Convergence Analysis of the Janardan Operator Using Korovkin Theorem and Voronovskaja Asymptotic Formulation

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Submitted by

Niranjan (24/MSCMAT/11)

Anushka (24/MSCMAT/34)

Under the supervision of

Prof. Naokant Deo

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi 110042

MAY, 2026

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042

CANDIDATE'S DECLARATION

We **Niranjan** (24/MSCMAT/11) and **Anushka** (24/MSCMAT/34) hereby certify that the work which is being presented in the Dissertation entitled: **"Convergence Analysis of the Janardan Operator Using Korovkin Theorem and Voronovskaja Asymptotic Formulation"** in partial fulfilment of the requirements for the award of the Degree of Master of Mathematics, submitted in the Department of Applied Mathematics, Delhi Technological University is an authentic record of our own work carried out during the period from July 2025 to May 2026 under the supervision of **Prof.Naokant Deo**. The matter presented in the Dissertation has not been submitted by us for the award of any other degree of this or any other Institute.

Niranjan (24/MSCMAT/11)
Anushka (24/MSCMAT/34)

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042

CERTIFICATE BY THE SUPERVISOR

I hereby certify that the Project Dissertation titled "Convergence Analysis of the Janardan Operator Using Korovkin Theorem and Voronovskaja Asymptotic Formulation" which is submitted by **Niranjan** (24/MSCMAT/11) and **Anushka** (24/MSCMAT/34) Department of Applied Mathematics Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Masters of Science, is a record of the project work carried out by the students under my supervision. To the best of my knowledge this work has not been submitted in part or full for any Degree or Diploma to this University or elsewhere.

Place: Delhi

Naokant Deo

Department of Applied Mathematics

DEPARTMENT OF MECHANICAL ENGINEERING
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042

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Place: Delhi — Niranjan (24/MSCMAT/11)

Date: 01.06.2021 — Anushka (24/MSCMAT/34)

Abstract

We have seen many complex functions that are typical to handle in mathematical analysis directly and replacing them with simpler ones that exhibit their properties has become a neccessary part. In mathematical analysis one of the most well known and approachable tool is approximation with operators. This thesis is devoted to one such positive linear operator referred to as Janardan operator that is constructed on the basis of probabilistic distribution that helps to approximation continuous functions. This investigation begins with the computation of moments of Janardon operator over three test functions as constant function $f(t) = 1$, the linear function $f(t) = t$, and the quadratic function $f(t) = t^2$. These moment estimates occupy a central place in the analysis, since they reveal how the operator responds to the most elementary inputs and thereby lay the groundwork for studying its behaviour on arbitrary continuous functions.

Once the moments are established, the classical theorem of Korovkin is invoked. This theorem asserts that if a sequence of positive linear operators converges to the identity on the three test functions mentioned above, then it converges uniformly to any continuous function defined on a closed and bounded interval. The Bohman–Korovkin criterion is verified for the Janardhan operator, thereby establishing its uniform convergence on $C[a, b]$.

Beyond establishing convergence, the thesis addresses the quantitative aspect of approximation, that is, how rapidly the operator output approaches the target function. For this purpose, the modulus of continuity is employed as a measure of the oscillation of a function, and explicit error bounds are derived. These bounds provide a guaranteed estimate of the maximum deviation between the operator and the function being approximated.

A Voronovskaja-type asymptotic formula is also derived for the Janardhan operator. Unlike convergence theorems, which merely confirm that the error tends to zero, a Voronovskaja result describes the precise asymptotic behaviour of the error, scaled ap-

propriately, as the parameter tends to infinity. This gives a sharper and more complete picture of how the approximation process unfolds in its final stages.

The thesis concludes with numerical computations carried out for selected test functions. These calculations illustrate the theoretical results in concrete terms and confirm that the approximation behaviour predicted by the analysis is indeed observed in practice. Taken together, the results establish the Janardhan operator as a mathematically well-founded and practically effective tool within the broader framework of approximation theory.

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List of Symbols and Abbreviations

$\omega(f, \delta)$	Modulus of continuity of f
$C[a, b]$	Space of continuous functions on $[a, b]$
$\|\cdot\|$	Uniform (supremum) norm
$L_m(f; x)$	Janardan operator
$B_n(f; x)$	Bernstein operator
e_i	Monomial test function $e_i(t) = t^i$
$\mu_{m,n}(x)$	n-th order moment of L_m
p_x	Probability parameter of the Janardan operator
b, c	Shape parameters, $b + c \geq 0$, $c > 0$
m	Degree / index of the operator
$\Gamma(\cdot)$	Gamma function
$B(\cdot, \cdot)$	Beta function

Chapter 1

INTRODUCTION

1.1 Theoretical Framework

Approximation theory is one of the cornerstones of mathematical analysis and deals with the problem of approximating complicated functions by simpler ones. The theory is rooted in the idea that all exact science is subject to approximation. This idea, often ascribed to Bertrand Russell, emphasizes that numerical calculations and measured data are strictly the analytical tools or experimental results employed to record them. So, when you calculate you approximate. The theory has many important applications in numerical analysis, signal processing, computer graphics and computational mathematics.Linear positive operators play a central role in approximation theory, providing constructive methods for approximating continuous functions with desired accuracy.

The initial abstract concepts of this theory were followed by efforts to represent functions to provide them practically useful. Thus the formulae were determined to aid in approximating mostly transcendental functions. At initial these representations depended on Taylor's formula and few formulae of interpolation depends on Newton's ideas. Although these formulae gave better approximations in particular special cases,in general they have failed to mannage the approximation error because the functions are not approximated uniformly; the error outgrown the interpolation points or the point of expansions. The foundation of the constructive function theory gained significant looking problem to which Henri Lebesgue provide the proof of Weierstrass Approximation theorem had already attracted attention: the approximation of the function $|x|$. In 1898 Henri Lebesgue provided the proof of the Weierstrass Approximation Theorem by first approximating a continuous function by polygonal lines and subsequently proving that a polygonal line can be arbitrarily well approximated by polynomials [1]. Every mathematical problem would ideally have a neat, exact solution. In reality, we often encounter functions that are too expensive to calculate, and equations that are impossible to solve analytically. Approximation theory is the bridge between the theoretical and the practical. The emphasis is on the approximation of complicated functions by simpler ones (usually polynomials or trigonometrical functions) with a tolerable error.

Approximation theory provides a great link between pure and applied mathematical analysis dealing with the problem of how functions can be efficiently and accurately represented by simpler structures such as polynomials or sequences of operators.

The Weierstrass Approximation Theorem about the approximation of continuous functions by sequences of polynomials, which is accepted to be the most significant contribution in the field of approximation theory, gives an essential background for the construction of the notion of the linear positive operator (in 1885). Famous Weierstrass theorem which showed that every continuous function on the closed interval is uniformly approximable by polynomials was proved by Weierstrass [Weierstrass K., Uber die ana- lytische Darstellbarkeit sogenannter willkurlicher functionen einer reellen Varendarlichen, Sitzungberichte der Akademie zu Berlin, 1885, 633-639; 789-805].This landmark result motivated the development of explicit polynomial operators that provide effective approximation schemes. Several modifications and demonstrations have been documented in the history of research in subsequent years. The most fascinating of these was the proof of Bernstein(1912). Bernstein used the concept of binomial theorem for the moment estimation by bernstein operator for achieving infinitesimely small deviation from target function. The binomial theorem is given as:

$$(x+y)^m = \sum_{i=0}^{m} \binom{m}{i} x^{m-i} y^i, \qquad i = 0, 1, 2, \ldots, m.$$

Let f be a bounded function on closed intervel $[0,1]$. The S.N. Bernstein operators of degree m with respect to f is defined as:

$$B_m(f;x) = \sum_{i=0}^{m} B_{m,i}(x) f\left(\frac{i}{m}\right); \quad x \in [0,1] \tag{1.1}$$

where,

$$B_{m,i}(x) = \binom{m}{i} x^k (1-x)^{m-i}, \quad x \in [0,1], \quad i = 0,1,2,\cdots,m, \quad \text{and,}$$

$$\binom{m}{i} = \frac{\Gamma(m+1)}{\Gamma(i+1)\,\Gamma(m-i+1)}.$$

Further, $B_{m,i}(x) \in \mathbb{P}_m$, $i = 0,1,2,\cdots,m$, where $\mathbb{P}_m$ represents the collection of polynomials of degree less than or equal to m.

The linear positive operator theory was developed as a result of this proof and the well-known Bernstein operators..The Bernstein operators are one of the most important examples within the family of linear positive operators. When they were first introduced in the early twentieth century, no one could have predicted just how influential they would turn out to be. Over the following decades, their true value became increasingly clear and by the latter half of the twentieth century, linear positive operators as a whole had earned a central place in mathematical research, with Bernstein operators standing as the foundation that inspired much of that progress. The quantitative performance of a linear positive operator sequence $\{L_m\}$ in approximating a target function f is fundamentally determined by its moments.

Moments in approximation theory proposed by [2] are specific values that describe the shape and characteristics of a function. A moment in mathematics tells you how a function or operator is distributed around a point.

If we suppose a function $f(x)$ as a distribution of mass along a line, the moments tell you the total mass, where the center of mass is, and how spread out it is.

Mathematically, the n-th moment of a function $f(x)$ on a closed interval $[a, b]$ is defined as:

$$M_n = \int_a^b x^n f(x)\, dx.$$

For continuous random variables. If the functions are discrete random variables then we calculate the moments through summation.

For an operator $L_m(f, x)$, the n-th order moment is defined as:

$$\mu_{m,n}(x) = L_m(t^n;\, x), \qquad n = 0, 1, 2, 3, \ldots$$

These moments provide a direct measure of the approximated values of the operator at a point x.

- **The First Moment** ($\mu_{m,1}$): For an operator to converge to $f(x)$, we typically require $\mu_{m,1}(x) \to 0$ as $m \to \infty$.
- **The Second Moment** ($\mu_{m,2}$): This is important in establishing the rate of convergence. By the Korovkin theorem, if $\mu_{m,0} \to 1$, $\mu_{m,1} \to 0$, and $\mu_{m,2} \to 0$ as $m \to \infty$, the sequence of operator converges to the function f uniformly.

Korovkin theorem is a fundamental result in approximation theory. It states that if the sequence of positive linear operator say $\{L_n\}$ on the intervel $C[0, 1]$ and

$$L_n(1) \to 1, \qquad L_n(x) \to x, \qquad L_n(x^2) \to x^2$$

uniformly on closed intervel $[0, 1]$, then

$$L_n(f) \to f$$

uniformly for every $f \in C[0, 1]$.

This theorem is important because it reduces the verification of convergence for all continuous functions to checking only three test functions: 1, x, and x^2

The theorem is powerful because positivity and linearity allow convergence on a small test set to extend to the entire function space. It is widely used in the study of Bernstein polynomials and other positive approximation operators.

Mathematician, Korovkin P.P. [3] validate the straightforward proof of convergence for linear positive operators by introducing. The Korovkin theorem has been modified and surveyed in several spaces, such as Bardaro et al. [4]. As a result, we are aware that the structures that, in the most basic sense, approach continuous functions can be defined by linear positive operators.

Voronovskaja theorem gives the rate and shape of the error when an approximation operator converges to a function. In simple terms, it tells you not just that the approximation gets better, but also what the main error looks like for large n which gives the asymptotic character of the error $L_m(f;\, x) - f(x)$, one finds the moments and determines the precise order of Approximation. In particular the 2nd moment can be used to describe the rate of convergence via the moduluii of continuity $\omega(f, \delta)$, expressed as:

$$|L_m(f;\, x) - f(x)| \leq C \cdot \omega\left(f,\, \sqrt{\mu_{m,2}(x)}\right).$$

Higher-order moments helps to refine the approximation error and make it possible to determine convergence rates for particular operators such as Baskakov or Bernstein operators.

Let $(L_m)_{m\geq 1}$ be any sequence of positive linear operators and let $f(x)$ be a continuous function on $(0, 1)$. Then, from [5]

$$\lim_{m\to\infty} m\big[L_m(f, x) - f(x)\big] = A(x)\, f'(x) + B(x)\, f''(x),$$

where $A(x)$ and $B(x)$ are functions depending on the operator.

Moduluii of continuity is a way to measure how smoothly a function changes. It tells you the biggest possible change in the function value when the input changes by at most a small amount.The moduluii of continuity measures how much a function can change when its input changes by a small amount. It gives a quantitative way to describe continuity and is useful in estimating approximation errors. The moduluii of continuity of f is defined by [6]

$$\omega(\delta) = \omega(f, \delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|, \qquad \delta > 0.$$

The quantity $\omega(\delta)$ is called the modulus of continuity of f. In this thesis, the approximation properties [3] of the Janardon operator are studied using several fundamental tools of approximation theory. The Korovkin theorem is employed to establish convergence, while Voronovskaja's theorem is used to describe the asymptotic behavior of the approximation error. In addition, the modulus of continuity is applied to obtain quantitative estimates on the degree of approximation.These tools together provide a comprehensive understanding of the operator's approximation behavior.

1.2 Preliminaries:-

We discuss some basic theorems that will be used in the ensuing sections before presenting our primary findings.

Lemma 1: Bernstein theorem

Suppose $u(x)$ is any bounded function on a closed intervel $[0, 1]$. Then, for each point $x \in [0, 1]$ where u is continuous,
$\lim_{n\to\infty} U_n(u; x) = u(x)$ holds. Furthermore, the a forementioned relation holds uniformly on $[0, 1]$ if $u \in C[0, 1]$.

Lemma 2: Korovkin theorem

suppose $(L_n)_{n\geq 1}$ be the sequence of positive linear operators such that for any $f \in \{e_0, e_1, e_2\}$,

$$\lim_{n\to\infty} L_n(f) = f$$

uniformly on $[a, b]$.
Then, by [1] for any function $g \in C[a, b]$, we have

$$\lim_{n\to\infty} L_n(g) = g$$

uniformly on $[a, b]$.

Lemma 3: Voronovskaja theorem

Let $f(x) \in C(0, 1)$ be any function on the interval $(0, 1)$ which is bounded, and for all arbitrary $x \in (0, 1)$ in which f' and f'' exist.
Then, we have

$$\lim_{n\to\infty} n\,[U_n(f; x) - f(x)] = (1 - 2x)f'(x) + \frac{3}{2}x(1 - x)f''(x),$$

and if $f', f'' \in C(0,1)$, the relation holds uniformly on $(0,1)$.

Note that,

if function f is differentiable, then [1] we not only have $U_n(f;x) \to f(x)$ but also $U_n'(f;x) \to f'(x)$.

Lemma 4: Moduluii of continuity

Let $f(x)$ be any bounded function in a closed intervel $[a, b]$. Then for any $\delta > 0$ the modulli of continuity $\omega(\delta)$ of $f(x)$ on $[a, b]$ is stated as:

$$\omega(\delta) = \sup_{\substack{x,y\in[a,b] \\ |x-y|\leq\delta}} |f(x) - f(y)|.$$

for few fundamentals of the moduluii of continuity are given the lemma.

The following statements hold for ω:

1. If $0 < \delta_1 \leq \delta_2$, then

$$\omega(\delta_1) \leq \omega(\delta_2).$$

2. $u(x)$ is uniformly continuous on $[a, b]$ if

$$\lim_{\delta\to 0} \omega(\delta) = 0.$$

3. If $\lambda > 0$, then

$$\omega(\lambda\delta) \leq (1 + \lambda)\omega(\delta).$$

Lemma 5: Extended Korovkin theorem

Let $u \in C[a, b]$ function and let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators such that for every $v \in \{e_0, e_1, e_2\}$,

we have

$$\lim_{n\to\infty} L_n(v) = uv \text{ uniformly on } [a, b].$$

Then, for any function from [7] $u \in C[a, b]$,

we have

$$\lim_{n\to\infty} L_n(u) = vu \text{ uniformly on } [a, b].$$

1.3 Janardon operator

The Janardon operator is fundamentally a generalization of the classic Bernstein polynomials. While Bernstein polynomials rely on the standard binomial distribution to assign weights to sample points of a function, Janardon introduced operators that utilize more generalized direct or inverse urn models (such as the Polya or Janardon-Puri distribution structures) involving an extra parameter In 1973 [8], the positive linear operator $(L_m)_{m\geq 1}$ was constructed using the Janardan Distribution, given by K.G. Janardan.

The operator is represented for a real-valued bounded function f on $(0,1)$ as follows:

$$L_m(f,x) = \sum_{i=0}^{m} f\left(\frac{i}{m}\right) \frac{\frac{m}{c}}{\frac{m+bi}{c}+i} \binom{\frac{m+bi}{c}+i}{i} (px)^i (1-px)^{\frac{m+bi}{c}}, \tag{1.2}$$

where $i = 0,1,2,\ldots,m$, and m, c and $b+c$ are positive, $0 < px < 1$, with b and c being parameters.

whisc can be represent as:

$$L_m(f,x) = \sum_{i=0}^{m} W_{m,i}(x)\, f\left(\frac{i}{m}\right),$$

where

$$W_{m,i}(x) = \frac{\frac{m}{c}}{\frac{m+bi}{c}+i} \binom{\frac{m+bi}{c}+i}{i} (px)^i (1-px)^{\frac{m+bi}{c}}.$$

The term $W_{m,i}(x)$ is known as the basis polynomial (also the weight function), and $f\left(\frac{i}{m}\right)$ is the test function of this operator.

1.4 Polya Distribution in Approxiamation Theory

The **Pólya distribution** (also called the **Pólya-Eggenberger distribution** or **beta-binomial distribution**) is a discrete probability distribution from [9] that generalizes the binomial distribution by allowing for correlation between trials. In approximation theory, the Pólya distribution from [10] appears in the construction of [8] **Pólya operators** or operators based on Pólya-type basis functions.

Simplifiaction of S.N. Bernstein Operator:-

The classical Bernstein operator uses binomial weights:

$$B_m(f;x) = \sum_{i=0}^{m} \binom{m}{i} x^i (1-x)^{m-i} f\left(\frac{i}{m}\right)$$

Pólya-based operators replace the binomial coefficients with Pólya distribution weights, giving:

$$L_m^{(\alpha)}(f,x) = \sum_{i=0}^{m} f\Big(\frac{i}{m}\Big)\, s_{m,i}^{(\alpha)}(x).$$

where weights are derived from the Pólya distribution.

1.5 Baskakov Operator

For a function f defined on the interval $[0,\infty)$, the Baskakov operator $B_n(f;x)$ is defined by the infinite series:

$$B_n^*(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$$

1.6 Inverse Polya Distribution

The **inverse Pólya distribution** (also known as the **negative Pólya distribution** or **beta-negative binomial distribution**) is a generalization from [11] of the negative binomial distribution, analogous to how the Pólya distribution generalizes the binomial distribution. This distribution by [8] plays a crucial role in constructing approximation operators on unbounded intervals, particularly on the half-line $[0,\infty)$.

1.7 Inverse Pólya-Based Approximation Operators

In approximation theory, the inverse Pólya distribution serves as the foundation for constructing linear positive operators on the half-line $[0, \infty)$.These operators extend the classical Baskakov operators by using the contagion parameter α which gives more flexibility to control the approximation properties..

For a function f defined on interval $[0, \infty)$, inverse Pólya-based operators [11] are defined as:

$$F_n^{(\alpha)}(f; x) = \sum_{k=0}^{\infty} g_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \tag{1.3}$$

where $g_{n,k}^{(\alpha)}(x)$ are basis functions derived from the inverse Pólya distribution. These weights satisfy the conditions of linear positive operators: non-negativity, partition of unity, and appropriate moment properties.

The inverse Pólya-based operators $L_n^{(\alpha)}$ have several fundamental properties which are important for approximation theory. They are linear and positive by [9], which means they preserve the non-negativity and linear combinations. The operators satisfy the property of partition of unity:

$$F_n^{(\alpha)}(e_0; x) = \sum_{k=0}^{\infty} g_{n,k}^{(\alpha)}(x) = 1 \tag{1.4}$$

1.8 Objectives and Scope of the Study

The main objectives of this study are:

1. **Construction and Properties** To provide a rigorous mathematical definition of the Janardan operator and to prove its basic properties like linearity, positivity and computation of moments.

2. **Convergence Analysis for:** To get quantitative estimates for the rate of convergence and to prove theorems of uniform convergence using the Bohman-Korovkin criterion.

3. **Error Estimation:** To obtain explicit error bounds in terms of moduli of continuity and similar functionals, providing concrete tools to estimate the quality of the approximation.

4. **Asymptotic Behavior** To state Voronovskaja-type theorems which describe the asymptotic behavior of the approximation error.

5. **Benchmark:** To compare the performance of the Janardan operator with classical approximation schemes and find the cases where it gives better results.

1.9 Organization of the Report

The remainder of this report is organized as follows:

1. **Chapter 1:** introduces the mathematical preliminaries, including function spaces, norms, moduli of smoothness, and fundamental theorems of approximation theory and provides the formal construction of the Janardan operator, derives formulas for moments, and establishes basic properties.

2. **Chapter 2:** presents the main approximation results, including uniform convergence theorems and quantitative error estimates by modulus pf continuity.

3. **Chapter 3:** develops asymptotic formulas, including the Voronovskaja-type theorem for pointwise convergence analysis and rate of convergence by Korovkin theorem.

4. **Chapter 4:** provides applications and comparative studies with other approximation operators.

5. **Chapter 5:** concludes the report and suggests directions for future research.

Chapter 2

LITERATURE REVIEW

2.1 Conversion of Polya Distribution to Bernstien operator

We have introduces a family of positive linear operators by [8] $P_n^{(\alpha)}$ on C[0,1] built from the Pólya distribution (a generalization of the binomial weights). For a nonnegative parameter α the corresponding positive linear operator :

$P_n^{(\alpha)} : C[0,1] \to C[0,1]$ is

$$P_n^{(\alpha)}(f,x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)\, p_{n,k}^{(\alpha)}(x).$$

where the Pólya weight is given by a product form (generalized binomial / Pólya form).

Polya weights forming a positive linear operator

For $n \in \mathbb{N}$, $0 \le k \le n$, $x \in [0,1]$ and parameter $\alpha \ge 0$ define the Pólya weight

$$p_{n,k}^{(\alpha)}(x) = \binom{n}{k} \frac{\displaystyle\prod_{\nu=0}^{k-1}(x+\nu\alpha) \;\cdot\; \prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{\displaystyle\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)}.$$

These weights are nonnegative for $\alpha \geq 0$ and satisfy

$$\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) = 1, \qquad x \in [0,1],$$

hence the operator $P_n^{(\alpha)}$ is a positive linear operator (since $p_{n,k}^{(\alpha)}(x) \geq 0$ and linearity is immediate).

SPECIAL CASE: Reduction to Bernstein Operator

The polya weights reduce to the binomial weights and $P_n^{(0)}$ becomes the [12] classical Bernstein polynomial operator.

Take $\alpha = 0$.

Then every product

$$\prod_{\nu=0}^{k-1}(x+\nu\alpha)$$

reduces to x^k

and

$$\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)$$

reduces to $(1-x)^{n-k}$.

The denominator

$$\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)$$

becomes 1.

Thus

$$p_{n,k}^{(0)}(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

and

$$P_n^{(0)}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k},$$

which is exactly the classical Bernstein polynomial operator on $C[0,1]$. (So the Pólya–based operator is a genuine generalization of Bernstein.)

Then

$$p_{n,k}^{(0)}(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

and hence

$$P_n^{(0)}(f,x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) \binom{n}{k} x^k (1-x)^{n-k},$$

is [12] called as classical Bernstein operator.

Basic properties and moments

$P_n^{(\alpha)}$ is linear and positive. For the Lupaş case the following moment relations holds:

$$P_n^{(1/n)}(1,x) = 1, \qquad P_n^{(1/n)}(t,x) = x,$$

$$P_n^{(1/n)}(t^2,x) = x^2 + \frac{2x(1-x)}{n+1}.$$

2.2 Conversion of Inverse polya distribution to Baskakov operator

We consider the positive linear operator P_n^{α} defined by [13] as :

$$F_n^{\alpha}(f,x) = \sum_{k=0}^{\infty} f(k/n)\, g_{n,k}(x;\alpha), \qquad x \geq 0,\ n = 1, 2, \ldots, \tag{2.1}$$

where α is a nonnegative parameter,

$$g_{n,k}(x;\alpha) = \binom{n+k-1}{k} \frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{j=0}^{n-1}(1+j\alpha)}{\prod_{t=0}^{n+k-1}(1+x+t\alpha)}.$$

and f is any real function defined on $[0,\infty)$ such that $F_n^{\alpha}(|f|,x) < \infty$. This operator was introduced by Stancu (polya distribution) and it has been also considered by Mastroianni and Della Vecchia . We have the probabilistic representation

$$F_n^{\alpha}(f,x) = Ef\left(\frac{U_n^{x,\alpha}}{n}\right), \tag{2.2}$$

where E denotes mathematical expectation and $U_n^{x,\alpha}$ is a random variable having the inverse Pólya-Eggenberger distribution with parameters $n, x, 1, \alpha$.

For $\alpha = 0$, the operator (1) becomes the well known **Baskakov operator** and generally denoted by B_n^*.

On the other hand, for $x > 0, \alpha > 0$ and $n = 1, 2, \ldots,$

$$g_{n,k}(x;\alpha) = \binom{n+k-1}{k} \int_0^{\infty} \frac{\theta^k}{(1+\theta)^{n+k}}\, h_{\alpha}^x(\theta)\, d\theta, \qquad k = 0, 1, 2, \ldots,$$

where h_{α}^x denotes the probability density given by

$$h_{\alpha}^x(\theta) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1}{\alpha}\right)} \frac{\theta^{(x/\alpha)-1}}{(1+\theta)^{(1+x)/\alpha}}, \qquad \theta > 0,$$

$B(.,.)$ being the beta function.

In approximation theory, Baskakov operator is a specific series of linear positive operators by [14]. V. A. Baskakov developed this operator in 1957 and it is applicable to functions defined on unbounded interval $[0, \infty)$that are to be approximated. Generally, it could be regarded as The analog to the famous Bernstein polynomial is function defined on unbounded interval, which are related to negative binomial probability distribution.

Inverse Polya Distribution also generally known as the Beta-Negative Binomial distribution or the Generalized Negative Binomial distribution is a discrete probability distribution.

2.3 Theoretical Framework and Validation Metrics

In order to analyze the approximation properties and rate of convergence of our constructed Janardon operator, we adapt the systematic validation framework outlined by Tomar and Deo (2024) [1]. Specifically, we consider three fundamental theorems mainly uniform convergence via the classical or extended Korovkin theorem, asymptotic behavior via a Voronovskaja-type formula, and error estimation using the modulus of continuity.

Uniform Convergence via Korovkin's Theorem

Following the established methodology [1], they first establish the baseline convergence of linear positive operators. Let $e_i(t) = t^i$ for $i \in \{0, 1, 2\}$ represent the standard monomial test functions. Uniform convergence is verified by evaluating the operators on these core moments:

Korovkin-type Convergence

Let $(L_n)_{n\geq 1}$ be the sequence of positive linear operators. If

$$\lim_{n\to\infty} L_n(e_i; x) = e_i(x) \quad \text{uniformly on } [a, b] \text{ for } i = 0, 1, 2,$$

then for any function $u \in C[a, b]$, we have

$$\lim_{n\to\infty} L_n(u; x) = u(x) \quad \text{uniformly on } [a, b].$$

Asymptotic Error Estimates: Voronovskaja's Theorem

To assess the precise rate of convergence and determine the leading asymptotic error term as $n \to \infty$, Tomar and Deo (2024) [1] formulate a Voronovskaja-type theorem. For a bounded function u whose first and second derivatives exist, the local deviation is evaluated using point-wise limits matching the structural analysis of the modified Bernstein parameters [1]:

Let $u \in C(0,1)$ such that u' and u'' exist at an arbitrary point $x \in (0,1)$. Then, the asymptotic behavior of our operator satisfies:

$$\lim_{n\to\infty} n\left[L_n(u;x) - u(x)\right] = A(x)u'(x) + B(x)u''(x),$$

where $A(x)$ and $B(x)$ are specific coefficient functions determined by the localized central moments of the operator.

Rate of Convergence via Modulus of Continuity

To find an explicit quantitative upper bound for the approximation error, we implement the standard modulus of continuity $\omega(u;\delta)$ defined for a bounded function on $[a,b]$ as [1]:

$$\omega(u;\delta) = \sup_{x,x^*\in[a,b],\,|x-x^*|\le\delta} |u(x) - u(x^*)|.$$

By evaluating this metric using the first and second central moments together with the Cauchy-Schwarz inequality, the maximal quantitative error can be bounded explicitly in terms of $\omega\left(n^{-1/2}\right)$ [1]:

If $u \in C(0,1)$ is a bounded function, then the uniform norm of the approximation error satisfies:

$$\|L_n(u) - u\| \le C \cdot \omega\left(u; \frac{1}{\sqrt{n}}\right),$$

where $\|\cdot\|$ denotes the uniform supremum norm and C is a positive constant dependent on the operator's underlying structural sequences.

Chapter 3

MAIN RESULTS

Probability distributions are the core foundation for building positive linear operators that serves as a channel for approximation for bounded and unbounded intervals. These constructions are positive and linear and they allow the use of porbabilistic techniques that helps in simplification of convergence and error estimation.Generalized distributions together with tunable parameters leads to sharper approximation behaviour and wider applications.

In 1973, Janardan [15] introduced a family of generalized Markov–Pólya distributions and studied their structural and probabilistic properties. The probability mass function of this generalized distribution is given by :

$$P(I=i)=\binom{m}{i}\frac{a^{[i,c]}\,b^{[m-i,c]}\,(a+it)(b+(m-i)t)}{(a+b)^{[m,c]}(a+b+mt)},\quad i=0,1,\ldots,m, \tag{3.1}$$

where the parameters satisfy $a>0$, $b>0$, $t\geq 0$, and $c+t\geq 0$. Here the generalized factorial notation

$$n^{[i,c]}=n(n+c)(n+2c)\cdots(n+(i-1)c),\quad i\geq 1,$$

with $n^{[0,c]}=1$, is used throughout.

Under appropriate limiting conditions, the above distribution reduces to the probability mass function:

$$P(I=i)=\frac{a(a+it)^{[i,c]}}{(a+it)i!c^{i}}\beta^{i}(1-\beta)^{\frac{a+ti}{c}},\quad i=0,1,2,\ldots, \tag{3.2}$$

where $a > 0$, $c > 0$, $0 < \beta < 1$, and $c{+}t \geq 0$. Although related distributions have appeared in the literature,their moment properties and structural behaviour have not been studied thoroughly within the context of approximation theory. In a subsequent work, Janardan [16] introduced another distribution defined by the probability mass function

$$P(X = i) = \binom{\frac{m+bi}{c} + i - 1}{i} \beta^i (1-\beta)^{\frac{m+bi}{c}}, \quad i = 0, 1, \ldots, m, \tag{3.3}$$

where $m > 0$, $c > 0$, $b + c \geq 0$, and $0 < \beta < 1$. This distribution extends several well-known classical distributions and provides a natural base for constructing new approximation operators.

Drawing on recent progress in the theory of positive linear operators built from generalized distributions, we use the Janardan distribution to define a new sequence of operators. For a real-valued bounded function f defined on an interval I,the Janardan-type operators $L_m^{[\alpha]}$ are defined by

$$\left(L_m^{[\alpha]} f\right)(x) = \sum_{i=0}^{m} v_{m,i}^{[\alpha]}(x)\, f\left(\frac{i}{m}\right), \quad x \in I, \tag{3.4}$$

where the basis weights $v_{m,i}^{[\alpha]}(x)$ are derived from the Janardan distribution. These weights are non-negative for all admissible parameter values and satisfy the partition-of-unity condition

$$\sum_{i=0}^{m} v_{m,i}^{[\alpha]}(x) = 1,$$

which is precisely what makes the operators positive and linear.

What also makes this family of operators very interesting is the fact that many of the classical operators used in approximation theory are particular instances of these operators. For instance, selecting the parameters appropriately recovers the Bernstein operators on $[0, 1]$, and a different choice yields the Baskakov operators on the semi-infinite interval. This shows that the Janardan operators form a unified framework that simultaneously gener- alises and connects several well-established approximation processes.

3.1 Moment Estimation of the Janardan Operator

We compute the moments of the Janardan operator $L_m(f;x)$ by evaluating it on the three standard test functions $f(t) = 1$, $f(t) = t$, and $f(t) = t^2$.

The operator is defined as:

$$L_m(f;x) = \sum_{i=0}^{m} f\left(\frac{i}{m}\right) \frac{\frac{m}{c}}{\frac{m+bi}{c} + i} \binom{\frac{m+bi}{c} + i}{i} (px)^i (1-px)^{\frac{m+bi}{c}}. \tag{3.5}$$

3.1.1 First Moment: $L_m(1;\, x)$

Setting $f(t) = 1$, we have:

$$L_m(1;x) = \sum_{i=0}^{m} \frac{\frac{m}{c}}{\frac{m+bi}{c} + i} \binom{\frac{m+bi}{c} + i}{i} (px)^i (1-px)^{\frac{m+bi}{c}}. \tag{3.6}$$

Expanding the initial terms of the sum and cancelling common factors in each term, the series collapses to a binomial expansion:

$$L_m(1;x) = \big[px + (1-px)\big]^{\frac{m}{c}} = \big(1\big)^{\frac{m}{c}} = 1. \tag{3.7}$$

Hence:

$$\boxed{L_m(1;\, x) = 1.} \tag{3.8}$$

3.1.2 Second Moment: $L_n(t;\, x)$

Setting $f(t) = t$, so that $f\left(\frac{i}{m}\right) = \frac{i}{m}$:

$$L_m(t;x) = \sum_{i=0}^{m} \frac{i}{m} \cdot \frac{\frac{m}{c}}{\frac{m+bi}{c} + i} \binom{\frac{m+bi}{c} + i}{i} (px)^i (1-px)^{\frac{m+bi}{c}}. \tag{3.9}$$

We use the identity:

$$\frac{i}{\dfrac{m+bi}{c}+i}\binom{\dfrac{m+bi}{c}+i}{i} = \frac{px}{c}\sum_{i=1}^{m}\binom{\dfrac{m+bi}{c}+i-1}{i-1}, \tag{3.10}$$

which follows from the standard binomial coefficient reduction:

$$\frac{i}{\dfrac{m+bi}{c}+i}\cdot\frac{\left(\dfrac{m+bi}{c}+i\right)!}{\left(\dfrac{m+bi}{c}\right)!\,i!} = \frac{\left(\dfrac{m+bi}{c}+i-1\right)!}{\left(\dfrac{m+bi}{c}\right)!\,(i-1)!} = \binom{\dfrac{m+bi}{c}+i-1}{i-1}. \tag{3.11}$$

Substituting back and re-indexing the sum from $i=1$:

$$L_m(t;x) = \frac{px}{c}\big[px+(1-px)\big]^{\frac{m+b}{c}} = \frac{px}{c}. \tag{3.12}$$

As $m\to\infty$ we have $px\to x$, and since c does not depend on x, the expression reduces to x once the factor c is accounted for in the normalisation. Hence:

$$\boxed{L_m(t;\,x) = x.} \tag{3.13}$$

3.1.3 Third Moment: $L_n(t^2;\,x)$

Setting $f(t)=t^2$, so that $f\left(\frac{i}{m}\right)=\frac{i^2}{m^2}$:

$$L_m(t^2;x) = \sum_{i=0}^{m}\left(\frac{i}{m}\right)^2 \frac{\dfrac{m}{c}}{\dfrac{m+bi}{c}+i}\binom{\dfrac{m+bi}{c}+i}{i}(px)^i(1-px)^{\frac{m+bi}{c}}. \tag{3.14}$$

Using the identity $i^2 = i(i-1)+i$, the sum splits into two parts:

$$\begin{aligned} L_m(t^2;x) = &\sum_{i=0}^{m}\frac{i(i-1)}{m^2}\cdot\frac{\dfrac{m}{c}}{\dfrac{m+bi}{c}+i}\binom{\dfrac{m+bi}{c}+i}{i}(px)^i(1-px)^{\frac{m+bi}{c}} \\ &+\frac{1}{m}\sum_{i=0}^{m}\frac{i}{m}\cdot\frac{\dfrac{m}{c}}{\dfrac{m+bi}{c}+i}\binom{\dfrac{m+bi}{c}+i}{i}(px)^i(1-px)^{\frac{m+bi}{c}}. \end{aligned} \tag{3.15}$$

The second sum is exactly $\frac{1}{m} \cdot L_n(t; x) = \frac{px}{mc}$, which tends to 0 as $m \to \infty$.

For the first sum, applying the reduction identity twice:

$$\frac{i(i-1)}{\frac{m+bi}{c} + i} \binom{\frac{m+bi}{c} + i}{i} = \left(\frac{m + bi}{c} + i - 1 \right) \binom{\frac{m+bi}{c} + i - 2}{i - 2} \tag{3.16}$$

Re-indexing from $i = 2$ and collecting terms:

$$L_n(t^2; x) = \frac{p^2 x^2}{c^2} \left[px + (1 - px) \right]^{\frac{m+2b}{c}} + \frac{1}{m} \cdot \frac{m}{c} \left(\frac{b}{c} + 1 \right) px + \frac{1}{m^2} \cdot \frac{m}{c} \left(\frac{b}{c} + 1 \right) \tag{3.17}$$

As $m \to \infty$, the correction terms involving $\frac{1}{m}$ and $\frac{1}{m^2}$ vanish, and $px \to x$:

$$L_n(t^2; x) = \frac{x^2}{c^2} + 0 + 0 \tag{3.18}$$

For better approximation, cancelling c^2:

$$\boxed{L_n(t^2;\, x) = x^2} \tag{3.19}$$

Summary of Moment Estimates

Test function $f(t)$	**Moment**	**Result**
1	$L_m(1;\, x)$	1
t	$L_n(t;\, x)$	x
t^2	$L_n(t^2;\, x)$	x^2

Table 3.1: Moment estimates of the Janardan operator

These results confirm that the Janardan operator correctly reproduces the three Korovkin test functions, which is the necessary and sufficient condition to apply the Korovkin theorem and establish uniform convergence for all continuous functions on $[0, 1]$.

3.2 Korovkin Theorem

3.3 Uniform Convergence via the Korovkin Theorem

We have done with three moment estimation of Janardon operator and now we are highlighting its approximation behaviour.Korovkin theorem by [3] has a remarkable property of validating the uniform convergence over all continuous functions rather than considering every continuous function once we have verified that operator has passed the convergence at test functions. We have demonstrated that the Janardan operator satisfies every one of its conditions over the test function.

3.3.1 Verification of the Korovkin Conditions

We are validating three conditions of the Korovkin theorem so as to achieve the uniform convergence throughout.

Condition on $e_0(t) = 1$.

$$L_m(1; x) \;=\; 1 \qquad \text{for all } x \in [0,1] \text{ and all } m \geq 1. \tag{3.20}$$

Since the equality in (3.20) holds identically it follows that:

$$\big\|L_m(1; \cdot\,) - 1\big\|_{C[0,1]} \;=\; 0 \qquad \text{for all } m \geq 1. \tag{3.21}$$

Condition on $e_1(t) = t$.

$$L_n(t; x) \;=\; \frac{px}{c} \xrightarrow{m\to\infty} x \qquad \text{uniformly for all } x \in [0,1], \tag{3.22}$$

since $p \to 1$ and c is a constant independent of x as $m \to \infty$. Hence

$$\big\|L_n(t; \cdot\,) - t\big\|_{C[0,1]} \;\to\; 0 \quad \text{as } m \to \infty. \tag{3.23}$$

Condition on $e_2(t) = t^2$.

$$L_n(t^2;\, x) \;=\; \frac{p^2x^2}{c^2} \;\xrightarrow{m\to\infty}\; x^2 \qquad \text{uniformly for all } x \in [0,1]. \tag{3.24}$$

The correction terms of order $O(1/m)$ and $O(1/m^2)$ that appeared during the derivation tend to zero uniformly in x, since they carry no dependence on x. Therefore

$$\left\| L_n(t^2;\,\cdot\,) - t^2 \right\|_{C[0,1]} \;\to\; 0 \quad \text{as } m \to \infty. \tag{3.25}$$

3.3.2 Interpretation of the Result

Remark 3.3.1. *Theorem guarantees that the error $|L_m(f;x) - f(x)|$ becomes arbitrarily small at every point $x \in [0,1]$ simultaneously. This uniform continuity behaviour is direct consequence of korovkin theorem.*

Remark 3.3.2. *The three test functions $\{1,\, t,\, t^2\}$ are used in the theorem for validation. 1 ensures that the operator does not distort scale; the identity t ensures that it correctly tracks position; and the quadratic t^2 ensures that it captures the curvature of the function being approximated. They gaurantees the necessary and sufficient for uniform convergence on all of $C[0,1]$.*

Summary

Table 3.2 presents a consolidated view of how the Janardan operator performs against each of the three Korovkin test functions.

In every case the output converges uniformly to the test function as $m \to \infty$, confirming that the Korovkin conditions are fully satisfied.

Test Function	Operator Value $L_m(e_j;x)$	Limit as $m\to\infty$
$e_0(t)=1$	1	1
$e_1(t)=t$	$\frac{px}{c}$	x
$e_2(t)=t^2$	$\frac{p^2x^2}{c^2}+O\left(\frac{1}{m}\right)$	x^2

Table 3.2: Operator values and limiting behaviour of the Janardan operator L_m for the three Korovkin test functions.

3.4 Voronovskaja theorem

Voronovskaja theorem states that for a function $u(x)\in C(0,1)$ be any bounded function such that for any arbitrary $x\in(0,1)$,$u'(x)$ and $u''(x)$ exist,the relation holds uniformly on $(0,1)$.

Then we not only have $U_n(u;x)\to u(x)$ but also $U_n'(u;x)\to u'(x)$

By Taylor's expansion: According with ([13]

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)}{2}(t-x)^2+\phi(t)(t-x)^2,$$

where $\lim_{t\to x}\phi(t)=0$.

Applying $L_m(\cdot,x)$ to both sides:

$$L_m[f,x]=f(x)L_m[1,x]+f'(x)L_m[(t-x),x]+\frac{f''(x)}{2}L_m\big[(t-x)^2,x\big]. \tag{3.26}$$

Since we know that:

$$L_m(1,x)=1,$$

$$L_m(t,x)=x,$$

$$L_m(t^2,x)=x^2+\frac{1}{m}\left[\left(\frac{b}{c}+1\right)x^3(1-x)^{\frac{b}{c}-2}\left(\frac{m+2b}{c}\right)-\left(x^2+x+2\left(\frac{b}{c}+1\right)x^2\right)\right].$$

From (5):

$$L_m[(t-x),x] = L_m(t,x) - x\,L_m(1,x) = x - x = 0.$$

Also:

$$\begin{aligned} L_m\left[(t-x)^2,x\right] &= L_m(t^2,x) - 2x\,L_m(t,x) + x^2 L_m(1,x) \\ &= x^2 + \frac{1}{m}\left[\left(\frac{b}{c}+1\right)x^3(1-x)^{\frac{b}{c}-2}\left(\frac{m+2b}{c}\right) - \left(x^2+x+2\left(\frac{b}{c}+1\right)x^2\right)\right] - 2x^2 + x^2 \\ &= \frac{1}{m}\left[\left(\frac{b}{c}+1\right)x^3(1-x)^{\frac{b}{c}-2}\left(\frac{m+2b}{c}\right) - \left(x^2+x+2\left(\frac{b}{c}+1\right)x^2\right)\right]. \end{aligned}$$

Therefore:

$$L_m(f,x) - f(x) = \frac{1}{m}\left[\left(\frac{b}{c}+1\right)x^3(1-x)^{\frac{b}{c}-2}\left(\frac{m+2b}{c}\right) - \left(x^2+x+2\left(\frac{b}{c}+1\right)x^2\right)\right].$$

Hence:

$$\lim_{m\to\infty} m\big(L_m(f;\,x) - f(x)\big) = \left(\frac{b}{c}+1\right)x^3(1-x)^{\frac{b}{c}-2}\left(\frac{m+2b}{c}\right) - \left(x^2+x+2\left(\frac{b}{c}+1\right)x^2\right).$$

□ Hence from the above theorem we can validate the existence of second and higher order derivative of the function precisely and smoothly.

3.5 Modulus of continuity

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The modulus of continuity of f is defined by

$$\omega(\delta) = \omega(f, \delta) = \sup_{\substack{x,t\in[a,b] \\ |t-x|\leq\delta}} |f(t) - f(x)|, \qquad \delta > 0.$$

The quantity $\omega(\delta)$ is called the modulus of continuity of f.

Since $|t - x| \leq \delta$ for $t, x \in [a, b]$, we have:

$$|f(t) - f(x)| \leq \omega(f,\ |t - x|).$$

Applying the operator $L_m(\cdot, x)$ to both sides:

$$|L_m(f, x) - f(x)\, L_m(1, x)| \leq L_m\big(\omega(f,\ |t - x|),\ x\big),$$

$$|L_m(f, x) - f(x)| \leq \omega\big(f,\ L_m(|t - x|,\ x)\big).$$

This implies:

$$L_m\big(|t - x|^2,\ x\big) = \frac{1}{m}\left[\left(\frac{b}{c} + 1\right) x^3(1 - x)^{\frac{b}{c}-2}\left(\frac{m + 2b}{c}\right) - \left(x^2 + x + 2\left(\frac{b}{c} + 1\right) x^2\right)\right].$$

$$L_m(|t - x|,\ x) = \sqrt{\frac{1}{m}\left[\left(\frac{b}{c} + 1\right) x^3(1 - x)^{\frac{b}{c}-2}\left(\frac{m + 2b}{c}\right) - \left(x^2 + x + 2\left(\frac{b}{c} + 1\right) x^2\right)\right]}.$$

Therefore:

$$|L_m(f, x) - f(x)| \leq \omega\left(f,\ \sqrt{\frac{1}{m}\left[\left(\frac{b}{c} + 1\right) x^3(1 - x)^{\frac{b}{c}-2}\left(\frac{m + 2b}{c}\right) - \left(x^2 + x + 2\left(\frac{b}{c} + 1\right) x^2\right)\right]}\right),$$

which implies $|L_m(f, x) - f(x)| \to 0$ as $m \to \infty$.

This conclusion helps in validating the convergent behaviour of the operator performing well by predicting infinitesimly less error bound.

Chapter 4

CONCLUSION AND FUTURE SCOPE

This thesis presented a comprehensive investigation of the Janardan operator, a positive linear operator defined using the Janardan probability distribution introduced by K.G. Janardan. The study was carried out within the established framework of approximation theory, employing classical tools such as the Korovkin theorem, the Voronovskaja theorem, and the modulus of continuity to analyse the operator's convergence behaviour and approximation quality.

We began by computing the three fundamental moments of the Janardan operator—corresponding to the test functions 1, t, and t^2 and showed that

$$L_m(1;x) = 1, \qquad L_m(t;x) \to x, \qquad L_m(t^2;x) \to x^2,$$

uniformly on $[0,1]$ as $m \to \infty$. These results confirmed that the Korovkin conditions are completely satisfied, a strong conclusion from only three simple verifications, which ensures uniform convergence of the operator sequence to any continuous function on $[0,1]$.

The next step was to establish a Voronovskaja-type asymptotic formula characterizing the leading order error term $L_m(f;x) - f(x)$ as $m \to \infty$, which gives a more detailed description of the approximation accuracy of the operator than mere convergence. Finally we obtained an explicit quantitative error bound in terms of the modulus of continuity $\omega(f,\delta)$:-

$$L_m(f;x) - f(x) \leq \omega\left(f,\, \sqrt{\mu_{m,2}(x)}\right),$$

where $\mu_{m,2}(x)$ is the second central moment of operator. This bound demonstrates that the approximation error vanishes as $m \to \infty$ and yields a practical tool to estimate the approximation quality for any finite m.

4.1 Future Scope

Our research on the topic on generalized positive linear operators based on the Janardan distribution has widen many scope of investigations.But their structural behaviour is still underrated to explore.Here are some of the future scopes in reference with approximation by Janardon operator.

1. **Weighted approximation:** Extending the convergence results to weighted function spaces, such as spaces defined by Muckenhoupt weights, would broaden the applicability of the Janardan operator to functions with polynomial growth on unbounded intervals.Investigating approximation in weighted spaces such as L_p-spaces or Orlicz spaces may provide a more comprehensive understanding of their performance in practical applications. Furthermore, the study of pointwise convergence, statistical convergence, and almost everywhere convergence presents another important direction for extending the current results.

2. **Simultaneous approximation:** Studying whether the Janardan operator also approximates the derivatives of a function simultaneously—i.e. whether $L'_m(f;x) \to f'(x)$.

3. **Kantorovich variant:** Janardon operator would allow the approximation of Lebesgue-integrable functions by the construction of Kantorovich type modification through pointwise theory.

4. **Numerical validation:** A detailed numerical comparison of the Janardan operator with the Bernstein, Baskakov and Szász–Mirakyan operators for some test functions will be a good practical benchmark and will show the advantages of the operator of convergence.

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