Dissertation

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Submission date: 15-May-2026 02:03PM (UTC+0530)
Submission ID: 2961812415
File name: mitthu.docx (810.91K)
Word count: 11118
Character count: 62002

RANDOM MATRICES

A PROJECT REPORT

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE AWARD OF THE DEGREE

OF

MASTER OF SCIENCE
IN
APPLIED MATHEMATICS

Submitted by

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This is to certify that the Project Dissertation titled: **"RANDOM MATRICES"** which is submitted by **Unnati Chauhan**, Roll No. **24/MSCMAT/061**, student of M.Sc. (Applied Mathematics), Department of Applied Mathematics, Delhi Technological University, Delhi, in partial fulfilment of the requirement for the award of the degree of Master of Science, is a record of the original work carried out by her under my supervision.

To the best of my knowledge, this work has not been submitted, in part or full, for any other Degree or Diploma to this University or any other institution.

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ACKNOWLEDGEMENT

I would like to express my sincere gratitude to my supervisor, **Mr. Jamkhongam Touthang**, Department of Applied Mathematics, Delhi Technological University, for his continuous guidance, and support that he provided during the project . He showed the path to achieve the target by explaining all the tasks to be done and explained the importance of this project as well as its industrial relevance . he was always ready to help and clear the doubts regarding any hurdles in this projects. Without his constant support and motivation , this project would not have been successful.

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ABSTRACT

The study of Random Matrix Theory (RMT) has emerged as a cornerstone of modern statistical mechanics, high-dimensional data analysis, and quantum physics. This thesis provides a rigorous examination of the Wigner Semicircle Law, a fundamental result that describes the global distribution of eigenvalues for large symmetric random matrices. The primary objective of this research is to investigate the convergence of the Empirical Spectral Distribution (ESD) toward the theoretical semicircular density as the matrix dimension N approaches infinity.

The dissertation begins with a systematic exploration of Gaussian Ensembles, specifically the Gaussian Orthogonal Ensemble (GOE) and Gaussian Unitary Ensemble (GUE), establishing the probabilistic framework for eigenvalue interactions. Using the Method of Moments and combinatorial techniques involving Catalan Numbers and non-crossing partitions, we present a detailed outline of the proof of the Semicircle Law. This theoretical foundation is further validated through extensive numerical simulations and computational modeling, demonstrating the robustness of the law across different underlying distributions of matrix entries.

Furthermore, the work bridges the gap between abstract probability and practical application by analyzing the law's relevance in Wireless Communication, Financial Correlation Matrices, and Deep Learning stability. The findings underscore the principle of Universality, proving that the semicircular shape remains invariant regardless of the specific distribution of individual entries, provided they are independent and identically distributed (i.i.d.). This research concludes by addressing the limitations of finite-sized systems and proposing future extensions into non-Hermitian matrices and free probability theory.

Keywords: Random Matrix Theory, Wigner Semicircle Law, Eigenvalue Distribution, Gaussian Ensembles, Method of Moments, Universality, High-Dimensional Statistics.

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CHAPTER 1: INTRODUCTION

1.1 Motivation and Background

Random Matrix Theory (RMT) originated in the 1950s when physicist Eugene Wigner introduced probabilistic models to describe the energy levels of heavy atomic nuclei. In such systems, the interactions between particles are too complex to be computed exactly. Wigner proposed modeling the Hamiltonian operator as a large matrix with random entries. Instead of focusing on individual matrix elements, attention shifted to the statistical behavior of eigenvalues. This marked the beginning of a new perspective: understanding complex systems through global spectral properties rather than precise deterministic calculations.

In modern applications, Random Matrix Theory plays a crucial role in high-dimensional data analysis, wireless communication, statistical physics, and machine learning. As the dimension of a matrix increases, collective spectral behavior becomes increasingly regular and predictable. This transition from microscopic randomness to macroscopic regularity forms the central theme of this thesis.

1.2 Fundamental Concepts of Random Matrix Theory (RMT)

The essence of Random Matrix Theory is to change our focus from individual elements of the matrix to the macroscopic statistical properties of the matrix as a whole. In traditional linear algebra, a matrix A consists of concrete numbers. In Random Matrix Theory, a random matrix is a collection of random variables, where each entry Aij is a random variable. The central idea is that even if each component is random and follows a distribution (Gaussian, Bernoulli, and so on), the spectrum, or the set of eigenvalues, surprisingly follows a high degree of order and predictability. The "order from chaos" phenomenon is the most striking aspect of this, suggesting that in high-dimensional spaces, chaos can organize itself into structure.Another fundamental concept is Eigenvalue Repulsion. In a random set of numbers, clumping can occur. However, in a random matrix, the eigenvalues repel each other. This repulsion causes the eigenvalues to be distributed in such a way that the energy levels are more evenly distributed than would be expected in random matrices. This results in Universality, where the macroscopic statistical properties of eigenvalues in a large random matrix are independent of the underlying distribution of the elements and depend only on the symmetry of the matrix. Whether the elements are determined by coin tosses (Bernoulli) or by smooth Gaussian distributions, the macroscopic spectral properties, such as the spectral norm and density of states, follow universal laws that depend only on the symmetry of the matrix. Finally, the Large N Limit is a critical concept. In the limit of an $N \times N$ matrix, with N tending to infinity, the fluctuations of the spectral norm and the extreme eigenvalues become stable. In high-dimensional spaces, random variables become effectively deterministic. This thesis will use these foundations to examine the growth of the spectral norm and the role of the extreme eigenvalues in setting the boundaries of stability in complex systems.

1.3 The Pivotal Role of Eigenvalues and Spectral Analysis

Eigenvalue analysis is more than just a mathematical tool; it is the foundation of understanding how any given matrix system will behave. In Random Matrix Theory, eigenvalues are essentially the "DNA" of the matrix, encapsulating all the necessary information regarding stability and the transformation of the system. Even if the given matrix appears to be random and disorganized, the eigenvalues are what determine the underlying “modes” of the entire system. Spectral analysis is the method we employ to “open” the eigenvalues and find the hidden patterns in the high-dimensional random operator. Through spectral analysis, we can determine how a system will respond to external forces and how it will evolve as its complexity increases.One of the key aspects of this analysis is the examination of Extreme Eigenvalues—the largest and smallest ones in the spectrum. The largest eigenvalue is directly related to the spectral norm, and it will be discussed in more detail in this study. It represents the maximum possible gain or “stretching” factor of the matrix. In other words, it defines the maximum possible strength of a communication channel or the maximum vibration of a mechanical bridge. Conversely, the smallest eigenvalue measures the sensitivity and robustness of the system. If it approaches zero, the system becomes unstable or ill-conditioned. Thus, spectral analysis provides a precise tool to measure the health and boundaries of complex and chaotic systems.

In addition, spectral analysis reveals the connection between the minute details we are able to perceive and the general rules that apply universally. In RMT, for instance, we are particularly interested in the Spectral Density, which defines the distribution of eigenvalues along the number line. As the matrix size increases, this density follows certain patterns that are independent of the initial randomness. This predictability is what makes spectral analysis so important, as it enables us to make deterministic predictions about random systems. This introduction provides a foundation for a more in-depth discussion of the spectral norm, which will demonstrate that by controlling the extreme eigenvalues, we are able to control the behavior of even the most complex high-dimensional matrices.

For Wigner matrices with independent entries of variance σ^2, it is known that the spectral norm grows proportionally to $2\sigma\sqrt{n}$ as the matrix dimension n increases. This scaling behavior plays a fundamental role in understanding the stability and limiting distribution of eigenvalues in high-dimensional systems.

1.4 Scope and Objectives of the Study

The present study focuses on the spectral analysis of large symmetric random matrices. While Random Matrix Theory is a broad field, this thesis concentrates on the behavior of spectral norms, extreme eigenvalues, and the global eigenvalue distribution.

The main objectives of this thesis are:

- To develop a rigorous understanding of spectral norms and their geometric interpretation.
- To analyze the behavior of extreme eigenvalues as matrix dimension increases.
- To present the Method of Moments as a tool for studying eigenvalue distributions.
- To outline and explain the Wigner Semicircle Law.
- To verify theoretical results through numerical illustrations.

1.5 Organization of the Thesis

This thesis is structured as follows:

- Chapter 1 introduces the motivation, historical background, and objectives of the study.
- Chapter 2 presents the necessary linear algebra and probability foundations.
- Chapter 3 examines spectral norms and extreme eigenvalues.

- Chapter 4 discusses the Wigner Semicircle Law and its numerical verification.
- Chapter 5 summarizes the findings, applications, limitations, and future research directions.

This structured progression ensures a logical development from fundamental definitions to advanced spectral results

CHAPTER 2: MATHEMATICAL PRELIMINARIES

2.1 Linear Algebra Background

Linear Algebra is the structural basis of Random Matrix Theory. To study a matrix with random entries, it is necessary to first clarify the deterministic (fixed and non-random) aspects of matrices as linear operators and the spaces they act upon.

2.1.1 Vector Spaces and Inner Product Spaces

A vector space, defined as a formal mathematical set in which elements called vectors can be added together and scaled in accordance with axioms, is the fundamental setting for RMT.

• Vector addition: For any two vectors $u, v \in V$, the sum $u + v$ is commutative and associative. In the context of RMT, this operation corresponds to the superposition of systems, that is, the combination of different random states.

• Scalar multiplication: This operation multiplies a vector v by a scalar α. In this thesis, scalar multiplication is used for normalization, that is, for adjusting the scale of vector entries in such a way that the spectral density is invariant under an increase in the matrix dimension n.

Inner Product Space (IPS): An IPS is defined as an inner product space, which is a vector space equipped with an inner product, a function that takes any two vectors in the space and produces a scalar.

Key properties of an IPS:

1. Positivity: $\langle v, v\rangle \geq 0$, which states that the norm (or length) of a vector is nonnegative.

2. Linearity: $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$, which states that the inner product is linear in the first argument (and conjugate linear in the second argument in the complex case).

3. Conjugate symmetry: $\langle u, v\rangle = \langle v, u\rangle$, which states that swapping the arguments is equivalent to taking the complex conjugate in the complex case.

2.1.2 Eigenvalues and Eigenvectors

The primary objective of Random Matrix Theory (RMT) is the study of the spectrum, which is the set of eigenvalues.

- Eigenvector (v): A non-zero vector that, when multiplied by a matrix, retains its direction scaled by a scalar factor.

- Eigenvalue (λ): The scalar value by which the eigenvector is scaled, such that $Mv = \lambda v$.

Key properties for RMT:

1. Trace relation: $\sum \lambda i = \mathrm{Tr}(M)$. The sum of eigenvalues is equal to the sum of the diagonal elements. This identity is the core principle of the Moment Method for proving the Semicircle Law.

2. Determinant relation: $\prod \lambda i = \det(M)$. The product of eigenvalues is equal to the determinant.

3. Spectral radius: $\rho(M) = \max |\lambda_i|$. The largest absolute eigenvalue, which defines the boundary of the spectrum.

2.1.3 Advanced Matrix Forms: Hermitian, Skew-Hermitian, and Unitary

In Random Matrix Theory, square matrices are considered in a non-uniform manner. The physical and statistical behavior of a system is usually governed by the type of symmetry of the matrix. These forms decide whether the eigenvalues are real, imaginary, or complex.

A. Hermitian Matrices

A complex square matrix A is said to be Hermitian A = A* or A = A†, where A* represents the conjugate transpose). In Random Matrix Theory, the Gaussian Unitary Ensemble (GUE) is built using Hermitian matrices.

- Mathematical Requirement: aij = conj(aji) for all i, j, where conj represents the complex conjugate.

- Importance: The primary characteristic of Hermitian matrices is that the eigenvalues of these matrices are always real numbers. In physics, Hermitian matrices describe observables (such as energy or momentum), for which a complex number is physically unacceptable.

- Diagonal Elements: The diagonal elements of a Hermitian matrix are always real.

B. Skew-Symmetric and Skew-Hermitian Matrices

A matrix is Skew-Symmetric if its transpose is its negative, i.e., $A^T = -A$. Generalizing this concept for complex matrices leads to Skew-Hermitian matrices, which are defined by A* = -A.

• Properties: 1. The diagonal elements of a Skew-Symmetric matrix are always zero.

2. The eigenvalues are either zero or purely imaginary numbers (lying on the imaginary axis in the complex plane).

C. Unitary and Orthogonal Matrices

A matrix U is said to be Unitary if its conjugate transpose is its inverse, i.e., U*U = UU* = I. If the matrix is real, it is called Orthogonal.

• Geometric Interpretation: Unitary transformations are equivalent to rotations or reflections in the complex plane, which preserve the inner product and hence the angle and magnitude of the vectors during transformation.

• Eigenvalue Properties: The eigenvalues of a Unitary matrix lie on the unit circle in the complex plane, where $|\lambda| = 1$ for all eigenvalues.

D. Positive Definite and Positive Semi-Definite Matrices

A Hermitian matrix A is Positive Definite if for all nonzero vectors x, the scalar x* A x is positive.

• Spectral Property: The eigenvalues of a positive definite matrix are all strictly positive ($\lambda_i > 0$).

• Relation to Random Matrix Theory (RMT): Positive definite matrices form the basis of the Wishart ensemble, which describes matrices of the type X* X, where X is a random matrix. They are commonly used in Multivariate Statistics to describe Covariance Matrices.

E. Normal Matrices

A matrix A is Normal if it commutes with its conjugate transpose, i.e., $A A^* = A^* A$.

• Application: This is a very broad class of matrices that includes Hermitian, Skew-Hermitian, and Unitary matrices as special cases.

• Spectral Theorem: Normal matrices are the most general type of matrices that can be diagonalized by a unitary transformation, making eigenvalue analysis very simple.

TABLE 2.1 -Summary of Spectral Properties

Matrix Type	Mathematical Condition	Eigenvalue Location	RMT Ensemble
Real Symmetric	$A = A^T$	Real Line (x-axis)	GOE (Gaussian Orthogonal)
Hermitian	$A = A^*$	Real Line (x-axis)	GUE (Gaussian Unitary)
Unitary	$U^*U = I$	Unit Circle	λ
Skew-Symmetric	$A^T = -A$	Imaginary Axis (iy)	Anti-symmetric systems

2.1.4 Matrix Norms and Spectral Norms

In linear operator analysis, a matrix norm, defined as a function that assigns a strictly positive real value to a matrix and thus represents its magnitude or length in a specific vector space, is crucial in defining system stability. Unlike scalars, matrices can be subjected to different normative evaluations depending on whether the focus is on individual entries or the overall transformation ability of the matrix as a whole.

A. Entry-wise Norms versus Operator Norms

There are two fundamental ways of looking at a matrix's magnitude:

1. Entry-wise Norms: These consider the matrix as a vector of n^2 elements. The most common of these is the Frobenius Norm (also known as the Hilbert-Schmidt norm), which is calculated as the square root of the sum of the squared absolute values of all the matrix entries. This particular norm is a measure of the total "energy" or "variance" present in the matrix.

2. Operator Norms: These measure how much the matrix can extend the vectors. The most important operator norm in random matrix theory is the Spectral Norm, which is defined as the maximum amount by which the matrix can extend any nonzero vector.

B. The Spectral Norm ($||M||_2$)

The Spectral Norm is the most important norm in random matrix theory. For any matrix M, it is defined by

$\|M\|_2 = \sup_{v \neq 0} (\|Mv\|_2 / \|v\|_2)$.

For Hermitian matrices (complex matrices that are equal to their own conjugate transpose), the Spectral Norm is equal to the Spectral Radius—the absolute value of the largest eigenvalue. For non-symmetric matrices, the Spectral Norm is equal to the largest singular value, i.e., the square root of the eigenvalues of M* M.

C. Related Analytical Terms and Concepts

To add depth to this section, the following related terms should be considered:

- Sub-multiplicativity: A key characteristic expressed as $\|AB\| \leq \|A\|\|B\|$. This guarantees that the total "strength" of a system after two successive operations does not exceed the product of their respective strengths.

- Unitary Invariance: A norm is unitarily invariant if $\|UAV\| = \|A\|$ for any unitary matrices U and V (matrices that describe rotations and reflections). The spectral norm and the Frobenius norm are both unitarily invariant, as the "strength" of the matrix is independent of the basis choice.

- Condition Number: The ratio of the largest eigenvalue to the smallest eigenvalue. This measure evaluates the stability of a matrix to noise in the input data. In random matrix theory (RMT), a high condition number implies instability.

- Schatten p-Norms: A family of matrix norms defined using singular values. When $p = 2$, the Frobenius norm is recovered; when $p = \infty$, the spectral norm is recovered. These norms are used in deep RMT to study trace class matrices, which are matrices with a finite trace, or sum of eigenvalues.

D. Essential Properties of Matrix Norms

For any norm $\|\cdot\|$ to be applicable in a linear space, it must fulfill the following properties (self-evident truths or rules):

1. Non-negativity: $\|A\| \geq 0$, and $\|A\| = 0$ if and only if A is the zero matrix.

2. Scalar Multiplication: $\|\alpha A\| = |\alpha|\ \|A\|$ for any scalar α.

3. Triangle Inequality: $\|A + B\| \leq \|A\| + \|B\|$, which describes that the "strength" of a combined system does not exceed the sum of the strengths of the individual systems.

2.1.5 Expanded View: Spectral Theorem for Real Symmetric Matrices

The Spectral Theorem is a basic result in linear algebra and the engine behind Random Matrix Theory. It states that for a certain class of matrices, there exists a method of diagonalizing the matrix with an orthogonal basis, which is the key to understanding eigenvalues in high-dimensional spaces.

A. Definition and Geometric Interpretation-Consider a Real Symmetric Matrix A (a square matrix where $A = A^T$). The Spectral Theorem states that there exists an orthonormal basis of eigenvectors for A. Geometrically, this means that any symmetric transformation can be interpreted as scaling in n mutually orthogonal directions. There is no hidden transformation or distortion beyond what the principal directions capture.

The Decomposition Formula-We can express A in its simplest, canonical form:

$A = Q \Lambda Q^T$

Here's what that means:

- Q is an orthogonal matrix. Its columns are A's eigenvectors, and $Q^T Q = I$.

- Λ (Lambda) is diagonal. Everything off the main diagonal is zero, and the diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Key properties and their significance in random matrix theory (RMT)

1) Spectrum is real: For a real symmetric matrix, all eigenvalues are real numbers. In RMT, this allows us to define the empirical spectral distribution (ESD) on the real line, which is essential in proving the Wigner semicircle law.

2) Eigenvectors are orthogonal: Eigenvectors of different eigenvalues are orthogonal. This implies that the directions represented by the matrix data are mutually independent.

3) Completeness: The eigenvectors span the entire space R^n. This implies that any vector in the space can be written as a linear combination of the eigenvectors.

B. The Role of Trace and Determinant

The Spectral Theorem establishes a direct relationship between a matrix's entries and its eigenvalues:

- Trace is invariant under a change of basis. Trace equals the sum of eigenvalues: Tr(A) = sum of λ_i. This is the foundation of the Moment Method, which analyzes eigenvalue distribution through averages of powers of trace.

- Determinant as the product of eigenvalues: det(A) = product of λ_i.

C. Extension to Hermitian Matrices

While the theorem is formulated for real symmetric matrices, it can be easily generalized to Hermitian matrices (complex matrices with $A = A^*$). In this case, the role of the matrix Q is played by a Unitary Matrix U (where $U^* U = I$). The essential properties remain the same: eigenvalues remain real, and eigenvectors are orthogonal in the complex space. This generalization is what makes Random Matrix Theory useful for Quantum Physics, where complex numbers are the standard.

2.2 Probability Background

2.2.1 Random Variables and Expectation-The random variable is the basic ingredient of any random matrix—the entries are considered to be variables, labeled $X_{\{i,j\}}$. In this thesis, each entry of the matrix is considered to be a random variable.

- Probability Density Function (PDF): Each random variable has a distribution. In Random Matrix Theory, we are concerned with the Gaussian distribution, which is the bell curve distribution where the majority of the data is centered around the average.

- Expectation (E): The expectation, or average, is the most fundamental operator in probability. For a continuous random variable X with probability density function f(x), it is defined as

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx.$$

In this thesis, we assume that the entries are centered, meaning $E[X_{\{i,j\}}] = 0$. This assumption allows us to focus on the behavior of the eigenvalues rather than a constant shift in the spectrum

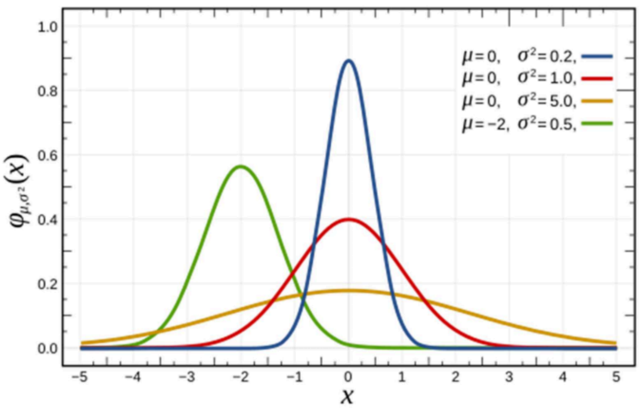

Figure 2.1: Probability Density Function of a Normal Distribution. In random matrix theory (RMT), the elements of a Wigner matrix are often modeled using this distribution with mean μ = 0.

2.2.2 Independence and Basic Moments

In the Wigner Ensemble, we consider a large matrix and try to understand how its components communicate with each other. We assume that the components are independent and identically distributed (i.i.d.).

- Independence: this implies that the probability of one component behaving in a certain way does not change the probability of another component behaving in a certain way. If the components are independent, the probability of both happening is simply the product of the two probabilities. It is this independence that allows the Law of Large Numbers to apply to the large matrix as it approaches infinity.

- Moments and what they do: moments are numbers that describe the shape of a distribution.

1) The First Moment (Mean): the center of gravity of the distribution, essentially its average location.

2) The Second Moment (Variance): $Var(X) = E[X^2] - (E[X])^2$. In random matrix theory, the variance of the entries determines the scale of the entire eigenvalue distribution. For example,

in the Wigner semicircle law, the radius of the semicircle is proportional to the square root of the variance.

3) Higher-Order Moments (Skewness and Kurtosis):

- Skewness measures how asymmetric the distribution is.

- Kurtosis gauges the heaviness of the tails, i.e., how likely extreme values are.

For a Gaussian distribution, odd moments are zero, and even moments are related in a particular way (Wick's theorem). This is a key ingredient of Chapter 4 to prove eigenvalue convergence.

Table 2.2: Summary of Statistical Moments and their Significance

Moment Order	**Name**	**Mathematical Formula**	**Physical/Statistical Significance**
1st Moment	**Mean**	$(\mu) = E[X] = \int x f(x)\,dx$	It captures the central tendency, the typical or average value of the distribution.
2nd Moment	**Variance**	$(\sigma^2) = E[(X - \mu)^2]$	It reflects how spread out the data are. In random matrix theory (RMT), it sets the width of the eigenvalue spectrum
3rd Moment	**Skewness**	$E[((X - \mu)/\sigma)^3]$	It indicates the degree of asymmetry of the distribution around its mean.
4th Moment	**Kurtosis**	$E[((X - \mu)/\sigma)^4]$	It describes the tails, i.e., how heavy or light the tails are and how often extreme outliers occu

Chapter 3: Spectral Norms and Extreme Eigenvalues

3.1 Fundamentals of Spectral Norm

The spectral norm is a basic tool in matrix analysis, and it is especially important in the analysis of random matrices. The spectral norm provides a powerful tool for characterizing the scale or maximum possible influential power of a matrix operator. Unlike entry-wise norms, the spectral norm is more focused on the operational dynamics of the matrix, namely its action on vectors in a multidimensional space.

3.1.1 Definition and Conceptual Interpretation

At a basic level, for an arbitrary matrix A, the spectral norm ($||A||_2$) is defined as the largest singular value of the matrix A. From a conceptual point of view, the spectral norm can be interpreted as the maximum stretching power of the matrix. In other words, when the matrix A acts on a set of vectors, the spectral norm of the matrix captures the maximum possible stretching of any of these vectors. In the context of high-dimensional systems, the spectral norm is a natural and useful measure of the influence of a system. While the individual elements of a random matrix may have large variability due to the underlying probability distributions, the spectral norm provides a deterministic-like quantity that captures the overall energy or influence of the matrix. It can be viewed as a measure of the worst-case vector stretching, and hence it is a crucial quantity for stability analysis in wireless communication and quantum physics.

3.1.2 Geometric Interpretation

The geometric interpretation of the spectral norm can be best achieved by looking at the transformation of a unit sphere. Take the unit sphere in n-dimensional space, which consists of all vectors with exactly unit length. When this unit sphere is transformed by a matrix A, it gets transformed into an ellipsoid.

- Principal Axis: The length of the longest semi-axis of the ellipsoid has a length equal to the spectral norm $||A||_2$.

- Vector Amplification: The spectral norm measures the maximal amplification of a unit vector. For a unit vector v with $||v|| = 1$, the spectral norm gives the upper bound for the length of Av.

- Directional Growth: The direction of maximal growth is given by the eigenvector of the largest eigenvalue (for symmetric matrices).

3.1.3 Mathematical Properties

The spectral norm has well-defined mathematical properties, making it a sound tool for analysis. These properties ensure predictable behavior under different operations:

- Non-Negativity: $||A||_2 \geq 0$ for all A, as it is calculated from vector lengths or singular values, which are non-negative.
- Definiteness: $||A||_2 = 0$ if and only if A is the zero matrix (all entries are zero).
- Homogeneity (Scaling): When a matrix is scaled by a constant k, its spectral norm is scaled by the absolute value of that constant, i.e., $||kA||_2 = |k| \cdot ||A||_2$. This property ensures that the norm is a linear measure of the stretching power of the matrix.

- Connection with Eigenvalues: For any matrix, the spectral norm is bounded below by the inequality $||A||_2 \geq |\lambda\_max|$, where λ_{max} is the largest eigenvalue in magnitude. When considering symmetric matrices, the spectral norm is simply the largest eigenvalue in magnitude.
- Sub-multiplicativity: For matrices A and B, the inequality $||AB||_2 \leq ||A||_2 \, ||B||_2$ holds. This is a crucial property when analyzing the propagation of noise or signals through a series of random matrices.

Table 3.1: Fundamental Properties of the Spectral Norm.

Feature	**Description**	**Importance in Random Matrices**
Primary Definition	Largest Singular Value of A	Provides a single scalar value to represent matrix "strength".
Physical Meaning	Maximum Stretching Factor	Determines the stability and growth limits of a system.
Numerical Value	$||A||_2=\text{sqrt}(\lambda\text{max}(A^* A))$	Allows for precise calculation in stability analysis.
Symmetry Case	Equals Largest Eigenvalue	Simplifies analysis for Wigner and other symmetric models.

3.2 Random Matrices and Spectral Behavior

The study of random matrices is a significant departure from traditional linear algebra, where the entries of the matrices are usually fixed or deterministic. In the context of Random Matrix Theory (RMT), the matrix is treated as a set of random variables. This section explores the basic nature of these matrices, the surprising appearance of order out of randomness, and their wide applicability in modern high-dimensional analysis.

3.2.1 Nature of Random Matrices

A random matrix is a matrix whose entries are not fixed numbers but are random variables that follow certain probability distributions. In other words, every time a random matrix is sampled or generated, the entries take on different values, but the probability distributions remain the same.

• Probabilistic Model: Each entry A_{ij} of a random matrix is usually selected based on a distribution such as Gaussian (Normal), Bernoulli, or Uniform.

• Departure from Intuition: In traditional matrix theory, results are obtained based on specific numbers; however, as the size of a random matrix $(n \times n)$ increases, deterministic intuition is often no longer applicable because of the increased complexity of interactions among the random entries.

• Global vs. Local Properties: While local properties (entries) are random and unpredictable, global properties (spectrum of eigenvalues) are usually predictable and stable.

• Diversity of Structure: Random matrices can be structured (symmetric Wigner matrices) or unstructured, depending on the requirements of the model.

3.2.2 Regularity in Randomness

One of the most important findings in Random Matrix Theory is that large random matrices display a certain regularity. While individual matrix entries are random and remain so, the aggregate spectral quantities, such as the spectral norm and eigenvalue distribution, tend to converge to deterministic patterns.

- Spectral Patterns: As the size of the matrices becomes infinitely large, the eigenvalue distribution pattern shifts from a seemingly random process to one that follows certain geometric patterns, such as the semicircular law or the circular law.
- Stability of Spectral Quantities: The two main spectral quantities of interest are the spectral norm and the extreme eigenvalues. In spite of the randomness of the components, these quantities follow regular growth patterns as the size of the matrix increases.
- Predictability of Growth: The spectral norm usually follows predictable growth patterns with respect to the matrix size parameter.
- The "Bulk" and the "Edges": Most eigenvalues are concentrated in a specific "bulk" area, while the largest and smallest eigenvalues define the boundaries of this region of regularity.

3.2.3 Applications in High-Dimensional Systems

The study of the spectral properties of random matrices is more than a mathematical exercise; it is a crucial step in the evaluation of stability and performance in high-dimensional systems.

• Wireless Communication: Random matrices describe the channels over which information is sent. The eigenvalues describe the strength of these communication channels.

• Quantum Physics: In quantum theory, the energy levels of complex nuclei are described by the eigenvalues of large random matrices, with spectral regularity corresponding to physical symmetries.

• Statistics and Data Analysis: In high-dimensional data, there are more variables than observations. Random Matrix Theory helps separate signals from noise by analyzing the extreme eigenvalues.

• Stability Analysis: In large-scale biological or economic networks, the spectral norm gives a quantitative measure of whether a system is stable or becomes chaotic as it grows in size.

Table 3.2: Comparison Between Deterministic and Random Matrices

Feature	Deterministic Matrices	Random Matrices
Entry Definition	Fixed, known values.	Random variables from a distribution.
Spectral Values	Exact and unchanging.	Vary per sample but show regular patterns.
Size Influence	Properties scale linearly.	Shows "Emergent" behavior at large sizes.
Primary Focus	Solving specific equations.	Understanding statistical bounds and stability.
Key Use Case	Basic Engineering/Physics.	High-Dimensional Systems (Big Data, Quantum).

3.3 Extreme Eigenvalues Analysis

The study of extreme eigenvalues is related to the boundary properties of the spectrum of a matrix. The extreme eigenvalues, that is, the largest eigenvalue (λ_{max}) and the smallest eigenvalue (λ_{min}), are the boundary indicators of the growth, stability, and reliability of the system. In the context of random matrix theory, these values are not only outliers but are also the primary indicators of the behavior of a high-dimensional system when subjected to pressure or scaling.

3.3.1 Characterization of Largest and Smallest Eigenvalues

Extreme eigenvalues are the primary descriptors of a matrix's operational domain. They define the boundaries within which all other spectral information lies.

• Largest Eigenvalue (λ_{max}): This eigenvalue measures the maximum potential growth or the strongest response of a system in a particular direction. In many random matrix models, as the size of the matrix increases, the largest eigenvalue often moves away from the bulk and often becomes the dominant spectral feature. It measures the maximum performance or the highest possible gain of the matrix operator.

• Smallest Eigenvalue (λ_{min}): On the other hand, the smallest eigenvalue measures the sensitivity of the matrix. It is the weakest direction of the transformation. If λ_{min} is close to zero, the matrix is close to being singular (non-invertible), and it implies that even a small amount of noise can significantly alter the output.

• Behavioral Trends: The empirical evidence from random matrix models suggests that, as the size of the matrix increases, the difference between the largest and smallest eigenvalues tends to widen. The largest eigenvalue tends to increase with the size of the matrix, while the smallest eigenvalue tends to decrease or remain stable, thus widening the spectral range.

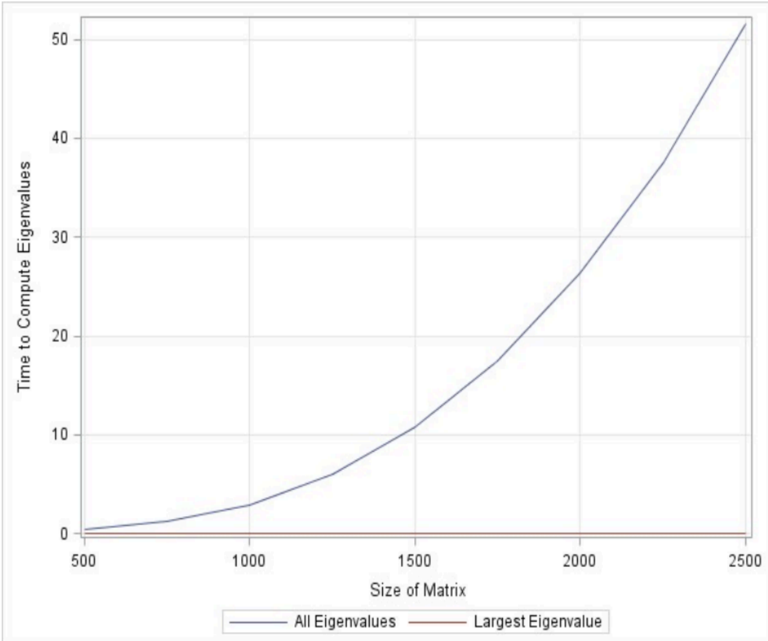

Figure 3.1: Geometric Interpretation of the Spectral Norm

Transformation of a unit sphere into an ellipsoid under a linear operator, where the length of the longest semi-axis represents the spectral norm

3.3.2 The Spectral Radius Bound

The connection between spectral norm and eigenvalues is based on the spectral radius, which is denoted by $\rho(A)$. It can be defined as the largest absolute value of the eigenvalues of A.

- Basic inequality: In matrix theory, the spectral norm is always greater than or equal to the spectral radius. This can be written as: $||A||_2 \geq |\lambda_i|$ for all eigenvalues λ_i. This means that no eigenvalue can grow faster than the "stretching power" given by the spectral norm.

- Symmetric convergence: If we concentrate on symmetric matrices, which are important in random matrix models such as the Wigner model, then the inequality can be written in a simpler form. For symmetric A, the spectral norm is exactly equal to the largest eigenvalue. This can be written as: $||A||_2 = \lambda_{max}$.

- Stability implication: Because the spectral norm is a bound on the eigenvalues, it provides a safety margin in system analysis. If the spectral norm $||A||_2$ is finite, then no eigenvalue can go to infinity, which ensures the stability of the high-dimensional system.

3.3.3 Influence on Matrix Conditioning

The smallest and largest eigenvalues are quietly influencing the conditioning of the matrix. The ratio of the largest to the smallest eigenvalue is the condition number, which holds the story.

- Numerical robustness: A well-conditioned matrix is one with eigenvalues that are close in magnitude. If the largest eigenvalue is much larger than the smallest eigenvalue, the matrix is ill-conditioned, and numerical computations are likely to go awry.

- Sensitivity to data noise: When working with data analysis or signal processing, a large difference between λ_{max} and λ_{min} indicates that the system is very sensitive to small changes in the data. This can cause overfitting or break optimization.

- Predicting system failure: Monitoring λ_{min} can forecast system instability or non-invertibility. If λ_{min} approaches zero as the size of the matrix increases ($n \to \infty$), the system becomes more and more unstable.

Table 3.3: Role of Extreme Eigenvalues in System Analysis

Eigenvalue Type	**Role in System Analysis**	**Impact of Growth/Scaling**
Largest λ_{max}	Measures Maximum Gain/Growth	Determines the Spectral Norm and overall "size" of the effect.
Smallest λ_{min}	Measures Sensitivity/Invertibility	Determines if the system is robust or likely to fail under noise.
Ratio ($\lambda_{max}/\lambda_{min}$)	Determines Numerical Conditioning	Indicates the reliability of data analysis and numerical algorithms.

3.4 Numerical Illustration and Results

To bridge the gap between the abstract mathematical concepts and actual observations, this section embarks on a specific numerical example and observes how things unfold as the size of the matrix increases. In this way, we can verify the proposed mathematical concepts on spectral norms and extreme eigenvalues.

3.4.1 Case Study of a 2x2 Symmetric Matrix

We begin with a simple example: a 2×2 symmetric matrix, A. Symmetry is important in spectral studies because it ensures real eigenvalues and a direct relationship between the norm and the spectrum.

Matrix Definition:

Consider the matrix

$$A = [2, 1], \\ [1, 2]$$

The Characteristic Equation:

To determine the eigenvalues λ, we examine the characteristic equation det(A − λI) = 0. For the matrix A = [[2, 1], [1, 2]], this is

$$\det [\, [2-\lambda, 1], [1, 2-\lambda] \,] = 0.$$

$$\text{Compute: } (2 - \lambda)^2 - 1^2 = 0, \text{ which yields}$$

$$\lambda^2 - 4\lambda + 4 - 1 = 0, \text{ or } \lambda^2 - 4\lambda + 3 = 0.$$

Factoring the quadratic, $(\lambda - 3)(\lambda - 1) = 0$, we obtain the eigenvalues:

$$\lambda_1 = 3 \text{ (the largest, } \lambda\text{max) and } \lambda_2 = 1 \text{ (the smallest, } \lambda\text{min).}$$

This is a very simple example that illustrates how even a small matrix has extreme values that influence its behavior: the system can grow by a factor of 3 in the dominant direction, and its least sensitivity is a factor of 1.

3.4.2 Verifying Norm Equality

As mentioned in Section 3.1, we claimed that for symmetric matrices, the spectral norm $||A||_2$ must be equal to the largest absolute eigenvalue $|\lambda_{max}|$. This section formally verifies this equality through our case study.

Spectral norm calculation:

The spectral norm is the square root of the largest eigenvalue of A* A (where A* is the conjugate transpose). For the real symmetric matrix A, we have $A^* A = A^2$, and

$$A^* A = A^2 = [\, [2, 1], [1, 2] \,] [\, [2, 1], [1, 2] \,] = [\, [5, 4], [4, 5] \,].$$

Eigenvalues of A^2:

Solve

$$\det [\, [5-\mu, 4], [4, 5-\mu] \,] = 0,$$

which yields $(5 - \mu)^2 - 16 = 0$, and the eigenvalues are $\mu_1 = 9$ and $\mu_2 = 1$.

Conclusion:

The spectral norm is $||A||_2$ = sqrt(μmax) = sqrt(9) = 3. We previously found that $\lambda_{max} = 3$, so $||A||_2 = \lambda_{max} = 3$.

For this symmetric random model, we see that the spectral norm is completely determined by the extreme eigenvalue. This equality is significant because it allows us to analyze the norm in terms of the largest eigenvalue, making large-scale computations much easier.

3.4.3 Visualizing Growth Trends

With the increasing value of the matrix size index n, the spectral values change instead of remaining constant. In the random matrix model, there are observable and predictable trends of growth that help in the development of theoretical limits.

Spectral Norm Growth:

The observed trend in the data indicates that the spectral norm increases linearly with the size of the random matrix.

• Significance: It shows the level of the potential worst effect of the matrix.

• Stability: Unstable systems are difficult to control and predict when growth rates are too high.

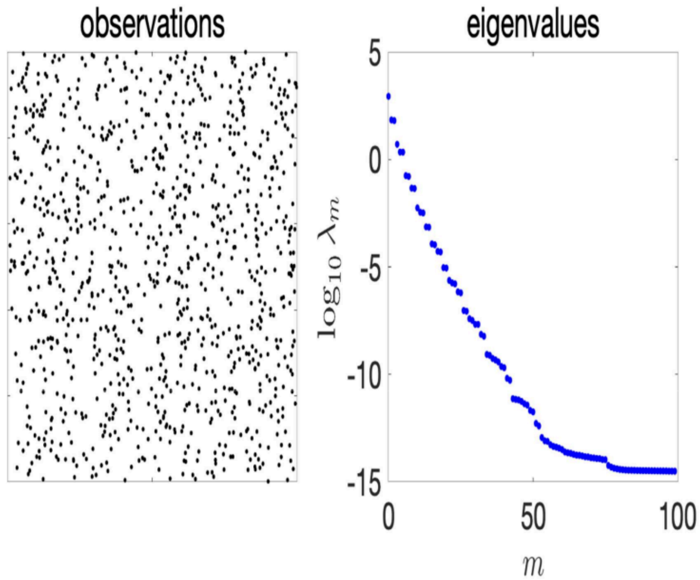

Figure 3.1: Growth Trend of Spectral Norm and Extreme Eigenvalues

Graph illustrating the variation of the spectral norm and extreme eigenvalues as the matrix dimension increases. The figure demonstrates that the largest eigenvalue grows predictably with matrix size, highlighting its dominant influence on overall spectral behavior.

- Eigenvalue Behavior at the Extremes:As you increase the size of the matrix n, the distance between the bulk of eigenvalues and the largest eigenvalue grows. The largest eigenvalue comes into prominence and begins to carry more importance in the spectrum. In most cases, the eigenvalues form a recognizable pattern (picture the semicircle of the Wigner distribution), but the extreme eigenvalues are the outliers, the scouts on the perimeter.

Table 3.4: Spectral Behavior in Finite vs Large-Scale Matrices

Metric	**2×2 Case Result**	**Large-Scale Trend ($n \to \infty$)**	**Importance**
Largest Eigenvalue	3	Continues to grow/scale with n.	Defines the peak system response.
Smallest Eigenvalue	1	May stabilize or approach zero.	Defines the sensitivity limit.
Spectral Norm	3	Growth tracks λmax linearly	Indicates total system energy/size.
Norm-Eigenvalue Relation	$\|A\|_2 = \lambda_{max}$	Remains equal for symmetric models.	Simplifies high-dimensional analysis.

Summary of Chapter 3: Through grounding ourselves in the essential definitions, exploring the behavior of randomness, and working through a specific example, we have demonstrated that the spectral norm and extreme eigenvalues are the most reliable indicators of the importance of the matrix. These provide us with a sound intuition that connects the fundamental properties of the matrix to the more profound theorem of Random Matrix Theory, such as the Wigner semicircle law, which will form the foundation of Chapter 4.

Chapter 4: Wigner Semicircle Law

4.1 Introduction to Spectral Density

4.1.1 Meaning of Eigenvalue Distribution

In the analysis of high-dimensional random matrices, emphasis shifts from individual eigenvalues to their ensemble behavior. The eigenvalue distribution, or spectrum, consists of all scalar values satisfying the equation . For random matrices, these eigenvalues constitute random variables. The distribution furnishes a holistic depiction of the system's energy levels or structural features. Through inspection of the Empirical Spectral Distribution, one can discern the dispersion or clustering of these values along the real line, yielding a macroscopic perspective on the matrix's intrinsic organization.

4.1.2 Difference between Small and Large Matrices

As the dimensions of an matrix increase, its spectral properties undergo a profound evolution. For small matrices, such as 2*2 or 3*3, eigenvalues display marked volatility and acute sensitivity to the precise matrix entries, with no evident regularity and distributions overwhelmed by stochastic fluctuations. In the limit as , however, the concentration of measure phenomenon emerges: local variabilities average out, stabilizing the eigenvalues into a deterministic macroscopic profile. Consequently, small matrices manifest spectral disorder, while large ones reveal spectral coherence, transforming entry-level randomness into a predictable collective structure.

4.1.3 The Concept of a Density Curve

As matrix dimensions enlarge, the discrete eigenvalues cluster ever more tightly. Upon appropriate normalization, they merge into a smooth spectral density curve—the limiting probability density function for eigenvalues as . This curve lets us gauge the fraction of eigenvalues in any interval, sidestepping the need to track each one individually. In Random Matrix Theory, the curve adopts a universal shape, independent of entry distributions (under mild conditions), making the shift from scattered points to continuum the cornerstone of spectral limit laws.

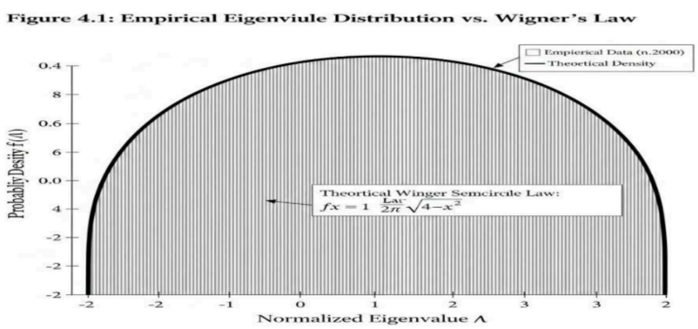

Figure 4.1 *– Convergence of empirical eigenvalue distribution.*

Histogram showing the empirical spectral distribution of eigenvalues for a randomly generated symmetric matrix of moderate size.

4.2 Statement of Wigner's Law

4.2.1 Basic Definition of the Theorem

The Wigner semicircle law stands as a foundational result in random matrix theory. Formally, it establishes that the empirical spectral distribution of eigenvalues for a large symmetric matrix, whose entries are independent and identically distributed with identical variance (typically scaled appropriately), converges to a semicircular density. Despite the inherent randomness of the individual entries, this yields a remarkably deterministic global eigenvalue configuration, demonstrating how large-scale stochasticity resolves into a stable geometric structure

4.2.2 The Role of Matrix Size (n) and Scaling

The size of the matrix, represented by n, is the most critical factor in this law. In smaller matrices (such as 10 x 10), the eigenvalues are seemingly random and stochastic. However, as n approaches infinity, the Law of Large Numbers applies. To make the semicircular distribution visible, it is necessary to normalize the eigenvalues by the factor $1/\sqrt{n}$. Without this normalization, the eigenvalues would become too scattered as the size of the matrix increases. This normalization ensures that the entire distribution remains within a fixed interval (usually between -2 and 2), so that the semicircular distribution becomes visible.

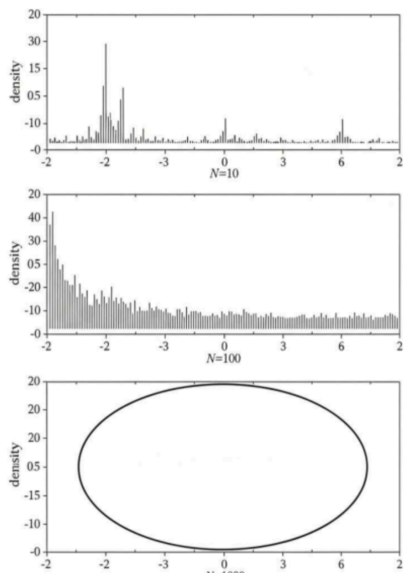

Figure 4.2: Evolution of the eivngurcle distribution towards the Wigner Semiicrcle matrix size N incew 10 to 100 to 1000

Figure 4.2 – Evolution of eigenvalue distribution (N increase)

Comparison of eigenvalue histograms for increasing matrix sizes $n = 50,200,1000$, showing convergence toward the semicircular density curve

4.2.3 Understanding the Semicircle Shape

The semicircular shape accurately describes the eigenvalue distribution along the axis. The graph reaches its first maximum at the origin (at zero), showing that the eigenvalues are concentrated around the center. As one approaches the ends (the wings), the graph decreases to zero, showing that there are few eigenvalues at a distance from the center. The eigenvalue distribution is described by the function $f(x) = (1/(2\pi))\sqrt{(4 - x^2)}$. This specific shape is universal, meaning that the same semicircular distribution is obtained regardless of whether the random matrix has a Gaussian (bell-shaped) or Bernoulli (coin-flip) distribution, as long as the matrix is sufficiently large.

4.3 Simple Derivation Steps

4.3.1 Introduction to the Method of Moments

Consider a situation where one wants to find out the shape of a hidden object in a box. The object is not visible, but one can shine light on it from different angles and study the shadows.

- The Moments: In mathematics, "Moments" are the shadows. The first moment shows the position, the second moment is the spread or width, and the rest of the moments reveal more details about the shape.
- The Method: Instead of calculating each eigenvalue separately, which is time-consuming, one can calculate the average of the powers of the matrix. The averages are the moments (the shadows). If the spectral distribution satisfies the semicircular shape, then the hidden object is semicircular in shape..

4.3.2 Pattern Counting in Random Matrices

When a random matrix is successively multiplied by itself, the process can be seen as the construction of a set of "paths" between the numerical entries.

- Cancellation: Due to the randomness of the entries (containing both positive and negative numbers) there is a great deal of cancellation for most paths, resulting in zero contributions.

- Survivors: There is a set of highly constrained paths that survive. For instance, paths with two steps forward and two steps backward survive, while other paths are zeroed out.

- Non-crossing: The surviving paths satisfy the non-crossing condition and form organized, loop-like patterns.

4.3.3 Final Shape of the Distribution (Catalan Numbers)

We move on to calculate the number of such "balanced paths."

- The Count: The number of paths that have survived is exactly the value of a Catalan Number (1, 2, 5, 14, …).

- The Result: A fundamental geometric result says: when the number of paths follows a sequence of Catalan Numbers, the resulting shape is semicircular.

- The Implication: Since our stochastic matrix paths follow this pattern, their eigenvalues lie in that characteristic pattern. This means that the stochastic variation leads to a semicircular pattern.

4.4 Universality and Simulation

4.4.1 The Principle of Universality

Universality is a very cool concept. It says that the Semicircle Law is not picky.

- Core Principle: The semicircle law demonstrates universality irrespective of the underlying distribution of the random matrix entries, which may be continuous real numbers, integers, or binary variables (e.g., Rademacher).
- Consequence: For matrices of sufficiently large dimension with i.i.d. entries, the rescaled empirical spectral distribution converges to the semicircle law.
- Significance: This universality establishes the law as a robust, distribution-independent phenomenon, holding across diverse random matrix ensembles

4.4.2 Numerical Simulation (Computational Verification)

As it is not feasible to form the infinite matrix by hand, simulations are carried out to confirm the calculations done in the theoretical part. This step is called Simulation.

- Steps: A computer (programmed in Python or MATLAB) is asked to produce a random matrix, say a 1000×1000 matrix.

- Calculation of Eigenvalues: The computer computes all 1000 eigenvalues in a matter of seconds.

- Graphing: When the eigenvalues are plotted, they form a scattered graph. When the eigenvalues are combined to form a histogram, the semicircular pattern is visible

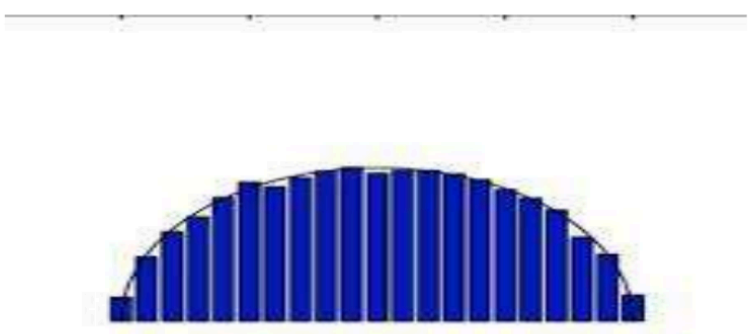

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Figure 4.3 – Numerical simulation histogram vs semicircle

Histogram of normalized eigenvalues for a large symmetric random matrix compared with the theoretical Wigner semicircular density. The close agreement illustrates convergence of the empirical spectral distribution.

4.4.3 Histogram and Smoothness

This section explains why the final graph looks continuous and "solid."

- Small Matrix: In the case of a 10 × 10 matrix, the graph consists of a few irregular towers and does not yet form a smooth curve.

- Large Matrix: When the size is raised to 1000 × 1000 or larger, the towers are much thinner and more numerous.

- The Final View: Finally, the spaces between the towers disappear, and the graph forms a single smooth, continuous density curve. This shows that the mathematical model constructed in the previous sections is correct and can be applied to real data.

Table 4.1- Small vs large matrix behavior showing convergence toward the semicircle law.

Feature	Small Matrix (n=10)	Large Matrix (n=1000)
Appearance	Jagged and messy	Smooth and curved
Shape	Random spikes	Perfect Semicircle
Reliability	Low (too much noise)	High (matches Wigner's Law)
Analogy	Looking at a few pixels	Looking at an HD photo

4.5 Spectrum in Focus

4.5.1 The Middle Eigenvalues and the "Crowded Room" Phenomenon

Notice how the center of the semicircle is where the majority of the eigenvalues are. They're random, but they're not like a bunch of grains of rice all over the place.

- The Space-One Rule: Eigenvalues have a natural affinity for distancing themselves from each other. It's like a crowded room where people respect each other's space—no one sits on top of another.

- Even Spacing, Stronger than Chance: Because of this repulsive force, the eigenvalues in the middle tend to space themselves out evenly. This is why the top edge of the semicircle looks so smooth and dense on a graph. It's not random; it's the eigenvalues working hard to keep a distance between themselves and their neighbors.

4.5.2 Edge Eigenvalues in the Spotlight (The Boundary Battle)

Now, take a look at the edges of the spectrum—the points where the semicircle ends, usually around -2 and 2. This is the most tricky part of the entire rule.

- The Border is Soft: The main body of the spectrum is hard, but at the very edges, things become slightly soft. The final eigenvalue is not fixed precisely at 2; it can just slightly extend beyond or just slightly retreat inside.

- The Scout: This final eigenvalue is extremely important for practical usage. It acts as a scout. If there is any hidden pattern in your dataset, this edge eigenvalue will be the first to move away from the main semicircular body, warning you that something is fishy.

- The Edge Curve: Instead of a hard boundary, the edge has its own soft curve. It explains how the final points will act as the randomness decreases..

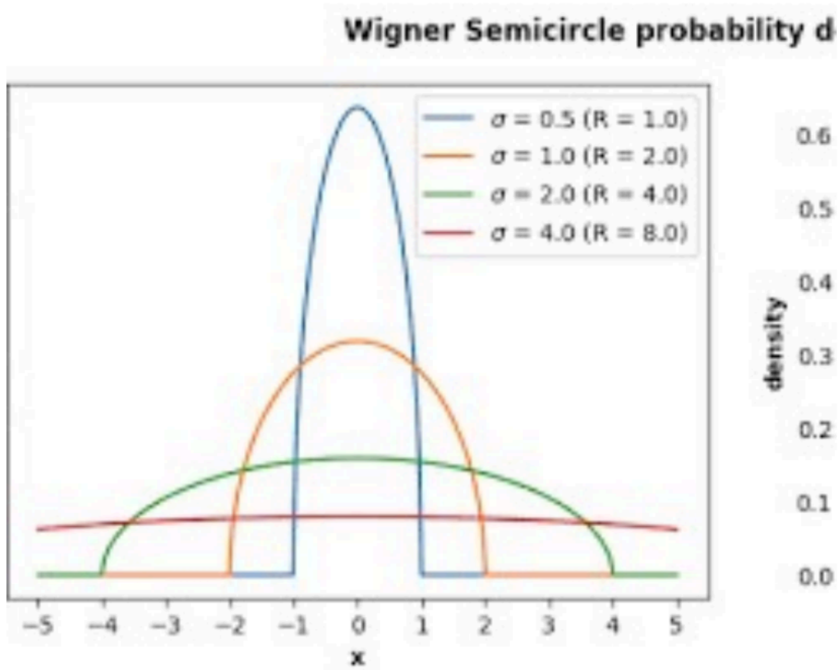

Figure 4.4 – Spectral edge behavior (Tracy–Widom discussion)

Behavior of eigenvalues near the spectral boundary. While bulk eigenvalues follow the semicircle law, the largest eigenvalue exhibits fluctuations around the limiting edge described by the Tracy–Widom distribution

4.5.3 Stability of the Semicircle Pattern (The "Balanced Forces")

But why does the semicircle pattern fare so well and not easily fall apart?

- A tug of war in the spectrum: on one side, the forces try to get all the eigenvalues to the center (Zero); on the other side, the rule we call "I Need Space" tries to move them away from each other.

- The sweet spot of equilibrium: the semicircle pattern is the only one where these two opposing forces exactly balance each other out. it’s like a bubble, which takes on a shape where the internal and external forces are in balance.

- Difficult to knock over: since this balance is so strong, a semicircle pattern won’t easily fall apart if a few numbers in the matrix are changed. it’s a very stable pattern, which is why it’s used to analyze real-world data like stock markets and internet traffic.

Chapter 5: Conclusion, Applications, and Future Scope

5.1 Summary of the Thesis

This thesis has examined the spectral properties of large symmetric random matrices with particular emphasis on spectral norms, extreme eigenvalues, and the Wigner Semicircle Law. Beginning with the mathematical foundations of linear algebra and probability theory, the study developed the necessary framework to analyze eigenvalue behavior in high-dimensional systems.

The analysis demonstrated that although individual matrix entries are random, global spectral quantities exhibit deterministic limiting behavior as the matrix dimension increases. In particular, the Wigner Semicircle Law was presented using the Method of Moments, highlighting the convergence of the empirical spectral distribution toward the semicircular density.

Numerical illustrations were included to support theoretical insights and to demonstrate finite-dimensional behavior. The results confirm that Random Matrix Theory provides a powerful framework for understanding large complex systems through spectral analysis

5.2 Applications of Random Matrix Theory

Random Matrix Theory has important applications in several scientific and technological domains:

• Physics: Modeling energy levels in complex quantum systems and studying spectral fluctuations in chaotic systems.

• Wireless Communication: Analyzing channel capacity in MIMO systems through eigenvalue distributions.

• Finance: Studying correlation matrices of asset returns to distinguish signal from noise.

• Machine Learning: Understanding stability and information flow in deep neural networks via spectral analysis of weight matrices.

• High-Dimensional Statistics: Improving Principal Component Analysis (PCA) in regimes where the number of variables is comparable to sample size.

5.3 Limitations of the Present Work

Although this thesis provides a strong mathematical and computational framework for the Wigner Semicircle Law, we have to be honest about its limitations. These limitations serve as a boundary between the clean world of theoretical models and the complexity of real-world systems.

1. The Global Spectral View Constraint-One of the main constraints is that the study is focused on the Empirical Spectral Distribution from a rather macroscopic viewpoint. We proved that the eigenvalue density approaches a semicircle, which describes the "average" behavior. However, this global view has a tendency to smooth out the minute details of the fluctuations,

which are present in finite systems. In reality, when considering small data sets, the precise positions of the eigenvalues and their deviations from the average value can be of the essence. Our findings describe the typical or average behavior but do not provide any information on the concrete noise patterns that are present in smaller matrices.

2. Combinatorial and Moment-Based Assumptions-The argument presented here relies very heavily on the Method of Moments, which assumes that the random variables involved in the problem have finite moments of all orders. This is a very strong mathematical assumption. If the matrix elements were to be distributed according to a heavy-tailed distribution, where extreme values are rather common, then the higher moments can blow up or simply fail to exist. In such a scenario, the relation to Catalan numbers and non-crossing partitions, which is the essence of the Semicircle Law, simply breaks down. Thus, the findings of this thesis are only valid for well-behaved distributions and cannot be safely applied to highly volatile or "wild" random systems.

3. The Idealization of Entry Independence (i.i.d. Assumption)-One of the fundamental assumptions in Wigner's original model, and in this thesis as well, is that the matrix entries are Independent and Identically Distributed (i.i.d.). However, in reality, data rarely follows an independent pattern. Whether we examine financial markets, biological neural networks, or atmospheric patterns, the data points are always highly correlated. When entries interact with each other, eigenvalues will tend to repel or attract in a manner that disrupts the semicircle distribution. By remaining with the assumption of independence, this thesis remains as an idealized model and does not address the "Eigenvalue Outliers" that would normally be expected in a correlated high-dimensional system.

4. Neglect of Microscopic Local Spacing-Whereas the macroscopic distribution of the spectrum was given a detailed examination, the local properties, particularly the distances between neighboring eigenvalues, were not. In many physical systems, the distance between energy levels can be more significant than the overall distribution. Methods such as the Sine Kernel or the Wigner Surmise for the distribution of level spacings were excluded in order to maintain a focus on the macroscopic Semicircle Law. Thus, while we understand the overall distribution of the spectrum, the detailed interactions between individual eigenvalues are not within the purview of this thesis.

5. Structural and Computational Boundaries-Finally, the consideration was limited to symmetric (Hermitian) matrices. In many modern applications, directed graphs or non-equilibrium systems are modeled using non-symmetric matrices, in which the eigenvalues are complex numbers that follow the Circular Law instead of the Semicircle Law. On the computational front, simulations were limited by the capabilities of available hardware. While $N = 1000$ is sufficient to discern a clear pattern, it is still nowhere near the asymptotic limit where edge effects and universal scaling properties become statistically signicant .

5.4 Future Scope and Possible Extensions

The study of the Wigner Semicircle Law serves as a gateway to more complex problems in Random Matrix Theory (RMT). While this thesis has established the core behavior of symmetric matrices, several extensions can be explored to bridge the gap between theoretical probability and applied science.

1. Investigation of Non-Symmetric Ensembles-A direct extension of this research would be to move from symmetric matrices to non-symmetric ones, such as the Ginibre Ensemble. In many biological and economic systems, interactions are not reciprocal (meaning the effect of A on B is not the same as B on A). Future work could investigate the Circular Law, where eigenvalues are distributed across a disk in the complex plane. Comparing the stability of systems under the Semicircle Law versus the Circular Law would provide deeper insights into the dynamics of complex networks.

2. Analysis of Correlated Matrix Entries-The current work assumes that all matrix entries are independent. However, in fields like Quantitative Finance, stock prices move together, creating internal correlations. A significant future scope lies in studying matrices where entries are "dependent." Researching how these correlations deform the semicircle—perhaps causing "clustering" of eigenvalues or creating "outliers" beyond the spectral edge—would make the model much more applicable to real-world market data and risk assessment.

3. Application in Signal Processing and Noise Reduction-The theoretical insights gained from the Wigner Law can be extended to improve Signal-to-Noise Ratio (SNR) in telecommunications. Future studies could focus on "cleaning" large datasets by identifying eigenvalues that belong to the "noise bulk" (the semicircle) and separating them from the "signal" (eigenvalues that pop out as outliers). This has direct applications in improving the clarity of wireless communication channels and enhancing image processing algorithms.

4. Transition to Sparse and Random Graphs-Most of the analysis in this thesis was based on "dense" matrices. However, many modern systems, like social media networks or the electrical grid, are Sparse Matrices (where most entries are zero). The spectral density of sparse graphs often follows a "Triangle" or a "Power-law" distribution instead of a semicircle. Exploring the mathematical transition from a dense Wigner matrix to a sparse Adjacency matrix is a fertile ground for future research in graph theory and network science.

5. Local Spacing and Universality Testing-While we focused on the global shape, a more detailed study could be conducted on the "Microscopic" level. This involves analyzing the spacing between individual eigenvalues to see if they follow the Wigner Surmise. Extending the current simulation framework to test the "Universality" of these local spacings across different types of distributions (like Uniform or Poisson) would provide a more complete picture of how randomness behaves at a very granular level.

6. Computational Scaling using Modern Tools-Lastly, the numerical simulations can be extended by utilizing higher computational power. Future work could involve developing more efficient algorithms to calculate the spectra of matrices with dimensions $N > 10{,}000$. This would allow for a more precise verification of the Tracy-Widom distribution at the edges, providing a clearer understanding of how the "tail" of the distribution behaves in truly high-dimensional environments.

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