FIXED POINT THEORETIC ANALYSIS OF GAME THEORY

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by

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Abstract

The existence of Nash equilibrium is one of the most fundamental results in game theory with wide applications across economics, optimization, and computer science. This study examines the existence of Nash equilibrium in games with finite strategies using fixed point theorems. The paper partly focuses on Brouwer's fixed-point theorem, Kakutani's fixed-point theorem, and Sperner's lemma, which pave the core foundation in proving the existence of the Nash equilibrium. This paper focuses on the importance of the Nash equilibrium being a fixed point of the best response correspondence. This paper also contains the statement and proof of the important lemma of Sperner in combinatorics and applies it in the proof of Brouwer's fixed-point theorem, which demonstrates an intriguing interrelation between combinatorics, topology, and game theory. Finally, the paper discusses the existence of the Nash equilibrium in infinite strategic games using Schauder's fixed-point theorem.

Keywords: Nash equilibrium, Brouwer's fixed Point Theorem, Kakutani's fixed point theorem, Sperner's lemma, Combinatorics, and Schauder's fixed point theorem.

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Chapter 1

Introduction

In the vast development of modern-day applied mathematics and economics literature, the theory of fixed points and the theory of games are the major contributors, as they facilitate the inquiry of whether rational agents engaged in strategic interaction can be expected to reach a stable outcome. John Nash, in his pair of seminal papers published in 1950 and 1951, answered this by exhibiting that a mixed-strategy Nash equilibrium will always exist for a game with countable finite strategies. It is a deep and surprising mathematical fact, the proof of which relies entirely on fixed-point theorems. Since then, it has become apparent that this was no coincidence. Fixed-point theorems provide a general analytical tool to prove the existence of equilibria to many different types of games and other mathematical constructs used in economics. Thus, a foundation for an exponential surge at the relative point between mathematics and economics is previously determined by observations.

In mathematics, in a particular sense, a formal model of a conflict or cooperation is called a "game"—an interaction—among rational decision-making agents. Respondents are allowed to participate in it, and the possible actions that can be taken within the framework of the game are specified by the numerical and structural framework of the game. The agents are motivated to achieve some set of objectives, usually referred to as preferences. The outcome for any player results from the joint action of all the players, which is defined by each player complying with his or her strategy. In dynamic multi-actor games future relationships and expectations also affect current actions. All things being equal, it is desirable if all else remains unchanged.

The proof of Nash's existence theorem proceeds through a remarkable chain of mathematical reasoning. It begins with Sperner's Lemma, a fundamental result in combinatorics on labeled subdivisions of simplices, which provides the discrete foundation for the Brouwer Fixed Point Theorem and illustrates a dense yet gibbous-like convex set on which every self-map is continuous in its attributes and encompasses a fixed point. An extension of upper-hemicontinuous set-valued correspondences for the latter is provided by Kakutani's generalization, which is precisely the form taken by the joint best-response map in a game. Thus, there is an extant congruity between a fixed point and Nash equilibrium. This logical chain — from Sperner through Brouwer and Kakutani to Nash — forms the mathematical backbone of the present thesis.

Beyond finite games, Nash equilibria exist in games with an infinite number of strategies where players choose from continuous strategy spaces such as the interval [0, 1]—this requires more powerful topological tools. With the aid of Schauder's Fixed Point Theorem, which itself provides an extension of Brouwer's theorem to countably infinite dimensions, an extension of Nash's equilibrium result was given by Glicksberg's theorem (1952), which incorporates it further into continuous games over compact metric spaces. These extensions make explicit that compactness and continuity are not merely convenient assumptions but are in fact necessary conditions for equilibrium existence, as shown through explicit counterexamples. This allows us to refine and strengthen the connection between functional analysis, topology, and equilibrium theory.

The document includes an extensive investigation related to the systematic fixed-point theoretic aspects of both game theory and Nash equilibrium. To be more specific, it starts with a strict presentation of game theory, including strategic games, pure and mixed strategies, payoff matrices, and motivation for equilibrium analysis. Classical literature observations such as the Prisoner's Dilemma and matching pennies are deployed as core motivations. The thesis goes on to present in detail the basic mathematical tools that are needed to study the subject matter, i.e., correspondences, upper hemicontinuity, and fixed-point theorems. Then two distinct full proofs of the Nash Existence Theorem utilizing Brouwer's theorem and Kakutani's theorem are introduced. Further, at the end of this thesis, infinite strategic games are investigated, and some conclusions along with prospective future work are described.

1.1 Objectives

The fundamental objectives of this thesis are

1. Structural burgeoning game theory, strategic games, pure and mixed strategies, and Nash equilibrium in a mathematically detailed way.

2. To build the foundation of vital topological tools, fixed-point theory, including Sperner's Lemma and the Brouwer, Kakutani, and Schauder Fixed Point Theorems.

3. Use Brouwer and Kakutani fixed-point theorem to give the proof for existence of Nash equilibrium

4. To expand the analysis of finite strategic games into infinite strategic games using Glicksberg's theorem and the Schauder fixed-point theorem.

5. To show examples illustrating the importance of compactness and continuity for the Nash equilibria to exist in infinite games.

1.2 Literature Review

1.2.1 Overview

The intersection of fixed-point theory and game theory constitutes a mathematically elegant area of modern economics as well as applied mathematics. The foundational question—whether rational agents in strategic interaction can be expected to reach a stable outcome—was answered by John Nash (1950, 1951) through the landmark observation that a minimum of Nash equilibrium mixed strategies was encompassed by each respondent with a countable, finite number of strategies under the particular game, a foundational statement whose proof is entirely dependent on fixed-point theorems.

The chronological order of reasoning initiates with Sperner's combinatorial lemma on labeled triangulations, then passes on to Brouwer's fixed-point theorem for non-breaking continuous self-maps of dense gibbous-like convex sets, to Kakutani's generalization to upper-hemicontinuous set-valued correspondences, and finally to the Nash equilibrium, which is precisely the existence of a fixed point of the best-response multi-valued functions.

Beyond finite games, Glicksberg's (1952) theorem extends Nash's result to continuous games with compact metric strategy spaces via approximation by finite games and weak-topology compactness arguments, while Dasgupta and Maskin (1986) further relax continuity to semicontinuity conditions that accommodate the discontinuous payoffs that arise naturally in models of price competition and congestion.

pure-strategy Nash equilibrium uniqueness constraints in concave N-person games were characterized by Rosen (1965) using the notion of diagonal strict concavity, a condition verifiable through the negative definiteness of a certain symmetrized Jacobian matrix.

Collectively, the works reviewed here span the combinatorial foundations (Sperner's lemma), the topological machinery (Brouwer and Kakutani theorems), the game-theoretic applications (existence and uniqueness of a Nash equilibrium in a countably finite, continuous as well as discontinuous game), and the pedagogical synthesis (Osborne's textbook treatment), providing a comprehensive mathematical scaffold for a thesis on the fixed-point theoretic analysis of game theory.

1.2.2 Overview of Individual Sources

Osborne (2003) — Nash Equilibrium, Mixed Strategies, and Rationalizability

Osborne's *An Introduction to Game Theory* offers a comprehensive and well-taught treatment of strategic games, Nash equilibrium, mixed strategies, and rationalizability. Chapters 2, 3, and 4 of the book develop both the theoretical and epistemic underpinnings of equilibrium analysis in modern game theory.

Key Contributions:

- Formal definition of strategic games and equivalence of Nash equilibrium to a fixed point of the best response multi-valued transformation.
- Best-response characterization of equilibrium along with applications to Cournot duopoly, Bertrand competition, public-goods provision, and the War of Attrition.

- Discussion of strict and weak dominance, symmetric equilibrium, and coordination games.
- Introduction of mixed strategies as probability distributions over pure strategies using the expected utility framework under von Neumann-Morgenstern utility theory.
- Development of the indifference condition in overall pure strategies where each respondent must not be different in the underpinning of a mixed-(tactics) strategy equilibrium.
- Investigation of classical game-theoretic models such as Matching Pennies, Battle of the Sexes, and first-price auctions.
- Rationalizability as iterated elimination of dominated strategies: the epistemic foundations of strategic reasoning.
- To prove that each Nash equilibrium strategy can be rationalized, but not every rationalizable strategy is a Nash equilibrium.
- Implicit fixed-point interpretation of rationalizable strategy sets as sets closed under best responses.
- Discussion of correlated equilibrium as a broader solution concept extending beyond independent randomization assumptions.

Narahari (2012) – Game Theory Lecture Notes: Existence of Nash Equilibrium (IISc Bangalore)

These lecture notes provide a detailed and self-contained treatment of Nash equilibrium for graduate students in computer science.

Key Contributions:

- Formal development of correspondences and upper hemicontinuity.
- Brouwer's and Kakutani's Fixed Point Theorems and proving how Nash equilibrium exists.
- Debreu showed how pure-strategy Nash equilibrium exists with sufficient conditions.
- Detailed proof of Sperner's Lemma.
- Glicksberg's theorem for epsilon-Nash equilibria and true Nash equilibria.

Wright (2005) — Sperner's Lemma and Brouwer's Fixed Point Theorem

The given expository note presents elegant proofs of the lemma given by Sperner and the theorem given by Brouwer.

Key Contributions:

- Path-based proof of Sperner's Lemma for $n = 2$
- Strong parity result:

$$A - B = 1$$

- Proof of Brouwer's theorem using barycentric coordinates.
- Topological invariance of fixed-point properties under homeomorphism.

Yuan (n.d.) — Fixed Point Theorems and Applications to Game Theory (UChicago REU)

This paper provides one of the most mathematically rigorous proof chains among the reviewed literature.

Main Contributions:

- Inductive proof of the Sperner Lemma for arbitrary dimension.
- Brouwer's theorem for standard closed simplices.
- Homeomorphism argument for compact convex sets.
- Kakutani's theorem for simplices.
- Theorem to show pure-strategy Nash equilibrium exists.
- A theorem to show mixed-strategy Nash equilibrium existence exists for finite games.

Ozdaglar (2010) — MIT 6.254 Lecture 6: Continuous and Discontinuous Games

This MIT lecture note provides a rigorous graduate-level treatment of Nash equilibrium existence in games with infinite strategy spaces.

Key Contributions:

1. Glicksberg's Theorem and Continuous Games.
2. Every continuous game admits a mixed-strategy Nash equilibrium.
3. Development of epsilon-equilibrium framework and stability results.
4. Dasgupta–Maskin theorem for discontinuous games.
5. Rosen's diagonal strict concavity condition for uniqueness of pure-strategy NE.
6. Example of mixed-strategy equilibrium for Bertrand competition with capacity constraints:

Chapter 2

Introduction to Game Theory

2.1 Definition of a Game

A game is a structured interaction or activity involving two or more decision-making agents, known as players, who act according to a set of predefined rules.

The goal of each player is to increase one's benefit or decrease the maximum loss it can make, and any player's outcome is not just about what they decide to do but also what others will decide too.

Mathematically, a game can be shown in the form of the following tuple structure:

$$J = \langle M, (P_r)_{r \in M}, (\rho_r)_{r \in M} \rangle$$

where:

- M is a set containing players from 1 to m.
- $P_r \neq \phi$ set that contains all the strategies of player r.
- $$P = \prod_{r \in M} P_i$$
  represents a set containing strategy sets of all players .
- $$\rho_r : P \to \mathbb{R}$$
  represents a function (utility) that maps each strategy to a real-valued number for player r.

A conflict of interest among players is an essential characteristic of a game; depending on the type of game, one player's loss can lead to another player's benefit.

2.2 Definition of Game Theory

Game theory is a field that studies mathematical foundations for examining interaction based on strategies between rational players.

It examines how the success of one player is affected by the strategy other players will choose.

Formally, game theory presents an approach to forecast and analyze the behavior of players in interactive decision-making scenarios, where each player chooses strategies to increase one's payoff considering the possible decisions of others.

Game theory serves as a foundation for several disciplines — economics, political science, operations research, computer science, and evolutionary biology.

2.3 Types of Games

Games can be classified in various ways, depending on the number of players, the nature of payoffs, and the available information.

2.3.1 Two-Person and m-Person Games

A game may involve any number of players:

- When there is a number of players $m = 2$, such a game is referred to as a two-person game.
- An m-person game is when $m > 2$.

Each player may have multiple choices or strategies, but the structure of interaction differs:

- In two-person games, each player's gain or loss depends directly on the opponent's strategy.
- The choices all the players make in an m-person game affect the gain or loss of other players.

Mathematically, for player $r \in M = \{1, 2, \ldots, m\}$, their payoff is given by

$$\rho_r(p_1, p_2, \ldots, p_m),$$

where $p_j \in P_j$ for each $j \in M$.

2.3.2 Zero-Sum Games

This is a kind of game where the net summation of gains of all players is zero.

That is, any gain made by one player corresponds to an equal loss suffered by others.

For a two-player game, if $\rho_1(p_1, p_2)$ and $\rho_2(p_1, p_2)$ represent their payoffs, then:

$$\rho_1(p_1, p_2) + \rho_2(p_1, p_2) = 0, \quad \forall (p_1, p_2) \in P_1 \times P_2.$$

Hence,

$$\rho_2(p_1, p_2) = -\rho_1(p_1, p_2).$$

Clearly the gain of one player is equal to the loss other player will make.

Zero-sum games model competitive situations such as military conflicts, market duopolies, and betting scenarios.

Example:

If Player A gains +10, Player B necessarily loses −10.

Thus, the sum of all payoffs remains zero:

$$+10 + (-10) = 0.$$

2.3.3 Two-Person Zero-Sum Game

This is a special case where only two players are involved and their total payoff remains constant at zero.

Thus, one player's advantage represents the other's exact disadvantage:

$$\text{Gain of one player} = \text{Loss of the other}.$$

Such games are often called rectangular games, that is so because the outcome (payoff) can be shown in a matrix representation, with rows corresponds to Player α's strategies and columns corresponds to Player β's strategies.

2.4 Strategies

A strategy represents a complete set of action that a player may adopt throughout the game. It specifies what choice a player will make for every possible situation that may arise.

2.4.1 Pure Strategies

A pure strategy is a fixed rule of action — a player decides in advance what move to make and follows it deterministically, regardless of the opponent's behavior.

Formally,for a player r a single element is contained in it's pure strategy.

$$p_r \in P_r.$$

The strategy profile for pure strategy is:

$$p = (p_1, p_2, \ldots, p_m)$$

uniquely determines the game outcome and payoffs

$$(\rho_1(p), \rho_2(p), \ldots, \rho_m(p)).$$

Pure strategies are generally used when one strategy clearly dominates others or when a saddle point exists (stable outcome).

2.4.2 Mixed Strategies

A mixed strategy is the set of probabilities that sum to 1, where probabilities are assigned to each strategy.

Player does not choose one obvious strategy but randomly choose their action based on the assigned probabilities.

Mathematically, mixed strategy of player r can be represented in a vector form, where m_r are total strategies(pure):

$$x_r = (x_{r1}, x_{r2}, \ldots, x_{rm_r}),$$

where

$$x_{rj} \geq 0 \quad \forall j,$$

and

$$\sum_{j=1}^{m_i} x_{rj} = 1.$$

Here, probability of player r to select strategy j is x_{rj} .

The collection of all the mixed strategies can be represented in form of a simplex for player r :

$$\Delta_r = \left\{ x_r \in \mathbb{R}^{m_r} : x_{rj} \geq 0, \sum_j x_{rj} = 1 \right\}.$$

The expected payoff to player r, given mixed strategies $x_1, x_2, \ldots, x_m$, is:

$$U_r(x_1, x_2, \ldots, x_m) = \sum_{p \in P} \left(\prod_{k=1}^{m} x_k(p_k) \right) \rho_r(p),$$

where $x_k(p_k)$ denotes the probability assigned by player k to pure strategy p_k.

Mixed strategies are essential in games lacking a pure equilibrium, as they ensure the existence of stable outcomes (as proved later by Nash's theorem).

2.5 Payoff Matrix

In a two-player game, outcomes can be conveniently represented in a payoff matrix.

Let Player α has a total of q strategies

$$\alpha_1, \alpha_2, \ldots, \alpha_q$$

and Player β has a total of f strategies

$$\beta_1, \beta_2, \ldots, \beta_f.$$

Payoff matrix can be represented like:

	β_1	β_2	$\cdots$	β_f
α_1	v_{11}	v_{12}	$\cdots$	v_{1f}
α_2	v_{21}	v_{22}	$\cdots$	v_{2f}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_q	v_{q1}	v_{q2}	$\cdots$	v_{qf}

Here:

- The gain of Player α when it selects α_i and β selects β_j is represented by v_{ij}.
- In zero-sum game, Player β's payoff is $-v_{ij}$, so both players' interests are exactly opposed.

The matrix format is compact and aids in computing equilibria, dominance relations, and expected payoffs in both pure and mixed strategy cases.

2.6 Motivation for Equilibrium Analysis

In strategic interactions the outcome obtained by a player is a function not only of its own decisions but also of decisions made by other players. Since each player acts rationally and tries to maximize his own payoff, it's important to determine whether there exists an outcome such that there is not better outcome for the player.

In many games, players are constantly adjusting their strategies to what others do. This brings up a fundamental question in game theory:

Can there be a stable configuration of strategies , where everyone is happy with their choice given the choices of the other players ?

This search for stable outcomes is in accordance with the idea of equilibrium. A situation where the strategy of each player gives a stable outcome knowing the plan of action of other players is equilibrium. In such a state, no player can unilaterally change his strategy to get a higher payoff.

Presence of Nash Equilibrium is a fundamental result in game theory and was introduced by John Nash. In game theory, a Nash equilibrium is very vital in analyzing the stability of strategies in games with rational players.

"Basis of Modern game theory is analysis of Nash equilibrium, it's applications in economics, political science , computer science , evolutionary biology and optimization theory . The existence of an equilibrium is not always obvious and therefore mathematical tools like theorems fixed point are of great importance to prove the that it exists. In the following chapter we formally present the concept of Nash equilibrium .

Chapter 3

Nash Equilibrium

3.1 Definition: Nash Equilibrium

Let a countable player set encompasses in a game of strategic form:

$$M = \{1, \dots, m\}$$

where r has $P_r \neq \phi$ and payoff function defined as:

$$\rho_r : P_1 \times \cdots \times P_m \to \mathbb{R}.$$

Actions of players can be shown in form of a tuple structure:

$$p = (p_1, \dots, p_m)$$

where $p_r \in P_r$ denotes action r chooses.

For any player r, write p_{-r} for the actions of all players except r.

Tuple structure of action of a player -

$$p^* = (p_1^*, \dots, p_m^*)$$

is a Nash equilibrium, satisfying

$$\rho_r(p_r^*, p_{-r}^*) \geq \rho_r(p_r, p_{-r}^*)$$

$\forall\ p_r \in P_r$ and $\forall\ r \in M$.

Another way of saying this is that, p^* is a best response .

No player can improve it's gain choosing another profile other than this.

Thus, a Nash equilibrium represents a stable configuration of strategic choices: once the players reach p^*, no individual has an incentive to depart from it.

3.2 Pure Strategy Nash Equilibrium

Let us look at a game:

$$J = \langle M, (P_r)_{r \in M}, (\rho_r)_{r \in M} \rangle$$

here:

- M is a set containing players players from 1 to m.
- $P_r \neq \phi$ set that contains all the strategies of player r.
- ρ_r : represents a function (utility) that maps each strategy a real valued number for player r.

The set containing 'm' number of strategies is

$$p^* = (p_1^*, p_2^*, \ldots, p_n^*)$$

is said to be following a pure strategy Nash equilibrium of game J if

$$\rho_r(p_r^*, p_{-r}^*) \geq \rho_i(p_r, p_{-r}^*)$$

for all

$$p_r \in P_r, \qquad r = 1, 2, \ldots, n.$$

Here, p_{-r}^* depicts the strategies at equilibrium which are adopted by all players apart from player r.
i.e., whatever the return a player gets from playing p_r^* is at least as good as the payoff obtained by playing any other strategy p_r.

3.2.1 Using Best Response Correspondence to define Pure Strategy Nash Equilibrium

Imagine a player r. Conditional on the actions of other players, except for r, we choose the most optimal actions—those that provide player r with the highest payoff. We will denote the set containing player r's ideal responses by $BR_r(p_{-r})$ where p_{-r} is the list of other player's action. Mathematically,

$$BR_r(p_{-r}) = \{p_r \in P_r : \rho_r(p_r, p_{-r}) \geq \rho_r(p_r', p_{-r}) \text{ for all } p_r' \in P_r\} \, [7]$$

A profile of actions

$$p^* = (p_1^*, p_2^*, \ldots, p_m^*)$$

is a **Nash equilibrium** if each player's action is a best response he can give to the moves chosen by the other significant players[7]. Equivalently, for every player r,

$$p_r^* \in BR_r(p_{-r}^*)$$

under which BR_r is the appropriate response function of strategist r, and p^*_{-r} indicates the point of stable actions of all players leaving player r.

3.2.2 Prisoner's Dilemma

Two individuals suspected are convicted for a meagre crime by police, but the law enforcements presumes that both of them have committed a more serious crime and does not have enough evidence regarding it.

Now the police is trying to force a confession out of these two prisoners.

They are locked up in two different cells, while the police gives them the following choices:

1. If both deny the crime, they will be jailed for 2 years.
2. If one admits the crime and the partner denies the crime, the one who admits gets 1 year in prison on the other hand the partner attains 10 years in prison.
3. If one denies and the partner admits the crime, then the one who denies gets 10 years in prison and the partner gets freed in 1 year.
4. If both confess to the crime, then both will be jailed for 3 years.

The matrix formed by the returns of the two suspects is:

	Admit	Deny
Admit	$(3, 3)$	$(1, 10)$
Deny	$(10, 1)$	$(2, 2)$

The pair (Admit, Admit) is a Nash equilibrium because:

- If suspect 2 denies the crime, suspect 1 is more likely to benefit from admitting.
- If suspect 2 admits the crime, suspect 1 still acheives a better outcome by choosing Admit.
- Similarly, if suspect 1 admits the crime, suspect 2 is more in advantage by choosing Admit.
- If suspect 1 denies the crime, suspect 2 would still benefit more by selecting Admit.

Thus, the sole point of Nash equilibrium in Prisoner's Dilemma is when each of the two parties choose to admit the crime.

3.3 Mixed Strategy Nash Equilibrium

Take a look at the game

$$J = \langle M, (P_r)_{r \in M}, (\rho_r)_{r \in M} \rangle$$

where:

- M is a set containing players players from 1 to m.
- $P_r \neq \phi$ set that contains all the strategies of player r.
- ρ_r : represents a function (utility) that maps each strategy a real valued number for player r.

A mixed strategy for player r represents the probability distribution in relation to the set P_r. Let $\Delta(P_r)$ depicts the set consisting of all possible mixed strategies of player r.

The profile showcasing mixed strategies

$$\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_m^*)$$

is recognised as a mixed strategy Nash equilibrium of game J under the condition

$$\rho_r(\sigma_r^*, \sigma_{-r}^*) \geq \rho_r(\sigma_r, \sigma_{-r}^*)$$

for all

$$\sigma_r \in \Delta(P_r), \qquad r = 1, 2, \ldots, m.$$

Here, σ_{-r}^* represents the equilibrium of mixed strategies which was selected by all players leaving out player r.

i.e., the mixed strategies handed to us of the other players, none of the players can enhance their expected returns by unilaterally modifying the mixed strategy of their own.

3.3.1 Using Best Response set valued function to characterise Mixed Strategy Nash Equilibrium

Let us take the player r's best response correspondence by BR_r. For it to be following a strategic game containing mixed strategies, $BR_r(\sigma_{-r})$ is the set comprising best mixed strategies of player r's whereas the catalogue of the other players' mixed strategies is σ_{-r}. A profile of mixed strategy σ^* is referred to as a mixed strategy Nash equilibrium only in the case where every player's mixed strategy is a most appropriate response against the other player's mixed strategies. Mathematically,

$$\sigma_r^* \in BR_r(\sigma_{-r}^*) \quad \text{for every player } r.$$

[7]

3.3.2 Matching Pennies

Consider two players, Player 1 (P_1) and Player 2 (P_2).

Both players have two choices: Head (H) or Tail (T).

1. If both choose Head or both choose Tail, Player 1 gains Rs. 1 and Player 2 loses Rs. 1.
2. If the choices mismatch, Player 1 loses Rs. 1 and Player 2 gains Rs. 1.

The matrix representing the payoff is:

	Head	Tail
Head	$(+1, -1)$	$(-1, +1)$
Tail	$(-1, +1)$	$(+1, -1)$

This game does not hold any pure strategy Nash equilibrium.
Both competitors randomize between Head and Tail.
The P_1 be the probability of player 1 selecting head be a.
Then the probability of not electing head be $(1 - a)$.
Correspondingly, the P_2 be the probability of player 2 choosing Head be b.
implying the probability of opting tail is $(1 - b)$.
The expected payoff table for Player 1 is:

Sample Space	Probability	Payoff of P_1
(H, H)	ab	$+1$
(H, T)	$a(1 - b)$	-1
(T, H)	$(1 - a)b$	-1
(T, T)	$(1 - a)(1 - b)$	$+1$

Therefore, the combined total expected payoff of Player 1 is

$$E_{P_1} = (2b - 1)a + (1 - b).$$

We observe:

- If
$$b > \frac{1}{2},$$
then E_{P_1} increases, so it is best for P_1 to choose Head, i.e.,
$$a = 1.$$

- If
$$b < \frac{1}{2},$$
then E_{P_1} decreases, so it is best for P_1 to choose Tail, i.e.,
$$a = 0.$$

- If
$$b = \frac{1}{2},$$
then E_{P_1} remains the same for all a.

  Hence, P_1 can randomly choose Head or Tail.

Similarly, for Player 2, if

$$a = \frac{1}{2},$$

then Player 2 is equally favourable toward all choices.

Hence, the mixed strategy Nash equilibrium is:

$$P_1 : (H, T) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

and

$$P_2 : (H, T) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

3.4 The Nash Occurrence Theorem

Statement:

Every game based on strategies having countable number of players and limited number of tactics for each player,

$$J = \langle M, (P_r)_{r \in M}, (\rho_r)_{r \in M} \rangle$$

has the very least one mixed strategy Nash equilibrium.

Chapter 4

Correspondences and Fixed Point Theorems

4.1 Correspondences

" Let $W \subseteq \mathbb{R}^m$and $Q \subseteq \mathbb{R}^l$ be a correspondance from W till Q. If every element w belongs to W given a subset $S(w)$ subset of Q. Also, known as set valued function and is denoted by $S : W \rightrightarrows Q$. So, we have an example like"[6]

Example: $S(w) = \{q \in \mathbb{R} : q^2 = w\}$. Then, $S(9) = \{-3, 3\}$.

4.1.1 Graph of a Correspondence

"Let $W \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^l$, where W and Q are set and closed set respectively. For $S : W \rightrightarrows Q$ the graph of a correspondence can be shown as:

$$G(S) = \{(w, q) \in W \times Q : w \in W \text{ and } q \in S(w)\}, \forall w \in W$$

."[6]

4.1.2 Closed Graph of a Correspondence

"Let $W \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^l$, where W and Q are set and close set respectively. If $G(S)$ is a subset od $W \times Q$ remains closed the the set valued function $S : W \rightrightarrows Q$ is observed to have a closed graph. Equivalently, two sequences $w^n \to w \in W$ where $w^n \in W$ and $q^n \to q \in Q$ with $q^n \in S(w^n)$, then $q \in S(w)$."[6]

4.1.3 Upper Hemicontinuity

"Let $W \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^l$, where W and Q are set and closed set respectively. S will be called upper hemicontinuous if graph of correspondence will form a closed set and after applying S the compact sets will map to bounded sets. This concept is nothing but an extension of concept of continuity in an ordinary function." [6]

4.2 Fixed Point Theory

4.2.1 Fixed Point of a Function

"Define a function $k : C \to C$ by taking C subset of $\mathbb{R}^l$. Now for a point to be considered as a fixed point of self-map ($c \in C$) if:

$$k(c) = c.\text{”}$$

[6] Intuitively, it is a point that does not move when the function k is applied.

4.2.2 Fixed point of a correspondence

"Suppose $W \subseteq \mathbb{R}^l$ and correspondence given as $S : W \rightrightarrows W$. A point is considered to be as a fixed point of correspondence S if $w \in S(w)$, given the point w belongs to W."[6]

4.2.3 Fixed Point Theorems

"Now, to prove the existence of fixed points under certain given conditions, we use theorems also known as Fixed point theorems. Few widely used fixed point theorems are Kakutani's , Brouwer's and Schauder's fixed point theorem.

4.3 Sperner's Lemma

In combinatorics a very vital lemma given by Sperner which is extremely significant in proving Brouwer's Theorem on fixed points and further the Nash Theorem.

4.3.1 Geometric Setup for Sperner's Lemma

Take a triangle and denote it's corners by P, Q, and R, now take the midpoint of edges PQ, RP, and QR and join these midpoints to form a triangle. This gives us four small triangles inside the big triangle PQR. Now when we repeat the same process after the first subdivision, we get sixteen smaller triangles, for reference check the below given $figure$. This process of dividing triangle into smaller triangles is called triangulation. Triangulating further we get 64, 256, ... triangles.
Labeling rules for the vertices of baby triangles (see Fig 1):

- Vertices lying on the edge PQ can be labeled by either P or Q and not Q.
- Vertices lying on the edge RP can be labeled by either P or R and not B.
- Vertices lying on the edge QR can be labeled by either Q or R and not P.
- Vertices lying inside the big triangle PQR can be labeled by P, Q or R.

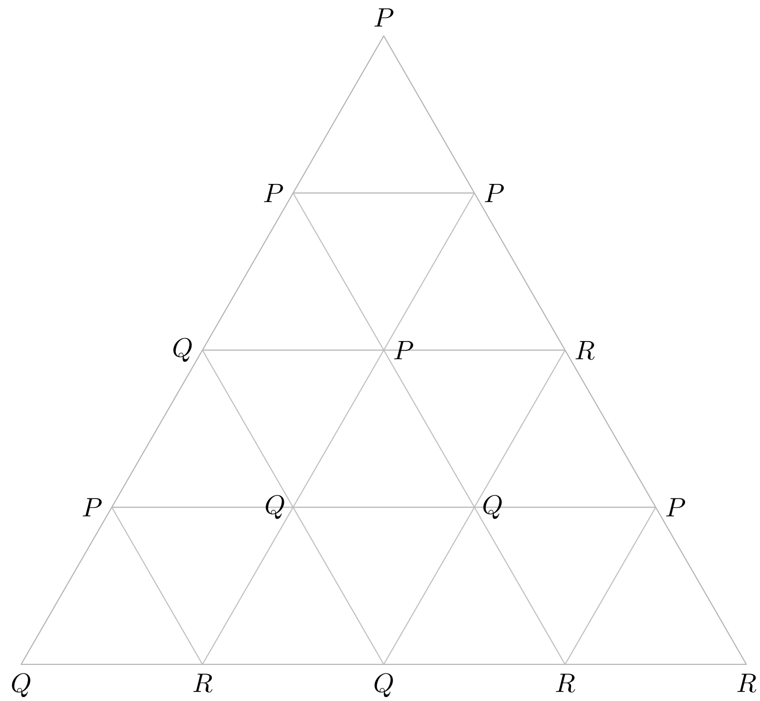

figure: $N_C = 2$; $N_{AC} = 3$

Identify all baby triangles that have different labeling on all three vertices i.e. P, Q, R. Such triangles are of two kinds, one in which P, Q, R are in clockwise order and other in which P, Q, R are in anti-clockwise order. Let us denote the number of baby triangles in which P, Q, R appears in clockwise order as N_C and number of baby triangles in which P, Q, R appears in anti-clockwise order as N_{AC}.

4.3.2 Statement and proof of Sperner's Lemma

We will be following proof given in [6]. Sperner's Lemma says that, all the smaller triangles inside the original triangle that have different labeling on all three vertices i.e. P, Q, R are always odd in number[9].Also, ; $N_{AC} = N_C + 1$; To prove Sperner's Lemma, let us first label the sides of triangles using the below given rules:

1. If the corners of a side is marked with the same label, then label the side with 0.
2. If the corners of a side is marked with the different label but in anti-clocwise sense, then label the side with 1.
3. If the corners of an side have different labels but are in clockwise manner, label the side by -1.

Now, for all the baby triangles, take the sum of all three edges to get a number which is called the *Sperner number*.
Four cases arise:
Case I: When all the three vertices are differently labelled but in clockwise fashion, for such triangles

the Sperner number is

$$-1 - 1 - 1 = -3.$$

Case II: When all the three vertices are differently labelled but in anti-clockwise fashion, for such triangles the Sperner number is

$$1 + 1 + 1 = 3.$$

Case III: When two vertices have the same label, then one edge contributes +1, the second edge contributes −1, and the third edge contributes 0. Thus, the Sperner number for such triangles is

$$-1 + 1 + 0 = 0.$$

Case IV: When all the three vertices are labeled the same, then every edge contributes 0. Thus, the Sperner number of such triangles is 0. Hence, we observe that only triangles with all three differently labeled vertices give non-zero Sperner numbers. Every interior edge of the big triangle PQR belongs to two adjacent triangles. If for one triangle the orientation is clockwise, for the other it will be anti-clockwise, thus their contributions cancel each other. Hence, the total sum of Sperner numbers depends solely on the boundary sides of the original triangle. Now, on the side PQ, overall the labels change from P to Q, giving the sum of all numbers assigned along PQ equal to 1. In the same way, the sums on the edges QR and PR are also 1. Hence, taking Sperner numbers of all the smaller triangles and then taking it's summation is

$$1 + 1 + 1 = 3.$$

But we know that all triangles labeled differently in anti-clockwise (clockwise) fashion contribute 3 (−3). Thus,

$$3N_{AC} - 3N_C = 3$$

$$\Rightarrow N_{AC} - N_C = 1$$

$$\Rightarrow N_{AC} = 1 + N_C$$

Hence,

$$N_{AC} + N_C$$

is an odd number, which means that all the baby triangles with differently labeled vertices are odd in number.

4.4 Brouwer Fixed Point Theorem

"Take T to be a subset of finite euclidean space $\mathbb{R}^l$ which is a compact set and convex set. Define a self map e such that e is continuous, then, e must have a fixed point."[9]

Compact Set: "A subset T of $\mathbb{R}^l$ is said to be compact if and only if it is closed and bounded (Heine–Borel Theorem)."[9]

Convex Set: "Take T to be a subset of finite euclidean space $\mathbb{R}^l$ is said to be a convex if the line that joins two arbitrary points of that set also lie in the set." [9]

This famous theorem from the late nineteenth century can be used to give the proof of a very important theorem of Game theory i.e, The Nash theorem in a zero-sum game.

4.5 Proof of Brouwer's Fixed Point theorem using Sperner's Lemma

To prove fixed point theory given by Brouwer one first needs to prove a lemma that uses Sperner's Lemma. We will first define $m - Simplex$.

Definition: Mathematically, m-Simplex is :

$$\Delta^m = \left\{ w \in \mathbb{R}^m : w_l \geq 0 \text{ for } l = 1, 2, \ldots, m, \ \sum_{l=1}^{m} w_l = 1 \right\} [9].$$

An $m - Simplex$ is known to be a compact and convex set.

- For $m = 2$, $m - Simplex$ is a line segment.
- For $m = 3$, $m - Simplex$ is a triangle.
- For $m = 4$, $m - Simplex$ is a tetrahedron.

Lemma 1. *Consider a continuous self map t on an $m - simplex$, then this self map t will have a fixed point.*

Proof. The proof is adapted from [6]. Prove this lemma for $3 - simplex$ by making use of Sperner's Lemma, i.e., if $t : \Delta^3 \to \Delta^3$ is continuous, then t does not have a fixed point. We will prove this using contradiction. Suppose that t has no fixed point.

Take $t(w) = (t_1(w), t_2(w), t_3(w))$. We know that Δ^3 is a triangle. Thus, triangulate it using the usual method. Define the labeling rule for each vertex $(1, 2, 3)$ in the triangulation as follows:

$$l(w) = \min\{k : t_k(w) < w_k\}.$$

Now, we will show that the coloring is well defined. Let, for some $w \in \Delta^3$,

$$t_k(w) \geq w_k \qquad \forall k = 1, 2, 3.$$

Clearly $w, t(w) \in \Delta^3, \implies w_1 + w_2 + w_3 = 1$ and $t_1(w) + t_2(w) + t_3(w) = 1$.
Hence, $t_k(w) = w_k \quad \forall k = 1, 2, 3$. Therefore,

$$t(w) = w,$$

but we assumed that there is no fixed point hence, contradiction.
Thus, $\exists\ k$ such that $t_k(w) < w_k$. Hence, the labeling rule is well defined. The above given labeling rule fulfills the requirement for using Sperner's lemma. We can check that: $l(1, 0, 0) = 1, \quad l(0, 1, 0) = 2, \quad l(0, 0, 1) = 3$. i.e.,the original triangle has distinct labels on all three corners.(Δ^3) .
Let w be any point on the side joining points $(1, 0, 0)$ and $(0, 1, 0)$. We know that w will be of the form: $w = \zeta(1, 0, 0) + (1 - \zeta)(0, 1, 0) = (\zeta, 1 - \zeta, 0)$, where $0 \leq \zeta \leq 1$.
Here, third coordinate is zero, i.e., $w_3 = 0$.
Since $t(w) \in \Delta^3, \implies t_3(w) \geq 0$.
Therefore, $t_3(w) < w_3 = 0$ is not possible.
Hence, we can say that the labeling rules fulfill the requirements for Sperner's Lemma.
Consider the m^{th} subdivision of Δ^3. Applying Sperner's Lemma, we know that $\exists$ at least one baby triangle having three distinct labels. Let these vertices be denoted by (s_1^m, s_2^m, s_3^m), and without loss of generality, let $l(s_1^m) = 1, \quad l(s_2^m) = 2, \quad l(s_3^m) = 3$.
As $m \to \infty$, the sequence $\{s_1^m\}_{m \geq 1}$ will have a convergent subsequence, as Δ^3 is compact. Thus, for some $w \in \Delta^3$, $s_1^m \to w$.
Similarly, the sequences $\{s_2^m\}_{m \geq 1}$ and $\{s_3^m\}_{m \geq 1}$ will also have convergent subsequences. As $m \to \infty$, the subdivisions become finer, so that the baby triangles become arbitrarily small, and therefore, $s_2^m \to w$ and $s_3^m \to w$.
Given that $t : \Delta^3 \to \Delta^3$ is continuous,

$$t(s_1^m) \to t(w), \qquad t(s_2^m) \to t(w), \qquad t(s_3^m) \to t(w).$$

Observe $l(s_1^m) = 1$, we have,

$$t_1(s_1^m) < (s_1^m)_1.$$

As $m \to \infty$, we get

$$t_1(w) \leq w_1.$$

Similarly, $t_2(w) \leq w_2$ and $t_3(w) \leq w_3$.
Take sum of all the above given inequalities, we get

$$t_1(w) + t_2(w) + t_3(w) \leq w_1 + w_2 + w_3.$$

But $w, t(w) \in \Delta^3$, therefore

$$t_1(w) + t_2(w) + t_3(w) = w_1 + w_2 + w_3 = 1.$$

This can occur iff,

$$t_k(w) = w_k \qquad \forall k = 1, 2, 3.$$

Therefore,

$$t(w) = w,$$

This contradicts what we assumed that t has no fixed point. Hence, t has a fixed point. □

Theorem 1 (Brouwer's Fixed point Theorem)**.** *"If Y is a compact and convex set then any continuous self map g on Y will have a fixed point ." [9].*

Proof. The following proof is adapted from[9] According to a topological result, if Y is a set which is convex and compact and has dimension $(m-1)$, then it is equivalent to an m-simplex (Δ^m). Thus, there exists a continuous mapping and inverse mapping between them, say $h : Y \to \Delta^m$ and $h^{-1} : \Delta^m \to Y$. Define a function, say

$$f : \Delta^m \to \Delta^m$$

such that, $f(y) = h(g(h^{-1}(y)))$, where $y \in Y$
Since g, h, and h^{-1} are continuous, their composition will be continuous as well and hence, f is continuous. From the previous lemma when a function which is continuous is applied on Δ^m we get a fixed point, i.e., $\exists\ y^* \in \Delta^m$ such that, $f(y^*) = y^*$. Therefore,

$$h(g(h^{-1}(y^*))) = y^*.$$

Now, apply h^{-1} on both sides to get,

$$g(h^{-1}(y^*)) = h^{-1}(y^*).$$

Hence, we can say that $g : Y \to Y$ has a fixed point $(h^{-1}(y^*))$. This proves the Theorem. □

4.6 Kakutani Fixed Point Theorem

"Let Z $(\neq \phi)$ be a convex compact subset of finite euclidean space $\mathbb{R}^n$. Let

$$G : Z \rightrightarrows Z$$

satisfies:

- $G(z)$ is nonempty for every $z \in Z$,
- $G(z)$ forms a convex set for every $z \in Z$,
- Z is upper hemicontinuous.

Then $\exists$

$$z^* \in Z$$

such that

$$z^* \in G(z^*).$$

From the definition of fixed point, z^* is is a fixed point of the multi-valued function G."[9]

4.7 Schauder Fixed Point Theorem

"Consider Z ($\neq \phi$) be a convex, bounded and closed set. Let Z be a subspace of K a Banach space, now let

$$T : Z \to Z$$

satisfies:

- $T(z)$ is continuous for every $z \in Z$,
- $T(z)$ is compact for every $z \in Z$,

Then $\exists$

$$z^* \in Z$$

such that

$$z^* \in T(z^*).$$

This satisfies the definition of fixed point and hence z^* is a fixed point of map T."[9]

Chapter 5

Nash Equilibrium via Fixed point theorems

5.1 The Nash Occurrence Theorem

Statement: Every game based on strategies having countable number of players and limited number of tactics for each player,

$$J = \langle M, (P_r)_{r \in M}, (\rho_r)_{r \in M} \rangle$$

has the very least one mixed strategy Nash equilibrium.

5.2 Existence of Nash Equilibrium via Brouwer's Fixed Point Theorem

Proof. Proof adapted from [11] Let us look at a game:

$$J = \langle M, (P_r)_{r \in M}, (\rho_r)_{r \in M} \rangle$$

here:

- M is a set containing players players from 1 to m.
- $P_r \neq \phi$ set that contains all the strategies of player r.
- ρ_r : represents a function (utility) that maps each strategy a real valued number for player r.

Let $\Delta(P_r)$ is a collection of all probabilities that sum upto 1 such that probabilities are assigned to each strategy for r.

Now define the set,

$$G = \prod_{r=1}^{n} \Delta(P_r).$$

Since each $\Delta(P_r)$ is convex and compact ($\because$ simplex), after taking product of all it will still remain non-empty, convex and compact. Thus, $G \neq \emptyset$ is compact and convex.

Every element $\delta \in G$ is of the form

$$\delta = (\delta_1, \delta_2, \ldots, \delta_m),$$

where,

$$\delta_r = (\delta_{r1}, \delta_{r2}, \ldots, \delta_{r|P_r|}),$$

and δ_{rk} is the probability that player r chooses the pure strategy p_{rk}.

Now, define a map

$$h : G \to G$$

such that

$$h_{rk}(\delta) = \frac{\delta_{rk} + \max\{\rho_r(p_{rk}, \delta_{-r}) - \rho_r(\delta_r, \delta_{-r}), 0\}}{\sum_{p_{rj} \in P_r} [\delta_{rj} + \max\{\rho_r(p_{rj}, \delta_{-r}) - \rho_r(\delta_r, \delta_{-r}), 0\}]}.$$

Now let us show that the map is well-defined. We know that the sum of all probabilities is 1, i.e.,

$$\sum_{p_{rj} \in P_r} \delta_{rj} = 1.$$

Thus, the denominator is positive and moreover,

$$\sum_{k=1}^{|P_r|} h_{rk}(\delta) \text{ sums to one, where } r \text{ varies from 1 to } m.$$

Hence, $h(\delta) \in G$, and therefore h is well-defined.

Observe that h is continuous and $G(\neq \emptyset)$ is compact and convex. Therefore, by Brouwer Fixed Point Theorem, h is going to have a fixed point, say

$$\delta^* \in G,$$

i.e.,

$$h(\delta^*) = \delta^*.$$

For each player r and it's pure strategy p_{rk} ,

$$\delta_{rk}^* = \frac{\delta_{rk}^* + \max\{\rho_r(p_{rk}, \delta_{-r}^*) - \rho_r(\delta_r^*, \delta_{-r}^*), 0\}}{\sum_{p_{rj} \in P_r} \left[\delta_{rj}^* + \max\{\rho_r(p_{rj}, \delta_{-r}^*) - \rho_r(\delta_r^*, \delta_{-r}^*), 0\}\right]}.$$

Now, for

$$\sum_{p_{rj} \in P_r} \max\{0, \rho_r(p_{rj}, \delta_{-r}^*) - \rho_r(\delta_r^*, \delta_{-r}^*)\},$$

two cases arise. **Case I:** Let

$$\sum_{p_{rj} \in P_r} \max\{0, \rho_r(p_{rj}, \delta_{-r}^*) - \rho_r(\delta_r^*, \delta_{-r}^*)\}, \text{will be } 0$$

$\forall\ r$ from 1 to m. The sum is 0 and each term is non-negative,

$$\max\{\rho_r(p_{rj}, \delta_{-r}^*) - \rho_r(\delta_r^*, \delta_{-r}^*), 0\} \text{is } 0$$

$\forall\ p_{rj} \in P_r$. Therefore,

$$\rho_r(p_{rj}, \delta_{-r}^*) - \rho_i(\delta_r^*, \delta_{-r}^*) \leq 0,$$

which implies

$$\rho_r(\delta_r^*, \delta_{-r}^*) \geq \rho_r(p_{rj}, \delta_{-r}^*)$$

for all $p_{rj} \in P_r$ and $\forall\ r$ from 1 to m.

Thus, any player won't be able to achieve a better payoff by choosing another strategy unilaterally. Hence, according to the definition of Nash equilibrium,

$$\delta^* = (\delta_1^*, \delta_2^*, \ldots, \delta_m^*)$$

is a Nash equilibrium.

Case II: Suppose that for some player r and some strategy $p_{rk} \in P_r$,

$$\max\{\rho_r(p_{rk}, \rho_{-r}^*) - \rho_r(\rho_r^*, \rho_{-r}^*), 0\} \text{ is strictly greater than } 0.$$

$\Longrightarrow$

$$\rho_r(p_{rk}, \rho_{-r}^*) > \rho_r(\rho_r^*, \rho_{-r}^*).$$

Then,

$$\delta_{ik}^* = \frac{\max\{\rho_r(p_{rk}, \delta_{-r}^*) - \rho_r(\delta_r^*, \delta_{-r}^*), 0\}}{\sum_{p_{rj} \in P_r} \max\{\rho_r(p_{rj}, \delta_{-r}^*) - \rho_r(\delta_r^*, \delta_{-r}^*), 0\}},$$

which implies

$$\delta^*_{rk} > 0.$$

We know that

$$\rho_r(\delta^*_r, \delta^*_{-r}) = \sum_{p_{rj} \in P_r} \delta^*_{rj} \rho_r(p_{rj}, \delta^*_{-r}).$$

The right-hand side of the above equation is a convex combination. But for some $p_{rk} \in P_r$, we have

$$\rho_r(p_{rk}, \delta^*_{-r}) > \rho_r(\delta^*_r, \delta^*_{-r}),$$

with δ^*_{rk}. strictly greater than 0
But this is not true because it $\rho_r(\delta^*_r, \delta^*_{-r})$ is the weighted average of these payoffs. Hence, Case II is not possible. Therefore, only Case I can occur. And hence, it δ^* is a Nash equilibrium. □

5.3 Existence of Nash Equilibrium via Kakutani's Fixed-Point Theorem

Lemma 2 (Lemma on Best Response Correspondence)**.** *The best response multi-valued function ($\neq \phi$)satisfies convexity and upper hemicontinuity. It can be denoted by:*

$$BR_r(p_{-r}) = \{p_r \in P_r \ : \rho_r(p_r, p_{-r}) \geq \rho_r(p'_r, p_{-r}) \ \forall p'_r \in P_r\},$$

for player r.

Proof. Proof follows approach in [6]
We first make the following assumptions:

1. Let the strategy sets $P_1, P_2, \ldots, P_n$ $(\neq \phi)$, satisfying compactness and convex.
2. The utility function of r,
$$\rho_r : P_1 \times P_2 \times \cdots \times P_n \to \mathbb{R},$$
satisfies continuity and quasi-concavity in p_r.

We'll begin with showing that $BR_r(p_{-r}) \neq \phi$. The set $BR_r(p_{-r})$ collects all points in the domain where the payoff function ρ_r reaches its highest value on $P = P_1 \times P_2 \times \cdots \times P_n$. Since, ρ_r is a continuous function and P is compact since $P_1, P_2, \ldots, P_n$ are compact. Weierstrass's theorem says that a function that is continuous reaches a maximum value when applied to a set that is compact. Thus, $\exists$ some

$$p^*_r \in P_r$$

so that

$$\rho_r(p^*_r, p_{-r}) \geq \rho_r(p_r, p_{-r}) \qquad \forall p_r \in P_r.$$

Therefore,

$$p_r^* \in BR_r(p_{-r}),$$

and we have shown that $BR_r(p_{-r}) \neq \phi$.

Now prove it $BR_r(p_{-r})$ satisfies convexity. Since it $\rho_r(p_r, p_{-r})$ satisfies quasi concavity in p_r, its upper contour set

$$U(s) = \{y \in P_r : \rho_r(y, p_{-r}) \geq s\}$$

satisfies convexity $\forall\ s \in \mathbb{R}$.

As previously shown using Weierstrass's theorem, it ρ_r attains a maximum value. Let

$$s = \max_{y \in P_r} \rho_r(y, p_{-r}).$$

Then we observe that $BR_r(p_{-r})$ and $U(s)$ are the same sets. Since it $U(s)$ is convex, it follows that it $BR_r(p_{-r})$ satisfies convexity as well.

Finally, prove it $BR_r(p_{-r})$ satisfies upper hemicontinuity. For this, see if it BR_r has a graph that is a closed set.

Take it

$$(p_r^m, p_{-r}^m) \to (p_r, p_{-m})$$

such that

$$p_r^m \in BR_r(p_{-r}^m).$$

According to the best response multi-valued function, it

$$\rho_r(p_r^m, p_{-r}^m) \geq \rho_r(p_r', p_{-r}^m) \qquad \forall p_r' \in P_r.$$

ρ_r satisfies continuity; thus,

$$\rho_r(p_r^m, p_{-r}^m) \to \rho_r(p_r, p_{-r}),$$

and

$$\rho_r(p_r', p_{-r}^m) \to \rho_r(p_r', p_{-r}) \qquad \forall p_r' \in P_r.$$

Taking limits in the above inequality, we obtain

$$\rho_r(p_r, p_{-r}) \geq \rho_r(p_r', p_{-r}) \qquad \forall p_r' \in P_r.$$

thus,

$$p_r \in BR_r(p_{-r}).$$

Hence, it $BR_r(p_{-r})$ has a graph that is closed. Also, after applying BR_r on compact sets, the image set comes out to be bounded. Hence, it $BR_r(p_{-r})$ is an upper hemicontinuous.

Theorem 2 (Nash Theorem via Kakutani Fixed Point Theorem). *"Games having finite strategies will have at least one mixed strategy equilibrium." [7]*

Proof. Take a look at a game having finite strategies:

$$J = \langle M, (P_r)_{r\in M}, (\rho_r)_{r\in M}\rangle, \qquad \text{where } r = 1, 2, \ldots, m.$$

LET $P_r = \{p_{r1}, p_{r2}, \ldots, p_{rv_r}\}$ is a collection of strategies for player r. The mixed strategy set for the player r is defined by

$$\Delta(P_r) = \left\{(z_1, z_2, \ldots, z_{v_r}) : 0 \le z_l \le 1, \quad \sum_{l=1}^{v_r} z_l = 1\right\}.$$

We know that $\Delta(P_r)$ is non-empty because it is the collection of all probability distributions assigned to P_r.

Next, we show that $\Delta(P_r)$ is convex. Let $z, w \in \Delta(P_r)$, where

$$z = (z_1, z_2, \ldots, z_{v_r}) \quad \text{and} \quad w = (w_1, w_2, \ldots, w_{v_r}).$$

Take $\zeta \in [0, 1]$. Then,

$$\zeta z + (1-\zeta)w = \zeta(z_1, \ldots, z_{v_r}) + (1-\zeta)(w_1, \ldots, w_{v_r})$$

$$= (\zeta z_1 + (1-\zeta)w_1, \ldots, \zeta z_{v_r} + (1-\zeta)w_{v_r}).$$

Now,

$$\sum_{l=1}^{v_r}\big(\zeta z_l + (1-\zeta)w_l\big) = \zeta\sum_{l=1}^{v_r} z_l + (1-\zeta)\sum_{l=1}^{v_r} w_l.$$

Since

$$\sum_{l=1}^{v_l} z_l = 1 \quad \text{and} \quad \sum_{l=1}^{v_r} w_l = 1,$$

we obtain

$$\sum_{l=1}^{v_r}\big(\zeta z_l + (1-\zeta)w_l\big) = \zeta + (1-\zeta) = 1.$$

Hence,

$$\zeta z + (1-\zeta)w \in \Delta(P_r),$$

and therefore $\Delta(P_r)$ is convex.

We now show compactness of $\Delta(P_r)$. It suffices to show that $\Delta(P_r)$ is closed and bounded (Heine-Borel Theorem). Observe that $\Delta(P_r) \subseteq [0, 1]^{v_r}$, hence $\Delta(P_r)$ is bounded.

For proving closedness of $\Delta(P_r)$ take

$$z^m = (z_1^m, \ldots, z_{v_r}^m) \in \Delta(P_r)$$

a sequence where,

$$z^m \to z = (z_1, \ldots, z_{v_r}).$$

Since $z_j^m \geq 0, \qquad 1 \leq j \leq v_r$, taking limits gives

$$z_j \geq 0, \qquad 1 \leq j \leq v_r.$$

Also,

$$\sum_{j=1}^{v_r} z_j^m = 1 \qquad \forall m.$$

Taking limits, we obtain

$$\lim_{m\to\infty} \sum_{j=1}^{v_r} z_j^m = \sum_{j=1}^{v_r} \lim_{m\to\infty} z_j^m = 1.$$

Hence,

$$\sum_{j=1}^{v_r} z_j = 1.$$

Therefore, $z \in \Delta(P_r)$, and hence $\Delta(P_r)$ is closed. Thus, $\Delta(P_r)$ is compact.

Next, we show that the payoff function ρ_r is continuous. We know that

$$\rho_r(\delta) = \sum_{p\in P} \left(\prod_{l=1}^{m} \delta_l(p_l) \right) \rho_r(p).$$

Since right-hand side is a finite weighted sum and $\rho_r(p)$ is continuous, it follows that $\rho_r(\delta)$ is continuous.

Finally, $\rho_r(\delta)$ is quasi-concave since $\rho_r(p)$ is quasi-concave. Best response correspondence is:

$$BR_r(\delta_{-r}) = \left\{ \delta_r \in \Delta(P_r) : \rho_r(\delta_r, \delta_{-r}) \geq \rho_r(\delta_r', \delta_{-r}) \quad \forall \delta_r' \in \Delta(P_r) \right\}.$$

From above proved lemma, BR_r $(\neq \phi)$ satisfies upper hemicontinuity and convexity. Thus Kakutani's theorem's hypothesis are met $\implies$ BR_r must have a fixed point, i.e.,

$$\delta_r^* \in BR_r(\delta_{-r}^*).$$

Thus,

$$\rho_r(\delta_r^*, \delta_{-r}^*) \geq \rho_r(\delta_r, \delta_{-r}^*) \qquad \forall \delta_r \in \Delta(P_r),\ \forall r.$$

Therefore,

$$\delta^* = (\delta_1^*, \ldots, \delta_m^*)$$

is a mixed strategy Nash equilibrium. □

The theorem provides sufficient but not necessary conditions for the occurrence of Nash equilibrium. □

Aspect	**Brouwer Approach**	**Kakutani Approach**
Type of Mapping	Single-valued continuous function $h : Y \to Y$	Multi-valued function $H : Y \rightrightarrows Y$
Fixed Point Condition	$h(y) = y$	$y \in H(y)$
Nature of Best Response	Requires converting the best response into a continuous function.	Directly uses the best-response correspondence
Continuity Requirement	Continuity	Upper hemicontinuity
Convexity Requirement	Requires subset of $\mathbb{R}^n$ satisfying convexity	Compact convex domain with convex-valued correspondence
Applicability	Finite-dimensional finite games	Finite and more general strategic games
Main Advantage	Conceptually simpler and constructive	More natural and mathematically general
Limitation	Cannot directly handle multiple best responses	Can handle multiple best responses naturally
Use in Nash Equilibrium Proof	Used in Nash's original 1950 proof	Used in most modern textbook proofs
Relation Between Theorems	Special case of Kakutani's theorem	Generalization of Brouwer's theorem
Extension to Infinite Games	Limited applicability	Forms the basis for extensions such as Glicksberg's theorem

Table 5.1: Comparison between Brouwer and Kakutani approaches in proving Nash Equilibrium existence

Chapter 6

Infinite Strategic Games

6.1 Infinite Strategic Games and Fixed Point Theorems

In this chapter we turn to infinite strategic games and the overall contribution of fixed-point theorems for laying the continuance of Nash equilibrium. There is at least one infinite strategy set in an infinite game of strategy. For example,

$$P_r = [0, 1]$$

a respondent in an infinite strategy space can opt out for any number from $[0, 1]$. Such games naturally arise in applications such as auction theory, pricing theory, economics, optimization theory, control theory, etc. From the overall discussion help above, it can be stated that there has to be at least one Nash equilibrium mixed strategy in each and every finite tactical game. However, for infinite strategic games, the existence of a Nash equilibrium is no longer immediate. Additional topological assumptions become necessary.

The deployment of Brouwer's and Kakutani's fixed-point theorems has been made in the latter-discussed chapters to provide the proof of Nash equilibrium occurrence in finite strategic games. These theorems are generally formulated for finite-dimensional spaces. However, in infinite strategic games, the strategy spaces may become infinite dimensional. Consequently, more general fixed-point methods are required.

6.1.1 Continuous Games

Take a look at the game

$$J = \langle M, (P_r)_{r \in M}, (\rho_r)_{r \in M} \rangle$$

where:

- M is a set containing players players from 1 to m.
- $P_r \neq \phi$ set that contains all the strategies of player r.

- ρ_r : represents a function (utility) that maps each strategy a real valued number for player r.

Such a game Γ is called a continuous game.

6.1.2 Glicksberg's Theorem

Theorem 3 (Glicksberg)**.** *Every continuous game possesses at least one mixed strategy Nash equilibrium.*

That is, there will exist at least one Nash equilibrium mixed strategy when the following conditions are met: if each strategy space P_r is compact and metrizable and each utility function ρ_r is continuous.

The theorem extends Nash's equilibrium existence theorem from finite strategic games to games with infinite strategy spaces.

The assumptions of compactness and continuity arise naturally in finite strategic games. Every transformation stated on a finite domain is continuous in nature, and every finite strategy set is compact under the discrete metric.

Kakutani's Fixed Point Theorem has been used in proving the occurrence of Nash equilibrium in finite strategic games. Moreover, Brouwer's fixed-point theorem is also extended from it, to set-valued correspondences from single-valued functions.

However, in infinite strategic games, the strategy spaces may become infinite dimensional. Therefore, more general fixed point theorems are required.

The theorem of Schauder's fixed point is one such theorem.

6.1.3 Schauder Fixed Point Theorem

Theorem 4 (Schauder Fixed Point Theorem)**.** *"Let it C be a non-empty compact convex subset of a Banach space and let it*

$$R : C \to C$$

be a continuous transformation. Here at least one fixed point is encompassed by R."[9]

Brouwer's Fixed Point has been extended to Banach spaces of infinite dimensions by the Schauder Fixed Point Theorem. Moreover, it exhibits a vital contribution to functional analysis, differential equations, equilibrium theory, and differential equations.

In infinite strategic games, the strategy spaces may consist of functions, probability distributions, or infinite-dimensional objects. In such settings, equilibrium problems can often be interpreted as fixed-point problems.

If this

$$R : C \to C$$

represents a continuous best response operator, then the latter discussed concept Nash equilibrium can be equivantly corresponds to a fixed point satisfying

$$R(c^*) = c^*.$$

Thus, Schauder-type fixed point methods are closely related to equilibrium existence problems in infinite strategic games.

Although the standard proof of Glicksberg's theorem generally uses Kakutani-type fixed-point theorems for set-valued correspondences, Schauder's theorem remains closely connected through the common framework of infinite-dimensional fixed-point theory.

Both Glicksberg's theorem and Schauder's theorem fundamentally rely upon compactness, continuity, and convexity to establish existence results.

6.1.4 Necessity of Continuity

The continuity assumption in Glicksberg's theorem is essential.

Consider the strategy space

$$P = [0, 1]$$

and define the utility function

$$\rho(p) = \begin{cases} p, & 0 \leq p < 1, \\ 0, & p = 1. \end{cases}$$

The strategy space is compact, but the utility function is not continuous at $p = 1$.

For every strategy $p < 1$, the player can improve the payoff by choosing a value closer to 1. However, choosing $p = 1$ gives payoff 0. Hence, no strategy maximizes the utility function.

Therefore, no Nash equilibrium exists.

This example shows that compactness alone is insufficient without continuity.

6.1.5 Necessity of Compactness

Consider the strategy space

$$P = [0, 1)$$

with utility function

$$\rho(p) = p.$$

In this case, the utility function is continuous, but the strategy space is not compact.

For every strategy $p \in [0, 1)$, there exists another strategy $p' > p$ producing a larger payoff.

Since the value 1 does not belong to the strategy space, no maximizing strategy exists.

Thus, no Nash equilibrium exists.

This example demonstrates that continuity alone is insufficient without compactness.

6.1.6 Parallel Between Brouwer, Kakutani, Schauder and Glicksberg

These theorems may be related through the development of fixed-point theory from finite-dimensional spaces to infinite-dimensional equilibrium theory.

Brouwer Fixed Point Theorem

Its direct implications is based on single-valued functions with continuity on subsets of finite-dimensional Euclidean spaces which are dense-compact and compact.

It forms the basic fixed-point theorem of finite-dimensional topology.

Kakutani Fixed Point Theorem

Brouwer's theorem has been generalized by Kakutani's Fixed Point Theorem to set-valued correspondences from single-valued functions.

This theorem is particularly important in game theory because best response mappings are generally set-valued.

Kakutani's theorem is one of the standard tools used in proving Nash's equilibrium theorem for finite strategic games.

Schauder Fixed Point Theorem

Brouwer's theorem has been generalized by Schauder's Fixed Point Theorem to Banach spaces of infinite dimensions from Euclidean finite-dimensional spaces.

Thus, Schauder's theorem plays an important role in the study of infinite-dimensional spaces and infinite strategic games.

Glicksberg's Theorem

Glicksberg generalized Nash's equilibrium existence theorem from finite strategic games to infinite strategic games under compactness and continuity assumptions.

The theorem relies heavily upon topological fixed-point techniques and infinite-dimensional analysis.

These relationships may be summarized schematically as follows:

$$\text{Brouwer} \to \text{Kakutani}$$

for finite strategic games, and

$$\text{Schauder} \to \text{Glicksberg}$$

for infinite strategic games.

Thus, this theory of infinite strategic games depicts the deep connection between topology, functional analysis, and game theory.

Chapter 7

Conclusion and Future Scope

7.1 Conclusion

This thesis studies in detail about the connection between game theory and fixed-point theory with special focus on proving Nash theorem using fixed-point theorems. After studying standard results in mathematical economics and topology, we bridged the gap between these two areas by providing mathematical results used in the equilibrium analysis of strategic games.

This thesis started studying introductory game theory that included pure strategy, payoff matrix, game, Nash equilibrium, mixed strategy etc. Also, examples of the prisoner's dilemma and matching pennies were offered to further clarify the concept of finite strategic games.

Then we move towards building a mathematical foundation for Nash equilibrium existence proofs. We defined important topological concepts such as correspondences, upper hemicontinuity, and fixed points of functions and correspondences. Most importantly, we have elaborated some well-known fixed point theorems, Kakutani's fixed-point theorem, Brouwer's fixed-point theorem, Schauder's fixed-point theorem and Sperner's lemma. We proved Sperner's lemma, a basic result in combinatorics, which we used to give an elaborated proof of Brouwer's fixed point theorem. The work demonstrated a very interesting relationship between topology and combinatorics.

Therefore by using Kakutani's fixed-point theorem and Brouwer's fixed point theorem, the thesis helped in proving the existence of Nash equilibrium, mixed strategy in finite strategic games. In establishing the existence of Nash equilibrium, we created a suitable best response mapping and analyzed its properties, emphasizing how compactness, convexity, continuity, and upper hemicontinuity contributed in proving the Nash theorem.

Finally, the thesis discussed infinite games (infinite strategic games). We analyzed how standard finite-dimensional fixed-point theorems can be extended to Schauder's and Glickberg's theorems for games with infinite strategies. Also with the help of examples, we discussed about the importance of assuming continuity and compactness in proving the existence of mixed strategy Nash equilibrium. Overall, this work showed that fixed-point theorems are not just another tool in topology but one of the important mathematical principles in modern equilibrium theory and strategic decision-making.

7.2 Contributions

The key achievements of this thesis are:

- Presented an interesting connection between fixed point theory and game theory.
- Developed the foundations of strategic games, mixed strategies, and Nash equilibrium in a mathematically detailed way.
- Used realistic examples to illustrate the idea of Nash equilibrium, such as the Prisoner's Dilemma and Matching Pennies.
- Studied foundational topological tools such as compactness, convexity, correspondences, and upper hemicontinuity to use equilibrium analysis.
- Sperner's Lemma, a famous combinatorial result was examined and utilised to prove the Brouwer's Fixed Point Theorem.
- Both Kakutani's and Brouwer's approaches were used to demonstrate the proof of The Nash Theorem.
- Used the Schauder and Glicksberg theorems to expand the analysis of finite strategic games into infinite contexts.
- Evaluated the difference between finite and infinite strategic games and highlighted the importance of continuity and compactness in the existence of equilibrium in infinite games.

7.3 Future scopes

This work presents various possible possibilities for additional research in fixed-point theory and game theory. Some future possible directions are the following:

- Investigating discontinuous games and equilibrium existence under reduced continuity assumptions.
- Generalizing equilibrium analysis to stochastic games, recurrent games, and dynamic games.
- In large scale strategic games, to approximate Nash equilibria, study of algorithms and computational approaches can be done.
- Studying other equilibrium concepts, including correlated equilibrium and evolutionary equilibrium.
- Applications of fixed-point approaches to auction theory, economic modeling, and network games. On the extension of the analysis of infinite strategic games with modern functional analytic techniques and infinite dimensional topology.
- Analysis of requirements for uniqueness and stability of equilibrium for concave and non-convex games.

7.4 Final Remarks

he study of Nash equilibrium by using fixed point theory is one of the most important connections between economics, topology and analysis. The mathematical framework created in this thesis ranges from Sperner's combinatorial arguments to Brouwer's and Kakutani's fixed-point theorems and shows how abstract notions in topology can be applied fruitfully to strategic decision-making problems. The results presented in this paper not only prove the existence of equilibrium in finite and infinite games but also show the potential of fixed-point methods in modern mathematical economics and applied mathematics. As game theory evolves and interacts with other subjects such as computer science, optimization, artificial intelligence, and economic theory, fixed-point techniques will remain essential tools for the analysis of strategic behavior and equilibrium occurrences.

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