A STUDY ON THE APPROXIMATION ORDER OF POSITIVE LINEAR OPERATORS BY CERTAIN APPROXIMATION METHODS

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by

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Dedicated to my beloved parents

Whose support has been the foundation of all I do.

To my dear grandmother in heaven,

Thank you for your guiding light.

And to my brother,

For always standing by me with strength and kindness.

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DECLARATION

I, Kanita, hereby declare that the work which is being presented in the thesis entitled "**A study on the approximation order of positive linear operators by certain approximation methods**" in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy, submitted in the Department of Applied Mathematics, Delhi Technological University is an authentic record of my own work carried out during the period from August 2021 to September 2025 under the supervision of ***Prof. Naokant Deo***, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

The matter presented in the thesis has not been submitted by me for the award of any other degree of this or any other Institute.

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CERTIFICATE

This is to certify that the research work embodied in the thesis entitled **"A study on the approximation order of positive linear operators by certain approximation methods"** submitted by **Ms. Kanita** with enrollment number **2K21/PHDAM/01** is the result of her original research carried out in the Department of Applied Mathematics, Delhi Technological University, Delhi, for the award of **Doctor of Philosophy** under the supervision of **Prof. Naokant Deo**, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

It is further certified that this work is original and has not been submitted in part or fully to any other university or institute for the award of any degree or diploma.

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

Date: November 2025

Place: Delhi, India

Prof. Naokant Deo

(Supervisor)

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(Head of Department)

A STUDY ON THE APPROXIMATION ORDER OF POSITIVE LINEAR OPERATORS BY CERTAIN APPROXIMATION METHODS

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ABSTRACT

Approximation theory, and in particular the study of positive linear operators, plays a fundamental role in both pure and applied mathematics. Classical operators such as those introduced by Bernstein and Stancu have provided powerful tools for approximating continuous functions on compact intervals. However, despite their elegance and historical significance, these operators come with certain well-known limitations. Their rate of convergence is generally quite slow, and in many cases they fail to reproduce even simple test functions such as quadratic polynomials, and sometimes even linear functions are not preserved. Moreover, while some attempts exist, they have not yet been systematically developed within more advanced frameworks such as fractional calculus, which accounts for memory effects, or fuzzy mathematics, which deals with uncertainty. These limitations highlight a research gap and provide strong motivation for the development of new families of operators with better approximation properties, wider applicability and closer connections to real-world mathematical models.

This thesis addresses these issues through the construction, analysis and application of several new classes of positive linear operators. Beginning with operators based on the Pólya-Eggenberger (contagion) distribution, parametric generalizations are introduced to provide greater flexibility in capturing approximation behaviour. Variants of King-type, Kantorovich-type and genuine-type are then developed, which not only improve convergence but also preserve key test functions that classical operators do not. Inspired by kernels arising in partial differential equations, the thesis further introduces semi-exponential operators, carefully analyzing their central moments, recurrence relations and generating functions. A Voronovskaya-type theorem is established, offering insights into the asymptotic behaviour of these operators.

The study then moves into the fractional domain, where fractional versions of Bernstein-Kantorovich operators are proposed using Caputo's fractional derivative. Their moments, using Laplace transforms, and convergence properties are derived,

and their potential for solving fractional differential and fractional integro-differential equations is demonstrated. This represents a significant step toward connecting operator theory with the modelling of systems that exhibit memory effects. Parallel to this, the theory of approximation is extended to the fuzzy domain by defining positive linear fuzzy operators and proving approximation results using the fuzzy Korovkin theorem.

Higher-order constructions are also investigated in depth. Second and third order semi-exponential operators are defined and analyzed, revealing that their improved moments lead to better rates of convergence compared to their first order counterparts. Numerical evidence supports these theoretical results, showing that there is an improvement in approximation as the order increases. Similarly, higher order Stancu-Bernstein operators based on the contagion distribution are developed, reducing the order of error from $O(1/n)$ to $O(1/n^2)$. These operators are studied using Korovkin's theorem, modulus of continuity and illustrative numerical examples, confirming that higher-order modifications are a powerful means of enhancing approximation accuracy. In addition, sequence-based operators are constructed that avoid the use of derivatives, making them applicable to non-differentiable functions while still maintaining convergence. A careful comparison reveals that while all such operators converge uniformly, their endpoint behaviour differs depending on the choice of sequences, with some operators interpolating the boundary values and others not.

Taken together, the contributions of this thesis provide advancements in approximation theory using sequences of positive linear operators. By addressing the shortcomings of classical operators, introducing higher-order and fractional variants, and extending the theory into fuzzy and non-differentiable settings, this work not only enriches the theoretical foundations of approximation theory but also broadens its applicability to modern mathematical models. These results lay the groundwork for future investigations into the optimization of operator constructions, the study of higher-order generalizations and their application in fields such as numerical analysis, differential equations and uncertainty modelling.

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$\mathbb{N}$	the set of natural numbers
$\mathbb{N}\cup\{0\}$	the set of natural number including zero
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{R}_{\mathscr{F}}$	the set of fuzzy numbers
$[\tilde{p}]^\ell$	the ℓ-level cut of a fuzzy number $\tilde{p}$
(a,b)	an open interval
$[a,b]$	a closed interval
$\langle a,b\rangle$	open, closed or semi-open intervals $(a,b),\ [a,b],\ (a,b]$ or $[a,b)$
Λ	index set
P_n	the set of all polynomials of degree at most n
e_n	the n^{th} degree monomial, $e_n : [a,b] \to \mathbb{R}$, where $e_n(x) = x^n$, $n \in \mathbb{N}_0$
$x^{[n,\nu]}$	the rising factorial $x^{[n,\nu]} := x\,(x-\nu)\,(x-2\nu)\ldots(x-(n-1)\nu)\,,\ x^{[0,\nu]} = 1$
$K_2\,(f;\delta)$	the Peetre's K-functional
$\omega(f;\delta)$	the first order modulus of continuity
$\omega_k(f;\delta)$	the k^{th} order modulus of continuity
$\omega^*(f;\delta)$	the exponential modulus of continuity
$\Omega(f;\delta)$	the weighted modulus of continuity

$\omega_k^{\mathscr{F}}(f;\delta)$	the k^{th} order fuzzy modulus of continuity
$\omega_*^{\mathscr{F}}(f;\delta)$	the exponential fuzzy modulus of continuity
$\Omega^{\mathscr{F}}(f;\delta)$	the weighted fuzzy modulus of continuity
$C[a,b]$	the set of all real-valued continuous function defined on $[a,b]$
$C^r[a,b]$	the set of all real-valued, r-times continuously differentiable function ($r \in \mathbb{N}$)
$C[0,\infty)$	the set of all continuous functions defined on $[0,\infty)$
$C_B[0,\infty)$	the set of all continuous and bounded functions on $[0,\infty)$
$C_B^r[0,\infty)$	the set of all r-times continuously differentiable functions in $C_B[0,\infty)$ ($r \in \mathbb{N}$)
$B_\rho[0,\infty)$	the set of all functions f defined on $[0,\infty)$ satisfying the condition : $\lvert f(x)\rvert \leq M\rho(x)$, M is a positive constant, and ρ is weight function.
$C_\rho[0,\infty)$	the set of all continuous function in $B_\rho[0,\infty)$
$C_{\mathscr{F}}[a,b]$	the set of all fuzzy continuous function on $[a,b]$
$L^p[0,\infty)$	the Lebesgue space on $[0,\infty)$

Chapter 1

Introduction

"A good approximation is often better than a bad exact solution."

-John von Neumann

In many fields, ranging from science and engineering to economics and decision-making, an excessive emphasis on precision can sometimes lead to unnecessary complexities. Although finding exact solutions is important and can help us understand problems more deeply, in many real situations, exact answers are often difficult or impossible to get and using good approximations can be more useful and practical. In fact, in many areas of mathematics, science and engineering, we frequently prioritize finding a solution that is "close enough" over seeking an exact answer. Striving for perfection can be time-consuming and may not significantly improve the outcome, e-specially when a good approximation can accomplish the task effectively. Therefore, the concept of approximation plays a central role in research, as we focus on improving these approximations and minimizing the errors introduced throughout the process.

For instance, in numerical analysis, the goal is often not to find a perfectly exact solution, but rather to obtain a solution that is sufficiently close to the true value for practical purposes. Algorithms in numerical linear algebra typically prioritize computational efficiency and robustness over exactness, employing approximation techniques such as iterative methods and error bounds to yield solutions that are close enough for practical use [90]. Zienkiewicz and Taylor [225], in their study, discussed how in finite element analysis, engineers commonly work with approximations, understanding that small deviations from exact solutions do not substantially affect the final result, but significant reductions in computational time and resources can be achieved through approximate methods. In many areas of applied science, researchers and practitioners routinely accept approximations because they allow for more practical solutions. For instance, in physics, the use of perturbation theory and asymptotic methods enables scientists to approximate solutions to complex differential equations when an exact solution is either unknown or intractable [143; 28]. In climate modeling, simulations often rely on approximations due to the chaotic nature of weather systems, and solutions are accepted within a range of uncertainty rather than an exact answer [190]. This acceptance of approximations and uncertainty is a central theme across various disciplines, particularly in mathematics, where the theory of approximation is crucial for simplifying complex problems and deriving practical solutions.

In mathematics, approximations can be seen almost everywhere. When solving transcendental or complex nonlinear equations, numerical methods such as Newton-Raphson, secant or bisection, approximate the true root through repeated evaluations of the function and, in some cases, its derivatives or interpolants [22]. Approximation is not just limited to values of the root, but also affects how function behaviour is modeled, for example, using polynomial or rational interpolation in Muller's and Brent's methods [22; 175]. Moreover, when solving a system of linear equations $Ax = b$, especially when the system is large or sparse, finding an exact solution using direct methods like Gaussian elimination can be computationally expensive. In such cases, iterative methods, such as the Jacobi method and Gauss-Seidel method, provide an efficient way to approximate the solution. These approximation techniques are widely used in scientific computing, particularly in solving systems arising from discretized differential equations [164; 185]. Other places where approximation can be seen is the Taylor's and Maclaurin's series expansion of infinitely differentiable functions near a point a, where $a = 0$ for Maclaurin's series. The Taylor series approximation locally represents a smooth function by a polynomial, providing increasingly accurate approximations near a point as more terms are added. It relies on derivatives and is most effective for functions that are analytic in a neighborhood [22; 37; 183]. In addition, the Fourier series approximation globally approximates periodic functions using sums of sines and cosines, capturing the overall behaviour of the function over an interval rather than just near a point. It is particularly effective for approximating functions that are piecewise smooth, even if they have discontinuities [198]. Approximation is also evident in numerical differentiation. Since exact derivatives are often unavailable for tabulated or experimentally obtained data, finite difference formulas such as forward, backward and central differences are used to approximate derivatives. These approximations are widely applied in engineering and applied sciences, for instance in computing rates of change from experimental measurements or in discretizing differential equations for numerical simulation [22; 37].

In many areas of mathematics and science, where exact functions are either too complicated or not explicitly known, approximation theory plays a crucial role by replacing them with functions that are easier to study and work with, while still giving a good idea of the original behaviour. The work done in this thesis is mainly focused on approximating functions defined over closed intervals. Here, approximating a function means finding another function, usually simpler, that stays close to it in value. The functions used for approximation are often polynomials, trigonometric series or rational functions, chosen mainly because they are easy to work with, quick to compute and generally behave in a smooth and predictable way. Moreover, our objective is not limited to merely finding simpler functions, but also involves an analysis of how accurate these approximations are, and a study of the error involved during the process. Understanding how the error behaves across the interval of approximation is essential for judging the effectiveness of the method. Therefore, a significant part of our study is devoted to explore ways to reduce this error, thereby improving the approximation process.

The field of approximation theory became more formal and well known in 1885, when the German mathematician Karl Weierstrass proved an important theorem which states that: *If a function f is continuous on a closed and bounded interval $[a,b]$, then for every $\varepsilon > 0$, there exists a polynomial, say $P(x)$, such that $|f(x) - P(x)| < \varepsilon, \forall x \in [a,b]$.* In other words, any continuous function over a compact subset of $\mathbb{R}$, can be uniformly approximated on that interval by a polynomial to any degree of accuracy. This result, known as the first Weierstrass approximation theorem, was a turning point in the development of this field. Weierstrass also showed a similar result for periodic functions using trigonometric polynomials, called the second Weierstrass approximation theorem [215].

In this thesis, we explore various classical and modern techniques of approximation, analyze the order of convergence and explore how to improve this order. This study adds to the knowledge of how approximation methods can solve a wide range of mathematical problems. These range from questions rooted in abstract theoretical frameworks, such as functional analysis and operator theory, to practical applications in fields like numerical analysis, signal processing and data approximation. By exploring both the construction of approximation operators and the analysis of their error behaviour, we aim to highlight the versatility and effectiveness of these methods across different mathematical and applied contexts.

1.1 Preliminaries

In this section, we recall some definitions and properties regarding approximation operators, that will be used throughout the thesis.

1.1.1 Positive Linear Operators

An operator is a mapping that acts on functions and produces another function as output. In approximation theory, we are particularly interested in linear operators, which preserve the structure of addition and scalar multiplication. A further important class is that of positive linear operators, which, in addition to linearity, also preserve the non-negativity of functions. That is, they map non-negative functions to non-negative outputs. These properties are especially important as they ensure that key features of the original function, such as shape, bounds and monotonicity, are preserved in approximation.

We now present the formal definition of a positive linear operator:

Definition 1.1.1 *Let X, Y be two linear spaces of real functions. Then the mapping $\mathscr{L}: X \to Y$ is a linear operator if:*

$$\mathscr{L}(\alpha f+\beta g;x)=\alpha\mathscr{L}(f;x)+\beta\mathscr{L}(g;x),$$

for all $f,g \in X$ and $\alpha,\beta \in \mathbb{R}$. Moreover, if for all $f \in X$ and $f \geq 0$, it follows that $\mathscr{L}(f;x) \geq 0$, then $\mathscr{L}$ is called a positive operator.

Positive linear operators form the foundation of many classical approximation processes. Operators like the Bernstein, Szász-Mirakyan and Baskakov operators are prominent examples. These are designed to approximate a given function as closely as possible, while maintaining their structural properties like positivity and smoothness.

Their convergence, error analysis and stability have been studied extensively, and they continue to be an active area of research, particularly when these classical forms are modified or generalized for better performance.

Next, we define the modulus of smoothness, which is mainly used to measure the uniform continuity and smoothness a function.

1.1.2 Quantitative Measures of Continuity and Smoothness

In approximation theory, a central question is not just whether an operator or a sequence of functions converges to a given function, but also how well and how fast this convergence occurs. To study this, we need tools that can quantitatively measure the smoothness or regularity of a function. One such important tool is the modulus of continuity, which provides a measure of the uniform continuity of a function. It provides a numerical measure of how much the value of a function can change over small intervals.

1.1.2.a Modulus of Continuity

Introduced first by Ditzian and Totik in 1987, modulus of smoothness, also known as modulus of continuity, describes how well a function can be approximated by simpler functions, such as polynomials. It measures the largest possible difference in function values over all pairs of points within a given distance. A smaller modulus implies that the function changes more slowly and smoothly. This is particularly useful when estimating how closely an operator can approximate a continuous function, especially in the context of positive linear operators, where preserving uniform continuity plays a key role in proving convergence results.

Definition 1.1.2 *Let $I \subseteq \mathbb{R}$, and let f be a continuous function on I. Then, for $\delta > 0$, the modulus of continuity of f is defined as*

$$\begin{aligned} \omega(\delta) &= \omega(f;\delta) \\ &= \sup_{\substack{|x-y|\leq\delta \\ x,y\in I}} |f(x)-f(y)|. \end{aligned}$$

The function $\omega(f;\delta)$ is always non-negative, non-decreasing in δ and tends to zero as $\delta \to 0$ if and only if f is uniformly continuous on I. These features make it a standard tool for providing quantitative estimates of approximation errors.

1.1.2.b Higher Order Modulus of Smoothness

The k^{th}-order modulus of smoothness $\omega_k(f;\delta)$ generalizes the concept of the usual modulus of continuity by incorporating higher-order finite differences. The forward difference $\Delta_h^k f(x)$ is a discrete analogue of the k^{th} derivative, defined recursively by:

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+ih).$$

It measures how the function behaves over $k+1$ equally spaced points. The modulus $\omega_k(f;\delta)$ then takes the supremum of the absolute value of these differences over all admissible points x and step sizes h with $|h| \leq \delta$. This quantity provides a numerical measure of the function's smoothness, and plays a crucial role in obtaining approximation error bounds in direct theorems.

Some of the error estimates in this thesis are given in terms of the modulus of smoothness of higher order. Therefore we now give the definition of higher order modulus of smoothness, ω_k for $k \in \mathbb{N}$.

Definition 1.1.3 *[70] Let function $f : \langle a,b\rangle \to \mathbb{R}$ be a real-valued function defined on an interval $\langle a,b\rangle$, which can be open, closed or semi-open, for $a,b \in \mathbb{R} \cup \{-\infty,\infty\}$. The k^{th} order modulus of smoothness of function f is given by*

$$\omega_k(f;\delta) = \sup_{|h|\leq\delta} \left\{ \sup \left\{ \left| \Delta_h^k f(x) \right| : x, x+kh \in \langle a,b\rangle \right\} \right\}, \quad \delta \geq 0,$$

where

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+ih), \quad k \in \mathbb{N}.$$

For $k=1$, the above definition coincides with the usual modulus of continuity, that is, $\omega_1(f;\delta) = \omega(f;\delta)$.

Understanding the behaviour of higher-order modulus of smoothness is essential for applying it effectively in approximation theory. The following properties describe how this measure responds to changes in scale, function smoothness and order. They form the theoretical basis for many approximation results.

Proposition 1.1.4 *The modulus of smoothness of order k satisfies the following:*

(i) $\omega_k(f;\delta)$ is a positive, monotonically increasing function on $(0,\infty)$.

(ii) f *is uniformly continuous* $\Leftrightarrow \lim_{\delta\to 0} \omega(f;\delta) = 0.$

(iii) $\omega_k(f;nx) \leq n^k \omega_k(f;x)$ *for all* $n \in \mathbb{N}$.

(iv) $\omega_k(f;\lambda\delta) \leq (1+\lambda)\,\omega_k(f;\delta)$, *for any* $\lambda > 0$.

(v) $\omega_{k+1}(f;\delta) \leq 2\omega_k(f;\delta)$.

For $k = 1$, *these properties are valid for the usual modulus of continuity* $\omega(f;.)$.

1.1.3 Weighted Spaces and Corresponding Modulus of Continuity

Classical function spaces often assume that functions remain bounded or behave uniformly across their domains. However, when dealing with functions defined on unbounded intervals, such as $[0,\infty)$, weighted spaces are used, where functions are measured relative to a positive weight function. Within these spaces, notions like continuity and smoothness are adapted using corresponding weighted measures. These spaces make use of a weight function $\rho(x)$, which is a positive and continuous function on the interval $I \subseteq \mathbb{R}$, to measure the growth of functions relative to ρ.

The weighted function space $B_\rho(I)$ consists of all functions f for which there exists a constant $C > 0$ such that $|f(x)| \leq C\rho(x)$ for every $x \in I$. This means the growth of f is bounded by the weight function. In 1974, Gadzhiev [79; 82] introduced the weighted space $C_\rho(I)$, which is the set of all continuous functions f on the interval $I \subseteq \mathbb{R}$ such that $f \in B_\rho(I)$. This space is a Banach space, endowed with the norm

$$||f||_\rho = \sup_{x\in I} \frac{|f(x)|}{\rho(x)}.$$

Specifically, for $I = [0,\infty)$, we get the weighted space $C^*_\rho[0,\infty)$, defined as:

$$C^*_\rho[0,\infty) = \{f \in C_\rho[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} = k < \infty\}.$$

1.1.3.a Weighted Modulus of Continuity

To study the smoothness of functions in the weighted space $C^*_\rho[0,\infty)$, the weighted modulus of continuity $\Omega(f;\delta)$ is defined as:

$$\Omega(f;\delta) = \sup_{x\in[0,\infty),|h|<\delta} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}.$$

Unlike the usual modulus of continuity, which simply captures the maximum difference in function values over small intervals, the weighted modulus of continuity includes a scaling factor, which ensures that the function values are appropriately controlled across its domain [6; 111].

1.1.4 Exponential Modulus of Continuity

In the preceding section, we discussed the weighted modulus of continuity, which is particularly effective in handling approximation on unbounded domains by incorporating a weight function. However, when we talk about $C^*[0,\infty)$, the Banach space of all real-valued continuous functions on $[0,\infty)$ such that $\lim_{x\to\infty} f(x)$ exists and is finite, which is endowed with the uniform norm, a different approach is often more appropriate. In such cases we define the exponential modulus of continuity of f. The exponential modulus of continuity deals with approximation problems involving exponential-type operators. The usual modulus of continuity, based on the Euclidean distance $|x-t|$ between points, works well when the approximating operators behave uniformly across the domain. However, many approximation processes, especially on the unbounded interval $[0,\infty)$, involve functions or operators that may be sensitive to changes on the exponential scale. In such cases, measuring smoothness using a linear scale becomes inadequate. The exponential modulus addresses this by adapting the notion of 'closeness' between points using the distance $|e^{-x}-e^{-t}|$ instead.

The exponential modulus of continuity is defined as

$$\omega^*(f;\delta) = \sup\left\{|f(x)-f(t)| : x,t \geq 0, \left|e^{-x}-e^{-t}\right| \leq \delta\right\},$$

where $\delta > 0$ and $f \in C^*[0,\infty)$. It captures how slowly or rapidly a function f changes when its argument changes slightly on the exponential scale. The smaller the value of $\omega^*(f;\delta)$, the more slowly the function varies with respect to changes in e^{-x}.

Proposition 1.1.5 *The exponential modulus of continuity has the following properties:*

(i) $\omega^*(f;\delta)$ *can be expressed in terms of usual modulus of continuity, by the relation*

$$\omega^*(f;\delta) = \omega(f^*;\delta),$$

where f^* *is the continuous function on* $[0,\infty)$ *given by:*

$$f^*(x) = \begin{cases} f(-\ln(x)), & x \in (0,\infty] \\ \lim_{t\to\infty} f(t), & x = 0. \end{cases}$$

(ii) For $f \in C^*[0,\infty)$ and $M > 0$, we have

$$\omega^*(f;\delta) \leq \left(1+e^M\right)\omega(f;\delta).$$

(iii) For all $x,t \geq 0$, the difference $|f(t)-f(x)|$ can be bounded as:

$$|f(t)-f(x)| \leq \left(1+\frac{\left(e^{-t}-e^{-x}\right)^2}{\delta^2}\right)\omega(f^*;\delta).$$

1.1.5 Peetre's K-functional

In approximation theory, it is often important to measure not only how close a function f is to a smoother function, but also how much smoothness can be imposed without losing accuracy. One of the most powerful tools for this purpose is the Peetre's K-functional, which provides a way to balance the trade-off between approximation and smoothness. It does so by considering all functions g from a sufficiently smooth class, and then measuring both the deviation $\|f-g\|$ and the smoothness quantity involving derivatives of g. The formal definition of Peetre's K-functional is as follows.

Definition 1.1.6 *[70] Let $C[0,1]$ denote the space of real-valued continuous and bounded functions on $[0,1]$ with the supremum norm, $\|f\| = \sup_{0\leq x\leq 1} |f(x)|$. Then for every $\delta > 0$, the Peetre's K-functional is defined by*

$$K_2(f;\delta) = \inf_{g\in C^2[0,1]} \left\{\|f-g\| + \delta \left\|g''\right\|\right\}.$$

Remark 1.1.7 *There exists a positive real number $M > 0$ such that*

$$K_2(f;\delta) \leq M\omega_2\left(f;\sqrt{\delta}\right).$$

This equivalence of Peetre's K-functional with the second-order modulus of continuity is particularly valuable. While the K-functional serves as a flexible and powerful tool in the theoretical analysis of approximation processes, the modulus of continuity is often more convenient for explicit computations and practical interpretation. This correspondence allows us to move seamlessly between the two frameworks, with the assurance that both capture the same smoothness characteristics of the underlying function.

1.1.6 Lipschitz Class

Some functions change very gradually, while others may vary sharply even over small intervals. To distinguish between these behaviours, we often describe how steadily a function responds to changes in points. One such description is given by the Lipschitz condition, which provides a bound on how much the value of a function can change based on the distance between two points. A function belongs to the Lipschitz class if it does not change too rapidly, that is, the difference in function values is bounded by a fixed power of the distance between the corresponding points in the domain.

Definition 1.1.8 *Let $f(x)$ be a real-valued function defined on an interval $\langle a,b\rangle$, which can be open, closed or semi-open. The function f is said to satisfy a Lipschitz condition of order $\beta \in (0,1]$ if there exists a constant $M > 0$ such that*

$$|f(y) - f(x)| \leq M|y-x|^{\beta} \quad \text{for all } x,y \in \langle a,b\rangle.$$

The collection of such functions is denoted by $Lip_M(\beta)$. When $\beta = 1$, the function is said to be Lipschitz continuous. Moreover, for $\beta > 1$, the only functions that satisfy the Lipschitz condition are the constant functions.

1.2 Developments of Positive Linear Operators in Approximation Theory

Now that we have defined positive linear operators and the various tools to study convergence, we need to study whether a given sequence of positive linear operators $\left\{\mathscr{L}_n(f(t);x)\right\}_{n\geq 1}$, defined for functions belonging to compact intervals, converges to a continuous function or not. It is obviously pointless to substitute every such function $f \in C[a,b]$ into the sequence and check for convergence individually. So, a natural question arises that is there a minimal set of functions such that convergence on this set guarantees convergence for all continuous functions? This is precisely where Korovkin's theorem provides an elegant and powerful answer. It asserts that if the sequence of positive linear operators converges on just three specific test functions, that is, the constant function 1, the identity function x, and the quadratic function x^2, then it converges uniformly on the entire space $C[a,b]$. This result significantly simplifies the convergence analysis and allows us to study approximation a sequence of positive linear operators in a far more manageable way.

Theorem 1.2.1 (Korovkin's Theorem for $C[a,b]$) *If the three conditions*

$$\mathscr{L}_n(1;x) = 1 + \alpha_n(x),$$
$$\mathscr{L}_n(t;x) = x + \beta_n(x),$$
$$\mathscr{L}_n\left(t^2;x\right) = x^2 + \gamma_n(x)$$

are satisfied for the sequence of positive linear operators $\mathscr{L}_n(f(t);x)$, where $\alpha_n(x)$, $\beta_n(x)$, $\gamma_n(x)$ converge uniformly to zero in the interval $a \le x \le b$, then the sequence $\mathscr{L}_n(f(t);x)$ converges uniformly to the function $f(x)$ in this interval, provided $f(t)$ is bounded and continuous in the interval $[a,b]$.

Here, $1, x$ and x^2, are known as the test functions, also denoted by e_i (for $i = 1,2,3$), and the values of $\mathscr{L}_n(f(t);x)$ obtained at these test functions are known as the moments of the given sequence of positive linear operators.

A result similar in spirit to Korovkin's theorem was obtained earlier by H. Bohman in 1952 [35]. He studied the positive linear operators on the space $C[0,1]$, defined by

$$\mathscr{L}_n(f;x) = \sum_{i\in I} f(a_i)\,\phi_i(x),$$

where $\{a_i\} \subset [0,1]$ is a finite set of points and each ϕ_i is a continuous function on $[0,1]$. Bohman showed that if such operators preserve the test functions 1, x, and x^2, then they converge uniformly to any function $f \in C[0,1]$. Although this result was more limited in form compared to the general Korovkin theorem, it was based on the same key idea: convergence on a few test functions is enough to ensure convergence on the whole space. Because of this, Korovkin's first theorem is often referred to as the Bohman-Korovkin theorem.

The classical Bohman-Korovkin theorem discussed above applies to continuous functions defined on compact intervals. However, in many practical scenarios, especially those involving periodic functions or functions defined on unbounded domains, the space $C[a,b]$ is no longer sufficient. To accommodate such cases, several extensions and generalizations of Korovkin's first theorem have been developed.

One such extension is Korovkin's second theorem, which is formulated on the space $C[0,2\pi]$, the space of all 2π-periodic continuous real-valued functions on $\mathbb{R}$. In this setting, the role of test functions is played by the basic trigonometric functions:

$$e_0(t) = 1, \quad e_1(t) = \cos t, \quad e_2(t) = \sin t.$$

While Korovkin's theorems provide a powerful criterion for uniform approximation on compact intervals and for periodic functions, they cannot be directly applied when the domain is unbounded, such as $[0,\infty)$. The difficulty arises because on an unbounded domain, uniform convergence alone is not sufficient and the function requires to have some finite limit at infinity. Thus, in 1974, A. D. Gadzhiev introduced a weighted approximation process that extends the idea of Korovkin-type theorems to non-compact settings [82]. He considered functions within the weighted space $C_\rho[0,\infty)$, defined in Section 1.1.3.

Theorem 1.2.2 (Korovkin-type Theorem on Unbounded Intervals [82]) *Let the function $\sigma:[0,\infty)\to[0,\infty)$ be continuous, strictly increasing and unbounded on $[0,\infty)$. Define the weight function as $\rho(x)=1+\sigma^2(x)$, and consider the weighted space*

$$C_\rho[0,\infty)=\left\{f\in C[0,\infty):\exists\, M>0 \text{ such that } |f(x)|\leq M\rho(x),\forall x\in[0,\infty)\right\},$$

equipped with the norm

$$\|f\|_\rho=\sup_{x\in[0,\infty)}\frac{|f(x)|}{\rho(x)}.$$

If a sequence of positive linear operators $\mathscr{L}_n:C_\rho[0,\infty)\to C_\rho[0,\infty)$ satisfies

$$\lim_{n\to\infty}\left\|\mathscr{L}_n\left(\sigma^i(t);x\right)-\sigma^i(x)\right\|_\rho=0,\quad \text{for } i=0,1,2,$$

then for every $f\in C_\rho[0,\infty)$ such that $\lim_{x\to\infty}\frac{f(x)}{\rho(x)}$ exists and is finite, we have

$$\lim_{n\to\infty}\|\mathscr{L}_n(f)-f\|_\rho=0.$$

The Korovkin-type theorems laid the foundation for analyzing the uniform convergence of sequences of positive linear operators. Building on these theorems, researchers began to construct specific families of such operators. These operators not only yielded constructive approximations of continuous functions but also found widespread applicability in numerical analysis, computer-aided geometric design and the solutions of differential and integral equations.

One of the earliest and most influential examples of such an operator was introduced by Sergei N. Bernstein in 1912 [29]. His goal was to provide a constructive proof of the Weierstrass approximation theorem using a sequence of polynomials. The resulting Bernstein operators for a function $f\in C[0,1]$ are defined by:

$$B_n(f;x)=\sum_{k=0}^{n}b_{n,k}(x)f\left(\frac{k}{n}\right),\quad x\in[0,1],\tag{1.1}$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0,1,2,\ldots,n,$$

and $\binom{n}{k}$ represents the binomial coefficient. The polynomials $b_{n,k}(x)$, for $k = 0,1,2,\ldots,n$ and $n \in \mathbb{N}$, are known as the Bernstein polynomials and belong to the space P_n, that is the space of all polynomials of degree at most n. These operators are linear, positive and converge to the test functions 1, x, and x^2. As a result, by Korovkin's theorem, we have

$$B_n(f;x) \to f(x) \quad \text{uniformly on } [0,1] \text{ as } n \to \infty,$$

for every $f \in C[0,1]$. The Bernstein operators thus provide an explicit and elegant sequence that approximates any continuous function uniformly on the unit interval.

Proposition 1.2.3 *Some important properties of the Bernstein polynomials $b_{n,k}(x)$, are listed as follows:*

(i) $b_{n,k}(x)$, *where* $k = 0,1,2\ldots n$, *form a basis for* P_n.

(ii) $b_{n,k}(x) = b_{n,n-k}(1-x)$ *for* $k = 0,1,2,\ldots,n$ *and* $n \in \mathbb{N}$.

(iii) $b_{n,k}(x) \geq 0, \forall x \in [0,1],\ k = 0,1,2,\ldots,n$ *and* $n \in \mathbb{N}$.

(iv) $\sum\limits_{k=0}^{n} b_{n,k}(x) = 1$ *for all* $x \in [0,1]$ *and* $n \in \mathbb{N}$.

(v) $b_{n,k}(x) = (1-x)\,b_{n-1,k}(x) + x b_{n-1,k-1}(x)$ *for all* $x \in [0,1]$.

However, the Bernstein operators are defined only for approximating functions in the compact interval $[0,1]$. To handle function approximation on unbounded intervals, other families of positive linear operators were developed. In the early 20th century, Mirakyan (1941) [148] and later Szász (1950) [202] independently proposed sequences of operators suitable for approximating functions defined on the semi-infinite interval $[0,\infty)$. The resulting Szász-Mirakyan operators are defined as:

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x)\, f\left(\frac{k}{n}\right), \quad x \in [0,\infty), \tag{1.2}$$

where

$$s_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}.$$

Around the same time, Baskakov introduced another operator designed for approximation on unbounded intervals [26]. The Baskakov operators are an extension of Bernstein operators to $[0,\infty)$ and are defined as

$$V_n(f;x) = \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty), \tag{1.3}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

As the study of approximation theory developed, mathematicians began looking for ways to improve existing operators and expand their use to wider classes of functions. A significant step in this direction was made by L. Kantorovich in 1930 [127]. Unlike the original operators, which evaluate the function f at specific points, the Kantorovich modification replaces these point evaluations with local averages, that is, integrals of the function over small intervals. This change allows the operators to approximate not just continuous functions, but also functions in L^pspaces, which may not be continuous everywhere. To approximate integrable functions on the compact interval $[a,b]$, Kantorovich was the first to define the integral variant of Bernstein operators by replacing the weight function with the average mean of the weight function in the vicinity of the point k/n. Thus, he defined the Bernstein-Kantorovich operators as

$$\hat{B}_n(f;x) = (n+1) \sum_{k=0}^{n} b_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt. \tag{1.4}$$

where $b_{n,k}(x)$ is defined by equation (1.1).

Similarly, the Szász-Kantorovich operators [206], defined on the semi-infinite interval $[0,\infty)$, are constructed using the Szász basis functions $s_{n,k}(x)$ as defined in equation (1.2). These operators are given by:

$$\hat{S}_n(f;x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t)\,dt. \tag{1.5}$$

The Baskakov-Kantorovich operators [78] are integral variants of the classical Baskakov operators. They are defined for functions $f \in L^p[0,\infty)$, where $p \geq 1$, as

$$\hat{V}_n(f;x) = (n-1) \sum_{k=0}^{\infty} v_{n,k}(x) \int_{k/(n-1)}^{(k+1)/(n-1)} f(t)\,dt. \tag{1.6}$$

where $v_{n,k}(x)$ is defined in equation (1.3). The factors $(n+1)$, n and $(n-1)$ in front of the Bernstein-Kantorovich, Szász-Kantorovich and Baskakov-Kantorovich operators respectively, ensures normalization of these operators as $n \to \infty$. Numerous researchers have contributed to the analysis of the approximation properties of Kantorovich operators, their rates of convergence and extensions in weighted or generalized function spaces. Some of the notabble works are given in [11; 61; 91; 124; 157; 159; 223].

A more refined generalization of these operators was introduced by J.L. Durrmeyer in 1967 [73]. He proposed an integral modification of the classical Bernstein operators, wherein instead of evaluating the function f at discrete points k/n, he replaced them with an integral of the function weighted by the Bernstein basis functions over the interval $[0,1]$. This construction led to what are now known as the Bernstein-Durrmeyer operators, which not only preserve positivity and linearity but also provide improved approximation properties for functions in L^p spaces. The Bernstein-Durrmeyer operators were first studied by Derrienic [62] in 1981, and are defined as

$$\tilde{B}(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x)\int_{0}^{1} b_{n,k}(t)\, f(t)\, dt. \tag{1.7}$$

Inspired by this approach, further generalizations of Durrmeyer-type operators were introduced for other classical operators. In 1985, Mazhar and Totik extended this idea to the Szász-Mirakyan operators, resulting in the Szász-Durrmeyer operators [146],and in the same year, Sahai and Prasad introduced the Baskakov-Durrmeyer operators [186] as the integral analogue of the classical Baskakov operators. The Szász-Durrmeyer operators are defined as

$$\tilde{S}_n(f;x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_{0}^{\infty} s_{n,k}(t)\, f(t)\, dt. \tag{1.8}$$

Similarly the Baskakov-Durrmeyer operators are defined as

$$\tilde{V}_n(f;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x)\int_{0}^{\infty} v_{n,k}(t)\, f(t)\, dt. \tag{1.9}$$

These Durrmeyer-type operators preserve many desirable properties of the classical operators while enhancing their approximation capability in integral norms. Their study has led to a rich body of literature on modified positive linear operators. For further details, readers are referred to [4; 8; 44; 83; 105].

The Bernstein operators hold significant importance in approximation theory, offering a powerful framework for constructing polynomial approximations used in computer-aided design and graphics through the creation of Bézier curves. Zeng and Piriou [224] were among the first to study the Bézier variant of Bernstein operators. Bézier curves, originally developed by the French engineer Pierre Étienne Bézier at Renault, are widely used in computer graphics, geometric design, interpolation, approximation, and curve fitting. In vector graphics, these curves are instrumental in modeling smooth shapes and are also employed in animation design due to their parametric flexibility and smoothness. Building on this idea, Chang [89] introduced Bézier variants for generalized Bernstein operators and investigated their approximation properties. He defined the generalized Bernstein-Bézier polynomial of $\phi(x)$ as

$$B_{n,\alpha}(\phi;x) = \sum_{i=0}^{n} \left(f_{n,i}^{\alpha}(x) - f_{n,i+1}^{\alpha}(x)\right) \phi\left(\frac{i}{n}\right),$$

where $f_{n,i}$; $i = 1,2\ldots n$; denote the n^{th} Bézier basis functions. Later, Zeng and Chen proposed the Bézier-Bernstein-Durrmeyer operators and analyzed their rate of convergence [222]. Since then, many researchers have contributed to this area by constructing Bézier-type variants of various positive linear operators and studying their approximation behaviour. For further developments and detailed discussions, the reader may refer to the works in [12; 63; 102; 108; 161; 172; 192; 193].

The employment of Bernstein polynomials in spline construction serves as a linchpin connecting various disciplines. Thus, to widen their applicability, especially for functions outside the compact interval $[0,1]$, numerous generalizations and modifications have been proposed. One of the earliest generalizations was proposed by Chlodovsky in 1937 [54], who constructed a version of Bernstein polynomials, extending them from $[0,1]$ to the semi-infinite interval $[0,b_n]$, where $b_n \to \infty$ and $\frac{b_n}{n} \to 0$. He proved that under these conditions, the Chlodovsky operators preserve the convergence behaviour of the classical Bernstein polynomials and can approximate continuous functions on the semi-infinite interval $[0,\infty)$. Moreover, in 1964, Cheney and Sharma observed that positive linear operators are often constructed using classical identities or distributions [52]. For instance, the Bernstein operators are based on binomial distribution and the Szász operators are based on Poisson distribution. Building upon this fact, they developed a one-parameter generalization of the Bernstein operators based on the Jensen's identity $(x+y+n\beta)^n$ for $0 \leq \beta = O\left(n^{-1}\right)$, defined by

$$P_n(f;x) = (1+n\beta)^{-n} \sum_{v=0}^{n} \binom{n}{v} x(x+v\beta)^{v-1}[1-x+(n-v)\beta]^{n-v} f\left(\frac{v}{n}\right).$$

This work has since been studied and extended by several authors, as seen in [9; 39; 156; 163; 165; 173].

Inspired by this parametric generalization of the Bernstein operators, several authors tried to generalize the family of positive linear operators to include a wider class of functions. One such generalization was made by D.D. Stancu in 1972, wherein he introduced some new classes of positive linear operators, depending on some real parameters [196]. Recently, another notable extension of the classical Bernstein operators has emerged in the form of the α-Bernstein operators, introduced by Chen et al. in 2017 [51]. The idea behind the α-Bernstein operators is to modify the original Bernstein basis functions by incorporating this parameter in a way that preserves positivity and linearity while enhancing the approximation properties in certain contexts. These operators reduce to the classical Bernstein operators [29] for $\alpha = 0$. In this thesis, the α-Bernstein operators are further generalized by introducing a new parameter γ, through the incorporation of the Pólya distribution.

The Pólya distribution was first introduced into the classical Bersntein operators by D.D. Stancu in 1968 [194]. Since then, it has been extensively studied, leading to the development of several new classes of positive linear operators. For instance, Vijay Gupta established a link between the generalized Bernstein polynomials based on Pólya distribution and its Kantorovich type modification, using backward difference operator [98]. In 2023, Lipi and Deo extended the λ-Bernstein operators by integrating the Pólya distribution, aiming to improve their adaptability in approximation results [139]. The λ-Bernstein operators, introduced by Cai et al. [42], represent another important extension of the classical Bernstein operators and are defined as

$$B_{n,\lambda}(f;x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) f\left(\frac{k}{n}\right), \tag{1.10}$$

such that the basis, $\tilde{b}_{n,k}(\lambda;x)$, is given by

$$\begin{cases} \tilde{b}_{n,0}(\lambda;x) = b_{n,0}(x) - \dfrac{\lambda}{n+1} b_{n+1,1}(x) \\ \tilde{b}_{n,i}(\lambda;x) = b_{n,i}(x) + \lambda \left(\dfrac{n-2i+1}{n^2-1} b_{n+1,i}(x) - \dfrac{n-2i-1}{n^2-1} b_{n+1,i+1}(x) \right) \\ \tilde{b}_{n,k}(\lambda;x) = b_{n,n}(x) - \dfrac{\lambda}{n+1} b_{n+1,n}(x), \end{cases}$$

where $b_{n,i}(x)$ is defined in equation (1.1). These operators incorporate a real parameter λ, which plays a role in adjusting the weight distribution of the basis polynomials, thereby offering an additional degree of control in the approximation process. For $\lambda = 0$, the operators (1.10) reduce to the classical Bernstein operators.

1.3 Operator Modifications for Better Approximations

In approximation theory, the introduction of new classes or sequences of positive linear operators is often motivated by the aim to achieve better approximation results. However, such developments lose much of their significance if the resulting operators converge to the desired function with a poor order of convergence. In practical applications, the efficiency of an approximation method depends not merely on the fact of convergence, but also on how good the approximation is. Consequently, many researchers have focused on finding ways to either improve the order of existing operators or develop new ones that preserve desirable properties while achieving better rates of convergence. A part of this thesis is devoted to studying the order of convergence of sequences of positive linear operators and exploring ways to improve it.

Gadjiev and Ghorbanalizadeh [80] conducted one such study, introducing Bernstein-Stancu type polynomials with shifted knots. The use of shifted knots allows approximation on the interval $(0,1)$ and its subintervals, while also enhancing the flexibility of these operators for approximation.

Most positive linear operators do not exactly preserve the test functions x and x^2, instead, they produce expressions that converge to them. However, the ability of an operator to preserve or closely approximate these test functions plays a crucial role in determining its effectiveness. An influential approach for improving the order of approximation was introduced by King in his pioneering work [130]. He proposed a non-trivial sequence of positive linear operators defined on $C[0,1]$ that preserve the test functions e_0 and e_2, where $e_i(x) = x^i$. Let $\{r_n(x)\}$ be a sequence of continuous functions on $[0,1]$ satisfying $r_n(x) \in [0,1]$. The operators $V_{n,r_n} : C[0,1] \to C[0,1]$ are then defined by

$$V_{n,r_n}(f;x) = \sum_{k=0}^{n} \big(r_n(x)\big)^k \big(1 - r_n(x)\big)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where

$$r_n(x) = \begin{cases} x^2, & n = 1, \\ -\dfrac{1}{2(n-1)} + \sqrt{\dfrac{n}{n-1}x^2 + \dfrac{1}{4(n-1)^2}}, & n \geq 2. \end{cases}$$

These operators interpolate f at the endpoints $x = 0$ and $x = 1$, but unlike classical Bernstein operators, they are not polynomial. King demonstrated that, for $x \in \left[0, \frac{1}{3}\right]$, the order of approximation of V_{n,r_n} is at least as good as that of the Bernstein operators.

Motivated by King's idea, several researchers have proposed modifications of well-known operators to preserve specific functions and improve both approximation quality and shape-preserving properties. For example, in 2006, Cárdenas-Morales et al. [47] introduced a family of Bernstein-type operators $B_{n,\alpha}$, $n > 1$, depending on a real parameter $\alpha \geq 0$, that fix the polynomial $e_2 + \alpha e_1$. They proved that if f is convex and increasing on $[0,1]$, then $f(x) \leq B_{n,\alpha}(f;x) < B_n(f;x)$, $\forall x \in [0,1]$. In 2017, Acar et al. [5] modified the Szász-Mirakyan operators so that they preserve the function e^{2at}, and a similar modification was applied to Baskakov-Szász-Stancu type operators by Bodur et al. in 2018 [34]. In further developments, Acar et al. [5] introduced a version of the Szász-Mirakyan operators that simultaneously preserved both e^{at} and e^{2at}, extending the flexibility of these operators in preserving exponential behaviour. In a related direction, Gupta and Tachev [106] examined Phillips-type operators that could fix either e^{-t} or e^{At} for a real constant A, but noted that both functions could not be preserved at the same time. To overcome this limitation, Gupta and Moreno introduced a modified sequence of Phillips operators capable of fixing both e^{at} and e^{bt}, simultaneously, for any real numbers a and b, regardless of whether they are equal or distinct [103]. In 2020, Usta [209] introduced sequences of positive linear operators defined on both bounded and unbounded intervals, which preserve the function sets $\{1,\varphi\}$ and $\{1,\varphi^2\}$, where $\varphi \in C[0,1]$. He then constructed different operators by varying the choice of φ and associated parameters. Moreover, in 2021, Lipi and Deo presented a modification of the exponential-type operators originally introduced by Ismail and May, with a particular focus on constructing a sequence of operators that preserves both the constant functions and the function e^{-t} [137]. In the same year, Mishra and Deo extended this work to construct a modification of the Ismail-May operators which preserve exponential functions of the form e^{At}, where $A \in \mathbb{R}$. Their results demonstrated that these operators not only retain the exponential function but also exhibit an improved rate of convergence when $A > 0$ [150]. On a parallel track, attention has also been given to the preservation of the polynomial test functions. In 2022, Lipi and Deo modified certain gamma-type operators which preserve functions of the form t^{ϑ}, where ϑ is a non-negative integer [138]. Their analysis showed that their modified operators offered best approximation when preserving the function t^3. In this thesis, the parametric form of α-Bernstein operators, with parameters $\alpha, \gamma \in [0,1]$, are introduced and preserved for the test functions e_0 and e_2.

This thesis introduces an improved modification of the parametric form of α-Bernstein operators, that preserve the test functions e_0 and e_2, achieve reduced error and faster convergence. To enhance the order of approximation of operators where the linear test function e_1 is not preserved, we also introduce a genuine-type modification.

1.4 Chapter-wise Overview of the Thesis

The thesis consists of nine chapters, whose contents are described below:

Chapter 1 provides the introductory literature regarding the title of the thesis. Additionally, the already established definitions and lemmas are provided as a part of the preliminary section of this chapter. We also provide a concise overview of the structure of the thesis, outlining the content of subsequent chapters.

Chapter 2 is based on the Pólya-Eggenberger distribution, also known as contagion distribution, a discrete probability distribution that is associated with the Pólya-Eggenberger urn scheme. The contagion distribution was first utilized by D.D. Stancu to form a sequence of positive linear operators using integral operators for $n \geq 2$. This chapter deals with the extended form of these operators based on two real parameters, α and γ, such that, $\alpha,\ \gamma \in [0,1]$ and $\gamma = \gamma(n) \to 0$. It is divided into two sections. The first section of this chapter introduces the parametric form of Bernstein polynomials by incorporating contagion distribution into the α-Bernstein operators [51]. We observe the significance of both the parameters utilized. Subsequently, we calculate their moments and on the basis of Korovkin's approximation theorem we assert that our defined operators converge for a real valued continuous function h. We give some basic properties, study the Voronovskaya type asymptotic result and discuss the A-Statistical approximation of our operators. We also use the modulus of continuity to study the rate of convergence. Alongside our primary work, we have also defined the King-type modification which preserves the operators at e_0 and e_2.

The second section constructs the Kantorovich type generalization of these operators, which helps to approximate integrable functions. First, we give some auxiliary properties and then we study its asymptotic results with the help of Taylor's expansion. Like the classical Kantorovich-Bernstein operators, the Kantorovich variant defined in this section does not preserve e_1, that is, $K_{m,\alpha}^{\langle\gamma\rangle}(t;u) \neq u$. Rather, we get an expression which tends to u as m tends to ∞. As an opening at finding a better approximating linear operators, we try and preserve these operators at e_1 and propose a genuine-type modification. We also include graphical illustrations to analyze and compare the approximation results and properties of both the Kantorovich variant of the α-Bernstein operators using contagion distribution and its genuine-type modification.

Over the past few decades, researchers have extensively studied and worked on exponential-type operators. Very recently, the concept of semi-exponential operators, an extension of the exponential operators, came into light. Thus, **chapter 3** mainly focuses on the class of semi-exponential operators, particularly on semi-exponential Bernstein operators. The analysis begins by considering the partial differential equation,

$$\frac{\partial}{\partial x}W_\beta\left(r,x,t\right) = \frac{r\left(t-x\right)}{p(x)}W_\beta\left(r,x,t\right) - \beta W_\beta\left(r,x,t\right)$$

for the kernel of sequence of operators

$$S_r\left(f;x\right) = \sum W_\beta\left(r,x,\frac{k}{r}\right) f\left(\frac{k}{r}\right)$$

while satisfying the normalization condition $\sum W_\beta\left(r,x,\frac{k}{r}\right) = 1$. Carrying forward this notion, we examine these operators specifically for the function $p(x) = x(1-x)$ and focus our attention on the semi-exponential Bernstein operators. Our aim is to explore the properties and characteristics of these Bernstein type semi-exponential operators. This chapter delves into deriving a recurrence relation for the central moments of these operators, while also establishing a moment generating function for the same. Finally, we analyze the approximation of these operators using the Voronovskaya-type asymptotic result.

In **chapter 4**, we derive a new sequence of positive linear operators by using the concept of fractional calculus, an extension of the classical calculus for integrals and derivatives of non-integer order α. The formulation starts by taking the classical Bernstein operators and differentiating them using the Caputo fractional derivative of order α. Using the definition of fractional integral I^α and Caputo fractional derivative on the classical Bernstein operators, we get

$$I^{\lceil\alpha\rceil-\alpha}\frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}}B_{n+1}(F), \qquad \text{where } F(x)=\int_0^x f(t)dt.$$

Solving and appropriately modifying this expression, we arrive at a new sequence of positive linear operators which falls under the class of 'Positive Linear Fractional Operators'. This chapter specifically focuses on the Fractional Kantorovich Bernstein Operators. We give the moments of these operators with the help of Laplace transform. Using these moments and the Korovkin theorem we prove their convergence properties. Furthermore, the applications of these operators in solving linear fractional differential equations as well as fractional integro-differential equations using the basis polynomial of the fractional Kantorovich Bernstein operators is also shown. Various numerical illustrations are provided to better understand this approach. Further, the credibility of these approximate solutions is verified graphically and through error tables.

Chapter 5 extends the main results of classical approximation theory to fuzzy theory. We begin by defining fuzzy valued functions and then apply the fuzzy Korovkin theorem to approximate them. The study delves into the approximation of various linear positive fuzzy operators, utilizing them to approximate fuzzy-valued functions. This chapter is divided into two sections. The first section introduces fuzzy Bernstein, Szász-Mirakyan and Baskakov operators and by using them we approximate fuzzy-valued functions. In the second section, our focus shifts to fuzzy Boas-Buck operators inspired by the work of Ralph P. Boas and R. Creighton Buck [32; 33]. They considered the generalized Appell polynomials by means of generating function of the type

$$\mathscr{W}(t)\,\mathscr{P}(x\mathscr{Q}(t))=\sum_{k=0}^{\infty}a_k(x)t^k, \tag{1.11}$$

where $\mathscr{W}$, $\mathscr{P}$ and $\mathscr{Q}$ are analytic functions

$$\begin{aligned}\mathscr{W}(t) &= \sum_{k=0}^{\infty} w_k t^k,\ \ w_0\neq 0,\\ \mathscr{P}(t) &= \sum_{k=0}^{\infty} p_k t^k,\ \ p_k\neq 0,\\ \mathscr{Q}(t) &= \sum_{k=1}^{\infty} q_k t^k,\ \ q_1\neq 0.\end{aligned} \tag{1.12}$$

Based on this generating function, Sucu et al. [200] presented linear positive operators using Boas-Buck type polynomials that satisfy the following conditions:

$$\begin{aligned}
&(i)\ \mathscr{W}(1)\neq 0,\ \ \mathscr{Q}'(1)=1,\ \ a_k(x)\geq 0,\ k=0,1,2,\ldots,\\
&(ii)\ \mathscr{P}:\mathbb{R}\to(0,\infty),\\
&(iii)\ \text{For } |t|<\rho \text{ (where } \rho>1\text{), the equations (1.11) and (1.12) are convergent.}
\end{aligned} \tag{1.13}$$

This section broadens the scope of previous research on real operators to fuzzy sense. We define the fuzzy Boas Buck operators, based on the generating function (1.11) and assumptions (1.13). We prove that they are fuzzy linear and positive and give their moments and central moments. Using fuzzy Korovkin theorem, we prove their convergence property. Further, we study their rate of convergence using the fuzzy modulus of continuity and fuzzy weighted modulus of continuity. Since, the fuzzy Boas-Buck operators are a very generalized form of operators, so we also give two of its particular examples, namely Laguerre and Charlier operators, and further talk about their convergence properties. This chapter also presents special cases of the fuzzy Boas-Buck operators which includes the fuzzy Brenke, Sheffer, Appell and Szász operators for which further studies are possible in fuzzy sense.

Chapter 6 introduces a new sequence of positive linear operators with improved order of approximation with the help of semi-exponential operators. The classical case of semi-exponential operators has already been discussed in this thesis. However, it turns out that these operators have an approximation order of $O(1/n)$, at most (see [116; 208]). Many authors have made tremendous efforts to improve the approximation of continuous functions. This chapter aims to improve the order of approximation for the semi-exponential Bernstein operators by defining new operators involving the function as well as its higher order derivatives. First, we define the second order semi-exponential Bernstein operators, give their moments and prove their asymptotic results. Further, we define the third order semi-exponential Bernstein operators, give their moments and central moments and derive their Voronskaya type asymptotic result. We verify these results using numerical illustrations.

Chapter 7 talks about the order of approximation of sequence of positive linear operators based on Pólya-Eggenberger distribution, involving a positive real parameter $\alpha\in[0,1]$. The sequence of first order operators has already been defined by Stancu in 1968 [194], in the following manner:

$$P_n^{\langle\alpha\rangle}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{n}\right),$$

where $p_{n,k}^{\langle\alpha\rangle}(x) = \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}$ and α is a non-negative parameter. The factorial power is determined by $a^{[n,\nu]} = a(a-\nu)(a-2\nu)\ldots(a-(n-1)\nu)$ and $a^{[0,\nu]} = 1$. We extend this notation of the Pólya-Eggenberger distribution, with the parameter α, to approximate continuous real-valued function on $[0,1]$, with a better order of convergence. For a continuous real-valued function f defined on $[0,1]$ such that f'' exits, we define the higher order Stancu operators as

$$Q_n^{\langle\alpha\rangle}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\left[f\left(\frac{k}{n}\right) - \frac{cx(1-x)}{n(1+\alpha)} f''\left(\frac{k}{n}\right)\right],$$

where $c = \lim_{n\to\infty} \frac{1+n\alpha}{2}$. We prove that their order of approximation is $O(1/n^2)$, give their moments and central moments and using Korovkin theorem prove their uniform convergence on the compact interval $[0,1]$. Using first and second order modulus of continuity, the approximation properties of the defined second order operators (7.3) are also proved. Further, graphical and numerical illustrations are presented to support the theoretical findings of these operators.

Chapter 8 again focuses on the development of sequence of higher order positive linear operators based on Pólya-Eggenberger distribution, but without the use of higher order derivatives. This chapter aims to improve the order of approximation by defining a modification of the classical Stancu operators with the help of sequences of natural numbers. Initially, we introduce the first order modification of the Stancu operators and derive their moments. Using Korovkin theorem, we prove uniform convergence and with the help of Petrre's K-functional and modulus of continuity, we show various convergence properties of these operators as well. Thereafter, extending this concept of sequences, we also define a sequence of second order positive linear operators based on Pólya distribution. The uniform convergence of these operators using Korovkin theorem is analyzed and a necessary condition for their convergence is established. Based on this condition we develop particular second order operators and derive their moments and central moments. Moreover, the results established are verified graphically as well.

Following Chapter 8, the conclusion of the work, an outline of the future scope of the research and a discussion on the possible social impacts of the study are presented in **Chapter 9**. The thesis concludes with a bibliography and a list of the author's publications.

We now move on to our next chapter, which explores the extension of α-Bernstein operators based on the Pólya-Eggenberger distribution, also known as the contagion distribution.

Chapter 2

Parametric Bernstein operators based on contagion distribution

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This chapter proposes an extended form of the α-Bernstein operators incorporating the Pólya-Eggenberger (contagion) distribution, constructed using the integral operators for $m \geq 2$. We refer to this sequence as the Parametric Bernstein operators based on contagion distribution, or simply the modified α-Bernstein operators. These operators are defined with the help of two parameters, namely, α and γ. We observe the significance of both these parameters. Subsequently, we calculate their moments and on the basis of Korovkin's approximation theorem we assert that our defined operators converge for a real valued continuous function. We give some basic properties, study the Voronovskaya type asymptotic result and discuss the A-statistical approximation of our operators. We also use the modulus of continuity to study the rate of convergence. Alongside our primary work, we have also defined the King-type modification that p-reserves the operators at e_0 and e_2. Further, using modulus of continuity, we compare the properties and behavior of both the operators. The second section develops a Kantorovich variant of the proposed operators, extending their applicability to integrable functions. Since this variant does not preserve the linear test function e_1, we introduce a genuine-type modification as an attempt towards finding better approximating linear operators. Graphical illustrations are provided to analyze and compare the approximation results and properties of both the Kantorovich variant and its genuine-type modification.

2.1 Parametric Bernstein Operators

The Bernstein operators hold significant importance in approximation theory, offering a powerful framework for constructing polynomial approximations used in computer-aided design and graphics through the creation of Bézier curves. The employment of Bernstein polynomials in spline construction serves as a linchpin connecting various disciplines. Additionally, their role in probability and statistics underscores their significance in Bayesian statistics. Over the past years, several studies have been made on these operators and their applications. However, the Bernstein operators are not very much applicable in modeling some of the real-life scenarios. For instance, situations where the probability of occurrence of an outcome increases as more of that outcome occurs can not be captured by binomial distribution. This is where the Pólya-Eggenberger distribution comes into picture. In the year 1923, Pólya and Eggenberger formulated a discrete probability distribution that is associated with the Pólya-Eggenberger urn scheme. This distribution, commonly referred to as the contagion distribution or the Pólya distribution, was derived using an urn model [74]. The distribution represents a straightforward procedure: An urn holds U red balls and V blue balls, and one ball is selected randomly from it. The colour of the ball is recorded and is substituted along with X identical, same coloured balls. The aforementioned procedure is iterated m times. The occurrence of obtaining a red (or blue) ball in the j^{th} draw is represented by the random variable Y_j, which takes the value 1 (or 0) to indicate the event. The probability of observing a total of k red drawings, denoted as $k = \sum Y_j$, can be determined by the following expression:

$$\rho_{m,k} = \binom{m}{k} \frac{\prod_{i=0}^{k-1} (U+iX) \prod_{i=0}^{m-k-1} (V+iX)}{\prod_{i=0}^{m-1} (U+V+iX)}. \tag{2.1}$$

The year 1968 saw the introduction of linear positive operators $P_m^{\langle\gamma\rangle} : C[0,1] \to C[0,1]$, defined by D. D. Stancu [194; 195], based on the above defined contagion distribution (2.1) with density function, as:

$$P_m^{\langle\gamma\rangle}(f;x) = \sum_{k=0}^{m} \binom{m}{k} \frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}} f\left(\frac{k}{m}\right), \tag{2.2}$$

where, γ is a non-negative parameter and the factorial power is defined as $a^{[0,\nu]} = 1$ and $a^{[n,\nu]} = a(a-\nu)(a-2\nu)\ldots(a-(n-1)\nu)$. This represents a polynomial of degree n.

The operators (2.2) can actually be derived from a more general class of positive linear operators $\mathscr{N}_m^{\langle\gamma\rangle}$, constructed from an arbitrary sequence of positive linear operators $\mathscr{D}_m$. These generalized sequence of positive linear operators are defined as

$$\mathscr{N}_m^{\langle\gamma\rangle}(f;x) = \frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1 t^{\frac{x}{\gamma}-1}(1-t)^{\frac{1-x}{\gamma}-1}\mathscr{D}_m(f;t)\,dt, \tag{2.3}$$

where $\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$ is the Beta function and $\mathscr{D}_m(f;t)$ is a known sequence of some positive linear operators [196]. The operators $P_m^{\langle\gamma\rangle}$ given in (2.2) arise as a special case of this construction when $\mathscr{D}_m$ is chosen to be the classical Bernstein operators.

In this chapter, we focus on a particular case of the generalized framework $\mathscr{N}_m^{\langle\gamma\rangle}$ by selecting a specific sequence of base operators $\mathscr{D}_m$ and an appropriate parameter choice. This leads to the construction of a new sequence of operators which retain the desirable approximation properties of the contagion distribution. The formulation of these operators and an analysis of their fundamental approximation results are presented in the subsequent sections.

2.1.1 Construction of the proposed Operators

To enhance the accuracy and versatility of approximation techniques, researchers have pursued numerous modifications of Bernstein operators (see [14; 42; 139; 158]). Among these modifications and generalizations, a notable stride has been made by Chen et al. [51] in 2017, with the introduction of α-Bernstein operators, for any function f having its domain as $[0,1]$ and defined by

$$\begin{aligned}\mathrm{B}_{m,\alpha}(f;x) = \sum_{k=0}^{m}\Bigg[&\binom{m-2}{k}(1-\alpha)x + \binom{m-2}{k-2}(1-\alpha)(1-x)\\ &+\binom{m}{k}\alpha x(1-x)\Bigg]x^{k-1}(1-x)^{m-k-1}f_k, \qquad m\geq 2\end{aligned} \tag{2.4}$$

where $f_k = f\left(\frac{k}{m}\right)$ and f is defined on $[0,1]$. These α-Bernstein operators given by equation (2.4) can be simplified and written as

$$\mathrm{B}_{m,\alpha}(f;x) = (1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}x^k(1-x)^{m-k-1}g_k + \alpha\sum_{k=0}^{m}\binom{m}{k}x^k(1-x)^{m-k}f_k \tag{2.5}$$

where $f_k = f\left(\frac{k}{m}\right)$ and $g_k = \left(1-\frac{k}{m-1}\right)f_k + \frac{k}{m-1}f_{k+1}$.

In recent studies, several mathematicians (see the literature [40; 43; 61; 153]) gave various modifications of α-Bernstein operators (2.4). These modifications have been analyzed in terms of their auxiliary properties and used modulus of continuity to observe the rate of convergence, analyzed the direct local approximation theorem, function of bounded variation and Voronovskaya type asymptotic result.

Now, we consider another modification of the α-Bernstein operators with the help of contagion distribution as density function, defined for $m \geq 2$ as

$$\begin{aligned} \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) = \sum_{k=0}^{m} \Bigg[& \binom{m-2}{k}(1-\alpha)\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}} \\ & + \binom{m-2}{k-2}(1-\alpha)\frac{x^{[k-1,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m-1,-\gamma]}} + \binom{m}{k}\alpha\frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}}\Bigg] h_k, \end{aligned} \tag{2.6}$$

where $h_k = h\left(\frac{k}{m}\right)$. Let us denote the polynomial inside summation as $p_{m,k}^{\langle\alpha,\gamma\rangle}(x)$, that is,

$$\begin{aligned} p_{m,k}^{\langle\alpha,\gamma\rangle}(x) = & \binom{m-2}{k}(1-\alpha)\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}} + \binom{m-2}{k-2}(1-\alpha)\frac{x^{[k-1,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m-1,-\gamma]}} \\ & + \binom{m}{k}\alpha\frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}}. \end{aligned}$$

Some basic calculations will give us the alternate form of $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$ as

$$\begin{aligned} \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) = & (1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}} g_k \\ & + \alpha\sum_{k=0}^{m}\binom{m}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}} h_k, \end{aligned} \tag{2.7}$$

where $h_k = h\left(\frac{k}{m}\right)$ and $g_k = \left(1 - \frac{k}{m-1}\right)h_k + \frac{k}{m-1}h_{k+1}$.

2.1.1.a Special cases

1. For $\gamma = 0$ and $\alpha = 1$, the proposed operators $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$ transform into the classical Bernstein operators.

2. For $\alpha = 1$, the proposed operators transform into the operators (2.2), proposed by D. D. Stancu.

3. For $\gamma = 0$, we get the operators (2.4) as defined by Chen et al. for $m \geq 2$.

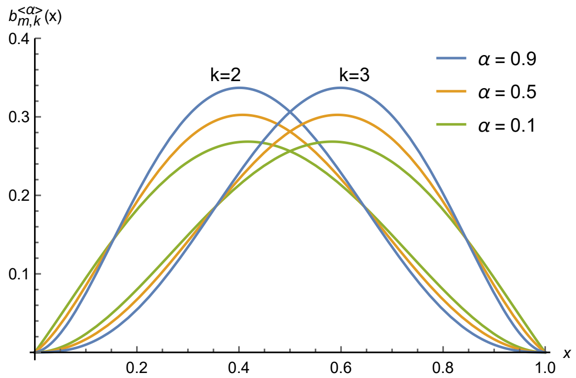


Figure 2.1: Graph of the basis function of α-Bernstein operators (2.4), for $m = 5$, $k = 2$ and $k = 3$ and different values of α.

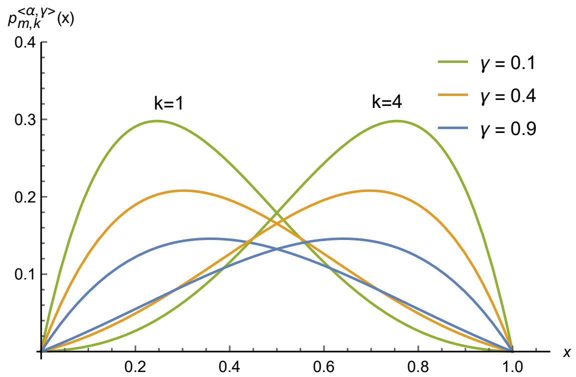


Figure 2.2: Graph of the basis function of the parametric Bernstein operators (2.6), for $m = 5$, $k = 1$ and $k = 4$, $\alpha = 0.5$ and different values of γ.

From Figures 2.1 and 2.2 it is clear that like in the classical Bernstein, α-Bernstein and the Stancu-Bernstein operators, the basis polynomial, $p_{m,k}^{\langle\alpha,\gamma\rangle}(x)$ as defined with the help of operators (2.6) are also symmetric in $[0,1]$. Contagion distribution is particularly important and applicable in various real-world scenarios where the probability of an event depends on its past occurrences.

2.1.1.b Comparison of the parameters α and γ

This chapter aims to analyze the results of approximations for the sequence of operators (2.7), which have been defined by the utilization of two parameters, namely α and γ. Both entities are restricted to exist within the closed interval $[0,1]$. In order to gain a better understanding of the impact of these parameters on the process of approximation, an analysis is performed using a specific example, with the function $h(x) = \cos(2\cos 5x)$.

Initially, we fix the value of γ and then observe how varying α affects the results (as shown in Fig. 2.3). Remarkably, it is observed that despite substantial variations in α, its impact on the enhancement of the approximation process is not significant. In contrast, modifying the values of γ exerts a substantial impact on the precision of the approximations made by the operators. Notably, smaller values of γ results in more precise approximations, suggesting that as m approaches ∞, γ tends to converge towards 0 (Fig. 2.4).

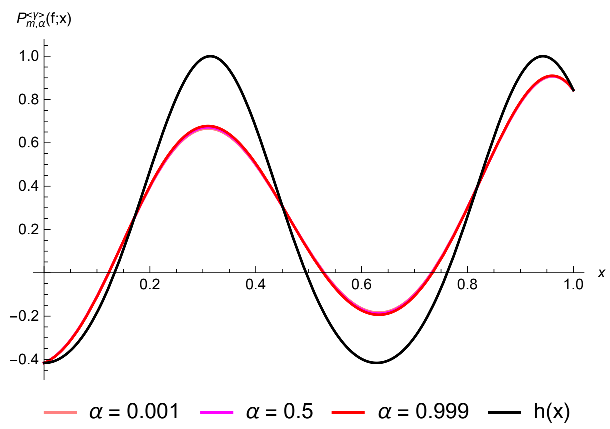


Figure 2.3: Approximation process of $\mathscr{P}^{\langle\gamma\rangle}_{m,\alpha}(\cos(2\cos 5t);x)$ for $m = 30$, when $\gamma = 0.01$ is fixed and α is varying in $[0,1]$.

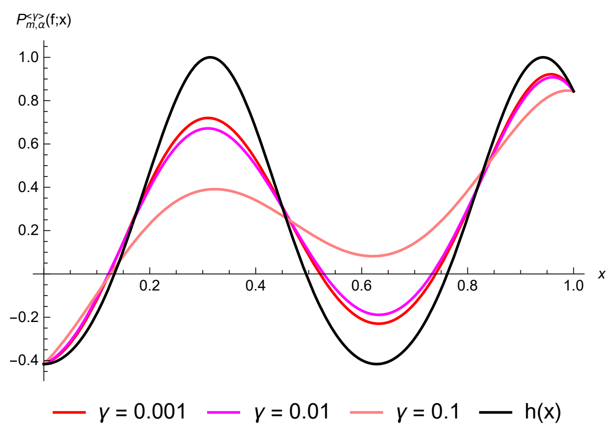


Figure 2.4: Approximation process of $\mathscr{P}^{\langle\gamma\rangle}_{m,\alpha}(\cos(2\cos 5t);x)$ for $m = 30$, when $\alpha = 0.5$ is fixed and γ is varying in $[0,1]$.

Thus we can conclude a relationship between γ and m as $\gamma = O\left({}^{1}\!/_{m}\right)$. This can also be studied in the subsequent section concerning the Korovkin theorem. Nevertheless, it is worth noting that there exists potential of further research by considering simultaneous operators for $\gamma = O\left({}^{1}\!/_{m^a}\right)$, where $a > 0$, or any other function $\gamma(m) \to 0$, which opens door for analysis on the convergence properties in future. Our choice of $\gamma = O\left({}^{1}\!/_{m}\right)$ reduces further complexities.

2.1.2 Auxiliary Results

Before proceeding to the main approximation results, we first establish some supporting lemmas and results that serve as essential tools for establishing the approximation properties of the proposed operators.

Theorem 2.1.1 *The parametric Bernstein operators* (2.7) *are a form of the* α*-Bernstein operators derived from equation* (2.3).

Proof. The operators $\mathscr{N}_m^{\langle\gamma\rangle}(f;x)$, defined by Stancu, can be obtained form an already existing sequence of operators $\mathscr{D}_m(f;t)$. Taking $\mathscr{D}_m(f;t)$ to be the α-Bernstein operators, with $h_k = h\left(\frac{k}{m}\right)$ and $g_k = \left(1 - \frac{k}{m-1}\right)h_k + \frac{k}{m-1}h_{k+1}$, we get the sequence of operators (2.7), and the same can be proved as follows:

$$
\begin{aligned}
&\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) \\
&= (1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}g_k \\
&\quad + \alpha\sum_{k=0}^{m}\binom{m}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}}h_k \\
&= (1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x(x+\gamma)...(x+(k-1)\gamma)(1-x)(1-x+\gamma)...(1-x+(m-k-2)\gamma)}{(1+\gamma)...(1+(m-2)\gamma)}g_k \\
&\quad + \alpha\sum_{k=0}^{m}\binom{m}{k}\frac{x(x+\gamma)...(x+(k-1)\gamma)(1-x)(1-x+\gamma)...(1-x+(m-k-1)\gamma)}{(1+\gamma)...(1+(m-1)\gamma)}h_k \\
&= (1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{\frac{\gamma^k\Gamma\left(k+\frac{x}{\gamma}\right)}{\Gamma\left(\frac{x}{\gamma}\right)}\frac{\gamma^{m-k-1}\Gamma\left(m-k-1+\frac{1-x}{\gamma}\right)}{\Gamma\left(\frac{1-x}{\gamma}\right)}}{\frac{\gamma^{m-1}\Gamma\left(m-1+\frac{1}{\gamma}\right)}{\Gamma\left(\frac{1}{\gamma}\right)}}g_k + \alpha\sum_{k=0}^{m}\binom{m}{k}\frac{\frac{\gamma^k\Gamma\left(k+\frac{x}{\gamma}\right)}{\Gamma\left(\frac{x}{\gamma}\right)}\frac{\gamma^{m-k}\Gamma\left(m-k+\frac{1-x}{\gamma}\right)}{\Gamma\left(\frac{1-x}{\gamma}\right)}}{\frac{\gamma^{m}\Gamma\left(m+\frac{1}{\gamma}\right)}{\Gamma\left(\frac{1}{\gamma}\right)}}h_k \\
&= (1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\beta\left(k+\frac{x}{\gamma}, m-k-1+\frac{1-x}{\gamma}\right)\frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}g_k \\
&\quad + \alpha\sum_{k=0}^{m}\binom{m}{k}\beta\left(k+\frac{x}{\gamma}, m-k+\frac{1-x}{\gamma}\right)\frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}h_k
\end{aligned}
$$

$$= \frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1\left[(1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}t^{k+\frac{x}{\gamma}-1}(1-t)^{m-k-1+\frac{1-x}{\gamma}-1}g_k\right.$$
$$\left.+\alpha\sum_{k=0}^{m-1}\binom{m}{k}t^{k+\frac{x}{\gamma}-1}(1-t)^{m-k+\frac{1-x}{\gamma}-1}h_k\right]dt$$
$$= \frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1 t^{\frac{x}{\gamma}-1}(1-t)^{\frac{1-x}{\gamma}-1}B_{m,\alpha}\,dt.$$

□

We now derive the moments of the Stancu-Bernstein operators (2.2), which will subsequently be employed in obtaining the moments of the proposed operators (2.6).

Lemma 2.1.2 *The moments of the linear positive operators $P_m^{\langle\gamma\rangle}$, defined by* (2.2) *are given by*

(i) $P_m^{\langle\gamma\rangle}(1;x)=1$

(ii) $P_m^{\langle\gamma\rangle}(t;x)=x$

(iii) $P_m^{\langle\gamma\rangle}(t^2;x)=\frac{x}{m}+\frac{(m-1)x}{m}\left(\frac{x+\gamma}{1+\gamma}\right)$

(iv) $P_m^{\langle\gamma\rangle}(t^3;x)=\frac{x}{m^2}+\frac{3(m-1)x}{m^2}\left(\frac{x+\gamma}{1+\gamma}\right)+\frac{(m-1)(m-2)x}{m^2}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)$

(v) $P_m^{\langle\gamma\rangle}(t^4;x)=\begin{cases}\frac{x}{m^3}+\frac{7(m-1)x}{m^3}\left(\frac{x+\gamma}{1+\gamma}\right)+\frac{6(m-1)(m-2)x}{m^3}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\\ +\frac{(m-1)(m-2)(m-3)x}{m^3}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right).\end{cases}$

Proof. We begin by recalling the moments of the classical Bernstein operators (1.1).

- $B_m(1;x)=1$
- $B_m(t;x)=t$
- $B_m\left(t^2;x\right)=x^2+\frac{x(1-x)}{m}$
- $B_m\left(t^3;x\right)=\frac{(m-1)(m-2)x^3}{m^2}+\frac{3(m-1)x^2}{m^2}+\frac{x}{m^2}$
- $B_m\left(t^4;x\right)=\frac{(m-1)(m-2)(m-3)x^4}{m^3}+\frac{6(m-1)(m-2)x^3}{m^3}+\frac{7(m-1)x^2}{m^3}+\frac{x}{m^3}.$

Observe that, in equation (2.3), $\mathscr{N}_m^{\langle\gamma\rangle}(f;x)$ reduces to the Stancu-Bernstein operators (2.2) under the choice $\mathscr{D}_m(f;t)$ = classical Bernstein operators. Therefore, by inserting the respective moments into (2.3), we obtain

$$P_m^{\langle\gamma\rangle}(1;x)=\frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1 t^{\frac{x}{\gamma}-1}(1-t)^{\frac{1-x}{\gamma}-1}dt=1,$$

and,

$$\begin{aligned}
P_m^{\langle\gamma\rangle}(t;x) &= \frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1 t^{\frac{x}{\gamma}}(1-t)^{\frac{1-x}{\gamma}-1}dt\\
&= \frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)\\
&= \frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\frac{\frac{x}{\gamma}}{\frac{x}{\gamma}+\frac{1-x}{\gamma}}\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)\\
&= x.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P_m^{\langle\gamma\rangle}\left(t^2;x\right) &= \frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1 t^{\frac{x}{\gamma}-1}(1-t)^{\frac{1-x}{\gamma}-1}B_m\left(u^2;t\right)dt\\
&= \frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1 t^{\frac{x}{\gamma}-1}(1-t)^{\frac{1-x}{\gamma}-1}\left(\frac{(m-1)t^2+t}{m}\right)dt\\
&= \frac{1}{m\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\left[\int_0^1 (m-1)t^{\frac{x}{\gamma}+1}(1-t)^{\frac{1-x}{\gamma}-1}dt+\int_0^1 (m-1)t^{\frac{x}{\gamma}}(1-t)^{\frac{1-x}{\gamma}-1}dt\right]\\
&= \frac{1}{m\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\left[(m-1)\,\beta\left(\frac{x}{\gamma}+2,\frac{1-x}{\gamma}\right)+\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)\right]\\
&= \frac{1}{m\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\left[\frac{(m-1)\left(\frac{x}{\gamma}+1\right)\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)}{\frac{x}{\gamma}+1+\frac{1-x}{\gamma}}+\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)\right]\\
&= \frac{1}{m\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)\left[\frac{(m-1)(x+\gamma)}{1+\gamma}+1\right]\\
&= \frac{1}{m\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\frac{\frac{x}{\gamma}}{\frac{x}{\gamma}+\frac{1-x}{\gamma}}\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)\left[\frac{(m-1)x+m\gamma+1}{1+\gamma}\right]\\
&= \frac{x}{m}\left[\frac{(m-1)x+m\gamma+1}{1+\gamma}\right]\\
&= \frac{x}{m}+\frac{(m-1)x}{m}\left[\frac{x+\gamma}{1+\gamma}\right],
\end{aligned}$$

$$
\begin{aligned}
P_m^{\langle\gamma\rangle}\left(t^3;x\right) &= \frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1 t^{\frac{x}{\gamma}-1}(1-t)^{\frac{1-x}{\gamma}-1}B_m\left(u^3;t\right)dt \\
&= \frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1 t^{\frac{x}{\gamma}-1}(1-t)^{\frac{1-x}{\gamma}-1}\left(\frac{(m-1)(m-2)t^3+3(m-1)t^2+t}{m^2}\right)dt \\
&= \frac{1}{m^2\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\left[(m-1)(m-2)\int_0^1 t^{\frac{x}{\gamma}+2}(1-t)^{\frac{1-x}{\gamma}-1}dt\right. \\
&\qquad \left.+3(m-1)\int_0^1 t^{\frac{x}{\gamma}+1}(1-t)^{\frac{1-x}{\gamma}-1}dt+\int_0^1 t^{\frac{x}{\gamma}}(1-t)^{\frac{1-x}{\gamma}-1}dt\right] \\
&= \frac{1}{m^2\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\left[(m-1)(m-2)\ \beta\left(\frac{x}{\gamma}+3,\frac{1-x}{\gamma}\right)+3(m-1)\ \beta\left(\frac{x}{\gamma}+2,\frac{1-x}{\gamma}\right)\right. \\
&\qquad \left.+\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)\right] \\
&= \frac{1}{m^2\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\left[(m-1)\ \beta\left(\frac{x}{\gamma}+2,\frac{1-x}{\gamma}\right)\left((m-2)\frac{\frac{x}{\gamma}+2}{\frac{1}{\gamma}+2}+3\right)+\beta\left(\frac{x}{\gamma}+,\frac{1-x}{\gamma}\right)\right] \\
&= \frac{1}{m^2\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\left[(m-1)\frac{\frac{x}{\gamma}+1}{\frac{x}{\gamma}+\frac{1-x}{\gamma}+1}\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)\left(\frac{(m-2)(x+2\gamma)}{1+2\gamma}+3\right)\right. \\
&\qquad \left.+\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)\right] \\
&= \frac{1}{m^2\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)\left[\frac{(m-1)(m-2)(x+\gamma)(x+2\gamma)}{(1+\gamma)(1+2\gamma)}\right. \\
&\qquad \left.+\frac{3(m-1)(x+\gamma)}{(1+\gamma)}+1\right] \\
&= \frac{1}{m^2\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\frac{\frac{x}{\gamma}}{\frac{x}{\gamma}+\frac{1-x}{\gamma}}\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)\left[\frac{(m-1)(m-2)(x+\gamma)(x+2\gamma)}{(1+\gamma)(1+2\gamma)}\right. \\
&\qquad \left.+\frac{3(m-1)(x+\gamma)}{(1+\gamma)}+1\right] \\
&= \frac{x}{m^2}+\frac{3(m-1)x}{m^2}\left(\frac{x+\gamma}{1+\gamma}\right)+\frac{(m-1)(m-2)x}{m^2}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right),
\end{aligned}
$$

and,

$$
\begin{aligned}
&P_m^{\langle\gamma\rangle}\left(t^4;x\right)\\
&=\frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1 t^{\frac{x}{\gamma}-1}(1-t)^{\frac{1-x}{\gamma}-1}B_m\left(u^4;t\right)dt\\
&=\frac{1}{\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\int_0^1 t^{\frac{x}{\gamma}-1}(1-t)^{\frac{1-x}{\gamma}-1}\left(\frac{\begin{array}{c}(m-1)(m-2)(m-3)t^4\\+6(m-1)(m-2)t^3+7(m-1)t^2+t\end{array}}{m^3}\right)dt\\
&=\frac{1}{m^3\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\left[(m-1)(m-2)(m-3)\int_0^1 t^{\frac{x}{\gamma}+3}(1-t)^{\frac{1-x}{\gamma}-1}dt\right.\\
&\quad+6(m-1)(m-2)\int_0^1 t^{\frac{x}{\gamma}+2}(1-t)^{\frac{1-x}{\gamma}-1}dt+7(m-1)\int_0^1 t^{\frac{x}{\gamma}+1}(1-t)^{\frac{1-x}{\gamma}-1}dt\\
&\quad\left.+\int_0^1 t^{\frac{x}{\gamma}}(1-t)^{\frac{1-x}{\gamma}-1}dt\right]\\
&=\frac{1}{m^3\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\left[(m-1)(m-2)(m-3)\,\beta\left(\frac{x}{\gamma}+4,\frac{1-x}{\gamma}\right)\right.\\
&\quad\left.+6(m-1)(m-2)\,\beta\left(\frac{x}{\gamma}+3,\frac{1-x}{\gamma}\right)+7(m-1)\,\beta\left(\frac{x}{\gamma}+2,\frac{1-x}{\gamma}\right)+\beta\left(\frac{x}{\gamma}+1,\frac{1-x}{\gamma}\right)\right]\\
&=\frac{1}{m^3\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\left[(m-1)(m-2)(m-3)\frac{\frac{x}{\gamma}+3}{\frac{x}{\gamma}+\frac{1-x}{\gamma}+3}\,\frac{\frac{x}{\gamma}+2}{\frac{x}{\gamma}+\frac{1-x}{\gamma}+2}\,\frac{\frac{x}{\gamma}+1}{\frac{x}{\gamma}+\frac{1-x}{\gamma}+1}\,\frac{\frac{x}{\gamma}}{\frac{x}{\gamma}+\frac{1-x}{\gamma}}\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)\right.\\
&\quad+6(m-1)(m-2)\frac{\frac{x}{\gamma}+2}{\frac{x}{\gamma}+\frac{1-x}{\gamma}+2}\,\frac{\frac{x}{\gamma}+1}{\frac{x}{\gamma}+\frac{1-x}{\gamma}+1}\,\frac{\frac{x}{\gamma}}{\frac{x}{\gamma}+\frac{1-x}{\gamma}}\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)\\
&\quad\left.+7(m-1)\frac{\frac{x}{\gamma}+1}{\frac{x}{\gamma}+\frac{1-x}{\gamma}+1}\,\frac{\frac{x}{\gamma}}{\frac{x}{\gamma}+\frac{1-x}{\gamma}}\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)+\frac{\frac{x}{\gamma}}{\frac{x}{\gamma}+\frac{1-x}{\gamma}}\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)\right]\\
&=\frac{1}{m^3\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)}\beta\left(\frac{x}{\gamma},\frac{1-x}{\gamma}\right)\left[(m-1)(m-2)(m-3)\frac{x(x+3\gamma)(x+2\gamma)(x+\gamma)}{(1+3\gamma)(1+2\gamma)(1+\gamma)}\right.\\
&\quad\left.+6(m-1)(m-2)\frac{x(x+2\gamma)(x+\gamma)}{(1+2\gamma)(1+\gamma)}+7(m-1)\frac{x(x+\gamma)}{(1+\gamma)}+x\right]\\
&=\frac{x}{m^3}+\frac{7(m-1)x}{m^3}\left(\frac{x+\gamma}{1+\gamma}\right)+\frac{6(m-1)(m-2)x}{m^3}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\\
&\quad+\frac{(m-1)(m-2)(m-3)x}{m^3}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right).
\end{aligned}
$$

□

2.1.3 Approximation Results

Having established the necessary auxiliary results, we now turn to the main approximation theorems of the proposed operators.

Lemma 2.1.3 *Let* $\alpha \in [0,1]$ *and* $\gamma = O\left({}^{1}\!/_{m}\right) \in [0,1]$. *Then the moments of* $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$ *are as follows:*

(i) $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(1;x) = 1$

(ii) $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(t;x) = x$

(iii) $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}\left(t^2;x\right) = \frac{1}{1+\gamma}\left[x^2 + \frac{m+2(1-\alpha)}{m^2}x(1-x)\right] + x\frac{\gamma}{1+\gamma}$

(iv) $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}\left(t^3;x\right) = \begin{cases} x\left[\frac{m+6(1-\alpha)}{m^3}\right] + x\left(\frac{x+\gamma}{1+\gamma}\right)\left[\frac{3m^2+3m(1-2\alpha)-18(1-\alpha)}{m^3}\right] \\ +x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left[\frac{m^3-3m^2-2m(2-3\alpha)+12(1-\alpha)}{m^3}\right] \end{cases}$

(v) $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}\left(t^4;x\right) = \begin{cases} x\left[\frac{m+14(1-\alpha)}{m^4}\right] + x\left(\frac{x+\gamma}{1+\gamma}\right)\left[\frac{7m^2+m(29-36\alpha)-86(1-\alpha)}{m^4}\right] \\ +6x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left[\frac{m^3-m^2(1+2\alpha)-2m(7-8\alpha)+24(1-\alpha)}{m^4}\right] \\ +x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)\left[\frac{m^4-6m^3-m^2(1-12\alpha)+6m(9-10\alpha)-72(1-\alpha)}{m^4}\right] \end{cases}.$

Proof. To calculate the moments of the contagion form of α-Bernstein operators, we will use the alternate form of these operators, given by equation (2.7), and the moments of the contagion form of the Bernstein operators (2.2), as given in Lemma 2.1.2. We know, $h_k = h\left(\frac{k}{m}\right)$ and $g_k = \left(1 - \frac{k}{m-1}\right)h_k + \frac{k}{m-1}h_{k+1}$. Thus, for $h(t) = 1$, we have $h_k = 1 = g_k$. Hence, the first moment of our proposed operators is given by

$$\begin{aligned} \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(1;x) = (1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}} \\ +\alpha\sum_{k=0}^{m}\binom{m}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}} = 1. \end{aligned}$$

Now for $h(t) = t$, $h_k = \frac{k}{m}$, and

$$\begin{aligned} g_k &= \left(1 - \frac{k}{m-1}\right)\frac{k}{m} + \frac{k}{m-1}\left(\frac{k+1}{m}\right) \\ &= \frac{k}{m}\left(\frac{m-1-k}{m-1} + \frac{k+1}{m-1}\right) \\ &= \frac{k}{m-1}. \end{aligned}$$

Thus, we have

$$\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(t;x) = (1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\frac{k}{m-1} + \alpha\sum_{k=0}^{m}\binom{m}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}}\frac{k}{m} = x.$$

For $h(t) = t^2$, $h_k = \frac{k^2}{m^2}$, and

$$\begin{aligned}
g_k &= \left(1-\frac{k}{m-1}\right)\frac{k^2}{m^2} + \frac{k}{m-1}\left(\frac{k+1}{m}\right)^2 \\
&= \frac{k}{m^2}\left(\frac{k(m-1-k)}{m-1} + \frac{(k+1)^2}{m-1}\right) \\
&= \frac{k}{m^2}\left(\frac{mk+k+1}{m-1}\right) \\
&= \frac{k^2}{m^2}\left(\frac{m+1}{m-1}\right) + + \frac{k}{(m-1)m^2} \\
&= \frac{k^2}{(m-1)^2}\left[\frac{m^2-1}{m^2}\right] + \frac{k}{(m-1)m^2}.
\end{aligned}$$

Thus, the third moment is given by

$$\begin{aligned}
\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(t^2;x) &= (1-\alpha)\left[\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\frac{k^2}{(m-1)^2}\left(\frac{m^2-1}{m^2}\right)\right. \\
&\quad \left. + \sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\frac{k}{(m-1)}\left(\frac{1}{m^2}\right)\right] \\
&\quad + \alpha\sum_{k=0}^{m}\binom{m}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}}\frac{k^2}{m^2} \\
&= (1-\alpha)\left[\frac{m^2-1}{m^2(1+\gamma)}\left(\frac{(m-2)x^2}{m-1} + \frac{((m-1)\gamma+1)x}{m-1}\right) + \frac{1}{m^2}x\right] \\
&\quad + \frac{\alpha}{1+\gamma}\left[\frac{(m-1)x^2}{m} + \frac{(m\gamma+1)x}{m}\right] \\
&= \frac{(1-\alpha)}{1+\gamma}\left[\frac{m+1}{m^2}\left((m-2)x^2 + ((m-1)\gamma+1)x\right)\right] + (1-\alpha)\frac{x}{m^2} \\
&\quad + \frac{\alpha}{1+\gamma}\left[\frac{(m-1)x^2}{m} + \frac{(m\gamma+1)x}{m}\right] \\
&= x^2\left[\frac{(m-1)(m-2)(1-\alpha)}{m^2(1+\gamma)} + \frac{(m-1)\alpha}{m(1+\gamma)}\right] \\
&\quad + x\left[\frac{((m-1)\gamma+1)(m+1)(1-\alpha)}{m^2(1+\gamma)} + \frac{1-\alpha}{m^2} + \frac{(m\gamma+1)\alpha}{m(1+\gamma)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{x^2}{1+\gamma}\left[(1-\alpha)\left(1-\frac{1}{m}-\frac{2}{m^2}\right)+\alpha\left(1-\frac{1}{m}\right)\right] \\
&\quad + \frac{x}{1+\gamma}\left[\frac{(m+1)\left((m-1)\gamma+1\right)(1-\alpha)}{m^2}+\frac{(m\gamma+1)\,\alpha}{m}\right]+\frac{x\,(1-\alpha)}{m^2} \\
&= \frac{x^2}{1+\gamma}\left[1-\frac{1}{m}-\frac{2\,(1-\alpha)}{m^2}\right]+\frac{x}{1+\gamma}\left[\frac{(m+1)\,(1-\alpha)}{m^2}\right. \\
&\quad \left. + \frac{(m+1)\,(m-1)\,(1-\alpha)\gamma}{m^2}+\frac{\alpha}{m}+\alpha\gamma\right]+\frac{x(1-\alpha)}{m^2} \\
&= \frac{x^2}{1+\gamma}\left[1-\frac{1}{m}-\frac{2\,(1-\alpha)}{m^2}\right]+\frac{x}{1+\gamma}\left[\frac{1}{m}+\frac{(1-\alpha)}{m^2}\right] \\
&\quad + \frac{\gamma x}{1+\gamma}\left[(1-\alpha)\left(1-\frac{1}{m^2}\right)+\alpha\right]+\frac{x\,(1-\alpha)}{m^2} \\
&= \frac{1}{1+\gamma}\left[x^2-\frac{x^2}{m}-\frac{(1-\alpha)\,x^2}{m^2}+\frac{x}{m}+\frac{x\,(1-\alpha)}{m^2}\right]-\frac{(1-\alpha)\,x^2}{m^2(1+\gamma)}+\frac{\gamma x}{1+\gamma} \\
&\quad - \frac{(1-\alpha)\,\gamma x}{(1+\gamma)m^2}+\frac{x\,(1-\alpha)}{m^2} \\
&= \frac{1}{1+\gamma}\left[x^2+\frac{m+2(1-\alpha)}{m^2}x\,(1-x)\right]+x\frac{\gamma}{1+\gamma}.
\end{aligned}
$$

Similarly, for $h(t)=t^3$, $h_k=\frac{k^3}{m^3}$ and

$$
\begin{aligned}
g_k &= \left(1-\frac{k}{m-1}\right)\frac{k^3}{m^3}+\frac{k}{m-1}\left(\frac{k+1}{m}\right)^3 \\
&= \frac{k}{m^3}\left(\frac{k^2(m-1-k)+(k+1)^3}{m-1}\right) \\
&= \frac{k}{m^3}\left(\frac{mk^2+2k^2+3k+1}{m-1}\right) \\
&= \frac{k}{m^3}\left(\frac{(m+2)k^2}{m-1}+\frac{3k}{m-1}+\frac{1}{m-1}\right) \\
&= \frac{k^3}{(m-1)^3}\left(\frac{(m+2)(m-1)^2}{m^3}\right)+\frac{3k^2}{(m-1)^2}\left(\frac{m-1}{m^3}\right)+\frac{k}{(m-1)m^2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}\left(t^3;x\right) \\
&\quad = (1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\left[\frac{k^3}{(m-1)^3}\frac{(m+2)(m-1)^2}{m^3}\right. \\
&\quad\quad \left. + \frac{3k^2}{(m-1)^2}\frac{(m-1)}{m^3}+\frac{k}{(m-1)m^3}\right]+\alpha\sum_{k=0}^{m}\binom{m}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}}\frac{k^3}{m^3}
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\alpha)\,\frac{(m+2)(m-1)^2}{m^3}\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\frac{k^3}{(m-1)^3}\\
&\quad +3\,(1-\alpha)\,\frac{(m-1)}{m^3}\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\frac{k^2}{(m-1)^2}\\
&\quad +(1-\alpha)\,\frac{1}{m^3}\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\frac{k}{(m-1)}+\alpha\sum_{k=0}^{m}\binom{m}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}}\frac{k^3}{m^3}\\
&= (1-\alpha)\,\frac{(m+2)(m-1)^2}{m^3}\left[\frac{x}{(m-1)^2}+\frac{3(m-2)x}{(m-1)^2}\left(\frac{x+\gamma}{1+\gamma}\right)+\frac{(m-2)(m-3)x}{(m-1)^2}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\right]\\
&\quad +3\,(1-\alpha)\,\frac{(m-1)}{m^3}\left[\frac{x}{(m-1)}+\frac{(m-2)x}{(m-1)}\left(\frac{x+\gamma}{1+\gamma}\right)\right]+(1-\alpha)\,\frac{x}{m^3}\\
&\quad +\alpha\left[\frac{x}{m^2}+\frac{3(m-1)x}{m^2}\left(\frac{x+\gamma}{1+\gamma}\right)+\frac{(m-1)(m-2)x}{m^2}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\right]\\
&= \frac{1-\alpha}{m^3}\left[(m+2)x+3(m^2-4)x\left(\frac{x+\gamma}{1+\gamma}\right)+(m^2-4)(m-3)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\right.\\
&\quad \left.+3(m-2)x\left(\frac{x+\gamma}{1+\gamma}\right)+4x\right]+\frac{\alpha}{m^2}\left[x+3(m-1)x\left(\frac{x+\gamma}{1+\gamma}\right)+(m-1)(m-2)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\right]\\
&= \frac{1-\alpha}{m^3}\left[(m+6)x+3(m-2)(m+3)x\left(\frac{x+\gamma}{1+\gamma}\right)+(m^2-4)(m-3)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\right]\\
&\quad +\frac{\alpha}{m^2}\left[x+3(m-1)x\left(\frac{x+\gamma}{1+\gamma}\right)+(m-1)(m-2)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\right]\\
&= x\left[(1-\alpha)\left(\frac{1}{m^2}+\frac{6}{m^3}\right)+\frac{\alpha}{m^2}\right]+x\left(\frac{x+\gamma}{1+\gamma}\right)\left[3(1-\alpha)\left(\frac{1}{m}+\frac{1}{m^2}-\frac{6}{m^3}\right)+3\alpha\left(\frac{1}{m}-\frac{1}{m^2}\right)\right]\\
&\quad +x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left[(1-\alpha)\left(1-\frac{3}{m}-\frac{4}{m^2}+\frac{12}{m^3}\right)+\alpha\left(1-\frac{3}{m}+\frac{2}{m^2}\right)\right]\\
&= x\left[\frac{1}{m^2}+\frac{6(1-\alpha)}{m^3}\right]+x\left(\frac{x+\gamma}{1+\gamma}\right)\left[\frac{3}{m}+\frac{6(1-2\alpha)}{m^2}-\frac{18(1-\alpha)}{m^3}\right]\\
&\quad +x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left[1-\frac{3}{m}-\frac{2(2-3\alpha)}{m^2}+\frac{12(1-\alpha)}{m^3}\right]\\
&= x\left[\frac{m+6(1-\alpha)}{m^3}\right]+x\left(\frac{x+\gamma}{1+\gamma}\right)\left[\frac{3m^2+3m(1-2\alpha)-18(1-\alpha)}{m^3}\right]\\
&\quad +x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left[\frac{m^3-3m^2-2m(2-3\alpha)+12(1-\alpha)}{m^3}\right].
\end{aligned}
$$

Lastly, For $h(t)=t^4$, $h_k=\frac{k^4}{m^4}$ and

$$
\begin{aligned}
g_k &= \left(1-\frac{k}{m-1}\right)\frac{k^4}{m^4}+\frac{k}{m-1}\left(\frac{k+1}{m}\right)^4\\
&= \frac{k}{(m-1)m^4}\left(k^3(m-1-k)+k^4+4k^3+6k^2+4k+1\right)\\
&= \frac{k}{m^4}\left((m+3)k^3+6k^2+4k+1\right)\\
&= \frac{k^4(m+3)}{(m-1)m^4}+\frac{6k^3}{(m-1)m^4}+\frac{4k^2}{(m-1)m^4}+\frac{k}{(m-1)m^4}\\
&= \frac{k^4}{(m-1)^4}\left(\frac{(m+3)(m-1)^3}{m^4}\right)+\frac{6k^3}{(m-1)^3}\left(\frac{(m-1)^2}{m^4}\right)+\frac{4k^2}{(m-1)^2}\left(\frac{m-1}{m^4}\right)+\frac{k}{(m-1)m^4}.
\end{aligned}
$$

And thus,

$$
\begin{aligned}
&\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}\left(t^4;x\right)\\
&= (1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\frac{k^4}{(m-1)^4}\frac{(m+3)(m-1)^3}{m^4}\\
&\quad+(1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\frac{6k^3}{(m-1)^3}\frac{(m-1)^2}{m^4}\\
&\quad+(1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\frac{4k^2}{(m-1)^2}\frac{m-1}{m^4}\\
&\quad+(1-\alpha)\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\frac{k}{(m-1)m^4}\\
&\quad+\alpha\sum_{k=0}^{m}\binom{m}{k}\frac{x^{[k,-\gamma]}(1-x)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}}\frac{k^4}{m^4}\\
&= (1-\alpha)\frac{(m+3)(m-1)^3}{m^4}\left[\frac{x}{(m-1)^3}+\frac{7(m-2)x}{(m-1)^3}\left(\frac{x+\gamma}{1+\gamma}\right)+\frac{6(m-2)(m-3)x}{(m-1)^3}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\right.\\
&\quad\left.+\frac{(m-2)(m-3)(m-4)x}{(m-1)^3}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)\right]+6(1-\alpha)\frac{(m-1)^2}{m^4}\left[\frac{x}{(m-1)^2}\right.\\
&\quad\left.+\frac{3(m-2)x}{(m-1)^2}\left(\frac{x+\gamma}{1+\gamma}\right)+\frac{(m-2)(m-3)x}{(m-1)^2}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\right]+4(1-\alpha)\frac{m-1}{m^4}\left[\frac{x}{m-1}\right.\\
&\quad\left.+\frac{(m-2)x}{m-1}\left(\frac{x+\gamma}{1+\gamma}\right)\right]+(1-\alpha)\frac{x}{m^4}+\alpha\left[\frac{x}{m^3}+\frac{7(m-1)x}{m^3}\left(\frac{x+\gamma}{1+\gamma}\right)\right.\\
&\quad\left.+\frac{6(m-1)(m-2)x}{m^3}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)+\frac{(m-1)(m-2)(m-3)x}{m^3}\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1-\alpha}{m^4}\Bigg[(m+3)x + 7(m-2)(m+3)x\left(\frac{x+\gamma}{1+\gamma}\right) + 6(m-2)(m^2-9)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right) \\
&\quad + (m-2)(m-4)(m^2-9)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right) + 6x + 18(m-2)x\left(\frac{x+\gamma}{1+\gamma}\right) \\
&\quad + 6(m-2)(m-3)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right) + 4x + 4(m-2)x\left(\frac{x+\gamma}{1+\gamma}\right) + x\Bigg] + \frac{\alpha}{m^3}\Bigg[x \\
&\quad + 7(m-1)x\left(\frac{x+\gamma}{1+\gamma}\right) + 6(m-1)(m-2)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right) \\
&\quad + (m-1)(m-2)(m-3)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)\Bigg] \\
&= \frac{1-\alpha}{m^4}\Bigg[(m+14)x + (7m+43)(m-2)x\left(\frac{x+\gamma}{1+\gamma}\right) + 6(m+4)(m-2)(m-3)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right) \\
&\quad + (m-2)(m-4)(m^2-9)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)\Bigg] + \frac{\alpha}{m^3}\Bigg[x + 7(m-1)x\left(\frac{x+\gamma}{1+\gamma}\right) \\
&\quad + 6(m-1)(m-2)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right) + (m-1)(m-2)(m-3)x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)\Bigg] \\
&= x\left[(1-\alpha)\left(\frac{1}{m^3}+\frac{14}{m^4}\right)+\frac{\alpha}{m^3}\right] + x\left(\frac{x+\gamma}{1+\gamma}\right)\left[(1-\alpha)\left(\frac{7}{m^2}+\frac{29}{m^3}-\frac{86}{m^4}\right)+7\alpha\left(\frac{1}{m^2}-\frac{1}{m^3}\right)\right] \\
&\quad + 6x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left[(1-\alpha)\left(\frac{1}{m}-\frac{1}{m^2}-\frac{14}{m^3}+\frac{24}{m^4}\right)+\alpha\left(\frac{1}{m}-\frac{3}{m^2}+\frac{2}{m^3}\right)\right] \\
&\quad + x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)\left[(1-\alpha)\left(1-\frac{6}{m}-\frac{1}{m^2}+\frac{54}{m^3}-\frac{72}{m^4}\right)+\alpha\left(1-\frac{6}{m}+\frac{11}{m^2}-\frac{6}{m^3}\right)\right] \\
&= x\left[\frac{1}{m^3}+\frac{14(1-\alpha)}{m^4}\right] + x\left(\frac{x+\gamma}{1+\gamma}\right)\left[\frac{7}{m^2}+\frac{29-36\alpha}{m^3}-\frac{86(1-\alpha)}{m^4}\right] \\
&\quad + 6x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left[\frac{1}{m}-\frac{1+2\alpha}{m^2}-\frac{2(7-8\alpha)}{m^3}+\frac{24(1-\alpha)}{m^4}\right] \\
&\quad + x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)\left[1-\frac{6}{m}-\frac{1-12\alpha}{m^2}+\frac{6(9-10\alpha)}{m^3}-\frac{72(1-\alpha)}{m^4}\right] \\
&= x\left(\frac{m+14(1-\alpha)}{m^4}\right) + x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{7m^2+m(29-36\alpha)-86(1-\alpha)}{m^4}\right) \\
&\quad + 6x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{m^3-m^2(1+2\alpha)-2m(7-8\alpha)+24(1-\alpha)}{m^4}\right) \\
&\quad + x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)\left(\frac{m^4-6m^3-m^2(1-12\alpha)+6m(9-10\alpha)-72(1-\alpha)}{m^4}\right).
\end{aligned}
$$

□

This result brings us to a conclusion that as $m \to \infty$, $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(e_s;x) \to e_s$, where $e_s = t^s$ for $s = 0,1,2$. From the Bohman-Korovkin Theorem [35; 132], we can hence state that for a real valued continuous function h, the parametric Bernstein operators, $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$ converges to $h(x)$ uniformly. That is,

$$\lim_{m\to\infty} \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle} h = h.$$

Lemma 2.1.4 *Let $\gamma = O(1/m)$. Considering Lemma 2.1.3 and computing the central moments of* (2.6)*, we obtain the following limits:*

(i) $\lim_{m\to\infty} m\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2(t);x) = \frac{2C}{1+\gamma}x(1-x)$

(ii) $\lim_{m\to\infty} m\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_4(t);x) = 0,$

where $\varphi_m(t) = (t-x)^m$ *and* $C = \lim_{m\to\infty} \frac{1+m\gamma}{2}$.

Proof. Let $\varphi_m(t) = (t-x)^m$. We can easily verify that the first central moment, $P_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1(t);x)$, is identically zero. For the second central moment,

$$\begin{aligned}
P_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2(t);x) &= P_{m,\alpha}^{\langle\gamma\rangle}\left(t^2;x\right) - 2xP_{m,\alpha}^{\langle\gamma\rangle}(t;x) + x^2 P_{m,\alpha}^{\langle\gamma\rangle}(1;x) \\
&= \frac{1}{1+\gamma}\left(x^2 + \frac{m+2(1-\alpha)}{m^2}x(1-x)\right) + \frac{x\gamma}{1+\gamma} - x^2 \\
&= \frac{1}{1+\gamma}\left(-\gamma x^2 + \frac{m+2(1-\alpha)}{m^2}x(1-x)\right) + \frac{x\gamma}{1+\gamma} \\
&= \frac{1}{1+\gamma}\left(\frac{m+2(1-\alpha)}{m^2}x(1-x)\right) + \frac{\gamma}{1+\gamma}x(1-x) \\
&= \frac{x(1-x)}{1+\gamma}\left(\frac{m+2(1-\alpha)}{m^2} + \gamma\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{m\to\infty} mP_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2(t);x) &= \lim_{m\to\infty} \frac{x(1-x)}{1+\gamma}\left(1 + \frac{2(1-\alpha)}{m} + m\gamma\right) \\
&= \frac{2C}{1+\gamma}x(1-x),
\end{aligned}$$

where $C = \lim_{m\to\infty} \frac{1+m\gamma}{2}$. Similarly,

$$\begin{aligned}
&P_{m,\alpha}^{\langle\gamma\rangle}(\varphi_4(t);x) \\
&= P_{m,\alpha}^{\langle\gamma\rangle}\left(t^4;x\right) - 4xP_{m,\alpha}^{\langle\gamma\rangle}\left(t^3;x\right) + 6x^2P_{m,\alpha}^{\langle\gamma\rangle}\left(t^2;x\right) - 4x^3P_{m,\alpha}^{\langle\gamma\rangle}(t;x) + x^4P_{m,\alpha}^{\langle\gamma\rangle}(1;x) \\
&= x\left(\frac{m+14(1-\alpha)}{m^4}\right) + x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{7m^2+m(29-36\alpha)-86(1-\alpha)}{m^4}\right) \\
&\quad + x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{6m^3-6m^2(1+2\alpha)-12m(7-8\alpha)+144(1-\alpha)}{m^4}\right) \\
&\quad + x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)\left(\frac{m^4-6m^3-m^2(1-12\alpha)+6m(9-10\alpha)-72(1-\alpha)}{m^4}\right)
\end{aligned}$$

$$-4x^2[\frac{m+6(1-\alpha)}{m^3}]-4x^2\left(\frac{x+\gamma}{1+\gamma}\right)[\frac{3m^2+3m(1-2\alpha)-18(1-\alpha)}{m^3}]$$
$$-4x^2\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)[\frac{m^3-3m^2-2m(2-3\alpha)+12(1-\alpha)}{m^3}]$$
$$+6x^3\left[\frac{m+2(1-\alpha)}{m^2}\right]+6x^3\left(\frac{x+\gamma}{1+\gamma}\right)\left[\frac{m^2-m-2(1-\alpha)}{m^2}\right]-4x^4+x^4.$$

In order to evaluate $\lim_{m\to\infty} mP^{\langle\gamma\rangle}_{m,\alpha}(\varphi_4(t);x)$, only the terms independent of m and those of order $1/m$ are relevant. Rest of the terms will vanish as $m\to\infty$.
Thus,

$$\lim_{m\to\infty} m\mathscr{P}^{\langle\gamma\rangle}_{m,\alpha}(\varphi_4(t);x)$$
$$=\lim_{m\to\infty}\begin{cases} m\Big[x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)-4x^2\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)+6x^3\left(\frac{x+\gamma}{1+\gamma}\right)-3x^4\Big] \\ +6x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)-6x\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right)\left(\frac{x+3\gamma}{1+3\gamma}\right)-12x^2\left(\frac{x+\gamma}{1+\gamma}\right)+12x^2\left(\frac{x+\gamma}{1+\gamma}\right)\left(\frac{x+2\gamma}{1+2\gamma}\right) \\ +6x^3-6x^3\left(\frac{x+\gamma}{1+\gamma}\right)\end{cases}$$
$$=\lim_{m\to\infty}\begin{cases} \frac{m}{(1+\gamma)(1+2\gamma)(1+3\gamma)}\Big[x^4+6x^3\gamma+11x^2\gamma^2+6x\gamma^3-4x^4-12x^4\gamma-12x^3\gamma-36x^3\gamma^2 \\ -8x^2\gamma^2-24x^2\gamma^3+6x^4+30x^4\gamma+36x^4\gamma^2+6x^3\gamma+30x^3\gamma^2+36x^3\gamma^3-3x^4 \\ -18x^4\gamma-33x^4\gamma^2-18x^4\gamma^3\Big]+\frac{1}{(1+\gamma)(1+2\gamma)(1+3\gamma)}\Big[-6x^4+6x^3-18x^3\gamma+18x^2\gamma \\ -12x^2\gamma^2+12x\gamma^2+12x^4+36x^4\gamma-12x^3-24x^3\gamma+36x^3\gamma^2-12x^2\gamma \\ -36x^2\gamma^2-6x^4-30x^4\gamma-36x^4\gamma^2+6x^3+30x^3\gamma+36x^3\gamma^2\Big]\end{cases}$$
$$=\lim_{m\to\infty}\frac{m\left(3x^2\gamma^2+6x\gamma^3-6x^3\gamma-24x^2\gamma^3+36x^4\gamma^2+36x^3\gamma^3-18x^4\gamma^3\right)}{(1+\gamma)(1+2\gamma)(1+3\gamma)}$$
$$+\lim_{m\to\infty}\frac{-12x^3\gamma+72x^3\gamma^2+6x^2\gamma-48x^2\gamma^2+12x\gamma^2+6x^4\gamma-36x^4\gamma^2}{(1+\gamma)(1+2\gamma)(1+3\gamma)}$$
$$=\lim_{m\to\infty}\frac{m\left(3x^2\gamma^2(1-2x+x^2)+6x\gamma^3(1-4x+6x^2-3x^3)\right)}{(1+\gamma)(1+2\gamma)(1+3\gamma)}$$
$$+\lim_{m\to\infty}\frac{6x^2\gamma(1-2x+x^2)+12x\gamma^2(1-4x+6x^2-3x^3)}{(1+\gamma)(1+2\gamma)(1+3\gamma)}$$
$$=\lim_{m\to\infty}\frac{3mx\gamma^2\left(x(x-1)^2-2\gamma(x-1)(3x^2-3x+1)\right)}{(1+\gamma)(1+2\gamma)(1+3\gamma)}$$
$$+\lim_{m\to\infty}\frac{6x\gamma\left(x(x-1)^2-2\gamma(x-1)(3x^2-3x+1)\right)}{(1+\gamma)(1+2\gamma)(1+3\gamma)}.$$

Given that $\gamma=O(1/m)$, the limits are readily seen to vanish as $m\to\infty$. □

Theorem 2.1.5 *Let $0 \leq \alpha \leq 1$ and h is a bounded function defined for all $x \in [0,1]$. Then for $\gamma = O(1/m)$,*

$$\lim_{m\to\infty} m\left[\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h(t);x) - h(x)\right] = \frac{C}{1+\gamma}x(1-x)h''(x),$$

where $C = \lim_{m\to\infty} \frac{1+m\gamma}{2}$, such that $h''(x)$ exists.

Proof. By means of Taylor's series expansion for $k \leq m$, we have

$$h(t) = h(x) + (t-x)h'(x) + \frac{1}{2}(t-x)^2 h''(x) + \eta(t-x)(t-x)^2,$$

where $\lim_{t\to x} \eta(t-x) = 0$. Taking $t = k/m$, we get,

$$\begin{aligned} m&\left[\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h(t);x) - h(x)\right] \\ &= m\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(x)\left[\left(\frac{k}{m}-x\right)h'(x) + \frac{1}{2}\left(\frac{k}{m}-x\right)^2 h''(x) + \eta\left(\frac{k}{m}-x\right)\left(\frac{k}{m}-x\right)^2\right] \\ &= \sum_{k=0}^{m}(k-mx)p_{m,k}^{\langle\alpha,\gamma\rangle}(x)h'(x) + \frac{1}{2m}\sum_{k=0}^{m}(k-mx)^2 p_{m,k}^{\langle\alpha,\gamma\rangle}(x)h''(x) + mR_m(x), \end{aligned}$$

where $R_m(x) = \sum_{k=0}^{m} \eta\left(\frac{k}{m}-x\right)\left(\frac{k}{m}-x\right)^2 p_{m,k}^{\langle\alpha,\gamma\rangle}(x)$.
The first summation will be zero and for the second summation, we have

$$\begin{aligned} \sum_{k=0}^{m}(k-mx)^2 p_{m,k}^{\langle\alpha,\gamma\rangle}(x) &= \sum_{k=0}^{m}k^2 p_{m,k}^{\langle\alpha,\gamma\rangle}(x) - 2mx\sum_{k=0}^{m}kp_{m,k}^{\langle\alpha,\gamma\rangle}(x) + m^2x^2\sum_{k=0}^{m}p_{m,k}^{\langle\alpha,\gamma\rangle}(x) \\ &= \frac{m^2}{1+\gamma}\left[x^2 + \frac{m+2(1-\alpha)}{m^2}x(1-x)\right] + m^2x\frac{\gamma}{1+\gamma} \\ &\quad - 2m^2x^2 + m^2x^2 \\ &= \frac{m^2\gamma + m + 2(1-\alpha)}{1+\gamma}x(1-x). \end{aligned}$$

Hence,

$$\begin{aligned} m\left[\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h(t);x) - h(x)\right] &= \frac{x(1-x)}{2m(1+\gamma)}\left\{m^2\gamma + m + 2(1-\alpha)\right\}h''(x) + mR_m(x) \\ &= \frac{1}{1+\gamma}\left(\frac{m\gamma}{2} + \frac{1}{2} + \frac{1-\alpha}{m}\right)x(1-x)h''(x) + mR_m(x). \end{aligned}$$

Now, all that is left to prove is that, $mR_m(x) \to 0$ as $m \to \infty$.

We know, for every $\varepsilon > 0$, $\exists\ \delta > 0$: $|\mathfrak{z}(t-x)| < \varepsilon$, whenever $|t-x| < \delta$. Also, since $0 \le |\mathfrak{z}(t-x)| \le m$, therefore $\exists\ M > 0$ such that $|\mathfrak{z}(t-x)| \le M$, whenever $|t-x| \ge \delta$. Hence, we get

$$|\mathfrak{z}(t-x)| \le \varepsilon + M\frac{(t-x)^2}{\delta^2}, \qquad \text{for every } t.$$

Thus,

$$\begin{aligned} m\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}\left(\mathfrak{z}(t-x)(t-x)^2;x\right) &\le m\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}\left(\left(\varepsilon + M\frac{(t-x)^2}{\delta^2}\right)(t-x)^2;x\right) \\ &\le m\varepsilon\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) + m\frac{M}{\delta^2}P_{n,\alpha}^{\langle\gamma\rangle}(\varphi_4;x). \end{aligned}$$

From Lemma 2.1.4, we can say, $\lim\limits_{m\to\infty} m\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}\left(\mathfrak{z}(t-x)(t-x)^2;x\right) = 0$. Hence, we get

$$\begin{aligned} \lim_{m\to\infty} m\left[\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h(t);x) - h(x)\right] &= \frac{x(1-x)}{1+\gamma}\left(\lim_{m\to\infty}\frac{m\gamma}{2} + \frac{1}{2} + \frac{(1-\alpha)}{m}\right)h''(x) \\ &= \frac{C}{1+\gamma}x(1-x)h''(x), \end{aligned}$$

where $C = \lim\limits_{m\to\infty}\frac{m\gamma+1}{2}$. □

Theorem 2.1.6 *Let the functions $g,h:[0,1]\to\mathbb{R}$. If $g,h\in C^2[0,1]$ then,*

$$\lim_{m\to\infty} m\left[\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(gh;x) - \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x)\right] = \frac{2C}{1+\gamma}x(1-x)g'(x)h'(x),$$

where $C = \lim\limits_{m\to\infty}\frac{m\gamma+1}{2}$.

Proof. We consider,

$$\begin{aligned} &\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(gh;x) - \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) \\ &\quad = \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(gh;x) - \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) - g(x)h(x) + g(x)h(x) - (gh)'\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1;x) \\ &\qquad + g'h\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1;x) + gh'\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1;x) - \frac{1}{2}(gh)''\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) + \frac{1}{2}g''h\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) \\ &\qquad + g'h'\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) + \frac{1}{2}gh''\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) - h(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x) + h(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) \\ &\qquad - h'(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1;x) + h'(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1;x) \\ &\qquad - \frac{1}{2}h''(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) + \frac{1}{2}h''(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) \end{aligned}$$

$$
\begin{aligned}
&= \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(gh;x) - g(x)h(x) - \frac{1}{2}(gh)'' \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) \\
&\quad - h(x)\Big(\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x) - \frac{1}{2}g''(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) - g(x)\Big) \\
&\quad - \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\Big(\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) - \frac{1}{2}h''(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) - h(x)\Big) \\
&\quad + \frac{1}{2}\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x)\Big(2g'(x)h'(x) + g(x)h''(x) - h''(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\Big). \\
&= \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(gh;x) - g(x)h(x) - (gh)'\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1;x) - \frac{1}{2}(gh)'' \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) \\
&\quad - h(x)\Big(\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x) - g(x) - g'(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1;x) - \frac{1}{2}g''(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x)\Big) \\
&\quad - \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\Big(\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) - h(x) - h'(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1;x) - \frac{1}{2}h''(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x)\Big) \\
&\quad + \frac{1}{2}\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x)\Big(g(x)h''(x) + 2g'(x)h'(x) - h''(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\Big) \\
&\quad + \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1;x)\Big(g(x)h'(x) - h'(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\Big). \qquad (2.8)
\end{aligned}
$$

From Theorem 2.1.5, we can write

$$
\lim_{m\to\infty} m\left[\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(gh;x) - g(x)h(x)\right] = \frac{2C}{1+\gamma}x(1-x)\left(g''(x)h(x) + 2g'(x)h'(x) + g(x)h''(x)\right).
$$

Multiplying both sides of equation (2.8) by m, taking $m \to \infty$ and using Lemma 2.1.4, we get

$$
\begin{aligned}
&\lim_{m\to\infty} m\left[\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(gh;x) - \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x)\right] \\
&\quad = \lim_{m\to\infty} m\left[\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(gh;x) - g(x)h(x)\right] - \frac{1}{2}(gh)'' \lim_{m\to\infty} m\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x) \\
&\qquad - h(x)\left[\lim_{m\to\infty} m\left(\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x) - g(x)\right) - \frac{1}{2}g''(x)\lim_{m\to\infty} m\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x)\right] \\
&\quad - \lim_{m\to\infty}\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\left[\lim_{m\to\infty} m\left(\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) - h(x)\right) - \frac{1}{2}h''(x)\lim_{m\to\infty} m\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x)\right] \\
&\quad + \lim_{m\to\infty}\frac{m}{2}\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2;x)\left[g(x)h''(x) + 2g'(x)h'(x) - h''(x)\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(g;x)\right] \\
&= \frac{C}{1+\gamma}x(1-x)(gh)'' - \frac{1}{2}(gh)''\frac{2C}{1+\gamma}x(1-x) \\
&\quad - h(x)\left[\frac{C}{1+\gamma}x(1-x)g''(x) - \frac{1}{2}g''(x)\frac{2C}{1+\gamma}x(1-x)\right] \\
&\quad - g(x)\left[\frac{C}{1+\gamma}x(1-x)h''(x) - \frac{1}{2}h''(x)\frac{2C}{1+\gamma}x(1-x)\right] \\
&\quad + \frac{2C}{2(1+\gamma)}x(1-x)\left[g(x)h''(x) + 2g'(x)h'(x) - h''(x)g(x)\right] \\
&= \frac{2C}{1+\gamma}x(1-x)g'(x)h'(x),
\end{aligned}
$$

where $C = \lim\limits_{m\to\infty} \frac{m\gamma+1}{2}$. □

We have seen that for any continuous real-valued function h, $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle} h \to h$ uniformly on $[0,1]$, giving us an explicit approximation to h. Our next focus shifts on knowing how good this approximation will be. For that we calculate the error that would have occurred during the approximation process. One way to determine this is using the modulus of continuity. Let us take into account the error bound

$$\left\|h - \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle} h\right\| = \max_{0\le x\le 1} \left|h(x) - \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x)\right|,$$

where $\|.\|$ is the uniform norm over the interval $[0,1]$. Our next attempt is to attain an upper bound to estimate the approximation error committed during the process.

Theorem 2.1.7 *Let* $0 \le \alpha \le 1$ *and* $\gamma = O\left({}^1\!/_m\right)$. *For a bounded function* h *on* $[0,1]$ *we deduce,*

$$\left\|h - \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle} h\right\| \le \frac{3}{2}\omega\left(\frac{\sqrt{m^2\gamma + m + 2(1-\alpha)}}{m\sqrt{1+\gamma}}\right),$$

where $\omega(.)$ *denotes modulus of continuity.*

Proof. From operators (2.7), we get

$$\begin{aligned}\left|h(x) - \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x)\right| &= \left|\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(x)\left(h(x) - h\left(\frac{k}{m}\right)\right)\right| \\ &\le \left|h(x) - h\left(\frac{k}{m}\right)\right| \sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(x) \\ &\le \sum_{k=0}^{m} \omega\left(\left|x - \frac{k}{m}\right|\right) p_{m,k}^{\langle\alpha,\gamma\rangle}(x).\end{aligned}$$

Using Schwarz's inequality, we can say

$$\begin{aligned}\sum_{k=0}^{m}\left|x - \frac{k}{m}\right| p_{m,k}^{\langle\alpha,\gamma\rangle}(x) &= \sum_{k=0}^{m}\left|x - \frac{k}{m}\right| \sqrt{p_{m,k}^{\langle\alpha,\gamma\rangle}(x)}.\sqrt{p_{m,k}^{\langle\alpha,\gamma\rangle}(x)} \\ &\le \left[\sum_{k=0}^{m}\left(x - \frac{k}{m}\right)^2 p_{m,k}^{\langle\alpha,\gamma\rangle}(x)\right]^{1/2} \left[\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(x)\right]^{1/2} \\ &= \left[\frac{1}{1+\gamma}\left(\frac{m+2(1-\alpha)}{m^2} + \gamma\right) x(1-x)\right]^{1/2} \\ &\le \left[\frac{m^2\gamma + m + 2(1-\alpha)}{4m^2(1+\gamma)}\right]^{1/2}.\end{aligned}$$

From Proposition 1.1.4,

$$\omega\left(\left|x-\frac{k}{m}\right|\right)=\omega\left(\frac{m\sqrt{1+\gamma}}{\sqrt{m^2\gamma+m+2(1-\alpha)}}\left|x-\frac{k}{m}\right|.\frac{\sqrt{m^2\gamma+m+2(1-\alpha)}}{m\sqrt{1+\gamma}}\right)$$
$$\leq\left[1+\frac{m\sqrt{1+\gamma}}{\sqrt{m^2\gamma+m+2(1-\alpha)}}\left|x-\frac{k}{m}\right|\right]\omega\left(\frac{\sqrt{m^2\gamma+m+2(1-\alpha)}}{m\sqrt{1+\gamma}}\right).$$

Thus, we get

$$\begin{aligned}&\left|h(x)-\mathscr{P}^{\langle\gamma\rangle}_{m,\alpha}(h;x)\right|\\&\leq\sum_{k=0}^{m}\left[1+\frac{m\sqrt{1+\gamma}}{\sqrt{m^2\gamma+m+2(1-\alpha)}}\left|x-\frac{k}{m}\right|\right]\omega\left(\frac{\sqrt{m^2\gamma+m+2(1-\alpha)}}{m\sqrt{1+\gamma}}\right)p^{\langle\alpha,\gamma\rangle}_{m,k}(x)\\&\leq\omega\left(\frac{\sqrt{m^2\gamma+m+2(1-\alpha)}}{m\sqrt{1+\gamma}}\right)\left[1+\frac{m\sqrt{1+\gamma}}{\sqrt{m^2\gamma+m+2(1-\alpha)}}\sum_{k=0}^{m}\left|x-\frac{k}{m}\right|p^{\langle\alpha,\gamma\rangle}_{m,k}(x)\right]\\&\leq\frac{3}{2}\omega\left(\frac{\sqrt{m^2\gamma+m+2(1-\alpha)}}{m\sqrt{1+\gamma}}\right),\end{aligned}$$

and thereafter, we arrive at our required result. □

Now that we have obtained an upper bound for the error function, we can also use the fact that $\lim_{\delta\to 0}\omega(\delta)=0$ iff h is uniformly continuous on $[a,b]$ to see that our modified α-Bernstein operators, $\mathscr{P}^{\langle\gamma\rangle}_{m,\alpha}h\to h$ uniformly on the interval $[0,1]$. Figure 2.5 gives us a representation of the approximation of our operators $\mathscr{P}^{\langle\gamma\rangle}_{m,\alpha}(h;x)$ defined in (2.6) for the function $h(x)=(3x-1)\sin\frac{7\pi}{3}x+\frac{5}{4}$.

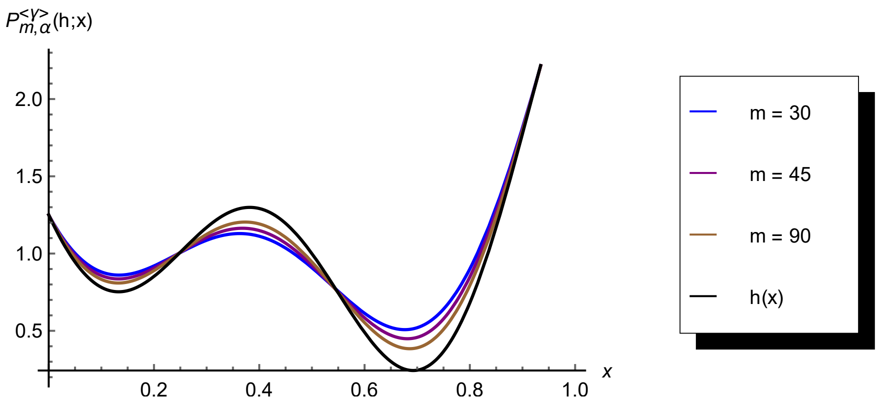


Figure 2.5: Convergence of $\mathscr{P}^{\langle\gamma\rangle}_{m,\alpha}(h;x)$ with parameters $\alpha=0.7$ and $\gamma=0.01$, for different values of m.

2.1.4 A-Statistical Approximation

In 1951, an alternative concept of convergence emerged, recognized as statistical convergence, introduced by Fast [75] and Steinhaus [199]. The term statistical convergence used henceforth is nothing but another name for "almost convergence", used by Antoni Zygmund in his book [226]. Unlike classical convergence, almost convergence accommodates divergent sequences, offering a framework to analyze and understand the behaviour of sequences that do not converge in the traditional sense. One of the ways for summing sequences and series using an infinite matrix is the matrix summation method (see, [36]). Taking an infinite matrix $A=(a_{mk})$ for $m,k\in\mathbb{N}$, a given sequence $x=\langle x_m\rangle$ can be transformed into the sequence $\langle Ax\rangle_m$.

$$\langle Ax\rangle_m=\sum_{k=1}^{\infty}a_{mk}x_k.$$

If $\lim\limits_{m\to\infty}\langle Ax\rangle_m=L$ and the series on the RHS, $\sum\limits_{k=1}^{\infty}a_{mk}x_k$, converges $\forall$ m, then the sequence $\langle x_m\rangle$ is said to be summable by the matrix A and L will be called its limit. Further, if the limit is preserved then this method is referred to as regular matrix summability method and the infinite matrix A is called a regular matrix.

Definition 2.1.8 *[81] For every $\varepsilon>0$, if $\lim\limits_{m}\sum\limits_{m:|x_m-L|\geq\varepsilon}a_{mk}=0$, then the sequence, $x=\langle x_m\rangle$ is said to be $A-$statistically convergent to L and is denoted by $st_A-\lim x=L$.*

Lemma 2.1.9 *[81] Let $\{\mathscr{L}_m\}$ be a sequence of linear positive operators and $A=(a_{mk})$ be a regular positive summability matrix. Then,*

$$st_A-\lim_{m}\|\mathscr{L}_mh-h\|=0;\qquad h\in C([a,b])$$

iff

$$st_A-\lim_{m}\|\mathscr{L}_me_r-e_r\|=0;\qquad \textit{where } e_r=t^r \textit{ for } r=0,1,2.$$

Theorem 2.1.10 *Let $A=(a_{mk})$ be a regular positive summability matrix. Then for each $h\in C([0,1])$, we get*

$$st_A-\lim_{m\to\infty}\left\|\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}h-h\right\|=0.$$

Proof. From Lemma 2.1.9, it is sufficient to prove that

$$\left\|\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}e_r-x^r\right\|=0,$$

where $e_r = t^r$ for $r = 0,1,2$. Using the moments of modified α-Bernstein operator (2.6), we have

$$st_A - \lim_{m\to\infty} \left\| \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle} e_0 - 1 \right\| = 0,$$

and,

$$st_A - \lim_{m\to\infty} \left\| \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle} e_1 - x \right\| = 0.$$

Also,

$$\begin{aligned}
&\left\| \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle} e_2 - x^2 \right\| \\
&\leq \sup \left\{ \left| \frac{x^2}{1+x^2} \left[\frac{(m+1)(m-2)}{m^2} \frac{1-\alpha}{1+\gamma} + \frac{m-1}{m} \frac{\alpha}{1+\gamma} - 1 \right] \right| \right. \\
&\quad \left. + \left| \frac{x}{1+x^2} \left[\frac{((m-1)\gamma+1)(m+1)}{m^2} \frac{1-\alpha}{1+\gamma} + \frac{m\gamma+1}{m} \frac{\alpha}{1+\gamma} + \frac{1-\gamma}{m^2} \right] \right| \right\} \\
&= \frac{(m+1)(m-2)}{m^2} \frac{1-\alpha}{1+\gamma} + \frac{m-1}{m} \frac{\alpha}{1+\gamma} - 1 \\
&\quad + \frac{((m-1)\gamma+1)(m+1)}{m^2} \frac{1-\alpha}{1+\gamma} + \frac{m\gamma+1}{m} \frac{\alpha}{1+\gamma} + \frac{1-\gamma}{m^2} \\
&= \frac{(1-\alpha)(-1-\gamma)+(1-\gamma)(1+\gamma)}{m^2(1+\gamma)} \\
&= \frac{\alpha-\gamma}{m^2}.
\end{aligned}$$

Taking the sets $M = \{k : \left\| \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle} e_2 - x^2 \right\| \geq \varepsilon\}$ and $M_1 = \left\{k : \frac{\alpha-\gamma}{m^2} \geq \varepsilon \right\}$. Since $M \subseteq M_1$, therefore

$$\sum_{m\in M} a_{mk} \leq \sum_{m\in M_1} a_{mk}.$$

Thus,

$$st_A - \lim_{m\to\infty} \left\| \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle} e_2 - x^2 \right\| = 0 \ \ as \ \ k \to \infty.$$

□

2.1.5 King-Type Modification

In the study of approximation processes, the preservation of certain test functions plays a crucial role. The Korovkin theorem establishes that in order to guarantee uniform convergence of a sequence of positive linear operators $\{\mathscr{L}_m\}$, it is sufficient to verify convergence on the three functions

$$e_0(x) = 1, \quad e_1(x) = x \quad \text{and} \quad e_2(x) = x^2.$$

Till date many operators have been defined and a majority of these operators preserve the constant and the linear functions exactly, i.e.,

$$\mathscr{L}_m(e_0;x) = e_0(x) \text{ and } \mathscr{L}_m(e_1;x) = e_1(x).$$

However, the same does not hold for e_2. i.e., $\mathscr{L}_m(e_2;x) \neq e_2(x)$. Rather, $\mathscr{L}_m(e_2;x)$ tends to $e_2(x)$ as m tends to infinity. The modified α-Bernstein operators defined in equation (2.6) is an example for the same. This limiting property is sufficient to ensure that the operators approximate every continuous function. However, if they also preserved the quadratic test function, the approximation would be more effective and precise.

In 2003, King [130] gave a method by which any sequence of operators can be modified to a new sequence that preserves the functions e_0 and e_2. In this case, the linear function is no longer preserved exactly, instead, the operators yield an expression that converges to $e_1(x)$ as m tends to infinity. Recently, Finta [76] refined and extended King's idea by proving the unique existence of functions r_n, as defined by King. With the help of these functions, Finta showed that it is possible to construct operators that preserve e_0 together with $e_t(x) = x^t$, where $t \in \{2,3,\ldots\}$ is fixed.

On the same grounds, we define $\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$, an alteration of $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$ such that $\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(1;x) = 1$, $\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(t^2;x) = x^2$ and $\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(t;x) = r_m(x) \to x$ as $m \to \infty$. Then, by Korovkin theorem, for all continuous functions h and $x \in [0,1]$, we have

$$\lim_{m\to\infty} \tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(h;x) = h(x).$$

Now, from Lemma 2.1.3, we have

$$\begin{aligned}
&\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(1;x) = 1,\ \mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(t;x) = x, \text{ and}\\
&\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(t^2;x) = \frac{1}{1+\gamma}\left[x^2 + \frac{m+2(1-\alpha)}{m^2}x(1-x)\right] + \frac{x\gamma}{1+\gamma}\\
&\qquad = \frac{x^2}{1+\gamma}\left[1 - \frac{m+2(1-\alpha)}{m^2}\right] + \frac{x}{1+\gamma}\left[\gamma + \frac{m+2(1-\alpha)}{m^2}\right]\\
&\qquad = a_m x^2 + b_m x,
\end{aligned}$$

where

$$a_m = \frac{1}{1+\gamma}\left[1 - \frac{m+2(1-\alpha)}{m^2}\right] \text{ and } b_m = \frac{1}{1+\gamma}\left[\gamma + \frac{m+2(1-\alpha)}{m^2}\right].$$

Let $y^2 = a_m x^2 + b_m x$. Solving for x in this quadratic equation gives us the value of y. Replacing x by $r_m(x)$ and y by x, gives us the value of $r_m(x)$, used to define $\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$.

Thus, we consider the modified operators as:

$$\begin{aligned}\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(h;x) &= (1-\alpha)\sum_{k=0}^{m-1} {}^{m-1}C_k \frac{(r_m(x))^{[k,-\gamma]}(1-r_m(x))^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}} g_k \\ &+ \alpha\sum_{k=0}^{m} {}^{m}C_k \frac{(r_m(x))^{[k,-\gamma]}(1-r_m(x))^{[m-k,-\gamma]}}{1^{[m,-\gamma]}} h_k, \end{aligned} \tag{2.9}$$

where $h_k = h\left(\frac{k}{m}\right)$, $g_k = \left(1-\frac{k}{m-1}\right)h_k + \frac{k}{m-1}h_{k+1}$ and

$$r_m(x) = \frac{-b_m + \sqrt{{b_m}^2 + 4a_m x^2}}{2a_m}.$$

Lemma 2.1.11 *For the operators* (2.9)*, we can prove the following identities:*

(i) $\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(e_0;x) = 1$

(ii) $\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(e_1;x) = r_m(x)$

(iii) $\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(e_2;x) = x^2$,

where $e_r = t^r$ *for* $r = 0,1,2$.

Lemma 2.1.12 *Let* $\gamma = O(1/m)$. *From Lemma 2.1.11, we have*

(i) $\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1(t);x) = r_m(x) - x \to 0$ *as* $m \to \infty$

(ii) $\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2(t);x) = -2x\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1;x)$,

where $\varphi_m(t) = (t-x)^m$.

Proof. Before proceeding, we assume that $\gamma = O(1/m)$.

(i) For the first central moment,

$$\begin{aligned}\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1(t);x) &= \tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}((t-x);x) \\ &= \tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(t;x) - x\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(1;x) \\ &= r_m(x) - x,\end{aligned}$$

where

$$\begin{aligned}r_m(x) &= \frac{-b_m + \sqrt{{b_m}^2 + 4a_m x^2}}{2a_m} \\ &= -\frac{b_m}{2a_m} + \sqrt{\frac{{b_m}^2}{4{a_m}^2} + \frac{x^2}{a_m}}.\end{aligned}$$

Now,

$$\begin{aligned}
-\frac{b_m}{2a_m} &= -\frac{1}{2}\left(\frac{m^2\gamma+m+2(1-\alpha)}{m^2-m-2(1-\alpha)}\right)\\
&= -\frac{1}{2}\left(\frac{\gamma+\frac{1}{m}+\frac{2(1-\alpha)}{m^2}}{1-\frac{1}{m}-\frac{2(1-\alpha)}{m^2}}\right)\\
&\to -\frac{\gamma}{2}, \text{ and}\\
\frac{x^2}{a_m} &= \frac{x^2(1+\gamma)}{1-\frac{m+2(1-\alpha)}{m^2}}\\
&\to x^2(1+\gamma).
\end{aligned}$$

Taking $\gamma = O\left(1/m\right)$, we have

$$\begin{aligned}
r_m(x) &\to \frac{-\gamma+\sqrt{\gamma^2+4\gamma x^2+4x^2}}{2}\\
&\to x \text{ as } m\to\infty.
\end{aligned}$$

(i) From Lemma 2.1.11,

$$\begin{aligned}
\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2(t);x) &= \tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}\left((t-x)^2;x\right)\\
&= \tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}\left(t^2;x\right) - 2x\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(t;x) + x^2\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(1;x)\\
&= 2x^2 - 2xr_m(x)\\
&= 2x\left(x-r_m(x)\right)\\
&= -2x\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1(t);x).
\end{aligned}$$

□

Lemma 2.1.13 *[65] For any sequence of linear positive operators $\{\mathscr{L}_m\}$ and $h \in C[a,b]$, we have the following relations:*

$$\begin{aligned}
|\mathscr{L}_m(h;x)-h(x)| \le{}& |h(x)|\,|1-\mathscr{L}_m(1;x)|\\
&+\omega(h,\delta)\left[\mathscr{L}_m(1;x)+\frac{1}{\delta}(\mathscr{L}_m(1;x))^{1/2}\alpha_m(x)\right],
\end{aligned}$$

where $\omega(h,\delta)$ is the modulus of continuity of h and ${\alpha_m}^2(x) = \mathscr{L}_m\left((t-x)^2;x\right)$.

Further, if h' is continuous, then

$$\begin{aligned}
|\mathscr{L}_m(h;x)-h(x)| \le{}& |h(x)|\,|1-\mathscr{L}_m(1;x)| + \left|h'(x)\right|\,|\mathscr{L}_m((t-x);x)|\\
&+\omega\left(h',\alpha_m(x)\right)\left[1+(\mathscr{L}_m(1;x))^{1/2}\right]\alpha_m(x).
\end{aligned}$$

Let us compare the contagion form of α-Bernstein operators (2.6) and its King-type modification (2.9), as defined above, on the basis of these relations.

Theorem 2.1.14 *Let* $\left\{\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}\right\}$ *and* $\left\{\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}\right\}$ *be the sequences of linear positive operators as defined in the above equations and* $h \in C([a,b])$. *Then for* $0 \leq \alpha \leq 1$ *and* $\gamma = O\left({}^{1}\!/\!{}_{m}\right)$, *we have the following:*

$$\left|\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) - h(x)\right| \leq \omega(h,\delta)\left[1 + \frac{1}{\delta}\sqrt{b_m x(1-x)}\right]$$
$$\left|\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(h;x) - h(x)\right| \leq \omega(h,\delta)\left[1 + \frac{1}{\delta}\sqrt{2x(x-r_m(x))}\right].$$

Further, if $h \in C^2([a,b])$, *then*

$$\left|\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) - h(x)\right| \leq 2\omega\left(h', \alpha_m(x)\right)\sqrt{b_m x(1-x)}$$
$$\left|\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(h;x) - h(x)\right| \leq 2\sqrt{2}\,\omega\left(h', \alpha_m(x)\right)\sqrt{x(x-r_m(x))},$$

where $\omega(h,\delta)$ *is the modulus of continuity of* h, ${\alpha_m}^2(x)$ *is the respective second central moment of that operators and* $r_m(x)$ *and* $b_m(x)$ *are as defined above.*

Proof. Using Lemma 2.1.11 and Lemma 2.1.13, with $\varphi_m(t) = (t-x)^m$, we have

$$\begin{aligned}
&\left|\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x) - h(x)\right| \\
&\quad \leq |h(x)|\left|\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(1;x) - 1\right| \\
&\qquad + \omega(h,\delta)\left[\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(1;x) + \frac{1}{\delta}\sqrt{\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(1;x)\,\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2(t);x)}\right] \\
&\quad = \omega(h,\delta)\left[1 + \frac{1}{\delta}\sqrt{\frac{1}{1+\gamma}\left(\frac{m+2(1-\alpha)}{m^2} + \gamma\right)x(1-x)}\right] \\
&\quad = \omega(h,\delta)\left[1 + \frac{1}{\delta}\sqrt{b_m x(1-x)}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\left|\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(h;x) - h(x)\right| \\
&\quad \leq |h(x)|\left|\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(1;x) - 1\right| \\
&\qquad + \omega(h,\delta)\left[\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(1;x) + \frac{1}{\delta}\sqrt{\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(1;x)\,\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2(t);x)}\right] \\
&\quad = \omega(h,\delta)\left[1 + \frac{1}{\delta}\sqrt{2x(x-r_m(x))}\right].
\end{aligned}$$

Now, if h' is continuous, then we have

$$\begin{aligned}\left|\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;x)-h(x)\right| &\leq |h(x)|\left|1-\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(1;x)\right|+\left|h'(x)\right|\left|\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1(t);x)\right| \\ &\quad+\omega\left(h',\alpha_m(x)\right)\left[1+\left(\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(1;x)\right)^{1/2}\right]\left(\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2(t);x)\right)^{1/2} \\ &= 2\omega\left(h',\alpha_m(x)\right)\sqrt{\frac{1}{1+\gamma}\left(\frac{m+2(1-\alpha)}{m^2}+\gamma\right)x(1-x)} \\ &= 2\omega\left(h',\alpha_m(x)\right)\sqrt{b_m x(1-x)},\end{aligned}$$

and

$$\begin{aligned}\left|\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(h;x)-h(x)\right| &\leq |h(x)|\left|1-\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(1;x)\right|+\left|h'(x)\right|\left|\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_1(t);x)\right| \\ &\quad+\omega\left(h',\alpha_m(x)\right)\left[1+\left(\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(1;x)\right)^{1/2}\right]\left(\tilde{\mathscr{P}}_{m,\alpha}^{\langle\gamma\rangle}(\varphi_2(t);x)\right)^{1/2} \\ &= \left|h'(x)\right||r_m(x)-x|+2\sqrt{2}\omega\left(h',\alpha_m(x)\right)\sqrt{x(x-r_m(x))} \\ &\leq 2\sqrt{2}\omega\left(h',\alpha_m(x)\right)\sqrt{x(x-r_m(x))}.\end{aligned}$$

□

2.2 Kantorovich Variant of the Proposed Operators

The Kantorovich variant is an important modification of linear positive operators that involves taking the integral of the operator rather than just the point-wise evaluation, which can lead to better approximation of integrable functions.

In Section 2.1, we introduced the parametric Bernstein operators, based on the parameters α and γ, which are an extension to the α-Bernstein operators, defined by Chen et al. in 2017 [51]. Simultaneously that year, Mohiuddine et al. [153] put forth the following Kantorovich form of these operators:

$$K_{n,\alpha}(f;u)=(n+1)\sum_{k=0}^{n}p_{n,k}^{(\alpha)}(u)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(t)\,dt. \tag{2.10}$$

Deo and Pratap [61] conducted a thorough analysis of several auxiliary properties of the operators given by equation (2.10), which included examining the Voronovskaya-type asymptotic behaviour, the direct local approximation theorem and functions of bounded variation.

In 2018, Cai and Xu [43] extended the research on Kantorovich operators by introducing a q-analog of the operators (2.4) and investigated some convexity and shape preserving properties, such as monotonicity, with respect to $f(u)$. In a separate work, Pratap and Deo [174] proposed the q-analog of the Kantorovich form of the α-Bernstein operators, as defined in equation (2.10). They computed the moments and analyzed its convergence rate by using the concept of modulus of continuity.

Several mathematicians have studied the Bernstein and Baskakov operators with the help of contagion distribution, defined by equation (2.1), and gave their variants (see [56; 57; 58; 67; 105]). Section 2.1 defined the α-Bernstein operators with respect to the contagion distribution by equation (2.6).

However, since discrete operators are not suitable for approximating functions which are not continuous, we introduce the Kantorovich type generalization, which helps to approximate integrable functions. The Kantorovich form of $\mathscr{P}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ is defined as,

$$K_{m,\alpha}^{\langle\gamma\rangle}(h;u) = (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle} \int_{k/m+1}^{(k+1)/m+1} h(t)dt. \tag{2.11}$$

When $\gamma = 0$, the Kantorovich operators described above gets simplified and reduced to the α-Bernstein-Kantorovich operators presented earlier in equation (2.10). This observation is a special case of a more general mathematical concept known as a limiting case, and so we can say that as m tends to ∞, $\gamma = O\left({}^{1}/_{m}\right)$ tends to zero. That is,

$$K_{m,\alpha}^{\langle\gamma\rangle}(h;u) \to K_{m,\alpha}(h;u).$$

The proposed operators (2.11) indeed have a complicated form, however taking γ of order $1/m$ reduces the complexities upto some extend. Contagion distribution is particularly important and applicable in various real-world scenarios where the probability of an event depends on its past occurrences. For instance, researchers in the field of bio-mathematics can approximate current and future population growth functions by studying the past environmental activities. Researchers in the field of applied mathematics can use this distribution to approximate functions and curves as per their requirements and availability of data.

2.2.1 Approximation Results

Lemma 2.2.1 *Let α be any fixed real number in $[0,1]$ and $\gamma = O(1/m)$. Then for $0 \leq u \leq 1$,*

$$K_{m,\alpha}^{\langle\gamma\rangle}(h;u) = \frac{1}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)} \int_0^1 s^{\frac{u}{\gamma}-1}(1-s)^{\frac{1-u}{\gamma}-1} K_{m,\alpha}(h;s)\,ds,$$

where $K_{m,\alpha}(h;u)$ is the Kantorovich form of α-Bernstein operators (2.10).

Proof. The Beta function $\beta(\mu,\nu)$ is described as:

$$\beta(\mu,\nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)},$$

where $\Gamma(\alpha)$ represents the Gamma function. Moreover, by the characteristics of Gamma function we can write,

$$\Gamma(m+\alpha) = \alpha(\alpha+1)(\alpha+2)...(\alpha+m-1)\Gamma(\alpha).$$

Thus,

$$\begin{aligned}
&\beta\left(\theta+\frac{u}{\gamma},\pi+\frac{1-u}{\gamma}\right)\\
&= \frac{\Gamma\left(\theta+\frac{u}{\gamma}\right)\Gamma\left(\pi+\frac{1-u}{\gamma}\right)}{\Gamma\left(\theta+\pi+\frac{1}{\gamma}\right)}\\
&= \frac{\frac{u}{\gamma}\left(\frac{u}{\gamma}+1\right)\cdots\left(\frac{u}{\gamma}+\theta-1\right)\Gamma\left(\frac{u}{\gamma}\right)\times\frac{1-u}{\gamma}\left(\frac{1-u}{\gamma}+1\right)\cdots\left(\frac{1-u}{\gamma}+\pi-1\right)\Gamma\left(\frac{1-u}{\gamma}\right)}{\frac{1}{\gamma}\left(\frac{1}{\gamma}+1\right)\cdots\left(\frac{1}{\gamma}+\theta+\pi-1\right)\Gamma\left(\frac{1}{\gamma}\right)}\\
&= \frac{u(u+\gamma)\cdots(u+(\theta-1)\gamma)(1-u)(1-u+\gamma)\cdots(1-u+(\pi-1)\gamma)}{(1+\gamma)\cdots(1+(\theta+\pi-1)\gamma)}\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)\\
&= \frac{u^{[\theta,-\gamma]}(1-u)^{[\pi,-\gamma]}}{1^{[\theta+\pi,-\gamma]}}\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right).
\end{aligned} \tag{2.12}$$

By substituting the expression for the Kantorovich form of α-Bernstein operators (2.10) in the corresponding equation, and by utilizing the relation given by equation

(2.12), we arrive at,

$$\begin{aligned}
&\frac{1}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)}\int_0^1 s^{\frac{u}{\gamma}-1}(1-s)^{\frac{1-u}{\gamma}-1}K_{m,\alpha}(h;s)\,ds\\
&\quad=\frac{(m+1)}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)}\sum_{k=0}^{m}\left[\binom{m-2}{k}(1-\alpha)\beta\left(k+\frac{u}{\gamma},m-k-1+\frac{1-u}{\gamma}\right)\right.\\
&\qquad+\binom{m-2}{k-2}(1-\alpha)\beta\left(k-1+\frac{u}{\gamma},m-k+\frac{1-u}{\gamma}\right)\\
&\qquad\left.+\binom{m}{k}\alpha\beta\left(k+\frac{u}{\gamma},m-k+\frac{1-u}{\gamma}\right)\right]\int_{k/m+1}^{(k+1)/m+1}h(s)\,ds\\
&\quad=(m+1)\sum_{k=0}^{m}\left[\binom{m-2}{k}\frac{(1-\alpha)u^{[k,-\gamma]}(1-u)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\right.\\
&\qquad+\binom{m-2}{k-2}\frac{(1-\alpha)u^{[k-1,-\gamma]}(1-u)^{[m-k,-\gamma]}}{1^{[m-1,-\gamma]}}\\
&\qquad\left.+\binom{m}{k}\frac{\alpha u^{[k,-\gamma]}(1-u)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}}\right]\int_{k/m+1}^{(k+1)/m+1}h(s)\,ds\\
&\quad=(m+1)\sum_{k=0}^{m}p_{m,k}^{\langle\alpha,\gamma\rangle}\int_{k/m+1}^{(k+1)/m+1}h(s)\,ds\\
&\quad=K_{m,\alpha}^{\langle\gamma\rangle}(h;u).
\end{aligned}$$

□

Lemma 2.2.2 *For the proposed Kantorovich form given in equation* (2.11)*, we get the following identities:*

(i) $K_{m,\alpha}^{\langle\gamma\rangle}(e_0;u)=1$

(ii) $K_{m,\alpha}^{\langle\gamma\rangle}(e_1;u)=\frac{mu}{m+1}+\frac{1}{2(m+1)}$

(iii) $K_{m,\alpha}^{\langle\gamma\rangle}(e_2;u)=\frac{m^2-m-2(1-\alpha)}{(m+1)^2}u\left(\frac{u+\gamma}{1+\gamma}\right)+\frac{2m+2(1-\alpha)}{(m+1)^2}u+\frac{1}{3(m+1)^2},$

where $e_r=t^r$ *for* $r=0,1,2.$

Proof. In order to evaluate the first and second moments of the proposed operators, it is convenient to make use of the known moment identities of the parametric Bernstein operators, which are summarized in Lemma 2.1.3. By substituting these formulae, we get

$$\begin{aligned}
K_{m,\alpha}^{\langle\gamma\rangle}(e_0;u) &= (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} dt \\
&= (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\left(\frac{1}{m+1}\right) \\
&= 1,
\end{aligned}$$

and,

$$\begin{aligned}
K_{m,\alpha}^{\langle\gamma\rangle}(e_1;u) &= (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} t dt \\
&= \frac{m+1}{2}\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\left(\frac{(k+1)^2}{(m+1)^2}-\frac{k^2}{(m+1)^2}\right) \\
&= \frac{1}{2(m+1)}\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\,(2k+1) \\
&= \frac{1}{2(m+1)}\left[2m\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\frac{k}{m}+\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\right] \\
&= \frac{mu}{m+1}+\frac{1}{2(m+1)}.
\end{aligned}$$

In the case of the third moment, a similar line of computation can be followed. Nevertheless, unlike the first and second moments, using the moments of the parametric Bernstein operators to evaluate $K_{m,\alpha}^{\langle\gamma\rangle}(e_2;u)$ leads to a much complicated expression.

It is important to observe that, just as the parametric Bernstein operators, discussed in Section 2.1, arise from the generalized sequence of operators given in equation (2.3) when built with the α-Bernstein operators, a similar idea works for the Kantorovich modification as well. If, in place of the α-Bernstein operators, we take their Kantorovich form (given in equation (2.10)), then the same operators (2.3) leads naturally to the proposed Kantorovich form of parametric Bernstein operators. This observation also allows us to compute the moments of the operators in (2.11) directly by combining the sequence (2.3) with the known moments of the Kantorovich form of α-Bernstein operators.

First, we state the third moment of operators (2.10).

$$K_{m,\alpha}(e_2;u) = \frac{3\left(m^2-m-2(1-\alpha)\right)u^2+3\left(2m+2(1-\alpha)\right)u+1}{3(m+1)^2}.$$

Thus,

$$\begin{aligned}
&K_{m,\alpha}^{\langle\gamma\rangle}(e_2;u)\\
&= \frac{1}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)}\int_0^1 t^{\frac{u}{\gamma}-1}(1-t)^{\frac{1-u}{\gamma}-1}K_{m,\alpha}(e_2;t)dt\\
&= \frac{1}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)}\int_0^1 t^{\frac{u}{\gamma}-1}(1-t)^{\frac{1-u}{\gamma}-1}\frac{3\left(m^2-m-2(1-\alpha)\right)t^2+3\left(2m+2(1-\alpha)\right)t+1}{3(m+1)^2}dt\\
&= \frac{\beta\left(\frac{u}{\gamma}+2,\frac{1-u}{\gamma}\right)}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)}\frac{m^2-m-2(1-\alpha)}{3(m+1)^2}+\frac{\beta\left(\frac{u}{\gamma}+1,\frac{1-u}{\gamma}\right)}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)}\frac{2m+2(1-\alpha)}{(m+1)^2}+\frac{\beta\left(\frac{u}{\gamma}+1,\frac{1-u}{\gamma}\right)}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)}\frac{1}{3(m+1)^2}\\
&= \frac{m^2-m-2(1-\alpha)}{(m+1)^2}u\left(\frac{u+\gamma}{1+\gamma}\right)+\frac{2m+2(1-\alpha)}{(m+1)^2}u+\frac{1}{3(m+1)^2}.
\end{aligned}$$

□

Lemma 2.2.3 *With respect to the Kantorovich operators $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$, we can determine the following expressions for the central moments,*

$$\begin{aligned}
\mu_1(u) &= K_{m,\alpha}^{\langle\gamma\rangle}((t-u);u)\\
&= \frac{1-2u}{2(m+1)}\\
\mu_2(u) &= K_{m,\alpha}^{\langle\gamma\rangle}((t-u)^2;u)\\
&= \frac{m^2\gamma-\gamma+m-2\alpha+1}{(m+1)^2(1+\gamma)}u(1-u)+\frac{1}{3(m+1)^2}
\end{aligned}$$

$$\begin{aligned}
\mu_4(u) &= K_{m,\alpha}^{\langle\gamma\rangle}((t-u)^4;u)\\
&= \frac{3m^2-4(2+3\alpha)m-(131-132\alpha)}{(m+1)^4}\left(\frac{u+3\gamma}{1+3\gamma}\right)\left(\frac{u+2\gamma}{1+2\gamma}\right)\left(\frac{u+\gamma}{1+\gamma}\right)u\\
&\quad-\frac{3m^2-4(2+3\alpha)m-(131-132\alpha)}{(m+1)^4}\left(\frac{u+2\gamma}{1+2\gamma}\right)\left(\frac{u+\gamma}{1+\gamma}\right)u\\
&\quad+\frac{3m^2-(13+12\alpha)m-2(80-81\alpha)}{(m+1)^4}\left(\frac{u+\gamma}{1+\gamma}\right)u+\frac{5m+(33-32\alpha)}{(m+1)^4}u\\
&\quad+\frac{1}{5(m+1)^4}.
\end{aligned}$$

Lemma 2.2.4 *Let $h\in C[0,1]$ and $0\le\alpha\le 1$. Then $\left|K_{m,\alpha}^{\langle\gamma\rangle}(h;u)\right|\le\|h\|$, where $\|.\|$ is the supremum norm of a function, defined as $\|h\|=\sup_{s\in[0,1]}|h(s)|$.*

Proof. From equation (2.11), we can write

$$\left|K_{m,\alpha}^{\langle\gamma\rangle}(h;u)\right| = \left|(m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}\int_{k/m+1}^{(k+1)/m+1} h(s)ds\right|,$$

which implies,

$$\begin{aligned}\left|K_{m,\alpha}^{\langle\gamma\rangle}(h;u)\right| &\leq \|h(u)\|\,(m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}\int_{k/m+1}^{(k+1)/m+1} ds\\ &= \|h(u)\|.\end{aligned}$$

Thus, we get the desired result. □

Theorem 2.2.5 *Let $h(u)$ be bounded on $[0,1]$ and $0 \leq \alpha \leq 1$. Then for any $u \in [0,1]$ and $\gamma = O(1/m)$ at which $h''(u)$ exists,*

$$\lim_{m\to\infty} m\left[K_{m,\alpha}^{\langle\gamma\rangle}(h(t);u) - h(u)\right] = \left(\frac{1-2u}{2}\right)h'(u) + \frac{u(1-u)}{2(1+\gamma)}h''(u).$$

Proof. By Taylor's theorem, there exist ξ between u and t such that,

$$h(t) = h(u) + (t-u)h'(u) + \frac{(t-u)^2}{2!}h''(u) + \eta(t,u)(t-u)^2,$$

where $\eta(t,u) = \frac{h''(\xi)-h''(u)}{2} \to 0$ as $t \to u$. Hence,

$$\begin{aligned}\lim_{m\to\infty} m\left[K_{m,\alpha}^{\langle\gamma\rangle}(h(t)-h(u);u)\right] &= h'(u)\lim_{m\to\infty} mK_{m,\alpha}^{\langle\gamma\rangle}((t-u);u)\\ &\quad + \frac{h''(u)}{2}\lim_{m\to\infty} mK_{m,\alpha}^{\langle\gamma\rangle}\left((t-u)^2;u\right)\\ &\quad + \lim_{m\to\infty} mK_{m,\alpha}^{\langle\gamma\rangle}\left(\eta(t,u)(t-u)^2;u\right).\end{aligned}$$

First we claim that $mK_{m,\alpha}^{\langle\gamma\rangle}\left(\eta(t,u)(t-u)^2;u\right) \to 0$ as $m \to \infty$.
For every $\varepsilon > 0$, we consider $\delta > 0$ such that $\eta(t,u) < \varepsilon$ for $|t-u| < \delta$. And for $|t-u| \geq \delta$, we can say that $\eta(t,u)$ is bounded above, say by $M > 0$. Thus,

$$mK_{m,\alpha}^{\langle\gamma\rangle}\left(\eta(t,u)(t-u)^2;u\right) \leq \varepsilon m\mu_2(u) + \frac{M}{\delta^2}m\mu_4(u),$$

where $\mu_r = K_{m,\alpha}^{\langle\gamma\rangle}((t-u)^r;u)$. Taking $m \to \infty$ proves our claim.

Now, from Lemma 2.2.3 we have,

$$\begin{aligned}&\lim_{m\to\infty} m\left[K_{m,\alpha}^{\langle\gamma\rangle}(h(t)-h(u);u)\right]\\ &= \lim_{m\to\infty} m\left[K_{m,\alpha}^{\langle\gamma\rangle}(h(t);u) - h(u)\right]\\ &= \lim_{m\to\infty} m\left(\frac{1-2u}{2(m+1)}\right)h'(u) + \lim_{m\to\infty} m\left(\frac{m-2\alpha+1}{(m+1)^2(1+\gamma)}u(1-u) + \frac{1}{3(m+1)^2}\right)\frac{h''(u)}{2}\\ &= \left(\frac{1-2u}{2}\right)h'(u) + \frac{u(1-u)}{2(1+\gamma)}h''(u).\end{aligned}$$

□

Theorem 2.2.6 *Consider a continuous function h on* $[0,1]$. *For* $\alpha \in [0,1]$ *and* $\gamma = O(1/m)$, *we have*

$$\left| K_{m,\alpha}^{\langle\gamma\rangle}(h(t);u) - h(u) \right| \leq 2\omega(h,\delta(u)),$$

where $\omega(h;\delta)$ *is the modulus of continuity of h with* $\delta(u) = (\mu_2(u))^{1/2}$.

Proof. Using Proposition 1.1.4 and the definition of operators (2.11), we have,

$$\begin{aligned}
&\left| K_{m,\alpha}^{\langle\gamma\rangle}(h(s);u) - h(u) \right| \\
&\leq (m+1) \sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} |h(s)-h(u)|\, ds \\
&\leq (m+1) \sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} \omega(h;|s-u|)\, ds \\
&\leq (m+1) \sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} \left(1+\frac{1}{\delta}|s-u|\right) \omega(h;\delta)\, ds \\
&\leq \left[1+\frac{1}{\delta}(m+1) \sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} |s-u|\, ds\right] \omega(h;\delta) \\
&\leq \left[1+\frac{1}{\delta}(m+1) \sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \left(\int_{k/m+1}^{(k+1)/m+1} 1^2 ds\right)^{\frac{1}{2}} \left(\int_{k/m+1}^{(k+1)/m+1} |s-u|^2 ds\right)^{\frac{1}{2}}\right] \omega(h;\delta) \\
&\leq \left[1+\frac{1}{\delta}\left((m+1) \sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} ds\right)^{\frac{1}{2}}\right. \\
&\qquad \left.\times \left((m+1) \sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} (s-u)^2 ds\right)^{\frac{1}{2}}\right] \omega(h;\delta) \\
&= \left[1+\frac{1}{\delta}\left(K_{m,\alpha}^{\langle\gamma\rangle}(1;u)\right)^{\frac{1}{2}} \left(K_{m,\alpha}^{\langle\gamma\rangle}\left((s-u)^2;u\right)\right)^{\frac{1}{2}}\right] \omega(h;\delta) \\
&= \left[1+\frac{1}{\delta}(\mu_2(u))^{\frac{1}{2}}\right] \omega(h;\delta) \\
&= 2\omega(h;\delta),
\end{aligned}$$

where $\delta = (\mu_2(u))^{\frac{1}{2}}$. □

2.2.2 Genuine-Type Modification

From Lemma 2.2.2, it is clear that the proposed operators (2.11) exactly preserve only the constant test function. For the linear test function $e_1(t)$, they do not produce it identically, instead, they yield an expression that converges to $e_1(u)$ as $m \to \infty$. While this asymptotic property ensures approximation, it also highlights a limitation of the operators in their present form. To enhance their effectiveness, our objective is to introduce a suitable modification so that both the constant and the linear test functions are preserved exactly. Thus, sticking to the assumption that $K_{m,\alpha}^{\langle\gamma\rangle}(e_1;u) = u$, we can define a new sequence of positive linear operators as

$$\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u) = (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(r_m(u))\int_{k/m+1}^{(k+1)/m+1} h(s)ds, \tag{2.13}$$

where,

$$r_m(u) = \left(\frac{m+1}{m}\right)u - \frac{1}{2m}.$$

We will now find the moments of this new modification. For our convenience, let us denote the central moments of (2.13) with $\tilde{\mu}_1$ and $\tilde{\mu}_2$.

Lemma 2.2.7 *Let $\alpha \in [0,1]$ and $\gamma = O(1/m) \in [0,1]$. Then the moments and central moments of $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ are as follows:*

(i) $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(1;u) = 1$

(ii) $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(t;u) = u$

(iii) $$\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(t^2;u) = \begin{cases} \frac{1}{(m+1)(1+\gamma)}\left[(m-2)(r_m(u))^2 + (m\gamma+2)r_m(u)\right] \\ -\frac{2\alpha}{(m+1)^2(1+\gamma)}r_m(u)(1-r_m(u)) + \frac{1}{3(m+1)^2} \end{cases}$$

(iv) $\tilde{\mu}_1 = 0$

(v) $$\tilde{\mu}_2 = \begin{cases} \frac{1}{1+\gamma}\left[\frac{(m-2)(m+1)}{m^2}u^2 - \frac{m-2}{m^2}u + \frac{m-2}{4m^2(m+1)} + \frac{m\gamma+2}{m}u - \frac{m\gamma+2}{2m(m+1)}\right] \\ -\frac{2\alpha}{1+\gamma}\left[\frac{u}{m(m+1)} - \frac{1}{2m(m+1)^2} - \frac{u^2}{m^2} + \frac{u}{m^2(m+1)} - \frac{1}{4m^2(m+1)^2}\right] + \frac{1}{3(m+1)^2} - u^2. \end{cases}$$

The moments and central moments of the Kantorovich operators (2.11) can be utilized to demonstrate the proof of the aforementioned lemma. Building upon this, we can establish the Voronovskya-type result for the newly modified operators $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ as well.

Theorem 2.2.8 *Let $h(u)$ be bounded on $[0,1]$ and $0 \leq \alpha \leq 1$. Then for any $u \in [0,1]$ and $\gamma = O(1/m)$ at which $h''(u)$ exists,*

$$\lim_{m\to\infty} m\left[\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h(t);u) - h(u)\right] = \frac{1}{2(1+\gamma)} cu(1-u)h''(u),$$

where $c = \lim_{m\to\infty}(m\gamma+1)$.

Proof. By proceeding in a similar manner as in Theorem 2.2.5, we obtain

$$\begin{aligned}\lim_{m\to\infty} m\left[\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h(t)-h(u));u\right] &= h'(u)\lim_{m\to\infty} m\tilde{\mu}_1 + \tfrac{h''(u)}{2}\lim_{m\to\infty} m\tilde{\mu}_2 \\ &\quad + \lim_{m\to\infty} m\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}\left(\imath(t,u)(t-u)^2;u\right), \qquad (2.14)\end{aligned}$$

where $m\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}\left(\imath(t,u)(t-u)^2;u\right) \to 0$ as $m\to\infty$. Moreover, we can simplify the expression on the right hand side of equation (2.14), as

$$\begin{aligned}\lim_{m\to\infty} m\tilde{\mu}_2 &= \frac{1}{1+\gamma}\left[\left(-1-\frac{2}{m}-m\gamma\right)u^2 + \left(m\gamma+\frac{2}{m}+1\right)u + \frac{m-2}{4m(m+1)} - \frac{m\gamma+2}{2(m+1)}\right] \\ &\quad - \frac{2\alpha}{1+\gamma}\left[\frac{u}{m+1} - \frac{1}{2(m+1)^2} - \frac{u^2}{m} + \frac{u}{m(m+1)} - \frac{1}{4m(m+1)^2}\right] + \frac{m}{3(m+1)^2} \\ &= \frac{1}{1+\gamma} cu(1-u),\end{aligned}$$

where $c = \lim_{m\to\infty}(m\gamma+1)$.

Thus,

$$\lim_{m\to\infty} m\left[\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h(t)-h(u);u)\right] = \frac{1}{2(1+\gamma)} cu(1-u)h''(u).$$

□

2.2.3 Graphical Study of the Approximation Process

In this section, we examine the approximation properties of the Kantorovich operators (2.11) along with their genuine-type modification (2.13). Our focus is to analyze how these operators approximate a given function $h(u)$. To achieve this, we present numerical examples supported by graphical representations. These graphs illustrate not only the action of the operators but also the corresponding approximation errors, thereby providing a comparative view of their performance. For this purpose, we define the error functions as

$$E_{m,\alpha}^{\langle\gamma\rangle}(u) = \left|K_{m,\alpha}^{\langle\gamma\rangle}(h;u) - h(u)\right|,$$

and

$$\tilde{E}_{m,\alpha}^{\langle\gamma\rangle}(u) = \left|\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u) - h(u)\right|.$$

From the moments of the proposed Kantorovich operators, given in Lemma 2.2.2, we claimed that for the operators to uniformly converge to the desired function, γ should tend to 0 as m tends to ∞. We verify this graphically in the subsequent subsection.

2.2.4 Effect of parameter γ

In this subsection, we investigate the influence of the Pólya parameter γ on the behaviour of the proposed Kantorovich operator $K_{m,\alpha}^{\langle\gamma\rangle}(h;x)$ for a fixed value of α and m. The parameter γ plays a crucial role in shaping the generalized factorial terms appearing in the definition of the operator, and thereby controls the skewness and spread of the associated basis functions. To illustrate this dependence, we consider the piecewise continuous function

$$h(u) = \begin{cases} u, & 0 \le u < 0.6 \\ 1-u, & 0.6 \le u < 0.8 \\ 0.2, & 0.8 \le u \le 1, \end{cases}$$

and compute $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ for a fixed value of $m = 200$ and $\alpha = 0.8$, while varying

$$\gamma \in \{0.001, 0.01, 0.1\}.$$

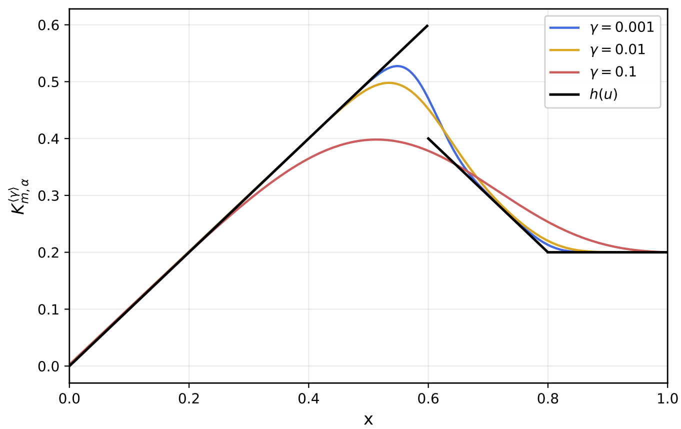
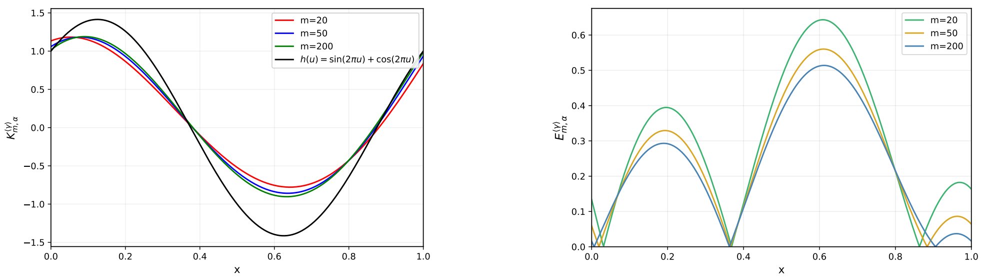


Figure 2.6: Effect of γ on the approximation process, with $\alpha = 0.8$ and $m = 200$.

From Figure 2.6 we can conclude that for smaller values of γ, the approximation follows the desired function, $h(u)$ more closely and for $\gamma = 0$, $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ will coincide with the classical α-Kantorovich-Bernstein form.

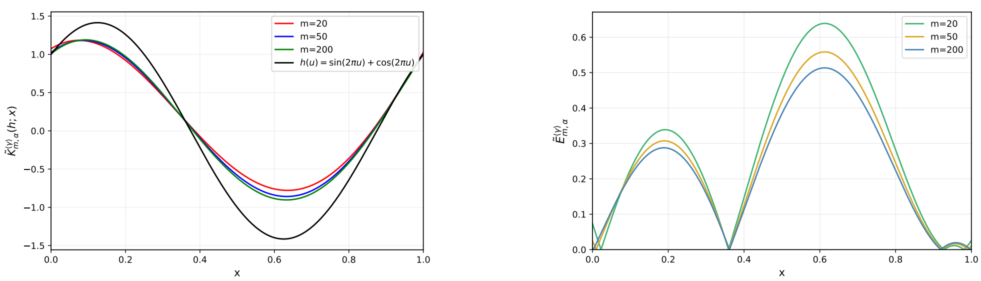
This confirms that γ acts as a shape control parameter. In other words, increasing its value leads to poorer approximations, whereas smaller γ values yield more accurate estimation of the desired function.

Example 2.2.9 *Let us consider the case when* $\alpha = 0.8$ *and* $\gamma = 0.1$. *For the function,* $h(u) = \sin 2\pi u + \cos 2\pi u$, *Figure 2.7 illustrates the graphs of the operators* $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ *and the error function* $E_{m,\alpha}^{\langle\gamma\rangle}(u)$ *for different values of m, respectively. Likewise, we graph the operators* $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ *and the error function* $\tilde{E}_{m,\alpha}^{\langle\gamma\rangle}(u)$ *in Figure 2.8.*



(a) Approximation Process for $K_{m,\alpha}^{\langle\gamma\rangle}(h;x)$ (b) Error Function, $E_{m,\alpha}^{\langle\gamma\rangle}(u)$

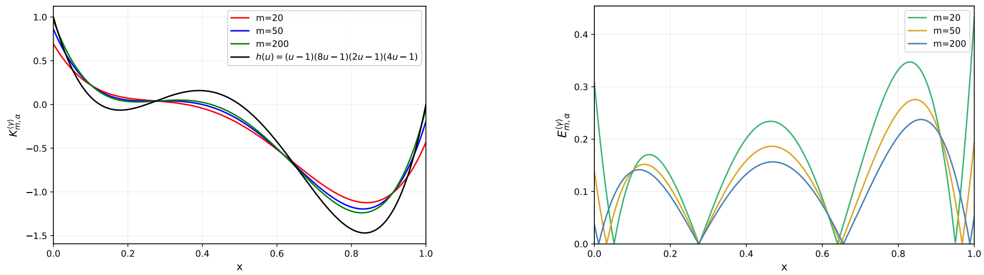
Figure 2.7: Kantorovich Operators $K_{m,\alpha}^{\langle\gamma\rangle}(\sin 2\pi u + \cos 2\pi u; x)$, for $\alpha = 0.8$ and $\gamma = 0.1$.



(a) Approximation Process for $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ (b) Error Function, $\tilde{E}_{m,\alpha}^{\langle\gamma\rangle}(u)$

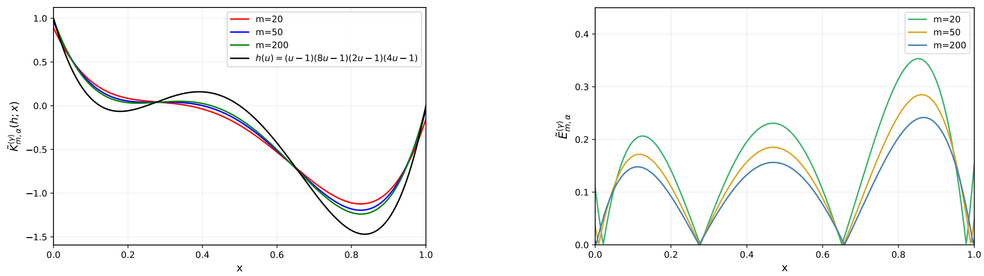
Figure 2.8: Genuine-type Modification $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$, corresponding to Figure 2.7.

Example 2.2.10 *For* $h(u) = (u-1)(8u-1)(2u-1)(4u-1)$ *let* $\alpha = 0.95$ *and* $\gamma = 0.05$, *with the error functions defined as before. We then examine graphs corresponding to different values of m, as shown in Figures 2.9 and 2.10.*



(a) Approximation Process for $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ (b) Error Function, $E_{m,\alpha}^{\langle\gamma\rangle}(u)$

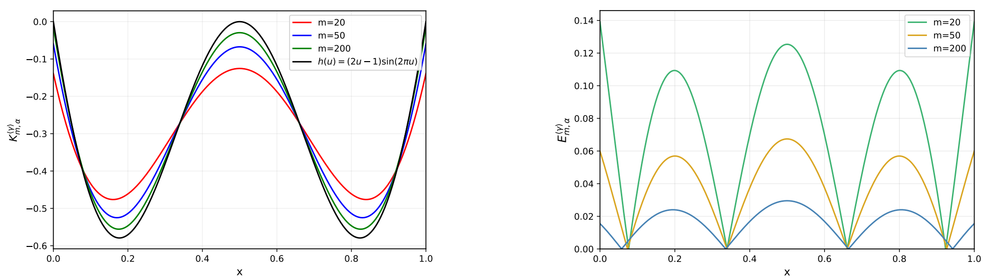
Figure 2.9: Kantorovich Operators $K_{m,\alpha}^{\langle\gamma\rangle}((u-1)(8u-1)(2u-1)(4u-1);x)$, for $\alpha = 0.95$ and $\gamma = 0.05$.



(a) Approximation Process for $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ (b) Error Function, $\tilde{E}_{m,\alpha}^{\langle\gamma\rangle}(u)$

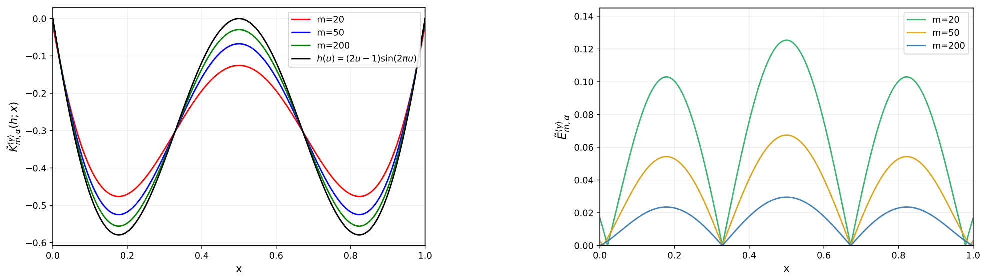
Figure 2.10: Genuine-type Modification $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$, corresponding to Figure 2.9.

Example 2.2.11 *Let $\alpha = 0.9$ and $\gamma = 0.005$. Figures 2.11 and 2.12 represent the approximation and error of the function $h(u) = (2u-1)\sin 2\pi u$ for the operators $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ and its genuine-type modification $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$, respectively.*



(a) Approximation Process for $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ (b) Error Function, $E_{m,\alpha}^{\langle\gamma\rangle}(u)$

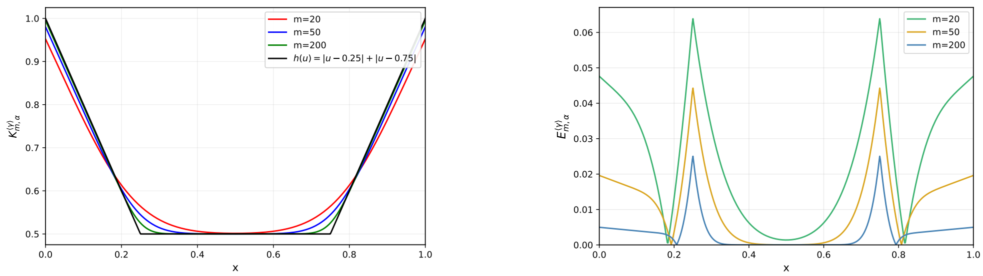
Figure 2.11: Kantorovich Operators $K_{m,\alpha}^{\langle\gamma\rangle}((2u-1)\sin 2\pi u;x)$, for $\alpha = 0.9$ and $\gamma = 0.005$.



(a) Approximation Process for $\tilde{K}^{\langle\gamma\rangle}_{m,\alpha}(h;u)$ (b) Error Function, $\tilde{E}^{\langle\gamma\rangle}_{m,\alpha}(u)$

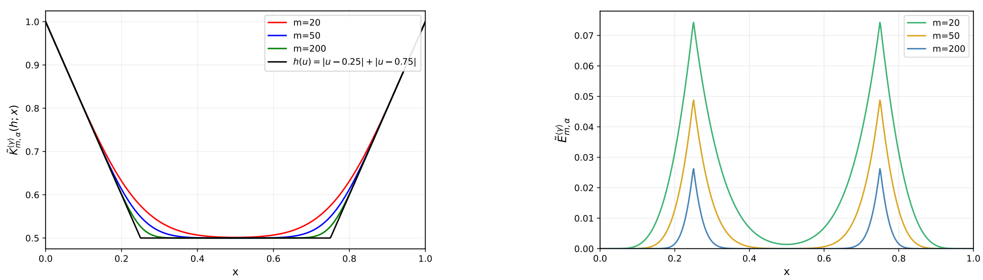
Figure 2.12: Genuine-type Modification $\tilde{K}^{\langle\gamma\rangle}_{m,\alpha}(h;x)$, corresponding to Figure 2.11.

Example 2.2.12 *Let* $\alpha = 0.85$ *and* $\gamma = 0.001$. *We approximate the non-differentiable function* $h(u) = |u-0.25| + |u-0.75|$ *on* $u \in [0,1]$, *represented graphically by Figures 2.13 and 2.14.*



(a) Approximation Process (b) Error Function

Figure 2.13: Kantorovich Operators $K^{\langle\gamma\rangle}_{m,\alpha}(|u-0.25| + |u-0.75|;x)$, for $\alpha = 0.85$ and $\gamma = 0.001$.



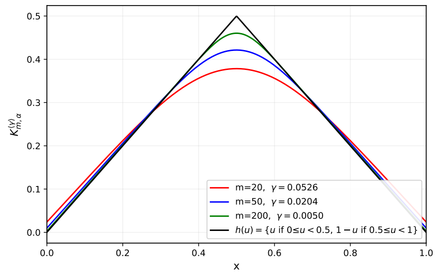
(a) Approximation Process (b) Error Function

Figure 2.14: Genuine-type Modification $\tilde{K}^{\langle\gamma\rangle}_{m,\alpha}(h;x)$, corresponding to Figure 2.13.

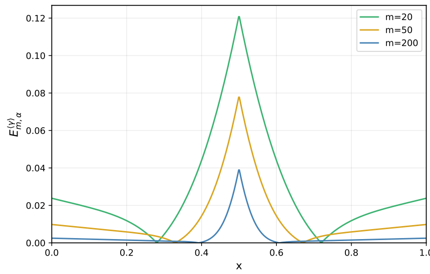
Example 2.2.13 *Consider the piece-wise function*

$$h(u) = \begin{cases} u, & 0 \leq u < 0.5 \\ 1-u, & 0.5 \leq u < 1, \end{cases}$$

for $\alpha = 0.75$ *and* $\gamma = 1/(m-1)$*. Figures 2.15 and 2.16 represent the approximation and error of the function* $h(u)$ *for the operators* $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ *and its genuine-type modification* $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$*, respectively.*

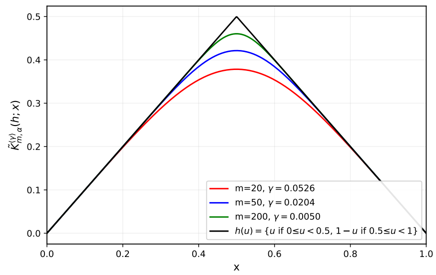


(a) Approximation Process

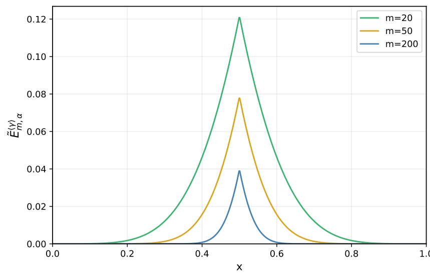


(b) Error Function

Figure 2.15: Kantorovich Operators $K_{m,\alpha}^{\langle\gamma\rangle}(h;x)$, for $\alpha = 0.75$ and $\gamma = 1/(m-1)$.



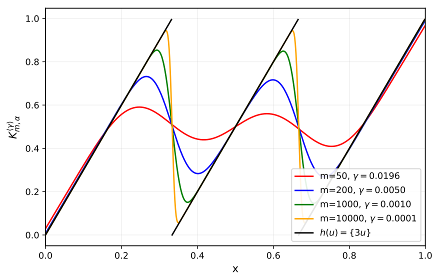
(a) Approximation Process



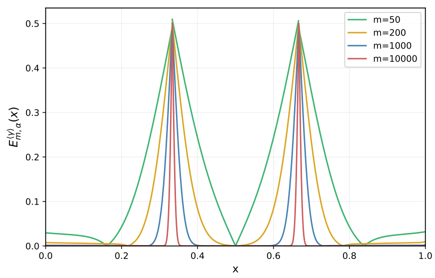
(b) Error Function

Figure 2.16: Genuine-type Modification $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$, corresponding to Figure 2.15.

Example 2.2.14 *Now, we take the non-continuous but integrable function* $h(u) = \{3u\}$ *for* $u \in [0,1]$*, where* $\{.\}$ *denotes the fractional-part function. Let* $\alpha = 0.95$ *and* $\gamma = 1/(m+1)$*. Then, Figures 2.17 and 2.18 represent the approximation and error of the function* $h(u)$ *for the operators* $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ *and its genuine-type modification* $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$*, respectively.*

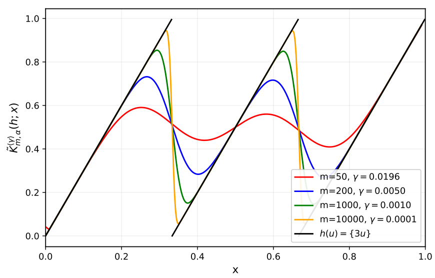


(a) Approximation Process

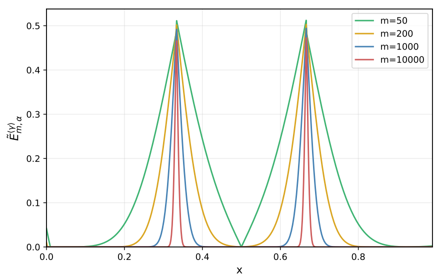


(b) Error Function

Figure 2.17: Kantorovich Operators $K_{m,\alpha}^{\langle\gamma\rangle}(\{3u\};x)$, for $\alpha = 0.95$ and $\gamma = 1/(m+1)$.



(a) Approximation Process



(b) Error Function

Figure 2.18: Genuine-type Modification $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$, corresponding to Figure 2.17.

Throughout this section, we have worked under the assumption that the parameter γ is of order $1/m$. This choice is not arbitrary, rather, it plays a crucial role in determining the efficiency of approximation. The influence of γ can be clearly observed in the graphical illustrations, where smaller values of γ consistently lead to improved approximation results. In other words, as γ decreases, the operators are able to capture the behaviour of the required function more accurately.

Moreover, Lemma 2.2.7 highlights a significant distinction between the proposed Kantorovich operators (2.11) and their genuine-type modification (2.13). For the operators (2.11), the first central moment only tends to zero as m tends to ∞, which means that the linear behaviour of the function is reproduced only approximately for finite m. In contrast, the genuine-type modification forces the first central moment to vanish completely for all values of m, i.e., it is exactly equal to zero. This structural improvement ensures that the modified operators preserve the linear function at every stage of approximation.

As a direct consequence, the genuine-type modification provides better approximation results near the boundary points $u = 0$ and $u = 1$, where the classical form typically shows weaker performance. This theoretical advantage is supported by the numerical and graphical results included in this section, which consistently demonstrate that the genuine-type operators achieve a visibly closer fit to the target function compared to the unmodified ones.

Chapter 3

Bernstein type Semi-Exponential Operators

The concept of semi-exponential operators is an extension of the exponential operators in approximation theory. The analysis begins by considering the partial differential equation,

$$\frac{\partial}{\partial x} W_{\beta}(r,x,t) = \frac{r(t-x)}{p(x)} W_{\beta}(r,x,t) - \beta W_{\beta}(r,x,t)$$

with domain of definition Ω*, for the kernel of sequence of operators*

$$S_r(f;x) = \sum_{k\in\Omega} W_{\beta}\left(r,x,\frac{k}{r}\right) f\left(\frac{k}{r}\right)$$

while satisfying the normalization condition $\sum_{k\in\Omega} W_{\beta}\left(r,x,\frac{k}{r}\right) = 1$*. Carrying forward this notion, we examine these operators specifically for the function* $p(x) = x(1-x)$ *and focus our attention on the semi-exponential Bernstein operators. The primary focus of this chapter is to explore the properties and characteristics of these Bernstein type semi-exponential operators. This chapter delves into deriving a recurrence relation for the central moments of these operators, while also establishing a moment generating function for the same. Finally, we analyze the approximation of these operators using the Voronovskaya-type asymptotic result. The objective of this analysis is to characterize functions by the degree of approximation.*

3.1 Class of Exponential Operators

The development of sequences of positive linear operators in approximation has always been closely tied to the search for faster convergence rates and applicability within a variety of functions and spaces. With the rapid advancements in approximation, researchers began to explore new classes of operators that went beyond classical polynomial-based constructions and extended naturally to integral forms with analytically defined kernels. A significant step in this direction was taken by C. P. May in 1976 [145], with the introduced of a family of operators of the form

$$S_r(f(u);t) = \int_{\Omega} W(r,t,u)f(u)du, \tag{3.1}$$

where Ω is the domain of definition and W is a positive function, known as the kernel of S_r. May termed these operators as exponential operators motivated by the fact that these operators are governed by a kernel function $W(r,t,u)$ which satisfies certain exponential-type conditions. The first condition requires the kernel to satisfy a homogeneous partial differential equation of the form

$$\frac{\partial}{\partial t}W(r,t,u) = \frac{r}{p(t)}W(r,t,u)(u-t), \tag{3.2}$$

such that $p(t)$ is both analytic and positive over the domain Ω. The second condition imposed on the kernel $W(r,t,u)$ is the normalization condition, which ensures that the operator preserves constant functions, that is,

$$S_r(1,t) = \int_{-\infty}^{\infty} W(r,t,u)\,du = 1.$$

Assigning particular values to the function $p(t)$ in equation (3.2), various operators can be obtained, some of which are listed in Table 3.1.

Exponential Operators	Choice of polynomial $p(t)$
Bernstein operators	$t(1-t)$
Szász-Mirakjan operators	t
Baskakov operators	$t(1+t)$
Post-Widder operators	t^2
Gauss-Weierstrass operators	1

Table 3.1: Classical operators obtained for different choices of the polynomial $p(t)$.

One important point to observe here is that, unlike the Post-Widder and Gauss-Weierstrass operators which can be obtained directly from the continuous formulation (3.1), the Bernstein, Szász-Mirakyan and Baskakov operators belong to the class of discrete operators. In these cases, even though we can formally substitute their respective choices of $p(t)$ into the differential equation (3.2), the resulting kernel function W cannot be used in equation (3.1), since the framework there is based on an integral representation with a continuous kernel. Discrete operators, on the other hand, are naturally defined in terms of series (summation kernels) rather than integrals. Therefore, to handle these cases appropriately, we work with the corresponding discrete exponential operators, which are defined as follows:

$$S_r(f;t) = \sum_{k=0}^{\infty} W\left(r,t,\frac{k}{r}\right) f\left(\frac{k}{r}\right), \tag{3.3}$$

where the kernel satisfies the homogeneous partial differential

$$\frac{\partial}{\partial t} W\left(r,t,\frac{k}{r}\right) = \frac{r\left(\frac{k}{r}-t\right)}{p(t)} W\left(r,t,\frac{k}{r}\right),$$

and the normalization condition

$$S_r(1;t) = \sum_{k=0}^{\infty} W\left(r,t,\frac{k}{r}\right) = 1.$$

The study initiated by May was soon extended by Ismail and May in 1978 [116], and they demonstrated that for a polynomial p of any degree, the corresponding exponential operators S_r can be uniquely determined while still satisfying both the differential condition and the normalization condition. Consequently, they recovered several classical operators corresponding to constant, linear and quadratic choices of p, and further introduced new operators for higher degree of p.

One such example arises when considering the cubic polynomial $p(t) = t(1+t)^2$. With this choice of p, Ismail and May employed the bilateral Laplace transform to introduce a new sequence of exponential operators, defined as:

$$R_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}}{k!} \left(xe^{-x}\right)^k f\left(\frac{k}{n+k}\right), \;\; n \in \mathbb{N} \;\; x \in (0,1).$$

The approximation properties of these operators, along with their Kantorovich variants, have been analyzed in [149], while their complete asymptotic expansions were later derived in [2]. Further studies introduced additional exponential operators corresponding to other choices of p, such as $p(t) = 2t^{3/2}$, which have been investigated by Abel and Gupta in [1]. Beyond specific examples, the global approximation behaviour of exponential operators has been a subject of wide interest. In 1981, Sato [187] examined their properties in weighted function spaces, while in 1988, Totik [207] provided a comprehensive study of their global uniform approximation properties. The theory of exponential operators and related results are discussed extensively in [104], together with further references for interested readers.

3.2 Class of Semi-Exponential Operators

For quite a few years following Ismail and May, no new work had been done on exponential operators up until the year 2005. It was during this time that A. Tyliba and E. Wachnicki [208] introduced the concept of semi-exponential operators. They expanded upon the findings of Ismail and May by considering a sequence of operators, denoted as V_r^{β}, such that $V_r^{\beta}(t,x) \neq x$. In equation (3.2), they introduced a non-negative real parameter β and worked on a new homogeneous partial different equation, thus obtaining a new family of operators, known as the semi-exponential operators, defined in the following way:

$$V_r^{\beta}(f(t);x) = \int_{\Omega} W_{\beta}(r,x,t)\, f(t)dt, \tag{3.4}$$

where Ω is the domain of definition. Here, we denote the kernel by $W_{\beta}(r,x,t)$, which satisfies:

1. The homogeneous partial differential equation

$$\frac{\partial}{\partial x} W_{\beta}(r,x,t) = \frac{r}{p(x)}(t-x)\, W_{\beta}(r,x,t) - \beta W_{\beta}(r,x,t).$$

2. The normalization condition

$$V_r^{\beta}(1,x) = \int_{\Omega} W_{\beta}(r,x,t)\, dt = 1.$$

For the case $\beta = 0$, the operators V_r^{β} can simply be regarded as the operators of the exponential type defined earlier in equation (3.1). These operators form the foundation of the subsequent work in this chapter. Recently, Herzog [110] conducted a study on the approximation properties of exponential-type operators when acting on functions from exponential weighted spaces. A range of several new semi-exponential operators were captured by Abel et al. recently [3], each corresponding to a different function $p(x)$. Further, in the past few years, numerous investigations have been carried out concerning the modification and preservation of exponential-type operators [20; 60; 136; 167].

The present chapter is devoted to the study of semi-exponential Bernstein operators with $p(x) = x(1-x)$ and the function f is discretely varied over the interval $[0,\infty)$. Due to the discrete nature of the distribution, the aforementioned definitions and normalization condition will vary when summed over the discrete range of k from 0 to ∞. Thus, we consider here the discrete form instead of the integral representation given in equation (3.4) and defined as

$$L_r(f;x) = \sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right) f\left(\frac{k}{r}\right), \tag{3.5}$$

satisfying the homogeneous partial differential equation

$$\frac{\partial}{\partial x} W_\beta\left(r,x,\frac{k}{r}\right) = \left(\frac{r}{p(x)}\left(\frac{k}{r}-x\right)-\beta\right) W_\beta\left(r,x,\frac{k}{r}\right), \tag{3.6}$$

and the normalization condition

$$\sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right) = 1. \tag{3.7}$$

First, we solve the homogeneous partial differential equation (3.6). Let

$$W_\beta\left(r,x,\frac{k}{r}\right) = A(r,k,\beta)\,y, \tag{3.8}$$

such that $A(r,k,\beta)$ is independent of x and y is a function of x. Substituting (3.8) in equation (3.6) with $p(x) = x(1-x)$, we get

$$y' = \left(\frac{r}{x(1-x)}\left(\frac{k}{r}-x\right)-\beta\right) y$$

which implies,

$$\begin{aligned}\frac{y'}{y} &= \frac{k}{x(1-x)} - \frac{r}{(1-x)} - \beta\\ &= k\left[\frac{1}{x}+\frac{1}{1-x}\right] - \frac{r}{(1-x)} - \beta.\end{aligned}$$

Integrating the above differential equation, we arrive at

$$\begin{aligned}\ln y = k\ln x - k\ln(1-x) + n\ln(1-x) - \beta x\\ = \ln e^{-\beta x} x^k (1-x)^{n-k},\end{aligned}$$

which gives the value of $y(x) = e^{-\beta x}x^k(1-x)^{n-k}$. Thus,

$$W_\beta\left(r,x,\frac{k}{r}\right) = A(r,k,\beta)\,e^{-\beta x}x^k(1-x)^{n-k}.$$

To find the value of $A(r,k,\beta)$, we use the normalization condition (3.7), which gives

$$\sum_{k=0}^{\infty} A(r,k,\beta)\,e^{-\beta x}x^k(1-x)^{n-k} = 1$$

which implies,
$$\sum_{k=0}^{\infty} A(r,k,\beta)\left(\frac{x}{1-x}\right)^k = e^{\beta x}(1-x)^{-r}.$$

Taking $z = \frac{x}{1-x}$, we get $x = \frac{z}{1+z}$, and thus

$$\begin{aligned}
\sum_{k=0}^{\infty} A(r,k,\beta)\,z^k &= e^{\beta\frac{z}{1+z}}(1+z)^r \\
&= \sum_{i=0}^{\infty}\frac{(\beta z)^i}{i!}(1+z)^{r-i} \\
&= \sum_{i=0}^{\infty}\frac{\beta^i z^i}{i!}\sum_{j=0}^{\infty}\binom{r-i}{j}z^j \\
&= \sum_{i=0}^{\infty}\frac{\beta^i z^i}{i!}\left[\binom{r-i}{0}+\binom{r-i}{1}z+\binom{r-i}{2}z^2+\dots\right] \\
&= \sum_{i=0}^{\infty}\binom{r-i}{0}\frac{\beta^i z^i}{i!}+\sum_{i=0}^{\infty}\binom{r-i}{1}\frac{\beta^i z^{i+1}}{i!}+\sum_{i=0}^{\infty}\binom{r-i}{2}\frac{\beta^i z^{i+2}}{i!}+\dots \\
&= \sum_{k=0}^{\infty} z^k \sum_{i+j=k}\binom{r-i}{j}\frac{\beta^i}{i!}.
\end{aligned}$$

Thus,

$$\begin{aligned}
A(r,k,\beta) &= \sum_{i+j=k}\binom{r-i}{j}\frac{\beta^i}{i!} \\
&= \sum_{i=0}^{\infty}\binom{r-i}{k-1}\frac{\beta^i z^i}{i!}.
\end{aligned}$$

Hence, we get the kernel of the semi-exponential Bernstein operators, $W_\beta(r,k,x)$ as,

$$W_\beta(r,k,x) = e^{-\beta x}x^k(1-x)^{r-k}\sum_{l=0}^{\infty}\binom{r-l}{k-l}\frac{\beta^l}{l!}. \tag{3.9}$$

Solving the partial differential equation (3.2) corresponding to exponential operators, we get the kernel of the classical Bernstein operators, known as the Bernstein polynomials, defined as $b_{r,k}(x) = {}^r c_k x^k(1-x)^{r-k}$. We see that the relation between the kernel of the semi-exponential Bernstein operators (3.9) and the kernel of the exponential Bernstein operators is given by

$$W_0(r,k,x) = b_{r,k}(x). \tag{3.10}$$

In other words, we can say that for $\beta = 0$, $W_\beta(r,k,x)$ reduces to the Bernstein polynomial basis, $b_{r,k}(x)$.

Employing $W_\beta(r,k,x)$ to the definition of semi-exponential operators (3.5), we get the Bernstein-type semi-exponential operators, given by

$$B_r^\beta(f;x) = e^{-\beta x} \sum_{k=0}^{\infty} x^k (1-x)^{r-k} \sum_{l=0}^{\infty} \binom{r-l}{k-l} \frac{\beta^l}{l!} f\left(\frac{k}{r}\right), \tag{3.11}$$

where $\binom{r}{k}$ is the binomial coefficient. The operators defined in (3.11) can equivalently be expressed in the following form, which is particularly convenient for computing their moments and central moments.

$$B_r^\beta(f;x) = e^{-\beta x} \sum_{l=0}^{\infty} \frac{\beta^l}{l!} x^l (1-x)^{r-l} \sum_{j=0}^{\infty} \binom{r-l}{j} \left(\frac{x}{1-x}\right)^j f\left(\frac{l+j}{r}\right). \tag{3.12}$$

In this chapter we will focus on a whole new class of kernels $W_\beta(r,k,x)$ depending upon r and k and the parameter β, unlike the classical Bernstein polynomial basis $b_{r,k}(x)$, which solely depends upon r and k. To further analyze these functions and determine the effect of the parameters on $W_\beta(r,k,x)$, we graph them for different values of r, k and β.

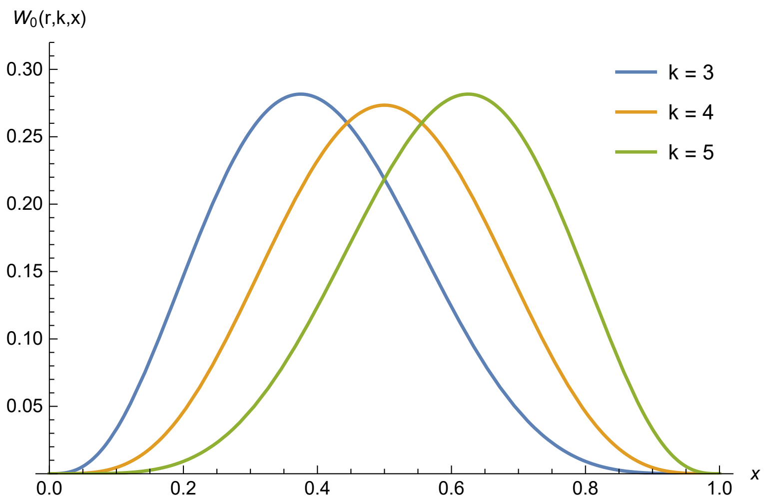
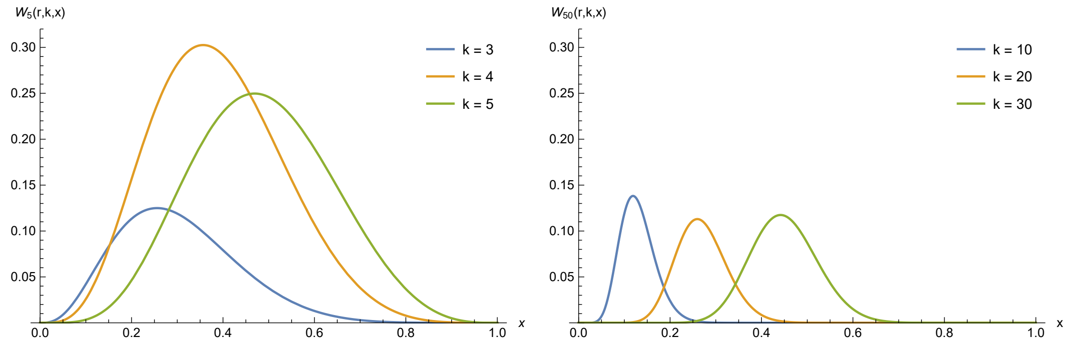


Figure 3.1: Graphs of the kernel of semi-exponential Bernstein operators with $\beta = 0$, given by equation (3.9), for $r = 8$ and $k = 3,4,5$.



(a) Graphs of the kernel of semi-exponential Bernstein operators with $\beta = 5$, for $r = 8$ and $k = 3, 4, 5$.

(b) Graphs of the kernel of semi-exponential Bernstein operators with $\beta = 50$, for $r = 40$ and $k = 10, 20, 30$.

Figure 3.2: Graphs of the kernel of semi-exponential Bernstein operators with $\beta > 0$, given by equation (3.9), for different values of r and k.

Figure 3.1 validates expression (3.10) as it is well known that the Bernstein polynomials are symmetric in the interval $[0,1]$, that is, $b_{r,k}(1-x) = b_{r,r-k}(x)$. However, from Figure 3.2, we can see that the kernel of the semi-exponential operators for $\beta > 0$ does not satisfy this property. Thus, we can conclude that unlike the kernels of classical Bernstein operators and its other modifications, such as the λ-Bernstein operators [42] and α-Bernstein [51], the kernel of semi-exponential operators defined by (3.4) for $\beta > 0$, are not symmetric over $[0, \infty]$. Though, one may observe that the Bernstein operators themselves are included in the class of exponential operators, however, their modified versions no longer satisfy the defining conditions of this class.

3.3 Auxiliary Results

In the following section, we present the properties of the operators mentioned in (3.12), which are utilized in the subsequent sections of this chapter to establish the main theorems.

Lemma 3.3.1 *Let $B_r^{\beta}(.;x)$ be the Bernstein type semi-exponential operators. Then for $x \in \left[0, \frac{1}{2}\right)$ the following equalities hold:*

(i) $B_r^{\beta}(1;x) = 1$

(ii) $B_r^{\beta}(t;x) = x + \beta \frac{x(1-x)}{r}$

(iii) $B_r^{\beta}(t^2;x) = x^2 + \frac{x(1-x)+2\beta x^2(1-x)}{r} + \frac{\beta x(1-3x+2x^2)+\beta^2 x^2(1-x)^2}{r^2}$

Proof. Applying the negative binomial expansion in (3.12) we get,

$$\begin{aligned}
B_r^{\beta}(1;x) &= e^{-\beta x}\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\sum_{j=0}^{\infty}\binom{r-l}{j}\left(\frac{x}{1-x}\right)^j \\
&= e^{-\beta x}\sum_{l=0}^{\infty}\frac{\beta^l x^l}{l!}(1-x)^{r-l}\left(1+\frac{x}{1-x}\right)^{r-l} \\
&= e^{-\beta x}\sum_{l=0}^{\infty}\frac{\beta^l x^l}{l!} \\
&= 1,
\end{aligned}$$

and,

$$\begin{aligned}
B_r^{\beta}(t;x) &= e^{-\beta x}\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\sum_{j=0}^{\infty}\binom{r-l}{j}\left(\frac{x}{1-x}\right)^j\left(\frac{l+j}{r}\right) \\
&= e^{-\beta x}\left[\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\sum_{j=0}^{\infty}\binom{r-l}{j}\left(\frac{x}{1-x}\right)^j\frac{l}{r}\right. \\
&\quad \left.+\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\sum_{j=0}^{\infty}\binom{r-l}{j}\left(\frac{x}{1-x}\right)^j\frac{j}{r}\right] \\
&= e^{-\beta x}\left[\sum_{l=0}^{\infty}\frac{\beta^l x^l}{l!}\frac{l}{r}+\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\sum_{j=0}^{\infty}\frac{(r-l)!}{j!(r-l-j)!}\left(\frac{x}{1-x}\right)^j\frac{j}{r}\right] \\
&= e^{-\beta x}\left[\sum_{l=1}^{\infty}\frac{\beta^l x^l}{(l-1)!}\frac{1}{r}+\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\sum_{j=1}^{\infty}\frac{(r-l)!}{(j-1)!(r-l-j)!}\left(\frac{x}{1-x}\right)^j\frac{1}{r}\right].
\end{aligned}$$

Replacing l with $m+1$ in the first summation and j with $n+1$ in the second summation, we get

$$\begin{aligned}
B_r^{\beta}(t;x) &= e^{-\beta x}\left[\sum_{m=0}^{\infty}\frac{\beta^{m+1}x^{m+1}}{m!}\frac{1}{r}+\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\sum_{n=0}^{\infty}\frac{r-l}{r}\binom{r-l-1}{n}\left(\frac{x}{1-x}\right)^{n+1}\right] \\
&= e^{-\beta x}\left[\frac{\beta x}{r}e^{\beta x}+\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\left(\frac{r-l}{r}\right)\left(\frac{x}{1-x}\right)\left(\frac{1}{1-x}\right)^{r-l-1}\right] \\
&= \frac{\beta x}{r}+e^{-\beta x}\sum_{l=0}^{\infty}\frac{\beta^l x^{l+1}}{l!}\left(1-\frac{l}{r}\right) \\
&= \frac{\beta x}{r}+e^{-\beta x}\sum_{l=0}^{\infty}\frac{\beta^l x^{l+1}}{l!}-e^{-\beta x}\sum_{l=0}^{\infty}\frac{\beta^l x^{l+1}}{l!}\frac{l}{r} \\
&= \frac{\beta x}{r}+x-xe^{-\beta x}\sum_{l=1}^{\infty}\frac{\beta^l x^l}{r(l-1)!}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\beta x}{r} + x - \frac{\beta x^2 e^{-\beta x}}{r}\sum_{l=1}^{\infty}\frac{\beta^{l-1}x^{l-1}}{(l-1)!}\\
&= \frac{\beta x}{r} + x - \frac{\beta x^2}{r}\\
&= x + \frac{\beta}{r}x(1-x).
\end{aligned}$$

Similarly,

$$\begin{aligned}
B_r^{\beta}\left(t^2;x\right) &= e^{-\beta x}\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\sum_{j=0}^{\infty}\binom{r-l}{j}\left(\frac{x}{1-x}\right)^j\left(\frac{l^2+2lj+j^2}{r^2}\right)\\
&= e^{-\beta x}\left[\sum_{l=0}^{\infty}\frac{\beta^l x^l}{l!}\frac{l^2}{r^2} + 2\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\frac{l}{r}\sum_{j=0}^{\infty}\binom{r-l}{j}\frac{j}{r}\left(\frac{x}{1-x}\right)^j\right.\\
&\left.+\sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\sum_{j=0}^{\infty}\binom{r-l}{j}\left(\frac{x}{1-x}\right)^j\left(\frac{j}{r}\right)^2\right]\\
&= e^{-\beta x}\left[\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3\right].
\end{aligned}$$

Solving the above summations, we will arrive at,

$$\begin{aligned}
\sum\nolimits_1 &= \sum_{l=0}^{\infty}\frac{\beta^l x^l}{l!}\left(\frac{l(l-1)}{r^2}+\frac{l}{r^2}\right)\\
&= \sum_{l=0}^{\infty}\frac{\beta^l x^l}{l!}\frac{l(l-1)}{r^2} + \sum_{l=0}^{\infty}\frac{\beta^l x^l}{l!}\frac{l}{r^2}\\
&= \frac{\beta^2x^2}{r^2}\sum_{l=2}^{\infty}\frac{\beta^{l-2}x^{l-2}}{(l-2)!} + \frac{\beta x}{r^2}\sum_{l=1}^{\infty}\frac{\beta^{l-1}x^{l-1}}{(l-1)!}\\
&= e^{\beta x}\frac{\beta x}{r^2}\left(1+\beta x\right),\\
\sum\nolimits_2 &= \sum_{l=0}^{\infty}\frac{\beta^l}{l!}x^l(1-x)^{r-l}\sum_{j=0}^{\infty}\binom{r-l}{j}\frac{j}{r}\left(\frac{x}{1-x}\right)^j\frac{2l}{r}\\
&= \sum_{l=0}^{\infty}\frac{\beta^l x^{l+1}}{l!}\left(\frac{r-l}{r}\right)\frac{2l}{r}\\
&= \sum_{l=0}^{\infty}\frac{\beta^l x^{l+1}}{l!}\frac{2l}{r} - \sum_{l=0}^{\infty}\frac{\beta^l x^{l+1}}{l!}\frac{2l^2}{r^2}\\
&= e^{\beta x}\left(\frac{2\beta x^2}{r} - 2x\left(\frac{\beta^2x^2}{r^2}+\frac{\beta x}{r^2}\right)\right)\\
&= e^{\beta x}\frac{2\beta x^2}{r}\left(1-\frac{1}{r}-\frac{\beta x}{r}\right),
\end{aligned}$$

and

$$\begin{aligned}
\sum\nolimits_3 &= \sum_{l=0}^{\infty} \frac{\beta^l}{l!} x^l (1-x)^{r-l} \sum_{j=0}^{\infty} \binom{r-l}{j} \left(\frac{x}{1-x}\right)^j \left(\frac{j(j-1)}{r^2} + \frac{j}{r^2}\right) \\
&= \sum_{l=0}^{\infty} \frac{\beta^l}{l!} x^l (1-x)^{r-l} \sum_{j=0}^{\infty} \binom{r-l}{j} \left(\frac{x}{1-x}\right)^j \frac{j(j-1)}{r^2} + \sum_{l=0}^{\infty} \frac{\beta^l}{l!} x^l (1-x)^{r-l} \sum_{j=0}^{\infty} \binom{r-l}{j} \left(\frac{x}{1-x}\right)^j \frac{j}{r^2} \\
&= \sum_{l=0}^{\infty} \frac{\beta^l}{l!} x^l (1-x)^{r-l} \frac{(r-l)(r-l-1)}{r^2} \sum_{j=0}^{\infty} \binom{r-l-2}{j} \left(\frac{x}{1-x}\right)^{j+2} + \frac{x}{r} - \frac{\beta x^2}{r^2} \\
&= \sum_{l=0}^{\infty} \frac{\beta^l x^{l+2}}{l!} \left(\frac{r-1}{r} - l\left(\frac{2r-1}{r^2}\right) + \frac{l^2}{r^2}\right) + e^{\beta x}\left(\frac{x}{r} - \frac{\beta x^2}{r^2}\right) \\
&= \sum_{l=0}^{\infty} \frac{\beta^l x^{l+2}}{l!} \left(\frac{r-1}{r} - l\left(\frac{2r-1}{r^2}\right) + \frac{l^2}{r^2}\right) + e^{\beta x}\left(\frac{x}{r} - \frac{\beta x^2}{r^2}\right) \\
&= \sum_{l=0}^{\infty} \frac{\beta^l x^{l+2}}{l!} \frac{r-1}{r} - \sum_{l=0}^{\infty} \frac{\beta^l x^{l+2}}{l!} \frac{l}{r} \left(\frac{2r-1}{r}\right) \left(- + \frac{l^2}{r^2}\right) + \sum_{l=0}^{\infty} \frac{\beta^l x^{l+2}}{l!} \frac{l^2}{r^2} + e^{\beta x}\left(\frac{x}{r} - \frac{\beta x^2}{r^2}\right) \\
&= e^{\beta x}\left(\frac{(r-1)x^2}{r} - \frac{\beta x^3(2r-1)}{r^2} - \frac{\beta^2 x^4}{r^2} + \frac{\beta x^3}{r^2} + \frac{x}{r} - \frac{\beta x^2}{r^2}\right) \\
&= e^{\beta x}\left(\frac{\beta^2 x^4}{r^2} - \frac{\beta x^2}{r^2} - 2\left(\frac{r-1}{r^2}\right)\beta x^3 + \left(\frac{r-1}{r}\right)x^2 + \frac{x}{r}\right).
\end{aligned}$$

Substituting these values in $B_r^{\beta}\left(t^2;x\right)$ we get the desired moment. □

Now, let us denote the ς^{th} central moment of $B_r^{\beta}\left(f;x\right)$ as,

$$\mu_{r,\varsigma}(x) = e^{-\beta x} \sum_{k=0}^{\infty} A\left(r,k,\beta\right) x^k (1-x)^{r-k} \left(\frac{k}{r} - x\right)^{\varsigma}, \tag{3.13}$$

where $A\left(r,k,\beta\right) = \sum\limits_{l=0}^{\infty} \binom{r-l}{k-l} \frac{\beta^l}{l!}$.

Theorem 3.3.2 *Based on equation* (3.13)*, the following recurrence relation can be derived for the central moments of semi-exponential Bernstein operators:*

$$\begin{aligned}
\mu_{r,1}(x) &= \frac{\beta}{r} x(1-x) \\
r\mu_{r,\varsigma+1}(x) &= x(1-x)\left[{\mu'}_{r,\varsigma}(x) + \beta\mu_{r,\varsigma}(x) + \varsigma\mu_{r,\varsigma-1}(x)\right]; \quad \textit{for } \varsigma \geq 1.
\end{aligned} \tag{3.14}$$

Proof. Let,

$$b_{r,k}^{\beta}(x) = A\left(r,k,\beta\right) x^k (1-x)^{r-k}.$$

This implies,

$$\begin{aligned}
{b_{r,k}^{\beta}}'(x) &= A\left(r,k,\beta\right) x^{k-1} (1-x)^{r-k-1} (k - rx). \\
&= rA\left(r,k,\beta\right) x^{k-1} (1-x)^{r-k-1} \left(\frac{k}{r} - x\right) \\
&= \frac{r}{x(1-x)} A\left(r,k,\beta\right) x^k (1-x)^{r-k} \left(\frac{k}{r} - x\right).
\end{aligned}$$

Thus,

$$e^{-\beta x}\sum_{k=0}^{\infty} {b_{r,k}^{\beta}}'(x)\left(\frac{k}{r}-x\right)^{\varsigma} = e^{-\beta x}\sum_{k=0}^{\infty}\frac{r}{x(1-x)}A(r,k,\beta)\,x^k(1-x)^{r-k}\left(\frac{k}{r}-x\right)^{\varsigma+1}$$
$$= \frac{r}{x(1-x)}\mu_{r,\varsigma+1}(x).$$

Differentiating the ς^{th} central moment of $B_r^{\beta}(f;x)$ for $\varsigma \geq 1$, we get

$$\mu_{r,\varsigma}'(x) = -\beta e^{-\beta x}\sum_{k=0}^{\infty} b_{r,k}^{\beta}(x)\left(\frac{k}{r}-x\right)^{\varsigma} + e^{-\beta x}\sum_{k=0}^{\infty} {b_{r,k}^{\beta}}'(x)\left(\frac{k}{r}-x\right)^{\varsigma} - \varsigma e^{-\beta x}\sum_{k=0}^{\infty} b_{r,k}^{\beta}(x)\left(\frac{k}{r}-x\right)^{\varsigma-1}$$
$$= -\beta\mu_{r,\varsigma}(x) - \varsigma\mu_{r,\varsigma-1}(x) + \frac{r}{x(1-x)}\mu_{r,\varsigma+1}(x),$$

which implies,

$$r\mu_{r,\varsigma+1}(x) = x(1-x){\mu'}_{r,\varsigma}(x) + \beta x(1-x)\mu_{r,\varsigma}(x) + \varsigma x(1-x)\mu_{r,\varsigma-1}(x)$$
$$= x(1-x)\left[{\mu'}_{r,\varsigma}(x) + \beta\mu_{r,\varsigma}(x) + \varsigma\mu_{r,\varsigma-1}(x)\right].$$

For $\varsigma = 1$, $\mu_{r,\varsigma}(x)$ is simply the first central moment. □

Theorem 3.3.3 *$A_{\varsigma}(r,x) = r^{\varsigma}\mu_{r,\varsigma}(x)$ is a polynomial of degree $\left[\frac{\varsigma}{2}\right]$ in r, where $[.]$ is the greatest integer function. Also, the coefficient of r^{ς} in*

(i) $A_{2\varsigma}(r,x)$ is $1\cdot 3\cdot 5\ldots(2\varsigma-1)(x(1-x))^{\varsigma}$, and

(ii) $A_{2\varsigma+1}(r,x)$ is $c_1\beta(x(1-x))^{\varsigma+1} + c_2(x(1-x))^{\varsigma}(1-2x)$

where c_1, c_2 are constants and $c_1 = 1\cdot 3\cdot 5\ldots(2\varsigma+1)$.

Proof. We have $A_{\varsigma}(r,x) = r^{\varsigma}\mu_{r,\varsigma}(x)$. Therefore, from Lemma 3.3.2, we can write

$$A_0(r,x) = 1,$$
$$A_1(r,x) = \beta x(1-x),$$
$$\text{and } A_{\varsigma+1}(r,x) = x(1-x)\left(\frac{d}{dx}A_{\varsigma}(r,x) + \beta A_{\varsigma}(r,x) + r\varsigma A_{\varsigma-1}(r,x)\right), \quad \varsigma \geq (3.15)$$

By repeatedly substituting the value of $A_{\varsigma}(r,x)$ in equation (3.15), it can be demonstrated that $A_{2\varsigma}(r,x)$ and $A_{2\varsigma+1}(r,x)$ are polynomials in r of degree ς. It follows that, the highest powered term appearing in their expression is r^{ς} with its coefficient as:

$$1\cdot 3\cdot 5\ldots(2\varsigma-1)(x(1-x))^{\varsigma} \qquad \text{for } A_{2\varsigma}(r,x)$$
$$c_1\beta(x(1-x))^{\varsigma+1} + c_2(x(1-x))^{\varsigma}(1-2x) \qquad \text{for } A_{2\varsigma+1}(r,x)$$

where $c_1 = 1\cdot 3\cdot 5\ldots(2\varsigma+1)$ and c_2 is a fixed constant. □

Remark 3.3.4 *From Theorem 3.3.3 we have,*

$$A_{\varsigma}(r,x) = O\left(r^{[\varsigma/2]}\right).$$

$$\textit{It follows } \mu_{r,\varsigma}(x) = O\left(r^{-\left[\frac{\varsigma+1}{2}\right]}\right),$$

where $[.]$ *is the greatest integer function.*

3.3.1 Moment Generating Function

One of the fundamental aspects in the study of positive linear operators is the analysis of their central moments, as these play a crucial role in understanding the approximation order and in establishing various direct and inverse theorems. For the sequence of semi-exponential Bernstein operators introduced in (3.12), there are multiple ways to determine their central moments.

One way relies on the recurrence relation already established in Theorem 3.3.2. This recursive formula provides a systematic way to compute the successive central moments by expressing higher-order moments in terms of lower-order ones. Such relations are often computationally efficient and well-suited for theoretical analysis. However, a limitation of this method is that the computation of the $(\varsigma+1)^{\text{th}}$ central moment requires prior knowledge of both the ς^{th} and $(\varsigma-1)^{\text{th}}$ central moments.

An alternative and often more elegant method involves the use of the exponential moment generating function (MGF). The moment generating function serves as a compact analytic representation that encodes all the moments of the operators within a single functional expression. By differentiating the MGF appropriately, we can obtain the central moments in a straightforward manner.

Therefore, as the next step in our discussion, we focus on deriving an explicit expression for the central moment generating function corresponding to the semi-exponential Bernstein operators defined in (3.12). This not only provides an alternative perspective but also establishes a deeper connection between exponential-type operators and probabilistic tools such as generating functions.

Consider the recurrence relation (3.14),

$$r\mu_{r,\varsigma+1}(x) = x(1-x)\left[\mu'_{r,\varsigma}(x) + \beta\mu_{r,\varsigma}(x) + \varsigma\mu_{r,\varsigma-1}(x)\right].$$

It follows

$$\frac{r}{x(1-x)}B_r^{\beta}\left((t-x)^{\varsigma+1};x\right) = \frac{d}{dx}B_r^{\beta}\left((t-x)^{\varsigma};x\right)+\beta B_r^{\beta}\left((t-x)^{\varsigma};x\right) +\varsigma B_r^{\beta}\left((t-x)^{\varsigma-1};x\right). \tag{3.16}$$

Now, the exponential moment generating function of $B_r^{\beta}\left((t-x)^{\varsigma};x\right)$ is defined as,

$$\begin{aligned} G(x,z) &= B_r^{\beta}\left(e^{z(t-x)};x\right) \\ &= \sum_{\varsigma=0}^{\infty}\frac{z^{\varsigma}}{\varsigma!}B_r^{\beta}\left((t-x)^{\varsigma};x\right). \end{aligned}$$

Multiplying equation (3.16) by $\frac{z^{\varsigma}}{\varsigma!}$ and taking summation over $\varsigma = 0,1,\ldots\infty$, we get,

$$\begin{aligned} &\frac{r}{x(1-x)}\sum_{\varsigma=0}^{\infty}\frac{z^{\varsigma}}{\varsigma!}B_r^{\beta}\left((t-x)^{\varsigma+1};x\right) \\ &= \sum_{\varsigma=0}^{\infty}\frac{z^{\varsigma}}{\varsigma!}\frac{d}{dx}B_r^{\beta}\left((t-x)^{\varsigma};x\right)+\beta\sum_{\varsigma=0}^{\infty}\frac{z^{\varsigma}}{\varsigma!}B_r^{\beta}\left((t-x)^{\varsigma};x\right)+\sum_{\varsigma=0}^{\infty}\varsigma\frac{z^{\varsigma}}{\varsigma!}B_r^{\beta}\left((t-x)^{\varsigma-1};x\right) \end{aligned}$$

Replacing ς with $\varsigma-1$ in the left hand side of the above equation, we get

$$\begin{aligned} \sum_{\varsigma=0}^{\infty}\frac{z^{\varsigma}}{\varsigma!}B_r^{\beta}\left((t-x)^{\varsigma+1};x\right) &= \sum_{\varsigma=1}^{\infty}\frac{z^{\varsigma-1}}{(\varsigma-1)!}B_r^{\beta}\left((t-x)^{\varsigma};x\right) \\ &= \sum_{\varsigma=0}^{\infty}\frac{\partial}{\partial z}\left[\frac{z^{\varsigma}}{\varsigma!}B_r^{\beta}\left((t-x)^{\varsigma};x\right)\right] \\ &= \frac{\partial}{\partial z}G(x,z). \end{aligned}$$

Thus, we can write

$$\frac{r}{x(1-x)}\sum_{\varsigma=1}^{\infty}\frac{z^{\varsigma-1}}{(\varsigma-1)!}B_r^{\beta}\left((t-x)^{\varsigma};x\right) = \frac{\partial}{\partial x}G(x,z)+\beta G(x,z)+zG(x,z),$$

which implies,

$$\frac{r}{x(1-x)}\frac{\partial}{\partial z}G(x,z) = \frac{\partial}{\partial x}G(x,z)+(\beta+z)G(x,z).$$

Thus,

$$\frac{r}{x(1-x)}\frac{\partial G}{\partial z}-\frac{\partial G}{\partial x} = (\beta+z)G.$$

To solve the above PDE, first we write the auxiliary equation, that is,

$$\frac{x(1-x)}{r}dz = -dx = \frac{1}{(\beta+z)G}dG.$$

Taking the first two equalities, we get

$$z = -r\ln\left(\frac{x}{1-x}\right)+c_1, \tag{3.17}$$

and taking the last two equalities, we get

$$\ln G + c' = -\beta x - c_1 x + r \int_{\alpha}^{x} \ln\left(\frac{\theta}{1-\theta}\right) d\theta.$$

Now, assuming $x(1-x) = p(x)$, then

$$\int_{\alpha}^{\theta} \frac{1}{p(\vartheta)} d\vartheta = \ln\left(\frac{\theta}{1-\theta}\right) = q(\theta), \ \ \text{(say)}.$$

Hence,

$$\begin{aligned}
\int_{\alpha}^{x} \ln\left(\frac{\theta}{1-\theta}\right) d\theta &= \int_{\alpha}^{x} q(\theta) d\theta \\
&= \int_{\theta=\alpha}^{x} \int_{\vartheta=\alpha}^{\theta} \frac{1}{p(\vartheta)} d\vartheta d\theta \\
&= \int_{\vartheta=\alpha}^{x} \int_{\theta=\vartheta}^{x} \frac{1}{p(\vartheta)} d\theta d\vartheta \\
&= \int_{\alpha}^{x} \left(\frac{x}{p(\vartheta)} - \frac{\vartheta}{p(\vartheta)}\right) d\vartheta \\
&= xq(x) - \int_{\alpha}^{x} \frac{1}{1-\vartheta} d\vartheta \\
&= x\ln\left(\frac{x}{1-x}\right) + \ln(1-x)
\end{aligned}$$

Thus, we get

$$\ln G + c' = -\beta x - c_1 x + r\left[x\ln\left(\frac{x}{1-x}\right) + \ln(1-x)\right],$$

which implies,

$$G = \phi(c_1)\exp\{r\ln(1-x) - (\beta+z)x\}.$$

By substituting the value of c_1 obtained from the first characteristic curve (3.17), we get

$$G(x,z) = \phi\left(z + r\ln\left(\frac{x}{1-x}\right)\right)\exp\{r\ln(1-x) - (\beta+z)x\}. \tag{3.18}$$

Putting the initial condition, $G(x,0) = 1$, we arrive at

$$\phi\left(r\ln\left(\frac{x}{1-x}\right)\right) = \exp\{\beta x - r\ln(1-x)\}.$$

It follows

$$\phi(ry) = \exp\left\{\beta\frac{e^y}{1+e^y} - r\ln\left(1-\frac{e^y}{1+e^y}\right)\right\}$$
$$= \exp\left\{\beta\frac{e^y}{1+e^y} + r\ln(1+e^y)\right\}.$$

Thus,

$$\phi(t) = \exp\left\{\beta\frac{e^{t/r}}{1+e^{t/r}} + r\ln\left(1+e^{t/r}\right)\right\}.$$

Substituting this value of the function ϕ in equation (3.18), we arrive at the final solution, thus obtaining our moment generating function

$$G(x,z) = \left(1-x+xe^{z/r}\right)^r \exp\left\{-zx + \frac{\beta x(1-x)(e^{z/r}-1)}{1-x+xe^{z/r}}\right\}. \tag{3.19}$$

3.4 Asymptotic Results

Theorem 3.4.1 *Let $R > 0$ and a, b be such that $0 < a, b < \infty$. Then for a sufficiently large r, we have,*

$$\left\|B_r^{\beta}\left(e^{R|t|};x\right)\right\|_{C[a,b]} < \infty,$$

where $C[a,b]$ is the space of all real- valued continuous functions on $[a,b]$ endowed with the supremum norm.

Proof. As per the definition of semi-exponential operators and equation (3.19), we can write,

$$B_r^{\beta}\left(e^{Rt};x\right) = B_r^{\beta}\left(e^{Rx}e^{R(t-x)};x\right)$$
$$= e^{Rx}\sum_{\varsigma=0}^{\infty}\frac{R^{\varsigma}}{\varsigma!}B_r^{\beta}\left((t-x)^{\varsigma};x\right)$$
$$= \left(1-x+xe^{R/r}\right)^r \exp\left\{\frac{\beta x(1-x)(e^{R/r}-1)}{1-x+xe^{R/r}}\right\}.$$

Taking the limit $r \to \infty$, we get

$$\lim_{r\to\infty} B_r^{\beta}\left(e^{Rt};x\right) = \lim_{r\to\infty}\left(1-x+xe^{R/r}\right)^r = e^{Rx}.$$

Hence, $\lim\limits_{r\to\infty} B_r^{\beta}\left(e^{Rt};x\right) = e^{Rx}$ uniformly in $C[a,b]$.

Thus, for a sufficiently large r, we get $\left\|B_r^{\beta}\left(e^{Rt};x\right)\right\|_{C[a,b]} < \infty$.

And similarly, we can show that $\left\|B_r^{\beta}\left(e^{-Rt};x\right)\right\|_{C[a,b]} < \infty$.

Hence, $\left\|B_r^{\beta}\left(e^{R|t|};x\right)\right\|_{C[a,b]} \leq \left\|B_r^{\beta}\left(e^{Rt};x\right)\right\|_{C[a,b]} + \left\|B_r^{\beta}\left(e^{-Rt};x\right)\right\|_{C[a,b]}$.

Thus, $\left\|B_r^{\beta}\left(e^{R|t|};x\right)\right\|_{C[a,b]} < \infty$. □

Theorem 3.4.2 *Let $x \in [0,\infty]$. Let f be a continuous function such that $f''(x)$ exists. Then,*

$$\lim_{r\to\infty} r\left[B_r^{\beta}\left(f(t);x\right) - f(x)\right] = \beta x(1-x)f'(x) + \frac{1}{2}x(1-x)f''(x).$$

Proof. Using Taylor's expansion, we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + \zeta(t,x)(t-x)^2,$$

where $\lim_{t\to x} \zeta(t,x) = 0$. Taking $t = k/r$ and operating the equation with B_r^{β}, we can write

$$\begin{aligned} B_r^{\beta}\left(f(t);x\right) - f(x) &= f'(x)B_r^{\beta}\left((t-x);x\right) + \frac{f''(x)}{2}B_r^{\beta}\left((t-x)^2;x\right) \\ &\quad + B_r^{\beta}\left(\zeta(t,x)(t-x)^2;x\right) \\ &= \frac{\beta x(1-x)}{r}f'(x) + \left[\frac{x(1-x)}{r} + \frac{\beta x(1-x)(1-x-x^2)}{r^2}\right]\frac{f''(x)}{2} \\ &\quad + \sum_{k=0}^{\infty} W_{\beta}(r,x,k)\, \zeta\left(\frac{k}{r},x\right)\left(\frac{k}{r}-x\right)^2, \end{aligned} \tag{3.20}$$

where $\lim_{t\to x} \zeta(t,x) = 0$.

Now, we will prove that $r\sum_{k=0}^{\infty} W_{\beta}(r,x,k)\, \zeta\left(\frac{k}{r},x\right)\left(\frac{k}{r}-x\right)^2 \to 0$ as $r\to\infty$. (3.21)

Using Holder's inequality,

$$\begin{aligned} & r\sum_{k=0}^{\infty} W_{\beta}(r,x,k)\, \zeta\left(\frac{k}{r},x\right)\left(\frac{k}{r}-x\right)^2 \\ &\quad \leq \left(\sum_{k=0}^{\infty} W_{\beta}(r,x,k)\, \zeta^3\left(\frac{k}{r},x\right)\right)^{1/3}\left(r^{3/2}\sum_{k=0}^{\infty} W_{\beta}(r,x,k)\left(\frac{k}{r}-x\right)^3\right)^{2/3}. \end{aligned}$$

Since $\lim_{t \to x} \varkappa(t,x) = 0$, we get that $\sum_{k=0}^{\infty} W_\beta(r,x,k)\, \varkappa^3\left(\frac{k}{r},x\right) \to 0$, as $r \to \infty$. Also, from Remark 3.3.4, we have

$$\begin{aligned}
\left(r^{3/2} \sum_{k=0}^{\infty} W_\beta(r,x,k) \left(\frac{k}{r}-x\right)^3\right)^{2/3} &= \left(r^{3/2}\mu_{r,3}(x)\right)^{2/3} \\
&= \left(r^{3/2} O\left(\frac{1}{r^2}\right)\right)^{2/3} \\
&= O\left(\frac{1}{r^{1/3}}\right) \to 0.
\end{aligned}$$

It follows that our claim, given by (3.21) is true. Thus, from equation (3.20) we get the desired result. □

3.5 Categorizing functions according to their level of approximation

The focus of this section is to categorize functions in a manner such that

$$\left\| B_r^\beta(f;x) - f(x) \right\| \leq O\left(r^{-\alpha/2}\right).$$

We know, $C[0,\infty)$ denotes the set of all real-valued continuous functions on $[0,\infty)$. We define the space $\left(C_R, \|.\|_{C_R}\right)$ as follows:

$$C_R = \left\{ f \in C[0,\infty) : \exists\, M > 0\, \forall\, t \in [0,\infty), |f(t)| \leq M e^{R|t|} \right\},$$

and

$$\|f\|_{C_R} = \sup\left\{ e^{-R|t|}\, |f(t)| : t \in [0,\infty) \right\}.$$

Lemma 3.5.1 *Suppose that ς and η are positive numbers, $R \geq 0$ and $[a,b] \subset [0,\infty)$. Then,*

$$\sum_{\mathrm{T}_r} W_\beta\left(r,x,\frac{k}{r}\right) e^{R\frac{k}{r}} = O\left(r^{-\varsigma}\right)$$

uniformly on $[a,b]$ as $r \to \infty$, where

$$\mathrm{T}_r = \left\{ k : \left|\frac{k}{r}-x\right| \geq \eta;\ k = 0,1,\ldots \right\}.$$

Proof. From the definition of T_r and Theorem 3.3.3, we can write

$$\begin{aligned}
A_{4\varsigma}(r,x) &= r^{4\varsigma} \sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right)\left(\frac{k}{r}-x\right)^{4\varsigma} \\
&\geq r^{4\varsigma} \sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right) \eta^{4\varsigma},
\end{aligned}$$

which implies

$$\sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right) \le \sum_{\mathrm{T}_r} W_\beta\left(r,x,\frac{k}{r}\right)$$
$$\le r^{-4\varsigma}\eta^{-4\varsigma}A_{4\varsigma}(r,x).$$

Also, by Theorem 3.4.1, as $r\to\infty$, we get

$$\sum_{\mathrm{T}_r} W_\beta\left(r,x,\frac{k}{r}\right)e^{2R\frac{k}{r}} < \infty.$$

Thus, by Cauchy-Schwarz's inequality and Remark 3.3.4,

$$\begin{aligned}\sum_{\mathrm{T}_r} W_\beta\left(r,x,\frac{k}{r}\right)e^{R\frac{k}{r}} &\le \left(\sum_{\mathrm{T}_r} W_\beta\left(r,x,\frac{k}{r}\right)\right)^{1/2}\left(\sum_{\mathrm{T}_r} W_\beta\left(r,x,\frac{k}{r}\right)e^{2R\frac{k}{r}}\right)^{1/2}\\ &\le \left(r^{-4\varsigma}\eta^{-4\varsigma}A_{4\varsigma}(r,x)\right)^{1/2}\left(B_r^\beta(e^{2R\frac{k}{r}};x)\right)^{1/2}\\ &= O\left(r^{-\varsigma}\right) \qquad \text{as } r\to\infty.\end{aligned}$$

This completes the proof. □

Theorem 3.5.2 *Suppose that $f\in C_R$ and $0<a<a_1<b_1<b<\infty$. Then, for every $\varsigma>0$ there exists a constant K_ς, such that*

$$\left\|B_r^\beta(f,x)-f(x)\right\|_{C_{[a_1,b_1]}} \le K_\varsigma\left[r^{-1/2}\omega\left(f;r^{-1/2},a,b\right)+\omega_2\left(f;r^{-1/2},a,b\right)+r^{-\varsigma}\|f\|_{C_R}\right].$$

Proof. Let $\delta>0$. Define,

$$g_\delta(x)=\frac{1}{2\delta^2}\int_{-\delta/2}^{\delta/2}\int_{-\delta/2}^{\delta/2}\left[f(x+u+v)+f(x-u-v)\right]du\,dv.$$

By the linearity of our operators B_r^β, it follows that

$$\begin{aligned}&\left\|B_r^\beta(f;x)-f(x)\right\|_{C_{[a_1,b_1]}}\\ &\le \left\|B_r^\beta(f-g_\delta;x)\right\|_{C_{[a_1,b_1]}}+\left\|B_r^\beta(g_\delta;x)-g_\delta(x)\right\|_{C_{[a_1,b_1]}}+\|g_\delta-f\|_{C_{[a_1,b_1]}}\\ &=I_1+I_2+I_3. \end{aligned} \tag{3.22}$$

For I_1, we consider

$$\left|B_r^\beta(f;x)-B_r^\beta(g_\delta;x)\right| = \sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right)\left|f\left(k/r\right)-g_\delta\left(k/r\right)\right|.$$

Taking $\eta = \frac{1}{2}\min\{a_1 - a, b - b_1\}$, let us define

$$\Gamma_r = \left\{k : \left|\frac{k}{r} - x\right| < \eta;\ \ k = 0, 1, ...\right\}$$

and

$$\mathrm{T}_r = \left\{k : \left|\frac{k}{r} - x\right| \geq \eta;\ \ k = 0, 1, ...\right\}.$$

Then,

$$\begin{aligned}
&\left|B_r^{\beta}(f;x) - B_r^{\beta}(g_\delta;x)\right| \\
&\leq \sum_{\Gamma_r} W_\beta\left(r, x, \frac{k}{r}\right)\left|f(k/r) - g_\delta(k/r)\right| + \sum_{\mathrm{T}_r} W_\beta\left(r, x, \frac{k}{r}\right)\left|f(k/r) - g_\delta(k/r)\right| \\
&= L_1 + L_2.
\end{aligned}$$

Since, $\left|\frac{k}{r} - x\right| < \eta$ for L_1, we can say that

$$\frac{k}{r} \in [x - \eta, x + \eta] \subset [a_1 - \eta, b_1 + \eta].$$

Hence, we can state that

$$L_1 \leq \frac{1}{2}\omega_2(f;\delta, a, b).$$

Using Lemma 3.5.1 for a constant M, we can write L_2 as

$$L_2 = \left|\sum_{\mathrm{T}_r} W_\beta\left(r, x, \frac{k}{r}\right) e^{R\frac{k}{r}}\left|f - g_\delta\right|(k/r)\, e^{-R\frac{k}{r}}\right|.$$

Now, since $f - g_\delta \in C_R$, it follows

$$\begin{aligned}
&\left|(f - g_\delta)\left(\frac{k}{r}\right)\right| e^{-R\frac{k}{r}} \\
&= e^{-R\frac{k}{r}}\left|\frac{1}{2\delta^2}\int_{-\delta/2}^{\delta/2}\int_{-\delta/2}^{\delta/2}\left[f\left(\frac{k}{r} + u + v\right) - 2f\left(\frac{k}{r}\right) + f\left(\frac{k}{r} - u - v\right)\right] du dv\right| \\
&\leq \frac{1}{2\delta^2}\int_{-\delta/2}^{\delta/2}\int_{-\delta/2}^{\delta/2}\left|f\left(\frac{k}{r} + u + v\right)\right| e^{-R\frac{k}{r}} du dv + \frac{1}{2\delta^2}\int_{-\delta/2}^{\delta/2}\int_{-\delta/2}^{\delta/2} 2\left|f\left(\frac{k}{r}\right)\right| e^{-R\frac{k}{r}} du dv \\
&\quad + \frac{1}{2\delta^2}\int_{-\delta/2}^{\delta/2}\int_{-\delta/2}^{\delta/2}\left|f\left(\frac{k}{r} - u - v\right)\right| e^{-R\frac{k}{r}} du dv \\
&\leq \|f\|_{C_R} + 2\|f\|_{C_R} + \|f\|_{C_R} \\
&= 4\|f\|_{C_R}.
\end{aligned}$$

Thus, $L_2 \leqslant 4Mr^{-\varsigma}\|f\|_{C_R}$. Hence,

$$\left\|B_r^{\beta}(f;x) - B_r^{\beta}(g_\delta;x)\right\|_{C_{[a_1,b_1]}} \leq \frac{1}{2}\omega_2(f;\delta,a,b) + 4Mr^{-\varsigma}\|f\|_{C_R},$$

which gives the following estimate of I_1:

$$I_1 \leq M_1\left(\omega_2(f;\delta,a,b) + r^{-\varsigma}\|f\|_{C_R}\right). \tag{3.23}$$

Now, to estimate I_2, we consider the first and second derivative of $g_\delta(x)$.

$$g_\delta{}'(x) = \frac{1}{2\delta^2}\int_{-\delta/2}^{\delta/2}\left[f\left(x+u+\frac{\delta}{2}\right) - f\left(x+u-\frac{\delta}{2}\right)\right]du - \frac{1}{2\delta^2}\int_{-\delta/2}^{\delta/2}\left[f\left(x-u+\frac{\delta}{2}\right) - f\left(x-u-\frac{\delta}{2}\right)\right]du.$$

Hence, for $x \in [a_1-\eta, b_1+\eta]$, we have

$$\begin{aligned}\left\|g_\delta{}'(x)\right\|_{C_{[a_1-\eta,b_1+\eta]}} &\leq \frac{\omega(f;\delta,a,b)}{\delta^2}\int_{-\delta/2}^{\delta/2} du \\ &\leq \frac{1}{\delta}\omega(f;\delta,a,b).\end{aligned}$$

Moreover,

$$g_\delta{}''(x) = \frac{1}{\delta^2}\left[f(x-\delta) - 2f(x) + f(x+\delta)\right],$$

and

$$\left\|g_\delta{}''(x)\right\|_{C_{[a_1-\eta,b_1+\eta]}} \leq \frac{1}{\delta^2}\omega_2(f;\delta,a,b).$$

By the definition of semi-exponential operators and Taylor's expansion, we have

$$\begin{aligned}&\left|B_r^{\beta}(g_\delta;x) - g_\delta(x)\right| \\ &= \left|\sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right)\left[g_\delta(k/r) - g_\delta(x)\right]\right| \\ &\leq \left|\sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right)\left(\frac{k}{r}-x\right)g_\delta{}'(x)\right| + \frac{1}{2}\left|\sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right)\left(\frac{k}{r}-x\right)^2 g_\delta{}''(x)\right| \\ &\quad + \left|\sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right)\mathscr{r}\left(\frac{k}{r}-x\right)\left(\frac{k}{r}-x\right)^2\right| \\ &\leq \left|\sum_{k=0}^{\infty} W_\beta\left(r,x,\frac{k}{r}\right)\left(\frac{k}{r}-x\right)g_\delta{}'(x)\right| + \left|\sum_{\Gamma_r} W_\beta\left(r,x,\frac{k}{r}\right)\left(\frac{k}{r}-x\right)^2 g_\delta{}''(x)\right| \\ &\quad + \left|\sum_{\mathrm{T}_r} W_\beta\left(r,x,\frac{k}{r}\right)\left(\frac{k}{r}-x\right)^2 g_\delta{}''(x)\right| \\ &= S_1 + S_2 + S_3.\end{aligned}$$

For $x \in [a_1 - \eta, b_1 + \eta]$ and some constant M, we obtain,

$$S_1 = \left| g_\delta{}'(x) r^{-1} A_1(r,x) \right| \le \frac{Mr^{-1}}{\delta} \omega(f;\delta,a,b).$$

Whereas, for $x \in [a_1, b_1]$ and some constant K,

$$S_2 = \left| g_\delta{}''(x) r^{-2} A_2(r,x) \right| \le \frac{Kr^{-1}}{\delta^2} \omega_2(f;\delta,a,b).$$

Following Theorem 3.3.3 and Lemma 3.5.1, we can write,

$$\begin{aligned} S_3 &= \left| \sum_{\mathrm{T}_r} W_\beta\left(r,x,\frac{k}{r}\right)\left(\frac{k}{r}-x\right)^2 g_\delta{}''(x) \right| \\ &\le 4\|f\|_{C_R} \left| \left(\sum_{\mathrm{T}_r} W_\beta\left(r,x,\frac{k}{r}\right) e^{2r} \right)^{1/2} \left(r^{-4} A_4(r,x)\right)^{1/2} \right| \\ &\le L r^{-\varsigma} \|f\|_{C_R}. \end{aligned}$$

Thus, finally there exists M_2, such that

$$I_2 \le M_2 \left[\frac{r^{-1}}{\delta} \omega(f;\delta,a,b) + \frac{r^{-1}}{\delta^2} \omega_2(f;\delta,a,b) + r^{-\varsigma} \|f\|_{C_R} \right]. \tag{3.24}$$

To estimate I_3, let $\eta = \frac{1}{2}\min\{a_1 - a, b - b_1\}$. Then, for $x \in [a_1 - \eta, b_1 + \eta]$, we have

$$|f(x) - g_\delta(x)| = \left| \frac{1}{2\delta^2} \int_{-\delta/2}^{\delta/2} \int_{-\delta/2}^{\delta/2} [f(x+u+v) - 2f(x) + f(x-u-v)]\, du\, dv \right|.$$

By the limits of u and v, we have $|u+v| \le \delta$. Thus, using the definition of second order modulus of continuity,

$$\begin{aligned} |f(x) - g_\delta(x)| &\le \frac{1}{2\delta^2} \int_{-\delta/2}^{\delta/2} \int_{-\delta/2}^{\delta/2} \omega_2(f;\delta,a,b)\, du\, dv \\ &= \frac{1}{2} \omega_2(f;\delta,a,b). \end{aligned}$$

It follows that,

$$I_3 \le \frac{1}{2} \omega_2(f;\delta,a,b). \tag{3.25}$$

Putting $\delta = r^{-1/2}$ in equations (3.23), (3.24), (3.25) and using (3.22) we get the conclusion. □

This theorem particularly highlights that, if $\omega_2(f;h,a,b) = O(h^\alpha)$ then we can say that $\left\| B_r^\beta(f;x) - f(x) \right\|_{C[a_1,b_1]} = O\left(r^{-\alpha/2}\right)$. The converse of the above statement is also true and is proved next. But first let us define the Zygmund class of functions.

Definition 3.5.3 *Let $0 < \alpha < 2$. The Zygmund class, $Lip^*\alpha$ is defined as*

$$Lip^*(\alpha;a,b) = \{f : \omega_2(f;h,a,b) \le Mh^\alpha\}.$$

Theorem 3.5.4 *Let $0 < a < a_1 < b_1 < b < \infty$, $0 < \alpha < 2$ and $f \in C_R$. Then the following are all equivalent statements:*

(i) $f \in Lip^*(\alpha;a_1,b_1)$

(ii) $\left\| B_r^\beta(f;x) - f(x) \right\|_{C[a_1,b_1]} = O\left(r^{-\alpha/2}\right)$.

Proof. We have already seen that (i) implies (ii). We are only left to prove that (ii) implies (i). Let,

$$\left\| B_r^\beta(f;x) - f(x) \right\|_{C[a_1,b_1]} \le M_1 r^{-\alpha/2}.$$

We need to find an upper bound for the first order modulus of continuity of f. Let $x,y \in [0,\infty)$ be such that $|y-x| \le h$. Then,

$$\begin{aligned}
&|f(y) - f(x)| \\
&\quad \le \left|f(y) - B_r^\beta(f;y)\right| + \left|B_r^\beta(f;y) - B_r^\beta(f;x)\right| + \left|B_r^\beta(f;x) - f(x)\right| \\
&\quad \le 2M_1 r^{-\alpha/2} + P.
\end{aligned}$$

To calculate P, let us use the PDE (3.6) and examine $\left|\left(B_r^\beta\right)'(f;x)\right|$. We have

$$\begin{aligned}
\left|\left(B_r^\beta\right)'(f;x)\right| &= \left|\sum_{k=0}^\infty \frac{\partial}{\partial x} W_\beta(r,x,k/r) f\left(\frac{k}{r}\right)\right| \\
&= \left|\frac{r}{x(1-x)} \sum_{k=0}^\infty W_\beta(r,x,k/r)\left(\frac{k}{r} - x\right) f\left(\frac{k}{r}\right) - \beta \sum_{k=0}^\infty W_\beta(r,x,k/r) f\left(\frac{k}{r}\right)\right| \\
&\le \left|\frac{r}{x(1-x)} \sum_{k=0}^\infty W_\beta(r,x,k/r)\left(\frac{k}{r} - x\right)\left(f\left(\frac{k}{r}\right) - f(x)\right)\right| \\
&\quad + \left|\frac{r}{x(1-x)} \sum_{k=0}^\infty W_\beta(r,x,k/r)\left(\frac{k}{r} - x\right) f(x) - \beta \sum_{k=0}^\infty W_\beta(r,x,k/r) f\left(\frac{k}{r}\right)\right|.
\end{aligned}$$

Also, from Theorem 3.3.3, we get

$$\begin{aligned}
\sum_{k=0}^\infty W_\beta(r,x,k/r)\left(\frac{k}{r} - x\right) f(x) &= \frac{A_1(r,x)}{r} f(x) \\
&= \frac{\beta}{r} x(1-x) f(x).
\end{aligned}$$

Using the property $\omega(\lambda\delta) \le [1+\lambda]\,\omega(\delta)$, we can write

$$\omega\left(\left|\frac{k}{r}-x\right|\right) \le \left[1+r\left|\frac{k}{r}-x\right|\right]\omega\left(r^{-1}\right). \tag{3.26}$$

Thus,

$$\begin{aligned}
\left|\left(B_r^{\beta}\right)'(f;x)\right| &\le \frac{r}{x(1-x)}\sum_{k=0}^{\infty} W_\beta(r,x,k/r)\left|\frac{k}{r}-x\right|\omega\left(f;\left|\frac{k}{r}-x\right|,a,b\right) \\
&\quad + \left|\beta f(x) - \beta\sum_{k=0}^{\infty} W_\beta(r,x,k/r) f\left(\frac{k}{r}\right)\right| \\
&\le \frac{r}{x(1-x)}\sum_{k=0}^{\infty} W_\beta(r,x,k/r)\left|\frac{k}{r}-x\right|\omega\left(f;\left|\frac{k}{r}-x\right|,a,b\right) + \beta M_1 r^{-\alpha/2} \\
&\le \frac{r}{x(1-x)}\sum_{k=0}^{\infty} W_\beta(r,x,k/r)\left|\frac{k}{r}-x\right|\left[1+r\left|\frac{k}{r}-x\right|\right]\omega\left(f;\left|\frac{1}{r}\right|,a,b\right) \\
&\quad + \beta M_1 r^{-\alpha/2} \\
&\le rM_2\,\omega\left(f;\left|\frac{1}{r}\right|,a,b\right)\sum_{k=0}^{\infty} W_\beta(r,x,k/r)\left|\frac{k}{r}-x\right|\left[1+r\left|\frac{k}{r}-x\right|\right] \\
&\quad + \beta M_1 r^{-\alpha/2} \\
&\le rM_2\,\omega\left(f;\left|\frac{1}{r}\right|,a,b\right)\left[A_1(r,x) + \frac{A_2(r,x)}{r}\right] + \beta M_1 r^{-\alpha/2} \\
&\le M_3\left[r\omega\left(f;\left|\frac{1}{r}\right|,a,b\right) + r^{-\alpha/2}\right].
\end{aligned}$$

Now, for $|y-x| \le h$,

$$\begin{aligned}
P &= \left|B_r^{\beta}(f;y) - B_r^{\beta}(f;x)\right| \\
&= \left|\int_x^y B_r^{\beta\prime}(f;\xi)\,d\xi\right| \\
&\le \int_x^y \left|B_r^{\beta\prime}(f;\xi)\right| d\xi \\
&\le \int_x^y M_3\left[r\omega\left(f;\left|\frac{1}{r}\right|,a,b\right) + r^{-\alpha/2}\right] d\xi \\
&= M_3 r\omega\left(f;\left|\frac{1}{r}\right|,a,b\right)|y-x| + M_3 r^{-\alpha/2}|y-x| \\
&\le M_3 rh\omega\left(f;\left|\frac{1}{r}\right|,a,b\right) + M_3 r^{-\alpha/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}\omega(f;h,a,b) &\le |f(y)-f(x)| \\ &\le M'\left[rh\omega\left(f;\left|\frac{1}{r}\right|,a,b\right)+r^{-\alpha/2}\right].\end{aligned}$$

Next we will prove, if $\omega(f;h,a,b) \le M'\left[rh\omega\left(f;\left|\frac{1}{r}\right|,a,b\right)+r^{-\alpha/2}\right]$ then, for $0<h\le 1$ and $r>1$, we have that $f \in Lip^*(\alpha;a,b)$.

Let $K>1$ be such that $2M' < K^{1-\alpha}$. Choose $M'' = \max\{\omega(f;1,a,b), 2M'K^{\alpha}\}$ and define $h_m = K^{-m}$ for $m=1,2,\ldots$.

Using the principle of mathematical induction, we can easily show that for every positive integer m,

$$\omega(f;h_m,a,b) \le M''h_m^{\alpha}.$$

Now, for every $0<h\le 1$, there exists an integer $m>0$ such that $h_m < h \le h_{m-1}$. Thus we can imply that,

$$\begin{aligned}\omega(f;h,a,b) &\le \omega(f;h_{m-1},a,b) \\ &\le M''h_{m-1}^{\alpha} \\ &= M''K^{\alpha}h_m^{\alpha} \\ &< M'''h^{\alpha}.\end{aligned}$$

Since,

$$\begin{aligned}|f(t)-2f(t+\delta)+f(t+2\delta)| &\le |f(t)-f(t+\delta)|+|f(t+\delta)-f(t+2\delta)| \\ &\le 2\omega(f;h,a,b) \\ &\le 2M'''h^{\alpha}.\end{aligned}$$

We get,

$$\omega_2(f;h,a,b) \le Ch^{\alpha}.$$

Thus, $f \in Lip^*(\alpha;a,b)$. With this, we have shown that the conclusion is true. □

3.6 Graphical Interpretations

In this section, we investigate how the parameters β and values r affect the behaviour of the semi-exponential Bernstein operators defined in (3.11). Since these parameter values play a central role in shaping the approximation process, it is important to examine their impact both individually and in combination. To this end, we study their effect numerically on test functions, complementing the theoretical convergence results established in the preceding sections.

3.6.0.a Effect of r

Figures 3.3 and 3.4 illustrate the effect of r on the approximation of the function

$$f(x) = x\sin(2\pi x) - x\cos(2\pi x)$$

by fixing the value of β at 5. As r increases, the curves generated by our proposed operators converge uniformly towards $f(x)$. This behaviour aligns with the theoretical results proven earlier, which guarantee uniform convergence of the proposed operators as $r \to \infty$. Thus, increasing r enhances the accuracy of approximation.

3.6.0.b Effect of β

Next, we analyze the role of the parameter β. For this purpose, we fix $r = 50$ and consider the polynomial function

$$f(x) = -27x^5 + 27x^4 + 21x^3 - 27x^2 + 6x,$$

while varying β over the values $5, 10$, and 15. The results are shown in Figures 3.5 and 3.6. From the figure, it is evident that smaller values of β yield better approximations, as the operator graphs lie closer to the curve of $f(x)$. Conversely, larger values of β tend to slow down convergence, thereby reducing the efficiency of approximation. Hence, the parameter β can be viewed as a refinement factor, with lower values favoring more accurate operator behaviour.

These observations highlight the roles of r and β. While increasing r enhances convergence in a uniform sense, adjusting β allows for finer control over the quality of approximation. As illustrated in Figures 3.7 and 3.8, increasing the value of r while simultaneously decreasing the value of β, for $f(x) = 4x^5 - 2x^4 + 2x^3 - 5x^2 + 2x$, leads to a significant enhancement in the approximation process. This combined effect results in operator curves that align more closely with the required function, thereby demonstrating the complementary influence of the two parameters on the overall accuracy.

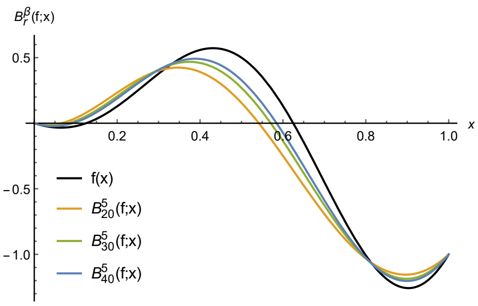


Figure 3.3: Graph of the semi-exponential Bernstein operators for $\beta = 5$ and different values of r.

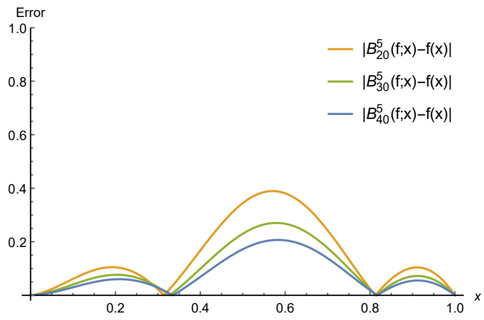


Figure 3.4: Graph of the error values of semi-exponential Bernstein operators corresponding to Figure 3.3.

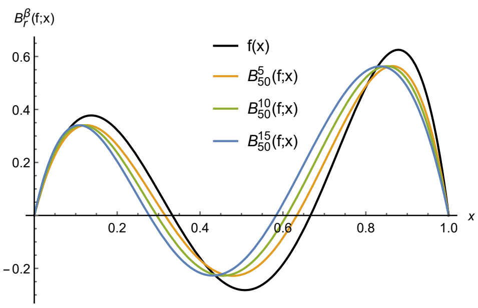


Figure 3.5: Graph of the semi-exponential Bernstein operators for $r = 50$ and different values of β.

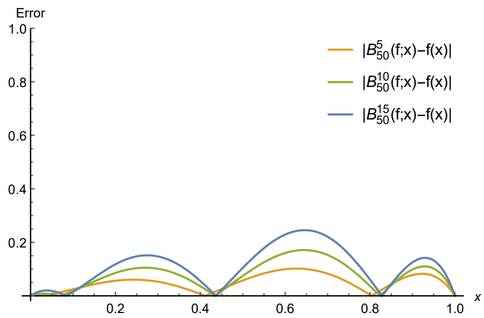


Figure 3.6: Graph of the error values of semi-exponential Bernstein operators corresponding to Figure 3.5.

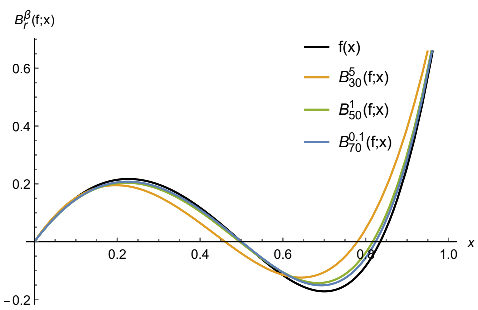


Figure 3.7: Graph of the semi-exponential Bernstein operators different values of r and β.

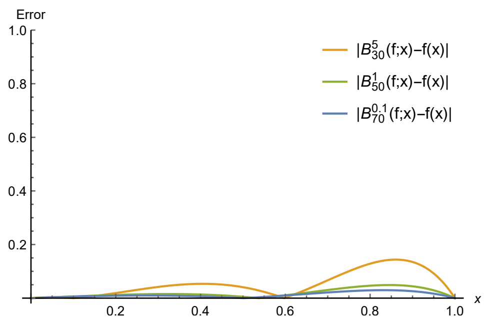


Figure 3.8: Graph of the error values of semi-exponential Bernstein operators corresponding to Figure 3.7.

Chapter 4

Solution of Fractional Differential and Integro-Differential Equations using Fractional Positive Linear Operators

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In this chapter, we derive a new sequence of positive linear operators by using the concept of fractional calculus, an extension of the classical calculus for integrals and derivatives of non-integer order α. The formulation starts by taking the classical Bernstein operators and differentiating them using the Caputo fractional derivative of order α. Such type of modification will fall under a new class of operators namely, Fractional Positive Linear Operators. We specifically focus on the 'Fractional Bernstein-Kantorovich Operators'. We give the moments of these operators with the help of Laplace transform. Using these moments and the Korovkin theorem we prove their convergence properties. Furthermore, the applications of these operators in solving linear fractional differential equations as well as fractional integro-differential equations using the basis polynomial of the fractional Bernstein-Kantorovich operators is also shown. Various Numerical illustrations are provided to better understand this approach. Further, the credibility of these approximate solutions is verified graphically and through error tables.

4.1 Introduction

The field of fractional calculus extends the principles of integer-order calculus to include non-integer orders [147; 170]. Fractional calculus may be regarded as both an old and a modern topic. This idea was first proposed by mathematicians L'Hopital and Leibniz in the 17^{th} century, who described derivatives of order $\alpha = 1/2$. However, the real breakthroughs came in the 19^{th} and 20^{th} centuries, when significant developments were made notably through the work of Liouville, Riemann, and later, Caputo and Grunwald [45; 46; 95; 135; 179]. Despite these early foundations, fractional calculus has continued to evolve over the centuries and remains an active field of research today, with new methods, interpretations and applications constantly emerging [68; 182].

Theoretically, it may not be of much importance to visualize $1/2^{th}$ order derivative of a function, however the properties of various real materials can be accurately described using derivatives and integrals of non-integer order [216; 170]. The integer order differential operator is a local operator while the fractional order differential operator is non-local. In other words, at each time t, the output of an integer order system depends solely on the input at that same time t, whereas the determination of the current output of a non-integer order system requires knowledge of its current input as well as all of its past input values. Thus, non-integer order derivatives and integrals can be utilized to explain processes with memory.

One significant place where fractional calculus finds its applications is in the formulation of fractional order differential equations. Fractional differential equations (FDEs) extend the concept of classical differential equations, where the orders of differentiation are extended to non-integer (fractional) values. This extension offers a more flexible and accurate framework for modeling various complex phenomena that cannot be adequately described by integer-order differential equations. In the last few decades, fractional differential equations have found a multitude of very interesting and novel applications in physics, mathematics, chemistry, engineering, finance, and other sciences (see [23; 69; 205] and references therein). In 1990, Koh and Kelly [131] applied fractional derivatives to model the stress-strain relationship in elastomers. In 1994, another significant milestone in the theory of fractional differential equations was achieved with the introduction of a new linear capacitor model based on Curie's empirical law, by Westerlund and Ekstam [217]. They incorporated fractional derivatives to describe the current-voltage relationship as $i(t) = C\frac{d^n u(t)}{dt^n}$, where

$u(t)$ represents the input voltage and $i(t)$ represents current. The model accounts for dielectric absorption and the capacitor's ability to 'remember' past voltages, offering solutions to problems that conventional theory cannot address. Chatterjee [49], in his work, explained that fractional order derivatives in linear viscoelastic materials arise from disordered dissipation mechanisms, which lead to power-law decay instead of the typical exponential decay. His study provided an engineering perspective on how multiple closely spaced exponential decay rates can produce fractional damping behaviour. Thus, literature suggests that fractional derivatives are an excellent tool for modeling memory, hereditary, and past-dependent properties of various materials and processes. This represents the main benefit of fractional derivative models over the traditional integer-order models, where such effects are overlooked.

While ordinary differential equations can be solved using methods like separation of variables, substitution and integration but solving fractional differential equations requires a combination of numerical, analytical, and specialized techniques based on the characteristics of the given equation. In 1999, Luchko and Gorenflo [141] used the Caputo fractional differential operator and developed operational calculus of Mikusiński's type to find the solution of an initial value problem of a general linear fractional differential equation with constant coefficients. Later in 1993, Samko et al. [197] studied fractional differentiation and various forms of multidimensional fractional integro-differentiation along with its application to differential equations. They sought a solution of ordinary fractional-order differential equations using the method of iterations and successive approximations. In 1995, Beyer and Kempfle [30] generalized an approach which uses Fourier transforms to solve a linear, not necessarily ordinary, differential equation by specifically taking the equation $\left(D^2 + aD^q + b\right)x = f$, where $0 < q < 2$. Inspired by the broad applicability of fractional differential equations (FDEs) in modeling physical phenomena, numerous other authors explored their mathematical aspects and solution methods, such as the Laplace transform technique [92; 142].

Motivated by the growing significance of fractional calculus and its applications in fractional differential equations, the objective of this chapter is to develop a new class of positive linear operators, hereafter known as 'Fractional Positive Linear Operators', to seek an approximate solution for linear fractional differential equations (FDEs) and fractional integro-differential equations (FIDEs).

First, we present a collection of fundamental definitions and lemmas that will be used throughout this chapter. These preliminaries include not only essential concepts from fractional calculus but also related mathematical tools, such as the hypergeometric function, which play a significant role in the analysis developed in the subsequent sections.

4.1.1 Preliminaries

To begin with, we recall the notion of the fractional integral, which serves as the foundation of fractional calculus. This operator extends the classical integral to arbitrary real or complex orders and provides the basis for defining fractional derivatives.

Definition 4.1.1 *[68] The fractional integral of order α of a function f is defined as,*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)dx, \quad t > 0, \ \alpha > 0.$$

Building upon the fractional integral, one of the most widely used formulations of a fractional derivative is the Caputo derivative. It is particularly useful in applications since it allows for the use of classical initial conditions, making it well-suited for physical and engineering problems.

Definition 4.1.2 *[68] The Caputo fractional derivative of order $\alpha > 0$ for a function $f(t) \in \mathbb{C}^n$ is given by*

$$D^{\alpha} f(t) = I^{(\lceil\alpha\rceil-\alpha)} \left(\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} f(t) \right) = \begin{cases} \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \int\limits_0^t \frac{f^{\lceil\alpha\rceil}(x)}{(t-x)^{\alpha-\lceil\alpha\rceil+1}} dx, & \alpha \notin N \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in N, \end{cases}$$

where $\lceil . \rceil$ is the ceiling function.

An important tool in the analysis of fractional differential equations is the Laplace transform of the Caputo derivative. This result provides a convenient way to handle fractional differential equations by transforming them into algebraic equations in the Laplace domain.

Definition 4.1.3 *[68] Assume that $f : [0,\infty) \to \mathbb{R}$ is such that Laplace transform $\mathscr{L} f$ exists on $[t_0,\infty)$ with some $t_0 \in \mathbb{R}$. Let $\alpha > 0$ and $n = \lceil\alpha\rceil$. Then for $t > \max\{0,t_0\}$, the Laplace transform formula for the Caputo fractional derivative $D^{\alpha} f(t)$ is given by,*

$$\mathscr{L}(D^{\alpha} f(t)) = s^{\alpha} F(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0),$$

where $F(s)$ denotes the Laplace transform of $f(t)$.

For completeness, we now state some basic properties of fractional integrals and derivatives. These properties not only generalize the familiar rules of integer-order calculus but also highlight the distinctive features of fractional operators.

- $D^{\alpha}(af(t)+bg(t)) = aD^{\alpha}f(t)+bD^{\alpha}g(t)$
- $I^{\alpha}(af(t)+bg(t)) = aI^{\alpha}f(t)+bI^{\alpha}g(t)$
- $I^{\alpha}D^{\alpha}f(t) = f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0), \quad x>0,\ n-1<\alpha\leq n$
- $I^{\alpha}I^{\beta} = I^{\alpha+\beta}$, for all $\alpha,\beta \geq 0$
- $I^{\alpha}D^{\beta} = I^{\alpha-\beta}$, where $\alpha > \beta$
- $I^{\alpha}x^{m} = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)}x^{\alpha+m}, \quad \alpha\geq 0,\ m>-1,\ x>0$
- $L[D^{\alpha}f(x)] = s^{\alpha}L[f(x)]-s^{\alpha-1}f(0), \quad 0<\alpha\leq 1.$

In addition to integral transforms, another powerful tool in fractional calculus is the generalization of classical series expansions. Such results extend the notion of Taylor's expansion to the fractional-order framework, thereby providing a way to represent functions in terms of their fractional derivatives. Thus, next we state the fractional order Taylor's expansion of a function f.

Theorem 4.1.4 *[162] Suppose that $D^{k\alpha}f(x)\in C[0,1]$ for $k=0,1,\ldots,n+1$, where $0<\alpha\leq 1$, then the fractional order Taylor's expansion of f is given by*

$$f(x)=\sum_{i=0}^{n}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}\left(D^{i\alpha}f\right)(a)+\frac{\left(D^{(n+1)\alpha}f\right)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha},$$

with $a\leq\xi\leq x$, $\forall x\in[a,b]$, where $D^{n\alpha}=D^{\alpha}\cdot D^{\alpha}\cdots D^{\alpha}$ (n-times).

The fractional Taylor expansion plays a crucial role in both theoretical and applied aspects of fractional calculus. It not only generalizes a well-known result from classical analysis but also provides a foundation for proving approximation results and establishing error estimates for the operators introduced later in this chapter.

In addition to the tools of fractional calculus discussed above, our analysis also makes use of the hypergeometric function. Although not a part of fractional calculus, it is employed here as a constructive tool in the formulation of the operators introduced in this chapter. We now provide its definition and basic properties.

Definition 4.1.5 *[18] A function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written as*

$$\frac{c_{k+1}}{c_k} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(k+b_2)...(k+b_q)(k+1)},$$

is known as the generalized hypergeometric function and is expressed as

$${}_pF_q(a_1,...,a_p;b_1,...,b_q;x) = \sum_k c_k x^k.$$

The hypergeometric functions can explicitly be expressed as

$${}_pF_q\left(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x\right) = \sum_{r=0}^{\infty} \frac{(a_1)_r \ldots (a_p)_r}{(b_1)_r \ldots (b_n)_r} \frac{x^r}{r!},$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the pochhammer symbol. Specifically, for $p = 2$ and $q = 1$, we get the Gauss's hypergeometric function, defined as

$$\begin{aligned} {}_2F_1\left(a_1,a_2;b_1;x\right) &= \sum_{r=0}^{\infty} \frac{(a_1)_r (a_2)_r}{(b_1)} \frac{x^r}{r!} \\ &= 1 + \frac{a_1 a_2}{b_1} x + \frac{a_1\left(1+a_1\right) a_2\left(1+a_2\right)}{2b_1\left(1+b_1\right)} x^2 + \ldots, \quad |x| < 1. \end{aligned} \tag{4.1}$$

Beyond its series representation, the Gauss hypergeometric function also admits various integral representations, which often prove more convenient in analysis and applications. One particularly useful representation connects the hypergeometric function with the Beta function, allowing us to express certain integrals in terms of ${}_2F_1\left(a,b;c;x\right)$. The following lemma states this result.

Lemma 4.1.6 *[24] Let $c > b > 0$ and $|x| < 1$. Then, an integral in terms of the Gauss hypergeometric function is given by*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt = \beta(b,c-b) {}_2F_1\left(a,b;c;x\right).$$

When $x = 1$, the integral on the right reduces to a beta function.

Having established the fundamental concepts of fractional integrals, Caputo derivatives, their Laplace transforms and the associated properties, we are now equipped with the essential mathematical tools required for the construction of our operators and subsequent analysis in this chapter.

4.2 Fraction Bernstein-Kantorovich Operators

The concept of positive linear operators has undergone significant enhancement since the early 20^{th} century in an effort to improve and understand approximation processes. Among the most widely recognized and frequently used positive linear operators are the Bernstein Kantorovich operators [127] which are based on a modification of the Bernstein operators and are defined to approximate functions f that are integrable over the interval $[0,1]$. These operators replace the sample values $f(k/n)$ in the Bernstein operators with the mean values of f over the sub-intervals $\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$. This is achieved by integrating the function $f(t)$ over each subinterval $\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$ and weighting it by the Bernstein basis polynomial $p_{n,k}(x) = x^k(1-x)^{n-k}$. The Bernstein Kantorovich operators K_n are defined by equation (1.4)

The factor $(n+1)$ ensures that the Bernstein-Kantorovich operators are positive linear operators that preserve constant functions. For more study on Bernstein and Kantorovich type operators, one can refer to [153; 85; 125; 113; 157].

Building upon these classical constructions, it is important to note that the basis functions in positive linear operators play a central role in determining the efficiency of approximation. By enriching these basis functions through the use of fractional-order derivatives and integrals, one can incorporate memory effects directly into the operator. Such memory-dependent basis functions enable the operator to capture long-range dependencies and non-local behaviours in the function being approximated.

This chapter is devoted to the construction and formulation of a new sequence of positive linear operators, namely Fractional Positive Linear Operators, with the help of fractional derivative in Caputo sense. Using the definition of Caputo fractional derivative on classical Bernstein operators, we get

$$I^{\lceil\alpha\rceil-\alpha}\frac{d^{\lceil\alpha\rceil}}{dx^{\lceil\alpha\rceil}}B_{n+1}(F), \qquad \text{where } F(x) = \int_0^x f(t)dt.$$

Without loss of generality, let $0 < \alpha < 1$, then

$$\begin{aligned}
D^{\alpha}\left(B_{n+1}(F)\right) &= I^{1-\alpha}\frac{d}{dx}B_{n+1}(F) \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}\frac{d}{dt}B_{n+1}(F)dt \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}(n+1)\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}\left(F\left(\frac{k+1}{n+1}\right)-F\left(\frac{k}{n+1}\right)\right)dt
\end{aligned}$$

$$= \frac{n+1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \binom{n}{k} \left(F\left(\frac{k+1}{n+1}\right) - F\left(\frac{k}{n+1}\right) \right) \int_0^x t^k (1-t)^{n-k} (x-t)^{-\alpha} dt.$$

$$= \frac{n+1}{\Gamma(1-\alpha)} x^{-\alpha} \sum_{k=0}^{n} \binom{n}{k} \left(F\left(\frac{k+1}{n+1}\right) - F\left(\frac{k}{n+1}\right) \right) \int_0^x t^k (1-t)^{n-k} \left(1-\frac{t}{x}\right)^{-\alpha} dt.$$

Substituting $t = xy$ and using Lemma 4.1.6, we can write

$$\begin{aligned} &D^{\alpha}\left(B_{n+1}(F)\right) \\ &= \frac{n+1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \binom{n}{k} x^{k-\alpha+1} \left(F\left(\frac{k+1}{n+1}\right) - F\left(\frac{k}{n+1}\right) \right) \int_0^1 y^k (1-y)^{-\alpha} (1-xy)^{n-k} dy \\ &= \frac{n+1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \binom{n}{k} x^{k-\alpha+1} \left(F\left(\frac{k+1}{n+1}\right) - F\left(\frac{k}{n+1}\right) \right) \beta\left(k+1, 1-\alpha\right) {}_2F_1\left(-n+k, k+1; k-\alpha+2; x\right) \\ &= (n+1) x^{-\alpha+1} \sum_{k=0}^{n} \binom{n}{k} x^k \frac{\Gamma(k+1)}{\Gamma(k-\alpha+2)} {}_2F_1\left(-n+k, k+1; k-\alpha+2; x\right) \left(F\left(\frac{k+1}{n+1}\right) - F\left(\frac{k}{n+1}\right) \right), \end{aligned}$$

where ${}_2F_1\left(a,b;c;x\right)$ is the Gauss's hypergeometric function given by equation (4.1). Substituting $F(x) = \int_0^x f(t)dt$, we arrive at

$$D^{\alpha}\left(B_{n+1}(F)\right) = (n+1) x^{-\alpha+1} \sum_{k=0}^{n} \binom{n}{k} x^k \frac{\Gamma(k+1)}{\Gamma(k-\alpha+2)} {}_2F_1\left(k+1, -n+k; k-\alpha+2; x\right) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt. \quad (4.2)$$

For $f(t) = 1$,

$$\begin{aligned} D^{\alpha}\left(B_{n+1}(F)\right) &= x^{-\alpha+1} \sum_{k=0}^{n} \binom{n}{k} x^k \frac{\Gamma(k+1)}{\Gamma(k-\alpha+2)} {}_2F_1\left(k+1, -n+k; k-\alpha+2; x\right) \\ &= \frac{x^{-\alpha+1}}{\Gamma(2-\alpha)}. \end{aligned}$$

The operators defined by equation (4.2) are positive linear operators. However, for the constant function $f(t) = 1$, we observe that $D^{\alpha}\left(B_{n+1}(F)\right) \neq 1$, but it converges to 1 as $\alpha \to 1$. To construct a positive linear operator that exactly preserves the test function $e_0(t) = 1$, it is necessary to normalize by dividing $D^{\alpha}\big(B_{n+1}(F)\big)$ with the factor $\frac{x^{-\alpha+1}}{\Gamma(2-\alpha)}$. With this adjustment, the Fractional Bernstein Kantorovich operators are defined as,

$$\begin{aligned} K_n^{\alpha}\left(f;x\right) &= \frac{\Gamma(2-\alpha)}{x^{-\alpha+1}} D^{\alpha}\left(B_{n+1}(F)\right) \\ &= (n+1)\Gamma(2-\alpha) \sum_{k=0}^{n} \binom{n}{k} x^k \frac{\Gamma(k+1)}{\Gamma(k-\alpha+2)} {}_2F_1\left(k+1, -n+k; k-\alpha+2; x\right) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt. \end{aligned} \quad (4.3)$$

For $\alpha = 1$, the fractional Bernstein Kantorovich operators (4.3) reduce to the classical Bernstein Kantorovich operators defined in equation (1.4). We now denote the corresponding basis functions of the proposed operators (4.3) by

$$B_{n,k}(x) = \binom{n}{k} x^k \frac{\Gamma(2-\alpha)\Gamma(k+1)}{\Gamma(k-\alpha+2)} {}_2F_1\left(k+1, -n+k; k-\alpha+2; x\right) \tag{4.4}$$

To gain further insight into the structure of the fractional Bernstein Kantorovich operators, it is useful to examine the behaviour of their basis functions. In particular, plotting these basis functions allows us to visualize how the fractional parameter α influences their shape and distribution. Figures 4.2, 4.3, 4.4 and 4.5 represent the graphs of these basis functions corresponding to $n = 8$ for different values of α, while Figure 4.1 represents the graphs of the Bernstein polynomials $b_{8,k}(x)$, defined in equation (1.1). These plots clearly illustrate how varying α modifies the spread and smoothness of the basis functions, thereby reflecting the impact of fractional order on the approximation process.

It can also be observed that as $\alpha \to 1$, the basis functions defined in equation (4.4) gradually recover the shape of Bernstein polynomials. In contrast, for smaller values of α, the basis functions become more flattened and widely spread across the interval, indicating a stronger memory effect embedded in the operator. This comparative behaviour highlights how the fractional parameter governs the transition between classical and memory-dependent approximation schemes.

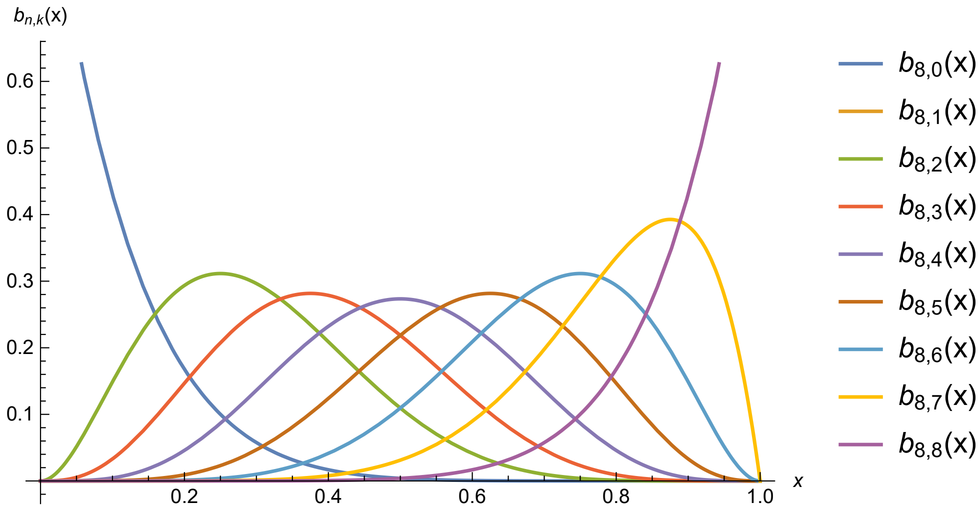


Figure 4.1: Graph of the Bernstein polynomials $b_{n,k}(x)$, for $n = 8$.

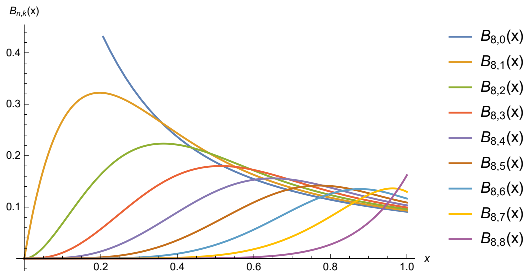


Figure 4.2: Graph of the Basis functions $B_{n,k}(x)$, for $n = 8$ and $\alpha = 0.2$.

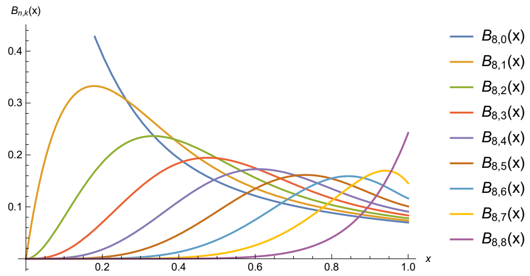


Figure 4.3: Graph of the Basis functions $B_{n,k}(x)$, for $n = 8$ and $\alpha = 0.4$.

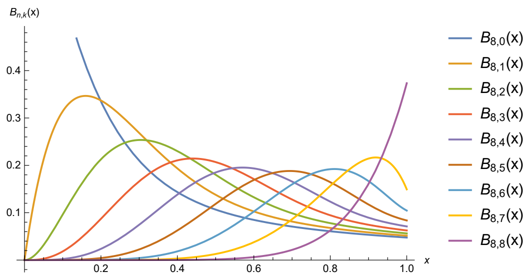


Figure 4.4: Graph of the Basis functions $B_{n,k}(x)$, for $n = 8$ and $\alpha = 0.6$.

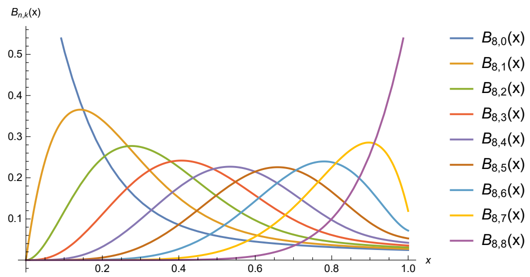


Figure 4.5: Graph of the Basis functions $B_{n,k}(x)$, for $n = 8$ and $\alpha = 0.8$.

4.2.1 Approximation Results

Remark 4.2.1 *The fractional Bernstein Kantorovich operators $K_n^{\alpha}(f;x)$ are a sequence of positive linear operators such that $K_n^{\alpha}(c;x) = c$, where c is any real constant.*

Theorem 4.2.2 *For $0 < \alpha \leq 1$ and the operators* (4.3)*, we get the following identities (also known as the moments of K_n^{α}):*

(i) $K_n^{\alpha}(e_0;x) = 1$

(ii) $K_n^{\alpha}(e_1;x) = \frac{1}{2(n+1)} + \frac{nx}{(n+1)(2-\alpha)}$

(iii) $K_n^{\alpha}(e_2;x) = \frac{1}{3(n+1)^2} + \frac{2nx}{(n+1)^2(2-\alpha)} + \frac{2n(n-1)x^2}{(n+1)^2(3-\alpha)(2-\alpha)}$

(iv) $K_n^{\alpha}(e_3;x) = \frac{1}{4(n+1)^3} + \frac{7nx}{2(n+1)^3(2-\alpha)} + \frac{9n(n-1)x^2}{(n+1)^3(3-\alpha)(2-\alpha)} + \frac{6n(n-1)(n-2)x^3}{(n+1)^3(4-\alpha)(3-\alpha)(2-\alpha)}$

(v) $$K_n^{\alpha}(e_4;x) = \begin{cases} \frac{1}{5(n+1)^4} + \frac{6nx}{2(n+1)^4(2-\alpha)} + \frac{30n(n-1)x^2}{(n+1)^4(3-\alpha)(2-\alpha)} + \frac{48n(n-1)(n-2)x^3}{(n+1)^4(4-\alpha)(3-\alpha)(2-\alpha)} \\ + \frac{24n(n-1)(n-2)(n-3)x^4}{(n+1)^4(5-\alpha)(4-\alpha)(3-\alpha)(2-\alpha)}, \end{cases}$$

where $e_r = t^r$ for $r \in \mathbb{N}$.

Proof. Traditionally, the moments of positive linear operators are derived by substituting the test functions $e_r(t) = t^r$ for $r = 0,1,2,3,4$ into the operator, simplifying the resulting expressions, and then evaluating their limits as $n \to \infty$. This direct approach, though effective, can sometimes involve lengthy algebraic manipulations.

In this chapter, we adopt an alternative technique to compute the moments of the fractional Bernstein Kantorovich operators $K_n^{\alpha}(f;x)$. Instead of proceeding through direct substitution, we make use of the Laplace transform. Specifically, we calculate the Laplace transforms of $K_n^{\alpha}(e_r;x)$ for $r = 0,1,2,3,4$ in the limiting case as $n \to \infty$, and subsequently obtain the required moments by applying the inverse Laplace transform.

From Definition 4.1.3, the Laplace transform of Caputo derivative can be written as,

$$\begin{aligned}\mathscr{L}\left[D^{\alpha} f(x)\right] &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \\ &= s^{\alpha} F(s) - s^{\alpha-1} f(0), \ \text{ for } 0 < \alpha \le 1,\end{aligned}$$

where $F(s)$ is the Laplace transform of $f(x)$. Thus,

$$\mathscr{L}\left[D^{\alpha} B_{n+1}(F)\right] = s^{\alpha} \mathscr{L}\left[B_{n+1}\left(F\left(\frac{k}{n+1}\right)\right)(x)\right] - s^{\alpha-1} B_{n+1}\left(F\left(\frac{k}{n+1}\right)\right)(0) \quad (4.5)$$

where $F(t) = \int_0^t f(\xi)d\xi$. Using equation (4.5), we can calculate the moments of our proposed operators.

(i) For $f(\xi) = 1$,

$$\begin{aligned}\mathscr{L}\left[D^{\alpha} B_{n+1}(F)\right] &= s^{\alpha} \mathscr{L}\left[B_{n+1}\left(F\left(\frac{k}{n+1}\right)\right)(x)\right] \\ &= s^{\alpha} \mathscr{L}\left[\sum_{k=0}^{n+1} \binom{n+1}{k} \frac{k}{n+1} x^k (1-x)^{n-k+1}\right] \\ &= \frac{1}{s^{2-\alpha}}.\end{aligned}$$

Applying the inverse Laplace transform and multiplying the expression by $\frac{\Gamma(2-\alpha)}{x^{-\alpha+1}}$ (as in (4.3)), we will arrive at $K_n^{\alpha}(e_0;x) = 1$.

(ii) For $f(\xi) = \xi$,

$$
\begin{aligned}
\mathscr{L}\left[D^{\alpha} B_{n+1}(F)\right] &= s^{\alpha} \mathscr{L}\left[\sum_{k=0}^{n+1} \binom{n+1}{k} \frac{k^2}{2(n+1)^2} x^k (1-x)^{n-k+1}\right] \\
&= s^{\alpha}\left(\frac{n}{s^3(n+1)} + \frac{1}{2s^2(n+1)}\right) \\
&= \frac{n}{s^{3-\alpha}(n+1)} + \frac{1}{2s^{2-\alpha}(n+1)} \\
\Rightarrow \qquad D^{\alpha} B_{n+1}(F) &= \mathscr{L}^{-1}\left[\frac{n}{s^{3-\alpha}(n+1)} + \frac{1}{2s^{2-\alpha}(n+1)}\right] \\
&= \frac{x^{1-\alpha}}{2(n+1)\Gamma(2-\alpha)} + \frac{n x^{2-\alpha}}{(n+1)\Gamma(3-\alpha)}.
\end{aligned}
$$

Again, multiplying the expression by $\frac{\Gamma(2-\alpha)}{x^{-\alpha+1}}$, we get the second moment as

$$
K_n^{\alpha}(e_1; x) = \frac{1}{2(n+1)} + \frac{nx}{(n+1)(2-\alpha)}.
$$

(iii) For $f(\xi) = \xi^2$,

$$
\begin{aligned}
\mathscr{L}\left[D^{\alpha} B_{n+1}(F)\right] &= s^{\alpha} \mathscr{L}\left[\sum_{k=0}^{n+1} \binom{n+1}{k} \frac{k^3}{3(n+1)^3} x^k (1-x)^{n-k+1}\right] \\
&= s^{\alpha}\left(\frac{2n(n-1)}{s^4(n+1)^2} + \frac{2n}{s^3(n+1)^2} + \frac{1}{3s^2(n+1)^2}\right) \\
&= \frac{2n(n-1)}{s^{4-\alpha}(n+1)^2} + \frac{2n}{s^{3-\alpha}(n+1)^2} + \frac{1}{3s^{2-\alpha}(n+1)^2}.
\end{aligned}
$$

Thus, the third moment is given by

$$
K_n^{\alpha}(e_2; x) = \frac{1}{3(n+1)^2} + \frac{2nx}{(n+1)^2(2-\alpha)} + \frac{2n(n-1)x^2}{(n+1)^2(3-\alpha)(2-\alpha)}.
$$

In a similar manner, we can find the moments at e_3 and e_4. □

Using Theorem 4.2.2, we can conclude that as $n \to \infty$ and $\alpha \to 1$, $K_n^{\alpha}(e_r; x) \to e_r$ where $e_r = t^r$ for $r = 0, 1, 2$. From the Bohman-Korovkin theorem [35; 132], we can hence state that for a real valued continuous function f, the fractional Bernstein Kantorovich operators $K_n^{\alpha}(f; x)$ converges to $f(x)$ uniformly. That is,

$$
\lim_{n\to\infty} K_n^{\alpha} f = f, \qquad \text{for } \alpha \to 1.
$$

Lemma 4.2.3 *For $0 < \alpha \leq 1$, the first and second central moments of the fractional Bernstein Kantorovich operators defined in equation* (4.3)*, and the asymptotic behaviour of higher-order moments as $n \to \infty$ and $\alpha \to 1$ is given by:*

(i) $K_n^{\alpha}((t-x);x) = \frac{1}{2(n+1)} + x\frac{\alpha(n+1)-(n+2)}{(n+1)(2-\alpha)}$

(ii) $K_n^{\alpha}((t-x)^2;x) = \frac{1}{3(n+1)^2} + x\frac{\alpha(n+1)-2}{(n+1)^2(2-\alpha)} + x^2\frac{\alpha^2(n+1)^2-\alpha(n+1)(3n+5)+2(n+1)^2+4}{(n+1)^2(3-\alpha)(2-\alpha)}$

(iii) $\lim\limits_{n\to\infty}\lim\limits_{\alpha\to 1} nK_n^{\alpha}((t-x)^4;x) = 0$

(iv) $\lim\limits_{n\to\infty}\lim\limits_{\alpha\to 1} n^2K_n^{\alpha}((t-x)^4;x) = 3x^2(1-x)^2.$

Having obtained the central moments and asymptotic properties of the operators, it is also important to establish uniform bounds that will be useful in the study of convergence. The following lemma provides such an estimate for $K_n^{\alpha}(f;x)$ in terms of the supremum norm of f.

Lemma 4.2.4 *Let f be a continuous function defined on $[0,1]$. For $\|f\| = \sup\limits_{x\in[0,1]} |f(x)|$ and $0<\alpha<1$,*

$$|K_n^{\alpha}(f;x)| \le \frac{x^{-\alpha+1}\|f\|}{\Gamma(2-\alpha)}.$$

Proof. Using $F(x) = \int\limits_0^x f(t)dt$ and definition of the Bernstein operators,

$$\begin{aligned}
\left|B'_{n+1}(F)\right| &\le (n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left|F\left(\frac{k+1}{n+1}\right)-F\left(\frac{k}{n+1}\right)\right| \\
&\le (n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int\limits_{k/(n+1)}^{(k+1)/(n+1)}|f(t)|\,dt \\
&\le \|f\|(n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int\limits_{k/(n+1)}^{(k+1)/(n+1)}1dt \\
&= \|f\|.
\end{aligned}$$

Now,

$$\begin{aligned}
|K_n^{\alpha}(f;x)| &= |D^{\alpha}(B_{n+1}(F))| \\
&\le \frac{1}{\Gamma(1-\alpha)}\int\limits_0^x (x-t)^{-\alpha}\left|B'_{n+1}(F)\right|dt \\
&\le \frac{\|f\|}{\Gamma(1-\alpha)}\int\limits_0^x (x-t)^{-\alpha}dt \\
&= \frac{x^{-\alpha+1}\|f\|}{\Gamma(2-\alpha)}.
\end{aligned}$$

□

Theorem 4.2.5 *Let $f, D^{\alpha}f$ and $D^{2\alpha}f$ belong to the class of continuous functions in* $[0,1]$. *Then for* $0<\alpha\leq 1$, *the following inequality holds:*

$$|K_n^{\alpha}(f;x)| \leq |f(x)| + \frac{x^{-\alpha+1}(1-x)^{\alpha}}{\Gamma(2-\alpha)}\left|\frac{(D^{\alpha}f)(x)}{\Gamma(\alpha+1)} + \frac{(1-x)^{\alpha}\left(D^{2\alpha}f\right)(\xi)}{\Gamma(2\alpha+1)}\right|,$$

where $x\leq\xi\leq 1$, $\forall x\in[0,1]$.

Proof. Using the generalized Taylor's formula from Theorem 4.1.4,

$$\begin{aligned} f(t) = f(x) &+ \frac{(t-x)^{\alpha}}{\Gamma(\alpha+1)}(D^{\alpha}f)(x) + \frac{(t-x)^{2\alpha}}{\Gamma(2\alpha+1)}\left(D^{2\alpha}f\right)(x) + \dots \\ &+ \frac{(t-x)^{m\alpha}}{\Gamma(m\alpha+1)}(D^{m\alpha}f)(x) + \frac{(t-x)^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)}\left(D^{(m+1)\alpha}f\right)(\xi), \end{aligned}$$

where $x\leq\xi\leq t$, $\forall t\in[0,1]$. Applying the operator K_n^{α} on both sides,

$$\begin{aligned} K_n^{\alpha}(f(t);x) &= f(x)K_n^{\alpha}(1;x) + \frac{(D^{\alpha}f)(x)}{\Gamma(\alpha+1)}K_n^{\alpha}\left((t-x)^{\alpha};x\right) + \frac{\left(D^{2\alpha}f\right)(x)}{\Gamma(2\alpha+1)}K_n^{\alpha}\left((t-x)^{2\alpha};x\right) \\ &+ \dots + \frac{(D^{m\alpha}f)(x)}{\Gamma(m\alpha+1)}K_n^{\alpha}\left((t-x)^{m\alpha};x\right) + \frac{(t-x)^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)}\left(D^{(m+1)\alpha}f\right)(\xi) \end{aligned} \quad (4.6)$$

Let $\mu(t)=(t-x)^{m\alpha}$, define

$$\|\mu\| = \sup_{t\in[0,1]}|t-x|^{m\alpha}.$$

Consider, $\mu'(t)=m\alpha(t-x)^{m\alpha-1}=0$. This implies that the critical point of $\mu(t)$ is at $t=x$. Now,
for $t>x$, $m\alpha(t-x)^{m\alpha-1}>0$, and
for $t<x$, $\begin{cases} m\alpha(t-x)^{m\alpha-1}>0, \text{ if } m\alpha-1 \text{ is even} \\ m\alpha(t-x)^{m\alpha-1}<0, \text{ if } m\alpha-1 \text{ is odd} \end{cases}$
Thus, either $\mu(t)$ is always increasing or has a minimum in $(0,1)$. Hence, the supremum will always be at $t=1$. So, for $f(t)=(t-x)^{m\alpha}$, we have $\|f\|=(1-x)^{m\alpha}$. Thus,

$$\left|K_n^{\alpha}\left((t-x)^{m\alpha}\right)\right| \leq \frac{x^{1-\alpha}(1-x)^{m\alpha}}{\Gamma(2-\alpha)}.$$

Subsequently, from (4.6)

$$|K_n^{\alpha}(f(t);x)| \leq |f(x)| + \frac{x^{1-\alpha}(1-x)^{\alpha}}{\Gamma(2-\alpha)}\left|\frac{(D^{\alpha}f)(x)}{\Gamma(\alpha+1)} + \frac{\left(D^{2\alpha}f\right)(\xi)(1-x)^{\alpha}}{\Gamma(2\alpha+1)}\right|.$$

□

Theorem 4.2.6 *For any $f \in C[0,1]$ and $0 < \alpha < 1$:*

$$|K_n^{\alpha}(f;x) - f(x)| \leq 2\omega\left(f;\, \sqrt{\mu_{n,2}^{\alpha}(x)}\right),$$

where $\omega(f;\delta)$ is the modulus of continuity of f and $\mu_{n,2}^{\alpha}(x) = K_n^{\alpha}\left((t-x)^2;x\right)$.

Proof. Using the definition of the fractional Bernstein Kantorovich operators and its moments, we can write

$$\begin{aligned}
&|K_n^{\alpha}(f;x) - f(x)| \\
&\leq (n+1)\Gamma(2-\alpha)\sum_{k=0}^{n}\binom{n}{k}x^k\frac{\Gamma(k+1)}{\Gamma(k-\alpha+2)}{}_2F_1\left(k+1,-n+k;k-\alpha+2;x\right)\int_{k/(n+1)}^{(k+1)/(n+1)}|f(t)-f(x)|\,dt.
\end{aligned}$$

Now, $|f(t) - f(x)| \leq \omega(f;|t-x|)$

$$\leq \left(1 + \frac{1}{\delta}|t-x|\right)\omega(f;\delta).$$

From the properties of modulus of continuity and Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
&|K_n^{\alpha}(f;x) - f(x)| \\
&\leq (n+1)\Gamma(2-\alpha)\sum_{k=0}^{n}\binom{n}{k}x^k\frac{\Gamma(k+1)}{\Gamma(k-\alpha+2)}{}_2F_1\left(k+1,-n+k;k-\alpha+2;x\right)\int_{k/(n+1)}^{(k+1)/(n+1)}\left(1+\frac{1}{\delta}|t-x|\right)\omega(f;\delta)\,dt \\
&= \left[\frac{n+1}{\delta}\sum_{k=0}^{n}B_{n,k}(x)\int_{k/(n+1)}^{(k+1)/(n+1)}|t-x|\,dt + 1\right]\omega(f;\delta) \\
&\leq \left[\frac{K_n^{\alpha}\left((t-x)^2;x\right)^{1/2}}{\delta} + 1\right]\omega(f;\delta) \\
&= \left[\frac{\sqrt{\mu_{n,2}^{\alpha}(x)}}{\delta} + 1\right]\omega(f;\delta).
\end{aligned}$$

Choosing $\delta = \sqrt{\mu_{n,2}^{\alpha}(x)}$ proves our result. □

Theorem 4.2.7 *Let $f \in C[0,1]$. If $f''(x)$ exists at a point $x \in [0,1]$. Then,*

$$\lim_{n\to\infty} 2n\left[K_n^{\alpha}(f;x) - f(x)\right] = (1-2x)f'(x) + x(1-x)f''(x).$$

Proof. Let $\mu_{n,m}^{\alpha}(x) = K_n^{\alpha}((t-x)^m;x)$. By Taylor's expansion,

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2}f''(x) + (t-x)^2\varepsilon(t),$$

where $\lim_{t \to x} \varepsilon(t) = 0$. Thus,

$$\lim_{n\to\infty}\lim_{\alpha\to 1} n\left[K_n^{\alpha}(f;x) - f(x)\right] = \lim_{n\to\infty}\lim_{\alpha\to 1}\left(n\mu_{n,1}^{\alpha}(x)f'(x) + \frac{n}{2}\mu_{n,2}^{\alpha}(x)f''(x)\right.$$
$$\left. + nK_n^{\alpha}\left((t-x)^2\varepsilon(t);x\right)\right).$$

After mathematical simplifications, we can deduce $\lim_{n\to\infty}\lim_{\alpha\to 1} n\mu_{n,2}^{\alpha}(x) = x(1-x)$. Moreover, since $\lim_{t\to x}\varepsilon(t) = 0$, therefore $\lim_{n\to\infty}\lim_{\alpha\to 1} nK_n^{\alpha}\left((t-x)^2\varepsilon(t);x\right) = 0$. Thus,

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\alpha\to 1} n\left[K_n^{\alpha}(f;x) - f(x)\right] \\
&\quad= \lim_{n\to\infty}\lim_{\alpha\to 1}\left(n\mu_{n,1}^{\alpha}(x)f'(x) + \frac{n}{2}\mu_{n,2}^{\alpha}(x)f''(x)\right) \\
&\quad= \lim_{n\to\infty}\left(f'(x)\frac{n(1-2x)}{2(n+1)} + \frac{f''(x)}{2}\frac{n+3xn(n-1)-3x^2n(n-1)}{3(n+1)^2}\right) \\
&\quad= f'(x)\frac{1-2x}{2} + f''(x)\frac{x(1-x)}{2}.
\end{aligned}$$

□

From Theorem 4.2.7 we can say that if $f'(x)$ and $f''(x)$ are both not zero simultaneously on $[0,1]$, then the order of convergence of $K_n^{\alpha}(f;x)$ to $f(x)$ is exactly $O\left(n^{-1}\right)$. Using the Voronoskaya theorem on classical Bernstein operators also gives the same order of convergence [213]. So, in a way, we can say that the fractional Bernstein Kantorovich operators (4.3), including a parameter α (which can be thought of as a fractional-order parameter possessing memory or heredity properties), approximates any continuous function defined on the interval $[0,1]$ with an order of convergence as good as that of the classical Bernstein operators, which do not contain any such parameter.

4.2.2 Graphical Representation of $K_n^{\alpha}(f;x)$

We approximate the function $f(x) = (1-2x)\sin\left(2\pi x - \frac{1}{2}\right) + \frac{1}{3}\cos(2\pi x)$ on the interval $[0,1]$ using $K_n^{\alpha}(f;x)$ (from equation (4.3)) whose value depends on the values of n and α. As shown in Figure 2.8, varying n while keeping α fixed at 0.8, the approximation improves as n increases. Furthermore, Figure 2.9 shows the convergence of $K_n^{\alpha}(f;x)$ to the function $f(x) = x\cos(3\pi x)$ with $n = 50$ and varying values of α. The figure indicates that lower values of α lead to slower convergence rates. This highlights the importance of selecting an appropriate value of α to achieve efficient and accurate approximations.

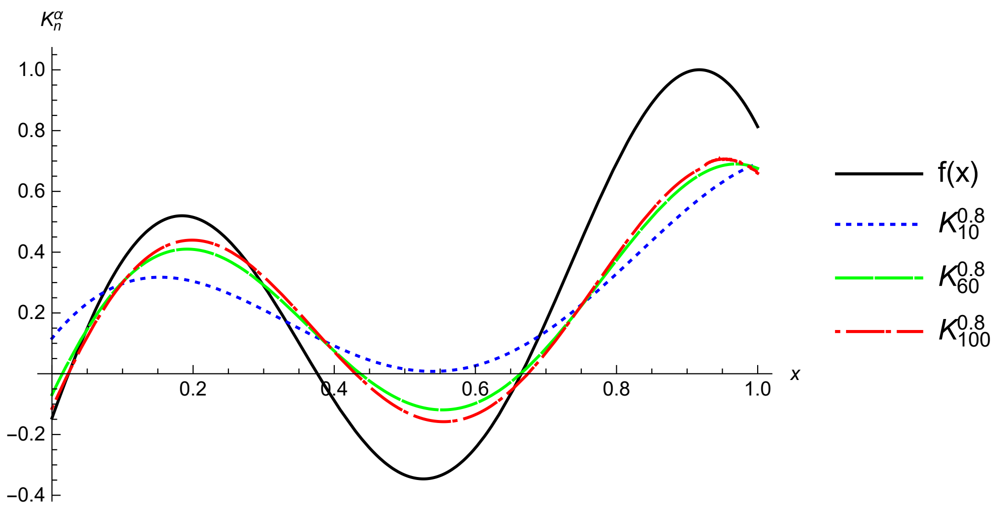


Figure 4.6: Approximation Process of $K_n^\alpha\left((1-2t)\sin\left(2\pi t-\frac{1}{2}\right)+\frac{1}{3}\cos\left(2\pi t\right);x\right)$ for $\alpha=0.8$ and different values of n.

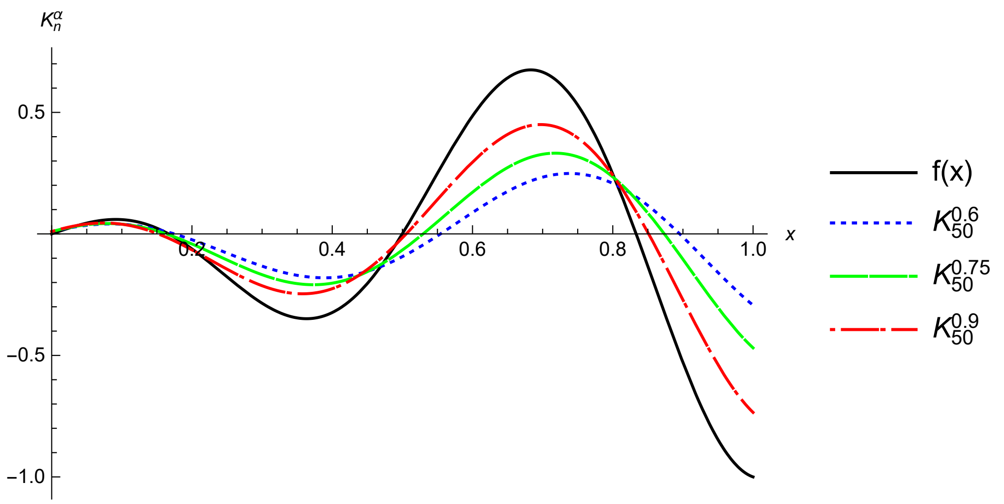


Figure 4.7: Approximation Process of $K_n^\alpha\left(t\cos(3\pi t);x\right)$ for $n=50$ and different values of α.

4.3 Solving Fractional Differential Equations

One of the distinctive features of fractional derivatives is their inherent memory effect. This suggests that the future state of a system described by fractional derivatives depends not only on its current state but also on its entire history. This property makes fractional derivatives particularly suitable for modeling processes with memory and hereditary characteristics, such as models involving viscoelastic materials, electrical circuits, and certain biological systems [55; 77; 144; 180; 188; 191]. Thus, extending the concept of integer-order differential equations to fractional order enhances their descriptive power by incorporating memory effects, long-range interactions, and anomalous behaviours.

One method for solving fractional differential equations (FDEs) is the FDE12 algorithm, introduced by R. Garrappa [88]. However, in most cases these solutions are not in a form which can be easily utilized for further analysis. Therefore, we aim to obtain a solution in a form, such as polynomials, that can be interpreted and used for subsequent analysis. This section is dedicated to finding an approximate polynomial solution to linear FDEs, utilizing the basis of fractional Bernstein-Kantorovich operators as described in equation (4.4).

4.3.1 Mathematical Framework

Consider the linear Fractional Differential equation of the form,

$$D^{\alpha}y(x)+P(x)y(x)=Q(x), \tag{4.7}$$

where D^{α} is the Caputo fractional derivative of order α.

Step 1: Take the fractional integral on both sides of equation (4.7), we get,

$$y(x)-y(0)+I^{\alpha}\left[P(x)y(x)\right]=I^{\alpha}\left[Q(x)\right]. \tag{4.8}$$

Step 2: To determine the approximate solution of (4.8), we use our fractional basis polynomial (4.4) on $[0,1]$ as

$$y_n(x)=\sum_{k=0}^{n}c_kB_{n,k}(x), \quad \text{where } c_k \text{ are unknown constants.} \tag{4.9}$$

Step 3: Substituting equation (4.9) into equation (4.8), we obtain,

$$\begin{aligned} &\sum c_kB_{n,k}(x)+I^{\alpha}\left[P(x)\sum c_kB_{n,k}(x)\right]=y(0)+I^{\alpha}\left[Q(x)\right]\\ \Rightarrow\quad &\sum c_k\left(B_{n,k}(x)+I^{\alpha}\left[P(x)B_{n,k}(x)\right]\right)=y(0)+I^{\alpha}\left[Q(x)\right]. \end{aligned} \tag{4.10}$$

Step 4: Put $x=x_j, j=0,1,\dots,n$ into equation (4.10), where $x_j{}'s$ are being chosen as suitable distinct points in $(0,1)$. This gives us a linear system

$$\sum_{k=0}^{n}a_{kj}^{T}c_k^{T}=f_j^{T}, \;\; j=0,1,...n, \tag{4.11}$$

where $a_{kj} = B_{n,k}(x_j) + I^{\alpha}\left[P(x_j)B_{n,k}(x_j)\right]$ and $f_j = y(0) + I^{\alpha}\left[Q(x_j)\right]$.

Step 5: The linear system (4.11) can easily be solved by standard methods for the unknown constants $c_k{}'s$. These c_k, for $k = 0, 1, \ldots n$, are then used in equation (4.9) to obtain the unknown function $y(x)$ approximately.

Using the above approach, we claim that a higher value of n leads to a better solution. Following are some numerical illustrations for better understanding.

4.3.2 Numerical Illustrations

Example 4.3.1 *Consider the fractional differential equation:*

$$D^{0.4}y(x) = y(x) + 1, \quad y(0) = 0. \tag{4.12}$$

Taking fractional integral on both sides, we get

$$y(x) = I^{0.4}\left[y(x)\right] + I^{0.4}\left[1\right]. \tag{4.13}$$

In order to get an approximate solution of the FDE (4.12), substitute $y_2(x) = \sum_{k=0}^{2} c_k B_{2,k}(x)$ in (4.13), we get

$$\sum_{k=0}^{2} c_k B_{2,k}(x) = I^{0.4}\left[\sum_{k=0}^{2} c_k B_{2,k}(x)\right] + I^{0.4}\left[1\right]$$

$$\Rightarrow \left.\begin{array}{l} \left(1 - 1.25x + 0.480769x^2\right)c_0 \\ +1.25x\left(1 - 0.769231x\right)c_1 \\ +0.480769x^2 c_2 \end{array}\right\} = \begin{array}{l} 0.450824\left(2.5x^{0.4} - 2.23214x^{1.4} + 0.71543x^{2.4}\right)c_0 \\ +0.450824\left(2.23214x^{1.4} - 1.43086x^{2.4}\right)c_1 \\ +0.322533x^{2.4}c_2 + \frac{x^{0.4}}{\Gamma(1.4)}. \end{array}$$

Taking $x = 0.1,\ 0.2$ and 0.3, we get the following system of linear equations

$$\begin{aligned} 0.469895c_0 + 0.0778911c_1 + 0.00352367c_2 &= 0.448691 \\ 0.276126c_0 + 0.119369c_1 + 0.0124536c_2 &= 0.592051 \\ 0.140545c_0 + 0.13782c_1 + 0.0253357c_2 &= 0.696299, \end{aligned}$$

and solving this system we get $c_0 = 0.315191,\ c_1 = 3.57448$ and $c_2 = 6.29011$. Hence, an approximate solution to the FDE $D^{0.4}y(x) = y(x) + 1$ is

$$\begin{aligned} y_2(x) &= 0.315191B_{2,0}(x) + 3.57448B_{2,1}(x) + 6.29011B_{2,2}(x) \\ &= 0.315191 + 4.07411x - 0.261377x^2. \end{aligned}$$

Let us increase the value of n and put $y_4(x) = \sum_{k=0}^{4} c_k B_{4,k}(x)$ in equation (4.13), to verify our claim. For $y_4(x) = \sum_{k=0}^{4} c_k B_{4,k}(x)$. The solution takes the form,

$$\sum_{k=0}^{4} c_k \left(B_{4,k}(x) - I^{0.4}\left[B_{4,k}(x)\right]\right) = \frac{x^{0.4}}{\Gamma(1.4)}.$$

Taking $x = 0.1, 0.2, 0.3, 0.4$ and 0.5 leads us to the following system of linear equations with $c_k{}'s$ as variables.

$$\begin{aligned}
0.401377c_0 + 0.131158c_1 + 0.0176315c_2 + 0.00111c_3 + 0.0000273748c_4 &= 0.448691\\
0.185683c_0 + 0.164012c_1 + 0.0506189c_2 + 0.00723459c_3 + 0.000399842c_4 &= 0.592051\\
0.0531708c_0 + 0.147756c_1 + 0.0809801c_2 + 0.0199106c_3 + 0.00188374c_4 &= 0.696299\\
-0.0311758c_0 + 0.107229c_1 + 0.099025c_2 + 0.038114c_3 + 0.00559195c_4 &= 0.781216\\
-0.0845752c_0 + 0.0581489c_1 + 0.100501c_2 + 0.058879c_3 + 0.012894c_4 &= 0.854152
\end{aligned}$$

Solving the system gives us the following solution:

$$c_0 = 0.271432,\ c_1 = 2.1806,\ c_2 = 2.60501,\ c_3 = 6.83978 \text{ and } c_4 = 6.65295.$$

Thus, another approximate solution to the FDE is,

$$\begin{aligned}
y_4(x) &= \sum_{k=0}^{4} c_k B_{4,k}(x)\\
&= 0.271432 + 4.77291x - 4.28293x^2 + 8.48573x^3 - 4.71259x^4.
\end{aligned}$$

Now, proceeding in a similar manner for $n = 6$. Let $y_6(x) = \sum_{k=0}^{6} c_k B_{6,k}(x)$, then we get

$$\sum_{k=0}^{6} c_k \left(B_{6,k}(x) - I^{0.4}\left[B_{6,k}(x)\right]\right) = \frac{x^{0.4}}{\Gamma(1.4)}$$

Taking $x = 0.1,\ 0.2,\ 0.3,\ 0,4,\ 0.5,\ 0.6$ and 0.7, we get the following system of equations:

$$\begin{aligned}
0.34362c_0 + 0.16591c_1 + 0.0367989c_2 + 0.00462706c_3 + 0.000339269c_4 + 0.0000135801c_5 + 2.30191\times 10^{-7}c_6 &= 0.448691\\
0.123324c_0 + 0.169862c_1 + 0.0860766c_2 + 0.0243152c_3 + 0.00399795c_4 + 0.000358967c_5 + 0.0000136569c_6 &= 0.592051\\
0.0062986c_0 + 0.118877c_1 + 0.108698c_2 + 0.0526613c_3 + 0.0147603c_4 + 0.00225921c_5 + 0.000146654c_6 &= 0.696299\\
-0.0571595c_0 + 0.0579537c_1 + 0.099986c_2 + 0.0763351c_3 + 0.0330738c_4 + 0.0078114c_5 + 0.000783232c_6 &= 0.781216\\
-0.0912533c_0 + 0.00609255c_1 + 0.0697091c_2 + 0.084545c_3 + 0.0547263c_4 + 0.0191739c_5 + 0.00285423c_6 &= 0.854152\\
-0.109266c_0 - 0.0.312824c_1 + 0.0312848c_2 + 0.0733539c_3 + 0.0718064c_4 + 0.0371624c_5 + 0.00816932c_6 &= 0.918772\\
-0.118611c_0 - 0.0553729c_1 - 0.00421616c_2 + 0.0464152c_3 + 0.0753874c_4 + 0.0593964c_5 + 0.0197942c_6 &= 0.977207.
\end{aligned}$$

Solving the above system we get,

$$c_0 = 0.252074,\ c_1 = 1.62695,\ c_2 = 1.96943,\ c_3 = 3.91579,\ c_4 = 4.28293,$$

$$c_5 = 7.33919 \text{ and } c_6 = 8.33301.$$

And thus, another approximate solution to the FDE (4.12) is,

$$\begin{aligned} y_6(x) &= \sum_{k=0}^{6} c_k B_{6,k}(x) \\ &= 0.252074 + 5.1558x - 7.4452x^2 + 21.1241x^3 - 30.4107x^4 + 24.7679x^5 \\ &\quad - 8.41046x^6. \end{aligned}$$ □

We do not claim that our method is superior or inferior to the FDE12 algorithm. Our approach simply provides a polynomial approximation to the solution of linear FDEs. The comparison of these approximate solutions with the solution obtained using the FDE12 algorithm can be made from Figure 4.8 and Table 4.1. As the value of n increases, the curve of approximate polynomial solution gets closer to the FDE12 algorithm solution and the absolute error values decreases.

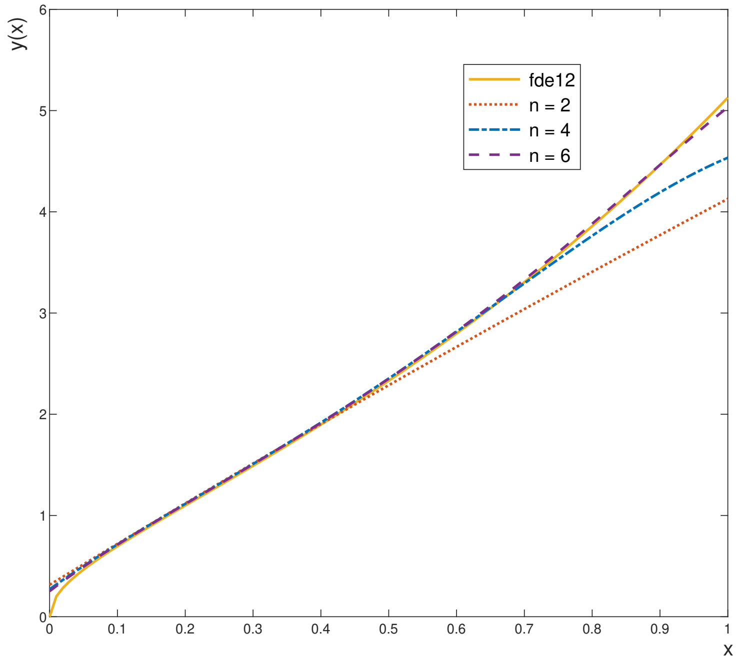


Figure 4.8: Comparison between the approximate solutions of the FDE (4.12) for $n = 2, 4$ and 6 with the solution obtained using FDE12.

x	$n=2$	$n=4$	$n=6$
0.1	0.744077	0.737997	0.735612
0.2	0.0215585	0.0170435	0.0151478
0.3	0.0229009	0.0177854	0.0158225
0.4	0.00501588	0.0197733	0.0177609
0.5	0.0450967	0.0213357	0.0190957
0.6	0.132437	0.0174911	0.0224064
0.7	0.266005	0.0120523	0.0251908
0.8	0.4508	0.0948966	0.0242811
0.9	0.693823	0.271954	0.000204418

Table 4.1: The absolute error values of the approximate solutions of (4.12) using fractional operators for $n = 2,4$ and 6 with the solution obtained using FDE12.

Example 4.3.2 *Consider the FDE:*

$$D^{0.7}y(x) = xy(x) + \cos x, \quad y(0) = 0. \tag{4.14}$$

Let $y_5(x) = \sum\limits_{k=0}^{5} c_k B_{5,k}(x)$ and for easier calculations let us consider the first 4 terms in the expansion of $\cos x$. Thus,

$$\begin{aligned}
&y(x) = I^{0.7}[xy(x)] + I^{0.7}[\cos x] \\
\Rightarrow &\sum_{k=0}^{5} c_k \left(B_{5,k}(x) - I^{0.7}[xB_{5,k}(x)]\right) = I^{0.7}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right] \\
&= \frac{x^{0.7}}{\Gamma(1.7)} - \frac{\Gamma(3)x^{2.7}}{2\Gamma(3.7)} + \frac{\Gamma(5)x^{4.7}}{24\Gamma(5.7)} - \frac{\Gamma(7)x^{6.7}}{720\Gamma(7.7)} \\
&= \frac{x^{0.7}}{\Gamma(1.7)} - \frac{x^{2.7}}{\Gamma(3.7)} + \frac{x^{4.7}}{\Gamma(5.7)} - \frac{x^{6.7}}{\Gamma(7.7)}.
\end{aligned}$$

Taking $x = 0.3, 0.4, 0.5, 0.6, 0.7$ and 0.8, we get a system of linear equation as follows:

$$0.26941c_0 + 0.32839c_1 + 0.220046c_2 + 0.081497c_3 + 0.015784c_4 + 0.001253c_5 = 0.464556$$

$$0.1639c_0 + 0.25797c_1 + 0.25265c_2 + 0.14179c_3 + 0.042152c_4 + 0.005165c_5 = 0.55948$$

$$0.09657c_0 + 0.175882c_1 + 0.23648c_2 + 0.192207c_3 + 0.084266c_4 + 0.015342c_5 = 0.641094$$

$$0.05401c_0 + 0.10353c_1 + 0.183601c_2 + 0.213174c_3 + 0.137053c_4 + 0.03697c_5 = 0.710558$$

$$0.02659c_0 + 0.04959c_1 + 0.11430c_2 + 0.19277c_3 + 0.1867c_4 + 0.076975c_5 = 0.768405$$

$$0.007884c_0 + 0.01394c_1 + 0.0493c_2 + 0.13253c_3 + 0.209567c_4 + 0.143759c_5 = 0.81488$$

The above system of equations gives us the following values of the unknown constants c_k:

$c_0 = 0.0713591,\ c_1 = 0.506416,\ c_2 = 0.740912,\ c_3 = 1.10491,\ c_4 = 1.46891$ and $c_5 = 2.2013$.

And hence, an approximation solution to the FDE problem (4.14) is

$$\begin{aligned}
y_5(x) &= 0.0713591B_{5,0}(x) + 0.506416B_{5,1}(x) + 0.740912B_{5,2}(x) \\
&\quad + 1.10491B_{5,3}(x) + 1.46891B_{5,4}(x) + 2.2013B_{5,5}(x) \\
&= 0.0713591\big(1 - 3.84615x + 6.68896x^2 - 6.08088x^3 + 2.82831x^4 - 0.533644x^5\big) \\
&\quad + 0.506416\big(3.84615x(1 - 3.47826x + 4.74308x^2 - 2.94145x^3 + 0.693737x^4)\big) \\
&\quad + 0.740912\big(6.68896x^2(1 - 2.72727x + 2.537x^2 - 0.797798x^3)\big) \\
&\quad + 1.10491\big(6.08088x^3(1 - 1.86047x + 0.877578x^2)\big) \\
&\quad + 1.46891\big(2.82831(1 - 0.943396x)x^4\big) + 2.2013\big(0.533644x^5\big) \\
&= 0.0713591 + 1.6733x - 1.34154x^2 + 2.00705x^3 - 1.29976x^4 \\
&\quad + 0.510923x^5. \qquad \square
\end{aligned}$$

Figure 4.9 presents the curve of the fifth-degree polynomial solution $y(x)$ to equation (4.14), alongside the solution obtained using the FDE12 algorithm.

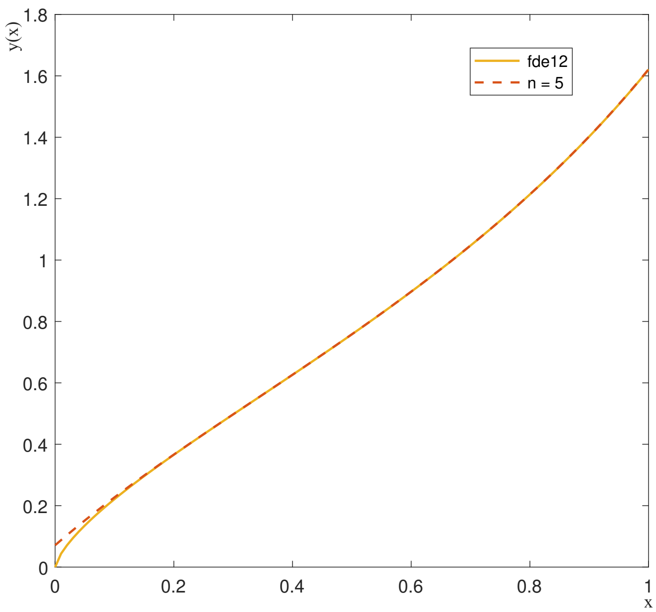


Figure 4.9: Comparison between the approximate solutions of the FDE (4.14) using fractional operators for $n = 5$ with the solution obtained using FDE12.

We can also evaluate an approximate solution of a linear fractional differential equation having fractional derivatives of more than 1 order. Let us consider a general example of such an equation.

Example 4.3.3 *Consider a linear fractional differential equation*

$$D^{\mu}y(x) - D^{\rho}y(x) = e^{x} + 2x; \quad y(0) = 0; \; \mu,\rho \in [0,1] \textit{ with } \mu > \rho. \tag{4.15}$$

Let $y_3(x) = \sum_{k=0}^{3} c_k B_{3,k}(x)$. For easier calculations and an approximate solution, we take the first 5 terms in the expansion of e^x. Then, an approximate solution of (4.15) is given by

$$y(x) - I^{\mu-\rho}[y(x)] = I^{\mu}[e^{x} + 2x]$$

$$\Rightarrow \sum_{k=0}^{3} c_k \left(B_{3,k}(x) - I^{\mu-\rho}\left[B_{3,k}(x)\right]\right) = I^{\mu}\left[1 + 3x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right].$$

Calculating the fractional integral of the required functions, we get

$$\left.\begin{array}{l} c_0\left[\frac{6x^3}{(\alpha-4)(\alpha-3)(\alpha-2)}\left(1-\frac{6x^{\mu-\rho}}{\Gamma(\mu-\rho+4)}\right)\right. \\ +\frac{6x^2}{(\alpha-3)(\alpha-2)}\left(1-\frac{2x^{\mu-\rho}}{\Gamma(\mu-\rho+3)}\right) \\ \left.+\frac{3x}{(\alpha-2)}\left(1-\frac{x^{\mu-\rho}}{\Gamma(\mu-\rho+2)}\right)+\left(1-\frac{x^{\mu-\rho}}{\Gamma(\mu-\rho+1)}\right)\right] \\ -3c_1\left[\frac{6x^3}{(\alpha-4)(\alpha-3)(\alpha-2)}\left(1-\frac{6x^{\mu-\rho}}{\Gamma(\mu-\rho+4)}\right)\right. \\ \left.+\frac{4x^2}{(\alpha-3)(\alpha-2)}\left(1-\frac{2x^{\mu-\rho}}{\Gamma(\mu-\rho+3)}\right)+\frac{x}{(\alpha-2)}\left(1-\frac{x^{\mu-\rho}}{\Gamma(\mu-\rho+2)}\right)\right] \\ +6c_2\left[\frac{3x^3}{(\alpha-4)(\alpha-3)(\alpha-2)}\left(1-\frac{6x^{\mu-\rho}}{\Gamma(\mu-\rho+4)}\right)\right. \\ \left.+\frac{x^2}{(\alpha-3)(\alpha-2)}\left(1-\frac{2x^{\mu-\rho}}{\Gamma(\mu-\rho+3)}\right)\right] \\ -6c_3\left[\frac{x^3}{(\alpha-4)(\alpha-3)(\alpha-2)}\left(1-\frac{6x^{\mu-\rho}}{\Gamma(\mu-\rho+4)}\right)\right] \end{array}\right\} = \begin{array}{l}\left[\frac{x^{\mu}}{\Gamma(\mu+1)}+\frac{3x^{\mu+1}}{\Gamma(\mu+2)}+\frac{x^{\mu+2}}{\Gamma(\mu+3)}\right. \\ \left.+\frac{x^{\mu+3}}{\Gamma(\mu+4)}+\frac{x^{\mu+4}}{\Gamma(\mu+5)}\right]\end{array}$$

$$\Rightarrow \left.\begin{array}{l} (c_0-3c_1+3c_2-c_3)\frac{6x^3}{(\alpha-4)(\alpha-3)(\alpha-2)}\left(1-\frac{6x^{\mu-\rho}}{\Gamma(\mu-\rho+4)}\right) \\ +(c_0-2c_1+c_2)\frac{6x^2}{(\alpha-3)(\alpha-2)}\left(1-\frac{2x^{\mu-\rho}}{\Gamma(\mu-\rho+3)}\right) \\ +(c_0-c_1)\frac{3x}{(\alpha-2)}\left(1-\frac{x^{\mu-\rho}}{\Gamma(\mu-\rho+2)}\right)+c_0\left(1-\frac{x^{\mu-\rho}}{\Gamma(\mu-\rho+1)}\right) \end{array}\right\} = \begin{array}{l}\frac{x^{\mu}}{\Gamma(\mu+1)}+\frac{3x^{\mu+1}}{\Gamma(\mu+2)}+\frac{x^{\mu+2}}{\Gamma(\mu+3)} \\ +\frac{x^{\mu+3}}{\Gamma(\mu+4)}+\frac{x^{\mu+4}}{\Gamma(\mu+5)}.\end{array}$$

For fixed values of $\alpha,\mu,\rho \in [0,1]$ we take different values of $x \in [0,1]$ and obtain a linear system. Solving the linear system for these constants gives us an approximate solution for the linear FDE.

□

4.3.3 Some real-life examples

Tautochrone problem: A tautochrone curve is the trajectory of an object, sliding under uniform gravity in the absence of friction, such that the time of descend of the object does not depend on its starting point [189]. If we assume that the curve is in the first quadrant of a 2-D plane such that the origin is its lowest point, (x,y) is the initial point of the object and (x^*,y^*) is any point between $(0,0)$ and (x,y), then according to the law of conservation of energy, we have

$$\frac{m}{2}\left(\frac{ds}{dt}\right)^2 = mg\left(y-y^*\right),$$

where m represents the mass of the object, s is the arc length and g represents acceleration due to gravity. After some calculations and taking integral on both sides, we arrive at the integral equation

$$\int_0^y \frac{s'(v)}{\sqrt{y-v}}dv = \sqrt{2g}T.$$

The right-hand side of the equation can be identified as the Caputo derivative operator and the equation can be re-written as

$$D^{0.5}s(y) = \frac{\sqrt{2g}T}{\Gamma(0.5)}.$$

Now this problem has reduced to a fractional differential equation problem whose solution can be approximated using fractional positive linear operators. Since, our fractional positive linear operators (4.3) are defined for the domain $[0,1]$, hence the tautochrone problem can be solved by the basis of fractional Bernstein-Kantorovich operators (4.4), provided the domain is $[0,1]$.

Fractional Relaxation-Oscillation Equation: For $0<\mu\leq 2$, we can illustrate the fractional relaxation-oscillation model as

$$D^{\mu}x(t)+Cx(t)=g(t), \qquad \begin{array}{l} x(0)=a \text{ if } 0<\mu\leq 1 \\ x(0)=b,\ x'(0)=c \text{ if } 1<\mu\leq 2, \end{array}$$

where $t>0$ and C is a positive constant. The model describes relaxation with power law attenuation when $0<\mu\leq 1$, and a damped oscillation with viscoelastic intrinsic damping of the oscillator when $1<\mu\leq 2$ [97]. Applications of this model can be seen in signal processing, electrical heart modeling, predator-prey system etc. [13; 50; 94; 178; 204; 211]. The solution of this FDE can again be approximated using fractional positive linear operators. Further, if the domain is $[0,1]$, it can specifically be solved using the basis of fractional BernsteinKantorovich operators (4.4).

4.4 Solving Fractional Integro-Differential Equations

The study and modeling of many physical phenomena relies heavily on fractional integro-differential equations (FIDEs) [152; 203]. To study more about FDIE, one can refer [10; 118; 155; 160] and references therein. Our method of finding an approximate solution using the basis polynomial of fractional Bernstein-Kantorovich operators can also be employed to compute an approximate solution to fractional integro-differential equations of the following form:

$$D^{\alpha}z(y) = p(y)z(y) + f(y) + \int_0^1 K(y,s)z(s)ds, \quad y \in [0,1].$$

To ensure the reliability and accuracy of this approach, we will compare our approximate solution for the fractional order $\alpha = 1$ with the exact solution of the corresponding integer-order integro-differential equation.

Example 4.4.1 *Consider the fractional order integro-differential equation*

$$D^{\alpha}y(x) = e^{2x} + \int_0^1 xy(t)dt, \quad y(0) = 0. \tag{4.16}$$

For an approximate solution, let $y_2(x) = \sum_{k=0}^{2} c_k B_{2,k}(x)$ and the series expansion of e^{2x} be considered upto 4 terms. Then,

$$y(x) - I^{\alpha}\left[\int_0^1 xy(t)dt\right] = I^{\alpha}\left[e^{2x}\right]$$

$$\Rightarrow \left.\begin{array}{l} c_0B_{2,0}(x) + c_1B_{2,1}(x) + c_2B_{2,2}(x) \\ -I^{\alpha}\left[c_0x\int_0^1 B_{2,0}(t)dt + c_1x\int_0^1 B_{2,1}(t)dt + c_2x\int_0^1 B_{2,2}(t)dt\right] \end{array}\right\} = I^{\alpha}\left[e^{2x}\right]$$

$$\Rightarrow \left.\begin{array}{l} c_0B_{2,0}(x) + c_1B_{2,1}(x) + c_2B_{2,2}(x) - I^{\alpha}\left[\frac{c_0x(3\alpha^2-12\alpha+11)}{3(\alpha-3)(\alpha-2)}\right. \\ \left.+\frac{c_1x(5-3\alpha)}{3(\alpha-3)(\alpha-2)} + \frac{c_2}{3}x(1-\alpha)\beta(1-\alpha,3)\right] \end{array}\right\} = I^{\alpha}\left[1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!}\right]$$

$$\Rightarrow \left.\begin{array}{l} c_0\left(\frac{6-6x+2x^2-5\alpha+2x\alpha+\alpha^2}{(\alpha-3)(\alpha-2)} - I^\alpha\left[\frac{x(3\alpha^2-12\alpha+11)}{3(\alpha-3)(\alpha-2)}\right]\right) \\ +c_1\left(\frac{6-4x-2\alpha}{(\alpha-3)(\alpha-2)} - I^\alpha\left[\frac{x(5-3\alpha)}{3(\alpha-3)(\alpha-2)}\right]\right) \\ +c_2\left(\frac{2}{(\alpha-3)(\alpha-2)} - I^\alpha\left[\frac{x}{3}(1-\alpha)\beta(1-\alpha,3)\right]\right) \end{array}\right\} = \begin{array}{l} \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{2x^{\alpha+1}}{\Gamma(\alpha+2)} \\ +\frac{4x^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{8x^{\alpha+3}}{\Gamma(\alpha+4)}. \end{array}$$

Taking different values of x and solving for the constants c_k, we will get an approximate solution to the FIDE (4.16). □

Example 4.4.2 *Consider the FIDE*

$$D^\alpha y(x) = 2e^x + \sin x + \int_0^1 y(t)dt; \quad y(0) = -1. \tag{4.17}$$

Since larger values of n, give better approximations, let $y_6(x) = \sum_{k=0}^{6} c_k B_{6,k}(x)$. By operating with the fractional integral operator I^α on both sides of (4.17), we get

$$y(x) - y(0) = I^\alpha\left[2e^x + \sin x\right] + I^\alpha\left[\int_0^1 y(t)dt\right]$$

$$\Rightarrow \sum_{k=0}^{6} c_k B_{6,k}(x) - I^\alpha\left[\int_0^1 \sum_{k=0}^{6} c_k B_{6,k}(t)dt\right] = \left\{\begin{array}{l} -1 + I^\alpha\Big[2 + 3x + x^2 + \frac{x^3}{3!} + \frac{2x^4}{4!} \\ \qquad + \frac{3x^5}{5!} + \frac{2x^6}{6!} + \frac{x^7}{7!}\Big] \end{array}\right.$$

$$\Rightarrow \sum_{k=0}^{6} c_k\left(B_{6,k}(x) - I^\alpha\left[\int_0^1 B_{6,k}(t)dt\right]\right) = \left\{\begin{array}{l} -1 + \frac{2x^\alpha}{\Gamma(\alpha+1)} + \frac{3x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{x^{\alpha+3}}{\Gamma(\alpha+4)} \\ + \frac{2x^{\alpha+4}}{\Gamma(\alpha+5)} + \frac{3x^{\alpha+5}}{\Gamma(\alpha+6)} + \frac{2x^{\alpha+6}}{\Gamma(\alpha+7)} + \frac{x^{\alpha+7}}{\Gamma(\alpha+8)}. \end{array}\right. \tag{4.18}$$

Rest of the solution of this problem for a particular value of α is trivial. Particularly, taking $\alpha = 1$ and for different values of x, we get a system of linear equations with unknown constants c_k, where $k = 0, 1, \ldots 6$, as follows:

$$0.517155c_o + 0.340008c_1 + 0.0841293c_2 + 0.000294286c_3 - 0.0130707c_4 - 0.012317c_5 - 0.0142847c_6 = -0.784662$$

$$0.233573c_o + 0.364645c_1 + 0.217189c_2 + 0.0533486c_3 - 0.0132114c_4 - 0.0270354c_5 - 0.0285076c_6 = -0.537261$$

$$0.0747919c_o + 0.259669c_1 + 0.281278c_2 + 0.142363c_3 - 0.0166779c_4 - 0.0326511c_5 - 0.0421281c_6 = -0.255619$$

$$-0.0104869c_o + 0.129481c_1 + 0.253897c_2 + 0.2193371c_3 + 0.0810971c_4 - 0.0202789c_5 - 0.0530469c_6 = 0.0625884$$

$$-0.0558036c_o + 0.0223214c_1 + 0.162946c_2 + 0.241071c_3 + 0.162946c_4 + 0.0223214c_5 - 0.0558036c_6 = 0.41986$$

$$-0.0816183c_o - 0.0488503c_1 + 0.052557c_2 + 0.190766c_3 + 0.225326c_4 + 0.10091c_5 - 0.0390583c_6 = 0.818902$$

$$-0.099271c_o - 0.089794c_1 - 0.040465c_2 + 0.08522c_3 + 0.224135c_4 + 0.202526c_5 + 0.017649c_6 = 1.26266$$

Solving for the constants, we get
$c_o = -1,\ c_1 = -0.468304,\ c_2 = 0.163397,\ c_3 = 0.911752,\ c_4 = 1.79625,\ c_5 = 2.84176$ and $c_6 = 4.08643$. Thus, an approximate solution of the FIDE for $\alpha = 1$ is given by

$$\begin{aligned} y_6(x) &= \sum_{k=0}^{6} c_k B_{6,k}(x) \\ &= -1 + 3.19018x + 1.50006x^2 + 0.333021x^3 + 0.0425638x^4 + 0.0152139x^5 \\ &\quad + 0.00539236x^6. \end{aligned}$$

Since, for $\alpha = 1$, the FIDE reduces to its corresponding integer-order integro-differential equation, given by

$$\frac{d}{dx}y(x) = 2e^x + \sin x + \int_0^1 y(t)dt;\ y(0) = -1, \tag{4.19}$$

so the solution obtained above is, in fact, the approximate solution to equation (4.19).

Now, to check the credibility of this approach let us compare our approximate solution with the exact solution of the integer-order integro-differential equation (4.19), which can be calculated using any mathematical software.

As demonstrated through Figure 4.10 and Table 4.2, the negligible difference between the exact solution of the corresponding integer-order integro-differential e-quation and the approximate solution of (4.17) strongly supports the accuracy and effectiveness of our method for approximating FIDEs.

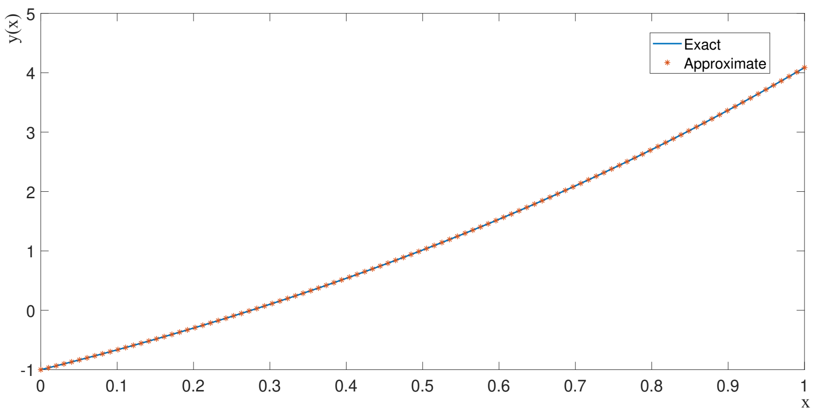


Figure 4.10: The exact and approximate solution of FIDE (4.17) with $\alpha = 1$.

x	Error
0.1	2.38668×10^{-7}
0.2	4.77339×10^{-7}
0.3	7.16118×10^{-7}
0.4	9.56206×10^{-7}
0.5	1.20494×10^{-6}
0.6	1.49274×10^{-6}
0.7	1.91746×10^{-6}
0.8	2.98732×10^{-6}
0.9	6.55793×10^{-6}

Table 4.2: The absolute error values between the exact and approximate solution.

Thus, we go ahead and approximate the solution of FIDE (4.17),

$$\text{for } \alpha = 0.6 \text{ as } y_6(x) = -0.489341 + 16.2561x - 32.6457x^2 + 79.5763x^3 - 117.246x^4 + 95.9771x^5 - 32.9542x^6 \text{ and}$$

$$\text{for } \alpha = 0.3 \text{ as } y_6(x) = 4.28456 + 47.7671x - 147.19x^2 + 367.721x^3 - 558.06x^4 + 463.38x^5 - 160.604x^6.$$

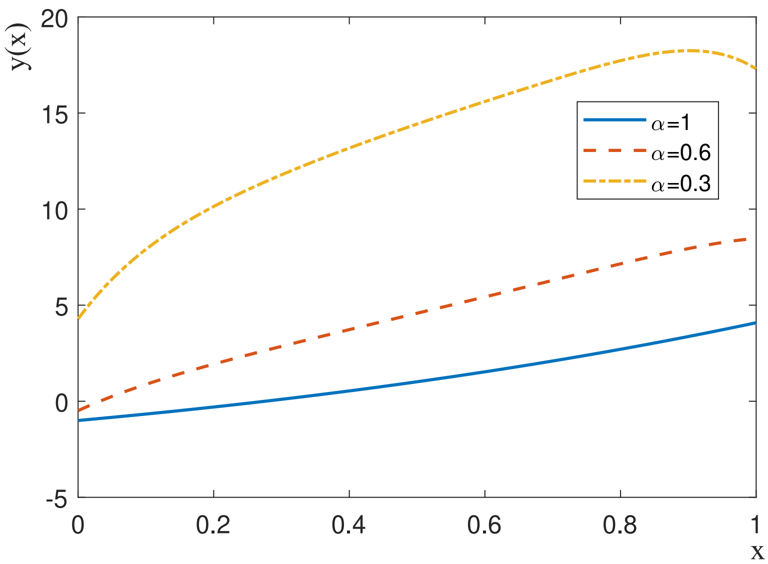


Figure 4.11: The approximate solutions of (4.17) for $\alpha = 1, 0.6$ and 0.3.

4.5 Conclusion

In this chapter, we derived a new class of positive linear operators, namely the fractional positive linear operators. We introduced the fractional Bernstein Kantorovich operators, obtained by differentiating the classical Bernstein operators using Caputo fractional derivative. Further, the moments and central moments for these operators are obtained using the Laplace transform and thus using Bohman-Korovkin theorem, the convergence properties of the fractional positive linear Bernstein Kantorovich operators are shown. We also established a Voronoskaya type asymptotic result which proves that the order of convergence of our fraction operators; incorporating a parameter for 'non-local' or 'hereditary' properties of real-life complex phenomenons; is the same as the order of convergence of the classical Bernstein operators which do not include any such parameter. Therefore, the formulation of various complex real-life phenomena can be more effectively captured using the fractional operators defined in this chapter, as presented in (4.3), along with their corresponding basis functions given in (4.4). The approximation process of these operators is also shown with the hep of graphs.

This chapter also gave a new approach to obtain an approximate solution to linear fractional differential equations (FDE) and fractional integro-differential equations (FIDE) using the basis polynomial of the defined fractional operators. A comparison between our approximate solution of FDE and the solution obtained using the FDE12 method shows that the error in the approximate solution reduces as we increase the value of n. Numerical illustrations are also provided to understand the applications of these fractional operators to solve FDE and FIDE. We also compared our approximate solutions of FIDE for fractional order $\alpha = 1$, with the exact solution of the corresponding first order integro-differential equation. Graphs of the solution of FIDE are also shown for different values of α.

While the proposed method demonstrates promising results for approximating a polynomial solution to FDEs, a limitation of this method is that it is only applicable to linear FDEs. Although fractional derivatives can accommodate multiple orders, as illustrated in Example 4.15, the linearity of the FDE remains a necessary condition for the applicability of this approach. Another limitation of using fractional Bernstein-Kantorovich operators to approximate polynomial solutions of FDEs and FIDEs is that they confine the solutions to the interval $[0,1]$. This constraint arises from the fact that

Bernstein polynomials are inherently defined on this interval. However, this constraint can be addressed by exploring alternative positive linear operators defined over larger domains. For instance, operators such as the Szász-Mirakyan operators and Baskakov operators are defined over the entire positive real line, $\mathbb{R}^+$ [202; 19; 26]. Future work could focus on deriving new fractional positive linear operators based on these or similar operators, which are defined over extended domains. Such advancements would not only overcome the current spatial limitations but also enhance the versatility of the method for solving FDEs and FIDEs in broader contexts.

Chapter 5

Approximation of Fuzzy-Valued Functions using Fuzzy Positive linear operators

This chapter extends the main results of classical approximation theory to fuzzy theory. We begin by defining fuzzy valued functions and then apply the fuzzy Korovkin theorem to approximate them. The study delves into the approximation of various linear positive fuzzy operators, utilizing them to approximate fuzzy-valued functions. Their Voronovskaya type asymptotic result is also proved using Taylor's theorem.

5.1 Introduction

In mathematical language, a fuzzy set is a notion used in fuzzy logic to represent a set where the membership of an element is described by a degree of belonging, ranging from 0 (not a member) to 1 (fully a member), rather than the traditional binary distinction of being either in the set or not. Unlike the classical set theory which focuses on black and white, fuzzy theory focuses on the grey portions. The inception of fuzzy sets dates back to Zadeh's groundbreaking study in 1965 [221]. Sugeno [201], in 1977, introduced a different concept of fuzziness where, for any object x in the universe X, a value $g_x(A) \in [0,1]$ is assigned to each nonfuzzy subset A of X representing the "grade of fuzziness" as a certainty measure of the assertion "x belongs to A". The framework formulated by Sugeno revolves around guessing whether x is a member of A, rather than addressing vagueness as characterized by Zadeh. Dubois and Prade in their study [72] extensively focused on a comprehensive exploration of mathematical concepts within the realm of fuzzy set theory. They thoroughly discussed the introduced mathematical notions, providing an in-depth analysis and presentation of the framework underlying fuzzy set theory. They defined various types of fuzzy sets, set operations, and properties and introduced the extension principle, applying it to mathematics and higher-order fuzzy sets. The findings of Dubois and Prade are considered crucial as they covered the definitions on fuzzy relations and fuzzy functions, emphasizing their extremum, integration, differentiation, fuzzy topology and categories of fuzzy objects.

From here on many researchers have worked upon this idea. Subsequently, numerous authors extended and explored the notions of fuzziness dealing in the area of real analysis. In 2000, Burgin [38] explored and examined the fuzzy limits of functions using two approaches: the first one relies on the concept of a fuzzy limit in the context of a sequence, while the other extends and generalizes the traditional ε-δ definition. In 1975, Kramosil and Michalek first introduced the concept of fuzzy metric spaces [133], derived from the expansion of probabilistic metric spaces to encompass fuzzy scenarios. They infused the traditional concepts of metrics and metric spaces with the idea of fuzziness and compared their adapted notions against those derived using other generalizations. Taking motivation from this work, Kaleva and Seikkala [126] expanded the notion of a metric space by defining the distance between two points as a non-negative fuzzy number. This introduced a more intuitive definition of fuzzy metric spaces. They presented various properties of fuzzy numbers, leading to the definition and exploration of fuzzy metric spaces. Their contribution extends to providing fixed point theorems specifically tailored for fuzzy metric spaces. For a comprehensive and intriguing exploration of these aspects, readers are encouraged to refer [21; 71; 214].

Fuzzy numbers are essential in a variety of fields with uncertainty, such as mathematics, engineering, economics, and artificial intelligence. Unlike their precise counterparts, these numbers have a particular modeling aspect that allows them to reflect the approximate values found in linguistic terms like “big", “small", “early" or “around." This attribute is extremely useful for modeling and analyzing systems or scenarios when it is difficult to obtain exact numerical numbers. Fuzzy numbers therefore aid in representing uncertainty in decision-making processes in a more realistic manner. Due to their flexibility, there has been a noticeable surge in the amount of research devoted to fuzzy number approximation over the past few years. In an important study towards fuzzy numbers in 2014, Ban and Coroianu [25] defined a set of real parameters correlated with a fuzzy number, demonstrating the existence of a unique trapezoidal fuzzy number preserving a fixed parameter. Roldan et al. [181] characterized fuzzy numbers based on their level sets extremes, establishing relationships between their images and usual operations while preserving its continuity.

Researchers in recent years have explored various approximations of fuzzy numbers, categorizing them into Euclidean and non-Euclidean distance types. Euclidean approximations can be formulaically calculated, while the non-Euclidean counterparts pose greater complexity. The study introduced by Yen and Chu [220] focused on LR-type fuzzy numbers for approximating fuzzy numbers, offering generalized approximations within the Euclidean class, specifically without constraints. Additionally, they introduced an efficient formula for calculation.

In this chapter, we will be dealing with the approximation of fuzzy numbered-valued functions. The fuzzy approximation theory was first dealt by George A. Anastassiou [17] in 2010, where he gave numerous applications, all consistently situated within the context of fuzzy mathematics. He extended his study to fuzzy differentiation and integration, covering topics such as fuzzy Taylor formulae and fuzzy Ostrowski inequalities. But what interests us more is his idea of fuzzy approximation using algebraic and trigonometric polynomials, wherein he developed a theoretical framework exploring how linear positive fuzzy operators approach the fuzzy unit operator with respect to convergence rates, thus giving rise to the fuzzy Korovkin theorem. Results regarding the approximation of fuzzy numbered-valued functions can be seen in [16; 27; 31; 84; 86; 140; 218].

The approximation of linear positive operators have been an important topic in the world of approximation theory. Owing to their real-world applications, the Bernstein polynomials constitute a focal point of intensive research, offering a wide scope of surprising findings; for more details, see [59; 61; 66; 119; 139]. We extend the work done on the classical Bernstein, Szász, and Baskakov operators to fuzzy-valued functions.

5.2 Preliminaries

In order to proceed with our paper, we must first lay the groundwork by introducing some basic definitions and concepts in fuzzy theory. Let $\tilde{p}$ be a fuzzy set with membership function $\tilde{p}(u) : \mathbb{R} \to [0,1]$ such that:

(i) Normality: There exists $u_0 \in \mathbb{R}$ such that $\tilde{p}(u_0) = 1$.

(ii) Convexity: $\forall\, s,t \in \mathbb{R}$ and $\forall\, \gamma \in [0,1]$, $\tilde{p}(\gamma s + (1-\gamma)t) \geq \min\{\tilde{p}(s), \tilde{p}(t)\}$.

(iii) Upper semi-continuity: $\forall\, u_0 \in \mathbb{R}$ and $\forall\, \varepsilon > 0$, $\exists$ a neighourhood $V(u_0) : \tilde{p}(u) \leq \tilde{p}(u_0) + \varepsilon$, $\forall\, u \in V(u)$.

(iv) Boundedness of $\overline{supp(\tilde{p})}$: Define $supp(\tilde{p}) = \{u \in \mathbb{R} : \tilde{p}(x) > 0\}$ as the support of $\tilde{p}$. Then the closure of the support of $\tilde{p}$, i.e. $\overline{supp(\tilde{p})}$ is bounded in $\mathbb{R}$.

Then $\tilde{p}$ is called a fuzzy real number. It may be interesting to note that a fuzzy set is not a collection of fuzzy numbers. Rather, a fuzzy number in itself is a fuzzy set with the above properties. Let $\mathbb{R}_{\mathscr{F}}$ represent the set of all fuzzy numbers $\tilde{p}$.

Firstly, let us recall the ℓ-level cut of $\tilde{p}$. For $0 \leq \ell \leq 1$ and $\tilde{p} \in \mathbb{R}_{\mathscr{F}}$ with membership function $\tilde{p}(u)$, the ℓ-level cut or simply ℓ-cut of $\tilde{p}$, denoted by $[\tilde{p}]^{\ell}$, is defined as,

$$[\tilde{p}]^{\ell} = \begin{cases} \{u : \tilde{p}(u) \geq \ell\}, & \text{if } 0 < \ell \leq 1 \\ \overline{\{u : \tilde{p}(u) > 0\}}, & \text{if } \ell = 0 \end{cases}$$

where $\overline{A}$ denotes the closure of set A.

For each $\ell \in [0,1]$, $[\tilde{p}]^{\ell} = [p_{-}^{(\ell)}, p_{+}^{(\ell)}]$ is a closed and bounded interval of $\mathbb{R}$. It is apparent that, if $p_{-}^{(\ell)} = p_{+}^{(\ell)}$ then $\tilde{p}$ will reduce to a crisp real number.

Definition 5.2.1 *[86] For any $\tilde{p}$ and $\tilde{q}$ belonging to the set $\mathbb{R}_{\mathscr{F}}$, and for any α in $\mathbb{R}$, we uniquely define the sum $\tilde{p} \oplus \tilde{q}$ and the product with real scalars $\alpha \odot \tilde{p}$ in $\mathbb{R}_{\mathscr{F}}$ by $\oplus : \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_{\mathscr{F}}$,*

$$(\tilde{p} \oplus \tilde{q})(u) = \sup_{v+w=u} \min\{\tilde{p}(v), \tilde{q}(w)\}$$

and by $\odot : \mathbb{R} \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_{\mathscr{F}}$,

$$(\alpha \odot \tilde{p})(u) = \begin{cases} \tilde{p}\left(\frac{u}{\alpha}\right) & \text{if } \alpha \neq 0 \\ \tilde{o} & \text{if } \alpha = 0 \end{cases}$$

where $\tilde{o} : \mathbb{R} \to [0,1]$ is $\tilde{o} = \chi_{\{0\}}$.

This can also be written as,

$$[\tilde{p} \oplus \tilde{q}]^{\ell} = [\tilde{p}]^{\ell} + [\tilde{q}]^{\ell}, \qquad \forall \ell \in [0,1]$$
$$[\alpha \odot \tilde{p}]^{\ell} = \alpha[\tilde{p}]^{\ell}, \qquad \forall \ell \in [0,1]$$

Here, $[\tilde{p}]^{\ell} + [\tilde{q}]^{\ell}$ represents the standard addition of intervals considered as real subsets, while $\alpha[\tilde{p}]^{\ell}$ denotes the standard multiplication between a scalar and a real subset. The positivity of a fuzzy number is defined in the sense of its ℓ-cut for $\ell = 0$.

(i) $\tilde{p}$ is defined as positive, for $\tilde{p}_{-}^{0} \geq 0$.

(ii) $\tilde{p}$ is defined as negative, for $\tilde{p}_{+}^{0} \leq 0$.

Definition 5.2.2 (Fuzzy modulus of continuity) *For a continuous fuzzy-valued function $\tilde{h}$ mapping from the interval $[a,b]$ to $\mathbb{R}_{\mathscr{F}}$, the first fuzzy modulus of continuity is defined as follows:*

$$\omega_1^{\mathscr{F}}\left(\tilde{h};\delta\right) := \sup_{\substack{u,v\in[a,b]\\ |u-v|\leq\delta}} D\left(\tilde{h}(u),\tilde{h}(v)\right)$$

Definition 5.2.3 (Fuzzy exponential modulus of continuity) *Consider a continuous fuzzy-valued function $\tilde{h} : [0,\infty) \to \mathbb{R}_{\mathscr{F}}$. We can define the exponential fuzzy modulus of continuity for $\tilde{h}$ in the following manner:*

$$\omega_*^{\mathscr{F}}\left(\tilde{h};\delta\right) := \sup_{\substack{u,v\geq 0\\ |e^{-u}-e^{-v}|<\delta}} D\left(\tilde{h}(u),\tilde{h}(v)\right)$$

Definition 5.2.4 (Fuzzy weighted modulus of continuity) *Consider a continuous fuzzy-valued function $\tilde{h}$. We can define the weighted fuzzy modulus of continuity for $\tilde{h}$ as:*

$$\Omega^{\mathscr{F}}\left(\tilde{h};\delta\right) := \sup_{\substack{u,v\geq 0\\ |u-v|\leq\delta}} \frac{1}{\left(1+(u-v)^2\right)(1+u^2)} D\left(\tilde{h}(u),\tilde{h}(v)\right).$$

Definition 5.2.5 (Hausdorff metric in $\mathbb{R}_{\mathscr{F}}$) *[86] The formula for Hausdorff distance between two fuzzy numbers is given by $D : \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}^{+} \cup \{0\}$ as*

$$D(\tilde{p},\tilde{q}) = \sup_{\ell\in[0,1]} \max\left\{\left|p_{-}^{(\ell)} - q_{-}^{(\ell)}\right|, \left|p_{+}^{(\ell)} - q_{+}^{(\ell)}\right|\right\},$$

where $[\tilde{p}]^{\ell} = [p_{-}^{(\ell)}, p_{+}^{(\ell)}]$ and $[\tilde{q}]^{\ell} = [q_{-}^{(\ell)}, q_{+}^{(\ell)}]$. Then, $(\mathbb{R}_{\mathscr{F}}, D)$ is a complete metric space in $\mathbb{R}_{\mathscr{F}}$, known as the Hausdorff metric space.

Some properties of $(\mathbb{R}_{\mathscr{F}}, D)$ are:

(i) $D(\tilde{p} \oplus \tilde{r}, \tilde{q} \oplus \tilde{r}) = D(\tilde{p}, \tilde{q}), \quad \forall\, \tilde{p}, \tilde{q}, \tilde{r} \in \mathbb{R}_{\mathscr{F}}$

(ii) $D(a \odot \tilde{p}, a \odot \tilde{q}) = |a| D(\tilde{p}, \tilde{q}), \quad \forall\, \tilde{p}, \tilde{q} \in \mathbb{R}_{\mathscr{F}},\ a \in \mathbb{R}$

(iii) $D(\tilde{p} \oplus \tilde{q}, \tilde{r} \oplus \tilde{s}) \leq D(\tilde{p}, \tilde{r}) + D(\tilde{q}, \tilde{s}), \quad \forall\, \tilde{p}, \tilde{q}, \tilde{r}, \tilde{s} \in \mathbb{R}_{\mathscr{F}}$

The symbol $\precsim$ represents a partial order on $\mathbb{R}_{\mathscr{F}}$ and is defined as follows:

$$\tilde{p} \precsim \tilde{q} \quad \text{iff} \quad p_{-}^{(\ell)} \leq q_{-}^{(\ell)},\ p_{+}^{(\ell)} \leq q_{+}^{(\ell)},$$

for all fuzzy numbers $\tilde{p}$ and $\tilde{q}$ and $\ell \in [0,1]$. Herein, $\leq$ represents the partial order on the real number set.

Lemma 5.2.6 *[86] Let $\eta, \mu \in \mathbb{R}$ and $\tilde{p}, \tilde{q}, \tilde{r} \in \mathbb{R}_{\mathscr{F}}$. Let $\tilde{o}$ be the characteristic function of the set $\{0\}$, i.e. $\tilde{o} = \chi_{\{0\}}$. Then, the following conditions hold on $\mathbb{R}_{\mathscr{F}}$:*

(i) $\tilde{p} \oplus \tilde{q} = \tilde{q} \oplus \tilde{p}$, $\tilde{p} \oplus (\tilde{q} \oplus \tilde{r}) = (\tilde{p} \oplus \tilde{q}) \oplus \tilde{r}$.

(ii) The element $\tilde{o} \in \mathbb{R}_{\mathscr{F}}$ serves as the identity element for $\oplus$ and hence, $\tilde{p} \oplus \tilde{o} = \tilde{o} \oplus \tilde{p} = \tilde{p}$.

(iii) For any $\tilde{p} \in \mathbb{R}_{\mathscr{F}} \backslash \mathbb{R}$, there is no opposite element with respect to $\tilde{o}$ under the operation $\oplus$.

(iv) When $\eta\mu \geq 0$, the expression $(\eta + \mu) \odot \tilde{p} = \eta \odot \tilde{p} \oplus \mu \odot \tilde{p}$ holds true. However, this condition will not be satisfied for $\eta, \mu \in \mathbb{R}$.

(v) $\eta \odot (\tilde{p} \oplus \tilde{q}) = \eta \odot \tilde{p} \oplus \eta \odot \tilde{q}$, $\eta \odot (\mu \odot \tilde{p}) = (\eta\mu) \odot \tilde{p}$.

(vi) Let us define the usual norm on $\mathbb{R}_{\mathscr{F}}$ as $\|\tilde{p}\|_{\mathscr{F}} = D(\tilde{p}, \tilde{o})$. Then $\|\tilde{p}\|_{\mathscr{F}}$ has the following properties:

$$\|\tilde{p}\|_{\mathscr{F}} = 0 \textit{ iff } \tilde{p} = \tilde{o}$$
$$\|\eta \odot \tilde{p}\|_{\mathscr{F}} = |\eta|\, \|\tilde{p}\|_{\mathscr{F}}$$
$$\|\tilde{p} \oplus \tilde{q}\|_{\mathscr{F}} \leq \|\tilde{p}\|_{\mathscr{F}} + \|\tilde{q}\|_{\mathscr{F}}$$
$$\|\tilde{p}\|_{\mathscr{F}} + \|\tilde{q}\|_{\mathscr{F}} \leq D(\tilde{p}, \tilde{q})$$

(vii) $D(\eta \odot \tilde{p}, \mu \odot \tilde{p}) = |\eta - \mu| D(\tilde{o}, \tilde{p})$, for all $\eta\mu \geq 0$. This equation does not hold when η and μ are of opposite sign.

Having covered the fundamental arithmetic operations of fuzzy numbers, our focus now shifts to the examination of fuzzy valued functions.

5.3 Fuzzy-Valued Function

When domain of a function is the set of real numbers, and the co-domain is the set of fuzzy numbers, the function can have some unique properties and characteristics that reflect the inherent uncertainty and imprecision in the co-domain. Fuzzy numbers in the co-domain allow the function to associate each real number in the domain with a degree of membership in different fuzzy sets. This means the output is not a single real number but a fuzzy value that can represent a range of possible values. The function may not yield a unique result for a given input; instead, it can produce a range of values with varying degrees of certainty. This continuous variation allows the function to capture nuances and gradual transitions in the output. This reflects the imprecision and variability often encountered in real-world applications. Fuzzy function is useful in scenarios where precise values are hard to define, such as linguistic terms like "very hot" or "moderately cold" or data with inherent imprecision.

Chang and Zadeh [48], in their paper defined the class of fuzzy bunches of functions as a function with a fuzzy parameter. A fuzzy bunch of functions, generally speaking, is the fuzzy subset of a classical space of functions. To put it mathematically, a fuzzy bunch W of functions mapping from X to Y can be described as a fuzzy set on Y^X, that is, each function $w : X \to Y$ within the fuzzy bunch W is assigned a membership value $\mu(w)$.

However, we will not be dealing with fuzzy bunches of functions or their corresponding fuzzy sets at x. In our study, we are simply concerned with a function $\tilde{h}$, that takes inputs from the real interval $[a,b]$ and maps them to the fuzzy field $\mathbb{R}_{\mathscr{F}}$, having the representation:

$$\left[\tilde{h}(x)\right]^{\ell} = \left[h_{-}^{(\ell)}(x), h_{+}^{(\ell)}(x)\right],$$

for every $x \in [a,b]$ and ℓ within the range $[0,1]$, where $h_{-}^{(\ell)}(x)$ and $h_{+}^{(\ell)}(x)$ represent the left and right endpoints of $\left[\tilde{h}(x)\right]^{\ell}$, respectively. Furthermore, $h_{-}^{(\ell)}$ and $h_{+}^{(\ell)}$ are real valued functions defined over the interval $[a,b]$.

Example 5.3.1 *In order to increase our understanding towards a fuzzy valued function, let us take an example. Let* $\tilde{h} : [0,1] \to \mathbb{R}_{\mathscr{F}}$ *such that*

$$\tilde{h}(x) = \sin^2 3x - \frac{1}{2}\cos 9x + x^6,$$

where, each $\tilde{h}(x)$ is a triangular fuzzy number with the membership function:

$$\mu_h\left(t;\tilde{h}(x)-1,\tilde{h}(x),\tilde{h}(x)+1\right)=\max\left(\min\left(t-\tilde{h}(x)+1,\tilde{h}(x)+1-t\right),0\right).$$

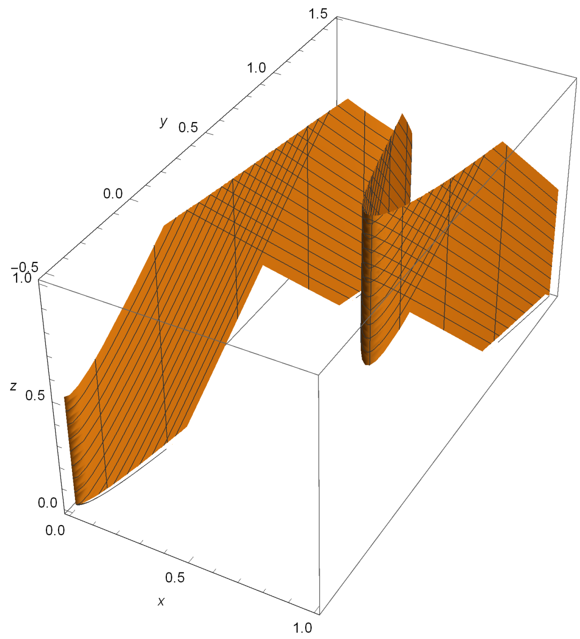
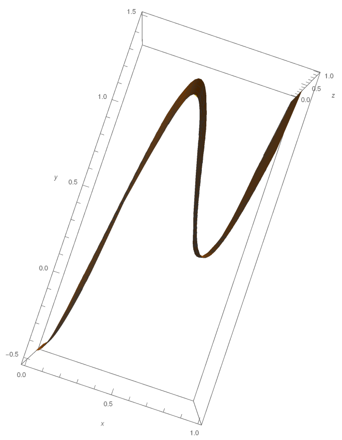
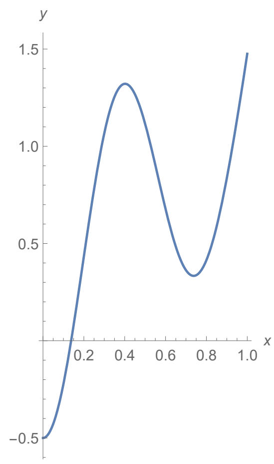


Figure 5.1: Graph of fuzzy number-valued function $\tilde{h}(x)=\sin^2 3x-\frac{1}{2}\cos 9x+x^6$ with triangular membership function.



(a) Aerial view of fuzzy valued function



(b) Crisp function

Figure 5.2: Association of a fuzzy valued function with its corresponding crisp function.

The graph of a fuzzy valued function is a 3-dimensional graph, as the Z-axis corresponds to the membership value corresponding to each $\tilde{h}(x)$. The above figure represents a function whose range is the triangular fuzzy number. Let us change this membership function and see what changes we get in our graph. Suppose each point in the range is a trapezoidal fuzzy number. Then the membership value corresponding to each $\tilde{h}(x)$ will be,

$$\mu_h\left(t;\tilde{h}(x)-1.5,\tilde{h}(x)-0.5,\tilde{h}(x)+\frac{1}{2},\tilde{h}(x)+1.5\right)=\max\left(\min\left(t-\tilde{h}(x)+1.5,1,\tilde{h}(x)+1.5-t\right),0\right).$$

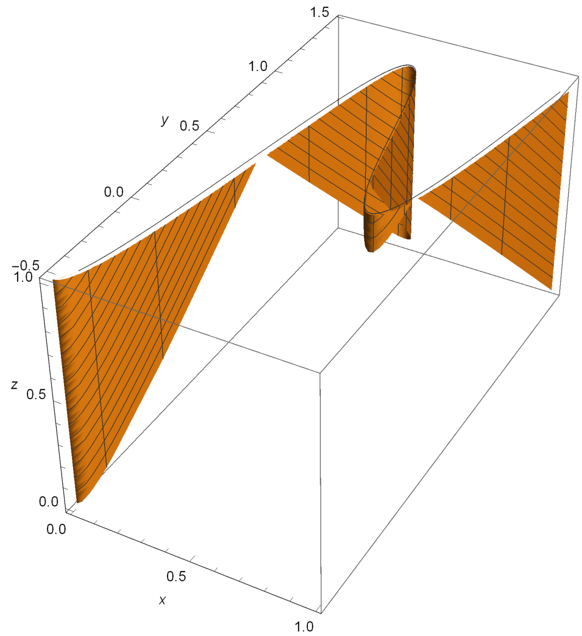


Figure 5.3: Graph of fuzzy number-valued function $\tilde{h}(x)=\sin^2 3x-\frac{1}{2}\cos 9x+x^6$ with trapezoidal membership function.

The aerial view will remain the same; only the degree of uncertainty, given by the membership function μ_h, changes along the z axis.

Consider two fuzzy numbered-valued functions, $\tilde{h}$ and $\tilde{g}$, defined on the interval $[a,b]\in\mathbb{R}$. We define the distance between $\tilde{h}$ and $\tilde{g}$ as follows:

$$D^*\left(\tilde{h},\tilde{g}\right):=\sup_{x\in[a,b]} D\left(\tilde{h}(x),\tilde{g}(x)\right).$$

We define $\tilde{h}$ as crisp when $\tilde{h}(x)$ assumes crisp values for all x in its domain. $\tilde{h}$ is defined to be a fuzzy continuous function if,

$$\lim_{x\to x_0} D\left(\tilde{h}(x),\tilde{h}(x_0)\right)=0,$$

holds for any $x_0 \in [a,b]$.

Represented by $C[a,b]$ and $C_{\mathscr{F}}[a,b]$ are the sets of continuous and fuzzy continuous functions on the interval $[a,b]$. If $\tilde{h}$ is a fuzzy continuous function over $[a,b]$, then the corresponding functions $h_-^{(r)}$ and $h_+^{(r)}$ become real-valued continuous functions defined on the same interval. Also, the space $(C_{\mathscr{F}}[a,b],\, D^*)$ forms a complete metric space. The operations of addition and scalar multiplication in $C_{\mathscr{F}}[a,b]$ are defined as follows:

$$(a \odot \tilde{g})(x) = a \odot \tilde{g}(x)$$
$$\left(\tilde{h} \oplus \tilde{g}\right)(x) = \tilde{h}(x) \oplus \tilde{g}(x),$$

for every $x \in [a,b]$, $a \in \mathbb{R}$ and $\tilde{h}, \tilde{g} \in C_{\mathscr{F}}[a,b]$. Moreover, a fuzzy number-valued function $\tilde{0}$ exists, defined on the interval $[a,b]$, satisfying the condition $\tilde{0}(x) = \tilde{o}$ for every $x \in [a,b]$ where $\tilde{o}$ represents the neutral element with respect to the operation $\oplus$ in $\mathbb{R}_{\mathscr{F}}$. We can also define the norm of $\tilde{h}$ as

$$\left\|\tilde{h}\right\|_{\mathscr{F}} = \sup_{a \leq x \leq b} D\left(\tilde{o}, \tilde{h}(x)\right).$$

Based on the above definitions we obtain the following properties.

Lemma 5.3.2 *For any $\tilde{h}, \tilde{g}, \tilde{f} \in C_{\mathscr{F}}[a,b]$ and real constants $\eta, \nu \in \mathbb{R}$, we have the following properties:*

(i) $\oplus$ is commutative and associative, that is,

$$\tilde{h} \oplus \tilde{g} = \tilde{g} \oplus \tilde{h},$$

$$\tilde{h} \oplus \left(\tilde{g} \oplus \tilde{f}\right) = \left(\tilde{h} \oplus \tilde{g}\right) \oplus \tilde{f}.$$

(ii) $\tilde{h} \oplus \tilde{0} = \tilde{0} \oplus \tilde{h}$.

(iii) Consider the function space $C_{\mathscr{F}}[a,b]$ with neutral element $\tilde{0}(x)$. If for any function $\tilde{h}$ in this space, the range of $\tilde{h}$ over the interval $[a,b]$ has a non-empty intersection with the set of real numbers, then there is no opposite member with respect to the operation $\oplus$ in $C_{\mathscr{F}}[a,b]$.

(iv) For all $\eta\nu \geq 0$, $(\eta + \nu) \odot \tilde{h} = \left(\eta \odot \tilde{h}\right) \oplus \left(\nu \odot \tilde{h}\right)$.
For general $\eta, \nu \in \mathbb{R}$, this property does not hold.

(v) For any $\tilde{h},\tilde{g}\in C_{\mathscr{F}}[a,b]$ and real constants $\eta,\nu\in\mathbb{R}$,

$$\eta\odot\left(\tilde{h}\oplus\tilde{g}\right)=\left(\eta\odot\tilde{h}\right)\oplus\left(\eta\odot\tilde{g}\right)$$

$$\eta\odot\left(\nu\odot\tilde{h}\right)=\left(\eta\nu\right)\odot\tilde{h}.$$

(vi) For any $\tilde{h},\tilde{g}\in C_{\mathscr{F}}[a,b]$,

$$\left\|\tilde{h}\right\|_{\mathscr{F}}=0 \text{ iff } \tilde{h}=\tilde{0}$$

$$\left\|\eta\odot\tilde{h}\right\|_{\mathscr{F}}=|\eta|\left\|\tilde{h}\right\|_{\mathscr{F}}$$

$$\left\|\tilde{h}\oplus\tilde{g}\right\|_{\mathscr{F}}\leq\left\|\tilde{h}\right\|_{\mathscr{F}}+\left\|\tilde{g}\right\|_{\mathscr{F}}$$

$$\left\|\tilde{h}\right\|_{\mathscr{F}}-\left\|\tilde{g}\right\|_{\mathscr{F}}\leq D^{*}\left(\tilde{h},\tilde{g}\right).$$

(vii) For $\eta\nu\geq 0$,

$$D^{*}\left(\eta\odot\tilde{h},\nu\odot\tilde{h}\right)=|\eta-\nu|\,D^{*}\left(\tilde{0},\tilde{h}\right).$$

(viii) For any $\tilde{h},\tilde{g},\tilde{f},\tilde{e}\in C_{\mathscr{F}}[a,b]$ and $\eta\in\mathbb{R}$,

$$D^{*}\left(\tilde{h}\oplus\tilde{f},\tilde{g}\oplus\tilde{f}\right)=D^{*}\left(\tilde{h},\tilde{g}\right)$$

$$D^{*}\left(\eta\odot\tilde{h},\eta\odot\tilde{g}\right)=|\eta|\,D^{*}\left(\tilde{h},\tilde{g}\right)$$

$$D^{*}\left(\tilde{h}\oplus\tilde{g},\tilde{f}\oplus\tilde{e}\right)\leq D^{*}\left(\tilde{h},\tilde{f}\right)+D^{*}\left(\tilde{g},\tilde{e}\right).$$

5.4 Fuzzy Positive Linear Operators

In the case of operators $\tilde{\mathscr{L}}$ from $C_{\mathscr{F}}[a,b]$ to itself, their representation takes the form,

$$\left[\tilde{\mathscr{L}}\left(\tilde{h}\right)(x)\right]^{\ell}=\left[\left(\tilde{\mathscr{L}}\left(\tilde{h}\right)(x)\right)_{-}^{(\ell)},\left(\tilde{\mathscr{L}}\left(\tilde{h}\right)(x)\right)_{+}^{(\ell)}\right].$$

Suppose $\tilde{\mathscr{L}}$ is an operator mapping the space of all continuous fuzzy-valued functions $C_{\mathscr{F}}[a,b]$ to itself, with the condition,

$$\tilde{\mathscr{L}}\left(a\odot\tilde{h}\right)=a\odot\tilde{\mathscr{L}}\left(\tilde{h}\right)$$

$$\tilde{\mathscr{L}}\left(\tilde{h}\oplus\tilde{g}\right)=\tilde{\mathscr{L}}\left(\tilde{h}\right)\oplus\tilde{\mathscr{L}}\left(\tilde{g}\right),$$

for any $a\in\mathbb{R}$ and $\tilde{h},\tilde{g}\in C_{\mathscr{F}}[a,b]$. In this case, $\tilde{\mathscr{L}}$ is identified as a fuzzy linear operator. Consider $\tilde{\mathscr{L}}$ mapping $C_{\mathscr{F}}[a,b]$ to itself such that for any $\tilde{h},\tilde{g}\in C_{\mathscr{F}}[a,b]$, $\tilde{\mathscr{L}}$ is linear and

$$\tilde{h}\precsim\tilde{g}\Rightarrow\tilde{\mathscr{L}}\left(\tilde{h}\right)\precsim\tilde{\mathscr{L}}\left(\tilde{g}\right).$$

Then $\tilde{\mathscr{L}}$ termed as a linear positive fuzzy operator.

Theorem 5.4.1 (Fuzzy analog of Shisha-Mond inequality) *[17] Consider a sequence $\{\tilde{\mathscr{L}}_m\}_{m\in\mathbb{N}}$ of positive fuzzy linear operators that map $C_{\mathscr{F}}[a,b]$ to itself. We posit the existence of a corresponding sequence $\{\mathscr{L}_m\}_{m\in\mathbb{N}}$, consisting of linear positive operators from $C[a,b]$ into itself, satisfying the property,*

$$\left(\tilde{\mathscr{L}}_m\left(\tilde{h}\right)\right)_{\pm}^{(\ell)} = \mathscr{L}_m\left(\tilde{h}_{\pm}^{(\ell)}\right),$$

respectively, $\forall \ell \in [0,1]$ and $\forall \tilde{h} \in C_{\mathscr{F}}[a,b]$. With the assumption that the sequence $\{\mathscr{L}_m(1)\}_{m\in\mathbb{N}}$ is bounded, we can deduce that, for any $m \in \mathbb{N}$,

$$D^*\left(\tilde{\mathscr{L}}_m\tilde{h},\tilde{h}\right) \leq \|\mathscr{L}_m(1)-1\| D^*\left(\tilde{h},\tilde{0}\right) + \|\mathscr{L}_m(1)+1\| \,\omega_1^{\mathscr{F}}\left(\tilde{h};\mu_m\right),$$

where $\mu_m = \left\|\mathscr{L}_m\left((t-x)^2\right)(x)\right\|^{1/2}$.

If $\mathscr{L}_m(1) = 1$, then, $D^*\left(\tilde{\mathscr{L}}_m\tilde{h},\tilde{h}\right) \leq 2\omega_1^{\mathscr{F}}\left(\tilde{h};\mu_m\right)$.

The theorem discussed above enables us to demonstrate the fuzzy counterpart of the Korovkin theorem in the closed and bounded interval $[a,b]$.

Theorem 5.4.2 (Fuzzy Korovkin Theorem) *Consider a sequence $\{\tilde{\mathscr{L}}_m\}_{m\in\mathbb{N}}$ of linear positive fuzzy operators, mapping $C_{\mathscr{F}}[a,b]$ into itself. We posit the existence of a corresponding sequence $\{\mathscr{L}_m\}_{m\in\mathbb{N}}$, consisting of linear positive operators from $C[a,b]$ into itself, satisfying the property,*

$$\left(\tilde{\mathscr{L}}_m\left(\tilde{h}\right)\right)_{\pm}^{(\ell)} = \mathscr{L}_m\left(\tilde{h}_{\pm}^{(\ell)}\right),$$

respectively, $\forall \ell \in [0,1]$ and $\forall \tilde{h} \in C_{\mathscr{F}}([a,b])$. Furthermore, assume that, as $m \to \infty$

$$\mathscr{L}_m(1) \to 1$$
$$\mathscr{L}_m(t) \to x$$
$$\mathscr{L}_m\left(t^2\right) \to x^2,$$

uniformly. Then,

$$D^*\left(\tilde{\mathscr{L}}_m\tilde{h},\tilde{h}\right) \to 0, \ \text{ as } m \to \infty,$$

for any $\tilde{h} \in C_{\mathscr{F}}[a,b]$. That is, we can say that,

$$\tilde{\mathscr{L}}_m\tilde{h} \xrightarrow{D^*} \tilde{h}.$$

This theorem proves an operator to approximate a function by using the test functions $1,t,t^2$. However, Altomare [15] in his study generalized the set of test functions, known as the Korovkin set. The following theorems are for using different test functions. In a real sense, we have the following theorem:

Theorem 5.4.3 *Given the metric space (X,d) and the linear space $C(X)$ of all continuous real-valued functions on X, take a subset E containing the constant function 1 and the functions d_x^2, where $d_x(y)=d(x,y)$ for $x,y\in X$. Assume $\{\mathscr{L}_m\}_{m\geq 1}$ is a sequence of linear positive operators mapping from E into $C(X)$. Then, for any uniformly continuous function h in E,*

$$|\mathscr{L}_m(h)(x)-h(x)\mathscr{L}_m(1)(x)|\leq \mathscr{L}_m(|h-h(x)|)(x)\leq \frac{2\|h\|_\infty}{\delta^2}\mathscr{L}_m\left(d_x^2\right)(x)+\varepsilon\mathscr{L}_m(1)(x).$$

Based on theorem 5.4.3, Altomare also stated the following remark.

Remark 5.4.4 *Assuming M is a subset of $C_0(X)$ and given that $f_0\in C_0(X)$ is strictly positive, the set $\{f_0\}\cup\{f_0M\}\cup\left\{f_0M^2\right\}$ serves as a Korovkin set within $C_0(X)$.*

By setting $M=\{e^{-x}\}$ and $f_0=1$, we form the Korovkin set $\left\{1,e^{-x},e^{-2x}\right\}$. It is necessary to broaden the findings established by Altomare to encompass fuzzy-valued functions. Consider a subset $E_{\mathscr{F}}$ of $F^{\mathscr{F}}(X)$ containing the fuzzy constant function 1 and all the fuzzy functions d_x^2, $x\in X$.

Theorem 5.4.5 *Consider a sequence $\left\{\tilde{\mathscr{L}}_m\right\}_{m\in\mathbb{N}}$ of linear positive fuzzy operators, mapping $C_{\mathscr{F}}(X)$ into itself, for the metric space (X,d). We posit the existence of a corresponding sequence $\{\mathscr{L}_m\}_{m\in\mathbb{N}}$, consisting of linear positive operators from $C(X)$ into itself, satisfying the property,*

$$\left(\tilde{\mathscr{L}}_m\left(\tilde{h}\right)\right)_\pm^{(\ell)}=\mathscr{L}_m\left(\tilde{h}_\pm^{(\ell)}\right),$$

respectively, $\forall\ell\in[0,1]$ and $\forall\tilde{h}\in C_{\mathscr{F}}(X)$. Then, we have

$$D^*\left(\tilde{\mathscr{L}}_m\left(\tilde{h}\right),\tilde{h}\mathscr{L}_m(1)\right)\leq\frac{2}{\delta^2}\mathscr{L}_m\left(d_x^2\right)D^*\left(\tilde{h},\tilde{o}\right)+\varepsilon\mathscr{L}_m(1),$$

where $d_x(y)=d(x,y)\ \ \forall x,y\in X$.

Proof. Let $\tilde{h} \in C_{\mathscr{F}}(X)$. Then,

$$\begin{aligned}
&D^*\left(\tilde{\mathscr{L}}_m\left(\tilde{h}\right), \tilde{h}\mathscr{L}_m(1)\right) \\
&\quad = \sup_{x\in X} D\left(\tilde{\mathscr{L}}_m\left(\tilde{h}\right)(x), \tilde{h}(x)\mathscr{L}_m(1)\right) \\
&\quad = \sup_{x\in X}\sup_{\ell\in[0,1]}\max\left\{\left|\mathscr{L}_m\left(\left(\tilde{h}\right)_-^{(\ell)}(x)\right) - \left(\tilde{h}\right)_-^{(\ell)}(x)\mathscr{L}_m(1)_-^{(\ell)}\right|,\right. \\
&\qquad \left.\left|\mathscr{L}_m\left(\left(\tilde{h}\right)_+^{(\ell)}(x)\right) - \left(\tilde{h}\right)_+^{(\ell)}(x)\mathscr{L}_m(1)_+^{(\ell)}\right|\right\} \\
&\quad = \sup_{\ell\in[0,1]}\max\left\{\left\|\mathscr{L}_m\left(\left(\tilde{h}\right)_-^{(\ell)}(x)\right) - \left(\tilde{h}\right)_-^{(\ell)}(x)\mathscr{L}_m(1)_-^{(\ell)}\right\|,\right. \\
&\qquad \left.\left\|\mathscr{L}_m\left(\left(\tilde{h}\right)_+^{(\ell)}(x)\right) - \left(\tilde{h}\right)_+^{(\ell)}(x)\mathscr{L}_m(1)_+^{(\ell)}\right\|\right\} \\
&\quad \leq \sup_{\ell\in[0,1]}\max\left\{2\frac{\left\|\left(\tilde{h}\right)_-^{(\ell)}\right\|}{\delta^2}\mathscr{L}_m\left(d_x^2\right) + \varepsilon\mathscr{L}_m(1), 2\frac{\left\|\left(\tilde{h}\right)_+^{(\ell)}\right\|}{\delta^2}\mathscr{L}_m\left(d_x^2\right) + \varepsilon\mathscr{L}_m(1)\right\} \\
&\quad \leq \frac{2}{\delta^2}\mathscr{L}_m\left(d_x^2\right)\sup_{\ell\in[0,1]}\max\left\{\left\|\left(\tilde{h}\right)_-^{(\ell)}\right\|, \left\|\left(\tilde{h}\right)_+^{(\ell)}\right\|\right\} + \varepsilon\mathscr{L}_m(1) \\
&\quad \leq \frac{2}{\delta^2}\mathscr{L}_m\left(d_x^2\right)D^*\left(\tilde{h}, \tilde{o}\right) + \varepsilon\mathscr{L}_m(1),
\end{aligned}$$

where $\tilde{o} = \chi_{\{0\}}$ is the neutral element for $\oplus$. □

Corollary 5.4.6 *Further, if we assume that,*

(i) $\lim_{m\to\infty} D^*\left(\tilde{\mathscr{L}}_m(1), 1\right) = 0$ *uniformly on* X

(ii) $\lim_{m\to\infty} D^*\left(\tilde{\mathscr{L}}_m\left(d_x^2(x)\right), 0\right) = 0$ *uniformly on* X.

Then for every fuzzy continuous $\tilde{h} \in C_{\mathscr{F}}(X)$,

$$\lim_{m\to\infty} D^*\left(\tilde{\mathscr{L}}_m\left(\tilde{h}\right), \tilde{h}\right) = 0,$$

uniformly on X.

Theorem 5.4.7 (Fuzzy Korovkin Theorem for $\mathbb{R}^+$) *Consider a sequence* $\left\{\tilde{\mathscr{L}}_m\right\}_{m\in\mathbb{N}}$ *of linear positive fuzzy operators, mapping* $C_{\mathscr{F}}\left(\mathbb{R}^+\right)$ *into itself. We posit the existence of a corresponding sequence* $\{\mathscr{L}_m\}_{m\in\mathbb{N}}$*, consisting of linear positive operators from* $C\left(\mathbb{R}^+\right)$ *into itself, satisfying the property,*

$$\left(\tilde{\mathscr{L}}_m\left(\tilde{h}\right)\right)_\pm^{(r)} = \mathscr{L}_m\left(\tilde{h}_\pm^{(r)}\right),$$

for all $\forall \ell \in [0,1]$ *and every* $\tilde{h} \in C_{\mathscr{F}}(\mathbb{R}^+)$, *respectively. Additionally, assume that, as m approaches* ∞,

$$\mathscr{L}_m(1) \to 1$$
$$\mathscr{L}_m\left(e^{-t}\right) \to e^{-x}$$
$$\mathscr{L}_m\left(e^{-2t}\right) \to e^{-2x},$$

uniformly. Then,

$$D^*\left(\tilde{\mathscr{L}}_m \tilde{h}, \tilde{h}\right) \to 0, \text{ as } m \to \infty,$$

for any $f \in C_{\mathscr{F}}(\mathbb{R}^+)$. *That is, we can say that,*

$$\tilde{\mathscr{L}}_m \tilde{h} \xrightarrow{D^*} \tilde{h}.$$

5.4.1 Fuzzy Bernstein Operators

Define the fuzzy Bernstein operators in the following way:

$$\left(B_m^{(\mathscr{F})} \tilde{h}\right)(x) = \sum_{k=0}^{m}{}^* \binom{m}{k} x^k (1-x)^{m-k} \odot \tilde{h}\left(\frac{k}{m}\right), \quad \forall x \in [0,1],$$

$\forall m \in \mathbb{N}$ and $\tilde{h} \in C_{\mathscr{F}}[0,1]$. Then the ℓ-cut of the function $\tilde{h}$ and its operator $B_m^{(\mathscr{F})} \tilde{h}$ will be the closed intervals,

$$\left[\tilde{h}\left(\frac{k}{m}\right)\right]^{\ell} = \left[\tilde{h}\left(\frac{k}{m}\right)_-^{(\ell)}, \tilde{h}\left(\frac{k}{m}\right)_+^{(\ell)}\right], \text{ and}$$
$$\left[B_m^{(\mathscr{F})} \tilde{h}\right]^{\ell} = \left[\left(B_m^{(\mathscr{F})} \tilde{h}\right)_-^{(\ell)}, \left(B_m^{(\mathscr{F})} \tilde{h}\right)_+^{(\ell)}\right].$$

Clearly, for any $\tilde{h} \in C_{\mathscr{F}}[0,1]$ and $\ell \in [0,1]$,

$$\left(B_m^{(\mathscr{F})} \tilde{h}\right)_{\pm}^{(\ell)} = B_m\left(\tilde{h}_{\pm}^{(\ell)}\right),$$

where B_m represents the traditional Bernstein operators.

Lemma 5.4.8 *The fuzzy Bernstein operators are characterized as linear positive fuzzy operators.*

Proof. Consider fuzzy continuous functions $\tilde{h}$ and $\tilde{g}$ defined on the interval $[0,1]$. We have,

$$\left(B_m^{(\mathscr{F})}\left(\tilde{h} \oplus \tilde{g}; x\right)\right)_{\pm}^{(\ell)} = B_m\left(\left(\tilde{h} \oplus \tilde{g}\right)_{\pm}^{(\ell)}; x\right),$$

for every $x \in [0,1]$ and $\ell \in [0,1]$, respectively. Given that $B_m^{(\mathscr{F})}\left(\tilde{h} \oplus \tilde{g}\right) \in C_{\mathscr{F}}[0,1]$, and taking into account the representation of $B_m^{(\mathscr{F})}$, we obtain

$$\left(B_m^{(\mathscr{F})}\left(\tilde{h} \oplus \tilde{g}\right)\right)_-^{\ell},\ \left(B_m^{(\mathscr{F})}\left(\tilde{h} \oplus \tilde{g}\right)\right)_+^{\ell} \in C[0,1].$$

Taking into account theorem 5.3.2 and the linearity of $B_m^{(\mathscr{F})}$, we can say,

$$\begin{aligned} B_m\left(\left(\tilde{h} \oplus \tilde{g}\right)_{\pm}^{(\ell)};x\right) &= B_m\left(\tilde{h}_{\pm}^{(\ell)} + \tilde{g}_{\pm}^{(\ell)};x\right) \\ &= B_m\left(\tilde{h}_{\pm}^{(\ell)};x\right) + B_m\left(\tilde{g}_{\pm}^{(\ell)};x\right). \end{aligned}$$

Thus we obtain,

$$\left(B_m^{(\mathscr{F})}\left(\tilde{h} \oplus \tilde{g};x\right)\right)_{\pm}^{(\ell)} = \left(B_m^{(\mathscr{F})}\left(\tilde{h};x\right)\right)_{\pm}^{(\ell)} + \left(B_m^{(\mathscr{F})}\left(\tilde{g};x\right)\right)_{\pm}^{(\ell)},$$

for each $x \in [0,1]$, and $\ell \in [0,1]$.

Using the above equation and taking into account the summation over the interval, we obtain,

$$\begin{aligned} &\left[B_m^{(\mathscr{F})}\left(\tilde{h} \oplus \tilde{g};x\right)\right]^{\ell} \\ &= \left[\left(B_m^{(\mathscr{F})}\left(\tilde{h} \oplus \tilde{g};x\right)\right)_-^{(\ell)}, \left(B_m^{(\mathscr{F})}\left(\tilde{h} \oplus \tilde{g};x\right)\right)_+^{(\ell)}\right] \\ &= \left[\left(B_m^{(\mathscr{F})}\left(\tilde{h};x\right)\right)_-^{(\ell)} + \left(B_m^{(\mathscr{F})}\left(\tilde{g};x\right)\right)_-^{(\ell)}, \left(B_m^{(\mathscr{F})}\left(\tilde{h};x\right)\right)_+^{(\ell)} + \left(B_m^{(\mathscr{F})}\left(\tilde{g};x\right)\right)_+^{(\ell)}\right] \\ &= \left[\left(B_m^{(\mathscr{F})}\left(\tilde{h};x\right)\right)_-^{(\ell)}, \left(B_m^{(\mathscr{F})}\left(\tilde{h};x\right)\right)_+^{(\ell)}\right] + \left[\left(B_m^{(\mathscr{F})}\left(\tilde{g};x\right)\right)_-^{(\ell)}, \left(B_m^{(\mathscr{F})}\left(\tilde{g};x\right)\right)_+^{(\ell)}\right] \\ &= \left[B_m^{(\mathscr{F})}\left(\tilde{h};x\right)\right]^{\ell} + \left[B_m^{(\mathscr{F})}\left(\tilde{g};x\right)\right]^{\ell} \\ &= \left[B_m^{(\mathscr{F})}\left(\tilde{h};x\right) \oplus B_m^{(\mathscr{F})}\left(\tilde{g};x\right)\right]^{\ell} \\ &= \left[\left(B_m^{(\mathscr{F})}\left(\tilde{h}\right) \oplus B_m^{(\mathscr{F})}\left(\tilde{g}\right);x\right)\right]^{\ell}, \end{aligned}$$

for each $x \in [0,1]$ and $\ell \in [0,1]$. Thus,

$$B_m^{(\mathscr{F})}\left(\tilde{h} \oplus \tilde{g}\right) = B_m^{(\mathscr{F})}\left(\tilde{h}\right) \oplus B_m^{(\mathscr{F})}\left(\tilde{g}\right), \quad \tilde{h}, \tilde{g} \in C_F\left[0,1\right].$$

Suppose that $k \geq 0$ be any real number. And, for each $x \in [0,1]$ and $\ell \in [0,1]$,

$$\left(B_m^{(\mathscr{F})}\left(k \odot \tilde{h};x\right)\right)_{\pm}^{(\ell)} = B_m\left(\left(k \odot \tilde{h}\right)_{\pm}^{(\ell)};x\right),$$

for each $x \in [0,1]$ and $\ell \in [0,1]$, respectively. Since, $B_m^{(\mathscr{F})}\left(k \odot \tilde{h}\right) \in C_{\mathscr{F}}[0,1]$, and taking into account the representation of $B_m^{(\mathscr{F})}$, we obtain $\left(B_m^{(\mathscr{F})}\left(k \odot \tilde{h}\right)\right)_-^{(\ell)}$, $\left(B_m^{(\mathscr{F})}\left(k \odot \tilde{h}\right)\right)_+^{(\ell)} \in C[0,1]$.

Taking into account Theorem 5.3.2 and the linearity of $B_m^{(\mathscr{F})}$, we can say,

$$\begin{aligned} B_m\left(\left(k \odot \tilde{h}\right)_\pm^{(\ell)};x\right) &= B_m\left(k\tilde{h}_\pm^{(\ell)};x\right) \\ &= kB_m\left(\tilde{h}_\pm^{(\ell)};x\right), \end{aligned}$$

for every $x \in [0,1]$ and $\ell \in [0,1]$. Thus we obtain,

$$\begin{aligned} \left(B_m^{(\mathscr{F})}\left(k \odot \tilde{h};x\right)\right)_\pm^{(\ell)} &= k\left(B_m^{(\mathscr{F})}\left(\tilde{h};x\right)\right)_\pm^{(\ell)} \\ &= \left(\left(k \odot B_m^{(\mathscr{F})}\right)\left(\tilde{h};x\right)\right)_\pm^{(\ell)}, \end{aligned}$$

for every $x \in [0,1]$, and $\ell \in [0,1]$.

Using the above equation we have,

$$\begin{aligned} &\left[B_m^{(\mathscr{F})}\left(k \odot \tilde{h};x\right)\right]^\ell \\ &= \left[\left(B_m^{(\mathscr{F})}\left(k \odot \tilde{h};x\right)\right)_-^{(\ell)}, \left(B_m^{(\mathscr{F})}\left(k \odot \tilde{h};x\right)\right)_+^{(\ell)}\right] \\ &= \left[\left(\left(k \odot B_m^{(\mathscr{F})}\right)\left(\tilde{h};x\right)\right)_-^{(\ell)}, \left(\left(k \odot B_m^{(\mathscr{F})}\right)\left(\tilde{h};x\right)\right)_+^{(\ell)}\right] \\ &= \left[\left(k \odot B_m^{(\mathscr{F})}\right)\left(\tilde{h};x\right)\right]^\ell. \end{aligned}$$

Therefore,

$$B_m^{(\mathscr{F})}\left(k \odot \tilde{h}\right) = \left(k \odot B_m^{(\mathscr{F})}\right)\left(\tilde{h}\right), \quad k \geq 0, \tilde{h} \in C_{\mathscr{F}}[0,1].$$

Similarly, we can prove for $k < 0$. Thus, the fuzzy Bernstein operators are fuzzy linear operators. Consider fuzzy continuous functions $\tilde{h}$ and $\tilde{g}$ defined on the interval $[0,1]$ with $\tilde{h} \precsim \tilde{g}$, where $\precsim$ is a partial order on $C_{\mathscr{F}}[0,1]$ previously defined. Then, $\tilde{h}_-^{(\ell)} \leq \tilde{g}_-^{(\ell)}$ and $\tilde{h}_+^{(\ell)} \leq \tilde{g}_+^{(\ell)}$, where $\leq$ is a partial order on $C[0,1]$.

Since, $\tilde{h}_-^{(\ell)}, \tilde{h}_+^{(\ell)}, \tilde{g}_-^{(\ell)}$ and $\tilde{g}_+^{(\ell)} \in C[0,1]$ and by the positivity of B_m, we have,

$$B_m\left(\tilde{h}_\pm^{(\ell)}\right) \leq B_m\left(\tilde{g}_\pm^{(\ell)}\right), \quad \ell \in [0,1].$$

Considering the above equation and Theorem 5.3.2, we obtain,

$$\left(B_m^{(\mathscr{F})}\left(\tilde{h}\right)\right)_\pm^{(\ell)} \leq \left(B_m^{(\mathscr{F})}\left(\tilde{g}\right)\right)_\pm^{(\ell)}, \quad \ell \in [0,1].$$

Thus,

$$B_m^{(\mathscr{F})}\left(\tilde{h};x\right) \precsim B_m^{(\mathscr{F})}\left(\tilde{g};x\right), \quad x \in [0,1],\ \ell \in [0,1].$$

This gives the positivity of $B_m^{(\mathscr{F})}$. ◻

Since,

$$B_m(1) \to 1$$
$$B_m(t) \to x$$
$$B_m\left(t^2\right) \to x^2,$$

thus, by fuzzy Korovkin theorem we can say that,

$$D^*\left(B_m^{(\mathscr{F})}\tilde{h}(t), \tilde{h}(x)\right) \to 0, \ \text{ as } m \to \infty.$$

Theorem 5.4.9 *Let $\tilde{h}(x)$ is a continuous, bounded and twice differential fuzzy function with membership value for each $\tilde{h}(x)$ as $\mu_h(x)$, defined for all $x \in [0,1]$. Then,*

$$\lim_{m\to\infty} mD^*\left(B_m^{(\mathscr{F})}\tilde{h}(t), \tilde{h}(x)\right) \leq \frac{1}{2}x(1-x)D^*\left(\tilde{h}''(x), \tilde{o}\right),$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathscr{F}}$.

Proof. Consider,

$$\begin{aligned} D^*\left(B_m^{(\mathscr{F})}\tilde{h}\left(\frac{k}{m}\right), \tilde{h}(x)\right) &= \sup_{x\in[0,1]} D\left(B_m^{(\mathscr{F})}\tilde{h}\left(\frac{k}{m}\right), \tilde{h}(x)\right) \\ &= \sup_{x\in[0,1]} D\left(\sum_{k=0}^{m}{}^* p_{m,k}(x) \odot \tilde{h}\left(\frac{k}{m}\right), \sum_{k=0}^{m}{}^* p_{m,k}(x) \odot \tilde{h}(x)\right) \\ &\leq \sup_{x\in[0,1]} \sum_{k=0}^{m} p_{m,k}(x) D\left(\tilde{h}\left(\frac{k}{m}\right), \tilde{h}(x)\right). \end{aligned}$$

Now,

$$D\left(\tilde{h}\left(\frac{k}{m}\right), \tilde{h}(x)\right) = \sup_{r\in[0,1]} \max\left\{\left|\tilde{h}\left(\frac{k}{m}\right)_-^{(\ell)} - \tilde{h}(x)_-^{(\ell)}\right|, \left|\tilde{h}\left(\frac{k}{m}\right)_+^{(\ell)} - \tilde{h}(x)_+^{(\ell)}\right|\right\}.$$

From Representation theorem/ Characteristic theorem,

$$\tilde{h}\left(\frac{k}{m}\right)_\pm^{(\ell)} = \tilde{h}(x)_\pm^{(\ell)} + \left(\frac{k}{m} - x\right)\tilde{h}'(x)_\pm^{(\ell)} + \frac{1}{2!}\left(\frac{k}{m} - x\right)^2 \tilde{h}''(x)_\pm^{(\ell)} + \frac{1}{2!}\sum_{x}^{k/m}\left(\frac{k}{m} - x\right)^2 \tilde{h}'''(x)_\pm^{(\ell)}.$$

Thus,

$$\left[\tilde{h}\left(\frac{k}{m}\right)\right]^{\ell} = \left[\tilde{h}(x)\right]^{\ell} + \left(\frac{k}{m}-x\right)\left[\tilde{h}'(x)\right]^{\ell} + \frac{1}{2!}\left(\frac{k}{m}-x\right)^{2}\left[\tilde{h}''(x)\right]^{\ell} + \frac{1}{2!}\left[\sum_{x}^{k/m}{}^{*}\left(\frac{k}{m}-x\right)^{2}\odot\tilde{h}'''(x)\right]^{\ell}$$

$$\Rightarrow \left[\tilde{h}\left(\frac{k}{m}\right)\right]^{\ell} - \left[\tilde{h}(x)\right]^{\ell}$$

$$= \left(\frac{k}{m}-x\right)\left[\tilde{h}'(x)\right]^{\ell} + \frac{1}{2!}\left(\frac{k}{m}-x\right)^{2}\left[\tilde{h}''(x)\right]^{\ell} + \frac{1}{2!}\left[\sum_{x}^{k/m}{}^{*}\left(\frac{k}{m}-x\right)^{2}\odot\tilde{h}'''(x)\right]^{\ell}$$

$$\Rightarrow \left[\tilde{h}\left(\frac{k}{m}\right)_{-}^{(\ell)}, \tilde{h}\left(\frac{k}{m}\right)_{+}^{(\ell)}\right] - \left[\tilde{h}(x)_{-}^{(\ell)}, \tilde{h}(x)_{+}^{(\ell)}\right]$$

$$= \left(\frac{k}{m}-x\right)\left[\tilde{h}'(x)_{-}^{(\ell)}, \tilde{h}'(x)_{+}^{(\ell)}\right] + \frac{1}{2!}\left(\frac{k}{m}-x\right)^{2}\left[\tilde{h}''(x)_{-}^{(\ell)}, \tilde{h}''(x)_{+}^{(\ell)}\right] + \frac{1}{2!}\left[\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^{2}\tilde{h}'''(x)_{-}^{(\ell)}, \sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^{2}\tilde{h}'''(x)_{+}^{(\ell)}\right]$$

$$\Rightarrow \left[\tilde{h}\left(\frac{k}{m}\right)_{-}^{(\ell)} - \tilde{h}(x)_{-}^{(\ell)}, \tilde{h}\left(\frac{k}{m}\right)_{+}^{(\ell)} - \tilde{h}(x)_{+}^{(\ell)}\right]$$

$$= \Bigg[\left(\frac{k}{m}-x\right)\tilde{h}'(x)_{-}^{(\ell)} + \frac{1}{2!}\left(\frac{k}{m}-x\right)^{2}\tilde{h}''(x)_{-}^{(\ell)} + \frac{1}{2!}\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^{2}\tilde{h}'''(x)_{-}^{(\ell)},$$
$$\left(\frac{k}{m}-x\right)\tilde{h}'(x)_{+}^{(\ell)} + \frac{1}{2!}\left(\frac{k}{m}-x\right)^{2}\tilde{h}''(x)_{+}^{(\ell)} + \frac{1}{2!}\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^{2}\tilde{h}'''(x)_{+}^{(\ell)}\Bigg]$$
$$= [A, B].$$

Thus, we have,

$$mD^{*}\left(B_{m}^{(\mathscr{F})}\tilde{h}\left(\frac{k}{m}\right), \tilde{h}(x)\right) = \sup_{x\in[0,1]} mD\left(\sum_{k=0}^{m}{}^{*}p_{m,k}(x)\odot\tilde{h}\left(\frac{k}{m}\right), \sum_{k=0}^{m}{}^{*}p_{m,k}(x)\odot\tilde{h}(x)\right)$$
$$\leq \sup_{x\in[0,1]} m\sum_{k=0}^{m} p_{m,k}(x)\, D\left(\tilde{h}\left(\frac{k}{m}\right), \tilde{h}(x)\right)$$
$$= \sup_{x\in[0,1]} m\sum_{k=0}^{m} p_{m,k}(x) \sup_{\ell\in[0,1]} \max\{A, B\}$$
$$= \sup_{x\in[0,1]} \sup_{\ell\in[0,1]} \max\left\{m\sum_{k=0}^{m} Ap_{m,k}(x), n\sum_{k=0}^{m} Bp_{m,k}(x)\right\}.$$

Now,

$$m\sum_{k=0}^{m}p_{m,k}(x)\left[\left(\frac{k}{m}-x\right)\tilde{h}'(x)_{-}^{(\ell)}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^{2}\tilde{h}''(x)_{-}^{(\ell)}+\frac{1}{2!}\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^{2}\tilde{h}'''(x)_{-}^{(\ell)}\right]$$
$$=\sum_{k=0}^{m}m\left(\frac{k}{m}-x\right)p_{m,k}(x)\tilde{h}'(x)_{-}^{(\ell)}+\frac{1}{2!}\sum_{k=0}^{m}m\left(\frac{k}{m}-x\right)^{2}p_{m,k}(x)\tilde{h}''(x)_{-}^{(\ell)}+mR_{m}(x)$$
$$=\frac{1}{2}x(1-x)\tilde{h}''(x)_{-}^{(\ell)}.$$

Similarly,

$$m\sum_{k=0}^{m}p_{m,k}(x)\left[\left(\frac{k}{m}-x\right)\tilde{h}'(x)_{+}^{(\ell)}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^{2}\tilde{h}''(x)_{+}^{(\ell)}+\frac{1}{2!}\sum_{x}^{k/m}\left(\frac{k}{m}-x\right)^{2}\tilde{h}'''(x)_{+}^{(\ell)}\right]$$
$$=\frac{1}{2}x(1-x)\tilde{h}''(x)_{+}^{(\ell)}.$$

That is,

$$\lim_{m\to\infty}mD^{*}\left(B_{m}^{(\mathscr{F})}\tilde{h}\left(\frac{k}{m}\right),\tilde{h}(x)\right)$$
$$\leq\sup_{x\in[0,1]}\sup_{\ell\in[0,1]}\max\left\{\frac{1}{2}x(1-x)\tilde{h}''(x)_{-}^{(\ell)},\frac{1}{2}x(1-x)\tilde{h}''(x)_{+}^{(\ell)}\right\}$$
$$=\sup_{x\in[0,1]}\frac{1}{2}x(1-x)\sup_{\ell\in[0,1]}\max\left\{\tilde{h}''(x)_{-}^{(\ell)},\tilde{h}''(x)_{+}^{(\ell)}\right\}$$
$$=\sup_{x\in[0,1]}\frac{1}{2}x(1-x)D\left(\tilde{h}''(x),\tilde{o}\right)$$
$$\leq\frac{1}{2}x(1-x)D^{*}\left(\tilde{h}''(x),\tilde{o}\right),$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathscr{F}}$. □

5.4.2 Fuzzy Szász-Mirakyan Operators

The fuzzy Szász-Mirakyan operators are defined as,

$$\left(S_{m}^{(\mathscr{F})}\tilde{h}\right)(x)=e^{-mx}\sum_{k=0}^{\infty}{}^{*}\frac{(mx)^{k}}{k!}\odot\tilde{h}\left(\frac{k}{m}\right),$$

where, $x\in[0,\infty)$ and $\tilde{h}$ is a fuzzy valued function from $\mathbb{R}^{+}$ to $\mathbb{R}_{F}^{+}$.

From Theorems 5.2.6 and 5.3.2, we can say that, there exists a corresponding sequence $S_{m}(h;x)$ of linear positive operators from $C[0,\infty)$ into itself with property,

$$\left(S_{m}^{(\mathscr{F})}(\tilde{h})\right)_{\pm}^{(\ell)}=S_{m}\left(\tilde{h}_{\pm}^{(\ell)}\right)$$

respectively, $\forall \ell \in [0,1]$ and $\forall \tilde{h} \in C_{\mathscr{F}}[0,\infty)$, where, $S_m(h;x)$ are the classical known Szász-Mirakyan operators.

By fuzzy Korovkin theorem, we can say that,

$$D^{*}\left(S_m^{(\mathscr{F})}\left(\tilde{h}\right), \tilde{h}\right) \to 0, \quad \text{as } m \to \infty.$$

Lemma 5.4.10 *The moment generating function of the Szász-Mirakyan operators is,*

$$S_m\left(e^{-\alpha t};x\right) = e^{-\alpha x \frac{1-e^{-\alpha/m}}{\alpha/m}}. \tag{5.1}$$

Based on (5.1)*, we can claim the following exponential moments of Szász operators:*

(i) $S_m(1;x) = 1$

(ii) $S_m(e^{-t};x) = e^{-mx\left(1-e^{-1/m}\right)} \to e^{-x}$

(iii) $S_m\left(e^{-2t};x\right) = e^{-mx\left(1-e^{-2/m}\right)} \to e^{-2x}$.

Since, $\left\{1, e^{-x}, e^{-2x}\right\}$ is a Korovkin set, the above lemma also confirms the Korovkin theorem. By simple calculation we can show that

$$\left|S_m\left(e^{-\alpha t};x\right) - e^{-\alpha x}\right| = e^{-\alpha x}\left|\left(e^{-x}\right)^{m-\alpha-me^{-\alpha/m}} - 1\right|. \tag{5.2}$$

Define a_m, b_m and c_m as the central moments corresponding to the Korovkin set $\left\{1, e^{-x}, e^{-2x}\right\}$ obtained by putting $\alpha = 0,1,2$, respectively in (5.2). Then, it is easy to verify that $a_m, b_m, c_m \to 0$ as $m \to \infty$.

Theorem 5.4.11 *If $f \in C_F[0,\infty)$, then for $m \geq 1$,*

$$D^{*}\left(S_m^{(\mathscr{F})}\left(\tilde{h}\right), \tilde{h}(x)\right) \leq 2\omega_{*}^{\mathscr{F}}\left(\tilde{h}; \frac{1}{\sqrt{m}}\right),$$

where $\omega_{*}^{\mathscr{F}}$ is the fuzzy exponential type modulus of continuity.

Proof. Let, $s_{m,k}(x) = e^{-mx}\frac{(mx)^k}{k!} \Rightarrow \sum_{k=0}^{\infty} s_{m,k}(x) = 1$. We can write,

$$\tilde{h}(x) = \left[\sum_{k=0}^{\infty} s_{m,k}(x)\right] \odot \tilde{h}(x) = \sum_{k=0}^{\infty}{}^{*}\left(s_{m,k}(x) \odot \tilde{h}(x)\right).$$

Thus,

$$
\begin{aligned}
D^{*}\left(S_m^{(\mathscr{F})}\left(\tilde{h}\right),\tilde{h}\left(x\right)\right) &= D^{*}\left(\sum_{k=0}^{\infty}{}^{*}s_{m,k}\left(x\right)\odot\tilde{h}\left(\frac{k}{m}\right),\sum_{k=0}^{\infty}{}^{*}s_{m,k}\left(x\right)\odot\tilde{h}\left(x\right)\right)\\
&\leq \sum_{k=0}^{\infty}s_{m,k}\left(x\right)D^{*}\left(\tilde{h}\left(\frac{k}{m}\right),\tilde{h}\left(x\right)\right)\\
&\leq \sum_{k=0}^{\infty}s_{m,k}\left(x\right)\left[1+\frac{\left(e^{-k/m}-e^{-x}\right)^2}{\delta^2}\right]\omega_*^{\mathscr{F}}\left(\tilde{h};\delta\right)\\
&= \omega_*^{\mathscr{F}}\left(\tilde{h};\delta\right)+\sum_{k=0}^{\infty}s_{m,k}\left(x\right)\frac{\left(e^{-k/m}-e^{-x}\right)^2}{\delta^2}\omega_*^{\mathscr{F}}\left(\tilde{h};\delta\right)\\
&\leq \omega_*^{\mathscr{F}}\left(\tilde{h};\delta\right)+\frac{a_m+2b_m+c_m}{\delta^2}\omega_*^{\mathscr{F}}\left(\tilde{h};\delta\right).
\end{aligned}
$$

Now, for every $m \geq 1$,

$$e^{-x\gamma_m}-e^{-x}<\frac{x_m}{2e},$$

where, $\gamma_m = \frac{1-e^{-x_m}}{x_m}$ and $x_m > 0$, for every $m \geq 1$. Hence, we get,

$$\left|S_m\left(e^{-\alpha t};x\right)-e^{-\alpha x}\right| \leq \frac{\alpha}{2em}.$$

It follows that,

$$b_m \leq \frac{1}{2em} \text{ and } c_m \leq \frac{1}{em}, \text{ for } m \geq 1,$$

which implies,

$$a_m+2b_m+c_m \leq \frac{1}{2em}+\frac{1}{em} \leq \frac{1}{m}, \text{ for } m \geq 1.$$

Taking $\delta = 1/\sqrt{m}$, we arrive at our result. □

Theorem 5.4.12 *Let $\tilde{h}(x)$ is a continuous, bounded and twice differential fuzzy function with membership value for each $\tilde{h}(x)$ as $\mu_h(x)$, defined for all $x \in [0,\infty)$. Then, for the fuzzy Szász-Mirakyan operators, we have the following Voronskya type asymptotic result:*

$$\lim_{m\to\infty} mD^{*}\left(S_m^{(\mathscr{F})}\left(\tilde{h}\right),\tilde{h}\left(x\right)\right) \leq \frac{1}{2}xD^{*}\left(\tilde{h}''\left(x\right),\tilde{o}\right),$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathscr{F}}$.

Proof. Following the proof of theorem 5.4.9, we arrive at,

$$
\begin{aligned}
&m\sum_{k=0}^{\infty}s_{m,k}\left(x\right)\left[\left(\frac{k}{m}-x\right)\tilde{h}'\left(x\right)_{\pm}^{(\ell)}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^2\tilde{h}''\left(x\right)_{\pm}^{(\ell)}+R_m\left(x\right)\right]\\
&\quad=\sum_{k=0}^{\infty}m\left(\frac{k}{m}-x\right)s_{m,k}\left(x\right)\tilde{h}'\left(x\right)_{\pm}^{(\ell)}+\frac{1}{2}\sum_{k=0}^{\infty}m\left(\frac{k}{m}-x\right)^2s_{m,k}\left(x\right)\tilde{h}''\left(x\right)_{\pm}^{(\ell)}+mR_m\left(x\right)\\
&\quad=\frac{1}{2}x\tilde{h}''\left(x\right)_{\pm}^{(\ell)}.
\end{aligned}
$$

Thus,

$$\begin{aligned}\lim_{m\to\infty} mD^*\left(S_m^{(\mathscr{F})}\tilde{h}\left(\frac{k}{m}\right),\tilde{h}(x)\right) &= \sup_{x\in[0,1]}\sup_{r\in[0,1]}\max\left\{\frac{1}{2}x\tilde{h}''(x)_-^{(\ell)},\frac{1}{2}x\tilde{h}''(x)_+^{(\ell)}\right\}\\ &= \sup_{x\in[0,1]}\frac{1}{2}xD\left(\tilde{h}''(x),\tilde{o}\right)\\ &\le \frac{1}{2}xD^*\left(\tilde{h}''(x),\tilde{o}\right),\end{aligned}$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathscr{F}}$. □

5.4.3 Fuzzy Baskakov Operators

The next fuzzy positive linear operators we will talk about are the fuzzy Baskakov operators. These operators are defined as

$$V_m^{(\mathscr{F})}\left(\tilde{h}(t);x\right)=\sum_{k=0}^{\infty}{}^*\binom{m+k-1}{k}\frac{x^k}{(1+x)^{m+k}}\odot\tilde{h}\left(\frac{k}{m}\right).$$

Again, using Theorems 5.2.6 and 5.3.2 we can claim that, there exists a corresponding sequence $V_m(h;x)$ of linear positive operators from $C[0,\infty)$ into itself with the property,

$$\left(V_m^{(\mathscr{F})}\left(\tilde{h}\right)\right)_{\pm}^{(\ell)}=V_m\left(\tilde{h}_{\pm}^{(\ell)}\right),$$

respectively, $\forall\ell\in[0,1]$ and $\forall\tilde{h}\in C_{\mathscr{F}}[0,\infty)$, where, $V_m(h;x)$ are the classical known Baskakov operators.

By fuzzy Korovkin theorem, we can say that,

$$D^*\left(V_m^{(\mathscr{F})}\left(\tilde{h}\right),\tilde{h}\right)\to 0,\quad \text{as } m\to\infty.$$

We know the moment generating function of the classical Baskakov operators is:

$$V_m\left(e^{-\alpha t};x\right)=\left(1+x\left(1-e^{-\alpha/m}\right)\right)^{-m}. \tag{5.3}$$

Lemma 5.4.13 *Using equation* (5.3)*, we can claim the following exponential moments of Baskakov operators:*

(i) $V_m(1;x)=1$

(ii) $V_m(e^{-t};x)=\left(1+x-xe^{-1/m}\right)^{-m}\to e^{-x}$

(iii) $V_m\left(e^{-2t};x\right)=\left(1+x-xe^{-2/m}\right)^{-m}\rightarrow e^{-2x}$.

Since, $\{1,e^{-x},e^{-2x}\}$ is a Korovkin set, the above lemma also confirms the Korovkin theorem. By simple calculation we can show that

$$\left|V_m\left(e^{-\alpha t};x\right)-e^{-\alpha x}\right|=e^{-\alpha x}\left|e^{-m\ln\left(1+x\left(1-e^{-\alpha/m}\right)\right)+\alpha x}-1\right|. \tag{5.4}$$

Define a_m,b_m and c_m as the expressions obtained by putting $\alpha=0,1,2$, respectively in equation (5.4). Clearly, $a_m,b_m,c_m\rightarrow 0$ as $m\rightarrow\infty$.

Theorem 5.4.14 *Let $\tilde{h}\in C_{\mathscr{F}}[0,\infty)$. Then, for $m\geq 2$,*

$$D^*\left(V_m^{(\mathscr{F})}\left(\tilde{h}\right),\tilde{h}\right)\leq 2\omega_*^{\mathscr{F}}\left(\tilde{h};\frac{5}{2\sqrt{m}}\right),$$

where $\omega_*^{\mathscr{F}}$ is the fuzzy exponential type modulus of continuity.

Proof. First, we have

$$D^*\left(V_m^{(\mathscr{F})}\tilde{h}\left(\frac{k}{m}\right),\tilde{h}(x)\right)\leq\omega_*^{\mathscr{F}}\left(\tilde{h};\delta\right)+\frac{a_m+2b_m+c_m}{\delta^2}\omega_*^{\mathscr{F}}\left(\tilde{h};\delta\right). \tag{5.5}$$

Now, using the inequality: $e^t-1\leq e^t$, for

$$t=-m\ln\left(1+x\left(1-e^{-\alpha/m}\right)\right)+\alpha x\geq -mx\left(1-e^{-\alpha/m}\right)+\alpha x\geq -mx\frac{\alpha}{m}+\alpha x=0,$$

we get,

$$\begin{aligned}\left|V_m\left(e^{-\alpha t};x\right)-e^{-\alpha x}\right|&=e^{-\alpha x}\left|e^{-m\ln\left(1+x\left(1-e^{-\alpha/m}\right)\right)+\alpha x}-1\right|\\&\leq\left[-m\ln\left(1+x\left(1-e^{-\alpha/m}\right)\right)+\alpha x\right]e^{-m\ln\left(1+x\left(1-e^{-\alpha/m}\right)\right)}.\end{aligned}$$

Since, $\ln(1+t)\geq\frac{t}{1+t}$, for every $t\geq 0$, we obtain

$$\begin{aligned}\left|V_m\left(e^{-\alpha t};x\right)-e^{-\alpha x}\right|&\leq\frac{-mx\left(1-e^{-\alpha/m}\right)+\alpha x+\alpha x^2\left(1-e^{-\alpha/m}\right)}{\left(1+x(1-e^{-\alpha/m})\right)^{m+1}}\\&\leq\frac{-mx\left(1-e^{-\alpha/m}\right)+\alpha x+\alpha x^2\left(1-e^{-\alpha/m}\right)}{1+(m+1)x(1-e^{-\alpha/m})+\frac{m(m+1)}{2}x^2(1-e^{-\alpha/m})^2}.\end{aligned}$$

Since, $1-e^{-\alpha/m}\geq\frac{\alpha}{m}-\frac{\alpha^2}{2m^2}$, thus, from the above inequality, we get

$$\sup_{x\geq 0}\left|V_m\left(e^{-\alpha t};x\right)-e^{-\alpha x}\right|\leq\frac{2\alpha}{m(m+1)(1-e^{-\alpha/m})}.$$

Using the same inequality, we obtain

$$b_m = \sup_{x\geq 0}\left|V_m\left(e^{-t};x\right)-e^{-x}\right| \leq \frac{2}{m(m+1)\left(\frac{1}{m}-\frac{1}{2m^2}\right)} \leq \frac{2}{m}, \quad \text{for } m\geq 1,$$

and using $1-e^{-2/m} \geq \frac{2}{m}-\frac{2}{m^2}+\frac{4}{3m^3}-\frac{2}{3m^4}$, we have

$$c_m = \sup_{x\geq 0}\left|V_m\left(e^{-2t};x\right)-e^{-2x}\right| \leq \frac{4}{m(m+1)\left(\frac{2}{m}-\frac{2}{m^2}+\frac{4}{3m^3}-\frac{2}{3m^4}\right)} = \frac{\zeta(m)}{m},$$

where $\zeta(t) = \frac{6t^4}{(t+1)\left(3t^3-3t^2+2t-1\right)}$. This implies,

$$\zeta'(t) = \frac{6t^3}{(t+1)^2\left(3t^3-3t^2+2t-1\right)}\left(-2t^2+3t-4\right) < 0, \quad t\geq 1$$

Thus, we obtain $\zeta(m) \leq \frac{32}{15}$, for $m\geq 2$. Finally, we obtain

$$\sqrt{a_m+2b_m+c_m} \leq \frac{1}{\sqrt{m}}\sqrt{4+\frac{32}{15}} \leq \frac{5}{2\sqrt{m}}.$$

Taking $\delta = \sqrt{a_m+2b_m+c_m} \leq \frac{5}{2\sqrt{m}}$, for $m\geq 2$, equation (5.5) becomes

$$D^*\left(V_m^{(\mathscr{F})}\left(\tilde{h}\right),\tilde{h}\right) \leq 2\omega_*^{\mathscr{F}}\left(\tilde{h};\frac{5}{2\sqrt{m}}\right).$$

□

Theorem 5.4.15 *For the fuzzy Baskakov operators and continuous fuzzy valued function f, whose ℓ^{th} level cut has continuous double derivative $\forall \ell \in [0,1]$, we have*

$$\lim_{m\to\infty} mD^*\left(V_m^{(\mathscr{F})}\left(\tilde{h}\right),\tilde{h}(x)\right) = \frac{1}{2}x(1+x)D^*\left(\tilde{h}''(x),\tilde{o}\right),$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathscr{F}}$.

Proof. Following the proof of Theorem 5.4.9, we have

$$\begin{aligned}
&m\sum_{k=0}^{\infty} v_{m,k}(x)\left[\left(\frac{k}{m}-x\right)\tilde{h}'(x)_{\pm}^{(\ell)}+\frac{1}{2!}\left(\frac{k}{m}-x\right)^2\tilde{h}''(x)_{\pm}^{(\ell)}+R_m(x)\right]\\
&=\sum_{k=0}^{\infty} m\left(\frac{k}{m}-x\right)v_{m,k}(x)\tilde{h}'(x)_{\pm}^{(\ell)}+\frac{1}{2}\sum_{k=0}^{\infty} m\left(\frac{k}{m}-x\right)^2 v_{m,k}(x)\tilde{h}''(x)_{\pm}^{(\ell)}+mR_m(x)\\
&=\frac{1}{2}x(1+x)\tilde{h}''(x)_{\pm}^{(\ell)},
\end{aligned}$$

where $v_{m,k}(x) = \binom{m+k-1}{k} \frac{x^k}{(1+x)^{m+k}}$. Thus, we get

$$\begin{aligned} \lim_{m\to\infty} mD^* &\left(V_m^{(\mathscr{F})} \tilde{h}\left(\frac{k}{m}\right), \tilde{h}(x) \right) \\ &= \sup_{x\in[0,1]} \sup_{r\in[0,1]} \max \left\{ \frac{1}{2} x(1+x) \tilde{h}''(x)_-^{(\ell)}, \frac{1}{2} x(1+x) \tilde{h}''(x)_+^{(\ell)} \right\} \\ &= \sup_{x\in[0,1]} \frac{1}{2} x(1+x) D\left(\tilde{h}''(x), \tilde{o}\right) \\ &\leq \frac{1}{2} x(1+x) D^*\left(\tilde{h}''(x), \tilde{o}\right), \end{aligned}$$

where $\tilde{o}$ is the zero element of $\mathbb{R}_{\mathscr{F}}$. □

5.4.4 Fuzzy Boas-Buck Operators

In 1956, Ralph P. Boas and R. Creighton Buck [32; 33] considered the generalized Appell polynomials by means of generating function of the type

$$\mathscr{W}(t)\,\mathscr{P}(x\mathscr{Q}(t)) = \sum_{k=0}^{\infty} a_k(x) t^k, \tag{5.6}$$

where $\mathscr{W}$, $\mathscr{P}$ and $\mathscr{Q}$ are analytic functions

$$\begin{aligned} \mathscr{W}(t) &= \sum_{k=0}^{\infty} w_k t^k, \;\; w_0 \neq 0, \\ \mathscr{P}(t) &= \sum_{k=0}^{\infty} p_k t^k, \;\; p_k \neq 0, \\ \mathscr{Q}(t) &= \sum_{k=1}^{\infty} q_k t^k, \;\; q_1 \neq 0. \end{aligned} \tag{5.7}$$

The generating function derived by Boas and Buck serves as a generalization of numerous other generating functions. This offers a framework for defining different types of polynomials as its special cases, listed in Table 5.1.

Table 5.1: Different generating functions obtained from Boas-Buck polynomials.

S.No.	Analytic Functions	Generating Function	Explicit polynomial
1.	$\mathscr{P}(t)=e^t$	Sheffer polynomials [114] $\mathscr{W}(t)e^{x\mathscr{Q}(t)}=\sum_{k=0}^{\infty}a_k(x)t^k$	–
2.	$\mathscr{Q}(t)=t$	Brenke polynomials [53] $\mathscr{W}(t)\mathscr{P}(xt)=\sum_{k=0}^{\infty}a_k(x)t^k$	$a_k(x)=\sum_{r=0}^{k}w_{k-r}p_rx^r$
3.	$\mathscr{P}(t)=e^t$ $\mathscr{Q}(t)=t$	Appell polynomials [117] $\mathscr{W}(t)e^{tx}=\sum_{k=0}^{\infty}a_k(x)t^k$	–
4.	$\mathscr{W}(t)=1$ $\mathscr{P}(t)=e^t$ $\mathscr{Q}(t)=t$	Exponential polynomials [202] $e^{tx}=\sum_{k=0}^{\infty}a_k(x)t^k$	$a_k(x)=x^k/k!$

Our study will be confined to the Boas-Buck type polynomials that satisfy:

$$\begin{aligned}&(i)\ \ \mathscr{W}(1)\neq 0,\ \ \mathscr{Q}'(1)=1,\ \ a_k(x)\geq 0,\ k=0,1,2,\ldots,\\&(ii)\ \ \mathscr{P}:\mathbb{R}\to(0,\infty),\\&(iii)\ \ \text{For } |t|<\rho \ (\text{where } \rho>1), \text{ the equations (5.6) and (5.7) are convergent.}\end{aligned}\tag{5.8}$$

With the constraints mentioned above in (5.8), Sucu et al. [200] presented the linear positive operators using Boas-Buck type polynomials in the subsequent manner:

$$\mathscr{B}_n f=\frac{1}{\mathscr{W}(1)\,\mathscr{P}(nx\mathscr{Q}(1))}\sum_{k=0}^{\infty}a_k(nx)\,f\left(\frac{k}{n}\right),\tag{5.9}$$

where $x\geq 0$ and $n\in\mathbb{N}$.

In this section, we broaden the scope of previous research on real operators to fuzzy sense. Our study begins by defining the fuzzy Boas Buck operators, based on the generating function (5.6) and assumptions (5.8), presented as follows:

$$\tilde{\mathscr{B}}_n\tilde{f}:=\frac{1}{\mathscr{W}(1)\,\mathscr{P}(nx\mathscr{Q}(1))}\sum_{k=0}^{\infty}{}^{*}\,a_k(nx)\odot\tilde{f}\left(\frac{k}{n}\right),\tag{5.10}$$

where $\sum^*$ and $\odot$ represent fuzzy addition and multiplication, respectively and $\tilde{f}$ is a fuzzy-valued function.

Distinct fuzzy operators emerge from different formulations of the defined fuzzy Boas-Buck operators (5.10), as detailed in Table 5.2. Researchers can further investigate the moments and convergence properties associated with these operators, employing analogous approaches within the fuzzy domain.

Table 5.2: Special cases of different fuzzy operators obtained from the fuzzy Boas-Buck operators.

S.No.	Analytic Functions	Fuzzy Operators
1.	$\mathscr{P}(t) = e^t$	Fuzzy Sheffer operators $\tilde{T}_n\tilde{f} = \frac{e^{-nx\mathscr{Q}(1)}}{\mathscr{W}(1)} \sum_{k=0}^{\infty}{}^{*} a_k(nx) \odot \tilde{f}\left(\frac{k}{n}\right)$
2.	$\mathscr{Q}(t) = t$	Fuzzy Brenke operators $\tilde{L}_n\tilde{f} = \frac{1}{\mathscr{W}(1)\mathscr{P}(nx)} \sum_{k=0}^{\infty}{}^{*} a_k(nx) \odot \tilde{f}\left(\frac{k}{n}\right)$
3.	$\mathscr{P}(t) = e^t$ $\mathscr{Q}(t) = t$	Fuzzy Appell operators $\tilde{P}_n\tilde{f} = \frac{e^{-nx}}{\mathscr{W}(1)} \sum_{k=0}^{\infty}{}^{*} a_k(nx) \odot \tilde{f}\left(\frac{k}{n}\right)$
4.	$\mathscr{W}(t) = 1$ $\mathscr{P}(t) = e^t$ $\mathscr{Q}(t) = t$	Fuzzy Szasz operators $\tilde{S}_n\tilde{f} = e^{-nx} \sum_{k=0}^{\infty} * \frac{(nx)^k}{k!} \odot \tilde{f}\left(\frac{k}{n}\right)$

Clearly, the Boas-Buck fuzzy operators represent the most general form among the operators mentioned above, thus we shall primarily focus on exploring the approximation properties of these operators. While the previously mentioned fuzzy analogs also warrant individual in-depth study, our discussion in this section centers only on operators (5.10). First we relate the fuzzy Boas Buck operators to the classical ones as defined by Sucu et al. [200] and then prove that the proposed operators are fuzzy positive linear operators. Further, the approximation properties of the fuzzy Boas-Buck operators are proved using the fuzzy Korovkin theorem and fuzzy weighted modulus of continuity.

Lemma 5.4.16 *For the fuzzy operators $\tilde{\mathscr{B}}_n$, we have*

$$\left(\tilde{\mathscr{B}}_n\tilde{h}\right)_{\pm}^{(\ell)} = \mathscr{B}_n\left(\tilde{h}_{\pm}^{(\ell)}\right),$$

where $\mathscr{B}_n$ are the classical Boas-Buck operators in real sense as defined by (5.9).

Proof. For fuzzy operators (5.10) and fuzzy function $\tilde{h}$, we can write

$$\begin{aligned}\left(\tilde{\mathscr{B}}_n\tilde{h}\right)_{\pm}^{(\ell)} &= \left(\frac{1}{\mathscr{W}(1)\,\mathscr{P}(nx\mathscr{Q}(1))}\sum_{k=0}^{\infty}{}^{*}a_k(nx)\odot\tilde{h}\left(\frac{k}{n}\right)\right)_{\pm}^{(\ell)} \\ &= \frac{1}{\mathscr{W}(1)\,\mathscr{P}(nx\mathscr{Q}(1))}\sum_{k=0}^{\infty}a_k(nx)\left(\tilde{h}\left(\frac{k}{n}\right)\right)_{\pm}^{(\ell)} \\ &= \mathscr{B}_n\left(\tilde{h}_{\pm}^{(\ell)}\right).\end{aligned}$$

□

Lemma 5.4.17 *The fuzzy Boas-Buck operators are characterized as fuzzy positive linear operators.*

Proof. Consider fuzzy continuous functions $\tilde{h}$ and $\tilde{g}$ defined on the interval $[0,\infty)$. Using the linearity of the real Boas-Buck operators and Lemma 5.4.16 we have,

$$\begin{aligned}\left(\tilde{\mathscr{B}}_n\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)\right)_{\pm}^{(\ell)} &= \mathscr{B}_n\left(\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)_{\pm}^{(\ell)}\right) \\ &= \mathscr{B}_n\left(\left(a\odot\tilde{h}\right)_{\pm}^{(\ell)}+\left(b\odot\tilde{g}\right)_{\pm}^{(\ell)}\right) \\ &= \mathscr{B}_n\left(a\tilde{h}_{\pm}^{(\ell)}+b\tilde{g}_{\pm}^{(\ell)}\right) \\ &= a\mathscr{B}_n\left(\tilde{h}_{\pm}^{(\ell)}\right)+b\mathscr{B}_n\left(\tilde{g}_{\pm}^{(\ell)}\right) \\ &= a\left(\tilde{\mathscr{B}}_n(\tilde{h})\right)_{\pm}^{(\ell)}+b\left(\tilde{\mathscr{B}}_n(\tilde{g})\right)_{\pm}^{(\ell)},\end{aligned} \tag{5.11}$$

for every $x\in[0,\infty)$ and $\ell\in[0,1]$.

Using equation (5.11), we obtain

$$\begin{aligned}&\left[\tilde{\mathscr{B}}_n\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)\right]^{\ell} \\ &\quad= \left[\left(\tilde{\mathscr{B}}_n\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)\right)_{-}^{(\ell)},\left(\tilde{\mathscr{B}}_n\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)\right)_{+}^{(\ell)}\right] \\ &\quad= \left[a\left(\tilde{\mathscr{B}}_n(\tilde{h})\right)_{-}^{(\ell)}+b\left(\tilde{\mathscr{B}}_n(\tilde{g})\right)_{-}^{(\ell)},a\left(\tilde{\mathscr{B}}_n(\tilde{h})\right)_{+}^{(\ell)}+b\left(\tilde{\mathscr{B}}_n(\tilde{g})\right)_{+}^{(\ell)}\right] \\ &\quad= \left[a\left(\tilde{\mathscr{B}}_n(\tilde{h})\right)_{-}^{(\ell)},a\left(\tilde{\mathscr{B}}_n(\tilde{h})\right)_{+}^{(\ell)}\right]+\left[b\left(\tilde{\mathscr{B}}_n(\tilde{g})\right)_{-}^{(\ell)},b\left(\tilde{\mathscr{B}}_n(\tilde{g})\right)_{+}^{(\ell)}\right] \\ &\quad= a\left[\tilde{\mathscr{B}}_n(\tilde{h})\right]^{\ell}+b\left[\tilde{\mathscr{B}}_n(\tilde{g})\right]^{\ell} \\ &\quad= \left[a\odot\tilde{\mathscr{B}}_n(\tilde{h})\right]^{\ell}+\left[b\odot\tilde{\mathscr{B}}_n(\tilde{g})\right]^{\ell} \\ &\quad= \left[a\odot\tilde{\mathscr{B}}_n(\tilde{h})\oplus b\odot\tilde{\mathscr{B}}_n(\tilde{g})\right]^{\ell};\qquad \forall\, x\in[0,\infty),\ \ell\in[0,1].\end{aligned}$$

This implies that for $\tilde{h},\tilde{g}\in C_{\mathscr{F}}[0,\infty)$,

$$\tilde{\mathscr{B}}_n\left(a\odot\tilde{h}\oplus b\odot\tilde{g}\right)=a\odot\tilde{\mathscr{B}}_n(\tilde{h})\oplus b\odot\tilde{\mathscr{B}}_n(\tilde{g}).$$

Thus, the fuzzy Boas-Buck operators are fuzzy linear operators. Consider fuzzy continuous functions $\tilde{h}$ and $\tilde{g}$ defined on the interval $[0,\infty)$ with $\tilde{h} \precsim \tilde{g}$. Then, $\tilde{h}_-^{(\ell)} \leq \tilde{g}_-^{(\ell)}$ and $\tilde{h}_+^{(\ell)} \leq \tilde{g}_+^{(\ell)}$, where $\leq$ is a partial order on $C[0,\infty)$.

Since $\tilde{h}_-^{(\ell)}$, $\tilde{h}_+^{(\ell)}$, $\tilde{g}_-^{(\ell)}$ and $\tilde{g}_+^{(\ell)}$ are continuous real functions on $[0,\infty)$ then by the positivity of $\mathscr{B}_n$, we have

$$\begin{aligned}
& \mathscr{B}_n\left(\tilde{h}_\pm^{(\ell)}\right) \leq \mathscr{B}_n\left(\tilde{g}_\pm^{(\ell)}\right), && \ell \in [0,1]\\
\Rightarrow\quad & \left(\tilde{\mathscr{B}}_n(\tilde{h})\right)_\pm^{(\ell)} \leq \left(\tilde{\mathscr{B}}_n(\tilde{g})\right)_\pm^{(\ell)}, && \ell \in [0,1]\\
\Rightarrow\quad & \tilde{\mathscr{B}}_n(\tilde{h}) \precsim \tilde{\mathscr{B}}_n(\tilde{g}), && \ell \in [0,1],\ x \in [0,\infty).
\end{aligned}$$

This gives the positivity of $\tilde{\mathscr{B}}_n$. □

Theorem 5.4.18 *[200] The moments of the classical sequence of operators* (5.9) *are given as follows:*

(i) $\mathscr{B}_n(1) = 1$

(ii) $\mathscr{B}_n(t) = \frac{\mathscr{P}'(nx\mathscr{Q}(1))}{\mathscr{P}(nx\mathscr{Q}(1))}x + \frac{\mathscr{W}'(1)}{n\mathscr{W}(1)}$

(iii) $\mathscr{B}_n(t^2) = \frac{\mathscr{P}''(nx\mathscr{Q}(1))}{\mathscr{P}(nx\mathscr{Q}(1))}x^2 + \frac{2\mathscr{W}'(1)+(1+\mathscr{Q}'(1))\mathscr{W}(1)}{n\mathscr{W}(1)\mathscr{P}(nx\mathscr{Q}(1))}\mathscr{P}'(nx\mathscr{Q}(1))x + \frac{\mathscr{W}''(1)+\mathscr{W}'(1)}{n^2\mathscr{W}(1)}$.

Assume that $\lim\limits_{y\to\infty}\frac{\mathscr{P}'(y)}{\mathscr{P}(y)} = 1$ and $\lim\limits_{y\to\infty}\frac{\mathscr{P}''(y)}{\mathscr{P}(y)} = 1$. Then, using fuzzy Korovkin theorem [17], $D^*\left(\tilde{\mathscr{B}}_n\tilde{h}, \tilde{h}(x)\right) \to 0$, as $n\to\infty$. Thus, $\tilde{\mathscr{B}}_n\tilde{h}$ are a sequence of fuzzy positive linear operators converging to $\tilde{h}$ as $n\to\infty$.

Lemma 5.4.19 *Let us denote* $b_{n,k}(x) = \frac{a_k(nx)}{\mathscr{W}(1)\mathscr{P}(nx\mathscr{Q}(1))}$. *Then by Theorem 5.4.18, we have*

(i) $\sum\limits_{k=0}^{\infty} b_{n,k}(x) = 1,$

(ii) $\sum\limits_{k=0}^{\infty}(t-x)^2 b_{n,k}(x) = \frac{\mathscr{W}''(1)+\mathscr{W}'(1)}{n^2\mathscr{W}(1)} + \frac{\mathscr{P}''(nx\mathscr{Q}(1))-2\mathscr{P}'(nx\mathscr{Q}(1))+\mathscr{P}(nx\mathscr{Q}(1))}{\mathscr{P}(nx\mathscr{Q}(1))}x^2 + \frac{\mathscr{W}(1)(1+\mathscr{Q}''(1))\mathscr{P}'(nx\mathscr{Q}(1))+2\mathscr{W}'(1)(\mathscr{P}'(nx\mathscr{Q}(1))-\mathscr{P}(nx\mathscr{Q}(1)))}{n\mathscr{W}(1)\mathscr{P}(nx\mathscr{Q}(1))}.$

Theorem 5.4.20 *If* $\tilde{h} \in C_{\mathscr{F}}[0,\infty)$*, then for* $n \geq 1$

$$D^*\left(\tilde{\mathscr{B}}_n\tilde{h}, \tilde{h}(x)\right) \leq 2\omega_1^{\mathscr{F}}\left(\tilde{h}; \sqrt{\varphi_{2,n}(x)}\right),$$

where $\omega_1^{\mathscr{F}}$ *is the fuzzy first order modulus of continuity and* $\varphi_{m,n}(x) = \mathscr{B}_n((t-x)^m; x)$.

Proof. Let $b_{n,k}(x) = \frac{a_k(nx)}{\mathscr{W}(1)\mathscr{P}(nx\mathscr{Q}(1))}$, then we can write,

$$\tilde{h}(x) = \left[\sum_{k=0}^{\infty} b_{n,k}(x)\right] \odot \tilde{h}(x) = \sum_{k=0}^{\infty}{}^{*} \left(b_{n,k}(x) \odot \tilde{h}(x)\right).$$

Thus,

$$\begin{aligned} D^*\left(\tilde{\mathscr{B}}_n\tilde{h}, \tilde{h}(x)\right) &= D^*\left(\sum_{k=0}^{\infty}{}^{*} b_{n,k}(x) \odot \tilde{h}\left(\frac{k}{n}\right), \sum_{k=0}^{\infty}{}^{*} b_{n,k}(x) \odot \tilde{h}(x)\right) \\ &\leq \sum_{k=0}^{\infty} b_{n,k}(x) D^*\left(\tilde{h}\left(\frac{k}{n}\right), \tilde{h}(x)\right) \\ &\leq \sum_{k=0}^{\infty} b_{n,k}(x)\, \omega_1^{\mathscr{F}}\left(\tilde{h}; |k/n - x|\right) \\ &\leq \sum_{k=0}^{\infty} b_{n,k}(x) \left[1 + \frac{|k/n - x|}{\delta}\right] \omega_1^{\mathscr{F}}\left(\tilde{h}; \delta\right) \\ &\leq \omega_1^{\mathscr{F}}\left(\tilde{h}; \delta\right) + \frac{\sqrt{\varphi_{2,n}(x)}}{\delta} \omega_1^{\mathscr{F}}\left(\tilde{h}; \delta\right). \end{aligned}$$

Substituting $\delta = \sqrt{\varphi_{2,n}(x)}$, we achieve our result. □

Lemma 5.4.21 *For fuzzy valued function $\tilde{h}$ and weighted modulus of continuity $\Omega^{\mathscr{F}}$,*

$$D\left(\tilde{h}(t), \tilde{h}(x)\right) \leq 4\left[1 + \frac{(t-x)^4}{\delta^4}\right] \left(1+\delta^2\right)^2 \left(1+x^2\right) \Omega^{\mathscr{F}}\left(\tilde{h}; \delta\right).$$

Proof. For a fuzzy-valued function $\tilde{h}$, taking values from the real line, we can say that

$$\begin{aligned} D\left(\tilde{h}(t), \tilde{h}(x)\right) &\leq \left(1 + (t-x)^2\right) \left(1+x^2\right) \Omega^{\mathscr{F}}\left(\tilde{h}; |t-x|\right) \\ &\leq 2\left(1 + \frac{|t-x|}{\delta}\right) \left(1 + (t-x)^2\right) \left(1+x^2\right) \left(1+\delta^2\right) \Omega^{\mathscr{F}}\left(\tilde{h}; \delta\right) \\ &\leq 4\left[1 + \frac{(t-x)^4}{\delta^4}\right] \left(1+\delta^2\right)^2 \left(1+x^2\right) \Omega^{\mathscr{F}}\left(\tilde{h}; \delta\right). \end{aligned}$$

□

Theorem 5.4.22 *Let $\tilde{h}(x)$ is a continuous, bounded and twice differential fuzzy function, defined for all $x \in [0, \infty)$. Then for the fuzzy Boas-Buck type operators, we have the following result:*

$$D^*\left(\tilde{\mathscr{B}}_n\tilde{h}, \tilde{h}(x)\right) \leq \varphi_{1,n}\left\|\tilde{h}'\right\| + \frac{1}{2}\varphi_{2,n}\left\|\tilde{h}''\right\| + 16\varphi_{2,n}\left(1+x^2\right) \Omega^{\mathscr{F}}\left(\tilde{h}'', \sqrt[4]{\frac{\varphi_{6,n}}{\varphi_{2,n}}}\right).$$

Proof. Consider,

$$D^*\left(\tilde{\mathscr{B}}_n\tilde{h}\left(\frac{k}{n}\right),\tilde{h}(x)\right)\leq\sum_{k=0}^{\infty}b_{n,k}(x)D\left(\tilde{h}\left(\frac{k}{n}\right),\tilde{h}(x)\right).$$

Now,

$$D\left(\tilde{h}\left(\frac{k}{n}\right),\tilde{h}(x)\right)=\sup_{\alpha\in[0,1]}\max\left\{\left|\tilde{h}\left(\frac{k}{n}\right)_-^{(\alpha)}-\tilde{h}(x)_-^{(\alpha)}\right|,\left|\tilde{h}\left(\frac{k}{n}\right)_+^{(\alpha)}-\tilde{h}(x)_+^{(\alpha)}\right|\right\}.$$

From Taylor's theorem,

$$h(u)=h(x)+(u-x)h'(x)+\frac{(u-x)^2}{2!}h''(x)+\frac{(u-x)^2}{2!}\varepsilon(u,x),$$

where $\varepsilon(u,x)=h''(\xi)-h''(x)\to 0$ as $u\to x$. Then, we can say that

$$\tilde{h}(u)_\pm^{(\ell)}=\tilde{h}(x)_\pm^{(\ell)}+(u-x)\tilde{h}'(x)_\pm^{(\ell)}+\frac{(u-x)^2}{2!}\tilde{h}''(x)_\pm^{(\ell)}+\frac{(u-x)^2}{2!}\varepsilon(u,x),$$

where $\varepsilon(u,x)=D\left(\tilde{h}''(\xi),\tilde{h}''(x)\right)\to 0$ as $u\to x$, with $x<\xi<u$. Thus,

$$\begin{aligned}
&\left[\tilde{h}(u)\right]^\ell-\left[\tilde{h}(x)\right]^\ell\\
&\quad=(u-x)\left[\tilde{h}'(x)\right]^\ell+\frac{(u-x)^2}{2!}\left[\tilde{h}''(x)\right]^\ell+\frac{(u-x)^2}{2!}\varepsilon(u,x)\\
\Rightarrow\quad&\left[\tilde{h}(u)_-^{(\ell)}-\tilde{h}(x)_-^{(\ell)},\tilde{h}(u)_+^{(\ell)}-\tilde{h}(x)_+^{(\ell)}\right]\\
&\quad=\left[(u-x)\tilde{h}'(x)_-^{(\ell)}+\frac{(u-x)^2}{2!}\tilde{h}''(x)_-^{(\ell)}+\frac{(u-x)^2}{2!}\varepsilon(u,x)_-^{(\ell)},\right.\\
&\qquad\left.(u-x)\tilde{h}'(x)_+^{(\ell)}+\frac{(u-x)^2}{2!}\tilde{h}''(x)_+^{(\ell)}+\frac{(u-x)^2}{2!}\varepsilon(u,x)_+^{(\ell)}\right]\\
&\quad=[A,B].
\end{aligned}$$

Since $\varepsilon(u,x)$ is crisp, using Lemma 5.4.21,

$$\begin{aligned}
\varepsilon(u,x)&=D\left(\tilde{h}''(\xi),\tilde{h}''(x)\right)\text{ with }x<\xi<u\\
&\leq 4\left[1+\frac{(u-x)^4}{\delta^4}\right]\left(1+\delta^2\right)^2\left(1+x^2\right)\Omega^F\left(\tilde{h};\delta\right).
\end{aligned}$$

Considering $\delta\leq 1$, we can say that

$$(u-x)^2\varepsilon(u,x)\leq 16\left(1+x^2\right)\left[(u-x)^2+\frac{(u-x)^6}{\delta^4}\right]\Omega^F\left(\tilde{h};\delta\right).$$

Thus, we have

$$
\begin{aligned}
D^*\left(\tilde{\mathscr{B}}_n\tilde{h},\tilde{h}(x)\right) &\leq \sum_{k=0}^{\infty} b_{n,k}(x)\, D\left(\tilde{h}\left(\frac{k}{n}\right),\tilde{h}(x)\right)\\
&= \sum_{k=0}^{n} b_{n,k}(x) \sup_{\ell\in[0,1]} \max\{A,B\}\\
&= \sup_{\ell\in[0,1]} \max\left\{\sum_{k=0}^{n} A b_{n,k}(x), \sum_{k=0}^{n} B b_{n,k}(x)\right\}\\
&\leq \sup_{\ell\in[0,1]} \max\left\{\varphi_{1,n}(x)\tilde{h}'(x)_-^{(\ell)} + \frac{1}{2}\varphi_{2,n}(x)\tilde{h}''(x)_-^{(\ell)}\right.\\
&\qquad + 16\left(1+x^2\right)\varphi_{2,n}(x)\Omega^{\mathscr{F}}\left(\tilde{h}''; \sqrt[4]{\frac{\varphi_{6,n}(x)}{\varphi_{2,n}(x)}}\right), \varphi_{1,n}(x)\tilde{h}'(x)_+^{(\ell)}\\
&\qquad \left. + \frac{1}{2}\varphi_{2,n}(x)\tilde{h}''(x)_+^{(\ell)} + 16\left(1+x^2\right)\varphi_{2,n}(x)\Omega^{\mathscr{F}}\left(\tilde{h}''; \sqrt[4]{\frac{\varphi_{6,n}(x)}{\varphi_{2,n}(x)}}\right)\right\}\\
&\leq \sup_{\ell\in[0,1]} \left[\varphi_{1,n}(x)\max\left\{\tilde{h}'(x)_-^{(\ell)}, \tilde{h}'(x)_+^{(\ell)}\right\}\right.\\
&\qquad \left. + \frac{1}{2}\varphi_{2,n}(x)\max\left\{\tilde{h}''(x)_-^{(\ell)}, \tilde{h}''(x)_+^{(\ell)}\right\}\right]\\
&\qquad + 16\left(1+x^2\right)\varphi_{2,n}(x)\Omega^{\mathscr{F}}\left(\tilde{h}''; \sqrt[4]{\frac{\varphi_{6,n}(x)}{\varphi_{2,n}(x)}}\right)\\
&= \varphi_{1,n}(x)\left\|\tilde{h}'(x)\right\|_{\mathscr{F}} + \frac{1}{2}\varphi_{2,n}(x)\left\|\tilde{h}''(x)\right\|_{\mathscr{F}}\\
&\qquad + 16\left(1+x^2\right)\varphi_{2,n}(x)\Omega^{\mathscr{F}}\left(\tilde{h}'', \sqrt[4]{\frac{\varphi_{6,n}}{\varphi_{2,n}}}\right),
\end{aligned}
$$

where $\tilde{o}$ is the zero element of $\mathbb{F}$. And thus, we arrive at our desired result. □

5.4.5 Particular examples

The Boas-Buck polynomials are a very general form of a special class of operators formed using the generating function (5.6). For particular values of $\mathscr{W}(t), \mathscr{P}(t)$ and $\mathscr{Q}(t)$ one can get different fuzzy linear operators. Mentioned below are two particular examples of the Boas-Buck fuzzy operators.

5.4.5.a Fuzzy Laguerre Operators

The Laguerre polynomials were first introduced in 1960 by Rainville [177] by means of its generating function. Later, these polynomials were explored by Gurland et al. [109] and Gupta [100] where the moments of these operators were found using the moment generating function and some direct convergence results were proved. Here, we consider the fuzzy analog of the Laguerre operators. Taking $\mathscr{W}(t) = (1-t)^{-\alpha-1}, \mathscr{P}(t) = e^t$ and $\mathscr{Q}(t) = \frac{-t}{1-t}$, for $\alpha > -1$ we get

$$\left(\tilde{G}_n^{\alpha}\tilde{h}\right)(x) = e^{-nx/2}2^{-\alpha-1}\sum_{k=0}^{\infty}{}^{*}2^{-k}L_k^{\alpha}\left(\frac{-nx}{2}\right)\odot\tilde{h}\left(\frac{k}{n}\right),$$

where

$$\begin{aligned} L_k^{\alpha}(-x) &= \frac{(\alpha+1)_k}{k!}{}_1F_1(-k;\alpha+1;-x) \\ &= \sum_{s=0}^{k}\frac{(\alpha+k)!}{(k-s)!(\alpha+s)!s!}x^s. \end{aligned}$$

Remark 5.4.23 $\left(\tilde{G}_n^{\alpha}\tilde{h}\right)(x)$ *are fuzzy positive linear operators with* $\left(\tilde{G}_n^{\alpha}(\tilde{h})\right)_{\pm}^{(\ell)} = G_n^{\alpha}\left(\tilde{h}_{\pm}^{(\ell)}\right)$, *where* G_n^{α} *are the real classical Laguerre operators [177].*

Theorem 5.4.24 *If* $\tilde{h} \in C_{\mathscr{F}}[0,\infty)$, *then for* $n \geq 1$

$$D^*\left(\tilde{G}_n^{\alpha}\tilde{h}, \tilde{h}(x)\right) \leq \left(1 + \sqrt{\alpha^2+4\alpha+3}\right)\omega_1^{\mathscr{F}}\left(\tilde{h};\frac{1}{n}\right),$$

where $\omega_1^{\mathscr{F}}$ *is the fuzzy modulus of continuity.*

Proof. Let $g_{n,k}(x) = e^{-nx/2}2^{-\alpha-k-1}L_k^{\alpha}\left(\frac{-nx}{2}\right)$. Then, we have

$$\begin{aligned} D^*\left(\tilde{G}_n^{\alpha}\tilde{h}, \tilde{h}(x)\right) &\leq \sum_{k=0}^{\infty} g_{n,k}(x)D^*\left(\tilde{h}\left(\frac{k}{n}\right), \tilde{h}(x)\right) \\ &\leq \sum_{k=0}^{\infty} g_{n,k}(x)\omega_1^{\mathscr{F}}\left(\tilde{h};\left|\frac{k}{n}-x\right|\right) \\ &\leq \sum_{k=0}^{\infty} g_{n,k}(x)\left[1+\frac{|k/n-x|}{\delta}\right]\omega_1^{\mathscr{F}}\left(\tilde{h};\delta\right) \\ &= \omega_1^{\mathscr{F}}\left(\tilde{h};\delta\right) + \frac{\omega_1^{\mathscr{F}}\left(\tilde{h};\delta\right)}{\delta}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|g_{n,k}(x). \end{aligned}$$

Simplifying the summation of the right hand side, we arrive at

$$\begin{aligned} \sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^2 g_{n,k}(x) &= \frac{\alpha^2+4\alpha+3}{n^2} + \frac{3x}{n} \\ &\leq \frac{\alpha^2+4\alpha+3}{n^2}. \end{aligned}$$

Thus,

$$\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right| g_{n,k}(x) \leq \frac{\sqrt{\alpha^2+4\alpha+3}}{n}.$$

Choosing $\delta = 1/n$, we get the required result. □

5.4.5.b Fuzzy Charlier Operators

Charlier polynomials [134] represent a significant family of orthogonal polynomials defined on the set of non-negative real numbers. They arise as a particular instance of the Boas-Buck operators characterized by $\mathscr{W}(t) = e^t$, $\mathscr{P}(t) = e^t$ and $\mathscr{Q}(t) = \ln\left(1 - \frac{t}{a}\right)$. Extensive research has been dedicated to explore the properties and applications of real classical Charlier operators. One can refer to [115; 166; 184; 210; 212] for the various mathematical contexts of these operators. We have extended these notations of Charlier operators into fuzzy approximation theory by defining the fuzzy analog of Charlier operators as follows:

$$\tilde{T}_n^a \tilde{h} = e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty}{}^{*} \frac{C_k^{(a)}\left(-(a-1)nx\right)}{k!} \odot \tilde{h}\left(\frac{k}{n}\right),$$

where
$$C_k^{(a)}(-u) = \sum_{r=0}^{k} \binom{k}{r} (u)_r \left(\frac{1}{a}\right)^r = \sum_{r=0}^{k} \binom{k}{r} \frac{\Gamma(u+r)}{\Gamma(u)} \left(\frac{1}{a}\right)^r.$$

Remark 5.4.25 *$\left(\tilde{T}_n^a \tilde{h}\right)(x)$ are fuzzy positive linear operators with $\left(\tilde{T}_n^a(\tilde{h})\right)_{\pm}^{(\ell)} = T_n^a \tilde{h}\left(\tilde{h}_{\pm}^{(\ell)}\right)$, where T_n^a are the real classical Charlier operators [134].*

Theorem 5.4.26 *If $\tilde{h} \in C_{\mathscr{F}}[0,\infty)$, then for $n \geq 1$*

$$D^*\left(\tilde{T}_n^a \tilde{h}, \tilde{h}(x)\right) \leq \left(1 + \sqrt{x\left(1 + \frac{1}{a-1}\right) + \frac{2}{n}}\right) \omega_1^{\mathscr{F}}\left(\tilde{h}; \frac{1}{\sqrt{n}}\right),$$

where $\omega_1^{\mathscr{F}}$ is the fuzzy modulus of continuity.

Proof. The proof of this theorem follows from the moments of Charlier operators and the properties of fuzzy modulus of continuity. □

Chapter 6

Semi-Exponential Operators with Improved Order of Convergence

The idea of semi-exponential operators is an extension of the exponential operators which derives a class of positive linear operators from the partial differential equation $\frac{\partial}{\partial x}W_\beta(r,x,t) = \frac{r(t-x)}{p(x)}W_\beta(r,x,t) - \beta W_\beta(r,x,t)$. *For* $p(x) = x(1-x)$ *we will get the Bernstein type semi-exponential operators which are used in approximating a continuous real-valued function* f. *However, the order of approximation of these operators is at most* $O(1/n)$. *This chapter focuses on deriving new sequences of positive linear Bernstein type semi-exponential operators of higher order. First, we define the second order semi-exponential Bernstein operators, give their moments and prove their asymptotic results. Further, we define the third order semi-exponential Bernstein operators, give their moments and central moments and derive their Voronskaya type asymptotic result. We verify these results using numerical illustrations.*

6.1 Introduction

The idea of approximation theory dates back to 1885 when K. Weierstrass [215] showed the uniform approximation of any real-valued continuous function by polynomials on a compact interval. Ever since the discovery of Weierstrass theorem, many researchers have worked on the approximation of continuous functions, not only on compact intervals, but on whole of the positive real line as well. In 1912, S.N. Bernstein [29] defined the famous Bernstein polynomials which approximate a continuous function on closed interval $[0,1]$. Later on, Szász and Mirakyan extended this concept to the positive real line by defining a sequence of positive linear operators, known as Szász-Mirakyan operators (see, [148; 202]), which approximate a real-valued continuous function on $[0,\infty)$.

In 1976, C.P. May [145] introduced another such class of positive linear operators, known as the exponential operators $S_m(\varphi;x)$, by establishing a connection between partial differential equations and positive linear operators. In his work, May proved that $S_m(\varphi;x)$ preserve linear functions and are defined as follows:

$$S_m(\varphi(u);t) = \int\limits_{\Omega} W(m,t,u)\varphi(u)du,$$

where, Ω is the domain of definition and W is a positive function, known as the kernel of S_m, satisfying the homogeneous PDE

$$\frac{\partial}{\partial t}W(m,t,u) = \frac{m}{p(t)}W(m,t,u)(u-t), \tag{6.1}$$

such that $p(t)$ is both analytic and positive over the domain Ω, and the kernel, $W(m,t,u)$ satisfies the normalization condition

$$S_m(1,t) = \int\limits_{-\infty}^{\infty} W(m,t,u)\,du = 1.$$

Different operators can be obtained by giving certain values to the function $p(t)$ in equation (6.1). For example, the function $p(t) = t(1-t)$ can be used to obtain the Bernstein polynomials. In a similar manner, one can obtain the Post-Widder operators for $p(t) = t^2$, the Gauss-Weierstrass operators for $p(t) = 1$, and the Szász-Mirakjan operators for the function $p(t) = t$. Moreover, Ismail and May [116] proved and established the uniqueness and existence properties of the class of these exponential operators.

Up until the year 2005, there had been no new developments on exponential operators for a considerable amount of time, after Ismail and May. Around this period, A. Tyliba and E. Wachnicki [208] developed the idea of semi-exponential operators and extended the results of Ismail and May. They did this by taking a sequence of operators, V_r^{β}, such that $V_r^{\beta}(t,x) \neq x$. They worked on a new homogeneous PDE by adding a non-negative real parameter β in (6.1). This derived a new family of operators, called the semi-exponential operators.

In the case where $\beta = 0$, the operators V_r^β simplify to the exponential type operators. Recently, Abel et al. [3] defined several new semi-exponential operators, each of which corresponds to a distinct function $p(x)$. For more work in exponential and semi-exponential operators, one can refer [93; 99; 101; 171]. A particular case of these semi-exponential operators have been studied extensively in Chapter 3 of this thesis.

However, it turns out that the rate of convergence of the classical Bernstein operators is relatively slow, since their approximation order does not exceed $O(1/n)$ (see [116; 208]). This limitation naturally motivated a large number of investigations aimed at improving their efficiency in the approximation of continuous functions. One of the earliest contributions in this direction was made by P.L. Butzer in 1953 [41], who introduced a refined construction by employing linear combinations of Bernstein polynomials of degree n together with those of higher degree $2n$. More precisely, if

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k},$$

denotes the classical Bernstein operator, then Butzer considered new operators of the form

$$C_n(f;x) = \alpha B_n(f;x) + \beta B_{2n}(f;x),$$

with suitably chosen constants $\alpha, \beta \in \mathbb{R}$, in order to accelerate the convergence to f. This technique allowed a better approximation behaviour while preserving positivity and linearity of the operators.

A significant breakthrough occurred with the introduction of q-calculus into approximation theory. In 1996, G.M. Phillips [169] defined the so-called q-Bernstein polynomials, which generalized the classical Bernstein polynomials by replacing the usual binomial coefficients and powers with their q-analogues. Specifically, the q-Bernstein operator of degree n is defined as

$$B_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \binom{n}{k}_q x^k (1-x)_q^{n-k},$$

where $[k]_q = \frac{1-q^k}{1-q}$ and $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ denote the q-integer and q-binomial coefficient, respectively. This extension not only added new ideas to the theory but also improved approximation properties. In particular, for $q \geq 1$, the q-Bernstein polynomials of degree n achieve an improved order of convergence, namely $O(1/q^n)$, which is considerably faster compared to the classical order $O(1/n)$ for polynomials of the same degree n.

Parallel to these developments, researchers have also sought to improve approximation order by drawing upon the recurrence relations satisfied by the Bernstein basis. By suitably modifying and combining terms arising from the Bernstein recurrence formula,

$$(n+1)B_{n+1,k}(x) = (n+1-k)xB_{n,k}(x) + (k+1)(1-x)B_{n,k+1}(x),$$

several authors have been able to design operators with higher rates of convergence. For instance, operators with approximation orders of $O(1/n^2)$ and even $O(1/n^3)$ have been proposed and studied in recent works (see [128; 129; 176]). These higher-order modifications not only accelerate the speed of convergence but also open pathways for more refined quantitative estimates in approximation theory.

Apart from their theoretical importance, these advancements have had a profound influence on applied fields. In computer-aided geometric design (CAGD), for example, the Bernstein basis forms the backbone of Bézier curves and B-splines, which are fundamental tools in computer graphics, animation, and CAD software. Improvements in approximation order directly translate to smoother curve generation, more accurate surface modeling, and better numerical stability. The concepts of positive linear operators and their improvements are also very useful in wavelet theory and signal processing. In these areas, signals are often represented in a simplified or transformed form, and it is important to both approximate the original signal accurately and then reconstruct it back to a form that closely resembles the original. The refinements of these operators help ensure that this process of approximation and reconstruction is done efficiently and reliably. Thus, the generalizations of classical Bernstein operators to positive linear operators with improved order not only highlight the journey of pure mathematics but also show how these approximation methods are closely connected to real-world applications.

Building upon the developments and generalizations of classical Bernstein operators discussed above, this chapter focuses on enhancing the order of approximation for semi-exponential Bernstein operators. In particular, we aim to define new operators that not only incorporate the function itself but also involve its higher-order derivatives. By doing so, these modified operators are expected to provide more accurate approximations, improving the convergence rate while preserving the essential properties of positivity and linearity.

6.2 Improving Order of Approximation of Semi-Exponential Bernstein Operators

The first-order semi-exponential Bernstein operators are defined as

$$B_n^{\beta}(\varphi;x) = e^{-\beta x}\sum_{s=0}^{\infty} A(n,s)x^s(1-x)^{n-s}\varphi\left(\frac{s}{n}\right), \tag{6.2}$$

where

$$A(n,s) = \sum_{i=0}^{\infty}\binom{n-i}{s-i}\frac{\beta^i}{i!}.$$

The construction and derivation of these operators have been discussed in detail in Chapter 3. In particular, from Theorem 3.4.2, it follows that the semi-exponential Bernstein operators approximate any continuous real-valued function with an order of $O(1/n)$.

While this order of convergence is consistent with classical Bernstein operators, there is significant interest in improving the approximation rate. In this chapter, we aim to define new sequences of semi-exponential Bernstein-type operators that achieve faster convergence, specifically with orders $O(1/n^2)$ and $O(1/n^3)$. To build upon this, we consider the first-order operators as the starting point and extend the construction to higher-order operators that incorporate derivatives of the required function. This method allows us to make the approximation more accurate while preserving the positivity and linearity of the operators.

6.2.1 Second order semi-exponential Bernstein operators

Building on the first-order operators defined in equation (6.2), we now introduce the second-order semi-exponential Bernstein operators, which are designed to improve the convergence rate by incorporating the first and second derivatives of the function. Let φ be a continuous function such that $\varphi''(x)$ exists. Then,

$$P_n^{\beta}(\varphi;x) = e^{-\beta x}\sum_{s=0}^{\infty} A(n,s)x^s(1-x)^{n-s}\left[\varphi\left(\frac{s}{n}\right) - \frac{\beta x(1-x)}{n}\varphi'\left(\frac{s}{n}\right) - \frac{x(1-x)}{2n}\varphi''\left(\frac{s}{n}\right)\right], \tag{6.3}$$

are defined as the second order semi-exponential Bernstein operators, where

$$A(n,s) = \sum_{i=0}^{\infty}\binom{n-i}{s-i}\frac{\beta^i}{i!}.$$

Having defined the second-order semi-exponential Bernstein operators in equation (6.3), we now proceed to study their approximation properties. In this subsection, we present several lemmas and theorems regarding their moments and convergence behaviour along with graphical representations that illustrate these approximation results.

6.2.1.a Approximation Results

We first compute the moments of the operators, which play a crucial role in understanding their approximation behaviour. Subsequently, we establish lemmas and theorems that characterize their convergence and other essential properties.

Theorem 6.2.1 *The moments of the second order semi-exponential Bernstein operators* (6.3) *are as follows:*

(i) $P_n^{\beta}(1;x) = 1$

(ii) $P_n^{\beta}(t;x) = x$

(iii) $P_n^{\beta}\left(t^2;x\right) = x^2 + \frac{\beta x(1-x)(1-2x)-\beta^2x^2(1-x)^2}{n^2}$.

Proof. The moments of operators (6.3) can be calculated with the help of moments of operators (6.2), given in Lemma 3.3.1.

(i) For the constant test function $\varphi(t) = 1$, all higher-order derivatives vanish. Thus, we get

$$P_n^{\beta}(1;x) = B_n^{\beta}(1;x) = 1.$$

(ii) For the test function $\varphi(t) = t$, $\varphi'(t) = 1$ and $\varphi''(t)$ will vanish. Thus,

$$P_n^{\beta}(t;x) = B_n^{\beta}(t;x) - \frac{\beta x(1-x)}{n} B_n^{\beta}(1;x) = x.$$

(iii) For $\varphi(t) = t^2$, we have $\varphi'(t) = 2t$ and $\varphi''(t) = 1$. Thus, the third moment of operators 6.3 is given by

$$\begin{aligned}
P_n^{\beta}(t^2;x) &= B_n^{\beta}(t^2;x) - \frac{2\beta x(1-x)}{n} B_n^{\beta}(t;x) - \frac{x(1-x)}{n} B_n^{\beta}(1;x) \\
&= x^2 + \frac{\beta x(1-x)(1-2x-\beta x+\beta x^2)}{n^2} \\
&= x^2 + \frac{\beta x(1-x)(1-2x) - \beta^2 x^2(1-x)^2}{n^2}.
\end{aligned}$$

□

Let us define the central moments of second-order semi-exponential Bernstein operators (6.3) by

$$\rho_{n,m} = P_n^{\beta}\left(\left(\frac{s}{n} - x\right)^m; x\right).$$

These central moments can be conveniently computed using the results of Theorem 6.2.1. In the following theorems, we present key results regarding these moments and central moments of the proposed operators.

Theorem 6.2.2 *Let $\psi : [0,1] \to \mathbb{R}$ be a continuous function such that ψ'' is bounded. Then for the supremum norm of ψ,*

$$\left|P_n^{\beta}(\psi; x) - \psi(x)\right| \leq \frac{\rho_{n,2}^*}{2}\left\|\psi''\right\|,$$

where $\rho_{n,2}^* = \sup_{0 \leq x \leq 1}\left|P_n^{\beta}\left((s/n - x)^2; x\right)\right|$.

Proof. By Taylor's expansion,

$$\psi(v) = \psi(x) + (v - x)\,\psi'(x) + \frac{(v - x)^2}{2}\psi''(x) + (v - x)^2\varepsilon(v),$$

such that $\lim_{v \to x}\varepsilon(v) = 0$. Taking $v = s/n$ and applying operators P_n^{β} on both sides, we get

$$P_n^{\beta}(\psi(s/n); x) = \psi(x) + P_n^{\beta}((s/n - x); x)\,\psi'(x) + \frac{P_n^{\beta}\left((s/n - x)^2; x\right)}{2}\psi''(x) + P_n^{\beta}\left((s/n - x)^2\varepsilon(s/n); x\right).$$

From Lemma 6.2.1, we can say that that $P_n^{\beta}((s/n - x); x) = 0$ and $P_n^{\beta}\left((s/n - x)^2\varepsilon(s/n); x\right)$ tends to 0, as $n \to \infty$. Thus for a sufficiently large value of n we can say that,

$$\begin{aligned}\left|P_n^{\beta}(\psi; x) - \psi(x)\right| &\leq \frac{P_n^{\beta}\left((s/n - x)^2; x\right)}{2}\left|\psi''(x)\right| \\ &\leq \frac{\rho_{n,2}^*}{2}\left\|\psi''\right\|.\end{aligned}$$

□

Now, in order to establish a connection between the operators $P_n^{\beta}(\varphi; x)$, the function φ, and its higher-order derivatives, it is convenient to employ the concept of the modulus of continuity. This will allow us to derive bounds that are essential for the subsequent approximation results.

Lemma 6.2.3 *Let $\varphi : \langle a,b\rangle \to \mathbb{R}$ be a function defined on $\langle a,b\rangle$ whose k^{th} derivative exists. Then, for $|h| \le \delta$*

$$\omega\left(\varphi^{(k)};\delta\right) \le \frac{1}{h^k}\omega_{k+1}(\varphi;\delta), \quad \delta \ge 0.$$

Proof. For $\delta \ge 0$, using Definition 1.1.3, we can write

$$\begin{aligned}
&\omega_k\left(\varphi^{(1)};\delta\right)\\
&= \sup_{|h|\le\delta} \Delta_h^k \varphi^{(1)}(x)\\
&= \sup_{|h|\le\delta} \Big|\varphi^{(1)}(x+kh) - \binom{k}{1}\varphi^{(1)}(x+(k-1)h) + \binom{k}{2}\varphi^{(1)}(x+(k-2)h) + \dots\\
&\qquad + (-1)^k\varphi^{(1)}(x)\Big|\\
&\le \frac{1}{h}\sup_{|h|\le\delta}\Big|\varphi(x+(k+1)h) - \left(\binom{k}{0}+\binom{k}{1}\right)\varphi(x+kh)\\
&\qquad + \left(\binom{k}{1}+\binom{k}{2}\right)\varphi(x+(k-1)h) + \dots + (-1)^{k+1}\varphi(x)\Big|\\
&= \frac{1}{h}\sup_{|h|\le\delta}\Big|\varphi(x+(k+1)h) - \binom{k+1}{1}\varphi(x+kh) + \binom{k+1}{2}\varphi(x+(k-1)h) + \dots\\
&\qquad + (-1)^{k+1}\varphi(x)\Big|\\
&= \frac{1}{h}\omega_{k+1}(\varphi;\delta).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\text{for } k=1,\ \ \omega_2(\varphi;\delta) \ge h\omega\left(\varphi^{(1)};\delta\right)\\
&\text{for } k=2,\ \ \omega_3(\varphi;\delta) \ge h\omega_2\left(\varphi^{(1)};\delta\right) \ge h^2\omega\left(\varphi^{(2)};\delta\right).
\end{aligned}$$

Proceeding in this manner, we can prove the stated lemma for all $k \in \mathbb{N}$. □

Theorem 6.2.4 *Let $\varphi \in C[0,1]$ such that $\varphi^{(2)}$ exists. Then for $|h| \le \delta$ and $n \in \mathbb{N}$,*

$$\begin{aligned}
&n\left|P_n^{\beta}(\varphi;x) - \varphi(x) + \beta x(1-x)\varphi^{(1)}(x) + \frac{x(1-x)}{2}\varphi^{(2)}(x)\right|\\
&\quad \le 2n\omega\left(\varphi;\sqrt{\mu_{n,2}^*}\right) + \frac{2\beta x(1-x)}{h}\omega_2\left(\varphi;\sqrt{\mu_{n,2}^*}\right) + \frac{x(1-x)}{h^2}\omega_3\left(\varphi;\sqrt{\mu_{n,2}^*}\right).
\end{aligned}$$

where $\mu_{n,2}^* = \sup_{0\le x\le 1}\left|B_n^{\beta}((t-x)^2;x)\right|$.

Proof. For simplicity, let $b_{n,s}^{\beta}(x) = A(n,s)e^{-\beta x}x^s(1-x)^{n-s}$. Thus,

$$\left|B_n^{\beta}(\varphi;x) - \varphi(x)\right| \le \sum_{s=0}^{\infty} b_{n,s}^{\beta}(x)\left|\varphi\left(\frac{s}{n}\right) - \varphi(x)\right|$$
$$\le \sum_{s=0}^{\infty} b_{n,s}^{\beta}(x)\,\omega\left(\varphi;\left|x-\frac{s}{n}\right|\right).$$

Using Cauchy-Schwarz's inequality

$$\sum_{s=0}^{\infty} b_{n,s}^{\beta}(x)\left|x-\frac{s}{n}\right| \le \sum_{s=0}^{\infty}\left|x-\frac{s}{n}\right|\sqrt{b_{n,s}^{\beta}(x)}\sqrt{b_{n,s}^{\beta}(x)}$$
$$\le \left[\sum_{s=0}^{\infty} b_{n,s}^{\beta}(x)\left(x-\frac{s}{n}\right)^2\right]^{1/2}\left[\sum_{s=0}^{\infty} b_{n,s}^{\beta}(x)\right]^{1/2}$$
$$= \sqrt{\mu_{n,2}(x)}$$
$$\le \sqrt{\mu_{n,2}^{*}},$$

where $\mu_{n,2}^{*} = \sup_{0\le x\le 1}\left|\mu_{n,2}(x)\right|$. From Proposition 1.1.4 we can write,

$$\left|B_n^{\beta}(\varphi;x) - \varphi(x)\right| \le \sum_{s=0}^{\infty} b_{n,s}^{\beta}(x)\,\omega\left(\varphi;\left|x-\frac{s}{n}\right|\right)$$
$$\le \omega\left(\varphi;\sqrt{\mu_{n,2}^{*}}\right)\sum_{s=0}^{\infty} b_{n,s}^{\beta}(x)\left(1+\frac{\left|x-\frac{s}{n}\right|}{\sqrt{\mu_{n,2}^{*}}}\right)$$
$$\le 2\omega\left(\varphi;\sqrt{\mu_{n,2}^{*}}\right).$$

Again, using Lemma 6.2.3 and Cauchy-Schwarz's inequality, we get

$$\frac{\beta x(1-x)}{n}\left|B_n^{\beta}\left(\varphi^{(1)};x\right) - \varphi^{(1)}(x)\right| \le \frac{2\beta x(1-x)}{n}\,\omega\left(\varphi^{(1)};\sqrt{\mu_{n,2}^{*}}\right)$$
$$\le \frac{2\beta x(1-x)}{nh}\,\omega_2\left(\varphi;\sqrt{\mu_{n,2}^{*}}\right),$$

and

$$\frac{x(1-x)}{2n}\left|B_n^{\beta}\left(\varphi^{(2)};x\right) - \varphi^{(2)}(x)\right| \le \frac{x(1-x)}{n}\,\omega\left(\varphi^{(2)};\sqrt{\mu_{n,2}^{*}}\right)$$
$$\le \frac{x(1-x)}{nh^2}\,\omega_3\left(\varphi;\sqrt{\mu_{n,2}^{*}}\right).$$

Thus, we get the inequality

$$\left|P_n^{\beta}(\varphi;x) - \varphi(x) + \frac{\beta x(1-x)}{n}\varphi^{(1)}(x) + \frac{x(1-x)}{2n}\varphi^{(2)}(x)\right|$$
$$\le 2\omega\left(\varphi;\sqrt{\mu_{n,2}^{*}}\right) + \frac{2\beta x(1-x)}{nh}\,\omega_2\left(\varphi;\sqrt{\mu_{n,2}^{*}}\right) + \frac{x(1-x)}{nh^2}\,\omega_3\left(\varphi;\sqrt{\mu_{n,2}^{*}}\right),$$

which proves the theorem. □

Theorem 6.2.5 *Let $\varphi \in C[0,1]$ such that $\varphi^{(1)}$ exists. Define $\mu_{n,2} = B_n^{\beta}((t-x)^2;x)$. Then,*

$$\left|P_n^{\beta}(\varphi;x) - \varphi(x)\right| \leq \frac{3}{2}\sqrt{\mu_{n,2}^{*}}\omega\left(\varphi', \mu_{n,2}^{*}\right),$$

where $\mu_{n,2}^{*} = \sup_{0\leq x\leq 1}\left|\mu_{n,2}\right|$.

Proof. For any function $\varphi \in C[0,1]$, we can write

$$\varphi(v) = \varphi(x) + (v-x)\varphi'(x) + \int_x^v \left(\varphi'(z) - \varphi'(x)\right) dz.$$

Applying the second order semi-exponential operator P_n^{β}, to both side of the above equation, we get

$$P_n^{\beta}(\varphi;x) - \varphi(x) = P_n^{\beta}\left(\int_x^v \left(\varphi'(z) - \varphi'(x)\right) dz;x\right). \tag{6.4}$$

From modulus of continuity and Proposition 1.1.4, we arrive at

$$\begin{aligned}
\left|\int_x^v \left(\varphi'(z) - \varphi'(x)\right) dz\right| &\leq \int_x^v \left|\varphi'(z) - \varphi'(x)\right| dz \\
&\leq \int_x^v \omega\left(\varphi', \delta\right)\left(\frac{|z-x|}{\delta} + 1\right) dz \\
&= \omega\left(\varphi', \delta\right)\left(\frac{|v-x|^2}{2\delta} + |v-x|\right).
\end{aligned}$$

Thus, equation (6.4) becomes

$$\begin{aligned}
\left|P_n^{\beta}(\varphi;x) - \varphi(x)\right| &\leq \omega\left(\varphi', \delta\right)\left(\frac{\left|\mu_{n,2}(x)\right|}{2\delta} + \left|\mu_{n,1}(x)\right|\right) \\
&= \omega\left(\varphi', \delta\right)\left(\frac{\left|\mu_{n,2}(x)\right|}{2\delta} + \sum_{s=0}^{\infty}\left|\frac{s}{n} - x\right|\sqrt{b_{n,s}^{\beta}(x)}\sqrt{b_{n,s}^{\beta}(x)}\right) \\
&= \omega\left(\varphi', \delta\right)\left(\frac{\left|\mu_{n,2}(x)\right|}{2\delta} + \left(\sum_{s=0}^{\infty}\left(\frac{s}{n} - x\right)^2 b_{n,s}^{\beta}(x)\right)^{1/2}\left(\sum_{s=0}^{\infty} b_{n,s}^{\beta}(x)\right)^{1/2}\right) \\
&= \omega\left(\varphi', \delta\right)\sqrt{\mu_{n,2}(x)}\left(\frac{\sqrt{\mu_{n,2}(x)}}{2\delta} + 1\right) \\
&\leq \omega\left(\varphi', \delta\right)\sqrt{\mu_{n,2}^{*}}\left(\frac{\sqrt{\mu_{n,2}^{*}}}{2\delta} + 1\right).
\end{aligned}$$

Choosing $\delta = \mu_{n,2}^{*}$ gives us the desired result. □

Theorem 6.2.6 *Let φ be a continuous function defined on $[0,1]$ and let $\varphi^{(n)}$ denote its n^{th} order derivative. For $\varphi^{(2)}, \varphi^{(3)}, \varphi^{(4)} \neq 0$ simultaneously, and for $\beta \geq 0$, we have*

$$\begin{aligned}
&\lim_{n\to\infty} n^2 \left[P_n^\beta(\varphi;x) - \varphi(x) \right] \\
&= \frac{\beta}{2}x(1-x)\left(1-2x-\beta x+\beta x^2\right)\varphi^{(2)}(x) + \frac{1}{6}x(1-x)\left(1-2x-3\beta x+3\beta x^2\right)\varphi^{(3)}(x) \\
&\quad - \frac{1}{8}x^2(1-x)^2\varphi^{(4)}(x).
\end{aligned}$$

Proof. Applying Taylor's expansion around x on $\varphi\left(\frac{s}{n}\right)$, $\varphi^{(1)}\left(\frac{s}{n}\right)$ and $\varphi^{(2)}\left(\frac{s}{n}\right)$, we get

$$\begin{aligned}
\varphi^{(2)}\left(\frac{s}{n}\right) &= \varphi^{(2)}(x) + \left(\frac{s}{n}-x\right)\varphi^{(3)}\left(\frac{s}{n}\right) + \frac{1}{2!}\left(\frac{s}{n}-x\right)^2\varphi^{(4)}\left(\frac{s}{n}\right) + \left(\frac{s}{n}-x\right)^2\chi(s/n) \\
\varphi^{(1)}\left(\frac{s}{n}\right) &= \varphi^{(1)}(x) + \left(\frac{s}{n}-x\right)\varphi^{(2)}\left(\frac{s}{n}\right) + \frac{1}{2!}\left(\frac{s}{n}-x\right)^2\varphi^{(3)}\left(\frac{s}{n}\right) + \left(\frac{s}{n}-x\right)^2\xi(s/n) \\
\varphi\left(\frac{s}{n}\right) &= \varphi(x) + \left(\frac{s}{n}-x\right)\varphi^{(1)}\left(\frac{s}{n}\right) + \frac{1}{2!}\left(\frac{s}{n}-x\right)^2\varphi^{(2)}\left(\frac{s}{n}\right) + \frac{1}{3!}\left(\frac{s}{n}-x\right)^3\varphi^{(3)}\left(\frac{s}{n}\right) \\
&\quad + \frac{1}{4!}\left(\frac{s}{n}-x\right)^4\varphi^{(4)}\left(\frac{s}{n}\right) + \left(\frac{s}{n}-x\right)^4\varepsilon(s/n),
\end{aligned}$$

where $\lim\limits_{s/n\to x}\varepsilon(s/n)=0$, $\lim\limits_{s/n\to x}\chi(s/n)=0$ and $\lim\limits_{s/n\to x}\xi(s/n)=0$.

Substituting the values of these expressions in (6.3) and taking summation from $s=0$ to ∞, we get

$$\begin{aligned}
P_n^\beta(\varphi;x) &= \varphi(x) + \mu_{n,1}(x)\varphi^{(1)}(x) + \frac{\mu_{n,2}(x)}{2!}\varphi^{(2)}(x) + \frac{\mu_{n,3}(x)}{3!}\varphi^{(3)}(x) + \frac{\mu_{n,4}(x)}{4!}\varphi^{(4)}(x) \\
&\quad - \frac{\beta x(1-x)}{n}\varphi^{(1)}(x) - (\mu_{n,1}(x))^2\varphi^{(2)}(x) - \frac{\mu_{n,1}(x)\mu_{n,2}(x)}{2!}\varphi^{(3)}(x) \\
&\quad - \frac{\mu_{n,1}(x)\mu_{n,3}(x)}{3!}\varphi^{(4)}(x) - \frac{x(1-x)}{2n}\varphi^{(2)}(x) - \frac{x(1-x)}{2n}\mu_{n,1}(x)\varphi^{(3)}(x) \\
&\quad - \frac{x(1-x)}{4n}\mu_{n,2}(x)\varphi^{(4)}(x) + B_n^\beta\left((t-x)^4\varepsilon(t);x\right) \\
&\quad - \frac{\beta x(1-x)}{n}B_n^\beta\left((t-x)^2\xi(t);x\right) - \frac{x(1-x)}{2n}B_n^\beta\left((t-x)^2\chi(t);x\right),
\end{aligned}$$

where $\mu_{n,r}(x)$ are the central moments of the semi-exponential Bernstein operators (6.2). Let us define,

$$\begin{aligned}
\mathrm{T}_n &= \left\{ s : \left|\frac{s}{n}-x\right| < \varepsilon, s=0,1,2\dots \right\}, \\
\text{and } \Gamma_n &= \left\{ s : \left|\frac{s}{n}-x\right| \geq \varepsilon, s=0,1,2\dots \right\}.
\end{aligned}$$

Taking into account the limits of ε, χ and ξ, we can say that for any $\delta>0$ if $\left|\frac{s}{n}-x\right|<\delta$ then $\exists\ \varepsilon,\chi,\xi>0$ such that $\varepsilon(s/n)<\varepsilon$, $\chi(s/n)<\chi$ and $\xi(s/n)<\xi$. Moreover, for $\left|\frac{s}{n}-x\right| \geq \delta$, define $M_1 = \sup\limits_{0\leq x\leq 1} \left(\frac{s}{n}-x\right)^2 \varepsilon(s/n)$, $M_2 = \sup\limits_{0\leq x\leq 1} \left(\frac{s}{n}-x\right)^2 \xi(s/n)$ and $M_3 = \sup\limits_{0\leq x\leq 1} \left(\frac{s}{n}-x\right)^2 \chi(s/n)$. Then, we have

$$\begin{aligned}
&\lim_{n\to\infty} n^2 B_n^{\beta}\left((t-x)^4 \varepsilon(t);x\right) \\
&= \sum_{s=0}^{\infty} n^2 b_{n,s}^{\beta}(x)\left(\frac{s}{n}-x\right)^4 \varepsilon(s/n) \\
&= \lim_{n\to\infty} \sum_{k\in \mathrm{T}_n} n^2 b_{n,s}^{\beta}(x)\left(\frac{s}{n}-x\right)^4 \varepsilon(s/n) + \lim_{n\to\infty} \sum_{k\in \Gamma_n} n^2 b_{n,s}^{\beta}(x)\left(\frac{s}{n}-x\right)^4 \varepsilon(s/n) \\
&\leq \lim_{n\to\infty} \left(\varepsilon + \frac{M_1}{\delta^2}\right) n^2 \mu_{n,4}(x) \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\lim_{n\to\infty} nB_n^{\beta}\left((t-x)^2 \xi(t);x\right) \\
&= \lim_{n\to\infty} \sum_{k\in \mathrm{T}_n} nb_{n,s}^{\beta}(x)\left(\frac{s}{n}-x\right)^2 \xi(s/n) + \lim_{n\to\infty} \sum_{k\in \Gamma_n} nb_{n,s}^{\beta}(x)\left(\frac{s}{n}-x\right)^2 \xi(s/n) \\
&\leq \lim_{n\to\infty} \left(\xi + \frac{M_2}{\delta^2}\right) n\mu_{n,2}(x) \\
&= \lim_{n\to\infty} \left(\xi + \frac{M_2}{\delta^2}\right) x(1-x) \\
&= 0,
\end{aligned}$$

and

$$\lim_{n\to\infty} nB_n^{\beta}\left((t-x)^2 \chi(t);x\right) = 0.$$

Thus, our equation becomes

$$\begin{aligned}
&\lim_{n\to\infty} n^2 \left|P_n^{\beta}(\varphi;x) - \varphi(x)\right| \\
&= \lim_{n\to\infty} \Big[n^2 \mu_{n,1}(x)\varphi^{(1)}(x) + \frac{n^2}{2!}\mu_{n,2}(x)\varphi^{(2)}(x) + \frac{n^2}{3!}\mu_{n,3}(x)\varphi^{(3)}(x) + \frac{n^2}{4!}\mu_{n,4}(x)\varphi^{(4)}(x) \\
&\quad - n\beta x(1-x)\varphi^{(1)}(x) - n^2(\mu_{n,1}(x))^2\varphi^{(2)}(x) - \frac{n^2}{2!}\mu_{n,1}(x)\mu_{n,2}(x)\varphi^{(3)}(x) \\
&\quad - \frac{n^2}{3!}\mu_{n,1}(x)\mu_{n,3}(x)\varphi^{(4)}(x) - \frac{n}{2}x(1-x)\varphi^{(2)}(x) - \frac{n}{2}x(1-x)\mu_{n,1}(x)\varphi^{(3)}(x) \\
&\quad - \frac{n}{4}x(1-x)\mu_{n,2}(x)\varphi^{(4)}(x) \Big]
\end{aligned}$$

$$= \lim_{n\to\infty} n^2\left(\frac{\beta x(1-x)(1-2x)}{2n^2} - \frac{\beta^2x^2(1-x)^2}{2n^2}\right)\varphi^{(2)}(x)$$
$$+ \frac{\left(x(1-x)(1-2x)+3\beta x^2(1-x)^2\right)}{6}\varphi^{(3)}(x) - \frac{\beta x^2(1-x)^2}{2}\varphi^{(3)}(x)$$
$$- \frac{\beta x^2(1-x)^2}{2}\varphi^{(3)}(x) + \frac{3x^2(1-x)^2}{4!}\varphi^{(4)}(x) - \frac{x^2(1-x)^2}{4}\varphi^{(4)}(x)$$
$$= \frac{\beta}{2}x(1-x)\left(1-2x-\beta x+\beta x^2\right)\varphi^{(2)}(x) + \frac{1}{6}x(1-x)\left(1-2x-3\beta x+3\beta x^2\right)\varphi^{(3)}(x)$$
$$- \frac{1}{8}x^2(1-x)^2\varphi^{(4)}(x).$$

This completes the proof. □

From Theorem 6.2.6, it follows that the sequence of operators $P_n^{\beta}(\varphi;x)$ provides an improved approximation to any continuous real-valued function. In particular, the rate of convergence is of order $O(1/n^2)$, which is significantly better than the classical first-order case where the approximation order is only $O(1/n)$. This result highlights the effectiveness of introducing higher-order derivatives into the construction of the semi-exponential Bernstein operators, thereby enhancing their approximation properties.

Having established that the second-order semi-exponential Bernstein operators achieve a significantly better approximation rate compared to the first-order case, it is natural to investigate whether this improvement can be extended further. To this end, we now turn our attention to the construction of third-order operators, which incorporate higher derivatives in order to potentially yield an even sharper rate of convergence.

6.2.2 Third order semi-exponential Bernstein operators

In continuation of the previous developments, the third-order semi-exponential Bernstein operators are introduced in this section. These operators are constructed by incorporating the first, second, third and fourth derivatives of the function. Our aim is to obtain an even higher order of approximation while preserving positivity and linearity. The third order semi-exponential Bernstein operators are defined as,

$$R_n^{\beta}(\gamma;x) = e^{-\beta x}\sum_{s=0}^{\infty} A(n,s)x^s(1-x)^{n-s}\Bigg[\gamma\left(\frac{s}{n}\right) - \frac{\beta x(1-x)}{n}\gamma^{(1)}\left(\frac{s}{n}\right)$$
$$-\frac{x(1-x)}{2n}\gamma^{(2)}\left(\frac{s}{n}\right) - \frac{\beta x(1-x)(1-2x-\beta x+\beta x^2)}{2n^2}\gamma^{(2)}\left(\frac{s}{n}\right)$$
$$-\frac{x(1-x)(1-2x-3\beta x+3\beta x^2)}{6n^2}\gamma^{(3)}\left(\frac{s}{n}\right) + \frac{x^2(1-x)^2}{8n^2}\gamma^{(4)}\left(\frac{s}{n}\right)\Bigg], \quad (6.5)$$

where $A(n,s) = \sum_{i=0}^{\infty} \binom{n-i}{s-i} \frac{\beta^i}{i!}$.

We now turn to analyze the mathematical properties of the third-order operators. In particular, we derive their moments and central moments, and then establish results that highlight their role in achieving higher-order approximation.

6.2.2.a Approximation Results

First and foremost, we study the moments and central moments of these third order semi-exponential Bernstein operators.

Lemma 6.2.7 *For the sequence of operators* (6.5)*, we get the following moments:*

(i) $R_n^{\beta}(1;x) = 1$

(ii) $R_n^{\beta}(t;x) = x$

(iii) $R_n^{\beta}\left(t^2;x\right) = x^2$.

Proof. The moments of operators (6.5) can be calculated with the help of moments of operators (6.2), given in Lemma 3.3.1.

(i) For the constant test function $\varphi(t) = 1$, all higher-order derivatives vanish. Thus, we get
$$R_n^{\beta}(1;x) = B_n^{\beta}(1;x) = 1.$$

(ii) For the test function $\varphi(t) = t$, $\varphi^{(1)}(t) = 1$ and rest of the higher order derivatives will vanish. Thus,
$$R_n^{\beta}(t;x) = B_n^{\beta}(t;x) - \frac{\beta x(1-x)}{n} B_n^{\beta}(1;x) = x.$$

(iii) For $\varphi(t) = t^2$, we have $\varphi^{(1)}(t) = 2t$ and $\varphi^{(2)}(t) = 1$. $\varphi^{(3)}(t)$ and $\varphi^{(4)}(t)$ will vanish, and thus, the third moment of operators 6.5 is given by
$$\begin{aligned} R_n^{\beta}(t^2;x) &= B_n^{\beta}(t^2;x) - \frac{2\beta x(1-x)}{n} B_n^{\beta}(t;x) - \frac{x(1-x)}{n} B_n^{\beta}(1;x) \\ &\quad - \frac{\beta x(1-x)(1-2x-\beta x+\beta x^2)}{n^2} B_n^{\beta}(1;x) \\ &= x^2. \end{aligned}$$

□

Lemma 6.2.8 *Using Lemma 6.2.7, we can derive the following central moments of $R_n^{\beta}(\gamma;x)$:*

(i) $R_n^{\beta}((t-x);x) = 0$

(ii) $R_n^{\beta}\left((t-x)^2;x\right) = 0$

(iii) $R_n^{\beta}\left((t-x)^3;x\right) = \beta x(1-x)\left(1-6x+6x^2-3\beta x(1-x)(1-2x)+\beta^2x^2(1-x)^2\right)\Big/n^3.$

By applying Korovkin's theorem to the obtained moments and central moments, we can conclude that the sequence of third order semi-exponential Bernstein operators (6.5), converges uniformly to the required function. Thus, we are now in a position to study the approximation behaviour of these operators in more detail.

Theorem 6.2.9 *Let $\psi : [0,1] \to \mathbb{R}$ be a continuous function such that ψ''' is bounded. Then for the supremum norm of ψ,*

$$\left|R_n^{\beta}(\psi;x) - \psi(x)\right| \leq \frac{\zeta_{n,3}^*}{6}\left\|\psi'''\right\|$$

where $\zeta_{n,3}^* = \sup_{0\leq x\leq 1}\left|R_n^{\beta}\left((s/n-x)^3;x\right)\right|.$

Proof. By Taylor's expansion,

$$\psi(v) = \psi(x) + (v-x)\psi'(x) + \frac{(v-x)^2}{2}\psi''(x) + \frac{(v-x)^3}{6}\psi'''(x) + (v-x)^3\varepsilon(v),$$

where $\lim_{v\to x}\varepsilon(v)$. Putting $v = s/n$ and taking operator $R_n^{\beta}(\psi;x)$ on both sides, we get

$$R_n^{\beta}(\psi(s/n);x) = \psi(x) + \frac{R_n^{\beta}\left((s/n-x)^3;x\right)(x)}{6}\psi'''(x) + R_n^{\beta}\left((s/n-x)^3\varepsilon(v);x\right).$$

So for a sufficiently large n, we can say that,

$$\left|R_n^{\beta}(\psi;x) - \psi(x)\right| \leq \frac{R_n^{\beta}\left((s/n-x)^3;x\right)}{6}\left|\psi'''(x)\right|$$
$$\leq \frac{\zeta_{n,3}^*}{6}\left\|\psi'''\right\|.$$

Thus, we get the desired result. □

While error bounds give a general sense of the approximation quality, they do not fully capture the precise asymptotic behaviour of the operators. To achieve a deeper understanding, we make use of a Voronovskaya-type theorem. This result describes the limiting behaviour of the operators as $n \to \infty$ and allows us to determine the exact order of convergence for the third-order case.

Theorem 6.2.10 *Let γ be a continuous function defined on $[0,1]$ and let $\gamma^{(n)}$ denote its n^{th} order derivative. For $\gamma^{(3)}, \gamma^{(4)}, \gamma^{(5)}, \gamma^{(6)} \neq 0$ simultaneously, and for $\beta \geq 0$, we have*

$$\begin{aligned}
&\lim_{n\to\infty} n^3 \left[R_n^{\beta}(\gamma;x) - \gamma(x)\right] \\
&= \frac{\beta x(1-x)\left(1-6x+6x^2-3\beta x(1-x)(1-2x)+\beta^2 x^2(1-x)^2\right)}{6}\gamma^{(3)}(x) \\
&\quad + \frac{x(1-x)\left(1-6x+6x^2-10\beta x(1-x)(1-2x)+6\beta^2 x^2(1-x)^2\right)}{24}\gamma^{(4)}(x) \\
&\quad - \left(\frac{x^2(1-x)^2(1-2x)}{12} - \frac{\beta x^3(1-x)^3}{8}\right)\gamma^{(5)}(x) + \frac{x^3(1-x)^3}{48}\gamma^{(6)}(x).
\end{aligned}$$

Proof. Applying Taylor's theorem, we get

$$\begin{aligned}
\gamma^{(4)}\left(\frac{s}{n}\right) &= \gamma^{(4)}(x) + \left(\frac{s}{n}-x\right)\gamma^{(5)}\left(\frac{s}{n}\right) + \frac{1}{2!}\left(\frac{s}{n}-x\right)^2\gamma^{(6)}\left(\frac{s}{n}\right) + \left(\frac{s}{n}-x\right)^2\eta(s/n) \\
\gamma^{(3)}\left(\frac{s}{n}\right) &= \gamma^{(3)}(x) + \left(\frac{s}{n}-x\right)\gamma^{(4)}\left(\frac{s}{n}\right) + \frac{1}{2!}\left(\frac{s}{n}-x\right)^2\gamma^{(5)}\left(\frac{s}{n}\right) + \left(\frac{s}{n}-x\right)^2\nu(s/n) \\
\gamma^{(2)}\left(\frac{s}{n}\right) &= \gamma^{(2)}(x) + \left(\frac{s}{n}-x\right)\gamma^{(3)}\left(\frac{s}{n}\right) + \frac{1}{2!}\left(\frac{s}{n}-x\right)^2\gamma^{(4)}\left(\frac{s}{n}\right) + \frac{1}{3!}\left(\frac{s}{n}-x\right)^3\gamma^{(5)}\left(\frac{s}{n}\right) \\
&\quad + \frac{1}{4!}\left(\frac{s}{n}-x\right)^4\gamma^{(6)}\left(\frac{s}{n}\right) + \left(\frac{s}{n}-x\right)^4\chi(s/n) \\
\gamma^{(1)}\left(\frac{s}{n}\right) &= \gamma^{(1)}(x) + \left(\frac{s}{n}-x\right)\gamma^{(2)}\left(\frac{s}{n}\right) + \frac{1}{2!}\left(\frac{s}{n}-x\right)^2\gamma^{(3)}\left(\frac{s}{n}\right) + \frac{1}{3!}\left(\frac{s}{n}-x\right)^3\gamma^{(4)}\left(\frac{s}{n}\right) \\
&\quad + \frac{1}{4!}\left(\frac{s}{n}-x\right)^4\gamma^{(5)}\left(\frac{s}{n}\right) + \left(\frac{s}{n}-x\right)^4\xi(s/n) \\
f\left(\frac{s}{n}\right) &= f(x) + \left(\frac{s}{n}-x\right)\gamma^{(1)}\left(\frac{s}{n}\right) + \frac{1}{2!}\left(\frac{s}{n}-x\right)^2\gamma^{(2)}\left(\frac{s}{n}\right) + \frac{1}{3!}\left(\frac{s}{n}-x\right)^3\gamma^{(3)}\left(\frac{s}{n}\right) \\
&\quad + \frac{1}{4!}\left(\frac{s}{n}-x\right)^4\gamma^{(4)}\left(\frac{s}{n}\right) + \frac{1}{5!}\left(\frac{s}{n}-x\right)^5\gamma^{(5)}\left(\frac{s}{n}\right) \\
&\quad + \frac{1}{6!}\left(\frac{s}{n}-x\right)^6\gamma^{(6)}\left(\frac{s}{n}\right) + \left(\frac{s}{n}-x\right)^6\varepsilon(s/n),
\end{aligned}$$

such that $\lim\limits_{s/n\to x} \varepsilon(s/n) = 0,$

$$\lim_{s/n\to x} \chi(s/n) = 0,$$

$$\lim_{s/n\to x} \xi(s/n) = 0,$$

$$\lim_{s/n\to x} \nu(s/n) = 0,$$

and $\lim\limits_{s/n\to x} \eta(s/n) = 0.$

Substituting these expansions in the definition of operators (6.5), we get

$$\begin{aligned}
&R_n^{\beta}(\gamma;x)\\
&= \gamma(x) - \mu_{n,1}(x)\gamma^{(1)}(x) + \left((\mu_{n,1}(x))^2 - \frac{\mu_{n,2}(x)}{2}\right)\gamma^{(2)}(x) - \frac{x(1-x)(1-2x-3\beta x+3\beta x^2)}{6n^2}\gamma^{(3)}(x)\\
&+ \frac{x^2(1-x)^2}{8n^2}\gamma^{(4)}(x) + \mu_{n,1}(x)\gamma^{(1)}(x) - (\mu_{n,1}(x))^2\gamma^{(2)}(x) + \left((\mu_{n,1}(x))^2 - \frac{\mu_{n,2}(x)}{2}\right)\mu_{n,1}(x)\gamma^{(3)}(x)\\
&- \frac{x(1-x)(1-2x-3\beta x+3\beta x^2)}{6n^2}\mu_{n,1}(x)\gamma^{(4)}(x) + \frac{x^2(1-x)^2}{8n^2}\mu_{n,1}(x)\gamma^{(5)}(x) + \frac{\mu_{n,2}(x)}{2}\gamma^{(2)}(x)\\
&- \frac{\mu_{n,1}(x)\mu_{n,2}(x)}{2}\gamma^{(3)}(x) + \left((\mu_{n,1}(x))^2 - \frac{\mu_{n,2}(x)}{2}\right)\frac{\mu_{n,2}(x)}{2}\gamma^{(4)}(x)\\
&- \frac{x(1-x)(1-2x-3\beta x+3\beta x^2)}{12n^2}\mu_{n,2}(x)\gamma^{(5)}(x) + \frac{x^2(1-x)^2}{16n^2}\mu_{n,2}(x)\gamma^{(6)}(x) + \frac{\mu_{n,3}(x)}{6}\gamma^{(3)}(x)\\
&- \frac{\mu_{n,1}(x)\mu_{n,3}(x)}{6}\gamma^{(4)}(x) + \left((\mu_{n,1}(x))^2 - \frac{\mu_{n,2}(x)}{2}\right)\frac{\mu_{n,3}(x)}{6}\gamma^{(5)}(x) + \frac{\mu_{n,4}(x)}{24}\gamma^{(4)}(x)\\
&- \frac{\mu_{n,1}(x)\mu_{n,4}(x)}{24}\gamma^{(5)}(x) + \left((\mu_{n,1}(x))^2 - \frac{\mu_{n,2}(x)}{2}\right)\frac{\mu_{n,4}(x)}{24}\gamma^{(6)}(x) + \frac{\mu_{n,5}(x)}{120}\gamma^{(5)}(x)\\
&+ \frac{\mu_{n,6}(x)}{720}\gamma^{(6)}(x) + B_n^{\beta}\left((t-x)^6\varepsilon(t);x\right) - \mu_{n,1}(x)B_n^{\beta}\left((t-x)^4\xi(t);x\right)\\
&+ \left((\mu_{n,1}(x))^2 - \frac{\mu_{n,2}(x)}{2}\right)B_n^{\beta}\left((t-x)^4\chi(t);x\right)\\
&- \frac{x(1-x)(1-2x-3\beta x+3\beta x^2)}{6n^2}B_n^{\beta}\left((t-x)^2\nu(t);x\right) + \frac{x^2(1-x)^2}{8n^2}B_n^{\beta}\left((t-x)^2\eta(t);x\right),
\end{aligned}$$

where $\mu_{n,r}(x)$ are the central moments of the classical semi-exponential operators, defined by Theorem 3.3.2. Again, let

$$\mathrm{T}_n = \left\{s : \left|\frac{s}{n} - x\right| < \varepsilon, s = 0,1,2\ldots\right\},$$
$$\text{and } \Gamma_n = \left\{s : \left|\frac{s}{n} - x\right| \geq \varepsilon, s = 0,1,2\ldots\right\}.$$

Taking into account the limits of ε, χ, ξ, ν and η, we can say that for any $\delta > 0$ if $\left|\frac{s}{n} - x\right| < \varepsilon$ then $\exists\ \varepsilon,\chi,\xi,\nu,\eta > 0$ such that $\varepsilon(s/n) < \varepsilon$, $\chi(s/n) <$

χ, $\xi(s/n) < \xi$, $\nu(s/n) < \nu$ and $\eta(s/n) < \eta$. Moreover, for $\left|\frac{s}{n} - x\right| \geq \varepsilon$, define $M = \sup_{0 \leq x \leq 1} \left(\frac{s}{n} - x\right)^2 \varepsilon(s/n)$.

$$
\begin{aligned}
&\lim_{n\to\infty} n^3 B_n^{\beta}\left((t-x)^6 \varepsilon(t); x\right) \\
&\quad = \lim_{n\to\infty} n^3 \sum_{s=0}^{\infty} e^{-\beta x} A(n,s) x^s (1-x)^{n-s} \left(\frac{s}{n} - x\right)^6 \varepsilon(s/n) \\
&\quad = \lim_{n\to\infty} \left(n^3 \sum_{s\in \mathrm{T}_n} e^{-\beta x} A(n,s) x^s (1-x)^{n-s} \left(\frac{s}{n} - x\right)^6 \varepsilon(s/n) \right. \\
&\qquad\qquad \left. + n^3 \sum_{s\in \Gamma_n} e^{-\beta x} A(n,s) x^s (1-x)^{n-s} \left(\frac{s}{n} - x\right)^6 \varepsilon(s/n) \right) \\
&\quad \leq \lim_{n\to\infty} \left(\varepsilon + \frac{M}{\delta^2}\right) n^3 \mu_{n,6}(x) \\
&\quad \leq \left(\varepsilon + \frac{M}{\delta^2}\right) 15x^3(1-x)^3 \; = \; 0.
\end{aligned}
$$

Similarly, using Theorem 3.3.2, we can prove that

$$
\begin{aligned}
&\lim_{n\to\infty} n^3 \mu_{n,1}(x) B_n^{\beta}\left((t-x)^4 \xi(t); x\right) = 0, \\
&\lim_{n\to\infty} n^3 \left((\mu_{n,1}(x))^2 - \frac{\mu_{n,2}(x)}{2} \right) B_n^{\beta}\left((t-x)^4 \chi(t); x\right) = 0, \\
&\lim_{n\to\infty} nx(1-x)(1-2x-3\beta x+3\beta x^2) B_n^{\beta}\left((t-x)^2 \nu(t); x\right) = 0, \text{ and} \\
&\lim_{n\to\infty} nx^2(1-x)^2 B_n^{\beta}\left((t-x)^2 \eta(t); x\right) = 0.
\end{aligned}
$$

Hence, we get the following expression

$$
\begin{aligned}
&\lim_{n\to\infty} n^3 \left[R_n^{\beta}(\gamma; x) - \gamma(x) \right] \\
&= \lim_{n\to\infty} \left[n^3 \left(-\frac{x(1-x)(1-2x)}{6n^2} + \frac{\beta x^2(1-x)^2}{2n^2} + (\mu_{n,1}(x))^3 - \mu_{n,1}(x)\mu_{n,2}(x) + \frac{\mu_{n,3}(x)}{6} \right) \gamma^{(3)}(x) \right. \\
&\quad + n^3 \left(-\frac{x(1-x)(1-2x)}{6n^2}\mu_{n,1}(x) + \frac{\beta x^2(1-x)^2}{2n^2}\mu_{n,1}(x) + \frac{(\mu_{n,1}(x))^2 \mu_{n,2}(x)}{2} - \frac{(\mu_{n,2}(x))^2}{4} \right. \\
&\quad \left. - \frac{\mu_{n,1}(x)\mu_{n,3}(x)}{6} + \frac{\mu_{n,4}(x)}{24} + \frac{x^2(1-x)^2}{8n^2} \right) \gamma^{(4)}(x) + n^3 \left(-\frac{x(1-x)(1-2x)}{12n^2}\mu_{n,2}(x) \right. \\
&\quad + \frac{\beta x^2(1-x)^2}{4n^2}\mu_{n,2}(x) + \frac{(\mu_{n,1}(x))^2 \mu_{n,3}(x)}{6} - \frac{\mu_{n,2}(x)\mu_{n,3}(x)}{12} - \frac{\mu_{n,1}(x)\mu_{n,4}(x)}{24} + \frac{\mu_{n,5}(x)}{120} \\
&\quad \left. + \frac{x^2(1-x)^2}{8n^2}\mu_{n,1}(x) \right) \gamma^{(5)}(x) + n^3 \left(\frac{(\mu_{n,1}(x))^2 \mu_{n,4}(x)}{24} - \frac{\mu_{n,4}(x)\mu_{n,2}(x)}{48} + \frac{\mu_{n,6}(x)}{720} \right. \\
&\quad \left. \left. + \frac{x^2(1-x)^2}{16n^2}\mu_{n,2}(x) \right) \gamma^{(6)}(x) \right]
\end{aligned}
$$

$$= \frac{\beta x(1-x)(1-6x+6x^2-3\beta x(1-x)(1-2x)+\beta^2 x^2(1-x)^2)}{6}\gamma^{(3)}(x)$$
$$+\frac{x(1-x)(1-6x+6x^2-10\beta x(1-x)(1-2x)+6\beta^2 x^2(1-x)^2)}{24}\gamma^{(4)}(x)$$
$$-\frac{x^2(1-x)^2(2-4x-3\beta x+3\beta x^2)}{24}\gamma^{(5)}(x)+\frac{x^3(1-x)^3}{48}\gamma^{(6)}(x).$$

This completes the proof. □

From Theorem 6.2.10, it is evident that the presence of the factor n^3 on the right-hand side of the result, characterizes the speed of convergence of the third order semi-exponential Bernstein operators. More precisely, this shows that the operators achieve an approximation order of $O(1/n^3)$, which marks a further improvement over the second-order case. Thus, the introduction of higher-order derivatives into the construction of these operators not only preserves their positivity and linearity but also substantially enhances their approximation properties. With the third-order operators, we now achieve a significantly better rate of convergence, reflecting the progressive effectiveness of this approach.

To complement the theoretical results established in the previous sections, it is essential to validate the performance of the semi-exponential Bernstein operators through numerical experiments. The following section is devoted to a detailed numerical analysis, where we study the behaviour of the operators for different functions. In particular, we illustrate their convergence through graphical comparisons and quantify the approximation quality using error tables.

6.3 Computational Study of the Operators

First, we provide graphical illustrations of the approximation process for the first, second and third order semi-exponential Bernstein operators. By plotting the operators alongside the targeted functions, we can visualize how fast and how well the operators converge to the given function. The graphs not only support our theoretical results but also give an easy-to-understand picture of how higher-order operators give better approximations.

6.3.1 Graphical Study of the Approximation Process

Example 6.3.1 *First we approximate the function $f(x) = x^2\cos(3\pi x) - x\sin(4\pi x)$ using second and third order semi-exponential Bernstein operators $P_n^{\beta}(f;x)$ and*

$R_n^{\beta}(f;x)$, respectively, keeping β constant and increasing the values of n. Figures 6.1 and 6.2 show the approximation process for $\beta = 5$ and $n = 10, 20, 30$.

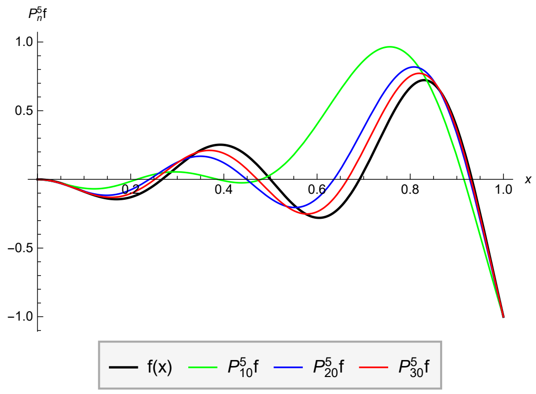


Figure 6.1: Approximation process of second order semi-exponential operators $P_n^{\beta}(t^2\cos(3\pi t) - t\sin(4\pi t);x)$ (equation (6.3)), for $\beta = 5$ and $n = 10, 20, 30$.

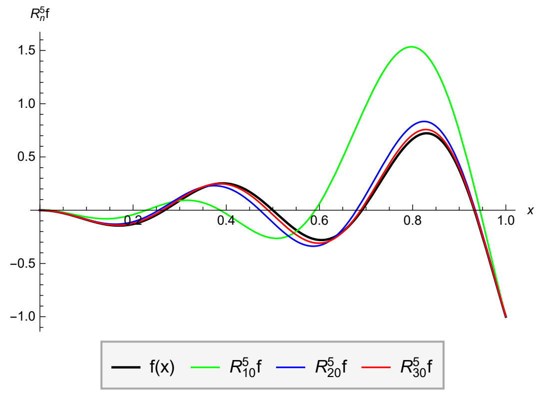


Figure 6.2: Approximation process of third order semi-exponential operators $R_n^{\beta}(t^2\cos(3\pi t) - t\sin(4\pi t);x)$ (equation (6.5)), for $\beta = 5$ and $n = 10, 20, 30$.

From Figures 6.1 and 6.2, it is observed that as n increases in the sequence of operators, while keeping the value of β fixed, both $P_n^{\beta}(f;x)$ and $R_n^{\beta}(f;x)$ converge u-niformly to the function $f(x)$. This graphical evidence is consistent with the theoretical results established earlier. Moreover, the figure clearly shows that $R_n^{\beta}(f;x)$ provides a faster convergence to $f(x)$ compared to $P_n^{\beta}(f;x)$, thereby confirming the advantage of the higher-order refinement.

Example 6.3.2 *In order to study the behaviour of parameter β on higher order semi-exponential Bernstein operators, $P_n^{\beta}(f;x)$ and $R_n^{\beta}(f;x)$, we fix the value of n and vary $\beta \geq 0$. Figures 6.3 and 6.4 show the approximation process of second and third order semi-exponential Bernstein operators for $g(x) = 8x^3 - 14x^2 + 7x - 1$ and $\beta = 5, 10, 15$, keeping $n = 20$ fixed.*

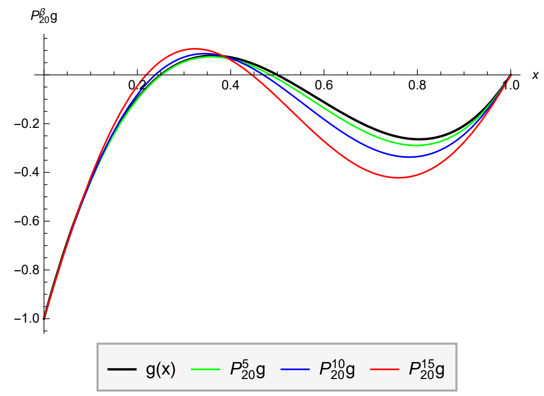


Figure 6.3: Approximation process of second order semi-exponential operators $P_n^\beta(8t^3-14t^2+7t-1;x)$ (equation (6.3)), for $\beta=5,10,15$ and $n=20$

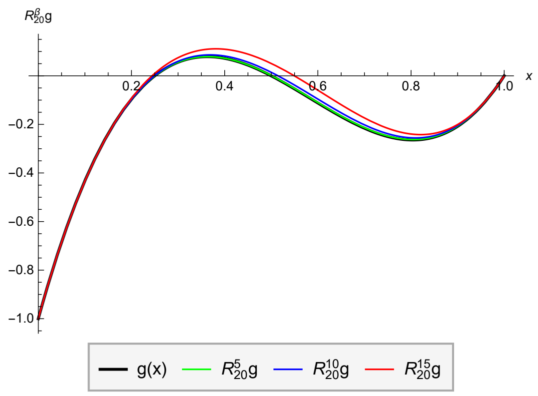


Figure 6.4: Approximation process of third order semi-exponential operators $R_n^\beta(8t^3-14t^2+7t-1;x)$ (equation (6.5)), for $\beta=5,10,15$ and $n=20$

In Figures 6.3 and 6.4, we fix the value of n and examine the effect of varying β on the quality of approximation. It is observed that smaller values of β lead to a closer approximation of the target function, while larger values of β result in a comparatively weaker approximation. However, it is also important to note that even when β takes relatively larger values, the third-order semi-exponential Bernstein operators still outperform the second-order operators obtained with smaller values of β. This clearly demonstrates the strength of higher-order operators, as they are capable of maintaining a superior rate of convergence across different parameter choices.

Example 6.3.3 *Figures 6.5 and 6.6 show the comparison between the approximation of the classical first order semi-exponential operators B_n^β, second order operators P_n^β and third order operators R_n^β for $\beta=5$, $n=20$ and $h(x)=\sin^2(2\pi x)+\cos(3\pi x)$.*

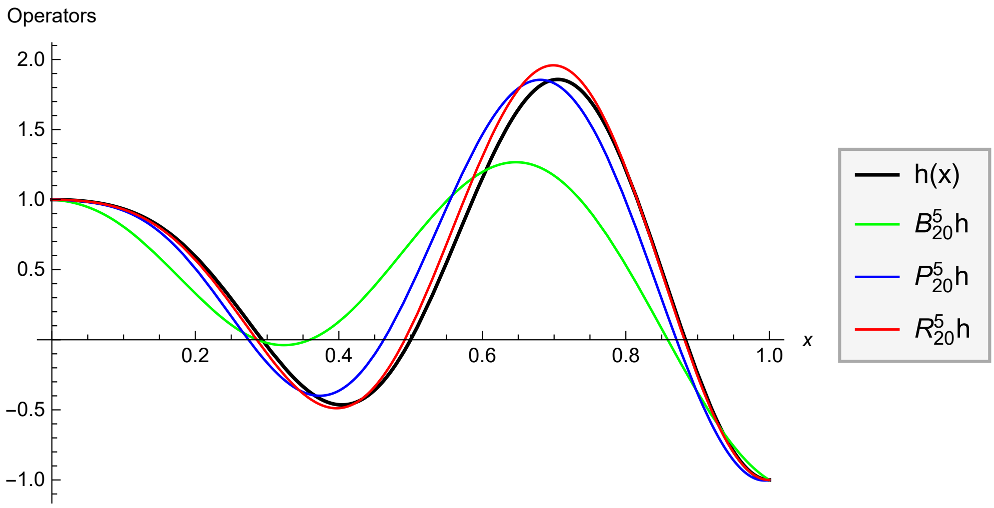


Figure 6.5: Comparison between the first, second and third order semi-exponential Bernstein operators, B_n^β, P_n^β and R_n^β respectively for $\beta = 5$, $n = 20$ and $h(x) = \sin^2(2\pi x) + \cos(3\pi x)$.

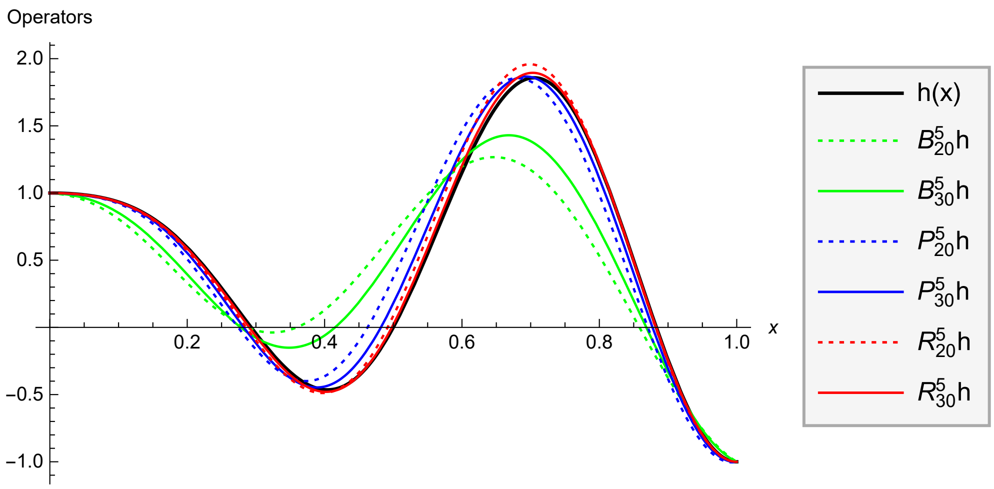


Figure 6.6: The approximation process of first order, second order and third order semi-exponential operators, B_n^β, P_n^β and R_n^β respectively.

Figure 6.5 compares the approximations produced by the first, second and third order semi-exponential Bernstein operators- $B_n^\beta(h;x)$, $P_n^\beta(h;x)$ and $R_n^\beta(h;x)$, respectively- for the same choice of n and β. The graph clearly confirms the theoretical expectation that the third order operators consistently provide the most accurate approximation to the target function, followed by the second order operators, while the first order operators offer the least precision.

Figure 6.6 shows how the approximations evolve as n increases in the sequence of operators, keeping β fixed. For each order, two curves are plotted for different n (for $n = 20$ and $n = 30$), demonstrating convergence toward the function $h(x)$. These curves, or successive approximants, are represented with dotted and solid lines- dotted for $n = 20$ and solid for $n = 30$. The spacing between successive approximants, from $n = 20$ to $n = 30$, is largest for the first order operators (represented with green), smaller for the second order operators (represented with red) and smallest for the third order operators (represented with blue). In other words, increment in n yields the greatest improvement for the third order, less for the second, and least for the first, graphically confirming the faster convergence of higher-order constructions.

All these graphical results clearly show that higher-order operators approximate the target functions more closely, thereby confirming the theoretical convergence rates. To further support these observations, we now turn to a quantitative study by computing and comparing the approximation errors.

6.3.2 Error Analysis

In this subsection, we calculate the approximation errors for different functions under varying values of n and β. These results enable us to compare the performance of the first, second and third order operators in a concrete way.

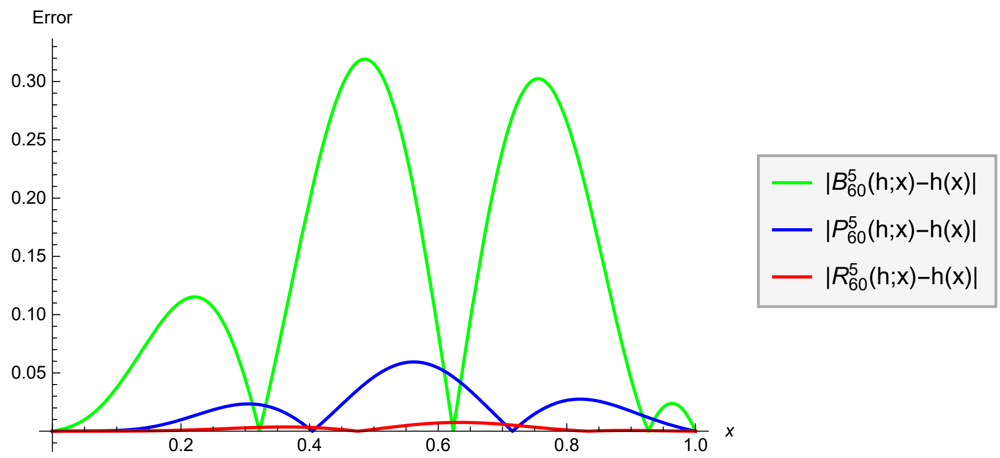


Figure 6.7: The error of first order, second order and third order semi-exponential operators, B_n^β, P_n^β and R_n^β respectively.

	$\left\|B_n^\beta(h;x)-h(x)\right\|$	$\left\|P_n^\beta(h;x)-h(x)\right\|$	$\left\|R_n^\beta(h;x)-h(x)\right\|$
$n=20$	0.696439	0.369022	0.0830424
$n=40$	0.439718	0.102866	0.00726765
$n=60$	0.315164	0.0465234	0.00162776
$n=80$	0.244815	0.02628	0.000554314
$n=100$	0.199943	0.0168343	0.000239278
$n=120$	0.168904	0.0116889	0.000120313

Table 6.1: The absolute error values between the operator approximation and function.

Figure 6.7 displays the absolute errors of the three operators B_n^β, P_n^β, and R_n^β for the function $h(x)=\sin^2(2\pi x)+\cos(3\pi x)$, with parameter $\beta=5$ and $n=60$. The error curves clearly highlight the difference in performance between the first, second and third order operators, with the third order operators producing the least error throughout the interval.

In addition, Table 6.1 provides the corresponding numerical values of the absolute error at a representative point, $x=0.5$, for varying values of n. From the table, we can observe that the absolute error decreases for all three operators. However, the third order operators R_n^β show the fastest decrease in error.

	$n=20$	$n=50$	$n=100$
$x=0.1$	0.0120176	0.00104047	0.000171453
$x=0.2$	0.0864522	0.0142362	0.00342182
$x=0.3$	0.117832	0.0321024	0.00915818
$x=0.4$	0.0999135	0.000224725	0.00210025
$x=0.5$	0.369022	0.0666179	0.0168343
$x=0.6$	0.309482	0.0749812	0.0212553
$x=0.7$	0.0252893	0.00964473	0.00390035
$x=0.8$	0.224883	0.0378357	0.00967448
$x=0.9$	0.141496	0.0244527	0.00630595

Table 6.2: The values of $\left|P_n^\beta(h;x)-h(x)\right|$

	$n = 20$	$n = 50$	$n = 100$
$x = 0.1$	0.00347376	0.00011565	8.46281×10^{-6}
$x = 0.2$	0.0242278	0.00152493	0.000178099
$x = 0.3$	0.0526182	0.00482326	0.000651934
$x = 0.4$	0.0242578	0.00513657	0.000831434
$x = 0.5$	0.0830424	0.00320301	0.000239278
$x = 0.6$	0.154404	0.0120135	0.0015759
$x = 0.7$	0.102684	0.00980138	0.0013885
$x = 0.8$	0.00955083	0.00173705	0.000273066
$x = 0.9$	0.0172267	0.00113401	0.000144761

Table 6.3: The values of $\left|R_n^{\beta}(h;x) - h(x)\right|$

Tables 6.2 and 6.3 present the numerical values of the approximation error for the second-order operators $P_n^{\beta}(h;x)$, as defined in (6.3), and the third-order operators $R_n^{\beta}(h;x)$, as defined in (6.5). The errors are computed for different values of n and x for the function $h(x) = \sin^2(2\pi x) + \cos(3\pi x)$.

From the numerical results and error analysis, we conclude that for approximating continuous real-valued functions, higher-order operators are more effective. They consistently yield better accuracy and significantly smaller errors compared to lower-order counterparts, making them a preferable choice in practical applications.

6.4 Conclusion

In this chapter, we focused on the construction and analysis of higher-order semi-exponential Bernstein operators and explored their role in improving approximation results. Specifically, we introduced operators of order $O(1/n^2)$ and $O(1/n^3)$, thereby extending the scope of the classical semi-exponential Bernstein operators which are of order $O(1/n)$. For each class of operators, we carefully derived their moments and central moments, which form the foundation for understanding their approximation behaviour. Using these, we employed the modulus of continuity to establish error estimates, and subsequently proved Voronovskaya-type asymptotic theorems. These results not only characterize the asymptotic nature of the approximation but also reveal the exact order of convergence associated with the newly defined operators.

A key insight emerging from this study is the increasing preservation of test functions, 1, t and t^2, as we move to higher-order operators. The first-order semi-exponential Bernstein operators (6.2) preserve only constant functions, that is,

- $B_n^{\beta}(1;x) = 1,$

while for the test functions t and t^2 the operators yield expressions that converge to x and x^2, respectively and $n \to \infty$. The second-order semi-exponential Bernstein operators (6.3) represent a clear improvement, since they preserve both 1 and t exactly and approximate t^2 by an expression converging to x^2. That is,

- $P_n^{\beta}(1;x) = 1$
- $P_n^{\beta}(t;x) = x.$

The third-order semi-exponential Bernstein operators (6.5) advance this progression further by preserving all three test functions $1, t, t^2$ exactly, that is,

- $R_n^{\beta}(1;x) = 1$
- $R_n^{\beta}(t;x) = x$
- $R_n^{\beta}(t^2;x) = x^2.$

This hierarchical structure highlights a systematic strengthening in the approximation power as the order increases, and explains why the convergence order improves from $O(1/n)$ in the first order, to $O(1/n^2)$ in the second order, and further to $O(1/n^3)$ in the third order.

Chapter 7

Higher Order Operators based on Contagion Distribution

This chapter talks about the order of approximation of sequence of positive linear operators based on contagion distribution, also known as the Pólya-Eggenberger distribution, involving a positive real parameter $\alpha \in [0,1]$. *The first order operators have already been defined by Stancu in* 1968. *We extend this notation of the contagion distribution, with the parameter* α, *to approximate continuous real-valued function on* $[0,1]$, *with a better order of convergence. We derive the higher order linear positive operators and prove that their order of approximation is* $O(1/n^2)$. *We give their moments and central moments and using Korovkin theorem prove their uniform convergence on the compact interval* $[0,1]$. *Using first and second order modulus of continuity, the approximation properties of the defined second order operators are also proved. Further, graphical and numerical illustrations are presented to support the theoretical findings of these operators.*

7.1 Introduction

The binomial distribution is one of the most fundamental concepts in probability and statistics, as it models situations where an experiment consists of a fixed number of independent trials, each having only two possible outcomes, often referred to as success and failure. A simple way to understand this is through the classic urn problem: suppose we have an urn containing identical balls of two different colours, say red and blue. A single trial consists of drawing a ball from the urn, noting its colour, and then placing it back into the urn so that the conditions of each trial remain unchanged. If the probability of drawing a red ball in one trial is x (hence the probability of drawing a blue ball is $1-x$), and this process is repeated n times independently, then we are interested in the probability of observing exactly k red balls in those n trials.

This probability is given by the binomial probability mass distribution function

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

where the binomial coefficient $\binom{n}{k}$ counts the number of distinct ways in which k successes (red balls) can occur among n trials. The factor x^k represents the probability of obtaining k successes, while $(1-x)^{n-k}$ accounts for the remaining $n-k$ failures. The binomial distribution not only plays a central role in probability theory and statistical inference but also forms the foundation for various approximation techniques. In particular, the weights $b_{n,k}(x)$ appear naturally in the definition of Bernstein polynomials, which are widely used in approximation theory to approximate continuous functions on the interval $[0,1]$.

In 1912, the Russian mathematician S. N. Bernstein gave a new proof of the famous Weierstrass Approximation Theorem [215], which states that any continuous real-valued function on a closed interval can be uniformly approximated by polynomials. Instead of relying on abstract arguments, Bernstein introduced what are now famously known as the Bernstein polynomials, thereby giving a concrete sequence of positive linear operators that converge to any continuous function on $[0,1]$. His approach was remarkable not only because it gave a simple and elegant proof of Weierstrass' result, but also because it laid the foundation for an entirely new direction in approximation theory.

To construct these operators, Bernstein employed the binomial distribution as the weight functions. In particular, he observed that the binomial probabilities $b_{n,k}(x)$ could serve as the coefficients in forming convex combinations of function values $f(k/n)$, thereby defining the Bernstein operators

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right). \tag{7.1}$$

Following the work of S. N. Bernstein, many researchers expanded the field of approximation theory by giving various modifications of these operators, like the Bernstein-Kantorovich operators, α-Bernstein operators, λ-Bernstein operators, Bernstein-Durrmeyer operators and more [7; 8; 42; 51; 120; 153].

Although the classical Bernstein operators and their generalizations converge to the target function with order at most $O(1/n)$, but they are built on the binomial distribution, which are not very much applicable in modeling the various real-life scenarios. For instance, in many natural and social phenomena, the probability of an outcome actually changes as the process evolves. In biology, the spread of a gene may become more likely once it has appeared a few times; in epidemiology, the likelihood of infection may increase as more individuals are already infected; in economics, consumer choice may become reinforced once a product is repeatedly chosen. In such cases, the simple binomial distribution cannot adequately capture the reinforcement or dependency between successive trials.

To address such situations, in 1923 Pólya and Eggenberger introduced a discrete probability distribution that has since become widely known as the Pólya-Eggenberger distribution, or the contagion distribution. Their construction is based on the Pólya-Eggenberger urn scheme [74]. Consider an urn initially containing X red and Y blue balls, identical in in shape and size. One ball is drawn at random from the urn and colour of the ball is noted. It is then substituted along with θ identical balls of the same colour. This reinforcement mechanism means that the chance of drawing a particular colour increases each time it is observed, modeling a form of 'contagion' or 'self-reinforcement.'

The probability of obtaining exactly k red balls after n such draws is given by

$$\upsilon_{n,k} = \binom{n}{k} \frac{\prod_{i=0}^{k-1}(X+i\theta)\prod_{j=0}^{n-k-1}(Y+j\theta)}{\prod_{l=0}^{n-1}(X+Y+l\theta)}.$$

This distribution reduces to the binomial distribution when $\theta = 0$, but for $\theta > 0$ it provides a much richer model where past outcomes influence future probabilities. Because of this flexibility, the contagion distribution has found applications in diverse fields such as genetics, epidemiology, social dynamics, and reliability theory.

From the perspective of approximation theory, replacing the binomial distribution in the definition of Bernstein operators with the contagion distribution gives rise to new classes of positive linear operators that can better capture the dynamics of reinforcement-based processes. These operators extend the reach of the classical Bernstein framework and allow us to approximate continuous functions while modeling more realistic probabilistic structures. Based on this idea, D. D. Stancu introduced a sequence of positive linear operators, now known as the Stancu-Bernstein operators [194; 195], which form one of the earliest and most influential generalizations of the Bernstein operators. These operators are defined as

$$P_n^{\langle\alpha\rangle}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{n}\right), \tag{7.2}$$

where

$$p_{n,k}^{\langle\alpha\rangle}(x) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}},$$

and α is a non-negative parameter. The factorial power is determined by $a^{[0,\nu]} = 1$ and $a^{[n,\nu]} = a(a-\nu)(a-2\nu)\ldots(a-(n-1)\nu)$. This representation gives rise to a polynomial of degree n. Since the introduction of the Pólya distribution and its incorporation into approximation theory, a large number of studies have been devoted to analyzing its properties and exploring its generalizations. Researchers have investigated convergence behavior, error estimates, Voronovskaya-type results and various modifications that extend its applicability to more complex settings. For a broader perspective and recent developments on this topic, one may refer to [64; 123; 139; 154; 161] and the references cited therein.

Even though Stancu introduced an additional parameter to generalize the classical Bernstein operators (7.1), the fundamental rate of convergence of these operators remains unchanged. That is, for the operators (7.2), the approximation process for continuous real-valued functions f still proceeds with an order of convergence $O(1/n)$, the same as in the case of the classical Bernstein operators. Although, the introduction of this new parameter α does provide extra flexibility in shaping the operators and adapting them to different situations, but does not accelerate the approximation in terms of order. Because these operators retain the same order of approximation as the classical case, we shall refer to them as the first order Stancu-Bernstein operators throughout this chapter.

Although the sequence of first order Stancu-Bernstein operators converges uniformly to the desired function f, their order of convergence is considered to be significantly low. In practice, this means that to get a good approximation one often needs to take very large values of n, which increases computational effort and makes these operators less efficient in real applications. This drawback becomes particularly noticeable in problems where accuracy and efficiency are both crucial.

To address this issue, the main aim of this chapter is to improve this order of approximation by defining a new sequence of operators, in continuation of the first order Stancu operators, with the parameter $\alpha \to 0$ as $n \to \infty$.

Thus, for a continuous real-valued function f defined on $[0,1]$, assuming that f'' exists, we define the higher order Stancu-Bernstein operators as

$$Q_n^{\langle\alpha\rangle}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\left[f\left(\frac{k}{n}\right) - \frac{cx(1-x)}{n(1+\alpha)}f''\left(\frac{k}{n}\right)\right], \tag{7.3}$$

where $c = \lim\limits_{n\to\infty} \frac{1+n\alpha}{2}$.

Having defined the higher order Stancu-Bernstein operators, we now focus on their approximation properties in the next section

7.2 Approximation Results

To establish the approximation properties of the higher order Stancu-Bernstein operators (7.3), we begin by examining the moments and central moments of the first order Stancu-Bernstein operators (7.2), which serve as the foundation for subsequent results.

Lemma 7.2.1 *Let $P_n^{\langle\alpha\rangle}(f;x)$ be the first order Stancu-Bernstein operators* (7.2)*. Taking $\alpha = O(1/n)$, we get the following equalities:*

(i) $P_n^{\langle\alpha\rangle}(1;x) = 1$

(ii) $P_n^{\langle\alpha\rangle}(t;x) = x$

(iii) $P_n^{\langle\alpha\rangle}(t^2;x) = x\left[\frac{(n-1)x+n\alpha+1}{n(1+\alpha)}\right]$

(iv) $P_n^{\langle\alpha\rangle}(t^3;x) = \frac{x}{n^2} + \frac{3(n-1)}{n^2}x\left(\frac{x+\alpha}{1+\alpha}\right) + \frac{(n-1)(n-2)}{n^2}x\left(\frac{x+\alpha}{1+\alpha}\right)\left(\frac{x+2\alpha}{1+2\alpha}\right)$

(v) $P_n^{\langle\alpha\rangle}(t^4;x) = \left\{ \begin{array}{l} \frac{x}{n^3} + \frac{7(n-1)}{n^3}x\left(\frac{x+\alpha}{1+\alpha}\right) + \frac{6(n-1)(n-2)}{n^3}x\left(\frac{x+\alpha}{1+\alpha}\right)\left(\frac{x+2\alpha}{1+2\alpha}\right) \\ +\frac{(n-1)(n-2)(n-3)}{n^3}x\left(\frac{x+\alpha}{1+\alpha}\right)\left(\frac{x+2\alpha}{1+2\alpha}\right)\left(\frac{x+3\alpha}{1+3\alpha}\right). \end{array} \right.$

The proof of this lemma is presented in Chapter 2, Lemma 2.1.2. Using this result, we can also compute the limits of the central moments for the first order Stancu-Bernstein operators, which play a key role in the subsequent analysis.

Lemma 7.2.2 *Let us define the central moments of the first order Stancu operators* (7.2) *as* $\mu_{n,r}^{\langle\alpha\rangle}(x) = P_n^{\langle\alpha\rangle}\left(\left(\frac{k}{n}-x\right)^r;x\right)$. *Then for* $\alpha = O(1/n)$, *we have the following limits:*

(i) $\lim\limits_{n\to\infty} n\mu_{n,2}^{\langle\alpha\rangle}(x) = \frac{2cx(1-x)}{1+\alpha}$

(ii) $\lim\limits_{n\to\infty} n^2\mu_{n,4}^{\langle\alpha\rangle}(x) = \frac{12c^2(1-6\alpha)x^2(1-x)^2}{(1+\alpha)(1+2\alpha)(1+3\alpha)}$,

where $c = \lim\limits_{n\to\infty} \frac{1+n\alpha}{2}$.

Proof. From Lemma 7.2.1, it follows that

$$\mu_{n,1}^{\langle\alpha\rangle}(x) = P_n^{\langle\alpha\rangle}(t;x) - xP_n^{\langle\alpha\rangle}(1;x) = 0,$$

and the second central moment is

$$\mu_{n,2}^{\langle\alpha\rangle}(x) = P_n^{\langle\alpha\rangle}\left(t^2;x\right) - 2xP_n^{\langle\alpha\rangle}(t;x) + xP_n^{\langle\alpha\rangle}(1;x) = \frac{(1+n\alpha)x(1-x)}{n(1+\alpha)}.$$

Thus, we get the limit, $\lim\limits_{n\to\infty} n\mu_{n,2}^{\langle\alpha\rangle}(x) = \lim\limits_{n\to\infty}(1+n\alpha)\frac{x(1-x)}{1+\alpha}$, which implies the desired result. Now, calculating the fourth central moment

$$\begin{aligned}
&\mu_{n,4}^{\langle\alpha\rangle}(x)\\
&= P_n^{\langle\alpha\rangle}\left((t-x)^4;x\right)\\
&= \frac{x(1-x)(1+n\alpha)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}\Big[1+(6n-1)\alpha+6n^2\alpha^2-3x\left(2+n(6\alpha-1)+n^2\alpha(6\alpha-1)\right)\\
&\qquad +3x^2\left(2+n(6\alpha-1)+n^2\alpha(6\alpha-1)\right)\Big]\\
&= \frac{x(1-x)(1+n\alpha)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}\Big[1-\alpha+6n\alpha(1+n\alpha)-3x(2+n(6\alpha-1)(1+n\alpha))\\
&\qquad +3x^2(2+n(6\alpha-1)(1+n\alpha))\Big]\\
&= \frac{x(1-x)(1+n\alpha)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}\Big[1-\alpha+6n\alpha(1+n\alpha)-3x(1-x)(2+n(1+n\alpha)(6\alpha-1))\Big]\\
&= \frac{1}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}\Big[x(1-x)(1+n\alpha)(1-\alpha)+6n(1+n\alpha)^2\alpha x(1-x)\\
&\qquad -3(1+n\alpha)x^2(1-x)^2(2+n(1+n\alpha)(6\alpha-1))\Big]\\
&= \frac{1}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}\Big[x(1-x)(1+n\alpha)(1-\alpha)+6n(1+n\alpha)^2\alpha x(1-x)\\
&\qquad -6(1+n\alpha)x^2(1-x)^2-3n(1+n\alpha)^2(6\alpha-1)x^2(1-x)^2\Big].
\end{aligned}$$

Since $\alpha = O(1/n)$, we can say that $\alpha \to 0$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} n^2 \mu_{n,4}^{\langle\alpha\rangle}(x) = \lim_{n\to\infty} (1+n\alpha)^2(1-6\alpha)\frac{3x^2(1-x)^2}{(1+\alpha)(1+2\alpha)(1+3\alpha)}$$
$$= \frac{12c^2(1-6\alpha)x^2(1-x)^2}{(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

□

The moments and central moments of the first order Stancu-Bernstein operators provide a foundation for extending the analysis to higher order operators. Using these results, we can systematically define and compute the moments of the higher order operators, which are crucial for studying their convergence, approximation order and error behaviour.

Lemma 7.2.3 *For $\alpha = O(1/n)$, the moments of higher order Stancu-Bernstein operators* (7.3) *are given as:*

(i) $Q_n^{\langle\alpha\rangle}(1;x) = 1$

(ii) $Q_n^{\langle\alpha\rangle}(t;x) = x$

(iii) $Q_n^{\langle\alpha\rangle}(t^2;x) = \frac{n-1+2c}{n(1+\alpha)}x^2 + \frac{n\alpha+1-2c}{n(1+\alpha)}x$

(iv) $Q_n^{\langle\alpha\rangle}(t^3;x) = \begin{cases} \left[\frac{6c}{n(1+\alpha)} + \frac{(n-1)(n-2)}{n^2(1+\alpha)(1+2\alpha)}\right]x^3 - \left[\frac{6c}{n(1+\alpha)} - \frac{3(n-1)}{n^2(1+\alpha)} + \frac{3\alpha(n-1)(n-2)}{n^2(1+\alpha)(1+2\alpha)}\right]x^2 \\ + \left[\frac{1}{n^2} + \frac{3\alpha(n-1)}{n^2(1+\alpha)} + \frac{2\alpha^2(n-1)(n-2)}{n^2(1+\alpha)(1+2\alpha)}\right]x \end{cases}$

(v) $Q_n^{\langle\alpha\rangle}(t^4;x) = \begin{cases} \left[\frac{12c(n-1)}{n^2(1+\alpha)^2} + \frac{(n-1)(n-2)(n-3)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}\right]x^4 + \left[\frac{12c(n\alpha-n+2)}{n^2(1+\alpha)^2} + \frac{6(n-1)(n-2)}{n^3(1+\alpha)(1+2\alpha)} \right. \\ \left. + \frac{6\alpha(n-1)(n-2)(n-3)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}\right]x^3 + \left[\frac{7(n-1)}{n^3(1+\alpha)} + \frac{18\alpha(n-1)(n-2)}{n^3(1+\alpha)(1+2\alpha)} + \frac{11\alpha^2(n-1)(n-2)(n-3)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)} \right. \\ \left. - \frac{12c(1+n\alpha)}{n^2(1+\alpha)^2}\right]x^2 + \left[\frac{1}{n^3} + \frac{7\alpha(n-1)}{n^3(1+\alpha)} + \frac{12\alpha^2(n-1)(n-2)}{n^3(1+\alpha)(1+2\alpha)} + \frac{6\alpha^3(n-1)(n-2)(n-3)}{n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}\right]x \end{cases}$

Proof. Using Lemma 7.2.1, the moments of the higher order StancuâĂŞBernstein operators (7.3) can be obtained directly by substituting the test functions $f(t) = t^r$ for $r = 0,1,2,3,4$.

(i) For $f(t) = 1$, $f''(t) = 0$. Thus,

$$Q_n^{\langle\alpha\rangle}(1;x) = \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x) = 1,$$

where $p_{n,k}^{\langle\alpha\rangle}(x)$ is the basis function of the first order Stancu-Bernstein operators, defined in (7.2).

(ii) Again for $f(t)=t$, $f''(t)=0$. Thus, the second moment is given by

$$Q_n^{\langle\alpha\rangle}(t;x)=\sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\frac{k}{n}=x.$$

(iii) For $f(t)=t^2$, $f''(t)=2$. Thus, the third moment of the operator takes the form

$$\begin{aligned}Q_n^{\langle\alpha\rangle}\left(t^2;x\right)&=\sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\left[\frac{k^2}{n^2}-\frac{2cx(1-x)}{n(1+\alpha)}\right]\\&=\frac{(n-1)}{n(1+\alpha)}x^2+\frac{n\alpha+1}{n(1+\alpha)}x-\frac{2cx(1-x)}{n(1+\alpha)}\\&=\frac{(n-1+2c)x^2+(n\alpha+1-2c)x}{n(1+\alpha)}\end{aligned}$$

Moreover, for $n\to\infty$, it follows that $Q_n^{\langle\alpha\rangle}\left(t^2;x\right)\to x^2$.

In a similar manner, we can calculate $Q_n^{\langle\alpha\rangle}\left(t^3;x\right)$ and $Q_n^{\langle\alpha\rangle}\left(t^4;x\right)$. □

Lemma 7.2.3 leads us to the conclusion that, for any real-valued continuous function f, the higher order Stancu-Bernstein operators $Q_n^{\langle\alpha\rangle}(f;x)$ converge uniformly to $f(x)$ on $[0,1]$. That is,

$$\lim_{n\to\infty} Q_n^{\langle\alpha\rangle}(f;x)=f(x).$$

We now proceed to calculate certain limits of the central moments of these operators, which will be essential for analyzing their approximation properties.

Lemma 7.2.4 *For the second order Stancu operators* (7.3)*, we have the following expressions for central moments:*

(i) $Q_n^{\langle\alpha\rangle}((t-x);x)=0$

(ii) $\lim_{n\to\infty} Q_n^{\langle\alpha\rangle}\left((t-x)^2;x\right)=\frac{\alpha x(1-x)}{1+\alpha}$

While the moments and central moments of the higher order Stancu-Bernstein operators give the desired convergence result to guarantee their validity as approximation operators, but they do not provide information about the order of convergence. To better understand the efficiency and practical usefulness of these operators, we now proceed to analyze the order of convergence and derive quantitative estimates that highlight the effectiveness of these higher order Stancu operators.

Theorem 7.2.5 *Let f be a continuous function defined on $[0,1]$ and let $f^{(n)}$ denote its n^{th} order derivative. Let $c = \lim\limits_{n\to\infty} \frac{1+n\alpha}{2}$ where $\alpha \to 0$ as $n \to \infty$. If $f^{(3)}$ and $f^{(4)}$ do not vanish simultaneously on $x \in [0,1]$, then*

$$\lim_{n\to\infty} n^2 \left[Q_n^{\langle\gamma\rangle}(f;x) - f(x)\right] = \frac{c(4c-1)x(1-x)(1-2x)}{3(1+\alpha)(1+2\alpha)} f^{(3)}(x) + \left[A\frac{x(1-x)}{24} - B\frac{x^2(1-x)^2}{4(1+\alpha)}\right] f^{(4)}(x),$$

where $A = \frac{\alpha(24c^2-\alpha+1)}{(1+\alpha)(1+2\alpha)(1+3\alpha)}$ and $B = \frac{\alpha(1+\alpha)+2c^2(18\alpha^2+15\alpha+1)}{(1+\alpha)(1+2\alpha)(1+3\alpha)}$.

Proof. Let $c = \lim\limits_{n\to\infty} \frac{1+n\alpha}{2}$, then by Taylor's expansion, we can write

$$f^{(2)}\left(\frac{k}{n}\right) = f^{(2)}(x) + \left(\frac{k}{n} - x\right) f^{(3)}(x) + \frac{1}{2!}\left(\frac{k}{n} - x\right)^2 f^{(4)}(x) + \left(\frac{k}{n} - x\right)^2 \mu\,(k/n) \tag{7.4}$$

and

$$f\left(\frac{k}{n}\right) = f(x) + \left(\frac{k}{n} - x\right) f^{(1)}(x) + \frac{1}{2!}\left(\frac{k}{n} - x\right)^2 f^{(2)}(x) + \frac{1}{3!}\left(\frac{k}{n} - x\right)^3 f^{(3)}(x) + \frac{1}{4!}\left(\frac{k}{n} - x\right)^4 f^{(4)}(x) + \left(\frac{k}{n} - x\right)^4 \varepsilon\,(k/n). \tag{7.5}$$

Equations (7.4) and (7.5) imply that

$$f\left(\frac{k}{n}\right) - \frac{cx(1-x)}{n(1+\alpha)} f^{(2)}\left(\frac{k}{n}\right) = \begin{cases} f(x) - \frac{cx(1-x)}{n(1+\alpha)} f^{(2)}(x) + \left(\frac{k}{n} - x\right) f^{(1)}(x) \\ -\frac{cx(1-x)}{n(1+\alpha)}\left(\frac{k}{n} - x\right) f^{(3)}(x) + \frac{1}{2!}\left(\frac{k}{n} - x\right)^2 f^{(2)}(x) \\ -\frac{cx(1-x)}{2n(1+\alpha)}\left(\frac{k}{n} - x\right)^2 f^{(4)}(x) + \frac{1}{3!}\left(\frac{k}{n} - x\right)^3 f^{(3)}(x) \\ +\frac{1}{4!}\left(\frac{k}{n} - x\right)^4 f^{(4)}(x) + \left(\frac{k}{n} - x\right)^4 \varepsilon\,(k/n) \\ -\frac{cx(1-x)}{n(1+\alpha)}\left(\frac{k}{n} - x\right)^2 \xi\,(k/n), \end{cases} \tag{7.6}$$

where both $\lim\limits_{k/n\to x} \varepsilon\,(k/n)$ and $\lim\limits_{k/n\to x} \xi\,(k/n)$ are equal to 0. By multiplying both sides of equation (7.6) with $p_{n,k}^{\langle\alpha\rangle}(x)$, summing over k from 0 to n, and applying Lemma 7.2.2, we obtain

$$\begin{aligned} &\sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\left[f\left(\frac{k}{n}\right) - \frac{cx(1-x)}{n(1+\alpha)} f^{(2)}\left(\frac{k}{n}\right)\right] \\ &= f(x) - \frac{cx(1-x)}{n(1+\alpha)} f^{(2)}(x) + \frac{x(1-x)(1+n\alpha)}{2n(1+\alpha)} f^{(2)}(x) \\ &\quad + \frac{x(1-x)(1-2x)(1+n\alpha)(1+2n\alpha)}{6n^2(1+\alpha)(1+2\alpha)} f^{(3)}(x) - \frac{cx^2(1-x)^2(1+n\alpha)}{2n^2(1+\alpha)^2} f^{(4)}(x) \\ &\quad + \frac{x(1-x)(1+n\alpha)}{24n^3(1+\alpha)(1+2\alpha)(1+3\alpha)}\big[(1-\alpha) + 6n\alpha(1+n\alpha) - 6x(1-x) \end{aligned}$$

$$-3n(1+n\alpha)(6\alpha-1)x(1-x)\big]f^{(4)}(x)+P_n^{\langle\alpha\rangle}\left(\left(\frac{k}{n}-x\right)^4\varepsilon\left(k/n\right);x\right)$$
$$-\frac{cx(1-x)}{n(1+\alpha)}P_n^{\langle\alpha\rangle}\left(\left(\frac{k}{n}-x\right)^2\xi\left(k/n\right);x\right).$$

We claim that both terms, $n^2P_n^{\langle\alpha\rangle}\left(\left(\frac{k}{n}-x\right)^4\varepsilon\left(k/n\right);x\right)$ and $nP_n^{\langle\alpha\rangle}\left(\left(\frac{k}{n}-x\right)^2\xi\left(k/n\right);x\right)$ tend to 0, as $n\to\infty$. Let us define,

$$\mathrm{T}_n=\left\{k:\left|\frac{k}{n}-x\right|<\varepsilon,k=0,1,2\ldots,n\right\},$$
$$\text{and}\ \ \Gamma_n=\left\{k:\left|\frac{k}{n}-x\right|\geq\varepsilon,k=0,1,2\ldots,n\right\}.$$

Taking into account the limits of $\varepsilon\left(k/n\right)$ and $\xi\left(k/n\right)$, we observe that for any $\delta>0$, if $\left|\frac{k}{n}-x\right|<\delta$, then there exist positive constants ε and ξ such that $\varepsilon\left(k/n\right)<\varepsilon$ and $\xi\left(k/n\right)<\xi$. On the other hand, for the case $\left|\frac{k}{n}-x\right|\geq\delta$, we define

$$M_1=\sup_{0\leq x\leq 1}\left(\frac{k}{n}-x\right)^2\varepsilon\left(k/n\right),\ \ \text{and}\ M_2=\sup_{0\leq x\leq 1}\left(\frac{k}{n}-x\right)^2\xi\left(k/n\right).$$

Thus, from Lemma 7.2.1, we can write

$$\begin{aligned}
&\lim_{n\to\infty}n^2P_n^{\langle\alpha\rangle}\left((t-x)^4\varepsilon\left(t\right);x\right)\\
&\quad=\sum_{k=0}^{n}n^2p_{n,k}^{\langle\alpha\rangle}(x)\left(\frac{k}{n}-x\right)^4\varepsilon\left(k/n\right)\\
&\quad=\lim_{n\to\infty}\sum_{k\in\mathrm{T}_n}n^2p_{n,k}^{\langle\alpha\rangle}(x)\left(\frac{k}{n}-x\right)^4\varepsilon\left(k/n\right)+\sum_{k\in\Gamma_n}n^2p_{n,k}^{\langle\alpha\rangle}(x)\left(\frac{k}{n}-x\right)^4\varepsilon\left(k/n\right)\\
&\quad\leq\lim_{n\to\infty}\left(\varepsilon+\frac{M_1}{\delta^2}\right)n^2\mu_{n,4}^{\langle\alpha\rangle}(x)\\
&\quad=\lim_{n\to\infty}\left(\varepsilon+\frac{M_1}{\delta^2}\right)\frac{12c^2x^2(1-x)^2}{(1+\alpha)(1+2\alpha)(1+3\alpha)}\\
&\quad=0,
\end{aligned}$$

and,

$$\begin{aligned}
&\lim_{n\to\infty}nP_n^{\langle\alpha\rangle}\left((t-x)^2\xi\left(t\right);x\right)\\
&\quad=\lim_{n\to\infty}\sum_{k\in\mathrm{T}_n}np_{n,k}^{\langle\alpha\rangle}(x)\left(\frac{k}{n}-x\right)^2\xi\left(k/n\right)+\sum_{k\in\Gamma_n}np_{n,k}^{\langle\alpha\rangle}(x)\left(\frac{k}{n}-x\right)^2\xi\left(k/n\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{n\to\infty}\left(\xi+\frac{M_2}{\delta^2}\right) n\mu_{n,2}^{\langle\alpha\rangle}(x)\\
&= \lim_{n\to\infty}\left(\xi+\frac{M_2}{\delta^2}\right)\frac{2cx(1-x)}{1+\alpha}\\
&= 0.
\end{aligned}
$$

This proves our claim. Hence, we can say that

$$
\begin{aligned}
&\lim_{n\to\infty} n^2\left[Q_n^{\langle\alpha\rangle}(f;x)-f(x)\right]\\
&\quad= \lim_{n\to\infty}\left[\frac{x(1-x)(1-2x)(1+n\alpha)(1+2n\alpha)}{6(1+\alpha)(1+2\alpha)}f^{(3)}(x)-\frac{cx^2(1-x)^2(1+n\alpha)}{2(1+\alpha)^2}f^{(4)}(x)\right.\\
&\qquad+\frac{x(1-x)}{24(1+\alpha)(1+2\alpha)(1+3\alpha)}\big(\alpha(1-\alpha)+6\alpha(1+n\alpha)^2-6\alpha x(1-x)\\
&\qquad\left.-3(1+n\alpha)^2(6\alpha-1)x(1-x)\big)f^{(4)}(x)\right]\\
&\quad= \frac{c(4c-1)x(1-x)(1-2x)}{3(1+\alpha)(1+2\alpha)}f^{(3)}(x)+\left[A\frac{x(1-x)}{24}-B\frac{x^2(1-x)^2}{4(1+\alpha)}\right]f^{(4)}(x),
\end{aligned}
$$

where

$$
A=\frac{\alpha(24c^2-\alpha+1)}{(1+\alpha)(1+2\alpha)(1+3\alpha)},\; B=\frac{\alpha(1+\alpha)+2c^2(18\alpha^2+15\alpha+1)}{(1+\alpha)(1+2\alpha)(1+3\alpha)}
$$

and

$$
c=\lim_{n\to\infty}\frac{1+n\alpha}{2}.
$$

This completes the proof. □

Since $n^2\left[Q_n^{\langle\alpha\rangle}(f;x)-f(x)\right]$ approaches a finite limit as $n\to\infty$, we deduce that the operators $Q_n^{\langle\alpha\rangle}$ provide an approximation to any continuous real-valued function f on $[0,1]$ with an order of approximation $\mathscr{O}(1/n^2)$. This is a remarkable improvement when compared with the classical Bernstein operators and their first-order Stancu-Bernstein modification, both of which are restricted to an approximation order of at most $\mathscr{O}(1/n)$.

The enhancement from $\mathscr{O}(1/n)$ to $\mathscr{O}(1/n^2)$ is not merely a quantitative improvement but also a qualitative one. It places the second-order Stancu-Bernstein operators in the broader family of higher-order approximation processes that aim to reduce the error significantly while retaining the positive linear structure of the original operators. Such an improvement is especially valuable in applications where faster convergence is critical, for example, in numerical solutions of differential equations, data fitting and computer-aided geometric design, where the efficiency of the approximating process directly impacts computational cost and accuracy.

An important feature of these operators is the presence of the parameter α, which is absent in the classical Bernstein operators. This additional parameter enriches the structure of the operators and makes them more flexible for modeling and capturing various real-life processes, while still preserving the improved order of convergence $\mathscr{O}(1/n^2)$. In this sense, the parameter α broadens the applicability of the operators without compromising their approximation order. Thus, the second-order Stancu-Bernstein operators strike a balance between generality (through parameter dependence) and efficiency (through improved order).

In light of these properties, we shall henceforth refer to the operators defined in (7.3) as the second-order Stancu-Bernstein operators. These operators not only extend the classical theory but also lay the foundation for studying even higher-order generalizations and their applications in approximation theory.

Theorem 7.2.6 *Let $f \in C[0,1]$ such that f'' exists. Then for $\alpha \to 0$ and $c = \lim_{n\to\infty} \frac{1+n\alpha}{2}$,*

$$\left|Q_n^{\langle\alpha\rangle}(f;x) - f(x) + \frac{cx(1-x)}{n(1+\alpha)} f''(x)\right| \le 2\omega\left(f;\, \sqrt{\mu_{n,2}^{\langle\alpha\rangle}(x)}\right) + \frac{2cx(1-x)}{n(1+\alpha)}\omega\left(f'';\, \sqrt{\mu_{n,2}^{\langle\alpha\rangle}(x)}\right),$$

where $\mu_{n,r}^{\langle\alpha\rangle}(x)$ denotes the r^{th} central moment of the first order Stancu-Bernstein operators (7.2).

Proof. Using the Cauchy-Schwarz's inequality and Proposition 1.1.4, we can write,

$$\begin{aligned}
\left|P_n^{\langle\alpha\rangle}(f;x) - f(x)\right| &= \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\left|f\left(\frac{k}{n}\right) - f(x)\right| \\
&\le \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\,\omega\left(f;\left|\frac{k}{n} - x\right|\right) \\
&\le \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\left(1 + \frac{|k/n - x|}{\sqrt{\mu_{n,2}^{\langle\alpha\rangle}(x)}}\right)\omega\left(f;\, \sqrt{\mu_{n,2}^{\langle\alpha\rangle}(x)}\right) \\
&= \omega\left(f;\, \sqrt{\mu_{n,2}^{\langle\alpha\rangle}(x)}\right)\left(1 + \frac{\sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\,|k/n - x|}{\sqrt{\mu_{n,2}^{\langle\alpha\rangle}(x)}}\right) \\
&\le 2\omega\left(f;\, \sqrt{\mu_{n,2}^{\langle\alpha\rangle}(x)}\right).
\end{aligned}$$

Similarly,

$$\left|P_n^{\langle\alpha\rangle}\left(f'';x\right)-f''(x)\right|\leq 2\omega\left(f'';\sqrt{\mu_{n,2}^{\langle\alpha\rangle}(x)}\right).$$

Thus,

$$\begin{aligned}&\left|Q_n^{\langle\alpha\rangle}(f;x)-f(x)+\frac{cx(1-x)}{n(1+\alpha)}f''(x)\right|\\&=\left|P_n^{\langle\alpha\rangle}(f;x)-\frac{cx(1-x)}{n(1+\alpha)}P_n^{\langle\alpha\rangle}\left(f'';x\right)-f(x)+\frac{cx(1-x)}{n(1+\alpha)}f''(x)\right|\\&\leq 2\omega\left(f;\sqrt{\mu_{n,2}^{\langle\alpha\rangle}(x)}\right)+\frac{cx(1-x)}{n(1+\alpha)}\omega\left(f'';\sqrt{\mu_{n,2}^{\langle\alpha\rangle}(x)}\right).\end{aligned}$$

Hence, the proof is established. □

Theorem 7.2.7 *Let $h:[0,1]\to\mathbb{R}$ be a continuous function such that h'' is bounded. Then for the supremum norm of h*

$$\left|Q_n^{\langle\alpha\rangle}(h;x)-h(x)\right|\leq\frac{\upsilon_{n,2}(x)}{2}\left\|h''\right\|,$$

where $\upsilon_{n,2}(x)=\lim_{n\to\infty}Q_n^{\langle\alpha\rangle}\left((k/n-x)^2;x\right)$.

Proof. By Taylor's expansion,

$$h(t)=h(x)+(t-x)h'(x)+\frac{(t-x)^2}{2}h''(x)+(t-x)^2\varepsilon(t),$$

such that $\lim_{t\to x}\varepsilon(t)=0$. By substituting $t=k/n$ into the expression and then applying the operators $Q_n^{\langle\alpha\rangle}$ to both sides, we obtain

$$Q_n^{\langle\alpha\rangle}(h(k/n);x)=h(x)+Q_n^{\langle\alpha\rangle}\left((k/n-x)^2;x\right)\frac{h''(x)}{2}+Q_n^{\langle\alpha\rangle}\left((k/n-x)^2\varepsilon(k/n);x\right).$$

Since, Lemma 7.2.4 guarantees the vanishing of such weighted central moments, we deduce that

$$Q_n^{\langle\alpha\rangle}\left(\left(\tfrac{k}{n}-x\right)^2\varepsilon\left(\tfrac{k}{n}\right);x\right)\to 0,\quad \text{as } n\to\infty.$$

Thus for a sufficiently large value of n we can say that,

$$\begin{aligned}\left|Q_n^{\langle\alpha\rangle}(h;x)-h(x)\right|&\leq\frac{Q_n^{\langle\alpha\rangle}\left((k/n-x)^2;x\right)}{2}\left|h''(x)\right|\\&\leq\frac{\upsilon_{n,2}(x)}{2}\left\|h''\right\|.\end{aligned}$$

□

Theorem 7.2.8 *For $f,g \in C([0,1])$, such that f'' and g'' are bounded on $\mathbb{R}$, we have:*

(i) $\left|Q_n^{\langle\alpha\rangle}(f;x) - Q_n^{\langle\alpha\rangle}(g;x)\right| \le C_1\omega_2\left(f;\sqrt{\delta}\right) + C_2\omega_2\left(g;\sqrt{\delta}\right)$

(ii) $\left|Q_n^{\langle\alpha\rangle}(f;x) - Q_n^{\langle\alpha\rangle}(g;x)\right| \le \frac{\upsilon_{n,2}(x)}{2}\|f''-g''\| + \|f-g\|,$

where C_1 and C_2 are constants, and $\delta = \upsilon_{n,2}(x)$.

Proof. For the first order Stancu-Bernstein operators (7.2), we can write,

$$\begin{aligned}\left|P_n^{\langle\alpha\rangle}(f;x)\right| &\le \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\left|f\left(\frac{k}{n}\right)\right| \\ &\le \|f\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|Q_n^{\langle\gamma\rangle}(f;x)\right| &\le \sum_{k=0}^{n} p_{n,k}^{\langle\alpha\rangle}(x)\left|f\left(\frac{k}{n}\right) - \frac{cx(1-x)}{n(1+\alpha)}f''\left(\frac{k}{n}\right)\right| \\ &\le \left|f\left(\frac{k}{n}\right)\right| + \frac{cx(1-x)}{n(1+\alpha)}\left|f''\left(\frac{k}{n}\right)\right| \\ &\le \|f\| + \frac{c}{4n(1+\alpha)}\left\|f''\right\|.\end{aligned}$$

Now consider,

$$\begin{aligned}\left|Q_n^{\langle\alpha\rangle}(f;x) - Q_n^{\langle\alpha\rangle}(g;x)\right| &\le \left|Q_n^{\langle\alpha\rangle}(f;x) - f(x)\right| + \left|Q_n^{\langle\alpha\rangle}(g;x) - g(x)\right| + |f(x)-g(x)| \\ &\le \frac{\upsilon_{n,2}(x)}{2}\left\|f''\right\| + \frac{\upsilon_{n,2}(x)}{2}\left\|g''\right\| + \|f-g\| \\ &= \frac{1}{2}\left[\upsilon_{n,2}(x)\left\|f''\right\| + \upsilon_{n,2}(x)\left\|g''\right\| + \|f-g\| + \|f-g\|\right].\end{aligned}$$

Taking infimum on the right hand side of the inequality and using Remark 1.1.7, we get

$$\begin{aligned}\left|Q_n^{\langle\alpha\rangle}(f;x) - Q_n^{\langle\alpha\rangle}(g;x)\right| &\le \frac{1}{2}K_2(f;\delta) + \frac{1}{2}K_2(g;\delta) \\ &\le C_1\omega_2\left(f;\sqrt{\delta}\right) + C_2\omega_2\left(g;\sqrt{\delta}\right),\end{aligned}$$

where $\delta = \upsilon_{n,2}(x)$. Alternatively, we can write

$$\begin{aligned}\left|Q_n^{\langle\alpha\rangle}(f;x) - Q_n^{\langle\alpha\rangle}(g;x)\right| &\le \left|Q_n^{\langle\alpha\rangle}(f-g;x) - (f-g)(x)\right| + |f(x)-g(x)| \\ &\le \frac{\upsilon_{n,2}(x)}{2}\left\|f''-g''\right\| + \|f-g\|.\end{aligned}$$

This completes the proof. □

To summarize, the previous section established that the proposed higher-order Stancu operators preserve the desirable features of classical Bernstein-type operators while achieving a stronger rate of convergence of order $O(1/n^2)$. This represents a significant theoretical improvement and highlights their potential for practical applications where accuracy and efficiency are crucial. Having developed the analytical framework, we now proceed to examine numerical evidence that supports and illustrates these findings.

7.3 Computational Study of the Operators

In this section, we provide a numerical investigation of the proposed operators, alongside with the first order Stancu-Bernstein operators. The study includes graphical illustrations and error tables that demonstrate the convergence behaviour. These visual and quantitative illustrations serve as a practical validation of the theory, offering a clearer picture of how effectively the operators approximate functions as n increases.

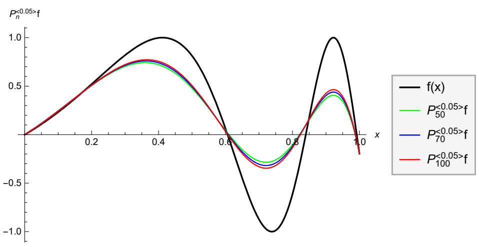


Figure 7.1: Approximation process for the First order operators $P_n^{\langle\alpha\rangle}(f;x)$ (7.2) for $\alpha = 0.05$, $n = 50, 70, 100$ and $f(x) = \sin\left(2xe^{\pi x/2}\right)$.

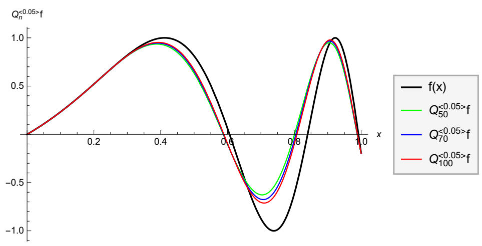


Figure 7.2: Approximation process for the Second order operators $Q_n^{\langle\alpha\rangle}(f;x)$ (7.3) for $\alpha = 0.05$, $n = 50, 70, 100$ and $f(x) = \sin\left(2xe^{\pi x/2}\right)$.

	$n=20$	$n=50$	$n=100$
$x=0.1$	0.0145421	0.0126529	0.011515
$x=0.2$	0.0314551	0.0145826	0.0101498
$x=0.3$	0.172764	0.116513	0.097632
$x=0.4$	0.363173	0.27585	0.241918
$x=0.5$	0.393535	0.333742	0.30561
$x=0.6$	0.0265156	0.016621	0.0301905
$x=0.7$	0.734555	0.601632	0.542777
$x=0.8$	0.56636	0.51103	0.480129
$x=0.9$	0.662236	0.526114	0.473075

Table 7.1: Error values corresponding to $P_n^{\langle\alpha\rangle}\left(\sin\left(2xe^{\pi x/2}\right);x\right)$ for $\alpha=0.05$ and different values of n.

	$n=20$	$n=50$	$n=100$
$x=0.1$	0.00146599	0.000742853	0.000594385
$x=0.2$	0.00173452	0.0000401649	0.0000702987
$x=0.3$	0.0277156	0.00805916	0.00452139
$x=0.4$	0.135266	0.0651192	0.0457725
$x=0.5$	0.275654	0.179682	0.142212
$x=0.6$	0.138471	0.166017	0.156329
$x=0.7$	0.501782	0.246977	0.168013
$x=0.8$	0.8202	0.607639	0.510759
$x=0.9$	0.094062	0.047708	0.0713646

Table 7.2: Error values corresponding to $Q_n^{\langle\alpha\rangle}\left(\sin\left(2xe^{\pi x/2}\right);x\right)$ for $\alpha=0.05$ and different values of n.

Figures 7.1 and 7.2 illustrate the approximation of the function $f(x)=\sin\left(2xe^{\pi x/2}\right)$, for a fixed value of α and for different values of n ($n=50,70,100$). Figure 7.1 corresponds to the first-order Stancu-Bernstein operators (7.2), while Figure 7.2 depicts the behaviour of their second-order counterparts (7.3).

As expected, both operators converge uniformly to the desired function f as n increases, confirming their approximation properties. However, a clear distinction can be observed between the two. For the same parameter α and identical values of n, the second-order Stancu-Bernstein operators achieve a noticeably faster convergence and produce graphs that are mostly indistinguishable from the original function even for moderate values of n. By contrast, the first-order operators still exhibit visible deviations from f at these values.

This observation is consistent with the theoretical analysis which proved that while the first order operators attain an approximation rate of at most $O(1/n)$, the second order construction improves this rate to $O(1/n^2)$.

To complement these graphical results, Tables 7.1 and 7.2 provide the numerical errors between the exact function values and those obtained from $P_n^{\langle\alpha\rangle}(f;x)$ and $Q_n^{\langle\alpha\rangle}(f;x)$, respectively, for selected points $x \in [0,1]$ with $\alpha = 0.05$. These tabulated values reinforce the graphical evidence, highlighting the superior accuracy and efficiency of the second-order Stancu-Bernstein operators over their first-order analogues.

By far, we have kept the parameter α fixed while varying n. To further investigate the influence of α, we now consider different values of this parameter.

	$n = 20$	$n = 50$	$n = 100$
$x = 0.1$	0.0124136	0.00628134	0.00387778
$x = 0.2$	0.0054333	0.000751393	0.00108517
$x = 0.3$	0.0879942	0.0349556	0.0196959
$x = 0.4$	0.229295	0.108654	0.0653643
$x = 0.5$	0.292611	0.163329	0.104453
$x = 0.6$	0.0206416	0.0418855	0.0350697
$x = 0.7$	0.531589	0.287539	0.181659
$x = 0.8$	0.423093	0.265802	0.180337
$x = 0.9$	0.52753	0.294337	0.19156

Table 7.3: Error values corresponding to $P_n^{\langle\alpha\rangle}\left(\sin\left(2xe^{\pi x/2}\right);x\right)$ for $\alpha = 0.005$ and different values of n.

	$n = 20$	$n = 50$	$n = 100$
$x = 0.1$	0.000424243	0.0000796755	0.0000229677
$x = 0.2$	0.00341629	0.000750881	0.000239992
$x = 0.3$	0.00241455	0.00167568	0.000665737
$x = 0.4$	0.0350036	0.0040388	0.000968976
$x = 0.5$	0.122688	0.0283786	0.0100398
$x = 0.6$	0.119582	0.0480118	0.0207142
$x = 0.7$	0.181522	0.0253843	0.00491898
$x = 0.8$	0.386119	0.12476	0.0524003
$x = 0.9$	0.0920456	0.00799706	0.000278563

Table 7.4: Error values corresponding to $Q_n^{\langle\alpha\rangle}\left(\sin\left(2xe^{\pi x/2}\right);x\right)$ for $\alpha = 0.005$ and different values of n.

It is observed that smaller values of α lead to a more refined approximation, thereby reducing the overall error significantly. In particular, Tables 7.3 and 7.4 present the same set of numerical comparisons as Tables 7.1 and 7.2, but this time with $\alpha = 0.005$. The reduction in error values is quite substantial, demonstrating that decreasing α enhances the efficiency of the approximation without altering the order of convergence.

A closer comparison of the results reveals that for identical choices of n and α, the error $\left|Q_n^{\langle\alpha\rangle}(f;x) - f(x)\right|$ approaches zero much more rapidly than $\left|P_n^{\langle\alpha\rangle}(f;x) - f(x)\right|$, which confirms the superior performance of the second-order operators over their first-order counterparts. This dual dependency on both n and α provides additional flexibility, showing that not only does increasing n improve accuracy, but also tuning α appropriately can further accelerate convergence and reduce computational error.

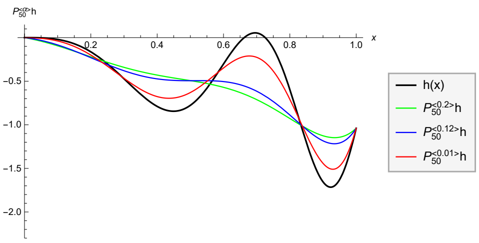


Figure 7.3: Approximation process for the First order operators $P_n^{\langle\alpha\rangle}(f;x)$ (7.2) for $\alpha = 0.2, 0.12, 0.01$, $n = 50$ and $h(x) = x\cos(2\pi e^x) - \sin x$.

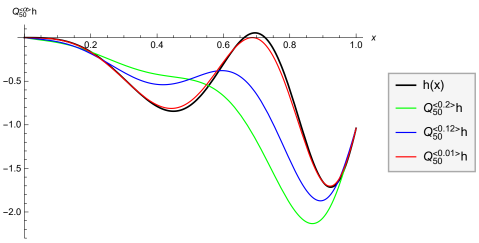


Figure 7.4: Approximation process for the Second order operators $Q_n^{\langle\alpha\rangle}(f;x)$ (7.3) for for $\alpha = 0.2, 0.12, 0.01$, $n = 50$ and $h(x) = x\cos(2\pi e^x) - \sin x$.

From Figures 7.3 and 7.4, together with the numerical evidence provided in Tables 7.1, 7.2, 7.3, and 7.4, it becomes clear that the choice of the parameter α plays a crucial role in determining the accuracy of approximation. In particular, smaller values of α consistently yield superior results, as reflected in the significant reduction of error values. Among the two families of operators, the second-order operators $Q_n^{\langle\alpha\rangle}$ demonstrate markedly better performance by converging more rapidly to the desired function. This reinforces the fact that higher-order constructions, when coupled with an appropriate parameterization, provide a more powerful and flexible framework for approximation than their first-order counterparts.

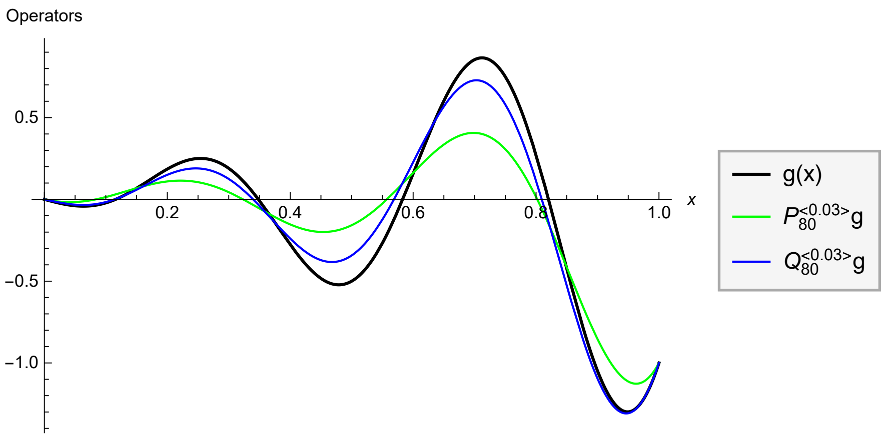


Figure 7.5: Comparison between the first and second order Stancu operators, $P_n^{\langle\alpha\rangle}(g;x)$ and $Q_n^{\langle\alpha\rangle}(g;x)$ respectively for $\alpha = 0.03$, $n = 80$ and $g(x) = x^2\sin(4\pi x) - x\cos(4\pi x)$.

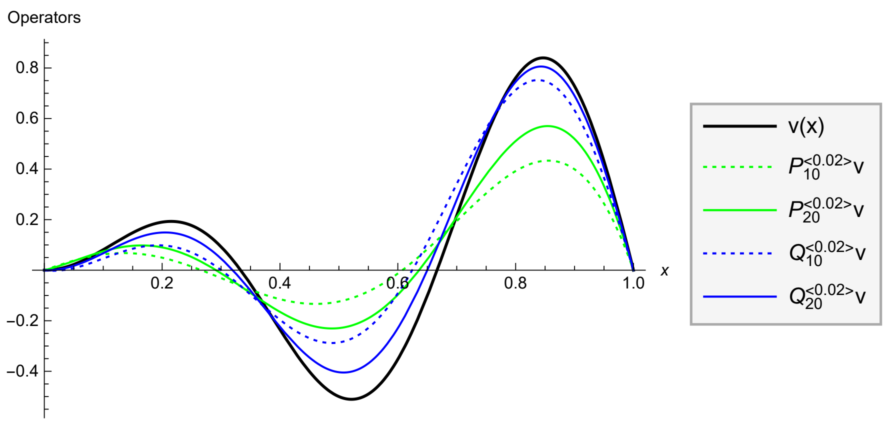


Figure 7.6: The approximation process for first order and second order Stancu operators, $P_n^{\langle\alpha\rangle}(v;x)$ and $Q_n^{\langle\alpha\rangle}(v;x)$, respectively for $v(x) = x\sin(3\pi x)$.

Figures 7.5 and 7.6 provide a comparative illustration of the approximation behaviour of the first-order operators $P_n^{\langle\alpha\rangle}$ and the second-order operators $Q_n^{\langle\alpha\rangle}$. To complement the graphical evidence, Table 7.5 reports the corresponding numerical error values computed at a specific point $x = 0.5$. This combined graphical-numerical analysis highlights the distinct advantage of the second-order operators in terms of accuracy and rate of convergence.

In addition, Figure 7.6 illustrates the simultaneous approximation process of both operators as n increases from 10 to 20. The plot clearly shows that the convergence achieved by the first-order operators $P_n^{\langle\alpha\rangle}$ (depicted in green), over this range of n is slower compared to that of the second-order operators $Q_n^{\langle\alpha\rangle}$ (depicted in blue).

	$\left\|P_n^{\langle\alpha\rangle}(v;x) - v(x)\right\|$	$\left\|Q_n^{\langle\alpha\rangle}(v;x) - v(x)\right\|$
$n = 10$	0.351321	0.178846
$n = 20$	0.229725	0.0654934
$n = 30$	0.173487	0.0352061
$n = 40$	0.141525	0.022711
$n = 50$	0.120967	0.0162774
$n = 60$	0.106647	0.0124888
$n = 70$	0.0961051	0.0100473
$n = 80$	0.0880213	0.00836885

Table 7.5: The error values corresponding to the first and second order operators, $P_n^{\langle\alpha\rangle}(x\sin(3\pi x);x)$ and $Q_n^{\langle\alpha\rangle}(x\sin(3\pi x);x)$, for $\alpha = 0.005$, at $x = 0.5$.

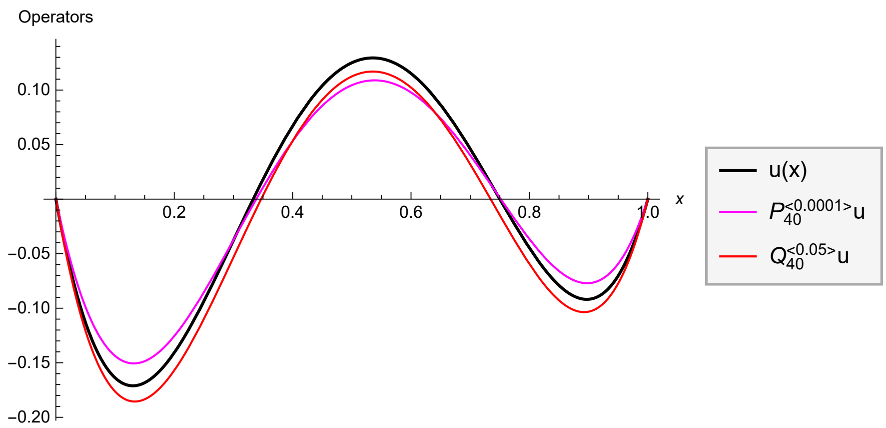


Figure 7.7: Comparison between the values of α in $P_n^{\langle\alpha_1\rangle}(u;x)$ and $Q_n^{\langle\alpha_2\rangle}(u;x)$ for $u(x) = 12x^4 - 25x^3 + 16x^2 - 3x$.

It is worth observing from Figure 7.7 that the approximations obtained from $P_{40}^{\langle\alpha_1\rangle}(u;x)$ and $Q_{40}^{\langle\alpha_2\rangle}(u;x)$ produce almost identical error values. However, in order for the first-order operators $P_n^{\langle\alpha_1\rangle}(u;x)$ to achieve convergence comparable to that of the second-order operator $Q_n^{\langle\alpha_2\rangle}(u;x)$, one must take $\alpha_1 = \frac{\alpha_2}{500}$. This striking disparity demonstrates that the second-order Stancu-Bernstein operators maintain a superior rate of convergence even when the parameter α is relatively large. In other words, while both operators benefit from the presence of the parameter α, the second-order operators achieves a far more efficient approximation without requiring α to be excessively small, thereby highlighting its robustness and practical advantage over the first-order counterpart.

7.4 Conclusion

In this chapter, we investigated the construction and approximation properties of higher order Stancu-Bernstein operators. The theoretical analysis demonstrated that these operators achieve an approximation rate of order $O(1/n^2)$, which is a significant improvement over the classical first order Stancu operators that only attain an order of $O(1/n)$. This enhancement underscores the advantage of introducing additional parameters in the operator design, as it allows for more refined control over convergence without compromising the fundamental stability and positivity inherent in the Bernstein-type framework.

The role of the parameter α was carefully examined. The theoretical results also established that the improved convergence order remains unaffected by the choice of

α, while the numerical investigations highlighted how smaller values of α lead to a visibly faster reduction in the approximation error.

The numerical experiments, presented through a series of figures and error tables, confirmed the theoretical predictions. Graphical illustrations showed the uniform convergence of both the first and second order operators, while quantitative error analysis clearly indicated the superiority of the second order operators. Notably, the comparison between the two classes of operators revealed that the second order version consistently outperformed the first, even when larger values of α were considered. This confirms the robustness of the proposed generalization.

Chapter 8

Modifications of Operators to construct Higher Order Operators with sequences of real numbers

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It is known that the contagion distribution models self-reinforcing processes where the probability of selecting an item increases as that item is selected more frequently. Based on this distribution D. D. Stancu derived positive linear operators with a real parameter $\alpha \in [0,1]$, which approximate a continuous real-valued function on the interval $[0,1]$. However, these operators converge with an order of convergence $1/m$ which is not very desirable in situations where we necessarily prefer faster and better convergence. This chapter aims to improve this order of approximation by defining a modification of the first order Stancu-Bernstein operators. However, in contrast to the previous chapter, here we do not rely on the derivatives of the function being approximated, instead, we construct the operators using sequences of real numbers. First, we introduce a first order modification of the Stancu-Bernstein operators using sequences of real numbers, and derive their moments. Using Korovkin theorem we then prove uniform convergence and with the help of Peetre's K-functional and modulus of continuity, we show various convergence properties of these operators as well. Extending this concept of sequences, we have also defined a sequence of second order positive linear operators based on contagion distribution. The uniform convergence of these operators using Korovkin theorem is analyzed and a necessary condition for their convergence is established. Based on this condition we have developed particular second

order operators and derived their moments and central moments. Moreover, the results established have been verified graphically as well.

8.1 Introduction

In the preceding chapters of this thesis, we have explored both the binomial distribution and the contagion distribution, highlighting their properties, differences and applicability to various real-life scenarios. In particular, we discussed how these distributions serve as a foundation for constructing positive linear operators for function approximation. Moreover, the Bernstein polynomials are proved to be non-negative and symmetric in the compact interval $[0,1]$, making them well-suited for approximation tasks. Building on these properties, S.N. Bernstein [29] established that any continuous and bounded function defined on $[0,1]$ can be uniformly approximated by a sequence of Bernstein operators, as defined in equation (1.1), thereby providing a constructive proof of the Weierstrass approximation theorem.

Since the introduction of Bernstein operators, numerous researchers have developed a variety of related operators and modifications aimed at approximating continuous real-valued functions on closed and bounded intervals [4; 42; 51; 120; 122; 219]. Despite their success, a significant limitation of these operators is that they are restricted to approximating only continuous functions. To overcome this limitation, L. Kantorovich introduced the Kantorovich modification of Bernstein operators, defined by equation (1.4), which extends the approximation framework to integrable functions. Unlike the classical discrete operators, these modified operators work effectively on a broader class of functions, thereby increasing their practical applicability. For further developments and studies on positive linear operators and their various modifications, see [91; 107; 112; 124; 125; 151; 168] and the references therein.

Since the Bernstein operators are based on binomial distribution, they cannot be used to model most of the real-life scenarios, mainly situations where the probability of occurrence of an outcome increases as more of that outcome occurs. For instance in opinion dynamics where a population holds different opinions, say opinion A and B, individuals might interact with each other and these interactions might influence their opinions. Such type of distributions are modeled very well by the contagion distribution, based on the Pólya-Eggenberger urn scheme [74]. The Pólya-Eggenberger urn scheme is a generalization of the classical urn model used in probability theory, where the composition of the urn changes with each draw.

Looking at the positive side of these operators and their ability to capture real-life situations, a lot of research has been done on this distribution, starting with D.D. Stancu with the introduction of Stancu operators [194; 195] as defined by,

$$P_m^{\langle\alpha\rangle}(f;x) = \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle}(x) f\left(\frac{s}{m}\right), \quad \forall x \in [0,1], \tag{8.1}$$

where

$$p_{m,s}^{\langle\alpha\rangle}(x) = \frac{x^{[s,-\alpha]}(1-x)^{[m-s,-\alpha]}}{1^{[m,-\alpha]}}, \tag{8.2}$$

such that α is a non-negative parameter and the modified Pochhammer symbol is determined by $\kappa^{[m,\nu]} = \kappa(\kappa-\nu)(\kappa-2\nu)\ldots(\kappa-(m-1)\nu)$ and $\kappa^{[0,\nu]} = 1$.

In recent times, many researchers have worked on the developments and modifications of the contagion distribution. In [139], Lipi and Deo proposed a Pólya distribution-based generalization of the λ-Bernstein operators and studied the convergence as well as order of approximation of these operators. Moreover, Kajla et al. [121] extended the Pólya and inverse Pólya distribution to Baskakov and Szász basis functions and studied the rate of convergence for functions with derivatives of bounded variation. For more work, refer [11; 64; 87; 154] and reference within.

However, another major drawback of these operators is their slow order of convergence. The Bernstein operators (7.1) and the operators involving contagion distribution (8.1), both approximate a real-valued continuous function, but with an order of approximation $1/m$. Situations where a faster or more effective approximation is desired may find it difficult to work with these classical operators. Some techniques have already been developed in order to improve this slow order of approximation. For instance, with the introduction of q-Bernstein polynomials by G.M. Phillips in 1996 [169], it was proved that q-positive linear operators have an improved order of convergence of $1/q^m$ for $q \geq 1$, which is still better than the best order of convergence of $1/m$ of the classical Bernstein polynomials of degree m. In 1953, P.L. Butzer [41] claimed that the existence of higher order derivatives can not improve the order of approximation. He introduced certain linear combinations of Bernstein polynomials involving not only the Bernstein polynomials of degree m but also degree of $2m$, which under definite conditions, approximate $f(x)$ more closely than the Bernstein polynomials. But this approach included polynomials of higher degree as well. To avoid this, Z Guan [96] proposed the iterated Bernstein polynomials of degree m for continuous function using values of the function at $s/m,\ s=0,1,\ldots,m$. He improved the rate of convergence of the classical Bernstein polynomials significantly without increasing the degree of the polynomials. He proposed a simple procedure to generalize and improve the classical Bernstein polynomial approximation by repeatedly approximating the errors using the Bernstein polynomial approximations. For more studies concerning the order of approximation of positive linear operators, readers can refer [128; 129; 176].

This chapter aims to enhancing the order of approximation of the first order operators based on the contagion distribution, as defined in (8.1). To achieve this improvement, we first introduce a modified form of these operators that incorporates sequences of real numbers, allowing us to systematically construct higher-order operators.

8.2 First Order Modification

We now present a modified version of the operators defined in (8.1), which incorporates sequences of real numbers to enhance the order of approximation. This modification serves as a foundation for constructing higher-order operators and provides greater flexibility in capturing the behaviour of continuous functions while preserving the positive linear structure of the original operators. We define,

$$P_{m,1}^{\langle\alpha\rangle}(f;x)=\sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(1)}(x) f\left(\frac{s}{m}\right), \tag{8.3}$$

where

$$p_{m,s}^{\langle\alpha\rangle(1)}(x)=u(m,x)p_{m-1,s}^{\langle\alpha\rangle}(x)+u(m,1-x)p_{m-1,s-1}^{\langle\alpha\rangle}(x), \tag{8.4}$$

such that the distribution $p_{m,s}^{\langle\alpha\rangle}(x)$ is defined by equation (8.2), and

$$u(m,x)=u_1(m)x+u_0(m),$$

where $u_1(m)$ and $u_0(m)$ are sequences of real numbers.

Using the moments of the classical Stanu-Bernstein operators (8.1), from Chapter 2 Lemma 2.1.2 and the definition of $u(m,x)$, we can say that

$$\begin{aligned}
P_{m,1}^{\langle\alpha\rangle}(1;x) &= \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(1)}(x)\\
&= u(m,x)\sum_{s=0}^{m} p_{m-1,s}^{\langle\alpha\rangle}(x) + u(m,1-x)\sum_{s=0}^{m} p_{m-1,s}^{\langle\alpha\rangle}(x)\\
&= u(m,x)+u(m,1-x)\\
&= u_1(m)+2u_0(m). \qquad (8.5)
\end{aligned}$$

For $P_{m,1}^{\langle\alpha\rangle}(1;x)$ to form a sequence of positive linear operators that converges uniformly to a given function f, it is necessary that

$$u_1(m)+2u_0(m)\to 1, \quad \text{as } m\to\infty.$$

This condition, together with the convergence of the second and third moments of $P_{m,1}^{\langle\alpha\rangle}(1;x)$, provides the necessary and sufficient criteria for establishing the convergence of these operators. However, our goal is to specifically study the convergence of sequences for which

$$\lim_{m\to\infty} n\left[P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right]=C(f;x),$$

where $C(f;x)$ depends solely on the derivatives of f. For this purpose, it is not enough for the first moment to merely converge, instead, the operators must preserve it exactly. Consequently, we impose the stronger requirement, that

$$u_1(m)+2u_0(m)=1. \qquad (8.6)$$

8.2.1 Approximation Results

Consider now the modification of the classical Stancu-Bernstein operators defined in equation (8.3), along with the condition (8.6). We derive their moments and central moments, which serve as the foundation for studying their approximation behaviour and convergence properties.

8.2.1.a Case Analysis for $u_1(m)$ and $u_0(m)$

From the condition $u_1(m)+2u_0(m)=1$, we obtain $u_0(m)=\frac{1-u_1(m)}{2}$. Under this constraint, we have the following seven possibilities for the sequences $u_1(m)$ and $u_0(m)$.

Case 1: If $u_1(m)<-1$, then $u_0(m)>0$ and $u_1(m)+u_0(m)<0$.

Case 2: If $u_1(m) = -1$, then $u_0(m) = 1$ and $u(m,x) = 1-x, u(m,1-x) = x$.

Case 3: If $-1 < u_1(m) < 0$, then $u_0(m) > 0$ and $u_1(m) + u_0(m) > 0$.

Case 4: If $u_1(m) = 0$, then $u_0(m) = \frac{1}{2}$ and $u(m,x) = u(m,1-x) = \frac{1}{2}$.

Case 5: If $0 < u_1(m) < 1$, then $u_0(m) > 0$.

Case 6: If $u_1(m) = 1$, then $u_0(m) = 0$ and $u(m,x) = x, u(m,1-x) = 1-x$.

Case 7: If $u_1(m) > 1$, then $u_0(m) < 0$ and $u_1(m) + u_0(m) > 0$.

Since $p_{m,s}^{\langle\alpha\rangle}(x) \geq 0$, the modified basis function

$$p_{m,s}^{\langle\alpha\rangle(1)}(x) = u(m,x)\, p_{m-1,s}^{\langle\alpha\rangle}(x) + u(m,1-x)\, p_{m-1,s-1}^{\langle\alpha\rangle}(x)$$

is non-negative whenever $u(m,x) \geq 0$ and $u(m,1-x) \geq 0$ for all $x \in [0,1]$. This condition is satisfied if and only if

$$-1 \leq u_1(m) \leq 1,$$

which corresponds precisely to Cases 2 – 6. Hence, for the sequence of positive linear operators $P_{m,1}^{\langle\alpha\rangle}(f;x)$ to be well defined and to converge uniformly to f, the admissible values of the sequences $u_1(m)$ and $u_0(m)$ are exactly those described in Cases 2 – 6.

Lemma 8.2.1 *The moments of the sequence of operators* (8.3) *are given by,*

(i) $P_{m,1}^{\langle\alpha\rangle}(1;x) = 1$

(ii) $P_{m,1}^{\langle\alpha\rangle}(t;x) = x - 2x\left(\frac{u_1(m)+u_0(m)}{m}\right) + \frac{u_1(m)+u_0(m)}{m}$

(iii) $$P_{m,1}^{\langle\alpha\rangle}\left(t^2;x\right) = \left\{\begin{array}{l} \frac{x^2}{1+\alpha}\left(1 - \frac{3+2u_1(m)+2\alpha u_1(m)}{m} + \frac{2+2u_1(m)+2\alpha u_1(m)}{m^2}\right) \\ +\frac{x}{1+\alpha}\left(\alpha(1+\alpha) + \frac{2-2\alpha-2\alpha^2+2\alpha u_0(m)+2\alpha u_1(m)+u_1(m)}{m}\right. \\ \left. -\frac{2-\alpha-\alpha^2+2\alpha u_0(m)+3\alpha u_1(m)+2u_1(m)}{m^2}\right) + \frac{u_1(m)+u_0(m)}{m^2}, \end{array}\right.$$

where $u_1(m) + 2u_0(m) = 1$ *and* $\alpha = O(1/n) \in [0,1]$.

Proof. The first moment of $P_{m,1}^{\langle\alpha\rangle}(1;x)$ can be obtained from equations (8.5) and (8.6) as,

$$P_{m,1}^{\langle\alpha\rangle}(1;x) = u_1(m) + 2u_0(m) = 1.$$

For the second moment, consider

$$\begin{aligned}\sum_{s=0}^{m} p_{m-1,s}^{\langle\alpha\rangle}(x)\frac{s}{m} &= \frac{m-1}{m}\sum_{s=0}^{m-1} p_{m-1,s}^{\langle\alpha\rangle}(x)\frac{s}{m-1}\\ &= x\left(\frac{m-1}{m}\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{s=0}^{m} p_{m-1,s-1}^{\langle\alpha\rangle}(x)\frac{s}{m} &= \sum_{s=1}^{m} p_{m-1,s-1}^{\langle\alpha\rangle}(x)\frac{s}{m}\\ &= \frac{m-1}{m}\sum_{s=0}^{m-1} p_{m-1,s}^{\langle\alpha\rangle}(x)\frac{s+1}{m-1}\\ &= \frac{m-1}{m}\left(x+\frac{1}{m-1}\right)\\ &= x\left(\frac{m-1}{m}\right)+\frac{1}{m}.\end{aligned}$$

Thus,

$$\begin{aligned}P_{m,1}^{\langle\alpha\rangle}(t;x) &= \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(1)}(x)\frac{s}{m}\\ &= u(m,x)x\left(\frac{m-1}{m}\right)+u(m,1-x)\left[x\left(\frac{m-1}{m}\right)+\frac{1}{m}\right]\\ &= x\left(u_1(m)+2u_0(m)\right)+\frac{(1-2x)\left(u_1(m)+u_0(m)\right)}{m}\\ &= x\left(u_1(m)+2u_0(m)-\frac{2u_1(m)+2u_0(m)}{m}\right)+\frac{u_1(m)+u_0(m)}{m}.\end{aligned}$$

Similarly, from the moments of the classical Stanu-Bernstein operators, we can write

$$\begin{aligned}\sum_{s=0}^{m} p_{m-1,s}^{\langle\alpha\rangle}(x)\frac{s^2}{m^2} &= \frac{(m-1)^2}{m^2}\sum_{s=0}^{m-1} p_{m-1,s}^{\langle\alpha\rangle}(x)\frac{s^2}{(m-1)^2}\\ &= \frac{(m-1)^2}{m^2}\left[x^2\left(\frac{m-2}{(m-1)(1+\alpha)}\right)+x\left(\frac{(m-1)\alpha+1}{(m-1)(1+\alpha)}\right)\right]\\ &= x^2\left(\frac{(m-1)(m-2)}{m^2(1+\alpha)}\right)+x\left(\frac{(m-1)^2\alpha+(m-1)}{m^2(1+\alpha)}\right),\end{aligned}$$

and

$$\begin{aligned}
\sum_{s=0}^{m} p_{m-1,s-1}^{\langle\alpha\rangle}(x)\frac{s^2}{m^2} &= \sum_{s=1}^{m} p_{m-1,s-1}^{\langle\alpha\rangle}(x)\frac{s^2}{m^2} \\
&= \frac{(m-1)^2}{m^2}\sum_{s=0}^{m-1} p_{m-1,s}^{\langle\alpha\rangle}(x)\frac{s^2+2s+1}{(m-1)^2} \\
&= x^2\left(\frac{(m-1)(m-2)}{m^2(1+\alpha)}\right)+x\left(\frac{(m-1)^2\alpha+(m-1)}{m^2(1+\alpha)}\right)+2x\left(\frac{m-1}{m^2}\right)+\frac{1}{m^2} \\
&= x^2\left(\frac{(m-1)(m-2)}{m^2(1+\alpha)}\right)+x\left(\frac{(m-1)^2\alpha+(m-1)+2(m-1)(1+\alpha)}{m^2(1+\alpha)}\right)+\frac{1}{m^2} \\
&= x^2\left(\frac{(m-1)(m-2)}{m^2(1+\alpha)}\right)+x\left(\frac{(m-1)^2\alpha+(m-1)(3+2\alpha)}{m^2(1+\alpha)}\right)+\frac{1}{m^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&P_{m,1}^{\langle\alpha\rangle}\left(t^2;x\right) \\
&\quad= u(m,x)\left[x^2\left(\frac{(m-1)(m-2)}{m^2(1+\alpha)}\right)+x\left(\frac{(m-1)^2\alpha+(m-1)}{m^2(1+\alpha)}\right)\right] \\
&\qquad+u(m,1-x)\left[x^2\left(\frac{(m-1)(m-2)}{m^2(1+\alpha)}\right)+x\left(\frac{(m-1)^2\alpha+(m-1)(3+2\alpha)}{m^2(1+\alpha)}\right)+\frac{1}{m^2}\right] \\
&\quad= x^2\left(\frac{(m-1)(m-2)}{m^2(1+\alpha)}\right)(u(m,x)+u(m,1-x))+\frac{x}{m^2(1+\alpha)}\Big((m-1)^2\alpha u(m,x) \\
&\qquad+(m-1)u(m,x)+(m-1)^2\alpha u(m,1-x)+(m-1)(3+2\alpha)u(m,1-x)\Big)+\frac{u(m,1-x)}{m^2} \\
&\quad= x^2\left(\frac{(m-1)(m-2)}{m^2(1+\alpha)}\right)(u_1(m)+2u_0(m))+\frac{x}{m^2(1+\alpha)}\Big((m-1)^2\alpha\left(u_1(m)+2u_0(m)\right) \\
&\qquad+(m-1)\left(-2u_1(m)x-2\alpha u_1(m)x+u_0(m)+3u_1(m)+3u_0(m)+2\alpha u_1(m)+2\alpha u_0(m)\right)\Big) \\
&\qquad+\frac{u_1(m)(1-x)+u_0(m)}{m^2} \\
&\quad= x^2\left(\frac{(m-1)(m-2)}{m^2(1+\alpha)}\right)(u_1(m)+2u_0(m))+\frac{x}{m^2(1+\alpha)}\Big((m-1)^2\alpha\left(u_1(m)+2u_0(m)\right) \\
&\qquad+(m-1)\left(u_1(m)x+u_0(m)+3u_1(m)(1-x)+3u_0(m)+2\alpha u_1(m)(1-x)+2\alpha u_0(m)\right)\Big) \\
&\qquad+\frac{u_1(m)+u_0(m)-xu_1(m)}{m^2} \\
&\quad= x^2\left(\frac{(m-1)(m-2)}{m^2(1+\alpha)}\right)(u_1(m)+2u_0(m))+x\left(\frac{m-1}{m^2(1+\alpha)}\right)\Big(\alpha(m-1)\left(u_1(m)+2u_0(m)\right) \\
&\qquad-2u_1(m)x-2\alpha u_1(m)x+3u_1(m)+4u_0(m)+2\alpha u_1(m)+2\alpha u_0(m)\Big) \\
&\qquad+\frac{u_1(m)+u_0(m)-xu_1(m)}{m^2}.
\end{aligned}$$

□

For sequences $u_0(m)$ and $u_1(m)$ satisfying the condition $u_1(m)+2u_0(m)=1$, the modified operators defined in (8.3) form a positive linear operator sequence. By applying the Bohman-Korovkin theorem[132], it follows that these operators converge uniformly to a continuous real-valued function f on the interval $[0,1]$. This convergence result ensures that, despite the modification introduced via the sequences $u_0(m)$ and $u_1(m)$, the operators retain the essential approximation properties of the classical Stancu-Bernstein operators. Furthermore, the condition on the sequences guarantees that the first moment of these operators is preserved exactly

Lemma 8.2.2 *For the proposed modification $P_{m,1}^{\langle\alpha\rangle}(f;x)$, we have the following expressions for central moments:*

(i) $P_{m,1}^{\langle\alpha\rangle}((t-x);x)=\frac{(1-2x)(u_1(m)+u_0(m))}{m}$

(ii) $\lim\limits_{m\to\infty} nP_{m,1}^{\langle\alpha\rangle}\left((t-x)^2;x\right)=\frac{x^2(2\alpha u_1(m)-c-1)+x(1-\alpha-\alpha^2)}{1+\alpha}$,

where $\alpha\to 0$ *and* $c=\lim\limits_{m\to\infty} m\alpha$.

With the moments and central moments of the modified operators established, we are now prepared to study their asymptotic behaviour. These results form the basis for deriving the Voronovskaya-type theorem, which characterizes the rate at which the operators converge to the desired function. The following theorem formalizes this asymptotic approximation and provides a precise description of the order of convergence of the operators.

Theorem 8.2.3 *Let f be a continuous function defined on $[0,1]$. Then for the positive linear operators* (8.3) *such that $u_1(m)+2u_0(m)=1$ and $\alpha\to 0$ as $m\to\infty$,*

(i) If $u_1(m)+u_0(m)\neq 0$ and f' exists at $x\in[0,1]$, then

$$\lim_{m\to\infty} n\left[P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right]=\lim_{m\to\infty}(1-2x)(u_1(m)+u_0(m))f'(x).$$

(ii) If $u_1(m)+u_0(m)=0$ and f'' exists at $x\in[0,1]$, then

$$\lim_{m\to\infty} n\left[P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right]=\left(\frac{c(1-x)+(1-2x)-x(3+2\alpha)}{2(1+\alpha)}\right)xf''(x),$$

where $c=\lim\limits_{m\to\infty} m\alpha$.

Proof. For the case when $u_1(m)+u_0(m) \neq 0$ and f' exists, by Taylor's expansion we can write

$$f\left(\frac{s}{m}\right) = f(x) + \left(\frac{s}{m} - x\right) f'(x) + \left(\frac{s}{m} - x\right) \varepsilon\left(s/m\right).$$

where $\lim\limits_{s/m \to x} \varepsilon\left(s/m\right) = 0$. Multiplying both sides by $p_{m,s}^{\langle\alpha\rangle(1)}(x)$, taking summation over s from 0 to m, and applying Lemma 8.2.2, we arrive at

$$P_{m,1}^{\langle\alpha\rangle}\left(f;x\right) = f(x) + \frac{(1-2x)(u_1(m)+u_0(m))}{m} f'(x) + P_{m,1}^{\langle\alpha\rangle}\left((t-x)\varepsilon\left(t\right);x\right).$$

Define,

$$\mathrm{T}_m = \left\{s : \left|\frac{s}{m} - x\right| < \varepsilon, s = 0,1,2\ldots,m\right\},$$
$$\text{and } \Gamma_m = \left\{s : \left|\frac{s}{m} - x\right| \geq \varepsilon, s = 0,1,2\ldots,m\right\}.$$

Taking into account the limiting behaviour of $\varepsilon\left(s/m\right)$, we observe that for any given $\delta > 0$, whenever $\left|\frac{s}{m} - x\right| < \delta$, there exists a positive number ε, such that $\varepsilon(s/m) < \varepsilon$. On the other hand, for values satisfying $\left|\frac{s}{m} - x\right| \geq \delta$, we define

$$M = \sup_{0 \leq x \leq 1} \left[\left(\frac{s}{m} - x\right) \varepsilon\left(\frac{s}{m}\right)\right].$$

Thus from Lemma 8.2.2, we can write

$$\begin{aligned}
&\lim_{m\to\infty} mP_m^{\langle\alpha\rangle(1)}\left((t-x)\varepsilon\left(t\right);x\right) \\
&\quad = \lim_{m\to\infty} \sum_{s=0}^{m} mp_{m,s}^{\langle\alpha\rangle(1)}(x)\left(\frac{s}{m} - x\right)\varepsilon\left(s/m\right) \\
&\quad = \lim_{m\to\infty} \sum_{s\in \mathrm{T}_m} mp_{m,s}^{\langle\alpha\rangle(1)}(x)\left(\frac{s}{m} - x\right)\varepsilon\left(s/m\right) + \sum_{s\in\Gamma_m} mp_{m,s}^{\langle\alpha\rangle(1)}(x)\left(\frac{s}{m} - x\right)\varepsilon\left(s/m\right) \\
&\quad \leq \lim_{m\to\infty} \left(\varepsilon + \frac{M}{\delta}\right) m\mu_{m,2}(x) \\
&\quad = \lim_{m\to\infty} \left(\varepsilon + \frac{M}{\delta}\right) (1-2x)(u_1(m)+u_0(m)) = 0.
\end{aligned}$$

This implies that,

$$\lim_{m\to\infty} n\left[P_{m,1}^{\langle\alpha\rangle}\left(f;x\right) - f(x)\right] = (1-2x)(u_1(m)+2u_0(m))f'(x).$$

Now, applying Taylor's theorem again for the case when $u_1(m)+u_0(m) = 0$ and f'' exists, we can write

$$\lim_{m\to\infty} n\left[P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right]=\lim_{m\to\infty} n\left(\frac{mx^2}{1+\alpha}-\frac{3x^2}{1+\alpha}+\frac{mx\alpha}{1+\alpha}-\frac{2x}{1+\alpha}-2u_1(m)x^2\right.$$
$$\left.+\frac{x}{1+\alpha}(u_1(m)+2)-mx^2\right)\frac{f''(x)}{2}$$
$$=\lim_{m\to\infty} n\left(\frac{mx^2}{1+\alpha}-mx^2-\frac{3x^2}{1+\alpha}-2u_1(m)x^2+\frac{mx\alpha}{1+\alpha}+\frac{u_1(m)x}{1+\alpha}\right)\frac{f''(x)}{2}$$
$$=\left(-cx^2-3x^2-2u_1(m)x^2(1+\alpha)+cx+u_1(m)x\right)\frac{f''(x)}{2(1+\alpha)},$$

where $c=\lim_{m\to\infty} m\alpha$. This proves the required result. □

Theorem 8.2.3 establishes that the modified Stancu-Bernstein operators $P_{m,1}^{\langle\alpha\rangle}(f;x)$, as defined in equation (8.3), provide uniform approximation to any continuous real-valued function on the interval $[0,1]$ with an order of convergence of $O(1/m)$. In this sense, these operators retain the same rate of approximation as the classical Stancu-Bernstein operators based on the Pólya-Eggenberger distribution, defined in equation (8.1).

Next, we focus on deriving some quantitative bounds in terms of the norm of f, for the modified operators $P_{m,1}^{\langle\alpha\rangle}(f;x)$.

Lemma 8.2.4 *For the supremum norm of h,* $\|h\|=\sup_{0\le x\le 1}|h(x)|$, *we have the following two inequalities*

(i) $\left|P_{m,1}^{\langle\alpha\rangle}(h;x)\right|\le\|h''\|$

(ii) $\left|P_{m,1}^{\langle\alpha\rangle}(h;x)-h(x)\right|\le\frac{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}{2}\|h''\|.$

Proof. The first inequality follows trivially from the definition of $P_{m,1}^{\langle\alpha\rangle}(h;x)$ and Lemma 8.2.1. Using Taylor's expansion for the second inequality,

$$h(t)=h(x)+(t-x)h'(x)+\frac{(t-x)^2}{2}h''(x)+(t-x)^2\varepsilon(t),$$

such that $\lim_{t\to x}\varepsilon(t)=0$. Taking $t=s/m$ and applying operators $P_{m,1}^{\langle\alpha\rangle}$ on both sides,

$$P_{m,1}^{\langle\alpha\rangle}(h(s/m);x)=h(x)+\frac{P_{m,1}^{\langle\alpha\rangle}\left((s/m-x)^2;x\right)}{2}h''(x)+P_{m,1}^{\langle\alpha\rangle}\left((s/m-x)^2\varepsilon(s/m);x\right).$$

From Lemma 8.2.2, we can see that

$$P_{m,1}^{\langle\alpha\rangle}\left((s/m-x)^2\varepsilon(s/m);x\right)\to 0 \text{ as } m\to\infty.$$

Thus, for a sufficiently large value of m, we can say that

$$\begin{aligned}\left|P_{m,1}^{\langle\alpha\rangle}(h;x)-h(x)\right| &\le \frac{P_{m,1}^{\langle\alpha\rangle}\left((s/m-x)^2;x\right)}{2}\left|h''(x)\right|\\ &\le \frac{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}{2}\left\|h''\right\|.\end{aligned}$$

□

Theorem 8.2.5 *Define* $\mu_{m,r}^{\langle\alpha\rangle(1)}(x)=P_{m,1}^{\langle\alpha\rangle}((t-x)^r;x)$. *Then for the first and second order modulus of continuity,* $\omega(f;\delta)$ *and* $\omega_2(f;\delta)$, *of* f, *we have*

(i) $\left|P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right|\le 2\omega\left(f;\sqrt{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}\right)$

(ii) $\left|P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right|\le C\omega_2\left(f;\sqrt{\frac{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}{2}}\right)$.

Proof. For modulus of continuity, using Proposition 1.1.4 and the Cauchy-Schwarz's inequality, we can say that

$$\begin{aligned}\left|P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right| &= \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(1)}(x)\left|f\left(\frac{s}{m}\right)-f(x)\right|\\ &\le \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(1)}(x)\,\omega\left(f;\left|\frac{s}{m}-x\right|\right)\\ &\le \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(1)}(x)\left(1+\frac{|s/m-x|}{\sqrt{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}}\right)\omega\left(f;\sqrt{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}\right)\\ &\le \omega\left(f;\sqrt{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}\right)\left[1+\frac{\sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(1)}(x)\,|s/m-x|}{\sqrt{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}}\right]\\ &= 2\omega\left(f;\sqrt{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}\right).\end{aligned}$$

Now, from Lemma 8.2.4, we consider

$$\begin{aligned}
&\left|P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right| \\
&\quad\le \left|P_{m,1}^{\langle\alpha\rangle}(f-h;x)\right|+|(f-h)(x)|+\left|P_{m,1}^{\langle\alpha\rangle}(h;x)-h(x)\right| \\
&\quad\le \|f-h\|+\|f-h\|+\frac{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}{2}\|h''\| \\
&\quad\le \|f-h\|+\frac{\mu_{m,2}^{\langle\alpha\rangle(1)}(x)}{2}\|h''\|.
\end{aligned}$$

Taking infimum for all $h\in C^2[0,1]$ and using Peetre's K-functional, we get

$$\left|P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right|\le K_2\left(f;\mu_{m,2}^{\langle\alpha\rangle(1)}(x)\big/2\right).$$

Applying Remark 1.1.5 will give us the desired result. □

This provides an estimate of the approximation error of the modified operators $P_{m,1}^{\langle\alpha\rangle}(f;x)$ in terms of the smoothness of the function f, as captured by its modulus of continuity.

Theorem 8.2.6 *Let f belong to the class of Lipschitz functions Lip_K^{β}, that is, there exists a positive real constant K such that for all $x_1,x_2\in\mathbb{R}$, $|f(x_1)-f(x_2)|\le K|x_1-x_2|^{\beta}$, where $0\le\beta<1$. Then for $\mu_{m,r}^{\langle\alpha\rangle(1)}(x)=P_{m,1}^{\langle\alpha\rangle}((t-x)^r;x)$,*

$$\left|P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right|\le K\left(\mu_{m,2}^{\langle\alpha\rangle(1)}(x)\right)^{\beta/2}.$$

Proof. Using Lemma 8.2.1 and the definition of Lipschitz class, we get

$$\begin{aligned}
&\left|P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right| \\
&\quad\le P_{m,1}^{\langle\alpha\rangle}(|f(t)-f(x)|;x) \\
&\quad\le P_{m,1}^{\langle\alpha\rangle}\left(K|t-x|^{\beta};x\right) \\
&\quad= K\sum_{s=0}^{m}p_{m,s}^{\langle\alpha\rangle(1)}(x)\left|\frac{s}{m}-x\right|^{\beta} \\
&\quad\le K\left[\sum_{s=0}^{m}\left(\left(p_{m,s}^{\langle\alpha\rangle(1)}(x)\right)^{\beta/2}\left|\frac{s}{m}-x\right|^{\beta}\right)^{\frac{2}{\beta}}\right]^{\frac{\beta}{2}}\left[\sum_{s=0}^{m}\left(\left(p_{m,s}^{\langle\alpha\rangle(1)}(x)\right)^{\frac{2-\beta}{2}}\right)^{\frac{2}{2-\beta}}\right]^{\frac{2-\beta}{2}}.
\end{aligned}$$

Applying Holder's inequality, we arrive at

$$\begin{aligned}
&\left|P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right| \\
&=K\left[\sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(1)}(x)\left|\frac{s}{m}-x\right|^{2}\right]^{\frac{\beta}{2}}\left[\sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(1)}(x)\right]^{\frac{2-\beta}{2}} \\
&=K\left[\mu_{m,2}^{\langle\alpha\rangle(1)}(x)\right]^{\beta/2}.
\end{aligned}$$

□

The above theorems and lemmas show that the modified Stnacu-Bernstein operators $P_{m,1}^{\langle\alpha\rangle}(f;x)$ approximate a continuous function f just as well as the classical operators $P_{m}^{\langle\alpha\rangle}(f;x)$. This means that introducing sequences of real numbers does not reduce the accuracy or convergence of the operators. In fact, this modification provides more flexibility in designing operators. Using this idea, we can construct higher-order operators that converge faster and give better approximation results than many classical operators found in the literature. These higher-order operators are therefore useful for applications where more accurate and efficient approximations are needed.

8.3 Higher Order Approximation

Building upon the idea of using sequences of real numbers to modify the classical operators, we now extend this concept to define a new family of positive linear operators. These operators are constructed in such a way that the choice of sequences allows for greater flexibility in improving the approximation process. These higher-order operators are defined as

$$P_{m,2}^{\langle\alpha\rangle}(f;x)=\sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(2)}(x)f\left(\frac{s}{m}\right), \tag{8.7}$$

where

$$p_{m,s}^{\langle\alpha\rangle(2)}(x)=v(m,x)p_{m-2,s}^{\langle\alpha\rangle}(x)+z(m,x)p_{m-2,s-1}^{\langle\alpha\rangle}(x)+v(m,1-x)p_{m-2,s-2}^{\langle\alpha\rangle}(x), \tag{8.8}$$

and

$$\begin{aligned}
v(m,x)&=v_2(m)x^2+v_1(m)x+v_0(m)\\
z(m,x)&=z_0(m)x(1-x),
\end{aligned}$$

such that $v_2(m)$, $v_1(m)$, $v_0(m)$, and $z_0(m)$ are sequences of real numbers. These sequences allow us to generalize the classical operators and introduce additional flexibility in approximating continuous functions. In particular, if we set the parameter $\alpha = 0$ and choose the sequences as $v_2(m) = 1 = v_0(m)$, $v_1(m) = -2$ and $z_0(m) = 2$, then the generalized operators defined in equation (8.7) reduce exactly to the classical Bernstein operators, defined in equation (1.1). This shows that the classical Bernstein operators appear as a special case of our more general construction, which demonstrates that our approach is both consistent with known results and flexible enough to cover new cases.

8.3.1 Approximation Results

In order to study the approximation properties of the newly defined operators (8.7), it is essential to first calculate their moments and central moments. These will serve as key tools in analyzing how well the operators approximate a continuous function. Thus, building on these results, we will then establish convergence theorems that describe the behaviour and accuracy of the operators on the interval $[0,1]$.

8.3.1.a Case Analysis for the sequences

In order that $P_{m,2}^{\langle\alpha\rangle}$ satisfy the Korovkin condition $P_{m,2}^{\langle\alpha\rangle}(t^i;x) \to x^i$, for $i = 0,1,2$, the sequences $v_2(m)$, $v_1(m)$, $v_0(m)$ and $z_0(m)$ must satisfy

$$z_0(m) = 2v_2(m), \qquad \text{and} \qquad v_2(m) + v_1(m) + 2v_0(m) \to 1.$$

The admissible cases for the sequences $v_2(m)$, $v_1(m)$, $v_0(m)$ and $z_0(m)$ are given below:

Case 1: If $v_2(m) < 0$, then $z_0(m) = 2v_2(m) < 0$ and $z(m,x) = 2v_2(m)x(1-x) < 0$.

Case 2: If $v_2(m) = 0$, then $z_0(m) = 0$ and $v_1(m) + 2v_0(m) = 1$. In this case $v(m,x) = v_1(m)x + v_0(m)$ and $p_{m,s}^{\langle\alpha\rangle(2)}(x) = v(m,x)p_{m-2,s}^{\langle\alpha\rangle}(x) + v(m,1-x)p_{m-2,s-2}^{\langle\alpha\rangle}(x)$. Thus, $P_{m,2}^{\langle\alpha\rangle}$ reduces to a first-order modification of the type $P_{m,1}^{\langle\alpha\rangle}$ with

$$u_1(m) = v_1(m), \qquad u_0(m) = v_0(m).$$

The cases of the sequences are then exactly the same as those in Subsection 8.2.1.a.

Case 3: If $v_2(m) > 0$ and $0 \le v_0(m) \le 1$, then $v(m,0) = v_0(m) \ge 0, v(m,1) = 1 - v_0(m) \ge 0$ and $z(m,x) = 2v_2(m)x(1-x) \ge 0$. Moreover, if $(v_2(m), v_0(m))$ satisfies

$$(v_2(m)-1)^2 + (2v_0(m)-1)^2 \le 1,$$

then $v(m,x)$ and $v(m,1-x)$ are non-negative on $[0,1]$.

Case 4: All remaining choices of $v_2(m), v_0(m)$ (i.e. $v_2(m) > 0$ but either $v_0(m) \notin [0,1]$ or $(v_2(m)-1)^2 + (2v_0(m)-1)^2 > 1$) lead to $v(m,x) < 0$ or $v(m,1-x) < 0$ for some $x \in (0,1)$.

From the above analysis, we can see that Cases 1 and 4 yield non-positive basis and therefore do not produce positive linear operators. Hence, the positivity of the second-order modification $P_{m,2}^{\langle\alpha\rangle}$ is ensured precisely when

$$v_2(m) \geq 0, \quad 0 \leq v_0(m) \leq 1 \quad \text{and} \quad (v_2(m)-1)^2 + (2v_0(m)-1)^2 \leq 1.$$

Considering the normalization conditions $z_0(m) = 2v_2(m)$ and $v_2(m) + v_1(m) + 2v_0(m) = 1$, along with Cases 2 and 3, the sequence $P_{m,2}^{\langle\alpha\rangle}$ forms a sequence of positive linear operators which preserves constants and approximates continuous functions on $[0,1]$.

Theorem 8.3.1 *For $\alpha = \alpha(m) \to 0$ let the sequences $v_2(m)$, $v_1(m)$, $v_0(m)$ and $z_0(m)$ be such that $2v_2(m) = z_0(m)$. Then, for the sequence of operators $P_{m,2}^{\langle\alpha\rangle}(f;x)$, the following equalities hold:*

(i) $P_{m,2}^{\langle\alpha\rangle}(1;x) = 2v_0(m) + v_1(m) + v_2(m)$

(ii) $P_{m,2}^{\langle\alpha\rangle}(t;x) = x(2v_0(m) + v_1(m) + v_2(m)) + \frac{2}{m}(1+2x)(v_0(m) + v_1(m) + v_2(m))$

(iii) $$P_{m,2}^{\langle\alpha\rangle}(t^2;x) = \begin{cases} \frac{x^2(2v_0(m)+v_1(m)+v_2(m))}{1+\alpha} + \frac{x\alpha(2v_0(m)+v_1(m)+v_2(m))}{1+\alpha} + \frac{x(1-5x-4\alpha)(2v_0(m)+v_1(m)+v_2(m))}{m(1+\alpha)} \\ + \frac{4x(v_0(m)+v_1(m)+v_2(m)-x(v_1(m)+v_2(m)))}{m} - \frac{2x(1-3x-2\alpha)(2v_0(m)+v_1(m)+v_2(m))}{m^2(1+\alpha)} \\ + \frac{4(v_0(m)+v_1(m)+v_2(m))-2x(4v_0(m)+6v_1(m)+7v_2(m))+2x^2(4v_1(m)+5v_2(m))}{m^2}. \end{cases}$$

Proof. For the first moment,

$$\begin{aligned} P_{m,2}^{\langle\alpha\rangle}(1;x) &= \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(2)}(x) \\ &= v(m,x)\sum_{s=0}^{m} p_{m-2,s}^{\langle\alpha\rangle}(x) + z(m,x)\sum_{s=0}^{m} p_{m-2,s-1}^{\langle\alpha\rangle}(x) + v(m,1-x)\sum_{s=0}^{m} p_{m-2,s-2}^{\langle\alpha\rangle}(x) \\ &= x^2v_2(m) + xv_1(m) + v_0(m) + x(1-x)z_0 + (1-x)^2v_2(m) + (1-x)v_1(m) + v_0(m) \\ &= x^2(2v_2(m) - z_0(m)) - x(2v_2(m) - z_0(m)) + 2v_0(m) + v_1(m) + v_2(m) \qquad (8.9) \\ &= 2v_0(m) + v_1(m) + v_2(m). \end{aligned}$$

Now, for the second moment, consider

$$\sum_{s=0}^{m} p_{m-2,s}^{\langle\alpha\rangle}(x)\frac{s}{m} = x - \frac{2x}{m},$$

$$\sum_{s=0}^{m} p_{m-2,s-1}^{\langle\alpha\rangle}(x)\frac{s}{m} = \sum_{s=0}^{m-2} p_{m-2,s}^{\langle\alpha\rangle}(x)\frac{s+1}{m}$$
$$= x - \frac{2x}{m} + \frac{1}{m},$$

and

$$\sum_{s=0}^{m} p_{m-2,s-2}^{\langle\alpha\rangle}(x)\frac{s}{m} = \sum_{s=0}^{m-2} p_{m-2,s}^{\langle\alpha\rangle}(x)\frac{s+2}{m}$$
$$= x - \frac{2x}{m} + \frac{2}{m}.$$

Thus, the second moment of the proposed operators is given by

$$\begin{aligned}
&P_{m,2}^{\langle\alpha\rangle}(t;x)\\
&= \left(x - \frac{2x}{m}\right)v(m,x) + \left(x - \frac{2x}{m} + \frac{1}{m}\right)z(m,x) + \left(x - \frac{2x}{m} + \frac{2}{m}\right)v(m,1-x)\\
&= \left(x - \frac{2x}{m}\right)(v(m,x) + z(m,x) + v(m,1-x)) + \frac{z(m,x) + 2v(m,1-x)}{m}\\
&= \left(x - \frac{2x}{m}\right)(2v_0(m) + v_1(m) + v_2(m)) + \frac{2(v_0(m) + v_1(m) + v_2(m)) - 2x(v_1(m) + v_2(m))}{m}\\
&= x\,(2v_0(m) + v_1(m) + v_2(m)) + \frac{2(v_0(m) + v_1(m) + v_2(m))(1+2x)}{m}.
\end{aligned}$$

Similarly, for the third moment, consider

$$\sum_{s=0}^{m} p_{m-2,s}^{\langle\alpha\rangle}(x)\frac{s^2}{m^2} = x^2\frac{(m-2)(m-3)}{m^2(1+\alpha)} + x\frac{(m-2)^2\alpha + m - 2}{m^2(1+\alpha)},$$

$$\sum_{s=0}^{m} p_{m-2,s-1}^{\langle\alpha\rangle}(x)\frac{s^2}{m^2} = \sum_{s=0}^{m-2} p_{m-2,s}^{\langle\alpha\rangle}(x)\frac{(s+1)^2}{m^2}$$
$$= x^2\frac{(m-2)(m-3)}{m^2(1+\alpha)} + x\frac{(m-2)^2\alpha + m - 2}{m^2(1+\alpha)} + \frac{2x}{m} - \frac{4x}{m^2} + \frac{1}{m^2},$$

and

$$\sum_{s=0}^{m} p_{m-2,s-2}^{\langle\alpha\rangle}(x)\frac{s^2}{m^2} = \sum_{s=0}^{m-2} p_{m-2,s}^{\langle\alpha\rangle}(x)\frac{(s+2)^2}{m}$$
$$= x^2\frac{(m-2)(m-3)}{m^2(1+\alpha)} + x\frac{(m-2)^2\alpha + m - 2}{m^2(1+\alpha)} + \frac{4x}{m} - \frac{8x}{m^2} + \frac{4}{m^2}.$$

This brings us to,

$$
\begin{aligned}
&P_{m,2}^{\langle\alpha\rangle}\left(t^2;x\right)\\
&\quad=\left(x^2\frac{(m-2)(m-3)}{m^2(1+\alpha)}+x\frac{(m-2)^2\alpha+m-2}{m^2(1+\alpha)}\right)v(m,x)\\
&\qquad+\left(x^2\frac{(m-2)(m-3)}{m^2(1+\alpha)}+x\frac{(m-2)^2\alpha+m-2}{m^2(1+\alpha)}+\frac{2x}{m}-\frac{4x}{m^2}+\frac{1}{m^2}\right)z(m,x)\\
&\qquad+\left(x^2\frac{(m-2)(m-3)}{m^2(1+\alpha)}+x\frac{(m-2)^2\alpha+m-2}{m^2(1+\alpha)}+\frac{4x}{m}-\frac{8x}{m^2}+\frac{4}{m^2}\right)v(m,1-x)\\
&\quad=\left(x^2\frac{(m-2)(m-3)}{m^2(1+\alpha)}+x\frac{(m-2)^2\alpha+m-2}{m^2(1+\alpha)}\right)\big(v(m,x)+z(m,x)+v(m,1-x)\big)\\
&\qquad+\left(\frac{2x}{m}-\frac{4x}{m^2}\right)\big(z(m,x)+2v(m,1-x)\big)+\frac{z(m,x)+4v(m,1-x)}{m^2}\\
&\quad=\left(x^2\frac{(m-2)(m-3)}{m^2(1+\alpha)}+x\frac{(m-2)^2\alpha+m-2}{m^2(1+\alpha)}\right)\big(2v_0(m)+v_1(m)+v_2(m)\big)\\
&\qquad+\frac{4x(m-2)\left(v_0(m)+v_1(m)+v_2(m)-x\left(v_1(m)+v_2(m)\right)\right)}{m^2}\\
&\qquad+\frac{4\left(v_0(m)+v_1(m)+v_2(m)\right)-2x\left(2v_1(m)+3v_2(m)\right)+x^2v_2(m)}{m^2}\\
&\quad=\frac{x^2\left(2v_0(m)+v_1(m)+v_2(m)\right)}{1+\alpha}+\frac{x\alpha\left(2v_0(m)+v_1(m)+v_2(m)\right)}{1+\alpha}\\
&\qquad+\frac{x(1-5x-4\alpha)\left(v_0(m)+v_1(m)+v_2(m)\right)}{m(1+\alpha)}+\frac{4x\left(2v_0(m)+v_1(m)+v_2(m)-x\left(v_1(m)+v_2(m)\right)\right)}{m}\\
&\qquad+\frac{4(v_0(m)+v_1(m)+v_2(m))-2x(4v_0(m)+6v_1(m)+7v_2(m))+2x^2(4v_1(m)+5v_2(m))}{m^2}\\
&\qquad-\frac{2x(1-3x-2\alpha)\left(v_0(m)+v_1(m)+v_2(m)\right)}{m^2(1+\alpha)}.
\end{aligned}
$$

□

However, from equation (8.9), we observe that the first moment of the operators $P_{m,2}^{\langle\alpha\rangle}(f;x)$, without imposing any specific restrictions on the sequences of real numbers, is given by

$$P_{m,2}^{\langle\alpha\rangle}(1;x)=x^2(2v_2(m)-z_0(m))-x(2v_2(m)-z_0(m))+2v_0(m)+v_1(m)+v_2(m).$$

While this expression does define a sequence of positive linear operators, these operators do not satisfy the conditions of the Bohman-Korovkin theorem. Consequently, without further restrictions, they may fail to converge uniformly to the required function f.

To ensure uniform convergence to f on the interval $[0,1]$, it is necessary that the operators converge to the test functions x^r for $r=0,1,2$, that is,

$$\lim_{m\to\infty} P_{m,2}^{\langle\alpha\rangle}(t^r;x) = x^r \quad \text{for } r=0,1,2.$$

Accordingly, the sequences $v_0(m), v_1(m), v_2(m), z_0(m)$ must satisfy certain necessary constraints to ensure that $P_{m,2}^{\langle\alpha\rangle}(f;x)$ not only remains positive but also converges uniformly to the given continuous function f while exactly preserving the first moment. These necessary conditions are

(i) $2v_2(m) = z_0(m)$, and

(ii) $2v_0(m) + v_1(m) + v_2(m) \to 1$ as $m \to \infty$.

These operators are defined using four sequences of real numbers, namely $v_0(m)$, $v_1(m)$, $v_2(m)$, and $z_0(m)$. Working with all of them in their generalized form can make calculations complicated. To make things simpler, we choose specific values for the sequences:

$$v_0(m) = 1, \quad v_1(m) = -1-\frac{m}{2}, \quad v_2(m) = \frac{m}{2}, \quad z_0(m) = m.$$

This choice helps simplify the operators and makes it easier to study their properties. Thus, our operators become

$$\begin{aligned} P_{m,2}^{\langle\alpha\rangle}(f;x) = \sum_{s=0}^{m} \Bigg[&(1-x)\left(1-\frac{mx}{2}\right) p_{m-2,s}^{\langle\alpha\rangle}(x) + mx(1-x) p_{m-2,s-1}^{\langle\alpha\rangle}(x) \\ &+ x\left(1-\frac{m(1-x)}{2}\right) p_{m-2,s-2}^{\langle\alpha\rangle}(x)\Bigg] f\left(\frac{s}{m}\right), \end{aligned} \tag{8.10}$$

where $p_{m,s}^{\langle\alpha\rangle}(x)$ is defined in equation (8.2).

Using Theorem 8.3.1, we can now compute the moments of the proposed operators (8.10), and based on these, determine their central moments as well. This provides the foundation for studying their convergence and approximation properties.

Lemma 8.3.2 *The moments of* $P_{m,2}^{\langle\alpha\rangle}(f;x)$ *defined by* (8.10) *are given by:*

(i) $P_{m,2}^{\langle\alpha\rangle}(1;x) = 1$

(ii) $P_{m,2}^{\langle\alpha\rangle}(t;x) = x$

(iii) $P_{m,2}^{\langle\alpha\rangle}(t^2;x) = \frac{2x(1-x)+8x\alpha(1-x)}{m^2(1+\alpha)} - \frac{5x\alpha(1-x)}{m(1+\alpha)} + \frac{x(x+\alpha)}{1+\alpha}$

(iv) $P_{m,2}^{\langle\alpha\rangle}\left(t^3;x\right)=\begin{cases}\frac{6x(1-x)(1-2x)(1+6\alpha)}{m^3(1+\alpha)(1+2\alpha)}-\frac{x(1-x)\left(2+21\alpha-18\alpha^2-2x(5+39\alpha+6\alpha^2)\right)}{m^2(1+\alpha)(1+2\alpha)}\\+\frac{3x\alpha(1-x)(1-4\alpha-x(7+2\alpha))}{m(1+\alpha)(1+2\alpha)}+\frac{x(x+\alpha)(x+2\alpha)}{(1+\alpha)(1+2\alpha)}\end{cases}$

Lemma 8.3.3 *Let us define the central moments of $P_{m,2}^{\langle\alpha\rangle}(f;x)$, given by* (8.10)*, as $\mu_{m,r}^{\langle\alpha\rangle(2)}(x)=P_{m,2}^{\langle\alpha\rangle}((t-x)^r;x)$. Then the following equalities hold:*

(i) $\mu_{m,1}^{\langle\alpha\rangle(2)}(x)=0$

(ii) $\mu_{m,2}^{\langle\alpha\rangle(2)}(x)=\frac{x(1-x)}{1+\alpha}\left(\alpha-\frac{5\alpha}{m}+\frac{2(1+4\alpha)}{m^2}\right)$

(iii) $\mu_{m,3}^{\langle\alpha\rangle(2)}(x)=\frac{x(1-x)(1-2x)}{(1+\alpha)(1+2\alpha)}\left(2\alpha^2+\frac{3\alpha(1-4\alpha)}{m}-\frac{2+21\alpha-18\alpha^2}{m^2}+\frac{6(1+6\alpha)}{m^3}\right).$

With the results established in Lemmas 8.3.2 and 8.3.3, we can now establish a Voronskaya-type theorem to understand the convergence behaviour of the sequence of operators $P_{m,2}^{\langle\alpha\rangle}(f;x)$ towards the function f.

Theorem 8.3.4 *For a continuous real-valued function f defined on $[0,1]$ with a continuous third derivative f''', let the real parameter α be chosen such that $\alpha\to 0$ and $\lim_{m\to\infty}m^2\alpha$ exists and is finite. Then,*

$$\lim_{m\to\infty}m^2\left[P_{m,2}^{\langle\alpha\rangle}(f;x)-f(x)\right]=\frac{x(1-x)}{2(1+\alpha)}f''(x)\left(cx+2x(1+4\alpha)\right)+\frac{x(1-x)(1-2x)}{6(1+\alpha)(1+2\alpha)}f'''(x)\left(18\alpha^2+(2c-21)\alpha-2\right),$$

where $c=\lim\limits_{m\to\infty}m^2\alpha$.

Proof. By Taylor's theorem,

$$f(\vartheta)=f(x)+(\vartheta-x)f'(x)+\frac{1}{2}(\vartheta-x)^2f''(x)+\frac{1}{6}(\vartheta-x)^3f'''(x)+(\vartheta-x)^3\varepsilon(\vartheta),$$

where $\lim\limits_{s/m\to x}\varepsilon(s/m)=0$. Let $\vartheta=s/m$. Multiplying both sides by $p_{m,s}^{\langle\alpha\rangle(2)}(x)$, then taking summation from $s=0$ to m and using Lemma 8.3.3, we get

$$\begin{aligned}P_{m,2}^{\langle\alpha\rangle}(f;x)&=f(x)+\frac{x(1-x)}{2(1+\alpha)}\left(\alpha-\frac{5\alpha}{m}+\frac{2+8\alpha}{m^2}\right)f''(x)\\&+\frac{x(1-x)(1-2x)}{6(1+\alpha)(1+2\alpha)}\left(2\alpha^2+\frac{3\alpha(1-4\alpha)}{m}-\frac{2+21\alpha-18\alpha^2}{m^2}+\frac{6+36\alpha}{m^3}\right)f'''(x)\\&+P_{m,2}^{\langle\alpha\rangle}\left((s/m-x)^3\varepsilon(s/m);x\right).\end{aligned}$$

First, we claim that

$$\lim_{m\to\infty} m^2 P_{m,2}^{\langle\alpha\rangle}\left(\left(s/m-x\right)^3 \varepsilon(s/m);x\right)=0.$$

To proceed, we define

$$\mathrm{T}_m=\left\{s:\left|\frac{s}{m}-x\right|<\varepsilon, s=0,1,2\ldots,m\right\},$$
$$\text{and }\ \Gamma_m=\left\{s:\left|\frac{s}{m}-x\right|\geq\varepsilon, s=0,1,2\ldots,m\right\}.$$

Taking into account the limiting behaviour of $\varepsilon(s/m)$, we observe that for any given $\delta>0$, if $\left|\frac{s}{m}-x\right|<\delta$, then there exists $\varepsilon>0$ such that $\varepsilon(s/m)<\varepsilon$. On the other hand, for $\left|\frac{s}{m}-x\right|\geq\delta$, we define

$$M=\sup_{0\leq x\leq 1}\left(\frac{s}{m}-x\right)^2\varepsilon(s/m).$$

Thus from Lemma 8.2.1, we can write

$$\begin{aligned}
&\lim_{m\to\infty} m^2 P_{m,2}^{\langle\alpha\rangle}\left((\vartheta-x)^3\varepsilon(\vartheta);x\right)\\
&=\sum_{s=0}^{m} m^2 p_{m,s}^{\langle\alpha\rangle}(x)\left(\frac{s}{m}-x\right)^3\varepsilon(s/m)\\
&=\lim_{m\to\infty}\sum_{s\in\mathrm{T}_m} m^2 p_{m,s}^{\langle\alpha\rangle}(x)\left(\frac{s}{m}-x\right)^3\varepsilon(s/m)+\sum_{s\in\Gamma_m} m^2 p_{m,s}^{\langle\alpha\rangle}(x)\left(\frac{s}{m}-x\right)^3\varepsilon(s/m)\\
&\leq\left(\varepsilon+\frac{M}{\delta^2}\right)\lim_{m\to\infty} m^2 P_{m,2}^{\langle\alpha\rangle}\left((\vartheta-x)^3;x\right)\\
&=0.
\end{aligned}$$

This proves our claim. Thus, we can say

$$\begin{aligned}
&\lim_{m\to\infty} m^2\left[P_{m,2}^{\langle\alpha\rangle}(f;x)-f(x)\right]\\
&=\lim_{m\to\infty}\left(\frac{x(1-x)\left(m^2\alpha x-5m\alpha x+2x(1+4\alpha)\right)}{2(1+\alpha)}f''(x)\right.\\
&\qquad\left.+\frac{x(1-x)(1-2x)\left(2m^2\alpha^2+18\alpha^2-21\alpha-2+3m\alpha(1-4\alpha)\right)}{6(1+\alpha)(1+2\alpha)}f'''(x)\right).
\end{aligned}$$

Thus, $\lim_{m\to\infty} m^2\alpha$ exists and is equal to c concludes the theorem. □

From Theorem 8.3.4, it follows that the operators $P_{m,2}^{\langle\alpha\rangle}(f;x)$, defined in (8.10), approximate any continuous real-valued function f with an order of approximation $O(1/m^2)$. This represents a significant improvement over the first-order operators (8.1) and (8.3), which only achieve an approximation order of $O(1/m)$. The enhancement in the rate of convergence demonstrates the effectiveness of incorporating higher-order sequences in the construction of positive linear operators. Consequently, the operators (8.10) will henceforth be referred to as the second-order Stancu-type operators, highlighting their superior approximation capabilities.

Next, we calculate error bounds for these proposed operators with the help of modulus of continuity and supremum norm of the function.

Theorem 8.3.5 *Let $\omega(f;\delta)$ be the modulus of continuity of f. Then*

$$\left|P_{m,2}^{\langle\alpha\rangle}(f;x)-f(x)\right| \leq 2\omega\left(f;\sqrt{\mu_{m,2}^{\langle\alpha\rangle(2)}(x)}\right)$$

Proof. Using modulus of continuity and Lemma 8.3.2, we can write

$$\begin{aligned}\left|P_{m,2}^{\langle\alpha\rangle}(f;x)-f(x)\right| &= \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(2)}(x)\left|f\left(\frac{s}{m}\right)-f(x)\right| \\ &\leq \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(2)}(x)\,\omega\left(f;\left|\frac{s}{m}-x\right|\right)\end{aligned}$$

Applying Cauchy-Schwarz's inequality and Proposition 1.1.4, we arrive at

$$\begin{aligned}\left|P_{m,2}^{\langle\alpha\rangle}(f;x)-f(x)\right| &\leq \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(2)}(x)\left(1+\frac{|s/m-x|}{\sqrt{\mu_{m,2}^{\langle\alpha\rangle(2)}(x)}}\right)\omega\left(f;\sqrt{\mu_{m,2}^{\langle\alpha\rangle(2)}(x)}\right) \\ &\leq \omega\left(f;\sqrt{\mu_{m,2}^{\langle\alpha\rangle(2)}(x)}\right)\left[1+\frac{\sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(2)}(x)\,|s/m-x|}{\sqrt{\mu_{m,2}^{\langle\alpha\rangle(2)}(x)}}\right] \\ &= 2\omega\left(f;\sqrt{\mu_{m,2}^{\langle\alpha\rangle(2)}(x)}\right).\end{aligned}$$

□

Theorem 8.3.6 *For the sequence of operators* (8.10) *and using the supremum norm, we can express the error bound as*

$$\left|P_{m,2}^{\langle\alpha\rangle}(h;x)-h(x)\right| \leq \frac{\mu_{m,2}^{\langle\alpha\rangle(2)}(x)}{2}\left\|h''\right\|.$$

Proof. By Taylor's expansion

$$h(t)=h(x)+(t-x)\,h'(x)+\frac{(t-x)^2}{2}h''(x)+(t-x)^2\varepsilon(t),$$

such that $\lim_{t\to x}\varepsilon(t)=0$. Taking $t=s/m$ and applying operators $P_{m,2}^{\langle\alpha\rangle}$ on both sides, we get

$$P_{m,2}^{\langle\alpha\rangle}(h(s/m);x)=h(x)+\frac{P_{m,2}^{\langle\alpha\rangle}\left((s/m-x)^2;x\right)}{2}h''(x)+P_{m,2}^{\langle\alpha\rangle}\left((s/m-x)^2\varepsilon(s/m);x\right).$$

From Lemma 8.3.3, we can say that $P_{m,2}^{\langle\alpha\rangle}\left((s/m-x)^2\varepsilon(s/m);x\right)\to 0$, as $m\to\infty$. Thus for a sufficiently large value of m, we have

$$\begin{aligned}\left|P_{m,2}^{\langle\alpha\rangle}(h;x)-h(x)\right| &\le \frac{P_{m,2}^{\langle\alpha\rangle}\left((s/m-x)^2;x\right)}{2}\left|h''(x)\right|\\ &\le \frac{\mu_{m,2}^{\langle\alpha\rangle(2)}(x)}{2}\left\|h''\right\|.\end{aligned}$$

□

Having established the convergence properties, moments, and order of approximation of the proposed higher-order Stancu-Bernstein operators, it is now natural to investigate their behaviour through numerical experiments. While the preceding results confirm uniform convergence and improved order theoretically, visualizing the operators and computing the actual errors for specific functions provides practical insight into their efficiency and performance.

In the following section, we present numerical examples illustrating the convergence of the first and second-order operators for different values of m and α, accompanied by error tables that quantify the approximation accuracy. These examples allow us to compare the speed of convergence of the operators and observe the impact of the parameter α on the approximation process.

8.4 Computational Study of the Operators

This section demonstrates the behaviour of the proposed Stancu-Bernstein operators in order to verify the theoretical results proved and claimed in the preceding sections.

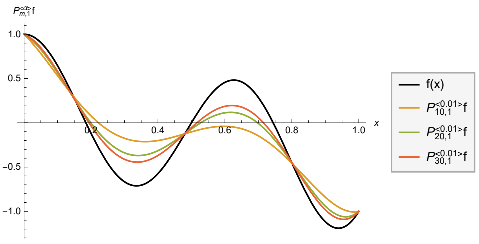

Figure 8.1: Convergence of first order $P_{m,1}^{\langle\alpha\rangle}(f;x)$ for different values of m and $\alpha = 0.01$ $\left(P_{m,1}^{\langle 0.01\rangle}(f;x)\right)$

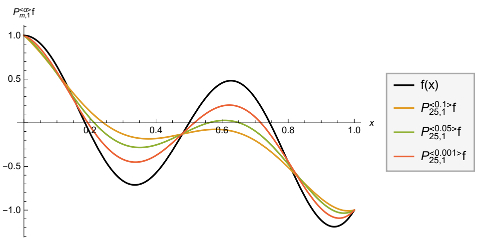

Figure 8.2: Convergence of first order $P_{m,1}^{\langle\alpha\rangle}(f;x)$ for different values of α and $m = 25$ $\left(P_{25,1}^{\langle\alpha\rangle}(f;x)\right)$

Since our modified operators $P_{m,1}^{\langle\alpha\rangle}(f;x)$ are constructed using the sequences $u_1(m)$ and $u_0(m)$ such that $u_1(m) + 2u_0(m) = 1$, we first illustrate a specific case where $u_1(m) = -1$ and $u_0(m) = 1$. Figures 8.1 and 8.2 display the convergence behaviour of this first-order modification, defined by (8.3), for various values of m and α.

It is evident from the graphs that as m increases while keeping α fixed, the operators progressively approach the required function f. Likewise, for a fixed m, reducing the value of α enhances the approximation, leading the operators closer to f. Therefore, we can conclude that in the limit as $m \to \infty$ and $\alpha \to 0$, the sequence of operators $P_{m,1}^{\langle\alpha\rangle}(f;x)$ converges uniformly to f, confirming both the effectiveness and stability of this first-order modification.

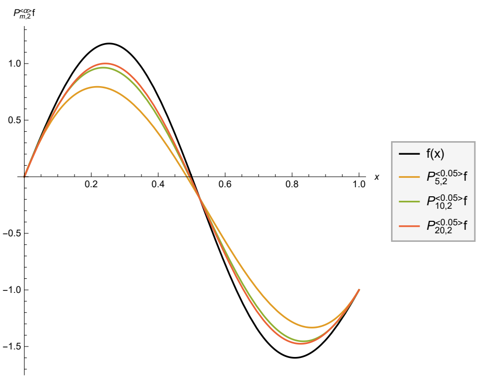

Figure 8.3: Convergence of second order operators $P_{m,2}^{\langle\alpha\rangle}(f;x)$ for different values of m and $\alpha = 0.05$ $\left(P_{m,2}^{\langle 0.05\rangle}(f;x)\right)$

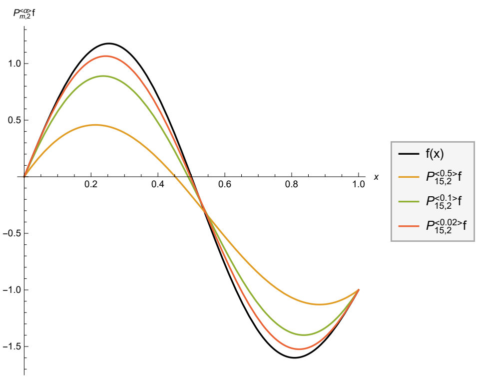

Figure 8.4: Convergence of second order operators $P_{m,2}^{\langle\alpha\rangle}(f;x)$ for different values of α and $m = 15$ $\left(P_{15,2}^{\langle\alpha\rangle}(f;x)\right)$

Similarly, Figures 8.3 and 8.4 illustrate the convergence behaviour of the second-order modification, $P_{m,2}^{\langle\alpha\rangle}(f;x)$, defined by (8.10), for various values of m and α. In this case, we have chosen the sequences as $v_0(m) = 1$, $v_1(m) = -1 - m/2$, $v_2(m) = m/2$, and $z_0(m) = m$.

From the graphical results, it is clear that the operators approximate the target function f increasingly well as m becomes larger. Moreover, for a fixed m, smaller values of α enhance the convergence, allowing the operators to more closely match the function f. This confirms that, in the limit as $m \to \infty$ and $\alpha \to 0$, the sequence $P_{m,2}^{\langle\alpha\rangle}(f;x)$ converges uniformly to f. Compared to the first-order operators, the second-order modification demonstrates faster convergence and reduced approximation error, validating the effectiveness of introducing higher-order sequences in the operator construction.

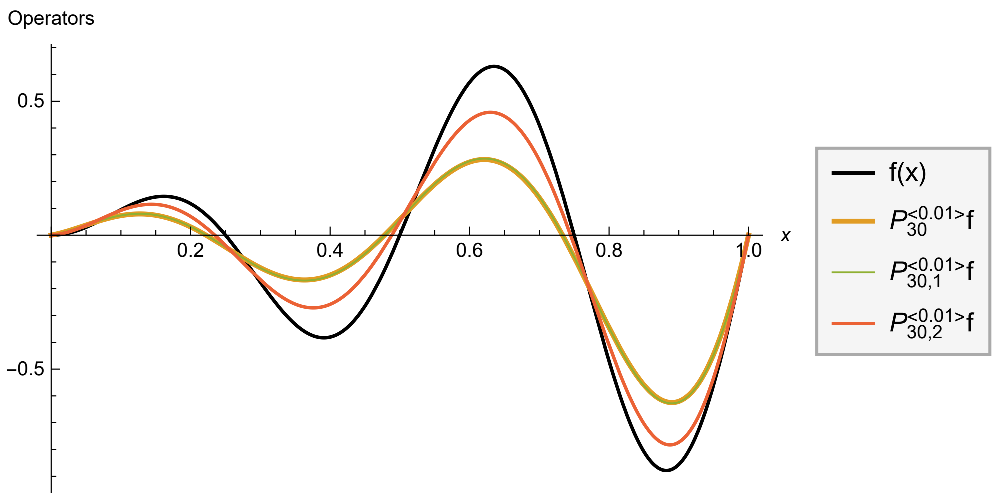

Figure 8.5: Comparison between the classical, first and second order operators $P_m^{\langle\alpha\rangle}$, $P_{m,1}^{\langle\alpha\rangle}$ and $P_{m,2}^{\langle\alpha\rangle}$, respectively, for $m = 30$ and $\alpha = 0.01$ where $f(x) = x\sin(4\pi x)$.

Figure 8.5 presents a comparison between the three types of operators: the classical Stancu-Bernstein operator $P_m^{\langle\alpha\rangle}$, the first-order modified operator $P_{m,1}^{\langle\alpha\rangle}$ and the second-order operator $P_{m,2}^{\langle\alpha\rangle}$, for a fixed value of m and α. It is evident from the figure that both $P_m^{\langle\alpha\rangle}$ and $P_{m,1}^{\langle\alpha\rangle}$ approximate the desired function f at a comparable rate. In contrast, the second-order operator $P_{m,2}^{\langle\alpha\rangle}$ achieves a significantly better approximation, converging more closely to the function.

Following this comparison, we now proceed to analyze the behaviour of these operators under a different choice of sequences. Specifically, we redefine the first-order modification as

$$Q_{m,1}^{\langle\alpha\rangle}(f;x) = \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(1)}(x) f\left(\frac{s}{m}\right), \tag{8.11}$$

where $p_{m,s}^{\langle\alpha\rangle(1)}(x)$ is defined as in equation (8.4), with the sequences of real numbers selected as follows, ensuring the necessary conditions for convergence of the operator:

$$u_1(m) = 1 - 1/m \quad \text{and} \quad u_0(m) = 1/(2m).$$

Next, we turn our attention to the second-order operators, defined as

$$Q_{m,2}^{\langle\alpha\rangle}(f;x) = \sum_{s=0}^{m} p_{m,s}^{\langle\alpha\rangle(2)}(x) f\left(\frac{s}{m}\right), \tag{8.12}$$

where $p_{m,s}^{\langle\alpha\rangle(2)}(x)$ is defined by equation (8.8), with the sequences chosen as

$$v_2(m) = m, \;\; v_1(m) = -m, \;\; v_0(m) = 1/2 \;\text{ and }\; z_0(m) = 2m.$$

This particular choice allows us to explore the behaviour of the second-order operators under a different set of sequences, highlighting their approximation properties.

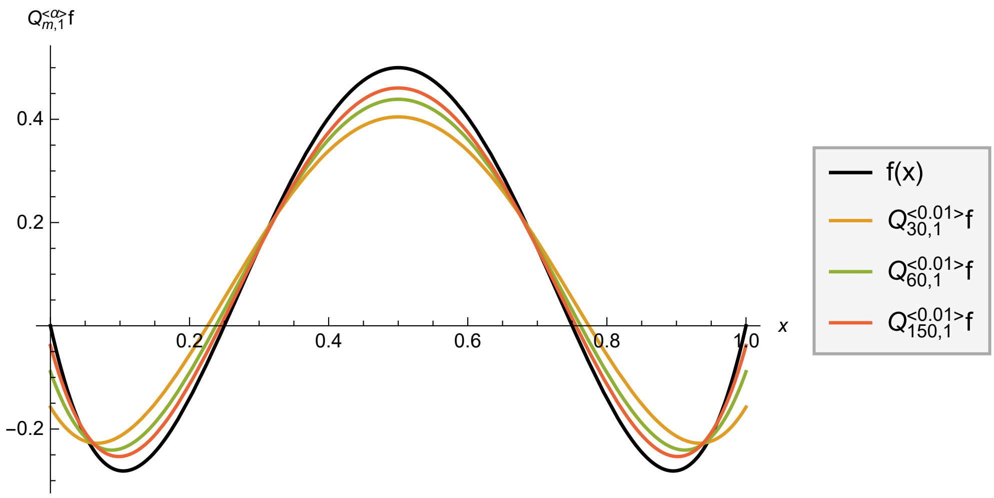

Figure 8.6: Approximation by $Q_{m,1}^{\langle\alpha\rangle}$ for $f(x) = 32x^4 - 64x^3 + 38x^2 - 6x$.

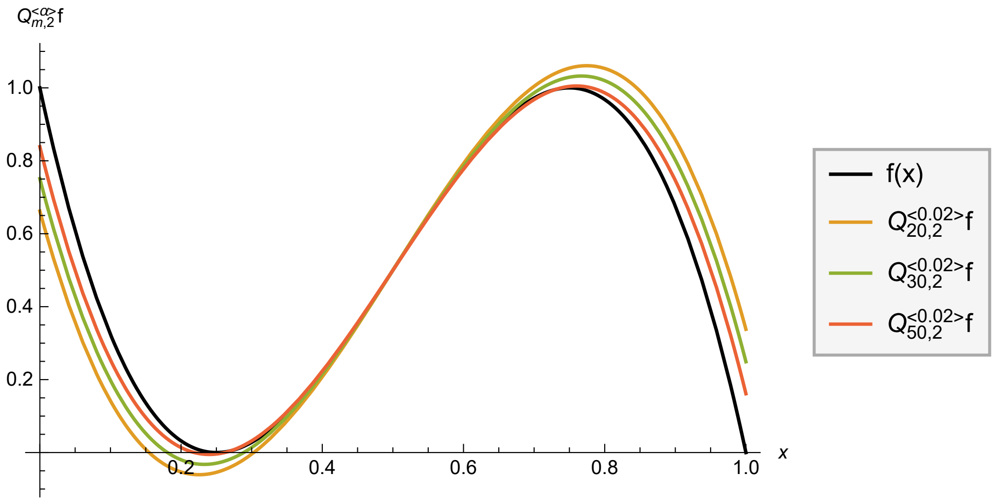

Figure 8.7: Approximation by $Q_{m,2}^{\langle\alpha\rangle}$ for $f(x) = -16x^3 + 24x^2 - 9x + 1$.

Figures 8.6 and 8.7 depict the approximation process for the first and second order operators $Q_{m,1}^{\langle\alpha\rangle}(f;x)$ and $Q_{m,2}^{\langle\alpha\rangle}(f;x)$, defined by equations (8.11) and (8.12), respectively, with a fixed value of α and varying m. As the graphs show, increasing m improves the accuracy of the approximation, and both operators converge uniformly to the required function f, confirming their effectiveness as positive linear operators.

However, unlike the earlier operators, $P_{m,1}^{\langle\alpha\rangle}(f;x)$ and $P_{m,2}^{\langle\alpha\rangle}(f;x)$, defined by equations (8.3) and (8.10), respectively, these operators do not interpolate the endpoints 0 and 1. In other words, the operators approximate the function well throughout the interval $(0,1)$ but do not interpolate the function values at the boundaries. This is a direct consequence of the specific sequences chosen for these operators:$u_0(m) = 1/2m$ and $u_1(m) = 1 - 1/m$ for the first order operators; and $v_2(m) = m$, $v_1(m) = -m$, $v_0(m) = 1/2$ and $z_0(m) = 2m$ for the second order operators, which alter the weight distribution in the operators. The same can be observed in Tables 8.3 and 8.4, which present the error values for different points $x \in [0,1]$.

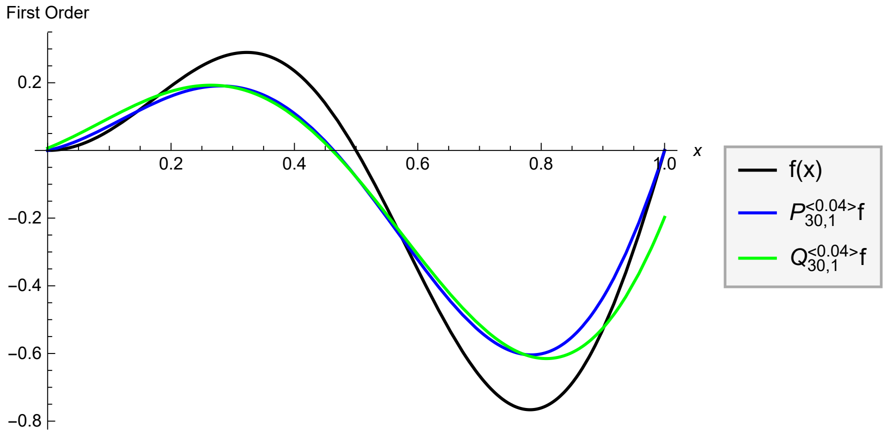

Figure 8.8: Comparison of the first order modification for different set of sequences and $f(x) = x\sin(2\pi x)$.

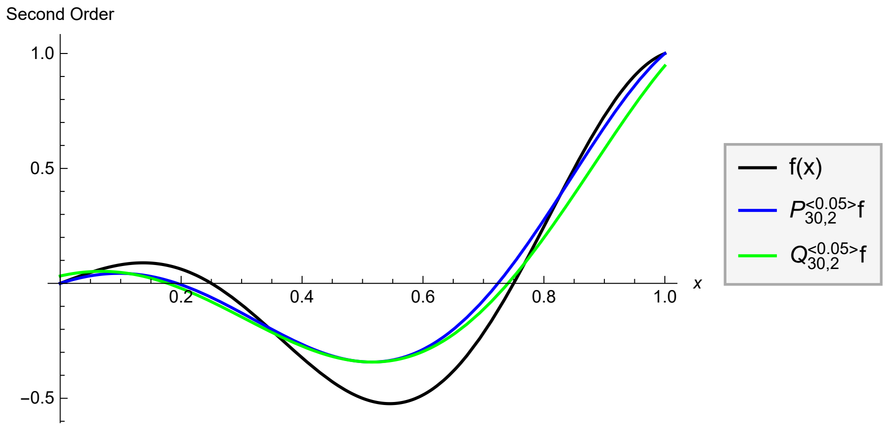

Figure 8.9: Comparison of the second order modification for different set of sequences and $f(x) = x\cos(2\pi x)$.

Figures 8.8 and 8.9 compare the approximation of the first and second order proposed operators, for the two different choices of sequences considered earlier. The overall convergence of the operators to the function f is very similar across the domain, however, a noticeable difference occurs at the endpoints. Both $P_{m,1}^{\langle\alpha\rangle}$ and $P_{m,2}^{\langle\alpha\rangle}$ exactly interpolate the end points $x = 0$ and $x = 1$, while the corresponding operators $Q_{m,1}^{\langle\alpha\rangle}$ and $Q_{m,2}^{\langle\alpha\rangle}$ do not. This illustrates that the choice of sequences primarily affects endpoint behaviour rather than the overall convergence rate, highlighting that the convergence is somewhat better when $u_1(m) + u_0(m) = 0$ (for first order) and $v_2(m) + v_1(m) + v_0(m) = 0$ (for second order).

	$\left\|P_m^{\langle\alpha\rangle}(f;x)-f(x)\right\|$	$\left\|P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right\|$	$\left\|P_{m,2}^{\langle\alpha\rangle}(f;x)-f(x)\right\|$
$m=20$	0.228672	0.227228	0.155885
$m=40$	0.174144	0.172665	0.0979084
$m=60$	0.145934	0.144693	0.0815196
$m=80$	0.129297	0.128255	0.0749659
$m=100$	0.118399	0.117506	0.0717722
$m=120$	0.110724	0.109945	0.07001

Table 8.1: Error values at $x=0.45$ as m increases.

	$\left\|P_m^{\langle\alpha\rangle}(f;x)-f(x)\right\|$	$\left\|Q_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right\|$	$\left\|Q_{m,2}^{\langle\alpha\rangle}(f;x)-f(x)\right\|$
$m=20$	0.228672	0.235491	0.086946
$m=40$	0.174144	0.178714	0.0270038
$m=60$	0.145934	0.14925	0.0221019
$m=80$	0.129297	0.131885	0.0250671
$m=100$	0.118399	0.120516	0.0290642
$m=120$	0.110724	0.112514	0.0327852

Table 8.2: Error values at $x=0.45$ as m increases.

	$\left\|P_m^{\langle\alpha\rangle}(f;x)-f(x)\right\|$	$\left\|P_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right\|$	$\left\|P_{m,2}^{\langle\alpha\rangle}(f;x)-f(x)\right\|$
$x=0$	0	0	0
$x=0.2$	0.0867318	0.0860204	0.0478877
$x=0.4$	0.231388	0.229236	0.122233
$x=0.6$	0.297587	0.294554	0.14178
$x=0.8$	0.124689	0.123656	0.0591589
$x=1$	0	0	0

Table 8.3: Error values for $m=30$ and for different $x\in[0,1]$.

	$\left|P_m^{\langle\alpha\rangle}(f;x)-f(x)\right|$	$\left|Q_{m,1}^{\langle\alpha\rangle}(f;x)-f(x)\right|$	$\left|Q_{m,2}^{\langle\alpha\rangle}(f;x)-f(x)\right|$
$x=0$	0	0.0133319	0.0247715
$x=0.2$	0.0867318	0.110113	0.0354948
$x=0.4$	0.231388	0.23558	0.0169478
$x=0.6$	0.297587	0.299916	0.0122273
$x=0.8$	0.124689	0.222753	0.0979446
$x=1$	0	0.386626	0.346801

Table 8.4: Error values for $m=30$ and for different $x\in[0,1]$.

Tables 8.1, 8.2, 8.3 and 8.4 present a comparative analysis of the error values for the first and second order operators defined by (8.1), (8.3), and (8.7) when applied to the function $f(x)=x\sin(4\pi x)$. From these numerical results, it is evident that the second order operators provide a more accurate approximation of the function compared to the first order operators. Specifically, Tables 8.1 and 8.2 illustrate that for a fixed point, $x=0.45$, the approximation error decreases consistently as m increases. This trend demonstrates that the operators converge to the function uniformly, and for sufficiently large m, the error becomes negligible. These results confirm the theoretical findings regarding the improved order of approximation achieved by the second order operators and highlight their practical effectiveness in approximating continuous real-valued functions.

8.5 Conclusion

In this chapter, we introduced improved approximation methods using first and second order positive linear operators based on the contagion distribution. These operators use sequences of real numbers, and they converge to a continuous function only when certain conditions on the sequences are satisfied: for the first order operators $P_{m,1}^{\langle\alpha\rangle}(f;x)$, we require $u_1(m)+2u_0(m)\to 1$, and for the second order operators $P_{m,2}^{\langle\alpha\rangle}(f;x)$, the conditions are $2v_2(m)=z_0(m)$ and $2v_0(m)+v_1(m)+v_2(m)\to 1$ as $m\to\infty$. To simplify computations, we considered specific choices of these sequences for the second order operators.

Both the theoretical results and numerical examples show that the second order operators provide a better approximation than the first order operators. While the operators generally converge, only certain sequences, specifically, $u_1(m)+u_0(m)=0$ for first order and $v_2(m) + v_1(m) + v_0(m) = 0$ for second order, ensure that the operators match the function interpolates the endpoints, as seen in Figures 8.8 and 8.9, and Tables 8.3 and 8.4.

This study demonstrates that using sequences of real numbers to modify operators can improve their approximation properties. It also provides a foundation for future work, such as designing higher order operators, optimizing the choice of sequences and applying these methods in numerical computations or data fitting.

Chapter 9

Conclusion, Future scope and Social Impact

9.1 Conclusion

The central theme of this thesis has been the study and development of improved approximation techniques using positive linear operators. The work brings together several perspectives, like probabilistic distributions, higher-order constructions, fractional calculus and fuzzy mathematics, into a unified framework of approximation theory. Both theoretical analysis and numerical verification have been carried out, showing how carefully designed modifications can significantly enhance the approximation process.

The first chapter of this thesis provided a comprehensive introduction to approximation theory and the role of positive linear operators within it. It began by outlining the fundamental concepts and motivation behind approximation processes, highlighting their importance in both pure and applied mathematics. To establish a solid foundation, the chapter included essential definitions, basic results and lemmas that are frequently used in later chapters of the thesis. In addition, it presented a review of relevant literature, tracing the historical development of approximation theory and summarizing key contributions by various researchers in the field. This literature review not only situated the present work within the broader context of existing research but also identified the gaps and open problems that motivate the contributions made in this thesis.

Chapter 2 focused on the construction and study of positive linear operators based on the contagion distribution. These operators were constructed with the introduction of parameters, α and γ, which provided an additional degree of flexibility and enabled them to capture a wider range of approximation behaviours. The moments and and central moments of these operators were calculated, followed by an analysis of their approximation results. To enhance their approximations and applicability, modifications of these operators were also introduced, namely the King-type, Kantorovich-type, and genuine-type modifications. Each of these played a distinct role: the King-type modification was useful in preserving the constant and well as the quadratic test functions, the Kantorovich-type modification extended their applicability to integrable functions, and the genuine-type modification ensured better structural properties by preserving the constant and linear test function of the Kantorovich operators. Overall, these modifications not only made the operators converge more effectively but also addressed the limitations of classical operators, particularly by preserving test functions.

Building upon this, Chapter 3 focused on introducing a new class of semi-exponential operators, which were inspired by kernels arising from homogeneous partial differential equations, incorporating a real parameter β. These operators represent a natural extension of the classical framework, combining exponential-type features with polynomial structures. A detailed study was done on their moments, central moments, recurrence relations and moment generating function, which helped us understand their approximation properties. Furthermore, a Voronovskaya-type theorem was established, providing insights into the asymptotic behaviour of these operators and showing how well they approximate functions as $m \to \infty$. To complement the theoretical analysis, numerical experiments were performed, which confirmed the effectiveness of the operators. The results showed that semi-exponential operators form a promising alternative to classical operators, offering flexible approximation properties and demonstrating their usefulness in the broader framework of approximation theory and its applications.

The study then extended into the domain of fractional calculus in Chapter 4, where we constructed fractional versions of the Bernstein-Kantorovich operators. These operators were developed using the Caputo fractional derivative, which is particularly well-suited for modeling processes with memory and hereditary effects. Their fundamental properties, such as moments using Laplace transforms, and convergence behaviour, were carefully examined. Beyond their mathematical analysis, we demonstrated how these fractional positive linear operators can be employed to solve fractional differential and fractional integro-differential equations, thereby connecting approximation theory with practical problem-solving techniques. Thus, the chapter highlights how approximation operators can enrich the study of fractional models and contribute meaningfully to applied sciences.

A different direction was explored through the concept of fuzzy theory in Chapter 5, where approximation of fuzzy-valued functions was considered. Here, the classical Korovkin theorem was adapted to the fuzzy context, and convergence of positive linear operators was established. A Voronovskaya-type theorem in the fuzzy environment was also derived. This part of the work expands approximation theory beyond crisp functions, aligning it with decision-making processes and uncertainty modelling. Moreover, in this chapter we have defined several positive linear fuzzy operators and investigated their approximation properties, thereby extending the theory of operators into the fuzzy setting.

In Chapter 6, the thesis then shifted its focus to higher-order constructions, where second and third-order semi-exponential operators were introduced and studied in depth. These operators were analyzed through their moments, central moments and asymptotic results, which helped to understand how they behave for as $n \to \infty$. The results showed that these higher-order operators give better approximations than the first-order case, since they reduce the error more effectively and capture the function with greater accuracy. The numerical experiments also supported this observation, showing that as the order of the operator increases, the convergence becomes faster and the approximation becomes smoother. This means that moving to higher orders is not just a theoretical idea but a practical way to improve the performance of positive linear operators. These results make it clear that higher-order versions are a very promising direction for future research, both in theory and in applications.

A similar idea was pursued with Stancu-Bernstein operators based on the contagion distribution in Chapter 7 by extending them to higher-order forms. The main goal was to improve their accuracy. Normally, the classical Stancu-Bernstein operators give an error of order $O(1/n)$, but the higher-order versions we introduced reduced the error to $O(1/n^2)$, which means they give much better approximations. To prove this, we used Korovkin's theorem and estimates based on the modulus of continuity. We also added numerical examples, which confirmed the theory and made the improvement clearly visible. In short, the higher-order Stancu-Bernstein operators turned out to be more powerful than the classical ones, both in theory and in practice, and they open up new possibilities for sharper results in approximation theory.

Finally, in Chapter 8, the thesis proposed sequence-based modifications of first and second-order operators. Instead of relying on derivatives, these operators were defined directly through sequences of natural numbers. This approach makes them applicable even to non-differentiable functions, while still improving convergence. We studied the conditions under which these operators converge uniformly to a continuous function. The analysis showed that all of these positive linear operators converge, but their behaviour at the endpoints depends on the choice of sequences. Some operators interpolate the required function at the end points $x = 0$ and $x = 1$, while others do not. Both graphical and numerical results supported these observations. Moreover, it was observed that the second-order sequence-based operators provide more accurate approximations than the first-order operators.

Overall, the findings of this thesis demonstrate that the introduction of parameters, higher-order constructions and sequence-based modifications significantly enrich the theory of approximation by positive linear operators. The parametric modifications built upon the contagion distribution provide a flexible mechanism to control convergence and allow preservation of functions that classical operators cannot handle. The semi-exponential operators proposed in this work open a connection between approximation theory and kernel-based methods arising from partial differential equations. Fractional extensions of Bernstein-Kantorovich type operators reveal that approximation operators are not only tools of classical integral calculus, but can also be effectively be applied to fractional calculus, incorporating memory effects, where they contribute to the numerical solutions of fractional differential and fractional integro-differential equations. Similarly, the fuzzy adaptation of classical operators demonstrates how approximation theory can be extended to functions that involve uncertainty and imprecision, which has direct relevance to decision-making and modeling in real-world problems. Moreover, higher-order operators constructed in this work consistently outperform first-order operators, both in theory and in numerical practice, highlighting the advantages of extending classical definitions to higher levels. Finally, the sequence-based operators introduced in the last chapter of this thesis show that convergence can be achieved under simple structural conditions, while the behaviour at the end points depends on the choice of sequences, thereby offering a balance between simplicity and approximation efficiency.

9.2 Academic future plans

In the future, I plan to continue this line of research by exploring several natural extensions of the operators introduced in this work. One of my immediate goals is to generalize the first and second order operators to higher-order versions and to study their approximation properties in greater depth. I am particularly interested in investigating how fractional and fuzzy adaptations of these operators can be extended to higher-order and multivariate cases, as this could significantly broaden their scope and practical relevance. Another direction I wish to pursue is the detailed study of simultaneous approximation and shape-preserving properties of the newly constructed operators, since such features are essential in applications such as curve fitting, computer-aided geometric design, and numerical analysis.

I also intend to further analyze the sequence-based approach, as the behaviour of the operators at the end points strongly depends on the choice of sequences. By carefully optimizing these sequences, I hope to design new families of operators that maintain global convergence while offering better control over boundary interpolation. Additionally, I am keen to apply these theoretical findings to real-world problems, especially in the numerical solution of partial differential equations, signal processing, and image reconstruction, where approximation plays a central role.

In the long term, I aim to combine different generalizations into hybrid constructions; for example, developing operators that are simultaneously fractional, fuzzy, and higher-order. Such hybrid operators could become powerful tools for modeling and analyzing complex systems that arise in applied sciences and engineering. With these directions, I see this research not as a closed chapter, but as the beginning of a broader program of study that I plan to carry forward in my future academic work.

9.3 Social Impact

The results of this thesis are not limited to pure mathematics, rather they carry potential to influence several aspects of science, technology and even everyday life. Approximation theory, though abstract in nature, serves as a foundation for many practical tools we rely on today. By improving positive linear operators through parameters, higher-order constructions, fractional extensions and fuzzy adaptations, this research contributes toward making these tools more precise and more versatile.

One important area of impact is computer graphics and geometric modeling. Bernstein-type operators are the mathematical backbone of Bézier curves and surfaces, which are widely used in computer-aided design, animation and industrial product development. The refinements presented in this thesis, such as higher-order and sequence-based operators, can potentially lead to smoother curves, more accurate shapes and better control in design processes. This translates into improved technologies in fields ranging from automobile and aerospace engineering to architecture and digital media.

The fractional operators studied in this work are useful for modeling processes that involve memory and long-term effects, such as population growth, disease spread or material behaviour under stress. More accurate approximation tools in these areas can support better predictions, which in turn can guide public policy, healthcare and resource management.

Similarly, the fuzzy operators connect directly to decision-making under uncertainty. This is relevant in fields like economics, social sciences and artificial intelligence, where real data is often vague or imprecise. By extending approximation theory into the fuzzy domain, the research contributes to building mathematical tools that can handle such uncertainty more effectively.

Higher-order and sequence-based operators also strengthen computational techniques in fields that rely on intensive calculations, such as image processing, signal reconstruction and computer-aided design. Improvements in approximation accuracy here can lead to smoother images, clearer signals and better geometric models, with direct impact on industries ranging from communication technology to medical imaging.

In the broader sense, this thesis shows how abstract mathematical developments have far-reaching consequences in society. Whether it is in designing a smoother curve on a car body using Bézier techniques, reconstructing a medical image with higher accuracy, or improving predictions in scientific models, the advances in approximation theory developed here highlight how mathematics quietly powers the technologies that shape our modern world.

This thesis aligns with SDG 4 (Quality Education) by advancing the theoretical foundations of approximation theory and enriching the mathematical tools used in higher learning and research. It supports SDG 8 (Decent Work and Economic Growth) by improving the computational techniques that drive modern industries, enabling more efficient workflows, higher accuracy and greater technological productivity in sectors such as digital media, communication technology and industrial design. Furthermore, it contributes to SDG 9 (Industry, Innovation and Infrastructure) through refined Bernstein-type and higher-order operators that enhance geometric modeling, computer-aided design and engineering computation, thereby strengthening digital infrastructure and fostering innovation across industrial applications.

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List of Publications

1. Kanita and Naokant Deo (2024). *Parametric Bernstein operators based on contagion distribution*. Mathematical Methods in the Applied Sciences, Pages 1-14. DOI- 10.1002/mma.10172 [**SCIE, Impact Factor: 2.1**]

2. Kanita and Naokant Deo (2025). *Bernstein type semi-exponential operators*. Applicable Analysis, Pages 1-17. DOI- 10.1080/00036811.2025.2480642 [**SCI-E, Impact Factor: 1.1**]

3. Kanita and Naokant Deo. *Analysis of Semi-Exponential Bernstein Operators with Improved Order of Approximation*. Mathematical Methods in the Applied Sciences [**Accepted**]

4. Kanita and Naokant Deo. *Kantorovich Variant of α-Bernstein Operators using Contagion Distribution* [**Revision Submitted**]

5. Kanita and Naokant Deo. *A Novel Class of Positive Linear Operators for Approximating Solutions to Fractional Differential and Integro-Differential Equations* [**Communicated**]

6. Kanita and Naokant Deo. *Developments in Fuzzy Operators using Boas-Buck Polynomials* [**Communicated**]

7. Kanita and Naokant Deo. *Approximation of Linear Positive Fuzzy Operators* [**Communicated**]

8. Kanita and Naokant Deo. *Extending the Pólya-Eggenberger Distribution: A Study of Higher Order Operators* [**Communicated**]

9. Kanita and Naokant Deo. *Study of Higher Order Approximation of Positive Linear Operators based on Polya Distribution: A Sequential Approach* [**Communicated**]

Papers presented in International Conferences

1. Presented a research paper entitled “α-Bernstein operators based on Pólya distribution" in *International Conference on “Role of Applied Sciences in Sciences in Social Implications (IC-RASSI)"*, organized by Faculty of Science, Govt. DigvÄşay Autonomous PG College, Chhattisgarh, India, during 6th-8th February, 2023.

2. Presented a research paper entitled “Parametric Bernstein Operators using Contagion Distribution" in the *International Conference on “Mathematics and Applications (ICMA)"*, organized by the Department of Mathematics, Mata Sundri College for Women, University of Delhi, Delhi, India, during January 10-12, 2024.

3. Presented a research paper entitled “Semi-exponential Bernstein Operators" in the “*International Conference on Multidisciplinary Approaches to Sustainability and Current Challenges (MACCS 2024)*", organized by G H Raisoni University, Amravati, India, during May 17-18, 2024.