

**FUZZY PORTFOLIO SELECTION VIA RANKING
MODELS IN DEA AND MULTI-CRITERIA DECISION
MAKING**

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CERTIFICATE

Certified that **Ms. Reenu Kumari** (2K19/PHD/AM/11), has carried out her research work presented in this thesis entitled “**Fuzzy Portfolio Selection via Ranking Models in DEA and Multi-criteria Decision Making**” for the award of **Doctor of Philosophy in Mathematics** from Delhi Technological University, New Delhi, under my supervision. The thesis embodies results of original work and studies are carried out by the student herself and the contents of the thesis do not form the basis for the award of any other degree to the candidate or to anybody else from this or any other University/Institution.

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CANDIDATE’S DECLARATION

I, Reenu Kumari (2K19/PHD/AM/11), a PhD student in the Department of Applied Mathematics, hereby declare that the thesis entitled “Fuzzy Portfolio Selection via Ranking Models in DEA and Multi-criteria Decision Making” which is submitted by me to the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree of “Doctor of Philosophy in Mathematics”, is original and not copied from any source without proper citation. This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship or other similar title or recognition.

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Abstract

A portfolio selection problem implemented through an optimization technique is called portfolio optimization. The mathematical model for portfolio optimization allocates total capital among various assets following investors' preferences about return/risk. Generally, investors seek to reduce risk while enhancing returns, yet attaining higher expected returns inevitably involves accepting greater levels of risk. Therefore, an investor faces a trade-off between risk and expected return. Hence, portfolio optimization is a technique used to construct an optimal basket of assets, where an optimal is understood in the context of an investor's objectives and desires. In many real-world situations, the return from an asset cannot be anticipated accurately based on historical data. The presence of vagueness and fuzziness in the input and output data can not be resolved by using probability theory. The unpredictable dynamic nature of the financial market motivates researchers to use the concept of fuzzy set theory in the field of portfolio selection. The possibility theory is an uncertainty theory devoted to the handling of incomplete information.

Besides an accurate determination of a risk measure of a return distribution, investors also wish to evaluate the performance of their portfolios concerning a benchmark index or to rank different portfolio strategies. Generally, the role of an asset's performance in optimal portfolio construction has not been considered so far. When selecting assets for a portfolio, an investor considers several factors. Data Envelopment Analysis (DEA) simultaneously accommodates multiple inputs and outputs, providing a composite efficiency score. As DEA measures the relative efficiency of several similar processing units, it also helps in asset selection before portfolio construction. However, the DEA allows each financial asset to evaluate its efficiency relative to other homogeneous financial assets by assigning favorable weights. This often results in unrealistic weight schemes. To address this issue, the DEA cross-efficiency framework is employed, which elim-

inates such unrealistic weight allocations. In financial markets, assets compete for higher efficiency scores, often leading to multiple optimal weights in standard cross-efficiency. DEA game cross-efficiency introduces a noncooperative framework where competing assets jointly determine balanced weights, reducing non-uniqueness and producing more stable and fair rankings for portfolio selection.

In certain instances, DEA models may yield an efficiency score of one for several decision-making units (DMUs), making it challenging to rank these DMUs. Further, in DEA, every approach uses a distinct theory and framework to rank the DMUs, so each DMU has a different ranking order. The decision-maker's reliability of the results is a critical consideration when choosing a ranking system. Multi-Criteria Decision Making (MCDM) approaches, which differ from DEA models, can be used to solve the problem of ranking efficient DMUs.

The challenge of aggregating self- and peer-evaluated cross-efficiencies into a final score has been widely discussed. Notably, using a simple arithmetic average inherently assumes that all DMUs' evaluations are equally valid or dependable, which is not always true. To address this issue, we propose a novel use of the Ordered Visibility Graph Averaging (OVGA) operator for more meaningful aggregation. Furthermore, in the same work, we introduce a portfolio selection model for constructing the most efficient optimal portfolio.

In DEA game cross-efficiency, the final score is obtained through an iterative algorithm, where each iteration aggregates the game cross-efficiency scores of all DMUs using the arithmetic averaging method. This thesis presents the OVGA aggregated DEA game cross-efficiency method, which considers the competition among DMUs in portfolio selection. These Game cross-efficiency scores serve as a tool for efficient portfolio selection. Further, a multi-objective portfolio selection model is proposed, where the Maverick index and variance of cross-efficiency are treated as risk metrics, and the OVGA game cross-efficiency scores are used as return characteristics.

The Semi-oriented radial measure (SORM) model of DEA effectively handles the negative input-output data. However, it has a limitation of producing negative cross-efficiencies. We propose a modified SORM model to deal with this issue. Also, a novel multi-objective portfolio selection model is introduced, using the maverick index to represent risk and the diversity index to represent return. The maverick index is calculated using the column average of the cross-efficiency

matrix, while the diversity index is determined using the row average.

In another research study in this thesis, an innovative approach is introduced to portfolio selection derived from the RDM cross-efficiency matrix. In practical applications, the column average of the cross-efficiency matrix is commonly employed for decision-making, as it helps identify efficient and consistent performers. However, the row average also provides valuable insight into how fairly or aggressively each DMU evaluates its peers. We provide a method for categorization of the assets, which utilizes both row and column averages of the RDM cross-efficiency matrix.

An essential aspect of investment management is the unique rating of portfolios, which enables investors to identify and assess the most effective portfolios based on criteria such as risk, return, etc. We present a hybrid approach for ranking investment portfolios by combining the Modified Slack-Based Measure (MSBM) of DEA with a multi-criteria decision-making method. Techniques like the MSBM and TOPSIS incorporate traditional performance metrics while adding flexibility to address fuzzy environments and handle imprecise data. aims to evaluate fuzzy portfolios using the MSBM model, with trapezoidal fuzzy numbers for returns and possibilistic measures for risk and mean return. Efficient portfolios are further ranked using the TOPSIS technique.

This thesis entitled “Fuzzy Portfolio Selection via Ranking Models in DEA and Multi-criteria Decision Making” aims to highlight the advantages of DEA as an innovative tool for portfolio optimization, contributing to the development of more robust and efficient investment strategies. The methodologies introduced and developed in this thesis are rigorously tested on real-world case studies, demonstrating their practical applicability and effectiveness in enhancing portfolio selection processes.

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Date :

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*To my daughter
Aaradhya,
who reminded me to stay strong and face challenges with courage.*

Contents

Certificate page	i
Declaration page	iii
Abstract	v
Acknowledgements	ix
List of tables	xvi
List of figures	xx
1 Introduction	1
1.1 Portfolio Selection	2
1.2 Data Envelopment Analysis	4
1.2.1 DEA Techniques to handle Negative Data	5
1.2.2 DEA Cross-efficiency Evaluation	6
1.3 Portfolio Selection Using DEA	8
1.3.1 Multi-Criteria Decision Making methods	10
1.4 Precursor models	11
1.4.1 Input-oriented CCR model for efficiency evaluation	11
1.4.2 DEA models with negative data	13
1.4.2.1 Semi-oriented radial measure (SORM) model	14
1.4.2.2 Modified slack-based measure (MSBM) model	15
1.4.3 Game Cross-efficiency Evaluation	16
1.4.4 Ordered Visibility Graph Averaging operator	17
1.4.5 MV portfolio selection model	20
1.5 Research Gaps	21
1.6 Motivation	23
1.7 Organization of the Thesis	24
2 Cross-efficiency aggregation by ordered visibility graph averaging: method, and application in portfolio selection	35

2.1	Introduction	36
2.1.1	Research motivation and novelty of the proposed approach	37
2.2	DEA cross-efficiency model to deal with alternative optima	38
2.3	Portfolio optimization model based on the cross-efficiency matrix: Proposed model	41
2.4	Numerical Illustration	45
2.4.1	A Case Study of the Indian Banking Sector	54
2.4.2	Managerial implications:	58
2.5	Conclusion and Future Directions	59
3	A Comprehensive mean-maverick-variance portfolio selection model based on OVGA aggregated game cross-efficiency approach	61
3.1	Introduction	62
3.1.1	Novelty of the work	63
3.2	Proposed OVGA game cross-efficiency approach	64
3.2.1	Algorithm for the proposed process	65
3.2.2	A simple illustrative case	66
3.3	A Comprehensive multiobjective portfolio selection model integrated with OVGA game cross-efficiency approach	72
3.4	An application to portfolio selection in the National Stock Exchange	73
3.4.1	Data collection and processing	74
3.4.2	Selection of input/output variables	75
3.4.3	Results and Discussion	77
3.4.4	Sensitivity analysis of the MMV model with different criteria	77
3.4.5	Comparison of weights with DEA-MV cross-efficiency	78
3.5	Conclusion and Future Research Directions	82
4	An Integrated Multi-Objective SORM Cross-Efficiency Model: An Ap- plication in Portfolio Selection	83
4.1	Introduction	84
4.2	The input-oriented SORM model to deal with negative data	86
4.3	Stage 1-Proposed input-oriented SORM cross-efficiency model	86
4.4	Stage 2-A multi-objective framework of portfolio selection based on SORM cross-efficiency evaluation	89
4.5	An application to stock portfolio selection	90
4.5.1	Results and Discussion	91
4.5.2	Comparative Analysis	94
4.6	Conclusions	95
5	Stock assessment using cumulative prospect theory in DEA cross- efficiency model: A case study of the Indian stock market	97

5.1	Introduction	98
5.2	DEA cross-efficiency evaluation with negative data	100
5.3	Proposed Methodology	102
5.3.1	Choice of Input and Output Variables	102
5.3.2	Assessing the cross-efficiency of the stocks and Categorization of stocks:	102
5.4	Empirical illustration	103
5.4.1	Stock management using the proposed method	104
5.5	Concluding Remarks	110
6	Ranking of Efficient Fuzzy Portfolios by Hybrid MSBM-TOPSIS Technique	111
6.1	Introduction	112
6.1.1	Literature Review of Ranking models in DEA	113
6.1.2	Contributions and Purpose	115
6.2	Preliminaries	116
6.2.1	Modified Slack Based Model to handle negative data	118
6.3	MSBM fuzzy portfolio evaluation models in the framework of possibilistic theory	119
6.3.1	MSBM fuzzy portfolio evaluation model in possibilistic mean-variance framework	120
6.3.2	MSBM fuzzy portfolio evaluation model in possibilistic mean semi-variance framework	121
6.4	TOPSIS method to rank the efficient portfolio	123
6.5	Numerical Illustration	125
6.5.1	Computation Complexity	132
6.6	Conclusion and Future Research Directions	133
	Summary and future scope of the work	134
	Bibliography	138
	List of Publications	161

List of Tables

1.4	Game cross-efficiency matrix for n DMUs corresponding to Model (1.5) for initial iteration $\tau = 0$	17
1.5	Degree and weights of ordered set $\hat{O}$	20
1.1	Summary of literature based on DEA models	28
1.1	(Continued)	29
1.2	Summary of literature based on the DEA cross-efficiency technique	30
1.2	(Continued)	31
1.3	Summary of literature based on DEA in portfolio selection	32
1.3	(Continued)	33
2.1	Cross efficiency matrix for n DMUs corresponding to Model (2.1)	40
2.2	Input and output data values for DMUs	46
2.3	Self-efficiency score e_{kk}^* of DMU_k	46
2.4	Degree and weights of DMU_p w.r.t. DMU_1	48
2.8	The aggregated minimum cross-efficiency score by simple average operator $\bar{e}_j^L$ and by OVGA operator $\tilde{e}_j^L$ of target DMU_j	48
2.5	The cross-efficiency matrix $\tilde{e}_{kj}^L$	49
2.6	Re-ordered cross-efficiency matrix $\tilde{e}_{kj}^L$	50
2.7	The weight w_p^j given by DMU_j to DMU_p on the basis of re-ordered cross-efficiency matrix $\hat{e}_{pj}^L$	51
2.9	The optimal portfolio strategies corresponding to Model (2.3) for equal weights	53
2.10	The optimal portfolio strategies corresponding to Model (2.4) for equal weights	54
2.11	The optimal portfolio strategies corresponding to Model (2.3) for unequal weights	55
2.12	The optimal portfolio strategies corresponding to Model (2.4) for unequal weights	56
2.13	Detailed description of the Banks	57
2.14	Optimal Portfolio Strategies for Banks under Models (2.3) and (2.4)	58
3.1	Weighted game Cross efficiency matrix for n DMUs corresponding to Model (1.5) for initial iteration $\tau = 0$	65

3.2	Input output data	66
3.3	The Cross efficiency matrix for 5 DMUs corresponding to Model (1.2) and equation 1.4.1	68
3.4	The game cross efficiency matrix for 5 DMUs corresponding to Model (1.6) for $\tau = 1$	69
3.5	Degree and weights of the ordered set corresponding to the data of column 1 game cross-efficiency matrix	69
3.6	The weights corresponding to the game cross-efficiency matrix for $\tau = 1$	70
3.7	The OVGA game cross efficiency matrix for 5 DMUs for $\tau = 1$	70
3.8	The iterative calculations of game cross-efficiency scores for $\epsilon = 0.001$	71
3.9	The efficiency scores of DMUs	71
3.10	The input and output description	75
3.11	Converted input-output and different efficiency values	76
3.12	Results of performance comparison of Pareto portfolios with a variation of h	79
3.13	Some Pareto portfolios using the proposed model and DEA-MV cross-efficiency methods for $h = 8$	80
3.14	The weights of the selected portfolios for $h = 10$ by different methods	81
4.1	The standardized input-output data of the selected stocks	92
4.2	The input-oriented SORM cross-efficiency matrix	93
4.3	Different Efficiency Values	93
4.4	The weights of the stocks in the selected portfolio for $h=5$	94
4.5	The statistics from obtained pareto solutions for $h = 5$	94
5.1	Input Output variable with Self-efficiency after solving Model (5.1)	105
5.2	The column and row average efficiency obtained from CEM	107
5.3	Statistics of group HC/HR	107
5.4	Statistics of group HC/LR	108
5.5	Statistics of group LC/LR	108
5.6	Statistics of group LC/HR	109
6.1	The possibility distribution of the return rates and their respective transaction cost of 5 stocks	126
6.2	The weight of each stock in different randomly generated portfolios	126
6.3	The efficiency score and the ranking of portfolios in possibilistic mean-variance framework	127
6.4	Ranking of efficient portfolios by different methods in possibilistic mean-variance framework	128
6.5	The efficiency score and the ranking of portfolios in possibilistic mean semi-variance framework	130

6.6	Ranking of efficient portfolios by in possibilistic mean semi-variance framework	131
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List of Figures

1.1	OVG of ordered set $\hat{O}$	20
2.1	Ordered visibility graph of ordered efficiency set $\hat{E}_1^L$	47
2.2	Efficient frontier of Model (2.3) and Model (2.4)	53
3.1	Approximate efficient frontier in case of (a) $h=5$, (b) $h=10$, (c) $h=12$, (d) $h=15$	82
5.1	The cross-efficiency matrix	106
5.2	Different group of stocks based on CA & RA	108
6.1	Rankings of efficient portfolios in possibilistic mean-variance frame- work	128
6.2	Rankings of efficient portfolios in possibilistic mean semi-variance framework	131

Chapter 1

Introduction

The introductory chapter starts with a review of the relevant literature, including studies related to Markowitz's mean-variance (MV) portfolio selection model. The chapter also presents a brief overview of DEA models and their extensions. Further, this chapter highlights the ability of DEA models to handle multiple inputs and outputs simultaneously, making them an effective tool for portfolio optimization, particularly in scenarios where a variety of factors beyond risk and return must be considered. Therefore, the concepts discussed in this chapter provide a solid foundation for the research presented in this thesis.

In modern financial markets, portfolio selection is a critical process that allows investors to allocate their resources efficiently across different assets. Traditional methods such as Markowitz's mean-variance (MV) model have provided a foundational approach to portfolio selection. However, the MV model often assumes simplified risk-return relationships and may fail to incorporate multiple performance criteria. The DEA approach provides a non-parametric, data-driven technique, allowing investors to select efficient portfolios across several financial performance dimensions. Let us begin the introduction of portfolio selection and the review related to the MV portfolio selection model.

1.1 Portfolio Selection

Portfolio selection refers to an investor's decision to diversify their financial assets to reduce investment risk while maintaining a specific level of projected return. Portfolio selection guides an investor to allocate resources among various investment opportunities. The breakthrough in the field came in 1952 when Harry Markowitz [117] published his modern portfolio theory, known as the mean-variance portfolio theory. Markowitz's mean-variance (MV) portfolio selection model is based on the trade-off between risk and return under various economic conditions. This model interprets asset returns as random variables, assessing a portfolio's performance based on its mean to represent returns and its variance to represent risk.

A key limitation of using variance as a measure of risk is that it treats high returns as equally undesirable as low returns [207]. Numerous researchers have used downside risk measures directly in place of variance in the MV model. Markowitz et al. [118] introduced the mean-semivariance model as an alternative to the MV model. Another measure is semi-absolute deviation, proposed by Speranza [168], which can be effortlessly calculated as it can be determined using linear programming models. Konno and Yamazaki [92] used the mean absolute deviation from the mean as a risk metric to approximate the nonlinear variance and covariances of stocks in the MV model. Chang et al. [25] proposed portfolio optimization models utilizing various risk metrics, including variance, semi-variance, and mean absolute deviation, combined with skewness. Value at Risk (VaR), introduced by Baumol [19], is a popular risk measure representing the maximum potential loss at a given confidence level. However, VaR lacks information on extreme losses beyond the threshold and fails to meet coherence properties. To overcome these issues, Rockafellar [150] proposed Conditional Value at Risk (CVaR), which averages VaR and losses exceeding it. Roman et al. [151] extended this by developing a portfolio model incorporating multiple risks, known as the random mean-variance-CVaR model. Guo et al. [71] examine the influence of background risk on an investor's portfolio decisions within the mean-variance, mean-VaR, and mean-CVaR frameworks. The study explores the characteristics of the mean-variance, mean-VaR, and mean-CVaR limitations and efficient frontiers in the context of background risk. Cui et al. [46] also proposed a time-

consistent and self-coordinating strategy for the multiperiod mean-CVaR portfolio selection model. Xu et al. [198] incorporated weight constraints into the standard CVaR-based portfolio selection model.

The MV portfolio selection model is very sensitive to variations in input data. If we were able to precisely estimate the future expected returns of each asset and the covariance matrix, the MV model would produce optimal portfolios. However, we live in an uncertain world where numerous unpredictable factors impact asset returns, making it difficult to obtain precise predictions for the input data required by the model. Fuzzy theory, developed by Zadeh [202], is among the most commonly used methods for addressing uncertainty. In portfolio selection research, many scholars have applied fuzzy theory to account for uncertainty in their models. Huang [76] developed portfolio selection models that treat security returns as stochastic variables with fuzzy information. Zhang et al. [210] introduced two credibilistic MV portfolio adjustment models with fuzzy returns. Mehlawat [123] addresses fuzzy multi-objective, multi-period portfolio selection problems, considering risk, wealth, liquidity, transaction cost, and asset diversity. Li et al. [97] proposed a multi-period credibilistic portfolio selection model that incorporates asymmetric return-risk ratios. Drawing inspiration from the Sharpe ratio and CVaR ratio, it defines lower and upper credibilistic Sharpe ratios and CVaR ratios with coherent perceptions to assess the impact of negative and positive volatility, as well as excessive risk, on portfolio selection. In recent years, numerous portfolio selection models have been developed based on credibility measures, such as [38, 116, 139]. Many scholars have also applied the possibility measure to explore fuzzy portfolio selection problems, including [111, 206], etc. Additional insights into portfolio selection problems can be found in [154] and [91].

It can be concluded from the above studies, that the MV portfolio selection model has at least three major limitations. First, it is a bi-criterion model focused on maximizing returns and minimizing risk, while the decision maker's objectives are typically more complex and involve multiple criteria. Second, the Markowitz model is inherently a quadratic optimization problem, which can be difficult to solve, especially for large-scale problems, even when linearization techniques are applied. Lastly, the data used in this model is often imprecise, ambiguous, or vague in real-world situations.

The following section provides an in-depth review of the literature on DEA mod-

els and their various extensions.

1.2 Data Envelopment Analysis

DEA is a technique that aims to compute efficiency scores of homogeneous DMUs by utilizing their input and output data and classifying them as efficient or inefficient based on their efficiency score. Efficiency fundamentally involves achieving the best outcomes with minimal resource use. Thus, measuring and enhancing efficiency is critical for an organization's growth [141]. While efficiency measurement is essential, it is often complex. Efficiency can be calculated as a ratio for single-input, single-output scenarios. However, the task becomes more challenging with multiple inputs and outputs, attracting significant attention from researchers in decision-making and operations research over the years. In 1957, Farrell laid the foundation for productivity efficiency assessment in his seminal paper [67]. Two decades later, in 1978, Charnes et al. [27] developed DEA to evaluate the relative efficiency of units with multiple inputs and outputs, defining efficiency as the ratio of the weighted sum of outputs to the weighted sum of inputs. The first DEA model was formalized under the assumption of constant returns to scale (CRS) meaning that a proportional change in all inputs will cause an identical proportional change in outputs and known as the input-oriented CCR model, and later expanded by Banker et al. [14] to accommodate variable returns to scale as the BCC model. The CCR and BCC models typically aim to minimize inputs while maintaining output levels (input-oriented), whereas their output-oriented versions aim to maximize outputs without increasing inputs.

Since its inception, various models have been proposed to enhance the effectiveness. The additive model, introduced by Charnes et al. [26], is a non-radial DEA approach that adjusts inputs and outputs simultaneously to move inefficient DMUs toward the efficient frontier. Both the BCC and CCR models can be expressed within the framework of the additive model. The additive model of DEA is non-radial and moves an inefficient DMU towards the efficient frontier by simultaneously reducing inputs and increasing outputs. Tone [174] introduced the Slack Based Measure (SBM)—a non-radial DEA model that directly reduces input excesses and output shortfalls, offering a more accurate efficiency evaluation.

1.2.1 DEA Techniques to handle Negative Data

The conventional DEA methods are traditionally developed under the condition that all inputs and outputs are non-negative. However, this condition is limiting and may not always apply in real-world situations. In practice, negative values may arise in certain applications, such as when net profit is treated as an output variable and shows a loss. Among the widely recommended methods for handling negative data, two key perspectives stand out: treating negative inputs as equivalent to positive outputs and negative outputs as equivalent to positive inputs [156, 157, 213]. The second involves employing DEA models with translation invariance, which adjust all negative values by adding a sufficiently large scalar to transform them into positive values. Lovell and Pastor [114] were the first to highlight that, utilizing the translation invariance property identified by Ali and Seiford [3], negative data can be accommodated under the assumption of VRS. Scheel [156] proposed an alternative approach that treats the absolute values of negative inputs as outputs and the absolute values of negative outputs as inputs. In this method, a smaller negative input is converted into a larger output, while a larger negative input becomes a smaller output.

Sharp et al. [160] proposed a Modified Slack-based measure (MSBM) model capable of handling both negative outputs and inputs. Similarly, the Range Directional Measure (RDM), introduced by Portela et al. [144], utilizes the directional distance function (DDF) based on Chambers et al. [24] and developed two variants of the RDM model to adapt variables that can take both positive and negative values and eliminates the need for any data transformation. Kerstens and Woestyne [88] present a straightforward modification to the conventionally defined proportional distance function that allows it to accommodate negative data. Tone et al. [175] introduced BP-SBM DEA models, a slacks-based measure (SBM) approach for handling negative data in DEA. Unlike existing methods, these models remain consistent with standard SBM, are unit-invariant, accommodate various returns to scale, and prevent division by zero. They achieve this by transforming negative inputs and outputs into positive values using a newly defined "base point." Emrouznejad et al. [59] examined the advantages and disadvantages of the aforementioned methods for handling negative data in DEA. They introduced a semi-oriented radial measure (SORM) to address situations where some DMUs

have negative inputs or outputs. The SORM approach offers the benefit of managing negative values without requiring a shift in the data origin. Emrouznejad and Yang [61] provide a comprehensive review of DEA-related articles, encompassing theoretical concepts, methodological developments, and practical applications across diverse contexts from 1978 through 2016. It also includes summary statistics on publication growth, the most frequently cited academic journals, authorship patterns, and keyword analysis.

Table 1.1 summarizes the literature primarily centered on the developments of basic DEA models along with their key advantages and limitations.

1.2.2 DEA Cross-efficiency Evaluation

In DEA, a DMU evaluates itself by optimizing input and output weights to attain maximum relative efficiency. However, the DEA has been criticized for its flexible weight assignment and reliance on self-assessment. To overcome this challenge, Sexton et al. [159] introduced a cross-efficiency method that integrates peer evaluation, shifting the focus from self-assessment. In self-evaluation, each DMU uses its most favorable weights to maximize its own efficiency. In contrast, peer evaluation assesses a DMU's performance using weights derived from other DMUs. The overall efficiency score is computed as the average of the self- and peer-evaluated efficiencies

Despite its benefits, cross-efficiency evaluation fails to generate a unique set of cross-efficiency scores for the DMUs, nor does it provide a definitive ranking. This arises because the DEA optimal weights are typically non-unique, leading to multiple possible cross-efficiency scores. To address this, Doyle and Green [56] introduced secondary goal models—aggressive and benevolent. The aggressive model selects weights that favor the evaluated DMU while minimizing the efficiency of others, whereas the benevolent model does the reverse, maximizing the overall efficiency of peer DMUs. Lim [102] proposed aggressive and benevolent cross-efficiency models using minimax and maximin objectives, respectively. These models select weights that maximize the evaluated DMU's efficiency while minimizing (or maximizing) the efficiency of its most (or least) efficient peer. Ramón et al. [146] proposed selecting the most similar optimal weights while preserving DMU efficiency. Wu et al. [187] addressed CCR infeasibility by introducing desirable and undesirable target scores in a secondary model. Chen

and Wang [36] extended this with target-setting based on cross-efficiency and reference sets to enhance DMU scores. Örkücü et al. [135] introduced the neutral secondary objective cross-efficiency evaluation model applicable to the overall system and sub-stages of two-stage production processes. This model accounts for the relative significance of all variables in the cross-efficiency evaluation. The benevolent, aggressive, and neutral approaches assume a single set of weights is used by the evaluated DMU for both self- and peer evaluations. Challenging this constraint, researchers have proposed extended cross-efficiency models. For example, Ramón et al. [148] developed models to identify all possible weights, avoiding the need to choose between benevolent and aggressive views. Additional secondary models, including neutral approaches, have been introduced by Wang et al. [179], Ramón et al. [147], Wang et al. [180], and others.

The issue of aggregating self-evaluated and peer-evaluated cross-efficiencies into a final cross-efficiency score for each DMU has received attention in the literature. In the conventional approach, the cross-efficiency score for each DMU is typically calculated as the average of the cross-appraisals it receives. Notably, using a simple arithmetic average inherently assumes that all DMUs' evaluations are equally valid or dependable, which is not always true. A range of arguments suggests that assigning equal aggregation weights to all cross-efficiency scores may be inadequate, as it might not fully capture the true performance of the evaluated units. Wang and Wang [183] proposed two approaches for aggregating cross-efficiency scores. The least-square dissimilarity metric down-weights DMUs with evaluations that differ from peers, while the least-square deviation metric reduces weights for DMUs with large gaps between self- and peer-assessments. Yang et al. [201] applied an evidential reasoning approach to obtain the final aggregated cross-efficiencies. Angiz et al. [9] proposed a two-stage sorting-based aggregation method. Initially, the cross-efficiency matrix was converted into a ranking order matrix, followed by the application of a first-order model to determine the aggregation weights. Liu [107] employed both aggressive and benevolent models to evaluate the interval composite efficiency scores for each DMU, providing a range that captures different perspectives of efficiency. To facilitate a comprehensive ranking of all DMUs, a signal-to-noise ratio index was applied, ensuring a robust and systematic comparison across the units. The method proposed by Yang et al. [199] evaluates the DEA weights for each DMU individually and estab-

lishes lower and upper bounds for cross-efficiencies rather than focusing solely on cross-efficiency scores.

Traditional DEA cross-efficiency assumes fully rational decision-makers, which may not reflect real-world behavior. To address this, scholars integrated behavioral decision theory. Liu et al. [109] introduced a prospect theory-based model that incorporates loss aversion by maximizing prospect utility to determine optimal weights. Shi et al. [163] expanded the above model by introducing a neutral cross-efficiency evaluation method that considers bounded rational behavior through interval reference points. Ning et al. [129] propose the EPCE model, an improved DEA prospect cross-efficiency method that incorporates decision-makers' risk attitudes and retains decision information. Li et al. [99] explored the reciprocal interactions among DMUs in cross-efficiency evaluation, introducing a new threshold value to classify behaviors as positive or negative.

Further, Table 1.2 presents a review of DEA cross-efficiency methods, outlining their key advantages and limitations.

1.3 Portfolio Selection Using DEA

Portfolio selection involves two key decisions: asset selection and capital allocation. However, many studies focus solely on allocation, treating it as an optimization problem while overlooking asset selection. Portfolio efficiency analysis addresses this gap by assessing how well a portfolio converts inputs (e.g., capital, risk) into outputs (e.g., returns), aiming to achieve an optimal risk-return balance within given constraints like budget, diversification, and allocation rules. It also helps identify underperforming assets, guiding improvements and informed investment decisions. The link between portfolio selection using quadratic optimization and DEA was first established by Sengupta [158]. However, it was not until the work of Murthi et al. [128] that DEA was acknowledged as an "extremely useful technique for measuring the efficiency" of mutual funds. Morey and Morey [127] introduced an innovative DEA application to recognize strictly optimal mutual funds, utilizing portfolio variance as input and mean return as output. Basso and Funari [17] applied the DEA methodology to evaluate fund performance, incorporating various risk factors as input indicators. Dia [53] presents a four-step DEA-based methodology for portfolio selection of decision-making units,

such as stocks or financial assets. Joro and Na [84] introduced a portfolio performance evaluation method using a non-parametric DEA approach. Tarnaud and Leleu [171] characterize the financial production process as the creation of a return distribution from an initial investment, treating the distribution of returns as input/output variables to assess portfolio performance using the DEA technique. Peykani et al. [142] presented a two-stage framework for portfolio selection and optimization. Xiao et al. [195] introduced a new input-output framework based on initial and terminal wealth, converting terminal wealth into the rate of return. Hosseinzadeh et al. [75] applied an SBM DEA model to preselect efficient assets for large-scale portfolio optimization. Liu et al. [112] conducted a systematic investigation into the theoretical justification for using DEA in portfolio performance evaluation. Zhu and Zhu [214] pioneered the concept of eco-efficiency of industrial investment (EELI) by evaluating energy consumption, economic benefits, and environmental impact through a slack-based DEA model. Additionally, Omid et al. [130] proposed a two-stage multiobjective optimization model incorporating network DEA, and Soltanfair et al. [164] developed DEA-R models for merger analysis capable of handling negative data, with applications in the Iranian banking sector. To assess Chinese banks' efficiency amid the financial crisis, Ghiyasi and Zhu [69] introduced an inverse semi-oriented DEA model for scenarios with negative outputs. Amirteimoori et al. [7] created a chance-constrained multi-stage DEA model for evaluating supply chain performance, while Emrouznejad et al. [60] reviewed eco-efficiency evaluations using DEA.

A significant contribution in this area comes from Lim et al. [103], who proposed incorporating DEA cross-efficiency evaluation into the mean-variance framework for the selection of optimal portfolios. Their method mitigates the "ganging together" issue common in DEA-based portfolio models. Cross-efficiency in portfolio selection offers key advantages—enhancing returns based on investor preferences while managing risk effectively. Mashayekhi and Omrani [119] applied the suggested MV-DEA cross-efficiency portfolio selection by assuming fuzzy returns. Recently, Amin and Hajjami [4] examined the effect of the alternative optimal solution in portfolio selection and indicated that a significant improvement in portfolio selection is obtained when alternative optimum solutions are used to obtain a cross-efficiency matrix (CEM). Chen et al. [33] describe a portfolio selection method using fuzzy DEA cross-efficiency assessment, which considers un-

desirable, ambiguous inputs and outputs. Deng and Fang [51] developed a novel mean-variance-maverick DEA cross-efficiency approach based on the prospect of selecting a fuzzy portfolio. Gong et al. [70] presented a method based on regret theory, in which they applied a fuzzy multiobjective portfolio optimization model with four objectives, namely, mean, variance, skewness, and efficiency, bounded by several realistic constraints. Essid et al. [63] offered an innovative framework for portfolio optimization that combines the Maverick index with a DEA game cross-efficiency model. Mehlawat et al. [125] looked at the portfolio selection issue by adding additional facts about the nonnormality of asset returns using skewness and kurtosis.

Table 1.3 outlines the literature on DEA-based portfolio selection, detailing the studies in chronological order along with their key contributions and limitations.

1.3.1 Multi-Criteria Decision Making methods

The MCDM method makes decisions when multiple alternatives are available for contradictory criteria. Numerous MCDM methods are available, each with its own set of approaches. Most MCDMs follow the same three-step procedure, which includes normalizing the data, calculating the criteria weights, and computing the scores for each alternative. These scores are then used to rank the options. Traditional techniques such as the Analytic Hierarchy Process (AHP) [152], the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [78], and the Preference Ranking Organization Method for Enrichment Evaluation (PROMETHEE) [21], Elimination et Choice Translating Reality (ELECTRE) [20], and VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) [132], have laid the foundation for MCDM research. DEA aims to compare multiple DMUs based on their efficiency in converting inputs into outputs, while MCDM analyzes and ranks alternatives across multiple criteria. DEA and MCDM align when inputs and outputs are treated as criteria. Unlike MCDM, which uses a common set of weights, DEA employs linear programming to determine weights for each DMU, maximizing its efficiency.

1.4 Precursor models

This section is organized into five subsections. Along with the notations and definitions, the first subsection reviews the efficiency evaluation model in DEA, the second examines DEA models addressing negative data, the third introduces the game cross-efficiency model for decision-making, the fourth subsection presents the ordered visibility graph averaging operator, and the final subsection introduces Markowitz's portfolio optimization model.

1.4.1 Input-oriented CCR model for efficiency evaluation

In 1978, Charnes et al. [27] presented DEA, a methodology for evaluating the relative efficiency of decision-making units by considering multiple inputs and outputs. They defined efficiency as the ratio of the weighted sum of outputs to the weighted sum of inputs, with the mathematical formulation outlined below.

$$\text{Efficiency} = \frac{\text{Weighted Sum of outputs}}{\text{Weighted Sum of inputs}}$$

Mathematically, suppose we have n DMUs with m inputs and s outputs. The input and output vectors of DMU_j , ($j = 1, 2, \dots, n$) are presented by $X_j = (x_{1j}, x_{2j}, \dots, x_{mj})$ and $Y_j = (y_{1j}, y_{2j}, \dots, y_{sj})$ respectively. Let DMU_k be the DMU under evaluation. Let $\omega_{ik}, i = 1, 2, \dots, m$ and $\mu_{rk}, r = 1, 2, \dots, s$ and be the weights given to the m inputs and s outputs of DMU_k respectively.

Then the relative efficiency of the k^{th} DMU is defined as

$$e_{kk} = \frac{\sum_{r=1}^s \mu_{rk} y_{rk}}{\sum_{i=1}^m \omega_{ik} x_{ik}}. \quad (1.4.1)$$

The goal is to assess each unit individually under the most favorable conditions. To achieve this, equation 1.4.1 is maximized for each unit while taking into account

a set of common constraints to ensure that the ratios of all units remain bounded.

$$\begin{aligned}
 \textbf{Model (1.1)} \quad & \max e_{kk} = \frac{\sum_{r=1}^s \mu_{rk} y_{rk}}{\sum_{i=1}^m \omega_{ik} x_{ik}} \\
 & \text{subject to,} \\
 & 0 \leq \frac{\sum_{r=1}^s \mu_{rk} y_{rj}}{\sum_{i=1}^m \omega_{ik} x_{ij}} \leq 1, \quad j = 1, 2, \dots, n, \\
 & \omega_{ik} \geq 0, \mu_{rk} \geq 0, \quad i = 1, 2, \dots, m, \quad r = 1, 2, \dots, s.
 \end{aligned}$$

The abovementioned model is known as the input-oriented CCR model, which is nonlinear and non-convex, featuring both linear and fractional constraints and a linear and fractional objective function. By applying a straightforward transformation introduced by Charnes et al. [27], this model can be converted into the following Linear Programming (LP) form (the Primal CCR model), allowing for the use of LP techniques.

$$\begin{aligned}
 \textbf{Model (1.2)} \quad & \max e_{kk} = \sum_{r=1}^s \mu_{rk} y_{rk} \\
 & \text{subject to,} \\
 & \sum_{i=1}^m x_{ik} \omega_{ik} = 1, \\
 & \sum_{r=1}^s \mu_{rk} y_{rj} - \sum_{i=1}^m \omega_{ik} x_{ij} \leq 0, \quad j = 1, 2, \dots, n, \\
 & \omega_{ik} \geq 0, \mu_{rk} \geq 0, \quad i = 1, 2, \dots, m, \quad r = 1, 2, \dots, s.
 \end{aligned}$$

Let μ_{rk}^* $r = 1, 2, \dots, s$ and ω_{ik}^* $i = 1, 2, \dots, m$ be the optimal solution of Model (1.2), then $e_{kk}^* = \sum_{r=1}^s \mu_{rk}^* y_{rk}$ is the optimal efficiency that DMU_k can attain, which is also referred to the CCR-efficiency of DMU_k under self-evaluation. If $e_{kk}^* = 1$ and all the optimal weights μ_{rk}^* $r = 1, 2, \dots, s$ and ω_{ik}^* $i = 1, 2, \dots, m$ are positive, then DMU_k is CCR-efficient; otherwise, it is CCR-inefficient.

The DEA cross-efficiency assessment approach incorporates peer review, rather than relying solely on self-evaluation, to determine efficiency. Thus, the peer evaluation of DMU_j , $j = 1, 2, \dots, n$ by the DMU_k is defined as follows:

$$e_{kj}^* = \frac{\sum_{r=1}^s \mu_{rk}^* y_{rj}}{\sum_{i=1}^m \omega_{ik}^* x_{ij}}, \quad (k, j = 1, 2, \dots, n). \quad (1.4.2)$$

Where μ_{rk}^* and ω_{ik}^* be the optimal solution of model (1.2). Lastly, the following formula yields the cross-efficiency score of DMU_j , defined as e_j^* and is determined by

$$e_j^* = \frac{1}{n} \sum_{k=1}^n e_{kj}^* \quad (j = 1, 2, \dots, n) \quad (1.4.3)$$

Cross-efficiency methods integrate the concepts of peer evaluation and self-evaluation. Model (1.2) is solved n times, each time for one particular DMU. As a result, we can get one CCR-efficiency value and $(n - 1)$ cross-efficiency values for each DMU. The n efficiency values constitute a cross-efficiency matrix (CEM), where $e_{kk} = e_{kk}^*$. The n efficiency values of each DMU are then averaged as its overall performance, which is called the average cross-efficiency value. Based on these overall performance values, the n DMUs can be compared or fully ranked. Cross-efficiency methods integrate the concepts of peer evaluation and self-evaluation.

The Maverick Index, proposed by Doyle and Green [56], quantifies the inconsistency between a DMU's self-evaluation and the average efficiency score assigned by its peers. A higher Maverick Index indicates that the DMU is potentially over-estimating its performance compared to peer assessments, suggesting it may be a false positive. The Maverick Index is computed as $M_k = \frac{e_{kk}^* - e_j^*}{e_j^*}$. In contrast, the diversity Index, introduced by Talluri et al. [170], captures how well a DMU performs not only under its weights but also relative to the weight schemes of other DMUs. The diversity index reflects the gap between a DMU's self-efficiency score and the row average score.

1.4.2 DEA models with negative data

Traditional DEA models lack translation invariance, rendering them unsuitable for managing negative input and output data. This section examines two approaches designed to address this limitation effectively: the SORM model introduced by Emrouznejad et al. [59], and the MSBM model developed by Sharp et al. [160].

1.4.2.1 Semi-oriented radial measure (SORM) model

Emrounzejad et al. [59] proposed a model to address negative data using the absolute value definition. They employed a partitioning approach to handle negative data and introduced the SORM model for evaluating the performance of observed production units. Here, we provide a brief overview of the SORM model.

Let us consider a positive input variable x_{ij} ($i \in I$) and an input variable x_{lj} ($l \in L$) that is positive for some DMUs and negative for others. Similarly, consider a positive output variable y_{rj} ($r \in R$) and y_{oj} ($o \in O$) be an output variable that takes positive values for some DMUs and negative values for others. Also $I \cup L = \{1, \dots, m\}$ with $I \cap L = \emptyset$, and $R \cup O = \{1, \dots, s\}$ with $R \cap O = \emptyset$.

To handle negative values, the negative input and output variables are decomposed as the difference of two non-negative components. The negative input variables is defined as $x_{lj} = x_{lj}^1 - x_{lj}^2 \forall l \in L$, where

$$x_{lj}^1 = \begin{cases} x_{lj} & \text{if } x_{lj} \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad x_{lj}^2 = \begin{cases} -x_{lj} & \text{if } x_{lj} \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1.4.4)$$

Similarly, the negative output variable is defined as $y_{oj} = y_{oj}^1 - y_{oj}^2 \forall o \in O$, where

$$y_{oj}^1 = \begin{cases} y_{oj} & \text{if } y_{oj} \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad y_{oj}^2 = \begin{cases} -y_{oj} & \text{if } y_{oj} \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1.4.5)$$

To access the DMU_k , the standard input-oriented SORM model is presented as follows:

Model (1.3)

$$\max e_{kk} = \sum_{r \in R} \mu_{rk} y_{rk} + \sum_{o \in O} \mu_{ok}^1 y_{ok}^1 - \sum_{o \in O} \mu_{ok}^2 y_{ok}^2 + u_0$$

subject to,

$$\begin{aligned} & - \sum_{i \in I} \omega_{ik} x_{ij} - \sum_{l \in L} \omega_{lk}^1 x_{lj}^1 + \sum_{l \in L} \omega_{lk}^2 x_{lj}^2 \\ & + \sum_{r \in R} \mu_{rk} y_{rk} + \sum_{o \in O} \mu_{ok}^1 y_{ok}^1 - \sum_{o \in O} \mu_{ok}^2 y_{ok}^2 + u_0 \leq 0 \end{aligned}$$

$$\sum_{i \in I} \omega_{ik} x_{ik} + \sum_{l \in L} \omega_{lk}^1 x_{lk}^1 - \sum_{l \in L} \omega_{lk}^2 x_{lk}^2 = 1$$

$$\omega_{ik} \geq 0, \forall i \in I \quad \omega_{lk}^1 \geq 0, \omega_{lk}^2 \geq 0 \quad \forall l \in L$$

$$\mu_{rk} \geq 0, \forall r \in R \quad \mu_{ok}^1 \geq 0, \mu_{ok}^2 \geq 0 \quad \forall o \in O, \quad u_0 \text{ is free.}$$

The optimal weights denoted as μ_{rk}^* , μ_{ok}^{1*} , μ_{ok}^{2*} , ω_{ik}^* , ω_{lk}^{1*} , ω_{lk}^{2*} and u_0^* determine the self-efficiency of DMU_k as e_{kk}^* . The efficiency measure in this model specifically indicates the radial contraction of the absolute input values. Therefore, the efficiency measure is referred to as the "input reduction semi-oriented radial measure (SORM)."

1.4.2.2 Modified slack-based measure (MSBM) model

To address the issue of negative data in conventional DEA models, Sharp et al. [160] introduced the MSBM model, incorporating a set of weights for inputs and outputs. This allows users to assign their preferred level of importance to improving the values of inputs and outputs. The MSBM model is presented as follows:

$$\begin{aligned}
 \text{Model (1.4)} \quad & \text{Min } \rho_k = \frac{1 - \frac{1}{m} \sum_{i=1}^m \frac{s_{ik}^-}{R_{ik}^-}}{1 + \frac{1}{s} \sum_{r=1}^s \frac{s_{rk}^+}{R_{rk}^+}} \\
 & \text{subject to,} \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_{rk}^+ = y_{rk}, \quad r = 1, 2, \dots, s \\
 & \sum_{j=1}^n \lambda_j x_{ij} + s_{ik}^- = x_{ik}, \quad i = 1, 2, \dots, m, \\
 & \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, \quad j = 1, 2, \dots, n, \quad s_{ik}^-, s_{rk}^+ \geq 0
 \end{aligned}$$

The objective function ρ_k^* represents the optimal efficiency score of DMU_k . The terms s_{ik}^- ($i = 1, \dots, m$) and s_{rk}^+ ($r = 1, \dots, s$) represent the i^{th} input excess and r^{th} output shortfall of DMU_k , referred to as slacks. The λ_j ($j = 1, \dots, n$) are intensity variables used to link inputs and outputs in a convex combination. Also, R_{ik}^- and R_{rk}^+ are the feasible directions to move and are defined as follows: where $R_{ik}^- = x_{ik} - \min_{j=1,2,\dots,n} x_{ij}$, $R_{rk}^+ = \max_{j=1,2,\dots,n} y_{rj} - y_{rk}$. Model (1.4) possesses unit invariance and translation invariance, two key properties of DEA models that enable them to accommodate negative inputs and outputs.

1.4.3 Game Cross-efficiency Evaluation

In real-world scenarios, DMUs often interact through competition, which may be adversarial (negative) or collaborative (positive). For example, in sectors like telecommunications and retail, they compete for market share, resources, or customers, affecting each other's outcomes. In contrast, in supply chains or cooperative networks, they collaborate to enhance collective efficiency.

Liang et al. [101] introduced the Game Cross-Efficiency DEA model, building on the foundational concept of cross-efficiency in DEA. The traditional cross-efficiency approach evaluates each DMU by its input-output performance and how it performs relative to others. However, this method assumes that DMUs operate independently, without considering potential interactions or interdependencies.

The game cross-efficiency of DMU_j regarding DMU_k in the form of the following optimization model:

Model (1.5)

$$\begin{aligned}
 \text{Max } \theta_j(\alpha_k^\tau) &= \sum_{r=1}^s \mu_{rj} y_{rj} \\
 \text{subject to} \\
 \sum_{i=1}^m \omega_{ij} x_{ij} &= 1 \\
 \sum_{r=1}^s \mu_{rj} y_{rj} - \sum_{i=1}^m \omega_{ij} x_{ij} &\leq 0, \quad j = 1, 2, \dots, n \\
 \alpha_k^\tau \sum_{i=1}^m \omega_{ij} x_{ik} - \sum_{r=1}^s \mu_{rj} y_{rk} &\leq 0 \\
 \mu_{rj} &\geq 0, r = 1, 2, \dots, s, \\
 \omega_{ij} &\geq 0, i = 1, 2, \dots, m.
 \end{aligned}$$

where α_k^τ is a parameter at τ^{th} iteration and $\alpha_k^\tau \leq 1$, initially take $\alpha_k^0 = e_k^*$. The algorithm for Model (1.5) is briefly described as follows.

Step 1: Initially assume $\alpha_k^0 = e_k^*$ ($\tau = 0$), where τ represents the number of iterations.

Step 2: Solve model (1.5) for each $k = 1, 2, \dots, n$, and $j = 1, 2, \dots, n$, and Calculate $\alpha_j^1 = \frac{1}{n} \sum_{k=1}^n \theta_j^*(\alpha_k^0)$ as described in Table 1.4

Step 3: Generally $\alpha_j^{\tau+1} = \frac{1}{n} \sum_{k=1}^n \theta_j^*(\alpha_k^\tau)$, where the optimal solutions at $(\tau+1)^{th}$ iteration

are represented by $\mu_{rj}^*(\alpha_k^\tau)$ and $\omega_{ij}^*(\alpha_k^\tau)$, and the optimal value of Model (1.5) is $\theta_j^*(\alpha_k^\tau)$.

Step 4: For a specific very small value of ϵ , if $|\alpha_j^{\tau+1} - \alpha_j^\tau| \leq \epsilon \forall j$, then $\alpha_j^{\tau+1}$ will be the final game cross-efficiency value.

Step 5: The final game cross-efficiency value is denoted as

$$\alpha_j^* = \alpha_j^{\tau+1} = \frac{1}{n} \sum_{k=1}^n \theta_j^*(\alpha_k^\tau)$$

Table 1.4: Game cross-efficiency matrix for n DMUs corresponding to Model (1.5) for initial iteration $\tau = 0$

k \ j	j			
	1	2	...	n
1	$\theta_1^*(\alpha_1^0)$	$\theta_2^*(\alpha_1^0)$	...	$\theta_n^*(\alpha_1^0)$
2	$\theta_1^*(\alpha_2^0)$	$\theta_2^*(\alpha_2^0)$	...	$\theta_n^*(\alpha_2^0)$
3	$\theta_1^*(\alpha_3^0)$	$\theta_2^*(\alpha_3^0)$	...	$\theta_n^*(\alpha_3^0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\theta_1^*(\alpha_n^0)$	$\theta_2^*(\alpha_n^0)$	...	$\theta_n^*(\alpha_n^0)$
α_j^1	$\frac{1}{n} \sum_{k=1}^n \theta_1^*(\alpha_d^0)$	$\frac{1}{n} \sum_{k=1}^n \theta_2^*(\alpha_d^0)$	...	$\frac{1}{n} \sum_{k=1}^n \theta_n^*(\alpha_d^0)$

The game cross-efficiency value is attained through a strategy where all DMUs, except the one under evaluation, maximize their efficiency weights without compromising the evaluated DMU's cross-efficiency. This approach ensures continuous improvement in the efficiency of the assessed and other DMUs, resulting in superior cross-efficiency values and richer information than alternative strategies.

1.4.4 Ordered Visibility Graph Averaging operator

The visibility graph introduced by Lacasa et al. [95] creates a logical connection between complex network theory and time series analysis. The values of a time

series are plotted using vertical bars in the visibility graph. Every bar is connected to others visible from the top of the one being considered, and then the associated graph is created. The VG operator was introduced by Chen et al. [31] based on the visibility graph. This aggregation operator converts time series data into the visibility graph, and then the weights are decided according to the degree of distribution. A Visibility Graph (VG) is an undirected graph of n nodes. The visibility graph method converts a time series into a graph. The associated graph derived from a time series has the following properties:

1. Connected: a node can see its nearest neighbors.
2. Undirected: the associated graph extracted from a time series
3. Invariant under affine transformations of the series data: Although rescale both horizontal and vertical axes, the visibility criterion is invariant.

Further, Wang et al. [181] introduced an algorithm for ordered visible graphs (OVG). The VG method is to convert a time series into a graph, while the OVG maps a set of ordered data into a network. Further to extend the use of the visibility graph for various series that are not influenced by the time factors, such as a set of heights, a set of test scores, an economic system, the performance parameters of the equipment, and so forth Wang et al. [181] introduced ordered visibility graph (OVG) averaging operator. Now we have the following definitions.

Definition 1.4.1 ([181]). Let $O = \{o_1, o_2, \dots, o_{\hat{n}}\}$ represents a set of $\hat{n}$ ordered data, where, $o_{\hat{j}}$ is the $\hat{j}^{th}$ largest element of the set. An ordered value $o_{\hat{j}}$ and its order $\hat{j}$ make up the coordinate $(\hat{j}, o_{\hat{j}})$ and represent a node $\hat{j}$ in the visibility graph.

Definition 1.4.2 ([181]). Two data values $(\hat{i}, o_{\hat{i}}), (\hat{j}, o_{\hat{j}})$ have visibility, if any other value $(\hat{k}, o_{\hat{k}})$ is placed between them fulfills

$$o_{\hat{k}} < o_{\hat{j}} + (o_{\hat{i}} - o_{\hat{j}}) \frac{\hat{j} - \hat{k}}{\hat{j} - \hat{i}}.$$

Definition 1.4.3 ([181]). Let $\{(1, o_1), (2, o_2), \dots, (\hat{n}, o_{\hat{n}})\}$ be the set of $\hat{n}$ ordered data values. The adjacency matrix $A = [a_{\hat{i}\hat{j}}]_{\hat{n} \times \hat{n}}$ on the basis of visibility between the node $\hat{i}$

and node $\hat{j}$ is defined as

$$a_{\hat{i}\hat{j}} = \begin{cases} 1 & \text{if there is visibility from node } \hat{i} \text{ to node } \hat{j}, \hat{i} \neq \hat{j} \\ 0 & \text{otherwise.} \end{cases}$$

Let $d_{\hat{i}}$ be the degree of node $\hat{i}$, and is given by $d_{\hat{i}} = \sum_{\hat{j}} a_{\hat{i}\hat{j}}$

Definition 1.4.4 ([181]). Ordered visibility graph aggregation (OVGA) operator: Let $\{a_1, a_2, \dots, a_{\hat{n}}\}$ be the set of $\hat{n}$ ordered values, where $a_{\hat{i}} \in R$. The OVGA operator is a mapping $F: I^{\hat{n}} \rightarrow I$, $I \in R$, such that $F(a_1, a_2, \dots, a_{\hat{n}}) = w_1 a_1 + w_2 a_2 + \dots + w_{\hat{n}} a_{\hat{n}}$, where $a_{\hat{i}}$ is the $\hat{i}^{th}$ value and $w_{\hat{i}}$ is the corresponding weight of the value $a_{\hat{i}}$, satisfying the conditions $w_{\hat{i}} \in [0, 1]$, $\sum_{\hat{i}=1}^{\hat{n}} w_{\hat{i}} = 1$, and is defined as follows:

$$w_{\hat{i}} = \frac{d_{\hat{i}}}{\sum_{\hat{i}=1}^{\hat{n}} d_{\hat{i}}}, \quad \hat{i} = 1, 2, \dots, \hat{n},$$

where $d_{\hat{i}}$ is the degree of the $\hat{i}^{th}$ node. Further the aggregated value of the ordered set $\{a_1, a_2, \dots, a_{\hat{n}}\}$ is $F(a_1, a_2, \dots, a_{\hat{n}}) = w_1 a_1 + w_2 a_2 + \dots + w_{\hat{n}} a_{\hat{n}}$.

To give more clarity, we consider an example given by Wang et al. [181].

Example 1.4.1 ([181]). Assume there are 8 ordered values, $\hat{O} = \{85, 75, 70, 70, 55, 50, 45, 40\}$.

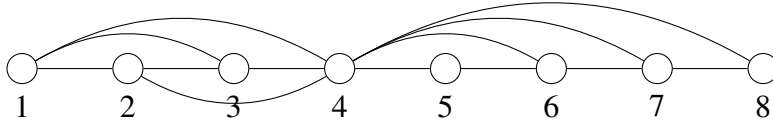
In the OVGA, every node represents the corresponding element in the ordered set, in the same order. The corresponding adjacency matrix A , degree, and weight of an ordered set $\hat{O}$ as follows:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Table 1.5: Degree and weights of ordered set $\hat{O}$

Nodes $\hat{i}$	1	2	3	4	5	6	7	8
degree (d_i)	3	3	3	7	2	3	3	2
weight (w_i)	$\frac{3}{26}$	$\frac{3}{26}$	$\frac{3}{26}$	$\frac{6}{26}$	$\frac{2}{26}$	$\frac{3}{26}$	$\frac{3}{26}$	$\frac{2}{26}$

According to the visibility criteria and on the lines of the adjacency matrix, 1.1 depicts the OVGA so obtained.

Figure 1.1: OVG of ordered set $\hat{O}$ 

Further, the aggregated value of the ordered data is

$$\begin{aligned}
 F(85, 75, 70, 70, 55, 50, 45, 40) &= 85 \times \frac{3}{26} + 75 \times \frac{3}{26} + 70 \times \frac{3}{26} + 70 \times \frac{6}{26} \\
 &\quad + 55 \times \frac{2}{26} + 50 \times \frac{3}{26} + 45 \times \frac{3}{26} + 40 \times \frac{2}{26} \\
 &= 60.96
 \end{aligned}$$

It should be noted that if a vertex is connected to another vertex by an edge, it is visible from that vertex since the two vertices have a direct line of sight. On the other hand, if a vertex is not connected to another vertex by an edge, it means that there is an obstruction in the way, and the vertex cannot be seen from that specific vertex. In a visible network, the vertex with the highest degree is the one with the greatest number of connections to other vertices. It resembles a graph's center that is visible from several other spots. This vertex is crucial to the network as a whole because it is connected to a larger number of other vertices.

1.4.5 MV portfolio selection model

The famous MV portfolio model by Markowitz [117] quantifies the portfolio return and risk, respectively, by the mean and variance of the return distribution. Let N denote the total number of feasible assets into which an investor may wish to invest. Let $R_t = (r_{1t}, r_{2t}, \dots, r_{Nt})$ represents the return from the assets at t^{th} scenario, $t = 1, 2, \dots, T$. A portfolio $w = (w_1, w_2, \dots, w_N)^t \in W \subseteq R^N$ refers to the vector of proportions of the initial budget to be invested in each asset,

where we assume W to be a polyhedron set of admissible portfolios. Let $W = \{(w_1, w_2, \dots, w_N)^t \mid \sum_{i=1}^N w_i = 1, w_i \geq 0, i = 1, 2, \dots, N\}$ be a polyhedron set of all admissible portfolios in the interest of an investor where $\sum_{i=1}^N w_i = 1$ is the normalized budget constraint and $w_i \geq 0$ prohibits short selling. Let p_t be the probability of the t^{th} scenario to occur $t = 1, 2, \dots, T$. Hence, the expected value of portfolio return $E(w)$ is $\sum_{t=1}^T (\sum_{i=1}^N r_{it} w_i) p_t$. Let $\hat{r} = \sum_{t=1}^T r_{it} p_t$ be the mean return i^{th} asset, then $E(w) = \sum_{i=1}^N \hat{r}_i w_i$. Let $\sigma^2 = \sum_{i=1}^N \sum_{j=1}^N \sigma_{ij} w_i w_j$ be the variance of portfolio return $R(w)$, where σ_{ij} is the covariance between returns of i^{th} and j^{th} assets.

The mathematical form of the Markowitz mean-variance model is as follows:

$$\begin{aligned} \textbf{Model (1.6)} \quad \text{Min } Z &= \sum_{i=1}^N \sum_{j=1}^N \sigma_{ij} w_i w_j \\ \text{subject to} \quad & \sum_{i=1}^N \hat{r}_i w_i = \mu_0, \\ & \sum_{i=1}^N w_i = 1, w_i \geq 0 \quad \forall i = 1, 2, \dots, N, \end{aligned}$$

where $\mu_0 = [\mu_{\min}, \mu_{\max}]$ is the interval of the expected return of an investor.

1.5 Research Gaps

DEA is particularly useful for portfolio selection, as it allows for the identification of the most efficient combinations of assets based on multiple criteria, such as risk and return. Although the reviewed literature provides valuable insights into the application of DEA in portfolio selection, it does not comprehensively address all relevant aspects. Consequently, several research gaps have surfaced, highlighting opportunities for further investigation to enhance DEA's theoretical framework, methodology, and practical applicability in this domain. The following research gaps have originated in the existing studies.

1. The use of cross-efficiency in portfolio selection is notably limited in the literature, particularly when negative inputs and outputs are present. However, the advantage of incorporating cross-efficiency in portfolio efficiency lies in its ability to offer a more thorough and competitive assessment by evaluating both self-performance and peer performance.
2. The use of equal weights for cross-efficiency aggregation presents a signif-

icant drawback: self-evaluated efficiencies are given much less weight than peer-evaluated ones.

3. The cross-efficiency aggregation methods found in the literature, such as regret theory, Ordered weighted Averaging (OWA), and prospect theory, involve numerous parameters. Therefore, there is a need to explore simpler aggregation methods. This thesis fills this gap by introducing a novel application of the OVGA operator in DEA cross-efficiency aggregation. Additionally, we use the OVGA-aggregated efficiencies to identify the optimal portfolio.
4. Very few studies have applied game cross-efficiency to portfolio selection problems, despite its effectiveness in addressing the competition and unpredictability associated with the cross-efficiency method.
5. Furthermore, in-game cross-efficiency evaluation, the final value is calculated as the simple average of the various game cross-efficiencies. There remains a gap in the literature concerning the aggregation of game cross-efficiency scores.
6. The SORM model is commonly employed in decision-making problems involving negative data. However, it has the limitation of producing negative cross-efficiencies when it is used for cross-evaluation.
7. Generally, the use of column averages of the cross-efficiency matrix for performance evaluation is well-established in the literature. However, the use of row average scores of the cross-efficiency matrix has not been explored in the literature, particularly for stock selection.
8. A few studies use the Maverick index as a risk parameter in portfolio optimization. However, no study has utilized the diversity index (calculated from the row average of the cross-efficiency matrix) as a return parameter.
9. Despite the widespread use of DEA and TOPSIS, few studies have tackled full portfolio ranking with negative input-output variables, leaving unique portfolio ranking largely unexplored.

1.6 Motivation

The rapid evolution of financial markets and the increasing complexity of investment strategies have led to a growing need for more advanced portfolio optimization techniques. Portfolio optimization identifies corporations and allocates shares to maximize investment benefits while minimizing risks for a fundamental investor. In mathematical terms, the MV portfolio selection model is the most fundamental model of portfolio optimization. Despite the importance of this model, it can be criticized due to the reliance on historical return, which does not necessarily reflect the actual situation. Also, the MV model often focuses narrowly on risk and return, overlooking other crucial factors such as performance evaluation and multi-criteria factors.

DEA offers a promising alternative to traditional optimization methods by providing a multi-criteria framework to evaluate portfolios based on various inputs and outputs. One of the advantages of using DEA is that for the DEA-inefficient securities, information is provided disclosing how much reduction of inputs or how much augmentation is needed for these inefficient securities to become DEA-efficient. In other words, DEA can inform the decision-maker which alternatives are consistently the best when several attributes are considered, but it also provides information as to how much improvement is needed for each alternative to become efficient with respect to inputs (input reduction) and outputs (output augmentation).

Overall, DEA cross-efficiency is a suitable method for solving portfolio selection problems as it enables explicit trade-offs and interactions. However, in peer evaluation, the tendency is to select assets that perform moderately well across all measures, potentially excluding those that excel in only a subset of criteria. This can result in a portfolio consisting of similar financial assets, leading to a lack of diversification. To address this limitation, we incorporate both cross-efficiency and game cross-efficiency into an optimization model, allowing for the selection of a diversified portfolio based on the investor's objectives. In this work, the first stage of the portfolio selection process utilizes the DEA cross-efficiency approach, while the second stage applies optimization models to reflect other investor preferences.

The motivation behind this thesis stems from the limitations of existing portfolio

optimization models and the potential of DEA to offer a more holistic approach. By incorporating multiple performance criteria, DEA can provide deeper insights into portfolio efficiency, enabling investors to make better-informed decisions. Furthermore, DEA's adaptability makes it applicable across a wide range of asset classes, including stocks, bonds, and derivatives, making it a versatile tool for portfolio selection in both traditional and alternative investments. Overall, these methodologies enhance the discrimination of financial assets in two key ways. In the first step, they assign unique ranks to the assets, enabling a more precise and distinct evaluation of each asset's performance. In the second step, the methodologies allow for portfolio selection that reflects the decision makers' underlying preferences, such as risk tolerance, return objectives, or other specific criteria.

1.7 Organization of the Thesis

The thesis, entitled “**Fuzzy Portfolio Selection via Ranking Models in DEA and Multi-criteria Decision Making**”, comprises six chapters, followed by a summary and delineation of future research directions. A summary of each chapter of the thesis is as follows:

The introductory **chapter 1** provides a comprehensive literature review, laying the foundational knowledge necessary for this research. It begins with exploring the basics of the MV portfolio selection model and related literature surveys. Further, this chapter discusses the DEA concept and related literature surveys. The chapter then follows the evolution of DEA over time, examining its various extensions and adaptations. Thus, the current chapter creates a background for this thesis's work and motivates the work carried out in this thesis.

Chapter 2 is based on the research paper entitled, “Cross-efficiency aggregation by ordered visibility graph averaging: method, and application in portfolio selection”, published in **IMA journal of management mathematics, (SCIE, Impact Factor: 1.9)**. In this chapter, we propose a novel use of the OVGA operator for aggregating DEA cross-efficiencies and apply this method to address the portfolio selection problem. The selection of assets before constructing an optimal portfolio is a critical step. An asset's efficiency is defined by its ability to convert inputs into outputs. When choosing assets for a portfolio, investors take into account multiple factors. By assessing the relative efficiency of similar processing units,

DEA aids in asset selection before portfolio construction. Next, the chapter introduces a model for constructing the most efficient optimal portfolio. It addresses various practical considerations, including budget constraints, cardinality, buy-in requirements, and restrictions on short selling. The proposed method is further validated using empirical data from the Indian banking industry. The findings from this study can be applied to generate highly effective portfolios for stock companies, financial institutions, and businesses in both the public and private sectors.

Chapter 3 is based on the research paper titled, “A Comprehensive mean-maverick- variance portfolio selection model based on OVG aggregated game cross-efficiency approach” (communicated in “**International Journal of Mathematics for Industry**”). This chapter makes two significant contributions to the field of portfolio selection and efficiency analysis. First, we introduce the aggregated DEA game cross-efficiency concept, which enhances traditional DEA cross-efficiency evaluation by incorporating a game-theoretic perspective. To determine the appropriate weight allocation for DMUs, we employ the OVGA approach, which structures and refines weight assignments based on a systematic ordering of game cross-efficiencies. Furthermore, we integrate these OVGA game cross-efficiency scores into a multi-objective mean-maverick-variance (MMV) portfolio selection model. In our proposed approach, the OVGA game cross-efficiency score is treated as a return characteristic, reflecting the overall effectiveness of an asset, while the Maverick index, a measure of how much an asset’s efficiency deviates from peer evaluations, along with the variance of cross-efficiency scores, is considered a risk characteristic. A key advancement in our approach lies in the refinement of game cross-efficiency as a tool for portfolio selection. By introducing OVGA game cross-efficiency scores, we provide a unique set of Nash equilibrium cross-efficiency values. Additionally, our methodology achieves faster convergence, making it computationally more efficient and practical for large-scale portfolio optimization problems. To validate the effectiveness of our approach, we conduct an empirical study using a dataset of 50 companies listed on the National Stock Exchange (NSE).

In **chapter 4**, we introduce a modified SORM model to address a major limitation of traditional SORM cross-efficiency analysis—the occurrence of negative cross-efficiency scores. The chapter is based on the research paper entitled, “An Integrated Multi-Objective SORM Cross-Efficiency Model: An Application in

Portfolio Selection, published in **Economic Computation and Economic Cybernetics Studies and Research**, (SCIE, Impact Factor: 0.9). In the conventional DEA method, cross-efficiency evaluation assesses DMUs by incorporating peer evaluations. However, when the traditional SORM model is applied for cross-evaluation, it often generates negative efficiency values, which undermine its applicability in financial and investment analysis, where efficiency scores must be non-negative. To resolve this issue, our modified SORM model refines the cross-efficiency calculation, ensuring that all efficiency scores remain positive.

The column averages of the cross-efficiency matrix are used to compute the Maverick index, which helps in measuring the false positivity of assets by quantifying how much an asset's self-evaluated efficiency deviates from the consensus of peer evaluations. Higher Maverick index values indicate greater discrepancies, signaling that an asset's perceived efficiency might be overestimated.

In contrast, the row averages of the cross-efficiency matrix reveal how a particular asset evaluates other assets. This insight allows us to construct a diversity index, which captures the difference between an asset's self-assessed efficiency score and the average score assigned by its peers. A higher diversity index means that an asset rates itself very differently from how others rate it. Thus, a novel multi-objective portfolio selection model is proposed, incorporating the maverick index as a risk factor and the diversity index as a representation of return. A case study highlights the model's ease of application, ability to distinguish between different stocks, and effectiveness in making informed and optimal portfolio decisions.

Chapter 5 is based on the research paper entitled, "Stock assessment using cumulative prospect theory in DEA cross-efficiency model: A case study of the Indian stock market", published in **Croatian Operational Research Review,(ESCI)**. In practice, the column average of the cross-efficiency matrix is frequently used for decision-making, but the row average of the cross-efficiency matrix is also an important factor. Row averages help assess the fairness or aggressiveness of DMU evaluations, while column averages assist decision-makers in identifying efficient and consistent performers. Therefore, in **chapter 5**, we present an innovative approach to portfolio selection derived from the RDM cross-efficiency matrix. The method utilizes the average scores of both columns and rows derived from the cross-efficiency matrix. The column average (CA) and row average (RA)

methods are employed for portfolio selection and the classification of different asset types. This classification aids investors in constructing a more resilient and well-balanced portfolio. To examine the psychological factors that influence the decisions of investors in the stock market, we integrate the value of the cumulative prospect theory value for each stock as an output and its variance as an input to assess overall asset efficiency.

Chapter 6, is based on a research paper entitled, “Ranking of Efficient Fuzzy Portfolios by Hybrid MSBM-TOPSIS Technique” published in **Foundations of Computing and Decision Sciences Journal (ESCI)**. **Chapter 6** aims to evaluate fuzzy portfolios using the MSBM model, with trapezoidal fuzzy numbers for returns and possibilistic measures for risk and mean return. Efficient portfolios are further ranked using the TOPSIS technique. A hybrid method is proposed for ranking investment portfolios by integrating the MSBM model of DEA with a multi-criteria decision-making approach. Techniques like the MSBM and TOPSIS incorporate traditional performance metrics while adding flexibility to address fuzzy environments and handle imprecise data. The ranking process is carried out in two stages: first, portfolio evaluation using the MSBM fuzzy model, and second, ranking through the TOPSIS method. This integrated MSBM-TOPSIS framework delivers a robust and distinctive ranking system, providing investors with a dependable method to identify high-performing portfolios.

A comprehensive summary of the key findings and contributions of the research is presented at the end of chapter 6, followed by a discussion on the potential directions for future research.

Table 1.1: Summary of literature based on DEA models

Title	Year	Paper	Description	Advantages/Limitations
RAM: A Range Adjusted Measure of Inefficiency for Use with Additive Models, and Relations to Other Models and Measures in DEA	1999	[44]	This study introduces the additive variable returns to scale DEA model incorporating a range-adjusted measure (RAM) of inefficiency.	It can be applied in the presence of negative variables also.
Negative data in DEA: A directional distance approach applied to bank branches	2004	[144]	The Range Directional Measure (RDM) utilizes an enhanced form of the generic directional distance model without requiring any transformation. RDM provides an efficiency measure similar to radial measures used in traditional DEA. There are two versions of the RDM model, one concentrates on improving variables where the DMU is most distant from the optimal achievable levels. Another one Prioritizes improvement in variables where the DMU is closest to the best attainable levels.	- RDM provides targets that reflect the priorities for improving the inputs and outputs of a DMU. - In contrast, the additive model yields targets that are furthest from the DMU to the efficient boundary. - RDM models generate efficiency measures similar to those of radial models. The additive model, however, does not produce an efficiency measure.
Returns to scale in different DEA models	2004	[15]	This paper examines returns to scale (RTS) in DEA across various existing model types. The BCC and CCR methodologies are analyzed in input-oriented version, while the multiplicative model is addressed in an output-oriented version.	It extends to both additive and multiplicative models, offering prospects for a deeper understanding of the nature of RTS and its application using DEA methods and concepts.
A modified slacks-based measure model for data envelopment analysis with 'natural' negative outputs and inputs	2007	[160]	The Modified Slack-Based Measure (MSBM) addresses both "natural" negative outputs and negative inputs. MSBM addresses the problem of translation invariance in the slack-based measure model. It incorporates concepts from the RDM to improve its functionality.	- MSBM allows for weighting different slacks based on their importance to decision-makers. - It considers individual input and output slacks, providing a more accurate evaluation of inefficient DMUs. - This approach is particularly effective in the presence of negative data.
Finding the most efficient DMUs in DEA: an improved integrated model	2007	[6]	This paper introduces an enhanced integrated DEA model for identifying the most efficient DMUs. The proposed model eliminates the need for a trial-and-error approach in the objective function.	- The improved integrated DEA model is demonstrated to be consistently feasible and effective in ranking the most efficient DMU. - Additionally, it can determine the most efficient DMUs without requiring the model to be solved n times, thereby enabling faster results.
A new integrated DEA model for finding most BCC-efficient DMU	2009	[173]	This paper introduces a novel integrated model designed to identify the most BCC-efficient DMU by solving a single linear programming (LP) problem.	This model is particularly advantageous in scenarios with variable returns to scale, offering a broader range of applications compared to models that determine the most CCR-efficient DMU.
A directional slacks-based measure of technical inefficiency	2009	[68]	This paper proposes a directional slacks-based measure of technical inefficiency to account for slack in technological constraints. Radial efficiency measures estimated using linear programming (LP) methods can be biased, as slack in the constraints suggests that at least one input could be reduced, or one output could be expanded, even when a firm is considered "technically efficient."	- The proposed measure generalizes existing slacks-based inefficiency measures. - When no slack exists, the directional slacks-based inefficiency reduces to the directional technology distance function. - The study suggests that slack is a significant source of inefficiency, which is often not captured by the directional technology distance function.
A semi-oriented radial measure for measuring the efficiency of decision making units with negative data, using DEA	2010	[59]	The SORM method is used to set targets for improved performance when some variables in a DEA framework have negative values. Unlike other methods such as RDM and MSBM, which also handle negative data, SORM treats each input-output variable as the sum of its negative and positive parts, with their sum reflecting the original value	- The negative part of a variable can be handled in absolute value terms, ensuring a positive format without arbitrary changes of origin. - The disadvantage of this approach is that treating the negative parts of a variable as a separate entity increases the problem's dimensionality, which can result in the exclusion of part of the original production possibility set. This may prevent the method from identifying Pareto-efficient targets. - However, the method typically ensures that the targets are not worse than the observed input-output levels of the unit.
Directional distance functions and slacks-based measures of efficiency: Some clarifications	2010	[66]	This study introduced a new variant of the SBM model utilizing the directional distance function without orientation. This model summarizes the SBM model under the variable returns to scale (VRS) framework.	Unlike the traditional SBM model, it accommodates cases where certain inputs and outputs are zero.
Improved efficiency measures through directional distance formulation of Data Envelopment Analysis	2013	[54]	This paper proposes a model to enhance efficiency measurement using a directional distance approach in Data Envelopment Analysis (DEA). It addresses scenarios where production factors include both positive and negative values.	- The measure is weakly monotonic; units and translation do not vary with respect to inputs and outputs. - The proposed model is validated using a case study done on Information Technology firms operating in India
A variant of radial measure capable of dealing with negative inputs and outputs in data envelopment analysis.	2013	[41]	In this paper, a variant of the traditional radial model is proposed, where original values are replaced with their absolute values to quantify the proportion of improvements required to reach the best-practice frontier. The VRM model is unit-invariant and capable of handling datasets with negative values in any scenario under assessment.	- The variant of the radial measure (VRM) retains the property of proportionate improvement found in the traditional radial model. - It produces identical results to the traditional radial model.

Table 1.1: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
Data envelopment analysis: Prior to choosing a model	2014	[43]	This paper examines various challenges associated with the application of data envelopment analysis (DEA), including model orientation, the selection and definition of inputs and outputs, handling mixed and raw data, and determining the appropriate number of inputs and outputs relative to the number of decision-making units (DMUs).	This paper provides guidance and clarification on determining the appropriate number of inputs and outputs.
A modified Semi-Oriented Radial Measure for target setting with negative data	2014	[121]	This paper addresses issues in efficiency evaluation and target setting within the standard SORM model and introduces a modified SORM model. The development of an axiomatic basis for the SORM production possibility set and the introduction of new radial and non-radial efficiency measures in DEA with negative data are identified as key challenges for future research.	- It explores the properties of the modified SORM. - The paper highlights problems in target setting with SORM models and presents a modified approach. - An empirical study in the banking sector demonstrates the applicability of the proposed model.
A two-phase approach for setting targets in DEA with negative data	2014	[?]	This study, proposes a new two-phase approach based on a modified version of the classical additive DEA model.	The proposed model is designed to generate a target with non-negative values for each observed unit, addressing negative components that are not handled by other methods.
Review of efficiency ranking methods in data envelopment analysis	2017	[2]	This paper reviews the literature on ranking methods in Data Envelopment Analysis (DEA) to enhance its discrimination power. The reviewed methods are categorized into ten distinct groups based on their structure. The paper includes approaches published up to 2016.	- It discusses the advantages and disadvantages of each ranking method. - Some methods offer a simple process for generating accurate results, while others achieve comparable accuracy through more complex procedures.
CCR or BCC: what if we are in the wrong model?	2018	[49]	This paper presents a comparative analysis of two concepts: the impact of activity scaling under constant BCC-efficiency on CCR-efficiency, and conversely, the behavior of BCC-efficiency when activity scaling is applied under constant CCR-efficiency.	- The observed mutual effects enhance a DMU's ability to reassess activity upsizing and downsizing, allowing for more informed decision-making. - Scale efficiency proves to be the most suitable concept for managing activity changes, as demonstrated through a numerical example and an illustrative application to a theater scenery scenario.
A directional semi-oriented radial DEA measure: an application on financial stability and the efficiency of banks	2018	[85]	This study introduces a new version of the modified SORM model, incorporating a directional distance function and selecting a relevant direction to handle variables with both positive and negative values. To demonstrate the model, the study estimates efficiency scores for a sample of banks in the Persian Gulf Council Countries (GCC) from 2002 to 2011. The efficiency scores are then used to examine the impact of financial system stability on the technical efficiency of individual banks.	- The proposed Directional SORM (DSORM) model is more efficient than previous models, both in terms of computational performance and target settings. - The DSORM model also allows for the dual formulation of linear programming.
DEA models with Russell measures	2019	[161]	This paper examines and compares various DEA models based on Russell measures, focusing on the convexity and free disposability of their PPS, as determined by the preferences incorporated into the PPS.	- This work systematically examines several DEA models based on the preferences used in their production possibility sets (PPS). - It identifies key differences among different models by analyzing the convexity and free disposability of their PPS.
A strong efficiency measure for CCR/BCC models	2021	[45]	This article introduces a robust efficiency measure that addresses the limitations, similar to the CCR/BCC models. It incorporates all non-zero input and output slacks after the proportional reduction in inputs is fully utilized.	- This approach enables accurate classification of production units in efficiency rankings and supports the measurement of inefficiencies in inputs and outputs, aiding business management decisions. - The model is extended to handle zero and negative input or output values.
Evaluating radial efficiency considering environmental factors: A generalization of classical DEA	2021	[1]	This paper extends the conventional DEA framework by developing a new radial model for efficiency assessment that accounts for undesirable outputs, allowing for more accurate problem modeling.	A key aspect of the new model is the distinction between different relationships among the factors. Inputs and desirable outputs establish the operational frontier, while desirable and undesirable outputs define the environmental frontier.
A data envelopment analysis model for performance evaluation and ranking of DMUs with alternative scenarios	2021	[204]	This paper contributes to DEA theory by addressing uncertainty in datasets through multiple alternative scenarios, enabling the analysis of how unit efficiency may evolve under various future development trajectories.	- The proposed models overcome the common limitations associated with other methods typically used for these issues and are always feasible. - They provide a complete ranking of all DMUs through a computationally efficient procedure.
Integrated data envelopment analysis, multi-criteria decision-making, and cluster analysis methods: Trends and perspectives	2023	[52]	This study provides a comprehensive review of the literature on DEA, MCDA, and CA models to identify existing applications, potential uses, and future development trends. The Methodi Ordinatio technique was employed to determine the most impactful publications, drawing from Scopus, ScienceDirect, and Web of Science databases.	- The study's limitations include the lack of detail on the most commonly used methods for each theme. - A search across Scopus, Science Direct, and Web of Science revealed no publications addressing all three techniques together, indicating a potential gap in the literature.

Table 1.2: Summary of literature based on the DEA cross-efficiency technique

Title	Year	Paper	Description	Advantages/Limitations
Determination of the weights for the ultimate cross efficiency using Shapley value in cooperative game	2009	[191]	To address the limitations of using average cross-efficiency values for evaluating DMUs, this approach removes the condition of averaging and instead treats DMUs as players in a cooperative game. Here, the characteristic function values of coalitions are defined to calculate each DMU's Shapley value. The common weights, linked to the Shapley value imputation, are then used to calculate the final cross-efficiency scores.	- The assumption of averaging for determining the final cross-efficiency is eliminated, and the cross-efficiency evaluation method is improved through a cooperative game perspective. - The Shapley value is applied as one approach to allocate contributions among DMUs in the cooperative game, serving as just one method for calculating the final cross-efficiency score. Other cooperative game solutions for determining cross-efficiency are not explored in this paper.
The use of OWA operator weights for cross-efficiency aggregation	2011	[178]	The ordered weighted averaging (OWA) technique is used to aggregate cross-efficiency scores.	Based on the subjective preferences of decision-makers, the OWA technique can provide different weights to each DMU's self and peer efficiencies.
Selecting symmetric weights as a secondary goal in DEA cross-efficiency evaluation	2011	[80]	This paper proposes the Symmetric Weight Assignment Technique (SWAT), which maintains feasibility and rewards decision-making units (DMUs) that make a symmetric selection of weights.	This method incorporates symmetric constraints into output weights to reward symmetric outputs while preserving linearity. It can be extended to include all input and output weights.
A ranking method for DMUs with interval data based on dea cross-efficiency evaluation and TOPSIS	2013	[194]	This study incorporates a DEA cross-efficiency approach to compute the range of cross-efficiency values. Based on this, a novel TOPSIS method is developed to rank the DMUs. Two case studies are provided to demonstrate and validate the proposed approach.	- The proposed method efficiently addresses multi-attribute problems involving interval data. - It effectively utilizes the original data, delivering comprehensive and fair results for all DMUs.
A cross-efficiency fuzzy Data Envelopment Analysis technique for performance evaluation of decision-making units under uncertainty	2015	[55]	This paper introduces a novel cross-efficiency fuzzy DEA methodology for assessing various DMUs under uncertainty. To assess the performance of multiple DMUs with uncertain input and output data, the proposed decision-making method utilizes triangular fuzzy numbers.	The model combines the DEA cross-efficiency technique with a fuzzy logic framework, preserving the discriminative power of cross-efficiency DEA while effectively handling uncertainty through the fuzzy framework.
DEA cross-efficiency evaluation based on satisfaction degree: an application to technology selection	2016	[188]	This paper proposed a max-min model to determine the optimal set of weights for each DMU, taking into account the satisfaction levels of all other DMUs. To solve the max-min model linearly and ensure the uniqueness of the final optimal weight set for each DMU, two algorithms were developed. Finally, the proposed approach was demonstrated through an example of technology selection.	- Integration of satisfaction degree, maximizing DMUs' satisfaction in weight selection. - Improved discrimination power compared to previous methods. - Guaranteed uniqueness of the final set of optimal weights for each DMU.
The cross-efficiency in the optimistic-pessimistic framework	2017	[89]	This study applies an optimistic-pessimistic approach to introduce a novel cross-efficiency methodology. The proposed method derives the estimated scores by accounting for both the strengths and weaknesses of the DMUs. Furthermore, the proposed models uniquely determine the optimal weights.	- The proposed method allows for a complete ranking of DMUs without the need for any secondary goals, using an optimistic-pessimistic technique. - In this method, the strengths and weaknesses of DMUs are incorporated into their evaluations, and the feasibility of the proposed models has been demonstrated, broadening their potential applications.
Cross-efficiency evaluation capable of dealing with negative data: A directional distance function-based approach	2019	[105]	Building on the RDM model, this study examines the duality relationship related to the maximum slacks formulation of RDM. It defines cross-efficiencies through equivalences with fractional programming problems. This cross-efficiency assessment operates under the framework variable returns to scale.	- The introduced VRS cross-efficiency assessment effectively handles negative input and output data. - It is non-oriented and provides self and peer assessment of DMUs by simultaneously measuring inefficiency in inputs and outputs. - Similar to classical cross-efficiency evaluation, it generates cross-efficiency values between 0 and 1 for all DMUs, and a benevolent model is introduced to select weights among multiple optimal solutions.
A slacks-based measure model for calculating cross-efficiency in data envelopment analysis	2020	[86]	This article introduces a slacks-based measure for calculating efficiency. The concept is similar to the radial measure, which uses the frontier associated with a DMU to assess the efficiency of all DMUs, but in this case, the efficiency measure is based on slacks rather than being radial.	- The first advantage of this method is that it avoids producing negative efficiencies, making it suitable for cases involving variable returns to scale. - Additionally, by considering inputs and outputs in the efficiency measurement, this model generates a combined efficiency score, rather than separate measures for inputs and outputs under variable returns to scale.
On Pareto-optimality in the cross-efficiency evaluation	2020	[47]	This paper proposes a new Pareto-optimality notion for cross-efficiency assessment, based on a self-prioritizing principle and less stringent postulates. A multi-objective model generates Pareto-optimal score sets, and a linear model using the weighted sum method determines optimal weights in a single step.	The proposed method is a linear model that yields the optimal solution, enabling the generation of Pareto-optimal cross-efficiency values under the self-prioritizing principle by solving a single linear model.

Table 1.2: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
An Improvement of DEA Cross-Efficiency Aggregation Based on BWM-TOPSIS	2020	[64]	This paper utilizes the Technique for Order Performance by Similarity to Ideal Solution (TOPSIS) and the Best Worst Method (BWM) to combine aggressive and benevolent cross-efficiency values. Additionally, the BWM method's high consistency helps address the TOPSIS method's limitations.	- It integrates TOPSIS, BWM, and cross-efficiency methods for the first time. - The BWM method significantly improves consistency, although it tends to be more subjective. - It addresses key challenges such as the non-uniqueness problem and the aggregation of cross-efficiency data.
Methods and applications of DEA cross-efficiency: Review and future perspectives	2021	[193]	This paper examines the theory and practical applications of Cross-Efficiency Evaluation (CREE).	- Applications of CREE in engineering management are explored in detail. - The paper also identifies and discusses potential directions for future research on CREE.
DEA Cross-efficiency Aggregation with Deviation Degree Based on Standardized Euclidean Distance	2021	[215]	This article introduces a standardized Euclidean distance to compare various optimal solutions for selecting appropriate input and output weights and to determine DMU weights for cross-efficiency aggregation by evaluating the disparity between individual preferences and the group ideal preference.	The results indicate that, compared to other methods, the weights derived from the proposed approach not only capture differences among DMUs but also ensure no efficiency scores are overlooked.
On the spectrum of achievable targets in cross-efficiency evaluation and the associated secondary goal models	2021	[48]	This paper presents a general secondary objective model using multi-objective programming, showing that cross-efficiencies from the MRA (LRA) model exceed (are lower than) those from any derived benevolent (aggressive) model. It also proposes benevolent, aggressive, neutral, and weighted average CE evaluation models.	- The proposed general secondary goal model extends several widely adopted secondary objective methods. - A neutral secondary goal model, based on the spectrum of achievable CE targets, is introduced, minimizing deviation discrepancies from upper and lower limits to treat all DMUs equally.
Environmental Efficiency Evaluation of the Xiangjiang River Basin: A DEA Cross-Efficiency Approach With Social Network	2021	[40]	This paper integrates social network analysis into cross-efficiency evaluation to enhance DMU ranking by explicitly modeling reference relationships. A pairwise comparison model identifies superiority and inferiority between DMUs, forming a non-weighted directed social network where edges represent learning from lower-performing to higher-performing DMUs.	- Compared to traditional cross-efficiency methods, the proposed approach incorporates reference relationships among DMUs rather than relying solely on cross-efficiency scores. - It also evaluates the importance of each DMU using its centrality within the network.
Efficiency evaluation with regret-rejoice cross-efficiency DEA models under the distributed linguistic environment	2022	[83]	This paper presents the development of a regret-rejoice cross-efficiency linguistic distribution DEA (RCE-LDDEA) method based on regret theory. This approach incorporates input and output data represented through linguistic distributions while accounting for the regret-aversion psychological characteristics of decision-makers (DMs).	- The RCE-LDDEA method offers the advantage of being more reasonable and better suited for handling complex cross-efficiency evaluation problems. - The advantage of the RCE-LDDEA method lies in its ability to account for the regret-aversion psychological characteristics of decision-makers, effectively capturing their regret-averse behavior.
A method to determine the integrated weights of cross-efficiency aggregation	2022	[98]	This study proposed a novel method for deriving aggregation weights from both subjective and objective perspectives. Prospect theory is used to capture the decision-maker's preferences, providing an interval reference point based on their goals. The variance concept reflects the deviation between peer and self-evaluation efficiency, yielding objective weights. An optimization technique is then constructed to calculate integrated weights, combining the DM's subjective preferences with the objective information in the cross-efficiency matrix.	- The method combines two complementary viewpoints: aggregation weights that capture decision-maker's subjective preferences and the intrinsic objective information within the cross-efficiency matrix. - Additionally, an optimization methodology is developed, aiming to minimize the total variation between the final decision result and the outcomes derived from subjective and objective weights, to calculate the integrated weights.
Improvement of cross-efficiency based on TODIM method	2022	[189]	This paper introduces an aggregated approach for the interval cross-efficiency matrix to evaluate the relative efficiency and achieve an accurate ranking of Decision-Making Units (DMUs). In this method, DMUs are treated as decision-makers, and their relative weights are determined using the Decision-Making Trial and Evaluation Laboratory (DEMATEL) technique. Additionally, relative advantage scores are calculated by incorporating decision-maker's risk attitudes, following the principles of the improved TODIM method.	- The proposed method employs the DEMATEL-cross efficiency model to determine DMU weights, uncovering decisive information from the cross-efficiency column vector and the relative relationships within the row vector for the first time. - This method also faces challenges, such as not addressing weight flexibility in the initial evaluation stage. While it solves the relative importance of DMUs, it overlooks attribute utility characteristics.
A modified DEA cross-efficiency method with negative data and its application in supplier selection	2022	[165]	This article proposed a novel non-radial model for evaluating DMU performance with negative data and introduced a secondary goal model to address the first limitation of the cross-efficiency methodology. To resolve the second limitation, a hybrid MADM-DEA approach incorporating the fuzzy VIKOR method is presented.	The proposed method provides a unique ranking of DMUs.
Cross-efficiency aggregation based on the denominator rule	2024	[87]	This paper introduced an alternative cross-efficiency aggregation technique based on self-appraisal efficiency theory. Using the denominator rule, cross-efficiency values from the DEA multiplier model are combined into a common weighted average.	- The proposed approach incorporates the relative importance of each matrix element, determined by either the resource opportunity cost or the output value, depending on the efficiency orientation. - Since this involves a common set of input and output weights across all DMUs, it enables a comprehensive ordering and comparison of both efficient and inefficient units.

Table 1.3: Summary of literature based on DEA in portfolio selection

Title	Year	Paper	Description	Advantages/Limitations
Measuring the performance of ethical mutual funds: a DEA approach	2003	[18]	This study introduces a DEA-based performance indicator that evaluates ethical funds by integrating expected return, investment risk, ethical components, and subscription/redemption costs. The indicator measures the relative efficiency of decision-making units in a multi-input, multi-output framework and can be adapted to various models based on the nature of the ethical criteria defining socially responsible funds.	All the proposed models can be naturally adapted or extended by incorporating additional input or output variables, enabling the consideration of other aspects that may be valuable for comparing ethical mutual funds.
Mutual fund performance evaluation using data envelopment analysis with new risk measures	2006	[35]	This applied study extends previous findings by incorporating new risk measures, value-at-risk (VaR) and conditional value-at-risk (CVaR), into the inputs of existing DEA models to better capture the pervasive skewness and leptokurtosis in the return distributions of actively managed funds. Additionally, it ensures a fair evaluation of a fund's relative performance across different periods.	Empirical analysis of the Chinese mutual fund market shows that combining VaR and CVaR with traditional risk measures effectively captures return distribution properties and asset allocation, enabling more accurate mutual fund performance evaluation.
Stock selection using data envelopment analysis	2008	[28]	This study utilized two DEA models to assess firm efficiency and construct portfolios by selecting stocks with high efficiency. Additionally, an empirical analysis was conducted to compare the return rates of portfolios formed by small-size firms, DEA models, and market indices.	- This study represents the first attempt to use DEA models for stock selection and compares the performance of portfolios constructed through DEA analysis, small-size firms, and stock market indices. - The proposed approach offers valuable managerial insights for stock selection and strategies to enhance the financial efficiency of corporations.
Portfolio selection under DEA-based relative financial strength indicators: case of US industries	2008	[57]	This paper proposes a Data Envelopment Analysis-based approach to derive a Relative Financial Strength (RFS) indicator, reflecting a firm's competitiveness within its industry. Correlating the RFS with historical stock returns enables informed equity portfolio decisions.	Developed a financial strength indicator using DEA to evaluate a firm's financial performance.
An integrated DEA-MODM methodology for portfolio optimization	2014	[77]	This study presents an integrated portfolio optimization method that includes stock screening, selection, and capital allocation. It starts with analyzing financial data of investment targets, selects the portfolio based on firms' relative efficiency using a DEA model, and allocates capital through a multi-objective decision-making (MODM) model.	The MODM mathematical model adopts the downside risk measure instead of the traditional risk measure for improved practicality.
An integrated multi-objective Markowitz-DEA cross-efficiency model with fuzzy returns for portfolio selection problem	2015	[119]	This paper presents a new multi-objective model for portfolio selection. The model integrates DEA cross-efficiency with the Markowitz model, addressing return, risk, and efficiency simultaneously. The proposed model is applied to a dataset of 52 firms listed on the Iranian stock exchange, and the results are analyzed.	- To account for uncertainty, asset returns are modeled as trapezoidal fuzzy numbers. - Due to the computational complexity of the model, the Non-Dominated Sorting Genetic Algorithm II (NSGA-II) is utilized for optimization. - Findings indicate that the proposed model outperforms traditional Markowitz and DEA models by simultaneously considering return, risk, and efficiency.
A fuzzy hybrid project portfolio selection method using data envelopment analysis, TOPSIS and integer programming	2015	[172]	This study utilizes DEA for preliminary screening, the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) for ranking projects, and linear Integer Programming to select the optimal project portfolio in a fuzzy environment, all aligned with organizational targets.	The proposed method guarantees optimal alignment between the final portfolio selection and the initial project rankings while considering various organizational objectives.
Cross Efficiency Evaluation with Negative Data in Selecting the Best of Portfolio Using OWA Operator Weights	2015	[13]	This paper examines the efficiency evaluation process within the cross-efficiency matrix containing negative data and introduces the application of ordered weighted averaging (OWA) operator weights for cross-efficiency analysis. The study aims to identify an optimal set of assets that minimizes risk while maximizing returns.	To reflect the investor's subjective requirements regarding different efficiencies, OWA operator weights are proposed for cross-efficiency evaluation.
Efficiency evaluation of fuzzy portfolio in different risk measures via DEA	2017	[30]	This paper addresses fuzzy portfolio efficiency evaluation under different risk measures. To overcome the practical challenges of the Real Frontier Approach (RFA), we propose three DEA-based fuzzy portfolio models incorporating possibilistic variance, semi-variance, and semi-absolute deviation for portfolio efficiency assessment.	- Simulations across various sample sizes validate the proposed models against RFA. - Results reveal that with adequate samples, proposed models closely approximate the real frontier, with highly correlated PE values.
Data envelopment analysis based fuzzy multi-objective portfolio selection model involving higher moments	2018	[125]	This paper introduces a multi-objective portfolio selection model that incorporates five objective functions: maximizing return, skewness, and efficiency while minimizing risk and kurtosis. The efficiency of the portfolio is assessed based on the performance of its assets, evaluated using DEA across six criteria: asset turnover, earnings per share (EPS), EPS growth rate, leverage ratio, price-to-earnings (P/E) ratio, and beta value.	To account for the uncertainty and vagueness inherent in real financial markets, the input data for all parameters are represented as (λ, ρ) interval-valued fuzzy numbers, constructed based on historical data.

Table 1.3: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
DEA frontier improvement and portfolio rebalancing: an application of China mutual funds on considering sustainability information disclosure	2018	[212]	This paper introduces a DEA frontier improvement technique within the mean-variance framework. This technique offers investors a rebalancing scheme and an enhanced DEA frontier that more accurately approximates the portfolio-efficient frontier compared to the conventional DEA model.	• 1. The improved DEA frontier, derived using proposed rebalancing strategies, offers a more accurate approximation of the real portfolio frontier compared to the traditional DEA frontier. • Simulation results confirm that the enhanced DEA frontier extensively outperforms the traditional DEA frontier.
Portfolio Optimization By Using MeanSharp-VaR and Multi-Objective MeanSharp-VaR Models	2018	[12]	This study develops portfolio optimization and asset allocation using a proposed three-phase approach. First, stock companies are screened based on financial data. Second, to address the limitations of conventional DEA models with non-negative data, the MeanSharp-VaR (MSHV) and Multi-Objective MeanSharp-VaR (MOMShV) models based on Range Directional Measure (RDM) are introduced, accommodating both positive and negative values while minimizing Value at Risk (VaR). Efficient stock companies are then selected for the portfolio. In the final phase, a Multi-Objective Decision-Making (MODM) model is used to allocate capital to the chosen stocks.	• Multi-objective functions provide greater accuracy, making the MOMShV model generally outperform the MSHV model. • Monte Carlo simulation was used to calculate VaR at different confidence levels, with higher confidence levels yielding more accurate results.
A comprehensive model for fuzzy multi-objective portfolio selection based on DEA cross-efficiency model	2018	[34]	This paper explores fuzzy portfolio selection in a multi-objective framework. A model combining the mean-semivariance approach with DEA cross-efficiency, based on the Sharpe ratio, is proposed, incorporating holding bounds and cardinality constraints.	• A multi-objective firefly algorithm is developed to address the nonlinear optimization complexity. • The model's validity is demonstrated through a practical example.
Performance Evaluation of Portfolios with Fuzzy Returns	2019	[211]	This paper redefines the input-output framework to align with real-world investment scenarios, considering initial wealth as the input and return as output. The efficiencies of fuzzy portfolios are determined using the possibilistic mean-variance-entropy criterion via a directional distance function.	• The diversification and DEA techniques can be considered as the nonlinear and linear estimates of the model based on the real frontier, respectively. • Results show that the portfolio efficiencies derived from both the diversification and DEA models can converge to the real frontier when the portfolio samples are sufficiently large.
Multi-Objective Fuzzy Portfolio Performance Evaluation Using Data Envelopment Analysis Under Credibilistic Framework	2020	[124]	Two multi-objective fuzzy portfolio optimization models are proposed, focusing on risk (variance/CVaR), return, entropy and liquidity. Trapezoidal fuzzy returns handle market uncertainty using credibility theory.	• Inefficient portfolios are rebalanced using a frontier improvement technique. • A simulation study validates the models with varying sample sizes.
Estimation of fuzzy portfolio efficiency via an improved DEA approach	2020	[196]	This paper introduces a method within the framework of fuzzy theory that enhances the DEA frontier while providing a replicable benchmark for investors. Initially, an improved DEA model is developed using the presented approach and then the relationships between the evaluation model based on a portfolio frontier, the standard DEA model, and the enhanced DEA model are explored.	• The empirical findings suggest that the improved DEA model offers a faster method for evaluating investment funds compared to the diversification DEA model, while also addressing the standard DEA model's limitation of overestimating portfolio efficiency.
Regret theory-based fuzzy multi-objective portfolio selection model Involving DEA cross-efficiency and higher moments	2021	[70]	This study presents a fuzzy multi-objective portfolio selection model based on regret theory that incorporates four key factors: mean, variance, skewness, and efficiency, while adhering to multiple realistic constraints. It allows investors to adjust their intrinsic behavioral tendencies, such as regret aversion and rejoice preference, to align with their actual preferences	• The developed model provides investors with comprehensive control over their priorities across all portfolio goals. • The study offers a comprehensive portfolio selection analysis from multi-objective and behavioral perspectives.
A credibilistic multiobjective multiperiod efficient portfolio selection approach using data envelopment analysis	2021	[94]	This article introduces a two-stage MPEPS technique, using trapezoidal fuzzy numbers for asset returns and credibility theory to handle fuzziness. Stage I employs relational network DEA to evaluate asset efficiency, and Stage II maximizes wealth and efficiency while minimizing CVaR, subject to various constraints.	• The proposed MPEPS approach is easy to understand, implement, and efficient for assets with positive returns, requiring less computation time. However, it fails when any asset's return rates are negative. • The approach is only valid in a credibilistic framework and cannot be applied in possibilistic, uncertain, intuitionistic fuzzy, or hesitant fuzzy environments.
Efficiency Evaluation of Assets and Optimal Portfolio Generation by Cross Efficiency and Cumulative Prospect Theory	2022	[169]	This research introduces a model for portfolio selection integrated with cross-efficiency and Cumulative Prospect Theory (CPT). The CPT-based model constructs efficient portfolios using the cross-efficiency matrix as the return matrix.	• This paper prioritizes the CPT value in portfolio construction to reflect the decision-maker's behavior, unlike the traditional approach. • The study reveals that portfolios constructed with the highest CPT values outperform those built with the lowest CPT values.
Efficiency Assessment of Mutual Fund With Risk and Negative Data: An Improved Two-Stage Network SBM Model	2024	[162]	This paper introduces an innovative improved two-phase network SBM model that can handle negative data and incorporates the CVaR indicator to represent risks in efficiency evaluation. The presented technique is applied to calculate the overall efficiency and subsystem efficiencies of 29 mutual funds in China from 2018 to 2022.	• Empirical results show that operational and portfolio management efficiencies significantly affect overall performance, with funds excelling in operational management more likely to achieve superior overall results.

Chapter 2

Cross-efficiency aggregation by ordered visibility graph averaging: method, and application in portfolio selection

In DEA, cross-efficiency ratings are often aggregated using the arithmetic average operator. Each DMU is thus accorded equal weight. This approach does not consider decision-maker preferences, giving all efficiency ratings equal weight, so it emphasizes self-evaluated efficiency because each DMU only has one self-evaluated score among many peer assessments [136, 178]. Thus, so many essential decision-making details are lost in the final aggregated cross-efficiency. The aim of this chapter is twofold. First it presents ¹ a novel application of the OVGA operator introduced by Jiang et al. [82] for DEA cross-efficiency aggregation, and then by employing it, a portfolio selection model in terms of an efficiency of an asset is presented here. The proposed OVGA aggregated cross-efficiency approach is explained through a numerical example, followed by the formulation of optimal portfolios based on these cross-efficiencies. The suggested method is also tested using empirical data from the Indian banking industry. The results of this study can be used to create the most acceptable portfolio in stock companies, financial institutions, and businesses in the public and private sectors.

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2.1 Introduction

The importance of aggregation methods in cross-efficiency assessment is widely recognized; however, there remain significant opportunities to improve these methods and their application in portfolio optimization. The first limitation of the arithmetic averaging approach is its disregard for the importance of self-efficiency. Each DMU has one self-efficiency value and $(n - 1)$ peer-efficiency values, yet the arithmetic average method assigns equal weight to all values. Secondly, this approach fails to consider the varying impacts of individual efficiency values on the aggregated results, neglecting their relative significance. To remedy these issues, Song et al. [167], and Li et al. [100] presented the aggregation methods that are motivated by the Shannon entropy weight and balanced adjustment weights, respectively. Chen et al. [32] addressed non-uniqueness in cross-efficiency assessments and introduced a meta-frontier analysis framework for a new cross-efficiency assessment method. In another approach, Oukil and Amin [138] derived the importance of each DMU from a common consensus among all DMUs, represented through a preference voting matrix. Here, self- and peer-evaluation scores reflect the mutual appreciation among DMUs, framing the cross-efficiency matrix as an election system where each DMU acts as a candidate and a voter. Oukil [137] proposed a novel approach leveraging the collective weight profile (CWP) for each DMU, utilizing a preference voting system embedded in the weight matrix. In this framework, the evaluating DMUs are considered voters, while the input/output factors serve as candidates. Liu and Chen [108] first applied regret theory to the cross-efficiency aggregation process to understand the regret aversion behavior of DMs, which plays a crucial role in the decision-making process. To determine the regret-based aggressive and benign cross-efficiency, weight matrices are built into their study. Wu et al. [192] suggested a common weight technique to aggregate the cross-efficiency by considering DMUs as players in a cooperative game sense and calculating their Shapley values to address these limitations. Li et al. [98] outlined a novel strategy for generating aggregation weights from subjective and objective viewpoints. To reflect DM preferences from a subjective standpoint, the authors established prospect theory. Their method

offers reference points as intervals that can be chosen based on the DMs' decision aims and preferences. Chen et al. [37] introduced a novel method based on prospect consensus (APC) theory for cross-efficiency aggregation. The APC method keeps as much of the cross-efficiency matrix's decision-making information as feasible and then aggregates the cross-efficiency that suits decision-making demands. Combining cumulative prospect theory, Fang and Yang [65] proposed an ordering method for interval cross-efficiency aggregation.

All the studies mentioned above have different methodologies, indicators, and characteristics, requiring deep knowledge to understand the various procedures. Additionally, the above-discussed methods require numerous parameters in the calculation process, which can be tedious to handle when working with large-scale data.

This study introduces an innovative cross-efficiency aggregation approach based on the OVGA operator, which facilitates a deeper analysis of DMUs. The OVGA operator determines weights by evaluating the significance of data within the visibility graph as pointed by Jiang et al. [82]. By assigning variable weights to cross-efficiency scores, this method ensures that DMUs are weighted according to the relative importance of their performance, providing a more nuanced and effective aggregation of cross-efficiency scores.

2.1.1 Research motivation and novelty of the proposed approach

Despite numerous studies exploring DEA cross-efficiency aggregation across different frameworks, there remains a gap for developing a novel aggregation method that relies solely on the cross-efficiency matrix while efficiently evaluating aggregated CE scores. Additionally, the application of DEA cross-efficiency in portfolio optimization problems is relatively underexplored. This study aims to address these gaps and makes the following key contributions:

- Our proposed OVGA method employs a unique approach by assigning distinct weights to cross-efficiencies. In the literature, the weights determined in the study of Wang et al. [178], León et al. [96] and Oukil [136] are based on the significance of the data values and are influenced by various factors, such as the decision maker's level of optimism and the orness degree of the data. The different orness degrees result in varying weight distributions, as

highlighted in [178] and [96].

- In contrast to other aggregation operators, the OVGA operator takes into account both the order of values and their interrelationships. A key advantage of OVGA is its data-driven nature, allowing the aggregation results to be directly derived from the original set of values.
- We introduce a novel application of the OVGA method for DEA cross-efficiency aggregation, distinct from the approaches presented by Jiang et al. [82], Jiang et al. [81], and Wang et al. [182].
- Furthermore, compared to Deng and Fang [51], Lim et al. [103], and Essid et al. [63], we integrate the MV-DEA cross-efficiency model with OVGA cross-efficiency scores and deal with the existence of an alternate optimal solution in cross-efficiency for portfolio optimization problems.

The remainder of this chapter unfolds as follows: Section 2.2 presents two DEA cross-efficiency models designed to address the problem of multiple optimal solutions. Section 2.3 begins with the terminology of the OVGA operator for cross-efficiency aggregation and then introduces the portfolio performance evaluation model. Section 2.4 uses a numerical dataset from the literature to demonstrate the applicability of the proposed model, followed by a case study of the Indian banking sector. Managerial implications of the proposed model are also discussed in this section. Finally, Section 2.5 presents some concluding remarks.

2.2 DEA cross-efficiency model to deal with alternative optima

Oral et al. [133] examine the issue of cross-evaluation within the DEA framework, highlighting the critical role of multiple optimal solutions. It demonstrates that the outcomes of traditional secondary goal DEA models can be enhanced by incorporating the most favorable individual evaluations into the cross-evaluation process. This enhancement is achieved through the introduction of a superior cross-evaluation DEA model, referred to as MRA.

Oral et al. [133], have shown that the cross-efficiency score of any DMU_j with respect to any DMU_k , $k = 1, 2, \dots, n$ can take any value between an interval. So

the minimum and the maximum cross-efficiency score of DMU_j , evaluated by DMU_k , $k = 1, 2, \dots, n$ can be computed by those optimal weights for which the self-efficiency score value will remain unchanged. They introduced two formulations, Models 2.1 and 2.2, to determine the cross-efficiency interval of DMU_j as assessed by DMU_k , where $k = 1, 2, \dots, n$.

Model (2.1)

$$\begin{aligned} \min \quad & \hat{e}_{kj}^L = \sum_{r=1}^s \mu_{rk} y_{rj} \\ \text{subject to,} \quad & \sum_{i=1}^m \omega_{ik} x_{ij} = 1 \\ & \sum_{r=1}^s \mu_{rk} y_{rj} - \sum_{i=1}^m \omega_{ik} x_{ij} \leq 0, \quad j = 1, 2, \dots, n, j \neq k \\ & \sum_{r=1}^s \mu_{rk} y_{rk} - e_{kk}^* \sum_{i=1}^m \omega_{ik} x_{ik} = 0, \\ & \mu_{rk}, \omega_{ik} \geq 0, \quad r = 1, 2, \dots, s; i = 1, 2, \dots, m, \end{aligned}$$

Let μ_{rk}^* , $r = 1, 2, \dots, s$ and ω_{ik}^* , $i = 1, 2, \dots, m$ be the optimal solutions of Model (2.1), then $\hat{e}_{kj}^L = \sum_{r=1}^s \mu_{rk}^* y_{rj}$ is the optimal efficiency that DMU_j can attain when evaluated by DMU_k . We can find the min cross-efficiency matrix for n DMUs corresponding to Model (2.1). The n efficiency values constitute a cross-efficiency matrix, as shown in Table 2.1. Similarly, the maximum cross-efficiency matrix for n DMUs can be obtained using the following Model (2.2).

Model (2.2)

$$\begin{aligned} \max \quad & \hat{e}_{kj}^U = \sum_{r=1}^s \mu_{rk} y_{rj} \\ \text{subject to,} \quad & \sum_{i=1}^m \omega_{ik} x_{ij} = 1 \\ & \sum_{r=1}^s \mu_{rk} y_{rj} - \sum_{i=1}^m \omega_{ik} x_{ij} \leq 0, \quad j = 1, 2, \dots, n, j \neq k \\ & \sum_{r=1}^s \mu_{rk} y_{rk} - e_{kk}^* \sum_{i=1}^m \omega_{ik} x_{ik} = 0, \\ & \mu_{rk}, \omega_{ik} \geq 0, \quad r = 1, 2, \dots, s; i = 1, 2, \dots, m, \end{aligned}$$

Let μ_{rk}^* , $r = 1, 2, \dots, s$ and ω_{ik}^* , $i = 1, 2, \dots, m$ be the optimal solutions of Model (2.2), then $\hat{e}_{kj}^U = \sum_{r=1}^s \mu_{rk}^* y_{rj}$ is the optimal efficiency that DMU_j can attain when evaluated

by DMU_k .

Table 2.1: Cross efficiency matrix for n DMUs corresponding to Model (2.1)

Target DMU_K	Cross Efficiency of DMU_j					
	1	2	...	1	...	n
1	$\hat{e}_{11}^L$	$\hat{e}_{12}^L$	...	$\hat{e}_{1l}^L$	...	$\hat{e}_{1n}^L$
2	$\hat{e}_{21}^L$	$\hat{e}_{22}^L$	...	$\hat{e}_{2l}^L$	...	$\hat{e}_{2n}^L$
3	$\hat{e}_{31}^L$	$\hat{e}_{32}^L$	...	$\hat{e}_{3l}^L$	...	$\hat{e}_{3n}^L$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\hat{e}_{n1}^L$	$\hat{e}_{n2}^L$	...	$\hat{e}_{nl}^L$	...	$\hat{e}_{nn}^L$
Average cross efficiency	$\frac{\sum_{k=1}^n \hat{e}_{k1}^L}{n}$	$\frac{\sum_{k=1}^n \hat{e}_{k2}^L}{n}$	...	$\frac{\sum_{k=1}^n \hat{e}_{kl}^L}{n}$	...	$\frac{\sum_{k=1}^n \hat{e}_{kn}^L}{n}$

Model (2.1) is designed to find the worst cross-efficiency score for DMU_k across all possible optimal solutions, whereas Model (2.2) focuses on determining the best cross-efficiency score for DMU_k .

Remark 2.2.1. The alternative optimal solutions of Model (1.1) corresponding to DMU_k are utilized to solve models (2.1) and (2.2). It is obvious that $\hat{e}_{kj}^L < \hat{e}_{kj}^U$ only holds if the corresponding self-evaluation DEA model has different optimal solutions. Oral et al. [133] have stated that models (2.1) and (2.2) take into consideration the issue of multiple optimal solutions from the CCR modeling of cross-efficiency. The results can be greatly impacted by selecting the values from these intervals. One can select extreme cross-efficiency (min cross-efficiency or max cross-efficiency) for pessimistic and optimistic decisions. We investigate the results of the OVGA aggregated min cross-efficiency score in this work.

The most popular method to determine a DMU's overall performance measurement as the final cross-efficiency score is to average the efficiency of each column in the cross-efficiency matrix with equal weights. Assigning equal weights for cross-efficiency aggregation is not ideal, as the self-evaluated efficiency score contributes less than the peer-evaluated efficiencies. In the following subsection, we outline the detailed steps for applying the Ordered Visibility Graph Averaging

operator to aggregate an ordered dataset.

2.3 Portfolio optimization model based on the cross-efficiency matrix: Proposed model

Lim et al. [103] studied the MV portfolio selection model based on the efficiencies of their assets and considered return and risk aspects of the MV model in terms of mean efficiencies and the covariance between the cross-efficiency scores of all DMU_s respectively. Before introducing the suggested model, it is necessary to comprehend the terminology from the previous part in terms of DMU efficiency.

Definition 2.3.1 (Adjacency matrix and degree of DMU_p with respect to DMU_j corresponding to min-cross-efficiency). Let $\hat{E}_j^L = \{\hat{e}_{1j}^L, \hat{e}_{2j}^L, \dots, \hat{e}_{pj}^L, \dots, \hat{e}_{nj}^L\}$ be the set of n ordered cross-efficiencies of a DMU_j , where $\hat{e}_{ij}^L$ is the i^{th} largest element of the set $\hat{E}_j^L$. Let $\{(1, \hat{e}_{1j}^L), (2, \hat{e}_{2j}^L), \dots, (p, \hat{e}_{pj}^L), \dots, (n, \hat{e}_{nj}^L)\}$ be the corresponding set of n ordered data nodes. The adjacency matrix for DMU_j , denoted by $A_j^L = [a_{pq}^j]_{n \times n}$ on the basis of visibility between DMU_p , $p = 1, 2, \dots, n$ and DMU_q , $q = 1, 2, \dots, n$ becomes as follows

$$a_{pq}^j = \begin{cases} 1, & \text{if there is visibility between the data values } (p, \hat{e}_{pj}^L) \text{ and } (q, \hat{e}_{qj}^L), p \neq q \\ 0, & \text{otherwise.} \end{cases}$$

Let d_p^j be the degree of DMU_p under the consideration of DMU_j , then $d_p^j = \sum_q a_{pq}^j$

Definition 2.3.2 (OVGA min-efficiency of DMU_j). Let $\hat{E}_j^L = \{\hat{e}_{1j}^L, \hat{e}_{2j}^L, \dots, \hat{e}_{kj}^L, \dots, \hat{e}_{nj}^L\}$ be the set of n ordered min-cross-efficiencies of a DMU_j , where $\hat{e}_{ij}^L$ is the i^{th} largest element of the set $\hat{E}_j^L$. Then OVGA min-efficiency of DMU_j , denoted by $\tilde{e}_j^L$ is defined as

$$\tilde{e}_j^L = w_{1j}\hat{e}_{1j}^L + w_{2j}\hat{e}_{2j}^L + \dots w_{pj}\hat{e}_{pj}^L + \dots w_{nj}\hat{e}_{nj}^L,$$

where,

$$w_{pj} = \frac{d_p^j}{\sum_{p=1}^n d_p^j}, p = 1, 2, \dots, n.$$

Definition 2.3.3 (Average of min-cross efficiency). Let $\hat{E}_j^L = \{\hat{e}_{1j}^L, \hat{e}_{2j}^L, \dots, \hat{e}_{kj}^L, \dots, \hat{e}_{nj}^L\}$ be the set of n cross-efficiencies of a DMU_j , where $\hat{e}_{kj}^L$ is the min-cross-efficiency of DMU_j

evaluated with respect to DMU_k . Then the average of min-cross-efficiencies of DMU_j , denoted by $\bar{e}_j$ is defined as

$$\bar{e}_j^L = \frac{\hat{e}_{1j}^L + \hat{e}_{2j}^L + \dots, \hat{e}_{kj}^L + \dots \hat{e}_{nj}^L}{n}$$

Definition 2.3.4 (Adjacency matrix and degree of DMU_p with respect to DMU_j corresponding to max-cross-efficiency). Let $\hat{E}_j^U = \{\hat{e}_{1j}^U, \hat{e}_{2j}^U, \dots, \hat{e}_{pj}^U, \dots, \hat{e}_{nj}^U\}$ be the set of n ordered max-cross-efficiencies of a DMU_j , where $\hat{e}_{ij}^U$ is the i^{th} largest element of the set $\hat{E}_j^U$. Let $\{(1, \hat{e}_{1j}^U), (2, \hat{e}_{2j}^U), \dots, (p, \hat{e}_{pj}^U), \dots, (n, \hat{e}_{nj}^U)\}$ be the corresponding set of n ordered data nodes. The adjacency matrix for DMU_j , denoted by $B_j^U = [b_{pq}^j]_{n \times n}$ on the basis of visibility between DMU_p , $p = 1, 2, \dots, n$ and DMU_q , $q = 1, 2, \dots, n$ becomes as follows

$$b_{pq}^j = \begin{cases} 1, & \text{if there is visibility between the data values } (p, \hat{e}_{pj}^U) \text{ and } (q, \hat{e}_{qj}^U), p \neq q \\ 0, & \text{otherwise.} \end{cases}$$

Let c_p^j be the degree of DMU_p under the consideration of DMU_j , then $c_p^j = \sum_q b_{pq}^j$

Definition 2.3.5 (OVGA max-efficiency of DMU_j). Let $\hat{E}_j^U = \{\hat{e}_{1j}^U, \hat{e}_{2j}^U, \dots, \hat{e}_{kj}^U, \dots, \hat{e}_{nj}^U\}$ be the set of n ordered max-cross-efficiencies of a DMU_j , where $\hat{e}_{ij}^U$ is the i^{th} largest element of the set $\hat{E}_j^U$. Then OVGA max-efficiency of DMU_j , denoted by $\tilde{e}_j^U$ and is defined as

$$\tilde{e}_j^U = w_{1j} \hat{e}_{1j}^U + w_{2j} \hat{e}_{2j}^U + \dots w_{pj} \hat{e}_{pj}^U + \dots w_{nj} \hat{e}_{nj}^U,$$

where,

$$w_{pj} = \frac{c_p^j}{\sum_{p=1}^n c_p^j}, p = 1, 2, \dots, n.$$

Definition 2.3.6 (Average of max-cross efficiency). Let $\hat{E}_j^U = \{\hat{e}_{1j}^U, \hat{e}_{2j}^U, \dots, \hat{e}_{kj}^U, \dots, \hat{e}_{nj}^U\}$ be the set of n cross-efficiencies of a DMU_j , where $\hat{e}_{kj}^U$ is the max-cross-efficiency of DMU_j evaluated with respect to DMU_k . Then the average of max-cross-efficiency of DMU_j , denoted by $\bar{e}_j^U$ is defined as

$$\bar{e}_j^U = \frac{\hat{e}_{1j}^U + \hat{e}_{2j}^U + \dots, \hat{e}_{kj}^U + \dots \hat{e}_{nj}^U}{n}$$

Remark 2.3.1. In the visibility graph, the weight of each node is determined by the degree distribution. Thus, if a node links with more nodes than others, its degree will be greater,

and it will be more significant. The following are the advantages of the OVGA operators over other methods for cross-efficiency aggregation:

1. In the OWA method, we obtain different aggregated results for different optimism levels and it is very difficult for decision-makers to select which value to make final decisions.
2. Compared with the classical OWA operator, the proposed OVGA operator not only takes the information of orders but also the argument values themselves into consideration to determine the weights.
3. Other cross-efficiency aggregation methods, such as prospect and regret theory, require many calculation parameters that will be very difficult to handle and will be very calculative for the large-scale data set. Prospect theory cross-efficiency aggregation requires at least four parameters, including risk attitude coefficients, the concave degree of the value function within the gain/loss region, and a psychological preference value. Similarly, the regret theory cross-efficiency aggregation method utilizes the regret-rejoice function based on the regret aversion coefficient.
4. The primary benefit of the OVGA operator is that it is based on the idea of human intuition. In simpler terms, a vertex with the highest degree can be considered a popular person who knows many people and has a lot of influence. Their connections and interactions provide valuable insights into the relationships and dynamics of the time series data.

Portfolio selection model corresponding to min cross-efficiency

One of the key challenges of the DEA cross-efficiency approach in portfolio selection is its tendency to assign equal weights to all DMUs or stocks in the optimal portfolio, which may not reflect their relative importance. To overcome this limitation, we proposed a portfolio selection model as follows.

Let us consider a portfolio Ω constituted with n DMUs say DMU_j , $j = 1, 2, \dots, n$. Let $[\hat{e}_{kj}^L]_{n \times n}$ be the min-cross-efficiency matrix on n DMUs taken under consideration for portfolio selection. Let w_j be the proportion of the total efficiency invested in DMU_j . The portfolio Ω of n DMUs by the OVGA operator is obtained by solving the following cross-efficiency portfolio optimization model:

Model (2.3)

$$\begin{aligned}
\min \quad & V_{\Omega} = \sum_{j=1}^n \sum_{d=1}^n w_d w_j \tilde{\sigma}_{dj}^L \\
\text{subject to,} \quad & E_{\Omega} = \sum_{j=1}^n w_j \tilde{e}_j^L \geq (1 - \gamma) \tilde{E}_{\Omega}^b \\
& \sum_{j=1}^n w_j = 1, \\
& \sum_{j=1}^n z_j = h, \quad j = 1, 2, \dots, n, \\
& \epsilon_j z_j \leq w_j \leq \delta_j z_j, \quad j = 1, 2, \dots, n, \\
& z_j \in \{0, 1\}, \quad j = 1, 2, \dots, n, \quad w_j \geq 0, \quad j = 1, 2, \dots, n,
\end{aligned}$$

where, $\tilde{e}_j^L$ is the mean cross- efficiency score of DMU_j by OVGA operator, defined in Definition 2.3.2 and $\tilde{\sigma}_{dj}^L = \frac{1}{n} \sum_{k=1}^n (\hat{e}_{kj}^L - \tilde{e}_d^L)(\hat{e}_{kj}^L - \tilde{e}_j^L)$ is the covariance between the cross-efficiency scores of DMU_d and DMU_j . Here, γ is the trade-off parameter between mean efficiency and covariance of their efficiency. Further, for a better selection of a DMU, the budget constraint $\sum_{j=1}^n w_j = 1$, cardinality constraint $\sum_{j=1}^n z_j = h$, $z_j \in \{0, 1\}$, $j = 1, 2, \dots, n$, buy-in threshold $\epsilon_j z_j \leq w_j \leq \delta_j z_j$, $j = 1, 2, \dots, n$, and no short-selling criteria $w_j \geq 0$, $j = 1, 2, \dots, n$, are also considered here. Here, E_{Ω}^b is the maximum OVGA efficiency achievable by the portfolio under the budget constraint, cardinality constraint, buy-in threshold, and no short-selling criteria.

Let $\tilde{E}_{\Omega}^*$ and $\tilde{V}_{\Omega}^*$ be the optimal mean efficiency and optimal risk corresponding to the optimal portfolio Ω^* so obtained.

Similarly, the mean-variance cross-efficiency model for simple average efficiency as defined in Definition 2.3.3 is as follows:

Model (2.4)

$$\min \quad V_{\Omega} = \sum_{j=1}^n \sum_{d=1}^n w_d w_j \bar{\sigma}_{dj}^L$$

subject to,

$$\sum_{j=1}^n w_j \bar{e}_j^L \geq (1 - \gamma) \bar{E}_{\Omega}^b$$

$$\sum_{j=1}^n w_j = 1$$

$$\sum_{j=1}^n z_j = h, \quad j = 1, 2, \dots, n,$$

$$\epsilon_j z_j \leq w_j \leq \delta_j z_j, \quad j = 1, 2, \dots, n,$$

$$z_j \in \{0, 1\}, \quad j = 1, 2, \dots, n, \quad w_j \geq 0, \quad j = 1, 2, \dots, n,$$

where, $\bar{e}_j^L$ is the mean cross- efficiency score of DMU_j defined in Definition 2.3.3 and $\bar{\sigma}_{dj}^L = \frac{1}{n} \sum_{k=1}^n (e_{kj}^L - \bar{e}_d^L)(e_{kj}^L - \bar{e}_j^L)$ is the covariance between the cross efficiency scores of DMU_d and DMU_j . E_{Ω}^b is the maximum mean efficiency, achievable by the portfolio E_{Ω} . Further, γ is the trade-off parameter between mean efficiency and covariance of their efficiency. Further, other constraints have the same interpretation as explained in Model (2.3).

Remark 2.3.2. Now on the lines of Definitions 2.3.4, 2.3.5, and 2.3.6, we can extend Model (2.1) and Model (2.2) to study the portfolio selection for max-cross-efficiency of DMUs.

Remark 2.3.3. It is imperative to bear in mind that the selection of the Minimum or Maximum cross-efficiencies, or the extraction of efficiencies from the interval between the min and max cross-efficiencies, will have a significant influence on the portfolio selection outcomes. In this work, we solely address the ideal portfolio derived from model (2.1). We can explore the OVGA max cross-efficiency and its impact on portfolio outcomes more thoroughly in future studies.

2.4 Numerical Illustration

In this section, we apply our proposed method to a numerical example consisting of 22 DMUs to demonstrate the empirical effectiveness of the method. The numerical dataset is taken from Lim et al. [103] which was further analyzed by Amin and Hajjani [4]. The input and output data for the DMUs are presented in Table 2.2.

Table 2.2: Input and output data values for DMUs

DMUs	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Input 1	5	6	4	8	5	8	4.4	2.6	3.4	3.6	2	3	3	2.6	4	5	6	4	7	6	8	9
Input 2	4	5	5	5	6	3	4.4	8	8	4.4	7	7	5.6	5	4	3.2	4	3.5	3	2.5	2	2
Output 1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

On solving the Model (1.2), for DMU_1 , we get $\mu_{11}^* = 0.85$, $\omega_{11}^* = 0.0769$, $\omega_{21}^* = 0.1538$, with the optimal value of the objective function as $e_{11}^* = 0.85$. Similarly on solving Model (1.2), for other $DMUs$, the self-efficiency of respective $DMUs$ can be acquired as given in Table 2.3.

Table 2.3: Self-efficiency score e_{kk}^* of DMU_k

k	1	2	3	4	5	6	7	8	9	10	11
e_{kk}^*	0.85	0.69	0.84	0.61	0.69	0.80	0.85	0.82	0.71	0.94	1
k	12	13	14	15	16	17	18	19	20	21	22
e_{kk}^*	0.80	0.88	1	0.94	0.96	0.79	1	0.85	1	1	1

For cross-efficiency interval evaluation of DMU_j , we have to solve Model (2.1) and Model (2.2) respectively for different values of k for a fixed value of j . For example if we take $k = 2$ and $j = 1$, we can find $\tilde{e}_{21}^L$ and $\bar{e}_{21}^U$. As in the cross-efficiency matrix, DMU_j is evaluated by DMU_k , so to determine the weights given by DMU_j to DMU_k to find out its aggregated efficiency by the OVGA operator, we need re-ordered minimum cross-efficiency matrix.

To get more clarity, let us calculate the weights given by DMU_1 to DMU_k , $k = 1, 2, \dots, n$ for the OVGA operator to get its aggregated efficiency. Similarly, we can find out the weights given by other $DMUs$ to DMU_k , $k = 1, 2, \dots, n$. The set of ordered cross-efficiencies of DMU_1 , $\hat{e}_{k1}^L$, $k = 1, 2, \dots, 22$ (as per Table 2.6) for

To calculate the weights given by DMU_1 to DMU_k , $k = 1, 2, \dots, 22$, for evaluating

its OVGA efficiency, the given Fig. 2.1 depicts the ordered visibility graph of all $DMUs$ with respect to ordered cross efficiency set $\hat{E}_1^L$. Following the approach outlined in Definition 2.3.1, the corresponding adjacency matrix A derived from Algorithm 2.2.1 is given. The weights are depicted in Table 2.4.

Table 2.4: Degree and weights of DMU_p w.r.t. DMU_1

p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
d_p^1	1	2	2	2	2	8	2	3	3	3	3	3	6	2	6	2	3	4	5	2	3	1
w_p^1	$\frac{1}{68}$	$\frac{2}{68}$	$\frac{2}{68}$	$\frac{2}{68}$	$\frac{2}{68}$	$\frac{8}{68}$	$\frac{2}{68}$	$\frac{3}{68}$	$\frac{3}{68}$	$\frac{3}{68}$	$\frac{3}{68}$	$\frac{3}{68}$	$\frac{6}{68}$	$\frac{2}{68}$	$\frac{6}{68}$	$\frac{2}{68}$	$\frac{3}{68}$	$\frac{4}{68}$	$\frac{5}{68}$	$\frac{2}{68}$	$\frac{3}{68}$	$\frac{1}{68}$

Hence, the aggregated cross-efficiency using the OVGA operator by simple average is $\bar{e}_1^L = \sum_{p=1}^{22} w_p^1 \hat{e}_{p1}^L = 0.76$ and $\bar{e}_1^L = \sum_{p=1}^{22} (1/n) \hat{e}_{p1}^L = 0.75$ respectively. These values are the same as given in Table 2.8, and similarly, we can calculate the aggregated cross-efficiency scores of other DMUs. For this, we first need to solve Model (2.1), for $k, j = 1, 2, \dots, n$ the minimum cross-efficiency matrix is depicted in Table 2.5. Further, Table 2.6 presents the re-ordered cross-efficiency matrix. As per the requirement of the calculation Table 2.7 illustrates the weights given by DMU_j to DMU_p for $j, p = 1, 2, \dots, 22$.

Table 2.8: The aggregated minimum cross-efficiency score by simple average operator $\bar{e}_j^L$ and by OVGA operator $\tilde{e}_j^L$ of target DMU_j

j	1	2	3	4	5	6	7	8	9	10	11
$\bar{e}_j^L$	0.75	0.61	0.73	0.53	0.6	0.66	0.76	0.63	0.58	0.83	0.75
$\tilde{e}_j^L$	0.76	0.62	0.74	0.54	0.60	0.69	0.76	0.66	0.60	0.82	0.77
j	12	13	14	15	16	17	18	19	20	21	22
$\bar{e}_j^L$	0.66	0.77	0.87	0.83	0.83	0.68	0.89	0.71	0.84	0.76	0.7
$\tilde{e}_j^L$	0.68	0.77	0.87	0.84	0.85	0.70	0.88	0.72	0.85	0.80	0.76

Table 2.5: The cross-efficiency matrix $\tilde{e}_{kj}^L$

$\begin{matrix} j \\ k \end{matrix}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
2	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
3	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
4	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
5	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
6	0.76	0.62	0.67	0.57	0.55	0.80	0.73	0.46	0.45	0.75	0.53	0.52	0.63	0.71	0.80	0.90	0.73	0.89	0.84	1.00	1.00	0.94
7	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
8	0.66	0.55	0.75	0.43	0.60	0.46	0.72	0.82	0.71	0.83	1.00	0.80	0.88	1.00	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
9	0.66	0.55	0.75	0.43	0.60	0.46	0.72	0.82	0.71	0.83	1.00	0.80	0.88	1.00	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
10	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
11	0.40	0.33	0.50	0.28	0.40	0.28	0.45	0.77	0.59	0.56	1.00	0.67	0.67	0.77	0.50	0.40	0.33	0.50	0.29	0.33	0.25	0.22
12	0.66	0.55	0.75	0.43	0.60	0.46	0.72	0.82	0.71	0.83	1.00	0.80	0.88	1.00	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
13	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
14	0.66	0.55	0.75	0.43	0.60	0.46	0.72	0.82	0.71	0.83	0.85	0.76	0.88	1.00	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
15	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
16	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
17	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
18	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.91	0.75	1.00	0.74	0.87	0.74	0.67
19	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
20	0.76	0.62	0.67	0.57	0.55	0.79	0.73	0.46	0.45	0.75	0.53	0.52	0.63	0.71	0.80	0.90	0.73	0.89	0.84	1.00	0.92	0.85
21	0.50	0.40	0.40	0.40	0.33	0.67	0.45	0.25	0.25	0.45	0.29	0.29	0.36	0.40	0.50	0.63	0.50	0.67	0.67	0.80	1.00	0.94
22	0.50	0.40	0.40	0.40	0.33	0.67	0.45	0.25	0.25	0.45	0.29	0.29	0.36	0.40	0.50	0.63	0.50	0.57	0.67	0.80	1.00	1.00

Table 2.6: Re-ordered cross-efficiency matrix $\tilde{e}_{kj}^L$

$\begin{matrix} j \\ \backslash k \end{matrix}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	0.85	0.69	0.84	0.61	0.69	0.80	0.85	0.82	0.71	0.94	1	0.80	0.88	1	0.94	0.96	0.79	1	0.85	1	1	1
2	0.85	0.69	0.84	0.61	0.69	0.79	0.85	0.82	0.71	0.94	1	0.80	0.88	1	0.94	0.96	0.79	1	0.85	1	1	0.94
3	0.85	0.69	0.84	0.61	0.69	0.79	0.85	0.82	0.71	0.94	1	0.8	0.88	1	0.94	0.96	0.79	1	0.85	1	1	0.94
4	0.85	0.69	0.84	0.61	0.69	0.79	0.85	0.77	0.67	0.94	1	0.76	0.88	1	0.94	0.96	0.79	1	0.85	1	0.92	0.85
5	0.85	0.69	0.84	0.61	0.69	0.79	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1	0.94	0.96	0.79	1	0.85	1	0.92	0.85
6	0.85	0.69	0.84	0.61	0.69	0.79	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1	0.94	0.96	0.79	1	0.85	1	0.92	0.85
7	0.83	0.68	0.79	0.57	0.65	0.79	0.83	0.72	0.67	0.89	0.85	0.76	0.88	1	0.92	0.91	0.75	1	0.84	1	0.92	0.85
8	0.83	0.68	0.79	0.57	0.65	0.79	0.83	0.72	0.67	0.89	0.85	0.76	0.88	1	0.92	0.91	0.75	1	0.84	1	0.92	0.85
9	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.72	0.67	0.89	0.85	0.76	0.88	1	0.92	0.91	0.75	1	0.74	0.87	0.92	0.85
10	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.72	0.67	0.89	0.85	0.76	0.88	1	0.92	0.91	0.75	1	0.74	0.87	0.92	0.85
11	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.72	0.59	0.89	0.85	0.67	0.77	0.87	0.92	0.91	0.75	1	0.74	0.87	0.74	0.67
12	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.91	0.75	1	0.74	0.87	0.74	0.67
13	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.91	0.75	1	0.74	0.87	0.74	0.67
14	0.76	0.62	0.75	0.57	0.6	0.67	0.73	0.59	0.57	0.83	0.69	0.65	0.77	0.87	0.8	0.9	0.73	0.89	0.74	0.87	0.74	0.67
15	0.76	0.62	0.75	0.57	0.6	0.67	0.73	0.59	0.57	0.83	0.69	0.65	0.77	0.87	0.8	0.9	0.73	0.89	0.74	0.87	0.74	0.67
16	0.66	0.55	0.75	0.43	0.6	0.67	0.72	0.59	0.57	0.83	0.69	0.65	0.77	0.87	0.79	0.69	0.57	0.81	0.67	0.8	0.74	0.67
17	0.66	0.55	0.75	0.43	0.6	0.67	0.72	0.59	0.57	0.83	0.69	0.65	0.77	0.87	0.79	0.69	0.57	0.81	0.67	0.8	0.74	0.67
18	0.66	0.55	0.67	0.43	0.55	0.46	0.72	0.59	0.57	0.75	0.69	0.65	0.67	0.77	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
19	0.66	0.55	0.67	0.43	0.55	0.46	0.72	0.46	0.45	0.75	0.53	0.52	0.63	0.71	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
20	0.5	0.4	0.5	0.4	0.4	0.46	0.45	0.46	0.45	0.56	0.53	0.52	0.63	0.71	0.5	0.63	0.5	0.57	0.52	0.61	0.48	0.43
21	0.5	0.4	0.4	0.4	0.33	0.46	0.45	0.25	0.25	0.45	0.29	0.29	0.36	0.4	0.5	0.63	0.5	0.57	0.52	0.61	0.48	0.43
22	0.4	0.33	0.4	0.25	0.33	0.25	0.45	0.25	0.25	0.45	0.29	0.29	0.36	0.4	0.5	0.4	0.33	0.5	0.29	0.33	0.25	0.22
$\bar{e}_j$	0.75	0.61	0.73	0.53	0.60	0.66	0.76	0.63	0.58	0.83	0.75	0.66	0.77	0.87	0.83	0.83	0.68	0.89	0.71	0.84	0.76	0.70

Table 2.7: The weight w_p^j given by DMU_j to DMU_p on the basis of re-ordered cross-efficiency matrix $\hat{e}_{pj}^L$

p	w_p^1	w_p^2	w_p^3	w_p^4	w_p^5	w_p^6	w_p^7	w_p^8	w_p^9	w_p^{10}	w_p^{11}	w_p^{12}	w_p^{13}	w_p^{14}	w_p^{15}	w_p^{16}	w_p^{17}	w_p^{18}	w_p^{19}	w_p^{20}	w_p^{21}	w_p^{22}
1	0.015	0.014	0.015	0.014	0.015	0.092	0.014	0.012	0.012	0.015	0.014	0.012	0.016	0.016	0.014	0.015	0.015	0.017	0.016	0.016	0.014	0.073
2	0.029	0.029	0.029	0.029	0.029	0.026	0.027	0.024	0.024	0.029	0.029	0.024	0.032	0.032	0.027	0.029	0.030	0.033	0.031	0.032	0.028	0.024
3	0.029	0.029	0.029	0.029	0.029	0.039	0.027	0.098	0.095	0.029	0.029	0.095	0.032	0.032	0.027	0.029	0.030	0.033	0.031	0.032	0.111	0.110
4	0.029	0.029	0.029	0.029	0.029	0.039	0.027	0.098	0.024	0.029	0.114	0.024	0.032	0.032	0.027	0.029	0.030	0.033	0.031	0.032	0.028	0.024
5	0.029	0.029	0.029	0.029	0.029	0.039	0.027	0.024	0.036	0.029	0.029	0.036	0.032	0.032	0.027	0.029	0.030	0.033	0.031	0.032	0.042	0.037
6	0.118	0.114	0.118	0.143	0.118	0.039	0.108	0.049	0.036	0.118	0.043	0.036	0.032	0.032	0.108	0.132	0.121	0.033	0.047	0.032	0.042	0.037
7	0.029	0.029	0.029	0.029	0.029	0.039	0.027	0.049	0.036	0.029	0.043	0.036	0.032	0.032	0.027	0.029	0.030	0.033	0.031	0.032	0.042	0.049
8	0.044	0.043	0.044	0.043	0.044	0.145	0.041	0.049	0.036	0.044	0.043	0.036	0.032	0.032	0.041	0.044	0.045	0.033	0.141	0.129	0.042	0.049
9	0.044	0.043	0.044	0.043	0.044	0.026	0.041	0.049	0.036	0.044	0.043	0.036	0.032	0.032	0.041	0.044	0.045	0.033	0.031	0.032	0.042	0.049
10	0.044	0.043	0.044	0.043	0.044	0.039	0.041	0.049	0.119	0.044	0.043	0.119	0.129	0.129	0.041	0.044	0.045	0.033	0.047	0.048	0.125	0.122
11	0.044	0.043	0.044	0.043	0.044	0.039	0.041	0.122	0.095	0.044	0.129	0.095	0.032	0.032	0.041	0.044	0.045	0.033	0.047	0.048	0.028	0.024
12	0.044	0.043	0.044	0.043	0.044	0.039	0.041	0.024	0.036	0.044	0.029	0.036	0.048	0.048	0.041	0.044	0.045	0.033	0.047	0.048	0.042	0.037
13	0.088	0.086	0.088	0.043	0.088	0.039	0.108	0.037	0.048	0.088	0.043	0.048	0.048	0.048	0.108	0.059	0.061	0.100	0.047	0.048	0.042	0.037
14	0.029	0.029	0.029	0.043	0.029	0.039	0.027	0.037	0.048	0.029	0.043	0.048	0.048	0.048	0.027	0.029	0.030	0.033	0.047	0.048	0.042	0.037
15	0.088	0.086	0.044	0.114	0.044	0.039	0.081	0.037	0.048	0.044	0.043	0.048	0.048	0.048	0.081	0.118	0.106	0.100	0.063	0.065	0.042	0.037
16	0.029	0.029	0.044	0.029	0.044	0.039	0.041	0.037	0.048	0.044	0.043	0.048	0.048	0.048	0.041	0.029	0.030	0.033	0.031	0.032	0.042	0.037
17	0.044	0.043	0.059	0.043	0.059	0.079	0.054	0.037	0.048	0.059	0.043	0.048	0.081	0.081	0.054	0.044	0.045	0.067	0.094	0.097	0.083	0.073
18	0.059	0.057	0.029	0.043	0.029	0.026	0.054	0.049	0.060	0.029	0.057	0.060	0.048	0.048	0.054	0.044	0.045	0.067	0.031	0.032	0.028	0.024
19	0.074	0.086	0.074	0.057	0.074	0.039	0.081	0.024	0.024	0.074	0.029	0.024	0.048	0.048	0.081	0.059	0.061	0.100	0.047	0.048	0.042	0.037
20	0.029	0.029	0.044	0.043	0.044	0.039	0.027	0.049	0.048	0.044	0.057	0.048	0.081	0.081	0.027	0.029	0.030	0.033	0.047	0.048	0.042	0.037
21	0.044	0.043	0.044	0.057	0.044	0.039	0.041	0.024	0.024	0.044	0.029	0.024	0.032	0.032	0.041	0.059	0.061	0.050	0.047	0.048	0.042	0.037
22	0.015	0.029	0.044	0.014	0.044	0.013	0.027	0.024	0.024	0.044	0.029	0.024	0.032	0.032	0.027	0.015	0.015	0.033	0.016	0.016	0.014	0.012

Portfolio selection

Here, optimal portfolios are selected in two ways:

Case I: The portfolio selection is performed with equal weights

The decision-maker aims to allocate investments equally across DMUs in the optimal portfolios. To highlight the benefits of the proposed portfolio selection model, portfolios are generated using the model under various cardinality constraints: $h = 5$, $h = 10$, and $h = 15$. The resulting portfolio strategies are presented in Table 2.9. For comparison, portfolios derived from the traditional method (Model (2.4)) under the same cardinality constraints are provided in Table 2.10.

Notably, the proposed model significantly reduces variance (risk) with only a minor decrease in mean returns. For instance, the portfolio generated by the proposed method (Table 2.9) for $h = 5$ and $\gamma = 0.1$ shows a 1.73% reduction in mean compared to a substantial 33.4% decrease in variance.

Case II: Optimal portfolio selection with unequal weights

Different portfolios are generated by varying the parameters h , γ , ϵ_j , and δ_j . The optimal portfolio strategies derived from the proposed model (Model (2.3)) are shown in Table 2.11, while those obtained using the traditional method (Model (2.4)) are presented in Table 2.12.

As noted earlier, the proposed model significantly reduces variance (risk) with only a slight decrease in mean returns. For instance, the portfolio generated by the proposed model (Table 2.11) for $h = 5$, $\gamma = 0.1$, $\epsilon_j = 0.03$, and $\delta_j = 0.4$ demonstrates a 1.1% reduction in mean compared to a substantial 27.5% decrease in variance.

It is also worth mentioning that larger values of γ can lead to a more pronounced reduction in the risk of a given expected return. The effectiveness of the proposed approach for portfolio selection is supported by the empirical results, demonstrating that the model can generate portfolios tailored to an investor's preferences for γ .

Remark 2.4.1. For risk-averse investors, Fig.2.2 shows that the return produced from the suggested model is higher than the standard model already present in the literature.

In the following section, we illustrate the advantages of the proposed method through a real-world application example.

Table 2.9: The optimal portfolio strategies corresponding to Model (2.3) for equal weights

DMU_k	$\gamma = 0.1$			$\gamma = 0.2$		
	$\epsilon_j = \delta_j = 0.2, h=5$	$\epsilon_j = \delta_j = 0.1, h=10$	$\epsilon_j = \delta_j = 0.1429, h=7$	$\epsilon_j = \delta_j = 0.2, h=5$	$\epsilon_j = \delta_j = 0.1, h=10$	$\epsilon_j = \delta_j = 0.1429, h=7$
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0.10	0
5	0	0	0	0	0	0
6	0	0	0	0	0.10	0.1429
7	0	0	0	0	0	0
8	0	0.10	0.1429	0.20	0.10	0.1429
9	0	0	0	0.20	0.10	0.1429
10	0	0	0	0	0	0
11	0.20	0.10	0.1429	0.20	0.10	0.1429
12	0	0	0	0	0.10	0.1429
13	0.20	0.10	0	0	0	0
14	0.20	0.10	0.1429	0	0.10	0
15	0	0.10	0.1429	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0.10	0	0	0	0
19	0	0.10	0	0	0.10	0
20	0	0.10	0.1429	0	0	0
21	0.20	0.10	0.1429	0.20	0.10	0.1429
22	0.20	0.10	0.1429	0.20	0.10	0.1429
$\bar{E}_\Omega$	0.794	0.792	0.793	0.718	0.709	0.709
$\bar{V}_\Omega$	0.0064	0.0094	0.0078	0.0046	0.0063	0.0052

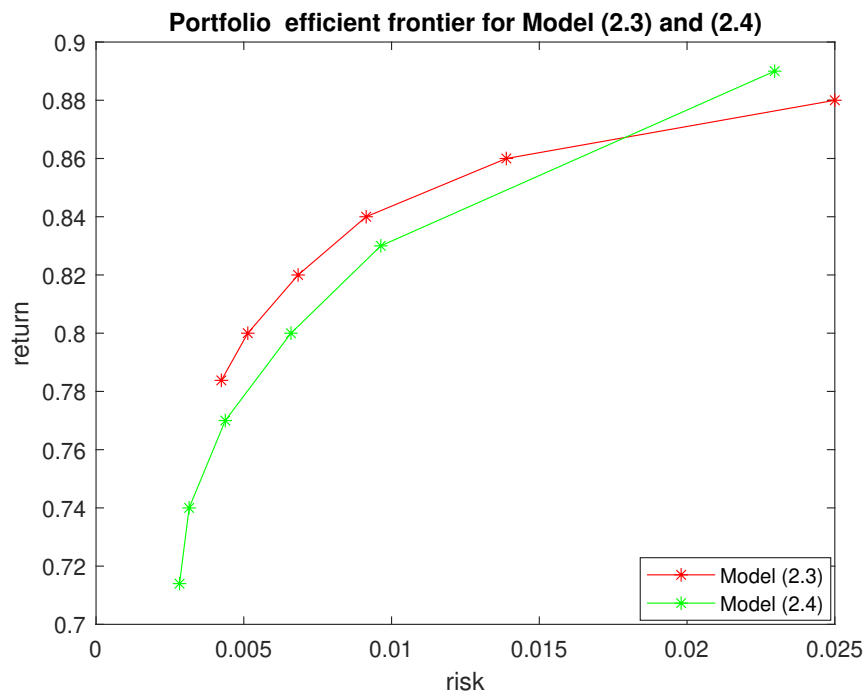


Figure 2.2: Efficient frontier of Model (2.3) and Model (2.4)

Table 2.10: The optimal portfolio strategies corresponding to Model (2.4) for equal weights

	$\gamma = 0.1$			$\gamma = 0.2$		
DMU_k	$\epsilon_j = \delta_j = 0.2, h = 5$	$\epsilon_j = \delta_j = 0.1, h = 10$	$\epsilon_j = \delta_j = 0.0667, h = 15$	$\epsilon_j = \delta_j = 0.2, h = 5$	$\epsilon_j = \delta_j = 0.1, h = 10$	$\epsilon_j = \delta_j = 0.0667, h = 15$
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0.10	0.1429
7	0	0	0	0	0	0
8	0	0	0	0.20	0.10	0.1429
9	0	0	0	0	0.10	0
10	0.20	0.10	0.1429	0	0	0
11	0.20	0.10	0.1429	0.20	0.10	0.1429
12	0	0	0	0	0.10	0.1429
13	0	0.10	0	0.20	0.20	0
14	0.20	0.10	0.1429	0	0.10	0.1429
15	0	0.10	0	0	0	0
16	0	0.10	0	0	0	0
17	0	0	0	0	0	0
18	0	0.10	0.1429	0	0	0
19	0	0	0	0	0.10	0
20	0	0.10	0.1429	0	0	0
21	0.20	0.10	0.1429	0.20	0.10	0.1429
22	0.20	0.10	0.1429	0.20	0.10	0.1429
$\bar{E}_\Omega$	0.808	0.807	0.805	0.721	0.709	0.718
$\bar{V}_\Omega$	0.0096	0.0116	0.0102	0.0043	0.0063	0.0052

2.4.1 A Case Study of the Indian Banking Sector

We present a case study with 20 banking companies listed on the NSE as a practical example of the suggested methodology. The data for the financial year 2021 is used here for the case study. In our analysis, we have randomly selected the following twenty banks listed in the Trade Brains portal (<https://portal.tradebrains.in>).

- | | | |
|-------------------|----------------------|-------------------------|
| 1. SBIN (B_1) | 5. ICICI (B_5) | 9. UNION (B_9) |
| 2. AXIS (B_2) | 6. KOTAK (B_6) | 10. UNCO (B_{10}) |
| 3. PNB (B_3) | 7. FEDERAL (B_7) | 11. BARODA (B_{11}) |
| 4. HDFC (B_4) | 8. CANARA (B_8) | 12. INDIAN (B_{12}) |

Table 2.11: The optimal portfolio strategies corresponding to Model (2.3) for unequal weights

	$\gamma = 0.1, \epsilon = 0.03, \delta = 0.4$			$\gamma = 0.2, \epsilon_j = 0.02, \delta_j = 0.3$		
DMU_k	h=5	h=10	h=7	h=5	h=10	h=7
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0.0529	0.02
4	0	0	0	0	0	0
5	0	0	0	0	0.037	0
6	0.03	0.03	0	0	0	0.02
7	0	0	0	0	0.02	0
8	0	0	0.03	0.2101	0.0316	0.1343
9	0	0	0	0.0282	0.09	0.1149
10	0	0.03	0	0	0	0
11	0.40	0.2886	0.40	0.2981	0.2819	0.30
12	0	0.03	0	0	0.021	0
13	0	0.03	0	0	0.047	0
14	0.1282	0.1423	0.0685	0	0	0
15	0	0	0	0	0	0
16	0	0.03	0.03	0	0	0
17	0	0	0	0	0	0
18	0	0	0.0397	0	0	0
19	0	0.03	0	0	0	0
20	0	0	0	0	0	0
21	0.400	0.3591	0.400	0.1635	0.1093	0.1108
22	0.0418	0.03	0.318	0.30	0.2996	0.30
$\bar{E}_\Omega$	0.792	0.792	0.792	0.744	0.739	0.734
$\bar{V}_\Omega$	0.0050	0.0061	0.0052	.0045	0.0050	0.0046

13. BANDHAN (B_{13})

16. INDIAN (B_{16})

19. DHAN (B_{19})

14. MAHA (B_{14})

17. CUB (B_{17})

20. SOUTH (B_{20})

15. IDBI (B_{15})

18. DCB (B_{18})

We denote the banks by B_k , $k = 1, 2, \dots, 20$, in order of the above listing. The description of input and output variables is as follows:

Inputs:

1. Input 1: Ratio of market price to the earnings per share (I_1)
2. Input 2: Ratio of the market price to the book value per share (I_2)

Outputs

1. Output 1: Ratio of net income to stockholders equity (O_1)

Table 2.12: The optimal portfolio strategies corresponding to Model (2.4) for unequal weights

	$\gamma = 0.1, \epsilon_j = 0.03, \delta_j = 0.4$			$\gamma = 0.2, \epsilon_j = 0.02, \delta_j = 0.3$		
DMU_k	h=5	h=10	h=7	h=5	h=10	h=7
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0.0204	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0.1924	0.0248	0.1809
7	0	0	0	0	0	0
8	0	0.03	0	0.1220	0.1479	0.1177
9	0	0	0	0	0.02	0
10	0	0.03	0.03	0	0.036	0.02
11	0.1569	0.1077	0.2052	0.30	0.2966	0.2977
12	0	0	0	0	0.0248	0
13	0	0.03	0	0	0	0.028
14	0.3583	0.3565	0.3253	0.0857	0	0
15	0	0.03	0	0	0	0
16	0	0.03	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0.03	0	0	0
19	0	0	0	0	0.02	0
20	0.03	0.03	0.0509	0	0	0
21	0.3959	0.3250	0.3285	0	0.1093	
22	0.0319	0.03	0.03	0.30	0.30	0.30
$\bar{E}_\Omega$	0.801	0.801	0.801	0.712	0.712	0.712
$\bar{V}_\Omega$	0.0069	0.0080	0.0073	0.0042	0.0037	0.0043

2. Output 2: percentage of a firm's ability to generate a return from its assets
(O_2)

Table 2.13 presents the input and output values of the 20 banking stocks and their self-evaluated DEA efficiency. Further, from Definition 2.3.2 and Definition 2.3.3, the minimum OVGA cross-efficiency $\bar{e}_j^L$ and the minimum average cross-efficiency $\bar{e}_j^L$ for Bank B_j , $j = 1, 2, \dots, 20$ are also presented there. Similarly, from Definitions 2.3.4, 2.3.5 and 2.3.6, the maximum OVGA cross-efficiency and maximum mean cross-efficiency $\bar{e}_j^U$ and $\bar{e}_j^U$ of respective Banks are also provided therein. We set $\epsilon_j = 0.02$, $\delta_j = 0.6$, to create portfolios with unequal weights. When we solve the proposed models (2.3) and (2.4) for a range of h and gamma values, we found that a small increase in portfolio return (from 0.72 to 0.75) would result in a sizable increase in risk (from 0.0128 to 0.0145). The results are shown in Table 2.14.

Table 2.13: Detailed description of the Banks

B_k	(I_1)	(I_2)	(O_1)	(O_2)	e_{kk}^*	$\bar{e}_k^L$	$\bar{e}_k^U$	$\bar{e}_k^L$	$\bar{e}_k^U$
B_1	14.1	1.29	10.13	0.54	0.3092	0.2361	0.2561	0.2489	0.2616
B_2	29.7	2.06	7.64	0.75	0.1877	0.1442	0.1568	0.1473	0.1610
B_3	14.99	0.45	2.98	0.2	0.2292	0.1569	0.1734	0.1533	0.1684
B_4	25.86	3.92	16.5	1.88	0.4737	0.2660	0.3112	0.2626	0.2909
B_5	21.87	2.6	14.85	1.38	0.4111	0.2732	0.3081	0.2799	0.2985
B_6	34.94	4.12	13.12	2.15	0.4009	0.2305	0.2704	0.2198	0.2457
B_7	9.09	0.92	10.52	0.85	0.6093	0.4619	0.5069	0.4849	0.5038
B_8	8.67	0.46	6.07	0.28	0.4494	0.3032	0.3361	0.3079	0.3238
B_9	7.62	0.37	6.25	0.35	0.5753	0.4185	0.4551	0.4008	0.4888
B_{10}	65.26	0.63	0.98	0.07	0.0573	0.0310	0.0367	0.0288	0.0374
B_{11}	24.78	0.5	1.98	0.12	0.1349	0.0822	0.0925	0.0702	0.0816
B_{12}	10.68	0.55	5.54	0.32	0.343	0.2585	0.2795	0.2536	0.2658
B_{13}	24.74	3.13	13.53	2.13	0.561	0.3157	0.3713	0.3059	0.3543
B_{14}	23.88	1.23	5.5	0.31	0.1523	0.1133	0.1228	0.1110	0.1165
B_{15}	27.38	1.32	4.85	0.48	0.1875	0.1251	0.1372	0.1320	0.1391
B_{16}	4.17	0.33	9.69	0.64	1	0.9511	1.0000	0.9592	1.0000
B_{17}	19.43	1.97	10.64	1.15	0.3856	0.2651	0.2980	0.2505	0.3163
B_{18}	9.5	0.9	10.01	0.86	0.5898	0.4537	0.4972	0.4572	0.4967
B_{19}	10.09	0.52	5.26	0.29	0.3445	0.2531	0.2751	0.2482	0.2618
B_{20}	28.02	0.32	1.16	0.06	0.1235	0.0642	0.0758	0.0520	0.0639

Table 2.14: Optimal Portfolio Strategies for Banks under Models (2.3) and (2.4)

	$\gamma = 0.25, \epsilon_j = 0.02, \delta_j = 0.6, h = 5$		$\gamma = 0.2, \epsilon_j = 0.02, \delta_j = 0.75, h = 7$		$\gamma = 0.3, \epsilon_j = 0.01, \delta_j = 0.5, h = 10$	
B_k	Model (2.3)	Model (2.4)	Model (2.3)	Model (2.4)	Model (2.3)	Model (2.4)
B_1	0	0	0	0	0.01	0
B_2	0	0	0.02	0.02	0.01	0
B_3	0	0	0	0	0.01	0
B_4	0	0	0	0	0	0
B_5	0	0	0	0	0.01	0.01
B_6	0	0	0	0	0	0
B_7	0.1335	0.313	0	0	0.0192	0.3068
B_8	0	0.02	0	0	0	0.01
B_9	0.0712	0.0469	0.02	0.1180	0.1572	0.1232
B_{10}	0	0	0	0	0	0
B_{11}	0	0	0	0	0	0
B_{12}	0.1752	0	0.957	0.0296	0.0914	0.01
B_{13}	0	0	0	0	0	0.01
B_{14}	0	0	0.02	0	0	0
B_{15}	0	0	0.0802	0.02	0.01	0
B_{16}	0.60	0.60	0.7441	0.75	0.5	0.5
B_{17}	0	0	0	0	0	0.01
B_{18}	0	0.02	0	0.0424	0	0.01
B_{19}	0.02	0	0.02	0.02	0.01	0.01
B_{20}	0	0	0	0	0	0
$\overline{E}_\Omega$	0.72	0.75	0.764	0.80	0.669	0.699
$\overline{V}_\Omega$	0.0128	0.0141	0.0145	0.0160	0.0112	0.0127

2.4.2 Managerial implications:

The proposed two-stage methodology has significant managerial implications in the field of DEA cross-efficiency aggregation and portfolio selection in terms of their efficiency. First, the developed approach can help managers to gain a better understanding of portfolio selection. Second, the approach's results are simple to grasp for managers. It can help investors and portfolio managers to build an efficient portfolio by incorporating only the best-performing stocks. Based on the implications discussed above, the proposed approach can be used in a financial market but also in other industries to examine the impact of long-term operational and decision-making on business performance. Furthermore, the problem addressed in this study is in its early stages of investigation, and further research can be conducted based on the findings of this work. Some options are as follows:

1. A similar study can be conducted with stochastic data.
2. In this study, the proposed model was applied to a problem in the banking industry. However, the same model could be used for a variety of other

selection-based problems like R&D and project selection, and the stock market. The methodology applies to other asset allocation problems, mutual fund portfolio selection, multi-period portfolio problems, and other case studies.

3. Furthermore, comparative studies in different industries or countries using the model and methods from this study would be necessary to validate its claims.
4. The OVGA aggregated efficiencies can be used in the different ranking problems, and the obtained results can be further compared with other aggregation methods in terms of complexity.

2.5 Conclusion and Future Directions

This chapter introduces a novel portfolio selection model that can help investors identify well-diversified equities based on numerous evaluation criteria. The unique primary contributions of this chapter can be seen in two ways. First, using OVGA in cross-efficiency aggregation may be viewed as an extension and refinement of the DEA cross-efficiency aggregation approaches. Meanwhile, this is the first time the OVGA aggregated cross-efficiencies have been successfully applied to portfolio management. It appears to be a promising tool for assessing financial assets. It is demonstrated herein that introducing the OVGA aggregation operator into the cross-efficiency portfolio selection yields superior portfolios with reduced variance and more significant or similar projected returns, at least for this particular study. To demonstrate the value of the suggested approach, a real-time example from the Indian banking industry is used.

Variance is unsuitable for asymmetric return distributions, leading researchers to adopt measures like semi-variance and semi-absolute deviation for more accurate risk evaluation. Therefore, the proposed method can be considered for these risk frameworks in future studies. The proposed OVGA cross-efficiency model can be extended to fuzzy portfolio selection and to assess DMUs with undesirable inputs and outputs.

Chapter 3

A Comprehensive mean-maverick-variance portfolio selection model based on OVGA aggregated game cross-efficiency approach

This chapter ¹ offers two key contributions. First, it introduces the concept of aggregated DEA game cross-efficiency, utilizing the OVGA method to determine weights. Second, it integrates the OVG game cross-efficiency approach into a multiobjective Mean-Maverick-Variance (MMV) portfolio selection model. In this framework, the OVGA game cross-efficiency score represents the return characteristic, while the Maverick index and variance of cross-efficiencies capture the risk characteristic. Using OVGA game cross-efficiency scores, the approach improves the cross-efficiency tool for portfolio selection, achieving unique Nash equilibrium scores and faster convergence. An empirical analysis, based on data from 50 companies listed on the National Stock Exchange, demonstrates the effectiveness of this method for portfolio selection.

¹The content of this chapter is based on the research paper “A Comprehensive mean-maverick-variance portfolio selection model based on OVGA aggregated game cross-efficiency approach”, accepted in *International Journal of Mathematics for Industry*, (SCIE)

3.1 Introduction

The traditional DEA method for calculating cross-efficiency has limitations, as it does not guarantee a unique optimal solution. However, incorporating game theory into efficiency analysis where cross-efficiency is strategic payoffs enables the determination of a Nash equilibrium as suggested by Lozano [115]. By integrating DEA with game theory, a more equitable and definitive allocation framework is established, ensuring a balanced approach for all DMUs engaged in both competition and cooperation. As a result, Liang et al. [101] suggested a novel cross-evaluation approach. They developed the DEA game cross-efficiency assessment by applying the non-cooperative game approach to achieve a balance between self-evaluation and peer evaluation. This model enables efficiency evaluation that converges to a Nash equilibrium point and can simultaneously consider direct or indirect competition among DMUs. In this model, DMUs act as players, while cross-efficiency scores represent their payoffs (potential gains or losses). Each DMU adopts a non-cooperative game approach, striving to maximize its minimum possible payoff. Consequently, integrating game theory into the DEA model and exploring DEA game research will advance DEA theory and significantly broaden the field of study for game theory's practical applications. To create a cross-game model, Örkücü et al. [134] blended pessimistic and optimistic viewpoints. Zou et al. [216] introduced a novel game cross-efficiency model that accounts for varying levels of importance assigned to different output indices. The proposed model groups output indices and allows DMUs to express their preferences for these groups. Additionally, an algorithm is formulated to solve the model, and its convergence is rigorously established. Zare et al. [205] introduced a hybrid DEA-game theory model to evaluate healthcare center performance, incorporating efficiency-related input and output variables identified through literature review and internal organizational data. Zuo and Guan [217] extended DEA game cross-efficiency framework to scenarios where the internal structure of a DMU operates in a parallel configuration. The proposed parallel DEA game cross-efficiency model considers both inter-DMU competition and competition among subprocesses, enabling a comprehensive evaluation of overall and subprocess efficiency. Xie et al. [197] integrated the DEA game cross-efficiency model with the Malmquist index approach to analyze the dynamic environmental efficiency

of China's power generation enterprises from 2007 to 2016. Liu et al. [110] suggested a novel aggressive game cross-efficiency model to demonstrate uniqueness independent of various initial values. Yang and Wei [200] employed game cross-efficiency to enhance the accuracy of urban energy efficiency measurement by accounting for competitive dynamics and environmental pollution. They applied this approach to assess the urban total factor energy efficiency (UTFEE) of 26 Chinese prefectural-level cities from 2005 to 2015 under environmental constraints. Chen et al. [39] analyze the energy efficiency of 26 cities in China by incorporating the DEA game cross-efficiency method. Wang et al. [184] use the game cross-efficiency to calculate the green efficiency of 18 ports in China.

In the studies mentioned above, the final cross-efficiency of the game was obtained using an iterative algorithm. Every iteration uses the arithmetic average method to aggregate the game cross-efficiency of each DMU in each iteration. In this study, we propose a new idea of aggregation of game cross-efficiencies in each iteration. We employ the Ordered visibility graph averaging method to find weights to aggregate the game cross-efficiencies due to the simplicity of this averaging method.

3.1.1 Novelty of the work

While numerous studies explore portfolio performance using game cross-efficiency and DEA, few have implemented a portfolio cross-efficiency model. The key highlights and contributions of this study are as follows:

1. The methodology used in this research is novel because it assesses the DMUs' performance in a non-cooperative game framework due to the direct or indirect competition among the DMUs studied. In this study, we develop a new game cross-efficiency method that is based on the aggregation of the game cross-efficiency, which is an entirely new approach and has not been implemented in the existing literature.
2. We use the newly developed OVGA operator to aggregate the game cross-efficiencies and obtain each DMU's final aggregated game cross-efficiency. This OVGA aggregated game cross-efficiency score also converges to a Nash equilibrium point in fewer iterations.
3. The present study extends the framework of stock assessment by integrat-

ing the game theory and visibility graph aggregation method in DEA evaluation. The OVGA game cross-efficiency is employed for assets evaluation instead of self-efficiency when compared to the prior work on portfolio selection using DEA [4, 34, 103].

4. The market portfolio is highly unpredictable, and asset returns cannot be fully characterized by just mean and variance. The maverick index effectively measures the false positivity of financial assets, where a higher value indicates a greater degree of maverick behavior. Therefore, minimizing the maverick index is essential, as it can be regarded as a risk factor alongside variance. In contrast to previous research on mean-variance portfolio selection with DEA [51, 63, 70], the current study expands the level of information by taking into account the mean, maverick, and variance based on OVGA game cross-efficiency scores of assets evaluation to address real-world investing scenarios.
5. We demonstrate the relevance and effectiveness of our methodology in portfolio development challenges using actual data from the National Stock Exchange.

The remainder of the chapter is set out in the following manner. We propose the OVGA aggregated game cross-efficiency approach in Section 3.2. Section 4 presents a comprehensive multi-objective portfolio selection model integrated with the OVG game cross-efficiency approach. Section 5 describes the application of the proposed method for the actual data extracted from NSE. Finally, section 6 offers the conclusions of the chapter.

3.2 Proposed OVGA game cross-efficiency approach

In the traditional game cross-efficiency model, the arithmetic average has been used to calculate the game cross-efficiency of the DMUs, and to obtain the Nash Equilibrium solution, this averaged value has been used to get the next game cross-efficiency value. The process continues until the Nash Equilibrium solution is obtained. In this study, we provide a weighted game cross-efficiency method to get a Nash equilibrium solution. The newly developed OVGA aggregation method has been used for aggregating the game cross-efficiency value. We present the

following algorithm to obtain aggregated OVGA game cross-efficiency values.

3.2.1 Algorithm for the proposed process

We denote the OVGA aggregated game cross-efficiency value by β_j^τ for DMU_j at the τ^{th} iteration. Initially take $\alpha_k^\tau = \beta_k^0 = e_{kk}^*$,

The following steps compute the OVGA aggregated game cross-efficiency value β_j^τ

Step 1: Initially take $\alpha_k^\tau = \beta_k^0 = e_k^*$, ($\tau = 0$), where τ represents the number of iterations. Solve model (1.5) discussed in section 1.4.3 for each $k = 1, 2, 3, \dots, n, j = 1, 2, \dots, n$.

Now arrange $\theta_j^*(\beta_k^0), k = 1, 2, \dots, n$ in increasing order and based on definitions 1.4.1, 1.4.2, 1.4.3 and 1.4.4 calculate the weights of the game cross-efficiency of DMU_j with respect to DMU_k .

Step 2: Calculate $\beta_j^1 = \sum_{k=1}^n (w_{jk}^0) \theta_j^*(\beta_k^0)$ where w_{jk}^0 be the respective weights of the optimal value $\theta_j^*(\beta_k^0)$ given in Table 3.1

Table 3.1: Weighted game Cross efficiency matrix for n DMUs corresponding to Model (1.5) for initial iteration $\tau = 0$

$\begin{matrix} j \\ \backslash \\ k \end{matrix}$	1	2	...	n
1	$(w_{11}^0) \theta_1^*(\beta_1^0)$	$(w_{21}^0) \theta_2^*(\beta_1^0)$	...	$(w_{n1}^0) \theta_n^*(\beta_1^0)$
2	$(w_{12}^0) \theta_1^*(\beta_2^0)$	$(w_{22}^0) \theta_2^*(\beta_2^0)$	...	$(w_{n2}^0) \theta_n^*(\beta_2^0)$
3	$(w_{13}^0) \theta_1^*(\beta_3^0)$	$(w_{23}^0) \theta_2^*(\beta_3^0)$	...	$(w_{n3}^0) \theta_n^*(\beta_3^0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$(w_{1n}^0) \theta_1^*(\beta_n^0)$	$(w_{2n}^0) \theta_2^*(\beta_n^0)$	...	$(w_{nn}^0) \theta_n^*(\beta_n^0)$
β_j^1	$\sum_{k=1}^n (w_{1k}^0) \theta_1^*(\beta_k^0)$	$\frac{1}{n} \sum_{k=1}^n (w_{2k}^0) \theta_2^*(\beta_k^0)$	...	$\sum_{k=1}^n (w_{nk}^0) \theta_n^*(\beta_k^0)$

Step 3: In a more general way, the weighted game cross-efficiency at $(\tau + 1)^{th}$ iteration is defined as $\beta_j^{\tau+1} = \sum_{k=1}^n (w_{jk}^\tau) \theta_j^*(\beta_k^\tau)$. Taking the weighted average in each iteration will ensure the faster convergence of the solution to the Nash

equilibrium because a proper weight has been assigned to each game's cross-efficiency score.

Step 4: If for some j , $|\beta_j^{\tau+1} - \beta_j^{\tau}| \neq \epsilon$, then repeat the step 2. If $|\beta_j^{\tau+1} - \beta_j^{\tau}| \leq \epsilon$ for all j then the algorithm stops. The best weighted game-cross-efficiency value is given by $\beta_j^{\tau+1}$.

Step 5: The final OVG game cross-efficiency is given by

$$\beta_j^{\star} = \beta_j^{\tau+1} = \sum_{k=1}^n (w_{jk}^{\tau}) \theta_j^{\star}(\beta_k^{\tau})$$

The proposed OVG game cross-efficiency strategy is utilized to solve a simple numerical problem in the following part to evaluate the methodology before it can be applied to address problems in the physical realm, such as the problem of portfolio selection.

3.2.2 A simple illustrative case

In the following section, we take a simple numerical example extracted from [101] using five DMUs with three inputs and two outputs to illustrate the proposed OVG game cross-efficiency approach. The details of input and output variables for the DMUs are stated in Table 3.2. The whole procedure is explained through

Table 3.2: Input output data

DMU	$Input_1$	$Input_2$	$Input_3$	$Output_1$	$Output_2$
DMU_1	7	7	7	4	4
DMU_2	5	9	7	7	7
DMU_3	4	6	5	5	7
DMU_4	5	9	8	6	2
DMU_5	6	8	5	3	6

the calculation for DMU_1 . We solve model (1.2), to calculate the efficiency of

DMU_1 , which is as follows

$$\begin{aligned}
 &\text{Max } 4\mu_{11} + 4\mu_{21} \\
 &\text{s.t.} \\
 &7\omega_{11} + 7\omega_{21} + 7\omega_{31} = 1, \\
 &4\mu_{11} + 4\mu_{21} - 7\omega_{11} - 7\omega_{21} - 7\omega_{31} \leq 0, \\
 &7\mu_{11} + 7\mu_{21} - 5\omega_{11} - 9\omega_{21} - 7\omega_{31} \leq 0, \\
 &5\mu_{11} + 7\mu_{21} - 4\omega_{11} - 6\omega_{21} - 5\omega_{31} \leq 0, \\
 &6\mu_{11} + 2\mu_{21} - 5\omega_{11} - 9\omega_{21} - 8\omega_{31} \leq 0, \\
 &3\mu_{11} + 6\mu_{21} - 6\omega_{11} - 8\omega_{21} - 5\omega_{31} \leq 0, \\
 &\mu_{11}, \mu_{21}, \omega_{11}, \omega_{21}, \omega_{31} \geq 0,
 \end{aligned}$$

The optimal solution of the above model gives $\mu_{11} = 0.1714, \mu_{21} = 0, \omega_{11} = 0, \omega_{21} = 0.1429, \omega_{31} = 0$ the self-efficiency of DMU_1 is $e_{11}^* = 0.6857$. Similarly, we solve model (1.2) and find the self-efficiency of other DMUs as $e_{22}^* = 1, e_{33}^* = 1, e_{44}^* = 0.8571, e_{55}^* = 0.8571$. The peer evaluation of $DMU_j, j = 2, 3, 4, 5$. by using optimal weights of DMU_1 are given as

$$\begin{aligned}
 e_{12}^* &= \frac{7 \times 0.1714 + 7 \times 0}{5 \times 0 + 9 \times 0.1714 + 7 \times 0} = 0.9333, & e_{13}^* &= \frac{5 \times 0.1714 + 7 \times 0}{4 \times 0 + 6 \times 0.1714 + 5 \times 0} = 1.0, \\
 e_{14}^* &= \frac{6 \times 0.1714 + 2 \times 0}{5 \times 0 + 9 \times 0.1714 + 8 \times 0} = 0.80, & e_{15}^* &= \frac{3 \times 0.1714 + 6 \times 0}{6 \times 0 + 8 \times 0.1714 + 5 \times 0} = 0.45.
 \end{aligned}$$

A similar process is repeated for other DMUs, and the obtained cross-efficiency matrix is shown in Table 3.3. The final cross-efficiency of each DMU is calculated by averaging each column of the cross-efficiency matrix and shown as e_j^* as the last row of the table.

OVGA game cross-efficiency calculation : The obtained cross-efficiencies are set as $\alpha_1^0 = \beta_1^0 = e_{11}^*$

To find the game cross-efficiency of DMU_j related to DMU_1 , model (1.6) be-

Table 3.3: The Cross efficiency matrix for 5 DMUs corresponding to Model (1.2) and equation 1.4.1

$\begin{matrix} \backslash & j \\ k \end{matrix}$	1	2	3	4	5
1	0.6857	0.9333	1.00	0.800	0.450
2	0.5714	1.00	1.000	0.8571	0.4286
3	0.4898	0.6667	1.00	0.1905	0.6429
4	0.5714	1.00	1.00	0.8571	0.4286
5	0.4082	0.7143	1.00	0.1786	0.8571
e_j^*	0.5453	0.8629	1.0	0.5767	0.5614

comes

$$\begin{aligned}
 \text{Max } \theta_j(\alpha_1^0 = 0.5453) &= \sum_{r=1}^s \mu_{rj} y_{rj} \\
 \text{s.t.} \\
 \sum_{i=1}^m \omega_{ij} x_{ij} &= 1 \\
 \sum_{r=1}^s \mu_{rj} y_{rj} - \sum_{i=1}^m \omega_{ij} x_{ij} &\leq 0, \quad j = 1, 2, \dots, 5 \\
 0.5453 \times \sum_{i=1}^m \omega_{ij} x_{ik} - \sum_{r=1}^s \mu_{rj} y_{rk} &\leq 0 \\
 \mu_{rj} &\geq 0, \quad r = 1, 2, \\
 \omega_{ij} &\geq 0, \quad i = 1, 2, 3.
 \end{aligned}$$

The game cross-efficiency of DMU_j , $j = 1, 2, 3, 4, 5$ related to DMU_1 are obtained by putting $j = 1, 2, 3, 4, 5$ in the above model and the obtained values are $\theta_1^*(\alpha_1^0) = 0.6857, \theta_2^*(\alpha_1^0) = 1.00, \theta_3^*(\alpha_1^0) = 1.00, \theta_4^*(\alpha_1^0) = 0.8571, \theta_5^*(\alpha_1^0) = 0.6412$.

Similarly, the game cross-efficiencies of DMU_j , $j = 1, 2, 3, 4, 5$ related to DMU_2, DMU_3, DMU_4 and DMU_5 are obtained, and the combined game cross-efficiency matrix for the first iteration is presented in Table 3.4 The traditional game cross-efficiency α_j^1 is provided in the last row of Table 3.4, as a simple averaging of the game cross-efficiency values. To obtain aggregated game cross-efficiency, we have to provide some weights to different game cross-efficiency, so

Table 3.4: The game cross efficiency matrix for 5 DMUs corresponding to Model (1.6) for $\tau = 1$

$\begin{matrix} j \\ \backslash \\ k \end{matrix}$	1	2	3	4	5
1	0.6857	1	1	0.8571	0.6412
2	0.6857	1	1	0.8571	0.7234
3	0.6857	1	1	0.8571	0.8571
4	0.6857	1	1	0.8571	0.678
5	0.5917	1	1	0.7665	0.8571
α_j^1	0.6669	1	1	0.83898	0.75136

for each column, we will find the weights using the OVGA approach. According to this approach, each column of the above game cross-efficiency matrix will be converted into an ordered data set. Then the weights are obtained corresponding to the degree of the ordered visibility graph. For example, column 1 of the above game cross-efficiency matrix (table 3.4) will be converted into an ordered data set and becomes $[0.6857, 0.6857, 0.6857, 0.6857, 0.5917]$. When we find an ordered visibility graph of this data, the weights are calculated using the definitions 1.4.1, 1.4.2, 1.4.3, and 1.4.4 described in Section 1.4.4. Table 3.5 shows the degree and weights for DMU_1 . The OVGA aggregated game cross-efficiency of

Table 3.5: Degree and weights of the ordered set corresponding to the data of column 1 game cross-efficiency matrix

Nodes ($\hat{i}$)	1	2	3	4	5
degree ($d_{\hat{i}}$)	1	2	2	2	1
weight ($w_{\hat{i}}$)	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

DMU_1 is calculated as

$$\beta_1^1 = 0.6857 \times \frac{1}{8} + 0.6857 \times \frac{2}{8} + 0.6857 \times \frac{2}{8} + 0.6857 \times \frac{2}{8} + 0.5917 \times \frac{1}{8} = 0.67395$$

A similar process is followed for each column of table 3.4, and weights obtained from the ordered visibility graph for the whole game cross-efficiency matrix are

given in the following Table 3.6. Table 3.7 represents the first iteration game cross-efficiencies.

Table 3.6: The weights corresponding to the game cross-efficiency matrix for $\tau = 1$

$\begin{smallmatrix} j \\ k \end{smallmatrix}$	1	2	3	4	5
1	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{14}$
2	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{4}{14}$
3	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{3}{14}$
4	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{14}$
5	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{14}$

Table 3.7: The OVGA game cross efficiency matrix for 5 DMUs for $\tau = 1$

$\begin{smallmatrix} j \\ k \end{smallmatrix}$	1	2	3	4	5
1	$\frac{1}{8} \times (0.6857)$	$\frac{1}{8} \times (1)$	$\frac{1}{8} \times (1)$	$\frac{1}{8} \times (0.8571)$	$\frac{1}{14} \times (0.8571)$
2	$\frac{2}{8} \times (0.6857)$	$\frac{2}{8} \times (1)$	$\frac{2}{8} \times (1)$	$\frac{2}{8} \times (0.8571)$	$\frac{4}{14} \times (0.8571)$
3	$\frac{2}{8} \times (0.6857)$	$\frac{2}{8} \times (1)$	$\frac{2}{8} \times (1)$	$\frac{2}{8} \times (0.8571)$	$\frac{3}{14} \times (0.7234)$
4	$\frac{2}{8} \times (0.6857)$	$\frac{2}{8} \times (1)$	$\frac{2}{8} \times (1)$	$\frac{2}{8} \times (0.8571)$	$\frac{3}{14} \times (0.678)$
5	$\frac{1}{8} \times (0.5917)$	$\frac{1}{8} \times (1)1$	$\frac{1}{8} \times (1)$	$\frac{1}{8} \times (0.7665)$	$\frac{3}{14} \times (0.6412)$
β_j^1	0.67395	1	1	0.845775	0.7438

Remark 3.2.1. In the traditional game cross-efficiency model the α_j^1 will be used for 2nd iteration calculation and in weighted game cross-efficiency the β_j^1 will be used in the second iteration.

The process is repeated until we reach a Nash equilibrium point for $\epsilon = 0.001$. The calculation of both simple averaged game cross-efficiency and OVGA aggregated game cross-efficiency for each iteration is presented in Table 3.8. Also, all efficiency scores are presented in Table 3.9.

Table 3.8: The iterative calculations of game cross-efficiency scores for $\epsilon = 0.001$

Iteration wise calculations of Game cross-efficiency score										
DMU	$\tau = 0$	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$	$\tau = 5$	$\tau = 6$	$\tau = 7$	$\tau = 8$	$\tau = 9$
DMU_1	0.5453	0.6669	0.60398	0.64832	0.62656	0.6416	0.63476	0.6393	0.6376	0.6386
DMU_2	0.8629	1	0.9548	0.98522	0.96918	0.97952	0.97428	0.9775	0.9759	0.9768
DMU_3	1	1	1	1	1	1	1	1	1	1
DMU_4	0.5767	0.83898	0.7585	0.81132	0.78512	0.80302	0.7945	0.8001	0.7974	0.7991
DMU_5	0.5614	0.75136	0.64614	0.69514	0.65948	0.6754	0.66426	0.66926	0.6660	0.6674
Iteration wise calculations of Weighted Game cross-efficiency score										
DMU	$\tau = 0$	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$	$\tau = 5$	$\tau = 6$			
DMU_1	0.5453	0.67395	0.61174	0.65395	0.62822	0.63691	0.635325			
DMU_2	0.8629	1	0.96483	0.98794	0.97855	0.98298	0.98179			
DMU_3	1	1	1	1	1	1	1			
DMU_4	0.5767	0.84578	0.79086	0.8286	0.81548	0.82034	0.8208			
DMU_5	0.5614	0.74381	0.62796	0.66214	0.65544	0.64999	0.647999			

Table 3.9: The efficiency scores of DMUs

DMU	CCR efficiency	Arbitrary cross-efficiency	Game cross-efficiency	OVGA Game cross-efficiency
DMU_1	0.6857	0.5453	0.6386	0.6353
DMU_2	1.00	0.8629	0.9768	0.9818
DMU_3	1.000	1.000	1.000	1.000
DMU_4	0.8571	0.5767	0.7991	0.8208
DMU_5	0.8571	0.5614	0.6660	0.6480

Remark 3.2.2. We calculate the Nash equilibrium solutions for each DMU using the given model and $\epsilon = 0.001$ for both simple averaged game cross-efficiency and OVG aggregated game cross-efficiency. We observe that the convergent approach provided in the OVG GCE method is stopped after 7 iterations for each DMU, and the GCE method stops after 10 iterations for each DMU. Table 3.8 describes the iteration-wise calculations of the Game cross-efficiency and OVG Game cross-efficiency values. If the DMUs are viewed as competitive, it is worth noting that, in a (non-cooperative) game sense, each "player" has a higher score than it would have gotten using standard cross-efficiency models (except in the case of the 100 percent efficient DMU).

In the next section, we step forward to propose a multiobjective MMV portfolio selection model integrated with the OVG game cross-efficiency approach.

3.3 A Comprehensive multiobjective portfolio selection model integrated with OVGA game cross-efficiency approach

We propose a novel MMV model for portfolio selection based on the OVGA game cross-efficiency evaluation introduced in section 3.2 in this part. In this model, we consider three objectives related to the portfolio efficiency, the OVG game cross-efficiency as the return of the portfolio, and the risk related to the portfolio efficiency as the variance of the cross-efficiencies and the maverick index of the efficiency. For this first we define maverick index as follows:

Definition 3.3.1. The Maverick Index (MI): MI shows how sensitive an individual is to environmental changes, which also serves as a reliable risk indicator. It calculates the difference between the DMU self-evaluation and Nash equilibrium scores. The MI score of any j^{th} stock is defined as follows:

$$M_j = \frac{e_{jj}^* - \beta_j^*}{\beta_j^*}$$

The MI calculates how far the self-evaluation score deviates from the OVGA game cross-efficiency score. Hence, we aim to maximize the OVGA game cross-efficiencies while reducing the maverick and variance of OVGA game cross-efficiencies.

Let us consider a portfolio Ω constituted with n stocks/assets say $stock_j$, $j = 1, 2, \dots, n$. Let $[e_{kj}^*]_{n \times n}$ be the cross-efficiency matrix on n assets taken under consideration for portfolio selection. Let w_j be the proportion of the total OVGA game efficiency invested in $stock_j$. E_Ω represents the return and I_Ω & V_Ω represents the risk (concerning alterations in DEA multipliers) of the portfolio Ω . To determine the optimal portfolio, we proposed the following multi-objective portfolio optimization

model:

Model (3.1)

$$\text{Max } E_{\Omega} = \sum_{j=1}^n w_j \beta_j^{\star}$$

$$\text{Min } I_{\Omega} = \sum_{j=1}^n w_j M_j$$

$$\text{Min } V_{\Omega} = \sum_{j=1}^n \sum_{k=1}^n w_j w_k \sigma_{kj}$$

subject to

$$\sum_{j=1}^n w_j = 1 \quad (a)$$

$$\sum_{j=1}^n z_j = h \quad y_j \in \{0, 1\} \quad (b)$$

$$\epsilon_j y_j \leq w_j \leq \delta_j y_j, \quad w_j \geq 0 \quad j = 1, 2, \dots, n \quad (c)$$

Where $\beta_j^{\star}$ is the OVGA game cross-efficiency score of j^{th} as described in section 3.2 and σ_{kj} is the covariance between the cross-efficiency scores of $stock_k$ and $stock_j$. Further, for a better selection of a stock, the budget constraint $\sum_{i=1}^n w_j = 1$, cardinality constraint $\sum_{j=1}^n y_j = h$, $w_j \in \{0, 1\}$, $j = 1, 2, \dots, n$, buy-in threshold $\epsilon_j y_j \leq w_j \leq \delta_j y_j$, $j = 1, 2, \dots, n$, and no short-selling criteria $w_j \geq 0$, $j = 1, 2, \dots, n$, are also considered here.

Remark 3.3.1. To determine a portfolio's sensitivity to environmental volatility, the MI and variations in OVGA game cross-efficiencies as two risk measures can be a useful tool. The proposed MMV model focuses on portfolio selection based solely on the efficiency of the stocks. In addition to return and variance of the return (risk), this chapter defines new measures of return and risk in terms of efficiency.

3.4 An application to portfolio selection in the National Stock Exchange

The following subsection describes the detailed analysis of the results and the importance of the work proposed in this study. In the first stage, we use the DEA cross-efficiency, game cross-efficiency, and OVGA game cross-efficiency frame-

works to assess the performance of the stocks. In the second stage, the portfolio is chosen by incorporating the proposed OVGA game cross-efficiency with the suggested model (3.1). Further, after examining the MMV model's diversity, we compare the outcomes using various cross-efficiency methodologies and the existing DEA MV cross-efficiency model.

3.4.1 Data collection and processing

The daily return of 50 heterogeneous stocks across different sectors enlisted with the National Stock Exchange (NSE), India for the period 1 January 2016 to 31 December 2021 was used as the sample data for this implementation and includes 1482 daily observations for each stock. The daily real-time returns of the stocks are obtained by using the Google Finance database. The 50 stocks are numbered as $S_1, S_2, S_3, \dots$ and S_{50} for ease of description. Following is the list of NIFTY 50 stocks from NSE.

- | | | |
|--------------------------|------------------------------|---------------------------|
| • ADANI PORTS (S_1) | • EICHERMOT (S_{14}) | • INFY (S_{26}) |
| • ASIANPAINT (S_2) | • GRASIM (S_{15}) | • ITC (S_{27}) |
| • AXIS-BANK (S_3) | • HCLTECH (S_{16}) | • JSWSTEEL (S_{28}) |
| • BAJAJ-AUTO (S_4) | • HDFCBANK (S_{17}) | • KOTAKBANK (S_{29}) |
| • BAJFINANCE (S_5) | • HDFCLIFE (S_{18}) | • LT (S_{30}) |
| • BAJAJFINSV (S_6) | • HEROMOTOCO
(S_{19}) | • M&M (S_{31}) |
| • BPCLB (S_7) | | • MARUTI (S_{32}) |
| • BHARTIARTL(S_8) | • HINDALCO (S_{20}) | • NESTLEIND (S_{33}) |
| • BRITANNIA (S_9) | • HINDUNILVR (S_{21}) | • TATACONSUM (S_{34}) |
| • CIPLA (S_{10}) | • HDFC (S_{22}) | • NTPC (S_{35}) |
| • COALINDIA (S_{11}) | • ICICIBANK (S_{23}) | • ONGC (S_{36}) |
| • DIVISLAB (S_{12}) | • IOC (S_{24}) | • POWERGRID (S_{37}) |
| • DRREDDY (S_{13}) | • INDUSINDBK (S_{25}) | • RELIANCE (S_{38}) |

- SBILIFE (S_{39}) • TCS (S_{43}) • TITAN (S_{47})
- SHREECEM (S_{40}) • TATAMOTORS (S_{44}) • ULTRACEMCO (S_{48})
- SBIN (S_{41}) • TATASTEEL (S_{45}) • UPL (S_{49})
- SUNPHARMA(S_{42}) • TECHM (S_{46}) • WIPRO(S_{50})

3.4.2 Selection of input/output variables

We evaluate the stock's performance by using the parameters that must be minimized as inputs and those that must be maximized as outputs. Higher moments have drawn more attention because they loosened the requirement that the return on the assets follow a normal distribution. The literature has grown over time with research on portfolio selection using the four-moment frameworks as given by Gong et al. [70]. We refer to the work of [63, 125] to select the input, and output variables based on the first four moments of the returns distribution. These four crucial moments are mean, variance, skewness, and kurtosis. Positive skewness is favored by the investor since it signifies a low possibility of earning a significant negative return, and also likes to maximize gain, which is represented by the arithmetic mean return [117], hence mean and skewness have been viewed as output. In contrast, to reduce the volatility roughly represented by the variance, and since kurtosis causes neutral noise to be transferred from the distribution's center to its tail, the investor must be against it. As a result, the variance and kurtosis should be viewed as inputs. We calculate the daily return (R_{jt}) of the j^{th} stock at day t by using the formula $R_{jt} = LN(P_{jt}/P_{jt-1}) * 100$ where P_{jt} & P_{jt-1} denotes the closing price at the end of the current day and the closing price at the end of the immediately preceding day. We follow Chen et al. [63] to calculate the mean, variance, skewness, and Kurtosis which are presented in Table 3.10.

Table 3.10: The input and output description

Input	output
Variance $\sigma_j^2 = \frac{1}{T} \sum_{t=1}^T (R_{jt} - R_j)^2$	mean return $R_j = \frac{1}{T} \sum_{t=1}^T R_{jt}$
Kurtosis $= \frac{1}{T} \sum_{t=1}^T \left(\frac{R_{jt} - R_j}{\sigma_j} \right)^4 - 3$	Skewness $= \frac{1}{T} \sum_{t=1}^T \left(\frac{R_{jt} - R_j}{\sigma_j} \right)^3$

Remark 3.4.1. It is important to note that the negative values are obtained based on the input and output criteria mentioned above. It is widely known that the CCR model cannot directly handle negative variables. Consequently, data transformation is required. The efficiency categorization would remain after applying the transformation to the data, therefore the model's formulation would not be impacted. So we apply the same transformation given by Ali and Seiford [3] to convert the negative values into positive values. Table 3.11 shows the converted results with different efficiency values.

Table 3.11: Converted input-output and different efficiency values

S.No.	Input 1	Input 2	Output 1	Output 2	Self-efficiency	Cross-efficiency	Game-efficiency.	OVGA-efficiency.
S ₁	5.558	7.976	0.125	0.813	0.4512	0.3344	0.4180	0.4248
S ₂	2.674	7.484	0.146	1.324	0.8885	0.6949	0.8127	0.8126
S ₃	5.653	29.00	0.085	0.016	0.2447	0.1201	0.1790	0.1811
S ₄	2.673	10.03	0.073	1.253	0.4491	0.3777	0.4249	0.4306
S ₅	6.600	10.58	0.220	0.637	0.6292	0.4174	0.5714	0.5748
S ₆	5.368	20.86	0.199	0.117	0.6033	0.3245	0.4724	0.4753
S ₇	5.589	11.20	0.073	0.553	0.2302	0.1796	0.2157	0.2168
S ₈	4.386	3.446	0.112	1.895	0.8971	0.6397	0.7494	0.7658
S ₉	2.565	9.811	0.115	1.488	0.7296	0.5587	0.6465	0.6536
S ₁₀	3.003	4.859	0.081	2.198	1.0000	0.6625	0.7904	0.8040
S ₁₁	3.648	4.748	0.000	1.200	0.5118	0.1829	0.2939	0.3213
S ₁₂	4.756	25.18	0.150	0.000	0.5132	0.2483	0.3722	0.3768
S ₁₃	3.208	6.584	0.087	1.712	0.6464	0.5132	0.5730	0.5810
S ₁₄	4.440	3.632	0.082	1.468	0.6708	0.4693	0.5520	0.5662
S ₁₅	4.274	9.816	0.131	1.112	0.5154	0.4128	0.4839	0.4855
S ₁₆	2.959	4.236	0.132	1.493	0.9159	0.7775	0.8797	0.8894
S ₁₇	2.233	12.49	0.124	1.094	0.9037	0.5425	0.7002	0.7154
S ₁₈	4.261	13.20	0.117	1.062	0.4468	0.3365	0.3983	0.3988
S ₁₉	3.429	5.724	0.050	1.721	0.6755	0.4056	0.5044	0.5216
S ₂₀	6.908	3.946	0.175	1.332	0.8347	0.5121	0.6703	0.7049
S ₂₁	2.046	9.342	0.123	2.405	1.0000	0.8023	0.9322	0.9429
S ₂₂	3.457	7.636	0.106	1.023	0.5228	0.4338	0.4973	0.4986
S ₂₃	5.118	7.956	0.133	1.410	0.4992	0.4304	0.4870	0.4904
S ₂₄	4.166	5.756	0.058	0.858	0.3271	0.2719	0.3033	0.3065
S ₂₅	8.221	33.85	0.051	1.720	0.1867	0.1115	0.1432	0.1490
S ₂₆	2.989	13.11	0.140	0.663	0.7622	0.4484	0.5999	0.6039
S ₂₇	2.923	9.093	0.055	1.059	0.3632	0.2952	0.3307	0.3368
S ₂₈	5.257	6.279	0.179	0.959	0.7646	0.5144	0.6762	0.6907
S ₂₉	3.154	8.326	0.118	1.262	0.6080	0.5097	0.5741	0.5741
S ₃₀	3.126	11.86	0.110	1.417	0.5726	0.4394	0.5079	0.5126
S ₃₁	4.008	6.715	0.075	1.844	0.6199	0.4297	0.5041	0.5156
S ₃₂	3.756	10.93	0.088	1.284	0.3813	0.3398	0.3672	0.3683
S ₃₃	2.262	5.674	0.139	2.090	1.0000	0.9574	1.0000	0.9987
S ₃₄	4.500	5.297	0.166	1.439	0.8355	0.6310	0.7635	0.7778
S ₃₅	2.813	2.512	0.058	1.484	1.0000	0.6138	0.7523	0.7849
S ₃₆	4.980	12.25	0.048	0.860	0.1891	0.1623	0.1762	0.1775
S ₃₇	2.545	3.749	0.100	1.383	0.8546	0.7297	0.8089	0.8205
S ₃₈	3.556	9.156	0.161	1.630	0.7368	0.6129	0.6955	0.6954
S ₃₉	3.723	10.46	0.105	0.584	0.4590	0.3226	0.4059	0.4068
S ₄₀	3.569	2.273	0.115	1.537	1.0000	0.7725	0.8925	0.9014
S ₄₁	5.458	10.76	0.105	2.168	0.4916	0.3786	0.4274	0.4354
S ₄₂	4.044	4.130	0.059	1.314	0.5825	0.3845	0.4593	0.4751
S ₄₃	2.531	3.942	0.133	1.374	1.0000	0.8625	0.9782	0.9783
S ₄₄	8.733	6.279	0.072	1.706	0.4231	0.2521	0.3124	0.3318
S ₄₅	5.975	2.560	0.157	1.480	1.0000	0.6068	0.7866	0.8351
S ₄₆	3.752	7.316	0.139	0.761	0.6589	0.4793	0.6062	0.6103
S ₄₇	4.336	9.616	0.188	1.952	0.7383	0.6258	0.7073	0.7087
S ₄₈	3.218	7.430	0.124	1.275	0.6467	0.5462	0.6178	0.6190
S ₄₉	5.314	7.177	0.119	1.260	0.4667	0.3895	0.4467	0.4520
S ₅₀	2.669	9.127	0.138	2.273	0.8473	0.7324	0.8106	0.8179

3.4.3 Results and Discussion

To begin with, the self-efficiency and the cross-efficiency values have been calculated using Model (1.2) and equation 1.4.1 respectively of the chosen financial stocks. Then, applying model (1.5), the game cross-efficiency scores as the Nash equilibrium point of the assets have been obtained after the 12th iteration, and the convergence criteria ϵ is set to 0.001 for this calculation. After that, the OVGA game cross-efficiency technique introduced in Section 3.2 is used to assess OVGA game cross-efficiency scores across the same number of chosen financial assets. We obtain the final OVGA game cross-efficiency after the 7th iteration only and for ($\epsilon = 0.001$). This OVGA game cross-efficiency is a Nash equilibrium point with less computation work and uses fewer iterations. As a result, it offers a reliable solution. All cross-efficiency values are shown in the table 3.11. As discussed by Lim et al. [103], the simple use of DEA cross-efficiency can lead to the problem of lack of diversification and ganging-together phenomena, so there is a need to develop a model that can justify the use of OVG cross-efficiency in portfolio selection problems. In the next step, we apply the MMV portfolio selection model provided in Section 3.3 for portfolio selection incorporating it with different cross-efficiency values. To solve the proposed multiobjective MMV portfolio selection model genetic algorithm with default parameters has been used in MATLAB 2022a. A set of Pareto optimal solutions is obtained in the implementation of GA to model (3.1) and we put all of the Pareto solutions together and select a suitable portfolio among them. Among different portfolios, a decision-maker can select his/her portfolio according to his/her preferences.

3.4.4 Sensitivity analysis of the MMV model with different criteria

To analyze the portfolio's performance, for different parameters of the MMV model, we set the criteria as return, efficiency, MI, and Variance of the portfolio. The formula calculates the mean return of the portfolio is $\sum_{j=1}^n R_j w_j^*$ where w_j^* are the optimal solution for model (3.1). The performance parameters, such as the maximum, minimum, and average value of the three objectives (portfolio efficiency, maverick, and risk related to efficiency) and return using MMV portfolio selection model with different cross-efficiency techniques such as DEA cross-efficiency, game cross-efficiency, and the OVGA game cross-efficiency proposed

in this study, are presented in Table 3.12. For this comparison, we set cardinality constraint $h=5, 10$ and 15 with buy-in threshold values $\epsilon_j = 0.02$ & $\delta_j = 0.25$ in model (3.1). It can be easily verified from Table 3.12 that the efficiency of the portfolio in the case of the OVG game cross-efficiency is higher than the efficiency obtained from the simple DEA cross-efficiency and game cross-efficiency. The results are also compared with the DEA-MV cross-efficiency model (without objective 2) [119] with DEA cross-efficiency. When we compare the MMV with OVGA GCE and DEA-MV cross-efficiency results the return and efficiency both are higher in the case of our proposed method. When we study the diversity factor we obtain a return between 0.082 to 0.1475 , efficiency from 0.8152 to 0.3652 , a maverick from 0.0453 to 0.1231 , and variance between 0.0035 to 0.0248 for a sample size of 10 .

We marked the best results in bold for each case. It can be observed that in most cases, the minimum, maximum, and mean values obtained by the MMV model with OVGA GCE and simple game cross-efficiency are better than those listed for the DEA-MV cross-efficiency method. Additionally, we can observe from Table 3.12 that very good diversified values are obtained in the case of the MMV model integrated with the OVGA GCE approach. For more analysis purposes and to discuss the sensitivity of the MMV model with the different values of ϵ_j & δ_j , we present some Pareto portfolios and compare the results with the DEA-MV cross-efficiency model. The results are presented in Table 3.13. The cardinality is set to 8 . In Table 3.13, portfolios have diversified returns with higher portfolio efficiency, very low maverick, and a little higher variances for the proposed MMV model with OVG game cross-efficiency as compared to the DEA-MV cross-efficiency model. We can conclude from Table 3.13 that in the case of DEA-MV cross-efficiency, all mentioned criteria have poor diversified values as compared to the MMV with OVGA cross-efficiency model.

3.4.5 Comparison of weights with DEA-MV cross-efficiency

Among many portfolios, we select the portfolio four times for comparison purposes: (1) by applying only the DEA-MV cross-efficiency model (by considering only objective 1 and objective 3 in Model (3.1)), (2) by applying the MMV model with DEA cross-efficiency, (3) by applying the MMV model with game cross-

Table 3.12: Results of performance comparison of Pareto portfolios with a variation of h

Results of performance comparison of Pareto portfolios when h=5					
		DEA-MV cross- efficiency	MMV model with DEA cross- efficiency	MMV model with game cross- efficiency	MMV model with OVGA game cross- efficiency
Return	Max	0.1309	0.1381	0.1378	0.1404
	Min	0.0759	0.0692	0.0642	0.0645
	Mean	0.1096	0.1063	0.1019	0.1069
Efficiency	Max	0.7502	0.7544	0.8476	0.8439
	Min	0.2240	0.2230	0.2660	0.2949
	Mean	0.5287	0.4879	0.5581	0.6015
Maverick	Max	0.4901	0.5671	0.1114	0.1258
	Min	0.2481	0.1491	0.0364	0.0367
	Mean	0.3177	0.3266	0.0712	0.0728
Variance	Max	0.0067	0.0072	0.0151	0.0275
	Min	0.0003	0.0003	0.00643	0.0019
	Mean	0.0016	0.0019	0.0064	0.0092
Results of performance comparison of Pareto portfolios when h=10					
Return	Max	0.1333	0.1288	0.1548	0.1475
	Min	0.0788	0.0759	0.082	0.082
	Mean	0.11605	0.1118	0.1062	0.1104
Efficiency	Max	0.7050	0.6923	0.8034	0.8152
	Min	0.2907	0.3030	0.3738	0.3652
	Mean	0.5683	0.5191	0.5751	0.6105
Maverick	Max	0.4572	0.4393	0.1182	0.1231
	Min	0.2427	0.1715	0.0462	0.0855
	Mean	0.3396	0.3024	0.0853	0.0453
Variance	Max	0.010	0.0098	0.0255	0.0248
	Min	0.0005	0.00047	0.0030	0.0105
	Mean	0.0032	0.0018	0.00852	0.0035
Results of performance comparison of Pareto portfolios when h=15					
Return	Max	0.1220	0.1279	0.1279	0.1313
	Min	0.0893	0.0881	0.0939	0.0919
	Mean	0.1116	0.1079	0.1099	0.1138
Efficiency	Max	0.6743	0.6743	0.7842	0.7943
	Min	0.3341	0.3341	0.4096	0.414
	Mean	0.5512	0.4868	0.5650	0.6133
Maverick	Max	0.4215	0.4215	0.1379	0.1401
	Min	0.2406	0.1916	0.0535	0.0494
	Mean	0.3169	0.2898	0.0745	0.0865
Variance	Max	0.0089	0.0089	0.021	0.031
	Min	0.0006	0.00047	0.0030	0.0035
	Mean	0.00256	0.00203	0.0074	0.0113

Table 3.13: Some Pareto portfolios using the proposed model and DEA-MV cross-efficiency methods for $h = 8$

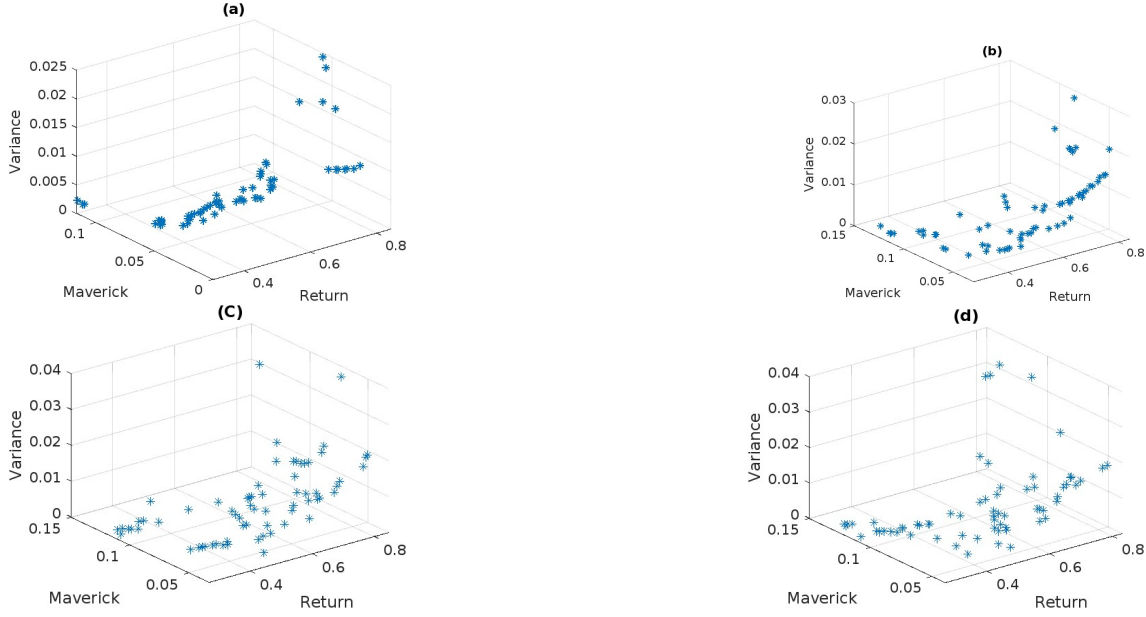
portfolio	DEA-MV cross-efficiency				MMV with OVGA game cross-efficiency				
	Return	Efficiency	Maverick	Variance	Return	Efficiency	Maverick	Variance	
$\epsilon_j = 0.01, \delta_j = 0.25$	1	0.128	0.5406	0.3654	0.0012	0.1155	0.6056	0.0513	0.0092
	2	0.0839	0.3057	0.3648	0.0004	0.1488	0.7876	0.0819	0.0246
	3	0.1113	0.6735	0.2785	0.0044	0.0848	0.3812	0.0691	0.0031
	4	0.0929	0.4377	0.2157	0.0008	0.0886	0.3993	0.1032	0.0036
	5	0.1247	0.533	0.3521	0.0012	0.1271	0.8343	0.1151	0.0254
	6	0.1226	0.6466	0.2751	0.0022	0.1233	0.6441	0.0537	0.0088
	7	0.1278	0.54	0.3644	0.0012	0.127	0.8328	0.117	0.0254
	8	0.1281	0.702	0.3542	0.0071	0.1064	0.4708	0.0387	0.0053
	9	0.1222	0.6401	0.2721	0.0021	0.1164	0.7138	0.0694	0.0111
	10	0.0848	0.3121	0.3499	0.0004	0.09	0.5505	0.1073	0.0048
$\epsilon_j = 0.01, \delta_j = 0.45$	1	0.1428	0.5817	0.3797	0.0017	0.1447	0.7714	0.0559	0.0166
	2	0.0771	0.2662	0.3987	0.0005	0.0823	0.3283	0.0629	0.0025
	3	0.1126	0.4704	0.2424	0.001	0.119	0.6410	0.0800	0.0093
	4	0.0968	0.2817	0.6318	0.0006	0.1328	0.8967	0.0373	0.0229
	5	0.1099	0.6591	0.2148	0.0028	0.1107	0.7512	0.0689	0.0109
	6	0.109	0.7459	0.328	0.009	0.126	0.6654	0.1021	0.0097
	7	0.109	0.7459	0.328	0.009	0.1123	0.8195	0.1316	0.0193
	8	0.0968	0.2817	0.6318	0.0006	0.1283	0.8828	0.0307	0.0204
	9	0.1106	0.676	0.2168	0.0032	0.109	0.6096	0.0636	0.0079
	10	0.1092	0.6922	0.3614	0.0062	0.0905	0.3862	0.0554	0.0038
$\epsilon_j = 0.01, \delta_j = 0.65$	1	0.1372	0.618	0.2974	0.0018	0.1581	0.8357	0.0327	0.0121
	2	0.1184	0.6759	0.231	0.0025	0.0991	0.4649	0.0627	0.0036
	3	0.0945	0.3999	0.1942	0.0007	0.0794	0.3211	0.0971	0.0021
	4	0.1299	0.8402	0.1626	0.0045	0.1307	0.9202	0.0296	0.0229
	5	0.1152	0.6304	0.2499	0.0021	0.101	0.5134	0.0469	0.0042
	6	0.0961	0.3785	0.302	0.0006	0.0802	0.3298	0.0934	0.0022
	7	0.1293	0.8322	0.1659	0.0044	0.081	0.3378	0.0903	0.0022
	8	0.1242	0.5925	0.3357	0.0018	0.101	0.5134	0.0469	0.0042
	9	0.103	0.4704	0.2205	0.0009	0.1166	0.4482	0.0331	0.0041
	10	0.1285	0.8206	0.1707	0.0042	0.1217	0.8484	0.0579	0.0072

efficiency, and (4) by applying the MMV model with OVGA game cross-efficiency. The parameters of the models are set as follows: $h=10$, $\epsilon_j = 0.02$ & $\delta_j = 0.25$. In this chapter, the portfolio that has average efficiency is selected for each model. The portfolio return, efficiency, mavericks, and variance are the performance parameters of the portfolio. The values of the mentioned criteria for the selected portfolio are presented in Table 3.14. When we compare the results of our proposed model with the DEA-MV cross-efficiency model, we observe that the selected portfolio in the case of the proposed model increases the portfolio return and efficiency and decreases the risk-related efficiency (maverick) as compared to the DEA-MV cross-efficiency model.

Finally, Figure 3.1 illustrates the approximate efficient frontiers of the MMV model with OVGA game cross-efficiency for $h = 5, 10, 12$, and 15 . The values of the buy-in threshold are taken as $\epsilon_j = 0$ & $\delta_j = 0.3$. Figure 3.1 shows that the portfolios are well-diversified in each case, and the investor can select the appropriate portfolio based on their choices among three different investment goals. In the portfolio chosen by our proposed model, there is a good trade-off between the criteria mentioned with diversified values.

Table 3.14: The weights of the selected portfolios for $h = 10$ by different methods

Stocks	DEA-MV Cross-Efficiency	MMV Cross-Efficiency	MMV Game Cross-Efficiency	MMV OVGA Game Cross-Efficiency
1	0	0	0	0
2	0	0	0	0
3	0.0217	0	0.02	0
4	0	0	0	0
5	0	0	0	0
6	0.02	0	0	0
7	0	0	0	0
8	0.25	0	0.25	0
9	0	0	0	0
10	0.02	0	0	0
11	0	0	0	0
12	0	0	0	0
13	0	0.02	0.02	0
14	0.02	0	0.02	0
15	0.02	0	0	0.2358
16	0	0.02	0.02	0.2352
17	0	0	0	0
18	0	0	0	0
19	0	0	0	0
20	0	0	0	0
21	0.0444	0.02	0.02	0.0294
22	0	0.25	0.0301	0.0348
23	0.02	0	0.02	0
24	0	0.02	0.02	0.02
25	0	0	0	0
26	0.02	0.02	0	0.02
27	0.02	0.02	0	0
28	0	0	0	0
29	0	0	0.25	0.0348
30	0.02	0.03	0	0.02
31	0	0	0	0
32	0.2339	0.25	0.2499	0.0348
33	0.02	0.02	0.02	0.02
34	0	0	0	0
35	0	0.02	0	0
36	0	0	0	0
37	0	0	0	0
38	0.25	0	0	0.25
39	0	0.02	0	0
40	0	0	0	0.02
41	0	0.037	0.02	0
42	0	0.02	0	0.02
43	0	0.02	0	0.02
44	0	0	0.02	0
45	0.02	0	0	0.02
46	0	0	0	0
47	0	0	0	0
48	0	0	0	0
49	0	0	0	0
50	0	0	0	0
Return	0.1207	0.1147	0.1041	0.1226
Efficiency	0.5318	0.4944	0.5571	0.6133
Maverick	0.2958	0.2043	0.0910	0.0497
Variance	0.0012	0.0035	0.0058	0.0100

Figure 3.1: Approximate efficient frontier in case of (a) $h=5$, (b) $h=10$, (c) $h=12$, (d) $h=15$ 

3.5 Conclusion and Future Research Directions

The significance of this study is highlighted by two key contributions. On the one hand, the DEA game cross-efficiency notion from Liang et al. [101] is expanded to provide the novel idea of aggregated game cross-efficiency scores utilizing the OVGA aggregation operator. We suggest an iterative method that ends at a unique Nash equilibrium point in fewer iterations. The proposed OVGA game cross-efficiency aggregation method may be viewed as an extension and refinement of the DEA game cross-efficiency framework. On the other hand, we apply the new OVGA game cross-efficiency approach presented in this chapter to the MMV framework to choose portfolios. From an empirical perspective, the real-time data from NSE has been selected, and the analytical results show that our method provides well-diversified portfolios for the mentioned criteria. Additionally, our findings demonstrate that, at least for this particular study, the MMV OVGA game cross-efficiency strategy is superior to the one based on the straightforward application of cross-efficiency and game cross-efficiency approaches.

For future research, the proposed work can be extended by employing alternative aggregation methods to combine the game cross-efficiency values. Further, the current study can be developed by considering various risk measures such as semi-variances and semi-absolute deviation of the cross-efficiencies.

Chapter 4

An Integrated Multi-Objective SORM Cross-Efficiency Model: An Application in Portfolio Selection

This chapter¹ introduces an innovative multi-objective framework for portfolio selection, integrating the cross-efficiency approach based on the SORM model of DEA. The proposed method follows a two-stage process for portfolio selection. In Stage 1, the SORM model, originally designed to overcome the limitations of traditional DEA models in dealing with negative data, is modified to include peer evaluation. Stage 2 builds on the SORM cross-efficiencies into a multi-objective framework, transforming it into an effective tool for portfolio optimization and selection. The multi-objective portfolio selection model incorporates two essential indices—the Maverick index (representing risk) and the Diversity index (representing return)—which provide critical insights into a portfolio's sensitivity to environmental volatility and its degree of diversification. These indices are instrumental in assessing asset performance throughout the portfolio selection process. A case study highlights the model's ease of application, ability to distinguish between different stocks, and effectiveness in making informed and optimal portfolio decisions.

¹The work presented in this chapter is based on the research paper, “An Integrated Multi-Objective SORM Cross-Efficiency Model: An Application in Portfolio Selection”, *Economic Computation and Economic Cybernetics Studies and Research*, Vol. 58, Issue 3/2024. <https://doi.10.24818/18423264/58.3.24.08> (SCIE, Impact Factor: 0.9)

4.1 Introduction

The literature discussed in section 1.2.2 fails to address a key limitation of cross-efficiency evaluation. Since it is based on the CCR model, traditional cross-efficiency evaluation cannot handle negative values in inputs and outputs. Few studies in the literature address cross-efficiency in the presence of negative data. The BCC model, which operates under the VRS assumption, was proposed by Banker et al. [14] can evaluate the efficiency of each DMU while accommodating negative inputs or outputs. However, it becomes inappropriate when both inputs and outputs have negative values. De Mello et al. [50] demonstrated in their paper that the input-oriented DEA BCC model can produce negative cross-efficiencies, which are typically concealed within the model. To deal with this issue, they introduced the modified input-oriented BCC model to ensure only positive efficiencies across all DMUs in cross-evaluation. Wu et al. [190] presented a BCC VRS cross-efficiency model that can handle the problem of negative cross-efficiency by adding a non-negativity restriction. Lim and Zhu [104] present a valid cross-efficiency method based on the assumption of VRS. Addressing limitations in prior research, Kao and Liu [86] have introduced a slacks-based measure model to calculate the cross-efficiency of DMUs, ensuring the absence of negative efficiencies. This model offers reasonable efficiencies for DMUs that exhibit only weak efficiency. Lim et al. [103] presented the cross-efficiency form of the RAM model capable of handling negative data. Lin [105] presented a DDF-based cross-efficiency evaluation method under the VRS technology framework that is based on the RDM model, presented by Portela et al. [144] and the duality theory.

The cross-efficiency evaluation in DEA evaluates the DMU's performance through a comprehensive perspective derived from its peers [186]. In cross-efficiency, each DMU receives n efficiency scores, computed using n different sets of weights chosen by all n DMUs, including itself. The final efficiency score for the DMU is determined by averaging these n efficiency scores (column average from cross-efficiency matrix). Additionally, Talluri et al. [170] suggested that the row averages in the cross-efficiency matrix (CEM) can also be utilized to measure the diversity of performance within the same evaluation system for the units.

Emrouznejad et al. [59] used a partitioning approach to model negative data, leading to the development of the SORM model for assessing the performance of observed production units. The SORM model is designed to handle cases where some DMUs show positive values while others show negative values for a given variable. It can also be used when DMUs have both negative inputs and outputs simultaneously. However, the application of the SORM model in cross-evaluation has not been extensively explored in the literature. Therefore, in this study, we apply the SORM model in cross-evaluation and propose a modified version of it.

This chapter makes four significant theoretical contributions:

- We propose a modified SORM model for cross-evaluation that handles negative data well. The modified model addresses the issue of negative cross-efficiencies found in the traditional SORM model. This improves the flexibility of the SORM model in practical portfolio applications.
- We expand the SORM cross-efficiency framework by integrating row and column average values of CEM for refined portfolio selection. This dual approach enhances decision-making by balancing individual asset evaluation with peer comparisons, offering a comprehensive view of asset performance.
- We introduce an original approach to defining return and risk indicators within a multi-objective portfolio selection model. By incorporating the diversity index as a measure of return and the Maverick index as a gauge of risk, the model addresses the complexities of portfolio optimization and offers a balanced risk-return trade-off.

Finally, the rest of the chapter is structured in the following manner: In 4.2, we delve into the traditional input-oriented SORM cross-efficiency model. 4.3 introduces the proposed SORM cross-efficiency model, marking the initial stage of the proposed methodology. 4.4 outlines the second stage of the work, presenting a multi-objective portfolio selection model. The practical application of the proposed approach is illustrated in 4.5 to underscore its validity and effectiveness. Finally, 4.6 serves as the conclusion.

4.2 The input-oriented SORM model to deal with negative data

Section 1.4.2 provides a comprehensive explanation of the SORM model (1.4). In this section, we extend the SORM model by introducing its cross-evaluation version, building upon the foundation laid in the previous section. This adaptation allows for a more refined assessment of efficiency, specifically in situations where negative data are present. The peer evaluation of DMU_j , $j = 1, 2, \dots, n$ by the DMU_k using the input-oriented SORM model (1.4) is defined as follows:

$$e_{kj}^* = \frac{\sum_{r \in R} \mu_{rk}^* y_{rj} + \sum_{o \in O} \mu_{ok}^{1*} y_{oj}^1 - \sum_{o \in O} \mu_{oj}^{2*} y_{oj}^2 + u_0^*}{\sum_{i \in I} \omega_{ik}^* x_{ij} + \sum_{l \in L} \omega_{lk}^{1*} x_{lj}^1 - \sum_{l \in L} \omega_{lk}^{2*} x_{lj}^2}. \quad (4.2.1)$$

Where μ_{rk}^* , μ_{ok}^{1*} , μ_{ok}^{2*} , ω_{ik}^* , ω_{lk}^{1*} , ω_{lk}^{2*} and u_0 are the optimal weights of model (1.4). Due to the presence of the free variable u_0 , the cross-efficiency calculated by equation 4.2.1 may become negative leading to a problematic situation as mentioned by de Mello et al. [50]. To address the issue of negative cross-efficiencies, we extend the SORM model in the following section.

4.3 Stage 1-Proposed input-oriented SORM cross-efficiency model

To avoid negative efficiencies, we must recognize that they are caused only by the potential negativity of the independent term u_0 . Therefore, a straightforward solution would be to constrain u_0 to be non-negative, i.e., $u_0 \geq 0$. However, it is important to note that not all negative values for u_0 result in negative efficiencies. In fact, negative efficiencies only arise if the absolute value of the negative u_0 exceeds the weighted sum of the outputs, i.e., negative efficiencies occur if and only if $u_0 \leq -\left(\sum_{r \in R} \mu_{rk} y_{rj} + \sum_{o \in O} \mu_{ok} y_{oj}^1 - \sum_{o \in O} \mu_{oj} y_{oj}^2\right)$. Therefore, we must ensure positive cross-efficiencies by preventing this condition from occurring. The modified SORM model with additional restriction is as follows:

Model (4.1)

$$\max e_{kk} = \sum_{r \in R} \mu_{rk} y_{rk} + \sum_{o \in O} \mu_{ok}^1 y_{ok}^1 - \sum_{o \in O} \mu_{ok}^2 y_{ok}^2 + u_0$$

subject to,

$$\begin{aligned} & - \sum_{i \in I} \omega_{ik} x_{ij} - \sum_{l \in L} \omega_{lk}^1 x_{lj}^1 + \sum_{l \in L} \omega_{lk}^2 x_{lj}^2 \\ & + \sum_{r \in R} \mu_{rj} y_{rk} + \sum_{o \in O} \mu_{oj}^1 y_{ok}^1 - \sum_{o \in O} \mu_{oj}^2 y_{ok}^2 + u_0 \leq 0 \\ & \sum_{i \in I} \omega_{ik} x_{ik} + \sum_{l \in L} \omega_{lk}^1 x_{lk}^1 - \sum_{l \in L} \omega_{lk}^2 x_{lk}^2 = 1 \\ & \sum_{r \in R} \mu_{rj} y_{rk} + \sum_{o \in O} \mu_{oj}^1 y_{ok}^1 - \sum_{o \in O} \mu_{oj}^2 y_{ok}^2 + u_0 \geq 0 \\ & \omega_{ik} \geq 0, \forall i \in I, \omega_{lk}^1 \geq 0, \omega_{lk}^2 \geq 0 \forall l \in L \\ & \mu_{rk} \geq 0, \forall r \in R, \mu_{ok}^1 \geq 0, \mu_{ok}^2 \geq 0 \forall o \in O, u_0 \text{ is free.} \end{aligned}$$

With the optimal solutions of the modified SORM model (4.1), the cross-efficiencies calculated using equation 4.2.1 will no longer produce negative cross-efficiencies. In the cross-efficiency framework, model (4.1) must be computed n times for each evaluated DMU_k to derive the cross-efficiency values for all DMUs.

A CEM has been obtained as $E = (e_{kj}^*), k, j = 1, 2, \dots, n$. The final cross-efficiency value of DMU_j , represented as the column average of the CEM, is given by:

$$e_j^C = \frac{1}{n} \sum_{k=1}^n e_{kj}^*. \quad (4.3.1)$$

While the column average scores reflect how a DMU is evaluated by its peers, it is equally important to analyze the row mean scores, which indicate how a DMU assesses others using its optimal weights. These row mean scores provide insights into a DMU's relative performance in comparison to others. The row average of the CEM is computed as:

$$e_j^R = \frac{1}{n} \sum_{k=1}^n e_{jk}^*. \quad (4.3.2)$$

This metric helps in understanding how a specific DMU evaluates its peers. A lower row average in the CEM suggests that the DMU applies stricter evaluations, indicating that it possesses unique capabilities that contribute to increased performance diversity within the group of DMUs.

Maverick Index and Diversity Index

The results derived from the CEM can be used to evaluate the Maverick index, which measures the difference between a DMU's self-evaluated efficiency and its column average cross-efficiency score. The maverick index is calculated as follows:

$$M_j = \frac{\tilde{e}_j - e_j^C}{e_j^C} \quad (4.3.3)$$

We denote the self-efficiency of the DMUs obtained by model (4.1) as $\tilde{e}_j$. DMUs with lower maverick index values (indicating higher mean cross-efficiency scores) are considered strong performers, demonstrating consistent efficiency across peer evaluations.

The diversity index is computed using a methodology similar to that of the maverick index. It evaluates how distinct a given Decision-Making Unit (DMU) or stock is in terms of its performance when compared to others. This is achieved by contrasting the self-evaluated efficiency score of the test DMU with the row average cross-efficiency score of all other DMUs, using the weights specific to the test DMU.

The diversity index is defined as:

$$D_j = \frac{\tilde{e}_j - e_j^R}{e_j^R} \quad (4.3.4)$$

This index highlights the unique strengths and competitive advantages of a DMU. A higher diversity index suggests that the DMU possesses distinctive capabilities that differentiate it from its peers. A DMU that demonstrates both high self-efficiency and a strong diversity index is considered to have unique resources and strategic advantages, positioning it to outperform other DMUs and achieve superior performance in its respective domain.

4.4 Stage 2-A multi-objective framework of portfolio selection based on SORM cross-efficiency evaluation

Essid et al. [63] emphasized that the maverick index is also a significant factor in evaluating the overall performance of financial assets. Stocks/assets with a low index are typically well-rounded performers, often demonstrating both self and peer efficiency. Choosing a financial asset with a high Maverick index may result in a highly vulnerable portfolio to environmental changes. In other words, a higher maverick index corresponds to a riskier portfolio. Similarly, we suggest that another useful index can be constructed using the row averages of the cross-efficiency matrix as a diversity index, providing insights into how a specific asset/stock evaluates its peers. This perspective is particularly valuable as it reflects how strictly or leniently a DMU assesses other units, shedding light on its relative performance standards. For optimal asset selection, an asset's diversity index represents the difference between its self-appraised score and its row average score should be high. Therefore, it would be beneficial to consider it as the portfolio's return.

Consider a portfolio Ω comprising n stocks (DMUs), represented by the proportion vector w_j ($j = 1, 2, \dots, n$). For the j -th stock, the return is characterized by the diversity index D_j , and the risk is measured by the Maverick index M_j , as defined in 4.3. We aim to maximize the weighted diversity index P_Ω (representing return) while minimizing the Maverick index I_Ω (representing risk). These are defined as follows:

$$P_\Omega = \sum_{j=1}^n D_j w_j \quad \text{and} \quad I_\Omega = \sum_{j=1}^n M_j w_j$$

The optimal portfolio Ω^* can then be determined using the following multi-objective portfolio selection model:

Model (4.2)

$$\begin{aligned}
\text{Min } I_{\Omega} &= \sum_{j=1}^n M_j w_j \\
\text{Max } P_{\Omega} &= \sum_{j=1}^n D_j w_j \\
\sum_{j=1}^n z_j &\leq h, \quad z_j \in \{0, 1\} \\
\epsilon_j z_j &\leq w_j \leq \delta_j z_j, \quad j = 1, 2, \dots, n, \\
w_j &\geq 0, \quad j = 1, 2, \dots, n,
\end{aligned}$$

The terms ϵ_j and δ_j denote the lower and upper bounds, respectively, for the proportion of the efficiency that can be allocated to the j -th asset. These bounds ensure that the allocation remains within acceptable limits for each asset. The first set of constraints in Model (4.2) imposes a cardinality constraint, which limits the total number of assets h included in the portfolio. The specific values of h , ϵ_j , and δ_j can vary based on investor preferences and risk tolerance.

By incorporating cross-efficiency analysis in stage one, the method ensures that only the most efficient assets are considered for inclusion, while stage two refines the portfolio by balancing risk and return metrics. This two-stage approach provides a structured and efficient framework for constructing portfolios that align with investor preferences and financial goals. The proposed method is more comprehensive as it evaluates stocks based on efficiency rather than absolute output, offering a more thorough rationalization of stocks.

4.5 An application to stock portfolio selection

To demonstrate the effectiveness of the proposed two-stage approach in handling scenarios with negative input-output data, we applied it to real-world data from the CSI 300 index, as presented in [33]. This analysis involves eleven carefully selected stocks, which are evaluated based on two key input parameters and seven output parameters. These stocks were chosen to provide a diverse representation of the market. The financial parameters associated with these inputs and outputs are detailed below, which highlights the nine specific parameters used in the evaluation.

- **Input**

- Solvency ratio-I (x_1): Total liabilities divided by total assets
- Solvency ratio-II (x_2): Total liabilities divided by shareholders' equity

- **Output**

- Return on equity (y_1): Net income divided by shareholders' equity
- Return on assets (y_2): Net income divided by total assets
- Net profit margin (y_3): Net income divided by revenue for the period
- Basic earnings per share (EPS) (y_4): Net income minus dividends and preferred stock dividends, divided by common shares
- Revenue growth rate (y_5): Current period revenue divided by previous period revenue minus one
- Net profit growth rate (y_6): Current period net income divided by previous period net income minus one
- Basic earning per share growth rated (y_7): Current period basic EPS divided by the previous period basic EPS minus-1

Furthermore, Table 4.1 displays the standardized values for both input and output variables, ensuring that comparisons are conducted using consistent and normalized metrics. This standardization facilitates a comprehensive evaluation of the model's capability to address negative relationships between inputs and outputs, offering valuable insights into its robustness and suitability for real-world financial applications.

4.5.1 Results and Discussion

We begin by applying Model (4.1) to evaluate the self-efficiency of the selected stocks. Next, the optimal weights derived from Model (4.2) and equation 4.2.1 are used to assess the CEM, as shown in Table 4.2. In addition, the column and row average efficiencies, calculated from the CEM, are also provided in Table 4.3. This concludes the first stage, yielding three types of efficiencies: self-efficiency, column average efficiency, and row average efficiency.

Table 4.1: The standardized input-output data of the selected stocks

Stocks	x_1	x_2	y_1	y_2	y_3	y_4	y_5	y_6	y_7
S_1	0.8135	0.3036	0.0855	0.0545	0.0189	0.0129	0.0031	0.0098	0.0290
S_2	0.4392	0.0936	0.6122	0.6862	0.1880	0.3365	0.0017	-0.0001	-0.0004
S_3	0.4655	0.105	0.3457	0.3495	0.2065	0.1054	0.0046	-0.0006	-0.0044
S_4	0.4481	0.0911	0.6088	0.6475	0.4377	0.3959	0.0133	0.0016	-0.0015
S_5	0.1235	0.0184	0.0384	0.0338	0.4418	0.0180	0.0063	0.0016	-0.0132
S_6	0.4384	0.0888	0.2430	0.2498	0.4978	0.2108	0.0061	0.0011	-0.0062
S_7	0.5197	0.1302	0.1559	0.1476	0.1948	0.0823	0.0193	-0.0026	-0.0059
S_8	0.8124	0.3278	0.0481	0.0292	0.0428	0.0090	0.0028	-0.0068	-0.0189
S_9	0.8275	0.3446	0.2248	0.1218	0.1296	0.0977	0.0051	-0.0005	-0.0014
S_{10}	0.3764	0.0758	0.4310	0.4924	0.2181	0.3332	0.0166	0.0034	-0.0058
S_{11}	0.4224	0.0835	0.5124	0.5424	0.4117	0.5116	0.0229	0.0097	-0.0012

In Stage 2, we proceed to solve the proposed multi-objective portfolio selection model (4.2). The stock selection process, facilitated by this multi-objective approach, stands out due to its unique resources and strengths, which allow it to either distinguish itself from or align with the performance characteristics of other assets. This capability ensures that the assets selected through this model contribute to the creation of a portfolio that not only reflects a balanced performance but also demonstrates long-term stability. By considering multiple objectives simultaneously, the model optimizes the selection of assets that can withstand market fluctuations and provide sustainable returns over time, thus contributing to the overall resilience and durability of the portfolio.

The solution to the multi-objective portfolio selection model is obtained by utilizing a genetic algorithm (GA) with default parameters in MATLAB 2022a. This approach leverages the genetic algorithm to generate a set of Pareto optimal solutions for Model (4.2), which represents the optimization of multiple objectives simultaneously. Each execution of the GA produces a collection of these Pareto

Table 4.2: The input-oriented SORM cross-efficiency matrix

Stocks	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
S_1	1.00	0.00	0.00	0.15	0.75	0.11	0.00	0.00	0.00	0.386	1.00
S_2	0.08	1.00	0.53	0.98	0.22	0.40	0.22	0.04	0.20	0.82	0.87
S_3	0.43	0.87	0.84	0.94	1.00	0.85	0.72	0.20	0.33	1.00	1.00
S_4	0.06	0.99	0.48	1.00	0.22	0.44	0.21	0.04	0.18	0.88	1.00
S_5	0.01	0.12	0.12	0.28	1.00	0.32	0.11	0.02	0.04	0.16	0.27
S_6	0.01	0.46	0.55	0.95	1.00	1.00	0.48	0.04	0.00	0.77	1.00
S_7	0.23	0.50	0.62	0.79	1.00	0.68	0.76	0.24	0.18	1.00	1.00
S_8	0.31	0.70	0.81	0.84	1.00	0.86	0.72	0.32	0.21	1.00	0.89
S_9	0.54	0.99	0.75	0.93	1.00	0.70	0.60	0.00	0.49	0.84	1.00
S_{10}	0.24	0.62	0.77	0.84	0.00	0.86	0.57	0.00	0.07	1.00	0.89
S_{11}	0.00	0.63	0.17	0.73	0.06	0.39	0.012	0.00	0.09	0.72	1.00

Table 4.3: Different Efficiency Values

Stocks	SORM self-efficiency	Column average efficiency	Row average efficiency	Stocks	SORM self-efficiency	Column average efficiency	Row average efficiency
S_1	1	0.2643	0.3084	S_7	0.7622	0.4096	0.6371
S_2	1	0.6254	0.4862	S_8	0.325	0.0825	0.6973
S_3	0.8387	0.5133	0.7436	S_9	0.4873	0.1630	0.7118
S_4	1	0.7637	0.5001	S_{10}	1	0.7796	0.5319
S_5	1	0.6588	0.2217	S_{11}	1	0.9023	0.3596
S_6	1	0.5994	0.5685				

optimal solutions, presenting a range of potential portfolios for the decision maker to choose from, based on their individual preferences and investment goals. In the context of the proposed model, investors can evaluate the portfolios generated by the algorithm, carefully considering the trade-offs between key factors such as the expected return rate and the associated risk. The flexibility of this multi-objective approach allows investors to select a portfolio that aligns with their risk tolerance and desired return outcomes. For example, an investor may prioritize a portfolio that maximizes returns (e.g., by achieving a higher diversity index) while ensuring that the risk remains within an acceptable threshold.

To solve Model (4.2), we impose a cardinality constraint of $h = 5$, which limits the number of assets in the portfolio to five. Additionally, we vary the values of ϵ_j and δ_j , which are key parameters influencing the optimization process. From the set of Pareto optimal solutions generated, we select the portfolio that offers the highest diversity index, reflecting its ability to provide a well-balanced and diversified selection of assets, while also maintaining the lowest Maverick index,

which indicates minimal risk or deviation from the desired risk-return profile. The outcomes of these two scenarios- one focused on maximizing diversity and the other on minimizing maverick behavior—are presented in Table 4.4. These results provide a comprehensive view of how the portfolio selection process, guided by the multi-objective model, achieves an optimal balance between diversification and risk control.

Additionally, the statistics of the obtained Pareto solutions in each case are presented in Table 4.5. Moreover, by adjusting the cardinality constraints, one can obtain more diversified portfolios.

Table 4.4: The weights of the stocks in the selected portfolio for $h=5$

Stocks	$\delta_j = 0.5, \epsilon_j = 0.05$		$\delta_j = 0.4, \epsilon_j = 0.1$		$\delta_j = 0.3, \epsilon_j = 0.06$	
S_1	0.35	0	0.3	0	0.3	0
S_2	0.05	0.05	0	0.1	0.1023	0.06
S_3	0	0	0.1	0	0	0
S_4	0.05	0.35	0.1	0.3	0.2377	0.28
S_5	0.5	0.05	0.4	0.1	0.3	0.3
S_6	0.05	0.05	0.1	0	0.06	0.06
S_7	0	0	0	0	0	0
S_8	0	0	0	0	0	0
S_9	0	0	0	0	0	0
S_{10}	0	0.5	0.1	0.4	0	0.3
S_{11}	0	0	0	0	0	0
Return	2.68(Max)	1.056	2.3(Max)	1.1215	2.2(Max)	1.706
Risk	1.318	0.34(Min)	1.1681	0.41(Min)	1.1652	0.40(Min)

Table 4.5: The statistics from obtained pareto solutions for $h = 5$

Criteria		$\delta_j = 0.5, \epsilon_j = 0.05$	$\delta_j = 0.4, \epsilon_j = 0.1$	$\delta_j = 0.3, \epsilon_j = 0.06$
Return	Max	0.26813	0.23411	2.1174
	Min	2.0744	1.121	1.7063
	Average	1.0563	2.083	1.9203
Risk	Max	1.3118	1.1681	1.1652
	Min	0.3389	0.4124	0.4028
	Average	0.6761	0.7909	0.7905

4.5.2 Comparative Analysis

In the existing literature, there are limited studies that address cross-evaluation in the context of negative input-output scenarios. The approach introduced by Lin [105] is non-oriented, providing self-evaluations and peer evaluations for DMUs

while simultaneously measuring inefficiencies in both inputs and outputs. However, the proposed SORM model, which calculates radial efficiency and cross-efficiency, only accounts for the radial contraction of input values. As a result, a direct comparison of the results obtained using this method with those from Lin's approach is not feasible, as they measure inefficiencies differently. Additionally, Wu et al. [190] proposed an input-oriented BCC VRS cross-efficiency model that can handle negative values when they occur exclusively in either the inputs or outputs. However, this model becomes unsuitable when negative values are present in both inputs and outputs simultaneously. In contrast, the method proposed in our study is specifically designed to handle such scenarios effectively, making it a more versatile solution in cases where both inputs and outputs exhibit negative values. Cooper et al. [44] introduced the RAM model to handle negative data, with its cross-evaluation model later provided by Lim et al. [103]. The RAM model is non-oriented and identifies the furthest targets on the frontier for inefficient DMUs, which limits its ability to accurately measure efficiency.

The proposed modified SORM model is input-oriented and evaluates the efficiency of DMUs. However, a limitation of this approach is that it is applicable only when at least one input has a positive value, while no restrictions are placed on the outputs [58].

4.6 Conclusions

This chapter presents two key novel contributions within the proposed framework. First, the enhanced SORM model serves as an advancement and extension of the DEA cross-efficiency evaluation. Additionally, this study pioneers the application of the SORM cross-efficiency approach in portfolio management, positioning it as a powerful tool for assessing financial assets. A distinct feature of this study is its unique multi-objective framework, which extends beyond the traditional mean-variance method. This framework integrates a sophisticated risk metric—derived from the Maverick index and calculated using column average cross-efficiency—with a return metric based on the diversity index, determined through row average cross-efficiency. The proposed portfolio selection model effectively incorporates real-world constraints, including maximum and minimum capital allocations and cardinality restrictions, offering meaningful benefits for portfolio man-

agement. Numerical results demonstrate that this two-stage approach enables decision-makers to achieve more balanced and well-diversified portfolios.

Chapter 5

Stock assessment using cumulative prospect theory in DEA cross-efficiency model: A case study of the Indian stock market

This chapter¹ explores stock assessment using cross-efficiency evaluation in scenarios involving negative data. To address this, we utilize an RDM DDF-based cross-efficiency model that effectively handles negative data. The approach extends RDM-based cross-efficiency analysis by utilizing row and column averages of the cross-efficiency matrix (CEM) to aid portfolio selection and group classification for stock management. To examine the psychological factors that influence the decisions of investors in the stock market, we integrate the value of the cumulative prospect theory value for each stock as an output and its variance as an input to assess overall asset efficiency. The empirical analysis focuses on 30 stocks from the Nifty-50 index on India's National Stock Exchange. The results validate the proposed method as a practical tool for stock selection, demonstrating its ability to create a diversified and balanced portfolio. This enhances logical decision-making for companies in stock selection and portfolio management.

¹The content of this chapter is based on the research paper “Stock assessment using cumulative prospect theory in DEA cross-efficiency model: A case study of the Indian stock market”, *Croatian Operational Research Review*, CRORR 15(2024), 1–11. DOI: 10.17535/crorr.2024.0001 (ESCI)

5.1 Introduction

The traditional cross-efficiency method relies on the CCR model and cannot handle negative input-output data. Cross-efficiency analysis is frequently employed in scenarios where the production frontier exhibits CRS, as negative efficiencies are uncommon across different situations. Lim et al. [103] and Wu et al. [187] introduced two notable methods for evaluations of the cross-efficiencies in cases involving the VRS production frontier. Their primary aims are either to reduce negative efficiencies or transform them into positive values. Recognizing the limitations in prior research, Kao and Liu [86] proposed a slack-based measure model for assessing the cross-efficiency value of DMUs without eliciting negative efficiencies. Their concept provides reasonable efficiencies for only weakly efficient DMUs. Soltanifar and Sharafi [164] presented a non-radial cross-efficiency model integrated with a hybrid MCDM technique. Banihashemi et al. [13] used the OWA operator to aggregate the RDM cross-efficiencies.

Lin [105] proposed an approach for cross-efficiency evaluation based on the DDF, utilizing the RDM model, and duality theory within the framework of VRS technology. The model introduced by Lin [105] can handle situations with negative input-output variables and successfully addresses the issue of negative cross-efficiency. Shrivastava et al. [169] introduced a portfolio selection approach that leverages cross-efficiency, by employing the model from [105] and Cumulative Prospect Theory (CPT). They calculated the CPT value of the assets from their weekly returns using the Cumulative probability (uniform probability) weighting function and the utility function in CPT. The CPT-based model then constructs effective portfolios by utilizing the cross-efficiency matrix as the basis for return calculations. Amin and Oukil [5] and Sanei and Banihashemi [155] effectively applied the cross-efficiency evaluation for the portfolio section. Kim et al. [90] assessed the efficiency and performance diversity of each Korean container terminal. They categorized all the containers into four groups superior, good overall, niche, and inferior based on these two scores.

Instead of utilizing DEA methods, researchers have also employed MCDM methods that focus on analyzing and ranking decision alternatives based on a set of criteria. Kovač and Podrug [93] provide practical perspectives on utilizing MCDM and the MPT theory for share selection and portfolio optimization. In addition,

Poklepovi and Babi [143] proposed a hybrid MCDM technique to evaluate the shares of the Zagreb stock exchange. DEA offers an advantage over MCDM because MCDM relies on a uniform set of weights. In contrast, DEA utilizes a linear programming model to determine individual weights for each DMU, along with an enhancement in efficiency.

This study addresses the issue of robust stock selection using a cross-efficiency evaluation that accommodates both positive and negative data. The CA and RA are used for robust portfolio selection and the identification of various types of assets. When implemented in a strategic stock selection process, this approach enables managers to pinpoint stocks/assets that not only deliver strong overall performance but also offer diverse capabilities to set them apart from other alternatives. The notable contributions of this research are as follows:

- Firstly, only a limited number of studies in the current body of literature make use of the DEA cross-efficiency model to handle negative input-output data in portfolio selection, as seen in the works of Chen et al. [33], Talluri et al. [170] and Shrivastava et al. [169], and among others.
- Additionally this study addresses asset cross-efficiency by incorporating the CPT value and return of each stock as outputs, and variance as an input. Our method combines cross-efficiency with CPT, exploring the psychological aspects that affect decision-makers when shaping portfolios. It is important to highlight that only a few studies (e.g., Shrivastava et al. [169] have employed CPT values as an output.
- Finally, in contrast to the methods employed by Talluri et al. [170], this study employs a DDF-based RDM cross-efficiency model to classify the assets. In contrast to, the classification used by Shrivastava et al. [169], this study classifies stocks into distinct categories, to enable the inclusion of well-performing stocks with both similarity and variation in their performance. This approach offers financial experts a more comprehensive and detailed understanding of the subject.

The rest of the chapter comprises four sections, including the preliminary Section 5.2, which discusses the RDM cross-efficiency techniques to deal with negative values in input/output. The proposed work methodology is described in section 5.3. Next, the practical application of the proposed approach is demonstrated

to highlight its validity and effectiveness in section 5.4. The chapter concludes with Section 5.5, which examines the results and proposes potential avenues for future research.

5.2 DEA cross-efficiency evaluation with negative data

Lin [105] introduced the VRS cross-efficiency evaluation method effectively handles negative values in both inputs and outputs. This approach was based on the RDM model and is non-oriented, meaning it assesses inefficiency in both inputs and outputs simultaneously, providing a comprehensive evaluation of each DMU from both self-assessment and peer perspectives. Similar to traditional cross-efficiency evaluation methods, the proposed VRS cross-efficiency evaluation assigns cross-efficiency scores to all DMUs, ranging between 0 and 1. The mathematical form of the RDM cross-efficiency evaluation model is as follows:

$$\begin{aligned}
 \textbf{Model (5.1)} \quad & \text{Min} \sum_{i=1}^m \omega_{ik} (x_{ik} + R_{ik}^-) - \sum_{r=1}^s \mu_{rk} (y_{rk} - R_{rk}^+) + \xi \\
 & \text{s.t.} \\
 & \sum_{i=1}^m \omega_{ik} x_{ij} - \sum_{r=1}^s \mu_{rk} y_{rj} + \xi \geq 0, \\
 & \sum_{i=1}^m \omega_{ik} R_{ik}^- + \sum_{r=1}^s \mu_{rk} R_{rk}^+ = 1, \quad j = 1, 2, \dots, n, \\
 & \omega_{ik}, \mu_{rk} \geq 0, \quad r = 1, 2, \dots, s \quad i = 1, 2, \dots, m.
 \end{aligned}$$

where the directional vectors are computed as

$$\begin{aligned}
 R_{rk}^+ &= \max_{j=1,2,\dots,n} y_{rj} - y_{rk}, \quad r = 1, 2, \dots, s, \\
 R_{ik}^- &= x_{ik} - \min_{j=1,2,\dots,n} x_{ij}, \quad i = 1, 2, \dots, m.
 \end{aligned}$$

are the possible non-negative improvements for inputs and outputs. Using the weights of DMU_k derived from Model (5.1), Lin [105] established the cross-efficiency

of DMU_j , in the following manner:

$$\beta_{kj} = \frac{\sum_{i=1}^m \omega_{ik}^* (-x_{ij} + R_{ik}^-) + \sum_{r=1}^s \mu_{rk}^* (y_{rj} + R_{rk}^+) - \xi^*}{\sum_{i=1}^m \omega_{ik}^* R_{ik}^- + \sum_{r=1}^s \mu_{rk}^* R_{rk}^+}. \quad (5.2.1)$$

The cross-efficiency matrix $E = (\beta_{kj})_{k,j=1}^n$ is constructed by computing the cross-efficiency scores of all DMUs from the perspective of every other unit using model (5.1) and equation 5.2.1. In this matrix, each column j ($j = 1, \dots, n$) represents the cross-efficiency scores of DMU j as evaluated by all other DMUs. The average value of this column, denoted as $\beta_j^c = \frac{1}{n} \sum_{k=1}^n \beta_{kj}$, is defined as the overall cross-efficiency score of DMU j . Similarly, each row k ($k = 1, \dots, n$) in the matrix E contains the cross-efficiency scores assigned to all DMUs based on the evaluation of DMU k and denoted as $\beta_j^r = \frac{1}{n} \sum_{k=1}^n \beta_{jk}$. The self-efficiency of DMU_k will be obtained by putting $j = k$ in the equation 5.2.1 and denoted as β_k^* . The row average cross-efficiency provides insights into the individual characteristics of each DMU. When the value of a DMU's row mean is relatively lower compared to other DMUs, it distinguishes that DMU from the rest, indicating a high level of performance diversity.

Calculation of the "Maverick index" (MI)

The cross-efficiency scores can be utilized even further by defining the extent to which there is variation between the peer appraisal and the self-appraisal efficiency of DMUs. In finance, the Maverick index serves as a valuable tool for assessing the false positives among financial assets. This measure of concordance between the two efficiencies is called the Maverick index [170] and it is defined as follows:

$$M_j = \frac{\beta_j^* - \beta_j^c}{\beta_j^c} \quad (5.2.2)$$

MI quantifies the extent of disparity between a self-evaluation score and a peer-evaluation score. A higher MI value suggests a higher likelihood of the DMU being mistakenly perceived as efficient. Conversely, a lower index value indicates a more favorable outcome.

Calculation of The "Performance Diversity Index" (PDI)

The calculation of the PDI follows a methodology similar to that of MI. To determine the performance of an evaluated DMU (stock), its basic efficiency is compared with the average cross-efficiency of all other DMUs, using the weights specific to the evaluated DMU. Consequently, all DMUs involved in the assessment procedure are subject to the same criteria. The PDI is computed as follows:

$$D_j = \frac{\beta_j^* - \beta_j^r}{\beta_j^r}. \quad (5.2.3)$$

5.3 Proposed Methodology

5.3.1 Choice of Input and Output Variables

Input: The risk associated with each asset is represented by its variance, which serves as the input in the proposed method.

Output: To incorporate potential irrational behaviors of decision-makers in stock selection, the CPT value of stocks calculated following Shrivastava et al. [169] is used as one output. The second output is the mean return of the stocks, determined by:

$$R_{jt} = \frac{C_{j(t+1)} - C_{jt}}{C_{jt}} \quad (5.3.1)$$

where C_{jt} and $C_{j(t+1)}$ denote the closing prices of the j^{th} stock at times t and $t + 1$, respectively.

5.3.2 Assessing the cross-efficiency of the stocks and Categorization of stocks:

This section presents the proposed approach that utilizes the row averages of the cross-efficiency matrix to assess performance diversity.

One of the most common applications of the CEM is to compute the average efficiency scores across each column of the matrix. This average is then designated as the peer appraisal score, often referred as the CA efficiency. This process helps evaluate the relative performance of each stock within the group, offering a more comprehensive understanding of its efficiency in comparison to others. The CA efficiency reflects how each stock is perceived by its peers, allow-

ing for a more nuanced view of stock performance than traditional self-efficiency measures. However, the row average shows how a stock evaluates its peers. If a stock consistently rates certain peers higher, it suggests that those peers perform well under similar weighting schemes. Conversely, lower ratings indicate that the evaluated peers do not align well with the stock's preferred criteria. By integrating the row and column averages, we group stocks into distinct categories, considering both their comparative performance and the diversity of their performance simultaneously.

Model (5.1) is employed to compute the cross-efficiency scores for the selected stocks. From the resulting CEM, we calculate the self-efficiency, RA efficiency, and column average CA efficiency.

Based on the CA and RA scores derived from Model (5.1) and the corresponding equation 5.2.1, stocks are classified into the following categories:

- **Group G1:** High Column Average / High Row Average (HC/HR)
- **Group G2:** High Column Average / Low Row Average (HC/LR)
- **Group G3:** Low Column Average / High Row Average (LC/HR)
- **Group G4:** Low Column Average / Low Row Average (LC/LR)

A stock with a high CA score excels not only under its own set of weights but also when evaluated using the weights of its peers. The RA value highlights the unique characteristics of each stock. When a stock's row average is lower compared to others, it indicates greater performance diversity. By plotting the row and column averages from the cross-efficiency analysis on a quadrant chart, stocks in G2 show higher efficiency than the others, setting them apart. Stocks in G1 and G3 should be moved to G2 for performance improvement. Conversely, stocks in G4 exhibit poor efficiency and fail to differentiate themselves from the rest.

5.4 Empirical illustration

To validate the effectiveness of the proposed RDM cross-efficiency evaluation method in handling negative data, we apply it to real-world data from the Indian

stock market. Specifically, we analyze the top 30 assets in the Nifty 50 index using data from Shrivastava et al. [169]. Each asset is assessed based on one input variable (variance) and two output variables (CPT value and long mean return). The long mean return is computed using data from 05-03-2018 to 25-02-2019, while the CPT value and variance are derived from weekly stock returns between 05-03-2018 and 02-08-2018. These outputs are chosen for their relevance to portfolio evaluation and their ability to incorporate decision-makers behavioral characteristics. To assess the performance of the listed stocks, model (5.1) is used. Table 5.1 presents the CPT value, mean return, variance, and self-efficiency obtained by Model (5.1) for the thirty stocks.

5.4.1 Stock management using the proposed method

Model (5.1) calculates the cross-efficiency of the stocks by assessing both their input and output values. Using the weights derived from Model (5.1) along with Equation 5.2.1, the model generates a CEM, which serves as a crucial tool in this analysis. The resulting CEM, presented in Figure 5.1, offers valuable insights into the performance and efficiency of the selected stocks.

A stock with a high CA efficiency performs well because it surpasses other units in terms of efficiency. Additionally, a low RA value indicates that the stock has unique capabilities or features that differentiate it from others. The CA and RA efficiencies derived from the CEM are presented in Table 5.2. Stocks are categorized based on their CA and RA values. Figure 5.2 illustrates how the 30 stocks are grouped according to their CA and RA scores. The data is classified as either high or low for both CA and RA, with the mean score used to define these categories (in this case, the average score is 0.641). While the classification thresholds are somewhat subjective, they can also be set by decision-makers or managers involved in the stock evaluation process.

Discussion on the Different Groups of Stocks: The proposed stock classification provides valuable insights for investors looking to develop portfolio optimization strategies. Stocks that stand out as top performers are ideal candidates for long-term investments due to their exceptional performance across various metrics and unique capabilities, which contribute to a diversified and resilient stock selection that can meet both current and future market demands. However, there are distinctions between all-round good performers and niche performers.

Table 5.1: Input Output variable with Self-efficiency after solving Model (5.1)

S.No	Stocks	Variance	CPT value	Return	Self-efficiency
S_1	BAJAJ-AUTO	0.0013	-0.0273	0.0001	0.4933
S_2	BRITANNIA	0.0007	-0.0065	0.0043	0.8914
S_3	HEROMOTOCO	0.0008	-0.0248	-0.0049	0.4717
S_4	ICICIBANK	0.0024	-0.0221	0.006	0.4848
S_5	INDUSINDBK	0.0004	-0.0067	0.0013	0.8383
S_6	KOTAK BANK	0.0007	-0.0080	0.0043	0.8914
S_7	L & T	0.0007	-0.0122	0.0016	0.7138
S_8	NESTLEIND	0.0009	-0.0041	0.007	1.0000
S_9	NTPC	0.0005	-0.0159	0.0001	0.7125
S_{10}	ULTRACEMCO	0.0009	-0.0176	0.0011	0.6200
S_{11}	WIPRO	0.0011	-0.0135	0.0041	0.6902
S_{12}	TCS	0.0009	0.0008	0.006	0.9569
S_{13}	ITC	0.0008	0.0042	0.0027	0.8092
S_{14}	TITAN	0.001	-0.0175	0.0059	0.8333
S_{15}	BAJFINANCE	0.002	0.0227	0.0125	1.0000
S_{16}	BAJAJFINSV	0.0011	-0.0038	0.0078	0.9756
S_{17}	SHREECEM	0.0009	-0.0134	0.0037	0.7399
S_{18}	TATASTEEL	0.0017	-0.0291	-0.0011	0.4177
S_{19}	BHARTIARTL	0.0011	-0.0267	-0.0013	0.4959
S_{20}	GRASIM	0.0011	-0.0213	-0.0037	0.4594
S_{21}	TATACONSUM	0.002	-0.0581	-0.0033	0.3361
S_{22}	M&M	0.0006	0.0086	-0.0010	1.0000
S_{23}	ONGC	0.0009	-0.0226	-0.0016	0.5309
S_{24}	MARUTI	0.0004	-0.0115	-0.0028	0.6319
S_{25}	RELIANCE	0.0016	0.0039	0.008	0.7770
S_{26}	TECHM	0.0012	-0.0059	0.0046	0.7109
S_{27}	CIPLA	0.0012	-0.0100	-0.0007	0.5287
S_{28}	COALINDIA	0.0014	-0.0324	-0.0040	0.4031
S_{29}	HINDALCO	0.0034	-0.0344	0.0008	0.2996
S_{30}	HDFCLIFE	0.001	-0.0180	-0.0010	0.5384

While the former excel across multiple areas compared to their peers, they may lack the specific, distinctive attributes that set them apart. Below is a detailed description of these groups:

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	RA
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Figure 5.1: The cross-efficiency matrix

1. All-round Good-Performing Stocks (HC/HR): Six stocks fall into the HC/HR quadrant (G1). These stocks are considered all-round good performers as they demonstrate strengths that align with the performance of other assets in the market. While a portfolio consisting of these stocks may perform well, it may not necessarily offer long-term stability. The RA and CA efficiencies, along with the Maverick Index (MI) and Portfolio Diversity Index (PDI), for this group are presented in Table 5.3. The average MI and PDI for this portfolio are 0.0364 and 0.1739, respectively.

2. Supreme and enduring stocks: Table 5.4, describes the list of stocks in the HC/LR (G2) group. The stocks within this category possess distinctive resources and strengths to set them apart or align them with the performance of other stocks. Consequently, assets from this group can be regarded as long-term stability portfolios. The portfolio obtained from this group is well-diversified with a PDI value of 0.4609.

3. Descent-Performing Stocks: Only Stock 4 is classified under the G3 category as shown in Table 5.5. This stock stands out due to its unique resources or

Table 5.2: The column and row average efficiency obtained from CEM

Stocks	S_1	S_1	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
CA efficiency	0.4816	0.8650	0.4521	0.4550	0.8027	0.8588	0.6902	0.9871	0.6750	0.6022
RA efficiency	0.6614	0.6332	0.6614	0.6182	0.6332	0.6332	0.6614	0.6182	0.6332	0.6614
Stocks	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}	S_{19}	S_{20}
CA efficiency	0.6759	0.9237	0.7644	0.7953	0.9591	0.9386	0.7219	0.4053	0.4816	0.4396
RA efficiency	0.6614	0.6614	0.6614	0.6182	0.6182	0.6182	0.6332	0.6614	0.6614	0.6614
Stocks	S_{21}	S_{22}	S_{23}	S_{24}	S_{25}	S_{26}	S_{27}	S_{28}	S_{29}	S_{30}
CA efficiency	0.3290	0.6737	0.5130	0.6007	0.7510	0.6877	0.5037	0.3894	0.2915	0.5179
RA efficiency	0.6182	0.4965	0.6614	0.6614	0.6614	0.6614	0.6614	0.6614	0.6182	0.6614

Table 5.3: Statistics of group HC/HR

Stocks	CA	RA	MI	PDI
S_{12}	0.9237	0.6614	0.0359	0.4469
S_{13}	0.7644	0.6614	0.0587	0.2236
S_{25}	0.7510	0.6614	0.0346	0.1749
S_7	0.6902	0.6614	0.0341	0.0793
S_{26}	0.6877	0.6614	0.0337	0.0749
S_{11}	0.6759	0.6614	0.0211	0.0436
Average	0.7488	0.6614	0.0364	0.1739

characteristics that differentiate it from others, giving it a distinct edge in certain aspects. However, despite these strengths, it fails to match the overall performance levels of its peers. This underperformance may be attributed to factors such as limited scalability, insufficient market presence, or other internal constraints that prevent it from achieving higher efficiency. Furthermore, this stock is less diversified compared to others, meaning it is more vulnerable to risks associated with specific market conditions or sectors. While it may offer specialized advantages, it does not have the breadth or robustness to compete effectively with other stocks in a diversified portfolio. Therefore, while it holds distinct potential, it may not be the best choice for those seeking consistent performance across a

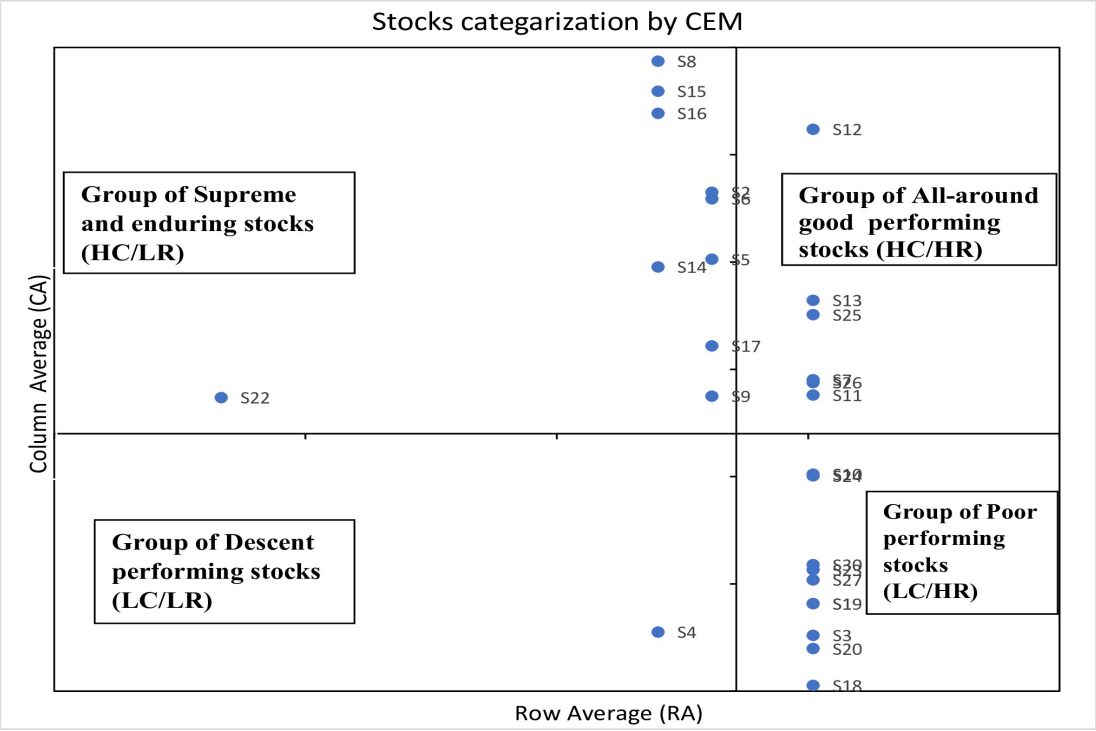


Figure 5.2: Different group of stocks based on CA & RA

Table 5.4: Statistics of group HC/LR

Stocks	CA	RA	MI	PDI
S_8	0.98713	0.6182	0.01304	0.61758
S_{15}	0.95908	0.6182	0.04266	0.61758
S_{16}	0.93862	0.6182	0.039398	0.57811
S_2	0.86501	0.6332	0.03050	0.40787
S_6	0.85878	0.6332	0.03798	0.40787
S_5	0.80265	0.6332	0.04441	0.32400
S_{14}	0.79534	0.6182	0.04772	0.34793
S_{17}	0.72189	0.6332	0.02495	0.16859
S_9	0.67497	0.6332	0.05560	0.12532
S_{22}	0.6737	0.4965	0.48431	1.0141
Average	0.8277	0.6135	0.0820	0.4609

range of market scenarios.

Table 5.5: Statistics of group LC/LR

Stocks	CA	RA	MI	PDI
S_4	0.4550	0.618	0.06541	-0.2158

4. Poor Performing Stocks: The high LC/HR (G4) quadrant contains thirteen stocks, which are generally regarded as poor performers. These stocks do not possess any distinctive resources or unique strengths that would enable them to outperform their market counterparts. As a result, they are often unable to deliver superior returns when compared to other assets in the market. Portfolios consisting primarily of assets from this group tend to struggle in terms of performance, as these stocks are not equipped with the competitive advantages needed for sustained growth or profitability. Furthermore, the overall diversification of such portfolios is typically very low, making them more susceptible to risks and less resilient to market fluctuations. These stocks lack the variety or strategic positioning to effectively balance risk and return, leading to a higher likelihood of underperformance. The detailed results of these poor performers are presented in Table 5.6.

Table 5.6: Statistics of group LC/HR

Stocks	CA	RA	MI	PDI
S_{10}	0.6022	0.6614	0.0296	-0.06253
S_{24}	0.6007	0.6614	0.05200	-0.04454
S_{30}	0.51788	0.6614	0.03962	-0.1859
S_{23}	0.5130	0.6614	0.034913	-0.19725
S_{27}	0.5037	0.6614	0.049605	-0.20058
S_{19}	0.4816	0.6614	0.02968	-0.25017
S_1	0.4816	0.6614	0.02435	-0.25411
S_3	0.45208	0.6614	0.04339	-0.28677
S_{20}	0.43963	0.6614	0.04498	-0.30536
S_{18}	0.40534	0.6614	0.03049	-0.368416
S_{28}	0.38944	0.6614	0.035076	-0.39049
S_{21}	0.32902	0.61821	0.02151	-0.45633
S_{29}	0.29147	0.61821	0.027893	-0.51537
Average	0.46212	0.65472	0.035624	-0.27060

Upon reviewing the tables, it is clear that the assets in G2 (HC/LR) exhibit the best performance, with an average Maverick Index (MI) score of 0.0820 and a higher average Portfolio Diversity Index (PDI) score of 0.4609. In contrast, G3 and G4 have notably lower PDI scores, indicating weaker overall diversification, along with lower MI values. It is important to note that a portfolio can be constructed

by selecting stocks from a combination of these groups or from the entire set of stocks. Depending on the specific goals, stocks with low MI or high PDI can also be considered for selection, aligning with the desired risk-return profile and diversification strategy.

Based on our extensive literature review, we found no existing approach that directly compares the outcomes of the method proposed in this research. In studies utilizing DEA cross-evaluations for portfolio selection, stocks are typically ranked in descending order of their cross-efficiency values, with the top n stocks selected, where n corresponds to the desired portfolio size. In contrast, our method categorizes stocks into distinct groups based on the average values of the rows and columns in the CEM. Consequently, a direct comparison between these two methods is not meaningful.

5.5 Concluding Remarks

In this chapter, we utilized the RDM DDF-based cross-efficiency analysis to categorize stocks with negative data. While there are several methods for categorizing and grouping stocks, the proposed classification approach stands out as unique and significant, particularly in the context of portfolio construction. In standard practice, CA values of CEM are commonly used to assess a unit's overall comparative performance. However, we introduced an alternative method by incorporating the RA values from CEM to gauge the diversity of performance. By combining these two concepts, we were able to categorize financial assets into different groups, facilitating a more comprehensive stock evaluation that accounts for both comparative performance and performance diversity.

The limitations of this study include its focus on the Indian stock market and the use of a restricted dataset. Future research could enhance the relevance of efficiency comparisons and benchmark selection by extending the analysis to major stock markets in other countries. Additionally, broadening the modeling framework to accommodate ordinal and interval data would allow for the consideration of fuzzy input-output data, enhancing the robustness of the analysis.

Chapter 6

Ranking of Efficient Fuzzy Portfolios by Hybrid MSBM-TOPSIS Technique

This chapter¹ focuses on ranking investment portfolios by integrating the MSBM model of DEA with a multi-criteria decision-making approach. Specifically, it expands the MSBM model to assess portfolios with positive and negative inputs and outputs in a fuzzy environment, using the possibilistic mean return of assets as the output, while employing possibilistic variance and semivariance as inputs. The ranking process is conducted in two stages: first, portfolios are assessed using an MSBM fuzzy portfolio model and then ranked using the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method. This hybrid MSBM-TOPSIS framework offers a robust and reliable ranking system, allowing investors to identify efficient portfolios leveraging the strengths of both methods. Detailed numerical examples are provided to validate the proposed approach, and the results are compared with other existing DEA methods to demonstrate the accuracy and feasibility of the technique.

¹The content of this chapter is based on the research paper “Ranking of Efficient Fuzzy Portfolios by Hybrid MSBM-TOPSIS Technique” **Accepted** in “Foundation of Computing and Decision Science” (ESCI)

6.1 Introduction

Possibility theory, introduced by Zadeh [203] is an uncertainty framework designed to address incomplete and imprecise information. The concept of possibility theory used in fuzzy set approaches is generally more applicable when examining an early stage of a decision process and when little information is available. Carlsson and Fuller [22] proposed the definition of lower and upper possibilistic mean values of a fuzzy number and considered them as possibility distributions. Besides, Zhang and Nie [209] defined the lower and upper possibilistic variances and covariances of fuzzy numbers. Carlsson et al. [23] developed a possibilistic approach to portfolio selection with the highest utility score by assuming the returns of assets as trapezoidal fuzzy numbers. Liu and Zhang [111] formulated two possibilistic mean-semivariance models for portfolio selection, in which market risk and liquidity risk are measured by lower possibilistic semi-variance. Chen et al. [30] proposed DEA-based fuzzy portfolio estimation models to determine fuzzy portfolio efficiency. They developed three DEA-based fuzzy models incorporating different risk measures: possibilistic variance, possibilistic semi-variance, and possibilistic semi-absolute deviation. Subsequently, fuzzy portfolio optimization models based on possibility and credibility theories were introduced by [29, 72, 177].

The MCDM methods makes decisions when multiple alternatives are available for contradictory criteria. Its principal goal is to maintain control over the decision-making process in case of various alternatives and criterion diversity. TOPSIS is a powerful MCDM approach developed by Hwang and Yoon [78] to evaluate the performance of different alternatives based on the similarity degree with the ideal solution. According to this technique, the best option should have the shortest distance from the ideal solution and the farthest from the negative ideal.

Following the work presented by Chen et al. [30], we offer the hybrid MSBM-TOPSIS technique that considers efficient fuzzy portfolios as the available alternative and different inputs and outputs of efficient portfolios as the evaluation attributes. Similar to the work [30], randomly generated fuzzy portfolios are considered in this study, where the return from the asset is a trapezoidal fuzzy number. Two types of risk measures, possibilistic variance, and possibilistic semi-variance, are considered here with numerical examples which effectively verify

the proposed approach.

6.1.1 Literature Review of Ranking models in DEA

In DEA, each DMU is self-evaluated with the most favorable weights for input and output to obtain relative efficiency. Consequently, DEA might assess too many DMUs as the best and cannot distinguish them further. To address this issue, Andersen et al. [8] proposed a VRS super-efficiency DEA model, where the efficient DMU under evaluation is excluded and assessed against a linear combination of the remaining DMUs within the production possibility set. Several extensions and adaptations of the super-efficiency DEA model have been proposed to overcome its limitations. For example, Lin and Chen [106] developed a DDF-based super-efficiency DEA model that builds upon the [8] framework. This approach introduces a suitable directional vector, enabling the model to handle datasets with negative values while addressing the infeasibility issue associated with the VRS radial super-efficiency DEA model. Similarly, Tran et al. [176] integrated the super-efficiency model with the SBM model to enhance its applicability. In applied research, Pan et al. [140] employed the super-efficiency model to analyze water use efficiency in cities across Shandong Province, illustrating its practical utility. Additionally, Hadi-Vencheh and Esmailzadeh [73] proposed the RDM super-efficiency model to provide a unique ranking for efficient DMUs, addressing challenges related to ranking ties among efficient units.

It is important to note that each method is based on a specific theory and framework, which can lead to different ranking outcomes for the same DMU. In practice, selecting a ranking method that the decision maker can trust is a critical concern. The challenge of ranking efficient DMUs can be addressed by employing MCDM techniques, which, although distinct from traditional DEA models, offer a complementary approach for more reliable rankings. Some studies have combined TOPSIS and DEA within a unified framework to generate a distinct ranking of DMUs. In these studies, one technique is typically used as the primary method, while the other serves as a supplementary tool. By leveraging both DEA and TOPSIS, decision-makers can effectively achieve the objectives of a corporate sourcing strategy through a comprehensive evaluation approach. Lotfi et al. [113] considered efficient DMUs from different DEA models and ranked them using the TOPSIS method. Banihashemi and Khalilzadeh [11] proposed a new method for

ranking efficient units and compared it with the other methods presented in the literature, such as the super-efficiency model in DEA, the concept of Shapley value in game theory, and the TOPSIS method in multi-criteria decision making. Rakhshan [145] utilized a hybrid approach based on TOPSIS and DEA methods to obtain a rational ranking of efficient DMUs. Jablonsky [79] presented two models for ranking efficient units. Both are based on MCDM methods; the first uses the goal programming technique for deriving a super-efficiency measure, and the second combines AHP and DEA principles. Chitnis and Vaidya [42] employed the DEA method to evaluate the efficiency of 15 Indian bank branches, using the SE-CCR-DEA model and TOPSIS for ranking. The study compares the outcomes of the CCR-DEA, SE-CCR-DEA, and TOPSIS methods. Rashid et al. [126] used the hybrid DEA-TOPSIS method for ranking of the 14 state water service providers in Malaysia. Omrani et al. [131] used a three-step DEA model to obtain four types of efficiency (technical efficiency, social efficiency, environmental efficiency scores, and sustainable efficiency) for Iranian airlines. They integrate DEA models with the TOPSIS method to aggregate the final efficiency score. Zhang et al. [208] presented a method to compare and rank the interval efficiencies of DMUs. Soltanifar et al. [166] recently introduced an integrated framework combining multi-attribute decision-making with DEA models to address heterogeneous attributes in evaluating European countries' SDG progress. This method reduces computational complexity and eliminates the need for expert-defined weights, with results compared to standard MCDM techniques. Aarbi et al. [10] introduced an innovative hybrid DEA-TOPSIS approach to identify the most sustainable refrigeration system technologies in the UK. Ersoy [62] focused on evaluating the performance of distance education departments at public universities in Turkey during the 2018–2019 academic year, utilizing both the DEA and TOPSIS methods. The study also aimed to compare and rank the efficient decision-making units based on their efficiency measurements. Bansal and Mehra [16] proposed an integrated interval DEA model that classifies DMUs into three categories: super-efficient, efficient, and inefficient in a single step, accommodating integer and negative data.

Despite these advancements, super-efficiency DEA models are criticized for infeasibility in certain scenarios. This limitation highlights the ongoing need to develop alternative methods to rank efficient DMUs effectively. Researchers continue to explore innovative approaches to refine ranking techniques and improve

the robustness of DEA models in various applications.

6.1.2 Contributions and Purpose

This study aims to advance the application of DEA by addressing the ranking of efficient fuzzy portfolios, incorporating two risk metrics, and handling negative input-output variables. The key contributions of this research are summarized as follows:

1. While the MSBM model is widely used to evaluate the efficiency of DMUs in the presence of negative data, its application in ranking portfolios under uncertainty remains unexplored. This study is the first to extend the MSBM model within a possibilistic framework, effectively addressing fuzzy variables. By integrating fuzziness into the model, this research fills a critical gap in portfolio efficiency analysis and introduces a novel approach to enhance decision-making under uncertainty.
2. Although DEA and TOPSIS are recognized as powerful tools across various domains, few studies such as Hatami et al. [74] and Lin et al. [105] have addressed the full ranking of portfolios with negative input-output variables. This research combines the MSBM fuzzy portfolio model with the TOPSIS method to leverage the strengths of both techniques, enabling comprehensive and reliable portfolio rankings.
3. Implementing MSBM fuzzy portfolio estimation models with TOPSIS to classify efficient portfolios is a new application in the field of DEA.
4. We demonstrate the applicability and effectiveness of our methodology using actual data from the Stock Exchange in portfolio construction challenges.

The remainder of the chapter is organized in the following manner. Section 6.2.1 presents some basic definitions and preliminaries regarding fuzzy numbers and their possibilistic mean, variance, and semi-variance for portfolio selection. Section 6.3 offers the MSBM fuzzy portfolio estimation model in a possibilistic framework. Section 6.4 presents the various steps of the TOPSIS method to rank the efficient portfolios. In Section 6.5, numerical examples are presented to justify

the proposed approach. Finally, Section 6.6 presents the conclusions with future directions.

6.2 Preliminaries

This section covers the basics of fuzzy sets and fuzzy numbers. Additionally, we present the mathematical models developed by Carlsson et al. [23], for computing the possibilistic mean, possibilistic variance, and possibilistic semi-variance of a fuzzy number are presented here.

Definition 6.2.1. Let X be a universal set. A fuzzy set $\tilde{A}$ in X is denoted by

$$\tilde{A} = \{ \langle x, \mu_{\tilde{A}}(x) \rangle \mid x \in X \},$$

where $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ describes the degree of belonging of an element $x \in X$ to the set $\tilde{A}$.

Definition 6.2.2 (γ -cut). Let $\tilde{A}$ be a fuzzy set in X and $\gamma \in (0, 1]$. The γ -cut of the fuzzy set $\tilde{A}$ is the crisp set A_γ given by

$$A_\gamma = \{x \in X : \mu_{\tilde{A}}(x) \geq \gamma\}.$$

Definition 6.2.3. A fuzzy set $\tilde{A}$ in $\mathbb{R}$ is called a fuzzy number if it satisfies the following conditions:

1. $\tilde{A}$ is normal.
2. $\tilde{A}_\gamma$ is a closed interval for every $\gamma \in (0, 1]$.
3. The support of $\tilde{A}$ is bounded.

Definition 6.2.4 (Trapezoidal fuzzy number (TrFN)). A fuzzy number $\tilde{A}$ represented by the quadruplet $\tilde{A} = [a, b, \alpha, \beta]$, where $[a, b]$ is the tolerance interval; α and β denote the left and right width of the TrFN- $\tilde{A}$, respectively, is called a trapezoidal fuzzy number if

its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a - \alpha, x \geq b + \beta, \\ \frac{x - a + \alpha}{\alpha}, & a - \alpha \leq x \leq a, \\ 1, & a \leq x \leq b, \\ \frac{b + \beta - x}{\beta}, & b \leq x \leq b + \beta, \end{cases}$$

Further, the γ -cut of the TrFN $\tilde{A} = (a, b, \alpha, \beta)$ is the closed interval and given by

$$A^\gamma = [a_1^\gamma, a_2^\gamma] = [(a - (1 - \gamma)\alpha, b + (1 - \gamma)\beta], \quad \gamma \in (0, 1].$$

Definition 6.2.5. Let $\tilde{A}$ be a fuzzy number with $A^\gamma = [a_1^\gamma, a_2^\gamma]$, $\gamma \in (0, 1]$. The (crisp) possibilistic mean (or expected value) and possibilistic variance of a fuzzy number $\tilde{A}$, denoted by $E(\tilde{A})$ and $Var(\tilde{A})$, is defined as

$$E(\tilde{A}) = \int_0^1 \gamma (a_1^\gamma + a_2^\gamma) d\gamma,$$

and

$$Var(\tilde{A}) = \frac{1}{2} \int_0^1 \gamma (a_2^\gamma - a_1^\gamma)^2 d\gamma.$$

respectively.

It is easy to see that for a TrFN $\tilde{A} = [a, b, \alpha, \beta]$

$$E(\tilde{A}) = \int_0^1 \gamma [a - (1 - \gamma)\alpha + b + (1 - \gamma)\beta] d\gamma = \frac{(a + b)}{2} + \frac{(\beta - \alpha)}{6},$$

and

$$Var(\tilde{A}) = \left[\frac{(b - a)}{2} + \frac{(\alpha + \beta)}{6} \right]^2 + \frac{(\alpha + \beta)^2}{72}.$$

Further, [153] defined the lower possibilistic semi-variance of a fuzzy number $\tilde{A}$ denoted by $Var^-(\tilde{A})$, as

$$Var^-(\tilde{A}) = \int_0^1 2\gamma (E(\tilde{A}) - a_1^\gamma)^2 d\gamma.$$

Thus for a TrFN $\tilde{A} = [a, b, \alpha, \beta]$, the lower possibilistic semi-variance is as follows

$$Var^-(\tilde{A}) = \left[\frac{(b-a)}{2} + \frac{(\alpha+\beta)}{6} \right]^2 + \frac{(\alpha)^2}{18}.$$

Assume there are N risky assets whose returns are TrFNs represented by $\tilde{r}_i = (a_i, b_i, \alpha_i, \beta_i)$, $i = 1, 2, \dots, n$ with investment proportion w_i , $i = 1, 2, \dots, N$, such that $w_i \geq 0$; $\sum_{i=1}^n w_i = 1$. Also, there is a fixed linear transaction cost K_i associated with each asset. As a result, the portfolio's probabilistic expected net return, variance, and lower semi-variance, denoted by R , σ^2 , and $(\sigma^2)^-$ are given as follows

$$R = E \left(\sum_{i=1}^N \tilde{r}_i w_i - \sum_{i=1}^N K_i w_i \right) = \sum_{i=1}^N \left[\left\{ \frac{1}{2} \left(a_i + b_i + \frac{1}{3} (\beta_i - \alpha_i) \right) w_i \right\} - K_i w_i \right],$$

$$\sigma^2 = Var \left(\sum_{i=1}^N \tilde{r}_i w_i \right) = \left[\left(\frac{(b_i - a_i)}{2} + \frac{(\alpha_i + \beta_i)}{6} \right) w_i \right]^2 + \frac{1}{72} \left[\sum_{i=1}^N (\alpha_i + \beta_i) w_i \right]^2,$$

and

$$(\sigma^2)^- = Var^- \left(\sum_{i=1}^N \tilde{r}_i w_i \right) = \left[\left(\frac{(b_i - a_i)}{2} + \frac{(\alpha_i + \beta_i)}{6} \right) w_i \right]^2 + \frac{1}{18} \left[\sum_{i=1}^N (\alpha_i w_i) \right]^2$$

respectively.

6.2.1 Modified Slack Based Model to handle negative data

Sharp et al. [160] proposed the MSBM model to handle the challenge of negative data in conventional DEA models. Wei et al. [185] modified the MSBM model to assess the performance of production units, adopting a different direction from [160].

Consider that there are n DMUs to be evaluated, each using m inputs to generate s outputs. Let x_{ij} , ($i = 1, \dots, m$) represent the i^{th} , input for the j^{th} DMU, and y_{rj} , ($r = 1, \dots, s$) denote the r^{th} output of the j^{th} DMU. The modified MSBM method is presented as follows:

Model (6.1)

$$\text{Min } \theta_k = \frac{1 - \frac{1}{m} \sum_{i=1}^m \frac{s_{ik}^-}{R_i^-}}{1 + \frac{1}{s} \sum_{r=1}^s \frac{s_{rk}^+}{R_r^+}}$$

subject to,

$$\sum_{j=1}^n \lambda_j y_{rj} - s_{rk}^+ = y_{rk}, \quad r = 1, 2, \dots, s$$

$$\sum_{j=1}^n \lambda_j x_{ij} + s_{ik}^- = x_{ik}, \quad i = 1, 2, \dots, m,$$

$$\sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, \quad j = 1, 2, \dots, n, \quad s_{ik}^-, s_{rk}^+ \geq 0$$

The objective function θ_k^* represents the optimal efficiency score of DMU_k . The terms s_{ik}^- ($i = 1, \dots, m$) and s_{rk}^+ ($r = 1, \dots, s$) represent the i^{th} input excess and r^{th} output shortfall of DMU_k , referred to as slacks. The λ_j ($j = 1, \dots, n$) are intensity variables used to link inputs and outputs in a convex combination. Also, R_i^- and R_r^+ are the feasible directions to move and are defined as follows:

$$R_i^- = \max_{j=1,2,\dots,n} x_{ij} - \min_{j=1,2,\dots,n} x_{ij}, \quad i = 1, 2, \dots, m$$

$$R_r^+ = \max_{j=1,2,\dots,n} y_{rj} - \min_{j=1,2,\dots,n} y_{rj}, \quad r = 1, 2, \dots, s$$

In the next section, we introduce the MSBM fuzzy portfolio estimation models proposed by Chen et al. [30].

6.3 MSBM fuzzy portfolio evaluation models in the framework of possibilistic theory

In this section, we present the MSBM fuzzy portfolio estimation model to handle the unpredictability of the variables. Assume that there are n portfolios to be evaluated. To evaluate portfolio performance, we consider the risk associated with the portfolio as the input indicator and the corresponding return as the output indicator. In this study, we assume N risky assets with fuzzy return as $\tilde{r}_\iota = (a_\iota, b_\iota, \alpha_\iota, \beta_\iota)$, $\iota = 1, 2, \dots, N$ with investment proportions $w_{\iota j}$, $\iota = 1, 2, \dots, N$;

$j = 1, 2, \dots, n$, such that $\sum_{i=1}^n w_{ij} = 1$, are considered for constructing j^{th} portfolio which is considered to be a DMU. Suppose K_i is the unit transaction cost of a unit for the i^{th} risky asset. Then the expected net return, variance, and semi-variance of the j^{th} portfolio denoted by $\bar{R}_j$, σ_j^2 and $(\sigma^2)_j^-$ respectively for $(j = 1, 2, \dots, n)$ are given respectively. To employ the MSBM models for the efficiency evaluation of all portfolios, in our study we consider one input and one output ($m = s = 1$) criteria as risk and return respectively.

6.3.1 MSBM fuzzy portfolio evaluation model in possibilistic mean-variance framework

We extend the above-presented model (6.1) in the possibilistic mean-variance framework. The mathematical form of the MSBM fuzzy portfolio model with possibilistic variances input (risk) and possibilistic mean as output (return) is given as follows:

$$\begin{aligned}
 \text{Model (6.2)} \quad \text{Min } \rho_k &= \frac{1 - \frac{s_{1k}^-}{R_1^-}}{1 + \frac{s_{1k}^+}{R_2^+}} \\
 \text{s. t. } \sum_{j=1}^n \lambda_j &\left(\left\{ \sum_{i=1}^N \frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right\}^2 + \frac{1}{72} \left[\sum_{i=1}^N (\alpha_i + \beta_i) w_{ij} \right]^2 \right) + s_{1k}^- \\
 &= \left(\left\{ \sum_{i=1}^N \frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ik} \right\}^2 + \frac{1}{72} \left[\sum_{i=1}^N (\alpha_i + \beta_i) w_{ik} \right]^2 \right) \\
 \sum_{j=1}^n \lambda_j &\left(\sum_{i=1}^N \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{ij} - \sum_{i=1}^n K_i w_{ij} \right) - s_{1k}^+ \\
 &= \sum_{i=1}^N \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{ik} - \sum_{i=1}^n K_i w_{ik},
 \end{aligned}$$

$$\sum_{j=1}^n \lambda_j = 1 \quad \lambda_j \geq 0, s_{1k}^-, s_{1k}^+ \geq 0, \quad j = 1, 2, \dots, n$$

Here, R_1^- and R_2^+ can be modified as follows :

$$R_1^- = \max_{j=1,2,\dots,n} \left(\sum_{i=1}^N \left(\frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right)^2 + \frac{1}{72} \left[\sum_{i=1}^N (\alpha_i + \beta_i) w_{ij} \right]^2 \right) \\ - \min_{j=1,2,\dots,n} \left(\sum_{i=1}^N \left(\frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right)^2 + \frac{1}{72} \left[\sum_{i=1}^N (\alpha_i + \beta_i) w_{ij} \right]^2 \right)$$

$$R_2^+ = \max_{j=1,2,\dots,n} \left(\sum_{i=1}^N \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{ij} - \sum_{i=1}^N K_i w_{ij} \right) \\ - \min_{j=1,2,\dots,n} \left(\sum_{i=1}^N \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{ij} - \sum_{i=1}^N K_i w_{ij} \right)$$

Let $(\rho^*, \lambda_1^*, \lambda_2^*, \dots, \lambda_n^*)$ be an optimum solution of (Model (6.2)). Further, ρ_k^* indicates the optimal efficiency score of the portfolio under evaluation (k^{th}) Here, s_{1k}^- and s_{1k}^+ , respectively denote the excess of input (risk) and shortfall of output (return) of k^{th} portfolio, and they are known as slacks.

6.3.2 MSBM fuzzy portfolio evaluation model in possibilistic mean semi-variance framework

The mathematical form of the MSBM fuzzy portfolio model with possibilistic semi-variance input (risk) and possibilistic mean as output (return) is given as follows:

$$\begin{aligned}
\textbf{Model (6.3)} \quad \min \quad \eta_k &= \frac{1 - \frac{s_{1k}^-}{R_3^-}}{1 + \frac{s_{1k}^+}{R_2^+}} \\
\text{s. t.} \quad \sum_{j=1}^n \lambda_j &\left(\left\{ \sum_{i=1}^N \frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right\}^2 + \frac{1}{18} \left[\sum_{i=1}^N (\alpha_i) w_{ij} \right]^2 \right) + s_{1k}^- \\
&= \left(\left\{ \sum_{i=1}^N \frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ik} \right\}^2 + \frac{1}{18} \left[\sum_{i=1}^N (\alpha_i) w_{ik} \right]^2 \right) \\
&\sum_{j=1}^n \lambda_j \left(\sum_{i=1}^N \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{ij} - \sum_{i=1}^N K_i w_{ij} \right) - s_{1k}^+ \\
&= \sum_{i=1}^n \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{ik} - \sum_{i=1}^N K_i w_{ik}, \\
&\sum_{j=1}^n \lambda_j = 1 \quad \lambda_j \geq 0, \quad s_{1k}^-, s_{1k}^+ \geq 0, \quad j = 1, 2, \dots, n
\end{aligned}$$

Here, R_2^+ is the same as described in the model (6.2), and R_3^- is defined as follows:

$$\begin{aligned}
R_3^- &= \max_{j=1,2,\dots,n} \left(\sum_{i=1}^N \left(\frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right)^2 + \frac{1}{18} \left[\sum_{i=1}^N (\alpha_i) w_{ij} \right]^2 \right) \\
&- \min_{j=1,2,\dots,n} \left(\sum_{i=1}^N \left(\frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right)^2 + \frac{1}{18} \left[\sum_{i=1}^N (\alpha_i) w_{ij} \right]^2 \right)
\end{aligned}$$

The models mentioned above can determine the efficiency values of all fuzzy portfolios; however, they do not provide a unique ranking of the portfolios. It is important to note that if multiple portfolios are deemed efficient, establishing a complete and unique ranking becomes impossible. Therefore, we apply the TOPSIS method to rank all efficient portfolios and obtain a unique ranking. It is a practical method for organizing the available options based on numerous

criteria. It outperforms other MCDM methods for solving a wide range of decision-making problems in the real world, due to its simple computational approach and systematic procedure.

6.4 TOPSIS method to rank the efficient portfolio

TOPSIS is founded on the notion of selecting the best alternative (here DMU) that is as close to the positive ideal solution while being as far away from the negative ideal solution as possible. The best performance values for each portfolio demonstrated by each criterion (here, the criterion is risk and return) in the decision matrix make up the perfect solution. As a result, we used a negative feature criterion for risk and a positive feature criterion for return.

Let L be the number of efficient portfolios. Suppose we have m inputs and s outputs for each portfolio, let D_1 and D_2 denote respectively the input and output matrix corresponding to L efficient portfolios. Therefore

$$D_1 = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1L} \\ x_{21} & x_{22} & \dots & x_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mL} \end{bmatrix} \quad D_2 = \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1L} \\ y_{21} & y_{22} & \dots & y_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ y_{s1} & y_{s2} & \dots & y_{sL} \end{bmatrix}$$

Now we have to follow the following steps:

Step 1. Normalization

Calculate the normalized decision matrix corresponding to the cost criterion (m inputs) and benefit criteria (s outputs). Let $p_{i\hat{l}}$ and $q_{r\hat{l}}$ ($i = 1, 2, \dots, m; r = 1, 2, \dots, s$) denotes the normalized value of i^{th} input and r^{th} output of $\hat{l}^{th}$ DMU respectively and calculated as follows:

$$p_{i\hat{l}} = \frac{x_{i\hat{l}}}{\sqrt{\sum_{\hat{o}=1}^L x_{i\hat{o}}^2}}, \quad \forall i = 1, 2, \dots, m, \hat{l} = 1, 2, \dots, L$$

and

$$q_{r\hat{l}} = \frac{y_{r\hat{l}}}{\sqrt{\sum_{\hat{o}=1}^L y_{r\hat{o}}^2}}, \quad \forall r = 1, 2, \dots, s, \hat{l} = 1, 2, \dots, L.$$

Step 2. Determine the positive ideal and negative-ideal points: Let p_{iw} and q_{rw} denote respectively the i^{th} input and r^{th} output of the worst DMU. Similarly, let p_{ib} and q_{rb} denote respectively the i^{th} input and r^{th} output of the best DMU. Then

$$p_{iw} = \max_{1 \leq \hat{l} \leq L} p_{i\hat{l}}, \quad i = 1, 2, \dots, m,$$

and

$$q_{rw} = \min_{1 \leq \hat{l} \leq L} q_{r\hat{l}}, \quad r = 1, 2, \dots, s.$$

Similarly

$$p_{ib} = \min_{1 \leq \hat{l} \leq L} p_{i\hat{l}}, \quad i = 1, 2, \dots, m,$$

and

$$q_{rb} = \max_{1 \leq \hat{l} \leq L} q_{r\hat{l}}, \quad r = 1, 2, \dots, s.$$

Step 3. Measuring the distance from the ideal and negative ideal points:

Let $d_{\hat{l}w}$ be the distance between the target DMU $\hat{l}$ and the worst DMU D_w (best DMU D_b), so

$$d_{\hat{l}w} = \sqrt{\sum_{i=1}^m (p_{i\hat{l}} - p_{iw})^2 + \sum_{r=1}^s (q_{r\hat{l}} - q_{rw})^2}, \quad \hat{l} = 1, 2, \dots, L,$$

and

$$d_{\hat{l}b} = \sqrt{\sum_{i=1}^m (p_{i\hat{l}} - p_{ib})^2 + \sum_{r=1}^s (q_{r\hat{l}} - q_{rb})^2}, \quad \hat{l} = 1, 2, \dots, L.$$

Step 4. Calculate the similarity to the worst condition:

Let s_w be the relative closeness of the target DMU $\hat{l}$ to the worst DMU D_w .

$$s_{\hat{l}w} = \frac{d_{\hat{l}w}}{d_{\hat{l}w} + d_{\hat{l}b}}, \quad \hat{l} = 1, 2, \dots, L.$$

Here, $0 \leq s_{\hat{l}w} \leq 1$, and

$$s_{\hat{l}w} = \begin{cases} 1, & \text{if and only if the alternative solution has the best condition} \\ 0, & \text{if and only if the alternative solution has the worst condition.} \end{cases}$$

Step 5. Ranking criterion:

Rank the alternatives or portfolios in the descending order of $s_{\hat{l}w}$, the alternative

with a maximum value of $s_{\hat{I}_w}$ is the best-performing alternative.

The working procedure of the proposed approach has been described in Algorithm 1.

Algorithm (6.1): Computation procedure of the proposed method

Input: The Fuzzy returns of Stocks with the different proportions invested in the stocks.

Output: The full ranking of efficient portfolios.

Steps:

1. Calculate the efficiency of portfolios by using MSBM fuzzy portfolio models.
 2. Rank the portfolios according to the efficiency score.
 3. If the no. of efficient portfolios ≥ 1 , then apply the TOPSIS method.
 4. Identification of attributes (Return, Risk) for all the alternatives (efficient portfolios).
 5. Form a weighted normalized decision matrix.
 6. Calculate the positive and negative ideal points.
 7. Evaluate the similarity to the worst condition as the measure of relative closeness $s_{\hat{I}_w}$.
 8. Rank the efficient portfolio in the decreasing order of $s_{\hat{I}_w}$.
-

6.5 Numerical Illustration

Two numerical examples are presented to show the practicality and validity of the proposed method. In the first example, the possibilistic mean return is used as the output, and the possibilistic variance is applied as the input to assess the portfolio's efficiency. In the second example, we continue using the possibilistic mean return as the output, but replace the variance with the possibilistic semi-variance as the input, to capture a more nuanced measure of risk. Both simulations were conducted using MATLAB software, ensuring a robust and accurate implementation of the proposed methodology. These examples validate the approach and demonstrate its effectiveness in real-world portfolio selection scenarios.

Example 6.5.1. In this example, the returns from the selected risky assets/stocks are presented in Table 6.1 as trapezoidal fuzzy numbers, consistent with the approach used by Mehlawat et al. [124] in their study. In their experimental analysis, Mehlawat et al. [124] used historical data from five assets, covering the duration between April 2009 - March 2015, with weekly closing prices employed to simulate future returns. For the current study, we replicate this setup by creating random sample portfolios using the asset weights shown in Table 6.2. For simplicity, we focus on five stocks that exhibit negative returns.

Model 6.2 is then applied to assess the efficiency and rank all the randomly generated portfolios. The results of this analysis are displayed in Table 6.3, providing a comprehensive evaluation of portfolio performance under the defined conditions.

Table 6.1: The possibility distribution of the return rates and their respective transaction cost of 5 stocks

Stocks	a_t	b_t	α_t	β_t	K_t
S_1	-0.1071	-0.0703	0.0105	0.0541	0.003
S_2	0.0883	0.1057	0.057	0.0772	0.003
S_3	0.0874	0.0948	0.0132	0.0928	0.003
S_4	-0.0907	-0.082	0.0055	0.0336	0.003
S_5	-0.0954	-0.0816	0.0093	0.013	0.003

Table 6.2: The weight of each stock in different randomly generated portfolios

w	1	2	3	4	5	6	7	8	9	10
w_1	0.1875	0.2660	0.2616	0.1956	0.2608	0.1677	0.0637	0.0948	0.3678	0.0891
w_2	0.1566	0.1102	0.2422	0.0962	0.3080	0.2472	0.2610	0.2547	0.2150	0.0561
w_3	0.3301	0.2483	0.1600	0.2975	0.2684	0.2437	0.1899	0.2408	0.0389	0.3180
w_4	0.1585	0.0805	0.2621	0.3064	0.1287	0.1964	0.2413	0.0050	0.2217	0.2013
w_5	0.1673	0.2905	0.0741	0.1043	0.0340	0.1450	0.2442	0.4048	0.1567	0.3355
w_p	11	12	13	14	15	16	17	18	19	20
w_1	0.0099	0.2206	0.3053	0.2608	0.1184	0.1714	0.0244	0.3602	0.0844	0.3571
w_2	0.2762	0.3457	0.2831	0.1016	0.3696	0.0105	0.2256	0.1147	0.0871	0.0746
w_3	0.2373	0.1664	0.0314	0.1039	0.0955	0.1306	0.2335	0.1934	0.2467	0.3654
w_4	0.2792	0.0005	0.1601	0.1654	0.2866	0.3579	0.1711	0.2879	0.2542	0.0149
w_5	0.1974	0.2669	0.2201	0.3684	0.1299	0.3296	0.1253	0.0438	0.3276	0.1879
w	21	22	23	24	25	26	27	28	29	30
w_1	0.2904	0.1345	0.0715	0.1815	0.1990	0.0479	0.2311	0.0904	0.2334	0.2311
w_2	0.1905	0.0337	0.2241	0.1996	0.2874	0.2777	0.2503	0.0134	0.0418	0.1278
w_3	0.2333	0.3373	0.2388	0.4058	0.0994	0.2683	0.1133	0.2684	0.4218	0.0272
w_4	0.0792	0.2580	0.1773	0.0647	0.1409	0.2410	0.0752	0.2950	0.1474	0.3668
w_5	0.2066	0.2366	0.2884	0.1484	0.2734	0.1651	0.3301	0.3328	0.1556	0.2452

Table 6.3: The efficiency score and the ranking of portfolios in possibilistic mean-variance framework

Portfolio	Return	Risk	Efficiency by model (6.2)	Ranking by Model (6.3)
1	0.0043	0.00052	0.4714	16
2	-0.0198	0.00048	0.4396	18
3	-0.0112	0.00057	0.3138	25
4	-0.0124	0.00045	0.5188	13
5	0.0212	0.00072	0.3558	23
6	0.0048	0.00055	0.4168	19
7	-0.0033	0.00045	0.5696	11
8	0.0040	0.00049	0.5376	12
9	-0.0392	0.00056	0.272	29
10	-0.0174	0.00034	0.672	5
11	0.0085	0.00045	0.6361	7
12	0.0076	0.00065	0.2256	30
13	-0.0286	0.00058	0.2756	28
14	-0.0488	0.00039	0.4689	17
15	-0.0004	0.00055	0.3918	20
16	-0.0605	0.00027	1.0000	1
17	0.0177	0.00031	1.0000	1
18	-0.0280	0.00055	0.3207	2
19	-0.0249	0.00033	0.6474	6
20	-0.0042	0.00060	0.3018	27
21	-0.0078	0.00059	0.3122	26
22	-0.0172	0.00036	0.6346	8
23	-0.0012	0.00044	0.5908	9
24	0.0263	0.00061	1.0000	1
25	-0.0155	0.00053	0.3816	21
26	0.0148	0.00050	0.5745	10
27	-0.0200	0.00052	0.3782	22
28	-0.0343	0.00028	1.0000	1
29	0.0005	0.00049	0.5065	14
30	-0.0577	0.00036	0.4788	15

Table 6.4: Ranking of efficient portfolios by different methods in possibilistic mean-variance framework

Portfolio	Return	Risk	Ranking by Model (6.2)	Ranking by Wei et al. [185]	Ranking by MSBM-TOPSIS method
16	-0.0605	0.00027	1	4	4
17	0.0177	0.00031	1	2	1
24	0.0263	0.00061	1	1	2
28	-0.0343	0.00028	1	3	3

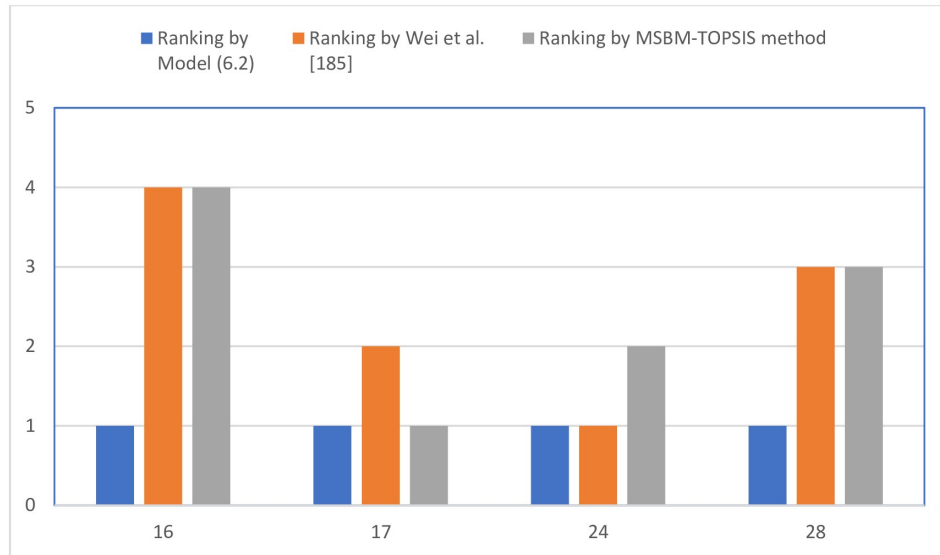


Figure 6.1: Rankings of efficient portfolios in possibilistic mean-variance framework

Remark 6.5.1. Table 6.3 demonstrates that the efficiency scores of portfolios 16, 17, 24, and 28 are identical, making it impossible to distinguish between these portfolios based on efficiency alone. To address this, the MSBM-TOPSIS approach is applied, as outlined earlier, to provide a comprehensive ranking of all the portfolios. The fourth, fifth, sixth, and seventh columns of Table 6.4 present the rankings of the efficient portfolios

as determined by Model (6.2), by [185], and by the proposed MSBM-TOPSIS method, respectively. A graphical representation of these rankings is shown in Figure 6.1. It is important to note that while the method proposed by Wei et al. [185] provides a complete ranking of all randomly generated portfolios, the Super Efficiency model introduced by Hadi-Vencheh and Esmailzadeh [73] faces feasibility issues. Interestingly, the rankings for portfolio 16 and portfolio 28 produced by the proposed MSBM-TOPSIS method align with those obtained by [185]. However, compared to the method in [185], the proposed approach is less computationally intensive and more user-friendly, offering a simpler yet effective solution for portfolio ranking.

Example 6.5.2. When asset return distributions are not symmetric, using variance as a risk measure can be misleading, as it can produce portfolio projections that significantly deviate from what would be expected in real-world scenarios. Variance treats extremely high and low returns as equally undesirable, which is a significant limitation. To address this issue, many researchers have explored alternative risk measures that better capture the nature of risk in asymmetric distributions. One such alternative is semi-variance, which focuses on downside risk, i.e., the risk of returns falling below a certain threshold.

In this example, we use the possibilistic semi-variance as an input parameter instead of the possibilistic variance, as it more accurately reflects the downside risk associated with the portfolio. Table 6.5 presents the rankings obtained using model (6.3), while Table 6.6 provides the complete ranking of efficient portfolios across different methods. This approach offers a more robust and realistic assessment of portfolio risk, especially in cases where asset return distributions are skewed or exhibit heavy tails.

Table 6.5: The efficiency score and the ranking of portfolios in possibilistic mean semi-variance framework

Portfolio	Return	Risk	Efficiency by Model (6.3)	Ranking by Model (6.3)
1	0.0043	0.00045	0.4694	15
2	-0.0198	0.00043	0.4175	18
3	-0.0112	0.00051	0.2866	25
4	-0.0124	0.00039	0.5195	12
5	0.0212	0.00064	0.3272	23
6	0.0048	0.00049	0.4014	19
7	-0.0033	0.00040	0.5524	11
8	0.0040	0.00044	0.5082	14
9	-0.0392	0.00052	0.2289	28
10	-0.0174	0.00029	0.6719	5
11	0.0085	0.00039	0.6316	8
12	0.0076	0.00060	0.167	30
13	-0.0286	0.00053	0.2242	29
14	-0.0488	0.00036	0.4413	17
15	-0.0004	0.00050	0.353	20
16	-0.0605	0.00025	1.00	1
17	0.0177	0.00027	1.00	1
18	-0.0280	0.00049	0.2996	24
19	-0.0249	0.00029	0.6418	6
20	-0.0042	0.00053	0.2853	26
21	-0.0078	0.00052	0.2826	27
22	-0.0172	0.00031	0.638	7
23	-0.0012	0.00039	0.5777	9
24	0.0263	0.00053	1.00	1
25	-0.0155	0.00048	0.3387	21
26	0.0148	0.00043	0.5724	10
27	-0.0200	0.00047	0.335	22
28	-0.0343	0.00025	1.00	1
29	0.0005	0.00042	0.5146	13
30	-0.0577	0.00033	0.4547	16

Table 6.6: Ranking of efficient portfolios by in possibilistic mean semi-variance framework

Portfolio	Return	Risk	Ranking by Model (6.3)	Ranking by [185]	Ranking by MSBM-TOPSIS method
16	-0.0605	0.00025	1	3	4
17	0.0177	0.00027	1	1	1
24	0.0263	0.00053	1	2	2
28	-0.0343	0.00025	1	4	3

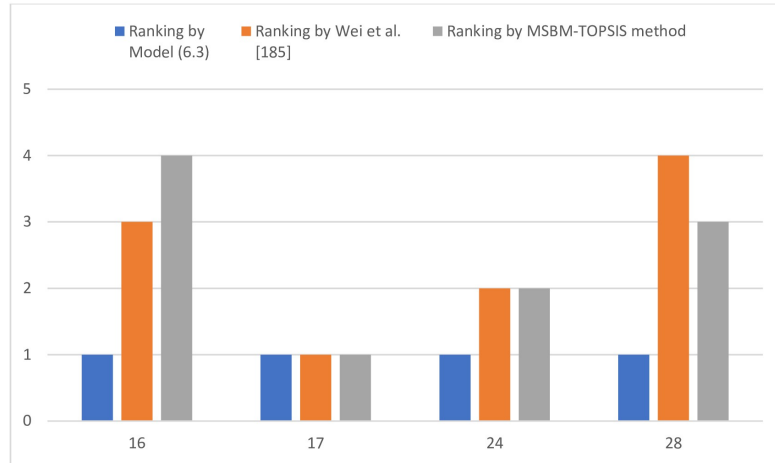


Figure 6.2: Rankings of efficient portfolios in possibilistic mean semi-variance framework

Remark 6.5.2. The primary advantage of the proposed method lies in its feasibility, especially when compared to super-efficiency methods, which, in certain situations, can be impractical or infeasible. Unlike superefficiency models, which may encounter issues related to the infeasibility of evaluating certain DMUs or portfolios under specific conditions, the proposed method offers a more reliable and workable solution. This ensures that it can be applied effectively across a broader range of portfolio selection scenarios, making it a more versatile tool for portfolio evaluation and ranking. By avoiding the lim-

itations of super-efficiency approaches, the suggested method enhances both the accuracy and applicability of portfolio decision-making in real-world contexts.

6.5.1 Computation Complexity

In this study, the computational complexity is evaluated by considering all essential logical and mathematical operations, including addition, subtraction, multiplication, division, exponentiation, and square root, as referenced in prior work by Rashidi and Cullinane [149]. Since the operations required to calculate return risk are uniform across all strategies analyzed in this research, they were not individually included in the computational complexity assessment. Assuming there are $\hat{n}$ efficient portfolios (treated as alternatives) and $\hat{m}$ criteria (e.g., return and risk), the following operations are required for the process:

1. $3\hat{n}\hat{m}$ operations to form the weighted normalized matrix.
2. $2(6\hat{n}\hat{m})$ operations to calculate the distance between each efficient portfolio and the ideal and negative ideal points.
3. $(\hat{n}\hat{m} - \hat{n})$ operations to calculate the positive and negative ideal points.
4. Finally, $2\hat{n}$ computations are necessary to determine each portfolio's closeness coefficient.

Thus, the overall computational complexity for the TOPSIS method, which is used to rank the efficient fuzzy portfolios in the second stage, is:

$$T_{\hat{n},\hat{m}} = 3\hat{n}\hat{m} + 12\hat{n}\hat{m} + \hat{n}\hat{m} - \hat{n} + 2\hat{n}.$$

The empirical analysis in this study demonstrates that the hybrid MSBM-TOPSIS method requires just 132 simple operations. This efficiency contrasts with other DEA models, which involve solving multiple mathematical models to fully rank efficient portfolios. As a result, the MSBM-TOPSIS approach considerably reduces the time needed to identify the optimal portfolio. Compared to other methods, such as the super-efficiency and modified MSBM approaches proposed by Wei et al. [185], which often require solving at least three models per portfolio, our method offers a significantly faster and less complex solution for portfolio selection.

6.6 Conclusion and Future Research Directions

This chapter explores the ranking of efficient fuzzy portfolios involving positive and negative input-output variables. It is recognized that when many portfolios are classified as efficient, the discrimination ability of DEA diminishes. To address this issue, we extend the MSBM model to handle fuzzy portfolio evaluation problems within the possibilistic mean-variance and possibilistic mean semi-variance frameworks. By integrating the MSBM fuzzy portfolio models with the TOPSIS method, we propose a hybrid strategy for ranking efficient portfolios in a fuzzy environment. This approach combines the strengths of the MSBM model for efficiency evaluation, fuzzy logic for managing uncertainty, and TOPSIS to effectively rank alternatives, offering a powerful and comprehensive tool for investors aiming to optimize their portfolios in uncertain and dynamic market conditions.

This approach can be explored in future research to examine the classification of fuzzy portfolios using inputs such as possibilistic semi-absolute deviation (SAD), possibilistic VaR, and CVaR. In the current study, only the 1st and 2nd-order moments (mean, variance, and semi-variance) have been utilized as outputs and inputs. Future work can extend this by incorporating higher-order moments as input-output values.

Conclusion, Future Scope and social impact

In real-life applications, DEA has become an essential tool, particularly in portfolio selection, which aids in evaluating the relative efficiency of different investment options. DEA offers a systematic approach for assessing portfolios by considering multiple inputs (such as risk measures) and outputs (such as returns), providing a clear understanding of the trade-offs between risk and return. Portfolio optimization is one of the most important and attractive financial and investment issues. The financial sector, characterized by its complexity, market volatility, and investor-specific constraints, faces challenges in constructing efficient portfolios. Traditional portfolio selection models often fail to address these challenges, particularly when dealing with negative data, uncertain market conditions, and the need for diversification. In this context, DEA stands out for its ability to analyze portfolios with uncertain or incomplete data, making it especially valuable in dynamic and unpredictable market environments. By integrating DEA with advanced techniques like cross-efficiency analysis, multi-objective optimization, game cross-efficiency, and multi-criteria decision-making methods, we have developed robust frameworks that allow for better risk-adjusted portfolio ranking and selection. Overall, the methodologies presented enhance the differentiation of financial assets by establishing a unique ranking in the first step and allowing portfolio selection based on the investor's underlying preferences in the second step. These methodologies not only contribute to the theoretical understanding of portfolio management but also provide real-world tools for constructing diversified, efficient, and resilient portfolios tailored to different investment goals and market conditions.

After introductory Chapter 1, **Chapter 2** introduces a novel application of the

OVGA operator for aggregating DEA cross-efficiency scores. Unlike traditional methods that rely on arithmetic averages, OVGA captures more decision-making nuances, enhancing the reliability of aggregated scores. Applied to portfolio selection, the study considers constraints such as budget, cardinality, buy-in requirements, and short-selling restrictions. The methodology is tested with empirical data from the Indian banking sector, demonstrating its utility in crafting optimal portfolios for diverse sectors, including public and private enterprises.

In **chapter 3**, the study is presented in two parts. First, a DMU's game cross-efficiency is aggregated using the OVGA operator. The resultant efficiency scores yield unique Nash equilibrium scores and achieve faster convergence. This game cross-efficiency score is used as a tool for efficient portfolio selection. The portfolio selection process reflects investors' preferences based on information about stocks. Selecting assets inherently creates competition among them. To account for this competitive factor, DEA cross-efficiency is applied in the first step of portfolio selection. Secondly, a multiobjective portfolio selection model is proposed, treating the Maverick index and cross-efficiency variance as risk characteristics, while OVGA game cross-efficiency scores are used as return characteristics. An empirical investigation uses a dataset of 50 companies listed on the National Stock Exchange to validate the methodology.

Chapter 4 proposes a SORM model of DEA, which addresses the issue of negative cross-efficiencies that arise in the traditional SORM cross-efficiency matrix. This proposed model is subsequently applied to calculate the cross-efficiency of various stocks and assets. While column averages of the cross-efficiency matrix are used in the Maverick index calculation, we can construct the other index with the row averages of the matrix, which reveals information on how a specific DMU is rating its peers. Similarly, for better asset selection, an asset's diversity index, which measures the difference between its self-appraised score and its row average score, should be high; hence, it will be advantageous to consider it as the portfolio's return.

In **chapter 5**, we introduce an innovative method based on the RDM cross-efficiency matrix for portfolio selection. The proposed method utilizes column and row average scores. Generally, an asset with a high column average score is considered a strong performer, as it performs well relative to the strengths of other assets. While column average scores from the CEM provide valuable insights,

we also focus on row mean scores (i.e., the average appraisal from peers). Analyzing row mean scores reveals how an asset outperforms others when using optimal weights. Typically, a low row average suggests that the asset has unique strengths or skills that other assets lack. Hence, we classify the stocks based on their column average (CA) and row average (RA) and categorize them. This classification process assists investors in building a more resilient portfolio.

Chapter 6 proposes a novel framework for ranking investment portfolios by integrating the MSBM model of DEA with a multi-criteria decision-making technique. By incorporating fuzzy logic, both MSBM and TOPSIS can address uncertainty and imprecision in data, going beyond conventional performance metrics. A key step in investment management is unique portfolio rating, which helps investors find and evaluate the most effective portfolios according to particular standards, including risk, return, and diversity. Hence, we proposed a hybrid MSBM-TOPSIS method to rank the efficient fuzzy portfolio in the presence of negative data. In this chapter, the possibilistic mean of return is used as the measure of return, while possibilistic variance and semi-variance are used as risk. The approach follows a two-step process: portfolios are first evaluated using the fuzzy MSBM model, and then ranked using the TOPSIS method.

Contribution to the UN sustainable development goals and Social impact

This thesis contributes to the United Nations Sustainable Development Goals (SDGs), particularly **SDG 8** (Decent Work and Economic Growth) and **SDG 12** (Responsible Consumption and Production). These goals emphasize the importance of promoting sustained economic growth and encouraging the efficient and sustainable use of resources—objectives that are directly supported by the methodologies and outcomes of this research. By applying DEA to the problem of portfolio optimization, this study advances the development of efficient, data-driven, and responsible investment strategies. DEA enables the assessment of multiple investment portfolios based on various performance indicators, facilitating the identification of relatively efficient portfolios that optimize returns while minimizing risks and inputs. This efficiency-based approach encourages optimal allocation of capital, ensuring that financial resources are directed toward high-

performing investments, thereby contributing to economic productivity and sustainable financial planning, in line with **SDG 8**. Moreover, by prioritizing efficiency and reducing resource waste in the investment process, the study contributes to **SDG 12**, supporting more responsible consumption and production patterns in the financial sector.

Further, this research helps bring more fairness and transparency to real trading by offering tools that evaluate stocks not just on profit, but also on efficiency and responsible performance. It supports smarter, more ethical investment choices by guiding traders with clear, data-driven scores and rankings. Helping traders manage risk and build diverse portfolios contributes to more stable and inclusive financial markets. Unlike traditional optimization techniques focusing solely on risk and return, DEA models can accommodate non-financial indicators such as Environmental, Social, and Governance (ESG) factors. This enables investors to align their portfolios with ethical standards and sustainable development goals, thus fostering socially responsible investing.

Future Scope

While the methodologies presented in this thesis offer distinct advantages and novel contributions, there is always room for further development. The proposed methodologies can be extended within a fuzzy environment to better address market uncertainty, using fuzzy logic to manage imprecise or incomplete data and enhance decision-making in volatile market conditions. Future research could enhance portfolio selection efficiency by incorporating additional risk metrics like semi-variance, semi-absolute deviation, VaR, and CVaR. Furthermore, there is an opportunity to expand DEA cross-efficiency and game cross-efficiency models by integrating time-varying data, enabling real-time adaptation to market fluctuations, and improving portfolio rebalancing effectiveness.

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List of Publications

1. **Reenu Kumari**, Abha Aggarwal, Anjana Gupta; *Cross-efficiency aggregation by ordered visibility graph averaging: method, and application in portfolio selection*, IMA journal of management mathematics, 36(1), 83-110. <https://doi.org/10.1093/imaman/dpae012>. (SCIE, **Impact Factor: 1.9**, **Publisher: Oxford University Press**).
2. **Reenu Kumari**, Anjana Gupta, Abha Aggarwal; *An Integrated Multi-Objective SORM Cross-Efficiency Model: An Application in Portfolio Selection*, Economic Computation and Economic Cybernetics Studies and Research, Vol. 58 Issue 3/2024, Doi 10.24818/18423264/58.3.24.08. ((SCIE, **Impact Factor: 0.9**).
3. **Reenu Kumari**, Anjana Gupta, Abha Aggarwal; *Stock assessment using cumulative prospect theory in DEA cross-efficiency model: A case study of the Indian stock market*, Croatian Operational Research Review CRORR 15(2024), 1–11.
4. Abha Aggarwal, Anjana Gupta, Rajkumar Verma, **Reenu Kumari**; *Ranking of Efficient Fuzzy Portfolios by Hybrid MSBM-TOPSIS Technique*, Foundations of Computing and Decision Sciences 50 (2025).
5. **Reenu Kumari**, Abha Aggarwal, Anjana Gupta; *A Comprehensive mean-maverick-variance portfolio selection model based on OVG aggregated game cross-efficiency approach* (accepted in "International Journal of Mathematics for Industry ").

Papers Presented in International Conferences

- Presented a research paper entitled **“Portfolio Selection using a new clustering approach based on DEA cross-efficiency in the presence of negative data ”** on *International Conference on Advancements in Mathematics (ICAM-2023)* held at Thapar Institute of Engineering and Technology, Patiala, Punjab, India during September 28-30, 2023.
 - Presented a research paper entitled **“DEA-TOPSIS Method for Ranking of Fuzzy Portfolios under Different Risk Measures ”** in *2nd International Conference on Mathematical Modeling, Computational Intelligence Techniques and Renewable Energy (MMCITRE-2021)* held online by conducted by Department of Mathematics, School of Technology, Pandit Deendayal Petroleum University (PDPU), Gandhinagar, Gujarat, India, during February 06-08, 2021.
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Curriculum Vitae

I, Reenu Kumari am a dedicated academic professional currently pursuing Ph.D. from Delhi Technological University in Mathematics. My research area is optimization techniques. With a strong passion for teaching and research, I serve as an Assistant Professor at Maharaja Surajmal Institute of Technology, Janakpuri, New Delhi. I come from a rural background and belong to a family of farmers. I have completed my secondary and higher secondary school from Janta Inter College Babri (U.P. Board) with first class. I earned my B.Sc. in PCM from R.K. College, Shamli (C.C.S. University, Meerut) with 65.2% and obtained my M.Sc. in Mathematics from V.V. Degree College, Shamli (C.C.S. University) with 63.4%. Additionally, I have over 14 years of teaching experience.

I was admitted to the Ph.D. (part-time) program at Delhi Technological University in 2019. I have published two research papers in reputed international journals indexed in SCI/SCIE, while two others have been published in ESCI. Additionally, one more research paper has been submitted for publication. I have presented two research papers at international conferences and have participated in numerous one-week short-term training programs on various topics aimed at enhancing teaching and research skills.

My aim in life to inspire and educate students in mathematics while conducting impactful research that advances knowledge in applied mathematics and decision science.

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Cross-efficiency aggregation by ordered visibility graph averaging: method, and application in portfolio selection

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In research and practice of data envelopment analysis (DEA), the arithmetic average is commonly used to aggregate cross-efficiency scores. For this, each decision-making unit contributes an equal weight, and many essential decision-making details are lost in the final aggregated cross-efficiency. We propose a novel application of the ordered visibility graph averaging (OVGA) operator for DEA cross-efficiency aggregation and apply the proposed method to study the portfolio selection problem. When solving this problem, several practical concerns, such as a budget, cardinality, buy-in requirements and restrictions against short selling, are also considered. The proposed OVGA aggregated cross-efficiency approach is explained through a numerical example, followed by the formulation of optimal portfolios based on these cross-efficiencies. The suggested method is also tested using empirical data from the Indian banking industry. The results of this study can be used to create the most acceptable portfolio in stock companies, financial institutions and businesses in the public and private sectors.

Keywords: data envelopment analysis; decision making units; aggregation method; portfolio optimization.

1. Introduction

Portfolio selection refers to an investor's decision to diversify their financial assets to reduce investment risk while maintaining a specific level of projected return. Portfolio selection offers guidance to an investor for allocating resources among various investment opportunities. A major breakthrough in the field came in 1952 when Harry Markowitz published his modern portfolio theory commonly known as mean-variance portfolio theory. Markowitz (1952) mean-variance portfolio selection model is based on the trade-off between risk and return under various economic conditions. To alleviate the limitations of the mean-variance framework, various alternative risk measures have been proposed in the literature. Although variance is widely accepted as a risk measure, it has limitations. Semi-variance is another risk

measure, originally introduced by [Markowitz \(1959\)](#), and is used in the mean-semi-variance portfolio selection model in [Guo & Ching \(2021\)](#).

Data envelopment analysis (DEA) is a non-parametric empirical approach for evaluating the relative efficiency of decision-making units (DMUs) in the presence of multiple inputs and multiple outputs. DEA model was first introduced by [Farrell \(1957\)](#), and then further developed by [Charnes *et al.* \(1978\)](#) (called CCR model) and [Banker *et al.* \(1984\)](#) (called BCC model). Due to the lack of translation invariance, the CCR and BCC models cannot handle the negative data values in inputs and outputs unless the data get transformed into positive values. The directional distance function based DEA models have been described in the literature to handle the non-positive data values without converting them into positive ones.

One of the applications of DEA in finance is portfolio selection. Few articles in the literature have studied the problem of portfolio selection using DEA. [Chen \(2008\)](#) presented an application of DEA in portfolio selection. [Dia \(2009\)](#) introduced a four-step methodology based on DEA for portfolio selection. [Chen *et al.* \(2018\)](#) constituted fuzzy portfolio assessment models based on DEA using various risk indicators to address the uncertainty involved in the financial market. [Zhou *et al.* \(2018\)](#) offered a DEA frontier improvement technique as a rebalancing strategy for investors. [Gupta *et al.* \(2020\)](#) introduced credibilitic fuzzy portfolio selection models based on DEA. [Zhu & Zhu \(2023\)](#) pioneered the concept of eco-efficiency of industrial investment in terms of energy consumption, economic benefits and environmental impact using a slack-based DEA model. [Omid *et al.* \(2023\)](#) introduced a two-stage multiobjective optimization problem incorporating network DEA. [Soltanifar *et al.* \(2023\)](#) developed DEA-R models for merger analysis that can deal with negative data and applied it to the Iranian banking sector. [Ghiyasi & Zhu \(2020\)](#) proposed an inverse semi-oriented DEA model to assess the efficiency of Chinese commercial banks following the global financial crisis, when negative outputs were present. [Amirteimoori *et al.* \(2022\)](#) developed the chance-constrained multistage DEA model for the relative performances of supply chains. [Emrouznejad *et al.* \(2023\)](#) recently presented a review of the eco-efficiency evaluation using DEA. Recently, [Lim *et al.* \(2014\)](#) introduced a novel way of portfolio selection using the DEA cross-efficiency. They considered the Markowitz's mean-variance optimization model, which identifies a portfolio consisting of DMUs with minimum variation in their cross-efficiency scores and having a desirable expected mean efficiency.

Conventional DEA approaches permit each DMU to assess its efficiency using the most favourable weights, which might lead to exaggerated weight schemes. One of the most prominent ways of dealing with limited discriminative power is the cross-efficiency assessment, which presented [Citesex-ton1986data](#) as an expansion of DEA and further improved by [Doyle & Green \(1994\)](#). These authors initiated aggressive and benevolent strategies as secondary objectives to resolve the problem of non-unique optimum weights in DEA. [Davtalab-Olyaie & Asgharian \(2021\)](#) discussed the problem of a non-Pareto optimum solution in CE evaluation. They proposed a multiobjective program to obtain a set of Pareto-optimal CE scores. [Chen *et al.* \(2020a\)](#) highlighted the problem of non-uniqueness in CE assessments and developed a meta-frontier analysis framework to develop a novel CE assessment method. The approach presented in the paper [Oukil & Amin \(2015\)](#) derives the importance of each DMU from a common consensus among all DMUs, which is then reflected via a preference voting matrix. Because self and peer evaluation scores represent DMU's mutual appreciation for one another, the CE matrix is viewed as an election framework in which each DMU is both a candidate and a voter. [Oukil \(2020\)](#) introduced a new method based on collective weight profile for each DMU using the preference voting system embedded within the weight matrix, which views the assessing DMUs as voters and the input/output factors as candidates. For a more detailed study of cross-efficiency innovations, limitations and extensions, see [Wu *et al.* \(2021\)](#).

DEA cross-efficiency is helpful for portfolio selection problems since it removes the disadvantage of shifting weights in traditional DEA models. To achieve the maximum efficiency value, the DMU can give some inputs and outputs exceptionally high weights. As a result, portfolio selection that uses the traditional DEA technique is not recommended. To resolve this issue, [Lim *et al.* \(2014\)](#) proposed an MV-DEA portfolio selection method that selects a portfolio of DMUs with the least volatility in their CE scores that meets a specified anticipated return. [Mashayekhi & Omrani \(2016\)](#) applied the suggested MV-DEA cross-efficiency portfolio selection assuming fuzzy returns. Recently, [Amin & Hajjami \(2021\)](#) examined the effect of the alternative optimal solution in portfolio selection and indicated that significant improvement in portfolio selection is obtained when alternate optimum solutions are used to obtain a cross-efficiency matrix. [Chen *et al.* \(2021\)](#) describe a portfolio selection method using a fuzzy DEA cross-efficiency assessment, which considers undesirable ambiguous inputs and outputs. [Deng & Fang \(2019\)](#) developed a novel mean-variance-maverick DEA cross-efficiency approach based on the prospect of selecting a fuzzy portfolio. [Gong *et al.* \(2021\)](#) presented a method based on regret theory, in which they applied a fuzzy multiobjective portfolio optimization model with four objectives, namely, mean, variance, skewness and efficiency, bounded by several realistic constraints. [Essid *et al.* \(2018\)](#) offered an innovative framework for portfolio optimization that combines the Maverick index with a DEA game cross-efficiency model.

[Mehlawat *et al.* \(2018\)](#) looked at the portfolio selection issue by adding additional facts about the non-normality of asset returns using skewness and kurtosis. Efficiency evaluation is also a critical factor to consider in the portfolio selection problem.

One of the drawbacks of the DEA cross-efficiency approach in portfolio selection is that all DMUs/stocks in the optimal portfolio are given identical weights. [Amin & Hajjami \(2021\)](#) suggested the development of a novel DEA cross-efficiency portfolio selection model to obtain more reasonable weights for DMUs/stocks. Although many scholars have realized the importance of aggregation methods for cross-efficiency assessment, there is still room for improvement in the existing aggregation methods and their applications in portfolio optimization problems. This paper uses the ordered visibility graph averaging (OVGA) operator to develop a cross-efficiency aggregation technique to analyse the units more thoroughly. The weights in the OVGA operator are determined by the relevance of the data in the obtained visibility graph (VG) ([Jiang *et al.*, 2016](#)). The OVGA operator is vital in aggregating CE scores since it allows values to be given varied weights depending on the importance of CE scores of different DMUs.

1.1. Literature review

The cross-efficiency technique, an extended form of DEA, considers both self and peer appraisal, resulting in a unique ranking of DMUs and removing the problem of artificial weight selections. Most cross-efficiency studies emphasize the cross-evaluation technique, and the arithmetic average is frequently used as a CE aggregation method in the existing research. The arithmetic average technique has certain drawbacks. First, self-evaluated efficiency does not play a sufficient role in the efficiency aggregation procedure, as each DMU has only one self-evaluated efficiency value with multiple peer-evaluated efficiency values. Second, there is no other methodology to consider the decision-maker's subjective preferences, as the arithmetic average approach provides equal weight to each efficiency score ([Wang & Chin \(2011\)](#); [Oukil \(2019\)](#)). To remedy these issues, [Song *et al.* \(2017\)](#) and [Li *et al.* \(2018\)](#) devised aggregation methods that are motivated by the Shannon entropy weight and balanced adjustment weights, respectively, which provide both a solution for the inadequacies of the average arithmetic technique and an acceptable theoretical justification for CE aggregation. [Wang & Chin \(2011\)](#) introduced the ordered weighted averaging (OWA) operator weight to cross-efficiency aggregation.

Using weights based on the OWA operator for CE aggregation permits the decision-maker's optimism level towards the best relative efficiencies. Liu & Chen (2022) pointed out that different efficiency scores affect integration outcomes differently. The relative importance of each efficiency value is not considered in an average aggregation of cross-efficiency. Liu & Chen (2022) first applied regret theory to the CE aggregation process to understand the regret aversion behaviour of DMs, which plays a crucial role in the decision-making process. To determine the regret-based aggressive and benign cross-efficiency, weight matrices are built into their study. Wu *et al.* (2012) suggested a common weight technique to aggregate the cross-efficiency by considering DMUs as players in a cooperative game sense and calculating their Shapley values to address these limitations. Chen *et al.* (2020b) developed prospect theory to represent DMs' subjective preferences in their innovative technique to aggregate CE scores based on the prospect-consensus procedure. Li *et al.* (2022) outlined a novel strategy for generating aggregation weights from subjective and objective viewpoints. To reflect DM preferences from a subjective standpoint, the authors established prospect theory. Their method offers reference points as intervals that can be chosen based on the DMs' decision aims and preferences. Chen *et al.* (2020b) introduced a novel method based on prospect consensus (APC) theory for CE aggregation. The APC method keeps as much of the cross-efficiency matrix's decision-making information as feasible and then aggregates the cross-efficiency that suits decision-making demands. Combining cumulative prospect theory, Fang & Yang (2019) proposed an ordering method for interval CE aggregation.

All the studies mentioned above have different methodologies, indicators and characteristics, requiring deep knowledge to understand the various procedures. Additionally, the above-discussed methods require many parameters in the calculation process that can be very tedious in handling large-scale data.

1.2. Research motivation and novelty of the proposed approach

To the best of our knowledge, DEA cross-efficiency aggregation has been the subject of numerous research studies using a variety of frameworks; however, a new aggregation technique that needs the information from the cross-efficiency matrix remains possible. Also, we find fewer studies in the literature where DEA cross-efficiency has been applied to portfolio optimization problems. The proposed study makes the following contributions:

1. Compared with the aggregated method existing in the literature (Wang & Chin (2011); León *et al.* (2014); Oukil (2019)), our proposed OVGA method aggregates the cross-efficiencies with unique weights and provides weightage according to the importance of the data value, while the weights depend upon different characteristics such as DMs' optimism level and orness degree value, which is uncertain; furthermore, different orness degrees provide different sets of weights, as seen in Wang & Chin (2011) and León *et al.* (2014).
2. Compared with different aggregating operators, the OVGA operator fully considers the information of orders and the correlation between the values. The main advantage of OVGA is that it is completely data-driven, and the aggregated results can be produced from a series of original values.
3. We develop a new application of OVGA compared with that seen in Jiang *et al.* (2016, 2017) and Wang *et al.* (2022) in regard to DEA cross-efficiency aggregation.
4. Furthermore, compared with Lim *et al.* (2014); Deng & Fang (2019) and Essid *et al.* (2018), we integrate the MV-DEA cross-efficiency model with OVGA cross-efficiency scores and deal with the existence of an alternate optimal solution in CE for portfolio optimization problems.

1.3. Organization of the paper

The remainder of this paper is structured as follows. Section 2 is separated into two subsections; its subsequent subsections review the fundamental concepts of the DEA models and OVGA operator.

Section 3 first presents the terminology of the OVGA operator for cross-efficiency aggregation, and then introduces the portfolio performance evaluation model. In Section 4, a numerical data set from the literature is used to demonstrate the applicability of the proposed model; further, a case study of the Indian banking sector is presented and the managerial implications are also discussed there. Section 5 contains the concluding remarks.

2. Preliminaries

This section presents the DEA models existing in the literature, used to investigate the efficiency and cross-efficiency of a DMU. Further, the ordered visibility graph weighted averaging (OVGWA) operator, developed by Wang *et al.* (2015), is also presented here. For basic definitions and notations regarding DEA models and OVGA operator, we may refer to Charnes *et al.* (1978) and Wang *et al.* (2015), respectively.

2.1. DEA models for efficiency evaluation

Suppose we have n DMUs to be evaluated with m inputs and s outputs. Let j^{th} DMU, DMU_j , $j = 1, 2, \dots, n$ use x_{ij} , $i = 1, 2, \dots, m$ inputs and produce y_{rj} , $r = 1, 2, \dots, s$ outputs. Let DMU_k , $k = 1, 2, \dots, n$ be the DMU that is under evaluation. Let ω_{ik} $i = 1, 2, \dots, m$ and μ_{rk} $r = 1, 2, \dots, s$ be the weight given by the DMU_k , $k = 1, 2, \dots, n$ to the i^{th} input and r^{th} outputs, respectively. The best relative self-efficiency e_{kk}^* , under the assumption of constant return to scale, can be determined using the following input-oriented CCR model (Charnes *et al.* (1978)).

$$\begin{aligned} \text{Model I} \quad & \max e_{kk} = \sum_{r=1}^s \mu_{rk} y_{rk} \\ & \text{subject to,} \\ & \sum_{i=1}^m x_{ik} \omega_{ik} = 1, \\ & \sum_{r=1}^s \mu_{rk} y_{rj} - \sum_{i=1}^m \omega_{ik} x_{ij} \leq 0, \quad j = 1, 2, \dots, n, \\ & \omega_{ik} \geq 0, \mu_{rk} \geq 0, \quad i = 1, 2, \dots, m, \quad r = 1, 2, \dots, s. \end{aligned}$$

Let μ_{rk}^* $r = 1, 2, \dots, s$ and ω_{ik}^* $i = 1, 2, \dots, m$ be the optimal solution of Model I, then $e_{kk}^* = \sum_{r=1}^s \mu_{rk}^* y_{rk}$ is the optimal efficiency that DMU_k can attain, which is also referred to the CCR-efficiency of DMU_k under self-evaluation. If $e_{kk}^* = 1$ and all the optimal weights μ_{rk}^* $r = 1, 2, \dots, s$ and ω_{ik}^* $i = 1, 2, \dots, m$ are positive, then DMU_k is CCR-efficient, otherwise, it is CCR-inefficient.

The DEA cross-efficiency assessment approach uses peer review in place of sole self-evaluation to assess efficiency. Thus, the peer evaluation of DMU_j , $j = 1, 2, \dots, n$ by the DMU_k is defined as follows:

$$e_{kj} = \frac{\sum_{r=1}^s \mu_{rk}^* y_{rj}}{\sum_{i=1}^m \omega_{ik}^* x_{ij}}, \quad (k = 1, 2, \dots, n). \quad (2.1)$$

It should be noted that the optimal weights derived from model I are rarely unique. As a result, the cross-efficiency score defined in (2.1) will not produce a unique set of cross-efficiency values. To deal with this issue Oral *et al.* (2015) introduced the maximum resonated appreciative model, which was

further discussed in [Amin & Hajjami \(2021\)](#) and [Amin & Oukil \(2019\)](#). In [Oral et al. \(2015\)](#), the authors have shown that the cross-efficiency score of any DMU_j with respect to any DMU_k , $k = 1, 2, \dots, n$ can take any value between the interval. So the minimum and the maximum cross-efficiency score of DMU_j , evaluated by DMU_k , $k = 1, 2, \dots, n$ can be computed by those optimal weights for which the self-efficiency score value will remain unchanged. They presented the following two formulations, Models II and Model III, to determine the cross-efficiency interval of DMU_j evaluated by DMU_k , $k = 1, 2, \dots, n$.

$$\begin{aligned}
 \text{Model II} \quad & \min \quad \hat{e}_{kj}^L = \sum_{r=1}^s \mu_{rk} y_{rj} \\
 & \text{subject to,} \\
 & \sum_{i=1}^m \omega_{ik} x_{ij} = 1 \\
 & \sum_{r=1}^s \mu_{rk} y_{rj} - \sum_{i=1}^m \omega_{ik} x_{ij} \leq 0, \quad j = 1, 2, \dots, n, j \neq k \\
 & \sum_{r=1}^s \mu_{rk} y_{rk} - e_{kk}^* \sum_{i=1}^m \omega_{ik} x_{ik} = 0, \\
 & \mu_{rk}, \omega_{ik} \geq 0, \quad r = 1, 2, \dots, s; i = 1, 2, \dots, m.
 \end{aligned}$$

Let μ_{rk}^* , $r = 1, 2, \dots, s$ and ω_{ik}^* , $i = 1, 2, \dots, m$ be the optimal solutions of Model II, then $\hat{e}_{kj}^L = \sum_{r=1}^s \mu_{rk}^* y_{rj}$ is the optimal efficiency that DMU_j can attain when evaluated by DMU_k . We can find out the Min cross-efficiency matrix for n $DMUs$ corresponding to Model II. The n efficiency values constitute a cross-efficiency matrix, as shown in [Table 1](#).

$$\begin{aligned}
 \text{Model III} \quad & \max \quad \hat{e}_{kj}^U = \sum_{r=1}^s \mu_{rk} y_{rj} \\
 & \text{subject to,} \\
 & \sum_{i=1}^m \omega_{ik} x_{ij} = 1 \\
 & \sum_{r=1}^s \mu_{rk} y_{rj} - \sum_{i=1}^m \omega_{ik} x_{ij} \leq 0, \quad j = 1, 2, \dots, n, j \neq k \\
 & \sum_{r=1}^s \mu_{rk} y_{rk} - e_{kk}^* \sum_{i=1}^m \omega_{ik} x_{ik} = 0, \\
 & \mu_{rk}, \omega_{ik} \geq 0, \quad r = 1, 2, \dots, s; i = 1, 2, \dots, m.
 \end{aligned}$$

Let μ_{rk}^* , $r = 1, 2, \dots, s$ and ω_{ik}^* , $i = 1, 2, \dots, m$ be the optimal solutions of Model III, then $\hat{e}_{kj}^U = \sum_{r=1}^s \mu_{rk}^* y_{rj}$ is the optimal efficiency that DMU_j can attain when evaluated by DMU_k .

Across all possible optimal solutions of CCR, Model II aims to determine the worst cross-efficiency score for DMU_k , and Model III aims to determine the best cross-efficiency score for DMU_k .

REMARK 1. The alternative optimal solutions of CCR Model I corresponding to DMU_k are utilized to solve Models II and III. It is obvious that $\hat{e}_{kj}^L < \hat{e}_{kj}^U$ only holds if the corresponding self-evaluation DEA model has different optimal solutions. [Oral et al. \(2015\)](#) have stated that Model III takes into consideration the issue of multiple optimal solutions in the CCR model for cross-efficiency evaluation. The results can be greatly impacted by selecting the values from these intervals. One can select extreme cross-efficiency (Min cross-efficiency or Max cross-efficiency) for pessimistic and optimistic decisions. We investigate the results of the OVGA aggregated min cross-efficiency score in this work.

TABLE 1. *Cross-efficiency matrix for n DMUs corresponding to Model II*

Target DMU_K	Cross-efficiency of DMU_j					
	1	2	...	1	...	n
1	$\hat{e}_{11}^L$	$\hat{e}_{12}^L$	...	$\hat{e}_{1l}^L$	...	$\hat{e}_{1n}^L$
2	$\hat{e}_{21}^L$	$\hat{e}_{22}^L$	...	$\hat{e}_{2l}^L$	...	$\hat{e}_{2n}^L$
3	$\hat{e}_{31}^L$	$\hat{e}_{32}^L$	...	$\hat{e}_{3l}^L$	...	$\hat{e}_{3n}^L$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\hat{e}_{n1}^L$	$\hat{e}_{n2}^L$	...	$\hat{e}_{nl}^L$	...	$\hat{e}_{nn}^L$
Average cross-efficiency	$\frac{\sum_{k=1}^n \hat{e}_{k1}^L}{n}$	$\frac{\sum_{k=1}^n \hat{e}_{k2}^L}{n}$	...	$\frac{\sum_{k=1}^n \hat{e}_{kl}^L}{n}$	...	$\frac{\sum_{k=1}^n \hat{e}_{kn}^L}{n}$

Similarly, we can find out the Max cross-efficiency matrix for n $DMUs$ corresponding to Model III.

The most popular method is averaging each row or column's efficiency in the cross-efficiency matrix with equal weights to determine a DMU 's overall performance measurement. Assigning identical weights for cross-efficiency aggregation is not the best option because the self-evaluated efficiency score contributes less than peer-evaluated efficiencies do. In the next section, to follow we have the notion of the OVGA operator for aggregation of efficiency of DMU 's.

2.2. OVGA operator

A VG is an undirected graph of n nodes. The VG method converts a time series into a graph. The associated graph derived from a time series has the following properties:

1. Connected: a node can see its nearest neighbours.
2. Undirected: the associated graph extracted from a time series.
3. Invariant under affine transformations of the series data: Although both the horizontal and vertical axes are rescaled, the visibility criterion remains unchanged. This means that rescaling does not affect the ability to clearly distinguish or identify the elements on the graph.

Further, Wang et al. (2015) introduced an algorithm for ordered visible graphs (OVGs). The VG method is to convert a time series into a graph, while the OVG maps a set of ordered data into a network.

DEFINITION 1. (Wang et al. (2015)) Let $O = \{o_1, o_2, \dots, o_n\}$ represent a set of n ordered data, where o_j is the j^{th} largest element of the set. An ordered value o_j and its order j make up the coordinate (j, o_j) and represent a node j in the graph VG.

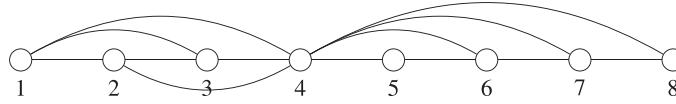
DEFINITION 2. (Wang et al. (2015)) Two data values $(i, o_i), (j, o_j)$ have visibility if any other value (k, o_k) is placed between them fulfils

$$o_k < o_j + (o_i - o_j) \frac{j - k}{j - i}.$$

These argument values are connected in a network of connections by the visibility criteria. These connections are made depending on both the argument values themselves and the order in which they are presented.

TABLE 2. Degree and weights of ordered set O

Nodes (i)	1	2	3	4	5	6	7	8
Degree (d_i)	3	3	3	7	2	3	3	2
Weight (w_i)	$\frac{3}{26}$	$\frac{3}{26}$	$\frac{3}{26}$	$\frac{6}{26}$	$\frac{2}{26}$	$\frac{3}{26}$	$\frac{3}{26}$	$\frac{2}{26}$

FIG. 1. OVG of ordered set O .

The visibility criteria of OVG are similar to VG and are defined as follows:

DEFINITION 3. (Wang et al. (2015)) Two ordered data values (i, o_i) , (j, o_j) have visibility, if any other value (k, o_k) placed between them fulfils

$$o_k < o_j + (o_i - o_j) \frac{j - k}{j - i}.$$

DEFINITION 4. Let $\{(1, o_1), (2, o_2), \dots, (n, o_n)\}$ be the set of n ordered data values. The adjacency matrix $A = [a_{ij}]_{n \times n}$ on the basis of visibility between the node i and node j is defined as

$$a_{ij} = \begin{cases} 1 & \text{if there is visibility from node } i \text{ to node } j, i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Let d_i be the degree of node i , and is given by $d_i = \sum_j a_{ij}$

DEFINITION 5. (OVGWA operator) Let $\{a_1, a_2, \dots, a_n\}$ be the set of n ordered values, where $a_i \in R$. OVGWA operator is a mapping $F : I^n \rightarrow I$, $I \in R$, such that $F(a_1, a_2, \dots, a_n) = w_1 a_1 + w_2 a_2 + \dots + w_n a_n$, where a_i is the i^{th} value and w_i is the corresponding weight of the value a_i , satisfying the conditions $w_i \in [0, 1]$, $\sum_{i=1}^n w_i = 1$, and is defined as follows:

$$w_i = \frac{d_i}{\sum_{i=1}^n d_i}, \quad i = 1, 2, \dots, n,$$

where d_i is the degree of the i^{th} node. Further the aggregated value of the ordered set $\{a_1, a_2, \dots, a_n\}$ is $F(a_1, a_2, \dots, a_n) = w_1 a_1 + w_2 a_2 + \dots + w_n a_n$.

The Python program that calculates the adjacency matrix, degree and weight of an ordered set, on the lines of Definitions 3, 4 and 5, is described as Algorithm 1 in 'Appendix'. To give more clarity, we consider an example considered in Wang et al. (2015).

EXAMPLE 1. (Wang et al. (2015)) Assume there are eight ordered values, $O = \{85, 75, 70, 70, 55, 50, 45, 40\}$. In the OVG, every node represents the corresponding element in the ordered set, in the same order.

From Algorithm 1, the corresponding adjacency matrix A , degree and weight of an ordered set O (depicted in Table 2) are as follows:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

According to the visibility criteria and on the lines of the adjacency matrix, Fig. 1 depicts the OVG so obtained.

Further, the aggregated value of the ordered data is

$$\begin{aligned} F(85, 75, 70, 70, 55, 50, 45, 40) &= 85 \times 3/26 + 75 \times 3/26 + 70 \times 3/26 + 70 \times 6/26 \\ &\quad + 55 \times 2/26 + 50 \times 3/26 + 45 \times 3/26 + 40 \times 2/26 \\ &= 60.96 \end{aligned}$$

It should be noted that if a vertex is connected to another vertex by an edge, it is visible from that vertex since the two vertices have a direct line of sight. On the other hand, if a vertex is not connected to another vertex by an edge, it means that there is an obstruction in the way and the vertex cannot be seen from that specific vertex. In a visible network, the vertex with the highest degree is the one with the greatest number of connections to other vertices. It resembles a graph's centre that is visible from several other spots. This vertex is crucial to the network as a whole because it is connected to a larger number of other vertices.

3. Portfolio optimization model based on the cross-efficiency matrix: proposed method

Lim *et al.* (2014) studied Markowitz (1952) portfolio selection model based on the efficiencies of their assets. Lim *et al.* (2014) considered return and risk aspects of Markowitz's model in terms of mean efficiencies and the co-variance between the cross-efficiency scores of all DMU 's, respectively. Before introducing the suggested model, it is necessary to comprehend the terminology from the previous part in terms of DMU efficiency.

DEFINITION 6. (Adjacency matrix and degree of DMU_p with respect to DMU_j corresponding to min-cross-efficiency) Let $\hat{E}_j^L = \{\hat{e}_{1j}^L, \hat{e}_{2j}^L, \dots, \hat{e}_{pj}^L, \dots, \hat{e}_{nj}^L\}$ be the set of n ordered cross-efficiencies of a DMU_j , where $\hat{e}_{ij}^L$ is the i^{th} largest element of the set $\hat{E}_j^L$. Let $\{(1, \hat{e}_{1j}^L), (2, \hat{e}_{2j}^L), \dots, (p, \hat{e}_{pj}^L), \dots, (n, \hat{e}_{nj}^L)\}$ be the corresponding set of n ordered data nodes. The adjacency matrix for DMU_j , denoted by $A_j^L = [a_{pq}^j]_{n \times n}$, on the basis of visibility between DMU_p , $p = 1, 2, \dots, n$ and DMU_q , $q = 1, 2, \dots, n$ becomes as

follows:

$$a_{pq}^j = \begin{cases} 1, & \text{if there is visibility between the data values } (p, \hat{e}_{pj}^L) \text{ and } (q, \hat{e}_{qj}^L), p \neq q \\ 0, & \text{otherwise.} \end{cases}$$

Let d_p^j be the degree of DMU_p under the consideration of DMU_j , then $d_p^j = \sum_q a_{pq}^j$

DEFINITION 7. (OVGA min-efficiency of DMU_j) Let $\hat{E}_j^L = \{\hat{e}_{1j}^L, \hat{e}_{2j}^L, \dots, \hat{e}_{kj}^L, \dots, \hat{e}_{nj}^L\}$ be the set of n ordered min-cross-efficiencies of a DMU_j , where $\hat{e}_{ij}^L$ is the i^{th} largest element of the set $\hat{E}_j^L$. Then OVGA min-efficiency of DMU_j , denoted by $\tilde{e}_j^L$, is defined as

$$\tilde{e}_j^L = w_{1j}\hat{e}_{1j}^L + w_{2j}\hat{e}_{2j}^L + \dots w_{pj}\hat{e}_{pj}^L + \dots w_{nj}\hat{e}_{nj}^L,$$

where

$$w_{pj} = \frac{d_p^j}{\sum_{p=1}^n d_p^j}, p = 1, 2, \dots, n.$$

DEFINITION 8. (Average of min-cross-efficiency) Let $\hat{E}_j^L = \{\hat{e}_{1j}^L, \hat{e}_{2j}^L, \dots, \hat{e}_{kj}^L, \dots, \hat{e}_{nj}^L\}$ be the set of n cross-efficiencies of a DMU_j , where $\hat{e}_{kj}^L$ is the min-cross-efficiency of DMU_j evaluated with respect to DMU_k . Then the average of min-cross-efficiencies of DMU_j , denoted by $\bar{e}_j^L$, is defined as

$$\bar{e}_j^L = \frac{\hat{e}_{1j}^L + \hat{e}_{2j}^L + \dots, \hat{e}_{kj}^L + \dots \hat{e}_{nj}^L}{n}$$

DEFINITION 9. (Adjacency matrix and degree of DMU_p with respect to DMU_j corresponding to max-cross-efficiency) Let $\hat{E}_j^U = \{\hat{e}_{1j}^U, \hat{e}_{2j}^U, \dots, \hat{e}_{pj}^U, \dots, \hat{e}_{nj}^U\}$ be the set of n ordered max-cross-efficiencies of a DMU_j , where $\hat{e}_{ij}^U$ is the i^{th} largest element of the set $\hat{E}_j^U$. Let $\{(1, \hat{e}_{1j}^U), (2, \hat{e}_{2j}^U), \dots, (p, \hat{e}_{pj}^U), \dots, (n, \hat{e}_{nj}^U)\}$ be the corresponding set of n ordered data nodes. The adjacency matrix for DMU_j , denoted by $B_j^U = [b_{pq}^j]_{n \times n}$, on the basis of visibility between DMU_p , $p = 1, 2, \dots, n$ and DMU_q , $q = 1, 2, \dots, n$ becomes as follows:

$$b_{pq}^j = \begin{cases} 1, & \text{if there is visibility between the data values } (p, \hat{e}_{pj}^U) \text{ and } (q, \hat{e}_{qj}^U), p \neq q \\ 0, & \text{otherwise.} \end{cases}$$

Let c_p^j be the degree of DMU_p under the consideration of DMU_j , then $c_p^j = \sum_q b_{pq}^j$

DEFINITION 10. (OVGA max-efficiency of DMU_j) Let $\hat{E}_j^U = \{\hat{e}_{1j}^U, \hat{e}_{2j}^U, \dots, \hat{e}_{kj}^U, \dots, \hat{e}_{nj}^U\}$ be the set of n ordered max-cross-efficiencies of a DMU_j , where $\hat{e}_{ij}^U$ is the i^{th} largest element of the set $\hat{E}_j^U$. Then OVGA

max-efficiency of DMU_j , denoted by $\tilde{e}_j^U$, is defined as

$$\tilde{e}_j^U = w_{1j}\hat{e}_{1j}^U + w_{2j}\hat{e}_{2j}^U + \dots w_{pj}\hat{e}_{pj}^U + \dots w_{nj}\hat{e}_{nj}^U,$$

where

$$w_{pj} = \frac{c_p^j}{\sum_{p=1}^n c_p^j}, \quad p = 1, 2, \dots, n.$$

DEFINITION 11. (Average of max-cross-efficiency) Let $\hat{E}_j^U = \{\hat{e}_{1j}^U, \hat{e}_{2j}^U, \dots, \hat{e}_{kj}^U, \dots, \hat{e}_{nj}^U\}$ be the set of n cross-efficiencies of a DMU_j , where $\hat{e}_{kj}^U$ is the max-cross-efficiency of DMU_j evaluated with respect to DMU_k . Then the average of max-cross-efficiency of DMU_j , denoted by $\bar{e}_j^U$, is defined as

$$\bar{e}_j^U = \frac{\hat{e}_{1j}^U + \hat{e}_{2j}^U + \dots, \hat{e}_{kj}^U + \dots \hat{e}_{nj}^U}{n}$$

REMARK 2. In the VG, the weight of each node is determined by the degree distribution. Thus if a node links with more nodes than others, its degree will be greater and it will be more significant. The following are the advantages of the OVGA operators over other methods for cross-efficiency aggregation:-

1. In the OWA method, we obtain different aggregated results for different optimism levels and it is very difficult for decision-makers to select which value to make final decisions.
2. Compared with the classical OWA operator, the proposed OVGA operator not only takes the information of orders but also the argument values themselves into consideration to determine the weights. The OVGA operator is inspired by the complex network theory and Newton's law of universal gravitation.
3. Other cross-efficiency aggregation methods, such as prospect and regret theory, require many calculation parameters that will be very difficult to handle and will be very calculative for the large-scale data set. Prospect theory cross-efficiency aggregation requires at least four parameters, including risk attitude coefficients, the concave degree of the value function within the gain/loss region and a psychological preference value. Similarly, the regret theory cross-efficiency aggregation method utilizes the regret-rejoice function based on the regret aversion coefficient.
4. The primary benefit of the OVGA operator is that it is based on the idea of human intuition. In simpler terms, a vertex with the highest degree can be considered a popular person who knows many people and has a lot of influence. Their connections and interactions provide valuable insights into the relationships and dynamics of the time series data.

Portfolio selection model corresponding to min-cross-efficiency

Let us consider a portfolio Ω constituted with n DMUs say DMU_i , $i = 1, 2, \dots, n$. Following the approach of Lim *et al.* (2014), this paper describes the return and risk characteristics of Markowitz's model (Markowitz 1952). Specifically, the mean cross-efficiency scores represent the model's return, while the variance of the cross-efficiencies represents the model's risk.

Let $[\hat{e}_{ij}^L]_{n \times n}$ be the min-cross-efficiency matrix on n DMUs taken under consideration for portfolio selection. Let x_i be the proportion of the total efficiency invested on DMU_i . The portfolio Ω of n DMUs by

the OVGA operator is obtained by solving the following cross-efficiency portfolio optimization model:

$$\begin{aligned}
 \text{Model IV} \quad & \min \quad V_{\Omega} = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \tilde{\sigma}_{ij}^L \\
 & \text{subject to,} \\
 & E_{\Omega} = \sum_{i=1}^n x_i \tilde{e}_i^L \geq (1 - \gamma) \tilde{E}_{\Omega}^b \\
 & \sum_{i=1}^n x_i = 1, \\
 & \sum_{i=1}^n z_i = d, \quad i = 1, 2, \dots, n, \\
 & \epsilon_i z_i \leq x_i \leq \delta_i z_i, \quad i = 1, 2, \dots, n, \\
 & x_i \in \{0, 1\}, \quad i = 1, 2, \dots, n, \\
 & x_i \geq 0, \quad i = 1, 2, \dots, n,
 \end{aligned}$$

where $\tilde{e}_i^L$ is the mean cross-efficiency score of DMU_i by OVGA operator, defined in Definition 7 and $\tilde{\sigma}_{ij}^L = \frac{1}{n} \sum_{k=1}^n (\hat{e}_{ki}^L - \tilde{e}_i^L)(\hat{e}_{kj}^L - \tilde{e}_j^L)$ is the covariance between the cross-efficiency scores of DMU_i and DMU_j . Here, γ is the trade-off parameter between mean efficiency and co-variance of their efficiency. Further, for a better selection of a DMU, the budget constraint $\sum_{i=1}^n x_i = 1$, cardinality constraint $\sum_{i=1}^n z_i = d$, $z_i \in \{0, 1\}$, $i = 1, 2, \dots, n$, buy-in threshold $\epsilon_i z_i \leq x_i \leq \delta_i z_i$, $i = 1, 2, \dots, n$, and no short-selling criteria $x_i \geq 0$, $i = 1, 2, \dots, n$, are also considered here. Here, E_{Ω}^b is the maximum OVGA efficiency achievable by the portfolio under the budget constraint, cardinality constraint, buy-in threshold and no short-selling criteria.

Let $\tilde{E}_{\Omega}^*$ and $\tilde{V}_{\Omega}^*$ be the optimal mean efficiency and optimal risk corresponding to the optimal portfolio Ω^* so obtained.

Similarly, the mean-variance cross-efficiency model for simple average efficiency as defined in Definition 8 is as follows:

$$\begin{aligned}
 \text{Model V} \quad & \min \quad V_{\Omega} = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \bar{\sigma}_{ij}^L \\
 & \text{subject to} \\
 & \sum_{i=1}^n x_i \bar{e}_i^L \geq (1 - \gamma) \bar{E}_{\Omega}^b \\
 & \sum_{i=1}^n x_i = 1 \\
 & \sum_{i=1}^n z_i = d, \quad i = 1, 2, \dots, n, \\
 & \epsilon_i z_i \leq x_i \leq \delta_i z_i, \quad i = 1, 2, \dots, n, \\
 & z_i \in \{0, 1\}, \quad i = 1, 2, \dots, n, \\
 & x_i \geq 0, \quad i = 1, 2, \dots, n,
 \end{aligned}$$

where $\bar{e}_i^L$ is the mean cross-efficiency score of DMU_i defined in Definition 8 and $\bar{\sigma}_{ij}^L = \frac{1}{n} \sum_{k=1}^n (e_{ki}^L - \bar{e}_i^L)(e_{kj}^L - \bar{e}_j^L)$ is the covariance between the cross-efficiency scores of DMU_i and DMU_j . E_{Ω}^b is the

TABLE 3. *Input and output data values for DMUs*

DMUs	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Input 1	5	6	4	8	5	8	4.4	2.6	3.4	3.6	2	3	3	2.6	4	5	6	4	7	6	8	9
Input 2	4	5	5	5	6	3	4.4	8	8	4.4	7	7	5.6	5	4	3.2	4	3.5	3	2.5	2	2
Output 1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

TABLE 4. *Self-efficiency score e_{kk}^* of DMU_k*

k	1	2	3	4	5	6	7	8	9	10	11
e_{kk}^*	0.85	0.69	0.84	0.61	0.69	0.80	0.85	0.82	0.71	0.94	1
k	12	13	14	15	16	17	18	19	20	21	22
e_{kk}^*	0.80	0.88	1	0.94	0.96	0.79	1	0.85	1	1	1

maximum mean efficiency, achievable by the portfolio E_{Ω} . Further, γ is the trade-off parameter between mean efficiency and co-variance of their efficiency. Further, other constraints have the same interpretation as explained in Model IV. Let $\tilde{E}_{\Omega^*}$ and $\tilde{V}_{\Omega^*}$ be the optimal mean efficiency and optimal risk corresponding to the optimal portfolio Ω^* so obtained.

REMARK 3. Now on the lines of Definitions 9, 10 and 11, we can extend Model IV and Model V to study the portfolio selection for max-cross-efficiency of DMUs.

REMARK 4. It is imperative to bear in mind that the selection of the minimum or maximum cross-efficiencies, or the extraction of efficiencies from the interval between the min and max cross-efficiencies, will have a significant influence on the portfolio selection outcomes. In this work, we solely address the ideal portfolio derived from model IV. We can explore the OVGA max cross-efficiency and its impact on portfolio outcomes more thoroughly in future studies.

4. Numerical illustration

In this section, we apply our proposed method to a numerical example consisting of 22 DMUs to demonstrate the empirical effectiveness of the method. The numerical dataset is taken from Lim *et al.* (2014), which was further analysed by Amin & Hajjani (2021). The input and output data for the DMUs are presented in Table 3.

On solving Model I, for DMU_1 , we get $\mu_{11}^* = 0.85$, $\omega_{11}^* = 0.0769$, $\omega_{21}^* = 0.1538$, with the optimal value of the objective function as $e_{11}^* = 0.85$. Similarly on solving Model I, for other DMUs, the self-efficiency of respective DMUs are presented in Table 4.

For cross-efficiency interval evaluation of DMU_j , we have to solve Model II and Model III, respectively, for different values of k for a fixed value of j. For example if we take $k = 2$ and $j = 1$, we can find $\tilde{e}_{21}^L$ and $\tilde{e}_{21}^U$. The minimum cross-efficiency matrix is shown in Table 6. As in the cross-efficiency matrix, DMU_j is evaluated by DMU_k , so to determine the weights given by DMU_j to DMU_k by the OVGA operator, we need a re-ordered minimum cross-efficiency matrix.

To get more clarity, let us calculate the weights given by DMU_1 to DMU_k , $k = 1, 2, \dots, n$ for the OVGWA operator to get its aggregated efficiency. In a similar fashion, we can find out the weights given by other DMUs to DMU_k , $k = 1, 2, \dots, n$. The set of ordered cross-efficiencies of DMU_1 , $\tilde{e}_{k1}^L$, $k =$

TABLE 5. Degree and weights of DMU_p w.r.t. DMU_1

p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
d_p^1	1	2	2	2	2	8	2	3	3	3	3	3	6	2	6	2	3	4	5	2	3	1
w_p^1	$\frac{1}{68}$	$\frac{2}{68}$	$\frac{2}{68}$	$\frac{2}{68}$	$\frac{2}{68}$	$\frac{8}{68}$	$\frac{2}{68}$	$\frac{3}{68}$	$\frac{3}{68}$	$\frac{3}{68}$	$\frac{3}{68}$	$\frac{3}{68}$	$\frac{6}{68}$	$\frac{2}{68}$	$\frac{6}{68}$	$\frac{2}{68}$	$\frac{3}{68}$	$\frac{4}{68}$	$\frac{5}{68}$	$\frac{2}{68}$	$\frac{3}{68}$	$\frac{1}{68}$

1, 2, ..., 22 (as per Table 7) for DMU_1 is

$$\hat{E}_1^L = \{0.85, 0.85, 0.85, 0.85, 0.85, 0.85, 0.83, 0.83, 0.83, 0.83, 0.83, 0.83, 0.83, 0.76, 0.76, 0.66, 0.66, 0.66, 0.66, 0.50, 0.50, 0.40\}.$$

To calculate the weights given by DMU_1 to DMU_k , $k = 1, 2, \dots, 22$, for evaluating its OVGWA efficiency, Fig. 2 depicts the ordered VG of all $DMUs$ with respect to ordered cross-efficiency set $\hat{E}_1^L$ and Table 5 presents the weights corresponding to the first DMU.

Now on lines of Definition 6, from Algorithm 1, the corresponding adjacency matrix is

$$A_1^L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

So the aggregated cross-efficiency by OVGWA operator and by simple average is $\tilde{e}_1^L = \sum_{p=1}^{22} w_p^1 \hat{e}_{p1}^L = 0.76$ and $\bar{e}_1^L = \sum_{p=1}^{22} (1/n) \hat{e}_{p1}^L = 0.75$, respectively. These values are the same as given in Table 9, and in a similar manner, we can calculate the aggregated cross-efficiency scores of other DMUs. For this, we first need to solve Model II; for $k, j = 1, 2, \dots, n$ the minimum

TABLE 6. The cross-efficiency matrix e_{kj}^L

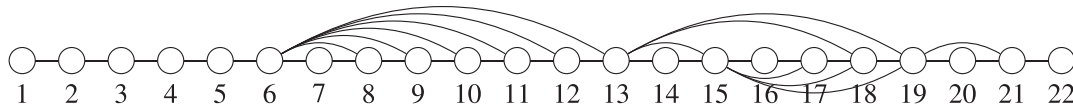
$k \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
2	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
3	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
4	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
5	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
6	0.76	0.62	0.67	0.57	0.55	0.80	0.73	0.46	0.45	0.75	0.53	0.52	0.63	0.71	0.80	0.90	0.73	0.89	0.84	1.00	1.00	0.94
7	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
8	0.66	0.55	0.75	0.43	0.60	0.46	0.72	0.82	0.71	0.83	1.00	0.80	0.88	1.00	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
9	0.66	0.55	0.75	0.43	0.60	0.46	0.72	0.82	0.71	0.83	1.00	0.80	0.88	1.00	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
10	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
11	0.40	0.33	0.50	0.28	0.40	0.28	0.45	0.77	0.59	0.56	1.00	0.67	0.67	0.77	0.50	0.40	0.33	0.50	0.29	0.33	0.25	0.22
12	0.66	0.55	0.75	0.43	0.60	0.46	0.72	0.82	0.71	0.83	1.00	0.80	0.88	1.00	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
13	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
14	0.66	0.55	0.75	0.43	0.60	0.46	0.72	0.72	0.67	0.83	0.85	0.76	0.88	1.00	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
15	0.83	0.68	0.84	0.57	0.69	0.67	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1.00	0.94	0.91	0.75	1.00	0.74	0.87	0.74	0.67
16	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
17	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
18	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.91	0.75	1.00	0.74	0.87	0.74	0.67
19	0.85	0.69	0.79	0.61	0.65	0.79	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.96	0.79	1.00	0.85	1.00	0.92	0.85
20	0.76	0.62	0.67	0.57	0.55	0.79	0.73	0.46	0.45	0.75	0.53	0.52	0.63	0.71	0.80	0.90	0.73	0.89	0.84	1.00	0.92	0.85
21	0.50	0.40	0.40	0.40	0.33	0.67	0.45	0.25	0.25	0.45	0.29	0.29	0.36	0.40	0.50	0.63	0.50	0.67	0.67	0.80	1.00	0.94
22	0.50	0.40	0.40	0.40	0.33	0.67	0.45	0.25	0.25	0.45	0.29	0.29	0.36	0.40	0.50	0.63	0.50	0.57	0.67	0.80	1.00	1.00

TABLE 7. *Re-ordered cross-efficiency matrix $\tilde{e}_{kj}^L$*

k \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	0.85	0.69	0.84	0.61	0.69	0.80	0.85	0.82	0.71	0.94	1	0.80	0.88	1	0.94	0.96	0.79	1	0.85	1	1	1
2	0.85	0.69	0.84	0.61	0.69	0.79	0.85	0.82	0.71	0.94	1	0.80	0.88	1	0.94	0.96	0.79	1	0.85	1	1	0.94
3	0.85	0.69	0.84	0.61	0.69	0.79	0.85	0.82	0.71	0.94	1	0.8	0.88	1	0.94	0.96	0.79	1	0.85	1	1	0.94
4	0.85	0.69	0.84	0.61	0.69	0.79	0.85	0.77	0.67	0.94	1	0.76	0.88	1	0.94	0.96	0.79	1	0.85	1	0.92	0.85
5	0.85	0.69	0.84	0.61	0.69	0.79	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1	0.94	0.96	0.79	1	0.85	1	0.92	0.85
6	0.85	0.69	0.84	0.61	0.69	0.79	0.85	0.72	0.67	0.94	0.85	0.76	0.88	1	0.94	0.96	0.79	1	0.85	1	0.92	0.85
7	0.83	0.68	0.79	0.57	0.65	0.79	0.83	0.72	0.67	0.89	0.85	0.76	0.88	1	0.92	0.91	0.75	1	0.84	1	0.92	0.85
8	0.83	0.68	0.79	0.57	0.65	0.79	0.83	0.72	0.67	0.89	0.85	0.76	0.88	1	0.92	0.91	0.75	1	0.84	1	0.92	0.85
9	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.72	0.67	0.89	0.85	0.76	0.88	1	0.92	0.91	0.75	1	0.74	0.87	0.92	0.85
10	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.72	0.67	0.89	0.85	0.76	0.88	1	0.92	0.91	0.75	1	0.74	0.87	0.92	0.85
11	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.72	0.59	0.89	0.85	0.67	0.77	0.87	0.92	0.91	0.75	1	0.74	0.87	0.92	0.85
12	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.91	0.75	1	0.74	0.87	0.74	0.67
13	0.83	0.68	0.79	0.57	0.65	0.67	0.83	0.59	0.57	0.89	0.69	0.65	0.77	0.87	0.92	0.91	0.75	1	0.74	0.87	0.74	0.67
14	0.76	0.62	0.75	0.57	0.6	0.67	0.73	0.59	0.57	0.83	0.69	0.65	0.77	0.87	0.92	0.91	0.75	1	0.74	0.87	0.74	0.67
15	0.76	0.62	0.75	0.57	0.6	0.67	0.73	0.59	0.57	0.83	0.69	0.65	0.77	0.87	0.92	0.91	0.75	0.89	0.74	0.87	0.74	0.67
16	0.66	0.55	0.75	0.43	0.6	0.67	0.72	0.59	0.57	0.83	0.69	0.65	0.77	0.87	0.79	0.69	0.57	0.81	0.67	0.8	0.74	0.67
17	0.66	0.55	0.75	0.43	0.6	0.67	0.72	0.59	0.57	0.83	0.69	0.65	0.77	0.87	0.79	0.69	0.57	0.81	0.67	0.8	0.74	0.67
18	0.66	0.55	0.67	0.43	0.55	0.46	0.72	0.59	0.57	0.75	0.69	0.65	0.67	0.77	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
19	0.66	0.55	0.67	0.43	0.55	0.46	0.72	0.46	0.45	0.75	0.53	0.52	0.63	0.71	0.79	0.69	0.57	0.81	0.52	0.61	0.48	0.43
20	0.5	0.4	0.5	0.4	0.4	0.46	0.45	0.46	0.45	0.56	0.53	0.52	0.63	0.71	0.5	0.63	0.5	0.57	0.52	0.61	0.48	0.43
21	0.5	0.4	0.4	0.4	0.33	0.46	0.45	0.25	0.25	0.45	0.29	0.29	0.36	0.4	0.5	0.63	0.5	0.57	0.52	0.61	0.48	0.43
22	0.4	0.33	0.4	0.25	0.33	0.25	0.45	0.25	0.25	0.45	0.29	0.29	0.36	0.4	0.5	0.4	0.33	0.5	0.29	0.33	0.25	0.22
$\bar{e}_j$	0.75	0.61	0.73	0.53	0.60	0.66	0.76	0.63	0.58	0.83	0.75	0.66	0.77	0.87	0.83	0.83	0.68	0.89	0.71	0.84	0.76	0.70

TABLE 8. The weight w_p^j given by DMU_j to DMU_p on the basis of re-ordered cross-efficiency matrix $\hat{e}_{pj}^L$

p	w_p^1	w_p^2	w_p^3	w_p^4	w_p^5	w_p^6	w_p^7	w_p^8	w_p^9	w_p^{10}	w_p^{11}	w_p^{12}	w_p^{13}	w_p^{14}	w_p^{15}	w_p^{16}	w_p^{17}	w_p^{18}	w_p^{19}	w_p^{20}	w_p^{21}	w_p^{22}
1	0.015	0.014	0.015	0.014	0.015	0.092	0.014	0.012	0.012	0.015	0.014	0.012	0.016	0.016	0.014	0.015	0.015	0.017	0.016	0.016	0.014	0.073
2	0.029	0.029	0.029	0.029	0.029	0.026	0.027	0.024	0.024	0.029	0.029	0.024	0.032	0.032	0.027	0.029	0.030	0.033	0.031	0.032	0.028	0.024
3	0.029	0.029	0.029	0.029	0.029	0.039	0.027	0.098	0.095	0.029	0.029	0.095	0.032	0.032	0.027	0.029	0.030	0.033	0.031	0.032	0.111	0.110
4	0.029	0.029	0.029	0.029	0.029	0.039	0.027	0.098	0.024	0.029	0.114	0.024	0.032	0.032	0.027	0.029	0.030	0.033	0.031	0.032	0.028	0.024
5	0.029	0.029	0.029	0.029	0.029	0.039	0.027	0.024	0.036	0.029	0.029	0.036	0.032	0.032	0.027	0.029	0.030	0.033	0.031	0.032	0.042	0.037
6	0.118	0.114	0.118	0.143	0.118	0.039	0.108	0.049	0.036	0.118	0.043	0.036	0.032	0.032	0.108	0.132	0.121	0.033	0.047	0.032	0.042	0.037
7	0.029	0.029	0.029	0.029	0.029	0.039	0.027	0.049	0.036	0.029	0.043	0.036	0.032	0.032	0.027	0.029	0.030	0.033	0.031	0.032	0.042	0.049
8	0.044	0.043	0.044	0.043	0.044	0.145	0.041	0.049	0.036	0.044	0.043	0.036	0.032	0.032	0.041	0.044	0.045	0.033	0.141	0.129	0.042	0.049
9	0.044	0.043	0.044	0.043	0.044	0.026	0.041	0.049	0.036	0.044	0.043	0.036	0.032	0.032	0.041	0.044	0.045	0.033	0.031	0.032	0.042	0.049
10	0.044	0.043	0.044	0.043	0.044	0.039	0.041	0.049	0.119	0.044	0.043	0.119	0.129	0.129	0.041	0.044	0.045	0.033	0.047	0.048	0.125	0.122
11	0.044	0.043	0.044	0.043	0.044	0.039	0.041	0.122	0.095	0.044	0.129	0.095	0.032	0.032	0.041	0.044	0.045	0.033	0.047	0.048	0.028	0.024
12	0.044	0.043	0.044	0.043	0.044	0.039	0.041	0.024	0.036	0.044	0.029	0.036	0.048	0.048	0.041	0.044	0.045	0.033	0.047	0.048	0.042	0.037
13	0.088	0.086	0.088	0.043	0.088	0.039	0.108	0.037	0.048	0.088	0.043	0.048	0.048	0.048	0.108	0.059	0.061	0.100	0.047	0.048	0.042	0.037
14	0.029	0.029	0.029	0.043	0.029	0.039	0.027	0.037	0.048	0.029	0.043	0.048	0.048	0.048	0.027	0.029	0.030	0.033	0.047	0.048	0.042	0.037
15	0.088	0.086	0.044	0.114	0.044	0.039	0.081	0.037	0.048	0.044	0.043	0.048	0.048	0.048	0.081	0.118	0.106	0.100	0.063	0.065	0.042	0.037
16	0.029	0.029	0.044	0.029	0.044	0.039	0.041	0.037	0.048	0.044	0.043	0.048	0.048	0.048	0.041	0.029	0.030	0.033	0.031	0.032	0.042	0.037
17	0.044	0.043	0.059	0.043	0.059	0.079	0.054	0.037	0.048	0.059	0.043	0.048	0.081	0.081	0.054	0.044	0.045	0.067	0.094	0.097	0.083	0.073
18	0.059	0.057	0.029	0.043	0.029	0.026	0.054	0.049	0.060	0.029	0.057	0.060	0.048	0.048	0.054	0.044	0.045	0.067	0.031	0.032	0.028	0.024
19	0.074	0.086	0.074	0.057	0.074	0.039	0.081	0.024	0.024	0.074	0.029	0.024	0.048	0.048	0.081	0.059	0.061	0.100	0.047	0.048	0.042	0.037
20	0.029	0.029	0.044	0.043	0.044	0.039	0.027	0.049	0.048	0.044	0.057	0.048	0.081	0.081	0.027	0.029	0.030	0.033	0.047	0.048	0.042	0.037
21	0.044	0.043	0.044	0.057	0.044	0.039	0.041	0.024	0.024	0.044	0.029	0.024	0.032	0.032	0.041	0.059	0.061	0.050	0.047	0.048	0.042	0.037
22	0.015	0.029	0.044	0.014	0.044	0.013	0.027	0.024	0.024	0.044	0.029	0.024	0.032	0.032	0.027	0.015	0.015	0.033	0.016	0.016	0.014	0.012

FIG. 2. Ordered VG of ordered efficiency set $\hat{E}_1^L$.TABLE 9. The aggregated minimum cross-efficiency score by simple average operator $\bar{e}_j^L$ and by OVGA operator $\tilde{e}_j^L$ of target DMU_j

j	1	2	3	4	5	6	7	8	9	10	11
$\bar{e}_j^L$	0.75	0.61	0.73	0.53	0.6	0.66	0.76	0.63	0.58	0.83	0.75
$\tilde{e}_j^L$	0.76	0.62	0.74	0.54	0.60	0.69	0.76	0.66	0.60	0.82	0.77
j	12	13	14	15	16	17	18	19	20	21	22
$\bar{e}_j^L$	0.66	0.77	0.87	0.83	0.83	0.68	0.89	0.71	0.84	0.76	0.7
$\tilde{e}_j^L$	0.68	0.77	0.87	0.84	0.85	0.70	0.88	0.72	0.85	0.80	0.76

cross-efficiency matrix is depicted in Table 9. Further, Table 7 presents the re-ordered cross-efficiency matrix. As per the requirement of the calculation Table 8 illustrates the weights given by DMU_j to DMU_p for $j, p = 1, 2, \dots, 22$.

Portfolio selection Here, optimal portfolios are selected in two ways:

Case I: the portfolio selection is performed with equal weights

The decision maker wishes to have an equal proportion of investment on the DMUs for the optimal portfolios. To present the advantages of the proposed portfolio selection model, we first obtain portfolios using the proposed model for different cardinality constraints i.e. $d=5$, $d=10$ and $d=7$. The obtained portfolio strategies are given in Table 10. To compare the portfolios obtained from the traditional method, i.e. after solving Model V, the portfolio strategies under the different cardinality constraints i.e. $d=5$, $d=10$ and $d=15$ are presented in Table 11. Here it is to be noted that the portfolios obtained through the proposed model have a considerable reduction in variance (risk) and a slight reduction in mean. For example, the portfolio obtained through the proposed method, presented in Table 10, corresponding to $d=5$ and $\gamma = 0.1$ shows a 1.73% decrease in mean in comparison with a 33.4% decrease in variance.

Case II: optimal portfolio selection with unequal weights

Here it is to be noted that different portfolios are generated by varying values of $d, \gamma, \epsilon_i, \delta_i$. The optimal portfolio strategies corresponding to the proposed model, i.e. after solving Model IV, are presented in Table 12, and with respect to the traditional method, i.e. after solving Model V, the optimal strategies are presented in Table 13. As discussed above, the portfolios obtained through the proposed model have a considerable reduction in variance (risk) against the slight reduction in mean. For example, the portfolio obtained through a proposed method, presented in Table 12, corresponding to $d = 5, \gamma = 0.1, \epsilon_i = 0.03$ and $\delta = 0.4$ shows a 1.1% decrease in mean against a considerable decrease of 27.5% in variance.

It is to be noted that a bigger value of γ can also result in a more significant decrease in risk for a given expected return. The effectiveness of the proposed approach for portfolio selection is empirically supported by the acquired portfolio selection strategies. The suggested model can generate portfolios based on the investor's choices corresponding to γ .

TABLE 10. *The optimal portfolio strategies corresponding to Model IV for equal weights*

DMU_k	$\gamma = 0.1$		$\gamma = 0.2$			
	$\epsilon =$	$\epsilon =$	$\epsilon =$	$\epsilon =$	$\epsilon =$	$\epsilon =$
	$\delta =$	$\delta =$	$\delta =$	$\delta =$	$\delta =$	$\delta =$
	0.2d=5	0.1d=10	0.1429d=7	0.2d=5	0.1d=10	0.1429d=7
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0.10	0
5	0	0	0	0	0	0
6	0	0	0	0	0.10	0.1429
7	0	0	0	0	0	0
8	0	0.10	0.1429	0.20	0.10	0.1429
9	0	0	0	0.20	0.10	0.1429
10	0	0	0	0	0	0
11	0.20	0.10	0.1429	0.20	0.10	0.1429
12	0	0	0	0	0.10	0.1429
13	0.20	0.10	0	0	0	0
14	0.20	0.10	0.1429	0	0.10	0
15	0	0.10	0.1429	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0.10	0	0	0	0
19	0	0.10	0	0	0.10	0
20	0	0.10	0.1429	0	0	0
21	0.20	0.10	0.1429	0.20	0.10	0.1429
22	0.20	0.10	0.1429	0.20	0.10	0.1429
$\bar{E}_\Omega$	0.794	0.792	0.793	0.718	0.709	0.709
$\bar{V}_\Omega$	0.0064	0.0094	0.0078	0.0046	0.0063	0.0052

REMARK 5. For risk-averse investors, Fig. 3 depicts that the return produced from the suggested model is higher than the standard model already present in the literature.

In the subsequent section to follow, we demonstrate the benefit of the proposed method by taking a real-world application problem.

4.1. A case study of the Indian Banking Sector

We present a case study with 20 banking companies listed on the National Stock Exchange as a practical example of the suggested methodology. The data for the financial year 2021 are used here for the case study. This is just for illustrative purposes; the analysis may easily be continued for several years.

TABLE 11. *The optimal portfolio strategies corresponding to Model V for equal weights*

DMU_k	$\gamma = 0.1$			$\gamma = 0.2$		
	$\epsilon = \delta = 0.2, d = 5$	$\epsilon = \delta = 0.1, d = 10$	$\epsilon = \delta = 0.0667, d = 15$	$\epsilon = \delta = 0.2, d = 5$	$\epsilon = \delta = 0.1, d = 10$	$\epsilon = \delta = 0.0667, d = 15$
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0.10	0.1429
7	0	0	0	0	0	0
8	0	0	0	0.20	0.10	0.1429
9	0	0	0	0	0.10	0
10	0.20	0.10	0.1429	0	0	0
11	0.20	0.10	0.1429	0.20	0.10	0.1429
12	0	0	0	0	0.10	0.1429
13	0	0.10	0	0.20	0.20	0
14	0.20	0.10	0.1429	0	0.10	0.1429
15	0	0.10	0	0	0	0
16	0	0.10	0	0	0	0
17	0	0	0	0	0	0
18	0	0.10	0.1429	0	0	0
19	0	0	0	0	0.10	0
20	0	0.10	0.1429	0	0	0
21	0.20	0.10	0.1429	0.20	0.10	0.1429
22	0.20	0.10	0.1429	0.20	0.10	0.1429
$\bar{E}_\Omega$	0.808	0.807	0.805	0.721	0.709	0.718
$\bar{V}_\Omega$	0.0096	0.0116	0.0102	0.0043	0.0063	0.0052

In our analysis, we have randomly selected the following 20 banks listed in trade brains portal (<https://portal.tradebrains.in>).

1. SBIN (B_1)
2. AXIS (B_2)
3. PNB (B_3)
4. HDFC (B_4)
5. ICICI (B_5)
6. KOTAK (B_6)
7. FEDERAL (B_7)
8. CANARA (B_8)

TABLE 12. *The optimal portfolio strategies corresponding to Model IV for unequal weights*

DMU_k	$\gamma = 0.1, \epsilon = 0.03, \delta = 0.4$			$\gamma = 0.2, \epsilon = 0.02, \delta = 0.3$		
	d=5	d=10	d=7	$\epsilon = \delta =$ 0.2 d=5	$\epsilon = \delta =$ 0.1 d=10	$\epsilon = \delta =$ 0.1429 d=7
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0.0529	0.02
4	0	0	0	0	0	0
5	0	0	0	0	0.037	0
6	0.03	0.03	0	0	0	0.02
7	0	0	0	0	0.02	0
8	0	0	0.03	0.2101	0.0316	0.1343
9	0	0	0	0.0282	0.09	0.1149
10	0	0.03	0	0	0	0
11	0.40	0.2886	0.40	0.2981	0.2819	0.30
12	0	0.03	0	0	0.021	0
13	0	0.03	0	0	0.047	0
14	0.1282	0.1423	0.0685	0	0	0
15	0	0	0	0	0	0
16	0	0.03	0.03	0	0	0
17	0	0	0	0	0	0
18	0	0	0.0397	0	0	0
19	0	0.03	0	0	0	0
20	0	0	0.	0	0	0
21	0.400	0.3591	0.400	0.1635	0.1093	0.1108
22	0.0418	0.03	0.318	0.30	0.2996	0.30
$\bar{E}_\Omega$	0.792	0.792	0.792	0.744	0.739	0.734
$\bar{V}_\Omega$	0.0050	0.0061	0.0052	.0045	0.0050	0.0046

9. UNION (B_9)
10. UNCO (B_{10})
11. BARODA (B_{11})
12. INDIAN (B_{12})
13. BANDHAN (B_{13})
14. MAHA (B_{14})
15. IDBI (B_{15})
16. INDIAN (B_{16})
17. CUB (B_{17})
18. DCB (B_{18})

TABLE 13. *The optimal portfolios strategies corresponding to Model V for unequal weights*

DMU_k	$\gamma = 0.1, \epsilon = 0.03, \delta = 0.4$			$\gamma = 0.2, \epsilon = 0.02, \delta = 0.3$		
	d=5	d=10	d=7	d=5	d=10	d=7
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0.0204	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0.1924	0.0248	0.1809
7	0	0	0	0	0	0
8	0	0.03	0	0.1220	0.1479	0.1177
9	0	0	0	0	0.02	0
10	0	0.03	0.03	0	0.036	0.02
11	0.1569	0.1077	0.2052	0.30	0.2966	0.2977
12	0	0	0	0	0.0248	0
13	0	0.03	0	0	0	0.028
14	0.3583	0.3565	0.3253	0.0857	0	0
15	0	0.03	0	0	0	0
16	0	0.03	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0.03	0	0	0
19	0	0	0	0	0.02	0
20	0.03	0.03	0.0509	0	0	0
21	0.3959	0.3250	0.3285	0	0.1093	
22	0.0319	0.03	0.03	0.30	0.30	0.30
$\bar{E}_\Omega$	0.801	0.801	0.801	0.712	0.712	0.712
$\bar{V}_\Omega$	0.0069	0.0080	0.0073	0.0042	0.0037	0.0043

19. DHAN (B_{19})20. SOUTH (B_{20})

We denote the banks by B_k , $k = 1, 2, \dots, 20$, in order of the above listing. The description of input and output variables considered here are as follows: **Inputs:**

1. Input 1: ratio of market price to the earning per share (I_1)
2. Input 2: ratio of the market price to the book value per share (I_2)

Outputs

1. Output 1: ratio of net income to stockholders equity (O_1)
2. Output 2: percentage of a firm's ability to generate a return from its assets (O_2)

Table 14 presents the input and output values of the 20 banking stocks and their self-evaluated DEA efficiency. Further, from Definition 7 and Definition 8, the minimum OVGWA cross-efficiency $\tilde{e}_j^L$ and the minimum average cross-efficiency $\bar{e}_j^L$ for Bank B_j , $j = 1, 2, \dots, 20$ are also presented there.

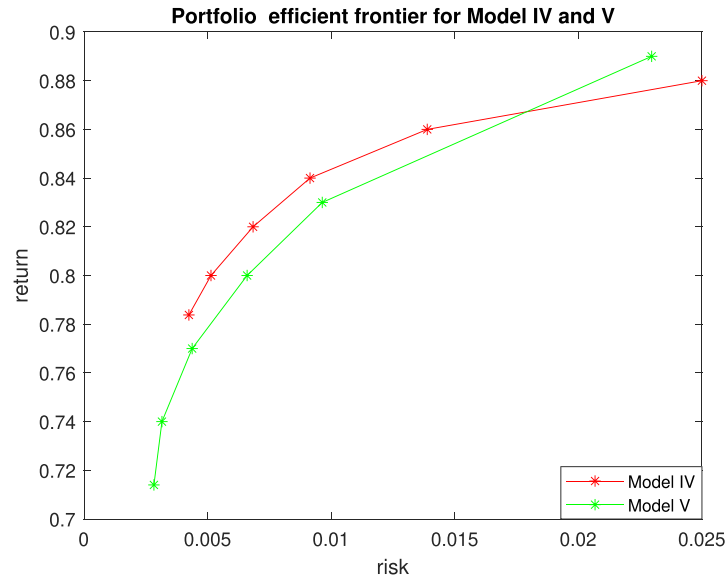


FIG. 3. Efficient frontier of Model IV and Model V.

Similarly, from Definitions 9, 10 and 11, the maximum OVGWA cross-efficiency and maximum mean cross-efficiency $\tilde{e}_j^U$ and $\bar{e}_j^U$ of respective Bank's are also provided therein. We set $\epsilon_i = 0.02$ and $\delta_i = 0.6$ to create portfolios with unequal weights. When we solve the proposed Models IV and V for a range of d and γ values, we found that a small increase in portfolio return (from 0.72 to 0.75) would result in a sizable increase in risk (from 0.0128 to 0.0145). The results are shown in Table 15.

4.2. Managerial implications

The proposed two-stage methodology has significant managerial implications in the field of DEA cross-efficiency aggregation and portfolio selection in terms of their efficiency. First, the developed approach can help managers to gain a better understanding of portfolio selection. Second, the approach's results are simple to grasp for managers. It can help investors and portfolio managers to build an efficient portfolio by incorporating only the best-performing stocks. Based on the implications discussed above, the proposed approach can be used not only in a financial market but also in other industries to examine the impact of operational and long-term decisions on business performance. Furthermore, the problem addressed in this study is in its early stages of investigation, and further research can be conducted based on the findings of this paper. Some options are as follows: 1. A similar study can be conducted to deal with stochastic data. 2. In this study, the proposed model was applied to a problem in the banking industry. However, the same model could be used for a variety of other selection-based problems like R&D and project selection, and the stock market. The methodology applies to other asset allocation problems, mutual fund portfolio selection, multi-period portfolio problems and other case studies. 3. Furthermore, comparative studies in different industries or countries using the model and methods from this study would be necessary to validate its claims. 4. The OVGA aggregated efficiencies can be used in different ranking problems and the obtained results can be further compared with other aggregation methods in terms of complexity.

TABLE 14. *Detailed description of the banks*

B_k	(I_1)	(I_2)	(O_1)	(O_2)	e_{kk}^*	$\bar{e}_k^L$	$\bar{e}_k^U$	$\tilde{e}_k^L$	$\tilde{e}_k^U$
B_1	14.1	1.29	10.13	0.54	0.3092	0.2361	0.2561	0.2489	0.2616
B_2	29.7	2.06	7.64	0.75	0.1877	0.1442	0.1568	0.1473	0.1610
B_3	14.99	0.45	2.98	0.2	0.2292	0.1569	0.1734	0.1533	0.1684
B_4	25.86	3.92	16.5	1.88	0.4737	0.2660	0.3112	0.2626	0.2909
B_5	21.87	2.6	14.85	1.38	0.4111	0.2732	0.3081	0.2799	0.2985
B_6	34.94	4.12	13.12	2.15	0.4009	0.2305	0.2704	0.2198	0.2457
B_7	9.09	0.92	10.52	0.85	0.6093	0.4619	0.5069	0.4849	0.5038
B_8	8.67	0.46	6.07	0.28	0.4494	0.3032	0.3361	0.3079	0.3238
B_9	7.62	0.37	6.25	0.35	0.5753	0.4185	0.4551	0.4008	0.4888
B_{10}	65.26	0.63	0.98	0.07	0.0573	0.0310	0.0367	0.0288	0.0374
B_{11}	24.78	0.5	1.98	0.12	0.1349	0.0822	0.0925	0.0702	0.0816
B_{12}	10.68	0.55	5.54	0.32	0.343	0.2585	0.2795	0.2536	0.2658
B_{13}	24.74	3.13	13.53	2.13	0.561	0.3157	0.3713	0.3059	0.3543
B_{14}	23.88	1.23	5.5	0.31	0.1523	0.1133	0.1228	0.1110	0.1165
B_{15}	27.38	1.32	4.85	0.48	0.1875	0.1251	0.1372	0.1320	0.1391
B_{16}	4.17	0.33	9.69	0.64	1	0.9511	1.0000	0.9592	1.0000
B_{17}	19.43	1.97	10.64	1.15	0.3856	0.2651	0.2980	0.2505	0.3163
B_{18}	9.5	0.9	10.01	0.86	0.5898	0.4537	0.4972	0.4572	0.4967
B_{19}	10.09	0.52	5.26	0.29	0.3445	0.2531	0.2751	0.2482	0.2618
B_{20}	28.02	0.32	1.16	0.06	0.1235	0.0642	0.0758	0.0520	0.0639

5. Conclusion and future directions

In this study, we introduce a novel portfolio selection model that can assist investors in identifying well-diversified equities based on numerous evaluation criteria. The unique primary contributions of this paper can be seen in two ways. First, the use of OVGA in cross-efficiency aggregation may be viewed as an extension and refinement of the DEA cross-efficiency aggregation approaches. Meanwhile, this is the first time that the OVGA aggregated cross-efficiencies have been successfully applied to portfolio management. It appears to be a promising tool for assessing financial assets. It is demonstrated herein that introducing the OVGA aggregation operator into the cross-efficiency portfolio selection yields superior portfolios with reduced variance and more significant or similar projected returns, at least for this particular study. To demonstrate the value of the suggested approach, a real-time example from the Indian banking industry is used. Further, the proposed strategy can also be completed by stock selection in larger stock markets.

It is worth noting that if asset return distributions are not symmetric, using variance as a risk measure is not recommended because doing so leads to portfolio predictions that are far from the actual target. Many researchers have utilized semi-variance and semi-absolute deviation as alternative risk metrics to qualify risk in such scenarios. Additionally, the OVGA cross-efficiency aggregation model may be extended to study portfolio selection problems in a fuzzy environment. Further, this study can be extended to study the efficiency of the DMUs having undesirable inputs and outputs.

TABLE 15. *The optimal portfolio strategies in the case study of banks corresponding to Model IV and Model V*

B_k	$\gamma = 0.25, \epsilon = 0.02, \delta = 0.6, d = 5$		$\gamma = 0.2, \epsilon = 0.02, \delta = 0.75, d = 7$		$\gamma = 0.3, \epsilon = 0.01, \delta = 0.5, d = 10$	
	Model IV	Model V	Model IV	Model V	Model IV	Model V
B_1	0	0	0	0	0.01	0
B_2	0	0	0.02	0.02	0.01	0
B_3	0	0	0	0	0.01	0
B_4	0	0	0	0	0	0
B_5	0	0	0	0	0.01	0.01
B_6	0	0	0	0	0	0
B_7	0.1335	0.313	0	0	0.0192	0.3068
B_8	0	0.02	0	0	0	0.01
B_9	0.0712	0.0469	0.02	0.1180	0.1572	0.1232
B_{10}	0	0	0	0	0	0
B_{11}	0	0	0	0	0	0
B_{12}	0.1752	0	0.957	0.0296	0.0914	0.01
B_{13}	0	0	0	0	0	0.01
B_{14}	0	0	0.02	0	0	0
B_{15}	0	0	0.0802	0.02	0.01	0
B_{16}	0.60	0.60	0.7441	0.75	0.5	0.5
B_{17}	0	0	0	0	0	0.01
B_{18}	0	0.02	0	0.0424	0	0.01
B_{19}	0.02	0	0.02	0.02	0.01	0.01
B_{20}	0	0	0	0	0	0
$\bar{E}_\Omega$	0.72	0.75	0.764	0.80	0.669	0.699
$\bar{V}_\Omega$	0.0128	0.0141	0.0145	0.0160	0.0112	0.0127

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Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

The data used for the numerical illustration in this study are included in this published article (<https://doi.org/10.1016/j.ejor.2013.12.002>). The data from the Indian banking sector used for the case study are available on the Trade Brains portal (<https://portal.tradebrains.in>).

Ethical statement

This article does not contain any studies with human participants or animals performed by the author.

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Appendix

The Python program that calculates the adjacency matrix, degree and weight of an ordered set is described as Algorithm 1.

Algorithm 1

```

1: import pandas as pd
2: from fractions import Fraction
3: function FUNC( $i, j, ls$ )
4:    $temp \leftarrow 1$ 
5:   for  $k$  in range( $i + 1, j$ ) do
6:      $ok \leftarrow ls[k]$ 
7:      $rhs \leftarrow ls[j] + (ls[i] - ls[j]) \cdot \left(\frac{j-k}{j-i}\right)$ 
8:     if  $ok \geq rhs$  then
9:       return 0
10:    end if
11:  end for
12:  return 1
13: end function
14:  $adj \leftarrow [[-1 \text{ for } \_ \text{ in range}(n)] \text{ for } \_ \text{ in range}(n)]$ 
15: for  $i$  in range( $n$ ) do
16:   for  $j$  in range( $i, n$ ) do
17:    if  $adj[i][j] \neq -1$  then
18:      print("asjndk", str( $i$ ), " : ", str( $j$ ))
19:      continue
20:    end if
21:    if  $i == j$  then
22:       $adj[i][j] \leftarrow 0$ 
23:      continue
24:    end if
25:    if  $i - j == 1$  or  $j - i == 1$  then
26:       $adj[i][j] \leftarrow 1$ 
27:      continue
28:    end if
29:     $adj[i][j] \leftarrow \text{func}(i, j, ls)$ 
30:  end for
31: end for
32: for  $j$  in range( $n$ ) do
33:   for  $i$  in range( $j, n$ ) do
34:     $adj[i][j] \leftarrow adj[j][i]$ 
35:   end for
36: end for
37: for  $i$  in range( $n$ ) do
38:   print( $adj[i]$ )
39: end for

```

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An Integrated Multi-Objective SORM Cross-Efficiency Model: An Application in Portfolio Selection

Abstract. *This paper presents a novel multi-objective framework for portfolio selection model integrated with the cross-efficiency approach of the Semi-oriented radial measure (SORM). This article suggests a two-stage method for choosing a portfolio. Although the SORM model was created as a workaround for the traditional DEA model's inability to handle negative data, in Stage 1, we extended it to peer evaluation as opposed to just the self-evaluation model. In the second stage, we enhance its application as a portfolio selection and optimisation tool by incorporating it into the multi-objective framework. We introduce the maverick index and diversity index as an innovative risk/return measure, that proves to be a valuable gauge of sensitivity to environmental volatility and diversification in the context of portfolio selection. The case study demonstrates the model's ease of application, its capacity to distinguish between stocks, and its effectiveness in portfolio selection.*

Keywords: *Data Envelopment Analysis, Semi-oriented radial measure, negative data, portfolio selection, cross-efficiency.*

JEL Classification : C02, C44, C60, C61, C67.

1. Introduction

Markowitz (1952) introduced the mean–variance (MV) model, which has significantly advanced the theory of modern portfolio selection. Many researchers have investigated and expanded the MV approach since the MV model was first introduced. Guo et al. (2019) examined how investors' decisions were affected by background risk using the MV, mean-VaR, and mean-CVaR frameworks. Xu et al. (2016) applied constraints on the weights in the standard CVaR-based portfolio selection model. Cui et al. (2019) introduced a strategy for the multiperiod mean–CVaR portfolio selection model that is both time-consistent and self-coordinating.

Most of the literature discussed above focusses on offering different models for portfolio selection. However, portfolio performance assessment is also crucial in the

financial market. To assess the relative performance of a particular group of decision-making units (DMUs), the DEA technology (CCR model), first introduced by Charnes et al. (1978) and later expanded by Banker et al. (1984), (BCC model) can manage multiple inputs and outputs simultaneously. As a result, it is widely utilised for performance assessment and portfolio selection. Chen et al. (2021) explored various DEA-based fuzzy portfolio evaluation models employing different risk measures. Gupta et al. (2020) formulated a comprehensive approach to streamlined portfolio evaluation, incorporating a dynamic rebalancing strategy. This empowers investors to construct optimal portfolios within a credibilistic framework. Zhou et al. (2018) introduced a DEA frontier improvement approach, offering investors a strategy for portfolio rebalancing. Traditional DEA models have a limitation—they only work with non-negative input and output values. However, in many cases, some inputs and/or outputs can be negative. Certain DEA models capable of handling negative data include those introduced by Cooper et al. (1999) known as Range Adjustment measure (RAM), Portela et al. (2004), known as Range Directional Measure (RDM), by Sharp et al. (2007) referred to as Modified Slack Based Measure (MSBM), and by Emrouznejad et al. (2010a). known as the SORM model. The radial DEA (BCC) model, under the VRS (variable return to scale) condition, offers an efficiency assessment for each DMU and accommodates negative values in inputs or outputs. However, it becomes inapplicable when negative values are present in both inputs and outputs.

To overcome the challenge of the weak discriminative power of DEA, a commonly used method is cross-efficiency evaluation, an extension to DEA introduced by Sexton (1986) which involves the utilisation of the weights chosen by each DMU to calculate its efficiency as a common set of weights for determining the efficiency of all other DMUs. This approach assesses the efficiency of each decision-making Unit (DMU) in two stages: self-evaluation and peer evaluation. Cross-efficiency assessment is commonly employed in the evaluation of production technologies exhibiting constant returns to scale, as negative efficiencies are not observed in such instances. In the case of variable returns to scale, de Mello et al. (2013) introduced the concept of constraining multiplier values to ensure only positive efficiencies across all DMUs. Wu et al. (2016) and Lim and Zhu (2015) represent notable methodologies for determining cross-efficiencies in the context of VRS production technology. Addressing limitations in prior research, Kao and Liu (2020) have introduced a slacks-based measure model for assessing cross-efficiency in DMUs, ensuring the absence of negative efficiencies. This model offers reasonable efficiencies for DMUs that exhibit only weak efficiency.

The literature ignores a cross-efficiency evaluation's inherent flaw. Negative values in inputs and outputs are not handled by the traditional cross-efficiency evaluation because it is built on the CCR model. Nonetheless, in many circumstances, inputs or outputs have negative values. There are not many studies in the literature that discuss cross-efficiency when dealing with negative data. Lim et al. (2014) presented the cross-efficiency form of the RAM model capable of handling negative data. Lin (2020) presented a DDF-based cross-efficiency evaluation method

under the VRS technology framework that is based on the RDM model, presented by Portela et al. (2004) and the duality theory. Wu et al. (2009) present a BCC VRS cross-efficiency model that can handle the problem of negative cross-efficiency by adding a non-negativity restriction.

Due to the continuous advancements in the evaluation of cross-efficiency using DEA and its exceptional characteristics, this approach has found widespread application across various fields. Researchers have explored the utilisation of DEA cross-efficiency methods in the realm of portfolio selection. Lim et al. (2014), for instance, introduced a DEA mean-variance cross-efficiency model to enhance the effectiveness of portfolio selection. Wu (2016) applied DEA cross-efficiency within the context of asset/stock organisation and portfolio optimisation, while Lim et al. (2014) proposed an innovative method for portfolio selection on the Korean stock exchange using DEA cross-efficiency. Amin and Hajjani (2021) demonstrated the impact of alternative optimal solutions in constructing a cross-efficiency matrix and verified the improved performance of the mean-variance portfolio selection method. Deng et al. (2019) integrated DEA prospect cross-efficiency evaluation into the novel Mean-Variance-Maverick (MVM) framework for fuzzy portfolio selection. Chen et al. (2021) delved into a portfolio selection problem that incorporates fuzzy Data Envelopment Analysis cross-efficiency evaluation, considering both undesirable fuzzy inputs and outputs.

Emrouznejad et al. (2010a) employed a partitioning approach to model negative data, leading to the introduction of the SORM model for evaluating the performance of observed production units. The SORM model is designed to address situations where some DMUs exhibit positive values, while others display negative values for a variable. It is also applicable when DMUs have both negative inputs and outputs simultaneously. A distinctive feature setting the SORM model apart from other approaches like RDM and MSBM is its conversion of each input-output variable into the sum of two components – one representing its negative value and the other its positive value. This allows for a positive format without requiring changes of origin that might otherwise be necessary to obtain positive values.

The above-mentioned studies on cross-efficiency evaluation in DEA evaluate the DMU's performance through a comprehensive perspective derived from its peers (Wu 2016). The average column values of the cross-efficiency matrix (CEM) are commonly used to evaluate the relative performance of a given DMU using a two-stage DEA model. Additionally, it is believed that the row averages in the cross-efficiency matrix can also be utilised to measure the diversity of performance within the same evaluation system for the unit (Talluri et al. (2013).

This paper makes four theoretical and novel contributions.: (i) We proposed a novel cross-efficiency technique that can handle negative data based on an input-oriented SORM model. (ii) Specifically, we extended the SORM cross-efficiency analysis in DEA that uses row and column means for optimal portfolio selection. (iii) We aim to develop a novel return indicator in the portfolio area and present a multi-objective model for portfolio selection by considering the diversity index as a return

and maverick as a risk. (iv) Furthermore, we covered the detailed sensitivity analysis by changing several realistic constraints such as cardinality and buy-in thresholds.

In this study, we broaden the SORM model into its cross-efficiency counterpart to gauge efficiency while accommodating variables that exhibit positive values for some DMUs and negative values for others. Here, we concurrently evaluate the performance diversity score (derived from the CEM row average) and the input-oriented SORM cross-efficiency score (based on the CEM column average). This approach provides a comprehensive method for identifying DMUs that excel not only in comparison to their peers, but also possess unique capabilities that differentiate them.

The remaining sections of the paper are organised as follows: In Section 2, we delve into the traditional input-oriented SORM cross-efficiency model. Section 3 introduces the proposed SORM cross-efficiency model, marking the initial stage of the proposed methodology. Section 4 outlines the second stage of the work, presenting a multi-objective portfolio selection model. The practical application of the proposed approach is illustrated in Section 5 to underscore its validity and effectiveness. Finally, Section 6 serves as the conclusion, where we analyse the results and suggest potential avenues for future research.

2. The input-oriented SORM model to deal with negative data

The traditional DEA models cannot be used directly to evaluate the efficiency of DMUs where some inputs and/or outputs may assume negative data. The semi-oriented radial measure (SORM) model was introduced by Emrouznejad et al. (2010a) to evaluate such DMUs. Without requiring any transformation, their method can be applied to negative data and produce an efficiency metric that is comparable to the radial measures in conventional DEA. We consider the input-oriented SORM model in dual form as described in Emrouznejad et al. (2010b). The fundamental concept of this approach involves replacing an input/output variable, which may have positive or negative data for different units, with the difference of two non-negative variables. The first part of this replacement involves exclusively positive values, while the second part incorporates the absolute values of the negative segment.

Let us assume there are n DMUs ($DMU_j, j = 1, 2, \dots, n$) that are associated with m inputs x_{ij} ($i = 1, 2, \dots, m$) and s outputs y_{rj} ($r = 1, 2, \dots, s$). Also, let

$$I = \{i \in (1, 2, \dots, m): x_{ij} \geq 0 \forall j = 1, 2, \dots, n\}$$

$$L = \{l \in (1, 2, \dots, m): j \in (1, 2, \dots, n) \text{ such that } x_{lj} < 0\} \text{ and}$$

$$R = \{r \in (1, 2, \dots, s): y_{rj} \geq 0 \forall j = 1, 2, \dots, n\}$$

$$K = \{k \in (1, 2, \dots, s): j \in (1, 2, \dots, n) \text{ such that } y_{kj} < 0\}$$

Emrouznejad et al. (2010a) define two output variables y_{kj}^1 and y_{kj}^2 such that

$$y_{kj}^1 = \begin{cases} y_{kj} & \text{if } y_{kj} \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad y_{kj}^2 = \begin{cases} -y_{kj} & \text{if } y_{kj} < 0 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, two input variable x_{kj}^1 and x_{kj}^2 are defined as

$$x_{kj}^1 = \begin{cases} x_{kj} & \text{if } x_{kj} \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad x_{kj}^2 = \begin{cases} -x_{kj} & \text{if } x_{kj} < 0 \\ 0 & \text{otherwise} \end{cases}$$

The standard input-oriented SORM model is presented as follows:

Model I

$$\begin{aligned} e_{dd} = \max & \sum_{r \in R} y_{rd} u_{rd} + \sum_{k \in K} y_{kd}^1 u_{kd}^1 - \sum_{k \in K} y_{kd}^2 u_{kd}^2 + u_0 \\ \text{s. t.} & - \sum_{i \in I} x_{ij} v_{id} - \sum_{l \in L} x_{lj}^1 v_{ld}^1 + \sum_{l \in L} x_{lj}^2 v_{ld}^2 + \sum_{r \in R} y_{rj} u_{rd} + \sum_{k \in K} y_{kj}^1 u_{kd}^1 \\ & - \sum_{k \in K} y_{kj}^2 u_{kd}^2 + u_0 \leq 0 \\ & \sum_{i \in I} x_{id} v_{id} + \sum_{l \in L} x_{ld}^1 v_{ld}^1 - \sum_{l \in L} x_{ld}^2 v_{ld}^2 = 1 \\ & u_{rd} \geq 0, \forall r \in R, \quad v_{id} \geq 0, \forall i \in I, \quad u_{kd}^1, u_{kd}^2 \geq 0, \forall k \in K, \quad v_{ld}^1, v_{ld}^2 \geq 0, \forall l \in L \end{aligned}$$

In Model I, the objective is to find a set of input-output weights that maximise the efficiency value of DMU_d . The optimal weights are referred as $u_{rd}^*, u_{kd}^{1*}, u_{kd}^{2*}, v_{id}^*, v_{ld}^{1*}, v_{ld}^{2*}$ and u_0^* and they determine the relative efficiency (self-efficiency) of DMU_d denoted as e_{dd} . The efficiency measure of the above model will only indicate the radial contraction of absolute input values. In the model I, the efficiency of DMU_j might become negative when assessed with DMU_d multipliers due to the free variable u_0^* . However, these negative efficiencies typically do not manifest in the results unless cross-evaluation models are utilised. Hence, we proposed a modified SORM model for cross-evaluation in the next section.

One of the best methods for choosing a portfolio is the DEA cross-efficiency method, since it examines stocks based on efficiency rather than absolute output and more fully rationalises stocks. The construction of the proposed method consists of two stages. Stage-I focuses with the evaluation of the efficiency of the stocks based on input-oriented SORM cross-efficiency model. Subsequently, the multi-objective portfolio selection model is formulated in Stage-II.

3. Stage 1-Proposed input-oriented SORM cross-efficiency model

As explained previously, the SORM model proves to be an efficient approach to address negative data in the input-output scenario. However, Model I is unsuitable for cross-efficiency due to its input orientation, which may result in negative cross-efficiencies, as noted by Lim and Zhou (2013). To overcome this limitation, we introduce an extension of the SORM model, referred to as Model II, specifically designed for cross-efficiency evaluation. This enhanced model effectively handles the issue of negative cross-efficiency as well as negative input output.

Model II

$$\begin{aligned}
 e_{dd}^{\sim} &= \max \sum_{r \in R} y_{rd} u_{rd} + \sum_{k \in K} y_{kd}^1 u_{kd}^1 - \sum_{k \in K} y_{kd}^2 u_{kd}^2 + u_0 \\
 \text{s. t. } & - \sum_{i \in I} x_{ij} v_{id} - \sum_{l \in L} x_{lj}^1 v_{ld}^1 + \sum_{l \in L} x_{lj}^2 v_{ld}^2 + \sum_{r \in R} y_{rj} u_{rd} + \sum_{k \in K} y_{kj}^1 u_{kd}^1 \\
 & - \sum_{k \in K} y_{kj}^2 u_{kd}^2 + u_0 \leq 0 \\
 & \sum_{r \in R} y_{rd} u_{rd} + \sum_{k \in K} y_{kd}^1 u_{kd}^1 - \sum_{k \in K} y_{kd}^2 u_{kd}^2 + u_0 \geq 0 \\
 & \sum_{i \in I} x_{id} v_{id} + \sum_{l \in L} x_{ld}^1 v_{ld}^1 - \sum_{l \in L} x_{ld}^2 v_{ld}^2 = 1 \\
 & u_{rd} \geq 0, \forall r \in R, \quad v_{id} \geq 0, \forall i \in I, \quad u_{kd}^1, u_{kd}^2 \geq 0, \forall k \in K, \quad v_{ld}^1, v_{ld}^2 \geq 0, \forall l \in L
 \end{aligned}$$

The 2nd constraint of model II will ensure the non-negativity of the cross-efficiency values. The optimal weights of Model II are referred as $u_{rd}^*, u_{kd}^{1*}, u_{kd}^{2*}, v_{id}^*, v_{ld}^{1*}, v_{ld}^{2*}$ and u_0^* and they determine the relative efficiency (self-efficiency) of DMU_d denoted as $e_{dd}^{\sim}$. In the above context of cross-efficiency evaluation, model II needs to be computed n times for each evaluated DMU_d to obtain the cross-efficiency values for all DMUs. Consequently, each DMU has both a self-efficiency value and $n - 1$ peer-efficiency value. Let $e_{dj}^{\sim}$ is the cross-efficiency of DMU_j obtained with the weights of DMU_d and defined as

$$e_{dj}^{\sim} = \frac{\sum_{r \in R} y_{rj} u_{rd}^* + \sum_{k \in K} y_{kj}^1 u_{kd}^{1*} - \sum_{k \in K} y_{kj}^2 u_{kd}^{2*} + u_0^*}{\sum_{i \in I} x_{ij} v_{id}^* + \sum_{l \in L} x_{lj}^1 v_{ld}^{1*} - \sum_{l \in L} x_{lj}^2 v_{ld}^{2*}} \quad (1)$$

A cross-efficiency matrix (CEM) has been obtained as $E = (e_{dj}^{\sim}), d, j = 1, 2, \dots, n$. Thus, the final cross-efficiency value of DMU_j (The column average of the CEM) is defined as $e_j^C = \frac{1}{n} \sum_{d=1}^n e_{dj}^{\sim}$. While the column average scores (averaged appraisal by peers) from the CEM offer valuable insights, our attention is also directed towards the row mean scores, representing the averaged appraisals of peers. Examining these rows mean scores enables us to understand how a DMU excels in comparison to other DMUs when utilising its optimal weights. The row

average of CEM is as follows: $e_j^r = \frac{1}{n} \sum_{d=1}^n e_{jd}^{\sim}$. The row averages in the CEM offer insights into how a specific DMU evaluates its peers. A lower row average in the CEM suggests that the DMU possesses distinctive capabilities, contributing to increased performance diversity within the group of DMUs.

Maverick Index and Diversity index

The results obtained from the CEM can be employed for assessing the maverick index, which contrasts the column average cross-efficiency score with the self-

evaluated efficiency. The calculation of the maverick index is given as $M_j = \frac{e_j^{\tilde{}} - e_j^c}{e_j^c}$.

DMUs characterised by low Maverick index scores (high mean cross-efficiency scores) are recognised as effective performers overall.

The calculation of the diversity index follows a similar methodology to that of the maverick index. To determine the performance of a test DMU (stock), its self-efficiency score is contrasted with the row average cross-efficiency score of all other DMUs, using the weights specific to the test DMU. The diversity index is computed as follows: $D_j = \frac{e_j^{\tilde{}} - e_j^r}{e_j^r}$. The diversity index brings to the table a set of capabilities

that sets it apart from its counterparts. A DMU that exhibits high self-efficiency and a robust diversity index is regarded as having unique resources, positioning it to outshine other DMUs and achieve superior performance.

4. Stage 2-A multi-objective framework of portfolio selection based on SORM cross-efficiency evaluation

The Lim et al. (2014) demonstrates that although the straightforward application of cross-efficiency evaluation in portfolio selection successfully considers the risk of changing the weights of specific DMUs chosen within a portfolio, it does not consider the risk for the portfolio. While inter-DMU risk is not considered, the straightforward application of cross-efficiency evaluation successfully lowers individual DMU risk. Thus, the SORM cross-efficiency approach alone can generate a moderately risk-consistent portfolio, particularly for individual risks. To solve this problem, we create the following multi-objective framework for portfolio selection based on cross-efficiency evaluation.

Let a portfolio Ω consist of n stocks with w_j ($j = 1, 2 \dots n$) proportion vector. For a given j^{th} stock the return and risk degree measure are defined as diversity index D_j and maverick index M_j respectively defined in section 3.

We maximise the weighted diversity index P_Ω (return) while minimise the maverick index (risk) I_Ω and define as $P_\Omega = \sum_{j=1}^n D_j w_j$ and $I_\Omega = \sum_{j=1}^n M_j w_j$ respectively. The optimal portfolio Ω^* can be obtained using the following multi-objective portfolio selection model:

Model III

$$\left\{ \begin{array}{l} \text{Min } I_{\Omega} = \sum_{j=1}^n M_j w_j \\ \text{Max } P_{\Omega} = \sum_{j=1}^n D_j w_j \\ \sum_{j=1}^n z_j \leq h \\ l_j z_j \leq w_j \leq \varepsilon_j z_j \\ \sum_{j=1}^n w_j = 1, \quad w_j \geq 0 \end{array} \right.$$

The terms l_j and ε_j correspond to the lower and upper bounds, respectively, representing a share of the capital budget available that can be invested in the j^{th} asset. The first constraints of Model III represent a cardinality constraint, imposing a limit on the number of assets (h) included in the portfolio. Investors can have varying opinions regarding the values of h , l_j and ε_j in this context.

In summary, the proposed method comprises two stages for portfolio selection. In the first stage, we apply model II and equation 1 to calculate the cross-efficiency (both row and column efficiency). In the second stage, we choose the portfolio by considering the maverick index for risk and the diversity index as return. A preferable portfolio exhibits a low maverick index for risk and a high diversity index.

5. An application to stock portfolio selection

To showcase the efficacy of the proposed two-stage approach in handling negative input-output scenarios, we have applied it to analyse authentic data sourced from the CSI 300 index. The evaluation of eleven specifically chosen stocks is conducted based on two inputs and seven outputs. The relevant financial input/output parameters are detailed in Table 1 for the selected nine parameters, and their standardised input output data are provided in Table 2.

Table 1. The Input-output parameters

Type	Parameter	Classification
Input	Solvency ratio-I (x1)	Total liability divided by total assets
	Solvency ratio-II (x2)	Total liability divided by shareholder's equity
Output	Return on equity (y1)	Net income divided by shareholders equity
	Return on assets (y2)	Net income divided by the total assets
	Net profit margin (y3)	Net income divided by revenue for the period
	Basic earnings per share (basic EPS) (y4)	Net income minus dividends and preferred stock dividend divided by common shares
	Revenue growth rate (y5)	Current period revenue divided by the previous period revenue minus one
	Net profit growth rate (y6)	Current period net income divided by the previous period net income minus one

Type	Parameter	Classification
	Basic earnings per share growth rate (y7)	Current period basic EPS divided by the previous period basic EPS minus one

Source: Sourced from Chen et al. (2021).

Table 2. The standardised input-output data with efficiency values

Stocks	x1	x2	y1	y2	y3	y4	y5	y6	y7
S_1	0.8135	0.3036	0.0855	0.0545	0.0189	0.0129	0.0031	0.0098	0.0290
S_2	0.4392	0.0936	0.6122	0.6862	0.1880	0.3365	0.0017	-0.0001	-0.0004
S_3	0.4655	0.105	0.3457	0.3495	0.2065	0.1054	0.0046	-0.0006	-0.0044
S_4	0.4481	0.0911	0.6088	0.6475	0.4377	0.3959	0.0133	0.0016	-0.0015
S_5	0.1235	0.0184	0.0384	0.0338	0.4418	0.0180	0.0063	0.0016	-0.0132
S_6	0.4384	0.0888	0.2430	0.2498	0.4978	0.2108	0.0061	0.0011	-0.0062
S_7	0.5197	0.1302	0.1559	0.1476	0.1948	0.0823	0.0193	-0.0026	-0.0059
S_8	0.8124	0.3278	0.0481	0.0292	0.0428	0.0090	0.0028	-0.0068	-0.0189
S_9	0.8275	0.3446	0.2248	0.1218	0.1296	0.0977	0.0051	-0.0005	-0.0014
S_{10}	0.3764	0.0758	0.4310	0.4924	0.2181	0.3332	0.0166	0.0034	-0.0058
S_{11}	0.4224	0.0835	0.5124	0.5424	0.4117	0.5116	0.0229	0.0097	-0.0012

Source: Sourced from Chen et al. (2021).

5.1 Results and discussion

Initially, we address model II to determine the self-efficiency of the chosen stocks. Subsequently, the optimal weights obtained from model II and equation (1) are utilised to assess the CEM, as presented in Table 3. Additionally, the column and row average efficiencies derived from the CEM are also outlined in Table 3. This marks the completion of the first stage, resulting in three types of efficiencies: self-efficiency, column average efficiency, and row average efficiency.

Table 3. Different efficiency values

Stocks	SORM self-efficiency	Column average efficiency	Row Average efficiency	Stocks	SORM self-efficiency	Column average efficiency	Row Average efficiency
S_1	1	0.2643	0.3084	S_7	0.7622	0.4096	0.6371
S_2	1	0.6254	0.4862	S_8	0.325	0.0825	0.6973
S_3	0.8387	0.5133	0.7436	S_9	0.4873	0.1630	0.7118
S_4	1	0.7637	0.5001	S_{10}	1	0.7796	0.5319
S_5	1	0.6588	0.2217	S_{11}	1	0.9023	0.3596
S_6	1	0.5994	0.5685				

Source: Sourced from MATLAB on solving Model II.

Table 4. The input-oriented SORM cross-efficiency matrix

Stocks	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
S_1	1.00	0.00	0.00	0.15	0.75	0.11	0.00	0.00	0.00	0.386	1.00
S_2	0.08	1.00	0.53	0.98	0.22	0.40	0.22	0.04	0.20	0.82	0.87
S_3	0.43	0.87	0.84	0.94	1.00	0.85	0.72	0.20	0.33	1.00	1.00
S_4	0.06	0.99	0.48	1.00	0.22	0.44	0.21	0.04	0.18	0.88	1.00
S_5	0.01	0.12	0.12	0.28	1.00	0.32	0.11	0.02	0.04	0.16	0.27
S_6	0.01	0.46	0.55	0.95	1.00	1.00	0.48	0.04	0.00	0.77	1.00
S_7	0.23	0.50	0.62	0.79	1.00	0.68	0.76	0.24	0.18	1.00	1.00
S_8	0.31	0.70	0.81	0.84	1.00	0.86	0.72	0.32	0.21	1.00	0.89
S_9	0.54	0.99	0.75	0.93	1.00	0.70	0.60	0.00	0.49	0.84	1.00
S_{10}	0.24	0.62	0.77	0.84	0.00	0.86	0.57	0.00	0.07	1.00	0.89
S_{11}	0.00	0.63	0.17	0.73	0.06	0.39	0.12	0.00	0.09	0.72	1.00

Source: Sourced from MATLAB on solving Model II and equation 1.

In stage 2, we solve the proposed multi-objective portfolio selection model (model III). The stock selection process through the proposed multi-objective model possesses distinctive resources and strengths, enabling it to differentiate or align with the performance of other assets. As a result, the assets chosen through this approach contribute to the formation of a portfolio with enduring stability in performance over the long term.

The solution of a multi-objective portfolio selection model is addressed by employing a genetic algorithm with default parameters in MATLAB 2022a. The implementation of the genetic algorithm resulted in a set of Pareto optimal solutions for model III. In each run of the GA, a collection of optimal Pareto solutions is generated, allowing the decision maker to choose a preferred portfolio based on their individual preferences. For example, in the model proposed herein, investors can evaluate the portfolios produced, weighing the trade-offs between

portfolio return rate and risk. They can then opt for the portfolio that offers the highest return rate (diversity index) while maintaining an acceptable level of risk. In this study, the portfolio that exhibits the highest return rate and the lowest risk is specifically chosen. Decision-makers can then choose a portfolio based on their individual preferences from this diverse set of portfolios.

To solve model III, we set the cardinality constraint $h=5$ while vary the values of l_j and ε_j . Within the set of Pareto solutions obtained, we choose the portfolio with the highest diversity index and the lowest maverick index. The results of both scenarios are presented in Table 5.

Table 5. The weights of the stocks in the selected portfolio for $h=5$

Stock	$l_j = 0.05,$ $\varepsilon_j = 0.5$		$l_j = 0.1,$ $\varepsilon_j = 0.4$		$l_j = 0.06,$ $\varepsilon_j = 0.3$	
S_1	0.35	0	0.3	0	0.3	0
S_2	0.05	0.05	0	0.1	0.1023	0.06
S_3	0	0	0	0.1	0	0
S_4	0.05	0.35	0.1	0.3	0.2377	0.28
S_5	0.5	0.05	0.4	0.1	0.3	0.3

Stock	$l_j = 0.05,$ $\varepsilon_j = 0.5$		$l_j = 0.1,$ $\varepsilon_j = 0.4$		$l_j = 0.06,$ $\varepsilon_j = 0.3$	
S_6	0.05	0.05	0.1	0	0.06	0.06
S_7	0	0	0	0	0	0
S_8	0	0	0	0	0	0
S_9	0	0	0	0	0	0
S_{10}	0	0.5	0.1	0.4	0	0.3
S_{11}	0	0	0	0	0	0
Return	2.68(max)	1.056	2.3(max)	1.1215	2.2(max)	1.706
Risk	1.318	0.34(min)	1.1681	0.41(min)	1.1652	0.40(min)

Source: Sourced from MATLAB on solving model III.

Table 6. The statistics from obtained pareto solutions for h=5

Criteria		$l_j = 0.05,$ $\varepsilon_j = 0.5$	$l_j = 0.1,$ $\varepsilon_j = 0.4$	$l_j = 0.06,$ $\varepsilon_j = 0.3$
return	Max	2.6813	2.3411	2.1174
	Min	2.0744	1.121	1.7063
	Average	1.0563	2.083	1.9203
Risk	Max	1.3118	1.1681	1.1652
	Min	0.3389	0.4124	0.4028
	Average	0.6761	0.7909	0.7905

Source: Sourced from MATLAB on solving model III.

Additionally, the statistics of the obtained pareto solutions in each case are presented in Table 6. Moreover, by adjusting the cardinality constraints, one can obtain more diversified portfolios.

Comparative study: In the literature, few studies address cross-evaluation in the presence of negative input-output scenarios. The approach proposed by Lin (2020) is non-oriented and offers self-evaluations and evaluations by peers for DMUs, simultaneously measuring inefficiencies in both inputs and outputs. While the proposed SORM model provides the radial efficiency, and cross-efficiency, which reflect radial contraction only of absolute input values, hence the direct comparison of the results obtained by these two methods is not possible.

The input-oriented BCC VRS cross-efficiency model by Wu et al. (2009) can also be utilised when negative values are present exclusively in either inputs or outputs, but it is not suitable for situations where negative values occur in both inputs and outputs. But our proposed method can be applied in such cases effectively. The detailed analysis presented in Table 7 further compares the proposed work with the existing literature available.

Table 7. Analysis of proposed work and existing work in the literature

Characteristic	Cross-evaluation	orientation	Limitation
Cooper et al. (1999) (RAM)	Yes	non-oriented	Resulting in the most distant targets on the frontier for

Characteristic	Cross-evaluation	orientation	Limitation
		(measure inefficiency)	inefficient DMUs and failing to offer a genuine measure of efficiency.
Wu et al. (2009)	Yes	Input/output oriented (Measure efficiency)	Not valid when both inputs and outputs exhibit negative values.
Lin (2020) (DDF based RDM)	Yes	non-oriented (measures inefficiency)	The efficiency score fails to accurately represent the actual performance of the DMU because it cannot encompass all the sources of inefficiency, as pointed out in Portela et al. (2004).
Sharp et al. (2007) (MSBM)	No cross-evaluation exist	Non-oriented	Applied in case of natural negative input output cases only
Proposed work	Yes	Input-oriented/measure efficiency	Applicable when there is at least one input with a positive value, with no conditions imposed on any of the outputs. (Emrouznejad et al. (2010b).)

Source: The table is compiled from an extensive survey of the literature.

To the best of our knowledge and following an extensive examination of the literature, there is currently no available method which takes the diversity index as a return indicator while selecting the optimal portfolio, so direct comparison of the proposed method with existing methods is not possible.

6. Conclusions

This study introduces two significant novel contributions within the proposed method. Firstly, the SORM model introduced can be viewed as enhancements and extensions to the DEA cross-efficiency evaluation. Additionally, this study marks the application of the SORM cross-efficiency approach to portfolio management, positioning it as a promising tool for the evaluation of financial assets. This study takes a unique approach to the problem, introducing an integrated multi-objective framework that surpasses the conventional mean-variance method. It introduces an additional layer of sophistication by melding a risk measure derived from the maverick index, computed through column average cross-efficiency, with a return component based on the diversity index, determined through row average cross-efficiency. Moreover, the proposed portfolio selection model adeptly addresses real-world constraints such as maximum and minimum capital fractions, as well as cardinal constraints, offering substantial advantages in portfolio management. The

numerical findings indicate that the proposed two-stage method offers a mechanism for decision-makers to achieve a more balanced and rationalised portfolio diversification.

The proposed work has the following limitations and possible future directions. First off, this study only looked at the input-oriented SORM cross-efficiency. Including the output-oriented SORM cross-efficiency would be beneficial to increase the significance of efficiency comparisons. The impact of fuzzy input-output data could be considered in future research by expanding the modelling scope to include ordinal and interval data values.

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Stock assessment using cumulative prospect theory in DEA cross-efficiency model: A case study of the Indian stock market

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Abstract. Market volatility is becoming increasingly common as numerous factors are implemented in the financial system. As a result, portfolio managers and individual investors require reliable methods to assess stock performance. This study examines stock assessments using cross-efficiency evaluations in cases where negative data is present. An alternative approach to achieve this goal is to use an RDM DDF-based cross-efficiency model which oversees the negative data. We expand the RDM-based cross-efficiency analysis, which uses row and column average values to select portfolios and identify different groups for stock management. To explore the psychological factors that influence the choices made by stock market investors, we incorporate the cumulative prospect theory value for each stock as an output and the variance as an input to evaluate the overall efficiency of the assets. For the empirical analysis, our study focuses on a sample of 30 stocks listed on the Nifty-50 on India's National Stock Exchange. The results of our empirical study verify that the proposed method can serve as an effective tool for stock selection. This demonstrates how the chosen portfolio gives companies a more diversified and well-balanced approach for selecting stocks, thus improving logical decision-making.

Keywords: cross-efficiency, cumulative prospect theory, data envelopment analysis, diversification, stock selection

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1. Introduction

After Markowitz [14] introduced the mean-variance optimization (MVO) model in the 1950s, the financial market grew exponentially and consistently. The Mean-Variance Optimization (MVO) model represents a crucial element in modern finance, offering a balance between portfolio returns and risks, thereby quantifying portfolio performance. With rapid advancements in artificial intelligence, the MVO selection process has now been integrated into diverse methodological frameworks within contemporary investment science. In the global market, asset assessment and ranking are vital for investors. In this study, we employ data envelopment analysis (DEA) with cross-efficiency analysis in the presence of negative input-output to improve portfolio selection by enhancing the stock evaluation and management process.

DEA stands out as one of the distinguished methodologies for examining the performance of decision-making units (DMUs) that manage a variety of inputs and outputs. After the

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groundbreaking research of Charnes et al. [3], who introduced non-parametric programming to approximate the production frontier, there has been a notable increase in publications concerning DEA principles and their practical implications in recent years. Pioneering studies, in this field, such as by Murthi et al. [15], were instrumental in applying DEA and crafting the DEA portfolio efficiency index (DPEI). Similarly, Chen et al. [4] introduced three distinct DEA models that incorporate various risk indicators for assessing portfolio efficiency in a possibilistic environment. Gupta et al. [7] developed an integrated approach for efficient portfolio evaluation, featuring a dynamic re-balancing strategy to empower investors in constructing optimal portfolios within a credibilistic framework. Kumar et al. [10] presented an innovative approach to multiperiod efficient portfolio selection (MPEPS), which involves a two-stage process. Additionally, Zhou et al. [26] proposed a portfolio re-balancing strategy that capitalizes on DEA frontier improvement within the mean-variance (MV) framework.

In DEA, a Decision Making Unit (DMU) evaluates itself by optimizing input and output weights to attain maximum relative efficiency. However, DEA has been criticized for its flexible weight assignment and reliance on self-assessment. To overcome this challenge, Sexton et al. [20] introduced a cross-efficiency method that integrates peer evaluation, shifting focus from self-assessment. Oukil and Amin [16] employed the concept of maximum cross-efficiency to evaluate the performance of baseball team players, presenting an alternative approach to team management. Wu [24] applied cross-efficiency to stock organization and portfolio optimization, and Lim et al. [11] introduced a novel methodology for portfolio selection on the Korean stock exchange using DEA cross-efficiency. Amin and Hajjani [1] demonstrated the effect of alternative optimal solutions in the construction of a cross-efficiency matrix (CEM) and validated that the performance of the MV portfolio selection method is enhanced. Deng and Fang [6] incorporated a DEA prospect cross-efficiency evaluation into an innovative MVM (mean-variance-maverick) framework for fuzzy portfolio selection. Amin and Oukil [2] and Sanei and Banihashemi [19] effectively applied the cross-efficiency evaluation for portfolio selection.

Instead of utilizing DEA methods, researchers have also employed multi-criteria decision-making (MCDM) methods that focus on analyzing and ranking decision alternatives based on a set of criteria. Kovač and Podrug [9] provide practical perspectives on utilizing MCDM and the MPT theory for share selection and portfolio optimization. In addition, Poklepovi and Babi [17] proposed a hybrid MCDM technique to evaluate the shares of the Zagreb stock exchange. DEA offers an advantage over MCDM because MCDM relies on a uniform set of weights, whereas DEA utilizes a linear programming model to determine individual weights for each DMU, along with an enhancement in efficiency.

Cross-efficiency analysis is frequently employed in scenarios where the production frontier exhibits constant returns to scale (CRS), as occurrences of negative efficiencies are uncommon across different situations. Lim et al. [12] and Wu et al. [25] introduced two notable methods for evaluations of the cross-efficiencies in cases involving variable returns to scale (VRS) production frontier. Their primary aims are either to reduce negative efficiencies or transform them into positive values. Recognizing the limitations in prior research, Kao and Liu [8] proposed a slack-based measure model for assessing the cross-efficiency value of DMUs without eliciting negative efficiencies. Their concept provides reasonable efficiencies for only weakly efficient DMUs. Soltanifar and Sharafi [21] presented a non-radial cross-efficiency model integrated with a hybrid MCDM technique. In contrast, Lin [13] proposed an approach for cross-efficiency evaluation based on the Directional Distance Function (DDF), utilizing the Range Directional Measure (RDM) model introduced by Portela et al. [18] and duality theory within the framework of VRS technology. The model introduced by Lin [13] can handle situations with negative input-output variables and successfully addresses the issue of negative cross-efficiency. Shrivastava et al. [22] introduced a portfolio selection approach that leverages cross-efficiency, by employing the model from [13] and Cumulative Prospect Theory (CPT). They calculated the CPT value of the assets from their weekly returns using the Cumulative probability (uniform

probability) weighting function and the utility function in CPT. The CPT-based model then constructs effective portfolios by utilizing the CEM as the basis for return calculations. Chen et al. [5] explored a portfolio selection challenge that incorporated fuzzy DEA cross-efficiency evaluation, considering both undesirable fuzzy inputs and outputs.

Typically, the mean values of the CEM columns are the standard for evaluating the performance of a specific DMU. The simultaneous assessment of both the conventional cross-efficiency score (from the CEM column average) and the performance diversity score (from the CEM row average) offers a holistic method to identify stocks that outperform their peers and possess distinctive capabilities absent in others.

This study addresses the issue of robust stock selection using a cross-efficiency evaluation, that accommodates both positive and negative data. The CA and RA are used for robust portfolio selection and the identification of various types of assets. When implemented in a strategic stock selection process, this approach enables managers to pinpoint stocks/assets that not only deliver strong overall performance but also offer diverse capabilities to set them apart from other alternatives. The notable contributions of this research are as follows:

- Firstly, only a limited number of studies in the current body of literature make use of the DEA cross-efficiency model to handle negative input-output data in portfolio selection, as seen in the works of Chen et al. [5] Talluri et al.[23] and Shrivastava et al. [22], and among others.
- Additionally this study addresses asset cross-efficiency by incorporating the CPT value and return of each stock as outputs, and variance as an input. Our method combines cross-efficiency with CPT, exploring the psychological aspects that affect decision-makers when shaping portfolios. It is important to highlight that only a few studies (e.g., Shrivastava et al. [22] have employed CPT values as an output.
- Finally, in contrast to the methods employed by Talluri et al. [23], this study employs a DDF-based RDM cross-efficiency model to classify the assets. In contrast to, the classification employed by Shrivastava et al.[22], this study classifies stocks into distinct categories, to enable the inclusion of well-performing stocks with both similarity and variation in their performance. This approach offers financial experts a more comprehensive and detailed understanding of the subject.

The rest of the study comprises four sections, including the preliminary section which discusses the DEA cross-efficiency techniques to deal with negative values in input/output. The proposed work methodology is described in section 3. Next, the practical application of the proposed approach is demonstrated to highlight its validity and effectiveness in section 4. The paper concludes with Section 5, which examines the results and proposes potential avenues for future research.

2. Preliminaries

This section provides an overview of the key studies that motivated us to initiate this research. Let a group of n homogeneous DMUs be evaluated. Assume that each of these DMUs consumes m different inputs x_{ij} ($i = 1, 2, \dots, m$) to generate s different outputs y_{rj} ($r = 1, 2, \dots, s$). Specifically, j^{th} DMU consumes amount x_{ij} of input i and generates amount y_{rj} of output r . Cross-efficiency is a concept employed in ranking to tackle the issue of inconsistency among the efficiencies of a set of DMUs assessed using different weightings in DEA. The traditional cross-efficiency method relies on the CCR model and cannot handle the negative input-output data.

To address this challenge, Lin [13] introduces a VRS cross-efficiency evaluation approach based on DDF, allowing input-output variables to take on negative values in line with the approach proposed by Portela et al. [18], as outlined below:

$$\begin{aligned}
 \textbf{Model I} \quad & \text{Min} \sum_{i=1}^m v_{ik}(x_{ik} + d_{ik}^-) - \sum_{r=1}^s \mu_{rk}(y_{rk} - d_{rk}^+) + \xi \\
 & \text{s.t.} \\
 & \sum_{i=1}^m v_{ik}x_{ij} - \sum_{r=1}^s \mu_{rk}y_{rj} + \xi \geq 0, \\
 & \sum_{i=1}^m v_{ik}d_{ik}^- + \sum_{r=1}^s \mu_{rk}d_{rk}^+ = 1, \quad j = 1, 2, \dots, n, \\
 & v_{ik}, \mu_{rk} \geq 0, \quad r = 1, 2, \dots, s \quad i = 1, 2, \dots, m.
 \end{aligned}$$

where the directional vectors are computed as

$$\begin{aligned}
 d_{rk}^+ &= \max_{j=1,2,\dots,n} y_{rj} - y_{rk}, \quad r = 1, 2, \dots, s, \\
 d_{ik}^- &= x_{ik} - \min_{j=1,2,\dots,n} x_{ij}, \quad i = 1, 2, \dots, m.
 \end{aligned}$$

are the possible non-negative improvements for inputs and outputs. Using the weights of DMU_k derived from M2, established the cross-efficiency of DMU_j, in the following manner:

$$\beta_{kj} = \frac{\sum_{i=1}^m v_{ik}^*(x_{ij} + d_{ik}^-) - \sum_{r=1}^s \mu_{rk}^*(y_{rj} + d_{rk}^+) - \xi^*}{\sum_{i=1}^m v_{ik}^*d_{ik}^- + \sum_{r=1}^s \mu_{rk}^*d_{rk}^+}. \quad (1)$$

Thus, the cross-efficiency score of j th DMU is given by $\beta_j = \frac{1}{n} \sum_{k=1}^n \beta_{kj}$ which is the simple column average of the cross-efficiency matrix. Similarly, we obtain the row average cross-efficiency as $\eta_j = \frac{1}{n} \sum_{k=1}^n \beta_{jk}$. The self-efficiency of DMU_k will be obtained by putting $j = k$ in the equation (1) and denoted as β_{kk}^* . The row average cross-efficiency provides insights into the individual characteristics of each DMU. When the value of a DMU's row mean is relatively lower compared to other DMUs, it distinguishes that DMU from the rest, indicating a high level of performance diversity.

Calculation of the "Maverick index" (MI)

The cross-efficiency scores can be utilized even further by defining the extent to which there is variation between the peer appraisal and the self-appraisal efficiency of DMUs. In finance, the Maverick index serves as a valuable tool for assessing the false positives among financial assets. This measure of concordance between the two efficiencies is called the Maverick index [?] and it is defined as follows:

$$MI_k = \frac{\beta_{kk}^* - \beta_k}{\beta_k} \quad (2)$$

MI quantifies the extent of disparity between a self-evaluation score and a peer-evaluation score. A higher MI value suggests a higher likelihood of the DMU being mistakenly perceived as efficient. Conversely, a lower index value indicates a more favorable outcome.

Calculation of The “Performance Diversity Index” (PDI)

The calculation of the PDI follows a methodology similar to that of MI. To determine the performance of an evaluated DMU (stock), its basic efficiency is compared with the average cross-efficiency of all other DMUs, using the weights specific to the evaluated DMU. Consequently, all DMUs involved in the assessment procedure are subject to the same criteria. The PDI is computed as follows:

$$PDI_k = \frac{\beta_{kk}^* - \eta_k}{\eta_k}. \quad (3)$$

3. Proposed Methodology

DEA cross-efficiency is the most appropriate approach for portfolio selection because it rationalizes equities more thoroughly and analyses stocks based on efficiency rather than absolute output. As financial decision-making often involves risks and uncertainties, decision analysis should consider the behavioral aspects of decisions. From this perspective, this study introduces a resilient approach to stock management, which encompasses the following stages:

Choice of input and output variables: Input: The variance of each asset is computed to assess its associated risk, which will serve as an input in the proposed method.

Output: To account for the irrational behaviors of decision-makers (DMs) in stock selection, we incorporate the CPT value of stocks using the procedures outlined by Shrivastava et al. [22] as an output. The second output is defined as the mean return of the stocks and is computed using the following formula: $R_{jt} = \frac{C_{j(t+1)} - C_{jt}}{C_{jt}}$ where C_{jt} and $C_{j(t+1)}$ are the closing price of the j^{th} stock at time t and $t + 1$ respectively.

Assessing the cross-efficiency of the stocks: Model (M2) is utilized to calculate the cross-efficiency scores for the chosen stocks. Subsequently, we determine the self-efficiency, the row average efficiency, and the column average efficiency from the obtained Cross-Efficiency Matrix (CEM).

Categorizing the Stocks: Based on the CA and RA scores obtained from M2 and equation, we categorize the stocks into the following groups: Group G1: High Column Average/High Row Average (HC/HR)

Group G2: High Column Average/Low Row Average (HC/LR)

Group G3: Low Column Average/High Row Average (LC/HR)

Group G4: Low Column Average/Low Row Average (LC/LR)

The RA value describes the unique characteristics of each stock. When a stock’s row average value is comparatively lower than that of other stocks, it stands out from the rest, indicating a higher degree of performance diversity. When we depict the means of the rows and columns derived from the cross-efficiency analysis in a quadrant format, stocks in G2 exhibit superior efficiency than the other stocks, setting them apart. Stocks originally in G1 and G3 should be relocated to G2 to improve their performance. In contrast, stocks situated in G4 demonstrate subpar efficiency and fail to distinguish themselves from other stocks.

4. Empirical illustration

To demonstrate the effectiveness of the proposed DEA cross-efficiency evaluation approach addressing negative data, we employ it to analyze real data from the Indian stock market. We use data from Shrivastava et al. [22] focusing on the performance of the top 30 assets included in the Nifty 50 index. For each asset, one input variable variance, and two output variables, CPT value and long mean return are considered. For the long mean return, we consider the data from 05-03-2018 to 25-02-2019. To calculate the CPT value and variance, we select the weekly return of stocks for the period from 05-03-2018 to 02-08-2018. We select the outputs

for stock evaluation because of their direct relevance to portfolio assessment, and, importantly, their ability to account for the DM's behavioral characteristics.

To evaluate the performance of the listed stocks, the M1 model is used to evaluate the assessment of the stocks. Table 1 shows the CPT value, mean return, and variance with self-efficiency obtained by M1 of the thirty stocks.

S.No	Stocks	Variance	CPT value	Return	Self-efficiency
S_1	BAJAJ-AUTO	0.0013	-0.0273	0.0001	0.4933
S_2	BRITANNIA	0.0007	-0.0065	0.0043	0.8914
S_3	HEROMOTOCO	0.0008	-0.0248	-0.0049	0.4717
S_4	ICICIBANK	0.0024	-0.0221	0.006	0.4848
S_5	INDUSINDBK	0.0004	-0.0067	0.0013	0.8383
S_6	KOTAK BANK	0.0007	-0.0080	0.0043	0.8914
S_7	L & T	0.0007	-0.0122	0.0016	0.7138
S_8	NESTLEIND	0.0009	-0.0041	0.007	1.0000
S_9	NTPC	0.0005	-0.0159	0.0001	0.7125
S_{10}	ULTRACEMCO	0.0009	-0.0176	0.0011	0.6200
S_{11}	WIPRO	0.0011	-0.0135	0.0041	0.6902
S_{12}	TCS	0.0009	0.0008	0.006	0.9569
S_{13}	ITC	0.0008	0.0042	0.0027	0.8092
S_{14}	TITAN	0.001	-0.0175	0.0059	0.8333
S_{15}	BAJFINANCE	0.002	0.0227	0.0125	1.0000
S_{16}	BAJAJFINSV	0.0011	-0.0038	0.0078	0.9756
S_{17}	SHREECEM	0.0009	-0.0134	0.0037	0.7399
S_{18}	TATASTEEL	0.0017	-0.0291	-0.0011	0.4177
S_{19}	BHARTIARTL	0.0011	-0.0267	-0.0013	0.4959
S_{20}	GRASIM	0.0011	-0.0213	-0.0037	0.4594
S_{21}	TATACONSUM	0.002	-0.0581	-0.0033	0.3361
S_{22}	M&M	0.0006	0.0086	-0.0010	1.0000
S_{23}	ONGC	0.0009	-0.0226	-0.0016	0.5309
S_{24}	MARUTI	0.0004	-0.0115	-0.0028	0.6319
S_{25}	RELIANCE	0.0016	0.0039	0.008	0.7770
S_{26}	TECHM	0.0012	-0.0059	0.0046	0.7109
S_{27}	CIPLA	0.0012	-0.0100	-0.0007	0.5287
S_{28}	COALINDIA	0.0014	-0.0324	-0.0040	0.4031
S_{29}	HINDALCO	0.0034	-0.0344	0.0008	0.2996
S_{30}	HDFCLIFE	0.001	-0.0180	-0.0010	0.5384

Table 1: *Input Output variable with Self-efficiency after solving Model I.*

4.1. Stock management using the proposed method

The model M1 computes the cross-efficiency of the stocks. The cross-efficiency matrix (CEM) is generated using the input and output weights from M1 and Equation (1). The resulting CEM offers unique insights pertinent to this analysis. A widely recognized use of CEM involves averaging the efficiency scores down a column and designating that average as the peer appraisal score, commonly known as the CA efficiency.

Stocks	S_1	S_1	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
CA efficiency	0.4816	0.8650	0.4521	0.4550	0.8027	0.8588	0.6902	0.9871	0.6750	0.6022
RA efficiency	0.6614	0.6332	0.6614	0.6182	0.6332	0.6332	0.6614	0.6182	0.6332	0.6614
Stocks	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}	S_{19}	S_{20}
CA efficiency	0.6759	0.9237	0.7644	0.7953	0.9591	0.9386	0.7219	0.4053	0.4816	0.4396
RA efficiency	0.6614	0.6614	0.6614	0.6182	0.6182	0.6182	0.6332	0.6614	0.6614	0.6614
Stocks	S_{21}	S_{22}	S_{23}	S_{24}	S_{25}	S_{26}	S_{27}	S_{28}	S_{29}	S_{30}
CA efficiency	0.3290	0.6737	0.5130	0.6007	0.7510	0.6877	0.5037	0.3894	0.2915	0.5179
RA efficiency	0.6182	0.4965	0.6614	0.6614	0.6614	0.6614	0.6614	0.6614	0.6182	0.6614

Table 2: The column and row average efficiency obtained from CEM.

A stock with a high CA performs well because it outperforms the other units. Also, the low Row average (RA) value, indicates that the specific stock possesses distinctive capabilities and expertise that other stocks lack. The CA efficiency and RA efficiency obtained from CEM are presented in Table 2. We categorize the stocks based on their CA and RA values. Figure 1 shows the grouping of stocks based on the CA and RA scores for all 30 stocks. The data is categorized as either high or low for both CA and RA. The mean of these scores is employed to split the categories into high and low (in this instance, the average resulted in a value of 0.641). Although the classification thresholds are subjective, they can also be determined by decision makers or managers engaged in stock evaluation..

Discussion on the different groups of stocks The suggested stock classification aids investors in formulating portfolio optimization approaches. Supreme or exceptional performers are prime candidates for long-term investments because of their outstanding performance in various aspects and unique skills that contribute to building a robust and diversified stock selection, capable of meeting the current and future demands of the financial market. However, all-round good performers and niche performers exhibit differences. While the former excel overall compared to their peers, they lack the specific distinctive skills that distinguish them. Detailed descriptions of these groups are as follows.

1. All-round good-performing stocks (HC/HR): Six stocks are assigned to the HC/HR quadrant (G1). Stocks in this group are referred to as all-round good performers because they possess strengths that align with the performance of other assets. A portfolio generated with these stocks may perform well, but it may not be stable. The RA and CA efficiencies along with MI and PDI are presented in Table 3. The average MI and PDI for this portfolio are 0.0364 and 0.1739, respectively.

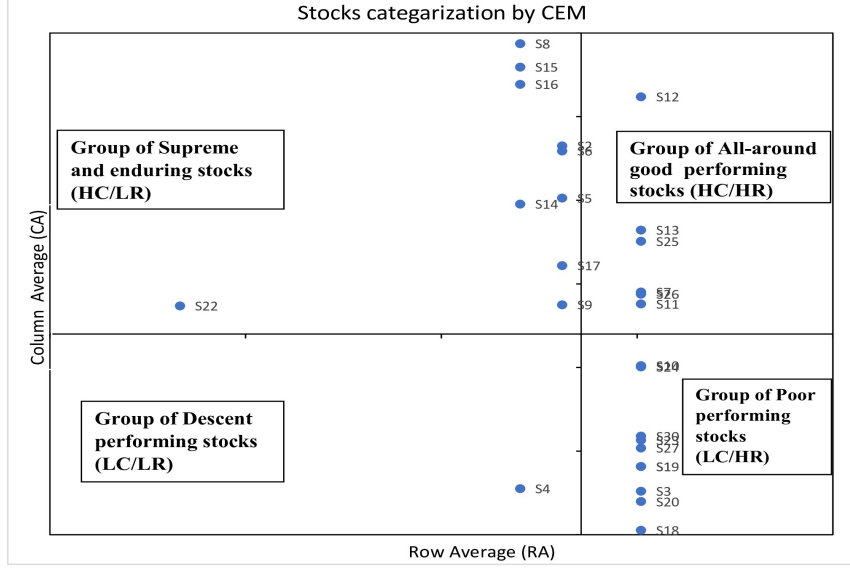


Figure 1: Different group of stocks based on CA & RA.

Stocks	CA	RA	MI	PDI
S_{12}	0.9237	0.6614	0.0359	0.4469
S_{13}	0.7644	0.6614	0.0587	0.2236
S_{25}	0.7510	0.6614	0.0346	0.1749
S_7	0.6902	0.6614	0.0341	0.0793
S_{26}	0.6877	0.6614	0.0337	0.0749
S_{11}	0.6759	0.6614	0.0211	0.0436
Average	0.7488	0.6614	0.0364	0.1739

Table 3: Statistics of group HC/HR.

2. Supreme and enduring stocks: Table 4, describes the list of stocks in the HC/LR (G2) group. The stocks within this category possess distinctive resources and strengths to set them apart or align them with the performance of other stocks. Consequently, assets from this group can be regarded as long-term stability portfolios. The portfolio obtained from this group is well-diversified with a PDI value of 0.4609.

3. Descent-performing stocks: Only stock 4 comes under the category G3. The stock in this group has unique resources that set them apart from other stocks but lack the strength to match the performance of their peers. But these are less diversified.

4. Poor performing stocks: Within the high LC/HR (G4) quadrant, there are thirteen assets, which are typically considered subpar performers. These assets lack distinctive resources or strengths that can outperform their market counterparts. Consequently, portfolios comprising assets from this group may not exhibit strong performance, and the overall diversification of such portfolios is notably low. The results are shown in Table 6.

After examining the tables, it is evident that G2 (HC/LR) assets have the best performance

Stocks	CA	RA	MI	PDI
S_8	0.98713	0.6182	0.01304	0.61758
S_{15}	0.95908	0.6182	0.04266	0.61758
S_{16}	0.93862	0.6182	0.039398	0.57811
S_2	0.86501	0.6332	0.03050	0.40787
S_6	0.85878	0.6332	0.03798	0.40787
S_5	0.80265	0.6332	0.04441	0.32400
S_{14}	0.79534	0.6182	0.04772	0.34793
S_{17}	0.72189	0.6332	0.02495	0.16859
S_9	0.67497	0.6332	0.05560	0.12532
S_{22}	0.6737	0.4965	0.48431	1.0141
Average	0.8277	0.6135	0.0820	0.4609

Table 4: *Statistics of group HC/LR.*

Stocks	CA	RA	MI	PDI
S_4	0.4550	0.618	0.06541	-0.2158

Table 5: *Statistics of group LC/LR.*

Stocks	CA	RA	MI	PDI
S_{10}	0.6022	0.6614	0.0296	-0.06253
S_{24}	0.6007	0.6614	0.05200	-0.04454
S_{30}	0.51788	0.6614	0.03962	-0.1859
S_{23}	0.5130	0.6614	0.034913	-0.19725
S_{27}	0.5037	0.6614	0.049605	-0.20058
S_{19}	0.4816	0.6614	0.02968	-0.25017
S_1	0.4816	0.6614	0.02435	-0.25411
S_3	0.45208	0.6614	0.04339	-0.28677
S_{20}	0.43963	0.6614	0.04498	-0.30536
S_{18}	0.40534	0.6614	0.03049	-0.368416
S_{28}	0.38944	0.6614	0.035076	-0.39049
S_{21}	0.32902	0.61821	0.02151	-0.45633
S_{29}	0.29147	0.61821	0.027893	-0.51537
Average	0.46212	0.65472	0.035624	-0.27060

Table 6: *Statistics of group LC/HR.*

with an average MI score of 0.0820 and a higher average PDI score of 0.4609. G3 and G4 have significantly lower PDI scores when assessing overall diversification with less MI. It should be

noted that the portfolio can also be chosen from a combination of the above groups of stocks or from the entire group. Similar to each category the stocks with low MI or high PDI can be selected.

To the best of our knowledge and based on our extensive literature review, there is currently no available approach to directly compare the outcomes of the method proposed in this research. Existing studies that utilize DEA cross-evaluations for portfolio selection, stocks are typically ranked in descending sequence of their cross-efficiency values, and the top n stocks are selected, where n corresponds to the number of stocks desired in portfolio. In contrast, our method involves the categorization of stocks into distinct groups based on the average values of the rows and columns in the Cross-Efficiency Matrix (CEM). Therefore, a direct comparison between these two methods is not meaningful.

5. Conclusion

We used the RDM DDF-based cross-efficiency analysis to categorize stocks with negative data. Although various approaches are available for categorizing and grouping stocks, the proposed classification methodology stands out as distinctive and holds significance in the context of portfolio construction. CA values in CEM are commonly employed in the standard practice of evaluating a unit's overall comparative performance. We established an alternative approach that involves the RA values of CEM to determine the breadth of performance diversity. These two ideas are merged to categorize financial assets into different groups, which further enables a more effective stock rationalization process that considers comparative performance and diversity of performance concurrently.

The limitations and potential future directions of this study include the Indian stock market with a restricted dataset. To enhance the significance of efficiency comparisons and benchmark selection, it would be valuable to encompass major markets in other countries. In future research, broadening the modeling scope to include ordinal and interval data values would enable the consideration of the impact of fuzzy input-output data.

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The screenshot shows a web browser window with multiple tabs open. The active tab is 'mjlcclarivate.com/search-results'. The page header includes the Clarivate logo and navigation links: 'Master Journal List', 'Search Journals', 'Match Manuscript', 'Downloads', and 'Help Center'. There are also 'Login' and 'Create Free Account' buttons.

The main content area is titled 'Refine Your Search Results'. It features a search bar with the text 'croatian operational', a 'Search' button, and a 'Sort By: Relevancy' dropdown. Below the search bar, it says 'Search Results' and 'Found 213 results (Page 1)'. A link 'Share These Results' is also present.

On the left side, there is a 'Filters' section with a 'Clear All' button. The filters include 'Web of Science Coverage', 'Open Access', 'Category', 'Country / Region', 'Language', and 'Frequency', each with a dropdown arrow.

The search results display the journal 'CROATIAN OPERATIONAL RESEARCH REVIEW'. It includes the following details:

- Publisher: CROATIAN OPERATIONAL RESEARCH SOC., C/O MARIJANA ZEKIC-SUSAC, EKON FAK & OSIJEKU, OSIJEK, CROATIA, 00000
- ISSN / eISSN: 1848-0225 / 1848-9931
- Web of Science Core Collection: Emerging Sources Citation Index

At the bottom of the journal entry, there are links for 'Share This Journal' and 'View profile page'. A small blue circle with a question mark and the number '5' is visible in the bottom right corner of the page.

The browser's taskbar at the bottom shows the system clock as 10:41 on 28-11-2024, and the weather as 16°C Sunny.

Ranking of Efficient Fuzzy Portfolios by Hybrid MSBM-TOPSIS Technique

Abha Aggarwal, *Anjana Gupta †, Rajkumar Verma ‡, Reenu Kumari §

Abstract. This study focuses on ranking of investment portfolios by integrating the Modified Slack Based Measure (MSBM) of Data Envelopment Analysis (DEA) with a multi-criteria decision-making method. Specifically, it extends the MSBM model to evaluate portfolios with positive and negative inputs and outputs in a fuzzy environment using possibilistic mean return of the assets as output and possibilistic variance and semi-variance as inputs. The ranking process involves two stages: first, portfolios are evaluated using an MSBM fuzzy portfolio model, followed by their ranking through the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method. This hybrid MSBM-TOPSIS approach provides a robust and reliable ranking system, enabling investors to identify efficient portfolios by leveraging the strengths of both methods. Detailed numerical illustrations are presented here to authenticate the proposed approach and the obtained results are compared with other existing DEA methods that validate the accuracy and feasibility of the proposed technique.

Keywords: Fuzzy Set, Modified Slack based Measure, Data Envelopment Analysis, Portfolio, TOPSIS method

1. Introduction

In financial terminology, a portfolio is simply a collection of financial assets such as stocks, bonds, cash, currency, derivatives, etc. Portfolio optimization employs a

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method to create an ideal asset mix, taking into account the goals and preferences of the investor. The major revolution in portfolio selection theory came in 1952 with the path-breaking work of [27]. His insight resulted in an optimization model now famous as the mean-variance model, where the portfolio's return and risk are quantified using the statistical measures of expected return and variance of the portfolio return distribution, respectively. Thus, an efficient portfolio has the minimum risk level for a given return or the maximum return for a given level of risk. The distance between the portfolio under evaluation and its projection on the efficient frontier can be used to calculate the efficiency of the portfolio by the M-V model [48].

Efficient frontiers are challenging, especially in large-scale evaluation scenarios, because of the many constraints imposed by real-world investment conditions [43]. To deal with this issue, the traditional DEA methodologies are widely used to approximate portfolio efficiency. DEA is one of the most prevalent approaches for evaluating the relative efficiency of concern portfolios based on multiple inputs and outputs. The first DEA model was initiated in 1978 by [9] to calculate the efficiency of an educational center and then further developed by [5] in 1984. Numerous researchers have utilized DEA models to assess portfolio efficiency. [14] addressed a portfolio performance evaluation problem by incorporating margin limitations, demonstrating through simulation how the DEA frontier increasingly aligns with the portfolio frontier as the sample size expands. [24] applied a DEA frontier improvement technique to suggest a re-balancing strategy for portfolios inside the MV framework. [48] presents a segmented DEA method that depends on data segment points to handle the problem of cardinality constraints in portfolio selection problems.

It is important to note that DEA models inherently assume non-negative input and output values. However, negative inputs or outputs may arise in many cases, such as losses relative to net profit, negative net income when expenses surpass revenues, or return rates on investments. [33] introduced the range directional model (RDM) in 2004, which is applicable in such circumstances. [37] proposed a modified slack-based measure (MSBM) capable of handling both negative outputs and negative inputs. Some other DEA-based models for dealing with negative inputs and outputs are the semi-oriented radial measure (SORM) model by [15], the Variant of the radial measure (VRM) model by [12], BP-SBM model by [39] etc.

The studies mentioned above rely on a probabilistic environment assuming future asset values are known ex-ante, but accurately predicting returns from historical data is impractical due to market complexity and volatility. Non-probabilistic factors like policies, regulations, and human behavior further complicate this. Fuzzy techniques, introduced by [44], offer a better fit for handling ambiguity in financial markets, with methods like possibility, credibility, and random fuzzy theories proposed for fuzzy portfolio selection. Possibility theory, introduced by [45] is an uncertainty framework designed to address incomplete and imprecise information. The concept of possibility theory used in fuzzy-set approaches is generally more applicable when examining an early stage of a decision process and when little information is available. [7] proposed the definition of lower and upper possibilistic mean values of a fuzzy number and considered them as possibility distributions. Besides, [46] defined the lower and upper possibilistic variances and covariances of fuzzy numbers. [8]

developed a possibilistic approach to portfolio selection with the highest utility score by assuming the returns of assets as trapezoidal fuzzy numbers. [25] formulated two possibilistic mean semi-variance models for portfolio selection, in which market risk and liquidity risk are measured by lower possibilistic semi-variance. [28] proposed a fuzzy multi-objective Markowitz-DEA cross-efficiency method in a possibilistic framework. [11] introduced DEA-based fuzzy portfolio estimation models to find the fuzzy portfolio efficiency. They construct three types of DEA-based fuzzy portfolio models assuming different risk indicators, i.e., possibilistic variance, possibilistic semi-variance, and possibilistic semi-absolute deviation. Later, possibility theory and credibility theory-based fuzzy portfolio optimization models were proposed by [10, 41, 29, 17].

The Multi-Criteria Decision-making (MCDM) method makes decisions when multiple alternatives are available for contradictory criteria. Its principal goal is to maintain control over the decision-making process in case of various alternative and criterion diversity. TOPSIS is a powerful MCDM approach developed by [20] to evaluate the performance of different alternatives based on the similarity degree with the ideal solution. According to this technique, the best option should have the shortest distance from the ideal solution and the farthest from the negative ideal.

Following the work presented in [11], we offer the hybrid MSBM-TOPSIS technique that considers efficient fuzzy portfolios as the available alternative and different inputs and outputs of efficient portfolios as the evaluation attributes. Similar to the work [11], randomly generated fuzzy portfolios are considered in this study, where the return from the asset is a trapezoidal fuzzy number. Two types of risk measures, possibilistic variance, and possibilistic semi-variance, are considered here with numerical examples which effectively verify the proposed approach.

1.1. Literature Review of Ranking models in DEA

In DEA, each DMU is self-evaluated with the most favorable weights for input and output to obtain relative efficiency. Consequently, DEA might assess too many DMUs as the best and cannot distinguish them further. To cope with this issue, [2] introduced a variable return to scale (VRS) based super efficiency (SE) DEA model in which the efficient DMU under SE evaluation is excluded and compared with a linear combination of the remaining DMUs in the production possibility set. Despite being infeasible in some cases, Super-efficiency models are widely applied and extended in literature. [23] developed a directional distance function (DDF) based super-efficiency DEA model, building on [2] approach, by designing a suitable directional vector. This model can accommodate negative values in the dataset and also resolves the infeasibility problem associated with the VRS radial super-efficiency DEA model. [40] Integrate the super-efficiency model with the SBM model. [32] applied the SE model to the water use efficiency of Shandong Province cities. [18] introduced RDM super efficiency model to provide a unique ranking for efficient DMUs. Although the DEA-super efficiency models provide a unique ranking to the efficient DMUs, researchers still argue for developing more methods for ranking ef-

efficient DMUs due to the infeasibility of DEA-SE models in many problems. It is important to note that each method is based on a specific theory and framework, which can lead to different ranking outcomes for the same DMU. In practice, selecting a ranking method that the Decision Maker (DM) can trust is a critical concern. The challenge of ranking efficient DMUs can be addressed by employing Multi-Criteria Decision-Making (MCDM) techniques, which, although distinct from traditional DEA models, offer a complementary approach for more reliable rankings. TOPSIS method is one of the MCDM methods widely used in the literature. Some studies have combined TOPSIS and DEA within a unified framework to generate a distinct ranking of DMUs. In these studies, one technique is typically used as the primary method, while the other serves as a supplementary tool. By leveraging both DEA and TOPSIS, decision-makers can effectively achieve the objectives of a corporate sourcing strategy through a comprehensive evaluation approach. [26] considered efficient DMUs from different DEA models and ranked them using the TOPSIS method. [4] proposed a new method for ranking efficient units and compared it with the other methods presented in the literature, such as the super-efficiency model in DEA, the concept of Shapley value in game theory, and the TOPSIS method in multi-criteria decision making. [34] utilized a hybrid approach based on TOPSIS and DEA methods to obtain a rational ranking of efficient DMUs. [21] presented two models for ranking efficient units. Both are based on MCDM methods; the first uses the goal programming technique for deriving a super-efficiency measure, and the second combines AHP and DEA principles. [13] employed the DEA method to evaluate the efficiency of 15 Indian bank branches, using the SE-CCR-DEA model and TOPSIS for ranking. The study compares the outcomes of the CCR-DEA, SE-CCR-DEA, and TOPSIS methods. [30] used the hybrid DEA-TOPSIS method for ranking of the 14 state water service providers in Malaysia. [31] used a three-step DEA model to obtain four types of efficiency (technical efficiency, social efficiency, environmental efficiency scores, and sustainable efficiency) for Iranian airlines. They integrate DEA models with the TOPSIS method to aggregate the final efficiency score. [47] presents a method to compare and rank the interval efficiencies of DMUs. [38] recently introduced an integrated framework combining multi-attribute decision-making with DEA models to address heterogeneous attributes in evaluating European countries' SDG progress. This method reduces computational complexity and eliminates the need for expert-defined weights, with results compared to standard MCDM techniques. [3] proposed a novel hybrid DEA-TOPSIS approach to identify the most sustainable refrigeration system technologies for the UK. [16] focused on evaluating the performance of distance education departments at public universities in Turkey during the 2018–2019 academic year, utilizing both the DEA and TOPSIS methods. The study also aimed to compare and rank the efficient decision-making units based on their efficiency measurements. [6] proposed an integrated interval DEA model that classifies DMUs into three categories—super-efficient, efficient, and inefficient in a single step, accommodating integer and negative data. A detailed review of DEA ranking models has been presented in [1].

From the detailed literature analysis, we conclude that using DEA and TOPSIS is a good idea for handling the problem of robust ranking of DMUs.

1.2. Contributions and Purpose

In this study, we have made an effort to add one more aspect to the DEA application, i.e. ranking of efficient fuzzy portfolios with two risk metrics in the presence of negative input-outputs. The following are the key highlights of the current study.

1. The MSBM model is commonly employed to assess the efficiency of DMUs in the presence of negative data. However, its application in evaluating the efficiency and ranking of portfolios with uncertainty has not been thoroughly explored in the existing literature. This research represents the first attempt to extend the MSBM model within a possibilistic framework, allowing it to address and manage fuzzy variables effectively. By incorporating fuzziness into the model, this study aims to fill the gap in portfolio efficiency analysis and provide a novel approach that enhances decision-making under uncertainty.
2. Based on previous discussions, researchers across different fields generally agree that utilizing DEA and TOPSIS is a valuable approach. Very few studies [19, 22] etc. address full ranking with negative input-output variables, so we combine the MSBM fuzzy portfolio model with TOPSIS to leverage the strengths of both methods.
3. Implementing the MSBM fuzzy portfolio estimation models with TOPSIS for ranking the efficient portfolios is a new application in the field of DEA.
4. We demonstrate the applicability and effectiveness of our methodology using actual data from the Stock Exchange in portfolio construction challenges.

The remainder of the paper is organized in the following manner. Section 2 presents some basic definitions and preliminaries regarding fuzzy numbers and their possibilistic mean, variance, and semi-variance for portfolio selection. Section 3 offers the MSBM fuzzy portfolio estimation model in a possibilistic framework. Section 4 presents the various steps of the TOPSIS method to rank the efficient portfolios. In Section 5, numerical examples are presented to justify the proposed approach. Finally, Section 6 presents the conclusions with some future directions.

2. Preliminaries

In this section, we present the basics of fuzzy sets and fuzzy numbers. Further, the mathematical models of [8], for computing the possibilistic mean, possibilistic variance, and possibilistic semi-variance of a fuzzy number are presented here.

Definition 1 Let X be a universal set. A fuzzy set $\tilde{A}$ in X is denoted by

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x) \rangle \mid x \in X\},$$

where $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ describes the degree of belongings of an element $x \in X$ to the set $\tilde{A}$.

Definition 2 (γ -cut) Let $\tilde{A}$ be a fuzzy set in X and $\gamma \in (0, 1]$. The γ -cut of the fuzzy set $\tilde{A}$ is the crisp set A_γ given by

$$A_\gamma = \{x \in X : \mu_{\tilde{A}}(x) \geq \gamma\}.$$

Definition 3 A fuzzy set $\tilde{A}$ in $\mathbb{R}$ is called a fuzzy number if it satisfies the following conditions

1. $\tilde{A}$ is normal.
2. $\tilde{A}_\gamma$ is a closed interval for every $\gamma \in (0, 1]$.
3. The support of $\tilde{A}$ is bounded.

Definition 4 (Trapezoidal fuzzy number (TrFN)) A fuzzy number $\tilde{A}$ represented by the quadruplet $\tilde{A} = [a, b, \alpha, \beta]$, where $[a, b]$ is the tolerance interval; α and β denote the left and right width of the TrFN- $\tilde{A}$, respectively, is called a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a - \alpha, x \geq b + \beta, \\ \frac{x - a + \alpha}{\alpha}, & a - \alpha \leq x \leq a, \\ 1, & a \leq x \leq b, \\ \frac{b + \beta - x}{\beta}, & b \leq x \leq b + \beta, \end{cases}$$

Further, the γ -cut of the TrFN $\tilde{A} = (a, b, \alpha, \beta)$ is the closed interval and given by

$$A^\gamma = [a_1^\gamma, a_2^\gamma] = [(a - (1 - \gamma)\alpha), b + (1 - \gamma)\beta], \quad \gamma \in (0, 1].$$

Definition 5 Let $\tilde{A}$ be a fuzzy number with $A^\gamma = [a_1^\gamma, a_2^\gamma]$, $\gamma \in (0, 1]$. The (crisp) possibilistic mean (or expected value) and possibilistic variance of a fuzzy number $\tilde{A}$, denoted by $E(\tilde{A})$ and $Var(\tilde{A})$, is defined as

$$E(\tilde{A}) = \int_0^1 \gamma (a_1^\gamma + a_2^\gamma) d\gamma,$$

and

$$Var(\tilde{A}) = \frac{1}{2} \int_0^1 \gamma (a_2^\gamma - a_1^\gamma)^2 d\gamma.$$

respectively.

It is easy to see that for a TrFN $\tilde{A} = [a, b, \alpha, \beta]$

$$E(\tilde{A}) = \int_0^1 \gamma [a - (1 - \gamma)\alpha + b + (1 - \gamma)\beta] d\gamma = \frac{(a + b)}{2} + \frac{(\beta - \alpha)}{6},$$

and

$$Var(\tilde{A}) = \left[\frac{(b - a)}{2} + \frac{(\alpha + \beta)}{6} \right]^2 + \frac{(\alpha + \beta)^2}{72}.$$

Further, [36] defined the lower possibilistic semi-variance of a fuzzy number $\tilde{A}$ denoted by $Var^-(\tilde{A})$, as

$$Var^-(\tilde{A}) = \int_0^1 2\gamma (E(\tilde{A}) - a_1^\gamma)^2 d\gamma.$$

Thus for a TrFN $\tilde{A} = [a, b, \alpha, \beta]$, the lower possibilistic semi-variance is as follows

$$Var^-(\tilde{A}) = \left[\frac{(b-a)}{2} + \frac{(\alpha+\beta)}{6} \right]^2 + \frac{(\alpha)^2}{18}.$$

Assume there are n risky assets whose returns are TrFNs represented by $\tilde{r}_i = (a_i, b_i, \alpha_i, \beta_i)$,

$i = 1, 2, \dots, n$ with investment proportion w_i , $i = 1, 2, \dots, n$, such that $w_i \geq 0$; $\sum_{i=1}^n w_i =$

1. Also, there is a fixed linear transaction cost k_i associated with each asset. As a result, the portfolio's probabilistic expected net return, variance, and lower semi-variance, denoted by R , σ^2 and $(\sigma^2)^-$ are given by as follows

$$R = E \left(\sum_{i=1}^n \tilde{r}_i w_i - \sum_{i=1}^n k_i w_i \right) = \sum_{i=1}^n \left[\left\{ \frac{1}{2} \left(a_i + b_i + \frac{1}{3} (\beta_i - \alpha_i) \right) w_i \right\} - k_i w_i \right],$$

$$\sigma^2 = Var \left(\sum_{i=1}^n \tilde{r}_i w_i \right) = \left[\left(\frac{(b_i - a_i)}{2} + \frac{(\alpha_i + \beta_i)}{6} \right) w_i \right]^2 + \frac{1}{72} \left[\sum_{i=1}^n (\alpha_i + \beta_i) w_i \right]^2,$$

and

$$(\sigma^2)^- = Var^- \left(\sum_{i=1}^n \tilde{r}_i w_i \right) = \left[\left(\frac{(b-a)}{2} + \frac{(\alpha+\beta)}{6} \right) w_i \right]^2 + \frac{1}{18} \left[\sum_{i=1}^n (\alpha_i w_i) \right]^2,$$

respectively.

2.1. Modified Slack Based Model to handle negative data

To address the issue of negative data in conventional DEA models, [37] introduced a modified slack-based measure (MSBM) model. The MSBM model is modified by [42] for evaluating the performance of production units by taking direction different from [37].

Consider that there are P DMUs to be evaluated, each using m inputs to generate s outputs. Let x_{ij} , ($t = 1, \dots, m$) represent the t^{th} , input for the j^{th} DMU, and y_{rj} , ($r = 1, \dots, s$) denote the r^{th} output of the j^{th} DMU. The modified MSBM method is presented as follows:

$$\begin{aligned}
(\text{Model I}) \quad \min \theta_0 &= \frac{1 - \frac{1}{m} \sum_{t=1}^m \frac{s_{t0}^-}{R_t^-}}{1 + \frac{1}{s} \sum_{r=1}^s \frac{s_{r0}^+}{R_r^+}} \\
\text{s. t.} \quad \sum_{j=1}^P \lambda_j x_{tj} + s_{t0}^- &= x_{t0} \quad t = 1, 2, \dots, m \\
\sum_{j=1}^P \lambda_j y_{rj} - s_{r0}^+ &= y_{r0} \quad r = 1, 2, \dots, s \\
\sum_{j=1}^P \lambda_j &= 1 \quad \lambda_j \geq 0, j = 1, 2, \dots, P, s_{t0}^-, s_{r0}^+ \geq 0
\end{aligned}$$

The objective function θ_0^* represents the optimal efficiency score of DMU_0 . The terms s_{t0}^- ($t = 1, \dots, m$) and s_{r0}^+ ($r = 1, \dots, s$) represent the t^{th} input excess and r^{th} output shortfall of DMU_0 , referred to as slacks. The λ_j ($j = 1, \dots, P$) are intensity variables used to link inputs and outputs in a convex combination. Also, R_t^- and R_r^+ are the feasible directions to move and are defined as follows:

$$\begin{aligned}
R_t^- &= \max_{j=1,2,\dots,P} x_{tj} - \min_{j=1,2,\dots,P} x_{tj}, \quad t = 1, 2, \dots, m \\
R_r^+ &= \max_{j=1,2,\dots,P} y_{rj} - \min_{j=1,2,\dots,P} y_{rj}, \quad r = 1, 2, \dots, s
\end{aligned}$$

In the following section, we present the MSBM fuzzy portfolio estimation models based on [11].

3. MSBM fuzzy portfolio evaluation models in the framework of possibilistic theory

In this section, we present the MSBM fuzzy portfolio estimation model to handle the unpredictability of the variables.

Assume that there are P portfolios to be evaluated. To evaluate a portfolio performance, we consider the risk associated with the portfolio as the input indicator, and the corresponding return as the output indicator. In this study, we assume n risky assets with fuzzy return as $\tilde{r}_i = (a_i, b_i, \alpha_i, \beta_i)$, $i = 1, 2, \dots, n$ with investment proportions

w_{ij} , $i = 1, 2, \dots, n$; $j = 1, 2, \dots, P$, such that $\sum_{i=1}^n w_{ij} = 1$, are considered for constructing j^{th} portfolio which is considered to be a DMU. Suppose k_i is the unit transaction cost

of a unit for the i^{th} risky asset. Then the expected net return, variance, and semi-variance of the j^{th} portfolio denoted by $\bar{R}_j$, σ_j^2 and $(\sigma^2)_j^-$ respectively for $(j = 1, 2, \dots, P)$ are given respectively. To employ the MSBM models for the efficiency evaluation of all portfolios, in our study we consider one input and one output ($m = s = 1$) criteria as risk and return respectively.

3.1. MSBM fuzzy portfolio evaluation model in possibilistic mean-variance framework

To handle the problem of negative data and to provide a full ranking of all DMUs, [42] suggested a novel MSBM (modified slacks-based model) model. We extend the above-presented model I in the possibilistic mean-variance framework. The mathematical form of the MSBM fuzzy portfolio model with possibilistic variances input (risk) and possibilistic mean as output (return) is given as follows:

$$\begin{aligned}
 \text{(Model II)} \quad \min \quad \rho_0 &= \frac{1 - \frac{s_{10}^-}{R_1^-}}{1 + \frac{s_{10}^+}{R_2^+}} \\
 \text{s. t.} \quad &\sum_{j=1}^P \lambda_j \left(\left\{ \sum_{i=1}^n \frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right\}^2 + \frac{1}{72} \left[\sum_{i=1}^n (\alpha_i + \beta_i) w_{ij} \right]^2 \right) + s_{10}^- \\
 &= \left(\left\{ \sum_{i=1}^n \frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{i0} \right\}^2 + \frac{1}{72} \left[\sum_{i=1}^n (\alpha_i + \beta_i) w_{i0} \right]^2 \right) \\
 &\sum_{j=1}^P \lambda_j \left(\sum_{i=1}^n \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{ij} - \sum_{i=1}^n k_i w_{ij} \right) - s_{10}^+ \\
 &= \sum_{i=1}^n \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{i0} - \sum_{i=1}^n k_i w_{i0}, \\
 &\sum_{j=1}^P \lambda_j = 1 \quad \lambda_j \geq 0, s_{10}^-, s_{10}^+ \geq 0, \quad j = 1, 2, \dots, P
 \end{aligned}$$

Here, R_1^- and R_2^+ can be modified as follows :

$$\begin{aligned}
 R_1^- &= \max_{j=1,2,\dots,P} \left(\sum_{i=1}^n \left(\frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right)^2 + \frac{1}{72} \left[\sum_{i=1}^n (\alpha_i + \beta_i) w_{ij} \right]^2 \right) \\
 &\quad - \min_{j=1,2,\dots,P} \left(\sum_{i=1}^n \left(\frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right)^2 + \frac{1}{72} \left[\sum_{i=1}^n (\alpha_i + \beta_i) w_{ij} \right]^2 \right) \\
 R_2^+ &= \max_{j=1,2,\dots,P} \left(\sum_{i=1}^n \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{ij} - \sum_{i=1}^n k_i w_{ij} \right) \\
 &\quad - \min_{j=1,2,\dots,P} \left(\sum_{i=1}^n \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{ij} - \sum_{i=1}^n k_i w_{ij} \right)
 \end{aligned}$$

Let $(\rho^*, \lambda_1^*, \lambda_2^* \dots, \lambda_P^*)$ be an optimum solution of (Model II). Further, ρ_0^* indicates the optimal efficiency score of the portfolio under evaluation (0^{th}). Here, s_{10}^- and s_{10}^+ , respectively denote the excess of input (risk) and shortfall of output (return) of 0^{th} portfolio, and they are identified as slacks.

3.2. MSBM fuzzy portfolio evaluation model in possibilistic mean semi-variance framework

The mathematical form of the MSBM fuzzy portfolio model with possibilistic semi-variance input (risk) and possibilistic mean as output (return) is given as follows:

$$\begin{aligned}
 \text{(Model III)} \quad \min \quad \eta_0 &= \frac{1 - \frac{s_{10}^-}{R_3^-}}{1 + \frac{s_{10}^+}{R_2^+}} \\
 \text{s. t.} \quad \sum_{j=1}^P \lambda_j &\left(\left\{ \sum_{i=1}^n \frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right\}^2 + \frac{1}{18} \left[\sum_{i=1}^n (\alpha_i) w_{ij} \right]^2 \right) + s_{10}^- \\
 &= \left(\left\{ \sum_{i=1}^n \frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{i0} \right\}^2 + \frac{1}{18} \left[\sum_{i=1}^n (\alpha_i) w_{i0} \right]^2 \right) \\
 \sum_{j=1}^P \lambda_j &\left(\sum_{i=1}^n \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{ij} - \sum_{i=1}^n k_i w_{ij} \right) - s_{10}^+
 \end{aligned}$$

$$= \sum_{i=1}^n \frac{1}{2} \left[(b_i + a_i) + \frac{\beta_i - \alpha_i}{3} \right] w_{i0} - \sum_{i=1}^n k_i w_{i0},$$

$$\sum_{j=1}^P \lambda_j = 1 \quad \lambda_j \geq 0, \quad s_{10}^-, s_{10}^+ \geq 0, \quad j = 1, 2, \dots, P$$

Here, R_2^+ is the same as described in the model II and R_3^- is defined as follows:

$$R_3^- = \max_{j=1,2,\dots,P} \left(\sum_{i=1}^n \left(\frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right)^2 + \frac{1}{18} \left[\sum_{i=1}^n (\alpha_i) w_{ij} \right]^2 \right) \\ - \min_{j=1,2,\dots,P} \left(\sum_{i=1}^n \left(\frac{1}{2} \left[(b_i - a_i) + \frac{\alpha_i + \beta_i}{3} \right] w_{ij} \right)^2 + \frac{1}{18} \left[\sum_{i=1}^n (\alpha_i) w_{ij} \right]^2 \right)$$

The models mentioned above can determine the efficiency values of all fuzzy portfolios; however, they do not provide a unique ranking of the portfolios. It is important to note that if multiple portfolios are deemed efficient, establishing a complete and unique ranking becomes impossible. Therefore, we apply the TOPSIS method to rank all efficient portfolios and obtain a unique ranking. The TOPSIS method was introduced by [20]. It is a practical method for organizing the available options based on numerous criteria. It outperforms other MCDM methods for solving a wide range of decision-making problems in real-world, due to its simple computational approach and systematic procedure.

4. TOPSIS method to rank the efficient portfolio

TOPSIS is founded on the notion of selecting the best alternative (here DMU) that is as close to the positive ideal solution at the same time being as far away from the negative ideal solution as possible. The best performance values for each portfolio demonstrated by each criterion (here criterion as risk and return) in the decision matrix make up the perfect solution. As a result, we used a negative feature criterion for risk and a positive feature criterion for return.

Let L be the number of efficient portfolios. Suppose we have m inputs and s outputs for each portfolio, let D_1 and D_2 denotes respectively the input and output matrix corresponding to L efficient portfolios. Therefore

$$D_1 = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1L} \\ x_{21} & x_{22} & \dots & x_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mL} \end{bmatrix} \quad D_2 = \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1L} \\ y_{21} & y_{22} & \dots & y_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ y_{s1} & y_{s2} & \dots & y_{sL} \end{bmatrix}$$

Now we have to follow the following steps:

Step 1. Normalization

Calculate the normalized decision matrix corresponding to the cost criterion (m inputs) and benefit criteria (s outputs). Let p_{tl} and q_{rl} ($t = 1, 2, \dots, m; r = 1, 2, \dots, s$) denotes the normalized value of t^{th} input and r^{th} output of l^{th} DMU respectively and calculated as follows:

$$p_{tl} = \frac{x_{tl}}{\sqrt{\sum_{k=1}^L x_{tk}^2}}, \quad \forall t = 1, 2, \dots, m, l = 1, 2, \dots, L$$

and

$$q_{rl} = \frac{y_{rl}}{\sqrt{\sum_{k=1}^L y_{rk}^2}}, \quad \forall r = 1, 2, \dots, s, l = 1, 2, \dots, L.$$

Step 2. Determine the positive ideal and negative-ideal points: Let p_{tw} and q_{tw} denotes respectively be the t^{th} input and r^{th} output of the worst DMU (D_w). Similarly let p_{tb} and q_{rb} denotes respectively be the t^{th} input and r^{th} output of the best DMU (D_b). Then

$$p_{tw} = \max_{1 \leq l \leq L} p_{tl}, \quad t = 1, 2, \dots, m,$$

and

$$q_{rw} = \min_{1 \leq l \leq L} q_{rl}, \quad r = 1, 2, \dots, s.$$

Similarly

$$p_{tb} = \min_{1 \leq l \leq L} p_{tl}, \quad t = 1, 2, \dots, m,$$

and

$$q_{rb} = \max_{1 \leq l \leq L} q_{rl}, \quad r = 1, 2, \dots, s.$$

Step 3. Measuring the distance from the ideal and negative ideal points:

Let d_{lw} be the distance between the target DMU l and the worst DMU D_w (best DMU D_b), so

$$d_{lw} = \sqrt{\sum_{t=1}^m (p_{tl} - p_{tw})^2 + \sum_{r=1}^s (q_{rl} - q_{rw})^2}, \quad l = 1, 2, \dots, L,$$

and

$$d_{lb} = \sqrt{\sum_{t=1}^m (p_{tl} - p_{tb})^2 + \sum_{r=1}^s (q_{rl} - q_{rb})^2}, \quad l = 1, 2, \dots, L.$$

Step 4. Calculate the similarity to the worst condition:

Let s_{lw} be the relative closeness of the target DMU l to the worst DMU D_w .

$$s_{lw} = \frac{d_{lb}}{d_{lw} + d_{lb}}, \quad l = 1, 2, \dots, L.$$

Here, $0 \leq s_{lw} \leq 1$, and

$$s_{lw} = \begin{cases} 1, & \text{if and only if the alternative solution has the best condition} \\ 0, & \text{if and only if the alternative solution has the worst condition.} \end{cases}$$

Step 5. Ranking criterion:

Rank the alternatives or portfolios in the descending order of s_{lw} , the alternative with a maximum value of s_{lw} is the best-performing alternative.

The working procedure of the proposed approach has been described in the algorithm 1.

Algorithm 1 Computation Procedure of the proposed method

Input: The Fuzzy returns of Stocks with the different proportions invested in the stocks.

Output: The full ranking of efficient portfolios.

Step 1. Calculate the Efficiency of portfolios by using MSBM fuzzy portfolio models.

Step 2. Rank the portfolios according to the efficiency score.

Step 3. If the no. of efficient portfolio ≥ 1 , then apply TOPSIS method.

Step 4. Identification of attributes (Return, Risk) for all the alternatives (efficient portfolios).

Step 5. Form a weighted normalized decision matrix.

Step 6. Calculate the positive and negative ideal points.

Step 7. Evaluate the similarity to the worst condition as the measure of relative closeness s_{lw} .

Step 8. Rank the efficient portfolio in the decreasing order of s_{lw} .

5. Numerical Illustration

To demonstrate the applicability and validity of the proposed method, two numerical examples are presented. In the first example, the possibilistic mean is considered as the output, and the possibilistic variance as the input. In the second example, the possibilistic mean is again used as the output, with the possibilistic semi-variance as the input. All simulations were implemented using MATLAB software.

Example 1

In this example, the returns from risky assets/stocks are provided in Table 1 as trapezoidal fuzzy numbers and are the same as taken by [29] in their study. In the experimental analysis of [29], the historical data of 5 assets from April 2009 to March 2015, in terms of the weekly closing prices are used to simulate the future returns. Furthermore, we create the random sample portfolios using the weights as shown in Table 2. For convenience, we select only those 5 stocks whose returns are negative. We employ Model II to calculate the efficiency and ranking of all randomly generated portfolios. The results are presented in Table 3.

Table 1. The possibility distribution of the return rates and their respective transaction cost of 5 stocks

Stocks	a_i	b_i	α_i	β_i	k_i
S_1	-0.1071	-0.0703	0.0105	0.0541	0.003
S_2	0.0883	0.1057	0.057	0.0772	0.003
S_3	0.0874	0.0948	0.0132	0.0928	0.003
S_4	-0.0907	-0.082	0.0055	0.0336	0.003
S_5	-0.0954	-0.082	0.0093	0.013	0.003

Table 2. The weight of each stock in different randomly generated portfolios

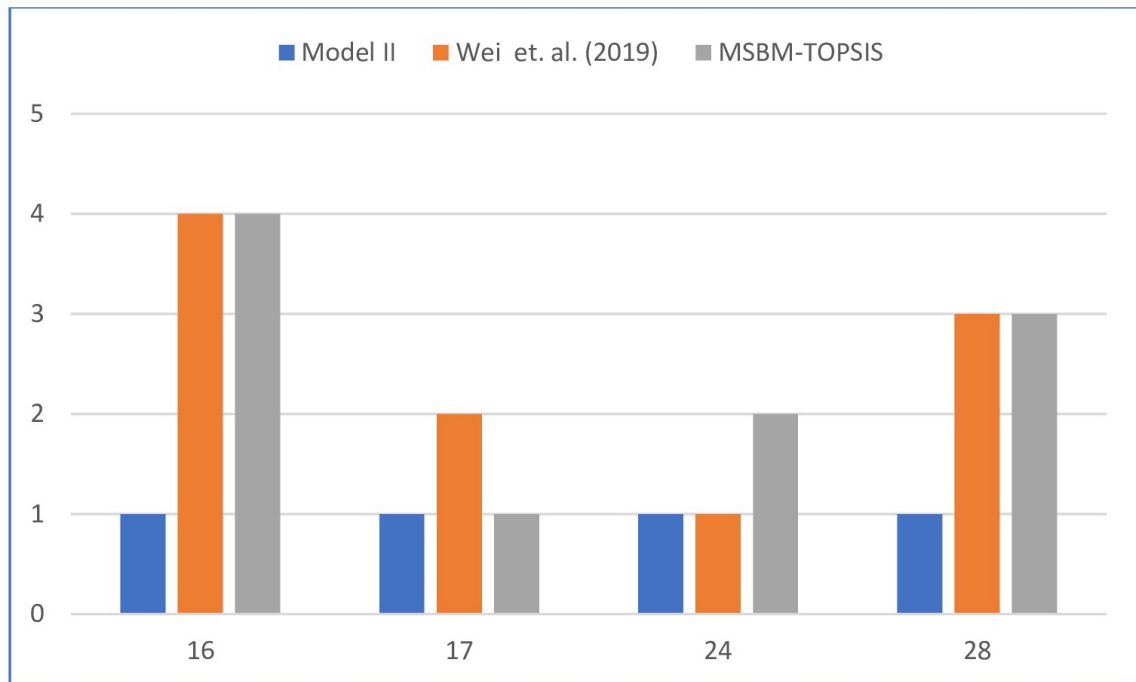
w	1	2	3	4	5	6	7	8	9	10
w_1	0.1875	0.2660	0.2616	0.1956	0.2608	0.1677	0.0637	0.0948	0.3678	0.0891
w_2	0.1566	0.1102	0.2422	0.0962	0.3080	0.2472	0.2610	0.2547	0.2150	0.0561
w_3	0.3301	0.2483	0.1600	0.2975	0.2684	0.2437	0.1899	0.2408	0.0389	0.3180
w_4	0.1585	0.0805	0.2621	0.3064	0.1287	0.1964	0.2413	0.0050	0.2217	0.2013
w_5	0.1673	0.2905	0.0741	0.1043	0.0340	0.1450	0.2442	0.4048	0.1567	0.3355
w_p	11	12	13	14	15	16	17	18	19	20
w_1	0.0099	0.2206	0.3053	0.2608	0.1184	0.1714	0.0244	0.3602	0.0844	0.3571
w_2	0.2762	0.3457	0.2831	0.1016	0.3696	0.0105	0.2256	0.1147	0.0871	0.0746
w_3	0.2373	0.1664	0.0314	0.1039	0.0955	0.1306	0.2335	0.1934	0.2467	0.3654
w_4	0.2792	0.0005	0.1601	0.1654	0.2866	0.3579	0.1711	0.2879	0.2542	0.0149
w_5	0.1974	0.2669	0.2201	0.3684	0.1299	0.3296	0.1253	0.0438	0.3276	0.1879
w_p	21	22	23	24	25	26	27	28	29	30
w_1	0.2904	0.1345	0.0715	0.1815	0.1990	0.0479	0.2311	0.0904	0.2334	0.2311
w_2	0.1905	0.0337	0.2241	0.1996	0.2874	0.2777	0.2503	0.0134	0.0418	0.1278
w_3	0.2333	0.3373	0.2388	0.4058	0.0994	0.2683	0.1133	0.2684	0.4218	0.0272
w_4	0.0792	0.2580	0.1773	0.0647	0.1409	0.2410	0.0752	0.2950	0.1474	0.3668
w_5	0.2066	0.2366	0.2884	0.1484	0.2734	0.1651	0.3301	0.3328	0.1556	0.2452

Table 3. The efficiency score and the ranking of portfolios in possibilistic mean-variance framework

portfolio	Return	Risk	Efficiency by model II	ranking by Model II
1	0.0043	0.00052	0.4714	16
2	-0.0198	0.00048	0.4396	18
3	-0.0112	0.00057	0.3138	25
4	-0.0124	0.00045	0.5188	13
5	0.0212	0.00072	0.3558	23
6	0.0048	0.00055	0.4168	19
7	-0.0033	0.00045	0.5696	11
8	0.0040	0.00049	0.5376	12
9	-0.0392	0.00056	0.272	29
10	-0.0174	0.00034	0.672	5
11	0.0085	0.00045	0.6361	7
12	0.0076	0.00065	0.2256	30
13	-0.0286	0.00058	0.2756	28
14	-0.0488	0.00039	0.4689	17
15	-0.0004	0.00055	0.3918	20
16	-0.0605	0.00027	1.0000	1
17	0.0177	0.00031	1.0000	1
18	-0.0280	0.00055	0.3207	2
19	-0.0249	0.00033	0.6474	6
20	-0.0042	0.00060	0.3018	27
21	-0.0078	0.00059	0.3122	26
22	-0.0172	0.00036	0.6346	8
23	-0.0012	0.00044	0.5908	9
24	0.0263	0.00061	1.0000	1
25	-0.0155	0.00053	0.3816	21
26	0.0148	0.00050	0.5745	10
27	-0.0200	0.00052	0.3782	22
28	-0.0343	0.00028	1.0000	1
29	0.0005	0.00049	0.5065	14
30	-0.0577	0.00036	0.4788	15

Table 4. Ranking of efficient portfolios by different methods in possibilistic mean-variance framework

Portfolio	Return	Risk	ranking by Model II	ranking by [42]	ranking by MSBM- TOPSIS method
16	-0.0605	0.00027	1	4	4
17	0.0177	0.00031	1	2	1
24	0.0263	0.00061	1	1	2
28	-0.0343	0.00028	1	3	3

**Figure 1.** Rankings of efficient portfolios in possibilistic mean-variance framework

Remark 1 Table 3 shows that the efficiency score of portfolios 16, 17, 24, and 28 is the same, and we cannot distinguish between efficient portfolios further. The MSBM-TOPSIS approach, as stated above, is used to determine the whole ranking. The fourth, fifth, sixth, and seventh column of Table 4 depicts the ranking of efficient portfolios by Model II, by [42], and by the proposed MSBM-TOPSIS method respectively. Figure 1 shows the ranking of efficient portfolios graphically. It should be highlighted that the process described in [42] gives us the whole ranking of all randomly produced portfolios. However, the Super efficiency model of [18] poses an in-feasibility concern. The ranking by the proposed method for Portfolio 16 and Portfolio 28 is the same as obtained by [42]. Compared to the method given by [42], the proposed approach is less calculative and easy to handle.

If asset return distributions aren't symmetric, using variance as a risk indicator isn't recommended because it initiates to portfolio's projections drastically different from what would be expected in real-life situations. Excessively high and low returns are both regarded as undesirable by variance, which is one of the key disadvantages. Many academics have utilized a variety of techniques to deal with such problems. Risk can be classified using semi-variance as an alternative risk measure. Therefore, a downside risk metric called possibilistic semi-variance is utilized here instead of possibilistic variance as an input parameter. Table 5 shows the ranking, obtained by model III and Table 6 provides the complete ranking of efficient portfolios using different methods.

Table 5. The efficiency score and the ranking of portfolios in possibilistic mean semi-variance framework

Portfolio	Return	Risk	efficiency by Model III	ranking by Model III
1	0.0043	0.00045	0.4694	15
2	-0.0198	0.00043	0.4175	18
3	-0.0112	0.00051	0.2866	25
4	-0.0124	0.00039	0.5195	12
5	0.0212	0.00064	0.3272	23
6	0.0048	0.00049	0.4014	19
7	-0.0033	0.00040	0.5524	11
8	0.0040	0.00044	0.5082	14
9	-0.0392	0.00052	0.2289	28
10	-0.0174	0.00029	0.6719	5
11	0.0085	0.00039	0.6316	8
12	0.0076	0.00060	0.167	30
13	-0.0286	0.00053	0.2242	29
14	-0.0488	0.00036	0.4413	17
15	-0.0004	0.00050	0.353	20
16	-0.0605	0.00025	1.00	1
17	0.0177	0.00027	1.00	1
18	-0.0280	0.00049	0.2996	24
19	-0.0249	0.00029	0.6418	6
20	-0.0042	0.00053	0.2853	26
21	-0.0078	0.00052	0.2826	27
22	-0.0172	0.00031	0.638	7
23	-0.0012	0.00039	0.5777	9
24	0.0263	0.00053	1.00	1
25	-0.0155	0.00048	0.3387	21
26	0.0148	0.00043	0.5724	10
27	-0.0200	0.00047	0.335	22
28	-0.0343	0.00025	1.00	1
29	0.0005	0.00042	0.5146	13
30	-0.0577	0.00033	0.4547	16

Remark 2 *The suggested method's most significant advantage is its feasibility when compared to super-efficiency methods, which are in some situations infeasible.*

Table 6. Ranking of efficient portfolios by in possibilistic mean semi-variance framework

Portfolio	Return	Risk	ranking by Model III	ranking by [42]	ranking by MSBM- TOPSIS method
16	-0.0605	0.00025	1	3	4
17	0.0177	0.00027	1	1	1
24	0.0263	0.00053	1	2	2
28	-0.0343	0.00025	1	4	3

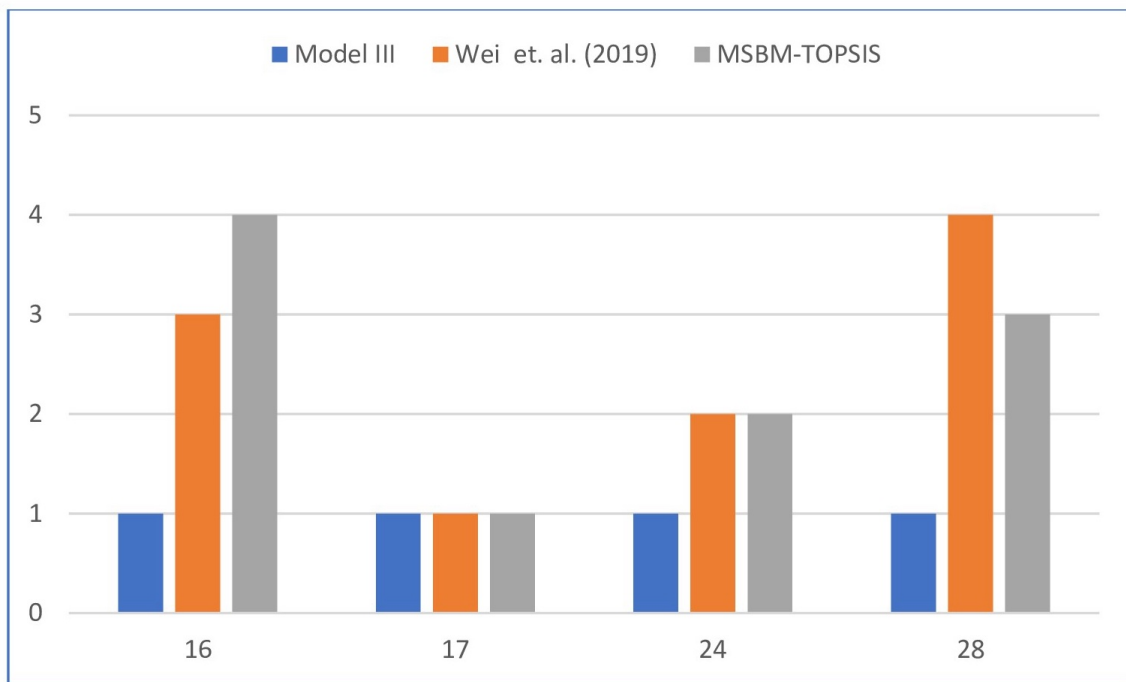


Figure 2. Rankings of efficient portfolios in possibilistic mean semi-variance framework

5.1. Computation Complexity

In this study, we compute the computational complexity by taking into account all logical-mathematical operations, such as addition, subtraction, multiplication, division, exponentiation and square root as described in [35]. The mathematical operations used for creating the return risk are the same for each strategy used in this study. As a result, they weren't considered while determining computational complexity. Suppose we have n efficient portfolio (to be considered as an alternative) and m criteria (return and risk) in the study, then it will take $3nm$ operations to reach at the weighted normalized matrix, $2(6nm)$ calculations required to determine the distance between efficient portfolio and the ideal and negative ideal points. $nm-n$ calculations are required for calculating the positive ideal and negative ideal points. Lastly, $2n$ computations are used to calculate each portfolio's closeness coefficient. As a result, the TOPSIS's computational complexity to determine the full ranking of the efficient fuzzy portfolio in the second stage is determined as follows:

$$T_{n,m} = 3nm + 12nm + nm - n + 2n$$

The hybrid MSBM-TOPSIS requires 132 simple operations for the empirical analysis included in this analysis, whereas other DEA models need to solve the mathematical models repeatedly to get the whole ranking for efficient portfolios. Hence, Our method is substantially less intricate than the existing methods, so finding the most effective portfolio will take less time. In contrast to our approach, which uses basic operations to find the best portfolio in the second phase, the super efficiency and modified MSBM method given in[42] often solves at least three models for each portfolio to determine the best portfolio.

6. Conclusion and Future Research Directions

This work investigates the ranking problem of all efficient fuzzy portfolios with positive and negative input-output variables. We know that if a sufficiently large number of portfolios are designated efficient, the DEA's discrimination ability decreases. We extend the MSBM model for handling fuzzy portfolio evaluation problems in the possibilistic mean-variance and possibilistic mean semi-variance framework. By combining the proposed MSBM fuzzy portfolio models with the TOPSIS method, the work has established a hybrid strategy for ranking efficient portfolios in a fuzzy environment. By combining the strengths of MSBM in efficiency evaluation, fuzzy logic in handling uncertainty, and TOPSIS in ranking alternatives, this approach provides a comprehensive tool for investors looking to optimize their portfolios in uncertain market environments.

This approach will be explored in future work to investigate the classification of fuzzy portfolios having input as possibilistic semi-absolute deviation (SAD), possibilistic VaR, and CvaR. In this research, only 1st and 2nd-order moments (mean, variance,

and semi-variance) have been used as output and input. The higher-order moments can be utilized as the input-output values for future work.

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