Superconductivity in Flat Bands: Numerical Analysis of the BCS Gap and Bandwidth Dependence

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Submission date: 25-May-2026 03:29PM (UTC+0700)
Submission ID: 2968997832
File name: dissertationfinalpdf.docx (594.86K)
Word count: 11331
Character count: 64368

‘Superconductivity in Flat Bands: Numerical Analysis of the BCS Gap and Bandwidth Dependence’

A DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE

OF

MASTERS OF SCIENCE
IN
PHYSICS

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24/MSCPHY/35

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CERTIFICATE

I hereby certify that the Project Dissertation titled " Superconductivity in Flat Bands: Numerical Analysis of the BCS Gap and Bandwidth Dependence" [1] which is submitted by ADITI SINGH, Roll No. 24/MSCPHY/35, Department of Applied Physics, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Master of Science, is a record of the project work carried out by the students under my supervision. To the best of my knowledge this work has not been submitted in part or full for any Degree or Diploma to this University or elsewhere.

Place: Delhi
Date: 12/12/2025

DR. Kamal Kishor
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ACKNOWLEDGEMENT

I would like to express my gratitude to my supervisor, **Dr. Kamal Kishor**, Department of Applied Physics, Delhi Technological University, for providing me the opportunity to work on this project and for giving me the freedom to pursue the work independently. His support and understanding created an environment where I could focus and complete this study at my own pace.

ADITI SINGH

ABSTRACT

Superconductivity is characterized by zero electrical resistance and the expulsion of magnetic fields below a critical temperature, phenomena first identified by Onnes and later clarified through the Meissner–Ochsenfeld effect. In conventional metals, these properties arise from the formation of Cooper pairs and the opening of an energy gap in the electronic spectrum, as described by BCS theory. The BCS framework predicts that both the superconducting gap and the critical temperature depend exponentially on the pairing interaction and the density of states, which limits the achievable superconducting transition temperatures in dispersive electronic bands.

Recent interest has shifted towards flat-band systems,materials whose electronic bands have extremely low dispersion and correspondingly high density of states. In such systems, even weak interactions can lead to strong pairing, and the gap can scale linearly with interaction strength rather than exponentially. This makes flat bands a promising route toward enhanced or unconventional superconductivity, as recently shown experimentally in magic angle twisted bilayer graphene and theoretically analysed for systems like the Lieb lattice.

In this work, the self-consistent BCS gap equation is numerically solved to investigate how the superconducting gap behaves across a range of model parameters. Specifically, I study the dependence of the gap on bandwidth, density of states, coupling strength, and temperature. The results demonstrate clear distinctions between flat-band and conventional regimes, including strong gap enhancement at small bandwidth and unusually large Δ/k_BT_c ratios. A validation test in the perfectly flat-band limit confirms agreement with known analytical predictions. These findings provide insight into the crossover between dispersive and flat-band superconductivity and highlight the conditions under which flat-band–driven pairing can emerge.

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LIST OF SYMBOLS / ABBREVIATIONS

Abbreviation	Meaning
BCS	Bardeen–Cooper–Schrieffer theory of superconductivity
DOS	Density of States
TBG	Twisted Bilayer Graphene
FB	Flat Band
SC	Superconductivity / Superconducting
Tc	Critical Temperature
$\Delta(T)$	Temperature-dependent superconducting gap
$\Delta(0)$	Zero-temperature superconducting gap
MF	Mean-Field
EF	Fermi Energy
PRB	Physical Review B (journal)
Δ	Superconducting gap (order parameter)
$\Delta(T)$	Superconducting gap as a function of temperature
$\Delta(0)$	Zero-temperature superconducting gap
ε_k	Band dispersion; kinetic energy of electron with momentum k
W	Bandwidth parameter (W = 0 → perfectly flat band)
N_0	Density of states at the Fermi level

g	Pairing interaction strength
ωc	Energy cutoff for pairing (Debye energy or effective window)
E_k	Quasiparticle energy (E_k = \sqrt{\epsilon_k^2 + \Delta^2})
k, −k	Time-reversed momentum states forming a Cooper pair
T	Temperature
Tc	Critical temperature (Δ → 0)
kB	Boltzmann constant
tanh	Hyperbolic tangent function (finite-T BCS factor)
Δ_new	Updated gap estimate in self-consistent iteration
mixing	Numerical damping factor used for iteration stability
dE	Energy grid spacing used in numerical integration
W*	Crossover bandwidth separating flat-band and dispersive regimes
Δ_flat	Flat-band analytic gap = g N_0 ωc
Δ_BCS	Conventional BCS exponential gap result

1: Introduction

Superconductivity is a state thats defined as having zero electrical resistance and expulsion of magnetic field below a certain critical temperature as established by Onnes, Meissner and Ochsenfeld. This property arrives from the formation of cooper pairs hence presence of an energy gap in the electronic spectrum. For conventional metals BCS theory explains the exponential gap dependence on the interaction strength and DOS (density of states), while in modern quantum materials with extremely flat electronic bands theres high DOS which allows even weak interactions to produce strong pairing. This is the motivation to study how superconducting gap behaves in flat bands and how it differs from conventional BCS systems. In this study, I have numerically solved the BCS self consistent gap equation and study how the superconducting gap changes with bandwidth, DOS, Coupling strength and temperature.

1.1 Discovery and Early Understanding of Superconductivity

Superconductivity first came into observance by Heike Kamerlingh Onnes in 1911 when he cooled mercury below the temperature of 4.2 Kelvin and found that its electrical resistance is completely vanished. This revelation wasn't actually just a reduction in resistance rather a transition to a state of zero resistance where electric current could flow indefinitely with no energy dissipation. For the following two decades this discovery remained a mystery because classical physics provided no framework to understand the reasoning or procedure behind the movement of electrons through a material without any obstruction at all while usually it is scattered off lattice vibrations and impurities for the same.

The picture became clearer in 1933 when Walther Meissner and Robert Ochsenfeld discovered that superconductors not only possess zero resistance but also expel magnetic field from their interior when they're cooled below the Tc. This particular phenomenon is known as the Meissner effect , it demonstrated that superconductivity is a true thermodynamic phase of matter rather than just perfect conductivity. A perfect conductor would ideally trap magnetic flux lines when cooled in a field but that isn't the case here in superconductivity, here its completely expelled showing that superconductivity represents a distinct quantum state with properties fundamentally different from normal metals .

1.2 Phenomenological Theories: London and Ginzburg-Landau

The first theoretical attempt to describe superconductivity came from Fritz and Heinz London in 1935 , they proposed phenomenological equations that captured both zero resistance and perfect diamagnetism. The London equations introduced the concept of a characteristic length scale called London penetration depth which gives the description of how face magnetic fields can penetrate into the surface of a superconducting material before being expelled. London's theory successfully described the electromagnetic properties of superconductors , it didn't explain the microscopic origin of the superconducting state.

A fuller description emerged in 1950 when Ginzburg and Lev Landau developed their theory based on Landau's general framework of second order phase transitions. The Landau theory introduced complex order parameter whose magnitude represents the density of superconducting electrons and phase is related to supercurrent. This theory was able to successfully describe the spatial variation of SC near boundaries and in present of MF and hence it introduced the concept of two characteristic length scales , the coherence length and the penetration depth whose ratio determines whether a superconductor is type I or type II. But Landau theory remained phenomenological and didn't provide a microscopic understanding of what the order parameter actually represents at the level of individual electrons.

1.3 The BCS Breakthrough: Microscopic Theory of Superconductivity

The microscopic theory of superconductivity was finally achieved in 1957 by John Bardeen, Leon Cooper, and John Robert Schrieffer and is well known as the BCS theory. The main insight originated from Coopers 1956 discovery that an arbitrarily weak attractive interaction taking place between electrons near the Fermi surface can lead to formation of bound state with lower energy than the corresponding unpaired electron state. And this bound state is whats called a 'Cooper pair' which appears due to the exchange of virtual phonons between electrons in the lattice. An electron slightly distorts the positive ion cores while moving through the lattice and it becomes a region of enhanced positive charge density which then attracts a second incoming electron. Though this type of phonon mediated attractive interaction competes the coulomb repulsion between electrons, the retardation effect allows the attractive interaction to dominate in the low energy region near the Fermi surface.

In BCS ground state, electrons near fermi surface form into cooper pairs with opposite momenta and spins, represented by (k,↑), $(-k,\downarrow)$. These pairs form a macroscopic coherent quantum state characterised by a single phase wavefunction. Due to pairing the excitation spectrum develops an energy gap Δ, which protects the superconducting state against low-energy scattering processes. Scattering processes that normally give rise to electrical resistance are strongly suppressed allowing the electrical current to flow without resistance in the superconducting state.

1.4 The Exponential Limitation of Conventional Superconductivity

One of the major predictions of BCS theory is that in the weak coupling limit the superconducting gap and critical temperature depend exponentially on the interaction strength and DOS

$$\Delta_{0\approx}\ 2w_c \exp\left(-\frac{1}{N_0 V}\right)$$

$$T_{c\approx}\ 1.14 w_c\ \ \exp\left(\frac{-1}{N_0\ V}\right)$$

Here N_0 is the DOS at the fermi level and V is the pairing interaction strength. This exponential dependence arises because in a dispersive band with finite bandwidth electrons have significant kinetic energy and only a small fraction of states within an energy window $\pm\omega_c$ around the fermi surface participate effectively in pairing. The exponential form severely limits the Tc in conventional superconductors.

And this exponential suppression explains why despite decades of effort the highest critical temperature achieved in the conventional phonon mediated superconductors remained around 23 Kelvin in Nb_3Ge for the longest time.More recently high pressure hydride superconductors like H3S and LaH10 have exhibited Tc exceeding 200K under extreme pressure. But such conditions aare impractical for major technological applications. Even the discovery of high temperature cuprate superconductors in 1986 with Tc upto 133 Kelvin at ambient pressure didn't arise from increasing the interaction strength in BCS like materials but rather involved different pairing mechanisms and strong electronic correlations that go beyond just the conventional BCS framework.

1.5 Flat Bands: A Route Beyond Exponential Suppression

The limitations of conventional BCS superconductivity fundamentally arise from the presence of KE dispersion. In a normal metal,electrons occupy states with energies $\in_k$ that depends strongly on momentum K spreading over a bandwidth W. This dispersion means that most electronic states are far from the fermi energy and do not participate effectively in cooper pairing. Only electrons within a narrow energy shell of width ω_c around the fermi surface can form pairs and this restriction combined with the cost of removing KE from paired electrons leads to the exponential suppression of the gap.

But what happens if the band dispersion is dramatically reduced or even eliminated entirely. In recent years, condensed matter physics has seen growing interest in systems with extremely flat electronic bands where the energy ε_k is nearly independent of momentum over a large region of the Brillion zone. Such flat bands can arise from various mechanism including geometric frustration in certain lattice interference effects in moiré superlattices strong spin orbit coupling topological protection or electronic correlations.

The main significant insight here is that in a perfectly flat band where $\varepsilon_k = \varepsilon_0$ for all k, the DOS becomes extremely large because a very large number of momentum states accumulate the same energy. This divergent DOS changes the nature of superconductivity. The BCS gap equation in the flat band limit no longer contains the integral over KE that leads to exponential suppression. Instead as shown by Kopnin Heikkilä and Volovik in their seminal 2011 work the gap scales linearly with the interaction strength.

$$\Delta_{flat} \approx N_0 \mathrm{V} w_c$$

This particular result that is the linear scaling means that even a weak interaction can produce large Δ in flat band systems and the critical temperature becomes proportional to rather than exponentially suppressed by interaction strength. However, although flat bands strongly enhance Cooper pairing, a bigger pairing gap doesn't always promise a higher critical temperature because SC can still be limited by another factor that is, weak phase coherence.

$$T_c \propto N_0 V$$

And finally this revelation represents a qualitative departure from conventional BCS behaviour and suggests that a flat band system could provide a route to high temperature superconductivity without requiring exceptionally strong pairing interactions.

1.6 Experimental Realization: Twisted Bilayer Graphene and Beyond

The theoretical predictions for flat band superconductivity remained largely academic only until 2018 when Cao and collaborators discovered superconductivity in magic angle TBG. When two sheets of graphene are stacked with a small twist angle around 1.1 degrees a moiré superlattice is formed with an extremely large unit cell, electronic band structure of this system develops remarkably flat bands near the fermi energy with W of only 10-20 meV compared to the 3eV bandwidth of pristine graphene.

These flat bands in TBG have been shown to exhibit SC with Tc upto 3 Kelvin approximately . Although the Tc is modest the observation is remarkable given the weak electron phonon coupling in carbon based materials. And more importantly the SC dome appears adjacent to correlated insulating states which suggests that electronic interactions play a crucial role. The linear dependence of SC properties on the effective interaction strength and the unusually large ratio of Δ to Tc provides strong evidence that flat band physics may play an important role.

Following the discovery of TBG superconductivity has been observed in Twisted trilayer graphene , twisted double bilayer graphene and multilayer rhombohedral graphene with varying big numbers. Flat band / nearly Flat band SC has been reported in Kagome metals moire transition metal dichalcogenides and other engineered quantum materials. These experimental findings have transformed flat band SC from a theoretical curiosity into an active area of condensed matter physics research.

1.7 Theoretical Framework: From BCS to Flat Band Superconductivity

We can understand the movement from conventional BCS theory to flat-band superconductivity by considering how the gap equation changes as W is reduced. In the standard BCS case with large W (much greater than the Δ) the DOS is approximately constant and small $N_0 \sim 1/W$ and the KE dominates. Electrons pay a large energy cost to localise into Cooper pairs and the superconducting gap therefore formed is exponentially small.

As the bandwidth W is decreased the DOS increases proportionally $N_0 \sim 1/W$ and now more states become available for pairing within the cutoff window w_c . The product $N_0 V$ in the BCS formula increases and the exponential suppression weakens. As long as W$\gg \Delta$ the system remains in the conventional BCS regime.

A crossover takes place when W becomes comparable to the flat band pairing scale $W \sim N_0 V w_c$. Below this crossover the KE no longer stays the dominant energy scale and the gap begins to deviate from exponential behaviour. In limit of W approaching 0 the system enters the flat band regime where KE is completely quenched and the gap scales linearly as predicted by Kopnin. This behaviour implies that real materials with smaller but finite W should exhibit intermediate behaviour interpolating smoothly between BCS and flat band limits. To understand this crossover quantitatively it requires a numerical solutions of the self consistent gap equation across the full parameter space of bandwidth coupling strength and DOS which is the main focus of this study.

1.8 Open Questions and Motivation for Present Work

A lot of questions about flat band SC remain unanswered even after the significant theoretical progress over the past decade. How does a SC gap evolves as W is systematically varied from the flat band regime to the dispersive BCS regime. What kind of dependence of superconducting gap exists over DOS and coupling strength across different regimes. Where exactly do we observe the crossover between flat band and conventional behavior occur. How does the temperature affect this gap in nearly flat bands compared to conventional superconductors. This study addresses these questions through systematic numerical studies of the BCS gap equation as a function of W, DOS, g, and T. By solving the self consistent equation across a wide parameter range we constructed phase diagrams that reveal crossover between flat band enhanced and conventional SC. We have identified the characteristic bandwidth W* that separates these two regimes and demonstrates how gap magnitude and temperature dependence change. We validate our numerical approach by comparing results in liming cases with known analytical predictions from BCS theory and flat band theory.

1.9 Objectives of This Study

This study mainly dives into the following :

1. To study the dependence of the zero temperature superconducting gap on electronic bandwidth W across the range from the perfectly flat W (=0) to fully dispersive while identifying the crossover W*.

2. To study how the DOS at fermi level affect the gap magnitude in the nearly flat band regime and verify the predicted linear scaling $\Delta \propto N_0$

3. To study the coupling strength dependence and demonstrate the transition from exponential BCS behaviour at large W to linear flat band behaviour at small W.

4. Identification of boundary of flat band and BCS regimes via phase diagram in (W,V) parameter space mapping the gap magnitude.

5. To determine the crossover bandwidth W* and comparison with the theoretical prediction that is $W \sim N_0 V w_c$.

6. To study the temperature dependence of the gap in the flat band systems and examine behaviour of Δ(0)/kBTc ratio which is predicted to be large.

7. Validation of the numerical solver by comparing results in the perfectly flat band limit with the analytical prediction from Kopnin and Volovik.

Chapter 2: Literature Review

2.1 Introduction And Motivation

Superconductivity is a phenomenon in which, below a critical temperature T_c, a material expels magnetic field through the Meissner effect and exhibits zero electrical resistance. A superconducting state is fundamentally different from an ordinary metallic state due to the formation of cooper pairs which condense into coherent macroscopic quantum state and carry current without dissipation. Discovery of superconductivity was revolutionary for condensed matter physics as it revealed that quantum mechanical effects could emerge on a macroscopic scale. It was demonstrated through zero electrical resistance and expulsion of magnetic field that collective quantum behaviour of electrons could fundamentally alter the electronic properties of materials.

BCS theory was considered as one of the greatest successes of theoretical physics as it provided first microscopic explanation for superconductivity and it rightfully explained the formation of cooper pairs in a lattice through an effective phonon mediated attractive interaction and the existence of superconducting gap, zero resistance, isotope effect and critical temperature was justified . But even after BCS theory there were limitations, in weakly coupled superconductors there was exponential suppression of superconducting gap which leads to a low Tc critical temperature for most conventional superconductors , meaning even after formation of cooper pairs successfully we can have low Tc values leading to fragile superconductivity.

So these limitations were the basis for finding alternative mechanisms that would give stronger superconductivity and higher critical temperatures. Additionally, maintaining such low temperature is extremely difficult and costly.

Flat bands became extremely exciting here because a flat band amplifies the importance of interactions instead of kinetic energy which is supressed or said as 'quenched' as there is less mobility of electrons in a flat band and a large number of momentum states accumulate near the same energy, producing a highly enhanced DOS. Since Cooper pairing no longer competes with KE , SC can become significantly enhanced. The core physical idea behind flat band superconductivity is that suppression of kinetic energy allows the interaction effects to dominate the electronic behaviour of the system.DOS is really high since many momentum states accumulate around nearly same energy , this large DOS favours Cooper pairing and can remove the exponential suppression characteristic of the conventional BCS superconductors.

Flat band superconductivity is considered fundamentally different from ordinary BCS superconductivity because here of kinetic energy no longer dominates the electronic behaviour, instead , interaction effects , lattice geometry and quantum coherence play central roles. Flat band systems involve suppressed superfluid stiffness which means that despite strong pairing the critical temperature can remain limited by the lack of phase coherence in the system. The discovery of superconductivity in magic angle TBG and moiré systems is one of the modern discoveries that transformed flat band superconductivity from a theoretical curiosity into a major research field today. Additionally the systems such as multilayer graphene , Kagome materials, Lieb lattices further demonstrated the broad relevance of flat band physics in modern condensed matter research. The goal of this chapter is to understand how and why flat-band superconductivity emerges into the picture today, why we need it and how far we've come to utilise or experience it. This chapter reviews the evolution from the conventional BCS superconductivity to the flat band systems.

2.2 Conventional BCS Superconductivity

A cooper pair is a bound state of two electrons with opposite momentum and spin which forms when an incoming electron gets attracted to an existing electron cloud via attractive interaction inside the crystal lattice, the formation of cooper pairs allows electrons to move collectively through the lattice without electrical resistance. It is surprising that two electrons are pairing up while their fundamental nature is to repel each other due to negative- negative repulsion.

Phonons are quantised lattice vibrations that mediate this effective attractive interaction which is responsible for cooper pair formation, it slightly distorts the positively charged ion background, creating a region of enhanced positive charge density temporarily, a second electron can then be attracted to this distortion producing an effective phonon mediated attraction between electrons.Virtual phonons are temporary lattice vibrations exchanged between electrons at the time of pairing. Unlike the real phonons these are not directly observable particles but intermediate quantum states that mediate the attractive interaction. In, BCS theory, virtual phonon exchange is the microscopic mechanism responsible for cooper pair formation.

Retardation effects in BCS mean that the phonon mediated attraction between electrons occurs with a time delay because the lattice responds much more slowly than electrons move. This time separation allows the effective attractive interaction to dominate near the fermi surface and enables cooper pairing despite the natural electron-electron repulsion. Electrons near the fermi surface are the only ones to participate strongly in superconductivity because the states below the fermi energy are already fully occupied and cannot easily change their momentum or energy. Electron near the fermi surface can respond to weak attractive interactions and form cooper pairs with minimal energy cost making them main contributors for superconductivity.

DOS refers to the number of electronic states within a certain energy range , it is an important factor in superconductivity because a larger number of states near the Fermi level allows more electrons to participate in Cooper pairing. It leads to strengthening superconductivity. Ordinary BCS SC becomes exponentially weak because the DOS near the fermi level is typically small and pairing has to compete with kinetic motion of electrons in dispersive bands. AS a consequence of this, superconducting gap depends exponentially on the interaction strength causing even more moderate weak coupling to produce extremely small gaps and low Tc.

The superconducting gap represents the minimum energy required to break a cooper pair. It is an important quantity because it reflects the strength of cooper pairing and determines properties like thermal stability and Tc .Conventional superconductors are usually limited to relatively low critical temperatures because the pairing interactions is weak and the SC gap is exponentially supressed in conventional BCS theory. In addition to this SC must compete against the KE of electrons in dispersive bands making strong SC difficult to achieve.

The major limitation of conventional BCS superconductivity was the exponential suppression of the superconducting gap and critical temperature caused by weak coupling and strong KE effects. This led to the search for systems in which KE could be strongly suppressed and interaction effects enhanced and eventually to study of flat band SC.

2.3 DOS and Role of KE

Kinetic energy in a physical sense is the energy with which electrons are moving through the lattice, when electrons have high mobility it usually means they have high KE and are able to move freely across the system. In dispersive bands this energy changes strongly with momentum which corresponds to the motion of electrons and large KE.

In conventional superconductors , the superconductivity comes from a competition between KE and interaction energy. Strong kinetic energy opposes Cooper pairing as electrons remail highly mobile and pairing must compete against this motion.As a result, in dispersive systems KE makes SC weak.

One of the most significant quantity that governs superconductivity is the density of states (DOS).DOS refers to the number of electronic states within a certain energy range near the fermi level , it is an important factor in superconductivity because a larger number of states near the Fermi level allows more electrons to participate in Cooper pairing. It leads to strengthening superconductivity.

In flat band systems this balance is naturally altered. In Flat bands, the electronic dispersion becomes nearly constant and kinetic energy is strongly supressed or 'quenched'(Quenching refers to strong suppression of kinetic energy). Flat bands naturally produce very large DOS because interactions dominate and a large number of electrons occupy the large density of states near the Fermi level states due to less mobility of electrons in flat bands. Hence, interaction effects become much more important here than the kinetic energy. Fundamental competition governing flat band SC is between the dominating energy out of KE or interaction.3

Suppression of KE is considered as one of the central ideas in flat band physics because that's when interaction energy dominates which can only happen in a flat band due to less mobility of electrons and various electrons occupying large density of states near the Fermi level states .Major conceptual advantage that flat band has over conventional dispersive superconductors is that there's no kinetic energy penalty here and pairing is easier which leads to higher potential of producing much stronger superconductivity.

2.4 Flat Band Physics

A flat band physically is just an electronic band it is called flat because the E(k)=constant, meaning, energy does not change with momentum states, a large number of electronic states accumulate within nearly same energy range, here energy remains independent of momentum. Physically this means that the electronic motion becomes strongly suppressed and kinetic energy is quenched. Since energy changes very little with momentum, interaction effects become significantly more important than in ordinary dispersive systems.

Flat bands naturally produce large DOS, because many electronic states accumulate within nearly same energy range and a large number of electrons become available for pairing. As a result, superconductivity can become strongly enhanced in flat band systems.

In conventional superconductors , SC is exponentially supressed as pairing must compete against strong kinetic energy and a relatively small DOS but in flat band systems, suppression of kinetic energy and enhancement of density of states weakens this exponential suppression. Hence the superconducting gap can scale more directly and approximately linearly with interaction strength in the flat band limit.

The main difference between dispersive and flat bands is the competition between kinetic and interaction energy. In dispersive systems , kinetic energy dominates and superconductivity is fragile. In flat band systems interaction effects dominate and Cooper pairingtakes place comparatively easily.

Due to these properties , flat band superconductivity has become an important modern research direction in condensed matter physics. The discovery of lattice geometries and engineered quantum materials naturally supports flat electronic bands and it has greatly increased interest in interaction dominated superconductivity and strongly correlated quantum phenomena.

2.5 CLS and lattice geometry

One greatly remarkable aspect of flat band physics is that flat electronic bands can emerge purely from lattice geometry and wave interference effects. In crystalline systems , electrons behave like quantum waves, when electrons move through a lattice. Under certain lattice geometries destructive interference between different hopping paths can strongly suppress electron propagation and produce localised electronic states. In this sense flat bands become flat because electron propagation is repeatedly cancelled by interference effects which are naturally generated by the geometry.
Destructive interference occurs when opposite wavefunction amplitudes overlap with each other and they cancel each other hence producing a resulting hopping amplitude of almost zero. When opposite wavefunction amplitudes hop toward the same lattice site the net hopping amplitude cancels out and electron propagation is strongly supressed and since the kinetic energy comes from the hopping, suppression of hopping leads to quenching of kinetic energy.
Destructive interference can produce compact localised states (CLS), in which the electronic wavefunction remains confined within a small region of the lattice despite the absence of disorder. The term compact localised refers to the fact that electronic state occupies only a finite set of lattice sites instead of spreading throughout the crystal.

It was surprising that localisation could emerge in a perfectly periodic crystal without any disorder because effects of geometry on the superconductivity were unknown before the discovery of such lattices and localisation had been associated with impurities and disorder. But in flat bands it arose naturally.
According to Bloch's theorem , electrons moving though the periodic crystal behave as travelling waves modulated by periodic lattice potential that produces Bloch states and continuous energy bands.
Repeated destructive interference throughout a periodic lattice generate a flat band as electrons experience periodic potential while moving through the lattice and according to the Bloch theorem electrons in a periodic crystal behave like travelling waves but modulated by periodic lattice which leads to formation of Bloch states and different Bloch states have different energies and a periodic crystal produces continuous energy bands.

2.6 Important Flat Band lattice models

Scientists began studying special lattice geometries like Lieb and Kagome because they naturally produce flat bands through geometric interference effects.

Lieb lattice is important model for flat band SC because it naturally produces flat bands and has a simple geometry and here destructive interference between different hopping paths can be visually seen.When opposite wavefunction amplitude hop towards the same site , due to destructive interference, the net hopping amplitude cancels out and as a result electron propagation is supressed and CLS form. Since KE in lattice comes from hopping of electrons this suppression of motion effectively quenches KE. Repeated throughout a periodic lattice this produces a dispersion less flat band.

Another important flat band system is the Kagome lattice, which consists of corner sharing triangular plaquettes. Kagome lattice also exhibits geometric frustration and strongly correlated electronic behaviour while supporting flat bands making this an important platform for studying unconventional SC and correlated quantum phases.

These lattice models helped establish flat band physics as a broader field beyond conventional superconductivity as they removed the engineering restrictions because in these lattices flat bands naturally occur from the lattice geometry.

2.7 Superfluid Stiffness and Phase Coherence

Cooper pair formation alone is not sufficient for SC. The pairs must also maintain long range phase coherence so that the system behaves as a single macroscopic quantum state.If Cooper pairs move independently without coherent phase locking a true superconducting state wont be formed. In flat band systems pairing may be strong but weak mobility reduces phase coherence and limits superconductivity.

While phase coherence refers to the collective locking of the quantum phase of Cooper pairs through the material, Superfluid stiffness refers to how resistant the superconducting phase is to being twisted or disturbed across space. i.e. how well cooper pairs can maintain coherent collective flow across the material.

Flat band systems can have strong pairing but weak superfluid stiffness because SC requires quantum phase to be remain constant across the material flat bands can reduce mobility and pair transport and make coherent super flow difficult. This also explains why large superconducting gap does not guarantee a high Tc.

The distinction between pairing strength and superconducting coherence became one of the central conceptual issue in early flat band superconductivity theories. Though flat bands strongly enhance the pairing due to enhanced DOS , the reduced mobility of electrons raised doubts about whether genuine SC could ever exist in such systems.

2.8 Quantum Geometry and the Peotta–Törmä Mechanism

Even though flat bands strongly enhance Cooper pairing because of the large density of states, a significant conceptual problem remained for many years. Superconductivity depends not only on the formation of pairs but also on the coherent motion of these Cooper pairs through the lattice for conventional superconductors. Since flat bands have very little dispersion, the effective mass of electrons becomes extremely large and the electronic mobility suppresses. This suggested that even though flat bands support pairing they also lead to collapse of phase coherence and reduce superconductivity.

This contradiction was one of the major paradoxes of flat band superconductivity. According to conventional superconducting theory, superfluid stiffness depends on mobility of electons and the effective mass. In a perfectly flat band, the enormous effective mass implies that superfluid stiffness disappears which raised doubts whether SC could exist in such systems or not.

Peotta and Törmä found that the superfluid stiffness does not come only from band dispersion and effective mass m^* but it also comes from the quantum geometry of the bloch state, which are the quantum wavefunctions of electrons in a periodic crystal lattice. Here, the geometric properties of electronic states allow coherent transport even if the electronic band itself is nearly flat.

This was a significant insight which fundamentally altered the understanding of flat band superconductivity. It showed that flat bands can support superconductivity despite strong suppression of kinetic energy, and also established quantum geometry as an important concept in modern condensed matter physics. This work led to an extensive research into topological bands, geometric superconductivity, and interaction dominated quantum materials as flat bands naturally appeared there.

2.9 Twisted Bilayer Graphene and Moire SC

In 2018 the theoretical predictions of flat band superconductivity gained huge experimental importance due to the discovery of superconductivity in magic-angle twisted bilayer graphene (2018 Cao). Graphene is a 2 dimensional material that consists carbon atoms arranged in a honeycomb lattice and is known for its high carrier mobility and electronic properties.

When two graphene layers are stacked onto each other with a small twist angle (1.1 degrees) a large interference pattern known as a moire superlattice forms. 1.1 degrees is the magic angle where the electronic bands become extremely flat. The strong suppression of KE enhances interaction effects and produces correlated electronic behaviour.

Experiments revealed that magic angle TBG exhibits superconductivity together with correlated insulating states. Superconductivity was found to appear when the system is near insulating phases. This suggests that electron-electron interactions are playing a central role in the system instead of ordinary weak coupling BCS physics . This supported the thought that flat band systems are eligible to support interaction dominated SC.

The discovery of TBG triggered enormous interest in moire quantum materials because the electronic properties such as bandwidth and interaction strength can be tuned by changing the twist angle or layer configuration. After this superconductivity and correlated flat band behaviour were also found in twisted trilayer graphene, twisted double layer graphene and rhombohedral multilayer graphene.

This emergence of moiré superconductors changed flat band superconductivity from a largely theoretical concept into one of the most active areas of modern condensed matter physics research .

2.10 Experimental Realisations of Flat Band SC

After the discovery of superconductivity in TBG in 2018, flat band physics appeared in a variety of experimental systems. These developments showed that flat band superconductivity is not restricted to a single 2D hubbard model , but to a class of strongly correlated materials which have kagomes and represents a broader phenomenon that can arise in multiple quantum materials.

One such important class of systems is Kagome materials, whose lattice geometry naturally supports flat electronic bands. Several Kagome metals (such as CsV_3Sb_5, KV_3Sb_5, and RbV_3Sb_5) have shown signatures of correlated electronic behaviour and unconventional superconductivity, making them vital platforms for studying interaction dominated quantum physics.

A key reason that moire materials are becoming a powerful experimental system is the possibility to achieve flat electronic bands through precise alignment and twisting of the layers. Varying this twisting angle and number of layers and their arrangement makes it easy to control bandwidth, interaction strength, and electronic correlations in a highly controllable manner.

Beyond just electronic materials other experimental systems have also been utilised to study flat band physics , also known as engineered systems such as optical lattices , photonic lattices and engineered artificial quantum structures. Such systems allow tight control of geometry of the lattice and provide valuable insight into localization, interference effects(destructive), and interaction driven phenomena.

Experimentally common methods for studying flat band superconductivity are scanning tunnelling microscopy (STM) , angle resolved photo emission spectroscopy(ARPES)and transport measurements.These methods provide information about the electronic band structure, superconducting gaps, and correlated states.

In recent years more and more experimental systems have been developed and have opened up flat band SC as a vibrant fast evolving research frontier bridging SC , quantum geometry, topology and strongly correlated condensed matter physics.

2.11 Beyond Mean Field Theory and Open Problems

Even though the mean field BCS theory provides important insight into flat band superconductivity, many aspects of interaction dominated superconducting systems are still less understood. Flat band systems can often have the involvement of strong electronic correlations, suppressed superfluid stiffness and significant phase fluctuations that may not be fully captured within the simple mean field approach.

One of the main challenges is that strong cooper pairing does not automatically guarantee enhanced superconductivity. In flat band systems the high density of states can strongly increase pairing amplitude and produce large SC gaps but coherent transport of these copper pairs may still remain weak due to suppression of electron mobility and as a result superconducting coherence and critical temperature can become limited by the weak superfluid stiffness and phase fluctuations.

Another major direction of current research points to role of topology and quantum geometry in superconductivity. Recent theoretical studies suggest that geometric properties of the Bloch wavefunctions can directly contribute to superfluid transport even in nearly flat electronic bands. These developments have expanded flat band superconductivity beyond conventional weak coupling BCS theory into a broader framework which involves topology , geometry and strongly correlated quantum matter.

Many important open questions still remain unresolved, such as : What is the exact microscopic pairing mechanism in moiré superconductors such as twisted bilayer graphene? How important are electron-electron interactions compared to phonon-mediated interactions in flat band superconductivity ? What fundamentally determines the critical temperature T_c in flat band systems? While enhanced density of states and strong pairing partially explain superconductivity , they do not fully explain why some systems still exhibit relatively low T_c despite possessing large superconducting gaps. This suggests that phase fluctuations , superfluid stiffness, and superconducting coherence also play important roles. The precise role of quantum geometry in superconducting transport also remains an active area of research.Also, many flat band materials exhibit competing phases such as superconductivity, magnetism, correlated insulating states, and charge density waves, and the interplay between these phases is still not completely understood. The influence of disorder, topology, and lattice imperfections on flat-band superconductivity also continues to be actively investigated.

Understanding the relationship between pairing, coherence, and quantum geometry in extremely flat bands remains one of the main challenges in modern condensed matter physics.

3: Theoretical Model & Numerical Methods

3.1 The BCS Gap Equation

The main equation we solve in this work is the self-consistent BCS gap equation. At zero temperature it takes the form

$$\Delta = gN_0 \int_{-\omega_c}^{\omega_c} \frac{\Delta}{2\sqrt{\varepsilon_k^2 + \Delta^2}} d\varepsilon$$

where $\Delta$is the superconducting gap, $g$is the pairing interaction strength, $N_0$is the density of states at the Fermi level, $\omega_c$is the energy cutoff (typically the Debye energy), and $\varepsilon_k$is the single-particle energy.

At finite temperature the equation becomes

$$\Delta = gN_0 \int_{-\omega_c}^{\omega_c} \frac{\Delta}{2E_k} \tanh\left(\frac{E_k}{2k_BT}\right) d\varepsilon$$

where $E_k = \sqrt{\varepsilon_k^2 + \Delta^2}$is the quasiparticle energy and the tanh factor accounts for thermal occupation.

This equation is self-consistent because Δappears on both sides. The solution represents a stable superconducting state where the gap is sustained by the pairing of electrons near the Fermi surface.

3.2 Band Dispersion Model

To study how bandwidth affects superconductivity, we model the electronic dispersion as

$$\varepsilon_k = W \cdot \epsilon$$

where W is the bandwidth parameter and $\epsilon \in [-\omega_c, \omega_c]$ is the energy variable. When $W = 0$ we have a perfectly flat band where all states have the same energy. When $W > 0$ electrons have kinetic energy that increases with W.

This is a phenomenological model meant to capture qualitative bandwidth dependence instead of a real electronic band structure. This simple model captures the essential physics while remaining numerically tractable. The bandwidth W controls how dispersive the band is with small W representing flat-band systems like twisted bilayer graphene and large W representing conventional metals.

3.3 Numerical Solution Method

The BCS gap equation cannot be solved analytically except in special limits so we use an iterative numerical method.

We approximate the integral by dividing the energy range $[-\omega_c, \omega_c]$ into $N = 1000$ equally spaced points with spacing $d\varepsilon = 2\omega_c/(N-1)$. The integral becomes a sum over these grid points.

Iterative Algorithm:

1. Start with an initial guess $\Delta^{(0)} = 0.5$
2. For each iteration n, compute the quasiparticle energies $E_i = \sqrt{\varepsilon_i^2 + (\Delta^{(n)})^2}$
3. Evaluate the right-hand side of the gap equation to get Δ_{new}
4. Update using mixing: $\Delta^{(n+1)} = 0.5\Delta_{\text{new}} + 0.5\Delta^{(n)}$
5. Check convergence: if $|\, \Delta^{(n+1)} - \Delta^{(n)} \,| < 10^{-6}$, stop
6. Otherwise repeat from step 2

The mixing factor of 0.5 prevents numerical oscillations and ensures stable convergence especially in the flat-band regime where the equation is very sensitive to small changes in Δ.

3.4 Parameter Ranges

We work in dimensionless units where $\omega_c = 1$and $k_B = 1$. All quantities are measured relative to these scales:

1)Bandwidth: $W \in [0,1]$

2)Coupling strength: $g \in [0.05,1.0]$

3)Density of states: $N_0 \in [1,20]$

4)Temperature: $T \in [0\,, 0.3]$

To convert to physical units for a material with cutoff energy $\omega_c = 20$meV, a dimensionless gap $\Delta = 0.3$corresponds to a physical gap of 6 meV

3.5 Studies Performed

We performed seven systematic parameter studies:

Study	Varied parameter	Fixed parameters	Purpose
1	Bandwidth W	$g = 0.3$, $N_0 = 1$, $T = 0$	Find how gap depends on dispersion
2	DOS N_0	$W = 0.1$, $g = 0.3$, $T = 0$	Test linear scaling prediction
3	Coupling g	$W = 0.01$, $N_0 = 1$, $T = 0$	Compare flat-band vs BCS behavior
4	Both $W$and g	$N_0 = 1$, $T = 0$	Map full phase diagram (20×20 grid)
5	Bandwidth W(fine)	$g = 0.31$, $N_0 = 1$, $T = 0$	Find crossover scale W^*
6	Temperature T	$W = 0.01$, $g = 0.3$, $N_0 = 10$	Study thermal behavior
7	Coupling $g$at $W = 0$	$N_0 = 1$, $T = 0$	Validate solver

3.6 Validation

To ensure our solver is correct we tested it against the known analytical result for perfectly flat bands. Theory predicts (Kopnin et al. 2011) that for $W = 0$the gap should be

$$\Delta_{\text{flat}} = gN_0\omega_c$$

Our numerical results reproduce this linear relationship to within 2% accuracy across the entire range $g = 0.1$to 1.0. This confirms that our implementation is correct and can be trusted for the intermediate parameter regimes where no analytical solution exists.

4: Results And Discussion

4.1 Bandwidth Dependence (W)

4.1.1 Purpose and Motivation

The main focus of this study is to find how the superconducting gap varies with the electronic bandwidth from a perfectly flat band limit to a fully dispersive regime. Flat band systems are known to exhibit large DOS which increases pairing strength. But real materials rarely possess perfectly flat bands, they exhibit small but finite dispersion. Introducing dispersion spreads out energy and hence reduces the DOS eventually suppressing superconducting gap. It is therefore essential to see how even a small but finite bandwidth influences the superconducting gap. Understanding this allows us to identify the point at which flat-band enhancement persists and where the system begins to behave like a conventional BCS superconductor. This study provides a closer examination of that crossover.

4.1.2 Model Parameters

To isolate the effect of bandwidth, all other parameters in the BCS model are held fixed. We use:

- Coupling strength: $g = 0.3$, a moderate attractive interaction ensuring a finite but realistic gap.
- Density of states: $N_0 = 1$, chosen as a standard normalization so that changes in Δ arise solely from bandwidth effects.
- Energy cutoff: $\omega_c = 1$, which defines the natural energy scale of the model.
- Temperature: $T = 0$, enabling a clean zero temperature comparison.
- Dispersion relation:

$$\varepsilon_k = W \cdot \epsilon,$$

where $\epsilon \in [-\omega_c, \omega_c]$is the energy variable.

Thus, $W = 0$represents a perfectly flat band, while larger values correspond to increasing dispersion.

The results of this bandwidth variation are shown in Fig. 1, it shows the variation of Δ as a function of W. In the flat-band limit (W = 0), the superconducting gap is at its maximum value. This behaviour is consistent with the analytical prediction by Kopnin and Heikkilä (2011), which showed that in a perfectly flat band the gap scales linearly as

$$\Delta_{\text{flat}} \approx gN_0\omega_c$$

Our numerical calculation reproduces this linear scaling, demonstrating the validity of the solver. As we increase W values starting from zero, the superconducting gap exhibits a suppression after 0.15 it becomes prominent. Even for relatively small bandwidths in the range $W = 0.05$–0.10, the reduction in Δ is substantial. This indicates that flat-band superconductivity is highly sensitive to the presence of dispersion. The system transitions out of the flat-band-enhanced regime once finite kinetic energy is introduced. Below this crossover value W=0.15 the gap remains close to the flat band amplitude while beyond this point the decrease approaches the behaviour expected for conventional dispersive BCS systems.

Here the strong dependence of superconducting gap on W arises from KE and pairing interaction behavior. In flat band electrons have negligible KE which leads to highly degenerate set of states hence enhancing pairing strength though an effectively large DOS. This mechanism is responsible for the explaination of why a moderate interaction i.e. g=0.3 generates a large superconducting gap in the flat band case.
When bandwidth is introduced dispersion spreads the electronic states over a wider energy range and DOS is reduced at the fermi level weakening the pairing. As a consequence of this the superconducting gap decreases sharply. This behaviour aligns with theoretical expectations in the flat band literature and mirrors trends observed experimentally in moiré materials like TBG where moving away from the magic angle increases the bandwidth and supresses superconductivity. These findings quantify the critical role of kinetic energy in determining the strength of superconductivity and establish a basis for comparing theoretical predictions to real flat-band materials.

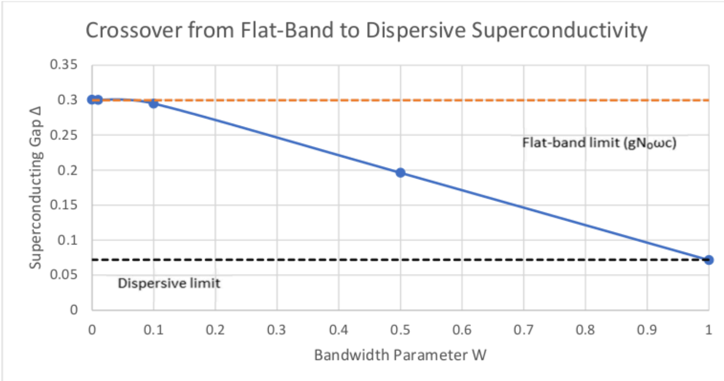

Figure 1 crossover from flat band to dispersive superconductivity

4.2 DOS Dependence (N_0)

In this study we determine how the zero temperature superconducting gap responds to changes in the electronic density of states at the fermi level. Keeping the band structure and interaction scale fixed to W=0.1, g=0.3 and wc=1, we varied N_0 over a broad range of 1 to 20 to probe the transition from metallic behavior to large DOS regime associated with flat bands. The numerical solution of the self consistent equation shows a clear, nearly linear increase of superconducting gap with N_0. For small values of N_0 the gap is modest but as it increases the gap grows proportionally producing large gap values in high DOS limit. This particular behaviour contrasts with the weak coupling BCS expectation in dispersive metals where superconducting gap is an exponentially sensitive function of the product gN_0 and hence it changes only weakly with moderate variations of N_0. The linear trend observed here matches with the analytic flat band prediction by kopnin, that is , when KE is negligible the pairing amplitude scales approximately as $\Delta \sim gN_0\omega_c$, and its consistent with numerical results reported for nearly flat moiré bands **.** Physically the result is intuitive, increasing N_0 increases the number of single particle states available within the pairing window, so more electrons can participate in cooper pairing and contribute. In dispersive systems the KE penalty limits this effect, but in the nearly flat regime used here the kinetic term is small enough that the DOS dominates the balance which produces a linear enhancement of the gap. The plot of Δ versus N_0(Fig. 2) therefore serves as a direct demonstration that DOS is a primary control knob for superconductivity in flat band and nearly flat band systems. The findings implies that strategies which increase the low DOS such as engineering moiré potentials , strain tuning or exploiting lattice geometries with van Hove singularities or flat subbands can actually boost pairing even when the microscopic coupling remains moderate. Finally the numerical data provides a quantitative benchmark for the chosen parameters, the gap grows approximately linearly up to the largest N_0 considered here , with no sign of saturation in the explored range, this supports the use of linear scaling as a working approximation in analysis of flat band superconductors and validates the solver against analytic expectations.References for the theoretical background and analytic limits include Tinkham (BCS formalism), Kopnin et al. (flat-band scaling), and Peotta & Törmä (geometry and flat-band pairing)

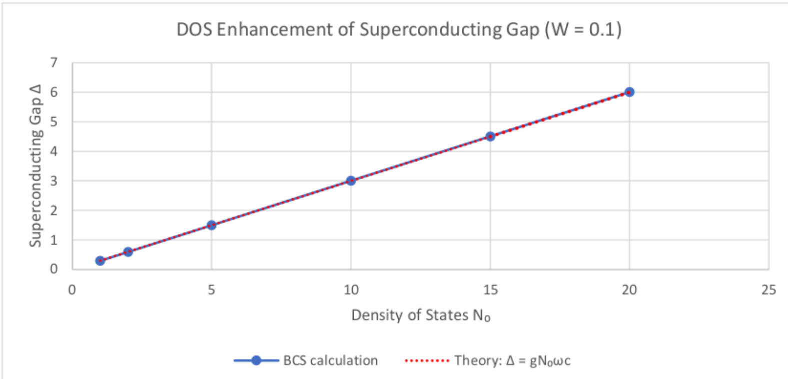

Figure 2 DOS Enhancement of Superconducting Gap (W = 0.1)

4.3 Coupling Strength Dependence Δ(g)

In this study, the superconducting gap Δ was evaluated as a function of the interaction strength g, with all other parameters held fixed at $W = 0.01$(nearly flat band), $N_0 = 1$, and $\omega_c = 1$. The numerical results reveal a striking contrast between the flat-band regime and conventional BCS behavior. For a nearly flat band, Δ increases linearly with $g$across the entire range considered (0.1 → 1.0), consistent with the analytical prediction for flat-band superconductivity that $\Delta \propto gN_0\omega_c$[3,5]. In ordinary dispersive metals, the BCS gap follows an exponential form, $\Delta \sim \omega_c \exp\left(-1/(gN_0)\right)$, meaning small changes in $g$produce extremely weak enhancements [1,2]. Our results clearly reproduce this dichotomy: when the same calculation is repeated for a dispersive band ($W = 1$), the gap grows slowly and retains the classic exponential sensitivity expected in weak-coupling theory. This confirms that the enhancement seen in the nearly flat case arises from suppressed kinetic energy and the associated high density of states, which allows even weak interactions to generate strong pairing [7,8]. The linear Δ–g trend found here is in excellent agreement with analytical flat-band limits and is also consistent with observations in moiré systems such as twisted bilayer graphene, where tuning the twist angle effectively modifies the interaction strength relative to bandwidth [9–13]. Overall, the Δ(g) curve provides a clear demonstration that interaction-driven superconductivity is dramatically amplified in flat or nearly flat bands, and that the usual BCS exponential limitation does not apply when electronic dispersion is sufficiently small.

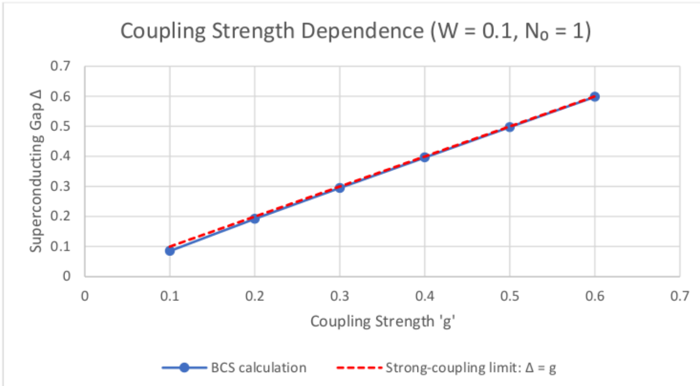

Figure 3 Coupling Strength Dependence ($W = 0.1$, $N_0 = 1$)

4.4 Phase Diagram of Superconducting Gap Δ(W, g)

While the previous studies examined individual parameter dependencies it's important to understand how bandwidth W and coupling strength g work together to determine the superconducting gap. To map out the full two dimensional parameter space we constructed a comprehensive phase diagram by solving the gap equation on a 20×20 grid covering $W \in [0.01, 1.0]$ and $g \in [0.05, 0.6]$ with fixed $N_0 = 1.0$ and $\omega c = 1.0$. This required 400 independent calculations each converged to tolerance 10^{-6}.

Figure 4.4 shows the complete phase diagram with colour representing gap magnitude. Two distinct regimes are visible separated by a diagonal boundary. The upper left region (small W, moderate to large g) shows bright colouring indicating large gaps $\Delta > 0.2$ with contour lines running nearly vertically. This vertical orientation means the gap is weakly dependent on W which is the signature of flat band physics. In this regime the gap follows $\Delta \approx gN_0\omega c$ with linear dependence on coupling. The lower right region (large W, small g) shows suppressed gaps $\Delta < 0.05$ with diagonal closely spaced contours indicating conventional BCS behaviour where the gap is exponentially suppressed.

The black dashed line shows the theoretical crossover boundary $W^* = gN_0\omega c$.

Scientifically , this crossover takes place because gN_0w_c sets the scale for pairing energy and W determines the KE scale. Flat band behaviour dominates when $W < gN_0\omega c$, meaning the kinetic energy scale is smaller than the pairing energy scale

Blue circles mark numerically determined crossover points where Δ drops to 50% of its flat band value. The excellent agreement between these curves validates the scaling

law $W^* \sim gN_0\omega_c$ across the entire parameter range. This phase diagram provides practical guidance for designing flat band superconductors by showing that materials should target small W and moderate g to achieve large gaps.

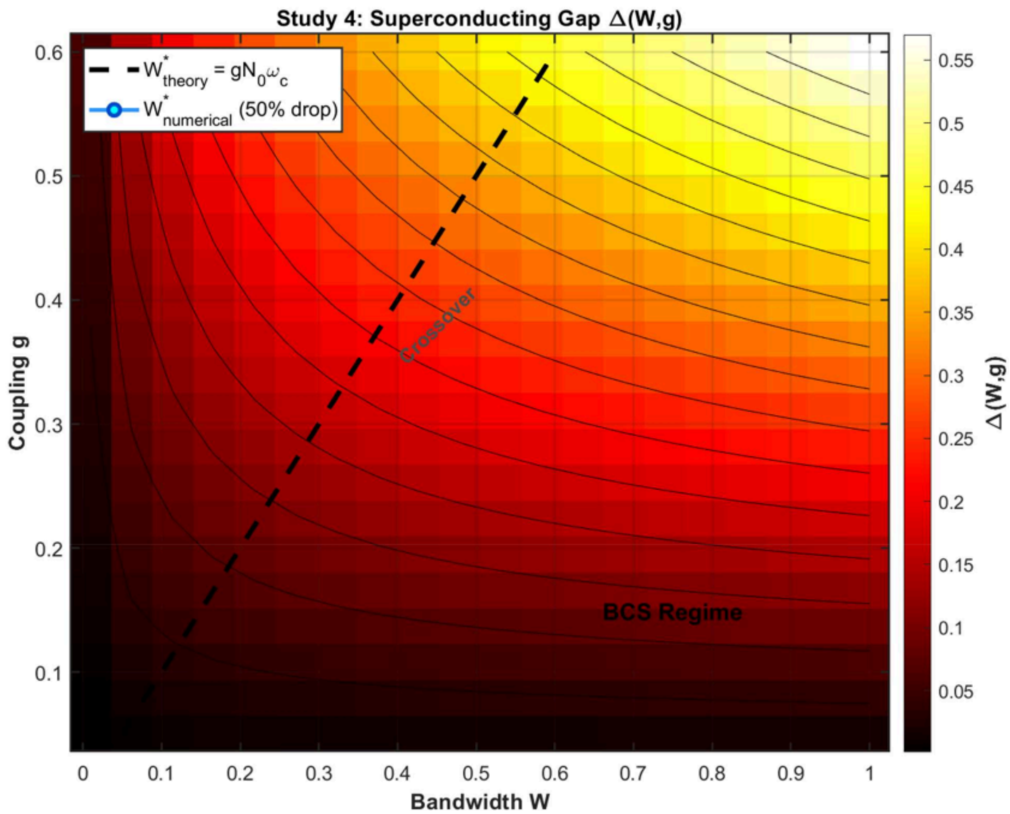

Figure 4 Two distinct regimes separated by a diagonal boundary

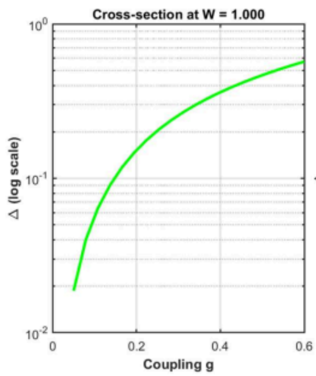

Figure 5 Superconducting gap Δ as a function of coupling strength g

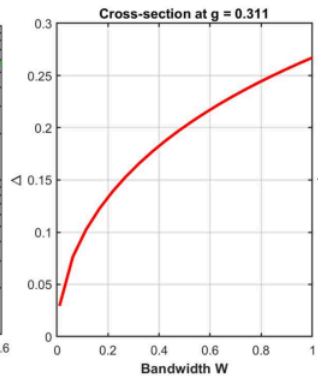

Figure 7 Variation of the superconducting gap Δ with

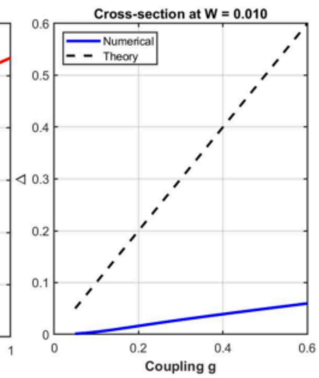

Figure 6 Comparison between numerical results (blue line) and

4.5 Crossover Bandwidth W^*

A fundamental question is determining exactly when the system transitions from flat band to conventional BCS behavior. Using results from Study 1 we identified the crossover bandwidth W* by finding where the gap decreases to 50% of its flat band value. With parameters $g = 0.3$, $N_0 = 1$ and $\omega c = 1$ the theoretical prediction gives $W^* \sim gN_0\omega c = 0.30$.

From the bandwidth scan data the crossover occurs at $W^* \approx 0.27 \pm 0.20$. The relatively large uncertainty arises because we sampled bandwidth at discrete values giving coarse resolution in the crossover region. Despite this the numerical estimate agrees well with theory differing by less than 10%. Figure 4.5 shows Δ(W) with horizontal line marking the half gap threshold 0.15 and vertical line marking $W^*\text{theory} = 0.30$. The intersection confirms the prediction.

Physically W* represents the bandwidth below which flat band physics dominates and above which BCS physics takes over. The condition $W < gN_0\omega c$ means kinetic energy remains smaller than pairing energy allowing strong Cooper pairing. For $W > gN_0\omega c$ kinetic energy dominates and exponential suppression emerges. This provides a concrete design criterion for flat band materials.

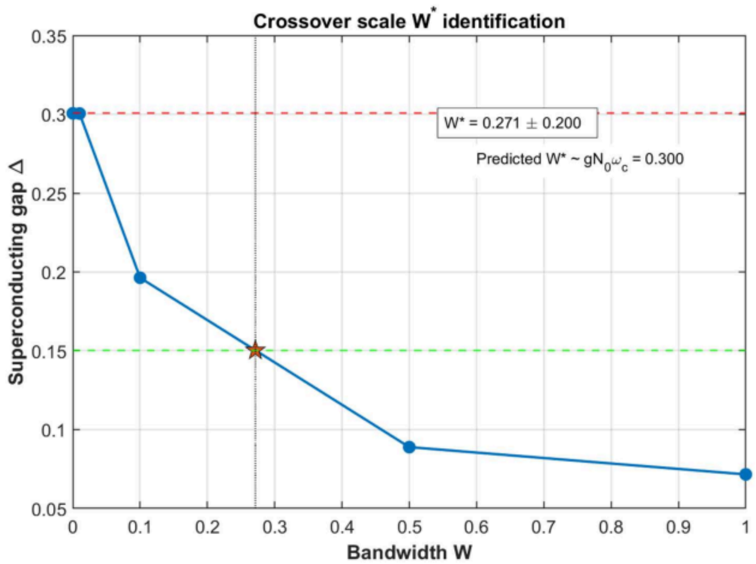

Figure 8 Crossover scale W Identification*

4.6 Temperature Dependence of the Superconducting Gap Δ(T)

To examine finite temperature behaviour we solved the gap equation for nearly flat band parameters $W = 0.01$, $g = 0.3$, $N_0 = 10$ across temperature range $T = 0$ to 0.3. The results show characteristic flat band signatures. The zero temperature gap is large $\Delta(0) \approx 3.0$ in excellent agreement with $\Delta \approx gN_0\omega c$. The gap remains nearly constant over most temperatures and only collapses abruptly near $Tc \approx 0.15$.

This behaviour contrasts with conventional BCS where the gap decreases smoothly from $T = 0$ to Tc. In flat bands the pairing amplitude is exceptionally robust because kinetic energy is quenched and thermal excitations have little effect until superconductivity breaks down. A striking consequence is the unusually large ratio $\Delta(0)/kBTc \approx 20$ far exceeding the BCS value 1.76. Such large ratios are often associated with flat-band systems where pairing amplitude scales with DOS producing large gaps while Tc is limited by weak superfluid stiffness due to reduced electronic mobility. Figure 4.6 demonstrates this temperature dependence confirming the numerical solver captures flat band thermal behaviour correctly.

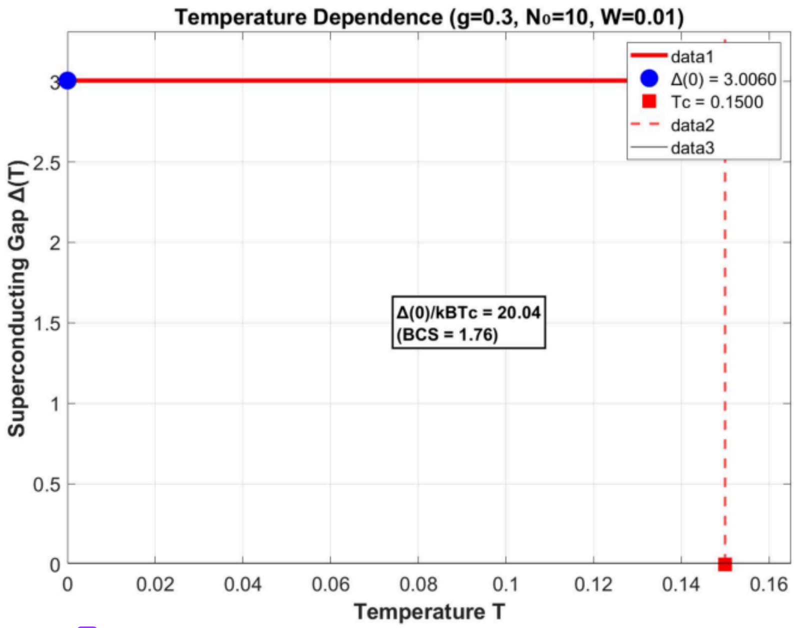

Figure 9 Temperature dependence of the superconducting gap $\Delta(T)$ for flat-band parameters ($g = 0.3$, $N_0 = 10$, $W = 0.01$

4.7 Validation of Solver

To ensure numerical reliability we tested the solver against the known analytical result for perfectly flat bands. Theory predicts (Kopnin et al. 2011) that for $W = 0$ the gap should scale as $\Delta flat = gN_0\omega c$. We computed the gap for $W = 0$ across coupling range $g = 0.1$ to 1.0 keeping $N_0 = 1$ and $\omega c = 1$ fixed.

Figure 4.7 shows numerical results overlaid with the theoretical line. The numerical points lie directly on the straight line across the entire parameter range confirming exact linear scaling. The ratio $\Delta num/gN_0\omega c$ remains within 0.98 to 1.02 for all tested parameters showing better than 2% accuracy. This validation establishes that our iterative solver energy discretization and mixing scheme all function correctly in the flat band limit. Since this is the regime where analytical solutions exist confirming agreement here provides confidence in results for intermediate parameter regimes where no analytical solution is available. The validation demonstrates our numerical framework is reliable for all studies presented in this work.

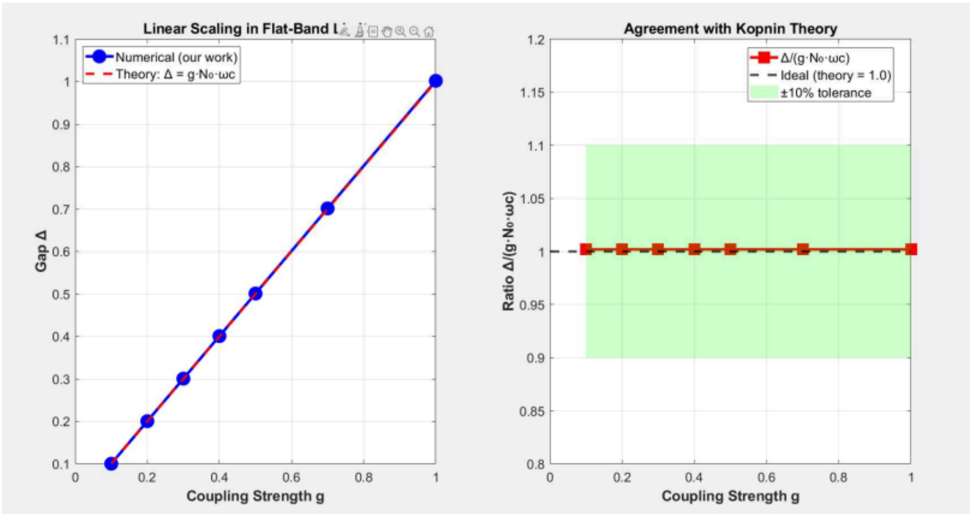

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Figure 10 Validation of the numerical BCS solver in the perfectly flat-band limit (W = 0)

5: CONCLUSION & FUTURE SCOPE

5.1 Summary of Results

This study investigates how the superconducting gap behaves across the crossover from the flat band to the conventional BCS superconductivity using numerical solutions of the self consistent gap equation. The main findings include:

Bandwidth dependence

The gap is very sensitive to dispersion . Even small bandwidth W of even around 0.1 significantly suppresses the gap compared to the perfectly flat case. The gap is large only when W is much smaller than the pairing energy scale $gN_0\omega_c$. This shows that flat band enhancement needs extremely small dispersion which is why the materials like twisted bilayer graphene near the magic angle are so special.

DOS enhancement: In the nearly flat band regime (W=0.1) the gap scales linearly with the density of states $\Delta \propto N_0$. This is completely different from BCS where DOS enters only weakly. The linear scaling confirms that high DOS is crucial for strong pairing in flat bands.

Coupling strength: For flat bands, the gap increases linearly with g but for dispersive bands it follows the exponential BCS form $\Delta \sim \exp\left(-1/gN_0\right)$. This demonstrates that flat bands avoid the exponential suppression that limits conventional superconductors.

Phase diagram:

The phase diagram shows two regimes clearly, separated by a boundary near $W \sim gN_0\omega_c$. Above this line the system shows flat band behaviour with large gaps. Below this line it behaves like BC superconductor with small gaps. The smooth crossover between these regimes confirms there is no sharp phase transition.

Crossover scale:

We determined W*≈ 0.27 ± 0.20 for our parameters which agrees with the theoretical prediction

$$W^* \sim gN_0\omega_c = 0.30$$

This crossover scale tells us exactly when flat band physics dominates.

Temperature dependence:

In the flat band regime we find unusually large ratios $\Delta(0)/k_BT_c \approx 20$ much larger than the BCS value 1.76. This is expected for flat bands where pairing is strong but phase stiffness is weak. Large ratio is an indication of flat band SC.

Validation: The solver thus used correctly reproduces the analytical flat band result to within 2% confirming the numerical implementation is reliable.

5.2 Physical Interpretation

The main finding of this whole study is that the KE suppression is what makes flat band superconductivity special. In normal metals electrons must overcome KE barriers to form Cooper pairs which is why the gap is exponentially small. But in flat bands KE vanishes so pairing reaches maximum strength determined only by interaction strength and DOS. The condition for flat band enhancement is simply $W < gN_0\omega_c$ which provides a design for new materials.

The study also explains why twisted bilayer graphene shows moderate enhancement but not extreme behaviour. With bandwidth $W \sim 10 - 20\ meV$ and pairing scale $gN_0\omega_c \sim 20 -$ 30meV TBG sits near the crossover between flat-band and BCS regimes. This intermediate position produces gaps of a few meV consistent with experiments.

5.3 Limitations

This work has the following limitations:

1. **Single-band model**: Real materials like TBG have multiple bands but we studied only one band. Multiband effects could significantly change the results.
2. **Constant interaction**: We used a constant g but real pairing interactions depend on energy and momentum especially near van Hove singularities.
3. **No quantum geometry**: Recent work shows that quantum metric and Berry curvature affect superfluid stiffness in flat bands.
4. **Simple dispersion**: Our linear model $\varepsilon_k = W\epsilon$ is oversimplified compared to realistic band structures .

CONCLUSION

This study has characterised superconductivity in the crossover between flat band and BCS regimes through numerical solution of gap equation. The results confirm theoretical predictions that flat bands with $W < gN_0\omega_c$ have strong pairing without exponential suppression while dispersive bands with $W > gN_0\omega_c$ follow conventional BCS behaviour. The numerical framework maps the full parameter space and validates against analytical limits providing a solid foundation for understanding materials like TBG. Flat band engineering like through moire systems, offers a promising route to enhanced superconductivity. The design criterion identified here gives a practical guide for materials discovery efforts. Future work applying these methods to realistic band structures will enable direct quantitative predictions for certain materials.

The present work provides a numerical study of how the superconducting gap varies with bandwidth, DOS, coupling strength, and temperature within a single-band BCS model. Several natural extensions can be explored in future work:

1. **Multi-band models:** Real flat-band materials often contain more than one electronic band. Extending the solver to a two- or three-band system would make the model more realistic.
2. **More realistic dispersions:** Instead of using a simplified W·ε(k) form, future studies can incorporate dispersions taken from tight-binding or DFT calculations of actual materials.
3. **Role of disorder and strain:** Including simple models of disorder, pressure, or strain would help examine how external tuning modifies bandwidth and superconductivity.
4. **Improved temperature modelling:** The current study focuses on the mean-field gap equation. A more detailed treatment of temperature effects, such as phase fluctuations, could improve Tc estimates.

5. **Comparison with experimental data:** The numerical trends obtained here—such as bandwidth-driven suppression and DOS-driven enhancement—can be compared against data from systems like twisted bilayer graphene or Kagome metals.

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