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AN ENHANCED DIVERGENCE-BASED DISTANCE MEASURE FOR INTUITIONISTIC FUZZY SETS WITH HESITATION INFORMATION AND ITS EXTENSIONS TO INTERVAL-VALUED AND PICTURE FUZZY ENVIRONMENTS

A Thesis Submitted
In Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE
IN
APPLIED MATHEMATICS

by

Vanita
(Roll No.: 24MSCMAT36)

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CANDIDATE'S DECLARATION

I, Vanita, hereby certify that the work presented in the thesis entitled *"An Enhanced Divergence-Based Distance Measure for Intuitionistic Fuzzy Sets with Hesitation Information and Its Extensions to Interval-Valued and Picture Fuzzy Environments"*, submitted in partial fulfillment of the requirements for the award of the degree of Master of Science in Mathematics, Department of Applied Mathematics, Delhi Technological University, is an authentic record of my own work carried out under the supervision of Dr. Dhirendra Kumar.

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CERTIFICATE BY THE SUPERVISOR

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VANITA

Abstract

Distance measures for intuitionistic fuzzy sets (IFSs) are central tools for pattern recognition, clustering, and multi-attribute decision making (MADM) under uncertainty. Although numerous divergence-based and geometric distance measures have been proposed in the literature, most of them either neglect the hesitation degree, which carries genuine epistemic information, or fail to discriminate between sets that exhibit equal membership-non-membership differences but distinct hesitation profiles. To overcome these limitations, this paper proposes a novel enhanced divergence-based distance measure D_L^+ for IFSs that incorporates an explicit hesitation term derived from a modified Kullback–Leibler divergence. The measure is constructed from a single, symmetric core function and is shown to satisfy all four axiomatic requirements of an IFS distance metric, namely boundedness, separability, symmetry, and monotonicity. The proposed measure is further extended to two important generalizations of intuitionistic fuzzy theory: a six-term version D_L^{IV+} for interval-valued intuitionistic fuzzy sets (IVIFSs) that fully exploits both the lower and upper bounds of membership, non-membership and hesitation intervals, and a three-component version D_L^P for picture fuzzy sets (PFSs) that handles positive, neutral, and negative memberships. Six classical benchmark cases and two innovation-management decision problems are recomputed entirely from scratch using the proposed measure. Comparative analysis with twelve existing measures shows that D_L^+ resolves the counter-intuitive ties that plague competing measures, distinguishes hesitation-sensitive cases that earlier divergence-based measures could not, and yields stable rankings in TOPSIS-based MADM. The results confirm that the enhanced measure is mathematically rigorous, computationally concise, and practically effective.

Keywords: Intuitionistic fuzzy set, interval-valued intuitionistic fuzzy set, picture fuzzy set, divergence measure, distance measure, multi-attribute decision making, innovation management.

Contents

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List of Symbols and Abbreviations

X	universe of discourse, $X = \{x_1, \ldots, x_n\}$
$\mu_A(x)$	degree to which x belongs to A
$\nu_A(x)$	degree to which x is excluded from A
$\pi_A(x)$	hesitation index, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$
$\mu^L_{\tilde{A}}, \mu^U_{\tilde{A}}$	endpoints of an IVIFS membership interval
$\eta_A(x)$	neutral membership grade (PFS setting)
$\rho_A(x)$	refusal grade (PFS setting)
$f(x, y)$	core kernel $(x - y) \ln \frac{1+x}{1+y}$
$L(A, B)$	modified Kullback–Leibler divergence between two IFSs
$D_L(A, B)$	divergence-based distance of Ju et al. (2020)
$D^+_L(A, B)$	enhanced IFS distance proposed in this work
$D^{IV+}_L(\tilde{A}, \tilde{B})$	proposed IVIFS extension
$D^P_L(A, B)$	proposed PFS extension
α^+_j, α^-_j	TOPSIS positive and negative ideal solutions
$T(x_i)$	TOPSIS closeness coefficient
IFS	intuitionistic fuzzy set
IVIFS	interval-valued intuitionistic fuzzy set
PFS	picture fuzzy set
IFV	intuitionistic fuzzy value
MADM	multi-attribute decision making
TOPSIS	Technique for Order Preference by Similarity to Ideal Solution
K–L	Kullback–Leibler

Chapter 1

Introduction

1.1 Background and Motivation

Uncertainty recurs throughout applied mathematics, and most acutely so in problems where the information is partial, qualitative, or filtered through expert judgement. Classical set theory leaves room for only two possibilities for an element relative to a set, namely full membership or full exclusion, which is too coarse for the situations met in practice, where belongingness is generally a matter of degree rather than a clean dichotomy. To accommodate such graded membership, [1] introduced fuzzy sets, in which each element carries a membership value $\mu(x) \in [0, 1]$. The framework has since found a home in control engineering, image analysis, pattern recognition, decision support, economic modelling and several other branches of computational intelligence.

Even so, an ordinary fuzzy set encodes its information through a single channel: once $\mu(x)$ is specified, the corresponding non-membership is read off automatically as $1 - \mu(x)$. While compact, this collapses two qualitatively different situations: the case in which the expert is positively confident that the object partly fails to belong, and the case in which the expert is simply uncertain about the object's status. To draw this distinction, [2] proposed the theory of intuitionistic fuzzy sets, in which both a membership grade $\mu_A(x)$ and a non-membership grade $\nu_A(x)$ are recorded, subject to the constraint

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

The leftover quantity $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is termed the hesitation degree; it captures the share of the assessment that the expert has neither committed to membership nor to non-membership. Hesitation is therefore not a mere algebraic remainder but a genuine information channel describing the uncertainty in the elicitation step.

Two extensions of this framework are likewise useful. [3] introduced interval-valued intuitionistic fuzzy sets, in which membership and non-membership are given as intervals rather than points, an appropriate model when the expert can only commit to a range. [4] introduced picture fuzzy sets, where positive, neutral and negative memberships are admitted, leaving the remainder to be read as refusal; this is well suited to settings such as voting or opinion analysis where neutrality is itself an active response.

A common task across all these frameworks is to compare two fuzzy objects. A classifier may compare an unidentified object with a prototype, a clustering algorithm may compare two data points, and a decision-making routine may compare an alternative with an ideal target. Such comparisons are carried out with a distance measure $D : \text{IFS}(X) \times \text{IFS}(X) \to [0, 1]$ obeying axioms such as boundedness, separability, symmetry and monotonicity. Similarity measures are then obtained from distance measures through $S = 1 - D$. The resulting tools play a central role in pattern recognition, clustering, image fusion and MADM methods such as TOPSIS [5] and VIKOR.

1.2 Existing Distance Measures and Their Limitations

A wide range of distance measures for intuitionistic fuzzy sets has been put forward. Early contributions include the Hamming and Euclidean families due to [6], Hausdorff-based constructions by Grzegorzewski, cosine-style measures by [7], transformation-based proposals by T. Y. Chen, and information-theoretic divergence measures, the most recent of which is the construction of [8].

While the literature is now sizeable, three difficulties keep reappearing. First, certain measures return a zero distance on pairs of IFSs that are demonstrably non-identical, which is at odds with what a distance measure is supposed to do. Second, formulae that look only at expressions in μ and ν can completely overlook changes in hesitation, even though hesitation is one of the principal reasons for working with IFSs in the first place. Third, some closed-form expressions become ill-defined at boundary values, owing either to zero divisors or to logarithms whose arguments collapse to zero. The benchmark problems used to compare IFS distances make these difficulties especially visible.

This dissertation takes innovation management as one practical setting where the foregoing issues matter. Innovation evaluations are typically expressed through linguistic criteria such as project design, deployment of new technology, application prospect and feasibility. Such assessments naturally accommodate hesitation, because experts are seldom fully certain. Two projects may then exhibit very similar membership and non-membership patterns while differing markedly in the amount of uncertainty attached to each. A distance measure that does not react to the hesitation component will therefore discard information that bears on the investment decision.

1.3 Problem Statement

The problem treated in this dissertation may be stated as follows. Given two intuitionistic fuzzy sets A and B on a finite universe X, we seek to construct a distance measure $D(A, B)$ such that

- (i) its values lie in $[0, 1]$;
- (ii) it vanishes precisely when A and B agree at every $x \in X$;
- (iii) it is symmetric in its two arguments;
- (iv) it respects inclusion chains in the suitable monotonic sense;
- (v) it uses the membership, non-membership and hesitation components directly;
- (vi) it remains well defined at boundary intuitionistic fuzzy values;
- (vii) it discriminates better on the standard benchmark cases; and
- (viii) it admits a natural extension to interval-valued intuitionistic fuzzy sets and to picture fuzzy sets.

1.4 Objectives

The principal objectives pursued in the dissertation are stated below.

1. To define an enhanced divergence-style distance measure for IFSs by attaching an explicit hesitation contribution to an information-theoretic kernel, and to prove the resulting measure satisfies the standard distance axioms.
2. To set the new construction alongside the divergence-based measure of [8], with a precise account of when the two measures coincide and when the proposed measure makes a strictly finer distinction.
3. To carry the construction over to interval-valued intuitionistic fuzzy sets and to picture fuzzy sets while keeping the same axiomatic scaffolding.
4. To embed the proposed distance in a TOPSIS-based MADM procedure and apply it to an innovation-management problem.

5. To benchmark the proposed measure against several representative existing measures on the classical benchmark pairs, examining its behaviour on counter-intuitive ties and on pathological boundary cases.

1.5 Contributions of the Dissertation

The dissertation's contributions are summarised as follows.

(i) Enhanced distance for IFSs. A new distance D_L^+ is constructed by applying the kernel $f(x,y) = (x-y)\ln\frac{1+x}{1+y}$ to all three IFS components. Section 3.5 establishes boundedness, separability, symmetry and monotonicity under strict inclusion. The link to the Ju–Yuan distance is expressed through a refinement proposition.

(ii) Interval-valued extension. Section 3.6 extends the construction to a six-term measure D_L^{IV+} for IVIFSs. The lower and upper bounds of membership, non-membership, and hesitation are all included, and the point-valued measure is recovered when the intervals degenerate.

(iii) Picture-fuzzy extension. A picture-fuzzy distance D_L^P is obtained by applying the same kernel to positive, neutral, and negative membership degrees. The construction treats the neutral component as a genuine information channel, not as an auxiliary term.

(iv) MADM procedure. A six-step TOPSIS procedure is developed using D_L^+ at the attribute level. The method is illustrated through a five-alternative and four-attribute investment example, with all intermediate computations reported.

(v) Numerical comparison. The proposed distance is compared with representative existing measures. The computations show that it separates benchmark cases that some earlier measures merge, avoids undefined boundary values, and gives a hesitation-sensitive ranking in decision-making examples.

1.6 Organisation of the Dissertation

Chapter 2 sets out the notions used throughout: fuzzy sets, intuitionistic fuzzy sets, interval-valued IFSs, picture fuzzy sets, the distance axioms, a selection of existing measures, and the Kullback–Leibler motivation behind the kernel we propose. Chapter 3 defines the enhanced measure, studies its relationship to the Ju–Yuan distance, proves its mathematical properties, and presents the IVIFS and PFS extensions. Chapter 4 develops the TOPSIS-based decision procedure and reports the comparative numerical study. Chapter 5 closes the dissertation and records limitations and directions for future work. A bibliography is included at the end.

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Chapter 2

Preliminaries and Literature Review

The present chapter pins down the notation and surveys the material called upon in the rest of the dissertation. It begins with fuzzy sets, intuitionistic fuzzy sets, interval-valued IFSs and picture fuzzy sets, then records the distance axioms used here, presents a selection of distance measures from the literature, and finishes by reviewing the divergence construction that underpins the measure we propose.

2.1 Fuzzy Sets and Their Generalisations

Definition 2.1 (Fuzzy Set, [1]). *Let X denote a non-empty universe. A fuzzy set A over X is described by*

$$A = \{\langle x, \mu_A(x)\rangle : x \in X\}, \tag{2.1}$$

where the map $\mu_A : X \to [0, 1]$ is its membership function.

The number $\mu_A(x)$ quantifies how strongly the element x is taken to be part of A. The classical case keeps only the two extreme values 0 and 1, whereas the fuzzy formulation permits intermediate grades.

Definition 2.2 (Intuitionistic Fuzzy Set, [2]). *An intuitionistic fuzzy set A over X has the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}, \tag{2.2}$$

where the maps $\mu_A, \nu_A : X \to [0, 1]$ are subject to

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \quad \text{for every } x \in X. \tag{2.3}$$

The associated hesitation degree is given by

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x). \tag{2.4}$$

When the universe is finite, $X = \{x_1, \ldots, x_n\}$, it is convenient to abbreviate

$$A_i = \langle \mu_A(x_i), \nu_A(x_i)\rangle,$$

and refer to A_i as an intuitionistic fuzzy value (IFV).

Definition 2.3 (Inclusion of IFSs). *Given two IFSs A, B over X, we write $A \subseteq B$ provided*

$$\mu_A(x) \leq \mu_B(x) \quad \text{and} \quad \nu_A(x) \geq \nu_B(x), \qquad \text{for every } x \in X. \tag{2.5}$$

Intuitively, the order asserts that, pointwise on X, B allows at least the membership of A while admitting no greater non-membership.

Definition 2.4 (Interval-Valued Intuitionistic Fuzzy Set, [3]). *An interval-valued intuitionistic fuzzy set A over X takes the form*

$$A = \{\langle x, [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]\rangle : x \in X\}, \tag{2.6}$$

with intervals lying inside $[0,1]$ and obeying

$$0 \leq \mu_A^U(x) + \nu_A^U(x) \leq 1. \tag{2.7}$$

Its hesitation interval is conventionally written

$$\pi_A(x) = [1 - \mu_A^U(x) - \nu_A^U(x),\ 1 - \mu_A^L(x) - \nu_A^L(x)]. \tag{2.8}$$

Definition 2.5 (Picture Fuzzy Set, [4]). *A picture fuzzy set A over X is the collection*

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle : x \in X\}, \tag{2.9}$$

where the three grades $\mu_A(x)$, $\eta_A(x)$ and $\nu_A(x)$ carry positive, neutral and negative meanings respectively, subject to

$$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1. \tag{2.10}$$

The leftover quantity $\rho_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ is called the refusal degree.

2.2 Axiomatic Distance and Similarity Measures

A distance measure attaches a real number to each pair of IFSs. The number is expected to be small for IFSs that resemble each other and large for those that differ markedly.

Definition 2.6 (Distance Measure for IFSs). *A map $D : \mathrm{IFS}(X) \times \mathrm{IFS}(X) \to [0,1]$ is termed a distance measure provided, for any IFSs A, B, C on X, the following four conditions hold:*

(D1) $0 \leq D(A,B) \leq 1$;

(D2) $D(A,B) = 0$ *exactly when* $A = B$;

(D3) $D(A,B) = D(B,A)$;

(D4) whenever $A \subseteq B \subseteq C$, *the inequalities* $D(A,B) \leq D(A,C)$ *and* $D(B,C) \leq D(A,C)$ *both hold.*

A similarity measure S is then linked to a distance measure via $S(A,B) = 1 - D(A,B)$. The present work concentrates on distance measures, since these fit naturally inside TOPSIS and related decision-making procedures.

Remark 2.1 (Metric versus distance). The word distance is used here in the standard IFS sense. The triangle inequality is not part of Definition 2.6. Therefore, an IFS distance measure need not be a metric in the strict metric-space sense.

2.3 Existing Distance and Similarity Measures

For ease of comparison we now list the measures used later for benchmarking. For brevity, write

$$\Delta\mu_i = \mu_A(x_i) - \mu_B(x_i), \quad \Delta\nu_i = \nu_A(x_i) - \nu_B(x_i), \quad \Delta\pi_i = \pi_A(x_i) - \pi_B(x_i).$$

The measure of Chen (1995) is

$$D_C(A,B) = \frac{1}{2n} \sum_{i=1}^{n} \left| (\mu_A(x_i) - \nu_A(x_i)) - (\mu_B(x_i) - \nu_B(x_i)) \right|. \tag{2.11}$$

The measure of Hong and Kim (1999) is

$$D_{HK}(A,B) = \frac{1}{2n} \sum_{i=1}^{n} \left| (\mu_A(x_i) - \mu_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) \right|. \tag{2.12}$$

The Li–Cheng (2002) measure, in the $p = 1$ case we shall need, reads

$$D_{DC}(A,B) = \frac{1}{n} \sum_{i=1}^{n} |\varphi_A(x_i) - \varphi_B(x_i)|, \tag{2.13}$$

where $\varphi_A(x_i) = (\mu_A(x_i) + 1 - \nu_A(x_i))/2$.

The similarity family of [9] yields the three distance variants D^E_{LS}, D^S_{LS} and D^H_{LS}. We refrain from reproducing their explicit forms here; the numerical values are taken up directly in Section 4.6. The Mitchell (2003) construction gives

$$D_{HB}(A,B) = 1 - \tfrac{1}{2}(\rho_\mu(A,B) + \rho_\nu(A,B)), \tag{2.14}$$

where ρ_μ and ρ_ν are L_p-style averages built from $|\Delta\mu_i|$ and $|\Delta\nu_i|$.

The measures of [6] are noteworthy because they use the hesitation degree explicitly. Two of the more commonly used variants are

$$D_{NH}(A,B) = \frac{1}{2n}\sum_{i=1}^{n}\big(|\Delta\mu_i| + |\Delta\nu_i| + |\Delta\pi_i|\big), \tag{2.15}$$

$$D_{NE}(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\big((\Delta\mu_i)^2 + (\Delta\nu_i)^2 + (\Delta\pi_i)^2\big)}. \tag{2.16}$$

Their accompanying ratio-style measure D_{SK} is also part of the comparison table of Section 4.6.

The distance of [10] is

$$D_{WX}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{|\Delta\mu_i| + |\Delta\nu_i|}{4} + \frac{\max\{|\Delta\mu_i|, |\Delta\nu_i|\}}{2}\right]. \tag{2.17}$$

The divergence-based proposal of [11] is

$$D_{VS}(A,B) = \frac{1}{2n\ln 2}\big(I_{\mathrm{IFS}}(A,B) + I_{\mathrm{IFS}}(B,A)\big), \tag{2.18}$$

in which

$$I_{\mathrm{IFS}}(A,B) = \sum_{i=1}^{n}\mu_A(x_i)\ln\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)} + \nu_A(x_i)\ln\frac{2\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}.$$

The cosine distance of [7] reads

$$D_Y(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n}\frac{\mu_A(x_i)\mu_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu_A(x_i)^2 + \nu_A(x_i)^2}\sqrt{\mu_B(x_i)^2 + \nu_B(x_i)^2}}. \tag{2.19}$$

The transformed-isosceles construction D_{JJ} of [12] and the Ju–Yuan divergence D_L of [8] round out the list of measures used in the numerical study.

These measures do not behave uniformly well across the benchmark cases. Score-based forms can collapse several distinct IFSs onto a single value; the cosine and entropy-style families may break down at zero boundary values; and the Ju–Yuan measure, although in many respects robust, does not treat hesitation as a separate term. Such observations motivate the construction laid out in the following chapter.

2.4 Divergence Measures and Kullback–Leibler Construction

The measure we propose is built on a regularised Kullback–Leibler-type kernel. We collect here the relevant information-theoretic notions.

Definition 2.7 (Shannon Entropy, [13]). *For a probability distribution* $P = (p_1, \ldots, p_k)$ *on a finite alphabet, the Shannon entropy is*

$$H(P) = -\sum_{j=1}^{k} p_j \log_2 p_j, \tag{2.20}$$

with the convention $0 \log_2 0 = 0$.

This quantity records the average uncertainty present in the distribution.

Definition 2.8 (Kullback–Leibler Divergence, [14]). *For probability distributions* P_1 *and* P_2 *on a finite alphabet, the Kullback–Leibler divergence of* P_1 *relative to* P_2 *is*

$$K(P_1, P_2) = \sum_{j=1}^{k} p_{1j} \log \frac{p_{1j}}{p_{2j}}. \tag{2.21}$$

The K–L divergence is non-negative and vanishes only when the two distributions coincide. It is not symmetric in its arguments, and it may diverge whenever a probability in the denominator equals zero. A regularised version that sidesteps this difficulty is

$$K_\lambda(P_1, P_2) = \sum_{j=1}^{k} (\lambda + p_{1j}) \log \frac{\lambda + p_{1j}}{\lambda + p_{2j}}, \quad 0 < \lambda \leq 1. \tag{2.22}$$

The corresponding symmetric quantity is

$$L_\lambda(P_1, P_2) = K_\lambda(P_1, P_2) + K_\lambda(P_2, P_1) = \sum_{j=1}^{k} (p_{1j} - p_{2j}) \log \frac{\lambda + p_{1j}}{\lambda + p_{2j}}. \tag{2.23}$$

Because IFS components lie inside $[0, 1]$, the choice $\lambda = 1$ yields the kernel

$$f(x, y) = (x - y) \log \frac{1 + x}{1 + y}, \tag{2.24}$$

which is the principal analytic object of the dissertation.

2.5 The Divergence-Based Measure of Ju et al.

The most direct predecessor of the measure we shall propose is the divergence-style distance of [8].

Definition 2.9 (Divergence between IFSs, [8])**.** *For any two IFSs A and B on X, the modified Kullback–Leibler divergence is given by*

$$L(A,B) = \sum_{i=1}^{n} \Big[(\mu_A(x_i) - \mu_B(x_i)) \ln \frac{1+\mu_A(x_i)}{1+\mu_B(x_i)} + (\nu_A(x_i) - \nu_B(x_i)) \ln \frac{1+\nu_A(x_i)}{1+\nu_B(x_i)} \Big]. \tag{2.25}$$

The distance measure associated with L is then

$$D_L(A,B) = \frac{1}{2n\ln 2} L(A,B). \tag{2.26}$$

The factor $1/(2n \ln 2)$ normalises the value into the unit interval. Ju *et al.* verify the four distance axioms and report improved behaviour relative to several earlier measures on the standard six-case benchmark.

For this dissertation, the relevant shortcoming is that D_L involves only the membership and non-membership terms. Hesitation enters the values only indirectly, through the identity $\mu + \nu + \pi = 1$. As a consequence, two pairs of IFSs may yield the same D_L value even when the changes in their hesitations are quite different. In the benchmark comparison of [8], Cases 1 and 2 both produce $D_L = 0.0107$; in Case 1, hesitation drops from 0.4 to 0.2, whereas in Case 2 hesitation stays at 0.3 in both IFSs. It is precisely this gap that the present work fills by introducing an explicit hesitation term.

2.6 Critical Summary of Gaps

The foregoing review points to three persistent gaps.

- **Zero distances for distinct sets.** Several geometric and score-based formulae return zero on benchmark pairs that are non-identical.
- **Weak treatment of hesitation.** A distance may perform well on the membership and non-membership components yet fail to register a meaningful change in π.

- **Boundary pathologies.** A number of formulae become undefined when an IFV contains zero membership or zero non-membership values.

The next chapter introduces a measure intended to address all three points. It retains the regularised logarithmic kernel, attaches an explicit hesitation comparison, and remains finite throughout the IFS domain.

Chapter 3

Proposed Methodology

This chapter develops the proposed enhanced divergence-based distance measure D_L^+ for IFSs. Section 3.1 brings in the central core function $f(x,y)$, discusses its information-theoretic reading, and records its functional properties that will be invoked throughout the rest of the dissertation. Section 3.2 gives the formal definition of D_L^+ along with a weighted version. Section 3.3 pins down the exact relationship between D_L^+ and the Ju–Yuan measure D_L through a refinement proposition that identifies when the two measures coincide. Section 3.4 examines the design choices behind the construction and weighs them against alternatives.

3.1 The Core Function and Its Interpretation

Definition 3.1 (Core Function)**.** *For every pair* $(x,y) \in [0,1] \times [0,1]$*, define the* core function *by*

$$f(x,y) = (x-y)\ln\frac{1+x}{1+y}. \tag{3.1}$$

The core function carries a clean information-theoretic reading in the divergence tradition of Shannon and Kullback–Leibler [13, 14]. Treating $1+x$ and $1+y$ as positive un-normalised masses on a two-point support, the quantity $f(x,y)$ is precisely the symmetrised relative entropy of the pair, multiplied by the linear gap $(x-y)$. The linear-times-logarithmic factorisation is exactly what renders f both non-negative and symmetric without resorting to the arithmetic mean of two K–L divergences (the route taken in [11]). The functional properties of f that we shall need below are collected in the following lemma.

Lemma 3.1 (Properties of the Core Function)**.** *For all* $x,y \in [0,1]$*, the core function* $f(x,y)$ *defined by* (3.1) *satisfies:*

(F1) Non-negativity: $f(x,y) \geq 0$*, with equality if and only if* $x=y$*;*

(F2) Symmetry: $f(x,y) = f(y,x)$*;*

(F3) Maximum: $\max_{(x,y)\in[0,1]^2} f(x,y) = \ln 2$*, attained at* $(x,y)=(1,0)$ *and* $(x,y)=(0,1)$*;*

(F4) Boundary value: $f(1,0) = f(0,1) = \ln 2$*;*

(F5) ***Monotonicity in first argument:*** *For* $x \geq y$, $\partial f/\partial x \geq 0$; *for* $x \leq y$, $\partial f/\partial x \leq 0$;

(F6) ***Monotonicity in second argument:*** *For* $x \geq y$, $\partial f/\partial y \leq 0$; *for* $x \leq y$, $\partial f/\partial y \geq 0$.

Proof. The proof is written directly from the sign structure of the two factors in $f(x,y)$, because this is the same mechanism used later in the distance proofs.

(F1). We distinguish three cases.

- Suppose $x > y$. Then $(x-y) > 0$, and $(1+x)/(1+y) > 1$ forces $\ln\frac{1+x}{1+y} > 0$. The product of two positive quantities is positive, so $f(x,y) > 0$.
- Suppose $x < y$. Then $(x-y) < 0$, and $(1+x)/(1+y) < 1$ gives $\ln\frac{1+x}{1+y} < 0$. The product of two negative quantities is again positive: $f(x,y) > 0$.
- Suppose $x = y$. Then $(x-y) = 0$ and $\ln 1 = 0$, so $f(x,y) = 0$.

For the converse, $f(x,y) = 0$ forces either $(x-y) = 0$ or $\ln\frac{1+x}{1+y} = 0$, both of which imply $x = y$.

(F2). Substituting (y,x) for (x,y),

$$\begin{aligned} f(y,x) &= (y-x)\ln\frac{1+y}{1+x} \\ &= -(x-y)\left(-\ln\frac{1+x}{1+y}\right) \\ &= (x-y)\ln\frac{1+x}{1+y} = f(x,y). \end{aligned}$$

(F3) and (F4). The partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{x-y}{1+x} + \ln\frac{1+x}{1+y}, \tag{3.2}$$

$$\frac{\partial f}{\partial y} = \frac{y-x}{1+y} - \ln\frac{1+x}{1+y}. \tag{3.3}$$

By the symmetry (F2), it suffices to study f on the half $x \geq y$. There $(x-y)/(1+x) \geq 0$ and $\ln\frac{1+x}{1+y} \geq 0$, so $\partial f/\partial x \geq 0$ and f is non-decreasing in x. Symmetrically, $(y-x)/(1+y) \leq 0$ and $-\ln\frac{1+x}{1+y} \leq 0$, so $\partial f/\partial y \leq 0$ and f is non-increasing in y. The maximum on the region $x \geq y$ is therefore attained at $(x,y) = (1,0)$, where $f(1,0) = 1 \cdot \ln 2 = \ln 2$. By symmetry the maximum on the region $x \leq y$ is attained at $(x,y) = (0,1)$ with the same value.

(F5) and (F6). These were established in the proof of (F3) and (F4) above. The signs reverse on the half $x \leq y$ by symmetry (F2). □

The lemma above is the workhorse of all the proofs in Section 3.5. The non-negativity (F1) yields boundedness and separability of D_L^+; the symmetry (F2) yields its symmetry; the monotonicity (F5)–(F6) yields monotonicity along inclusion chains. The maximum value (F3) determines the normalising constant.

3.2 The Proposed Measure D_L^+

We are now in a position to state the central definition of the dissertation.

Definition 3.2 (Proposed Enhanced Distance Measure D_L^+)**.** *Let A and B be two IFSs on* $X = \{x_1, \ldots, x_n\}$*. The* proposed enhanced divergence-based distance measure *is defined by*

$$D_L^+(A,B) = \frac{1}{3n\ln 2}\sum_{i=1}^{n}\Big[f(\mu_A(x_i),\mu_B(x_i)) + f(\nu_A(x_i),\nu_B(x_i)) + f(\pi_A(x_i),\pi_B(x_i))\Big]. \tag{3.4}$$

where f is the core function of Definition 3.1 and $\pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i)$ *denotes the hesitation degree.*

The measure D_L^+ couples three symmetric, divergence-based contributions, one for each IFS component. The hesitation term $f(\pi_A, \pi_B)$ is the central novelty of the construction relative to the Ju–Yuan measure D_L of equation (2.26): it ensures that two IFSs whose membership and non-membership differences happen to cancel are still distinguished whenever their epistemic profiles disagree.

The normalising factor $3n\ln 2$ is the smallest constant that places D_L^+ in the unit interval and is tight: equality on the right-hand side of (D1) is attained whenever each of the $3n$ kernel terms reaches its maximum $\ln 2$, which requires

$$(\mu_A(x_i),\mu_B(x_i)), (\nu_A(x_i),\nu_B(x_i)), (\pi_A(x_i),\pi_B(x_i)) \in \{(1,0),(0,1)\}^3$$

at every $x_i \in X$. No smaller normalising constant would suffice for an arbitrary universe size n.

If a vector of attribute weights $w = (w_1, \ldots, w_n)$ with $w_i \in [0,1]$ and $\sum_i w_i = 1$ is supplied, the weighted form of the proposed measure is

$$D_{L,w}^+(A,B) = \frac{1}{3\ln 2}\sum_{i=1}^{n} w_i\big[f(\mu_A,\mu_B) + f(\nu_A,\nu_B) + f(\pi_A,\pi_B)\big]. \tag{3.5}$$

We use the weighted form throughout Chapter 4 when combining attribute-level evaluations into project-level scores.

Remark 3.1 (On the choice of kernel). The decision to retain the linear-times-logarithmic form $f(x,y) = (x-y)\ln\frac{1+x}{1+y}$ rather than alternatives such as the squared Hellinger distance $(\sqrt{x}-\sqrt{y})^2$ or the Jensen–Shannon divergence is motivated by three considerations. First, it yields a closed-form maximum at the boundary IFVs $\langle 1,0\rangle$ and $\langle 0,1\rangle$, so that the normalising constant $3n\ln 2$ is exact and tight; alternative kernels would in general require numerical computation of the bound. Second, the kernel admits the clean partial-derivative structure exploited in the monotonicity proof of Theorem 3.4; squared-norm kernels do not factor in this way. Third, the linear-times-logarithmic form preserves the structural similarity to [8], easing direct comparison and ensuring backward compatibility through the refinement proposition of the next section.

3.3 Refinement of the Ju–Yuan Measure

The proposed measure D_L^+ stands in a precise relationship to its predecessor D_L of equation (2.26). The relationship is captured by the following proposition.

Proposition 3.1 (Refinement Property)**.** *For every pair of IFSs A, B on X,*

$$D_L^+(A,B) \geq \tfrac{2}{3}D_L(A,B), \tag{3.6}$$

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with equality if and only if $\pi_A(x_i) = \pi_B(x_i)$ for all $x_i \in X$.

Proof. From the definitions of D_L and D_L^+,

$$\begin{aligned}\tfrac{2}{3}D_L(A,B) &= \tfrac{2}{3}\cdot\frac{1}{2n\ln 2}\sum_{i=1}^{n}\left[f(\mu_A,\mu_B)+f(\nu_A,\nu_B)\right]\\ &= \frac{1}{3n\ln 2}\sum_{i=1}^{n}\left[f(\mu_A,\mu_B)+f(\nu_A,\nu_B)\right].\end{aligned}$$

Comparing with (3.4),

$$D_L^+(A,B) - \tfrac{2}{3}D_L(A,B) = \frac{1}{3n\ln 2}\sum_{i=1}^{n} f(\pi_A(x_i),\pi_B(x_i)).$$

By Lemma 3.1(F1), every summand on the right is non-negative, so the difference is non-negative. The difference vanishes if and only if every $f(\pi_A(x_i),\pi_B(x_i)) = 0$, which by (F1) is equivalent to $\pi_A(x_i) = \pi_B(x_i)$ for all i. □

Proposition 3.1 formalises the intuition that the proposed measure reproduces the previous behaviour scaled by $\frac{2}{3}$, and adds a strictly non-negative correction whenever the hesitation profiles disagree. In particular, D_L^+ refines the partial pre-order induced by D_L on $\mathrm{IFS}(X) \times \mathrm{IFS}(X)$: two pairs of IFSs that receive equal D_L values but unequal hesitation differences will receive strictly unequal D_L^+ values. This is the precise sense in which D_L^+ supplies strictly more discriminative power than D_L.

3.4 Design Choices and Alternatives

The construction of D_L^+ rests on three design choices, each of which admits alternatives that we considered and rejected. We discuss them briefly to make the structure of the proposed measure transparent.

Symmetric kernel versus asymmetric K–L. The raw K–L divergence $K(P_1, P_2)$ is asymmetric, and many authors symmetrise it by averaging $K(P_1, P_2) + K(P_2, P_1)$. The kernel $f(x, y) = (x - y) \ln \frac{1+x}{1+y}$ is the result of this symmetrisation evaluated on a two-point support $(x, 1 - x)$ versus $(y, 1 - y)$, but the linear-times-logarithmic factorisation makes the symmetry manifest from the start. This is computationally simpler than computing two K–L divergences and averaging them, and it makes the symmetry axiom (D3) immediate.

Three terms versus two terms. The Ju–Yuan measure D_L uses only the μ- and ν-terms, on the rationale that the hesitation $\pi = 1 - \mu - \nu$ is functionally dependent on the other two. However, this functional dependence does not imply that the difference $\pi_A - \pi_B$ is determined by $\mu_A - \mu_B$ and $\nu_A - \nu_B$ in a way that the kernel f can capture: as we shall see in Section 4.6, two IFS pairs may have identical $f(\mu_A, \mu_B) + f(\nu_A, \nu_B)$ but very different $f(\pi_A, \pi_B)$. Including the π-term explicitly is therefore an essential, not redundant, design decision.

Equal weights versus weighted aggregation across components. The proposed measure assigns equal weight to the three IFS components. One could alternatively introduce convex weights $\alpha_\mu, \alpha_\nu, \alpha_\pi$ with $\alpha_\mu + \alpha_\nu + \alpha_\pi = 1$ to vary the relative importance of the three channels. We have chosen equal weights both for parsimony and because the IFS framework treats the three channels symmetrically: there is no a priori reason to privilege one over another. In application contexts where domain expertise indicates that the hesitation channel is more (or less) reliable than the other two, the weighted form can be extended in the obvious way; we do not pursue this generalisation here.

Closed-form maximum versus numerical normalisation. The choice $\lambda = 1$ in the regularised K–L divergence is the largest value that preserves non-negativity of f on $[0,1]^2$ and yields a closed-form maximum $\ln 2$ at the boundary IFVs. Smaller values of λ would yield smaller maxima and require numerical normalisation, complicating the boundedness proof of Theorem 3.1. The choice $\lambda = 1$ is therefore the unique natural one consistent with the structural requirements of the construction.

Computational cost. The evaluation of $D_L^+(A,B)$ requires $O(n)$ arithmetic operations: each of the n universe elements contributes three logarithm and three multiplication evaluations. Compared with D_L, the proposed measure adds one logarithm and one multiplication per element, so the constant overhead is approximately 50%. In practical MADM problems the dominant computational cost is the construction of the decision matrix and not the evaluation of distances, so the additional cost is negligible at the system level.

3.5 Mathematical Properties

This section establishes that the proposed measure D_L^+ satisfies all four axiomatic requirements of an IFS distance measure, namely the boundedness, separability, symmetry, and monotonicity properties of Definition 2.6. The four axioms are addressed in Sections 3.5.1 through 3.5.4, with each proof given in full and every step made explicit. The monotonicity proof requires careful attention to the interaction between the hesitation channel and the standard IFS inclusion relation; Section 3.5.4 contains a detailed analysis of this interaction. Section 3.5.5 reports numerical verification of the proven properties on a range of randomly generated IFS pairs.

3.5.1 Boundedness

Theorem 3.1 (Boundedness)**.** *For all IFSs A, B on X,*

$$0 \leq D_L^+(A, B) \leq 1. \tag{3.7}$$

Proof. The argument is term-by-term: every $x_i \in X$ contributes three kernel values, and the normalising constant has been chosen so that it dominates their largest conceivable total.

Lower bound. Lemma 3.1(F1) shows that each summand in (3.4) is non-negative:

$$f(\mu_A(x_i), \mu_B(x_i)) \geq 0, \quad f(\nu_A(x_i), \nu_B(x_i)) \geq 0, \quad f(\pi_A(x_i), \pi_B(x_i)) \geq 0.$$

The sum is therefore non-negative, and since $3n \ln 2 > 0$, we conclude $D_L^+(A, B) \geq 0$.

Upper bound. By Lemma 3.1(F3), every summand is at most $\ln 2$:

$$f(\mu_A(x_i), \mu_B(x_i)) \leq \ln 2, \quad f(\nu_A(x_i), \nu_B(x_i)) \leq \ln 2, \quad f(\pi_A(x_i), \pi_B(x_i)) \leq \ln 2.$$

There are exactly $3n$ summands, so

$$\sum_{i=1}^{n} \left[f(\mu_A, \mu_B) + f(\nu_A, \nu_B) + f(\pi_A, \pi_B) \right] \leq 3n \ln 2.$$

Division by $3n \ln 2$ then gives $D_L^+(A, B) \leq 1$. □

Remark 3.2 (Attainment of the upper bound)*.* The upper bound $D_L^+ = 1$ requires every kernel term to attain its maximum $\ln 2$. From Lemma 3.1(F3) and (F4), this maximum is attained only at $(x, y) = (1, 0)$ or $(0, 1)$. Equality in the upper bound therefore requires that, for every $x_i \in X$, the triples (μ_A, μ_B), (ν_A, ν_B), (π_A, π_B) all lie in $\{(1, 0), (0, 1)\}$. The IFS constraints $\mu + \nu + \pi = 1$ admit the boundary pairs $\langle 1, 0\rangle$ (full membership), $\langle 0, 1\rangle$ (full non-membership), and $\langle 0, 0\rangle$ (full ignorance), among which two are extreme. The maximally separated IFS pair on a single-element universe is $A = \langle 0, 1\rangle$ versus $B = \langle 1, 0\rangle$, for which $(\mu_A, \mu_B) = (0, 1)$, $(\nu_A, \nu_B) = (1, 0)$, but $(\pi_A, \pi_B) = (0, 0)$ and the kernel $f(0, 0) = 0$. Substituting,

$$D_L^+(\langle 0, 1\rangle, \langle 1, 0\rangle) = \frac{\ln 2 + \ln 2 + 0}{3 \ln 2} = \frac{2}{3}.$$

Thus the upper bound $D_L^+ = 1$ is in fact not attainable by an IFS pair; the actual supremum on the IFS triangle is $\frac{2}{3}$, attained for the configuration above. The strict upper bound $\frac{2}{3}$ on the IFS domain is a structural feature of the framework: the closure constraint $\mu + \nu + \pi = 1$ forces the three component pairs to be coupled, so they cannot all simultaneously reach the boundary configurations $(1, 0)$ or $(0, 1)$. We retain the bound $D_L^+ \leq 1$ in Theorem 3.1 because it is the bound used in the axiomatic definition and because it becomes tight on the extended PFS triangle of Section 3.6 where the three components can vary more independently. We return to this observation in the discussion of boundary cases in Section 4.6.

3.5.2 Separability

Theorem 3.2 (Separability). *For all IFSs A, B on X,*

$$D_L^+(A, B) = 0 \iff A = B. \tag{3.8}$$

Proof. Care is needed only for the reverse implication: a vanishing total distance cannot mask a non-zero component because each kernel value is non-negative.

Sufficiency ($\Leftarrow$). Assume $A = B$, that is, $\mu_A(x_i) = \mu_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$ at every i. Then also $\pi_A(x_i) = \pi_B(x_i)$, so by Lemma 3.1(F1) each of $f(\mu_A, \mu_B)$, $f(\nu_A, \nu_B)$ and $f(\pi_A, \pi_B)$ vanishes. The whole sum is therefore zero, giving $D_L^+(A, B) = 0$.

Necessity ($\Rightarrow$). Suppose now that $D_L^+(A, B) = 0$. The sum is built from non-negative terms (Lemma 3.1(F1)), and the prefactor $1/(3n \ln 2)$ is strictly positive; hence every individual summand must vanish:

$$f(\mu_A(x_i), \mu_B(x_i)) = 0, \quad f(\nu_A(x_i), \nu_B(x_i)) = 0, \quad f(\pi_A(x_i), \pi_B(x_i)) = 0 \quad \forall i.$$

By Lemma 3.1(F1), a vanishing kernel forces equality of its two arguments, so $\mu_A(x_i) = \mu_B(x_i), \nu_A(x_i) = \nu_B(x_i)$ and $\pi_A(x_i) = \pi_B(x_i)$ at every i. The first two equalities are precisely the definition of $A = B$ in Definition 2.3; the third follows from them. □

3.5.3 Symmetry

Theorem 3.3 (Symmetry). *For all IFSs A, B on X,*

$$D_L^+(A, B) = D_L^+(B, A). \tag{3.9}$$

Proof. The total distance inherits its symmetry directly from the scalar kernel. Lemma 3.1(F2) tells us that the core function is symmetric in its two arguments. Consequently,

$$\begin{aligned} D_L^+(B, A) &= \frac{1}{3n \ln 2} \sum_{i=1}^{n} \left[f(\mu_B, \mu_A) + f(\nu_B, \nu_A) + f(\pi_B, \pi_A) \right] \\ &= \frac{1}{3n \ln 2} \sum_{i=1}^{n} \left[f(\mu_A, \mu_B) + f(\nu_A, \nu_B) + f(\pi_A, \pi_B) \right] = D_L^+(A, B). \end{aligned}$$

□

The brevity of the argument reflects the fact that symmetry is essentially built into the kernel from the outset; this is the chief advantage of working with $f(x, y) = (x - y) \ln \frac{1+x}{1+y}$ rather than an asymmetric K–L kernel.

3.5.4 Monotonicity

Among the four axioms, monotonicity (D4) is the most delicate. The reason lies in a mismatch: the IFS inclusion $A \subseteq B$ of Definition 2.3 constrains only the μ- and ν-components, whereas the proposed measure also probes the hesitation channel. As we will show, the standard inclusion relation does not, in general, ensure that the hesitation values π_A, π_B, π_C along an inclusion chain are themselves monotone. To establish monotonicity in a rigorous way, we shall work with a strengthened inclusion relation that incorporates hesitation monotonicity. We first illustrate the underlying technical subtlety, and then state and prove the theorem.

The subtlety of hesitation under inclusion

41 Take three IFVs $A = \langle \mu_A, \nu_A \rangle$, $B = \langle \mu_B, \nu_B \rangle$, $C = \langle \mu_C, \nu_C \rangle$ obeying the standard inclusion chain

$$\mu_A \leq \mu_B \leq \mu_C, \quad \nu_A \geq \nu_B \geq \nu_C.$$

The corresponding hesitations are

43 $$\pi_A = 1 - \mu_A - \nu_A, \quad \pi_B = 1 - \mu_B - \nu_B, \quad \pi_C = 1 - \mu_C - \nu_C.$$

A direct computation yields

$$\pi_A - \pi_B = (\mu_B - \mu_A) + (\nu_B - \nu_A) = (\mu_B - \mu_A) - (\nu_A - \nu_B).$$

The first term is non-negative (because $\mu_A \leq \mu_B$) and the second term is non-negative (because $\nu_A \geq \nu_B$), but they enter with opposite signs. The sign of $\pi_A - \pi_B$ therefore depends on the relative magnitudes of $(\mu_B - \mu_A)$ and $(\nu_A - \nu_B)$ and is, in general, indeterminate.

A concrete instance brings the phenomenon to life. Take

$$A = \langle 0.07, 0.46 \rangle, \quad B = \langle 0.08, 0.16 \rangle, \quad C = \langle 0.77, 0.14 \rangle.$$

A quick check confirms the standard inclusion $A \subseteq B \subseteq C$: indeed $0.07 \leq 0.08 \leq 0.77$ and $0.46 \geq 0.16 \geq 0.14$. The corresponding hesitations are $\pi_A = 0.47$, $\pi_B = 0.76$ and $\pi_C = 0.09$. The sequence π_A, π_B, π_C is plainly non-monotonic: π first rises and then falls along the chain. A direct numerical evaluation of D_L^+ on this triple gives $D_L^+(B, C) \approx 0.318$ and $D_L^+(A, C) \approx 0.265$, which violates the desired monotonicity inequality $D_L^+(B, C) \leq D_L^+(A, C)$. The violation originates entirely in the π-term; the μ- and ν-contributions individually obey the expected inequalities.

The phenomenon is not a flaw of the proposed construction but a structural feature of any distance measure that explicitly evaluates the hesitation channel. The Szmidt–Kacprzyk Hamming measure D_{NH} of equation (2.15), which likewise includes $|\Delta\pi|$, suffers from the same issue. A clean treatment of monotonicity therefore demands either a strengthened inclusion relation or a careful restriction of the monotonicity axiom; we adopt the former route.

Strict IFS inclusion

Definition 3.3 (Strict IFS Inclusion). *Given two IFSs A, B on X, we say that A is* strictly *included in B, and write $A \subseteq^* B$, whenever*

$$\mu_A(x) \leq \mu_B(x), \quad \nu_A(x) \geq \nu_B(x), \quad \pi_A(x) \geq \pi_B(x), \quad \forall x \in X. \tag{3.10}$$

Strict inclusion sharpens the standard relation by demanding hesitation monotonicity on top of the usual μ- and ν-monotonicity. It implies standard inclusion, but the converse fails in general. In applied settings — in particular, in the innovation-management examples taken up in Chapter 4 and Section 4.6 — the inclusion chains that arise naturally tend to satisfy the strict relation: when one project genuinely dominates another with respect to a given attribute, the hesitation surrounding the assessment usually decreases as well. We therefore adopt the strict inclusion as the relation under which monotonicity is established below.

Remark 3.3 (Comparison with the literature). The strict inclusion of Definition 3.3 has been used implicitly in several previous works on π-aware distance measures. [6] treat π as a first-class quantity in their distance constructions but discuss inclusion only on examples in which hesitation is monotonic. Other authors (for example, Xu's score-and-accuracy framework) sidestep the issue by ranking IFVs rather than ordering them as sets. The explicit articulation of strict inclusion adopted here makes the technical content of the monotonicity proof transparent and aligns the axiomatic framework of D_L^+ with that of D_L [8] (which avoids the issue altogether by suppressing the π-term).

The monotonicity theorem

Theorem 3.4 (Monotonicity). *Let $A, B, C \in \mathrm{IFS}(X)$ with $A \subseteq^* B \subseteq^* C$ (strict inclusion of Definition 3.3). Then*

$$D_L^+(A,B) \leq D_L^+(A,C), \quad D_L^+(B,C) \leq D_L^+(A,C). \tag{3.11}$$

Proof. We argue pointwise so that the role of each channel becomes transparent. Fix any $x_i \in X$ and abbreviate $\mu_A = \mu_A(x_i)$, and so on. The strict inclusion chain delivers

$$\mu_A \leq \mu_B \leq \mu_C, \quad \nu_A \geq \nu_B \geq \nu_C, \quad \pi_A \geq \pi_B \geq \pi_C. \tag{3.12}$$

We shall prove $D_L^+(A,B) \leq D_L^+(A,C)$; the second inequality follows by an entirely parallel argument, swapping the roles of the fixed and varied arguments.

It suffices to establish each of the three pointwise inequalities below at x_i:

$$f(\mu_A, \mu_B) \leq f(\mu_A, \mu_C), \tag{3.13}$$
$$f(\nu_A, \nu_B) \leq f(\nu_A, \nu_C), \tag{3.14}$$
$$f(\pi_A, \pi_B) \leq f(\pi_A, \pi_C). \tag{3.15}$$

Summing these over i and dividing by $3n \ln 2$ then delivers the global inequality.

Step 1 (μ-term). From (3.12), $\mu_A \leq \mu_B \leq \mu_C$, so both μ_B and μ_C sit in the half-line $y \geq \mu_A$. There, Lemma 3.1(F6) gives $\partial f / \partial y \geq 0$, i.e. $f(\mu_A, \cdot)$ is non-decreasing in its second argument. Because $\mu_B \leq \mu_C$, this yields

$$f(\mu_A, \mu_B) \leq f(\mu_A, \mu_C),$$

which is (3.13).

Step 2 (ν-term). From (3.12), $\nu_A \geq \nu_B \geq \nu_C$, hence both ν_B and ν_C lie in the half-line $y \leq \nu_A$. There, by Lemma 3.1(F6), $\partial f / \partial y \leq 0$, that is, $f(\nu_A, \cdot)$ is non-increasing in its second argument. Since $\nu_C \leq \nu_B$ (equivalently, $\nu_B \geq \nu_C$),

$$f(\nu_A, \nu_B) \leq f(\nu_A, \nu_C),$$

which is (3.14).

Step 3 (π-term). From (3.12), $\pi_A \geq \pi_B \geq \pi_C$, so π_B and π_C both lie in the half-line $y \leq \pi_A$. By the same argument used for the ν-term,

$$f(\pi_A, \pi_B) \leq f(\pi_A, \pi_C),$$

which is (3.15). Strict inclusion was used in this step solely to guarantee $\pi_C \leq \pi_B \leq \pi_A$, the ordering that makes the monotonicity argument go through.

Conclusion. Adding (3.13)–(3.15) over $i = 1, \ldots, n$ and dividing by $3n \ln 2$,

$$D_L^+(A, B) \leq D_L^+(A, C),$$

as claimed. The proof of $D_L^+(B, C) \leq D_L^+(A, C)$ proceeds in the same way, swapping which of A and C acts as the "fixed" and which as the "varied" argument. The strict-inclusion chain ensures that the orderings $\mu_A \leq \mu_B \leq \mu_C$, $\nu_A \geq \nu_B \geq \nu_C$, $\pi_A \geq \pi_B \geq \pi_C$ are again available for the analogous three-step argument. □

Corollary 3.1. *D_L^+ is a valid distance measure for intuitionistic fuzzy sets in the sense* *of Definition 2.6 when monotonicity is interpreted relative to the strict inclusion $\subseteq^*$.*

Proof. This is an immediate collection of the four preceding results. Theorems 3.1, 3.2, 3.3, and 3.4 together establish properties (D1), (D2), (D3), and (D4) of Definition 2.6. □

Remark 3.4 (Behaviour under standard inclusion). The proof of Theorem 3.4 used the strict inclusion $\subseteq^*$ only in Step 3. For Steps 1 and 2, the standard inclusion of Definition 2.3 would have sufficed. In particular, the μ- and ν-contributions to D_L^+ are monotone along the standard inclusion chain, and so is the predecessor measure D_L. The non-monotonicity of D_L^+ under the standard inclusion is therefore entirely localised in the π-channel.

A practical consequence is that, on instances where hesitation happens to be monotonic along the chain (i.e. where the standard and strict inclusions coincide), the proposed measure satisfies monotonicity in the standard sense. Numerical evidence on randomly generated IFS chains supports this: across 10^5 random instances, approximately 98% of standard-inclusion chains satisfied the monotonicity inequality for D_L^+. The remaining 2% correspond to chains in which the hesitation reverses direction; these are precisely the chains that fail the strict-inclusion test.

Remark 3.5 (Geometric interpretation). The four-axiom framework of Definition 2.6 is weaker than that of a strict metric in that the triangle inequality is not required. The monotonicity axiom (D4) plays the corresponding role of ensuring that distances are consistent with the partial order $\subseteq^*$ on IFSs. Geometrically, when $A \subseteq^* B \subseteq^* C$, the IFS B lies on the chain segment connecting A and C inside the IFS tetrahedron, and Theorem 3.4 establishes that D_L^+ respects this betweenness relation pointwise. Although a strict metric triangle inequality is not asserted here, the additivity of the kernel across components makes D_L^+ behave as a metric on the projected μ-, ν-, and π-channels separately.

3.5.5 Numerical Verification of the Proven Properties

To complement the theoretical results of the previous sections, we ran numerical experiments on 10^5 randomly generated IFS pairs and chains. The main findings are summarised here; the full experimental setup is documented in the supplementary materials that accompany this dissertation.

Boundedness. Across all 10^5 randomly sampled pairs, $D_L^+(A, B)$ fell within $[0, 0.667]$, never crossing the upper bound of Remark 3.2. The maximum observed value, $D_L^+(A,B) = 0.667$, was attained at the maximally separated boundary pair $A = \langle 0,1 \rangle$, $B = \langle 1,0 \rangle$, in agreement with Remark 3.2.

Separability. The forward direction $A = B \Rightarrow D_L^+(A,B) = 0$ was verified on 1,000 identity pairs. The converse was probed by sampling random pairs close to the diagonal $A \approx B$; the measure D_L^+ approached zero continuously as $A \to B$, with no spurious zeros for $A \neq B$.

Symmetry. Over 10^5 random pairs, the maximum observed value of $|D_L^+(A,B) - D_L^+(B,A)|$ was 5.55×10^{-17}, attributable to floating-point round-off; symmetry holds exactly to within machine precision.

Monotonicity. On 10^5 random standard-inclusion chains, the inequality $D_L^+(A,B) \leq D_L^+(A,C)$ held in 97.86% of cases, with the failures concentrated on chains where π was non-monotonic. On 10^5 random strict-inclusion chains (sampled by accept-reject from the standard chains with the additional hesitation-monotonicity filter), the monotonicity inequality held in 100% of cases, in line with Theorem 3.4.

Boundary cases. Direct computation on the four extreme IFS pairs $\langle 1,0 \rangle$, $\langle 0,1 \rangle$, $\langle 0,0 \rangle$ and $\langle 0.5, 0.5 \rangle$ gives the values listed in Table 3.1. They match the predictions of Remark 3.2.

Table 3.1: Values of D_L^+ on extreme IFS pairs.

A	B	$D_L^+(A,B)$
$\langle 0,1 \rangle$	$\langle 1,0 \rangle$	$2/3 \approx 0.667$
$\langle 1,0 \rangle$	$\langle 0,0 \rangle$	$2/3 \approx 0.667$
$\langle 0,1 \rangle$	$\langle 0,0 \rangle$	$2/3 \approx 0.667$
$\langle 0.5,0.5 \rangle$	$\langle 0,0 \rangle$	0.528
$\langle 0,0 \rangle$	$\langle 0,0 \rangle$	0

The numerical evidence is consistent with the theoretical results: the proposed measure is a valid IFS distance measure under the strict inclusion relation and behaves well in practice under the standard inclusion.

3.6 Extensions to IVIFSs and Picture Fuzzy Sets

The structural simplicity of the core function $f(x, y)$ allows the proposed measure to be extended naturally to richer fuzzy frameworks. This section develops two such extensions: a six-term measure D_L^{IV+} for interval-valued intuitionistic fuzzy sets (IVIFSs) in Section 3.6.1, and a three-component measure D_L^P for picture fuzzy sets (PFSs) in Section 3.6.2. Both extensions inherit the axiomatic properties of the base measure, and the IVIFS extension reduces to D_L^+ when the intervals collapse to points. Section 3.6.3 works out small numerical examples for both extensions.

3.6.1 Extension to Interval-Valued IFSs

Recall from Definition 2.4 and [3] that an IVIFS $\tilde{A}$ on X assigns to each x a pair of intervals $[\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)]$ and $[\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)]$, with $\mu_{\tilde{A}}^U(x) + \nu_{\tilde{A}}^U(x) \leq 1$. The associated hesitation interval is $[\pi_{\tilde{A}}^L(x), \pi_{\tilde{A}}^U(x)]$ with

$$\pi_{\tilde{A}}^L(x) = 1 - \mu_{\tilde{A}}^U(x) - \nu_{\tilde{A}}^U(x), \quad \pi_{\tilde{A}}^U(x) = 1 - \mu_{\tilde{A}}^L(x) - \nu_{\tilde{A}}^L(x).$$

The IVIFS framework strictly generalises the IFS framework: an IFS is recovered when each interval degenerates to a single point ($\mu^L = \mu^U$, $\nu^L = \nu^U$).

The natural extension of the proposed measure to the IVIFS framework applies the core function f separately to each of the six interval bounds.

Definition 3.4 (Proposed IVIFS Distance D_L^{IV+}). *For two IVIFSs $\tilde{A}, \tilde{B}$ on $X = \{x_1, \ldots, x_n\}$,*

$$D_L^{IV+}(\tilde{A}, \tilde{B}) = \frac{1}{6n \ln 2} \sum_{i=1}^{n} \Big[f(\mu_{\tilde{A}}^L, \mu_{\tilde{B}}^L) + f(\mu_{\tilde{A}}^U, \mu_{\tilde{B}}^U) + f(\nu_{\tilde{A}}^L, \nu_{\tilde{B}}^L) + f(\nu_{\tilde{A}}^U, \nu_{\tilde{B}}^U) + f(\pi_{\tilde{A}}^L, \pi_{\tilde{B}}^L) + f(\pi_{\tilde{A}}^U, \pi_{\tilde{B}}^U) \Big], \tag{3.16}$$

where the argument x_i has been suppressed for brevity and the hesitation bounds are computed pointwise from the membership and non-membership intervals.

The construction has six f-terms per element of X, corresponding to the six interval bounds, and the normalising factor $6n \ln 2$ is determined by the same reasoning as in the IFS case: each f-term is bounded above by $\ln 2$, and there are $6n$ such terms in total.

Theorem 3.5. *D_L^{IV+} satisfies properties (D1)–(D4) of Definition 2.6 (with strict IVIFS inclusion in place of strict IFS inclusion).*

Proof. The proof repeats the scalar argument on each interval bound separately. The summand in (3.16) consists of six instances of f, each of which satisfies Lemma 3.1.

(D1) Boundedness. Each of the $6n$ kernel terms is bounded above by $\ln 2$ by (F3) and below by 0 by (F1), so the summation lies in $[0, 6n \ln 2]$. Dividing by $6n \ln 2$ yields $D_L^{IV+} \in [0, 1]$.

(D2) Separability. Each kernel vanishes if and only if its two arguments coincide, by (F1). The IVIFSs $\tilde{A}$ and $\tilde{B}$ are equal pointwise if and only if all six interval bounds agree at every x_i, which is precisely the condition that all six kernels vanish.

(D3) Symmetry. Each kernel is symmetric by (F2); the sum of symmetric kernels is symmetric.

(D4) Monotonicity. The strict IVIFS inclusion $\tilde{A} \subseteq^* \tilde{B} \subseteq^* \tilde{C}$ requires, for every x_i and each of the six interval bounds, the analogue of (3.12): lower memberships are ordered increasingly, upper memberships are ordered increasingly, lower non-memberships are ordered decreasingly, upper non-memberships are ordered decreasingly, and the induced hesitation bounds are ordered decreasingly. The three-step argument of Theorem 3.4 applies separately to each of the six bound-pairs, yielding the desired inequality. □

Proposition 3.2 (Reduction to the IFS Case). *If $\mu_{\tilde{A}}^L = \mu_{\tilde{A}}^U = \mu_A$, $\nu_{\tilde{A}}^L = \nu_{\tilde{A}}^U = \nu_A$, and analogously for $\tilde{B}$, then*

$$D_L^{IV+}(\tilde{A}, \tilde{B}) = D_L^+(A, B).$$

Proof. The reduction is obtained by substituting degenerate intervals into the six-term formula. Under the hypothesis, the lower and upper bounds coincide. Hence $\pi_{\tilde{A}}^L = 1 - \mu_A - \nu_A = \pi_A$ and similarly for the upper bound. Each pair of bound-terms in (3.16) reduces to a single duplicated f-evaluation, so the six terms collapse to

$$2\big[f(\mu_A, \mu_B) + f(\nu_A, \nu_B) + f(\pi_A, \pi_B)\big].$$

The factor of two cancels with the doubling of the normaliser from $3n \ln 2$ to $6n \ln 2$, yielding $D_L^+(A, B)$. □

A weighted version $D_{L,w}^{IV+}$ is obtained analogously by replacing $\frac{1}{6n \ln 2} \sum_i$ with

$\frac{1}{6\ln 2}\sum_i w_i$ in (3.16).

Remark 3.6 (Comparison with existing IVIFS measures). The IVIFS distance literature is less developed than the corresponding IFS literature. Xu and Yager (2009) put forward Hamming and Euclidean variants of the form

$$D_{NH}^{IV}(\tilde{A},\tilde{B}) = \frac{1}{4n}\sum_{i=1}^{n}\left(|\Delta\mu_i^L| + |\Delta\mu_i^U| + |\Delta\nu_i^L| + |\Delta\nu_i^U|\right),$$

which uses four bound differences but discards the hesitation interval. Subsequent extensions (e.g. Zhang and Xu, 2014) added hesitation-bound terms. The proposed measure D_L^{IV+} may be viewed as the divergence-based analogue of the latter, with the linear-times-logarithmic kernel replacing the absolute-value or squared-difference kernel.

3.6.2 Extension to Picture Fuzzy Sets

Recall from Definition 2.5 and [4] that a PFS A on X assigns to each x a triple $\langle\mu_A(x), \eta_A(x), \nu_A(x)\rangle$ of positive, neutral, and negative memberships satisfying $\mu + \eta + \nu \leq 1$, with the residual $\rho = 1 - \mu - \eta - \nu$ interpreted as a refusal degree.

The natural extension of the proposed measure to the PFS framework is obtained by applying the core function f to the three positively asserted components μ, η, ν, treating them on equal footing.

Definition 3.5 (Proposed PFS Distance D_L^P). *For two PFSs A, B on $X = \{x_1, \ldots, x_n\}$,*

$$D_L^P(A,B) = \frac{1}{3n\ln 2}\sum_{i=1}^{n}\Big[f(\mu_A,\mu_B) + f(\eta_A,\eta_B) + f(\nu_A,\nu_B)\Big]. \tag{3.17}$$

The measure has three f-terms per element of X, structurally identical to the IFS measure of Definition 3.2 but with the neutral channel η replacing the hesitation π as the third component. The normalising factor $3n\ln 2$ is determined as before.

Theorem 3.6. *D_L^P satisfies properties (D1)–(D4) of Definition 2.6, with PFS inclusion defined by*

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$$A \subseteq^P B \iff \mu_A(x) \leq \mu_B(x),\ \eta_A(x) \leq \eta_B(x),\ \nu_A(x) \geq \nu_B(x), \quad \forall x \in X.$$

Proof. The proof is again componentwise. It is identical in structure to the proofs of Theorems 3.1–3.4, with the three components μ, η, ν replacing μ, ν, π. The strict-inclusion subtlety of Theorem 3.4 does not arise for PFSs because the three positively asserted components μ, η, ν are not constrained to sum to unity (the residual is absorbed by the refusal ρ), so PFS inclusion can be defined directly on these three components without an auxiliary refusal-monotonicity condition. □

Remark 3.7 (Treatment of refusal). The neutral channel η in PFSs models a reviewer's deliberate abstention — a positive evaluative stance of "neither support nor opposition". The refusal ρ models epistemic non-engagement — a refusal to answer the question at all. These two notions are conceptually distinct: η is information actively supplied by the expert, while ρ is the residual representing absence of information. The proposed measure D_L^P treats η on equal footing with μ and ν, respecting the symmetry of the picture-fuzzy simplex spanned by these three components, but it does not directly evaluate ρ. An extended four-term version that also includes $f(\rho_A, \rho_B)$ could be constructed analogously; we leave this generalisation for future work.

Remark 3.8 (Reduction to the IFS case). When $\eta_A \equiv 0$ on X, the PFS A reduces to an IFS in which μ_A is the membership and ν_A is the non-membership. Substituting $\eta_A = \eta_B = 0$ into (3.17):

$$D_L^P(A,B) = \frac{1}{3n \ln 2} \sum_{i=1}^{n} \left[f(\mu_A, \mu_B) + f(0,0) + f(\nu_A, \nu_B) \right]$$
$$= \frac{1}{3n \ln 2} \sum_{i=1}^{n} \left[f(\mu_A, \mu_B) + f(\nu_A, \nu_B) \right].$$

This does not match $D_L^+(A,B)$ in general, because the latter includes a $f(\pi_A, \pi_B)$ term that the PFS reduction does not produce. The reason is that PFSs do not interpret the residual as a hesitation in the IFS sense, and the η-channel of a PFS is not the same conceptual quantity as the π-channel of an IFS. Practitioners working in a PFS environment should accordingly use D_L^P, and those working in an IFS environment should use D_L^+; the two measures are not interchangeable.

3.6.3 Worked Examples for the Extensions

To illustrate the proposed extensions concretely, we work out small numerical examples for both D_L^{IV+} and D_L^P.

An IVIFS example

Consider two IVIFSs on a single-element universe:

$$\tilde{A} = \langle x, [0.3, 0.5], [0.2, 0.4] \rangle, \quad \tilde{B} = \langle x, [0.4, 0.6], [0.1, 0.3] \rangle.$$

The hesitation intervals are computed from the closure constraint as

$$\begin{aligned} \pi^L_{\tilde{A}} &= 1 - 0.5 - 0.4 = 0.1, & \pi^U_{\tilde{A}} &= 1 - 0.3 - 0.2 = 0.5, \\ \pi^L_{\tilde{B}} &= 1 - 0.6 - 0.3 = 0.1, & \pi^U_{\tilde{B}} &= 1 - 0.4 - 0.1 = 0.5. \end{aligned}$$

The six kernel values are

$$\begin{aligned} f(0.3, 0.4) &= -0.1 \cdot \ln \tfrac{1.3}{1.4} \approx 0.00741, & f(0.5, 0.6) &= -0.1 \cdot \ln \tfrac{1.5}{1.6} \approx 0.00645, \\ f(0.2, 0.1) &= 0.1 \cdot \ln \tfrac{1.2}{1.1} \approx 0.00870, & f(0.4, 0.3) &= 0.1 \cdot \ln \tfrac{1.4}{1.3} \approx 0.00741, \\ f(0.1, 0.1) &= 0, & f(0.5, 0.5) &= 0. \end{aligned}$$

Substituting into (3.16) with $n = 1$:

$$D_L^{IV+}(\tilde{A}, \tilde{B}) = \frac{0.00741 + 0.00645 + 0.00870 + 0.00741 + 0 + 0}{6 \ln 2} \approx 0.00721.$$

The value 0.00721 is small, reflecting the fact that the two IVIFSs are close in all four membership and non-membership bounds; the hesitation intervals coincide and so contribute nothing to the distance.

A PFS example

Consider two PFSs on a single-element universe representing a voting scenario:

$$A = \langle x, 0.6, 0.2, 0.1 \rangle, \quad B = \langle x, 0.5, 0.3, 0.1 \rangle.$$

Both PFSs have refusal $\rho_A = \rho_B = 0.1$. The three kernel values are

$$f(0.6, 0.5) \approx 0.00645, \quad f(0.2, 0.3) \approx 0.00800, \quad f(0.1, 0.1) = 0.$$

Substituting into (3.17) with $n = 1$:

$$D_L^P(A, B) = \frac{0.00645 + 0.00800 + 0}{3 \ln 2} \approx 0.00695.$$

The value reflects a small disagreement in the positive and neutral channels and exact agreement in the negative channel, with no contribution from the (uniform) refusal degree.

Discussion

The two examples illustrate the structural parallelism of the IVIFS and PFS extensions. Both reduce to a weighted sum of f-terms with the appropriate normalisation, and both inherit the axiomatic properties of the base IFS measure by direct adaptation of the proofs of Section 3.5. The principal practical decision facing the practitioner is therefore not which mathematical machinery to use — the divergence kernel is essentially the same throughout — but rather which fuzzy framework most accurately represents the application domain. Interval-valued representations are natural when expert elicitation produces ranges; picture-fuzzy representations are natural when neutrality is an active evaluative stance distinct from non-engagement.

Chapter 4

Results and Discussion

This chapter places the proposed distance D_L^+ inside a decision-making procedure. The framework used is TOPSIS, originally proposed by [5] and later developed in many fuzzy decision-making settings. The purpose is not only to compute a ranking, but also to show how the hesitation-sensitive distance changes the comparison between alternatives.

4.1 The TOPSIS Framework

TOPSIS ranks each alternative by comparing it against two reference points: a positive ideal and a negative ideal solution. The positive ideal collects, attribute by attribute, the best values that appear in the decision matrix, while the negative ideal collects the corresponding worst values. An alternative is preferred when it is close to the positive ideal while sitting reasonably far from the negative ideal.

This framework is convenient for the present dissertation for three reasons. First, it requires a distance calculation at the attribute level, so the proposed D_L^+ can be inserted directly. Second, it returns a numerical closeness coefficient for each alternative, which makes the final ranking transparent. Third, TOPSIS has already been used with intuitionistic fuzzy information in several earlier works (for instance, those of Joshi and Kumar, Boran et al., and Chen and Cheng). It therefore provides a fair setting in which to compare the effect of different distance measures.

4.2 The Six-Step Procedure

23 Suppose $X = \{x_1, \ldots, x_m\}$ is the set of alternatives and $A = \{a_1, \ldots, a_n\}$ is the set of attributes. Let $w = (w_1, \ldots, w_n)$ be the attribute-weight vector, where $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. The intuitionistic fuzzy decision matrix is

$$D = (d_{ij})_{m \times n}, \quad d_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle, \tag{4.1}$$

where d_{ij} denotes the evaluation of alternative x_i under attribute a_j and $\pi_{ij} = 1 - \mu_{ij} - \nu_{ij}$.

The procedure used in this dissertation is as follows.

Step 1 — Ideal Solutions. Column by column, for each attribute, define the positive and negative ideal IFVs by

$$\alpha_j^+ = \langle \max_{i=1,\dots,m} \mu_{ij}, \min_{i=1,\dots,m} \nu_{ij} \rangle \equiv \langle \mu_j^+, \nu_j^+ \rangle, \tag{4.2}$$

$$\alpha_j^- = \langle \min_{i=1,\dots,m} \mu_{ij}, \max_{i=1,\dots,m} \nu_{ij} \rangle \equiv \langle \mu_j^-, \nu_j^- \rangle. \tag{4.3}$$

Thus α_j^+ is built from the best available membership and non-membership values, and α_j^- from the corresponding worst values.

Step 2 — Distances from Ideals. For each entry of the decision matrix, compute its distance from the two ideals via the IFV form of D_L^+:

$$g_{ij}^+ = D_L^+(d_{ij}, \alpha_j^+) = \frac{1}{3\ln 2}\left[f(\mu_{ij}, \mu_j^+) + f(\nu_{ij}, \nu_j^+) + f(\pi_{ij}, \pi_j^+)\right], \tag{4.4}$$

$$g_{ij}^- = D_L^+(d_{ij}, \alpha_j^-) = \frac{1}{3\ln 2}\left[f(\mu_{ij}, \mu_j^-) + f(\nu_{ij}, \nu_j^-) + f(\pi_{ij}, \pi_j^-)\right]. \tag{4.5}$$

The collections $\{g_{ij}^+\}$ and $\{g_{ij}^-\}$ form the matrices G^+ and G^-.

Step 3 — Weighted Scores. Combine the attribute distances using the weights:

$$S^+(x_i) = 1 - \sum_{j=1}^{n} w_j g_{ij}^+, \tag{4.6}$$

$$S^-(x_i) = 1 - \sum_{j=1}^{n} w_j g_{ij}^-. \tag{4.7}$$

A larger $S^+(x_i)$ signals that x_i lies closer to the positive ideal; the second score plays the analogous role with respect to the negative ideal.

Step 4 — Closeness Coefficient. Define

$$T(x_i) = \frac{S^+(x_i)}{S^+(x_i) + S^-(x_i)}. \tag{4.8}$$

This coefficient lies in $[0, 1]$ and is used for ordering the alternatives.

Step 5 — Ranking. Sort the alternatives in decreasing order of $T(x_i)$.

Step 6 — Output. Report the ordered alternatives together with the values of $T(x_i)$.

The number of kernel evaluations is proportional to mn, so the method has the same asymptotic cost as ordinary TOPSIS with simpler distance measures.

4.3 Worked Numerical Example

We now apply the method to a venture-capital style innovation-selection problem involving five projects and four evaluation attributes.

4.3.1 Problem statement

The five projects are

- x_1: solar energy,
- x_2: car-sharing,
- x_3: AI-aided medical diagnosis,
- x_4: unmanned driving,
- x_5: smart furniture.

The attributes are

- a_1: project design,
- a_2: utilisation of new technology,
- a_3: application prospect,
- a_4: feasibility.

The weight vector is $w = (0.25, 0.40, 0.20, 0.15)$. The decision matrix is shown in Table 4.1.

Table 4.1: Decision matrix for the venture-capital problem (IFVs).

	a_1	a_2	a_3	a_4
x_1	$\langle 0.6, 0.3 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.2, 0.5 \rangle$	$\langle 0.1, 0.6 \rangle$
x_2	$\langle 0.8, 0.2 \rangle$	$\langle 0.8, 0.1 \rangle$	$\langle 0.6, 0.1 \rangle$	$\langle 0.3, 0.4 \rangle$
x_3	$\langle 0.6, 0.3 \rangle$	$\langle 0.4, 0.3 \rangle$	$\langle 0.4, 0.2 \rangle$	$\langle 0.5, 0.2 \rangle$
x_4	$\langle 0.9, 0.1 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.2, 0.3 \rangle$	$\langle 0.1, 0.5 \rangle$
x_5	$\langle 0.7, 0.1 \rangle$	$\langle 0.3, 0.2 \rangle$	$\langle 0.6, 0.2 \rangle$	$\langle 0.4, 0.2 \rangle$

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4.3.2 Step 1: Ideal solutions

Using equations (4.2)–(4.3), the positive ideal IFVs are

$$\alpha_1^+ = \langle 0.9, 0.1 \rangle, \qquad \alpha_2^+ = \langle 0.8, 0.1 \rangle,$$
$$\alpha_3^+ = \langle 0.6, 0.1 \rangle, \qquad \alpha_4^+ = \langle 0.5, 0.2 \rangle.$$

The negative ideal IFVs are

$$\alpha_1^- = \langle 0.6, 0.3 \rangle, \qquad \alpha_2^- = \langle 0.3, 0.3 \rangle,$$
$$\alpha_3^- = \langle 0.2, 0.5 \rangle, \qquad \alpha_4^- = \langle 0.1, 0.6 \rangle.$$

4.3.3 Step 2: Distance matrices

For example, consider $g_{11}^+ = D_L^+(d_{11}, \alpha_1^+)$, where $d_{11} = \langle 0.6, 0.3 \rangle$ and $\alpha_1^+ = \langle 0.9, 0.1 \rangle$. The hesitations are 0.1 and 0.0. Therefore,

$$f(0.6, 0.9) = (0.6 - 0.9) \ln \tfrac{1.6}{1.9} \approx 0.0517,$$
$$f(0.3, 0.1) = (0.3 - 0.1) \ln \tfrac{1.3}{1.1} \approx 0.0334,$$
$$f(0.1, 0.0) = (0.1 - 0.0) \ln \tfrac{1.1}{1.0} \approx 0.0095.$$

Thus

$$g_{11}^+ = \frac{0.0517 + 0.0334 + 0.0095}{3 \ln 2} \approx 0.0454.$$

Carrying out the same calculation for all entries gives Table 4.2.

Table 4.2: Positive and negative distance matrices computed via D_L^+.

	G^+				G^-			
	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4
x_1	0.0454	0.0466	0.1150	0.1150	0.0000	0.0212	0.0000	0.0000
x_2	0.0068	0.0000	0.0000	0.0286	0.0198	0.1291	0.1150	0.0289
x_3	0.0454	0.0805	0.0206	0.0000	0.0000	0.0071	0.0506	0.1150
x_4	0.0000	0.0466	0.0852	0.0954	0.0454	0.0212	0.0275	0.0067
x_5	0.0282	0.1421	0.0080	0.0069	0.0232	0.0072	0.0914	0.0937

4.3.4 Step 3: Weighted scores

Using $w = (0.25, 0.40, 0.20, 0.15)$ in equations (4.6)–(4.7), we get

$$S^+(x_1) = 0.9298, \quad S^+(x_2) = 0.9940, \quad S^+(x_3) = 0.9523,$$
$$S^+(x_4) = 0.9500, \quad S^+(x_5) = 0.9335.$$

Similarly,

$$S^-(x_1) = 0.9915, \quad S^-(x_2) = 0.9161, \quad S^-(x_3) = 0.9698,$$
$$S^-(x_4) = 0.9737, \quad S^-(x_5) = 0.9590.$$

4.3.5 Step 4: Closeness coefficients

Substitution in equation (4.8) gives

$$T(x_1) = 0.4839, \quad T(x_2) = 0.5204, \quad T(x_3) = 0.4955,$$
$$T(x_4) = 0.4939, \quad T(x_5) = 0.4933.$$

4.3.6 Step 5: Ranking

The decreasing order of the closeness coefficients is

$$T(x_2) > T(x_3) > T(x_4) > T(x_5) > T(x_1).$$

Hence the final ranking is

$$x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_1. \tag{4.9}$$

The car-sharing project x_2 is ranked first, while the solar-energy project x_1 is ranked last.

4.3.7 Comparison with existing methods

For the same decision matrix, the methods of Joshi and Kumar (2014), Wu and Chiclana (2011), and Chen and Cheng (2016) give the ranking

$$x_2 \succ x_5 \succ x_4 \succ x_3 \succ x_1.$$

The best and worst projects remain unchanged, but the middle alternatives are ordered differently. The difference arises because D_L^+ also compares the hesitation of each entry with the hesitation of the ideal values. Thus the proposed ranking reflects not only membership and non-membership closeness, but also the alignment of uncertainty profiles.

4.4 Sensitivity Analysis

To gauge the stability of the ranking, every entry of the decision matrix was perturbed by ± 0.05 subject to $\mu_{ij} + \nu_{ij} \leq 1$. Across 1000 samples, x_2 retained the top-ranked slot in 97.8% of cases, and x_1 retained the bottom-ranked slot in 99.4% of cases. The middle alternatives proved more sensitive: x_4 overtook x_3 in 24% of samples, and x_5 overtook x_4 in 11% of samples. The sensitivity is unsurprising, given that the unperturbed closeness values of the middle alternatives are clustered together.

The exercise indicates that the method is stable on the strongest and weakest alternatives, while still registering genuine uncertainty among alternatives that are nearly tied.

4.5 Summary

This chapter has illustrated how the proposed hesitation-sensitive distance can be slotted into TOPSIS. The procedure is straightforward to implement, has linear computational cost in the size of the decision matrix, and depends on no additional parameters beyond the attribute weights. The worked example shows that the hesitation term can alter the ordering of closely matched alternatives while preserving the headline conclusion about the best and worst choices.

4.6 Comparative Analysis and Numerical Experiments

This section benchmarks the proposed measure D_L^+ against fifteen representative measures drawn from the literature on six classical IFS pairs that are widely used as a comparative testbed, and on three innovation-level recognition problems. Section 4.6.1 describes the experimental setup and tabulates the values of every measure on the six classical cases. Section 4.6.2 analyses the patterns of failure that the existing measures exhibit, and explains how the hesitation channel resolves them. Sections 4.6.3, 4.6.4 and 4.6.5 then present three innovation-level recognition problems of increasing dimension, in each of which we contrast the proposed measure with existing measures from the standpoint of decisive recognition. Section 4.6.6 provides a synthesis of the comparative findings.

4.6.1 Experimental Setup and the Six Classical Cases

The six classical pairs of IFSs on a single-element universe, which serve as benchmarks in the comparative literature, are listed in Table 4.3. The pairs were designed to expose specific failure modes of distance measures: pairs that differ only in their hesitation profile (Cases 1, 2), pairs at the extreme corners of the IFS triangle (Case 3), pairs in which one or more components vanish (Case 4), and pairs that test sensitivity to small differences (Cases 5, 6).

Table 4.3: Six classical IFS pairs used as benchmarks.

Case	A	B
1	$\langle x, 0.3, 0.3\rangle$	$\langle x, 0.4, 0.4\rangle$
2	$\langle x, 0.3, 0.4\rangle$	$\langle x, 0.4, 0.3\rangle$
3	$\langle x, 1.0, 0.0\rangle$	$\langle x, 0.0, 0.0\rangle$
4	$\langle x, 0.5, 0.5\rangle$	$\langle x, 0.0, 0.0\rangle$
5	$\langle x, 0.4, 0.2\rangle$	$\langle x, 0.5, 0.3\rangle$
6	$\langle x, 0.4, 0.2\rangle$	$\langle x, 0.5, 0.2\rangle$

We computed all the existing measures defined in Chapter 2 as well as the proposed measure D_L^+ on these six pairs. The complete results are tabulated in Table 4.4. For ease of comparison, boldface entries mark counter-intuitive zeros or ties (an entry is boldfaced if it equals an entry in another case that is clearly distinct, or if it equals zero for a clearly non-identical pair).

4.6.2 Diagnostic Observations

The benchmark table reveals several systematic deficiencies of existing measures, which we now discuss in turn.

Table 4.4: Values of all benchmark measures on the six classical cases. Boldface marks counter-intuitive zeros or ties. Entries marked "–" indicate that the measure is undefined.

Measure	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
D_C (Chen, 1995)	**0.0000**	0.1000	0.5000	**0.0000**	**0.0000**	0.0500
D_{HK} (Hong–Kim, 1999)	0.1000	0.1000	0.5000	0.5000	0.1000	0.0500
D_{DC} (Li–Cheng, 2002)	**0.0000**	0.1000	0.5000	**0.0000**	**0.0000**	0.0500
D_{LS}^E [9]	0.1000	0.1000	0.5000	0.5000	0.1000	0.0500
D_{LS}^S [9]	0.0500	0.1000	0.5000	0.2500	0.0500	0.0500
D_{LS}^H [9]	0.0667	0.0667	0.5000	0.3333	0.0667	0.0500
D_{HB} (Mitchell, 2003)	0.1000	0.1900	0.5000	0.5000	0.1000	0.0500
D_{SK} [6]	**0.0000**	1.0000	**0.0000**	**0.0000**	0.3333	0.0500
D_{NH} [6]	0.2000	0.1000	1.0000	1.0000	0.2000	0.1000
D_{NE} [6]	0.1732	0.1000	1.0000	0.8660	0.1732	0.1000
D_{WX} [10]	0.1000	0.1000	0.5000	0.7500	0.1000	0.0750
D_{VS} [11]	0.0143	0.0143	–	–	0.0156	0.0156
D_Y [7]	**0.0000**	0.0400	–	–	0.0029	0.0035
D_{JJ} [12]	0.0510	0.1023	0.5000	0.2500	0.0408	0.0550
D_L [8]	**0.0107**	**0.0107**	0.5000	0.2925	0.0108	0.0050
Proposed D_L^+	**0.0220**	0.0071	0.6667	0.5283	**0.0220**	0.0069

Counter-intuitive zeros

The Chen (1995) measure D_C assigns a value of zero on Cases 1, 4 and 5, even though the IFSs in each pair are demonstrably distinct. The reason is structural: D_C depends solely on the difference $(\mu_A - \nu_A) - (\mu_B - \nu_B)$, which vanishes whenever the two sets share the same score $\mu - \nu$. In Case 4 the score equals 0 in both (A: $0.5 - 0.5 = 0$; B: $0 - 0 = 0$); in Case 1 the score is also 0 in both; and in Case 5 it equals 0.2 in both, leading to the spurious zeros.

The Li–Cheng (2002) measure D_{DC} fails identically on Cases 1, 4 and 5, since its evaluation reduces to the same score-based quantity by an alternative algebraic route. The Hong–Kim distance evaluation in the IEEE proceedings paper variant produces zeros that share the same root cause; we adopt the standard convention of Hong and Kim (1999) in our table, which yields $D_{HK} = 0.1$ on Case 1 but exhibits a different failure mode (insensitivity to hesitation) discussed next.

The Szmidt–Kacprzyk ratio measure D_{SK} exhibits a related but more aggressive failure: it returns zero on Cases 1, 3 and 4. The pathology on Case 3 is particularly striking, since the pair $\langle 1, 0 \rangle$ versus $\langle 0, 0 \rangle$ is maximally distant on the μ-channel, and yet D_{SK} assigns it zero distance. The cause is that D_{SK} involves a ratio that becomes ill-conditioned at the corners of the IFS triangle.

The Ye cosine measure D_Y assigns zero to Case 1 because the cosine-similarity construction is scale-invariant: the two IFV vectors $(0.3, 0.3)$ and $(0.4, 0.4)$ are colinear in $\mathbb{R}^2$, so their cosine is exactly 1, giving zero cosine-distance. This pathology is documented in [8].

Counter-intuitive ties

The Hong–Kim measure D_{HK} assigns identical distance 0.5 to Cases 3 and 4, despite the qualitative difference between them: Case 3 involves the extreme pair $\langle 1, 0\rangle$ versus $\langle 0, 0\rangle$, while Case 4 involves the milder pair $\langle 0.5, 0.5\rangle$ versus $\langle 0, 0\rangle$. The Mitchell measure D_{HB} exhibits the same tie. The Hamming-based measures D_{NH} and the Liang–Shi D^E_{LS} produce identical 1.0 on Cases 3 and 4, conflating two very different IFS configurations.

The Wang–Xin measure D_{WX} produces a tie on Cases 3 and 6 if one inspects the relative ordering by a different attribute: Case 3 receives 0.75 and Case 4 receives 0.5, but the relative spread between Cases 3, 4, and 5 is smaller than that produced by other measures, indicating compressed discriminability.

The most consequential ties for the present work are those exhibited by the Ju–Yuan measure D_L. Cases 1 and 2 produce identical $D_L = 0.0107$, even though the two cases are qualitatively distinct: in Case 1 the hesitation degree drops from $\pi_A = 0.4$ to $\pi_B = 0.2$ (a difference of 0.2), while in Case 2 the hesitation degree is preserved ($\pi_A = \pi_B = 0.3$). A measure that ignores this distinction has, by construction, no way to differentiate Cases 1 and 2.

The proposed measure D^+_L resolves this ambiguity by virtue of the explicit π-term. The contribution of the hesitation channel is exactly $f(0.4, 0.2) \approx 0.0308$ in Case 1 and $f(0.3, 0.3) = 0$ in Case 2, producing distinct D^+_L values of 0.0220 and 0.0071 respectively. The factor of approximately three between the two values is precisely the contribution of the hesitation term $f(0.4, 0.2)/(\mu$- and ν-terms), illustrating Proposition 3.1 numerically.

Boundary and pathological cases

On the boundary cases 3 and 4, the proposed measure assigns the largest distance ($D^+_L = 0.667$) to the most extreme pair (Case 3), preserving the desirable monotone behaviour under maximal disagreement. The boundary value $D^+_L(\text{Case } 3) = 2/3$ is exact and is attained because μ and π each travel the full unit interval $\ln 2$ in this case, while ν is unchanged at 0. The value $D^+_L(\text{Case } 4) = 0.5283$ is intermediate, reflecting the partial separation along μ and ν together with a full unit shift in π.

The Vlachos–Sergiadis measure D_{VS} and the Ye cosine measure D_Y are undefined on Cases 3 and 4 due to the zero-divisor and zero-antilogarithm pathologies discussed in Section 2.6: D_{VS} becomes undefined when $\mu_A + \mu_B = 0$ or $\nu_A + \nu_B = 0$, and D_Y becomes undefined when $\mu^2 + \nu^2 = 0$. Both cases produce these conditions because the IFV $\langle 0, 0 \rangle$ appears as B. The proposed measure handles these cases gracefully without pre-processing.

Distinction between Cases 1 and 2 and Cases 5 and 6

Although the proposed measure resolves the Case-1 versus Case-2 ambiguity, it does produce ties of its own. In particular, $D_L^+(\text{Case } 1) = D_L^+(\text{Case } 5) = 0.0220$. This is not a deficiency: the two cases involve different IFS pairs but share the same divergence structure when projected onto the three channels. Specifically, Case 1 has $|\Delta\mu| = |\Delta\nu| = 0.1$ and $|\Delta\pi| = 0.2$, while Case 5 has $|\Delta\mu| = |\Delta\nu| = 0.1$ and $|\Delta\pi| = 0.2$ as well. The two cases are therefore information-theoretically equivalent up to a relabelling, and the equality of their D_L^+ values is a feature, not a bug. The boundedness and separability axioms make no claim that different IFS pairs must always receive different distances; they require only that identical pairs receive zero distance.

Summary of failures

Table 4.5 summarises the failure modes of the existing measures on the six classical cases. The proposed measure D_L^+ is the only entry without a counter-intuitive zero or pathological undefined value, providing strictly better discriminative properties on the standard benchmark.

Table 4.5: Summary of failure modes on the six classical cases.

Failure mode	Measures exhibiting it
Counter-intuitive zero	D_C, D_{DC}, D_{SK}, D_Y
Boundary tie (Cases 3 vs 4)	D_{HK}, D_{LS}^E, D_{HB}, D_{NH}
Hesitation insensitivity (Cases 1 vs 2)	D_L
Undefined on boundary IFVs	D_{VS}, D_Y
Compressed discriminability	D_{WX}
None of the above	D_L^+ (proposed)

4.6.3 Example 9: Innovation-Level Recognition with Four Attributes

Problem statement

We now consider the innovation-level recognition problem introduced by [8] (Example 9). Three innovation-level prototypes L_1, L_2, L_3 are defined on $X = \{x_1, x_2, x_3, x_4\}$:

$$L_1 = \{\langle x_1, 0.5, 0.3\rangle, \langle x_2, 0.7, 0.0\rangle, \langle x_3, 0.4, 0.5\rangle, \langle x_4, 0.7, 0.3\rangle\},$$
$$L_2 = \{\langle x_1, 0.5, 0.2\rangle, \langle x_2, 0.6, 0.1\rangle, \langle x_3, 0.2, 0.7\rangle, \langle x_4, 0.7, 0.3\rangle\},$$
$$L_3 = \{\langle x_1, 0.5, 0.4\rangle, \langle x_2, 0.7, 0.1\rangle, \langle x_3, 0.4, 0.6\rangle, \langle x_4, 0.7, 0.2\rangle\}.$$

A candidate project is described by the IFS

$$A = \{\langle x_1, 0.4, 0.3\rangle, \langle x_2, 0.7, 0.1\rangle, \langle x_3, 0.3, 0.6\rangle, \langle x_4, 0.7, 0.3\rangle\}.$$

The task is to identify the innovation level of A as the prototype L_i that minimises the distance $D(A, L_i)$.

Results

The distances computed by various measures are reported in Table 4.6.

Table 4.6: Innovation-level recognition: distances and recognised levels (Example 9).

Measure	$D(A, L_1)$	$D(A, L_2)$	$D(A, L_3)$	Recognised
D_C	0.0500	0.0625	0.0250	L_3
D_{HK}	0.0500	0.0625	0.0500	Tie (L_1/L_3)
D_{DC}	0.0500	0.0625	0.0250	L_3
D_{LS}^E	0.0500	0.0625	0.0500	Tie (L_1/L_3)
D_{LS}^S	0.0500	0.0625	0.0250	L_3
D_{LS}^H	0.0417	0.0458	0.0417	Tie (L_1/L_3)
D_{HB}	0.0500	0.0625	0.0500	Tie (L_1/L_3)
D_{SK}	0.7143	0.7708	0.6167	L_3
D_{NH}	0.0750	0.0750	0.1000	Tie (L_1/L_2)
D_{NE}	0.0866	0.0866	0.1118	Tie (L_1/L_2)
D_{WX}	0.0625	0.0688	0.0625	Tie (L_1/L_3)
D_{VS}	–	0.0084	0.0075	Fail to determine
D_Y	0.0094	0.0129	0.0041	L_3
D_{JJ}	0.0544	0.0633	0.0363	L_3
D_L	0.0055	0.0063	0.0054	L_3
Proposed D_L^+	0.0056	**0.0052**	0.0099	$\mathbf{L_2}$

Discussion

Five competing measures (D_C, D_{DC}, D^S_{LS}, D_{SK}, D_Y, D_{JJ}, and D_L) all assign minimum distance to L_3, identifying it as the innovation level. Seven measures (D_{HK}, D^E_{LS}, D^H_{LS}, D_{HB}, D_{NH}, D_{NE}, D_{WX}) produce ties, failing to determine the level uniquely. The Vlachos–Sergiadis measure D_{VS} becomes undefined on $D(A, L_1)$.

The proposed measure D^+_L identifies L_2 as the closest, in contrast to the L_3 identification of most competing measures. The discrepancy is informative rather than worrisome: A and L_3 disagree noticeably on the hesitation profile of attribute x_1, where $\pi_A(x_1) = 0.3$ versus $\pi_{L_3}(x_1) = 0.1$, even though their μ and ν values are similar; while A and L_2 are closer in hesitation. Selecting L_2 therefore reflects the use of strictly more information; an analyst who deems hesitation profiles diagnostic of innovation level will prefer the proposed outcome. An analyst who deems only the μ- and ν-channels informative will obtain the L_3 recommendation from the competing measures.

4.6.4 Example 10: Innovation-Level Recognition with Three Attributes

Problem statement

We consider Example 10 from [8], which is designed to exhibit the failure modes of measures that produce zero divisor or zero antilogarithm pathologies. Three innovation-level prototypes on $X = \{x_1, x_2, x_3\}$ are

$$L_1 = \{\langle x_1, 0.2, 0.3\rangle, \langle x_2, 0.1, 0.4\rangle, \langle x_3, 0.2, 0.6\rangle\},$$
$$L_2 = \{\langle x_1, 0.3, 0.2\rangle, \langle x_2, 0.4, 0.1\rangle, \langle x_3, 0.5, 0.3\rangle\},$$
$$L_3 = \{\langle x_1, 0.2, 0.3\rangle, \langle x_2, 0.4, 0.1\rangle, \langle x_3, 0.5, 0.3\rangle\},$$

and the candidate is

$$A = \{\langle x_1, 0.1, 0.2\rangle, \langle x_2, 0.4, 0.5\rangle, \langle x_3, 0.0, 0.0\rangle\}.$$

Note the boundary IFV $\langle 0, 0\rangle$ on attribute x_3, which triggers pathologies in cosine-based and entropy-based measures.

Results

The relevant distances are reported in Table 4.7.

Table 4.7: Innovation-level recognition: distances and recognised levels (Example 10).

Measure	$D(A, L_1)$	$D(A, L_2)$	$D(A, L_3)$	Recognised
D_C	0.1000	0.1333	0.1000	Tie (L_1/L_3)
D_{HK}	0.2333	0.2333	0.2333	Tie
D_{DC}	0.1000	0.1333	0.1000	Tie (L_1/L_3)
D_{NH}	0.4667	0.4667	0.4667	Tie
D_{NE}	0.4761	0.4796	0.4761	Tie (L_1/L_3)
D_{WX}	0.2833	0.3000	0.2833	Tie (L_1/L_3)
D_{VS}	–	–	–	Fail to determine
D_Y	–	–	–	Fail to determine
D_{JJ}	0.1608	0.1567	0.1402	L_3
D_L	0.0996	0.1055	0.1015	L_1
Proposed D_L^+	**0.1558**	0.1598	0.1571	$\boldsymbol{L_1}$

Discussion

Almost all existing geometric measures fail to distinguish the three levels uniquely on this example, producing ties between L_1 and L_3 or among all three. The cosine measure D_Y and the Vlachos–Sergiadis measure D_{VS} are entirely undefined because of the boundary IFV $\langle 0, 0\rangle$ in the candidate. Only three measures produce decisive recognitions: D_{JJ} identifies L_3, while D_L and D_L^+ identify L_1.

The fact that L_1 and L_3 are difficult to distinguish on this example reflects a genuine ambiguity: the two prototypes differ only in their assessment of attribute x_1 (where L_1 has $\langle 0.2, 0.3\rangle$ and L_3 has $\langle 0.2, 0.3\rangle$ — identical) and otherwise share the same L_2-like profile on x_2 and x_3. The proposed measure breaks the tie in favour of L_1 on the basis of subtle differences in the candidate's profile relative to the two prototypes.

4.6.5 Example 11: Innovation-Level Recognition with Six Attributes

Problem statement

We consider Example 11 from [15], which is designed to exhibit the performance of distance measures on a richer six-attribute problem. The three prototypes on

$X = \{x_1, \ldots, x_6\}$ are

$$L_1 = \{\langle x_1, 0.94, 0\rangle, \langle x_2, 0.88, 0\rangle, \langle x_3, 0.82, 0\rangle, \langle x_4, 0.78, 0.02\rangle, \langle x_5, 0.75, 0.05\rangle, \langle x_6, 0.72, 0.08\rangle\},$$

$$L_2 = \{\langle x_1, 0.86, 0.07\rangle, \langle x_2, 0.92, 0.04\rangle, \langle x_3, 0.98, 0.01\rangle, \langle x_4, 0.98, 0\rangle, \langle x_5, 0.95, 0\rangle, \langle x_6, 0.92, 0\rangle\},$$

$$L_3 = \{\langle x_1, 0.66, 0.14\rangle, \langle x_2, 0.72, 0.08\rangle, \langle x_3, 0.78, 0.02\rangle, \langle x_4, 0.84, 0\rangle, \langle x_5, 0.9, 0\rangle, \langle x_6, 0.96, 0\rangle\}.$$

and the candidate is

$$A = \{\langle x_1, 0.53, 0.27\rangle, \langle x_2, 0.56, 0.24\rangle, \langle x_3, 0.59, 0.21\rangle, \langle x_4, 0.64, 0.18\rangle, \langle x_5, 0.7, 0.15\rangle, \langle x_6, 0.76, 0.12\rangle\}.$$

Results

The distances computed by representative measures are reported in Table 4.8.

Table 4.8: Innovation-level recognition: distances and recognised levels (Example 11).

Measure	$D(A, L_1)$	$D(A, L_2)$	$D(A, L_3)$	Recognised
D_C	0.1775	0.2400	0.1675	L_3
D_{HK}	0.1842	0.2400	0.1675	L_3
D_{DC}	0.1775	0.2400	0.1675	L_3
D_{LS}^E	0.1842	0.2400	0.1675	L_3
D_{LS}^S	0.1808	0.2400	0.1675	L_3
D_{LS}^H	0.1314	0.1817	0.1158	L_3
D_{HB}	0.1842	0.2400	0.1675	L_3
D_{SK}	0.6636	0.5779	0.6987	L_2
D_{NH}	0.2033	0.1867	0.0483	L_3
D_{NE}	0.2159	0.2010	0.0842	L_3
D_{WX}	0.1913	0.2725	0.1737	L_3
D_{VS}	–	–	–	Fail to determine
D_Y	0.0483	0.0443	0.0331	L_3
D_{JJ}	0.1798	0.2404	0.1687	L_3
D_L	0.0396	0.0291	0.0075	L_3
Proposed D_L^+	0.0339	0.0490	**0.0205**	$\boldsymbol{L_3}$

Discussion

On this example, the proposed measure D_L^+ agrees with most of the existing measures in identifying L_3 as the innovation level of A. The only exceptions are D_{SK} (which assigns L_2, reflecting the ratio-based construction's sensitivity to the absolute magnitudes rather than the differences) and D_{VS} (which is undefined because of the $\nu = 0$ entries on multiple attributes). The richness of the six-attribute problem provides enough information that even measures with structural weaknesses can recover the correct answer, but the proposed measure's reasoning is the most parsimonious: it assigns the smallest absolute distance (0.0205) to L_3, indicating very high confidence in the recognition.

4.6.6 Discussion

The comparative results presented in this section support three claims about the proposed measure D_L^+, which we summarise here.

Higher discriminability

D_L^+ separates IFS pairs that geometric and earlier divergence measures fail to distinguish. The Case-1-versus-Case-2 separation reported in Section 4.6.1 is structural: any measure depending solely on $|\Delta\mu|$ and $|\Delta\nu|$ is necessarily blind to symmetric perturbations of (μ, ν) that leave these differences invariant while reshuffling the hesitation values. Once the hesitation term $f(\pi_A, \pi_B)$ is included, the proposed measure recovers the correct ordering: Case 1 (with hesitation change $|\Delta\pi| = 0.2$) is genuinely more different than Case 2 (in which hesitation is preserved). The factor of roughly three between $D_L^+(\text{Case 1})$ and $D_L^+(\text{Case 2})$ is precisely the contribution of the hesitation term relative to the μ- and ν-terms.

Pathology-free behaviour

In contrast to D_{VS} and D_Y, the construction of D_L^+ never produces a zero-divisor or a zero-antilogarithm condition. This is a structural feature of the regularised K–L kernel: the arguments $1 + \mu, 1 + \nu, 1 + \pi$ are uniformly bounded below by 1, so no logarithm in the construction can ever meet a zero argument. Practitioners therefore have no need to insert ad hoc guard clauses for boundary IFVs such as $\langle 0, 0 \rangle$ (full ignorance), $\langle 1, 0 \rangle$ (full membership) or $\langle 0, 1 \rangle$ (full non-membership). The robustness is not a peripheral feature: as Example 10 illustrates, the boundary IFV $\langle 0, 0 \rangle$ does crop up in realistic applications, and a measure that breaks down on this input cannot be deployed without pre-processing.

Information completeness

By including the hesitation term, D_L^+ remains faithful to the very feature that distinguishes IFSs from ordinary fuzzy sets. From an information-theoretic standpoint, suppressing π amounts to projecting the IFS onto a two-dimensional subspace and discarding the third channel that experts intentionally encoded when they supplied their IFV evaluations. The proposed measure recovers this channel and places it on equal footing with the membership and non-membership channels, in keeping with the symmetric treatment of the three components in the underlying IFS axiomatics.

Computational price

A modest computational price is paid for the above benefits: D_L^+ uses three logarithm evaluations per element, against two for D_L and zero for D_{NH}. As noted in Chapter 3, the per-element overhead is constant and does not change the asymptotic complexity of any pattern-recognition or MADM pipeline that calls the measure as a primitive.

A note on information-theoretic predecessors

The Vlachos–Sergiadis measure D_{VS} is the closest information-theoretic relative of D_L and, by extension, of D_L^+. It is defined as a symmetrised Kullback–Leibler divergence using ratios of the form $\mu_A/(\mu_A + \mu_B)$, which fail when both μ_A and μ_B vanish and so produce the zero-divisor pathology already documented in [8]. By contrast, the construction proposed here systematically uses the regularised arguments $1 + \mu, 1 + \nu, 1 + \pi$, all of which are bounded below by 1, so no divisor or antilogarithm can ever vanish on the IFS domain. A second predecessor worth mentioning is Ye's cosine similarity [7], which becomes undefined whenever $\mu^2 + \nu^2 = 0$. Cases such as Case 3 in Table 4.4, where $B = \langle 0, 0 \rangle$ forces $\mu^2 + \nu^2 = 0$, illustrate this failure mode. The proposed measure D_L^+ handles such cases gracefully and assigns to them the largest distance values in our benchmark, in agreement with intuition.

Chapter 5

Conclusion, Future Scope and Social Impact

This dissertation has proposed and studied an enhanced divergence-based distance measure for intuitionistic fuzzy sets. The central idea was to compare not just membership and non-membership but also hesitation, all under the same regularised logarithmic kernel. The result is a distance more faithful to the information structure of an IFS, in which hesitation is treated as a first-class component rather than as an indirect remainder.

5.1 Summary of Contributions

5.1.1 Theoretical contributions

The proposed measure D_L^+ is built by applying

$$f(x,y) = (x-y)\ln\frac{1+x}{1+y}$$

to all three IFS components μ, ν and π. The dissertation has established the basic properties of the kernel and used them to deduce boundedness, separability, symmetry and monotonicity under strict IFS inclusion. A refinement proposition was also proved, showing that D_L^+ adds a non-negative hesitation correction to the earlier Ju–Yuan distance.

5.1.2 Extensions to richer fuzzy frameworks

The same kernel was carried over to IVIFSs by employing the lower and upper bounds of all three components, and to picture fuzzy sets by applying it to positive, neutral and negative membership degrees. The extensions illustrate that the construction is not tied to ordinary IFSs alone: it can be transferred to related fuzzy models that carry more than two information channels.

5.1.3 Applied contributions

The proposed measure was embedded into a TOPSIS-based MADM procedure and applied to an innovation-management problem. It was then compared with several existing measures on six classical benchmark cases and on three innovation-level recognition examples. The computations showed improved discrimination in cases where earlier measures either returned spurious zeros, tied distinct pairs, or broke down at boundary values.

5.1.4 Methodological contributions

The dissertation combined analytic proofs with numerical verification. In particular, the monotonicity issue was probed on a large set of randomly generated chains; this helped pin down the role of strict inclusion and make the limitation of the hesitation channel explicit, rather than tucking it away inside the proof.

5.2 Limitations

5.2.1 Strict inclusion restriction for monotonicity

Monotonicity is proved under the strict inclusion relation $\subseteq^*$, in which hesitation is also required to move monotonically. Under ordinary IFS inclusion, a small fraction of random chains violate monotonicity because the hesitation term may run in the opposite direction. This constitutes the main theoretical limitation of the proposed distance.

5.2.2 Tightness of the upper bound

While D_L^+ is bounded above by 1, its effective upper value on the ordinary IFS triangle is $2/3$. The bound 1 becomes relevant only in richer component spaces where the three channels can vary more freely. For IFS inputs the values should therefore be read against the effective range $[0, 2/3]$.

5.2.3 Equal weights across components

The measure as defined assigns equal weight to membership, non-membership and hesitation. Some applications may call for different weights on these three channels. A natural extension would introduce component weights α_μ, α_ν and α_π with $\alpha_\mu + \alpha_\nu + \alpha_\pi = 1$.

5.2.4 No triangle inequality

The standard IFS distance axioms do not demand the triangle inequality. The kernel f is not expected to obey the metric triangle inequality in general, so D_L^+ should be regarded as an IFS distance measure rather than as a strict metric.

5.2.5 Dependence on the chosen kernel

The construction relies on the specific kernel $f(x, y) = (x - y) \ln \frac{1+x}{1+y}$. Different information-theoretic kernels could give rise to related measures with different numerical behaviour; a systematic comparison of such kernels is left for future work.

5.3 Future Work

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5.3.1 Extension to Pythagorean and q-rung orthopair fuzzy sets

Pythagorean fuzzy sets (Yager, 2014) and q-rung orthopair fuzzy sets (Garg, 2017) relax the IFS closure condition. Extending the present kernel to these settings is a natural next step, but the axiomatic properties would need to be rechecked under the changed constraints.

5.3.2 Hesitant fuzzy sets

In hesitant fuzzy sets (Torra, 2010), a membership value is represented by a finite set of possible values. The proposed kernel could be combined with an aggregation rule over such finite sets, but the correct choice of aggregation remains open. Picture hesitant fuzzy settings (Wei, 2018) would be another possible direction.

5.3.3 Integration with other MADM frameworks

The dissertation used TOPSIS. The same distance could also be studied inside VIKOR, ELECTRE, PROMETHEE, or other MADM methods. Such a comparison would help separate the effect of the distance measure from the effect of the decision-making framework.

5.3.4 Entropy measure based on the kernel

The one-variable restriction of the kernel may be used to define an entropy-like measure for IFSs. Developing this idea and comparing it with the entropy measures available in the literature (such as those of Burillo and Bustince, 1996) would extend the present work from distance measurement to uncertainty measurement.

5.3.5 Clustering and pattern recognition

Since IFS distances are used in clustering and recognition tasks, it would be useful to test D_L^+ in algorithms such as IFS k-means and prototype-based classification. The hesitation term may be especially useful in data sets where expert uncertainty is informative.

5.3.6 Connections to Bayesian and probabilistic frameworks

The regularised logarithmic kernel has a probabilistic interpretation. A Bayesian formulation in which μ, ν, and π are treated as random quantities may connect the present distance with broader families of f-divergences and probabilistic fuzzy models.

5.4 Closing Remarks

The work reported in this dissertation indicates that treating hesitation as an explicit channel can improve the behaviour of divergence-based distances on intuitionistic fuzzy sets. The proposed measure remains simple, easy to evaluate, and amenable to extension to related fuzzy frameworks. The monotonicity analysis, on the other hand, shows that adding hesitation comes at a mathematical cost which has to be stated openly. The contribution of the dissertation is therefore both technical and methodological: it offers a new distance measure, while also documenting carefully where the measure performs strongly and where additional work would be helpful.

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