CONVERGENCE ANALYSIS OF SOME APPROXIMATION OPERATORS

A Thesis Submitted to

Delhi Technological University

for the Award of Degree of

Doctor of Philosophy

in

Mathematics

By

Sandeep Kumar

(Enrollment No.: 2K18/Ph.D./AM/15)

Under the Supervision of

Prof. Naokant Deo

Department of Applied Mathematics

Delhi Technological University (Formerly DCE)

Bawana Road, Delhi-110042, India.

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ACKNOWLEDGEMENTS

First and foremost, I am grateful to my supervisor, Prof. Naokant Deo, Department of Applied Mathematics, Delhi Technological University (DTU), for his valuable advice, support, and inspiration during my research journey. His thoughts and support have greatly influenced my academic and personal development.

I am grateful to my research committee members, Dr. Dinesh Udar and Prof. R.C. Singh, for their positive feedback and continuous support, which have greatly aided the advancement of my study.

I am thankful to Prof. R. Srivastava, Prof. S. S. P. Kumar, Prof. C. P. Singh, Prof. L. N. Das, Prof. Sangita Kansal, Prof. V. P. Kaushik, Dr. Satyabrat Adhikari, Mr. Rohit Kumar, and other faculty of the Department of Applied Mathematics for their help and encouragement.

Special thanks to Dr. Durvesh Kumar Verma, Dr. Chandra Prakash, Dr. Chandrawati Sindhi, and Dr. Ram Pratap for their constant support and advantageous recommendations during my Ph.D.

I also like to thank the DTU administration and academic branch for creating a positive research atmosphere, as well as the department office staff for their help.

I am thankful to my colleagues and friends, Mr. Kapil Kumar, Ms. Sarika Saini, Mr. Itendra Kumar, Mr. Suresh Kumar Yadav, Mr. Gaurav Kumar, Mr. Sanyam Gupta, Mr. Rahul Singh, Dr. Lipi, Dr. Nav Shakti Mishra, Dr. Neha, Mr. Vijay Kumar Sharma, Mr. Sachin Kumar, and Mr. Krishna Gopal Saini for their unwavering support and encouragement.

I am deeply grateful to my family— my parents, Mr. Desh Pal Singh and Mrs. Raj Kumari; my brothers, Mr. Pradeep Kumar and Mr. Adesh Kumar and most importantly, my wife, Mrs. Geetanjali Chauhan, along with my cherished daughter Ms. Kaashvi Singh and beloved son Mr. Ayush Singh— for their everlasting love, support, and sacrifices.

Finally, I express gratefulness to the Almighty God for the divine blessings and direction that enabled me to accomplish this work.

Date : **(Sandeep Kumar)**

Place : Delhi, India.

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DECLARATION

I, Sandeep Kumar, hereby declare that the work which is being presented in the thesis entitled "**Convergence Analysis of Some Approximation Operators**" in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy, submitted in the Department of Applied Mathematics, Delhi Technological University is an authentic record of my own work carried out during the period from July 2018 to July 2025 under the supervision of ***Prof. Naokant Deo***, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

The matter presented in the thesis has not been submitted by me for the award of any other degree of this or any other Institute.

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CERTIFICATE

This is to certify that the research work embodied in the thesis entitled "**Convergence Analysis of Some Approximation Operators**" submitted by **Mr. Sandeep Kumar** with enrollment number **2K18/PHD/AM/15** is the result of his original research carried out in the Department of Applied Mathematics, Delhi Technological University, Delhi, for the award of **Doctor of Philosophy** under the supervision of **Prof. Naokant Deo**, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

It is further certified that this work is original and has not been submitted in part or fully to any other University or Institute for the award of any degree or diploma.

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

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Abstract

The thesis is divided into seven chapters, the contents of which are organized as follows:

Chapter1 of the thesis covers the literature and historical foundation of certain important approximation operators. We provide a brief overview of the chapters that constitute this thesis and discuss some of the preliminary tools we will use to delve into the subject's depth.

Chapter 2 introduces a new sequence of operators involving Apostol-Genocchi polynomials and Baskakov operators and their integral variants. We estimate some direct convergence results using the second-order modulus of continuity, Voronovskaja type approximation theorem. Moreover, we find weighted approximation results of these operators.

Next **Chapter** 3 is mainly focused on the difference operators of two positive linear operators (generalized păltănea type operators $L_{n,c}^{\lambda}(f;x)$ and M. Heilmann type operators $M_{n,c}(f;x)$) with same basis functions. First, we estimate quantitative difference of these operators in terms of modulus of continuity and Peetre's $K-$functional

In **Chapter** 4, we present a recurrence relation for the semi-exponential Post-Widder operators and provide estimates for their moments. We then examine convergence results within Lipschitz-type spaces, analyzing the convergence rate using the Ditzian-Totik modulus of smoothness and the weighted modulus of continuity. Finally, we estimate the convergence rate for functions whose derivatives are of bounded variation.

Chapter 5 introduces a novel Bézier variant within the family of Phillips-type generalized positive linear operators. The moments of these operators are derived to enhance understanding of their fundamental properties. The chapter further explores convergence properties in Lipschitz-type spaces, with particular focus on the Ditzian-Totik modulus of smoothness. Finally, it provides a rigorous analysis of the convergence rate for functions whose derivatives are of bounded variation, contributing valuable insights to the field of approximation theory.

The aim of **Chapter** 6 is to introduce the sequence of Baskakov-Durrmeyer type operators linked with the generating functions of Boas-Buck type polynomials. After calculating the moments, including the limiting case of central moments of the constructed sequence of operators, in the subsequent sections, we estimate the convergence rate using the modulus of continuity and Ditzian-Totik modulus of smoothness and some convergence results in Lipchitz-type space and the end we estimates the convergence for the functions of bounded variations.

The thesis is summarised in **Chapter 7**, before providing some insight into the author's thoughts about the future research.

List of Publications and Conferences

Published

1. Naokant Deo, **Sandeep Kumar**, Durrmeyer variant of Apostol-Genocchi-Baskakov operators. **Quaestiones Mathematicae**, 44(1), (2021), 1817–1834. https://doi.org/10.2989/16073606.2020.1834000, **(SCIE, Impact Factor: 1.474)**.

2. **Sandeep Kumar**, Naokant Deo, Approximation of Generalized Păltănea and Heilmann-Type Operators, **Matematički Vesnik**, 74(2), (2022),101–109. **(ESCI, Impact Factor: 0.78)**.

3. Naokant Deo, **Sandeep Kumar**, Convergence analysis of semi exponential Post-Widder operators, **Miskolc Mathematical Notes**, 26(1), (2025), 367–382. https://doi.org/10.18514/MMN.2025.4600, **(SCIE, Impact Factor: 1.22)**.

4. Naokant Deo, **Sandeep Kumar**. Bezier Variant of Phillips-type generalised positive linear operators, **Palestine Journal of Mathathematics**, 14(3), (2025), 47–55. (Scopus indexed).

Communicated

1. Naokant Deo, **Sandeep Kumar**. Durrmeyer type operators linked with Boas-Buck type polynomial, Under Review. https://doi.org/10.48550/arXiv.2503.02243.

Papers presented in International Conferences

1. Rate of convergence of Durrmeyer variant of Apostol-Genochhi Baskakov operators; International Conference on Evolution in Pure and Applied Mathematics (ICEPAM-2022), Department of Mathematics, Akal University Talwandi Sabo, Punjab. November 16–18, 2022.

2. Approximation of Difference operators; International Conference on Analysis and Its Applications (ICAA-2023), Department of Mathematics, Shivaji College, University of Delhi, Delhi. February 27–28, 2023.

Contents

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List of Symbols and Notations

$\mathbb{N}$	set of natural numbers.
$\mathbb{N}\cup\{0\}$	the set of natural numbers including zero,
$\mathbb{R}$	set of real numbers,
$\mathbb{R}^+$	the set of positive real numbers,
$[a,b]$	a closed interval,
(a,b)	an open interval,
$C[a,b]$	the set of all real-valued and continuous function defined in $[a,b]$
$C^r[a,b]$	the set of all real-valued, r-times continuously differentiable function $(r\in\mathbb{N})$,
$C^2[0,\infty)$	the set of all real-valued, 2-times continuously differentiable function on $[0,\infty)$,
$C[0,\infty)$	the set of all continuous functions defined on $[0,\infty)$,
$C_B[0,\infty)$	class of real-valued functions defined on $[0,\infty)$, which are uniformly continuous and bounded.
$C_B^r[0,\infty)$	the set of all r-times continuously differentiable functions in $C_B[0,\infty)$ $(r\in\mathbb{N})$ $\lvert f(x)\rvert\leq M(1+x^2)$, M is a positive constant,
$C_\tau^*[0,\infty)$	class of all the functions $\psi\in C_B[0,\infty)$ for which $\lim\limits_{x\to\infty}\frac{f(x)}{1+x^2}$ exists and finite.
$\lVert f\rVert$	$\lVert f\rVert=\sup\{\lvert f(x)\rvert:\ x\in[a,b]\}$,
$Lip_M(\sigma)$	the set of all $C[a,b]$- functions which holds the Lipschitz condition $\lvert f(t)-f(x)\rvert\leq M\lvert t-x\rvert^\sigma$ for all $t,x\in[a,b]$, $0<\sigma\leq 1$, $M>0$.
e_r	the test function with $e_r(x)=x^r$, $r\in\mathbb{N}\cup\{0\}$.
$C^\gamma[0,\infty)$	Class of functions $\{\psi\in C[0,\infty):\psi(t)=O(t^\gamma),\gamma>0\}$.
W^2	Class of functions $g(x)\in C_B[0,\infty); g',g''\in C_B[0,\infty)$.

Chapter 1

Introduction

1.0.1 Historical Backdrop Review

Approximation theory has ancient origins, rooted in early mathematical efforts to solve complex problems using simpler forms. As far back as ancient Greece, mathematicians such as Eudoxus and Archimedes developed geometric methods to approximate areas and volumes. Notably, Archimedes method of exhaustion was a pioneering concept that foreshadowed the modern idea of limits, a cornerstone in approximation theory. During the 17th and 18th centuries, the development of calculus by Isaac Newton and Gottfried Wilhelm Leibniz marked a major advancement in approximation techniques. Newton's work on interpolation and the binomial series served as foundational tools for approximating functions. Later, Joseph-Louis Lagrange and Leonhard Euler, introduced polynomial interpolation, establishing it as a core method. In the 20th century, approximation theory matured into a formal and rigorous discipline, increasingly influenced by functional analysis and operator theory. Key contributions came from mathematicians like Chebyshev, Karl Weierstrass, and Sergei Bernstein. The Weierstrass approximation theorem, articulated by Karl Weierstrass in 1885, is a foundational statement in approximation theory. This theorem asserts that for any continuous function $\psi(x)$ defined on the interval $[a,b]$ and for any given $\varepsilon > 0$, there exists a polynomial $p_n(x)$ of sufficiently high degree n such that $|\psi(x) - p_n(x)| < \varepsilon$ for all $x \in [a,b]$. The polynomial sequence can uniformly approximate any continuous function $\psi(x)$. A trigonometric version also holds for periodic functions ψ with period 2π on the real axis.

Subsequent mathematicians like Runge, Lebesgue, Landau, Vallée-Poussin, Fejér, Jackson, and Bernstein expanded on Weierstrass' results. In particular, Russian mathematician Bernstein [21] developed the Bernstien polynomial and this polynomial sequence uniformly converges to ψ on $[0,1]$, also its derivative converges to ψ as $n \to \infty$, providing useful proof of the Weierstrass theorem. Bernstein Operators $B_n : C[0,1] \to \mathbb{R}$ are given by

$$B_n(\psi,x) = \sum_{k=0}^{n} p_{n,k}(x)\psi\left(\frac{k}{n}\right), \quad x \in [0,1], \tag{1.1}$$

where the basis function $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$. $B_n(\psi,x)$ are a convex combination of $\psi(0)$, $\psi(\frac{1}{n})$, $\cdots\psi(1)$. These operators are linear and positive.

Inspired by Bernstein's work, many other positive linear operators were developed, such as, Durrmeyer polynomials [37], Szász operators [101], Baskakov operators [17], Cheney-Sharma operators [27], Stancu polynomials [99] and Kantorovich operators [68]. These operators help approximate functions on different intervals, with modifications improving their behavior or extending their domains. The linear positive operator $K_n : C[0,1] \to C[0,1]$ was proposed and investigated in the year 1930, by L. V. Kantorovich [68], which are defined as:

$$K_n(\psi;x) = n\sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(t)dt. \tag{1.2}$$

Szász introduced the extension of the Bernstein polynomials on the interval $[0,\infty)$ in 1950 and obtained convergence findings. S. Mirakyan [76] and O. Szász [101] proposed the extension of the Bernstein operators for $[0,\infty)$ in 1941 and 1950, defined as:

$$S_n(\psi,x) = \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\psi\left(\frac{k}{n}\right), \quad x \in [0,\infty), \tag{1.3}$$

where ψ is a continuous function on $[0,\infty)$. These operators are called Szász-Mirakyan operators in (1.3). Generalization of Szász operators for the approximation in the infinite interval was presented by Jakimovski et al.. Through the use of a Lebesgue integral function in the interval $[0,1]$, Durrmeyer [37] provided an integral modification of the Bernstein polynomials in 1967 [37], which are as follows: $D_n : C[0,1] \to C[0,1]$

$$D_n(\psi,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\psi(t)dt,$$

where $0 \le x \le 1$, $\psi \in C[0,1]$.

The Apostol-Genocchi polynomials are a generalization of the classical Genocchi polynomials, which were introduced in the 19th century and are closely related to Bernoulli and Euler polynomials. The original Genocchi numbers and polynomials were studied by the Italian mathematician Angelo Genocchi $(1817-1889)$, primarily in the context of number theory and special functions. In the mid-20th century, the Romanian mathematician Tom M. Apostol made significant contributions to the theory of special functions and generalized various classical polynomials, including the Bernoulli, Euler, and Genocchi families. In 1951, Apostol introduced a new family of polynomials by modifying the generating functions of existing polynomials through the addition of a complex parameter λ. The generating function of the Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order α is given by:

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \qquad |t|<\pi$$

Where $\lambda \in \mathbb{C}\backslash\{-1\}$ is a complex parameter and α is a real or complex. For $\alpha = 1$ and $\lambda = 1$ they reduce to the classical Genocchi polynomials.

Bous-Buck polynomials are a less well-known but mathematically significant type of polynomial introduced in the study of generalized approximation operators and orthogonal polynomials. They were first examined by D. Bous and R.C. Buck [22] as part of their research into approximation theory and combinatorial analysis. These polynomials are a generalization of traditional polynomial families such as Bernstein, Bernoulli, and Appell polynomials. They are primarily used in the fields of positive linear operators, approximation theory, and operator theory.

Bézier curves, created by French engineer Pierre Bézier in 1972 while working at Renault, are frequently used in computer graphics, animation, and CAD for generating smooth and scalable shapes. Their symmetrical characteristics make them very useful in computer-aided geometric design (CAGD). Chang [43] investigated the convergence behavior of generalized Bernstein-Bézier polynomials in 1983. Zeng and Piriou [109] conducted additional study on the convergence rate and approximation features of Bézier-type operators, particularly for functions with constrained variation derivatives. These studies have significantly contributed to the development and application of Bézier techniques in approximation theory and design. Bézier curves are a popular choice in many areas, including automotive and aerospace engineering, due to their ability to easily approximate complex shapes. Their versatility and efficiency have transformed how geometric shapes are made and altered in modern technology. Several writers have examined the approximation behavior of Bézier-type operators (see [1, 47, 60, 96, 98]).

Researchers have proposed various modifications of classical operators and analyzed, Degree of approximation, Rate of convergence, Asymptotic behavior. These concepts are essential when studying ordinary and partial differential equations (PDEs) in modeling real-world systems. The moments of positive linear operators (expectations of monomials under the operator) play a key role in understanding convergence. These can be computed using direct formulas, recurrence relations, or hypergeometric series. In recent decades, the theory has continued to expand, incorporating positive linear operators and examining their convergence properties.

In the current trends, approximation theory is a critical component of both theoretical and applied mathematics, combining classical analysis with modern computing methods. A general approximation approach seeks to address the key concepts of approximation theory while maintaining accuracy and decreasing computation. A direct theorem specifies the order of approximation for functions with a certain level of smoothness. Over the last few years, several mathematicians have worked on new modifications of such operators and studied their approximation properties, like the degree of approximation and the asymptotic formula (cf. [31, 32, 52, 55, 57], and [66]).

1.1 Useful definitions and inequalities

1.1.1 Definitions

Definition 1.1.1 (Positive linear operators). Let X and Y denote two linear spaces of real functions. A linear operator is characterized as a mapping $L : X \to Y$ if it satisfies the condition $L(a\psi + bg;x) = aL(\psi;x) + bL(g;x)$ for $\psi, g \in X$ and $a, b \in \mathbb{R}$. The operator L is classified as a positive linear operator if, for all non-negative functions $\psi \in X$, it holds that $L(\psi;x) \geq 0$.

Proposition 1.1.1. *Consider $L : X \to Y$ as a positive linear operator. The subsequent inequalities are valid:*

- *If $\psi, g \in X$, with $\psi \leq g$, then $L(\psi;x) \leq L(g;x)$.*
- *$|L(\psi;x)| \leq L(|\psi|;x)$.*

Definition 1.1.2 (Modulus of continuity). The modulus of continuity has its roots in the development of real analysis and the concept of uniform continuity, which emerged in the 19th century. The idea of continuity itself was rigorously defined by mathematicians like Augustin-Louis Cauchy and Karl Weierstrass, who formalized the $\varepsilon - \delta$ definitions. However, the formal definition and widespread application of the modulus of continuity became prominent in the 20th century, particularly through the work of S. N. Bernstein and other pioneers of approximation theory.

For a function ψ defined on an interval $[a,b]$, the modulus of continuity $\omega(\psi,\delta)$ is defined as

$$\omega(\psi,\delta) = \sum_{|x-y|<\delta} |\psi(x) - \psi(y)|, \quad x,y \in [a,b]\,.$$

For $k \in \mathbb{N}$, $\delta \in \mathbb{R}^+$ and $\psi \in C[a,b]$, then the modulus of smoothness of order k is defined by

$$\omega_k(\psi,\delta) = \sup\{|\Delta_h^k \psi(x)| \,|\, 0 \leq h \leq \delta, x, x+kh \in [a,b]\},$$

where

$$\Delta_\delta^k \psi(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \psi(x + j\delta).$$

Definition 1.1.3 (**Peetre's K-functional**)**.** In 1963, Swedish mathematician Jaak Peetre [82] introduced the Peetre's K-functional, a fundamental concept in interpolation and approximation theory. It measures a function's ability to be approximated by smoother functions in function spaces. It's crucial for determining convergence rates and understanding operator behavior in function spaces.

Let $\psi \in C[a,b]$, the Peetre's $\mathscr{K}$-functional is defined by

$$\mathscr{K}_2(\psi;t) = \inf_{g \in C^2[a,b]} \{\|\psi - g\| + t\|g'\| + t^2\|g'\|\},\ t > 0$$

where, $C^2[a,b] = \{g \in C_B[a,b] : g', g'' \in C[a,b]\}$. By [36], there exists an absolute constant $C > 0$ such that

$$\mathscr{K}_2(f;t) \leq C\omega_2(\psi;\sqrt{t}). \tag{1.4}$$

For $\psi \in C_B[0,\infty)$ and $\delta > 0$, Peetre-K functional, a different technique to estimate the smoothness of a functions is given by

$$\mathscr{K}_2(\psi,\delta) = \inf_{g \in W^2} \{\|\psi - g\| + \delta\|g''\|\},$$

where $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$ and $\|\psi\| = \sup\{|\psi(x)| : x \in [0,\infty)\}$. By ([33] p.177, Thm. 2.4), there exists an absolute constant $C > 0$ such that

$$\mathscr{K}_2(\psi,\delta) \leq C\,\omega_2(\psi,\sqrt{\delta}), \tag{1.5}$$

where

$$\omega_2(\psi,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |\psi(x+2h) - 2\psi(x+h) + \psi(x)|$$

is the second order modulus of smoothness of $\psi \in C_B[0,\infty)$.

Definition 1.1.4 (**Ditzian-Totik modulus of smoothness**)**.** The Ditzian-Totik modulus of smoothness is an enhanced variant of the classical modulus of smoothness, expressly formulated to more accurately reflect the local behavior of functions, particularly around the interval's endpoints. It was established in 1987 by Z. Ditzian and V. Totik [36] and is widely utilized in research on polynomial approximation in weighted function spaces and the rate of convergence of linear positive operators.

The Ditzian-Totik modulus of smoothness $\omega_\phi(\psi,t)$, $t \in [0,1]$ (cf. [36]). Let $\phi(x)$ be a weight function given by $\phi(x) = \sqrt{x(1-x)}$, $\psi \in C[0,1]$, then the first order modulus of smoothness is given by

$$\omega_\phi(\psi;t) = \sup_{0<h\leq t}\left\{\left|\psi\left(x+\frac{h\phi(x)}{2}\right) - \psi\left(x-\frac{h\phi(x)}{2}\right)\right|, x\pm\frac{h\phi(x)}{2} \in [0,1]\right\},$$

and the appropriate Peetre's $K-$functional is defined by

$$\mathscr{K}_\phi(\psi;t) = \inf_{g\in W_\phi}\{\|\psi-g\| + t\|\phi g'\| + t^2\|g'\|\}\ (t>0),$$

where $W_\phi = \{g : g \in AC_{loc}, \|\phi g'\| < \infty, \|g'\| < \infty\}$ and $\|.\|$ is the uniform norm on $C[0,1]$. Where, $g \in AC_{loc}$ means that g is differentiable and absolutely continuous in every closed finite interval $[c,d] \subset [0,1]$. It is known from ([36], Thm. 3.1.2) that $\overline{K}_\phi(\psi;t) \sim \omega_\phi(\psi;t)$ which means there exists a constant $M > 0$ such that

$$M^{-1}\omega_\phi(f;t) \leq \mathscr{K}_\phi(\psi;t) \leq M\omega_\phi(\psi;t). \tag{1.6}$$

Definition 1.1.5 (Lipschitz class). The Lipschitz class, also known as the Lipschitz space, is a family of function spaces used to describe the degree of smoothness or regularity of functions. Spaces of the Lipschitz type are defined as:

$$Lip_M(\sigma) = \left\{\psi \in C_B[0,\infty) : |\psi(t)-\psi(x)| \leq\ M\frac{|t-x|^\sigma}{(t+x)^{\frac{\sigma}{2}}}\right\}.$$

Özarslan and Aktuğlu [80], consider the Lipschitz-type space with two parameters $a,b > 0$, we have

$$Lip_M^{a,b}(\sigma) = \left\{\psi \in C_B[0,\infty) : |\psi(t)-\psi(x)| \leq\ M\frac{|t-x|^\sigma}{(t+ax^2+bx)^{\frac{\sigma}{2}}}\right\},$$

where $0 < x,t < \infty$, $M > 0$ is a constant and $0 < \sigma \leq 1$.

Definition 1.1.6 (Weighted Approximation). Weighted approximation is crucial, especially when uniform approximation is insufficient due to singularities or unbounded behavior. By incorporating weights into the approximation process, these methods can provide more accurate results in challenging scenarios.

Consider a class of functions as:

$$C_\tau^*[0,\infty) = \left\{\psi \in C_B[0,\infty) : \lim_{x\to\infty}\frac{|\psi(x)|}{\tau(x)} \text{ exists and is finite}\right\}.$$

Yüksel and Ispir defined the weighted modulus of continuity $\Omega^*(\psi;\delta)$, as

$$\Omega^*(\psi;\delta) = \sup_{x\in[0,\infty),0<h<\delta}\frac{|\psi(x+h)-\psi(x)|}{1+(x+h)^2},$$

where $\psi \in C_\tau^*[0,\infty)$, $\tau(x) = 1+x^2$.

Proposition 1.1.2. *Let $\psi \in C_\tau^*[0,\infty)$. Then the following results hold:*

1. $\Omega^*(\psi;\delta)$ *is a monotonically increasing function of* δ;

2. $\lim\limits_{\delta\to 0^+} \Omega^*(\psi;\delta) = 0$;

3. *For each* $m \in \mathbb{N}$, $\Omega^*(\psi;m\delta) \leq m\Omega^*(\psi;\delta)$;

4. *For each* $\lambda \in (0,\infty)$, $\Omega^*(\psi;\lambda\delta) \leq (1+\lambda)\Omega^*(\psi;\delta)$.

Definition 1.1.7 (Derivative of Bounded Variation (DBV)). A function whose derivative is of bounded variation lies at the intersection of classical smoothness and irregular behavior. This concept plays an important role in approximation theory and the theory of functions of bounded variation.

Let DBV$[a,b]$ be the class of all functions in $C[a,b]$ having a derivative that is local of bounded variation on $[a,b]$. The function $\psi \in DBV[a,b]$ is defined as

$$\psi(x) = \int_0^x g(t)dt + \psi(0),$$

where g is a function of bounded variation on every finite subintervals of $[a,b]$.

Definition 1.1.8 (Bohman-Korovkin result). The Bohman-Korovkin theorem, or just a Bohman's theorem[24] (also known as Korovkin's first theorem) is a most important result in approximation theory, proposed by Russian mathematician P. P. Korovkin [69] in 1953. It defines a simple and elegant criterion for determining whether a sequence of positive linear operators converges to the identity operator in the space of continuous functions. Let T_n be the sequence of positive linear operators defined as $T_n : C[a,b] \to C[a,b]$. For the test function $e_r = t^r$, where r can take the values 0, 1, and 2 with the conditions $T_n(e_0;x) = 1+\alpha_n(x)$, $T_n(e_1;x) = x+\beta_n(x)$, and $T_n(e_2;x) = x^2+\gamma_n(x)$ such that $\alpha_n(x)$, $\beta_n(x)$, and $\gamma_n(x)$ uniformly converges to zero in $[a,b]$. Under these conditions, the following result holds:

$$\lim_{n\to\infty} \|T_n(e_r;x) - e_r\| = 0 \quad \text{for } r = 0,1,2, \tag{1.7}$$

This implies that for every function $\psi \in C[a,b]$, it follows that $\lim\limits_{n\to\infty} \|T_n(\psi;x) - \psi\| = 0$.

Furthermore, Korovkin's second theorem state that the sequence $T_n(\psi;x)$ converges uniformly to the periodic function $\psi(x)$ if $\psi(t)$ is continuous and bounded on $[a,b]$ and has a period of 2π.

Definition 1.1.9 (Hölder's inequalities). For $1 < p < \infty$, $1 < q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, the Hölder inequality for summation is defined as:

$$\sum_{r=0}^{n} |\xi_r \eta_r| \leq \left(\sum_{r=0}^{n} |\xi_r|^p\right)^{\frac{1}{p}} \left(\sum_{r=0}^{n} |\eta_r|^q\right)^{\frac{1}{q}},$$

where $\xi_r, \eta_r \in \mathbb{R}$. If f and g are real functions such that $|f|^p$ and $|g|^q$ are integrable on $[a,b]$ then the

Hölder's inequality for integration is defined as:

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}.$$

Definition 1.1.10 (Cauchy-Schwarz inequality). When $p = q = 2$, the Hölder inequality becomes the Cauchy-Schwarz inequality, which is given as:

$$\sum_{r=0}^{n} |\xi_r \eta_r| \leq \left(\sum_{r=0}^{n} |\xi_r|^2\right)^{\frac{1}{2}} \left(\sum_{r=0}^{n} |\eta_r|^2\right)^{\frac{1}{2}},$$

where $\xi_l, \eta_l \in \mathbb{R}$. The Cauchy-Schwarz inequality for integration is given by

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_a^b |g(x)|^2 dx\right)^{\frac{1}{2}}.$$

Chapter 2

Convergence analysis of Durrmeyer variant of Apostol-Genocchi-Baskakov operators

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The Apostol-Genocchi polynomials are a generalization of the classical Genocchi polynomials introduced by the Italian mathematician Angelo Genocchi in the $19th$ century and are closely related to Bernoulli numbers. In 1951, Tom M. Apostol generalized Bernoulli polynomials by introducing a complex parameter β into their generating functions, and this idea was later extended to Genocchi polynomials, giving rise to the Apostol-Genocchi polynomials. In this chapter, We study the approximation behavior of the Durrmeyer form of Apostol-Genocchi polynomials with Baskakov type operators including K-functional and second-order modulus of smoothness, Lipchitz space and find the rate of convergence for continuous functions whose derivative satisfies the condition of bounded variation. In the last section, we estimate weighted approximation behavior for these operators.

2.1 Introduction

The Baskakov operators, established by V.A. Baskakov [17] in 1957, are a traditional family of positive linear operators extensively utilized in approximation theory. In contrast to Bernstein operators, which are defined on the interval $[0,1]$, Baskakov operators are specified on the semi-infinite interval $[0,\infty)$, making them especially suitable for approximating functions defined on unbounded domains. For a function $\psi \in [0,\infty)$, the Baskakov operators is defined as follows

$$\mathscr{B}_n(\psi;x) = \sum_{r=0}^{\infty} s_{n,r}(x)\psi\left(\frac{r}{n}\right), \tag{2.1}$$

where

$$s_{n,r}(x) = \frac{(n)_r}{r!}\frac{x^r}{(1+x)^{n+r}},$$

and the represented symbol $(n)_r$ is defined as:

$$(n)_0 = 1,\ (n)_r = n(n+1)\cdots(n+r-1),\ r = 1,2,\cdots$$

Later, many scholars studied the various approximation properties of so-called Baskakov operators and their different generalized modifications. For more details, we refer to (cf. [3, 5, 7, 34, 51, 57, 59, 64, 67, 86, 105]).

Recently, Prakash et al.[85] conducted significant research on Apostol-Genocchi polynomials and summation integral type operators for real continuous functions. They proposed a novel sequence associated with Apostol-Genocchi polynomials, which is defined as follows:

$$\mathscr{G}_n^{\alpha,\beta}(\psi;x) = \sum_{r=0}^{\infty}\vartheta_{n,r}^{(\alpha,\beta)}(x)\psi\left(\frac{r}{n}\right) = e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\sum_{r=0}^{\infty}\frac{\mu_r^{\alpha}(nx;\beta)}{r!}\psi\left(\frac{r}{n}\right),$$

where $\alpha,\beta \in \mathbb{C}$ (set of complex numbers) and $\mu_r^{\alpha}(x;\beta)$ is generalized Apostol-Genocchi polynomials, together with generating function of the form

$$\left(\frac{2t}{1+\beta e^t}\right)^{\alpha} e^{xt} = \sum_{r=0}^{\infty}\mu_r^{\alpha}(x;\beta)\frac{t^r}{r!}, \quad (r \in \mathbb{N}\cup\{0\}, |t| < \pi).$$

The Apostol-Genocchi polynomials and their properties are studied by many researchers for more details, we refer to (cf. [16, 65, 73, 74, 75, 78, 81, 97]).

In 2015 Gupta and Greubel [53], introduced a new class of Szász-Mirakyan-Durrmeyer operators, a variant of the classical Szász-Mirakyan operators blended with the Durrmeyer-type approach. These operators are designed to improve the approximation of continuous functions on the interval $[0,\infty)$. In this article, The authors investigate the rate of convergence and provide quantitative approximation results via moduli of continuity and Peetre's K-functional.

Motivated by the remarkable work of Gupta and Greubel [53] and Prakash et al. [85], we proposed Durrmeyer Apostol-Genocchi polynomials with Baskakov-type operators (2.1), defined as

$$\begin{aligned}\varphi_n^{\alpha,\beta}(\psi;x) &= \sum_{r=0}^{\infty}\left(\int_0^{\infty}s_{n,r}(t)dt\right)^{-1}\vartheta_{n,r}^{(\alpha,\beta)}(x)\int_0^{\infty}s_{n,r}(t)\psi(t)dt \\ &= \sum_{r=0}^{\infty}\frac{\langle s_{n,r}(t),\psi(t)\rangle}{\langle s_{n,r}(t),1\rangle}\vartheta_{n,r}^{(\alpha,\beta)}(x),\end{aligned} \tag{2.2}$$

By simple computation we may write

$$\frac{\langle s_{n,r}(t), t^k\rangle}{\langle s_{n,r}(t), 1\rangle} = \frac{\Gamma(r+k+1)\Gamma(n-k-1)}{\Gamma(r+1)\Gamma(n-1)},$$

where $< f, g >= \int_0^\infty f(v)g(v)dv$.

2.2 Approximation Using Apostol-Genocchi Polynomial-Based Operators

2.2.1 Preliminaries

In this section, we provide some useful lemmas and results that we will need in the next sections.

Lemma 2.2.1. *[85] The moments of the operators $\mathscr{G}_n^{\alpha,\beta}(e_m(x) = x^m; x)$, for $m = 0,1,2,$ are*

$$\begin{aligned}
\mathscr{G}_n^{\alpha,\beta}(e_0;x) &= 1,\\
\mathscr{G}_n^{\alpha,\beta}(e_1;x) &= x+\frac{\alpha}{n(1+e\beta)},\\
\mathscr{G}_n^{\alpha,\beta}(e_2;x) &= x^2+\frac{(1+2\alpha+e\beta)}{n(1+e\beta)}x+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{n^2(1+e\beta)^2}.
\end{aligned}$$

Lemma 2.2.2. *The moments of $\varphi_n^{\alpha,\beta}$ with $(e_m(x) = x^m; x)$, $m = 0,1,2,$ could be written as:*

$$\begin{aligned}
\varphi_n^{\alpha,\beta}(e_0;x) &= 1,\\
\varphi_n^{\alpha,\beta}(e_1;x) &= \frac{1}{n-2}\left[nx+1+\frac{\alpha}{(1+e\beta)}\right],\\
\varphi_n^{\alpha,\beta}(e_2;x) &= \frac{1}{(n-2)(n-3)}\left[n^2x^2+\frac{4n+4en\beta+2n\alpha)}{1+e\beta}x+\frac{\alpha^2+2(1+e\beta)^2+\alpha(3+e\beta-e^2\beta^2)}{(1+e\beta)^2}\right].
\end{aligned}$$

Lemma 2.2.3. *We may write central moments of $\varphi_n^{\alpha,\beta}$, with the help of Lemma 2.2.2 as:*

$$\begin{aligned}
\varphi_n^{\alpha,\beta}((t-x);x) &= \frac{1}{n-2}\left[2x+1+\frac{\alpha}{1+e\beta}\right],\\
\varphi_n^{\alpha,\beta}((t-x)^2;x) &= \frac{1}{(n-2)(n-3)}\left[(6+n)x^2+\frac{2(n+en\beta+3(1+\alpha+e\beta))}{(1+e\beta)^2}x\right.\\
&\quad\left.+\frac{\alpha^2+2(1+e\beta)^2+\alpha(3+e\beta-e^2\beta^2)}{(1+e\beta)^2}\right].
\end{aligned}$$

Lemma 2.2.4. *Using Lemma 2.2.3, we observe that*

$$\lim_{n\to\infty} n\varphi_n^{\alpha,\beta}((t-x);x) = \frac{1+2x+\alpha+e\beta+2xe\beta}{1+e\beta},$$

and

$$\lim_{n\to\infty} n\varphi_n^{\alpha,\beta}((t-x)^2;x) = 2x+x^2.$$

Moreover, for $n \in \mathbb{N}$, we have

$$\varphi_n^{\alpha,\beta}((t-x)^2;x) \leq \frac{C}{n}\delta_n^2(x),$$

where $C > 0$ and $\delta_n^2(x) = (1+x)^2$.

Lemma 2.2.5. *For the operators $\varphi_n^{\alpha,\beta}$ and real valued bounded continuous function ψ on $[0,\infty)$, we have*

$$|\varphi_n^{\alpha,\beta}(\psi;x)| \leq \|\psi\|,$$

where norm of the function on the positive half real line is given by $\|\psi\| = \sup_{x\in[0,\infty)} |\psi(x)|$.

Proof. From (2.2) and Lemma (2.2.3), we have

$$\begin{aligned}
|\varphi_n^{\alpha,\beta}(\psi;x)| &\leq \sum_{r=0}^{\infty}\left(\int_0^{\infty} s_{n,r}(t)dt\right)^{-1} \vartheta_{n,r}^{(\alpha,\beta)}(x)\int_0^{\infty} s_{n,r}(t)dt|\psi(t)| \\
&\leq \|\psi\|.
\end{aligned}$$

□

2.3 Notable Results

Theorem 2.3.1. *Let $\psi \in C[0,\infty)\bigcap \mathscr{T}$, where*

$$\mathscr{T} := \{\psi : x\in[0,\infty), \frac{\psi(x)}{1+x^2} \text{ exists finitely as } x\to\infty\}.$$

Then, $\varphi_n^{\alpha,\beta}$ converges uniformly with respect to each compact subset of $[0,\infty)$, i.e.

$$\lim_{n\to\infty} \varphi_n^{\alpha,\beta}(\psi;x) = \psi(x), \text{ where } \alpha\to 0 \text{ as } n\to\infty.$$

Proof. From the Lemma 2.2.2, we have

$$\lim_{n\to\infty} \varphi_n^{\alpha,\beta}(t^k;x) = x^k,\ k = 0,1,2,\ldots$$

In the view of Korovkin's theorem, $\varphi_n^{\alpha,\beta}$ converges uniformly with respect to each compact subset of $[0,\infty)$. □

Theorem 2.3.2. *Let* $\psi \in C_B[0,\infty)$*, and for each* $x \in [0,\infty)$*, we have*

$$|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)| \leq 2\omega(\psi,\zeta),$$

where ω *considered as modulus of continuity of* ψ *on* $[0,\infty)$ *and, we choose* ζ *in such a way that* $\omega(\psi,x)$ *approaches to zero as* $\zeta \to 0$.

Proof. Using modulus of continuity and Lemma 2.2.2, we have

$$\begin{aligned}
|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)| &\leq \sum_{r=0}^{\infty}\left(\int_0^\infty s_{n,r}(t)dt\right)^{-1}\vartheta_{n,r}^{(\alpha,\beta)}(x)\int_0^\infty s_{n,r}(t)|\psi(t)-\psi(x)|dt\\
&\leq \left[\sum_{r=0}^{\infty}\left(\int_0^\infty s_{n,r}(t)dt\right)^{-1}\vartheta_{n,r}^{(\alpha,\beta)}(x)\int_0^\infty s_{n,r}(t)\left(1+\frac{1}{\zeta}|t-x|\right)dt\right]\omega(\psi;\zeta)\\
&\leq \left[1+\frac{1}{\zeta}\sum_{r=0}^{\infty}\left(\int_0^\infty s_{n,r}(t)dt\right)^{-1}\vartheta_{n,r}^{(\alpha,\beta)}(x)\int_0^\infty s_{n,r}(t)|t-x|dt\right]\omega(\psi;\zeta).
\end{aligned}$$

On applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)| &\leq \left[1+\frac{1}{\zeta}\sum_{r=0}^{\infty}\left(\int_0^\infty s_{n,r}(t)dt\right)^{-1}\vartheta_{n,r}^{(\alpha,\beta)}(x)\left(\int_0^\infty s_{n,r}(t)dt\right)^{\frac{1}{2}}\right.\\
&\quad\left.\times\left(\int_0^\infty s_{n,r}(t)(t-x)^2dt\right)^{\frac{1}{2}}\right]\omega(\psi;\zeta)\\
&\leq \left[1+\frac{1}{\zeta}\left(\sum_{r=0}^{\infty}\left(\int_0^\infty s_{n,r}(t)dt\right)^{-1}\vartheta_{n,r}^{(\alpha,\beta)}(x)\int_0^\infty s_{n,r}(t)dt\right)^{\frac{1}{2}}\right.\\
&\quad\left.\times\left(\sum_0^{\infty}\left(\int_0^\infty s_{n,r}(t)dt\right)^{-1}\vartheta_{n,r}^{(\alpha,\beta)}(x)\int_0^\infty s_{n,r}(t)(t-x)^2dt\right)^{\frac{1}{2}}\right]\omega(\psi;\zeta)\\
&\leq \left[1+\frac{1}{\zeta}\left(\varphi_n^{\alpha,\beta}(1;x)\right)^{\frac{1}{2}}\{\varphi_n^{\alpha,\beta}\left((t-x)^2;x\right)\}^{\frac{1}{2}}\right]\omega(\psi;\zeta).
\end{aligned}$$

We choose $\zeta = \varphi_n^{\alpha,\beta}\left((t-x)^2;x\right)\}^{\frac{1}{2}}$ to get required result. □

Theorem 2.3.3 (Lipschitz class)**.** *Suppose that* $\psi \in Lip_K^{\sigma}$*, then*

$$|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)| \leq K\left(\frac{2+x+\frac{1}{x}}{n}\right)^{\frac{\sigma}{2}}.$$

Proof. Since $\psi \in Lip_K^{\sigma}$ and $0 < \sigma \leq 1$, we have

$$\begin{aligned} |\varphi_n^{\alpha,\beta}(\psi;x)-\psi(x)| &\leq \varphi_n^{\alpha,\beta}(|\psi(t)-\psi(x)|;x) \\ &\leq K\varphi_n^{\alpha,\beta}\left(\frac{|t-x|^{\sigma}}{(t+x)^{\frac{\sigma}{2}}};x\right). \end{aligned}$$

Now, we have

$$\varphi_n^{\alpha,\beta}\left(\frac{|t-x|^{\sigma}}{(t+x)^{\frac{\sigma}{2}}};x\right) = \sum_{r=0}^{\infty}\left(\int_0^{\infty}s_{n,r}(t)dt\right)^{-1}\vartheta_{n,r}^{(\alpha,\beta)}(x)\int_0^{\infty}s_{n,r}(t)\frac{|t-x|^{\sigma}}{(t+x)^{\frac{\sigma}{2}}}dt.$$

By Holder's inequality $p=\frac{2}{\sigma}$ and $q=\frac{2}{2-\sigma}$ and the fact $\frac{1}{t+x}<\frac{1}{x}$, in view of Lemma 2.2.4, we get

$$\begin{aligned} \varphi_n^{\alpha,\beta}\left(\frac{|t-x|^{\sigma}}{(t+x)^{\frac{\sigma}{2}}};x\right) &\leq \{\varphi_n^{\alpha,\beta}(1;x)\}^{\frac{2-\sigma}{2}}\left[\frac{1}{x}\varphi_n^{\alpha,\beta}((t-x)^2;x)\right]^{\frac{\sigma}{2}} \\ &\leq \left[\frac{1}{x}\varphi_n^{\alpha,\beta}((t-x)^2;x)\right]^{\frac{\sigma}{2}} \\ &\leq \left[\frac{1}{x}\frac{C}{n}(1+x)^2\right]^{\frac{\sigma}{2}}. \end{aligned}$$

□

Theorem 2.3.4. *For real valued continuous function $\psi \in C_B^2[0,\infty)$, we have*

$$|\varphi_n^{\alpha,\beta}(\psi;x)-\psi(x)| \leq C_0\omega_2(\psi,\zeta)+\omega\left(\psi;\left|\frac{1}{n-2}\left(nx+1+\frac{\alpha}{1+e\beta}\right)\right|\right).$$

Proof. For any real and continuous function $\xi \in C_B[0,\infty)$, by Taylor's expansion, we have

$$\xi(t)=\xi(x)+(t-x)\xi'(x)+\int_t^x(t-v)\xi''(v)dv. \tag{2.3}$$

Consider an auxiliary operators associated with $\varphi_n^{\alpha,\beta}$

$$\tilde{\varphi}_n^{\alpha,\beta}(\psi;x)=\varphi_n^{\alpha,\beta}(\psi;x)-\psi\left(\frac{1}{n-2}\left[nx+1+\frac{\alpha}{1+e\beta}\right]\right)+\psi(x), \tag{2.4}$$

for $\psi(t)=1$, we have $\tilde{\varphi}_n^{\alpha,\beta}(1;x)=1$, and for $\psi(t)=t$

$$\tilde{\varphi}_n^{\alpha,\beta}(t;x) = \varphi_n^{\alpha,\beta}(t;x)-\frac{1}{n-2}\left[nx+1+\frac{\alpha}{1+e\beta}\right]+x=x.$$

Immediately, we may write

$$\tilde{\varphi}_n^{\alpha,\beta}((t-x);x) \quad = \quad 0. \tag{2.5}$$

Now, applying the operators $\tilde{\varphi}_n^{\alpha,\beta}$ on (2.3) and using (2.4), we have

$$\begin{aligned}
\tilde{\varphi}_n^{\alpha,\beta}(\xi;x)-\xi(x) &= \tilde{\varphi}_n^{\alpha,\beta}((t-x);x)\xi'(x)+\tilde{\varphi}_n^{\alpha,\beta}\left(\int_x^{\infty}(t-v)\xi''(v)dv;x\right)\\
&= \tilde{\varphi}_n^{\alpha,\beta}\left(\int_x^{\infty}(t-v)\xi''(v)dv;x\right)\\
&= \varphi_n^{\alpha,\beta}\left(\int_x^{\infty}(t-v)\xi''(v)dv;x\right)\\
&\quad -\int_x^{\frac{1}{n-2}\left[nx+1+\frac{\alpha}{1+e\beta}\right]}\left(\frac{1}{n-2}\left[nx+1+\frac{\alpha}{1+e\beta}\right]\right)\xi''(v)dv.
\end{aligned}$$

Now,

$$\begin{aligned}
|\tilde{\varphi}_n^{\alpha,\beta}(\xi;x)-\xi(x)| &\leq \varphi_n^{\alpha,\beta}\left(\frac{1}{2}(t-x)^2\|\xi''\|;x\right)+\frac{1}{2}\left(\frac{1}{n-2}\left[nx+1+\frac{\alpha}{1+e\beta}\right]\right)^2\|\xi''\|\\
&\leq \frac{1}{2}\left|\varphi_n^{\alpha,\beta}\left((t-x)^2;x\right)+\left(\frac{1}{n-2}\left[nx+1+\frac{\alpha}{1+e\beta}\right]\right)^2\right|\|\xi''\|\\
&\leq \zeta\|\xi''\|,
\end{aligned}$$

where

$$\zeta=\frac{1}{2}\left|\varphi_n^{\alpha,\beta}\left((t-x)^2;x\right)+\left(\frac{1}{n-2}\left[nx+1+\frac{\alpha}{1+e\beta}\right]\right)^2\right|.$$

Again, from (2.3) and (2.4), we have

$$\begin{aligned}
\varphi_n^{\alpha,\beta}(\psi;x) &= \tilde{\varphi}_n^{\alpha,\beta}(\psi;x)+\psi\left(\frac{1}{n-2}\left(nx+1+\frac{\alpha}{1+e\beta}\right)\right)-\psi(x)\\
&= \tilde{\varphi}_n^{\alpha,\beta}\left((\psi-\xi);x\right)+\tilde{\varphi}_n^{\alpha,\beta}((\xi;x)+\psi\left(\frac{1}{n-2}\left(nx+1+\frac{\alpha}{1+e\beta}\right)\right)-\psi(x)\\
&= \tilde{\varphi}_n^{\alpha,\beta}\left((\psi-\xi);x\right)-(\psi-\xi)(x)+\tilde{\varphi}_n^{\alpha,\beta}(\xi;x)-\xi(x)+f\left(\frac{1}{n-2}\left(nx+1+\frac{\alpha}{1+e\beta}\right)\right)-\psi(x).
\end{aligned}$$

Now, we have

$$\begin{aligned}
|\varphi_n^{\alpha,\beta}(\psi;x)-\psi(x)| &\leq 2\|\psi-\xi\|+\zeta\|\xi''\|+\left|\psi\left(\frac{1}{n-2}\left(nx+1+\frac{\alpha}{1+e\beta}\right)\right)-\psi(x)\right|\\
&\leq 2\|\psi-\xi\|+\zeta\|\xi''\|+\omega\left(\psi;\left|\frac{1}{n-2}\left(nx+1+\frac{\alpha}{1+e\beta}\right)\right|\right)\\
&\leq 2K_2(\psi,\zeta)+\omega\left(\psi;\left|\frac{1}{n-2}\left(nx+1+\frac{\alpha}{1+e\beta}\right)\right|\right).
\end{aligned}$$

From (1.3), we get the required result. □

Theorem 2.3.5 (Point-wise Estimate)**.** *For $\psi \in Lip_M^{(a,b)}(\sigma)$. Then, for all $x > 0$, we have*

$$|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)| \leq M \left(\frac{C\delta_n^2(x)}{n(ax^2+bx)} \right)^{\frac{\sigma}{2}}.$$

Proof. We start the proof by considering $\sigma = 1$. Then for $\psi \in Lip_M^{(a,b)}(1)$ and $x \in [0,\infty)$, we have

$$\begin{aligned}
|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)| &\leq \varphi_n^{\alpha,\beta}(|\psi(t) - \psi(x)|;x) \\
&\leq M \left\{ \varphi_n^{\alpha,\beta} \left(\frac{|t-x|}{(t+ax^2+bx)^{\frac{1}{2}}};x \right) \right\} \\
&\leq \frac{M}{(ax^2+bx)^{\frac{1}{2}}} \varphi_n^{\alpha,\beta}(|t-x|;x).
\end{aligned}$$

By applying Cauchy-Schwartz inequality and using Lemma 2.2.4

$$\begin{aligned}
|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)| &\leq \frac{M}{(ax^2+bx)^{\frac{1}{2}}} \{\varphi_n^{\alpha,\beta}(|t-x|^2;x)\}^{\frac{1}{2}} \\
&\leq M \left(\frac{C\delta_n^2(x)}{n(ax^2+bx)} \right)^{\frac{\sigma}{2}}.
\end{aligned}$$

Above inequality shows that result is true for $\sigma = 1$.

Next we prove the stated result for $0 < \sigma < 1$, we have

$$|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)| \leq \frac{M}{(ax^2+bx)^{\frac{\sigma}{2}}} \varphi_n^{\alpha,\beta}(|t-x|^{\sigma};x).$$

Assume $p = \frac{1}{\sigma}, q = \frac{p}{p-1}$, on applying the Hölder's inequality, we obtain

$$|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)| \leq \frac{M}{(ax^2+bx)^{\frac{\sigma}{2}}} \left(\varphi_n^{\alpha,\beta}(|t-x|;x) \right)^{\sigma}.$$

Finally, by Cauchy-Schwartz inequality we get

$$\begin{aligned}
|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)| &\leq \frac{M}{(ax^2+bx)^{\frac{\sigma}{2}}} \left(\varphi_n^{\alpha,\beta}(|t-x|^2;x) \right)^{\frac{\sigma}{2}} \\
&\leq M \left(\frac{C\delta_n^2(x)}{n(ax^2+bx)} \right)^{\frac{\sigma}{2}}.
\end{aligned}$$

Hence, we get the stated inequality. □

2.4 Rate of Convergence

We may write operators (2.2) in an alternate form as

$$\varphi_n^{\alpha,\beta}(\psi;x)=\int_0^\infty \mathscr{L}_n^{\alpha,\beta}(x,t)\psi(t)dt \tag{2.6}$$

where

$$\mathscr{L}_n^{\alpha,\beta}(x,t)=\sum_{r=0}^{\infty}\left(\int_0^\infty s_{n,r}(t)dt\right)^{-1}\vartheta_{n,r}^{(\alpha,\beta)}(x)s_{n,r}(t).$$

Lemma 2.4.1. *For every $x\in(0,\infty)$ and n to be very large. Then*

1. *For $0\le y<x$, we may write*

$$\xi_n(x,y)=\int_0^y \mathscr{L}_n^{\alpha,\beta}(x,t)dt\le \frac{C\delta_n^2(x)}{n(x-y)^2}.$$

2. *If $x<z<\infty$, then*

$$1-\xi_n(x,z)=\int_z^\infty \mathscr{L}_n^{\alpha,\beta}(x,t)dt\le \frac{C\delta_n^2(x)}{n(z-x)^2}$$

Proof. Using Lemma 2.2.3 and 2.2.4, for a very large n and $0\le y<x$, we have

$$\begin{aligned}\xi_n(x,y) &= \int_0^y \mathscr{L}_n^{\alpha,\beta}(x,t)dt\\ &\le \int_0^y \frac{(x-t)^2}{(x-y)^2}\mathscr{L}_n^{\alpha,\beta}(x,t)dt\\ &\le \frac{1}{(x-y)^2}\varphi_n^{\alpha,\beta}((x-t)^2;x)\\ &\le \frac{C\delta_n^2(x)}{n(x-y)^2}.\end{aligned}$$

In the similar way we can obtain second result. □

Theorem 2.4.2 (Bounded Variation)**.** *Let $\psi\in DBV(0,\infty)$, for all $x\in(0,\infty)$ and a very large n, and let $\bigvee_a^b(\psi'_x)$ be the total variation of ψ'_x, we have*

$$\begin{aligned}\left|\varphi_n^{\alpha,\beta}(\psi;x)-\psi(x)\right| \le\ & \frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\varphi_n^{\alpha,\beta}((x-t);x)+\frac{1}{2}\sqrt{\frac{C}{n}}\delta_n(x)\left|\psi'(x^+)+\psi'(x^-)\right|\\ &+\frac{C\delta_n^2(x)}{nx}\sum_{r=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{r}}^{x+\frac{x}{r}}(\psi'_x)\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}(\psi'_x)\right)+\sqrt{\frac{C}{n}}\delta_n(x)\psi'(x^+)\\ &+\frac{C\delta_n^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right|+M(\gamma,\lambda,x)\frac{C\delta_n^2(x)|\psi(x)|}{nx^2}.\end{aligned}$$

Where, $\bigvee_a^b(x)$ *denotes the total variation of* ψ_x *on* $[a,b]$ *and*

$$M(\gamma,\lambda,x) = M2^{\gamma}\left(\int_0^{\infty}(t-x)^{2\lambda}\mathscr{L}_n^{(\alpha)}(x,t)dt\right)^{\frac{\gamma}{2\lambda}}.$$

Proof. Using the operators (2.6) and for all x lies on positive real line, we obtain

$$\begin{aligned}\varphi_n^{\alpha,\beta}(\psi;x)-\psi(x) &= \int_0^{\infty}\mathscr{L}_n^{\alpha,\beta}(x,t)(\psi(t)-\psi(x))dt \\ &= \int_0^{\infty}\left(\mathscr{L}_n^{\alpha,\beta}(x,t)\int_x^t\psi(v)dv\right)dt. \qquad (2.7)\end{aligned}$$

For $\psi \in DBV(0,\infty)$, we define an auxiliary function ψ_x as:

$$\psi_x(t) = \begin{cases}\psi(t)-\psi(x^-) & \text{if } 0 \le t < x \\ 0 & \text{if } t = x \\ \psi(t)-\psi(x^+) & \text{if } x < t < \infty\end{cases}$$

Now, we have

$$\begin{aligned}\psi'(v) &= \frac{1}{2}(\psi'(x^+)+\psi'(x^-))+\psi'_x(x)+\frac{1}{2}(\psi'(x^+)-\psi'(x^-))sgn(x) \\ &\quad +\delta_x(v)(\psi'-\frac{1}{2}(\psi'(x^+)+\psi'(x^-)). \qquad (2.8)\end{aligned}$$

Where,

$$\delta_x(v) = \begin{cases}1, & v = x \\ 0, & v \neq x.\end{cases}$$

It can be easily seen that

$$\int_0^{\infty}\left(\int_x^t\left(\psi'(v)-\frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\right)\delta_x(v)dv\right)\mathscr{L}_n^{\alpha,\beta}(x,t)dt = 0. \qquad (2.9)$$

Using the equation (2.6), we obtain

$$\begin{aligned}&\int_0^{\infty}\left(\int_x^t\frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)dv\right)\mathscr{L}_n^{\alpha,\beta}(x,t)dt \\ &= \frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\varphi_n^{\alpha,\beta}\left((x-t);x\right). \qquad (2.10)\end{aligned}$$

Again using (2.6), we have

$$\int_0^{\infty}\left(\int_x^t\frac{1}{2}\left(\psi'(x^+)-\psi'(x^-)\right)sgn(v-x)dv\right)\mathscr{L}_n^{\alpha,\beta}(x,t)dt$$

$$
\begin{aligned}
&= \int_0^\infty \frac{1}{2}\left(\psi'(x^+) - \psi'(x^-)\right)(t-x)\mathscr{L}_n^{\alpha,\beta}(x,t)dt \\
&\leq \int_0^\infty \frac{1}{2}\left|\psi'(x^+) - \psi'(x^-)\right||t-x|\mathscr{L}_n^{\alpha,\beta}(x,t)dt \\
&\leq \frac{1}{2}\left|\psi'(x^+) - \psi'(x^-)\right|\left(\varphi_n^{\alpha,\beta}((x-t)^2;x)\right)^{\frac{1}{2}}. \qquad (2.11)
\end{aligned}
$$

From (2.7), (2.11) and Lemma 2.2.4, we have

$$
\begin{aligned}
\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x) \quad &\leq \quad \frac{1}{2}\left(\psi'(x^+) + \psi'(x^-)\right)\varphi_n^{\alpha,\beta}((x-t);x) \\
&\quad + \frac{1}{2}\left|\psi'(x^+) - \psi'(x^-)\right|\left(\varphi_n^{\alpha,\beta}((x-t)^2;x)\right)^{\frac{1}{2}} \\
&\quad + \int_0^\infty \left(\int_x^t \psi_x'(v)dv\right)\mathscr{L}_n^{\alpha,\beta}(x,t)dt \\
&\leq \quad \frac{1}{2}\left(\psi'(x^+) + \psi'(x^-)\right)\varphi_n^{\alpha,\beta}((x-t);x) + \frac{1}{2}\sqrt{\frac{C}{n}}\delta_n(x)\left|\psi'(x^+) + \psi'(x^-)\right| \\
&\quad + \int_0^\infty \left(\int_x^t \psi_x'(v)dv\right)\mathscr{L}_n^{\alpha,\beta}(x,t)dt.
\end{aligned}
$$

We obtain,

$$
\begin{aligned}
\left|\varphi_n^{\alpha,\beta}(\psi;x) - \psi(x)\right| \quad &\leq \quad \frac{1}{2}\left(\psi'(x^+) + \psi'(x^-)\right)\varphi_n^{\alpha,\beta}((x-t);x) \\
&\quad + \frac{1}{2}\sqrt{\frac{C}{n}}\delta_n(x)\left|\psi'(x^+) + \psi'(x^-)\right| + I_{n1}(x) + I_{n2}(x). \qquad (2.12)
\end{aligned}
$$

Here, we denote

$$
I_{n1}(x) = \left|\int_0^x \left(\int_x^t \psi_x'(v)dv\right)\mathscr{L}_n^{\alpha,\beta}(x,t)dt\right|,
$$

and

$$
I_{n2}(x) = \left|\int_x^\infty \left(\int_x^t \psi_x'(v)dv\right)\mathscr{L}_n^{\alpha,\beta}(x,t)dt\right|.
$$

Now, applying Lemma 2.4.1, integrating by parts and taking $y = x - \frac{x}{\sqrt{n}}$, we get

$$
\begin{aligned}
I_{n1}(x) \quad &= \quad \left|\int_0^x \left(\int_x^t \psi_x'(v)dv\right)d_t\xi_n(x,t)dt\right| = \left|\int_0^x \xi_n(x,t)\psi_x'(t)dt\right| \\
&\leq \quad \int_0^y |\xi_n(x,t)|\left|\psi_x'(t)\right|dt + \int_y^x |\xi_n(x,t)|\left|\psi_x'(t)\right|dt \\
&= \quad \int_0^{x-\frac{x}{\sqrt{n}}} \xi_n(x,t)\left|\psi_x'(t)\right|dt + \int_{x-\frac{x}{\sqrt{n}}}^x \xi_n(x,t)\left|\psi_x'(t)\right|dt.
\end{aligned}
$$

Since $|\xi_n(x,t)| \leq 1$ and $\psi_x'(x) = 0$, we get

$$
\int_{x-\frac{x}{\sqrt{n}}}^x \xi_n(x,t)\left|\psi_x'(t)\right|dt \quad = \quad \int_{x-\frac{x}{\sqrt{n}}}^x \xi_n(x,t)\left|\psi_x'(t) - \psi_x'(x)\right|dt
$$

$$\leq \int_{x-\frac{x}{\sqrt{n}}}^{x} \bigvee_{t}^{x}(\psi'_x)dt \leq \frac{x}{\sqrt{n}} \bigvee_{t-\frac{x}{\sqrt{n}}}^{x}(\psi'_x).$$

Again, using Lemma 2.4.1 and substituting $t = x - \frac{x}{v}$.

$$\begin{aligned}
\int_0^{x-\frac{x}{\sqrt{n}}} \xi_n(x,t)\left|\psi'_x(t)\right|dt &\leq \frac{C\delta_n^2(x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{|\psi'_x(t)|}{(x-t)^2}dt \\
&\leq \frac{C\delta_n^2(x)}{nx}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{v}}^{x}(\psi'_x)dv \\
&\leq \frac{C\delta_n^2(x)}{nx}\sum_{r=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{r}}^{x}(\psi'_x).
\end{aligned}$$

Therefore,

$$I_{n1}(x) \leq \frac{C\delta_n^2(x)}{nx}\sum_{r=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{r}}^{x}(\psi'_x)\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(\psi'_x)\right). \tag{2.13}$$

Now, on applying the property of integration and Lemma 2.4.1, we get

$$\begin{aligned}
I_{n2}(x) &= \int_x^\infty \mathscr{L}_n^{\alpha,\beta}(x,t)\left(\int_x^t \psi'_x(v)dv\right)dt \\
&\leq \left|\int_x^{2x} \mathscr{L}_n^{\alpha,\beta}(x,t)\left(\int_x^t \psi'_x(v)dv\right)dt\right| + \left|\int_{2x}^\infty \mathscr{L}_n^{\alpha,\beta}(x,t)\left(\int_x^t \psi'_x(v)dv\right)dt\right| \\
&\leq J_{n1}(x) + J_{n2}(x).
\end{aligned} \tag{2.14}$$

Here, we denote

$$J_{n1}(x) = \left|\int_x^{2x} \mathscr{L}_n^{\alpha,\beta}(x,t)\left(\int_x^t \psi'_x(v)dv\right)dt\right|,$$

and

$$J_{n2}(x) = \left|\int_{2x}^\infty \mathscr{L}_n^{\alpha,\beta}(x,t)\left(\int_x^t \psi'_x(v)dv\right)dt\right|.$$

Using integration by parts, applying Lemma 2.4.1, $1 - \xi_n(x,z) \leq 1$ and taking $t = x + \frac{x}{v}$, we have

$$\begin{aligned}
J_{n1}(x) &= \left|\int_x^{2x}\psi'_x(v)dv\,(1-\xi_n(x,2x)) - \int_x^{2x}(1-\xi_n(x,t))\,\psi'_x(t)dt\right| \\
&\leq \left|\int_x^{2x}\left(\psi'(v)-\psi'(x^+)\right)dv\right||1-\xi_n(x,2x)| \\
&\quad + \int_x^{2x}|\psi'_x(t)|\,|1-\xi_n(x,t)|\,dt \\
&\leq \frac{C\delta_n^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right| + \int_x^{x+\frac{x}{\sqrt{n}}}|\psi'_x(t)|\,|1-\xi_n(x,t)|\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{x+\frac{x}{\sqrt{n}}}^{2x} |\psi_x'(t)|\,|1-\xi_n(x,t)|\,dt\\
\leq\ & \frac{C\delta_n^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right| + \int_x^{x+\frac{x}{\sqrt{n}}} \bigvee_t^x(\psi_x')dt\\
&+\frac{C\delta_n^2(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{\bigvee_x^t(\psi_x')}{(t-x)^2}dt\\
\leq\ & \frac{C\delta_n^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right| + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}}(\psi_x')\\
&+\frac{C\delta_n^2(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{1}{(t-x)^2}\bigvee_x^t(\psi_x')dt\\
\leq\ & \frac{C\delta_n^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right| + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{n}}(\psi_x')\\
&+\frac{C\delta_n^2(x)}{nx}\sum_{r=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{r}}(\psi_x'),
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
J_{n2}(x) &= \left|\int_{2x}^{\infty}\left(\int_x^t \left(\psi'(v)-\psi'(x+)\right)dv\right)\mathscr{L}_n^{\alpha,\beta}(x,t)dt\right|\\
&\leq \int_0^{\infty}|\psi(t)-\psi(x)|\mathscr{L}_n^{\alpha,\beta}(x,t)dt + \int_{2x}^{\infty}|t-x|\psi'(x^+)\mathscr{L}_n^{\alpha,\beta}(x,t)dt\\
&\leq M\int_{2x}^{\infty}t^{\gamma}\mathscr{L}_n^{\alpha,\beta}(x,t)dt + |\psi(x)|\int_{2x}^{\infty}\mathscr{L}_n^{\alpha,\beta}(x,t)dt + \sqrt{\frac{C}{n}}\delta_n(x)\psi'(x^+).
\end{aligned}
$$

Here, we observe that

$$\frac{t}{2} \leq (t-x) \text{ and } x \leq t-x.$$

Using Holder's inequality, we get

$$
\begin{aligned}
J_{n2}(x) &\leq M2^{\gamma}\left(\int_0^{\infty}(t-x)^{2\lambda}\mathscr{L}_n^{\alpha,\beta}(x,t)dt\right)^{\frac{\gamma}{2\lambda}} + \frac{C\delta_n^2(x)}{nx^2}|\psi(x)|\\
&\quad+\sqrt{\frac{C}{n}}\delta_n(x)\psi'(x^+)\\
&\leq M(\gamma,\lambda,x) + \frac{C\delta_n^2(x)}{nx^2}|\psi(x)| + \sqrt{\frac{C}{n}}\delta_n(x)\psi'(x^+).
\end{aligned}
\tag{2.16}
$$

From (2.15) and (2.16), we obtained

$$
\begin{aligned}
I_{n2}(x) &\leq \frac{C\delta_n^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right| + \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{n}}(\psi_x')\\
&\quad+\frac{C\delta_n^2(x)}{nx}\sum_{r=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{r}}(\psi_x') + M(\gamma,\lambda,x)\frac{C\delta_n^2(x)|\psi(x)|}{nx^2} + \sqrt{\frac{C}{n}}\delta_n(x)\psi'(x^+).
\end{aligned}
\tag{2.17}
$$

On combining (2.12)-(2.14) and (2.17), we get required result. □

2.5 Weighted Approximation

In this section, we discuss the convergence of the operators $\varphi_n^{\alpha,\beta}$ in the weighted space using Korovkin's theorem.

Theorem 2.5.1. *For $n \geq 3$ and each real function $\psi \in C_\tau^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \|\varphi_n^{\alpha,\beta}(\psi;.) - \psi\|_{x^2} = 0.$$

Proof. To prove this theorem, it is acceptable to verify the following conditions [39]

$$\lim_{n\to\infty} \|\varphi_n^{\alpha,\beta}(t^r;x) - x^r\|_{x^2} = 0,\ \ r = 0,1,2. \tag{2.18}$$

Since $\varphi_n^{\alpha,\beta}(1;x) = 1$, therefore for $r = 0$ holds.
Using Lemma 2.2.3, we have

$$\begin{aligned}
\|\varphi_n^{\alpha,\beta}(e_1;\cdot) - e_1\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{|\varphi_n^{\alpha,\beta}(t;x) - x|}{1+x^2} \\
&\leq \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{1}{n-2}\left(nx + 1 + \frac{\alpha}{(1+e\beta)}\right) - x\right| \\
&\leq \frac{1}{n-2}\left(3 + \frac{\alpha}{(1+e\beta)}\right).
\end{aligned}$$

As $n \to \infty$, the condition (2.18) holds good for $r = 1$.

Again by Lemma 2.2.3, we obtain

$$\begin{aligned}
\|\varphi_n^{\alpha,\beta}(e_2;\cdot) - e_2\|_{x^2} = &\sup_{x\in[0,\infty)} \frac{|\varphi_n^{\alpha,\beta}(t^2;x) - x^2|}{1+x^2} \\
\leq &\left|\frac{1}{(n-2)(n-3)}\left(n^2x^2 + \frac{2n(2+2e\beta+\alpha)}{1+e\beta}x \right.\right. \\
&\left.\left. + \frac{\alpha^2 + 2(1+e\beta)^2 + \alpha(3+e\beta-e^2\beta^2)}{(1+e\beta)^2}\right) - x^2\right| \\
\leq &\frac{1}{(n-2)(n-3)}\left(6 + 5n + \frac{2n(2+2e\beta+\alpha)}{1+e\beta}\right. \\
&\left. + \frac{\alpha^2 + 2(1+e\beta)^2 + \alpha(3+e\beta-e^2\beta^2)}{(1+e\beta)^2}\right).
\end{aligned}$$

It follows that the condition (2.18) holds for $r = 2$ as $n \to \infty$.
Thus, on applying Korovkin's theorem required result follows. □

Corollary 2.5.2. *For each* $\psi \in C^*_\alpha[0,\infty)$, *and* $\alpha > 1$, *we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|\varphi_n^{\alpha,\beta}(\psi;x)-\psi(x)|}{(1+x^2)^\alpha} = 0.$$

Proof. For any fixed $\varepsilon_0 > 0$, we have

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|\varphi_n^{\alpha,\beta}(\psi;x)-\psi(x)|}{(1+x^2)^\alpha} &\le \sup_{x\le\varepsilon_0} \frac{|\varphi_n^{\alpha,\beta}(\psi;x)-\psi(x)|}{(1+x^2)^\alpha} + \sup_{x\ge\varepsilon_0} \frac{|\varphi_n^{\alpha,\beta}(\psi;x)-\psi(x)|}{(1+x^2)^\alpha} \\ &\le \|\varphi_n^{\alpha,\beta}(\psi;.)-\psi\|_{C[0,\varepsilon_0]} + \|\psi\|_\tau \sup_{x\ge\varepsilon_0} \frac{|\varphi_n^{\alpha,\beta}\left((1+t^2);x\right)|}{(1+x^2)^\alpha} \\ &\quad + \sup_{x\ge\varepsilon_0} \frac{|\psi(x)|}{(1+x^2)^\alpha}. \end{aligned} \tag{2.19}$$

Since $|\psi(x)| \le \|\psi\|_\tau(1+x^2)$, we have

$$\sup_{x\ge\varepsilon_0} \frac{|\psi(x)|}{(1+x^2)^\alpha} \le \sup_{x\ge\varepsilon_0} \frac{\|\psi(x)\|_\tau}{(1+x^2)^{\alpha-1}} \le \frac{\|\psi(x)\|_\tau}{(1+\varepsilon_0^2)^{\alpha-1}} \tag{2.20}$$

Let $\varepsilon > 0$>. In the view of Theorem 2.3.1 there exists $n_1 \in \mathbb{N}$ such that

$$|\varphi_n^{\alpha,\beta}\left(1+t^2;x\right) - (1+x^2)| < \frac{\varepsilon}{3\|\psi\|_\tau},\ \forall n \ge n_1$$

here, we may write,

$$\varphi_n^{\alpha,\beta}\left(1+t^2;x\right) < (1+x^2) + \frac{\varepsilon}{3\|\psi\|_\tau},\ \forall n \ge n_1 \tag{2.21}$$

Hence,

$$\begin{aligned}\|\psi\|_\tau \frac{|\varphi_n^{\alpha,\beta}\left((1+t^2);x\right)|}{(1+x^2)^\alpha} &< \|\psi\|_\tau \frac{1}{(1+x^2)^\alpha}\left((1+x^2)+\frac{\varepsilon}{3\|\psi\|_\tau}\right), \forall n \ge n_1 \\ &< \frac{\|\psi\|_\tau}{(1+\varepsilon_0{}^2)^{\alpha-1}} + \frac{\varepsilon}{3},\ \forall n \ge n_1 \end{aligned} \tag{2.22}$$

Now, let us choose ε_0 to be very large that $\frac{\|\psi\|_\tau}{(1+\varepsilon_0{}^2)^{\alpha-1}} < \frac{\varepsilon}{6}$. For $\varepsilon > 0$, we can find $n_2 \in \mathbb{N}$ such that $\|\varphi_n^{\alpha,\beta}(\psi;.)-\psi\|_{C[0,\varepsilon_0]} < \frac{\varepsilon}{3}$, $\forall n \ge n_2$. On choosing $n_0 = max(n_1,n_2)$ and combining (2.19)- (2.22), we quickly see that

$$\sup_{x\in[0,\infty)} \frac{|\varphi_n^{\alpha,\beta}(\psi;x)-\psi(x)|}{(1+x^2)^\alpha} < \varepsilon,\ \forall n \ge n_0.$$

Above fact concludes the proof.

□

Chapter 3

Approximation of generalized Pǎltǎnea and Heilmann-type operators

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Difference operators are fundamental tools in approximation theory, particularly in the study of discrete analogs of derivatives and their applications in function approximation. They are used to analyze the smoothness of functions, derive error estimates, and study the convergence behavior of approximation operators. In this chapter, we explore the approximation of the difference between generalized Pǎltǎnea-type operators and M. Heilmann-type operators. A detailed representation of M. Heilmann-type operators is given in terms of hypergeometric series. In the next section, these operators' convergence rate is examined through various tools, including the modulus of continuity, Peetre's $K-$ functional, weighted approximation, and quantitative estimates for their difference.

3.1 Introduction

In order to discuss the approximation properties of linear positive operators the rate of convergence is one of the important characteristics. Several methods and techniques have been applied to investigate and estimate the rate of convergence of positive linear operators. Motivated by an article of Acu and Rasa [10], we establish some quantitative results on the difference of generalized Pǎltǎnea type operators (3.1) and Heilmann type operators (3.4). In our estimation positive linear functionals in quantitative form playing an important role. So we are ready to obtain some information about the rate of convergence of the quantitative difference of these positive linear operators. Recently, the difference of positive linear operators have been discussed by many researchers ([9, 32, 48, 49, 50]).

For $\psi \in C^{\gamma}[0,\infty) = \{\psi \in C[0,\infty) : \psi(t) = O(t^{\gamma}), \gamma > 0\}$, Verma [104] defined the following generalization of Păltănea type operators based on certain parameter $\rho > 0$ in the following way:

$$L_{n,\rho}^{c}(\psi,x) = \sum_{k=1}^{\infty} p_{n,k}(x,c) F_{n,k}^{c}(\psi,x) \tag{3.1}$$

where,

$$F_{n,k}^{c}(\psi,x) = \int_{0}^{\infty} \Theta_{n,k}^{\rho}(t,c)\psi(t)dt + p_{n,0}(x,c)\psi(0), \tag{3.2}$$

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x), \tag{3.3}$$

$$\Theta_{n,k}^{\rho}(t,c) = \begin{cases} \dfrac{n\rho}{\Gamma(k\rho)} e^{-n\rho t}(n\rho t)^{k\rho-1}, & c=0 \\ \dfrac{\Gamma\left(\frac{n\rho}{c}+k\rho\right)}{\Gamma(k\rho)\Gamma\left(\dfrac{n\rho}{c}\right)} \dfrac{c^{k\rho}t^{k\rho-1}}{(1+ct)^{\frac{n\rho}{c}+k\rho}}, & c\in\mathbb{N}, \end{cases}$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c=0, \\ (1+cx)^{-\frac{n}{c}}, & c\in\mathbb{N} \end{cases}$$

where, $\{\phi_{n,c}(x)\}_{n=1}^{\infty}$ is a sequence of functions defined on the closed as well as bounded interval $[0,b]$, $b>0$ and satisfying the following properties for each $n\in\mathbb{N}$ and $k\in\mathbb{N}^0 := \mathbb{N}\cup\{0\}$:

(i) $\phi_{n,c} \in C^{\infty}[a,b]$, $b > a \geq 0$,

(ii) $\phi_{n,c}(0) = 1$,

(iii) $\phi_{n,c}$ is completely monotone so that $(-1)^k\phi_{n,c}^{(k)}(x) \geq 0;\ x\in[0,b]$,

(iv) there exists an integer c such that

$$\phi_{n,c}^{(k+1)}(x) = -n\phi_{n+c,c}^{(k)}(x),\ n > max\{0,-c\};\ x\in[0,b].$$

It can be verified that the operators $L_{n,\rho}^{c}(f,x)$ are well defined for $\psi \in C^{\gamma}[0,\infty)$.

For $\rho = 1$, operators (3.1) reduce to the Srivastava-Gupta operators [95]. When $c=0$, $\rho>0$, we get generalized operators due to Păltănea [87]. Also, for $c=0$ and $\rho=1$ operators (3.1) reduce to Phillips operators [83].

By simple computation of (3.2), we can easily observed that:

$$\int_0^\infty \Theta_{n,k}^{\rho}(t,c)\, t^r dt = \frac{\Gamma(k\rho+r)}{\Gamma(k\rho)} \frac{1}{\prod_{i=1}^{r}(n\rho - ic)}. \tag{3.4}$$

For $r=0$, we get

$$\int_0^\infty \Theta_{n,k}^{\rho}(t,c)\, dt = 1.$$

In the year 1988, Heilmann [61] considered the following operators:

$$(M_{n,c}\psi)(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\Upsilon_{n,k}(\psi,x) \tag{3.5}$$

where,

$$\Upsilon_{n,k}(\psi,x) = (n-c)\int_0^\infty p_{n,k}(t)\psi(t)dt. \tag{3.6}$$

For $x \in [0,\infty)$

$$p_{n,k}(x) = (-1)^k \frac{x^k}{k!}\phi_n^k(x)$$

where,

$$\phi_n(x) = \begin{cases} (1-x)^n \text{ for the interval } [0,1] \text{ with } c=1, \\ e^{-nx} \text{ for the interval } [0,\infty) \text{ with } c=0, \\ (1+cx)^{\frac{-n}{c}} \text{ for the interval } [0,\infty) \text{ with } c>0. \end{cases}$$

By the simple calculation, we obtain

$$\Upsilon_{n,k}(\psi,x) = \frac{(k+r)!\Gamma(\frac{n}{c}-r-1)}{c^r k!\Gamma(\frac{n}{c}-1)}. \tag{3.7}$$

Here, we consider only two cases $c>0$ and $c=0$.

3.2 Basic Results

In this section, we discuss some lemmas which will be used in results.

Lemma 3.2.1. *For $e_r(t)=t^r$, $r \in \mathbb{N}\cup\{0\}$, the moments of operators (3.1) are:*

$$\begin{aligned} L_{n,\rho}^{c}(e_0,x) &= 1, \\ L_{n,\rho}^{c}(e_1,x) &= \frac{n\rho}{n\rho-c}x, \\ L_{n,\rho}^{c}(e_2,x) &= \frac{1}{(n\rho-c)(n\rho-2c)}[n(n+c)\rho^2x^2+n\rho(1+\rho)x], \end{aligned}$$

$$
\begin{aligned}
L_{n,\rho}^{c}(e_3,x) &= \frac{1}{(n\rho-c)(n\rho-2c)(n\rho-3c)}[n\rho^3(2c^2+3cn+n^2)x^3 \\
&+3n\rho^2(c+n)(1+\rho)x^2+n\rho(1+3\rho+\rho^2x)], \\
L_{n,\rho}^{c}(e_4,x) &= \frac{1}{(n\rho-c)(n\rho-2c)(n\rho-3c)(n\rho-4c)}[c(c+n)(2c+n)(3c+n)\rho^4x^4 \\
&+6(c+n)(2c+n)(n+c\rho)\rho^3x^3+(c+n)(7c\rho^2+n(11+18\rho))\rho^2x^2 \\
&+(c\rho^3+n(6+11\rho+6\rho^2))\rho x].
\end{aligned}
$$

Proof. From (3.2), we have

$$
\begin{aligned}
F_{n,k}^{c}(e_0) &= 1, \\
F_{n,k}^{c}(e_1) &= \frac{k\rho}{n\rho-c}, \\
F_{n,k}^{c}(e_2) &= \frac{k\rho(k\rho+1)}{(n\rho-c)(n\rho-2c)}, \\
F_{n,k}^{c}(e_3) &= \frac{k\rho(k\rho+1)(k\rho+2)}{(n\rho-c)(n\rho-2c)(n\rho-3c)}, \\
F_{n,k}^{c}(e_4) &= \frac{k\rho(k\rho+1)(k\rho+2)(k\rho+3)}{(n\rho-c)(n\rho-2c)(n\rho-3c)(n\rho-4c)}.
\end{aligned}
$$

In view of these equalities, we get the required result. □

Lemma 3.2.2. *The moments of operators (3.5) with $e_r(t)=t^r,\ r\in\mathbb{N}\cup\{0\}$ are:*

$$
\begin{aligned}
M_{n,c}(e_0;x) &= 1, \\
M_{n,c}(e_1;x) &= \frac{1}{n-2c}[nx+1], \\
M_{n,c}(e_2;x) &= \frac{1}{(n-2c)(n-3c)}[(cn+n^2)x^2+4nx+2], \\
M_{n,c}(e_3;x) &= \frac{1}{(n-2c)(n-3c)(n-4c)}[n(2c^2+3cn+n^2)x^3+9n(c+n)x^2+18nx+6], \\
M_{n,c}(e_4;x) &= \frac{1}{(n-2c)(n-3c)(n-4c)(n-5c)}[n(6c^3+11c^2n+6cn^2+n^3)x^4 \\
&+16n(2c^2+3cn+n^2)x^3+72n(c+n)x^2+96nx+24].
\end{aligned}
$$

Proof. From the equation (3.7), we obtain

$$
\begin{aligned}
\Upsilon_{n,k}(e_0,x) &= 1, \\
\Upsilon_{n,k}(e_1,x) &= \frac{k+1}{(n-2c)}, \\
\Upsilon_{n,k}(e_2,x) &= \frac{(k+1)(k+2)}{(n-2c)(n-3c)}, \\
\Upsilon_{n,k}(e_3,x) &= \frac{(k+1)(k+2)(k+3)}{(n-2c)(n-3c)(n-4c)}, \\
\Upsilon_{n,k}(e_4,x) &= \frac{(k+1)(k+2)(k+3)(k+4)}{(n-2c)(n-3c)(n-4c)(n-5c)}.
\end{aligned}
$$

In view of these equalities, we get the required result. □

Remark 3.2.3. *Using Lemma(3.2.1), we have*

$$\begin{aligned} L_{n,\rho}^{c}\left((t-x);x\right) &= \frac{cx}{n\rho-c}, \\ L_{n,\rho}^{c}\left((t-x)^2;x\right) &= \frac{x(2c^2x+n\rho(1+\rho)+cnx\rho(1+\rho))}{(n\rho-c)(n\rho-2c)}. \end{aligned}$$

Moreover, for $n\in\mathbb{N}$, we may write

$$\begin{aligned} \lim_{n\to\infty} nL_{n,\rho}^{c}\left((t-x);x\right) &= \frac{cx}{\rho} \\ \lim_{n\to\infty} nL_{n,\rho}^{c}\left((t-x)^2;x\right) &= \frac{x(1+cx)(1+\rho)}{\rho}. \end{aligned}$$

3.3 Hypergeometric form

The hypergeometric function is a special function that generalizes many known functions such as exponential, logarithmic, trigonometric, and Bessel functions. The hypergeometric series and the confluent hypergeometric series are defined as

$${}_2F_1(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k \quad \text{and} \quad {}_1F_1(a,b;x)=\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k k!}x^k$$

respectively. Where, the Pochhammer symbol $(n)_k$ is defined as

$$(n)_k=n(n+1)(n+2)(n+3)...(n+k-1).$$

Lemma 3.3.1. *For $n>0$ and $r\geq 1$, we have*

$$M_{n,c}(e_r;x)=\frac{\Gamma(r+1)}{c^r}\frac{\Gamma(\frac{n}{c}-r-1)}{\Gamma(\frac{n}{c}-1)}{}_2F_1\left(\frac{n}{c},-r;1;-cx\right).$$

Proof. Using the equations (3.5) and (3.7) and from Pochhammer symbol $(n)_k$, we have

$$\begin{aligned} M_{n,c}(e_r;x) &= (n-c)\sum_{k=0}^{\infty}p_{n,k}(x,c)\int_0^{\infty}\frac{(\frac{n}{c})_k}{k!}\frac{(tc)^k}{(1+ct)^{\frac{n}{c}+k}}t^r dt \\ &= (n-c)\sum_{k=0}^{\infty}p_{n,k}(x,c)\frac{(\frac{n}{c})_k}{k!}\frac{1}{c^r}\int_0^{\infty}\frac{(tc)^{k+r}}{(1+ct)^{\frac{n}{c}+k}}dt \\ &= (n-c)\sum_{k=0}^{\infty}p_{n,k}(x,c)\frac{(\frac{n}{c})_k}{k!}\frac{1}{c^{r+1}}\beta\left(r+k+1,\frac{n}{c}-r-1\right) \\ &= \frac{r!}{c^r}\frac{\Gamma(\frac{n}{c}-r-1)}{\Gamma(\frac{n}{c}-1)}(1+cx)^{\frac{-n}{c}}\sum_{k=0}^{\infty}\frac{(r+1)_k}{k!(1)_k}\left(\frac{cx}{1+cx}\right)^k \\ &= \frac{\Gamma(r+1)}{c^r}\frac{\Gamma(\frac{n}{c}-r-1)}{\Gamma(\frac{n}{c}-1)}{}_2F_1\left(\frac{n}{c},-r;1;-cx\right). \end{aligned}$$

$\square$

3.4 Difference of Operators

Let $C_B[0,\infty)$ be the class of bounded and continuous functions defined on $[0,\infty)$ with the norm

$$\|.\| = \sup_{x\in[0,\infty)} |\psi(x)| < \infty.$$

In order to present the main results, we need some useful notations. For $i \in N$, let $e_i(x) = x^i,\ x \in [0,\infty)$. Let F be a positive linear functional defined on a linear subspace $\mathbb{B}$ of the closed space $C[0,\infty)$, such that $F(e_0) = 1$, $b^F := F(e_1)$ and $\mu_r^F = F(e_1 - b^F e_0)^r$, $r \in \mathbb{N}$.

On considering the operators (3.1) and (3.5), we have the following quantitative general results.

Remark 3.4.1. *From the generalised Păltănea type operators (3.1) we have,*

$$F_{n,k}(\psi) = \psi\left(\frac{k}{n}\right),\ \textit{such that}\ F_{n,k}(e_0) = 1,\ b^{F_{n,k}} := F_{n,k}(e_1)$$

and

$$\mu_r^{F_{n,k}} = F_{n,k}(e_1 - b^{F_{n,k}} e_0)^r,\ r \in \mathbb{N}.$$

By the simple calculations, we obtain

$$\begin{aligned}
F_{n,k}(e_1) &= \frac{\Gamma(k\rho+1)}{\Gamma(k\rho)}\frac{1}{n\rho-c} = \frac{k\rho}{n\rho-c},\\
\mu_2^{F_{n,k}} &= F_{n,k}\left(e_1 - b^{F_{n,k}} e_0\right)^2\\
&= F_{n,k}\left(e_2 + \left(\frac{k\rho}{n\rho-c}\right)^2 e_0^2 - 2e_1\frac{k\rho}{n\rho-c}\right)\\
&= \frac{k^2\rho^2 c + k\rho(n\rho-c)}{(n\rho-c)^2(n\rho-2c)},\\
\mu_4^{F_{n,k}} &= F_{n,k}\left(e_1 - b^{F_{n,k}} e_0\right)^4\\
&= \frac{1}{(n\rho-c)^4(n\rho-2c)(n\rho-3c)(n\rho-4c)}\left[-6k\rho(c-n\rho)^3 + 3k^4c^2\rho^4(5c+n\rho)\right.\\
&\quad \left. +6k^3c\rho^3(-5c^2+4cn\rho+n^2\rho^2) + 3k^2\rho^2(c-n\rho)^2(7c+n\rho)\right].
\end{aligned}$$

Remark 3.4.2. *From the M. Hielmann type operators (3.5), we have*

$$\Upsilon_{n,k}(e_0) = 1,\ b^{\Upsilon_{n,k}} = \Upsilon_{n,k}(e_1) = \frac{k+1}{n-2c}.$$

By simple computations, we obtain

$$\begin{aligned}
\mu_2^{\Upsilon_{n,k}} &= \Upsilon_{n,k}\left(e_1 - b^{\Upsilon_{n,k}} e_0\right)^2 \\
&= \frac{ck^2+nk+n-c}{(n-2c)^2(n-3c)}, \\
\mu_4^{\Upsilon_{n,k}} &= \frac{1}{(n-2c)^2(n-3c)(n-4c)(n-5c)}\left[3c^2k^4(4c+n)+6ck^3n(4c+n)\right. \\
&\quad \left. +3k^2n(-2c^2+8cn+n^2)-6k(c-2n)n^2+3(-4c^3+9c^2n-8cn^2+3n^3)\right].
\end{aligned}$$

Remark 3.4.3. *For positive linear functionals $F_{n,k}$ and $\Upsilon_{n,k}$, we estimate*

$$\delta_2 = \sum_{k=0}^{\infty} p_{n,k}\left(b^{F_{n,k}} - b^{\Upsilon_{n,k}}\right)^2.$$

Using remark 3.4.1 and 3.4.2, we get

$$\begin{aligned}
\delta_2 &= \sum_{k=0}^{\infty} \frac{(n/c)_k}{k!}\frac{(cx)^k}{(1+cx)^{n/c+k}}\left(\frac{k\rho}{n\rho-c} - \frac{k+1}{n-2c}\right)^2 \\
&= \frac{1}{(n\rho-c)^2(n-2c)^2}\sum_{k=0}^{\infty}\frac{(n/c)_k}{k!}\frac{(cx)^k}{(1+cx)^{n/c+k}}\Big(k\rho(n-2c)-(k+1)(n\rho-c)\Big)^2 \\
&= \frac{1}{(n\rho-c)^2(n-2c)^2}\sum_{k=0}^{\infty}\frac{(n/c)_k}{k!}\frac{(cx)^k}{(1+cx)^{n/c+k}} \\
&\quad \times[c^2k^2(1-2\rho)^2-2ck(-1+2\rho)(c-n\rho)+(c-n\rho)^2] \\
&= \frac{nx(1+cx+nx)c^2(1-2\rho)^2-2nxc(-1+2\rho)(c-n\rho)+(c-n\rho)^2}{(n\rho-c)^2(n-2c)^2}.
\end{aligned}$$

Theorem 3.4.4. *Let $\psi^s \in C_B[0,\infty)$, $s \in \{0,1,2\}$ and $x \in [0,\infty)$ then for $n \in \mathbb{N}$, we have*

$$|(L_{n,\rho}^c - M_{n,c})(\psi;x)| \leq \|\psi''\|\alpha(x) + \omega(\psi'',\delta_1)(1+\alpha(x)) + 2\omega(\psi,\delta_2)$$

where,

$$\alpha(x) = \frac{1}{2}\sum_{k=0}^{\infty} p_{n,k}\left(\mu_2^{F_{n,k}} + \mu_2^{\Upsilon_{n,k}}\right),$$

$$\delta_1^2 = \frac{1}{2}\sum_{k=0}^{\infty} p_{n,k}\left(\mu_4^{F_{n,k}} + \mu_4^{\gamma_{n,k}}\right),$$

and

$$\delta_2^2 = \sum_{k=0}^{\infty} p_{n,k}\left(b^{F_{n,k}} - b^{\Upsilon_{n,k}}\right)^2.$$

Now, we establish quantitative estimates for the difference of generalized Păltănea operators (3.1) and Heilmann operators (3.5) with the help of Theorem 3.4.4.

Theorem 3.4.5 (Difference estimation)**.** *Let $\psi^{\lambda} \in C_B[0,\infty)$, $\lambda \in \{0,1,2\}$ and $x \in [0,\infty)$ then for $n \in \mathbb{N}$,* *we have the following*

$$|(L^c_{n.\rho} - M_{n,c})(\psi;x)| \leq \|\psi''\|\alpha(x) + \omega(\psi'',\delta_1)(1+\alpha(x)) + 2\omega(\psi,\delta_2)$$

where,

$$\begin{aligned}
\alpha(x) &= \frac{1}{2(n-2c)^2(n-3c)(n\rho-c)^2(n\rho-2c)}\left[(c-n)(c-n\rho)^2(2c-n\rho)\right.\\
&\quad +cnx(1+cx+nx)(-11cn^2\rho^2+n^3\rho^2(1+\rho)+c^2n\rho(5+16\rho)-2c^3(1+6\rho^2))\\
&\quad \left.+nx(12c^4\rho+n^4\rho^2(1+\rho)+4c^2n^2\rho(3+4\rho)-cn^3\rho(1+11\rho)-2c^3n(1+8\rho+6\rho^2))\right],\\
\delta_1^2 &= \frac{1}{(n-2c)^4(n-3c)(n-4c)(n-5c)(n\rho-c)^4(n\rho-2c)(n\rho-3c)(n\rho-4c)}\\
&\quad \times\left[s_1n\left((n+c)(n+2c)(n+3c)x^4+6(n+c)(n+2c)x^3+7(n+c)x^2+x\right)\right.\\
&\quad \left.+s_2nx\left(1+3cx+3nx+2c^2x^2+3cnx^2+n^2x^2\right)+s_3nx(1+cx+nx)+s_4nx+s_5\right],\\
\delta_2^2 &= \frac{nx(1+cx+nx)c^2(1-2\rho)^2-2cnx(-1+2\rho)(c-n\rho)+(c-n\rho)^2}{(n-2c)^2(n\rho-c)^2},
\end{aligned}$$

and

$$\begin{aligned}
s_1 &= 3c^2k^4\left((3c-n)(4c-n)(5c-n)(-2c+n)^4(5c\rho^4+n\rho^5)\right.\\
&\quad \left.+(c-n\rho)^4(2c-n\rho)(3c-n\rho)(4c-n\rho)(4c+n)\right),\\
s_2 &= 6ck^3\left((3c-n)(4c-n)(5c-n)(-2c+n)^4(-5c^2\rho^3+4cn\rho^4+n^2\rho^5)\right.\\
&\quad \left.+(c-n\rho)^4(2c-n\rho)(3c-n\rho)(4c-n\rho)(4cn+n^2)\right),\\
s_3 &= 3k^2\left((3c-n)(4c-n)(5c-n)(-2c+n)^4(7c^3\rho^2-13c^2n\rho^3+5cn^2\rho^4+n^3\rho^5)\right.\\
&\quad \left.+(c-n\rho)^4(2c-n\rho)(3c-n\rho)(4c-n\rho)(-2c^2n+8cn^2+n^3)\right),\\
s_4 &= k\left((3c-n)(4c-n)(5c-n)(-2c+n)^4(-6c^3\rho+18c^2n\rho^2-18cn^2\rho^3+6n^3\rho^4)\right.\\
&\quad \left.+(c-n\rho)^4(2c-n\rho)(3c-n\rho)(4c-n\rho)(-6cn^2+12n^3)\right),\\
s_5 &= 3(-4c^3+9c^2n-8cn^2+3n^3)(c-n\rho)^4(2c-n\rho)(3c-n\rho)(4c-n\rho).
\end{aligned}$$

Proof. Using the remarks (3.4.1)-(3.4.3) we obtain our required results by direct computations. □

3.5 Estimate with K-functional

In this section, our measuring tool will be $K-$functional. Let $\omega_2(\psi,\delta)$ denote the modulus of smoothness of ψ on the closed and bounded interval $[0,b]$, $b>0$.

Lemma 3.5.1. *Let $\psi \in W^2$. Then*

$$|F_{n,k}(\psi)-\psi(b^{F_{n,k}})| \leq 2\mathscr{K}_2\left(\psi;\frac{1}{4}\mu_2^{F_{n,k}}\right).$$

Proof. For $\phi \in W^2$, by Taylor expansion, we may write

$$\left|\phi(t) - \phi(b^{F_{n,k}}) - \phi'\left(b^{F_{n,k}}\right)\left(t - b^{F_{n,k}}\right)\right| \leq \frac{1}{2}\|\phi''\|\left(t - b^{F_{n,k}}\right)^2, \tag{3.8}$$

on applying the functional $F_{n,k}$ to both sides of (3.8):

$$\begin{aligned}\left|F_{n,k}(\phi) - \phi(b^{F_{n,k}})F_{n,k}(e_0) - \phi'(b^{F_{n,k}})\left(F_{n,k}(e_1) - b^{F_{n,k}}F_{n,k}(e_0)\right)\right| \\ \leq \frac{1}{2}\|\phi''\|F_{n,k}\left(\left(e_1 - b^{F_{n,k}}e_0\right)^2\right).\end{aligned}$$

Since $F_{n,k}(e_1) = b^{F_{n,k}}$, it follows that

$$\left|F_{n,k}(\phi) - \phi\left(b^{F_{n,k}}\right)\right| \leq \frac{1}{2}\|\phi''\|\mu_2^{F_{n,k}}.$$

Now let $\psi \in C_B[0,\infty)$. $F_{n,k}(e_0) = 1$ implies that

$$\begin{aligned}\left|F_{n,k}(\psi) - \psi(b^{F_{n,k}})\right| &\leq \left|F_{n,k}(\psi) - F_{n,k}(\phi)\right| + \left|F_{n,k}(\phi) - \phi\left(b^{F_{n,k}}\right)\right| + \left|\phi\left(b^{F_{n,k}}\right) - \psi\left(b^{F_{n,k}}\right)\right| \\ &\leq 2\|\psi - \phi\| + \frac{1}{2}\|\phi''\|\mu_2^{F_{n,k}}.\end{aligned}$$

Taking the infimum over $\phi \in W^2$ concludes the proof. □

Theorem 3.5.2. *Let $\psi \in W^2$ and $\psi' \in C_B^2[0,\infty)$. Then*

$$\left|\left(L_{n,\rho}^c - M_{n,c}\right)(\psi;x)\right| \leq 4\mathscr{K}_2\left(\psi : \frac{1}{8}\eta(x)\right) + \|\psi'\|\mu(x)$$

where,

$$\eta(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\mu_2^{F_{n,k}} + \mu_2^{\Upsilon_{n,k}}\right)$$

and

$$\mu(x) = \sum_{k=0}^{\infty} p_{n,k}\left|b^{F_{n,k}} - b^{\Upsilon_{n,k}}\right| \leq \sum_{k=0}^{\infty} p_{n,k}\left|b^{F_{n,k}}\right| + \left|b^{\Upsilon_{n,k}}\right|.$$

Proof. Similar to the Theorem 3.4.4, we can write

$$\begin{aligned}\left|(L_{n,\rho}^c - M_{n,c})(\psi;x)\right| &\leq \sum_{k=0}^{\infty}\left|F_{n,k}(\psi) - \Upsilon_{n,k}(\psi)\right|p_{n,k}(x) \\ &\leq \sum_{k=0}^{\infty} p_{n,k}(x)\left\{\left|F_{n,k}(\psi) - \psi\left(b^{F_{n,k}}\right)\right| + \left|\Upsilon_{n,k}(\psi) - \psi\left(b^{\Upsilon_{n,k}}\right)\right|\right. \\ &\qquad \left.\left|\psi\left(b^{F_{n,k}}\right) - \psi\left(b^{\Upsilon_{n,k}}\right)\right|\right\}.\end{aligned}$$

We can write

$$\left|\psi\left(b^{F_{n,k}}\right)-\psi\left(b^{\Upsilon_{n,k}}\right)\right| \leq \|\psi'\|.\left|b^{F_{n,k}}-b^{\Upsilon_{n,k}}\right|.$$

From Lemma 3.5.1, we have

$$\begin{aligned}\left|(L_{n,\rho}^{c}-M_{n,c})(\psi;x)\right| &\leq 2\sum_{k=0}^{\infty} p_{n,k}(x)\left\{\mathscr{K}_2\left(\psi;\frac{1}{4}\mu_2^{F_{n,k}}\right)+\mathscr{K}_2\left(\psi;\frac{1}{4}\mu_2^{\Upsilon_{n,k}}\right)\right\}\\ &+\sum_{k=0}^{\infty} p_{n,k}(x)\|\psi'\|.\left|b^{F_{n,k}}-b^{\Upsilon_{n,k}}\right|.\end{aligned}$$

Using $K-$functional for fixed $\phi \in W^2$, the above inequality reduces into:

$$\begin{aligned}\left|(L_{n,\rho}^{c}-M_{n,c})(\psi;x)\right| &\leq 4\|\psi-\phi\|\sum_{k=0}^{\infty} p_{n,k}(x)+\frac{1}{2}\|\phi''\|\sum_{k=0}^{\infty} p_{n,k}(x)\left(\mu_2^{F_{n,k}}+\mu_2^{\Upsilon_{n,k}}\right)\\ &+\sum_{k=0}^{\infty} p_{n,k}(x)\|\psi'\|.\left|b^{F_{n,k}}-b^{\Upsilon_{n,k}}\right|\\ &=4\|\psi-\phi\|+\frac{1}{2}\|\phi''\|\eta(x)+\|\psi'\|\mu(x).\end{aligned}$$

Taking the infimum over $g \in W^2$, the required result follows. □

Now, we give an important result in terms of $K-$functional by Theorem(3.5.2).

Theorem 3.5.3. *Let $\psi \in W^2$ and $\psi' \in C_B^2[0,\infty)$, then*

$$\left|\left(L_{n,\rho}^{c}-M_{n,c}\right)(\psi;x)\right| \leq 4K_2\left(\psi:\frac{1}{8}\eta(x)\right)+\|\psi'\|\left(\frac{n\rho(1+2nx)-nxc(1+2\rho)-c}{(n-2c)(n\rho-c)}\right)$$

where,

$$\begin{aligned}\eta(x) &= \frac{1}{(n-2c)^2(n-3c)(n\rho-c)^2(n\rho-2c)}\left[(c-n)(c-n\rho)^2(2c-n\rho)\right.\\ &+cnx(1+cx+nx)(-11cn^2\rho^2+n^3\rho^2(1+\rho)+c^2n\rho(5+16\rho)-2c^3(1+6\rho^2))\\ &\left.+nx(12c^4\rho+n^4\rho^2(1+\rho)+4c^2n^2\rho(3+4\rho)-cn^3\rho(1+11\rho)-2c^3n(1+8\rho+6\rho^2))\right].\end{aligned}$$

Proof. For $\psi \in W^2$, using Operators (3.1) and (3.5), we have

$$\begin{aligned}\mu(x) &\leq \sum_{k=0}^{\infty} p_{n,k}(x)\left|b^{F_{n,k}}\right|+\left|b^{\Upsilon_{n,k}}\right|\\ &\leq \sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k\rho}{n\rho-c}+\frac{k+1}{n-2c}\right)\\ &\leq \frac{nx\rho}{n\rho-c}+\frac{1+nx}{n-2c}\\ &\leq \frac{n\rho(1+2nx)-ncx(1+2\rho)-c}{(n-2c)(n\rho-c)}. \qquad (3.9)\end{aligned}$$

In the view of remark (3.4.1) and (3.4.2), we have

$$\begin{aligned}
\eta(x) &= \sum_{k=0}^{\infty} p_{n,k}(x)\left(\mu_2^{F_{n,k}} + \mu_2^{\Upsilon_{n,k}}\right) \\
&= \sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k^2\rho^2 c + k\rho(n\rho - c)}{(n\rho - c)^2(n\rho - 2c)} + \frac{ck^2 + nk + n - c}{(n-2c)^2(n-3c)}\right) \\
&= \frac{1}{(n\rho - c)^2(n\rho - 2c)(n-2c)^2(n-3c)} \\
&\quad \times \sum_{k=0}^{\infty} p_{n,k}(x)\left(ck^2\left(-11cn^2\rho^2 + n^3\rho^2(1+\rho) + c^2 n\rho(5+16\rho) - 2c^3(1+6\rho^2)\right)\right. \\
&\quad +k\left(12c^4\rho + n^4\rho^2(1+\rho) + 4c^2n^2\rho(3+4\rho) - cn^3\rho(1+11\rho) - 2c^3 n(1+8\rho+6\rho^2)\right) \\
&\quad \left.+(c-n)(c-n\rho)^2(2c-n\rho)\right) \\
&= \frac{1}{(n-2c)^2(n-3c)(n\rho - c)^2(n\rho - 2c)}\left[(c-n)(c-n\rho)^2(2c-n\rho)\right. \\
&\quad +cnx(1+cx+nx)(-11cn^2\rho^2 + n^3\rho^2(1+\rho) + c^2 n\rho(5+16\rho)) \\
&\quad -2c^4 nx(1+cx+nx)(1+6\rho^2)) + nx(12c^4\rho + n^4\rho^2(1+\rho) + 4c^2n^2\rho(3+4\rho)) \\
&\quad \left.-cn^4 x\rho((1+11\rho) - 2c^3 n(1+8\rho+6\rho^2))\right]. \qquad (3.10)
\end{aligned}$$

We easily see that, (3.9) and (3.10) concludes the required result. □

Now, we establish the rate of convergence of generalized Păltănea operators (3.1) using second order of modulus of continuity of $\psi \in C_B[0,\infty)$ and Peetre's K-functional.

Theorem 3.5.4. *Let $\psi \in C_B[0,\infty)$ and for all $x \in [0,\infty)$, then*

$$\left|L_{n,\rho}^{c}(\psi;x) - \psi(x)\right| \leq C\omega_2(\psi;\delta) + \omega\left(\psi;\left|\frac{cx}{n\rho - c}\right|\right)$$

where, C is an arbitrary constant.

Proof. Let $g \in C_B^2[0,\infty)$ and applying Taylor's expansion of g, we have

$$g(t) = g(x) - (t-x)g'(x) + \int_x^t (t-u)g''(u)du. \qquad (3.11)$$

We introduce an auxiliary operators $L_{n,\rho}^{\sim}$ as follows

$$L_{n,\rho}^{\sim}(\psi;x) = L_{n,\rho}^{c}(\psi;x) - \psi\left(x + \frac{cx}{n\rho - c}\right) + \psi(x).$$

For $\psi(t) = t$, we have

$$L_{n,\rho}^{\sim}(t;x) = L_{n,\rho}^{c}(t;x) - \psi\left(x + \frac{cx}{n\rho - c}\right) + x = x,$$

From above relation, immediately we have $L_{n,\rho}^{\sim}(t-x;x) = 0$.

Now, applying the operator $\tilde{L_{n,\rho}}$ on (3.11), we get

$$\begin{aligned}
\tilde{L_{n,\rho}}(g;x)-g(x) &= \tilde{L_{n,\rho}}\left(\int_x^t (t-u)g''(u)du;x\right)\\
&= L^c_{n,\rho}\left(\int_x^t (t-u)g''(u)du;x\right)-\int_x^{x+\frac{cx}{n\rho-c}}\left(x+\frac{cx}{n\rho-c}-u\right)g''(u)du\\
&= L^c_{n,\rho}\left(\int_x^t (t-u)g''(u)du;x\right)-\frac{1}{2}\left[x+\frac{cx}{n\rho-c}-x\right]^2 g''\\
&= L^c_{n,\rho}\left[\frac{1}{2}(t-x)^2\|g''\|\right]-\frac{1}{2}\left(\frac{cx}{n\rho-c}\right)^2 g''.
\end{aligned}$$

Therefore,

$$\left|\tilde{L_{n,\rho}}(g;x)-g(x)\right| \le \frac{1}{2}\left|L^c_{n,\rho}((t-x)^2;x)|+\left(\frac{cx}{n\rho-c}\right)^2\right|\|g''\| = \delta\|g''\|$$

where, $\delta=\frac{1}{2}\left|L^c_{n,\rho}((t-x)^2;x)|+\left(\frac{cx}{n\rho-c}\right)^2\right|$.

Now,

$$\begin{aligned}
\left|L^c_{n,\rho}(\psi,x)-\psi(x)\right| &\le \tilde{L_{n,\rho}}(|\psi-g|;x)+|\psi-g|+\left|\tilde{L_{n,\rho}}(g;x)-g(x)\right|\\
&\quad + \left|\psi\left(\frac{n\rho x}{n\rho-c}\right)-\psi(x)\right|\\
&\le 2\|\psi-g\|+\delta\|g''\|+\left|\psi\left(\frac{n\rho x}{n\rho-c}\right)-\psi(x)\right|\\
&\le 2\|\psi-g\|+\delta\|g''\|+\omega\left(\psi;\left|\frac{cx}{n\rho-c}\right|\right)\\
&\le 2\mathscr{K}_2(\psi,\delta)+\omega\left(\psi;\left|\frac{cx}{n\rho-c}\right|\right)\\
&\le C\omega_2(\psi,\sqrt{\delta})+\omega\left(\psi;\left|\frac{cx}{n\rho-c}\right|\right).
\end{aligned}$$

□

Theorem 3.5.5. *Let $\psi\in C_B[0,\infty)$ and $n\in\mathbb{N}$, then we have*

$$\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right| \le C\omega^2_{\varphi}(\varphi)(\psi,\delta)+\omega\left(\psi;\frac{cx}{n\rho-c}\right),\ c>0.$$

Proof. For $g\in W^2_{\varphi}[0,\infty)=\{g\in AC_{\log}[0,\infty);\varphi^2 g''\in C_B[0,\infty)$, we have

$$|\tilde{L_{n,\rho}}(g;x)-g(x)| \le \delta\|g''\|$$

and Taylor's formula, we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t (t-u)g''(u)du,\ \forall\, t\in[0,\infty),$$

on applying auxiliary operators $\tilde{L_{n,\rho}}$, we have

$$
\begin{aligned}
\tilde{L_{n,\rho}}(g;x)-g(x) &= \tilde{L_{n,\rho}}\left(\int_x^t (t-u)g''(u)du;x\right)\\
&= L^c_{n,\rho}\left(\int_x^t (t-u)g''(u)du;x\right)+\int_x^{x+\frac{cx}{n\rho-c}}\left(x+\frac{cx}{n\rho-c}-u\right)g''(u)du\\
&= \frac{1}{2}L^c_{n,\rho}\left((t-x)^2\|g''\|;x\right)+\frac{1}{2}\left(\frac{cx}{n\rho-c}\right)^2 g''(u)\\
&= \frac{1}{2}\left[L^c_{n,\rho}\left((t-x)^2;x\right)+\left(\frac{cx}{n\rho-c}\right)^2\right]g''(u)\\
&= \frac{\left[L^c_{n,\rho}\left((t-x)^2;x\right)+\left(\frac{cx}{n\rho-c}\right)^2\right]}{2\varphi^2(x)}\|\varphi^2 g''\|\\
&= \frac{\left[L^c_{n,\rho}\left((t-x)^2;x\right)+\left(\frac{cx}{n\rho-c}\right)^2\right]}{2x(1+cx)}\|\varphi^2 g''\|=\eta_{\rho,c,x}\|\varphi^2 g''\|
\end{aligned}
$$

where,

$$\eta_{\rho,c,x}=\frac{L^c_{n,\rho}\left((t-x)^2;x\right)+\left(\frac{cx}{n\rho-c}\right)^2}{2x(1+cx)}.$$

Now, for $\psi\in C_B[0,\infty)$, we have

$$
\begin{aligned}
\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right| &\leq 2\|\psi-g\|+\left|\tilde{L_{n,\rho}}(g;x))-g(x)\right|+\psi\left(\frac{n\rho x}{n\rho-c}\right)-\psi(x)\\
&\leq 2\|\psi-g\|+\eta_{\rho,c,x}\|\varphi^2 g''\|+\omega\left(\psi;\left|\frac{cx}{n\rho-c}\right|\right)\\
&\leq \mathscr{K}_{2,\varphi}(\psi,\delta^2)+\omega\left(\psi;\frac{cx}{n\rho-c}\right)\\
&\leq C\omega^2_{\varphi}(\varphi)(\psi,\delta)+\omega\left(\psi;\frac{cx}{n\rho-c}\right),\ c>0.
\end{aligned}
$$

□

3.6 Weighted Approximation

Theorem 3.6.1. *Let $\psi\in C_2[0,\infty)$, then we have*

$$\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right|\leq 4M_\psi(1+b^2)\Omega^2_n(x)+2\omega_{b+1}(\psi;\Omega^1_n(x))$$

where, $\Omega^i_n(x)=L^c_{n,\rho}\left((t-x)^i;x\right)\ i=1,2.$

Proof. For $x \in [0,b]$ and $t \geq 0$, we have

$$|\psi(t)-\psi(x)| \leq 4M_{\psi}(1+b^2)(t-x)^2 + \left(1+\frac{(t-x)}{\delta}\right)\omega_{b+1}(\psi;\delta).$$

On applying $L^c_{n,\rho}$, we get

$$\begin{aligned}\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right| &\leq 4M_{\psi}(1+b^2)L^c_{n,\rho}((t-x)^2;x) + \left(1+\frac{L^c_{n,\rho}(|t-x|;x)}{\delta}\right)\omega_{b+1}(\psi;\delta)\\ &\leq 4M_{\psi}(1+b^2)\delta_n^2(x) + \left(1+\frac{1}{\delta}\delta_n^1(x)\right)\omega_{b+1}(\psi;\delta)\end{aligned}$$

where $\delta_n^i(x) = L^c_{n,\rho}\left((t-x)^i;x\right)\ i=1,2.$

After choosing $\delta = \Omega_n(x)$, we obtain the required result. □

Theorem 3.6.2. *Let $\psi \in C_2'[0,\infty)$, we have*

$$\lim_{n\to\infty} \|L^c_{n,\rho}(\psi;x)-\psi(x)\|_2 = 0.$$

Proof. From, By using Bohman-Korovkin theorem we can verify as:

$$\lim_{n\to\infty} \|L^c_{n,\rho}(t^i;x)-x^i\|_2 = 0,\ i=0,1,2.$$

From the Lemma 3.2.1, the result is true for $i=0$.

First we prove the theorem for $i=1$, we have

$$\begin{aligned}\|L^c_{n,\rho}(t;x)-x\|_2 &= \sup_{x\in[0,\infty)} \frac{|L^c_{n,\rho}(t;x)-x|}{1+x^2}\\ &\leq \left|\frac{c}{n\rho-c}\right| \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\\ &\leq \left|\frac{c}{n\rho-c}\right|.\end{aligned}$$

Thus theorem holds for $i=1$.

Now we prove the result for $i=2$

$$\begin{aligned}\|L^c_{n,\rho}(t^2;x)-x^2\|_2 &= \sup_{x\in[0,\infty)} \frac{|L^c_{n,\rho}(t^2;x)-x^2|}{1+x^2}\\ &= \sup_{x\in[0,\infty)} \left|\frac{n(n+c)\rho^2}{(n\rho-c)(n\rho-2c)}x^2 + \frac{\rho(1+\rho)n}{(n\rho-c)(n\rho-2c)}x - x^2\right| \left(\frac{1}{1+x^2}\right)\\ &\leq \left|\frac{n(n+c)\rho^2}{(n\rho-c)(n\rho-2c)}-1\right| \sup_{x\in[0,\infty)} \left(\frac{x^2}{1+x^2}\right)\\ &+ \left|\frac{\rho(1+\rho)n}{(n\rho-c)(n\rho-2c)}\right| \sup_{x\in[0,\infty)} \left(\frac{x}{1+x^2}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\frac{n(n+c)\rho^2}{(n\rho-c)(n\rho-2c)}-1\right|+\left|\frac{\rho(1+\rho)n}{(n\rho-c)(n\rho-2c)}\right| \\
&= 0, \text{ as } n\to\infty.
\end{aligned}
$$

Thus, holds for $i=2$. proved. $\square$

Theorem 3.6.3. *Let* $\psi \in C_2'[0,\infty)$ *and* $\eta > 0$*, we have*

$$
\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|L_{n,\rho}^c(\psi;x)-\psi(x)\right|}{(1+x^2)^{1+\eta}}=0.
$$

Proof. Let $x>0$ be any arbitrary fixed value, then we have

$$
\begin{aligned}
\sup_{x\in[0,\infty)}\left|\frac{L_{n,\rho}^c(\psi;x)-\psi(x)}{(1+x^2)^{1+\eta}}\right| &= \sup_{x<0}\frac{\left|L_{n,\rho}^c(\psi;x)-\psi(x)\right|}{(1+x^2)^{1+\eta}}+\sup_{x\geq x_0}\frac{\left|L_{n,\rho}^c(\psi;x)-\psi(x)\right|}{(1+x^2)^{1+\eta}} \\
&= \sup_{x<x_0}\frac{\left|L_{n,\rho}^c(\psi;x)-\psi(x)\right|}{(1+x^2)^{1+\eta}} \\
&+ \|\psi\|_2\sup_{x>x_0}\frac{\left|L_{n,\rho}^c(\psi;x)-\psi(x)\right|}{(1+x^2)^{1+\eta}}+\sup_{x>x_0}\frac{|\psi(x)|}{(1+x^2)^{1+\eta}}.
\end{aligned}
$$

Since $|\psi(x)|\leq\|\psi\|_2(1+x^2)$, we have

$$
\sup_{x>x_0}\frac{|\psi(x)|}{(1+x^2)^{1+\eta}}\leq\frac{\|\psi\|}{(1+x^2)^{\eta}}.
$$

Let $\varepsilon>0$ be arbitrary and choose x_o very big, then

$$
\frac{\|\psi\|}{(1+x_0^2)^{\eta}}<\frac{\varepsilon}{8}. \tag{3.12}
$$

Since $\lim_{n\to\infty}\sup_{x>x_0}\frac{L_{n,\rho}^c(1+t^2;x)}{1+x^2}=1$, we have

$$
\lim_{n\to\infty}\sup_{x>x_0}\frac{L_{n,\rho}^c(1+t^2;x)}{1+x^2}\leq 1+\frac{(1+x_0^2)^{\eta}}{\|\psi\|_2}\frac{\varepsilon}{8}, \text{ as } n\in\infty.
$$

Therefore, using equation (3.12), we get

$$
\|\psi\|_2\sup_{x>x_0}\frac{L_{n,\rho}^c(1+t^2;x)}{(1+x^2)^{1+\eta}} \leq \frac{\|\psi\|_2}{(1+x^2)^{\eta}}+\frac{\varepsilon}{8}<\frac{\varepsilon}{4}. \tag{3.13}
$$

Using Theorem (3.6.2) and take

$$
|L_{n,\rho}^c(\psi;x)-\psi(x)|<\frac{\varepsilon}{8}. \tag{3.14}
$$

From equations (3.13) and (3.14), we get

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right|}{(1+x^2)^{1+\eta}}<\frac{\varepsilon}{2}.$$

□

Theorem 3.6.4 (Pointwise Estimate)**.** *For $\psi\in Lip_M^{(a,b)}(r)$. Then, for all $x>0$, we have*

$$\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right|\le M\left(\frac{(n\rho c(1+\rho)+2c^2)x^2+n\rho(1+\rho)x}{(ax^2+bx)(n\rho-c)(n\rho-2c)}\right)^{\frac{1}{2}}.$$

Proof. Prove the theorem for $r=1$, let $\psi\in Lip_M^{(a,b)}(1)$ and $x\in[0,\infty)$, we have

$$\begin{aligned}\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right| &\le L^c_{n,\rho}\left(|\psi(t)-\psi(x)|;x\right)\\ &\le ML^c_{n,\rho}\left(\frac{|t-x|^r}{(t+ax^2+bx)^{\frac{1}{2}}};x\right)\\ &\le \frac{M}{(ax^2+bx)^{\frac{1}{2}}}L^c_{n,\rho}\left(|t-x|;x\right).\end{aligned}$$

By applying Cauchy-Schwartz inequality

$$\begin{aligned}\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right| &\le \frac{M}{(ax^2+bx)^{\frac{1}{2}}}\{L^c_{n,\rho}\left(|t-x|^2;x\right)\}^{\frac{1}{2}}\\ &\le \frac{M}{(ax^2+bx)^{\frac{1}{2}}}\left(\frac{n\rho c(1+\rho)+2c^2}{(n\rho-c)(n\rho-2c)}x^2+\frac{n\rho(1+\rho)}{(n\rho-c)(n\rho-2c)}x\right)^{\frac{1}{2}}.\end{aligned}$$

Shows that result is true for $r=1$.

Next, prove the stated result for $0<r<1$, we have

$$\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right| \le \frac{M}{(ax^2+bx)^{\frac{r}{2}}}L^c_{n,\rho}\left(|t-x|^r;x\right).$$

Assume $p=\frac{1}{r}, q=\frac{p}{p-1}$, on applying the Hölder's inequality, we have

$$\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right| \le \frac{M}{(ax^2+bx)^{\frac{r}{2}}}\left(L^c_{n,\rho}\left(|t-x|;x\right)\right)^r.$$

Finally, by Cauchy-Schwartz inequality

$$\left|L^c_{n,\rho}(\psi;x)-\psi(x)\right| \le \frac{M}{(ax^2+bx)^{\frac{r}{2}}}\left(L^c_{n,\rho}\left(|t-x|^2;x\right)\right)^{\frac{r}{2}},$$

by using the Lemma 3.2.1 we get the required result. □

Chapter 4

Convergence analysis of semi- exponential Post-Widder operators

In approximation theory, exponential and semi-exponential positive linear operators are crucial tools, especially when it comes to approximating continuous functions on unbounded intervals like $[0,\infty)$. These operators preserve positivity and linearity, making them suitable for applications in Approximation theory, numerical analysis and functional analysis. In this chapter, we present a recurrence relation for the semi-exponential Post-Widder operators and provide estimates for their moments. We then examine convergence results within Lipschitz-type spaces, analyzing the convergence rate using the Ditzian-Totik smoothness modulus and the weighted continuity modulus. Finally, we estimate the convergence rate for functions whose derivatives are of bounded variation.

4.1 Introduction

In the literature on approximation theory, various generalizations of exponential-type operators have been explored by numerous authors (cf. [2, 57, 70, 83, 102]). These researchers have focused on analyzing the rate of convergence using the modulus of smoothness and Petree's K-functional, which are associated with these generalizations. Recently, some scholars (cf. [4, 54, 63]) have introduced the concept of semi-exponential operators derived from exponential-type operators.

The Post-Widder operators for $n \in \mathbb{N}$ and $x \in [0,\infty)$ considered by D.V. Widder [106] are defined as

$$P_n(\psi;x) = \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1}\int_0^\infty \lambda^n e^{-\frac{n\lambda}{x}}\psi(\lambda)\,d\lambda.$$

Following [106], for $x \in [0,\infty)$ and a parameter $p>0$ Rathore and Singh [90] proposed an integral representation of Post-Widder operators as

$$P_{n,p}(\psi;x) = \frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1}\int_0^\infty \lambda^{n+p}e^{-\frac{n\lambda}{x}}\psi(\lambda)\,d\lambda.$$

Recently, for $x\in[0,\infty)$ and $\rho>0$, using Laplace transformation, M. Herzog [63] defined the semi-exponential Post-Widder operators as

$$W_n^{\rho}(\psi;x) = \frac{n}{\lambda^n e^{\rho\lambda}}\int_0^\infty \frac{\left(\frac{nu}{\rho}\right)^{\frac{n-1}{2}} I_{n-1}\left(2\sqrt{n\rho u}\right)}{e^{\frac{nu}{\lambda}}}\psi(\lambda)\,d\lambda.$$

Following [63], an alternative approach of semi-exponential Post-Widder operators is given by U. Abel et al. [4], which is defined as

$$\mathscr{P}_{n,m}^{\rho}(\psi;x) = \frac{n^n}{e^{\rho x}x^n}\sum_{m=0}^{\infty}\frac{(n\rho)^m}{m!}\frac{1}{\Gamma(n+m)}\int_0^\infty \lambda^{n+m-1}e^{-\frac{n\lambda}{x}}\psi(\lambda)\,d\lambda \tag{4.1}$$

and an alternative form of operators (4.1) is defined as follows

$$\mathscr{P}_{n,m}^{\rho}(\psi;x) = \frac{n}{e^{\rho x}x^n}\int_0^\infty \left(\frac{n\lambda}{\rho}\right)^{\frac{n-1}{2}} e^{\frac{-n\lambda}{x}} I_{n-1}\left(2\sqrt{n\rho\lambda}\right)\psi(\lambda)\,d\lambda, \tag{4.2}$$

I_n represents the modified Bessel function of the first kind:

$$I_n(x) = \sum_{k=0}^{\infty}\frac{x^{n+2k}}{2^{n+2k}k!\Gamma(n+k+1)}.$$

4.2 Basic Properties

In this section, we discuss some useful lemmas and results.

Remark 4.2.1. *For $\rho>0$, if we denote $H_{n,r}^{\rho} = \mathscr{P}_{n,m}^{\rho}(e_r;x)$, $\quad e_r(\lambda)=\lambda^r$, $\quad r=1,2,3\cdot$, $\quad x>0$, then*

$$nH_{n,r+1}^{\rho}(x) = x(\rho x+n)H_{n,r}^{\rho}(x) + x^2 H_{n,r}^{\rho\,\prime}(x).$$

Lemma 4.2.2. *Using the moments of the operators $\mathscr{P}_{n,k}^{\rho}$ could be written as*

$$\mathscr{P}_{n,m}^{\rho}(e_0;x) = 1,$$
$$\mathscr{P}_{n,m}^{\rho}(e_1;x) = \frac{x}{n}(n+\rho x),$$
$$\mathscr{P}_{n,m}^{\rho}(e_2;x) = \frac{x^2}{n^2}\left[n+n^2+2\rho x+2\rho n x+\rho^2x^2\right],$$
$$\mathscr{P}_{n,m}^{\rho}(e_3;x) = \frac{x^3}{n^3}\left[2n+3n^2+n^3+6\rho x+9\rho n x+3\rho n^2 x+6\rho^2x^2+3\rho^2 n x^2+\rho^3x^3\right],$$

$$\mathscr{P}^{\rho}_{n,m}(e_4;x) = \frac{x^4}{n^4}\left[6n+11n^2+6n^3+n^4+24\rho x+44\rho nx+24\rho n^2x+4\rho n^3x+36\rho^2x^2\right.$$
$$\left.+30\rho^2nx^2+6\rho^2n^2x^2+12\rho^3x^3+4\rho^3nx^3+\rho^4x^4\right].$$

Lemma 4.2.3. *Central moments of the operators $\mathscr{P}^{\rho}_{n,m}$, with the help of Lemma 4.2.2, are as follows:*

$$\mathscr{P}^{\rho}_{n,m}((\lambda-x);x) = \frac{\rho x^2}{n}$$
$$\mathscr{P}^{\rho}_{n,m}((\lambda-x)^2;x) = \frac{x^2}{n^2}\left[n+2\rho x+\rho^2x^2\right]$$
$$\mathscr{P}^{\rho}_{n,m}((\lambda-x)^4;x) = \frac{x^4}{n^4}\left[6n+3n^2+24\rho x+20\rho nx+36\rho^2x^2+6\rho^2nx^2+12\rho^3x^3+\rho^4x^4\right].$$

Lemma 4.2.4. *Using Lemma 4.2.3, we have*

$$\begin{aligned}\mathscr{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right) &= \frac{x^2}{n^2}\left(n+2\rho x+\rho^2x^2\right)\\ &\le \frac{x^2}{n}\left(1+2\rho x+\rho^2x^2\right) \le \frac{\chi^2(x)}{n},\end{aligned}$$

and

$$\mathscr{P}^{\rho}_{n,m}((\lambda-x)^4;x) \le C\frac{\chi^4(x)}{n^2},$$

where C is a positive constant and $\chi^2(x) = x(1+\rho x)$.

$$\lim_{n\to\infty} n\mathscr{P}^{\rho}_{n,m}((\lambda-x);x) = \rho x^2, \quad \textit{and} \quad \lim_{n\to\infty} n\mathscr{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right) = x^2.$$

Lemma 4.2.5. *For the operators $\mathscr{P}^{\rho}_{n,m}$ and $\psi \in C_B[0,\infty)$, we have*

$$|\mathscr{P}^{\rho}_{n,m}(\psi;x)| \le \|\psi\|,$$

where the norm of the function on the positive half real line is given by $\|\psi\| = \sup_{x\in[0,\infty)} |\psi(x)|$.

Proof. From (4.2) and Lemma 4.2.2, we have

$$\begin{aligned}|\mathscr{P}^{\rho}_{n,m}(\psi;x)| &\le \frac{n^n}{e^{-\rho x}x^n}\sum_{m=0}^{\infty}\frac{(n\rho)^m}{m!}\frac{1}{\Gamma(n+m)}\int_0^{\infty}\lambda^{n+m-1}e^{-\frac{n\lambda}{x}}|\psi(\lambda)|\,d\lambda,\\ &\le \|\psi\|.\end{aligned}$$

□

Theorem 4.2.6. *Let $\psi \in C_B[0,\infty)$, then $\lim_{n\to\infty}\mathscr{P}^{\rho}_{n,m}(\psi;x) = \psi(x)$, uniformly in every closed interval in*

$[0,\infty)$.

Proof. From Lemma 4.2.2, $\mathscr{P}^{\rho}_{n,m}(e_0;x)=1, \quad \mathscr{P}^{\rho}_{n,m}(\lambda;x)=x, \quad \mathscr{P}^{\rho}_{n,m}(\lambda^2;x)=x^2$, as $n\to\infty$. Therefore, by the Bohman-Korovkin theorem, we get $\mathscr{P}^{\rho}_{n}(\psi(\lambda);x)=\psi(x)$ as $n\to\infty$, uniformly in every closed subinterval of $[0,\infty)$. □

4.3 Main Results

Here, we assess the rate of convergence using the Ditzian-Totik modulus of smoothness $\omega_{\chi^{\gamma}}(\psi,\delta)$ and Peetre's $K-$functional $K_{\chi^{\gamma}}(\psi,\delta), \quad 0\le\gamma\le 1$.

Theorem 4.3.1. *For $\psi\in C_B[0,\infty)$ then, we have*

$$\left|\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x)\right|\le\mathscr{D}\omega_{\chi^{\gamma}}\left(\psi;\frac{\chi^{2-\gamma}(x)}{\sqrt{n}}\right).$$

Proof. For $\varphi\in W_{\gamma}$, and calling the representation

$$\varphi(\lambda)=\varphi(x)+\int_x^{\lambda}\varphi'(s)\,ds.$$

Applying $\mathscr{P}^{\rho}_{n,m}$ and using Hölder's inequality, we have

$$\begin{aligned}\left|\mathscr{P}^{\rho}_{n,m}(\varphi(\lambda);x)-\varphi(x)\right| &\le\mathscr{P}^{\rho}_{n,m}\left(\int_x^{\lambda}\left|\varphi'\right|ds;x\right)\\ &\le\|\psi^{\gamma}\varphi'\|\mathscr{P}^{\rho}_{n,m}\left(\left|\int_x^{\lambda}\frac{ds}{\chi^{\gamma}(s)}\right|;x\right)\\ &\le\|\psi^{\gamma}\varphi'\|\mathscr{P}^{\rho}_{n,m}\left(|\lambda-x|^{1-\gamma}\left|\int_x^{\lambda}\frac{ds}{\chi(s)}\right|^{\gamma};x\right). \qquad (4.3)\end{aligned}$$

Let $I=\left|\int_x^{\lambda}\frac{ds}{\chi(s)}\right|$, now first we simplify expression I

$$\begin{aligned}I&\le\left|\int_x^{\lambda}\frac{ds}{\sqrt{s}}\right|\left|\left(\frac{1}{\sqrt{1+\rho x}}+\frac{1}{\sqrt{1+\rho\lambda}}\right)\right|\\ &\le 2\left|\sqrt{\lambda}-\sqrt{x}\right|\left(\frac{1}{\sqrt{1+\rho x}}+\frac{1}{\sqrt{1+\rho\lambda}}\right)\\ &\le 2\frac{|\lambda-x|}{\sqrt{x}}\left(\frac{1}{\sqrt{1+\rho x}}+\frac{1}{\sqrt{1+\rho\lambda}}\right). \qquad (4.4)\end{aligned}$$

Now, we use the inequality $|p+q|^{\gamma}\le|p|^{\gamma}+|q|^{\gamma}, \quad 0\le\gamma\le 1$ then from (4.4), we get

$$\left|\int_x^{\lambda}\frac{ds}{\chi(s)}\right|^{\gamma}\le 2^{\gamma}\frac{|\lambda-x|^{\gamma}}{x^{\frac{\gamma}{2}}}\left(\frac{1}{(1+\rho x)^{\frac{\gamma}{2}}}+\frac{1}{(1+\rho\lambda)^{\frac{\gamma}{2}}}\right). \qquad (4.5)$$

From (4.3) and (4.5) and using Cauchy inequality, we get

$$\begin{aligned}\left|\mathscr{P}^{\rho}_{n,m}(\varphi(\lambda);x)-\varphi(x)\right| &\leq \frac{2^{\gamma}\|\chi^{\gamma}\varphi'\|}{x^{\frac{\gamma}{2}}}\mathscr{P}^{\rho}_{n,m}\left(|\lambda-x|\left(\frac{1}{(1+\rho x)^{\frac{\gamma}{2}}}+\frac{1}{(1+\rho\lambda)^{\frac{\gamma}{2}}}\right);x\right)\\ &= \frac{2^{\gamma}\|\chi^{\gamma}\varphi'\|}{x^{\frac{\gamma}{2}}}\left(\frac{1}{(1+\rho x)^{\frac{\gamma}{2}}}\left(\mathscr{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)\right)^{\frac{1}{2}}\right.\\ &\qquad \left.+\left(\mathscr{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)\right)^{\frac{1}{2}}.\left(\mathscr{P}^{\rho}_{n,m}\left((1+\rho\lambda)^{-\gamma};x\right)\right)^{\frac{1}{2}}\right).\end{aligned}$$

From Lemma 4.2.3, we may write

$$\left(\mathscr{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)\right)^{\frac{1}{2}} \leq \frac{\chi^2(x)}{\sqrt{n}}, \tag{4.6}$$

where $\chi(x)=x(1+\rho x)$.

For $x\in[0,\infty)$, $\mathscr{P}^{\rho}_{n,m}((1+\rho\lambda)^{-\gamma};x)\to(1+\rho x)^{-\gamma}$ as $n\to\infty$. Thus for $\varepsilon>0$, we find a number $n_0\in\mathbb{N}$ such that

$$\mathscr{P}^{\rho}_{n,m}\left((1+\rho\lambda)^{-\gamma};x\right)\leq(1+\rho x)^{-\gamma}+\varepsilon, \quad \text{for all} \quad n\geq n_0.$$

By choosing $\varepsilon=(1+\rho x)^{-\gamma}$, we obtain

$$\mathscr{P}^{\rho}_{n,m}\left((1+\rho\lambda)^{-\gamma};x\right)\leq 2(1+\rho x)^{-\gamma}, \quad \text{for all} \quad n\geq n_0. \tag{4.7}$$

From (4.6) and (4.7), we have

$$\begin{aligned}\left|\mathscr{P}^{\rho}_{n,m}(\varphi(\lambda);x)-\varphi(x)\right| &\leq 2^{\gamma}\|\chi^{\gamma}\varphi'\|\frac{\chi^2(x)}{\sqrt{n}}\left(\chi^{-\gamma}(x)+\sqrt{2}x^{-\frac{\gamma}{2}}(1+\rho x)^{-\frac{\gamma}{2}}\right)\\ &\leq 2^{\gamma}(1+\sqrt{2})\|\chi^{\gamma}\varphi'\|\frac{1}{\sqrt{n}}\chi^{2-\gamma}(x).\end{aligned} \tag{4.8}$$

We may write

$$\begin{aligned}\left|\mathscr{P}^{\rho}_{n,m}(\psi(\lambda);x)-\psi(x)\right| &\leq \left|\mathscr{P}^{\rho}_{n,m}(\psi(\lambda)-\varphi(\lambda);x)\right|\\ &\qquad+\left|\mathscr{P}^{\rho}_{n,m}(\varphi(\lambda);x)-\varphi(x)\right|+|\varphi(x)-\psi(x)|\\ &\leq 2\|\psi-\varphi\|+\left|\mathscr{P}^{\rho}_{n,m}(\varphi(\lambda);x)-\varphi(x)\right|.\end{aligned} \tag{4.9}$$

From (4.8) and (4.9) and for sufficiently large n, we obtain

$$\begin{aligned}\left|\mathscr{P}^{\rho}_{n,m}(\psi(\lambda);x)-\psi(x)\right| &\leq 2\|\psi-\varphi\|+2^{\gamma}\left(1+\sqrt{2}\right)\|\chi^{\gamma}\varphi'\|\frac{1}{\sqrt{n}}\chi^{2-\gamma}(x)\\ &\leq C_1\{\|\psi-\varphi\|+\chi^{2-\gamma}(x)\frac{1}{\sqrt{n}}\|\chi^{\gamma}\varphi'\|\}\end{aligned}$$

$$\leq CK_{\chi^\lambda}\left(\psi;\chi^{2-\gamma}(x)\frac{1}{\sqrt{n}}\right), \tag{4.10}$$

where $C_1=\max\{2,2^\lambda(1+\sqrt{2})\}$ and $C=2C_1$. From (4.3) and (4.10) we may conclude the required result. □

Theorem 4.3.2. *(Peetre's K− functional) For real valued continuous function $\psi\in C_B[0,\infty)$, we have*

$$|\mathscr{P}^\rho_{n,m}(\psi;x)-\psi(x)|\leq C_0\omega_2(\psi,\zeta)+\omega\left(\psi;\left|\frac{x}{n}(n+\rho x)\right|\right).$$

Proof. For any real and continuous function $\xi\in C^B[0,\infty)$, by Taylor's expansion, we have

$$\xi(\lambda)=\xi(x)+(\lambda-x)\xi'(x)+\int_\lambda^x(\lambda-s)\xi''(s)\,ds. \tag{4.11}$$

Consider an auxiliary operators associated with $\mathscr{P}^\rho_{n,m}$

$$\tilde{\mathscr{P}}^\rho_{n,m}(\psi;x)=\mathscr{P}^\rho_{n,m}(\psi;x)-\psi\left(\frac{x}{n}(n+\rho x)\right)+\psi(x), \tag{4.12}$$

for $\psi(\lambda)=1$, we have $\tilde{\mathscr{P}}^\rho_{n,m}(1;x)=1$, and for $\psi(\lambda)=\lambda$

$$\tilde{\mathscr{P}}^\rho_{n,m}(\lambda;x)=\mathscr{P}^\rho_{n,m}(\lambda;x)-\frac{x}{n}(n+\rho x)+x=x.$$

Immediately, we may write

$$\tilde{\mathscr{P}}^\rho_{n,m}((\lambda-x);x)=0.$$

Now, applying the operators $\tilde{\mathscr{P}}^\rho_{n,m}$ on (4.11) and using (4.12), we have

$$\begin{aligned}\tilde{\mathscr{P}}^\rho_{n,m}(\xi;x)-\xi(x)&=\tilde{\mathscr{P}}^\rho_{n,m}((\lambda-x);x)\xi'(x)+\tilde{\mathscr{P}}^\rho_{n,m}\left(\int_x^\infty(\lambda-s)\xi''(s)\,ds;x\right)\\&=\tilde{\mathscr{P}}^\rho_{n,m}\left(\int_x^\infty(\lambda-s)\xi''(s)\,ds;x\right)\\&=\mathscr{P}^\rho_{n,m}\left(\int_x^\infty(\lambda-s)\xi''(s)\,ds;x\right)-\int_x^{\frac{x}{n}(n+\rho x)}\left(\frac{x}{n}(n+\rho x)\right)\xi''(s)\,ds.\end{aligned}$$

Now,

$$\begin{aligned}|\tilde{\mathscr{P}}^\rho_{n,m}(\xi;x)-\xi(x)|&\leq\mathscr{P}^\rho_{n,m}\left(\frac{1}{2}(\lambda-x)^2\|\xi''\|;x\right)+\frac{1}{2}\left(\frac{x}{n}(n+\rho x)\right)^2\|\xi''\|\\&\leq\frac{1}{2}\left|\mathscr{P}^\rho_{n,m}\left((\lambda-x)^2;x\right)+\left(\frac{x}{n}(n+\rho x)\right)^2\right|\|\xi''\|\leq\zeta\|\xi''\|,\end{aligned}$$

where

$$\zeta = \frac{1}{2}\left|\mathscr{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)+\left(\frac{x}{n}(n+\rho x)\right)^2\right|.$$

Again, from equation (4.11) and (4.12), we have

$$\begin{aligned}\mathscr{P}^{\rho}_{n,m}(\psi;x) &= \tilde{\mathscr{P}}^{\rho}_{n,m}(\psi;x)+\psi\left(\frac{x}{n}(n+\rho x)\right)-\psi(x)\\ &= \tilde{\mathscr{P}}^{\rho}_{n,m}\left((\psi-\xi);x\right)+\tilde{\mathscr{P}}^{\rho}_{n,m}(\xi;x)+\psi\left(\frac{x}{n}(n+\rho x)\right)-\psi(x)\\ &= \tilde{\mathscr{P}}^{\rho}_{n,m}\left((\psi-\xi);x\right)-(\psi-\xi)(x)+\tilde{\mathscr{P}}^{\rho}_{n,m}(\xi;x)-\xi(x)\\ &\quad+\psi\left(\frac{x}{n}(n+\rho x)\right)-\psi(x).\end{aligned}$$

Now, we have

$$\begin{aligned}|\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x)| &\leq \|\psi-\xi\|+\zeta\|\xi''\|+\left|\psi\left(\frac{x}{n}(n+\rho x)\right)-\psi(x)\right|\\ &\leq 2\|\psi-\xi\|+\zeta\|\xi''\|+\omega\left(\psi;\left|\frac{x}{n}(n+\rho x)\right|\right)\\ &\leq 2K_2(\psi,\zeta)+\omega\left(\psi;\left|\frac{x}{n}(n+\rho x)\right|\right).\end{aligned}$$

In the view of (1.3), we get the required result. □

In the following theorem, we obtain the rate of convergence of the operators $\mathscr{P}^{\rho}_{n,m}$ for functions in $Lip^*_M(\alpha)$.

Theorem 4.3.3. *Let $\psi \in Lip^*_M(\alpha)$, $\alpha \in (0,1]$, all $x \in (0,\infty)$, we have*

$$|\mathscr{P}^{\rho}_{n,m}(\psi(\lambda);x)-\psi(x)| \leq M\left(\frac{x\left(n+2\rho x+\rho^2x^2\right)}{n^2}\right)^{\frac{\alpha}{2}}.$$

Proof. From the Lemma 4.2.5, we have

$$\begin{aligned}|\mathscr{P}^{\rho}_{n,m}(\psi(\lambda);x)-\psi(x)| &\leq \mathscr{P}^{\rho}_{n,m}\left(|\psi(\lambda)-\psi(x)|;x\right)\\ &\leq M\mathscr{P}^{\rho}_{n,m}\left(\frac{|\lambda-x|^{\alpha}}{(\lambda+x)^{\frac{\alpha}{2}}};x\right)\\ &\leq \frac{M}{x^{\frac{\alpha}{2}}}\mathscr{P}^{\rho}_{n,m}\left(|\lambda-x|^{\alpha};x\right).\end{aligned} \tag{4.13}$$

Taking $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$ and applying Hölder's inequality, we obtain

$$\begin{aligned}\mathscr{P}^{\rho}_{n,m}(\psi)\left(|\lambda-x|^{\alpha};x\right) &\leq \left\{\mathscr{P}^{\rho}_{n,m}\left(|\lambda-x|^2;x\right)\right\}^{\frac{\alpha}{2}}.\left\{\mathscr{P}^{\rho}_{n,m}\left(1^{\frac{2}{2-\alpha}};x\right)\right\}^{\frac{2-\alpha}{2}}\\ &\leq \left\{\mathscr{P}^{\rho}_{n,m}\left(|\lambda-x|^2;x\right)\right\}^{\frac{\alpha}{2}}.\end{aligned} \tag{4.14}$$

On combining (4.13), (4.14) and using the Lemma 4.2.3, we get the required result. □

For c, $d>0$, Özarslan and Aktuğlu [80] considered the Lipschitz-type space with two parameters, as follows

$$Lip_M^{(c,d)}(\alpha)=\left(\psi\in C[0,\infty):|\psi(\lambda)-\psi(x)|\leq M\frac{|\lambda-x|^{\alpha}}{(\lambda+cx^2+dx)^{\frac{\alpha}{2}}}\right),$$

where M is a positive constant and $0<\alpha\leq 1$.

Theorem 4.3.4 (Point-wise Estimate)**.** *For* $\psi\in Lip_M^{(c,d)}(\alpha)$*. Then, for all* $x>0$*, we have*

$$|\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x)|\leq M\frac{\chi(x)}{(cx^2+dx)^{\frac{\alpha}{2}}}.$$

Proof. First we prove the theorem for $\alpha=1$. Then for $\psi\in Lip_M^{(c,d)}(1)$ and $x\in[0,\infty)$, we have

$$\begin{aligned}|\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x)|&\leq\mathscr{P}^{\rho}_{n,m}(|\psi(\lambda)-\psi(x)|;x)\\&\leq M\left\{\mathscr{P}^{\rho}_{n,m}\left(\frac{|\lambda-x|}{(\lambda+cx^2+dx)^{\frac{1}{2}}};x\right)\right\}\\&\leq\frac{M}{(cx^2+dx)^{\frac{1}{2}}}\mathscr{P}^{\rho}_{n,m}(|\lambda-x|;x).\end{aligned}$$

By applying Cauchy-Schwartz inequality and using Lemma 4.2.4

$$\begin{aligned}|\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x)|&\leq\frac{M}{(cx^2+dx)^{\frac{1}{2}}}\{\mathscr{P}^{\rho}_{n,m}\left(|\lambda-x|^2;x\right)\}^{\frac{1}{2}}\\&\leq M\left(\frac{\chi^2(x)}{cx^2+dx}\right)^{\frac{1}{2}}.\end{aligned}$$

This shows that result is true for $\alpha=1$.

Next,Prove the stated result for $0<\alpha<1$, we have

$$|\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x)|\leq\frac{M}{(cx^2+dx)^{\frac{\alpha}{2}}}\mathscr{P}^{\rho}_{n,m}(|\lambda-x|^{\alpha};x).$$

Assume $p=\frac{1}{\alpha}$, $q=\frac{1}{1-\alpha}$, on applying the Hölder's inequality, we have

$$|\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x)|\leq\frac{M}{(cx^2+dx)^{\frac{\alpha}{2}}}\left(\mathscr{P}^{\rho}_{n,m}(|\lambda-x|;x)\right)^{\alpha}.$$

Again, by Cauchy-Schwartz inequality and Lemma 4.2.4 required result follows. □

Theorem 4.3.5. *Let* $\psi\in C_B[0,\infty)$ *and second order derivative of* ψ *exists in* $[0,\infty)$*, then we have*

$$\lim_{n\to\infty}n\left[\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x)\right]=x^2\left(\rho\psi'(x)+\psi''(x)\right)$$

4.4 Bounded Variation

In the next theorem, we estimate the operators' convergence rate. Let $\psi \in DBV[0,\infty)$ be a continuous function taken from the class of absolutely continuous functions $DBV(0,\infty)$ and having a derivative ψ' on the interval $(0,\infty)$ coincides a.e. with a function which is of bounded variation on every finite partition of $[0,\infty)$. We may write the operators (4.1) in alternate form as

$$\mathscr{P}^{\rho}_{n,m}(\psi;x) = \int_0^{\infty} \mathscr{K}^{\rho}_{n,m}(x,\lambda)\psi(\lambda)\,d\lambda \tag{4.15}$$

where

$$\mathscr{K}^{\rho}_{n,m}(x,\lambda) = \sum_{m=0}^{\infty} \frac{n^n}{e^{\rho x}x^n}\frac{(n\rho)^m}{m!}\frac{1}{\Gamma(n+m)}\lambda^{n+m-1}e^{-\frac{n\lambda}{x}}.$$

Lemma 4.4.1. *For $x \in (0,\infty)$ and sufficiently large n, we have*

1. *Since $0 \le y < x$, therefore*

$$\eta_n(x,y) = \int_0^{y} \mathscr{K}^{\rho}_{n,m}(x,\lambda)\,d\lambda \le \frac{\chi^2(x)}{n(x-y)^2}.$$

2. *If $x < z < \infty$, then*

$$1-\eta_n(x,z) = \int_z^{\infty} \mathscr{K}^{\rho}_{n,m}(x,\lambda)\,d\lambda \le \frac{\chi^2(x)}{n(z-x)^2}.$$

Proof. By simple computations and using Lemma 4.2.3 and Lemma 4.2.4 we get required results. □

Theorem 4.4.2. *Let $\psi \in DBV(0,\infty)$ then for all $x \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}\left|\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x)\right| \le{}& \frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\mathscr{P}^{\rho}_{n,m}((x-\lambda);x)\\ &+\frac{1}{2}\frac{\chi(x)}{\sqrt{n}}\left|\psi'(x^+)+\psi'(x^-)\right|+\frac{\chi^2(x)}{nx}\sum_{m=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{m}}^{x+\frac{x}{m}}(\psi'_x)\right)\\ &+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}(\psi'_x)\right)+\frac{\chi(x)}{\sqrt{n}}\psi'(x^+)\\ &+\frac{\chi^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right|+M(\gamma,r,x)\frac{\chi^2(x)|\psi(x)|}{nx^2},\end{aligned}$$

$\bigvee_a^b(x)$ denotes the total variation of ψ_x on $[a,b]$ and

$$M(\gamma,r,x) = M2^{\gamma}\left(\int_0^{\infty}(\lambda-x)^{2r}\mathscr{K}^{\rho}_{n,m}(x,\lambda)\,d\lambda\right)^{\frac{\gamma}{2r}}.$$

Proof. Using the operator (4.15) for all $x \in [0,\infty)$, we obtain

$$\begin{aligned}\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x) &= \int_0^\infty \mathscr{K}^{\rho}_{n,m}(x,\lambda)\,(\psi(\lambda)-\psi(x))\,d\lambda \\ &= \int_0^\infty \left(\mathscr{K}^{\rho}_{n,m}(x,\lambda)\int_x^\lambda \psi(s)\,ds\right)d\lambda. \end{aligned} \tag{4.16}$$

Let us define an auxiliary function ψ_x by

$$\psi_x(\lambda)=\begin{cases}\psi(\lambda)-\psi(x^-) & \text{if } 0\le\lambda<x\\ 0 & \text{if } \lambda=x\\ \psi(\lambda)-\psi(x^+) & \text{if } x<\lambda<\infty.\end{cases}$$

For $\psi \in DBV\,[0,\infty)$, we can write

$$\begin{aligned}\psi'(s) &= \frac{1}{2}(\psi'(x^+)+\psi'(x^-))+\psi'_x(x)+\frac{1}{2}(\psi'(x^+)-\psi'(x^-))sgn(x)\\ &\quad+\delta_x(s)(\psi'-\frac{1}{2}(\psi'(x^+)+\psi'(x^-)).\end{aligned} \tag{4.17}$$

$$\delta_x(s)=\begin{cases}1, & s=x\\ 0, & s\neq x.\end{cases}$$

It can be easily seen that

$$\int_0^\infty\left(\int_x^\lambda\left(\psi'(s)-\frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\right)\delta_x(s)\,ds\right)\mathscr{K}^{\rho}_{n,m}(x,\lambda)\,d\lambda=0.$$

Using the equation (4.15),we obtain

$$\begin{aligned}&\int_0^\infty\left(\int_x^\lambda\frac{1}{2}(\psi'(x^+)+\psi'(x^-))\,ds\right)\mathscr{K}^{\rho}_{n,m}(x,\lambda)\,d\lambda\\ &\qquad=\frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\mathscr{P}^{\rho}_{n,m}\left((x-\lambda);x\right).\end{aligned}$$

Again using (4.15), we have

$$\begin{aligned}&\int_0^\infty\left(\int_x^\lambda\frac{1}{2}(\psi'(x^+)-\psi'(x^-))sgn(s-x)\,ds\right)\mathscr{K}^{\rho}_{n,m}(x,\lambda)\,d\lambda\\ &\qquad=\int_0^\infty\frac{1}{2}(\psi'(x^+)-\psi'(x^-))(\lambda-x)\mathscr{K}^{\rho}_{n,m}(x,\lambda)\,d\lambda\\ &\qquad\le\frac{1}{2}\left|\psi'(x^+)-\psi'(x^-)\right|\left(\mathscr{P}^{\rho}_{n,m}\left((x-\lambda)^2;x\right)\right)^{\frac{1}{2}}.\end{aligned} \tag{4.18}$$

From (4.16), (4.18) and Lemma 4.2.4, we have

$$\begin{aligned}\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x) &\le \frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\mathscr{P}^{\rho}_{n,m}((x-\lambda);x)\\ &\le \frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\mathscr{P}^{\rho}_{n,m}((x-\lambda);x)+\frac{1}{2}\frac{\chi(x)}{\sqrt{n}}\left|\psi'(x^+)+\psi'(x^-)\right|\\ &\quad+\int_0^\infty\left(\int_x^\lambda \psi_x'(s)\,ds\right)\mathscr{K}^{\rho}_{n,m}(x,\lambda)\,d\lambda.\end{aligned}$$

We obtain

$$\begin{aligned}\left|\mathscr{P}^{\rho}_{n,m}(\psi;x)-\psi(x)\right| &\le \frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\mathscr{P}^{\rho}_{n,m}((x-\lambda);x)\\ &\quad+\frac{1}{2}\frac{\chi(x)}{\sqrt{n}}\left|\psi'(x^+)+\psi'(x^-)\right|+A_{n1}(x)+A_{n2}(x),\end{aligned} \tag{4.19}$$

$$A_{n1}(x)=\left|\int_0^x\left(\int_x^\lambda \psi_x'(s)\,ds\right)\mathscr{K}^{\rho}_{n,m}(x,\lambda)\,d\lambda\right|,$$

and

$$A_{n2}(x)=\left|\int_x^\infty\left(\int_x^\lambda \psi_x'(s)\,ds\right)\mathscr{K}^{\rho}_{n,m}(x,\lambda)\,d\lambda\right|.$$

Now applying Lemma 4.4.1, integrating by parts and taking $y=x-\frac{x}{\sqrt{n}}$, we get

$$\begin{aligned}A_{n1}(x) &= \left|\int_0^x\left(\int_x^\lambda \psi_x'(s)\,ds\right)d_\lambda\eta_n(x,\lambda)\,d\lambda\right| = \left|\int_0^x \eta_n(x,\lambda)\psi_x'(\lambda)\,d\lambda\right|\\ &\le \int_0^y |\eta_n(x,\lambda)|\left|\psi_x'(\lambda)\right|d\lambda+\int_y^x |\eta_n(x,\lambda)|\left|\psi_x'(\lambda)\right|d\lambda\\ &= \int_0^{x-\frac{x}{\sqrt{n}}} \eta_n(x,\lambda)\left|\psi_x'(\lambda)\right|d\lambda+\int_{x-\frac{x}{\sqrt{n}}}^x \eta_n(x,\lambda)\left|\psi_x'(\lambda)\right|d\lambda.\end{aligned}$$

Since $|\eta_n(x,\lambda)|\le 1$ and $\psi_x'(x)=0$, we get

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x \eta_n(x,\lambda)\left|\psi_x'(\lambda)\right|d\lambda &= \int_{x-\frac{x}{\sqrt{n}}}^x \eta_n(x,\lambda)\left|\psi_x'(\lambda)-\psi_x'(x)\right|d\lambda\\ &\le \int_{x-\frac{x}{\sqrt{n}}}^x \bigvee_\lambda^x(\psi_x')\,d\lambda \le \frac{x}{\sqrt{n}}\bigvee_{\lambda-\frac{x}{\sqrt{n}}}^x(\psi_x').\end{aligned}$$

Again using Lemma 4.4.1 and substituting $\lambda=x-\frac{x}{s}$.

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}} \eta_n(x,\lambda)\left|\psi_x'(\lambda)\right|d\lambda &\le \frac{\chi^2(x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{|\psi_x'(\lambda)|}{(x-\lambda)^2}\,d\lambda\\ &\le \frac{\chi^2(x)}{nx}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{s}}^x(\psi_x')\,ds\end{aligned}$$

$$\leq \frac{\chi^2(x)}{nx}\sum_{m=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{s}}^{x}(\psi_x').$$

Therefore,

$$A_{n1}(x) \leq \frac{\chi^2(x)}{nx}\sum_{m=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{m}}^{x}(\psi_x')\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(\psi_x')\right). \tag{4.20}$$

Now we observe, integration by parts and applying Lemma 4.4.1, we have

$$\begin{aligned}
A_{n2}(x) &= \int_x^{\infty}\mathscr{K}_{n,m}^{\rho}(x,\lambda)\left(\int_x^{\lambda}\psi_x'(s)\,ds\right)d\lambda \\
&\leq \left|\int_x^{2x}\mathscr{K}_{n,m}^{\rho}(x,\lambda)\left(\int_x^{\lambda}\psi_x'(s)\,ds\right)d\lambda\right|+\left|\int_{2x}^{\infty}\mathscr{K}_{n,m}^{\rho}(x,\lambda)\left(\int_x^{\lambda}\psi_x'(s)\,ds\right)d\lambda\right| \\
&\leq B_{n1}(x)+B_{n2}(x), \qquad (4.21)
\end{aligned}$$

$B_{n1}(x)=\left|\int_x^{2x}\mathscr{K}_{n,m}^{\rho}(x,\lambda)\left(\int_x^{\lambda}\psi_x'(s)\,ds\right)d\lambda\right|$, and
$B_{n2}(x)=\left|\int_{2x}^{\infty}\mathscr{K}_{n,m}^{\rho}(x,\lambda)\left(\int_x^{\lambda}\psi_x'(s)\,ds\right)d\lambda\right|$. Applying integration by parts, using (4.17) and Lemma 4.4.1. Since $1-\eta_n(x,\lambda)\leq 1$, substituting $\lambda = x+\frac{x}{s}$, we have

$$\begin{aligned}
B_{n1}(x) &= \left|\int_x^{2x}\psi_x'(s)\,ds\,(1-\eta_n(x,2x))-\int_x^{2x}(1-\eta_n(x,\lambda))\,\psi_x'(\lambda)\,d\lambda\right| \\
&\leq \left|\int_x^{2x}\left(\psi'(s)-\psi'(x^+)\right)ds\right||1-\eta_n(x,2x)|+\int_x^{2x}|\psi_x'(\lambda)|\,|1-\eta_n(x,\lambda)|\,d\lambda \\
&\leq \frac{\chi^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right|+\int_x^{x+\frac{x}{\sqrt{n}}}|\psi_x'(\lambda)|\,|1-\eta_n(x,\lambda)|\,d\lambda \\
&\quad+\int_{x+\frac{x}{\sqrt{n}}}^{2x}|\psi_x'(\lambda)|\,|1-\eta_n(x,\lambda)|\,d\lambda \\
&\leq \frac{\chi^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right|+\int_x^{x+\frac{x}{\sqrt{n}}}\bigvee_{\lambda}^{x}(\psi_x')\,d\lambda+\frac{\chi^2(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{\bigvee_x^{\lambda}(\psi_x')}{(\lambda-x)^2}\,d\lambda \\
&\leq \frac{\chi^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right|+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(\psi_x') \\
&\quad+\frac{\chi^2(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(\lambda-x)^2}\bigvee_{x}^{\lambda}(\psi_x')\,d\lambda \\
&\leq \frac{\chi^2(x)}{nx^2}\left|\psi(2x)-\psi(x)-x\psi'(x^+)\right|+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{n}}(\psi_x')+\frac{\chi^2(x)}{nx}\sum_{m=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}(\psi_x'). \qquad (4.22)
\end{aligned}$$

And

$$\begin{aligned}
B_{n2}(x) &= \left|\int_{2x}^{\infty}\left(\int_x^{\lambda}\psi'(s)-\psi'(x+)\,ds\right)\mathscr{K}_{n,m}^{\rho}(x,\lambda)\,d\lambda\right| \\
&\leq \int_0^{\infty}|\psi(\lambda)-\psi(x)|\mathscr{K}_{n,m}^{\rho}(x,\lambda)\,d\lambda+\int_{2x}^{\infty}|\lambda-x||\psi'(x^+)\mathscr{K}_{n,m}^{\rho}(x,\lambda)\,d\lambda
\end{aligned}$$

$$\leq M\int_{2x}^{\infty}\lambda^{\gamma}\mathscr{K}_{n,m}^{\rho}(x,\lambda)\,d\lambda+|\psi(x)|\int_{2x}^{\infty}\mathscr{K}_{n,m}^{\rho}(x,\lambda)\,d\lambda+\frac{\chi(x)}{\sqrt{n}}\psi'(x^{+}).$$

It is obvious that

$$\lambda\leq 2(\lambda-x)\quad\text{and}\quad x\leq\lambda-x,\quad\text{when}\quad\lambda\geq 2x.$$

Applying Holder's inequality, we get

$$\begin{aligned}B_{n2}(x)&\leq M2^{\gamma}\left(\int_{0}^{\infty}(\lambda-x)^{2r}\mathscr{K}_{n,m}^{\rho}(x,\lambda)\,d\lambda\right)^{\frac{\gamma}{2r}}+\frac{\chi^{2}(x)}{nx^{2}}|\psi(x)|+\sqrt{\frac{1}{n}}\chi(x)\psi'(x^{+})\\&\leq M(\gamma,r,x)+\frac{\chi^{2}(x)}{nx^{2}}|\psi(x)|+\frac{\chi(x)}{\sqrt{n}}\psi'(x^{+}).\end{aligned}\tag{4.23}$$

From (4.22) and (4.23), we get

$$\begin{aligned}A_{n2}(x)\leq&\frac{\chi^{2}(x)}{nx^{2}}\left|\psi(2x)-\psi(x)-x\psi'(x^{+})\right|+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{n}}(\psi_{x}')\\&+\frac{\chi^{2}(x)}{nx}\sum_{m=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{m}}(\psi_{x}')+M(\gamma,r,x)\frac{\chi^{2}(x)|\psi(x)|}{nx^{2}}\\&+\sqrt{\frac{1}{n}}\chi(x)\psi'(x^{+}).\end{aligned}\tag{4.24}$$

On combining (4.19)-(4.21) and (4.24), we get required result. □

Example 1. The Graphical representation of the convergence of the operators $\mathscr{P}_{n,k}^{\beta}(f(t);x)$ to the test function $f(x)=x^{2}-x+1$ are given in the Figure 4.1, for $\beta=5$, and $n=\{2,10,50\}$. Figure 4.2 shows the convergence of the operators $\mathscr{P}_{n,k}^{\beta}(f(t);x)$ for $\beta=2$, and $n=\{2,10,50\}$. And Figure-3 represents the convergence of the operators $\mathscr{P}_{n,k}^{\beta}(f(t);x)$ for $\beta=1$, and $n=\{2,10,50\}$. From the graphical representation we easily seen that the operators converges fast when β decreases.

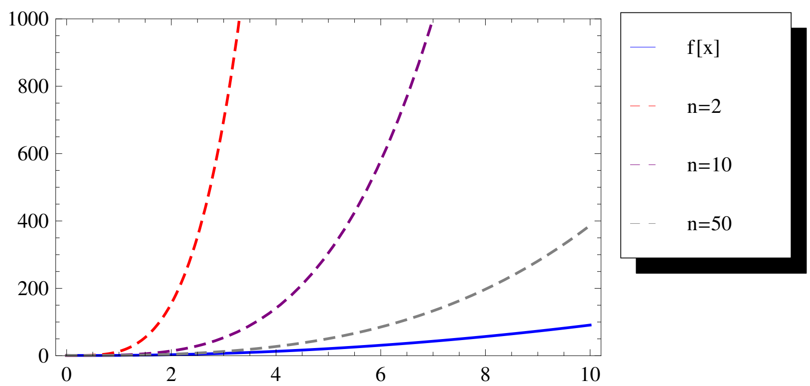

Figure 4.1: Convergence of the operators $\mathscr{P}_{n,k}^{\beta}(f(t);x)$ for $\beta=5$

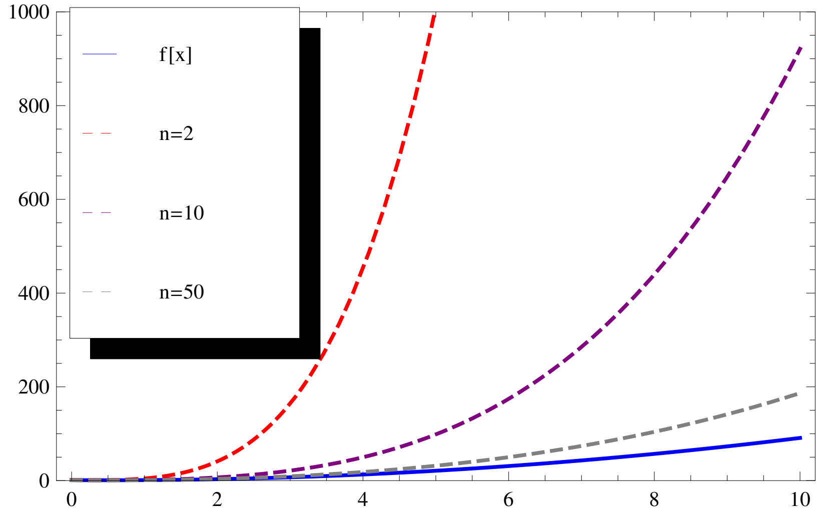

Figure 4.2: Convergence of the operators $\mathscr{P}_{n,k}^{\beta}(f(t);x)$ for $\beta = 2$

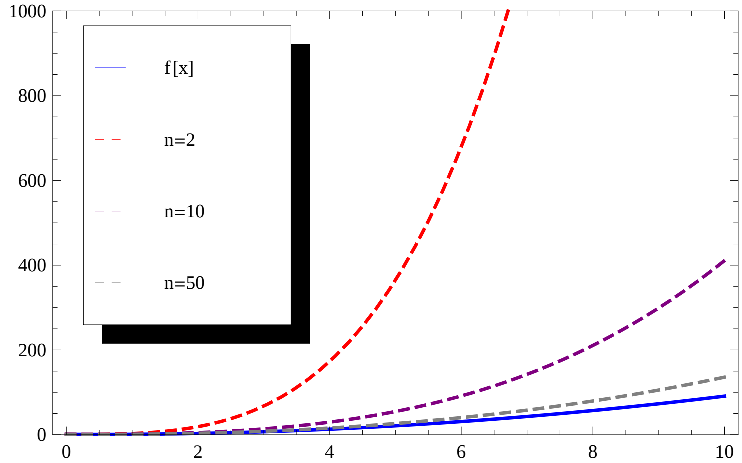

Figure 4.3: Faster convergence of the operators $\mathscr{P}_{n,k}^{\beta}(f(t);x)$ for $\beta = 1$

Chapter 5

Bézier variant of Phillips-type generalised positive linear operators

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This chapter introduces a novel Bézier variant within the family of Phillips-type generalized positive linear operators. The moments of these operators are derived to enhance understanding of their fundamental properties. The chapter further explores convergence properties in Lipschitz-type spaces, with particular focus on the Ditzian-Totik modulus of smoothness. Finally, it provides a rigorous analysis of the convergence rate for functions whose derivatives are of bounded variation, contributing valuable insights to the field of approximation theory.

5.1 Introduction

Over the course of the last thirty years, numerous researchers have introduced and examined the approximation properties of the summation-integral of various operators. In 2019, Gupta [57] presented a comprehensive collection of operators, which encompass various types of Durrmeyer operators, hybrid operators, and specific discrete operators that were examined as exceptional instances. The operators are defined for $\psi \in C_B[0,\infty)$

$$\mathscr{H}_{n,r}^{(\mu,\nu)}(\psi;x) = \sum_{r=1}^{\infty} p_{n,r}^{(\mu)}(x) \int_0^{\infty} p_{n,r}^{\nu+1,\rho}(\upsilon)\psi(\upsilon)d\upsilon + p_{n,0}^{(\mu)}\psi(0), \tag{5.1}$$

where,

$$p_{n,r-1}^{(\nu+1,\rho)}(\upsilon) = \frac{n}{\nu B(r\rho,\nu\rho+1)} \frac{(\frac{n\upsilon}{\nu})^{r\rho-1}}{(1+\frac{n\upsilon}{\nu})^{\nu\rho+r\rho+1}},$$

$p_{n,r}^{(\mu)}(x) = \frac{(\mu)_r}{r!}\frac{(\frac{nx}{\mu})^r}{(1+\frac{nx}{\mu})^{\mu+r}}$, $(\mu)_r = \mu(\mu+1)...(\mu+r-1)$ and $B(l,m)$ is a Beta function.

For $x \geq 0$ operators (5.1) can alternatively be written as

$$\mathscr{H}_{n,r}^{(\mu,\nu)}(\psi;x) = \sum_{r=1}^{\infty} p_{n,r}^{(\mu)}(x)\mathscr{G}_{n,r}^{\nu,\rho}(\psi),$$

where,

$$\mathscr{G}_{n,r}^{\nu,\rho}(\psi) = \begin{cases} \int_0^\infty p_{n,r-1}^{\nu+1,\rho}(\upsilon)\psi(\upsilon)d\upsilon, & 1 \leq r < \infty \\ \psi(0), & r = 0. \end{cases}$$

Here, we have some of the following special case of operators (5.1)

1. If $\mu = \nu \to \infty, \rho = 1$, the operators (5.1) reduces to Phillips type operators [83].
2. If $\mu = \nu = n, \rho = 1$, we obtain Baskakov-Durrmeyer type operators(see [38],[17]).
3. If $\mu \neq \nu$ and $\mu = n, \nu \to \infty, \rho = 1$, the operators (5.1) changes into Baskakov-Szász type operators considered in [25].
4. If $\mu \neq \nu$ and $\mu \to \infty, \nu = n, \rho = 1$, we receive Szász-Beta type operators proposed in [56].
5. If $\mu \neq \nu$ and $\mu = nx, \nu = n, \rho = 1$, we get Lupaş-Beta type operators introduced in [58].
6. If $\mu \neq \nu$ and $\mu = nx, \nu \to \infty, \rho = 1$, we have Lupaş-Szász type operators defined in [45].
7. If $\mu = \nu = -n$ and $\rho = 1$, we realize genuine Bernstein-Durrmeyer operators, proposed by Chen [26] and also by Goldman-Sharma [44].
8. If $\mu = \nu = n, \rho > 0$, the operators (5.1) follow the link operators due to Heilmann and Rasa [62] for $c = 1$.

Recently, Neha et al. [79] examined the Bézier variant of Păltănea operators associated with the inverse Pòlya-Eggenberger distribution. They also assessed the approximation properties of these operators, focusing on the first and second-order moduli of smoothness. Additionally, several studies (cf. [31, 35, 77, 84]) have explored the generalized form of positive linear operators.

Inspired by the aforementioned notable research, we examine the Bézier variation of the set of operators (5.1) in the following manner:

$$\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi;x) = \sum_{r=1}^{\infty} Q_{n,k}^{(\Theta)}(x)\int_0^\infty p_{n,r-1}^{\nu,\rho}(\upsilon)\psi(\upsilon)d\upsilon + Q_{n,0}^{(\Theta)}(x)\psi(0), \tag{5.2}$$

where,

$$Q_{n,r}^{(\Theta)}(x)=[J_{n,r}(x)]^{\Theta}-[J_{n,r+1}(x)]^{\Theta}\text{ and }J_{n,r}(x)=\sum_{j=r}^{\infty}p_{n,j}^{(\mu)}(x). \tag{5.3}$$

Alternatively, the operators (5.2) could be written as

$$\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi;x)=\int_0^{\infty}\mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)\psi(\upsilon)d\upsilon, \tag{5.4}$$

where,

$$\mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)=\sum_{r=1}^{\infty}Q_{n,r}^{(\Theta)}(x)p_{n,r-1}^{\nu,\rho}(\upsilon)+Q_{n,0}^{(\Theta)}(x)\psi(0).$$

5.2 Basic Properties

In this section, we discuss some useful lemmas and results.

Remark 5.2.1. *For $x\geq 0, m\in\mathbb{N}\cup\{0\}$ and $e_m(x)=x^m$, $m=0,1,2\ldots$, we have*

$$\begin{aligned}\mathscr{G}_{n,r}^{\nu,\rho}(e_m) &= \int_0^{\infty}p_{n,r-1}^{\nu+1,\rho}(\upsilon)\upsilon^m d\upsilon\\ &= \int_0^{\infty}\frac{n}{\nu.B(r\rho,\nu\rho+1)}\frac{(\frac{n\upsilon}{\nu})^{r\rho-1}}{(1+\frac{n\upsilon}{\nu})^{\nu\rho+r\rho+1}}\upsilon^m d\upsilon\\ &= \frac{\Gamma(\nu\rho-m+1)\Gamma(r\rho+m)}{\Gamma(\nu\rho+1)\Gamma(r\rho)}\left(\frac{\nu}{n}\right)^m.\end{aligned}$$

Remark 5.2.2. *For $\Theta\geq 1$, we have*

$$\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(e_0;x) = \sum_{r=0}^{\infty}Q_{n,r}^{(\Theta)}(x)=[J_{n,0}(x)]^{\Theta}=\left[\sum_{j=0}^{\infty}p_{n,j}^{(\mu)}(x)\right]^{\Theta}=1.$$

Lemma 5.2.3. *Using the Remark 5.2.1, the moments of the operators $\mathscr{H}_{n,r}^{(\mu,\nu)}$ (5.1) may be written as*

$$\begin{aligned}\mathscr{H}_{n,r}^{(\mu,\nu)}(e_0;x) &= 1,\\ \mathscr{H}_{n,r}^{(\mu,\nu)}(e_1;x) &= x,\\ \mathscr{H}_{n,r}^{(\mu,\nu)}(e_2;x) &= \frac{\nu}{\mu n(\nu\rho-1)}\left[x^2(1+\mu)n\rho+x(1+\rho)\mu\right],\\ \mathscr{H}_{n,r}^{(\mu,\nu)}(e_3;x) &= \frac{\nu^2}{\mu^2n^2(\nu\rho-1)(\nu\rho-2)}\left[x^3(2+3\mu+\mu^2)\rho^2n^2\right.\\ &\quad \left.+x^2(3+3\mu+3\rho+3\mu\rho)n\rho\mu+x(2+3\rho+\rho^2)\mu^2\right],\\ \mathscr{H}_{n,r}^{(\mu,\nu)}(e_4;x) &= \frac{\nu^3}{\mu^3n^3(\nu\rho-1)(\nu\rho-2)(\nu\rho-3)}\left[x^4(6+11\mu+6\mu^2+\mu^3)\rho^3n^3\right.\\ &\quad +x^3(12+18\mu+6\mu^2+12\rho+18\mu\rho+6\mu^2\rho)\rho^2n^2\mu\end{aligned}$$

$$+x^2(11+11\mu+18\rho+18\mu\rho+7\rho^2+7\mu\rho^2)\rho n\mu^2$$
$$+x(6+11\rho+6\rho^2+\rho^3)\mu^3\big].$$

Lemma 5.2.4. *Using Lemma 5.2.3, we get*

$$\begin{aligned}
\mathscr{H}_{n,\mu}^{(\nu,\rho)}((\upsilon-x)^2;x) &= \frac{\nu}{\mu n(\nu\rho-1)}\left[x^2(1+\mu)n\rho+x(1+\rho)\mu-x^2\right]\\
&= \frac{\mu+\nu\rho}{\nu\rho-1}x^2+\frac{\nu\rho+\nu}{(\nu\rho-1)n}x\\
&\leq x\left(\frac{\mu+\nu\rho}{\nu\rho-1}x+1\right)\\
&\leq x(1+\delta x)=\Phi^2(x).
\end{aligned}$$

where $\delta=\frac{\mu+\nu\rho}{\nu\rho-1}$.
Note: For $\nu>1, \quad \nu\rho+\nu>\nu\rho-1 \implies \frac{\nu\rho+\nu}{\nu\rho-1}>1.$

Lemma 5.2.5. *For real-valued continuous and bounded function* $\psi\in C_B[0,\infty)$ *and the operators* $\mathscr{H}_{n,\mu}^{(\nu,\rho)}$*, we have*

$$|\mathscr{H}_{n,\mu}^{(\nu,\rho)}(\psi;x)|\leq\|\psi\|,$$

where the norm of the function on the positive half real line is given by $\|\psi\|=\sup_{x\in[0,\infty)}|\psi(x)|$.

Lemma 5.2.6. *Let* $\psi\in C_B[0,\infty)$*, then for any* $x\in[0,\infty)$

$$\left|\tilde{\mathscr{H}}_{n,k}^{(\mu,\nu,\Theta)}(\psi)\right|\leq\Theta\|\psi\|.$$

Proof. Applying the well-known property $|a^\lambda-b^\lambda|\leq\lambda|a-b|$, with $0\leq a,\ b\leq 1,\ \lambda\geq 1$, and by using (5.3), we get

$$0<[J_{n,r}(x)]^\Theta-[J_{n,r+1}(x)]^\Theta \quad\leq\quad \Theta\left(J_{n,r}(x)-J_{n,r+1}(x)\right)\leq\Theta\left[p_{n,r}^{(\mu)}(x)\right].$$

Now, by the definition of Bézier variant operators (5.2) and Lemma 5.2.3, we obtain

$$\left|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi)\right|\leq\Theta\|\mathscr{H}_{n,r}^{(\mu,\nu)}(\psi)\|\leq\Theta\|\psi\|.$$

□

5.3 Convergence Results

Theorem 5.3.1. *Let $\psi \in Lip^*_{\mathscr{K}}(m)$ and $m \in (0,1]$. Then for all $x \in (0,\infty)$, we have*

$$|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi(\upsilon);x) - f(x)| \leq \Theta M \left(\frac{\mathscr{H}_{n,\mu}^{(\nu,\rho)}((\upsilon-x)^2;x)}{x} \right)^{\frac{m}{2}}.$$

Proof. From the remark 5.2.2, we have

$$\begin{aligned} |\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi(\upsilon);x) - \psi(x)| &\leq \tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(|\psi(\upsilon)-\psi(x)|;x) \\ &\leq \Theta \mathscr{H}_{n,r}^{(\mu,\nu)}(|\psi(\upsilon)-\psi(x)|;x) \\ &\leq \Theta M \mathscr{H}_{n,r}^{(\mu,\nu)}\left(\frac{|\upsilon-x|^m}{(\upsilon+x)^{\frac{m}{2}}};x \right) \\ &\leq \frac{\Theta M}{x^{\frac{m}{2}}} \mathscr{H}_{n,r}^{(\mu,\nu)}(|\upsilon-x|^m;x). \end{aligned} \tag{5.5}$$

Taking $p = \frac{2}{m}$, $q = \frac{2}{2-m}$ and applying Hölder's inequality, we obtain

$$\begin{aligned} \mathscr{H}_{n,r}^{(\mu,\nu)}(\psi)(|\upsilon-x|^m;x) &\leq \left\{ \mathscr{H}_{n,r}^{(\mu,\nu)}(|\upsilon-x|^2;x) \right\}^{\frac{m}{2}} . \left\{ \mathscr{H}_{n,r}^{(\mu,\nu)}(1^{\frac{2}{2-m}};x) \right\}^{\frac{2-m}{2}}, \\ &\leq \left\{ \mathscr{H}_{n,r}^{(\mu,\nu)}(|\upsilon-x|^2;x) \right\}^{\frac{m}{2}}. \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6), we get

$$|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi(\upsilon);x) - \psi(x)| \leq \Theta M \left(\frac{\mathscr{H}_{n,r}^{(\mu,\nu)}((\upsilon-x)^2;x)}{x} \right)^{\frac{m}{2}}.$$

□

Theorem 5.3.2 (Ditzian-Totik modulus of smoothness)**.** *For $\psi \in C_B[0,\infty)$ and $\lambda \in [0,1]$ then, we have*

$$\left| \tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi;x) - \psi(x) \right| \leq \mathscr{C} \omega_{\Phi^\lambda}\left(\psi; \Phi^{1-\lambda}(x) \sqrt{\frac{\Theta}{n}} \right).$$

Proof. For $h \in W_\lambda$, we consider

$$h(\upsilon) = h(x) + \int_x^\upsilon h'(u)du.$$

Applying $\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}$ and Hölder's inequality then, we have

$$\begin{aligned} \left| \tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(h(\upsilon);x) - h(x) \right| &\leq \tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left(\int_x^\upsilon |h'|\, du;x \right) \\ &\leq \|\Phi^\lambda h'\| \tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left(\left| \int_x^\upsilon \frac{du}{\Phi^\lambda(u)} \right|;x \right) \end{aligned}$$

$$\leq \|\Phi^{\lambda}h'\|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left(|\upsilon-x|^{1-\lambda}\left|\int_x^{\upsilon}\frac{du}{\Phi(u)}\right|^{\lambda};x\right)$$

$$. \tag{5.7}$$

Let $I=\left|\int_x^{\upsilon}\frac{du}{\Phi(u)}\right|$, now first we simplify expression I

$$\begin{aligned} I &\leq \left|\int_x^{\upsilon}\frac{du}{\sqrt{u}}\right|\left|\left(\frac{1}{\sqrt{1+\delta x}}+\frac{1}{\sqrt{1+\delta\upsilon}}\right)\right| \\ &\leq 2\left|\sqrt{\upsilon}-\sqrt{x}\right|\left(\frac{1}{\sqrt{1+\delta x}}+\frac{1}{\sqrt{1+\delta\upsilon}}\right) \\ &\leq 2\frac{|\upsilon-x|}{\sqrt{\upsilon}+\sqrt{x}}\left(\frac{1}{\sqrt{1+\delta x}}+\frac{1}{\sqrt{1+\delta\upsilon}}\right) \\ &\leq 2\frac{|\upsilon-x|}{\sqrt{x}}\left(\frac{1}{\sqrt{1+\delta x}}+\frac{1}{\sqrt{1+\delta\upsilon}}\right). \end{aligned} \tag{5.8}$$

Now, we use the inequality $|p+q|^{\lambda}\leq|p|^{\lambda}+|q|^{\lambda}, 0\leq\lambda\leq1$ then from (5.8), we get

$$\left|\int_x^{\upsilon}\frac{du}{\Phi(u)}\right|^{\lambda}\leq 2^{\lambda}\frac{|\upsilon-x|^{\lambda}}{x^{\frac{\lambda}{2}}}\left(\frac{1}{(1+\delta x)^{\frac{\lambda}{2}}}+\frac{1}{(1+\delta\upsilon)^{\frac{\lambda}{2}}}\right). \tag{5.9}$$

From (5.7) and (5.9) and using Cauchy inequality, we get

$$\begin{aligned} &\left|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(h(\upsilon);x)-h(x)\right| \\ &\leq \frac{2^{\lambda}\|\Phi^{\lambda}h'\|}{x^{\frac{\lambda}{2}}}\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left(|\upsilon-x|\left(\frac{1}{(1+\delta x)^{\frac{\lambda}{2}}}+\frac{1}{(1+\delta\upsilon)^{\frac{\lambda}{2}}}\right);x\right) \\ &\qquad= \frac{2^{\lambda}\|\Phi^{\lambda}h'\|}{x^{\frac{\lambda}{2}}}\left(\frac{1}{(1+\delta x)^{\frac{\lambda}{2}}}\left(\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left((\upsilon-x)^2;x\right)\right)^{\frac{1}{2}}\right. \\ &\left.+\left(\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left((\upsilon-x)^2;x\right)\right)^{\frac{1}{2}}.\left(\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left((1+\delta\upsilon)^{-\lambda};x\right)\right)^{\frac{1}{2}}\right). \end{aligned} \tag{5.10}$$

From Lemma 5.2.4, we may write

$$\left(\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left((\upsilon-x)^2;x\right)\right)^{\frac{1}{2}}\leq\sqrt{\frac{\Theta}{n}}\Phi(x), \tag{5.11}$$

For $x\in[0,\infty)$, $\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left((1+\delta\upsilon)^{-\lambda};x\right)$ approaches to $(1+\delta x)^{-\lambda}$ when n approaches to ∞ . Thus for any $\varepsilon>0$, we find $n_0\in\mathbb{N}$ such that

$$\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left((1+\delta\upsilon)^{-\lambda};x\right)\leq(1+\delta x)^{-\lambda}+\varepsilon, \text{ for all } n\geq n_0.$$

By assuming $\varepsilon=(1+\delta x)^{-\lambda}$, above inequality reduces to the following

$$\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}\left((1+\delta\upsilon)^{-\lambda};x\right)\leq 2(1+\delta x)^{-\lambda}, \text{ for all } n\geq n_0. \tag{5.12}$$

From (5.11) and (5.12), we have

$$\begin{aligned}\left|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(h(\upsilon);x)-h(x)\right| &\leq 2^{\lambda}\|\Phi^{\lambda}h'\|\sqrt{\frac{\Theta}{n}}\Phi(x)\left(\Phi^{-\lambda}(x)\right.\\ &\quad\left.+\sqrt{2}x^{-\frac{\lambda}{2}}(1+\delta x)^{-\frac{\lambda}{2}}\right)\\ &\leq 2^{\lambda}(1+\sqrt{2})\|\Phi^{\lambda}h'\|\sqrt{\frac{\Theta}{n}}\Phi^{1-\lambda}(x). \qquad (5.13)\end{aligned}$$

We may write

$$\begin{aligned}\left|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(f(\upsilon);x)-f(x)\right| &\leq \left|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(f(\upsilon)-h(\upsilon);x)\right|\\ &\quad+\left|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(h(\upsilon);x)-h(x)\right|+|h(x)-f(x)|\\ &\leq 2\|f-h\|+\left|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(h(\upsilon);x)-h(x)\right|. \qquad (5.14)\end{aligned}$$

From (5.13) and (5.14) and for very large n, we obtain

$$\begin{aligned}\left|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(f(\upsilon);x)-f(x)\right| &\leq 2\|f-g\|+2^{\lambda}(1+\sqrt{2})\|\Phi^{\lambda}h'\|\sqrt{\frac{\Theta}{n}}\Phi^{1-\lambda}(x)\\ &\leq C_1\{\|f-h\|+\Phi^{1-\lambda}(x)\sqrt{\frac{\Theta}{n}}\|\Phi^{\lambda}h'\|\}\\ &\leq \mathscr{C}K_{\Phi^{\lambda}}\left(f;\Phi^{1-\lambda}(x)\sqrt{\frac{\Theta}{n}}\right), \qquad (5.15)\end{aligned}$$

where $C_1=\max\{2,2^{\lambda}(1+\sqrt{2})\}$ and $\mathscr{C}=2C_1$. From (5.13) and (5.15), we follow the required result.

□

In the following theorem, we estimate the convergence rate of the operators (5.2) within the function class $DBV[0,\infty)$. This class consists of all absolutely continuous functions ψ defined on the interval $[0,\infty)$, which have bounded derivatives on the same interval.

Lemma 5.3.3. *For $x\in[0,\infty)$ and very large n, we have*

(i) Take $0\leq y<x$, therefore

$$\xi_n(x,y)=\int_0^y \mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)d\upsilon\leq\frac{\Theta x(1+\delta x)}{n(x-y)^2}.$$

(ii) If $x<z<\infty$, then

$$1-\xi_n(x,z)=\int_z^{\infty}\mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)d\upsilon\leq\frac{\Theta x(1+\delta x)}{n(z-x)^2}.$$

Proof. Using Lemma 5.2.3, 5.2.4 and the alternative form (5.4) of the operators (5.2), for sufficiently large n and $0 \leq y < x$, we have

$$\begin{aligned}
\xi_n(x,y) &= \int_0^y \mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)d\upsilon \\
&\leq \int_0^y \frac{(x-\upsilon)^2}{(x-y)^2}\mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)d\upsilon \\
&\leq \frac{1}{(x-y)^2}\mathscr{H}_{n,\mu}^{(\nu,\rho)}((\upsilon-x)^2;x) \\
&\leq \frac{\Theta x(1+\delta x)}{n(x-y)^2}.
\end{aligned}$$

In the similar way, we can prove the second part of the Lemma. □

Theorem 5.3.4 (Bounded variation)**.** *Let $\psi \in DBV[0,\infty)$ for all $x \in [0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}
\left|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi;x)-\psi(x)\right| &\leq \frac{\sqrt{\Theta}}{1+\Theta}\left|(\psi'(x^+)+\Theta\psi'(x^-))\right|\frac{\Phi(x)}{\sqrt{n}} \\
&+\frac{\Theta^{\frac{3}{2}}}{\Theta+1}\left|\psi'(x^+)-\psi'(x^-)\right|\frac{\Phi(x)}{\sqrt{n}} \\
&+\frac{\Theta\Phi(x)}{nx}\sum_{r=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{r}}^{x}(\psi'_x)+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(\psi'_x) \\
&+\frac{\Theta\Phi(x)}{nx}\sum_{r=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{r}}(\psi'_x)+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(\psi'_x),
\end{aligned}$$

where the auxiliary function ψ'_x is defined as:

$$\psi'_x(\upsilon)=\begin{cases}\psi(\upsilon)-\psi(x^-), & 0\leq\upsilon<x\\ 0, & \upsilon=x\\ \psi(\upsilon)-\psi(x^+), & x<\upsilon<\infty.\end{cases}$$

Proof. From Remark 5.2.2, and using (5.4) second form of the operators (5.2) for every $x \in [0,\infty)$, we have

$$\begin{aligned}
\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi(\upsilon);x)-\psi(x) &= \int_0^\infty \mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)(\psi(\upsilon)-\psi(x))d\upsilon \\
&= \int_0^\infty \mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)\left(\int_x^\upsilon \psi'(u)du\right)d\upsilon. \qquad (5.16)
\end{aligned}$$

For each function $\psi \in DBV[0,\infty)$, we can write

$$\psi'(u) = \psi'_x(u)+\frac{1}{\Theta+1}(\psi'(x^+)+\Theta\psi'(x^-))+\frac{1}{2}(\psi'(x^+)-\psi'(x^-))$$

$$\times\left(sgn(u-x)+\frac{\Theta-1}{\Theta+1}\right)+\delta_x(u)(\psi'(u)-(\psi'(x^+)+\psi'(x^-)),\tag{5.17}$$

where

$$\delta_x(u)=\begin{cases}1, & u=x\\ 0, & u\neq x.\end{cases}$$

It can be easily seen that

$$\int_0^\infty\left(\int_x^\upsilon\left(\psi'(u)-\frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\right)\delta_x(u)du\right)\mathscr{M}_{n,k}^{(\nu,\rho,\Theta)}(x,\upsilon)d\upsilon=0.\tag{5.18}$$

Using the operators (5.4) and (5.17), we have

$$\begin{aligned}I_1 &= \int_0^\infty\mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)\int_x^\upsilon\frac{1}{\Theta+1}(\psi'(x^+)+\Theta\psi'(x^-))dud\upsilon\\ &= \frac{1}{\Theta+1}\left|(\psi'(x^+)+\Theta\psi'(x^-))\right|\int_0^\infty\mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)\left|\upsilon-x\right|d\upsilon\\ &\le \frac{1}{\Theta+1}(\psi'(x^+)+\Theta\psi'(x^-))\left(\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}((\upsilon-x)^2;x)\right)^{\frac{1}{2}}\\ &\le \frac{\sqrt{\Theta}}{1+\Theta}\left|(\psi'(x^+)+\Theta\psi'(x^-))\right|\frac{\Phi(x)}{\sqrt{n}}.\end{aligned}\tag{5.19}$$

And

$$\begin{aligned}I_2 &= \int_0^\infty\mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)\int_x^\upsilon\frac{1}{2}\left(\psi'(x^+)-\psi'(x^-)\right)\left(sgn(u-x)+\frac{\Theta-1}{\Theta+1}\right)dud\upsilon\\ &\le \frac{\Theta}{\Theta+1}\left|\psi'(x^+)-\psi'(x^-)\right|\int_0^\infty\mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)\left|\upsilon-x\right|d\upsilon\\ &\le \frac{\Theta}{\Theta+1}\left|\psi'(x^+)-\psi'(x^-)\right|\left(\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}((\upsilon-x)^2;x)\right)^{\frac{1}{2}}\\ &\le \frac{\Theta^{\frac{3}{2}}}{\Theta+1}\left|\psi'(x^+)-\psi'(x^-)\right|\frac{\Phi(x)}{\sqrt{n}}.\end{aligned}\tag{5.20}$$

From the relations (5.16 - 5.20), we get the following estimate

$$\begin{aligned}\left|\tilde{\mathscr{H}}_{n,r}^{(\mu,\nu,\Theta)}(\psi;x)-\psi(x)\right| &\le \frac{\sqrt{\Theta}}{1+\Theta}\left|(\psi'(x^+)+\Theta\psi'(x^-))\right|\frac{\Phi(x)}{\sqrt{n}}\\ &\quad+\frac{\Theta^{\frac{3}{2}}}{\Theta+1}\left|\psi'(x^+)-\psi'(x^-)\right|\frac{\Phi(x)}{\sqrt{n}}\\ &\quad+A_{n,r}^{(\mu,\nu,\Theta)}(\psi'_x;x)+B_{n,r}^{(\mu,\nu,\Theta)}(\psi'_x;x),\end{aligned}\tag{5.21}$$

where,

$$A_{n,r}^{(\mu,\nu,\Theta)}(\psi'_x;x) \quad = \quad \int_0^x \mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)\left(\int_x^\upsilon \psi'(u)du\right)d\upsilon,$$

and,

$$B_{n,r}^{(\mu,\nu,\Theta)}(\psi'_x;x) \quad = \quad \int_x^\infty \mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)\left(\int_x^\upsilon \psi'(u)du\right)d\upsilon.$$

The terms $A_{n,r}^{(\mu,\nu,\Theta)}(\psi'_x;x)$ and $B_{n,r}^{(\mu,\nu,\Theta)}(\psi'_x;x)$ are to be estimated for a complete proof of the theorem. Using integration by parts and applying the Lemma 5.3.3 with $y = x - \frac{x}{\sqrt{n}}$, it follows

$$\begin{aligned}
\left|A_{n,r}^{(\mu,\nu,\Theta)}(\psi'_x;x)\right| &= \left|\int_0^x \left(\int_x^\upsilon \psi'(u)du\right)d_\upsilon \xi_{n,\Theta}(x,\upsilon)d\upsilon\right| \\
&= \left|\int_0^x \xi_{n,\Theta}(x,\upsilon)\psi'_x(\upsilon)d\upsilon\right| \\
&\leq \int_0^y \left|\psi'_x(\upsilon)\right||\xi_{n,\Theta}(x,\upsilon)|d\upsilon + \int_y^x \left|\psi'_x(\upsilon)\right||\xi_{n,\Theta}(x,\upsilon)|d\upsilon \\
&\leq \frac{\Theta\Phi(x)}{n}\int_0^y \bigvee_\upsilon^x(\psi'_x)(x-\upsilon)^{-2}d\upsilon + \int_y^x \bigvee_\upsilon^x(\psi'_x)d\upsilon \\
&\leq \frac{\Theta\Phi(x)}{n}\int_0^y \bigvee_\upsilon^x(\psi'_x)(x-\upsilon)^{-2}d\upsilon + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(\psi'_x). \qquad (5.22)
\end{aligned}$$

Again, using Lemma 5.3.3 and taking $u = \frac{x}{x-\upsilon}$, we get

$$\begin{aligned}
\frac{\Theta\Phi(x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_\upsilon^x(\psi'_x)(x-\upsilon)^{-2}d\upsilon &= \frac{\Theta\Phi(x)}{nx}\int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x(\psi'_x)du, \\
&\leq \frac{\Theta\Phi(x)}{nx}\sum_{r=1}^{[\sqrt{n}]}\int_r^{r+1} \bigvee_{x-\frac{x}{u}}^x(\psi'_x)du, \\
&\leq \frac{\Theta\Phi(x)}{nx}\sum_{r=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{r}}^x \psi'_x). \qquad (5.23)
\end{aligned}$$

On combining (5.22) and (5.23), we get following

$$\left|A_{n,r}^{(\mu,\nu,\Theta)}(\psi'_x;x)\right| \quad \leq \quad \frac{\Theta\Phi(x)}{nx}\sum_{r=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{r}}^x(\psi'_x) + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(\psi'_x). \qquad (5.24)$$

Now, using the integration by parts and applying the Lemma 5.3.3 with $z = x + \frac{x}{\sqrt{n}}$, we get

$$\begin{aligned}
\left|B_{n,r}^{(\mu,\nu,\Theta)}(\psi'_x;x)\right| &= \left|\int_x^\infty \mathscr{M}_{n,r}^{(\nu,\rho,\Theta)}(x,\upsilon)\left(\int_x^\upsilon \psi'_x(u)du\right)d\upsilon\right| \\
&= \left|\int_x^z \left(\int_x^\upsilon \psi'_x(u)du\right)d_\upsilon(1-\xi_{n,\Theta}(x,\upsilon))d\upsilon\right.
\end{aligned}$$

$$
\begin{aligned}
& +\int_z^\infty \left(\int_x^\upsilon \psi'_x(u)du\right) d_\upsilon(1-\xi_{n,\Theta}(x,\upsilon))d\upsilon\Bigg| \\
= & \left|\left[\left(\int_x^\upsilon \psi'_x(u)du\right)(1-\xi_{n,\Theta}(x,\upsilon))\right]_x^z - \int_x^z \psi'_x(\upsilon)(1-\xi_{n,\Theta}(x,\upsilon))d\upsilon\right. \\
& +\int_z^\infty \left(\int_x^\upsilon \psi'(u)du\right) d_\upsilon(1-\xi_{n,\Theta}(x,\upsilon))d\upsilon\Bigg| \\
= & \left|\left(\int_x^z \psi'_x(u)du\right)(1-\xi_{n,\Theta}(x,z)) - \int_x^z \psi'_x(\upsilon)(1-\xi_{n,\Theta}(x,\upsilon))d\upsilon\right. \\
& +\left[\left(\int_x^\upsilon \psi'_x(u)du\right)(1-\xi_{n,\Theta}(x,\upsilon))\right]_z^\infty - \int_z^\infty \psi'_x(\upsilon)(1-\xi_{n,\Theta}(x,\upsilon))d\upsilon\Bigg| \\
= & \left|\int_x^z \psi'_x(\upsilon)(1-\xi_{n,\Theta}(x,\upsilon))d\upsilon + \int_z^\infty \psi'_x(\upsilon)(1-\xi_{n,\Theta}(x,\upsilon))d\upsilon\right| \\
< & \frac{\Theta\Phi(x)}{n}\int_x^\infty \bigvee_x^\upsilon(\psi'_x)(\upsilon-x)^{-2}d\upsilon + \int_x^z \bigvee_x^\upsilon(\psi'_x) \\
\leq & \frac{\Theta\Phi(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^\infty \bigvee_x^\upsilon(\psi'_x)(\upsilon-x)^{-2}d\upsilon + \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}(\psi'_x), \qquad (5.25)
\end{aligned}
$$

on substituting $u=\frac{x}{\upsilon-x}$, we get

$$
\begin{aligned}
\frac{\Theta\Phi(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^\infty \bigvee_x^\upsilon(\psi'_x)(\upsilon-x)^{-2}d\upsilon &= \frac{\Theta\Phi(x)}{nx}\int_0^{\sqrt{n}} \bigvee_x^{x+\frac{x}{u}}(\psi'_x)du \\
&\leq \frac{\Theta\Phi(x)}{nx}\sum_{r=1}^{[\sqrt{n}]}\int_k^{k+1} \bigvee_x^{x+\frac{x}{u}}(\psi'_x)du \\
&\leq \frac{\Theta\Phi(x)}{nx}\sum_{r=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{r}}(\psi'_x). \qquad (5.26)
\end{aligned}
$$

Using (5.25) and (5.26), we get the following

$$
\left|B_{n,r}^{(\mu,\nu,\Theta)}(\psi'_x;x)\right| \leq \frac{\Theta\Phi(x)}{nx}\sum_{r=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{r}}(\psi'_x) + \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}(\psi'_x). \qquad (5.27)
$$

The relations (5.21), (5.24) and (5.27), leads to the required result. □

Chapter 6

Convergence analysis of Durrmeyer type operators linked with Boas-Buck type polynomials

In 1956, Boas and Buck [22] studied a class of generating functions of polynomial sets. Bous-Buck polynomials are a class of special polynomials introduced in approximation theory and operational methods. The present chapter introduces the sequence of Baskakov-Durrmeyer-type operators linked with the generating functions of Boas-Buck-type polynomials. In the subsequent sections, after calculating the moments, including the limiting case of central moments for the constructed sequence of operators, we estimate the convergence rate using the modulus of continuity, the Ditzian-Totik modulus of smoothness, and some convergence results in Lipschitz-type space. Finally, we also estimate the convergence for the functions of bounded variations.

6.1 Introduction

Suppose that $\xi, \mathscr{S}, \mathscr{T}, \mathscr{U}$ and $\mathscr{V}$ are analytic functions defined on some disc $|z| < \rho$ with $\rho > 1$. Let the power series expansions of these analytic functions are

$$\xi(s) = \sum_{j=0}^{\infty} p_j s^j, \qquad \mathscr{S}(s) = \sum_{j=0}^{\infty} q_j s^j, \qquad \mathscr{T}(s) = \sum_{j=0}^{\infty} r_j s^{j+1},$$

$$\mathscr{U}(s) = \sum_{j=0}^{\infty} u_j s^{j+2}, \qquad \mathscr{V}(s) = \sum_{j=0}^{\infty} v_j s^{j+3},$$

where the coefficients of the above series representation meet with $p_j \neq 0, q_j \neq 0$ and $r_j \neq 0$. Now, according to R. Goyal [46] and E. D. Rainville [88], we have polynomials generating function $\Theta_j(x)s^j$ as:

$$\mathscr{S}(s).\xi\left(x^2\mathscr{T}(s) + x\mathscr{U}(s) + \mathscr{V}(s)\right) = \sum_{j=0}^{\infty} \Theta_j(x)s^j.$$

Using the polynomial generating function Θ_j, S. Verma and S. Sucu [103] described the following operators:

$$\mathscr{B}_n(\psi;x) = \frac{1}{\mathscr{S}(1).\xi\left(n^2x^2\mathscr{T}(1) + nx\mathscr{U}(1) + \mathscr{V}(1)\right)} \sum_{j=0}^{\infty} \Theta_j(nx)\psi\left(\frac{j}{n}\right), \tag{6.1}$$

with some restrictions for the positivity like $\xi(s) \geq 0,\ \forall\ s \in \mathbb{R}$, and $\Theta_j(x) \geq 0;\ \ j = 0, 1, 2\ldots$ for all $x \geq 0$ and $\mathscr{S}(1) > 0$. In the reference of convergence of the sequence (6.1) we assume that $\mathscr{T}'(1) = 0, \mathscr{T}''(1) = 0$ and $\mathscr{U}'(1) = 1$ and apply the Korovkin-type theorem.

Many authors have discussed the convergence of related operators (cf. [8, 22, 23, 42, 57, 100]). The remarkable work of [92, 103] and [91] allow us to define a sequence of Baskakov-Durrmeyer-type operators linked with the generating function of Boas-Buck-type polynomials under the same assumptions and restrictions.

For $\sigma > 0$, consider a class of functions $C_\sigma[0,\infty) = \{\psi \in C[0,\infty) : |\psi(s)| \leq M(1+s^\sigma)\}$, for some $M > 0$ with the norm $\|\psi\| = \sup\limits_{s\in[0,\infty)} \dfrac{|\psi(s)|}{1+s^\sigma}$.

Now, for $\psi \in C_\sigma[0,\infty)$, we construct

$$\begin{aligned}\tilde{\mathscr{B}}_n(\psi;x) = \frac{1}{\mathscr{S}(1).\xi\left(n^2x^2\mathscr{T}(1) + nx\mathscr{U}(1) + \mathscr{V}(1)\right)} &\sum_{j=0}^{\infty} \frac{\Theta_j(nx)}{B(j,n+1)} \\ &\times \int_0^{\infty} \frac{s^{j-1}}{(1+s)^{n+j+1}}\psi(s)ds,\end{aligned} \tag{6.2}$$

where B is a beta function, defined as $B(j,n+1) = \frac{\Gamma(j)\Gamma(n+1)}{\Gamma(j+n+1)}$.
Alternatively, the operators (6.2) may be written as:

$$\tilde{\mathscr{B}}_n(\psi;x) = \int_0^{\infty} \mathscr{H}_n(x,s)(\psi;x)\psi(s)ds, \tag{6.3}$$

where

$$\mathscr{H}_n(x,s) = \frac{1}{\mathscr{S}(1).\xi\left(n^2x^2\mathscr{T}(1) + nx\mathscr{U}(1) + \mathscr{V}(1)\right)} \sum_{j=0}^{\infty} \frac{\Theta_j(nx)}{B(j,n+1)} \frac{s^{j-1}}{(1+s)^{n+j+1}}.$$

6.2 Some results on the operators $\mathscr{B}_n$ and $\tilde{\mathscr{B}}_n$

We develop several general lemmas that will be relevant throughout this chapter before moving on to our main results. Additionally, we computed the moments and central moments of the proposed operators using Mathematica software.

Lemma 6.2.1. *From [92], for $x \in [0,\infty)$ and $p(x) = n^2x^2\mathscr{T}(1) + nx\mathscr{U}(1) + \mathscr{V}(1)$ important results related to the operators (6.1) are follows:*

$$\begin{aligned}
\mathscr{B}_n(1;x) &= 1, \\
\mathscr{B}_n(s;x) &= \frac{\xi'(p(x))}{\xi(p(x))}x + \frac{1}{n}\left[\frac{\mathscr{S}'(1)}{\mathscr{S}(1)} + \mathscr{V}'(1)\frac{\xi'(p(x))}{\xi(p(x))}\right], \\
\mathscr{B}_n(s^2;x) &= \frac{\xi''(p(x))}{\xi(p(x))}x^2 + \left[\frac{\xi'(p(x))}{\xi(p(x))}\left(\frac{2\mathscr{S}'(1)}{\mathscr{S}(1)} + \mathscr{U}''(1) + 1\right)\right. \\
&\quad \left. +2\mathscr{V}'(1)\frac{\xi''(p(x))}{\xi(p(x))}\right]\frac{x}{n} + \left[\frac{\mathscr{S}''(1) + \mathscr{S}'(1)}{\mathscr{S}(1)} + \left(\frac{2\mathscr{S}'(1)\mathscr{V}'(1)}{\mathscr{S}(1)}\right.\right. \\
&\quad \left.\left. +\mathscr{V}''(1) + \mathscr{V}'(1)\right)\frac{\xi'(p(x))}{\xi(p(x))} + \left(\mathscr{V}'(1)\right)^2\frac{\xi''(p(x))}{\xi(p(x))}\right]\frac{1}{n^2}.
\end{aligned}$$

Lemma 6.2.2. *For $x \in [0,\infty)$, and with the help of Lemma 6.2.1, we have*

$$\begin{aligned}
\tilde{\mathscr{B}}_n(1;x) &= 1, \\
\tilde{\mathscr{B}}_n(s;x) &= \frac{\xi'(p(x))}{\xi(p(x))}x + \frac{1}{n}\left(\frac{\mathscr{S}'(1)}{\mathscr{S}(1)} + \mathscr{V}'(1)\frac{\xi'(p(x))}{\xi(p(x))}\right), \\
\tilde{\mathscr{B}}_n(s^2;x) &= \frac{n}{n-1}\frac{\xi''(p(x))}{\xi(p(x))}x^2 + \frac{x}{n-1}\left[2\mathscr{V}'(1)\frac{\xi''(p(x))}{\xi(p(x))} + \left(2 + 2\frac{\mathscr{S}'(1)}{\mathscr{S}(1)}\right.\right. \\
&\quad \left.\left. +\mathscr{U}''(1)\right)\frac{\xi'(p(x))}{\xi(p(x))}\right] + \frac{1}{n(n-1)}\left[\left(\mathscr{V}''(1)\right)^2\frac{\xi''(p(x))}{\xi(p(x))}\right. \\
&\quad \left. +\left(2\frac{\mathscr{S}'(1)}{\mathscr{S}(1)}\mathscr{V}'(1) + 2\mathscr{V}'(1) + \mathscr{V}''(1)\right)\frac{\xi'(p(x))}{\xi(p(x))} + \frac{2\mathscr{S}'(1) + \mathscr{S}''(1)}{\mathscr{S}(1)}\right].
\end{aligned}$$

Proof. By performing simple calculations, we obtain the following result:

$$\tilde{\mathscr{B}}_n(s^2;x) = \frac{1}{n-1}\left[n\mathscr{B}_n(s^2;x) + \mathscr{B}_n(s;x)\right].$$

Subsequently, by executing further computations and applying Lemma 6.2.1, we arrive at the required result. □

Lemma 6.2.3. *Using Lemma 6.2.1 and Lemma 6.2.2, central moments of the operators (6.2) are given by,*

$$\begin{aligned}
\tilde{\mathscr{B}}_n((s-x);x) &= \left[\frac{\xi'(p(x))}{\xi(p(x))} - 1\right]x + \frac{1}{n}\left(\frac{\mathscr{S}'(1)}{\mathscr{S}(1)} + \mathscr{V}'(1)\frac{\xi'(p(x))}{\xi(p(x))}\right), \\
\tilde{\mathscr{B}}_n((s-x)^2;x) &= \left[\frac{n}{n-1}\frac{\xi''(p(x))}{\xi(p(x))} - 2\frac{\xi'(p(x))}{\xi(p(x))} + 1\right]x^2
\end{aligned}$$

$$+\left[\frac{2}{n-1}\mathscr{V}'(1)\frac{\xi''(p(x))}{\xi(p(x))}+\frac{1}{n-1}\left(2+\mathscr{U}''(1)\right.\right.$$
$$\left.\left.+\frac{2\mathscr{S}'(1)}{\mathscr{S}(1)}\right)\frac{\xi'(p(x))}{\xi(p(x))}-\frac{2}{n}\left(\mathscr{V}'(1)\frac{\xi'(p(x))}{\xi(p(x))}+\frac{\mathscr{S}'(1)}{\mathscr{S}(1)}\right)\right]x$$
$$+\frac{1}{n(n-1)}\left[\left(\mathscr{V}'(1)\right)^2\frac{\xi''(p(x))}{\xi(p(x))}+\left(2\mathscr{V}'(1)\frac{\mathscr{S}'(1)}{\mathscr{S}(1)}\right.\right.$$
$$\left.\left.+2\mathscr{V}'(1)+\mathscr{V}''(1)\right)\frac{\xi'(p(x))}{\xi(p(x))}+\frac{2\mathscr{S}'(1)+\mathscr{S}''(1)}{\mathscr{S}(1)}\right].$$

Proof. Simple computations and the Lemma 6.2.2, leads the required proof. □

Now, to study the convergence properties of the operators (6.2), we assume that $\lim_{t\to\infty}\frac{\xi'(t)}{\xi(t)}=1$ and $\lim_{t\to\infty}\frac{\xi''(t)}{\xi(t)}=1$ that are point wise valid on the analytic functions $\mathscr{S}(s),\mathscr{T}(s),\mathscr{U}(s)$ and $\mathscr{V}(s)$. Also, we assume the following considerations:

$$\lim_{n\to\infty}n\left[\frac{\xi'(p(x))}{\xi(p(x))}-1\right]=\ell_1(x),$$

$$\lim_{n\to\infty}n\left[\frac{n}{n-1}\frac{\xi''(p(x))}{\xi(p(x))}-n\frac{\xi'(p(x))}{\xi(p(x))}+1\right]=\ell_2(x).$$

In the next Lemma, we discuss the limiting cases of the operators (6.2), by applying the above assumptions on the Lemma 6.2.3.

Lemma 6.2.4. *For the operators (6.2), we may write the following*

$$\lim_{n\to\infty}n\tilde{\mathscr{B}}_n((s-x);x)=\ell_1(x)x+\frac{\mathscr{S}'(1)}{\mathscr{S}(1)}+\mathscr{V}'(1),$$

$$\begin{aligned}\lim_{n\to\infty}n\tilde{\mathscr{B}}_n\left((s-x)^2;x\right) &= \ell_2(x)x^2+x\left(2+\mathscr{U}''(1)\right)\\ &= \eta_1(x)(say).\end{aligned}$$

Moreover, for $n\in\mathbb{N}$ and some constant c, we have

$$\begin{aligned}\tilde{\mathscr{B}}_n\left((s-x)^2;x\right) &\le \frac{\ell_2(x)x^2+x(2+\mathscr{U}''(1))}{n}\\ &\le \frac{\ell_2(x)x^2+cx}{n}.\end{aligned}$$

Since the limiting value $\ell_2(x)$ exists finitely, therefore for all $x\in[0,\infty)$, there exists $\mathscr{M}>0$, such that $\ell_2(x)\le\mathscr{M}$, taking it into the account, we may write:

$$\tilde{\mathscr{B}}_n\left((s-x)^2;x\right)\le\frac{\mathscr{M}.\chi^2(x)}{n},$$

where $\chi^2 = x(x+1)$.

6.3 Approximation Results

Theorem 6.3.1. *For any compact subset S of* $[0,\infty)$ *and a continuous function* ψ *defined on* $[0,\infty)$*, the sequence* $\tilde{\mathscr{B}}_n(\psi;x)_{n\geq 1}$ *converges uniformly to the function* ψ *on S.*

Proof. By simple computations and using the Lemma 6.2.2, we see that, for $i = 0,1,2$, the sequence $\tilde{\mathscr{B}}_n\left(t^i;x\right)_{n\geq 1}$ converges uniformly on S to the function $e_i(x) = x^i$. Hence, by using Bohman-Korovkin's theorem [39], the required result follows. ☐

Theorem 6.3.2 (Point-wise convergence)**.** *Let* ψ *be a function in a Lipchitz class* $Lip^*_{\mathscr{K}}$ *and* $r \in (0,1]$*. Then,*

$$\left|\tilde{\mathscr{B}}_n(\psi;x) - \psi(x)\right| \leq \frac{\mathscr{K}}{x^{\frac{r}{2}}}\left(\tilde{\mathscr{B}}_n\left((s-x)^2;x\right)\right)^{\frac{r}{2}}.$$

Proof.

$$\left|\tilde{\mathscr{B}}_n(\psi;x) - \psi(x)\right| \leq \int_0^\infty \mathscr{H}_n(x,s)\left|\psi(s)-\psi(x)\right| ds,$$

on applying Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2-r}{2}$ and Lemma 6.2.3, we have

$$\begin{aligned}
\left|\tilde{\mathscr{B}}_n(\psi;x) - \psi(x)\right| &\leq \left(\int_0^\infty \mathscr{H}_n(x,s)\left(\left|\psi(s)-\psi(x)\right|^{\frac{2}{r}}\right) ds\right)^{\frac{r}{2}}\left(\int_0^\infty \mathscr{H}_n(x,s)ds\right)^{\frac{2-r}{2}} \\
&\leq \left(\int_0^\infty \mathscr{H}_n(x,s)\left(\left|\psi(s)-\psi(x)\right|^{\frac{2}{r}}\right) ds\right)^{\frac{r}{2}} \\
&\leq \mathscr{K}\left(\int_0^\infty \mathscr{H}_n(x,s)\left(\frac{(s-x)^2}{s+x}\right) ds\right)^{\frac{r}{2}} \\
&\leq \frac{\mathscr{K}}{x^{\frac{r}{2}}}\left(\tilde{\mathscr{B}}_n\left((s-x)^2;x\right)\right)^{\frac{r}{2}}.
\end{aligned}$$

☐

Theorem 6.3.3. *Let* $C_B[0,\infty)$ *be the class of all real-valued continuous and bounded functions and* $\psi \in C_B[0,\infty)$*. Then, for* $x \in [0,\infty]$ *and* $\delta > 0$*, we have*

$$\left|\tilde{\mathscr{B}}_n(\psi;x) - \psi(x)\right| \leq 2\omega\left(\psi;\sqrt{\tilde{\mathscr{B}}_n\left((s-x)^2;x\right)}\right),$$

where, the modulus of continuity $\omega(\psi;\delta)$ *is given by*

$$\omega(\psi;\delta) := \sup_{|x-y|<\delta}\ \sup_{x,y\in[0,\infty)} \left|\psi(x)-\psi(y)\right|.$$

Proof. For $\psi \in C_B[0,\infty)$, we have

$$\begin{aligned}
\left|\tilde{\mathscr{B}}_n(\psi;x)-\psi(x)\right| &= \left|\frac{1}{\mathscr{S}(1).\xi(p(x))}\sum_{j=0}^{\infty}\frac{\Theta_j(nx)}{B(j,n+1)}\right. \\
&\quad \left.\times\int_0^{\infty}\frac{s^{j-1}}{(1+s)^{n+j+1}}(\psi(s)-\psi(x))\,ds\right| \\
&\leq \frac{1}{\mathscr{S}(1).\xi(p(x))}\sum_{j=0}^{\infty}\frac{\Theta_j(nx)}{B(j,n+1)} \\
&\quad \times\int_0^{\infty}\frac{s^{j-1}}{(1+s)^{n+j+1}}|\psi(s)-\psi(x)|\,ds \\
&\leq \frac{1}{\mathscr{S}(1).\xi(p(x))}\sum_{j=0}^{\infty}\frac{\Theta_j(nx)}{B(j,n+1)} \\
&\quad \times\int_0^{\infty}\frac{s^{j-1}}{(1+s)^{n+j+1}}\left(1+\frac{1}{\delta}|s-x|\right)\omega(\psi;\delta)ds \\
&\leq \left(1+\frac{1}{\delta}\frac{1}{\mathscr{S}(1).\xi(p(x))}\sum_{j=0}^{\infty}\frac{\Theta_j(nx)}{B(j,n+1)}\right. \\
&\quad \left.\times\int_0^{\infty}\frac{s^{j-1}}{(1+s)^{n+j+1}}|s-x|\,ds\right)\omega(\psi;\delta), \qquad (6.4)
\end{aligned}$$

on the account of Cauchy-Schwarz inequality and Lemma 6.2.4, we get

$$\begin{aligned}
&\sum_{j=0}^{\infty}\frac{\Theta_j(nx)}{B(j,n+1)}\int_0^{\infty}\frac{s^{j-1}}{(1+s)^{n+j+1}}|s-x|\,ds \\
&\leq \left(\sum_{j=0}^{\infty}\frac{\Theta_j(nx)}{B(j,n+1)}\int_0^{\infty}\frac{s^{j-1}}{(1+s)^{n+j+1}}(s-x)^2\,ds\right)^{\frac{1}{2}} \\
&\quad \times\left(\sum_{j=0}^{\infty}\frac{\Theta_j(nx)}{B(j,n+1)}\int_0^{\infty}\frac{s^{j-1}}{(1+s)^{n+j+1}}ds\right)^{\frac{1}{2}}. \qquad (6.5)
\end{aligned}$$

From (6.4) and (6.5), we may write

$$\left|\tilde{\mathscr{B}}_n(\psi;x)-\psi(x)\right| \leq \left(1+\frac{1}{\delta}\left(\tilde{\mathscr{B}}_n((1;x)\right)^{\frac{1}{2}}\left(\tilde{\mathscr{B}}_n\left((s-x)^2;x\right)\right)^{\frac{1}{2}}\right)\omega(\psi;\delta),$$

considering $\delta=\left(\tilde{\mathscr{B}}_n\left((s-x)^2;x\right)\right)^{\frac{1}{2}}$ and using the above inequality, we get the required result. □

Theorem 6.3.4. *For $\psi \in C_B[0,\infty)$ then, we have*

$$|\mathscr{B}_n(\psi;x)-\psi(x)| \leq \mathscr{D}\omega_{\chi^s}\left(\psi;\chi^{1-\gamma}(x)\frac{1}{\sqrt{n}}\right).$$

Proof. For $\varphi \in W_\gamma$, first we write the following

$$\varphi(s)=\varphi(x)+\int_x^s\varphi'(s)ds.$$

Applying $\mathscr{B}_n(.;x)$ and using Hölder's inequality, we have

$$\begin{aligned} \left|\tilde{\mathscr{B}}_n(\varphi(s);x)-\varphi(x)\right| &\leq \tilde{\mathscr{B}}_n\left(\int_x^s \left|\varphi'(u)\right| du;x\right) \\ &\leq \|\psi^\gamma\varphi'\|\tilde{\mathscr{B}}_n\left(\left|\int_x^s \frac{du}{\chi^\gamma(u)}\right|;x\right) \\ &\leq \|\psi^\gamma\varphi'\|\tilde{\mathscr{B}}_n\left(|s-x|^{1-\gamma}\left|\int_x^s \frac{du}{\chi(u)}\right|^\gamma;x\right). \end{aligned} \tag{6.6}$$

Let $I=\left|\int_x^s \frac{du}{\chi(u)}\right|$, now first we simplify expression I,

$$\begin{aligned} \left|\int_x^s \frac{du}{\chi(u)}\right| &\leq \left|\int_x^s \frac{du}{\sqrt{u}}\right|\left(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+s}}\right) \\ &\leq 2\left|\sqrt{s}-\sqrt{x}\right|\left(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+s}}\right) \\ &\leq 2\frac{|s-x|}{\sqrt{s}+\sqrt{x}}\left(\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+s}}\right). \end{aligned} \tag{6.7}$$

Now, we use the inequality $|p+q|^\gamma \leq |p|^\gamma+|q|^\gamma, 0\leq\gamma\leq 1$, then from (6.7), we get

$$\left|\int_x^s \frac{du}{\chi(u)}\right|^\gamma \leq 2^\gamma \frac{|s-x|^\gamma}{x^{\frac{\gamma}{2}}}\left(\frac{1}{(1+x)^{\frac{\gamma}{2}}}+\frac{1}{(1+s)^{\frac{\gamma}{2}}}\right). \tag{6.8}$$

From (6.6) and (6.8) and using Cauchy inequality, we get

$$\begin{aligned} \left|\tilde{\mathscr{B}}_n(\varphi(s);x)-\varphi(x)\right| &\leq \frac{2^\gamma\|\chi^\gamma\varphi'\|}{x^{\frac{\gamma}{2}}}\tilde{\mathscr{B}}_n\left(|s-x|\left(\frac{1}{(1+x)^{\frac{\gamma}{2}}}+\frac{1}{(1+s)^{\frac{\gamma}{2}}}\right);x\right) \\ &= \frac{2^\gamma\|\chi^\gamma\varphi'\|}{x^{\frac{\gamma}{2}}}\left(\frac{1}{(1+x)^{\frac{\gamma}{2}}}\left(\tilde{\mathscr{B}}_n\left((s-x)^2;x\right)\right)^{\frac{1}{2}}\right. \\ &\quad \left.+\left(\tilde{\mathscr{B}}_n\left((s-x)^2;x\right)\right)^{\frac{1}{2}}.\left(\tilde{\mathscr{B}}_n\left((1+s)^{-\gamma};x\right)\right)^{\frac{1}{2}}\right). \end{aligned} \tag{6.9}$$

From Lemma 6.2.2, we may write

$$\left(\tilde{\mathscr{B}}_n\left((s-x)^2;x\right)\right)^{\frac{1}{2}} \leq \sqrt{\frac{\mathscr{M}}{n}}.\chi(x), \tag{6.10}$$

where $\chi(x)=\sqrt{x(1+x)}$.

For $x\in[0,\infty), \tilde{\mathscr{B}}_n\left((1+s)^{-\gamma};x\right)\to(1+x)^{-\gamma}$ as $n\to\infty$. Thus for $\varepsilon>0$, we find a number $n_0\in\mathbb{N}$ such that

$$\tilde{\mathscr{B}}_n\left((1+s)^{-\gamma};x\right) \leq (1+x)^{-\gamma}+\varepsilon, \text{ for all } n\geq n_0.$$

By choosing $\varepsilon=(1+x)^{-\gamma}$, we obtain

$$\tilde{\mathscr{B}}_n\left((1+s)^{-\gamma};x\right)\leq 2(1+x)^{-\gamma},\ \text{for all } n\geq n_0. \tag{6.11}$$

From (6.9)-(6.11), we have

$$\begin{aligned}\left|\tilde{\mathscr{B}}_n(\varphi(\lambda);x)-\varphi(x)\right| &\leq 2^{\gamma}\|\chi^{\gamma}\varphi'\|\left(\tilde{\mathscr{B}}_n\left((s-x)^2;x\right)\right)^{\frac{1}{2}}\left\{\chi^{-\gamma}(x)\right.\\ &\quad\left.+x^{-\frac{\gamma}{2}}\left(\tilde{\mathscr{B}}_n\left((1+s)^{-\gamma};x\right)\right)^{\frac{1}{2}}\right\}\\ &\leq 2^{\gamma}\|\chi^{\gamma}\varphi'\|\sqrt{\frac{\mathscr{M}}{n}}\chi(x)\left(\chi^{-\gamma}(x)+\sqrt{2}x^{-\frac{\gamma}{2}}(1+x)^{-\frac{\gamma}{2}}\right)\\ &\leq 2^{\gamma}(1+\sqrt{2})\|\chi^{\gamma}\varphi'\|\sqrt{\frac{\mathscr{M}}{n}}\chi^{1-\gamma}(x), \end{aligned}\tag{6.12}$$

now, we write

$$\begin{aligned}\left|\tilde{\mathscr{B}}_n(\psi(s);x)-\psi(x)\right| &\leq \left|\tilde{\mathscr{B}}_n\left(\psi(s)-\varphi(s);x\right)\right|\\ &\quad+\left|\tilde{\mathscr{B}}_n\left(\varphi(s);x\right)-\varphi(x)\right|+|\varphi(x)-\psi(x)|\\ &\leq 2\|\psi-\varphi\|+\left|\tilde{\mathscr{B}}_n\left(\varphi(s);x\right)-\varphi(x)\right|. \end{aligned}\tag{6.13}$$

From (6.12) and (6.13) and for sufficiently large n, we obtain

$$\begin{aligned}\left|\tilde{\mathscr{B}}_n\left(\psi(s);x\right)-\psi(x)\right| &\leq 2\|\psi-\varphi\|+2^{\gamma}\left(1+\sqrt{2}\right)\|\chi^{\gamma}\varphi'\|\sqrt{\frac{\mathscr{M}}{n}}\chi^{1-\gamma}(x)\\ &\leq \mathscr{D}\left\{\|\psi-\varphi\|+\frac{\chi^{1-\gamma}(x)}{\sqrt{n}}\|\chi^{\gamma}\varphi'\|\right\}\\ &\leq \mathscr{D}\omega_{\chi^s}\left(\psi;\chi^{1-\gamma}(x)\frac{1}{\sqrt{n}}\right), \end{aligned}\tag{6.14}$$

where $\mathscr{D}\geq\max\{2,2^s(1+\sqrt{2})\sqrt{\mathscr{M}}\}$. From (6.14) the result is concluded. □

6.4 Weighted Approximation

The approximation results on the space $C_B^*[0,\infty)$ are examined in this section using the weighted modulus of continuity. We know that on an infinite interval, the classical modulus of continuity of first order generally does not tend to zero. The weighted space of the continuous function $C_B^*[0,\infty)$ of the provided operators $\tilde{\mathscr{B}}_n$ is then used to build an approximation theorem.

First we define a class of functions $C_B^{x^2}[0,\infty)=\{\psi\in C_B[0,\infty)\,\text{for each } x\in[0,\infty);|\psi(x)|\leq\mathscr{K}_{\psi}(1+x^2)\}$. Let $C_B^*[0,\infty)$ be a subspace of $C_B^{x^2}[0,\infty)$ consisting all the real functions existing the following limit $\lim\limits_{n\to\infty}\frac{|\psi(x)|}{1+x^2}$ and norm in the subspace $C_B^*[0,\infty)$ is defined by $\|\psi\|_{x^2}=\sup\limits_{x\in[0,\infty)}\frac{\psi(x)}{1+x^2}$.

The weighted modulus of continuity for any $\psi \in C_B^*[0,\infty)$ and $\delta > 0$, by [108] is explained as:

$$\Omega^*(\psi,\delta) = \sup_{x\in[0,\infty),0<h\leq\delta} \frac{\psi(x+h)-\psi(x)}{1+(x+h)^2}, \tag{6.15}$$

where $\Omega^*(\psi,\delta)$ holds the following properties:

1. $\Omega^*(\psi,\delta)$ is an increasing function of δ, and $\lim_{n\to\infty}\Omega^*(f,\delta)=0$.
2. For each $m\in\mathbb{N},\Omega^*(\psi,k\delta)\leq k\Omega^*(\psi,\delta)$ and for each $\alpha\in(0,\infty)$, $\Omega^*(\psi,\alpha\delta)\leq(1+\alpha)\Omega^*(\psi,\delta)$.

Theorem 6.4.1. *For each real function $\psi \in C_B^*[0,\infty)$, we have*

$$\lim_{n\to\infty}\|\tilde{\mathscr{B}}_n(\psi;.)-\psi\|_{x^2}=0.$$

Proof. To prove this theorem, it is acceptable to verify the following conditions,

$$\lim_{n\to\infty}\|\tilde{\mathscr{B}}_n(t^r;x)-x^r\|_{x^2}=0,\ r=0,1,2. \tag{6.16}$$

Since $\tilde{\mathscr{B}}_n(1;x)=1$, therefore for $r=0$ holds.
Using Lemma 6.2.3, we have

$$\begin{aligned}\|\tilde{\mathscr{B}}_n(t;x)-x\|_{x^2} &= \sup_{x\in[0,\infty)}\frac{|\tilde{\mathscr{B}}_n(t;x)-x|}{1+x^2}\\ &\leq \left|\frac{\xi'(p(x))}{\xi(p(x))}x+\frac{1}{n}\left(\frac{\mathscr{S}'(1)}{\mathscr{S}(1)}+\mathscr{V}'(1)\frac{\xi'(p(x))}{\xi(p(x))}\right)-x\right|\\ &\quad\times\sup_{x\in[0,\infty)}\frac{1}{1+x^2}.\end{aligned}$$

As $n\to\infty$ and using $\lim_{t\to\infty}\frac{\xi'(t)}{\xi(t)}=1$ and $\lim_{t\to\infty}\frac{\xi''(t)}{\xi(t)}=1$, the required inequality of the theorem holds good for $r=1$. In the similar way by using the Lemma 6.2.3, we may obtain $\|\tilde{\mathscr{B}}_n(t^2;x)-x^2\|_{x^2}=0$. It follows that the theorem holds for $r=2$ as $n\to\infty$. Thus, on applying Korovkin's theorem required result follows. □

Corollary 6.4.2. *Let $\alpha>0$ and $\psi\in C_B^*[0,\infty)$, then*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|\tilde{\mathscr{B}}_n(\psi;x)-\psi(x)|}{(1+x^2)^{1+\alpha}}=0.$$

Proof. For any fixed $\varepsilon_0>0$, we have

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|\tilde{\mathscr{B}}_n(\psi;x)-\psi(x)|}{(1+x^2)^{1+\alpha}} &\leq \sup_{x\leq\varepsilon_0}\frac{|\tilde{\mathscr{B}}_n(\psi;x)-\psi(x)|}{(1+x^2)^{1+\alpha}}+\sup_{x\geq\varepsilon_0}\frac{|\tilde{\mathscr{B}}_n(\psi;x)-\psi(x)|}{(1+x^2)^{1+\alpha}}\\ &\leq \|\tilde{\mathscr{B}}_n(\psi;.)-\psi\|_{C[0,\varepsilon_0]}+\|\psi\|_{x^2}\sup_{x\geq\varepsilon_0}\frac{|\tilde{\mathscr{B}}_n\left((1+t^2);x\right)|}{(1+x^2)^{1+\alpha}}\end{aligned}$$

$$+\sup_{x\geq\varepsilon_0}\frac{|\psi(x)|}{(1+x^2)^{1+\alpha}}.$$

Using Theorem (6.4.1) and for the fixed $\varepsilon_0 > 0$ (sufficiently large), we quickly seen that

$$\sup_{x\geq\varepsilon_0}\frac{|\tilde{\mathscr{B}}_n\left((1+t^2);x\right)|}{(1+x^2)^{1+\alpha}}$$

tends to zero as $n\to\infty$. Above fact concludes the proof. □

6.5 Rate of Convergence

Now we discuss the convergence rate of the operators (6.2) for functions with derivative of bounded variation.

Lemma 6.5.1. *For every $x\in[0,\infty)$ and n to be very large. Then*

1. *For $0\leq y<x$, we may write*

$$\xi_n(x,y)=\int_0^y \underline{\mathscr{H}}_n(x,s)ds\leq\frac{C_1|\eta_1(x)|}{(x-y)^2}.$$

2. *If $x<z<\infty$, then*

$$1-\xi_n(x,z)=\int_z^\infty \underline{\mathscr{H}}_n(x,s)ds\leq\frac{C_1|\eta_1(x)|}{(z-x)^2}.$$

Proof. Using Lemma 6.2.3 and 6.2.4, for a very large n and $0\leq y<x$, we have

$$\begin{aligned}
\xi_n(x,y) &= \int_0^y \underline{\mathscr{H}}_n(x,s)ds\\
&\leq \int_0^y \frac{(x-t)^2}{(x-y)^2}\underline{\mathscr{H}}_n(x,s)ds\\
&\leq \frac{1}{(x-y)^2}\tilde{\mathscr{B}}_n((x-s)^2;x)\\
&\leq \frac{C_1|\eta_1(x)|}{n(x-y)^2}.
\end{aligned}$$

In the similar way we can obtain second result. □

Theorem 6.5.2. *(Bounded Variation) Let $\psi\in DBV[0,\infty)$ then for all $x\in(0,\infty)$ and a very large n, we have*

$$\begin{aligned}
\left|\tilde{\mathscr{B}}_n(\psi;x)-\psi(x)\right| &\leq \left[\left(\frac{\xi'(p(x))}{\xi(p(x))}-1\right)x+\frac{1}{n}\left(\frac{\mathscr{S}'(1)}{\mathscr{S}(1)}+\mathscr{V}'(1)\frac{\xi'(p(x))}{\xi(p(x))}\right)\right]\\
&\quad\times\left|\frac{\psi'(x^+)+\psi'(x^-)}{2}\right|+\sqrt{C_1|\eta_1(x)|}\left|\frac{\psi'(x^+)+\psi'(x^-)}{2}\right|
\end{aligned}$$

$$+\frac{C_1|\eta_1(x)|}{x}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{j}}^{x}\psi_x'\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}\psi_x'\right)$$
$$\times\left(4\mathscr{K}_\psi+\frac{(\mathscr{K}_\psi+|\psi(x)|)}{x^2}\right)C_1|\eta_1(x)|+\left|\psi'(x^+)\right|$$
$$\times\sqrt{|C_1\eta_1(x)|}+\frac{C_1|\eta_1(x)|}{x^2}\left|\psi(2x)-x\psi'(x^+)-\psi(x)\right|$$
$$+\frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}\psi_x'\right)+\frac{C_1|\eta_1(x)|}{x}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{j}}\psi_x'\right).$$

Here, C_1 is a positive constant and $\bigvee_a^b f$ denotes the total variation of function ψ on $[a,b]$ and ψ_x' is defined as

$$\psi_x(t)=\begin{cases}\psi(t)-\psi(x^-) & \textit{if } 0\le t<x\\ 0 & \textit{if } t=x\\ \psi(t)-\psi(x^+) & \textit{if } x<t<\infty.\end{cases}$$

Proof. Using the operators (6.2) and for all x lies on positive real line, we obtain

$$\begin{aligned}\tilde{\mathscr{B}}_n(\psi;x)-\psi(x) &= \int_0^\infty \tilde{\mathscr{B}}_n(x,s)(\psi(s)-\psi(x))ds\\ &= \int_0^\infty\left(\tilde{\mathscr{B}}_n(x,s)\int_x^s\psi(v)dv\right)ds. \end{aligned}\tag{6.17}$$

For $\psi\in DBV[0,\infty)$, we have

$$\begin{aligned}\psi'(v) &= \frac{1}{2}(\psi'(x^+)+\psi'(x^-))+\psi_x'(v)+\frac{1}{2}(\psi'(x^+)-\psi'(x^-))sgn(x)\\ &\quad+\delta_x(v)(\psi'(v)-\frac{1}{2}(\psi'(x^+)+\psi'(x^-)),\end{aligned}\tag{6.18}$$

where

$$\delta_x(v)=\begin{cases}1, & v=x\\ 0, & v\neq x.\end{cases}$$

For every $x\in[0,\infty)$, using (6.18) and (6.3), we get

$$\begin{aligned}\tilde{\mathscr{B}}_n(\psi;x)-\psi(x) &= \int_0^\infty \underline{\mathscr{H}}_n(x,s)\,(\psi(s)-\psi(x))\,ds\\ &= \int_0^\infty \underline{\mathscr{H}}_n(x,s)\left(\int_s^x\psi'(u)du\right)ds\\ &= -\int_0^x \underline{\mathscr{H}}_n(x,s)\left(\int_s^x\psi'(u)du\right)ds\end{aligned}$$

$$+\int_x^\infty \underline{\mathscr{H}}_n(x,s)\left(\int_s^x \psi'(u)du\right)ds. \tag{6.19}$$

Now, we simplify the above expressions into two parts. Let

$$A_1=\int_0^x \underline{\mathscr{H}}_n(x,s)\left(\int_s^x \psi'(u)du\right)ds,$$

and

$$A_2=\int_x^\infty \underline{\mathscr{H}}_n(x,s)\left(\int_s^x \psi'(u)du\right)ds.$$

Since $\int_x^s \delta_x(s)ds=0$, using (6.18), we get

$$\begin{aligned}
A_1 &= \int_0^x\left\{\int_s^x\left(\frac{1}{2}(\psi'(x^+)+\psi'(x^-))+\psi_x'(u)\right.\right.\\
&\quad \left.\left.+\frac{1}{2}(\psi'(x^+)-\psi'(x^-))sgn(u-x)\right)du\right\}\underline{\mathscr{H}}_n(x,s)ds\\
&= \frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\int_0^x(x-s)\underline{\mathscr{H}}_n(x,s)ds\\
&\quad +\int_0^x \underline{\mathscr{H}}_n(x,s)\left(\int_s^x \psi_x'(u)du\right)ds\\
&\quad -\frac{1}{2}\left(\psi'(x^+)-\psi'(x^-)\right)\int_0^x(x-s)\underline{\mathscr{H}}_n(x,s)ds.
\end{aligned} \tag{6.20}$$

Similarly, we may write

$$\begin{aligned}
A_2 &= \frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\int_x^\infty(s-x)\underline{\mathscr{H}}_n(x,s)ds\\
&\quad +\int_x^\infty \underline{\mathscr{H}}_n(x,s)\left(\int_x^s \psi_x'(u)du\right)ds\\
&\quad +\frac{1}{2}\left(\psi'(x^+)-\psi'(x^-)\right)\int_x^\infty(s-x)\underline{\mathscr{H}}_n(x,s)ds.
\end{aligned} \tag{6.21}$$

Now, from the relations (6.19)-(6.21), we get

$$\begin{aligned}
\tilde{\mathscr{B}}_n(\psi;x)-\psi(x) &= \frac{1}{2}\left(\psi'(x^+)+\psi'(x^-)\right)\int_0^\infty(s-x)\underline{\mathscr{H}}_n(x,s)ds\\
&\quad +\frac{1}{2}\left(\psi'(x^+)-\psi'(x^-)\right)\int_0^\infty|x-s|\,\underline{\mathscr{H}}_n(x,s)ds\\
&\quad -\int_0^x\left(\int_t^x \psi_x'(u)du\right)\underline{\mathscr{H}}_n(x,s)ds\\
&\quad +\int_x^\infty\left(\int_x^t \psi_x'(u)du\right)\underline{\mathscr{H}}_n(x,s)ds.
\end{aligned}$$

Thus,

$$\left|\tilde{\mathscr{B}}_n(\psi;x)-\psi(x)\right| \le \left|\frac{\psi'(x^+)+\psi'(x^-)}{2}\right|\left|\tilde{\mathscr{B}}_n(s-x;x\right|$$

$$
+\left|\frac{\psi'(x^+)-\psi'(x^-)}{2}\right|\tilde{\mathscr{B}}_n(|s-x|;x)
+S_n(\psi',x)+T_n(\psi',x). \tag{6.22}
$$

Where,

$$
S_n(\psi',x)=\left|\int_0^x\left(\int_s^x\psi'_x(u)du\right)\underline{\mathscr{H}}_n(x,s)ds\right|,
$$

and

$$
T_n(\psi',x)=\left|\int_x^\infty\left(\int_x^s\psi'_x(u)du\right)\underline{\mathscr{H}}_n(x,s)ds\right|.
$$

Now, using the Lemma 6.5.1, and the properties of integration, we estimate $S_n(\psi',x)$ and $T_n(\psi',x)$.

$$
S_n(\psi',x)=\int_0^x\left(\int_s^x\psi'_x(u)du\right)\frac{\partial\xi_n(x,s)}{\partial s}ds=\int_0^x\psi'_x(s)\xi_n(x,s)ds
$$

$$
\begin{aligned}
\left|S_n(\psi',x)\right|=\int_0^x|\psi'_x(s)|\xi_n(x,s)ds \quad \leq \quad & \int_0^{x-\frac{x}{\sqrt{n}}}\left|\psi'_x(s)\right|\xi_n(x,s)ds \\
& +\int_{x-\frac{x}{\sqrt{n}}}^x\left|\psi'_x(s)\right|\xi_n(x,s)ds.
\end{aligned}
$$

Using the fact $\psi'_x(s)=0$, and $\xi_n(x,s)\leq 1$, we may write

$$
\begin{aligned}
\int_{x-\frac{x}{\sqrt{n}}}^x\left|\psi'_x(s)\right|\xi_n(x,s)ds &= \int_{x-\frac{x}{\sqrt{n}}}^x\left|\psi\prime_x(s)-\psi'_x(x)\right|\xi_n(x,s)ds \\
&\leq \int_{x-\frac{x}{\sqrt{n}}}^x\left(\bigvee_s^x\psi'_x\right)ds\leq\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x\psi'_x\right)\int_{x-\frac{x}{\sqrt{n}}}^x ds \\
&= \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x\psi'_x\right).
\end{aligned} \tag{6.23}
$$

Taking $s=x-\frac{x}{u}$, and use the Lemma 6.5.1, we obtain

$$
\begin{aligned}
\int_0^{x-\frac{x}{\sqrt{n}}}\left|\psi'_x(s)\right|\xi_n(x,s)ds &\leq C_1|\eta_1(x)|\int_0^{x-\frac{x}{\sqrt{n}}}\frac{|\psi'_x(s)|}{(x-s)^2}ds \\
&\leq C_1|\eta_1(x)|\int_0^{x-\frac{x}{\sqrt{n}}}\left(\bigvee_s^x\psi'_x\right)\frac{ds}{(x-s)^2} \\
&= \frac{C_1|\eta_1(x)|}{x}\int_1^{\sqrt{n}}\left(\bigvee_{x-\frac{x}{u}}^x\psi'_x\right)du \\
&\leq \frac{C_1|\eta_1(x)|}{x}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{j}}^x\psi'_x\right).
\end{aligned} \tag{6.24}
$$

From (6.23) and (6.24), we get

$$\left|S_n(\psi',x)\right| \leq \frac{C_1|\eta_1(x)|}{x}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{j}}^{x}\psi'_x\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}\psi'_x\right). \tag{6.25}$$

Now, using integration by parts and Lemma 6.5.1, we get

$$\begin{aligned} T_n(\psi',x) &\leq \left|\int_x^{2x}\left(\int_x^s \psi'_x(u)du\right)\frac{\partial}{\partial s}(1-\xi_n(x,s)ds)\right| \\ &\quad +\left|\int_{2x}^{\infty}\left(\int_x^s \psi'_x(u)du\right)\underline{\mathscr{H}}_n(x,s)ds\right| \\ &\leq \left|\int_x^{2x}\psi'_x(u)du\right||1-\xi_n(x,2x)|+\int_x^{2x}\left|\psi'_x(s)\right|(1-\xi_n(x,s))\,ds \\ &\quad +\left|\int_{2x}^{\infty}(\psi(s)-\psi(x))\underline{\mathscr{H}}_n(x,s)ds\right| \\ &\quad +|\psi'(x^+)|\left|\int_{2x}^{\infty}(s-x)\underline{\mathscr{H}}_n(x,s)ds\right|. \end{aligned} \tag{6.26}$$

Here, we simplify the following integration into two parts,

$$\begin{aligned} \int_x^{2x}\left|\psi'_x(s)\right|(1-\xi_n(x,s))\,ds &= \int_x^{x+\frac{x}{\sqrt{n}}}\left|\psi'_x(s)\right|(1-\xi_n(x,s))\,ds \\ &\quad +\int_{x+\frac{x}{\sqrt{n}}}^{2x}\left|\psi'_x(s)\right|(1-\xi_n(x,s))\,ds \\ &= I_1+I_2. \end{aligned} \tag{6.27}$$

Using the Lemma 6.5.1,

$$\begin{aligned} I_1 &= \int_x^{x+\frac{x}{\sqrt{n}}}\left|\psi'_x(s)-\psi'_x(x)\right|(1-\xi_n(x,s))\,ds \\ &\leq \int_x^{x+\frac{x}{\sqrt{n}}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}}\psi'_x\right)ds=\frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}}f'_x\right). \end{aligned} \tag{6.28}$$

Again, using the Lemma 6.5.1 and put $s=x+\frac{x}{u}$, we get

$$\begin{aligned} I_2 &\leq C_1|\eta_1(x)|\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(s-x)^2}\left|\psi'_x(s)-\psi'_x(x)\right|ds \\ &\leq C_1|\eta_1(x)|\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(s-x)^2}\left(\bigvee_x^s\psi'_x\right)ds \\ &\leq \frac{C_1|\eta_1(x)|}{x}\int_1^{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{u}}\psi'_x\right)du \\ &\leq \frac{C_1|\eta_1(x)|}{x}\sum_{j=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{j}}\psi'_x\right). \end{aligned} \tag{6.29}$$

Using (6.27)-(6.29), we get

$$\int_x^{2x} \left|\psi_x'(s)\right| (1-\xi_n(x,s))\, ds \;=\; \frac{C_1|\eta_1(x)|}{x} \sum_{j=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{x}{j}} \psi_x' \right) + \frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} \psi_x' \right). \tag{6.30}$$

Now, from (6.26) and (6.30) and the Cauchy-Schwarz inequality taking into the account, we get

$$\begin{aligned} T_n(\psi',x) \;\leq\; & \mathscr{K}_\psi \int_{2x}^{\infty} (s^2+1)\underline{\mathscr{H}}_n(x,s)ds + |\psi(x)| \int_{2x}^{\infty} \underline{\mathscr{H}}_n(x,s)ds \\ & + \left|\psi'(x^+)\right| \sqrt{|C_1\eta_1(x)|} + \frac{C_1|\eta_1(x)|}{x^2} \left|\psi(2x) - x\psi'(x^+) - \psi(x)\right| \\ & + \frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} \psi_x' \right) + \frac{C_1|\eta_1(x)|}{x} \sum_{j=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{x}{j}} \psi_x' \right). \end{aligned} \tag{6.31}$$

Since $x \leq s-x$, and $s \leq 2(s-x)$ when $s \geq 2x$, we have

$$\begin{aligned} & \mathscr{K}_\psi \int_{2x}^{\infty} (s^2+1)\underline{\mathscr{H}}_n(x,s)ds + |\psi(x)| \int_{2x}^{\infty} \underline{\mathscr{H}}_n(x,s)ds \\ & \quad \leq (\mathscr{K}_\psi + |\psi(x)|) \int_{2x}^{\infty} \underline{\mathscr{H}}_n(x,s)ds + 4\mathscr{K}_\psi \int_{2x}^{\infty} (s-x)^2 \underline{\mathscr{H}}_n(x,s)ds \\ & \quad \leq \frac{(\mathscr{K}_\psi + |\psi(x)|)}{x^2} \int_0^{\infty} (s-x)^2 \underline{\mathscr{H}}_n(x,s)ds + 4\mathscr{K}_\psi \int_0^{\infty} (s-x)^2 \underline{\mathscr{H}}_n(x,s)ds \\ & \quad \leq \left(4\mathscr{K}_\psi + \frac{(\mathscr{K}_\psi + |\psi(x)|)}{x^2} \right) C_1|\eta_1(x)|. \end{aligned} \tag{6.32}$$

Using (6.31) and (6.32), we have

$$\begin{aligned} T_n(\psi',x) \;\leq\; & \left(4\mathscr{K}_\psi + \frac{(\mathscr{K}_\psi + |\psi(x)|)}{x^2} \right) C_1|\eta_1(x)| + \left|\psi'(x^+)\right| \sqrt{|C_1\eta_1(x)|} \\ & + \frac{C_1|\eta_1(x)|}{x^2} \left|\psi(2x) - x\psi'(x^+) - \psi(x)\right| + \frac{x}{\sqrt{n}} \left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} \psi_x' \right) \\ & + \frac{C_1|\eta_1(x)|}{x} \sum_{j=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+\frac{x}{j}} \psi_x' \right). \end{aligned} \tag{6.33}$$

using (6.22), (6.25) and (6.33), we reach to the required result. □

6.6 Another extension of sequence of operators $\mathscr{B}_n(\psi;x)$

Here, we define the Sz'asz-type Durrmeyer variant of the operators $\mathscr{B}_n(\psi;x)$ under the same conditions and limitations as those outlined in the article's introduction. For $\psi \in C_B[0,\infty)$ and $p(x) = n^2x^2\mathscr{T}(1) + nx\mathscr{U}(1) + \mathscr{V}(1)$, we have

$$\mathscr{B}_n^*(\psi;x) = \frac{n}{\mathscr{S}(1).\xi(p(x))} \sum_{j=0}^{\infty} \Theta_j(nx) \int_0^{\infty} e^{-ns} \frac{(ns)^j}{j!} \psi(s)ds.$$

In a manner similar to that described in this article, one can prove the convergence results for the operators $\mathscr{B}_n^*(\psi;x)$.

Chapter 7

Conclusions and Future Prospects

In this chapter, findings of studies carried out in this thesis along with significant topics allied to the new aspects of analysis that identify present and future potential research aspects has been presented.

7.1 Conclusion

Approximation theory is a field within mathematical analysis that deals with representing complex functions using simpler or more computationally manageable functions, like polynomials or linear operators. The primary objective is to obtain solutions that closely resemble the exact values. This thesis focuses on the study of particular approximation operators and examines how well they converge to the original functions. Chapter **1** has covered the literature survey, definitions, tools, and historical background of some approximation operators.

Many researchers have studied Apostol-Genocchi and Appell polynomials, deriving various approximation results. Chapter **2** is based on the convergence analysis of the Apostol-Genocchi polynomial sequence involving Baskakov operators. By employing different analytical techniques, we analyze the rate of convergence and introduce a new class of operators. The chapter also explores the approximation behavior of these operators and presents an integral form modification.

Difference operators play a significant role in approximation theory, especially in examining discrete analogues of derivatives and in evaluating the behavior of functions and sequences. Chapter **3** primarily examines the difference operators of two positive linear operators: the generalized păltănea type operators $L_{n,c}^{\lambda}(f;x)$ and the M. Heilmann type operators $M_{n,c}(f;x)$, which possess identical basis functions. Initially, we assess the quantitative difference and convergence of these operators considering the modulus of smoothness and Peetre's K-functional.

In Chapter **4**, we develop a recurrence relation for the semi-exponential Post-Widder operators and demonstrate their moments. We also look at how well they converge in Lipschitz-type spaces, using the Ditzian-Totik modulus of smoothness and the weighted modulus of continuity to measure how fast they converge. Finally, we assess the convergence rate for functions with derivatives of constrained variation, providing significant insights into approximation theory.

Chapter **5**, presents a new Bézier-type modification belonging to the class of Phillips-type generalized positive linear operators. To better understand their essential characteristics, we compute the moments of these operators. The chapter then investigates their convergence behavior in Lipschitz-type function spaces, emphasizing the role of the Ditzian-Totik modulus of smoothness. Lastly, it offers a detailed study of the convergence rate for functions with derivatives of bounded variation.

Chapter **6**, focuses on introducing a sequence of Baskakov-Durrmeyer-type operators associated linked to the generating functions of Boas-Buck-type polynomials. The chapter commence with the computation of their moments, including the limiting case of central moments for the constructed sequence of operators. In this chapter, we employ both the Ditzian-Totik modulus of smoothness and the modulus of continuity to examine the rate of convergence. We also provide convergence results in Lipschitz-type spaces, Weighted space, and ultimately assess the convergence behavior for functions with derivatives of constrained variation.

7.2 Future scope for academics

Summability theory is a branch of mathematical analysis that aims to assign finite values to divergent series and improve the convergence of sequences and series. Recently, summability methods have become significant in approximation theory, especially for extending classical convergence results such as Korovkin-type theorems to broader and more complex frameworks, including cases where classical convergence fails.

Recently, Baxhaku et al.[18] applied power series summability methods to extend classical Korovkin-type approximation theorems into the fuzzy context. In their work, they established a Korovkin-type result for sequences of fuzzy-positive linear operators through the application of power series summability techniques. They also established a rate of convergence result utilizing a fuzzy modulus of smoothness, which quantifies the effectiveness of approximation convergence within a fuzzy context. In the field of approximation theory, researchers have recognized that a significant challenge in classical approaches lies in achieving exact or statistically precise limits. To address this, fuzzy versions of many classical approximation theorems have been developed(see [13], [89] and [93]). Looking ahead, readers may plan to further generalize approximation results on fuzzy positive linear operators by employing advanced summability techniques such as statistical summability, deferred Cesáro means, and power series summability.

Simultaneous approximation is a vital extension of classical approximation theory, aiming to approximate not only a function but also its derivatives or related functionals. This approach has attracted significant attention (see, [6], [15] [12], and [94]) in the mathematical community due to its relevance in applications where preserving essential features of a function such as smoothness, monotonicity, and curvature is critical.Traditional approximation frameworks, however, often fall short when dealing with uncertainty, vagueness, or incomplete information. In such cases, fuzzy-valued functions provide a powerful alternative, enabling more flexible and realistic modeling. The concept of fuzzy simultaneous approximation has become increasingly important across diverse fields, including biomedical signal analysis, financial forecasting, robotics, image processing, and pattern recognition where data is often imprecise or noisy. Looking ahead, my research will focus on developing Korovkin-type theorems for fuzzy-positive linear operators that simultaneously approximate functions and their derivatives. This work will include developing new ways to measure smoothness in fuzzy data, examining the convergence rate, and expanding the theory to different types of function spaces to improve the accuracy and usefulness of fuzzy approximation methods.

Approximation theory has traditionally focused on functions of a single variable, but with the growth of scientific and technological applications, there has been a need to extend these concepts to multivariable settings. Multivariable approximation theory aims to approximate multivariate functions using generalized forms of classical operators, particularly bivariate operators. These operators are natural extensions of well-known single-variable operators and are important for real-world uses like image processing, solving partial differential equations, and computer-aided geometric design. The extension to the bivariate setting was initiated by G.G. Lorentz [72] in 1953 with the introduction of bivariate Bernstein operators. Since then, many authors (see [11, 30, 41, 71, 107] have introduced numerous bivariate counterparts to classical univariate operators, retaining the desirable properties of positivity, linearity, and convergence. Theoretical advancements in this field have been greatly improved by extending Korovkin-type theorems to the bivariate context.

In the future, I plan to enhance my research by introducing fuzzy variants of bivariate positive linear operators, along with their q-analogues. My primary objective is to extend P.P. Korovkin-type theorems to establish the convergence of these operators and their q-analogues. Additionally, I intend to investigate the convergence rate by formulating suitable extensions of the modulus of smoothness and analyzing convergence in weighted function spaces.

Max-product operators are an important class of nonlinear operators in approximation theory, providing an alternative to traditional linear positive operators. Numerous authors have investigated convergence outcomes in their studies of these operators. Barnabás et al. [20] expanded the max-product technique by incorporating known operators, including Bernstein, Szász, Baskakov, and Picard. Sorin G. Gal [40] constructed max-product nonlinear (sublinear) operators in his work by replacing the summation operators $\sum$ with the max-product operators $\bigvee$. Subsequently, further scholars (see [14, 19, 28, 29]) have examined convergence qualities across various spaces. In the future, I plan to explore how these modified operators can enhance the accuracy of approximation in various function spaces, building upon the research conducted by previous scholars. This investigation may contribute a deeper understanding of the convergence properties of these operators and their applications in different mathematical contexts.

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Title of the Thesis: Convergence Analysis of Some Approximation Operators
Total Pages: 121
Name of the Scholar: Sandeep Kumar
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