# COEFFICIENT ESTIMATES, RADIUS CONSTANTS AND SUBORDINATION OF CERTAIN ANALYTIC FUNCTIONS

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**MATHEMATICS**

*by*

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I, **Mridula Mundalia**, hereby certify that the work which is being presented in the thesis entitled "**Coefficient Estimates, Radius Constants and Subordination of Certain Analytic Functions**" in partial fulfillment of the requirements for the award of the **Degree of Doctor of Philosophy** in Mathematics, submitted in the Department of **Applied Mathematics**, Delhi Technological University, is an authentic record of my own work carried out during the period from July, 2016 to February, 2025 under the supervision of **Prof. S. Sivaprasad Kumar**.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other institute.

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**February, 2025** **MRIDULA MUNDALIA**

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**Dedicated to**

**my Parents, Brother**

**&**

**Husband**

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# Abstract

Univalent Function Theory (UFT) is a fascinating branch of Complex Analysis. The elegance of UFT lies in its ability to derive significant insights from relatively simple geometric considerations. Owing to its strong focus on geometric interpretations, UFT constitutes a fundamental part of Geometric Function Theory (GFT), where such geometric properties play a central and significant role. The thesis provides a comprehensive study of subclasses of analytic functions, offering sharp coefficient estimates, radius constants, and inclusion relations across various classes. It introduces novel techniques, such as convolution-defined results, differential subordinations, and new Ma-Minda subclasses, while generalizing and extending known results in UFT. Chapter 2 introduces a unified class of analytic functions, providing sharp estimates for initial coefficients and the Fekete-Szegö functional, as well as results on convolution-defined classes and second order Hankel determinants for certain close-to-convex functions. Chapter 3 explores univalent as well as non-univalent analytic functions associated with a parabolic region, deriving radius constants (univalence, starlikeness) and sufficient conditions, supported by diagrams. In Chapter 4, sharp radius problems for the class $\mathscr{S}^*(\beta)$ and a product function involving tilted Carathéodory functions are determined, obtaining sharp radius constants and generalizing earlier known results. Chapter 5 introduces a new Ma-Minda subclass $\mathscr{S}^*_{\rho}$, associated with the hyperbolic cosine function $\cosh\sqrt{z}$, establishing inclusion relations and sharp radius results in context of various analytic classes. Finally, Chapter 6 uses Briot-Bouquet differential subordination and admissibility conditions to derive sufficient conditions for functions in the class $\mathscr{S}^*_{\rho}$. Also applications are provided, supported by diagrams. This study offers significant insights into analytic function subclasses, extending known results and introducing novel techniques in UFT.

# List of Symbols

| Notations | Brief Description |
|---|---|
| $\mathbb{C}$ | Set of complex numbers |
| $\mathbb{N}$ | Set of natural numbers |
| $\mathbb{R}$ | Set of real numbers |
| $\mathbb{D}$ | Open unit disc $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$ |
| $\mathbb{D}_r$ | Open disc $\{z \in \mathbb{C} : \lvert z\rvert < r \leq 1\}$ |
| $\mathscr{H}[a,n]$ | Class of analytic functions of the form $f(z) = a + \sum_{m=n}^{\infty} a_m z^m$ |
| $\mathscr{A}_n$ | Class of analytic functions of the form $f(z) = z + \sum_{m=n}^{\infty} a_{m+1} z^{m+1}$ |
| $\mathscr{A}$ | $\mathscr{A}_1$ |
| $\mathscr{S}$ | Class of univalent functions in $\mathscr{A}$ |
| $\mathscr{S}^*$ | Class of starlike functions in $\mathscr{A}$ |
| $\mathscr{C}$ | Class of convex functions in $\mathscr{A}$ |
| $\mathscr{P}$ | Class of Carathéodory functions defined as: |
| | $\{p \in \mathscr{H}[1,1] : \operatorname{Re} p(z) > 0\}$ |
| $\mathscr{P}_\lambda$ | Class of tilted Carathéodory functions defined as: |
| | $\{p \in \mathscr{H}[1,1] : \operatorname{Re}(e^{i\lambda} p(z)) > 0, -\pi/2 < \lambda < \pi/2\}$ |
| $\mathscr{K}$ | Class of close-to-convex functions |
| $f \prec g$ | $f(z)$ is subordinate to $g(z)$ |
| $\phi(z)$ | Ma-Minda function |
| $\mathscr{S}^*(\phi)$ | Class of Ma-Minda starlike functions, $\phi \in \mathscr{P}$ |
| | (Some special cases of $\mathscr{S}^*(\phi)$ are listed in Table 1.1 on page no. 6) |
| $\mathscr{C}(\phi)$ | Class of Ma-Minda convex functions, $\phi \in \mathscr{P}$ |
| $\mathscr{S}^*[A,B]$ | Class of analytic functions $f \in \mathscr{A}$ satisfying $zf'(z)/f(z) \prec (1+Az)/(1+Bz)$ |
| $\mathscr{S}^*(\beta)$ | Class of starlike functions of order $\beta, 0 \leq \beta < 1$ |
| $\mathscr{S}\mathscr{S}^*(\beta)$ | Class of strongly starlike functions of order $\beta, 0 < \beta \leq 1$ |
| $\mathscr{S}^k_{\gamma,\delta}(\phi)$ | This class is defined on page no. 16 |
| $\mathscr{S}^*_\rho$ | $\{f \in \mathscr{A} : zf'(z)/f(z) \prec \cosh\sqrt{z} =: \rho(z)\}$ |
| $\mathscr{F}_\varphi$ | $\{f \in \mathscr{A} : zf'(z)/f(z) \prec 1 - (2/\pi^2)(\log((1+\sqrt{z})/(1-\sqrt{z})))^2 =: \varphi(z)\}$ |

| | |
|---|---|
| $\mathscr{M}(\alpha)$ | Class of analytic functions $f \in \mathscr{A}$ satisfying $\operatorname{Re} zf'(z)/f(z) < \alpha, \alpha > 1$ |
| $\psi(z)$ | Analytic univalent function, starlike with respect to $0$ and $\psi(0) = 0$ |
| $\mathscr{F}(\psi)$ | $\mathscr{F}(\psi) := \left\{ f \in \mathscr{A} : \dfrac{zf'(z)}{f(z)} - 1 \prec \psi(z) \right\}$ (Some special cases of $\mathscr{F}(\psi)$ are listed in Table 3.1 on page no. 43) |

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# Chapter 1

# Introduction

This chapter gives a concise summary over the theory of Univalent functions, highlighting foundational techniques and key developments in the field. It begins by introducing basic terminologies and concepts, along with an exploration of fundamental aspects of univalent function theory. It covers definitions, concepts and results required for this study, and concludes with a brief synopsis of the thesis.

## 1.1 Introduction and Preliminaries

Complex analysis encompasses the geometric theory of functions, with a focus on analytic functions defined by their geometric properties. The theory of univalent functions primarily addresses the geometric aspects of these functions, placing it within the framework of Geometric Function Theory (GFT). The present study concentrates on exploring various subclasses of univalent functions, investigating their distinct geometric behaviors and properties, which play a crucial role in advancing the field. The theory of Univalent functions is an old subject and an active field of research. GFT finds applications in fluid dynamics, engineering, and digital image processing, where conformal mappings help solve boundary value problems and simulate physical phenomena, bridging theoretical insights with practical applications. It is also helpful in studying the time evolution of the free boundary of a viscous fluid in Hele-Shaw cells with either finite or infinite boundaries [169]. The origin of the theory of univalent functions traces back to a seminal treatise by Koebe [72]. Infact in 1914, Gronwall [49] gave proof of the Area theorem, which is fundamental to the theory of univalent functions. In 1916, Bieberbach [44] proposed a conjecture that remained an active area of research for over 69 years, inspiring numerous related problems. The theory of univalent functions has an extensive and vast variety of literature, which can be seen in books by Duren [33], Goodman [46, 47], Pommerenke [127], Graham and Kohr [48], Nehari [117], Hallenbeck and MacGregor [50], Thomas et al. [167], Jenkins [58] and reviewed articles of Hayman [51] and Duren [32]. The books on complex analysis authored by Conway [28] and Silverman [158] covers some topics on univalent function theory as well.

A single valued function $f(z)$ which is analytic except for atmost one simple pole is said to be *univalent* in a domain $\mathscr{D}$ if it is one-to-one in $\mathscr{D}$, or equivalently if $f(z_1) = f(z_2)$, then $z_1 = z_2$, for $z_1, z_2 \in \mathscr{D}$. The linear function $az+b$ (with $a \neq 0$) is the only univalent function in the entire complex plane $\mathbb{C}$. While an analytic univalent function requires $f'(z) \neq 0$, which alone does not guarantee univalence; for instance, $e^{bz}$, where $b \neq 0$ has a non-vanishing derivative, but not univalent in $\mathbb{C}$, and univalent only in the disc $|z| < \pi/|b|$. Furthermore, an analytic function is locally univalent in a neighborhood of a point in $\mathscr{D}$ if and only if its derivative does not vanish at that point, as seen with $e^{bz}$ in $\mathbb{C}$.

In 1851, Riemann enunciated the pivotal **Riemann Mapping Theorem**, which asserts that every proper simply connected domain can be conformally mapped onto the open unit disc $\mathbb{D} := \{z : |z| < 1\}$. In view of the Riemann Mapping Theorem, the study of univalent functions is restricted to the open unit disc $\mathbb{D}$.

Let $\mathscr{H}$ represent the class of all analytic functions defined on $\mathbb{D}$, and $\mathscr{H}[a,n] \subset \mathscr{H}$

denote the class of analytic functions $f(z)$ with a Taylor series expansion of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, where $n$ is a positive integer and $a \in \mathbb{C}$. Assume $\mathscr{H}_0 := \mathscr{H}[0,1]$ and $\mathscr{H}_1 := \mathscr{H}[1,1]$. Additionally, let $\mathscr{A}_n \subset \mathscr{H}$ denote the class of all functions whose Taylor series is given by $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$. Let $\mathscr{A}$ denote the class of all normalized analytic functions defined on $\mathbb{D}$ consisting of functions $f(z)$, such that $f(0) = 0$ and $f'(0) = 1$, given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + \cdots. \tag{1.1.1}$$

Assume $\mathscr{A} := \mathscr{A}_1$. Note that the importance of normalization is depicted by the existence of solution to the coefficient related problems and its relationship to compactness of a given function space, see [46, Chapter 4]. Let $\mathscr{S} \subset \mathscr{A}$ represent the class of all univalent functions. Examples in $\mathscr{S}$ includes functions such as, $z/(1-z^2)$, which maps $\mathbb{D}$ onto the region $\mathbb{C} \setminus \{w : |\operatorname{Im} w| \geq 1/2\}$ and the Koebe function $K(z) = z/(1-z)^2$, with image domain as $\mathbb{C} \setminus \{w : \operatorname{Re} w \leq -1/4\}$. The renowned Bieberbach's Theorem, a foundation for Bieberbach's Conjecture, asserts that *if $f \in \mathscr{S}$, then $|a_2| \leq 2$, equality occurs if and only if $f$ is some rotation of the Koebe function*. Infact an important consequence of the Bieberbach's inequality $|a_2| \leq 2$ is the Koebe distortion theorem, which provides sharp upper and lower bounds for $|f'(z)|$, where $f \in \mathscr{S}$, stating that: *for each $f \in \mathscr{S}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad |z| = r < 1.$$

*For each $z \in \mathbb{D}$, $z \neq 0$, equality occurs if and only if $f$ is a suitable rotation of the Koebe function.*

Analytic functions defined on $\mathbb{D}$ are categorized into various classes and subclasses based on certain geometries of the image domain. In the following section, we highlight several important classes that are relevant in this context.

## 1.2 Univalent Subfamilies of $\mathscr{A}$

Between the formulation of Bieberbach's conjecture in 1916 and its eventual proof in 1985, researchers identified and defined various classes of functions using specific geometrical conditions. These served as a tool to approach and analyze the conjecture systematically over the decades. Some of the notable classes studied during that time includes the class of starlike functions, convex functions and close-to-convex functions. Various subclasses of $\mathscr{S}$ are defined by natural geometric conditions. Chief subclasses of $\mathscr{S}$ are the classes of starlike and convex functions.

### Classes of Starlike and Convex functions

A domain is said to be **starlike** with respect to a point $w_0$ if every point of domain is visible from $w_0$. If every point is visible from each point of a domain, then such a domain is called **convex** domain. A function $f \in \mathscr{S}$ is said to be starlike if and only if $\operatorname{Re} zf'(z)/f(z) > 0$ and convex if and only if $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$, the corresponding classes are denoted by $\mathscr{S}^*$ and $\mathscr{C}$ respectively. Let $\mathscr{P}$ denote the *Carathéodory* class, consisting of functions $p(z)$ satisfying $\operatorname{Re} p(z) > 0$ and $p(0) = 1$. Thus $f \in \mathscr{S}^*$ if and only if $zf'(z)/f(z) \in \mathscr{P}$ and $f \in \mathscr{C}$ if and only if $1 + zf''(z)/f'(z) \in \mathscr{P}$. In 1915, Alexander [3] established a two way bridge between the classes $\mathscr{S}^*$ and $\mathscr{C}$, by stating: *a function $f \in \mathscr{C}$ if and only if $zf' \in \mathscr{S}^*$.*

The concept of *subordination* helps in comparing and analyzing the geometric properties of analytic functions, by providing insights into their behaviour and relationships. Subordination, first introduced by Lindelöf [94] in 1909, was later formalized and expanded by Littlewood [95, 96] and Rogosinski [143, 144]. Their work established a foundational framework for analyzing relationships between analytic functions, making subordination a pivotal concept in GFT.

**Definition 1.2.1.** Let $f$ and $g$ be analytic functions in $\mathbb{D}$, then $f$ is said to be *subordinate* ($\prec$) to $g$ if there exists a Schwarz function $\omega(z)$ with $|\omega(z)| < 1$ and $\omega(0) = 0$ such that $f(z) = g(\omega(z))$. In case if $g$ is univalent, then

$$f \prec g \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

In 1992, Ma and Minda [98] introduced a more generalized class of starlike and convex functions, unifying various subclasses of starlike and convex functions, respectively, given by

$$\mathscr{S}^*(\phi) := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\} \tag{1.2.2}$$

and

$$\mathscr{C}(\phi) := \left\{ f \in \mathscr{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\},$$

where $\phi(z)$ is an analytic function and possesses certain geometrical properties, listed below:

$$\left.\begin{array}{l} \phi \text{ is starlike with respect to } \phi(0) = 1, \\ \phi \text{ is univalent in } \mathbb{D}, \\ \phi \text{ is symmetric about the real-axis }, \\ \phi \in \mathscr{P} \text{ and } \phi'(0) > 0. \end{array}\right\} \tag{1.2.3}$$

For different choice of $\phi(z)$, the class $\mathscr{S}^*(\phi)$ and $\mathscr{C}(\phi)$ reduce to various subclasses of $\mathscr{S}^*$ and $\mathscr{C}$, respectively. Additionally, we also mention some important subclasses of $\mathscr{S}^*$ and

$\mathscr{C}$ which are studied on various aspects in our subsequent chapters as well. For $\phi(z) = (1+Az)/(1+Bz) =: \phi_{A,B}(z)$, where $-1 \leq B < A \leq 1$, we get the class of Janowski starlike and Janowski convex functions, introduced by Janwoski [55] and denoted by $\mathscr{S}^*[A,B] := \mathscr{S}^*((1+Az)/(1+Bz))$ and $\mathscr{C}[A,B] := \mathscr{C}((1+Az)/(1+Bz))$, respectively. Especially, if $A = 1-2\beta$ $(0 \leq \beta < 1)$ and $B = -1$, $\mathscr{S}^*[A,B]$ reduces to the class of starlike functions of order $\beta$, introduced by Robertson [140], defined as

$$\mathscr{S}^*(\beta) := \mathscr{S}^*\left(\frac{1+(1-2\beta)z}{1-z}\right) = \left\{f \in \mathscr{A} : \operatorname{Re}\frac{zf'(z)}{f(z)} > \beta,\ 0 \leq \beta < 1\right\}, \tag{1.2.4}$$

and $\mathscr{C}(\beta) := \mathscr{C}((1+(1-2\beta)z)/(1-z))$ represents the class of convex functions of order $\beta$ $(0 \leq \beta < 1)$. The class $\mathscr{S}\mathscr{S}^*(\beta)$ introduced by Stankiewicz [163], can be obtained when $\phi(z) = ((1+z)/(1-z))^\beta$, known as the class of strongly starlike functions of order $\beta$ $(0 < \beta \leq 1)$, where each $f \in \mathscr{A}$ is determined by the condition

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D}.$$

Further another interesting class introduced independently by Ma-Minda [99] and Rønning [147], involves a parabolic function representing the region

$$\Omega_p := \{w \in \mathbb{C} : \operatorname{Re} w > |w-1|\},$$

defined as

$$\mathscr{S}_P^* := \left\{f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \phi_P(z) := 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}, \tag{1.2.5}$$

where branch of $\sqrt{z}$ is chosen such that $\operatorname{Im}\sqrt{z} \geq 0$. Table 1.1 enlists several notable subclasses of $\mathscr{S}^*$ obtained for specific choices of $\phi(z)$.

Other prominent classes of univalent functions includes the class of close-to-convex functions. In 1952, Kaplan [66] introduced the class of close-to-convex functions, denoted by $\mathscr{K}$. A function $f \in \mathscr{A}$ lies in $\mathscr{K}$ if there exists a starlike function $\phi \in \mathscr{S}^*$ and a real number $\lambda \in (-\pi/2, \pi/2)$ such that

$$\operatorname{Re}\left(e^{-i\lambda}\frac{zf'(z)}{\phi(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{1.2.6}$$

Geometrically, a function $f(z)$ is Close-to-Convex if and only if $C_r := f(|\xi| = r)$, where $0 < r < 1$, has no large hairpin turns, i.e. no sections of the curve $f(C_r)$ in which tangent vector turns backward through an angle greater than or equal to $\pi$.

The problems addressed in (GFT) predominantly rely on subordination techniques. Key

Table 1.1: Special subclasses of Ma-Minda starlike functions for specific choices of $\phi(z)$

| **Class** $\mathcal{S}^*(\phi)$ | $\phi(z)$ | $\phi(\mathbb{D})$ | **References** |
|---|---|---|---|
| $\mathcal{S}^*_e$ | $\phi_e(z) := e^z$ | $\Omega_e$ | [106] Mendiratta and Nagpal |
| $\mathcal{S}^*_\wp$ | $\phi_\wp(z) := 1 + ze^z(z)$ | $\Omega_\wp$ | [79] Kumar and Kamaljeet |
| $\mathcal{S}^*_L$ | $\phi_L(z) := \sqrt{1+z}$ | $\Omega_L$ | [159] Sokół and Stankiewicz |
| $\mathcal{S}^*_{\sin}$ | $\phi_{\sin}(z) := 1 + \sin z$ | $\Omega_{\sin}$ | [23] Cho et al. |
| $\mathcal{S}^*_{SG}$ | $\phi_{SG}(z) := 2/(1+e^{-z})$ | $\Omega_{SG}$ | [39] Goel and Kumar |
| $\mathcal{S}^*_{\sinh}$ | $\phi_{\sinh}(z) := 1 + \sinh^{-1} z$ | $\Omega_{\sinh}$ | [11] Arora and Kumar |
| $\mathcal{S}^*_{\leftmoon}$ | $\phi_{\leftmoon}(z) := z + \sqrt{1+z^2}$ | $\Omega_{\leftmoon}$ | [133] Raina and Sokół |
| $\mathcal{S}^*_{N_e}$ | $\phi_{N_e}(z) := 1 + z - z^3/3$ | $\Omega_{N_e}$ | [173] Wani and Swaminathan |
| $\mathcal{S}^*_c$ | $\phi_c(z) := 1 + (4/3)z + (2/3)z^2$ | $\Omega_c$ | [156] Sharma et al. |
| $\mathcal{S}^*_{q_\kappa}$ | $\phi_{q_\kappa}(z) := \sqrt{1+\kappa z}$, $0 < \kappa \leq 1$ | $\Omega_{q_\kappa}$ | [10] Aouf et al. |
| $\mathcal{S}^*_{\alpha,e}$ | $\phi_{\alpha,e}(z) := \alpha + (1-\alpha)e^z$, $0 \leq \alpha < 1$ | $\Omega_{\alpha,e}$ | [71] Khatter et al. |
| $\mathcal{S}\mathcal{L}^*(\alpha)$ | $\phi_{\alpha,\mathcal{L}}(z) := \alpha + (1-\alpha)\sqrt{1+z}$, $0 \leq \alpha < 1$ | $\Omega_{\alpha,\mathcal{L}}$ | [71] Khatter et al. |
| $\mathcal{S}^*_L(s)$ | $\phi_s(z) := (1+sz)^2$, $s \in [-1/\sqrt{2}, 1/\sqrt{2}] \setminus \{0\}$ | $\Omega_s$ | [14] Bano and Raza |
| $\mathcal{S}^*_{RL}$ | $\phi_{RL}(z) := \sqrt{2} - (\sqrt{2}-1)\sqrt{(1-z)(1+2(\sqrt{2}-1)z)}$ | $\Omega_{RL}$ | [105] Mendiratta et al. |
| $\mathcal{S}^*_R$ | $\phi_R(z) := 1 + (z/k)((k+z)/(k-z))$, $k = 1+\sqrt{2}$ | $\Omega_R$ | [83] Kumar and Ravichandran |

areas of focus in this field include the estimation of coefficient functional bounds, the derivation of radius constants and establishing differential subordination implication results.

## 1.3 Some Problems in GFT

### Coefficient Problems

In 1916, Bieberbach [45] gave a conjecture stating that: *The coefficients of each function* $f \in \mathcal{S}$ *satisfies* $|a_n| \leq n$*, for* $n = 2, 3, \cdots$ *. Strict inequality holds for all n unless f is the Koebe function or one of it rotations.* This sparked a significant interest in the study of coefficient problems, eventually leading to the development of new methods and advancing the literature still further. These efforts not only led to the eventual proof by Louis De Branges in 1985 but also inspired the development of new techniques and results in GFT. Ever since, abundant theory has evolved and it richly contributed to the literature, see [32, 33, 44, 57, 70, 162, 174]. For instance, the second coefficient bound for functions in class $\mathcal{S}$

leads to growth and distortion estimates. The coefficient problems includes: estimation of bounds of initial coefficients, logarithmic coefficients, inverse coefficients, coefficient functionals such as, Fekete-Szegö, Hankel determinant, Toeplitz determinant and others. However, in the present work we focus on estimating bounds on initial coefficients, Fekete Szegö functional and Hankel determinant. In 1933, Fekete and Szegö [36] introduced a functional $|a_3 - \mu a_2^2|$, known as the Fekete-Szegö coefficient functional, and derived its sharp bound for the class $\mathscr{S}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3, & \mu \geq 1, \\ 1 + \exp(-2\mu/(1-\mu)), & 0 \leq \mu \leq 1, \\ 3 - 4\mu, & \mu \leq 0. \end{cases}$$

In 1960, coefficient functionals were first considered as determinants of the $q^{th}$ Hankel matrices $H_{q,n}(f)$ defined as:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}, \tag{1.3.7}$$

where $q, n \in \mathbb{N}$ and $f \in \mathscr{A}$. Hankel determinants are a sequence of determinants of matrices formed from the coefficients of a power series expansion of an analytic function. Hankel determinant of exponential polynomials was studied by Ehrenborg [35], and Hayman [53] examined some properties of Hankel transform of an integer sequence. The main objective of solving coefficient problems is to determine sharp bounds for coefficient functionals and identify the corresponding extremal functions, offering insights into optimal behavior and characterizing extremal cases under given constraints.

## Radius Problems

Let $f, g \in \mathscr{H}$ with series expansion of the form:$f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$, then the convolution (or Hadamard) product of $f$ and $g$, denoted by $f * g$, is given by

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

Let $\mathscr{G}$ and $\mathscr{H}$ be two subclasses of $\mathscr{A}$, then the $\mathscr{G}$*-radius* of $\mathscr{H}$, denoted by $R_{\mathscr{G}}(\mathscr{H})$ or simply $R_{\mathscr{G}}$ is the largest $R \in (0,1)$ such that $\rho^{-1} f(\rho z) \in \mathscr{G}$, whenever $0 < \rho \leq R$, for all $f \in \mathscr{H}$. Using convolution we can express the condition $f(\rho z)/\rho = f(z) * (z/(1-\rho z))$.

Note that for each positive number $r \leq 2-\sqrt{3}$, every member of $\mathcal{S}$ maps the disc $|z| \leq r$

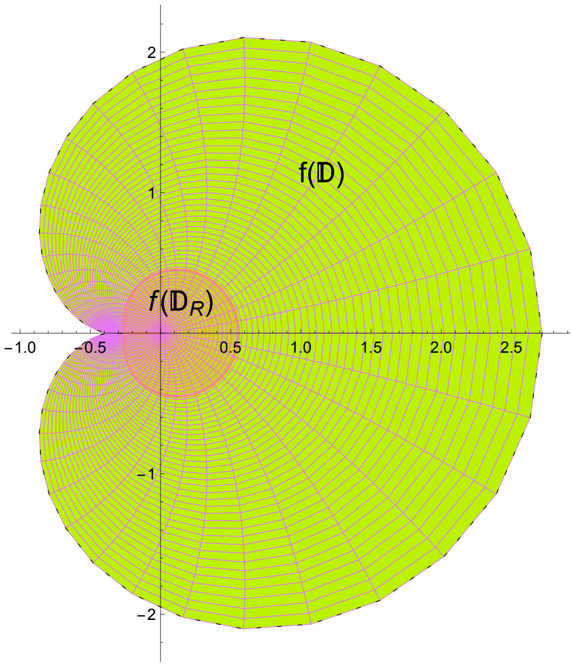


Figure 1.1: Cardioid function $f(z) = ze^z$ and $R = (3-\sqrt{5})/2 \approx 0.381\cdots$.

onto a convex domain. This is not true for $r > 2-\sqrt{3}$. Therefore, radius of convexity for the class $\mathcal{S}$ is $R_{\mathcal{C}} = 2-\sqrt{3}$. Another example is $f(z) = ze^z \in \mathcal{S}^*$, which maps $\mathbb{D}$ onto a cardioid domain, however, $f$ lies in $\mathcal{C}$, whenever $z \in \mathbb{D}_R$, where $\mathbb{D}_R := \{z : |z| < R \leq 1\}$ and $R = (3-\sqrt{5})/2 \approx 0.381\cdots$, which is the radius of convexity for $f(z)$ (see Figure 1.1). The study of radius problems has garnered significant attention in context of various subclasses of $\mathcal{A}$. Numerous researchers have investigated radius constants for starlike, convex, and other geometrically defined subclasses, as well as their relationships with specific functions. Determining radius constants, such as the radii of starlikeness, convexity and starlikeness of order $\beta$, is crucial for understanding the boundary behavior and inclusion relationships of various subclasses in the broader framework of analytic functions. Recently, sharp $\mathcal{S}^*_{\sin}$−radii, $\mathcal{S}^*_e$−radii and $\mathcal{S}^*_{SG}$−radii were derived for various classes, see [23, 39, 106]. The investigation of radius problems, provides new tools and results to analyze the behavior of functions lying in certain classes of analytic functions. Despite significant progress, many areas of radius problems remain unexplored, thereby making it a vibrant and intriguing field of study. For more details in this direction, one may refer to [46, 47, 167] and the references therein.

### Differential Subordination

The theory of Differential Subordination traces its origin to a treatise entitled *Differential subordinations and univalent functions* [109] in 1981. Their extensive study of the theory and groundbreaking work laid the foundation for the field. Another key contributor to this field is Bulboacă [17], who contributed significantly by presenting new results that enhances its theoretical framework, by making it more robust and comprehensive. Subsequent extensions and generalizations can be seen in [108, 109]. The foundational work on first-order differential subordination was pioneered by Goluzin [43] and Robinson [142], laying the groundwork for further exploration in this field. This powerful framework introduced novel approaches to analyzing and characterizing various classes, sparking widespread interest among researchers. By leveraging differential subordination techniques, mathematicians solved intricate problems related to univalent functions. The versatility and effectiveness of these methods inspired numerous extensions and applications across GFT and other areas of complex analysis. For instance, see [9, 12, 13, 16, 22, 34, 39, 63, 64, 69, 106, 155, 161, 173] and the references therein.

Interestingly, differential subordination is essentially an analogous extension of a differential inequality on the real line, adapted to the behavior of analytic functions in a more general setting. Further, a differential inequality determines the range of the original function, for instance if $f(0)=1$ and $f'(x)+f(x)\leq 1$, then $f(x)\leq 1$. Analogously, theory of differential subordination, deals with the differential implications leading to certain characterization of the function determined by conditions on differential expressions. For instance, *Noshiro-Warschawski Theorem* states that: *if $f$ is analytic in $\mathbb{D}$, then* $\operatorname{Re} f'(z)>0$ *implies $f(z)$ is univalent in $\mathbb{D}$*. Geometrically, every convex function is starlike, then with $K_f(z)=q_f(z)+zq'_f(z)/q_f(z)$, where $q_f(z)=zf'(z)/f(z)$, the implication: $\operatorname{Re}(q_f(z)+zq'_f(z)/q_f(z))>0 \Rightarrow \operatorname{Re} q_f(z)>0$, holds true for each $z\in\mathbb{D}$. This highly non-trivial observation made by Miller and Mocanu, was a foundation stone for the development of the theory of *differential subordination*.

Let $\Omega$ and $\Delta$ be any two sets in $\mathbb{C}$, assume $p(z)$ to be analytic in $\mathbb{D}$ with $p(0)=a$, and $\vartheta(r,s,t;z):\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$, then differential subordination involves generalizing the following implication

$$\left\{\vartheta(p(z),zp'(z),z^2p''(z);z)|z\in\mathbb{D}\right\}\subset\Omega\Rightarrow p(\mathbb{D})\subset\Delta.$$

**Definition 1.3.1.** [109] Let $\vartheta:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ and $h$ be univalent in $\mathbb{D}$. If $p$ is analytic in $\mathbb{D}$ and satisfies the second order differential subordination

$$\vartheta(p(z),zp'(z),z^2p''(z);z)\prec h(z), \tag{1.3.8}$$

then $p$ is called a solution of the differential subordination. Further, the univalent function $q$ is called a dominant of the solutions of (1.3.8), if $p\prec q$ for all $p$ satisfying (1.3.8). Furthermore, if $\tilde{q}\prec q$ for all dominants $q$ of (1.3.8), then $\tilde{q}$ is known as the best dominant of (1.3.8). Note that best dominant is unique up to a rotation of $\mathbb{D}$.

## 1.4 Synopsis of the Thesis

Geometric Function Theory is a branch of Complex Analysis that deals with geometric properties of analytic functions. Since the univalent functions, whose properties depends on certain geometrical aspects, is classified under GFT. It has deep implications both in pure mathematics and various applied disciplines. The contents of this thesis comprises of six chapters involving radius results, coefficient problems and differential subordination results, for several subclasses of analytic functions. An abstract at the beginning of every chapter gives a brief outline of the work presented in it followed by some highlights for the same. To progressively introduce the most important concepts leading up to the main results, the reported work is organized as follows:

In **chapter 2**, bounds for the second Hankel determinant of logarithmic coefficients are established for certain close-to-convex classes such as, $\mathscr{S}_s^*$ and other classes formed using expressions, $(1-z)f'(z)$, $(1-z^2)f'(z)$, $(1-z+z^2)f'(z)$ and $(1-z)^2 f'(z)$. This previously unexplored area motivated us to investigate it further, drawing inspiration from the works of Lecko [75] and Noor [122]. Additionally, motivated by the works of Ali and collaborators [5, 6], we establish bounds uptill fourth coefficient along with Fekete-Szegö bound for a newly defined class which includes starlike and convex class as special cases. Some of the contributions of this chapter are listed below:

1. Let $f \in \mathscr{S}^k_{\gamma,\delta}(\phi)$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$,

   (i) For any $\mu \in \mathbb{C}$, we have

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{M_1} \max\left\{1; \left|\frac{\gamma B_1 (2\mu M_1 - M_2)}{2(1+\delta)^2} - \frac{B_2}{B_1}\right|\right\},$$

   where

$$M_1 = 1 + k(1 + 4\alpha_1) + 2\alpha_2(1-k),$$
$$M_2 = k((3-k)(1+\alpha_1)^2 - (1-k)(1+\alpha_2)(1+2\alpha_1-\alpha_2)).$$

   (ii) For $H(q_1, q_2)$ as defined in [132, Lemma 2], where $q_1 = 2B_2/B_1, q_2 = B_3/B_1$ and $t = \gamma B_1 J_3 / 6 J_2 (1+\delta)^2 - B_2/B_1$,

$$|a_4| \leq \frac{|\gamma| B_1}{J_1} \left( H(q_1, q_2) + \frac{|\gamma J_2| B_1}{(1+\delta) M_1} \max\{1; |t|\} \right),$$

with

$$\begin{aligned}J_1 =&1+k(2+9\alpha_1)+3\alpha_2(1-k),\\ J_2 =&k\left(2\alpha_1^2(2k-5)+\alpha_1(6\alpha_2-6\alpha_2k+k-10)+\alpha_2(2\alpha_2-1)(k-1)-3\right),\\ J_3 =&-k(\alpha_1+1)((\alpha_1+1)^2(k-7)(k-2)M_1+3(2\alpha_1+1)(2k-5)M_2)\\ &+(k-1)k(k(1-\alpha_2+3\alpha_1(\alpha_1+1)+\alpha_2^2-3\alpha_1\alpha_2)+(1+\alpha_2)M_1((3\alpha_1+\alpha_2+4)\\ &\times(\alpha_2-3\alpha_1-2))+3M_2(\alpha_1(6\alpha_2+5)-2\alpha_2^2+\alpha_2+2)).\end{aligned}$$

This is a sharp result.

2. For $f\in\mathscr{A}$, let $F_f$ be given by (2.1.3), then
   (i) If $f\in\mathscr{S}_s^*$, then $|H_{2,1}(F_f)|\leq 1/4$. This bound is sharp.
   (ii) If $f\in\mathscr{F}_2$ and $a_2\geq 0$, then $|H_{2,1}(F_f)|\leq 1/4$.

In **chapter 3**, we consider a parabolic function in context of non-univalent functions, which is completely different from the way Rønning, Ma-Minda and Kanas (see [63,64,99,147]) handled parabolic regions. Certain geometric aspects and sharp radius constants for this class are established. Interestingly, the radius results are quite appealing, as the parabola given by

$$\varphi(z):=1-\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,$$

which maps $\mathbb{D}$ onto a domain that primarily lies in the left half-plane. Additionally, our findings also built a connection with several Ma-Minda type classes. Some of the contributions of this chapter are listed below:

1. Let $z\in\mathbb{D}_r$, then for each $0<r\leq 1$ and $-\pi<t\leq\pi$, we have

$$\varphi_0(r)\leq \operatorname{Re}\varphi_0(re^{it})\leq\varphi_0(-r).$$

2. If $c<3/2$ and $\zeta_{\eta_0}=\log(\sqrt{\eta_0}/\sqrt{1-\eta_0})$ with $\eta_0=(e^{-\pi\sqrt{1-2c}})/(1+e^{-\pi\sqrt{1-2c}})$, then $\varphi(\mathbb{D})$ satisfies $\mathscr{D}(c,r_c):=\{w\in\mathbb{C}:|w-c|<r_c\}\subset\Omega_\varphi:=\{w\in\mathbb{C}:|1-w|<2-\operatorname{Re}w\}$, where

$$r_c=\begin{cases}\sqrt{\left(c-\dfrac{3}{2}+\dfrac{2\zeta_{\eta_0}^2}{\pi^2}\right)^2+\dfrac{4\zeta_{\eta_0}^2}{\pi^2}}, & c\leq\dfrac{1}{2},\\ \dfrac{3}{2}-c, & \dfrac{1}{2}<c<\dfrac{3}{2}.\end{cases}$$

3. For $f\in\mathscr{A}$, the sharp $\mathscr{F}_\varphi$−radii for the classes $\mathscr{S}_p^*$, $\mathscr{S}_{\sin}^*$, $\mathscr{S}_{\mathbb{D}}^*$, $\mathscr{S}_{\sinh}^*$ and $\mathscr{S}_{\wp}^*$, are

(i) $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}^*_p) = \tanh^2(\pi/4)$,

(ii) $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}^*_{\sin}) = \pi/6$,

(iii) $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}^*_{\mathbb{D}}) = 5/12$,

(iv) $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}^*_{\sinh}) = \sinh(1/2)$,

(v) $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}^*_{\wp}) \approx 0.3517\cdots$,

respectively.

4. Let $0 \leq \alpha_0 < 1$, then sharp $\mathscr{S}^*(1+\alpha_0 z)-$ radius for the class $\mathscr{F}_\varphi$ is the unique positive root $r_{\alpha_0} = \tanh^2(\pi\sqrt{\alpha_0}/2\sqrt{2})$ of the equation $2\left(\log((1+\sqrt{r})/(1-\sqrt{r}))\right)^2 - \alpha_0\pi^2 = 0$.

In **chapter 4**, radius constants for the class of starlike functions of order $\beta$, where $0 \leq \beta < 1$, are determined, by employing a different technique. Further, we introduce a product class $\mathscr{S}_{\lambda,\beta}$ with $-1 \leq \beta \leq 1$, defined using functions lying in the tilted Carathéodory class $\mathscr{P}_\lambda$, where $\mathscr{P}_\lambda := \{p \in \mathscr{H}_1 : \mathrm{Re}(e^{i\lambda}p(z)) > 0, -\pi/2 < \lambda < \pi/2\}$. Thereby, determining sharp radius constants for functions in the class $\mathscr{S}_{\lambda,\beta}$ to lie in various subfamilies of $\mathscr{S}^*$. Some of the contributions of this chapter are listed below:

1. If $f \in \mathscr{S}^*(\beta)$, where $0 \leq \beta < 1$, and $0 < s \leq 1/\sqrt{2}$, then

(i) $\mathscr{R}_{\mathscr{S}^*_{SG}} = \dfrac{1}{1+2(1-\beta)\coth(1/2)}$.

(ii) $\mathscr{R}_{\mathscr{S}^*_{\mathbb{D}}} = \dfrac{1}{1+\sqrt{2}(1-\beta)}$.

(iii) $\mathscr{R}_{\mathscr{S}^*_L(s)} = \dfrac{s(s+2)}{2(1-\beta)+s^2+2s}$.

(iv) $\mathscr{R}_{\mathscr{S}^*_{\sin}} = \dfrac{\sin 1}{\sin 1 + 2(1-\beta)}$.

(v) $\mathscr{R}_{\mathscr{S}^*_{\sinh}} = \dfrac{\sinh^{-1} 1}{\sinh^{-1} 1 + 2(1-\beta)}$.

This result is sharp.

Based on the above result we deduce the following:

2. If $f \in \mathscr{S}^*(1/2)$, then

(i) $\mathscr{R}_{\mathscr{S}^*_{SG}} = \dfrac{1}{1+\coth(1/2)} \approx 0.316\cdots$.

(ii) $\mathscr{R}_{\mathscr{S}^*_{\mathbb{D}}} = \dfrac{\sqrt{2}}{1+\sqrt{2}} \approx 0.585\cdots$.

(iii) $\mathscr{R}_{\mathscr{S}^*_L(1/\sqrt{2})} = 4\sqrt{2}-5 \approx 0.656\cdots$.

(iv) $\mathscr{R}_{\mathscr{S}^*_{\sin}} = \dfrac{\sin 1}{1+\sin 1} \approx 0.456\cdots$.

(v) $\mathscr{R}_{\mathscr{S}^*_{\sinh}} = \dfrac{\sinh^{-1} 1}{1+\sinh^{-1} 1} \approx 0.468\cdots$.

This result is sharp.

In **chapter 5**, we investigate a new subclass of $\mathscr{A}$, denoted by $\mathscr{S}^*_\rho$,

$$\mathscr{S}^*_\rho := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \cosh\sqrt{z} =: \rho(z),\ z \in \mathbb{D} \right\},$$

involving a hyperbolic cosine function $\cosh\sqrt{z}$. We establish several key results, including certain inclusion relations that describe the relationship between $\mathcal{S}^*_\rho$ and other well-known subclasses of analytic functions. A detailed analysis of geometric properties for the class $\mathcal{S}^*_\rho$ are studied, thereby establishing sharp radius results for the class under study. The derived results not only enhance the understanding of the geometric properties of $\mathcal{S}^*_\rho$, but also contribute to the broader study of starlike and various Ma-Minda subclasses. Some of the contributions of this chapter are listed below:

1. For $\Omega_\rho := \rho(\mathbb{D}) = \cosh\sqrt{z}$, with $l_0 := \cos 1$ and $l_1 := \cosh 1$, then

   (i) $\{w \in \mathbb{C} : |w-c| < r_c\} = \Omega_\rho$, where

   $$r_c = \begin{cases} c - l_0, & l_0 < c \leq (l_1+l_0)/2 \\ l_1 - c, & (l_1+l_0)/2 \leq c < l_1. \end{cases}$$

   (ii) $\{w : |w-(l_0+l_1)/2| < (l_1-l_0)/2\} \subset \Omega_\rho$.

   (iii) $\Omega_\rho \subset \{w : |\arg w| < m\}$, where $m \approx 0.506053 \approx (0.322163)\ \pi/2 \approx 28.9947^\circ$.

   (iv) $\Omega_\rho \subset \{w : l_0 < \operatorname{Re} w < l_1\}$ and $\Omega_\rho \subset \{w : l_0 < |w| < l_1\}$.

   (v) $\Omega_\rho \subset \left\{w : |\operatorname{Im} w| < l_\rho\right\}$ and $\Omega_\rho \subset \left\{w : |w-(l_0+l_1)/2| < l_\rho\right\}$, where $l = |\operatorname{Im}(\cosh(e^{it_0/2}))|$ and $t_0$ is the solution of the equation $l_0 + l_1 - 2\cos(\sin(t/2))\cosh(\cos(t/2)) = 0$.

2. For $f \in \mathcal{S}^*_\rho$ with $l_0 := \cos 1$ and $l_1 := \cosh 1$, then following holds:

   (i) $\mathcal{S}^*_\rho \subset \mathcal{S}^*(\beta)$, where $\beta = l_0$.

   (ii) $\mathcal{S}^*_\rho \subset \mathcal{M}(\alpha)$, where $\alpha = l_1$.

   (iii) $\mathcal{S}^*_\rho \subset \mathcal{S}\mathcal{S}^*(\beta)$, where $\beta \approx 0.3222163$.

   (iv) $\mathcal{S}^*_{q_\kappa} \subset \mathcal{S}^*_\rho$, whenever $\kappa \leq 1 - l_0^2$.

   (v) $k - \mathcal{S}\mathcal{T} \subset \mathcal{S}^*_\rho$, whenever $k \geq l_1/(l_1-1)$.

   (vi) $\mathcal{S}^*_\rho \subset \mathcal{S}^*_{hpl}(\tilde{s})$, whenever $-\log l_0/\log 2 \leq \tilde{s} \leq 1$.

   (vii) $\mathcal{S}^*_\rho \subset \mathcal{S}^*_L(s)$, whenever $1 - \sqrt{l_0} \leq s \leq \frac{1}{\sqrt{2}}$.

   (viii) $\mathcal{S}^*_\rho \subset \mathcal{S}\mathcal{T}_p(\hat{\gamma})$, whenever $\hat{\gamma} \geq \hat{\gamma}_0 \approx 0.0654238$.

   This result is sharp.

3. If $f \in \mathcal{S}^*_\rho$, then

   (i) $f \in \mathcal{S}^*(\beta)$ for $|z| < r_\beta$, where $\beta \in [0,1)$ and $r_\beta < 1$ is the least positive root of the equation: $\cos\sqrt{r} = \beta$. This radius result is sharp.

(ii) $f \in \mathscr{C}(\beta)$ for $|z| \leq r_\beta$, where $\beta \in [0,1)$ and $r_\beta \in (0,1)$ is the least positive root of the equation: $2(1-r^2)\cos\sqrt{r} - \sqrt{r}\tan\sqrt{r} = \beta$.

Finally, **chapter 6** deals with certain differential subordination implication results pertaining to the class $\mathscr{S}_\rho^*$ of starlike functions introduced in chapter 5. Briot-Bouquet type differential subordination implication results are established for the class $\mathscr{S}_\rho^*$. Furthermore, results involving admissibility conditions are deduced, which provide sufficient conditions for the class under study. Listed below are some of the key contributions of this chapter:

1. Let $\eta, \gamma \in \mathbb{R}$ such that $\gamma \neq -\eta$, satisfy the following conditions:

$$\eta_2 \leq \eta \leq \eta_1 \text{ and } \eta_3 \leq \eta \leq \eta_4,$$

where

$$\eta_1 = -\frac{\gamma}{\cosh 1} + \frac{\sinh 1}{(2\cosh 1(1+\sqrt{2}-\cosh 1))}, \; \eta_2 = -\frac{\gamma}{\cosh 1},$$
$$\eta_3 = -\frac{\gamma}{\cos 1} - \frac{\sin 1}{(2\cos 1(1+\sqrt{2}-\cos 1))}, \; \eta_4 = -\frac{\gamma}{\cos 1}$$

and $p(z)$ be an analytic function with $p(0) = 1$, satisfying:

$$p(z) + \frac{zp'(z)}{\eta p(z) + \gamma} \prec z + \sqrt{1+z^2},$$

then $p(z) \prec \cosh\sqrt{z}$.
By taking $p(z) = zf'(z)/f(z)$, we deduce the following result:

2. Let $-\cos 1(1+(\tan 1)/(\sqrt{2}+1-\cos 1)) \leq 2\gamma \leq -\cos 1$, and $f \in \mathscr{A}$ satisfy

$$\frac{zf'(z)}{f(z)}\left(1 + \frac{1+2\left(\dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}\right)}{\dfrac{zf'(z)}{f(z)} + 2\gamma}\right) \prec z + \sqrt{1+z^2},$$

then $f \in \mathscr{S}_\rho^*$.

3. Suppose $A, B \in \mathbb{C}$, with $A \neq B$ and $|B| < 1$, $\eta$ be such that $|\eta| \geq 2|A-B|/(1-|B|)\tanh 1$. Let $p(z)$ be analytic in $\mathbb{D}$ with $p(0) = 1$, satisfying the subordination

$$1 + \eta\frac{zp'(z)}{p(z)} \prec \frac{1+Az}{1+Bz},$$

then $p(z) \prec \cosh\sqrt{z}$ and $\cosh\sqrt{z}$ is the best dominant. When $p(z) = zf'(z)/f(z)$, we deduce the following:

4. If $f \in \mathscr{A}$ satisfies $1 + \eta\left(1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}\right) \prec \dfrac{1+Az}{1+Bz}$, then $f \in \mathscr{S}_\rho^*$.

# Chapter 2

# Coefficient Problems for Certain Classes of Analytic Functions

---

We introduce and examine a unified class of analytic functions, providing sharp estimates for initial coefficients and Fekete-Szegö functional bounds. Further, we study more results for a class defined through convolution. Additionally, we determine bounds for second Hankel determinant of logarithmic coefficients of certain close-to-convex classes. Our findings generalize many earlier known results, which are explicitly pointed out in our study.

---

## 2.1 Introduction

Coefficient problems began to surface in GFT in 1916, when a conjecture by Bieberbach [45] was floated. Estimating coefficients bounds are significant due to its wide range of applications in engineering field such as machine learning, image and signal processing [120]. In the past some researchers made an attempt to establish bounds on coefficients pertaining to various subclasses of $\mathcal{A}$, namely Loewner [97], Nevanlinna [118] and Reade [138] proved the Bieberbach's conjecture for classes $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$, respectively. Further, the bounds for the Fekete-Szegö functional for the classes $\mathcal{K}$, $\mathcal{S}^*$ and $\mathcal{C}$ were estimated by Keogh and Merkes [68]. Although determining the sharp $n^{th}$ coefficient bounds for functions in subclass of $\mathcal{A}$ is quite challenging. So far sharp bounds for $|a_n|$ is known for classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ for $n = 2,3,4$. Infact Ma and Minda [98] handled the problem of estimating sharp Fekete-Szegö bound for the unified classes: $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$. Many authors [2, 25, 87, 123, 134] obtained sharp bound for various subclasses of $\mathcal{S}$.

Previously, several authors found coefficient bounds for functions lying in classes defined through subordination involving: $zf'(z)/f(z)$ or $f(z)/z$ or $1+zf''(z)/f'(z)$ or $f'(z)$ or their ratios or product of powers of these expressions or in terms of their weighted sum or product (see [33, 44, 57, 70, 98, 111, 113–115, 162, 174]). In light of this, below we make an attempt to unify all these analytic characterizing expressions into one, allowing various cross combinations of the above mentioned expressions, thereby clubbing together many well known classes and is given by:

$$\left(\frac{zf'(z)+\alpha_1 z^2 f''(z)}{\alpha_1 zf'(z)+(1-\alpha_1)f(z)}\right)^k \left(\alpha_2 f'(z)+(1-\alpha_2)\frac{f(z)}{z}\right)^{1-k}. \tag{2.1.1}$$

For brevity, we shall assume $F_m(z) := mzf'(z)+(1-m)f(z)$, so that the expression in (2.1.1) becomes: $\left(zF'_{\alpha_1}(z)/F_{\alpha_1}(z)\right)^k \left(F_{\alpha_2}(z)/z\right)^{1-k}$. Choose $\delta = \alpha_1 k + \alpha_2(1-k)$. Observe that $\alpha_1$ and $\alpha_2$ vanish along with $k$ and $1-k$, when they reduce to zero, respectively. Now we define a new class, which unifies several well known subclasses of $\mathcal{A}$.

**Definition 2.1.1.** Let $\gamma \in \mathbb{C}\setminus\{0\}$, then $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^k_{\gamma,\delta}(\phi)$, if:

$$1+\frac{1}{\gamma}\left(\left(\frac{zF'_{\alpha_1}(z)}{F_{\alpha_1}(z)}\right)^k \left(\frac{F_{\alpha_2}(z)}{z}\right)^{1-k} - 1\right) \prec \phi(z), \tag{2.1.2}$$

where $F_m(z) := mzf'(z)+(1-m)f(z)$, $m = \alpha_1$ or $\alpha_2$, $\delta = \alpha_1 k + \alpha_2(1-k)$, with $0 \le \alpha_1, \alpha_2, k \le 1$.

Define the class $\mathcal{S}^k_{\gamma,\delta,h}(\phi) := \{f \in \mathcal{A} : f * h \in \mathcal{S}^k_{\gamma,\delta}(\phi)\}$, where $f * h$ represents the convolution of $f,h\in\mathcal{A}$, with $f$ of the form (1.1.1) and $h(z)=z+\sum_{n=2}h_nz^n$, given by

$$(f*h)(z)=z+\sum_{n=2}^{\infty}a_nh_nz^n.$$

If $h(z)=z/(1-z)$, the class $\mathcal{S}^k_{\gamma,\delta,h}(\phi)$ reduces to $\mathcal{S}^k_{\gamma,\delta}(\phi)$. Observe that the class $\mathcal{S}^k_{\gamma,\delta}(\phi)$, reduces to several known classes for appropriate selection of $k$, $\alpha_1$, $\alpha_2$, $\gamma$ and $\phi$. We illustrate some of the important subclasses studied in the past. For instance, if $\phi(z)=(1+Az)/(1+Bz)$, where $-1\leq B<A\leq 1$ with $k=1$, we obtain the class $\mathcal{S}^1_{\gamma,\alpha_1}(A,B)=\mathcal{S}^1_{\gamma,\alpha_1}((1+Az)/(1+Bz))$. For $\alpha_1=0$ and $A=-B=1$, we get the class of starlike functions of complex order $\gamma$ $(\gamma\in\mathbb{C}\setminus\{0\})$, $\mathcal{S}^*(\gamma)\equiv\mathcal{S}^1_{\gamma,0}(1,-1)$, introduced by Nasr and Aouf [115]. Similarly, for $\phi(z)=(1+Az)/(1+Bz)$, with $A=-B=1$, and $k=\alpha_1=1$, we get the class of convex functions of complex order $\gamma\in\mathbb{C}\backslash\{0\}$, introduced by Wiatrowski [174], denoted by $\mathcal{C}(\gamma)\equiv\mathcal{S}^1_{\gamma,1}(1,-1)$. Further, if $k=0$ and $\alpha_2=1$, we get the class $\mathcal{S}^0_{\gamma,1}(\phi)\equiv\mathcal{R}_\gamma(\phi)$, containing functions that are closely related to the class of functions with positive real part. Note that $\mathcal{S}^0_{\gamma,1}(A,B)=\mathcal{R}_\gamma(A,B)$, introduced by Dixit et al. [31].

Now we discuss some special cases, when $\gamma=1$. By taking $k=1$ and $\alpha_1=0$, we obtain Ma-Minda class $\mathcal{S}^*(\phi)=\mathcal{S}^1_{1,0}(\phi)$. Further, if $\phi(z)=(1+Az)/(1+Bz)$, we obtain the class of Janowski starlike functions [98] and additionally, if $A=1-2\beta$ $(0\leq\beta<1)$ and $B=-1$, we obtain the class $\mathcal{S}^*(\beta)$. Further, if $\beta=0$ we have the class $\mathcal{S}^*=\mathcal{S}^1_{1,0}(1,-1)$ [44]. If $k=\alpha_1=1$, we obtain the Ma-Minda class $\mathcal{C}(\phi)=\mathcal{S}^1_{1,1}(\phi)$. Particularly, for $A=-B=1$, the class $\mathcal{S}^1_{1,1}(A,B)\equiv\mathcal{C}$ [44]. If $k=0$ and $\alpha_2=1$, the class $\mathcal{S}^0_{1,1}(\phi)$ coincides with $\mathcal{R}(\phi)$, a subclass of close-to-convex functions. Specifically, for $\phi(z)=(1+Az)/(1+Bz)$, the class $\mathcal{S}^0_{1,1}(A,B)=\mathcal{R}(A,B)$, studied by Goel and Mehrok [42]. MacGregor [101] systematically studied the class $\mathcal{R}=\{f\in\mathcal{A}:\operatorname{Re}f'(z)>0\}$. Note that $\mathcal{S}^0_{1,1}(1,-1)\equiv\mathcal{R}$, and $\mathcal{S}^0_{1,1}(1,0)\equiv\mathcal{R}(1,0)=\{f\in\mathcal{A}:|f'(z)-1|<1\}$ is a subclass of $\mathcal{R}$.

Several authors have studied coefficient problems in the past, such as, bounds on initial coefficients, Fekete-Szegö functional, Hankel determinant. Inspired by the works of Kowalczyk and Lecko [75], Lee et al. [91] and Noonan et al. [121], in section 2.2 we study the Fekete-Szegö and initial coefficient bounds. Apart from this, in section 2.3, we have handled the well-known Hankel determinant, defined in (1.3.7), for certain subclasses of $\mathcal{A}$. Note that, the problem of calculating $\max_{f\in\mathbb{F}}|H_{2,2}(f)|$ for various subfamilies $\mathbb{F}\subset\mathcal{A}$ was studied by Janteng et al. [57], Kowalczyk and Lecko [74] and Lee et al. [91]. Infact, Noor [122] studied Hankel determinant for close-to-convex functions. Now, for $f\in\mathcal{S}$, we define $\widetilde{F}_f(z):=2F_f(z)$, where

$$F_f(z)=\log\left(\frac{f(z)}{z}\right)=2\sum_{n=1}^{\infty}\gamma_n(f)z^n\quad(z\in\mathbb{D}\setminus\{0\},\ \log 1:=0).\tag{2.1.3}$$

The coefficients $\gamma_n := \gamma_n(f)$ in (2.1.3) are known as logarithmic coefficients of $f$. Sharp logarithmic coefficient estimates for the class $\mathcal{S}$ are already known for $n=1$ and $n=2$, given by $|\gamma_1| \leq 1$ and $|\gamma_2| \leq 1/2 + 1/e^2$. However, $|\gamma_n|$ for $n \geq 3$, is still an open problem. Logarithmic coefficients played a crucial role in Milin's conjecture [107], which states that if $f \in \mathcal{S}$, then

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left( k|\gamma_k|^2 - \frac{1}{k} \right) \leq 0.$$

For $q,n \in \mathbb{N}$ and $f \in \mathcal{A}$, $q^{th}$ Hankel determinant $H_{q,n}(F_f)$, with entries as logarithmic coefficients is defined as:

$$H_{q,n}(F_f) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

Kowalczyk et al. [75] pioneered the study of the Hankel determinant with entries defined by logarithmic coefficients. In this context, we study second Hankel determinant $H_{q,n}(F_f)$ for some special subclasses of $\mathcal{K}$. We obtain the following subclasses of close-to-convex functions, $\mathcal{F}_i's$ $(i=1,\ldots,4)$, by choosing $\phi(z)$ to be $1/(1-z), 1/(1-z^2), 1/(1-z+z^2)$ and $1/(1-z)^2$ respectively, when $\lambda = 0$ in (1.2.6):

$$\begin{aligned} \mathcal{F}_1 &:= \left\{ f \in \mathcal{A} : \operatorname{Re}(1-z)f'(z) > 0 \right\}, \\ \mathcal{F}_2 &:= \left\{ f \in \mathcal{A} : \operatorname{Re}(1-z^2)f'(z) > 0 \right\}, \\ \mathcal{F}_3 &:= \left\{ f \in \mathcal{A} : \operatorname{Re}(1-z+z^2)f'(z) > 0 \right\}, \\ \mathcal{F}_4 &:= \left\{ f \in \mathcal{A} : \operatorname{Re}(1-z)^2 f'(z) > 0 \right\}, \end{aligned} \tag{2.1.4}$$

where $z \in \mathbb{D}$. A function $f \in \mathcal{S}_s^*$ if for any $r < 1$ sufficiently close to 1, and any $\gamma$ lying on the circle $|z| = r$, the angular velocity of $f(z)$ about the point $f(-\gamma)$ is positive at $\gamma$ as $z$ traverses the circle $|z| = r$ in the positive direction, i.e. $\operatorname{Re}(2zf'(z)/(f(z) - f(-\gamma))) > 0$ for $|z| = r$ at $z = \gamma$. In 1959, Sakaguchi [149] introduced and examined the class $\mathcal{S}_s^*$, consisting of functions starlike with respect to symmetric points, characterized by

$$\operatorname{Re}\left( \frac{2zf'(z)}{f(z) - f(-z)} \right) > 0 \quad (z \in \mathbb{D}). \tag{2.1.5}$$

Observe that $\mathcal{S}_s^* \subset \mathcal{K}$. Notably, the classes defined in (2.1.4) and (2.1.5) have been extensively studied on different aspects earlier. The classes $\mathcal{F}_2$ and $\mathcal{F}_4$ have nice geometrical interpretations, as each member of $\mathcal{F}_2$ is convex in the direction of imaginary axis and every $f \in \mathcal{F}_4$ is convex in the positive direction of real axis. A variety of results have been established for these classes by several authors (see [18, 20]). The bounds of $\gamma_n$ for functions in $\mathcal{K}$ were examined in [167, p. 116], [166]. In 2018, Kumar and Vasudevrao [85] obtained bounds on early logarithmic coefficients for the subclasses $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of $\mathcal{K}$. Further, Kumar and Kumar [80] examined bounds on the third order Hermitian-Toeplitz determinants for the class $\mathcal{S}_s^*$. Recently, many authors derived bounds for various coefficient functionals for certain subclasses of close-to-convex functions, see [27, 76, 77, 80]. For more work in this direction refer to [8, 27, 41, 76, 77, 80, 89, 124].

Motivated by the above works, we chiefly study the coefficient related problems. Section 2.2, deals with estimation of sharp bounds of initial coefficients and Fekete-Szegö functional for the class $\mathcal{S}^k_{\gamma,\delta}(\phi)$ and section 2.3 focuses on deriving bounds on $|H_{2,1}(F_f)|$, for $f(z)$ lying in classes: $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ and $\mathcal{S}_s^*$. Several special cases of our results are also pointed out.

## 2.2 Estimation of Initial Coefficient Bounds

In this section, sharp bounds on initial coefficients for functions in the class $\mathcal{S}^k_{\gamma,\delta}(\phi)$ are determined. Further, sharp bound on the Fekete-Szegö functional $|a_3 - \mu a_2^2|$, for the class $\mathcal{S}^k_{\gamma,\delta}(\phi)$ is derived and various special cases are also pointed out. To begin with, consider functions of the form $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n$ such that $|\omega(z)| < 1$ for each $z \in \mathbb{D}$. Let $\mathcal{B}$ denote the collection of all such functions, referred to as the class of bounded analytic functions. We now present a few lemmas that will be essential for proving the results in this section.

**Lemma 2.2.1.** [6] If $\omega \in \mathcal{B}$, then

$$|\omega_2 - t\omega_1^2| \leq \begin{cases} -t, & \text{if } t \leq -1, \\ 1, & \text{if } -1 \leq t \leq 1, \\ t, & \text{if } t \geq 1. \end{cases} \tag{2.2.6}$$

When $t < -1$ or $t > 1$, equality holds if and only if $\omega(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $\omega(z) = z^2$ or one of its rotations. When $t = -1$ then equality holds if and only if $\omega(z) = z(\lambda + z)/(1 + \lambda z), (0 \leq \lambda \leq 1)$ or one of its rotations. For $t = 1$, equality holds if and only if $\omega(z) = -z(\lambda + z)/(1 + \lambda z)(0 \leq \lambda \leq 1)$ or one of its rotations. Also the above sharp upper bound can be improved as follows, when $-1 < t < 1$ :

$$|\omega_2 - t\omega_1^2| + (1+t)|\omega_1|^2 \leq 1 \ (-1 < t \leq 0),$$
$$|\omega_2 - t\omega_1^2| + (1-t)|\omega_1|^2 \leq 1 \ (0 < t \leq 1).$$

**Lemma 2.2.2.** [68] If $\omega \in \mathcal{B}$, then for any complex number t,

$$|\omega_2 - t\omega_1^2| \leq \max\{1; |t|\}.$$

This result is sharp for the functions $\omega(z)=z^2$ or $\omega(z)=z$.

### Coefficient Estimates for the Class $\mathscr{S}^k_{\gamma,\delta}(\phi)$

Using Lemma 2.2.1 and [132, Lemma 2], we establish the following bounds for functions in the class $\mathscr{S}^k_{\gamma,\delta}(\phi)$.

**Theorem 2.2.1.** *Let $f$ be in the class $\mathscr{S}^k_{\gamma,\delta}(\phi)$ and $\phi(z)=1+B_1z+B_2z^2+\cdots$. Then, for any $\mu\in\mathbb{C}$, we have*

$$|a_3-\mu a_2^2|\leq\frac{|\gamma|B_1}{M_1}\max\left\{1;\left|\frac{\gamma B_1(2\mu M_1-M_2)}{2(1+\delta)^2}-\frac{B_2}{B_1}\right|\right\},\tag{2.2.7}$$

*where*

$$M_1=1+k(1+4\alpha_1)+2\alpha_2(1-k)\tag{2.2.8}$$

*and*

$$M_2=k((3-k)(1+\alpha_1)^2-(1-k)(1+\alpha_2)(1+2\alpha_1-\alpha_2)).\tag{2.2.9}$$

*Further,*

$$|a_2|\leq\frac{|\gamma|B_1}{1+\delta}\quad\textit{and}\quad|a_3|\leq\frac{|\gamma|B_1}{M_1}\max\left\{1;\left|\frac{B_2}{B_1}+\frac{\gamma B_1M_2}{2(1+\delta)^2}\right|\right\}.$$

*These estimates are sharp.*

*Proof.* Since $f$ is in $S^k_{\gamma,\delta}(\phi)$, then there exists a Schwarz function $\omega\in\mathscr{B}$, such that

$$1+\frac{1}{\gamma}\left(\left(\frac{zF'_{\alpha_1}(z)}{F_{\alpha_1}(z)}\right)^k\left(\frac{F_{\alpha_2}(z)}{z}\right)^{1-k}-1\right)=\phi(\omega(z)).\tag{2.2.10}$$

Upon expressing $F_{\alpha_1}, F_{\alpha_2}$ in terms of $f$, and using power series expansion of $f$, we obtain:

$$\left(\frac{zF'_{\alpha_1}(z)}{F_{\alpha_1}(z)}\right)^k\left(\frac{F_{\alpha_2}(z)}{z}\right)^{1-k}=\left(\frac{zf'(z)+\alpha_1z^2f''(z)}{\alpha_1zf'(z)+(1-\alpha_1)f(z)}\right)^k\left(\alpha_2f'(z)+(1-\alpha_2)\frac{f(z)}{z}\right)^{1-k}\tag{2.2.11}$$

$$=1+(1+\delta)a_2z+\frac{1}{2}(a_2^2M_2+2a_3M_1)z^2+\cdots.$$

Also,

$$\phi(\omega(z))=1+B_1\omega_1z+(B_2\omega_1^2+B_1\omega_2)z^2+\cdots.$$

Therefore, using (2.2.10), we obtain the coefficients $a_2$ and $a_3$ as follows:

$$a_2=\frac{\gamma B_1\omega_1}{1+\delta}\quad\text{and}\quad a_3=\gamma\left(\frac{B_2}{M_1}+\frac{\gamma B_1^2M_2}{2(1+\delta)^2M_1}\right)\omega_1^2+\frac{\gamma B_1}{M_1}\omega_2.$$

On substituting these values in the Fekete-Szegö coefficient functional, it reduces to:

$$|a_3-\mu a_2^2| \le \frac{|\gamma|B_1}{M_1}\left|\omega_2-\left(\frac{\gamma B_1(2\mu M_1-M_2)}{2(1+\delta)^2}-\frac{B_2}{B_1}\right)\omega_1^2\right|. \tag{2.2.12}$$

Now, the desired result follows at once by applying Lemma 2.2.2. Sharp bounds for the first two coefficients $a_2$ and $a_3$ can be obtained directly from inequality (2.2.7). Following functions play the role of extremal functions:

$$f_1(z) = z+\frac{\gamma B_1}{1+\delta}z^2+\frac{\gamma B_1}{M_1}\left(\frac{B_2}{B_1}+\frac{\gamma M_2 B_1}{2(1+\delta)^2}\right)z^3,$$
$$f_2(z) = z+\frac{\gamma B_1}{M_1}z^3,$$

and can be obtained by substituting $\omega(z)=z$ and $z^2$, respectively, in Eq. (2.2.10). Equality in (2.2.12) and for bound on $|a_2|$, occurs when $f(z)=f_1(z)$. Further, a function $g(z)$ given by

$$g(z)=\begin{cases} f_1(z), & \text{if } |2(1+\delta)^2B_2+\gamma B_1^2M_2| > 2B_1(1+\delta)^2, \\ f_2(z), & \text{if } |2(1+\delta)^2B_2+\gamma B_1^2M_2| \le 2B_1(1+\delta)^2, \end{cases}$$

serves as the extremal for the bound on $|a_3|$. □

*Remark* 2.2.1. By taking $k=0$ and $\alpha_2=1$, in the above theorem, inequality in (2.2.7) reduces to an inequality given in [6, Theorem 3(for $p=1$)]. Further, with $\phi(z)=(1+Az)/(1+Bz)$, Theorem 2.2.1 reduces to [31, Theorem 4]. Also note that with $\gamma=k=1$ and $\alpha_1=0$, inequality (2.2.7) reduces to give the inequality in [6, Theorem 1 (for $p=1$)].

It is presumed that, $M_1$ and $M_2$ carry their expressions as stated in Eqs. (2.2.8) and (2.2.9), respectively. By choosing suitable values of $\alpha_1,\alpha_2$ and $k$, in Theorem 2.2.1 we obtain the following corollary for the class:

**Corollary 2.2.1.** Let $f$ belongs to the class $\mathscr{S}^1_{\gamma,\alpha_1}(\phi)$, then for $\mu\in\mathbb{C}$,

(i) If $k=1$, then $|a_3-\mu a_2^2| \le \dfrac{|\gamma|B_1}{2(1+2\alpha_1)}\max\left\{1;\left|\gamma B_1\left(\dfrac{2\mu(1+2\alpha_1)}{(1+\alpha_1)^2}-1\right)-\dfrac{B_2}{B_1}\right|\right\}$.

(ii) If $k=0$, then $|a_3-\mu a_2^2| \le \dfrac{|\gamma|B_1}{(1+2\alpha_2)}\max\left\{1;\left|\dfrac{\mu\gamma B_1(1+2\alpha_2)}{(1+\alpha_2)^2}-\dfrac{B_2}{B_1}\right|\right\}$.

Additionally, for $\gamma=1$ and $\alpha_1=1$, Theorem 2.2.1 reduces to give sharp bound for the class $\mathscr{S}^*(\phi)$.

This result is sharp.

*Remark* 2.2.2. For any $\mu\in\mathbb{R}$, if $\gamma=\alpha_1=1$, Corollary 2.2.1(ii) coincides to give a result derived by Ma and Minda in [98, Theorem 3]. Further by choosing $\phi(z)=(1+z)/(1-z)$, we have $|a_3-\mu a_2^2|\le\max\{1/3,|\mu-1|\}$, obtained for any $\mu\in\mathbb{C}$, a sharp estimate derived by Keogh et al. [68, Corollary 1]. The results obtained by Ali et al. [6, Theorem 7 and 8, pg 44, pg 45 (for $p = 1$ with $\alpha_1 = 1$ and $\alpha_1 = 0$, respectively)], were in fact not sharp. The corrected version of the sharp result has been illustrated in Corollary 2.2.1.

**Theorem 2.2.2.** *Let $f$ be in the class $\mathscr{S}^k_{\gamma,\delta}(\phi)$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. Then, we have*

$$|a_4| \leq \frac{|\gamma| B_1}{J_1}\left(H(q_1,q_2) + \frac{|\gamma J_2| B_1}{(1+\delta)M_1}\max\{1;|t|\}\right),$$

*where $H(q_1,q_2)$ is as defined in [132, Lemma 2], with*

$$q_1 = \frac{2B_2}{B_1}\ ,\ q_2 = \frac{B_3}{B_1} \quad \text{and} \quad t = \frac{\gamma B_1 J_3}{6J_2(1+\delta)^2} - \frac{B_2}{B_1},$$

*where*

$$\begin{aligned}
J_1 =& 1 + k(2+9\alpha_1) + 3\alpha_2(1-k),\\
J_2 =& k\left(2\alpha_1^2(2k-5) + \alpha_1(6\alpha_2 - 6\alpha_2 k + k - 10) + \alpha_2(2\alpha_2 - 1)(k-1) - 3\right),\\
J_3 =& -k(\alpha_1+1)((\alpha_1+1)^2(k-7)(k-2)M_1 + 3(2\alpha_1+1)(2k-5)M_2)\\
&+ (k-1)k(k(1-\alpha_2+3\alpha_1(\alpha_1+1)+\alpha_2^2-3\alpha_1\alpha_2) + (1+\alpha_2)M_1((3\alpha_1+\alpha_2+4)\\
&\times(\alpha_2-3\alpha_1-2)) + 3M_2(\alpha_1(6\alpha_2+5) - 2\alpha_2^2 + \alpha_2 + 2)).
\end{aligned}$$

*Proof.* Using equation (2.2.11), the fourth coefficient is given by

$$a_4 = \frac{\gamma B_1}{J_1}\left(\left(\omega_3 + \frac{2B_2}{B_1}\omega_1\omega_2 + \frac{B_3}{B_1}\omega_1^3\right) - \frac{\gamma B_1 J_2 \omega_1}{(1+\delta)M_1}(\omega_2 - \nu\omega_1^2)\right).$$

Now by applying Lemma 2.2.2 to the above expression together with [132, Lemma 2], bound on the fourth coefficient can be established. □

*Remark* 2.2.3. For $k = 0$ and $\beta_2 = 1$, inequality in Theorem 2.2.2 reduces to give the inequality in [6, Theorem 3 (for $p = 1$)].

We now derive the following result for functions in the class $\mathscr{S}^k_{1,\delta}(\phi) = \mathscr{S}^k_{\delta}(\phi)$.

**Theorem 2.2.3.** *If $f$ be in the class $\mathscr{S}^k_{\delta}(\phi)$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{-B_1 t}{M_1}, & \text{when } \mu \leq \sigma_1,\\[2ex] \dfrac{B_1}{M_1}, & \text{when } \sigma_1 \leq \mu \leq \sigma_2,\\[2ex] \dfrac{B_1 t}{M_1}, & \text{when } \mu \geq \sigma_2,\end{cases} \tag{2.2.13}$$

*where*

$$t = \frac{B_1(2\mu M_1 - M_2)}{2(1+\delta)^2} - \frac{B_2}{B_1}.$$

*Further if $\sigma_1 \leq \mu \leq \sigma_3$, then*

$$|a_3 - \mu a_2^2| + \frac{(1+\delta)^2}{B_1 M_1}\left\{1 - \frac{B_2}{B_1} + \frac{B_1(2\mu M_1 - M_2)}{2(1+\delta)^2}\right\}|a_2|^2 \leq \frac{B_1}{M_1}.$$

*If $\sigma_3 \leq \mu \leq \sigma_2$, then*

$$|a_3 - \mu a_2^2| + \frac{(1+\delta)^2}{B_1 M_1}\left\{1 - \frac{B_2}{B_1} - \frac{B_1(2\mu M_1 - M_2)}{2(1+\delta)^2}\right\}|a_2|^2 \leq \frac{B_1}{M_1}. \tag{2.2.14}$$

*where*

$$\sigma_1 = \frac{(1+\delta)^2}{B_1 M_1}\left(\frac{B_2}{B_1} - 1\right) + \frac{M_2}{2M_1}, \quad \sigma_2 = \frac{(1+\delta)^2}{B_1 M_1}\left(\frac{B_2}{B_1} + 1\right) + \frac{M_2}{2M_1}, \quad \sigma_3 = \frac{M_2}{2M_1} + \frac{(1+\delta)^2 B_2}{B_1^2 M_1}.$$

*Further,*

$$|a_2| \leq \frac{B_1}{1+\delta}$$

*and*

$$|a_3| \leq \begin{cases} \dfrac{B_2}{M_1} + \dfrac{B_1^2 M_2}{2M_1(1+\delta)^2}, & \text{when } 2(B_1 - B_2)(1+\delta)^2 \leq B_1^2 M_2, \\[2ex] \dfrac{B_1}{M_1}, & \text{when } 2(B_1 - B_2)(1+\delta)^2 \geq B_1^2 M_2 \text{ or} \\ & \quad -2(B_1 + B_2)(1+\delta)^2 \leq B_1^2 M_2, \\[2ex] -\dfrac{B_2}{M_1} - \dfrac{B_1^2 M_2}{2M_1(1+\delta)^2}, & \text{when } -2(B_1 + B_2)(1+\delta)^2 \geq B_1^2 M_2. \end{cases}$$

*These estimates are sharp.*

*Proof.* Proceeding as in Theorem 2.2.1, the bound in (2.2.13)-(2.2.16) can be established by applying Lemma 2.2.1. For sharpness, we define the functions $K_{\phi_n} : \mathbb{D} \to \mathbb{C}$ $(n = 2, 3, \ldots)$, satisfying:

$$1 + \frac{1}{\gamma}\left(\left(\frac{zK'_{\phi_n}(z) + \alpha_1 z^2 K''_{\phi_n}(z)}{\alpha_1 z K'_{\phi_n}(z) + (1-\alpha_1)K_{\phi_n}(z)}\right)^k \left(\alpha_2 K'_{\phi_n}(z) + (1-\alpha_2)\frac{K_{\phi_n}(z)}{z}\right)^{1-k} - 1\right) = \phi(z^{n-1}).$$

with $K_{\phi_n}(0) = 0, K'_{\phi_n}(0) = 1$, $H_\lambda$ and $G_\lambda$ $(0 \leq \lambda \leq 1)$ with $H_\lambda(0) = 0, H'_\lambda(0) = 1$ and $G_\lambda(0) = 0$, $G'_\lambda(0) = 1$, respectively, satisfying the following:

$$1 + \frac{1}{\gamma}\left(\left(\frac{zH'_\lambda(z) + \alpha_1 z^2 H''_\lambda(z)}{\alpha_1 z H'_\lambda(z) + (1-\alpha_1)H_\lambda(z)}\right)^k \left(\alpha_2 H'_\lambda(z) + (1-\alpha_2)\frac{H_\lambda(z)}{z}\right)^{1-k} - 1\right) = \phi\left(\frac{z(\lambda + z)}{1 + \lambda z}\right)$$

and

$$1+\frac{1}{\gamma}\left(\left(\frac{zG'_\lambda(z)+\alpha_1 z^2 G''_\lambda(z)}{\alpha_1 zG'_\lambda(z)+(1-\alpha_1)G_\lambda(z)}\right)^k \left(\alpha_2 G'_\lambda(z)+(1-\alpha_2)\frac{G_\lambda(z)}{z}\right)^{1-k}-1\right)=\phi\left(\frac{-z(\lambda+z)}{1+\lambda z}\right).$$

Clearly, functions $K_{\phi_n}$, $H_\lambda$, $G_\lambda \in \mathscr{S}_\delta^k(\phi)$. For $\mu < \sigma_1$ or $\mu > \sigma_2$ extremal function for inequality (2.2.13) is $K_\phi = K_{\phi_2}$ or one of its rotations. Extremal function for $\sigma_1 < \mu < \sigma_2$ is $K_{\phi_3}$ or any of its rotations. When $\mu = \sigma_1$, $H_\lambda$ or any of its rotations works as the extremal function. For $\mu = \sigma_2$, extremal function is $G_\lambda$ or any of its rotations. Bounds for $a_2$ and $a_3$ can be directly obtained from inequality (2.2.13). □

Next result can be derived by applying Lemma 2.2.2.

**Theorem 2.2.4.** *Let $f$ be in the class $\mathscr{S}_\delta^k(\phi)$ and $\phi(z) = 1+B_1z+B_2z^2+\cdots$. Then for any $\mu \in \mathbb{C}$, we have*

$$|a_3-\mu a_2^2| \leq \frac{B_1}{M_1}\max\left\{1;\left|\frac{B_1(2\mu M_1-M_2)}{2(1+\delta)^2}-\frac{B_2}{B_1}\right|\right\}.$$

*The result is sharp.*

*Remark* 2.2.4. Let $\mu$ be a real number. When $k = \alpha_1 = 1$, Theorem 2.2.3 reduces to [98, Theorem 3]. When $\alpha_2 = 1$ and $\alpha_1 = 0$, Theorem 2.2.3 coincides with [70, Theorem 2.11]. For $\alpha_1 = \alpha_2 = 1$, Theorem 2.2.3 reduces to a result in [70, Theorem 2.15]. If $\alpha_1 = \alpha_2 = 0$ and $\alpha_1 = 1$, $\alpha_2 = 0$, Theorem 2.2.3 reduces to [70, Theorem 2.19] and [70, Theorem 2.23] respectively. Further, all the special cases referred therein also become particular cases of our result.

**Theorem 2.2.5.** *Let $f$ be in the class $\mathscr{S}_\delta^k(\phi)$ and $\phi(z) = 1+B_1z+B_2z^2+\cdots$. Then, we have*

$$|a_4| \leq \frac{B_1}{J_1}H(q_1,q_2), \tag{2.2.15}$$

*where $H(q_1,q_2)$ is as defined in [132, Lemma 2], with*

$$q_1 = \frac{2B_2}{B_1}-\frac{B_1J_2}{(1+\delta)M_1} \quad and \quad q_2 = \frac{B_3}{B_1}-\frac{B_2J_2}{M_1(1+\delta)}+\frac{B_1^2J_3}{6M_1(1+\delta)^3},$$

*where $J_1$, $J_2$ and $J_3$ are as defined in Theorem 2.2.2. This result is sharp.*

*Proof.* Using (2.2.11) with suitable rearrangement of terms, we obtain the following expression for the fourth coefficient:

$$a_4 = \frac{B_1}{J_1}\left(\omega_3+\left(\frac{2B_2}{B_1}-\frac{B_1J_2}{(1+\delta)M_1}\right)\omega_1\omega_2+\left(\frac{B_3}{B_1}-\frac{B_2J_2}{M_1(1+\delta)}+\frac{B_1^2J_3}{6M_1(1+\delta)^3}\right)\omega_1^3\right).$$

Now by an application of [132, Lemma 2], we arrive at the bound of the fourth coefficient as stated above. □

*Remark* 2.2.5. When $k = 1$ and $\alpha_1 = 0$, inequality in (2.2.15) reduces to give the inequality [6, Theorem 1(for $p = 1$)]. For $k = 0$ and $\alpha_2 = 1$, inequality (2.2.15) of Theorem 2.2.10 reduces to give the inequality [6, Theorem 3(for $p = 1$)]. Infact, if $k = \alpha_1 = 1$, we obtain a sharp bound on $a_4$ for functions in the class $\mathscr{S}^1_{1,1}(\phi)$, given by:

$$|a_4| \leq \frac{B_1}{12} H(q_1, q_2), \text{ where } \ q_1 = \frac{4B_2 + 3B_1^2}{2B_1}, \ q_2 = \frac{2B_3 + 3B_2B_1 + B_1^3}{2B_1}.$$

Further, if $\phi(z) = (1+z)/(1-z)$, then $|a_4| \leq 1$ [44], which is a sharp estimate for the class $\mathscr{C}$.

### Coefficient Estimates for the Class $\mathscr{S}^k_{\gamma,\delta,h}(\phi)$

Proceeding as in the previous results we now state the following results for the class $\mathscr{S}^k_{\gamma,\delta,h}(\phi)$ without proofs.

**Theorem 2.2.6.** *Let $f$ be in the class $\mathscr{S}^k_{\gamma,\delta,h}(\phi)$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. Then, for any $\mu \in \mathbb{C}$, we have*

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{h_3 M_1} \max\left\{1; \left| \frac{\gamma B_1 (2\mu h_3 M_1 - h_2^2 M_2)}{2h_2^2 (1+\delta)^2} - \frac{B_2}{B_1} \right| \right\}.$$

*Further,*

$$|a_2| \leq \frac{|\gamma| B_1}{h_2(1+\delta)} \quad \text{and} \quad |a_3| \leq \frac{|\gamma| B_1}{h_3 M_1} \max\left\{1; \left| \frac{B_2}{B_1} + \frac{\gamma h_2^2 B_1 M_2}{2h_2^2(1+\delta)^2} \right| \right\}.$$

*These estimates are sharp.*

**Theorem 2.2.7.** *Let $f$ be in the class $\mathscr{S}^k_{\gamma,\delta,h}(\phi)$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. Then, we have*

$$|a_4| \leq \frac{|\gamma| B_1}{h_4 J_1} \left( H(q_1, q_2) + \frac{|\gamma J_2| B_1}{M_1(1+\delta)} \max\{1; |t|\} \right),$$

*where $q_1, q_2, H(q_1, q_1), J_1$ and $J_1$ are as defined in Theorem 2.2.2.*

We state the following results, with the assumption that $\mathscr{S}^k_{1,\delta,h}(\phi) =: \mathscr{S}^k_{\delta,h}(\phi)$.

**Theorem 2.2.8.** *Let $f$ be in the class $\mathscr{S}^k_{\delta,h}(\phi)$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{-B_1 t}{h_3 M_1}, & \text{when } \mu \leq \sigma_1, \\[2ex] \dfrac{B_1}{h_3 M_1}, & \text{when } \sigma_1 \leq \mu \leq \sigma_2, \\[2ex] \dfrac{B_1 t}{h_3 M_1}, & \text{when } \mu \geq \sigma_2, \end{cases}$$

*where*

$$t = \frac{B_1(2\mu h_3 M_1 - h_2^2 M_2)}{2h_2^2(1+\delta)^2} - \frac{B_2}{B_1}.$$

*Further, if* $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$|a_3 - \mu a_2^2| + \frac{(1+\delta)^2 h_2^2}{h_3 B_1 M_1}\left\{1 - \frac{B_2}{B_1} + \frac{B_1(2\mu h_3 M_1 - h_2^2 M_2)}{2h_2^2(1+\delta)^2}\right\}|a_2|^2 \leq \frac{B_1}{h_3 M_1}.$$

*If* $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$|a_3 - \mu a_2^2| + \frac{(1+\delta)^2 h_2^2}{h_3 B_1 M_1}\left\{1 + \frac{B_2}{B_1} - \frac{B_1(2\mu h_3 M_1 - h_2^2 M_2)}{2(h_2^2(1+\delta)^2}\right\}|a_2|^2 \leq \frac{B_1}{h_3 M_1}. \tag{2.2.16}$$

*where*

$$\sigma_1 = \frac{h_2^2(1+\delta)^2}{h_3 B_1 M_1}\left(\frac{B_2}{B_1} - 1\right) + \frac{h_2^2 M_2}{2h_3 M_1}, \quad \sigma_2 = \frac{h_2^2(1+\delta)^2}{h_3 B_1 M_1}\left(\frac{B_2}{B_1} + 1\right) + \frac{M_2}{2M_1} \tag{2.2.17}$$

*and*

$$\sigma_3 = \frac{h_2^2 M_2}{2h_3 M_1} + \frac{h_2^2(1+\delta)^2 B_2}{h_3 B_1^2 M_1}. \tag{2.2.18}$$

*Further,*

$$|a_2| \leq \frac{B_1}{h_2(1+\delta)}$$

*and*

$$|a_3| \leq \begin{cases} \dfrac{B_2}{h_3 M_1} + \dfrac{h_2^2 B_1^2 M_2}{2h_2^2 h_3(1+\delta)^2 M_1}, & \text{when } 2h_2^2(B_1 - B_2)(1+\delta)^2 \leq h_2^2 B_1^2 M_2, \\[2ex] \dfrac{B_1}{h_3 M_1}, & \text{when } 2h_2^2(B_1 - B_2)(1+\delta)^2 \geq B_1^2 M_2 \text{ or} \\ & \quad -2h_2^2(B_1 + B_2)(1+\delta)^2 \leq B_1^2 M_2, \\[2ex] -\dfrac{B_2}{h_3 M_1} - \dfrac{h_2^2 B_1^2 M_2}{2h_2^2 h_3 M_1(1+\delta)^2}, & \text{when } -2h_2^2(B_1 + B_2)(1+\delta)^2 \geq h_2^2 B_1^2 M_2. \end{cases}$$

**Theorem 2.2.9.** *Let* $f$ *be in the class* $\mathscr{S}_{\delta,h}^k(\phi)$ *and* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. *Then, for* $\mu \in \mathbb{C}$, *we have*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{h_3 M_1}\max\left\{1; \left|\frac{B_1(2\mu h_3 M_1 - h_2^2 M_2)}{2h_2^2(1+\delta)^2} - \frac{B_2}{B_1}\right|\right\}.$$

*The result is sharp.*

**Theorem 2.2.10.** *Let* $f$ *be in the class* $\mathscr{S}_{\delta,h}^k(\phi)$ *and* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. *Then, we have*

$$|a_4| \leq \frac{B_1}{h_4 J_1} H(q_1, q_2), \tag{2.2.19}$$

*where $q_1$, $q_2$, $J_1$ and $H(q_1,q_2)$ are as defined in Theorem 2.2.10. This result is sharp.*

## 2.3 Bounds on Hankel Determinant

In this section, we made an attempt to study the problem of establishing bound on $|H_{2,1}(F_f)| = |\gamma_1\gamma_3 - \gamma_2^2|$, where $H_{2,1}(F_f)$ has a striking resemblance to $H_{2,1}(f) = a_2a_4 - a_3^2$, for $f$ lying in specific subclass of $\mathcal{A}$. Further, from (2.1.3) logarithmic coefficients of $f \in \mathcal{S}$, can be expressed as

$$\gamma_1 = \frac{1}{2}a_2, \quad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \quad \gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right).$$

Then for $f \in \mathcal{S}$, the expression $H_{2,1}(F_f)$, becomes

$$H_{2,1}(F_f) := \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right).$$

Note that the functional $|H_{2,1}(F_f)|$ is rotationally invariant. However, the classes $\mathcal{F}_i's$ $(i = 1,\ldots,4)$ are not rotationally invariant. Listed below are some Lemmas that serve as a prerequisite for deriving our subsequent results in this section.

**Lemma 2.3.1.** [92, 93, 127] If $p \in \mathcal{P}$ of the form $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \ldots$, with $c_1 \geq 0$, then

$$c_1 = 2\zeta_1, \tag{2.3.20}$$

$$c_2 = 2\zeta_1^2 + 2(1-\zeta_1^2)\zeta_2, \tag{2.3.21}$$

$$c_3 = 2\zeta_1^3 + 4(1-\zeta_1^2)\zeta_1\zeta_2 - 2(1-\zeta_1^2)\zeta_1\zeta_2^2 + 2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3, \tag{2.3.22}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| < 1\}$. For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, there is a unique function $p \in \mathcal{P}$ with $c_1$ and $c_2$ as in (2.3.20)-(2.3.22), namely

$$p(z) = \frac{1 + (\overline{\zeta_1}\zeta_2 + \zeta_1)z + \zeta_2z^2}{1 + (\overline{\zeta_1}\zeta_2 - \zeta_1)z - \zeta_2z^2} \quad (z \in \mathbb{D}). \tag{2.3.23}$$

For $\zeta_1, \zeta_2 \in \mathbb{D}$ and $\zeta_3 \in \mathbb{T}$, there exists a unique $p \in \mathcal{P}$ with $c_1, c_2$ and $c_3$ as in (2.3.20)-(2.3.22), namely,

$$\frac{1 + (\overline{\zeta_1}\zeta_2 + \overline{\zeta_2}\zeta_3 + \zeta_1)z + (\overline{\zeta_1}\zeta_3 + \zeta_1\overline{\zeta_2}\zeta_3 + \zeta_2)z^2 + \zeta_3z^3}{1 + (\overline{\zeta_1}\zeta_2 + \overline{\zeta_2}\zeta_3 - \zeta_1)z + (\overline{\zeta_1}\zeta_3 - \zeta_1\overline{\zeta_2}\zeta_3 - \zeta_2)z^2 - \zeta_3z^3} \quad (z \in \mathbb{D}). \tag{2.3.24}$$

**Lemma 2.3.2.** [26] For $A_1, A_2, A_3 \in \mathbb{R}$, let

$$\mathcal{Y}(A_1,A_2,A_3) := \max\left\{|A_1 + A_2z + A_3z^2| + 1 - |z|^2 : z \in \overline{\mathbb{D}}\right\}.$$

1. If $A_1A_3 \geq 0$, then

$$\mathscr{Y}(A_1,A_2,A_3) = \begin{cases} |A_1|+|A_2|+|A_3|, & |A_2| \geq 2(1-|A_3|), \\ 1+|A_1|+\dfrac{A_2^2}{4(1-|A_3|)}, & |A_2| < 2(1-|A_3|). \end{cases}$$

2. If $A_1A_3 < 0$, then

$$\mathscr{Y}(A_1,A_2,A_3) = \begin{cases} 1-|A_1|+\dfrac{A_2^2}{4(1-|A_3|)}, & -4A_1A_3(A_3^{-2}-1) \leq A_2^2 \wedge |A_2| < 2(1-|A_3|), \\ 1+|A_1|+\dfrac{A_2^2}{4(1+|A_3|)}, & A_2^2 < \min\{4(1+|A_3|)^2, -4A_1A_3(A_3^{-2}-1)\}, \\ \mathscr{R}(A_1,A_2,A_3), & \text{otherwise}, \end{cases}$$

where

$$\mathscr{R}(A_1,A_2,A_3) = \begin{cases} |A_1|+|A_2|-|A_3|, & |A_3|(|A_2|+4|A_1|) \leq |A_1A_2|, \\ -|A_1|+|A_2|+|A_3|, & |A_1A_2| \leq |A_3|(|A_2|-4|A_1|), \\ (|A_3|+|A_1|)\sqrt{1-\dfrac{A_2^2}{4A_1A_3}}, & \text{otherwise}. \end{cases}$$

### Hankel Determinant of Logarithmic Coefficients

The results stated by Cho et al. [20] highlight the importance of functions given in (2.3.23) and (2.3.24). Based on the definition of each class in (2.1.4) and (2.1.5), functions in these classes can be represented in terms of Carathéodory class $\mathscr{P}$. For the purpose of computing the bounds for $H_{2,1}(F_f)$, we use the coefficient formula given in (2.3.20) for $c_1$ given in [19], for $c_2$ [127, pg. 166] and the formula for coefficient $c_3$ due to Libera and Złotkiewicz [92, 93]. We begin by determining sharp bound of $H_{2,1}(F_f)$ for $f \in \mathscr{S}_s^*$.

**Theorem 2.3.1.** *Let $F_f$ be given by (2.1.3). If $f \in \mathscr{S}_s^*$, then sharp bound on Hankel determinant for $F_f$ is given by*

$$|H_{2,1}(F_f)| \leq \frac{1}{4}. \tag{2.3.25}$$

*Above inequality is sharp due to the function*

$$\tilde{f}(z) = \int_0^z (1+t^2)/(1-t^2)^2 dt. \tag{2.3.26}$$

*Proof.* Suppose $f \in \mathscr{A}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots$ satisfy

$$\frac{2zf'(z)}{f(z)-f(-z)} = p(z), \tag{2.3.27}$$

where $p \in \mathscr{P}$. It is a well-known fact that, class $\mathscr{P}$ is invariant under rotation, we assume $c_1 \in [0,2]$. From equation (2.3.27) we express coefficients of $f(z)$, $a_i's$ $(i = 2,3,4)$ in terms of $c_i's$ $(i = 1,2,3)$ as follows

$$a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{2}c_2, \quad a_4 = \frac{1}{8}(c_1c_2 - 2c_3).$$

Lemma 2.3.1 yields the following expression in terms of $\zeta_i$ $'s$, where $\zeta_i \in \overline{\mathbb{D}}$ $(i = 1,2,3)$.

$$\begin{aligned}\mathscr{L} := \gamma_1\gamma_3 - \gamma_2^2 &= \frac{1}{4}\left(a_4a_2 - a_3^2 + \frac{1}{12}a_2^4\right)\\ &= \frac{1}{768}(c_1^4 + 12c_2c_1^2 - 24c_3c_1 - 48c_2^2)\\ &= \frac{1}{48}(6\zeta_2^2(5\zeta_1^2 - 3\zeta_1^4 - 2) - 11\zeta_1^4 - 6\zeta_1(1-\zeta_1^2)\zeta_3(1-|\zeta_2|^2)\\ &\quad - 30\zeta_2\zeta_1^2(1-\zeta_1^2)).\end{aligned} \tag{2.3.28}$$

As $|\zeta_2| \leq 1$, then the expression in (2.3.28) leads to,

$$|\mathscr{L}| \leq \begin{cases} 1/4, & \zeta_1 = 0, \\ 11/48, & \zeta_1 = 1. \end{cases}$$

For $\zeta_1 \in (0,1)$ and due to inequality $|\zeta_3| \leq 1$, the expression in (2.3.28) together with Lemma 2.3.1 results in the following inequality

$$|\mathscr{L}| \leq \frac{1}{8}\zeta_1(1-\zeta_1^2)\mathscr{Y}(A_1,A_2,A_3) \tag{2.3.29}$$

where

$$\mathscr{Y}(A_1,A_2,A_3) := |A_1 + A_2\zeta_2 + A_3\zeta_2^2| + 1 - |\zeta_2|^2$$

with

$$A_1 = -\frac{11\zeta_1^3}{6(1-\zeta_1^2)}, \quad A_2 = -5\zeta_1, \quad A_3 = 3\zeta_1 - \frac{2}{\zeta_1}. \tag{2.3.30}$$

In view of Lemma B, we now examine the following cases based on the expressions of $A_1, A_2$ and $A_3$ given in (3.1.3).

**I.** Let $\zeta_1 \in X = (0,\zeta^*]$, where $\zeta^* = \sqrt{2/3}$. It is easy to verify that $A_1A_3 \geq 0$. Since $|A_2| - 2(1 - |A_3|) = (4 - \zeta_1(2+\zeta_1))/\zeta_1$ is an increasing function on $X$, then $|A_2| - 2(1-|A_3|) \geq 5\zeta^* - 2 > 0$. Thus on applying Lemma 2.3.1 to inequality (2.3.29), we have

$$|\mathscr{L}| \leq \frac{1}{8}\zeta_1(1-\zeta_1^2)(|A_1| + |A_2| + |A_3|) = \frac{1}{4} - \frac{\zeta_1^4}{48} \leq \frac{1}{4}.$$

**II.** Assume $X^* = X^c \cap (0,1)$. Observe that $A_1A_3 < 0$ for each $\zeta_1 \in X^*$. This case demands the following subcases.

**A.** Simple calculation reveals that, for each $\zeta_1$ in $X^*$, we have

$$\begin{cases} T_1(\zeta_1) := |A_2| - 2(1-|A_3|) = \dfrac{1}{\zeta_1}(11\zeta_1^2 - 2\zeta_1 - 4) > 0, \\ \\ T_2(\zeta_1) := -4A_1A_3\left(\dfrac{1}{A_3^2} - 1\right) - A_2^2 = \dfrac{27\zeta_1^2 - 62}{\zeta_1^2(6-9\zeta_1^2)} < 0. \end{cases}$$

From above we conclude that the set $T_1(X^*) \cap T_2(X^*)$ is empty. Thus according to Lemma B, this case fails to exist for any $\zeta_1 \in X^*$.

**B.** For $\zeta_1 \in X^*$, the expressions $4(1+|A_3|)^2$ and $-4A_1A_3(A_3^{-2}-1)$ become

$$4\,T_3(\zeta_1) := 4(1+|A_3|)^2 \text{ and } T_4(\zeta_1) := -4A_1A_3\left(\frac{1}{A_3^2} - 1\right) = \frac{22\zeta_1^2(9\zeta_1^2-4)}{3(3\zeta_1^2-2)},$$

respectively, where $T_3(x) := (3x^2+x-2)^2/x^2$. It is easy to observe that $T_3'(x)$ is non-vanishing on $X^*$. Infact $T_3'(1) = 20$, therefore $T_3(x) < 4$. Since $\zeta^* > \zeta^{*2}$, this yields that $T_4(x)$ is positive on $X^*$. Thus inequality below is false for any $\zeta_1 \in X^*$,

$$A_2^2 = 25\zeta_1^2 < \min\{4\,T_3(x), T_4(x)\} = 4\,T_3(x).$$

**C.** The inequality below holds for $\zeta_1 \in X^*$,

$$|A_3|(|A_2| + 4|A_1|) - |A_1A_2| = -\frac{13\zeta_1^4 - 62\zeta_1^2 + 60}{6\left(1-\zeta_1^2\right)} \leq 0$$

if and only if $13\zeta_1^4 - 62\zeta_1^2 + 60 \geq 0$. Due to Lemma 2.3.1 and equations (2.3.28)-(2.3.29),

$$|\mathscr{L}| \leq \frac{1}{8}\zeta_1(1-\zeta_1^2)\left(|A_1| + |A_2| - |A_3|\right) = \frac{1}{4} - \frac{\zeta_1^4}{48} \leq \frac{1}{4}.$$

**D.** Finally the expression below is true for each $\zeta_1 \in X^*$,

$$|A_1A_2| - |A_3|(|A_2| - 4|A_1|) = \frac{277\zeta_1^4 - 238\zeta_1^2 + 60}{6(1-\zeta_1^2)} > 0.$$

Summarizing **I** and **II** inequality (2.3.1) follows. In Lemma 2.3.1, on replacing $\zeta_1 = 0, \zeta_2 = \zeta_3 = 1$ in (2.3.24) we get $\tilde{p}(z) = (1+z^2)/(1-z^2) \in \mathscr{P}$. The function $\tilde{f} \in \mathscr{A}$ given in (2.3.26) satisfies $2z\tilde{f}'(z)/(\tilde{f}(z) - \tilde{f}(-z)) = (1+z^2)/(1-z^2)$. Hence the result is sharp. □

**Theorem 2.3.2.** *Let $F_f$ be given by (2.1.3). If $f \in \mathscr{F}_2$ and $a_2 \geq 0$, then*

$$|H_{2,1}(F_f)| \leq \frac{1}{4}.$$

*Proof.* Since $f \in \mathscr{F}_2$, then there exists an analytic function $p \in \mathscr{P}$ such that

$$(1-z^2)f'(z) = p(z). \tag{2.3.31}$$

Then from (2.3.31) we obtain first three coefficients of $f(z)$ in terms of $c_i's$ $(i = 1,2,3)$ as follows:

$$a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{3}(1+c_2), \quad a_4 = \frac{1}{4}(c_1+c_3).$$

Using Lemma 2.3.1, we express $\gamma_1\gamma_3 - \gamma_2^2$ in terms of $\zeta_i's$ by replacing $c_i's$ with $\zeta_i's$ $(i = 1,2,3)$,

$$\begin{aligned}\mathscr{M} := \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{144}\big(&\zeta_1^4(5-4\zeta_2+2\zeta_2^2) + 2\zeta_1^2(1+10\zeta_2+7\zeta_2^2) - 4(1+2\zeta_2)^2 \\ &+ 18\zeta_1\zeta_3(1-\zeta_1^2)(1-|\zeta_2|^2)\big).\end{aligned} \tag{2.3.32}$$

Due to the fact that $|\zeta_2| \le 1$, inequality (2.3.32) gives

$$|\mathscr{M}| \le \begin{cases} (1/144)(11+32|\zeta_2|+32|\zeta_2|^2) \le 1/48, & \zeta_1 = 1, \\ (1/36)(1+2|\zeta_2|)^2 \le 1/4, & \zeta_1 = 0. \end{cases}$$

Since $|\zeta_3| \le 1$, then for $\zeta_1 \in (0,1)$, we have

$$|\mathscr{M}| \le \frac{1}{8}\zeta_1(1-\zeta_1^2)\mathscr{Y}(A_1,A_2,A_3), \tag{2.3.33}$$

where

$$\mathscr{Y}(A_1,A_2,A_3) := |A_1 + A_2\zeta_2 + A_3\zeta_2^2| + 1 - |\zeta_2|^2$$

with

$$A_1 = \frac{5\zeta_1^4 + 2\zeta_1^2 - 4}{18\zeta_1(1-\zeta_1^2)}, \quad A_2 = -\frac{2}{9\zeta_1}(4-\zeta_1^2), \quad A_3 = -\frac{1}{9\zeta_1}(8+\zeta_1^2). \tag{2.3.34}$$

In light of Theorem B and the expressions of $A_1, A_2$ and $A_3$ given in (3.2.1), we divide our result in two cases based on the value of $A_1A_3$.

**I.** Observe that $A_1A_3$ is non-negative over the interval $Y = (0, \zeta^*]$, where $\zeta^* = (1/5)\sqrt{\sqrt{21}-1}$. Since $|A_2| - 2(1-|A_3|) = \frac{8}{3\zeta_1} - 2 \ge 0$, for each $\zeta_1 \in Y$, therefore by applying Lemma B to (2.3.33) yields the following inequality:

$$|\mathscr{M}| \le \frac{1}{8}\zeta_1(1-\zeta_1^2)(|A_1|+|A_2|+|A_3|) = \frac{1}{48}(12 - 12\zeta_1^2 - \zeta_1^4) := S_1(\zeta_1).$$

It is clearly evident that $S_1(x)$ is a decreasing function of $x$, so $S_1(x) \to 1/4$ whenever $x \to 0$. Thus for this case $|\mathscr{M}| \le 1/4$.

**II.** Let $Y^* = Y^c \cap [0,1)$. Clearly, $A_1A_3 < 0$ for each $\zeta_1 \in Y^*$. For our suitability, we divide the computations in five subcases.

**A.** The expression in (2.3.35) is true for each $\zeta_1 \in Y^*$,

$$S_1(x) := -4A_1A_3\left(\frac{1}{A_3^2}-1\right) - A_2^2 = \frac{2(\zeta_1^4 - 106\zeta_1^2 - 12)}{27(8+\zeta_1^2)} \leq 0. \tag{2.3.35}$$

provided $\zeta_1^4 - 106\zeta_1^2 - 12 \leq 0$. Additionally, from Case **I** we have $|A_2| - 2(1-|A_3|) =: S_2(x) \geq 0$. Thus we conclude that $S_1(Y^*) \cap S_2(Y^*)$ is an empty set.

**B.** Moreover for $\zeta_1 \in Y^*$, it is easy to verify that

$$\begin{cases} S_3(\zeta_1) := 4(1+|A_3|)^2 = \dfrac{4}{81\zeta_1^2}(\zeta_1^2 + 9\zeta_1 + 8)^2 > 0, \\[2ex] S_4(\zeta_1) := -4A_1A_3\left(\dfrac{1}{A_3^2}-1\right) = \dfrac{2(\zeta_1^2-64)(5\zeta_1^4+2\zeta_1^2-4)}{81\zeta_1^2(8+\zeta_1^2)} \leq 0. \end{cases}$$

Therefore, $A_2^2 = 2(\zeta_1^2-4)/9\zeta_1 > \min\{S_3(\zeta_1), S_4(\zeta_1)\} = S_4(\zeta_1)$. This leads us to the next case.

**C.** Consider the following expression

$$|A_3|(|A_2| + 4|A_1| - |A_1A_2|) = \frac{S_5(\zeta_1)}{81\zeta_1^3(1-\zeta_1^2)},$$

where $S_5(x) := 2x^8 + 10x^7 + 11x^6 + 84x^5 - 90x^4 + 24x^3 + 52x^2 - 64x + 16$. Observe that for each $x \in Y^*$, the polynomial $S_5(x) \leq 0$ is true if and only if $x^2(2x^6 + 21x^4 + 76) \leq 6(8+x^4)$. Infact

$$\min_{x\in Y^*}(x^2(2x^6+21x^4+76)) = \frac{36}{625}(329\sqrt{21} - 419) \geq \max_{x \in Y^*}(6(8+x^4)) = 54.$$

Thus for each $x \in Y^*$, $S_5(x)$ must be positive.

**D.** Finally the following inequality

$$\begin{aligned} |A_1A_2| - |A_3|(|A_2| - 4|A_1|) = -\frac{1}{81\zeta_1^3(1-\zeta_1^2)}&(2\zeta_1^8 - 10\zeta_1^7 - 5\zeta_1^6 - 68\zeta_1^5 - 10\zeta_1^4 - 104\zeta_1^3 \\ &- 12\zeta_1^2 + 128\zeta_1 + 16) \leq 0, \end{aligned}$$

is equivalent to $\zeta_1^2(10\zeta_1^5 + 5\zeta_1^4 + 68\zeta_1^3 + 10\zeta_1^2 + 104\zeta_1 + 12) \leq 2(\zeta_1^8 + 64\zeta_1 + 8)$, provided $\zeta_1$ satisfies $\zeta^* < \zeta_1 \leq \sqrt{\sqrt{249} - 15}$. Note that the function $R(x)$ defined below is a decreasing function of $x$ $\left(\zeta^* < x \leq \sqrt{\sqrt{249}-15}\right)$. Thus Lemma B together with (2.3.33), gives

$$\begin{aligned} |\mathcal{M}| &\leq \frac{1}{8}\zeta_1(1-\zeta_1^2)(-|A_1| + |A_2| + |A_3|) \\ &= \frac{2\zeta_1^5 - 5\zeta_1^4 - 34\zeta_1^3 - 2\zeta_1^2 + 32\zeta_1 + 4}{144\zeta_1} =: R(\zeta_1) \\ &\leq \frac{1}{600}(169 - 29\sqrt{21}) \approx 0.0601755. \end{aligned}$$

**E.** Further for $\sqrt{\sqrt{249}-15}<\zeta_1<1$, inequality (2.3.33) and Lemma B, yields

$$\begin{aligned}
|\mathscr{M}| &\le \frac{1}{8}\zeta_1(1-\zeta_1^2)\left((|A_1|+|A_3|)\sqrt{1-\frac{A_2^2}{4A_1A_3}}\right)\\
&= \frac{1}{16\sqrt{3}}(2-\zeta_1^2)^2\sqrt{\frac{\zeta_1^2\left(\zeta_1^4+20\zeta_1^2-12\right)}{5\zeta_1^6+42\zeta_1^4+12\zeta_1^2-32}}\\
&\le \frac{1}{8}(3\sqrt{249}-47)\approx 0.0424002.
\end{aligned}$$

Based on all the above cases, we arrive at the required bound. □

**Theorem 2.3.3.** *Suppose* $X(x)=-48x^4-96x^3-392x^2+24x+357$ *and* $F_f$ *be given by* (2.1.3). *If* $f\in\mathscr{F}_1$ *and* $a_2\ge 0$, *then*

$$|H_{2,1}(F_f)|\le X(x_0)/2304,$$

*where* $x_0\approx 0.0302$ *is the unique real root of the equation* $X'(x)=0$.

*Proof.* For each $f\in\mathscr{F}_1$, there exists a function $p\in\mathscr{P}$ satisfying $(1-z)f'(z)=p(z)$. On comparing coefficients of like power terms, we obtain

$$a_2=\frac{1}{2}(1+c_1),\quad a_3=\frac{1}{3}(1+c_1+c_2),\quad a_4=\frac{1}{4}(1+c_1+c_2+c_3).$$

With the assumption that $c_1\in[0,2]$, and applying Lemma 2.3.1, we obtain,

$$\begin{aligned}
|\mathscr{N}|:=|\gamma_1\gamma_3-\gamma_2^2| = \frac{1}{2304}|&72(1+2\zeta_1)(1+2\zeta_3(1-|\zeta_2|^2)+2\zeta_1(1+2\zeta_2-\zeta_2^2)+2\zeta_2\\
&+2\zeta_1^3(1-\zeta_2)^2-2\zeta_1^2(\zeta_3(1-|\zeta_2|^2)+\zeta_2-1))+3(1+2\zeta_1)^4\\
&-64(1+2(\zeta_1+\zeta_2)+2\zeta_1^2(1-\zeta_2))^2|. \qquad (2.3.36)
\end{aligned}$$

**I.** In equation (2.3.36), on substituting $\zeta_1=1$, we get $|\mathscr{N}|=155/2304\approx 0.067274$.
**II.** Since $|\zeta_3|\le 1$, then for $\zeta_1\in[0,1)$, we obtain

$$|\mathscr{N}|\le\frac{1}{16}(1+2\zeta_1)(1-\zeta_1^2)\mathscr{Y}(A_1,A_2,A_3), \qquad (2.3.37)$$

where

$$\mathscr{Y}(A_1,A_2,A_3):=|A_1+A_2\zeta_2+A_3\zeta_2^2|+1-|\zeta_2|^2$$

with

$$A_1:=\frac{11+56\zeta_1-8\zeta_1^2+16\zeta_1^3+80\zeta_1^4}{144(1+2\zeta_1)\left(1-\zeta_1^2\right)},\quad A_2:=\frac{4\zeta_1^2+4\zeta_1-7}{9(1+2\zeta_1)}$$

and

$$A_3 := -\frac{2\zeta_1^2+9\zeta_1+16}{9(1+2\zeta_1)}.$$

Based on the range of $\zeta_1$ and the values of $A_1, A_2$ and $A_3$ defined above, we find that $A_1A_3 < 0$. Further as a consequence of Lemma B, we split our discussions into the following subcases.

**A.** For $\zeta_1 \in [0,1)$, consider the expression,

$$\mathscr{V}_1 := -4A_1A_3\left(\frac{1}{A_3^2}-1\right) - A_2^2 = \frac{T(\zeta_1)}{108\,(1+2\zeta_1)\left(2\zeta_1^2+9\zeta_1+16\right)},$$

where $T(x) = 32x^5 + 336^4 - 2720^3 + 856x^2 + 846x - 1687$. It is easy to verify that $\max_{x\in[0,1)} T(x) = -1687$, thus $T(x) \le 0$. Suppose $\zeta^* = (1/2)(2\sqrt{2}-1)$, then for $\zeta_1 \in W_1 \cup W_1^*$, where $W_1 := [0,\zeta^*]$ and $W_1^* := W_1^c \cap [0,1)$, define

$$\mathscr{V}_2 := |A_2| - 2(1-|A_3|) = \begin{cases} \dfrac{21-22\zeta_1}{18\zeta_1+9}, & \zeta_1 \in W_1, \\[2ex] \dfrac{8\zeta_1^2-14\zeta_1+7}{18\zeta_1+9}, & \zeta_1 \in W_1^*. \end{cases}$$

From the expression of $\mathscr{V}_2$, one can observe that, if $\zeta_1 \in W_1$, then $\mathscr{V}_2 \le 0$ leads to $\zeta_1 \ge 21/22$, this is not true. Clearly for $\zeta_1 \in W_1^*$, it is easy to verify that $\mathscr{V}_2 > 0$. Thus $\mathscr{V}_1 \cap \mathscr{V}_2$ is an empty set. Therefore according to Lemma B, this case fails to hold for any value of $\zeta_1 \in [0,1)$.

**B.** For $\zeta_1 \in [0,1)$, simple computations reveal that

$$\mathscr{V}_3 := 4(1+|A_3|)^2 = \frac{4(2\zeta_1^2+27\zeta_1+25)^2}{81(1+2\zeta_1)^2} > 0$$

and

$$\mathscr{V}_4 := -4A_1A_3\left(\frac{1}{A_3^2}-1\right) = \frac{(4\zeta_1^2+36\zeta_1-175)(80\zeta_1^4+16\zeta_1^3-8\zeta_1^2+56\zeta_1+11)}{324(2\zeta_1+1)^2(2\zeta_1^2+9\zeta_1+16)} < 0.$$

Above inequalities yield,

$$A_2^2 = \frac{(7-4\zeta_1-4\zeta_1^2)^2}{81\,(2\zeta_1+1)^2} > \min\{\mathscr{V}_3,\mathscr{V}_4\} = \mathscr{V}_4.$$

**C.** Furthermore for $\zeta_1 \in W_1 \cup W_1^*$, we have the following expression

$$\mathscr{V}_5 := |A_3|\left(|A_2|+4|A_1|\right) - |A_1A_2| = \begin{cases} \dfrac{Q_1(\zeta_1)}{1296\,(1+2\zeta_1)^2\left(1-\zeta_1^2\right)}, & \zeta_1 \in W_1, \\[2ex] \dfrac{Q_2(\zeta_1)}{1296\,(1+2\zeta_1)^2\left(1-\zeta_1^2\right)}, & \zeta_1 \in W_1^* \cup \{\zeta^*\}, \end{cases}$$

where

$$Q_1(x) := 1088x^6 + 4096x^5 + 6352x^4 + 576x^3 - 1252x^2 + 3616x + 2419$$

and

$$Q_2(x) := 192x^6 + 1920x^5 + 4912x^4 + 1792x^3 + 4436x^2 + 4344x - 1011.$$

A computation reveals that $Q_1'(x)$ is never zero on $W_1$ and $Q_1'(0) = 3616$, thus we deduce that $\mathscr{V}_5 \geq Q_1(0)/1296 > 0$. Similarly when $\zeta_1 \in W_1^*$, then $\mathscr{V}_5 \to Q_2(\zeta^*)/(8\sqrt{2} - 10) > 0$ as $\zeta_1 \to \zeta^*$. Therefore in view of Lemma B, this case fails to hold any $\zeta_1 \in [0,1)$.

**D.** Consider the expression

$$\mathscr{V}_6 := |A_1A_2| - |A_3|(|A_2| - 4|A_1|) = \frac{Q_2(\zeta_1)}{1296\,(1+2\zeta_1)^2\,(1-\zeta_1^2)}.$$

It can be easily verified that $Q_2'(x)$ is non-vanishing over the range $[0,1)$. Infact $Q_2(0) = -1011$ and $Q_2(1/2) = 2864$, thus by intermediate value property their exists $\gamma \in (0,1/2)$ such that $Q_2(\gamma) = 0$. Infact $Q_2(\gamma) = 0$, provided $\gamma \approx 0.190991$. Thus $Q_2(x) \leq 0$ for each $x \in [0,\gamma]$, this leads to $\mathscr{V}_6 \leq Q_2(\gamma)/(1296(1+2\gamma)^2(1-\gamma^2)) = 0$. Consequently, on applying Lemma B to inequality (2.3.37), we have

$$|\mathscr{N}| \leq \frac{1}{16}(1+2\zeta_1)(1-\zeta_1^2)(|A_2| + |A_3| - |A_1|) \leq T_1(x_0) \quad (x_0 \approx 0.0302689),$$

where

$$T_1(x) := \frac{X(x)}{2304} = \frac{-48x^4 - 96x^3 - 392x^2 + 24x + 357}{2304} \quad 0 \leq x \leq \gamma.$$

Observe that for $0 \leq x \leq \gamma$,

$$T_1'(x) = \frac{-192x^3 - 288x^2 - 784x + 24}{2304} = 0,$$

holds true only if $x = x_0 < \gamma$, also $T_1''(x_0) < 0$. Consequently, this leads us to the inequality $T_1(x) \leq T_1(x_0) \approx 0.155106$.

**E.** Furthermore, for $\gamma < \zeta_1 < 1$, from (2.3.37) and Lemma B, it follows that

$$|\mathscr{N}| \leq \frac{1}{16}(1+2\zeta_1)(1-\zeta_1^2)(|A_1| + |A_3|)\sqrt{1 - \frac{A_2^2}{4A_1A_3}} \leq T_2(\gamma) \quad (\gamma \approx 0.190991),$$

where

$$T_2(x) := \frac{1}{768}(48x^4 - 128x^3 - 232x^2 + 200x + 267) \times \sqrt{\frac{32x^6 + 208x^5 + 544x^4 + 216x^3 + 14x^2 + 257x + 124}{480x^6 + 2256x^5 + 4224x^4 + 888x^3 + 1194x^2 + 2985x + 528}}.$$

As $T_2'(x) = 0$ has no solution in $(\gamma, 1)$, also $T_2'(x) \to -(3079/15552) < 0$ when $x \to 1$. Thus $T_2(x)$ is a decreasing function and hence $T_2(x) \le T_2(\gamma) \approx 0.150413$. This leads us to the required bound. □

**Theorem 2.3.4.** *Suppose* $X(x) = -176x^4 - 224x^3 - 264x^2 + 328x + 469$. *Let* $F_f$ *be given by* (2.1.3). *If* $f \in \mathscr{F}_3$ *and* $a_2 \ge 0$, *then*

$$|H_{2,1}(F_f)| \le X(x_0)/2304,$$

*where* $x_0 \approx 0.3737$ *is the unique real root of the equation* $X'(x) = 0$.

*Proof.* Suppose $f \in \mathscr{A}$ such that $(1 - z + z^2)f'(z) = p(z)$, where $p \in \mathscr{P}$. Proceeding in a similar approach, we have

$$a_2 = \frac{1}{2}(1 + c_1), \quad a_3 = \frac{1}{3}(c_1 + c_2), \quad a_4 = \frac{1}{4}(c_2 + c_3 - 1).$$

The expression $\gamma_1\gamma_3 - \gamma_2^2$ in terms of $\zeta_i' s$ $(i = 1, 2, 3)$, becomes

$$\begin{aligned} \mathscr{G} := \gamma_1\gamma_3 - \gamma_2^2 = &\frac{1}{32}(1 + 2\zeta_1)(2\zeta_1^3(1 - \zeta_2)^2 + 2\zeta_1\zeta_2(2 - \zeta_2) + 2(\zeta_2 + \zeta_3) - 1 \\ &+ 2\zeta_1^2(1 - \zeta_2 - \zeta_3(1 - |\zeta_2|^2)) - 2\zeta_3|\zeta_2|^2) + \frac{1}{768}(1 + 2\zeta_1)^4 \\ &- \frac{1}{9}(\zeta_1 + \zeta_2 + (1 - \zeta_2)\zeta_1^2)^2. \end{aligned} \tag{2.3.38}$$

**I.** In expression (2.3.38) put $\zeta_1 = 1$, then $|\mathscr{G}| = 133/2304 \approx 0.0577257$.
**II.** If $\zeta_1 \in [0, 1)$, then from (2.3.38) together with $|\zeta_3| \le 1$, gives

$$|\mathscr{G}| \le \frac{1}{16}(1 + 2\zeta_1)(1 - \zeta_1^2)\mathscr{Y}(A_1, A_2, A_3), \tag{2.3.39}$$

where

$$\mathscr{Y}(A_1, A_2, A_3) := |A_1 + A_2\zeta_2 + A_3\zeta_2^2| + 1 - |\zeta_2|^2$$

with

$$A_1 := \frac{80\zeta_1^4 + 16\zeta_1^3 - 40\zeta_1^2 - 120\zeta_1 - 69}{144(1 + 2\zeta_1)(1 - \zeta_1^2)}, \quad A_2 := \frac{4\zeta_1^2 + 4\zeta_1 + 9}{9(1 + 2\zeta_1)}$$

and

$$A_3 := -\frac{2\zeta_1^2 + 9\zeta_1 + 16}{9(1 + 2\zeta_1)}.$$

The expressions of $A_1, A_2$ and $A_3$ defined above leads to $A_1A_3 > 0$. Now the inequality

$$|A_2| - 2(1 - |A_3|) = \frac{8\zeta_1^2 - 14\zeta_1 + 23}{9(1 + 2\zeta_1)} < 0, \tag{2.3.40}$$

is equivalent to $8\zeta_1^2 - 14\zeta_1 + 23 < 0$. Since $\min_{\zeta_1 \in [0,1)} 8\zeta_1^2 - 14\zeta_1 + 23 = 135/8 > 0$, then inequality (3.2.3) is false for each $\zeta_1 \in [0, 1)$. Therefore, $|A_2| - 2(1 - |A_3|) \ge 0$ for each $0 \le \zeta_1 < 1$. An application of Lemma B to inequality (2.3.39), gives

$$|\mathscr{G}| \leq \frac{1}{16}(1+2\zeta_1)(1-\zeta_1^2)(|A_1|+|A_2|+|A_3|) \leq S(x_0) \quad (x_0 \approx 0.373776), \tag{2.3.41}$$

where

$$S(x) := \frac{X(x)}{2304} = \frac{-176x^4-224x^3-264x^2+328x+469}{2304}. \tag{2.3.42}$$

Note that $S'(x)$ vanishes at $x = x_0$ and $S''(x_0) < 0$, where $x_0 \approx 0.373776$ is the only critical point of $S(x)$ in $[0,1)$. Hence (2.3.41) determines the desired bound. □

**Theorem 2.3.5.** *Suppose* $48(17+x)X(x) = (1+x)(x^4+20x^3-114x^2+4x+125)$. *Let* $F_f$ *be given by* (2.1.3). *If* $f \in \mathscr{F}_4$ *and* $a_2 \geq 0$, *then*

$$|H_{2,1}(F_f)| \leq X(x_0),$$

*where* $x_0 \approx 0.381$ *is the unique real root of the equation* $X'(x) = 0$.

*Proof.* Let $f \in \mathscr{A}$ satisfy $(1-z+z^2)f'(z) = p(z)$, where $p \in \mathscr{P}$. Following the same procedure as before, we have

$$a_2 = \frac{1}{2}(c_1+2), \quad a_3 = \frac{1}{3}(2c_1+c_2+3), \quad a_4 = \frac{1}{4}(3c_1+2c_2+c_3+4).$$

Due to Lemma 2.3.1, the expression $\gamma_1\gamma_3-\gamma_2^2$ in terms of $\zeta_i's$ $(i=1,2,3)$, becomes

$$\begin{aligned}\mathscr{H} := \gamma_1\gamma_3-\gamma_2^2 = \frac{1}{144}&(18(1+\zeta_1)^2(\zeta_1^2(1-\zeta_2)^2-\zeta_1(\zeta_2^2-1+\zeta_3(1-|\zeta_2|^2))\\ &-\zeta_3|\zeta_2|^2+2\zeta_2+\zeta_3+2)+3(1+\zeta_1)^4-4(3+4\zeta_1+2\zeta_2\\ &-2\zeta_1^2(\zeta_2-1))^2).\end{aligned} \tag{2.3.43}$$

**I**: Substitute $\zeta_1 = 1$ in (3.1.2), then $|\mathscr{H}| = 1/12$.
**II**: Since $|\zeta_3| \in \overline{\mathbb{D}}$, then for $\zeta_1 \in [0,1)$, we have

$$|\mathscr{H}| \leq \frac{1}{8}(1-\zeta_1)(1+\zeta_1)^2\mathscr{Y}(A_1,A_2,A_3), \tag{2.3.44}$$

where $\mathscr{Y}(A_1,A_2,A_3) := |A_1+A_2\zeta_2+A_3\zeta_2^2|+1-|\zeta_2|^2$, with

$$A_1 := \frac{5\zeta_1^4+2\zeta_1^3-4\zeta_1^2+6\zeta_1+3}{18(1-\zeta_1)(1+\zeta_1)^2}, \quad A_2 := -\frac{2(1-\zeta_1)(3+\zeta_1)}{9(1+\zeta_1)}, \quad A_3 := -\frac{1}{9}(8+\zeta_1).$$

The expressions of $A_1, A_2$ and $A_3$ defined above, demonstrates that $A_1A_3 < 0$. Moreover, based on the conditions in Lemma B, we now look into the following subcases.

**A.** Note that the inequality

$$|A_2|-2(1-|A_3|)=\frac{4(1-\zeta_1)}{9(1+\zeta_1)}\leq 0$$

is valid if $\zeta_1\geq 1$. This is false as $\zeta_1<1$. Infact

$$-4A_1A_3\left(\frac{1}{A_3^2}-1\right)-A_2^2=\frac{2(\zeta_1^5+21\zeta_1^4-10\zeta_1^3-2\zeta_1^2+93\zeta_1-31)}{27(1+\zeta_1)^2(8+\zeta_1)}\leq 0,$$

provided $\zeta_1\in[0,\zeta^*]$, where $\zeta^*\approx 0.336931$. Consequently, due to Lemma B this case does not hold for each $\zeta_1\in[0,1)$.

**B.** Note that the following inequalities

$$\mathscr{J}_1:=4(1+|A_3|)^2=\frac{4}{81}(17+\zeta_1)^2>0$$

and

$$\mathscr{J}_2:=-4A_1A_3\left(\frac{1}{A_3^2}-1\right)=\frac{2(17+\zeta_1)(5\zeta_1^4+2\zeta_1^3-4\zeta_1^2+6\zeta_1+3)}{81(1+\zeta_1)^2(8+\zeta_1)}>0,$$

hold simultaneously if $\zeta_1>\zeta^*$. Thus

$$A_2^2=\frac{4\,(1-\zeta_1)^2\,(3+\zeta_1)^2}{81\,(1+\zeta_1)^2}<\min\{\mathscr{J}_1,\mathscr{J}_2\}=\mathscr{J}_2.$$

Therefore in view of Lemma B, inequality (2.3.44), gives

$$|\mathscr{H}|\leq\frac{1}{8}(1-\zeta_1)(1+\zeta_1)^2\left(1+|A_1|+\frac{A_2^2}{4(1+|A_3|)}\right)\leq X(x_0)\quad(x_0\approx 0.381423),$$

where

$$X(x)=\frac{(1+x)(x^4+20x^3-114x^2+4x+125)}{48(17+x)}.$$

Since $x=x_0>\zeta^*$ is a unique real root of $X'(x)=0$ in $(\zeta^*,1)$ and $X''(x_0)<0$, where $x_0\approx 0.381423$. Then $X$ attains its maximum at $x=x_0$.

**C.** Further for $0\leq\zeta_1\leq\zeta^*$, simple computations reveal that

$$|A_3|(|A_2|+4|A_1|)-|A_1A_2|=\frac{17\zeta_1^6+128\zeta_1^5+137\zeta_1^4-80\zeta_1^3-17\zeta_1^2+160\zeta_1+87}{81\,(1-\zeta_1)\,(1+\zeta_1)^3}\geq m_0>0,$$

where $m_0\approx 1.05323$. Similarly, $|A_1A_2|-|A_3|\,(|A_2|-4\,|A_1|)>0$. Thus, Lemma B, together with (2.3.44), leads to $|\mathscr{H}|\leq M(\zeta_1)\leq n_0$, where

$$M(\zeta_1)=\frac{1}{48}(3\zeta_1^4-16\zeta_1^3-18\zeta_1^2+24\zeta_1+19)\sqrt{\frac{\zeta_1^5+12\zeta_1^4+8\zeta_1^3-2\zeta_1^2+3\zeta_1+14}{15\zeta_1^5+126\zeta_1^4+36\zeta_1^3-78\zeta_1^2+153\zeta_1+72}}$$

and $n_0\approx 0.183792$. By summarizing all the above cases, the desired bound can be obtained. □

Untill now we focused on investigating coefficient problems for some close-to-convex classes and a newly defined class of analytic functions. However, our study is not restricted to the study of coefficient problems. In the upcoming chapter, we proceed to establish certain radius results pertaining to a class of analytic functions containing non-univalent functions involving a parabolic function.

## Highlights of the Chapter

In this chapter, we establish bounds on the Hankel determinant involving logarithmic coefficients for specific subclasses of close-to-convex functions with integer coefficients. We derive sharp bounds on the initial coefficients and the Fekete-Szegö functional for the newly defined class of analytic functions, $\mathscr{S}^k_{\gamma,\delta}(\phi)$. Our findings demonstrate that this class generalizes various other classes, leading to results that encompass many previously known theorems, thereby highlighting the significance of our work.

The contents of this chapter is based on the findings presented in the following papers:

- *Mridula Mundalia and Shanmugam Sivaprasad Kumar: Coefficient bounds for a unified class of holomorphic functions, In Mathematical analysis I: Approximation theory, ICRAPAM 2018, Springer Proceedings in Mathematics & Statistics, vol.* ***306****, 197–210 (2020), Springer, Singapore.* https://doi.org/10.1007/978-981-15-1153-0_17
- *Mridula Mundalia and S. Sivaprasad Kumar: Coefficient Problems for Certain Close-to-Convex Functions, Bulletin of the Iranian Mathematical Society,* **49***(1), Article 5, pp. 1–19 (2023).* https://doi.org/10.1007/s41980-023-00751-1

# Chapter 3

# On a Class of Analytic Functions Associated with a Parabolic Region

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In this chapter, we study a class of analytic functions associated with a parabolic region. Since the parabolic domain extends leftward from the right side of the imaginary axis, the class includes both univalent and non-univalent functions. We establish various radius constants, including the radius of univalence and the radius of starlikeness. Additionally, we derive radius results for this class in context of other well-known subclasses of $\mathscr{A}$. To enhance understanding, pictorial illustrations of selected radius results are provided. Furthermore, we obtain sufficient conditions for functions belonging to the class under consideration.

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## 3.1 Introduction

A differential inequality involving a real-valued function can provide important insights into the nature of the function. For instance, the condition $f'(x) > 0$ characterizes $f$ as an increasing function. Similarly, in the theory of complex-valued functions, various differential conditions serve to characterize the behavior of a function. A classic result is the Noshiro-Warschawski Theorem, which states that *if $f$ is analytic in the unit disc $\mathbb{D}$ and satisfies $\mathrm{Re}(f'(z)) > 0$ for all $z \in \mathbb{D}$, then $f$ is univalent in $\mathbb{D}$.*

A differential subordination in the complex plane serves as a generalization of a differential inequality on the real line. Consequently, many characterizations of complex-valued functions are expressed through differential subordinations. In fact, nearly all classes in the theory of univalent functions are defined by such subordinations. For example, functions belonging to various subclasses of univalent functions, such as starlike, convex, strongly starlike, and others, as discussed in the introductory chapter of this thesis, are characterized by the condition $\omega(z) \prec \phi(z)$. Here, $\omega(z)$ often takes forms like $\omega(z) = zf'(z)/f(z)$, $1 + zf''(z)/f'(z)$, their ratios, powers of these expressions, or variants such as $1 + \alpha zf''(z)/f'(z)$, where $\alpha$ is a constant. The function $\phi(z)$ is typically a Carathéodory function or a Ma-Minda function in most of the cases, depending on the specific class of functions being studied.

It is important to note that when the condition $\mathrm{Re}(\phi(z)) > 0$ is relaxed or compromised, the corresponding function $f$ may no longer be univalent. Based on this observation, we encounter several classes that include non-univalent functions, which are listed below:

In 1994, Uralegaddi [168] introduced the class $\mathscr{M}(\alpha)$, by taking $\omega(z) = zf'(z)/f(z)$ and $\phi(z) = (1 + (2\alpha - 1)z)/(1 + z)$, defined by

$$\mathscr{M}(\alpha) := \left\{ f \in \mathscr{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha \text{ or } \frac{zf'(z)}{f(z)} \prec \frac{1 + (2\alpha - 1)z}{1 + z},\ \alpha > 1 \right\}.$$

Note that $\mathscr{M}(\alpha) \not\subseteq \mathscr{S}^*$. Infact, Kargar et al. [67] studied the following class by considering $\omega(z) = (zf'(z)/f(z)) - 1$ and $\phi(z) - 1 = \psi(z) = z/(1 - \alpha z^2)$,

$$\mathscr{BS}(\alpha) := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} - 1 \prec \frac{z}{1 - \alpha z^2},\ 0 < \alpha \leq 1 \right\}.$$

Motivated by the above classes, Kumar and Gangania [78] made a systematic study of the class $\mathscr{F}(\psi)$, defined as

$$\mathscr{F}(\psi) := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} - 1 \prec \psi(z) \right\},$$

where $\psi$ is analytic, univalent and starlike with respect to 0 and $\psi(0)=0$. In general, $\mathscr{F}(\psi)$ is not necessarily a subset of $\mathscr{S}^*$. Further if $\phi(z)=1+\psi(z)\prec(1+z)/(1-z)$, then $\mathscr{F}(\psi)$ reduces to $\mathscr{S}^*(1+\psi)$. By choosing different $\psi(z)$, the class $\mathscr{F}(\psi)$ reduces to certain known classes, which are listed below:

Table 3.1: The class $\mathscr{F}(\psi)$ for specific choices of $\psi(z)$

| **S. No.** | $\mathscr{F}(\psi)$ | $\psi(z)$ | **References** |
|---|---|---|---|
| 1. | $\mathscr{BS}(\alpha)$ | $z(1-\alpha z^2)^{-1}$ | Cho et al. [22] |
| 2. | $\mathscr{S}_{cs}(\beta)$ | $z(1-z)^{-1}(1+\beta z)^{-1}$ | Masih et al. [103] |
| 3. | $\mathscr{F}(A,B)$ | $(A-B)^{-1}\log((1+Az)/(1+Bz))$ | Kumar and Yadav [86] |

Where the range of various constants used in Table 3.1 are:

1. $0<\alpha<1$;
2. $0\leq\beta<1$;
3. $A=\alpha e^{i\tau}$, $B=\alpha e^{-i\tau}$ with $\tau\in(0,\pi/2]$ and $0<\alpha\leq 1$.

Note that the classes $\mathscr{BS}(\alpha)$ and $\mathscr{F}(A,B)$ contains non-univalent functions. Moreover, $1+z(1-z)^{-1}(1+\beta z)^{-1}\nprec(1+z)/(1-z)$ for $\beta>1/2$, therefore $\mathscr{S}_{cs}(\beta)\nsubseteq\mathscr{S}^*$ and infact $\mathscr{S}_{cs}(\beta)$ is a Ma-Minda subclass for $\beta\in[0,1/2]$.

Motivated essentially by the above classes, where $\phi(z)=1+\psi(z)\nprec(1+z)/(1-z)$, we consider the following function

$$\varphi(z):=1+\varphi_0(z), \tag{3.1.1}$$

where

$$\varphi_0(z):=-\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,$$

which maps $\mathbb{D}$ onto a parabolic region, symmetric about the real axis and $\varphi(z)\nprec(1+z)/(1-z)$ (see Figure 3.1), to define our class $\mathscr{F}_\varphi$, as follows:

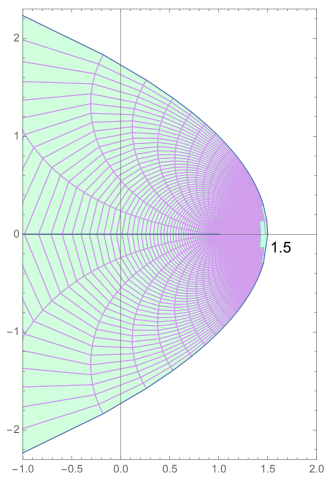

Figure 3.1: Image of $\varphi(\mathbb{D})$.

$$\mathscr{F}_\varphi:=\left\{f\in\mathscr{A}:\frac{zf'(z)}{f(z)}\prec\varphi(z)\right\}. \tag{3.1.2}$$

Since $\varphi(z)\nprec(1+z)/(1-z)$, all functions in the class $\mathscr{F}_\varphi$ need not be univalent, therefore, $\mathscr{F}_\varphi\nsubseteq\mathscr{S}^*$. Thus, $\mathscr{F}_\varphi$ includes non-univalent functions also. Moreover, if $f_0\in\mathscr{F}_\varphi$, then it can be expressed as

$$f_0(z)=z\left(\exp\int_0^z\frac{\varphi_0(t)}{t}dt\right), \tag{3.1.3}$$

which acts as an extremal function for many radius results. Given that the parabolic region $\varphi(\mathbb{D})$ spreads across both sides of the imaginary axis, it is intriguing to determine the optimal radius of the domain disc that ensures its complete mapping into the right half-plane. Taking this observation into account, we derive radius constants such as the radius of starlikeness for the class $\mathscr{F}_\varphi$. Furthermore, we establish radius results, along with illustrative examples of special cases.

In Section 3.2, we explore certain geometric properties of $\varphi(z)$. Additionally, we investigate radius results related to the class $\mathscr{F}_\varphi$, as well as the classes $\mathscr{S}^*(\phi)$ and $\mathscr{F}(\psi)$ for specific choices of $\phi(z)$ and $\psi(z)$, as outlined in Tables 1.1 and 3.1, respectively. Lastly, we provide sufficient conditions for the class under study.

## 3.2 Radius Problems for the Class $\mathscr{F}_\varphi$

We begin by establishing geometric properties of the function $\varphi(z)$, which serve as a foundation for our subsequent analysis. Following this, we derive sharp radius results for the function class $\mathscr{F}_\varphi$. In the next lemma, we determine explicit upper and lower bounds for the real part of $\varphi(z)$, which play a crucial role in our radius computations.

**Lemma 3.2.1.** Let $z \in \mathbb{D}_r$, then for each $0 < r \leq 1$ and $-\pi < t \leq \pi$, we have

$$\varphi_0(r) \leq \operatorname{Re} \varphi_0(re^{it}) \leq \varphi_0(-r).$$

*Proof.* Suppose $z = re^{it}$, where $-\pi < t \leq \pi$, then for $|z| = r < 1$,

$$\begin{aligned}
\operatorname{Re}(\varphi_0(z)) &= -\frac{2}{\pi^2}\left\{\operatorname{Re}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\} \\
&= -\frac{2}{\pi^2}\left(\log\left(\sqrt{\frac{\mu_1(r,c_t)}{\mu_2(r,c_t)}}\right)\right)^2 + \frac{2}{\pi^2}\left(\tan^{-1}\left(\frac{2\sqrt{1-c_t^2}\sqrt{r}}{1-r}\right)\right)^2 \\
&=: \mathscr{G}(r,c_t),
\end{aligned}$$

where $c_t := \cos(t/2)$ and

$$\mu_i(r,c_t) := \begin{cases} 1+r+2c_t\sqrt{r}, & i=1, \\ 1+r-2c_t\sqrt{r}, & i=2. \end{cases}$$

Observe that $c_t \in [-1,1]$, infact it is easy to check that $\partial\mathscr{G}(r,c_t)/\partial c_t = 0$ if and only if $c_t = 0$. Further, at $c_t = 0$, we have $\partial^2\mathscr{G}(r,0)/\partial c_t^2 < 0$ leads to

$$\max_{c_t\in[-1,1]} \mathscr{G}(r,c_t) = \mathscr{G}(r,0) = \varphi_0(-r) = \frac{2}{\pi^2}\left(\tan^{-1}\left(\frac{2\sqrt{r}}{1-r}\right)\right)^2. \tag{3.2.4}$$

Moreover, for each $R \le r < 1$, equation (3.2.4) leads to $\mathscr{G}(r,0) \ge \varphi_0(r) = \mathscr{G}(r,1)$. Since $\mathscr{G}(r,0)$ is an increasing function, whereas $\mathscr{G}(r,1)$ is a decreasing function of $r$, this leads to the inequality $\mathscr{G}(r,1) < \mathscr{G}(r,0)$, for each $r < 1$. Hence the required bound is achieved. □

As a consequence of Lemma 3.2.1 and [78, Theorem 2.1 & Corollary 2.2], we obtain the Growth and Covering Theorems for the class $\mathscr{F}_\varphi$, stated below:

**Theorem 3.2.1.** *Let $f \in \mathscr{F}_\varphi$, then the following holds:*

*I. (Growth Theorem) For $|z| = r < 1$, let*

$$\max_{|z|=r} \operatorname{Re} \varphi_0(z) = \varphi_0(-r) \text{ and } \min_{|z|=r} \operatorname{Re} \varphi_0(z) = \varphi_0(r),$$

*then for $|z| = r < 1$ the following sharp inequality holds*

$$r \exp\left(\int_0^r \frac{\varphi_0(t)}{t} dt\right) \le |f(z)| \le r \exp\left(\int_0^r \frac{\varphi_0(-t)}{t} dt\right).$$

*II. (Covering Theorem) Suppose $\min_{|z|=r} \operatorname{Re} \varphi_0(z) = \varphi_0(r)$ and $f \in \mathscr{F}_\varphi$. Let $f_0$ be given by (3.1.3), then $f(z)$ is a rotation of $f_0$ or $\{w \in \mathbb{D} : |w| \le -f_0(-1)\} \subset f(\mathbb{D})$, where $-f_0(-1) = \lim_{r\to 1} -f_0(-1)$.*

It may be noted that the boundary of the region $\varphi(\mathbb{D})$ represents a parabola $y^2 = 3 - 2x$ with focus $(1,0)$, where

$$\varphi(\mathbb{D}) = \Omega_\varphi := \{w \in \mathbb{C} : (\operatorname{Im} w)^2 < 3 - 2\operatorname{Re} w \text{ or } |1-w| < 2 - \operatorname{Re} w\}. \tag{3.2.5}$$

*Remark* 3.2.1. In view of Lemma 3.2.1, if $z \in \mathbb{D}_r$, then for each $0 < r \le 1$, we have $\varphi(r) \le \operatorname{Re} \varphi(z) \le \varphi(-r)$ and infact $\max_{|z|\le r} |\varphi(z)| = |\varphi(r)|$.

It is important to note that, Ravichandran et al. [136] computed the radius of starlikeness for the class $\mathscr{M}(\alpha)$. Cho et al. [22] dealt with certain sharp radius problems for the class $\mathscr{BS}(\alpha)$. Infact Masih et al. [103] studied the class $\mathscr{S}_{cs}(\beta)$, where $0 \le \beta < 1$, discussed the growth theorem and established sharp estimates of logarithmic coefficients for $0 \le \beta \le 1/2$. In 2022, Kumar and Yadav [86] introduced the class $\mathscr{F}(A,B)$ and established several radius results. In view of this, based on the definition of the class $\mathscr{F}_\varphi$ and pictorial representation of $\varphi(\mathbb{D})$ (see Figure 3.1), we conclude that

$$\max_{|z|\le 1} \operatorname{Re}(\varphi(z)) = \varphi(-1) = 3/2.$$

Thus if $f \in \mathscr{F}_\varphi$, we have $\operatorname{Re} zf'(z)/f(z) < 3/2$ and $f(z)$ can be non-univalent. Therefore, we find the largest radius $r_0 < 1$ such that each $f \in \mathscr{F}_\varphi$ is starlike in $|z| < r_0$. Here below, we provide a lemma that yields a maximal disc that can be subscribed within the parabolic region $\Omega_\varphi$.

**Lemma 3.2.2.** Suppose $c < 3/2$, and assume that $\zeta_{\eta_0}$ is defined as follows:

$$\zeta_{\eta_0} := \log\left(\frac{\sqrt{\eta_0}}{\sqrt{1-\eta_0}}\right) \text{ with } \eta_0 = \frac{e^{-\pi\sqrt{1-2c}}}{1+e^{-\pi\sqrt{1-2c}}},$$

then $\varphi(\mathbb{D})$ satisfies the following inclusion

$$\mathscr{D}(c, r_c) := \{w \in \mathbb{C} : |w-c| < r_c\} \subset \Omega_\varphi,$$

where $\Omega_\varphi$ is given by (3.2.5) and

$$r_c = \begin{cases} \sqrt{\left(c - \dfrac{3}{2} + \dfrac{2\zeta_{\eta_0}^2}{\pi^2}\right)^2 + \dfrac{4\zeta_{\eta_0}^2}{\pi^2}}, & c \leq \dfrac{1}{2}, \\ \dfrac{3}{2} - c, & \dfrac{1}{2} < c < \dfrac{3}{2}. \end{cases} \tag{3.2.6}$$

*Proof.* We obtain a maximal disc centered at $(c,0)$, where $c < 3/2$, that can be inscribed inside $\Omega_\varphi$. The distance from center $(c,0)$ to the boundary $f(\partial(\mathbb{D}))$ is given by square root of

$$\mathscr{D}_c(X) := \left(c + \frac{2}{\pi^2}\left(\log\left(\frac{\sqrt{X^2}}{\sqrt{1-X^2}}\right)\right)^2 - \frac{3}{2}\right)^2 + \frac{4}{\pi^2}\left(\log\left(\frac{\sqrt{X^2}}{\sqrt{1-X^2}}\right)\right)^2,$$

where $X = \cos t$. Now the critical points of $\mathscr{D}_c(X)$ are

$$X_0 := \begin{cases} \pm\dfrac{e^{\frac{1}{2}\pi\sqrt{1-2c}}}{\sqrt{1+e^{\pi\sqrt{1-2c}}}}, \pm\dfrac{e^{-\frac{1}{2}\pi\sqrt{1-2c}}}{\sqrt{1+e^{-\pi\sqrt{1-2c}}}}, & \text{if } c < \dfrac{1}{2}, \\ \pm\dfrac{1}{2}, & \text{if } \dfrac{1}{2} \leq c < \dfrac{3}{2}. \end{cases}$$

It can be verified that $\mathscr{D}_c''(X) > 0$ at $X = X_0$, whenever $c < 3/2$. Therefore, $X = X_0$ is the point of minima for $\mathscr{D}_c(X)$, which leads to the optimal disc centered at $c$ with radius $r_c$, as given in (3.2.6). □

In the next result, we determine sharp $\mathscr{F}_\varphi-$radii for several Ma-Minda classes.

**Theorem 3.2.2.** *Suppose $0 \leq \alpha < 1$ and $-1 < B < A \leq 1$. Then, for $f \in \mathscr{A}$, the sharp $\mathscr{F}_\varphi-$radii for the classes $\mathscr{S}_p^*$, $\mathscr{S}_{\sin}^*$, $\mathscr{S}_{\mathfrak{D}}^*$, $\mathscr{S}_{\sinh}^*$, $\mathscr{S}_{\wp}^*$, $\mathscr{BS}(\alpha)$, $\mathscr{S}_{\alpha,e}^*$ and $\mathscr{S}^*[A,B]$ (see Table 1.1), are given by*

*(i)* $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}_p^*) = \tanh^2(\pi/4)$.

*(ii)* $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}_{\sin}^*) = \pi/6$.

*(iii)* $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}_{\mathfrak{D}}^*) = 5/12$.

*(iv)* $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}_{\sinh}^*) = \sinh(1/2)$.

*(v)* $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}_{\wp}^*) \approx 0.3517\cdots$.

*(vi) For* $0 \leq \alpha < 1$, $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{BS}(\alpha)) = R_{\mathscr{BS}}$, *where*

$$R_{\mathscr{BS}} = \begin{cases} 1/2, & \alpha = 0, \\ (\sqrt{1+\alpha}-1)/\alpha, & 0 < \alpha < 1. \end{cases}$$

*(vii) For* $0 \leq \alpha < 1$, $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}^*_{\alpha,e}) = R_{\alpha,e}$, *where*

$$R_{\alpha,e} = \begin{cases} \log(1-1/2(\alpha-1)), & 0 \leq \alpha < 1-1/2(e-1), \\ 1, & 1-1/2(e-1) \leq \alpha < 1, \end{cases}$$

*and in particular,* $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}^*_e) = \log(3/2)$.

*(viii) For* $-1 < B < A \leq 1$, $\mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}^*[A,B]) = R_{A,B}$, *where*

$$R_{A,B} = \begin{cases} 1/(2A-3B), & \text{when } ((-1 < B \leq (2A-1)/3) \wedge (-1 < A < 0)) \\ & \quad \vee((-1 < B < (2A-1)/3) \wedge (0 \leq A \leq 1)), \\ 1, & \text{when } ((2A-1)/3 < B < A \leq 1) \wedge (-1 < A < 0)) \\ & \quad \vee(((2A-1)/3 \leq B < A \leq 1) \wedge (0 \leq A \leq 1)). \end{cases}$$

*Proof.* For part $(i)$, as $f \in \mathscr{S}^*_P$, then due to the geometry of the parabolic function

$$\phi_P(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,$$

given in (1.2.5), it can be observed that for $|z| < r \leq 1$,

$$\max_{|z|\leq r} \operatorname{Re}(\phi_P(z)) = \phi_P(r).$$

Now for $f(z)$ to lie in the class $\mathscr{F}_\varphi$, we must have

$$\phi_P(r) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{r}}{1-\sqrt{r}}\right)\right)^2 \leq \frac{3}{2},$$

which holds provided $r \leq \mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}^*_P)$. Clearly, $z\tilde{f}'(z)/\tilde{f}(z) = \phi_P(z)$ at $z = \mathscr{R}_{\mathscr{F}_\varphi}(\mathscr{S}^*_P)$, therefore, $\tilde{f}(z)$ is an extremal function. Further observe that, if $p \in \mathscr{P}[A,B]$, where

$$\mathscr{P}[A,B] := \left\{p \in \mathscr{H}_1 : p(z) = 1 + c_1 z + c_2 z^2 \ldots \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1\right\},$$

then for $|z| = r < 1$, it is a known fact that

$$\left|p(z) - \frac{1-ABr^2}{1-B^2r^2}\right| < \frac{|A-B|r}{1-B^2r^2}. \tag{3.2.7}$$

Now, in view of (3.2.7), for part $(viii)$ $f \in \mathcal{S}^*[A,B]$, where $f(z)$ satisfies $p(z) = zf'(z)/f(z)$, lies in $\mathcal{F}_\varphi$, if

$$\frac{(A-B)r + 1 - ABr^2}{1 - B^2r^2} \leq \frac{3}{2}.$$

Equivalently, we can say that $(1-Br)((2A-3B)r-1) \leq 0$, provided $r \leq \mathcal{R}_{\mathcal{F}_\varphi}(\mathcal{S}^*[A,B])$. Extremal function in this case is $\hat{f} \in \mathcal{A}$ as $z\hat{f}'(z)/\hat{f}(z) = (1+Az)/(1+Bz)$ at $z = R_{A,B}$. Further, we know that $\max_{|z|\leq r} \operatorname{Re}(1+\sin z) = 1+\sin r$, then for part $(ii)$ it is enough to find an $r<1$ satisfying the equation $\sin r = 1/2$, thus $r = \mathcal{R}_{\mathcal{F}_\varphi}(\mathcal{S}^*_{\sin}) = \pi/6$. Sharpness holds for the function $f_{\sin}(z)$ given by $zf'_{\sin}(z)/f_{\sin}(z) = 1+\sin z$. In all the subsequent parts, the proofs follow along the same lines, therefore they are omitted. □

Let $\mathcal{P}(\beta)$ consist of functions of the form $p(z) = 1+c_1z+c_2z^2+\dots$, satisfying $\operatorname{Re} p(z) > \beta$ for $0 \leq \beta < 1$, then we say $p(z)$ is a Carathéodory function of order $\beta$. Clearly, $\mathcal{P}(0) =: \mathcal{P}$, Further, assume $\mathfrak{P}_\varphi$ to be the class of functions of the form $p(z) = 1+c_1z+c_2z^2+\dots$, such that $p(z) \prec \varphi(z)$.

**Theorem 3.2.3.** *Let* $0 \leq \beta < 1$ *and* $0 < r \leq r_\beta$, *where* $r_\beta = \tanh^2(\pi\sqrt{1-\beta}/2\sqrt{2})$. *If* $p \in \mathfrak{P}_\varphi$, *then* $p \in \mathcal{P}(\beta)$ *for* $|z| < r_\beta$. *This result is sharp.*

*Proof.* Since $p \in \mathfrak{P}_\varphi$, then by definition of subordination and Schwarz Lemma, there exists an analytic function $\omega(z)$ with $|\omega(z)| \leq |z| < 1$ and $\omega(0) = 0$, such that $p(z) = \varphi(\omega(z))$. Suppose $\omega(z) = Re^{i\theta}$ $(-\pi < \theta \leq \pi)$, then $|\omega(z)| = R \leq |z| = r < 1$. In view of Remark 3.2.1, for each $p \in \mathfrak{P}_\varphi$, we have $\operatorname{Re} p(z) \geq \varphi(r)$, consequently, $p \in \mathfrak{P}_\varphi$ lies in $\mathcal{P}(\beta)$, if $\varphi(r) \geq \beta$, provided $r \leq r_\beta = \tanh^2(\pi\sqrt{1-\beta}/2\sqrt{2})$, where $0 \leq \beta < 1$. Thus $f_0(z)$ given by (3.1.3) is the extremal function. □

Upon replacing $p(z)$ with $zf'(z)/f(z)$ in Theorem 3.2.3, we deduce the next result.

**Corollary 3.2.1.** Let $0 \leq \beta < 1$ and $0 < r \leq r_\beta$, where $r_\beta$ is as given in Theorem 3.2.3. If $f \in \mathcal{F}_\varphi$, then $f \in \mathcal{S}^*(\beta)$ for $|z| < r_\beta$. This result is sharp.

*Remark* 3.2.2. Put $\beta = 0$ in Theorem 3.2.3, we get a sharp $\mathcal{P}-$ radius for the class $\mathfrak{P}_\varphi$. Infact for the class $\mathcal{F}_\varphi$, Corollary 3.2.1 gives sharp radius of starlikeness $r_0 = \tanh^2(\pi/2\sqrt{2})$. Moreover, $r = r_0 < 1$ serves as the sharp radius of univalence for the class $\mathcal{F}_\varphi$.

**Theorem 3.2.4.** *Assume* $0 < \alpha_0 \leq 1$, *then the sharp* $\mathcal{S}^*(1+\alpha_0 z)-$ *radius for the class* $\mathcal{F}_\varphi$ *is the smallest positive root* $r_{\alpha_0} = \tanh^2(\pi\sqrt{\alpha_0}/2\sqrt{2})$ *of the equation*

$$2\left(\log((1+\sqrt{r})/(1-\sqrt{r}))\right)^2 - \alpha_0\pi^2 = 0.$$

*Proof.* In view of Remark 3.2.1, for $|z| \leq r < 1$, we have

$$\max_{|z|\leq r<1} |\varphi(z)| = 1 - \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{r}}{1-\sqrt{r}}\right)\right)^2 = \varphi(r),$$

which is a decreasing function. Infact $\varphi(r) = 0$ if and only if $r = \tanh^2(\pi/2\sqrt{2}) \approx 0.6469\cdots$. As $f \in \mathcal{F}_\varphi$, then there exists a Schwarz's function $\omega(z)$ with $\omega(0) = 0$, so that

$$\frac{zf'(z)}{f(z)} = \varphi(\omega(z)).$$

Assume $\omega(z) = Re^{i\theta}$, where $R \leq r < 1$, then for $0 < \alpha_0 \leq 1$, and using the fact that $\max_{|z|\leq r} |\varphi_0(z)| = |\varphi_0(r)|$ is an increasing function of $r$, then we have

$$|\varphi(\omega(z)) - 1| \leq |\varphi(R) - 1| \leq |\varphi_0(R)| \leq |\varphi_0(r)| < \alpha_0,$$

provided $r \leq \tanh^2(\pi\sqrt{\alpha_0}/2\sqrt{2}) = r_{\alpha_0}$. Further, at $z_0 = r_{\alpha_0}$, the function $f_0(z)$ (defined in (3.1.3)) works as the extremal function. □

Following a similar approach as in Theorem 3.2.4, sharp $\mathcal{S}^*(1+\alpha_0 z)-$ radii for several well known Ma-Minda subclasses of starlike functions, namely, $\mathcal{S}^*_e$, $\mathcal{S}^*_{\sin}$, $\mathcal{S}^*_{\wp}$, $\mathcal{S}^*_{\sinh}$, $\mathcal{S}^*_{SG}$ and $\mathcal{S}^*_{N_e}$ (see Table 1.1) are stated in Corollary 3.2.2.

**Corollary 3.2.2.** Let $f \in \mathcal{F}_\varphi$, then the following radii are sharp for the class $\mathcal{F}_\varphi$ (see Figure 3.2):

(i) The $\mathcal{S}^*_e-$radius is $r_1 = \tanh^2(\lambda\pi)$, where $\lambda = (1/2)\sqrt{(e-1)/2e}$.

(ii) The $\mathcal{S}^*_{\sin}-$radius is $r_2 = \tanh^2(\pi/\lambda)$, where $\lambda = 2\sqrt{2\csc 1}$.

(iii) The $\mathcal{S}^*_{\wp}-$radius is $r_3 = \tanh^2(\pi/2\sqrt{2e})$.

(iv) The $\mathcal{S}^*_{\sinh}-$radius is $r_4 = \tanh^2(\pi\sqrt{\lambda}/2)$, where $\lambda = (1/2)\sinh^{-1} 1$.

(v) The $\mathcal{S}^*_{SG}-$radius is $r_5 = \tanh^2(\lambda\pi/2\sqrt{2})$, where $\lambda = \sqrt{(e-1)/(e+1)}$.

(vi) The $\mathcal{S}^*_{N_e}-$radius is $r_6 = \tanh^2(\pi/2\sqrt{3})$.

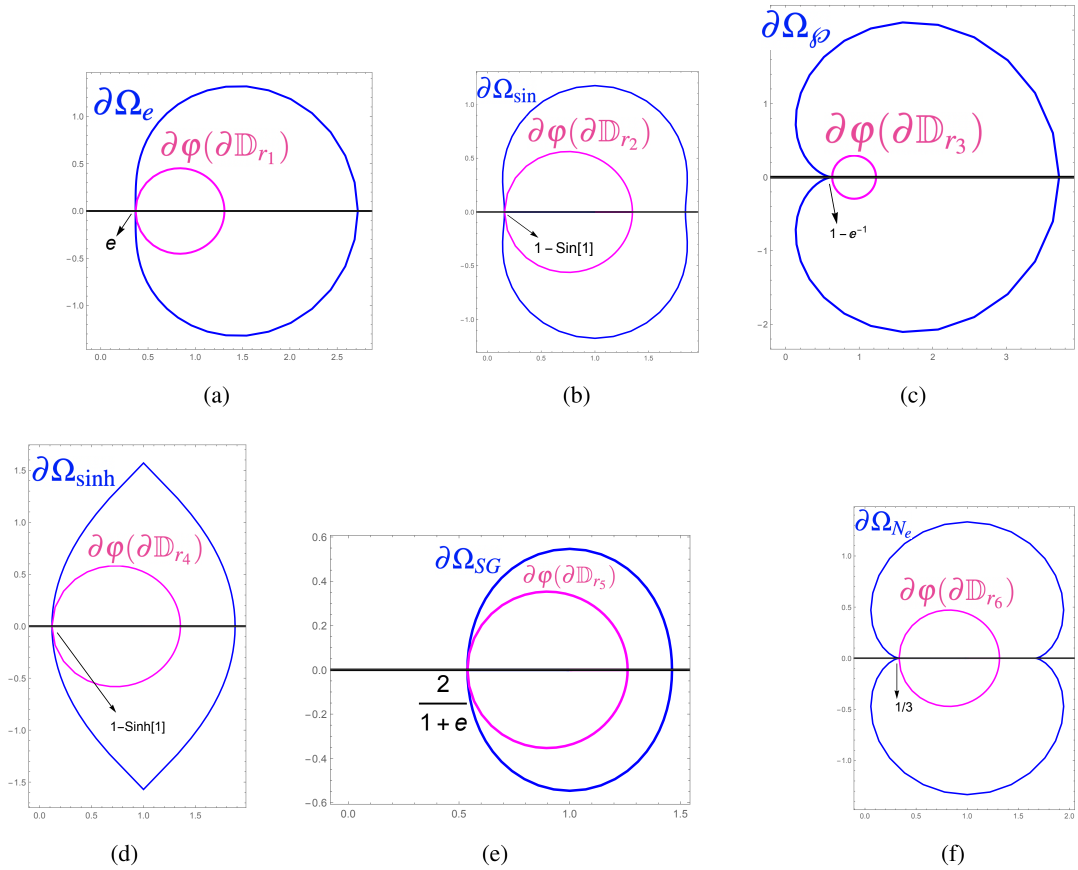


Figure 3.2: Images depicting sharpness of radii listed in Corollary 3.2.2.

**Corollary 3.2.3.** Let $f \in \mathscr{F}_{\varphi}$, then the following holds:

(i) $f \in \mathscr{S}^*_L$ in $|z| < \tanh^2\left(\pi \dfrac{\sqrt{\sqrt{2}-1}}{2\sqrt{2}}\right) \approx 0.376\cdots$.

(ii) $f \in \mathscr{S}^*_{RL}$ in $|z| < \tanh^2\left(\pi \dfrac{\sqrt[4]{\sqrt{2(\sqrt{2}-1)}\left(1-\sqrt{2(\sqrt{2}-1)}\right)}}{2\sqrt{2}}\right) \approx 0.283\cdots$.

We now define the following class constructed with the help of ratios of two analytic functions $f, g \in \mathscr{A}$, defined as

$$\mathfrak{F}_A := \left\{ f \in \mathscr{A} : \operatorname{Re}\frac{f(z)}{g(z)} > 0 \;\&\; \operatorname{Re}\frac{(1-z)^{1+A}g(z)}{z} > 0, -1 \leq A \leq 1 \right\}.$$

The class $\mathfrak{F}_A$ reduces to the following classes, if we choose $A = -1$ and $A = 1$ respectively,

$$\mathfrak{F}_{-1} := \left\{ f \in \mathscr{A} : \operatorname{Re}\frac{f(z)}{g(z)} > 0 \;\&\; \operatorname{Re}\frac{g(z)}{z} > 0 \right\}$$

and

$$\mathfrak{F}_{1} := \left\{ f \in \mathscr{A} : \operatorname{Re}\frac{f(z)}{g(z)} > 0 \;\&\; \operatorname{Re}\left(\frac{(1-z)^2 g(z)}{z}\right) > 0 \right\}.$$

For proving the next theorem, we require the following lemma given by Ravichandran et al. [135] for the classes defined below:

$$\mathscr{P}_n[A,B] := \left\{ p \in \mathscr{H}[1,n] : p(z) \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1 \right\}, \tag{3.2.8}$$

and particularly, for $A = 1-2\beta$, where $0 \le \beta < 1$ and $B = -1$,

$$\mathscr{P}_n(\beta) := \left\{ p \in \mathscr{H}[1,n] : p(z) \prec \frac{1+(1-2\beta)z}{1-z}, 0 \le \beta < 1 \right\}. \tag{3.2.9}$$

**Lemma 3.2.3.** [135] If $p \in \mathscr{P}_n[A,B]$, then for $|z| = r$

$$\left| p(z) - \frac{1-ABr^{2n}}{1-B^2r^{2n}} \right| \le \frac{|A-B|r^n}{1-B^2r^{2n}}.$$

Particularly, if $p \in \mathscr{P}_n(\beta)$, then

$$\left| p(z) - \frac{1+(1-2\beta)r^{2n}}{1-r^{2n}} \right| \le \frac{2(1-\beta)r^n}{1-r^{2n}}.$$

**Theorem 3.2.5.** *Let $-1 \le A \le 1$, and suppose $f \in \mathscr{F}_\varphi$, then the sharp $\mathfrak{F}_A-$radius is given by*

$$\mathscr{R}_{\mathfrak{F}_A}(\mathscr{F}_\varphi) = \frac{1}{2A+3}\left( \sqrt{A^2+12A+28} - (5+A) \right) =: R_{\mathfrak{F}_A}.$$

*Proof.* Since $f \in \mathfrak{F}_A$, then by definition of class $\mathfrak{F}_A$, we have $f(z) = p_1(z)g(z)$ and $g(z) = zp_2(z)(1-z)^{-(1+A)}$, where for each $i = 1,2$, $p_i : \mathbb{D} \to \mathbb{C}$ are analytic functions such that $p_i(0) = 1$ and $\operatorname{Re} p_i(z) > 0$. This leads to $f(z) = zp_1(z)p_2(z)(1-z)^{-(1+A)}$, and as a consequence of logarithmic differentiation, we obtain

$$\frac{zf'(z)}{f(z)} = \frac{1+Az}{1-z} + \frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)}.$$

For each $-1 \le A \le 1$, Lemma 3.2.3 leads to,

$$\left| \frac{zf'(z)}{f(z)} - \frac{1+Ar^2}{1-r^2} \right| \le \frac{(5+A)r}{1-r^2} = R. \tag{3.2.10}$$

Further, for each $|z| = r \le R_{\mathfrak{F}_A}$, one can observe that

$$\frac{1}{2} \le 1 \le c = \frac{1+Ar^2}{1-r^2} \le \frac{1+AR_{\mathfrak{F}_A}^2}{1-R_{\mathfrak{F}_A}^2} < \frac{3}{2}. \tag{3.2.11}$$

Infact inequalities (3.2.10) and (3.2.11) yields the inequality,

$$\frac{(5+A)r}{1-r^2} \le \frac{3}{2} - \frac{1+Ar^2}{1-r^2},$$

provided $r \leq R_{\mathfrak{F}_A}$. Due to Lemma 3.2.2, it is clear that the disc $|w-c| < R$ lies in $\Omega_\varphi$ and

$$f_{\mathfrak{F}_A}(z) = \frac{z(1+z)^2}{(1-z)^{3+A}}$$

acts as the extremal function. □

**Corollary 3.2.4.** Let $f \in \mathscr{F}_\varphi$, then sharp $\mathfrak{F}_{-1}-$ radius and $\mathfrak{F}_1-$ radius for the class $\mathscr{F}_\varphi$ are respectively, given as

(i) $\mathscr{R}_{\mathfrak{F}_{-1}}(\mathscr{F}_\varphi) = \sqrt{17} - 4 \approx 0.123\cdots$.

(ii) $\mathscr{R}_{\mathfrak{F}_1}(\mathscr{F}_\varphi) = (\sqrt{41} - 6)/5 \approx 0.080\cdots$.

**Theorem 3.2.6.** *Let $1 < \alpha < 3/2$, and suppose $f \in \mathscr{F}_\varphi$, then $\mathscr{M}(\alpha)-$ radius is*

$$r_\alpha = 1 + 2\left(\cot\left(\frac{\pi\sqrt{\alpha-1}}{\sqrt{2}}\right)\right)^2 - 2\left|\sec\left(\frac{\pi\sqrt{\alpha-1}}{\sqrt{2}}\right)\left(\tan\left(\frac{\pi\sqrt{\alpha-1}}{\sqrt{2}}\right)\right)^{-2}\right|.$$

*Proof.* From Remark 3.2.1, it can be viewed that

$$\operatorname{Re}\varphi(z) \leq \varphi(-r) = 1 - \frac{2}{\pi^2}\left(\log\left(\frac{1+i\sqrt{r}}{1-i\sqrt{r}}\right)\right)^2 = 1 + \frac{2}{\pi^2}\left(\tan^{-1}\left(\frac{2\sqrt{r}}{1-r}\right)\right)^2.$$

As $f \in \mathscr{F}_\varphi$, then assume that $zf'(z)/f(z) = p(z)$. Due to the above inequality $\operatorname{Re} p(z) \leq \varphi(-r)$. Moreover, $\varphi(-r) \leq \alpha$ provided $r \leq r_\alpha$, where $r_\alpha$ is the root of the equation

$$(1-\alpha)\pi^2 + 2\left(\tan^{-1}\left(\frac{2\sqrt{r}}{1-r}\right)\right)^2 = 0$$

for $1 < \alpha < 3/2$. Equality here occurs for the function $f_0 \in \mathscr{A}$, given by (3.1.3). □

In 2017, Peng and Zhong [125], introduced the class $\Omega_0 := \{f \in \mathscr{A} : |zf'(z) - f(z)| < 1/2\}$. In the next theorem, we determine sharp $\Omega_0-$radius for the class $\mathscr{F}_\varphi$.

**Theorem 3.2.7.** *Let $f \in \mathscr{F}_\varphi$, then $f \in \Omega_0$ for $|z| < r_{\mathscr{L}} \approx 0.522\ldots$, where $r_{\mathscr{L}}$ is the smallest positive root of*

$$4f_0(r)(\log((1+\sqrt{r})/(1-\sqrt{r})))^2 = \pi^2$$

*and*

$$g_0(z) = z\left(\exp\int_0^z \frac{\varphi_0(-t)}{t}dt\right) = z + \frac{8}{\pi^2}z^2 - \frac{8}{3\pi^4}(\pi^2-12)z^3 + \frac{8}{135\pi^6}(1440 - 360\pi^2 + 23\pi^4)z^4 - \cdots. \tag{3.2.12}$$

*This is a sharp estimate.*

*Proof.* Since $f \in \mathscr{F}_\varphi$, then as a consequence of Remark 3.2.1 for $|z| = r < 1$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < |\varphi(r) - 1| = |\varphi_0(r)|.$$

Due to the Growth theorem as mentioned in [78, Theorem 1] and Theorem 3.2.1, we observe that $|f(z)| \leq g_0(r)$, where $g_0(r)$ is given by (3.2.12). Further,

$$|zf'(z) - f(z)| = |f(z)|\left|\frac{zf'(z)}{f(z)} - 1\right| \leq g_0(r)|\varphi_0(r)|.$$

Eventually, $g_0(r)|\varphi_0(r)| \leq 1/2$ provided $|z| < r_{\mathscr{L}} \approx 0.522864$. Hence the result is sharp for the function $f_0(z)$. □

## Sufficient Conditions for Functions to be in $\mathscr{F}_\varphi$

We now establish some sufficient conditions for functions to be in the class $\mathscr{F}_\varphi$, for which we need the following Lemma given by Jack [54]:

**Lemma 3.2.4.** [54, Lemma 1, p.470] Let $\nu(z)$ be a non-constant analytic function in $\mathbb{D}$, such that $\nu(0) = 0$. If $|\nu(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0$, then $z_0\nu'(z_0) = k\nu(z_0)$, where $k$ is real and $k \geq 1$.

**Theorem 3.2.8.** *Suppose* $0 \leq \mu \leq 1$ *and let* $f \in \mathscr{A}$ *satisfy the following differential inequality*

$$\left|\mu\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\mu)\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{6}(3 + 2\mu), \quad z \in \mathbb{D}, \tag{3.2.13}$$

*then* $f \in \mathscr{F}_\varphi$.

*Proof.* Consider an analytic function $\nu(z)$ with $\nu(0) = 0$. Assume $f \in \mathscr{A}$ such that

$$\frac{zf'(z)}{f(z)} - 1 = \frac{1}{2}\nu(z),$$

then we show that $|\nu(z)| < 1$ in $\mathbb{D}$. Suppose on the contrary $|\nu(z)| \geq 1$, then by an application of Lemma 3.2.4, there exists $z_0 \in \mathbb{D}$ such that for $k \geq 1$, $|\nu(z_0)| = 1$ and $z_0\nu'(z_0) = k\nu(z_0)$. Substituting $\nu(z_0) = e^{it}$, $-\pi < t \leq \pi$ leads to

$$\begin{aligned}
&\left|\mu\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\mu)\frac{zf'(z)}{f(z)} - 1\right| \\
&= \left|\mu\left(1 + \frac{\nu(z_0)}{2} + \frac{k\nu(z_0)}{2 + \nu(z_0)}\right) + (1-\mu)\left(1 + \frac{\nu(z_0)}{2}\right) - 1\right| \\
&= \left|\frac{\mu k e^{it}}{2 + e^{it}} + \frac{e^{it}}{2}\right| \geq \frac{1}{6}(3 + 2\mu).
\end{aligned}$$

This is a contradiction to the assumption given in (3.2.13). Thus $|\nu(z)| < 1$, which means that

$zf'(z)/f(z)$ lies in the disc $|(zf'(z)/f(z))-1|<1/2$. Hence in view of Lemma 3.2.2, (with $c=1$) required result is achieved. □

For $\mu = 1/2$, $\mu = 0$ and $\mu = 1$ in Theorem 3.2.8, we obtain the following corollary:

**Corollary 3.2.5.** If $f \in \mathscr{A}$ satisfy any of the following differential inequalities:

(i) $\left|\dfrac{zf'(z)}{f(z)} + \dfrac{zf''(z)}{f'(z)} - 1\right| < \dfrac{4}{3}$,

(ii) $\left|\dfrac{zf'(z)}{f(z)} - 1\right| < \dfrac{1}{2}$,

(iii) $\left|\dfrac{zf''(z)}{f'(z)}\right| < \dfrac{5}{6}$,

then $f \in \mathscr{F}_{\varphi}$.

Till now we have studied radius problems for a special type of class that includes non-univalent analytic functions. In the forthcoming chapter, we provide more radius results for the classes: $\mathscr{S}^*(\beta)$ and a product class defined using funtions lying in tilted Carathéodory class $\mathscr{P}_{\lambda}$, where $\mathscr{P}_{\lambda} := \{p \in \mathscr{H}_1 : \operatorname{Re}(e^{i\lambda}p(z)) > 0, -\pi/2 < \lambda < \pi/2\}$.

## Highlights of the Chapter

This chapter introduces a novel class of functions associated with a parabolic region—a topic that has been scarcely explored in the realm of non-univalent functions. We derived significant radius results, including the radius of univalence and starlikeness for the class $\mathscr{F}_{\varphi}$, contextualizing our findings within the framework of Ma-Minda and several other related classes. Our results, which are accompanied by illustrative examples and figures as special cases, are shown to be sharp, underscoring the precision and depth of our analysis.

The contents of this chapter is based on the findings presented in the paper:

*S. Sivaprasad Kumar and Mridula Mundalia: On a Class of Non-Univalent functions Associated with a Parabolic Region, (2023), (Under review).*

# Chapter 4

# Radius Problems for the Class $\mathscr{S}^*(\beta)$ and a Product Function

---

We determine various radius constants for the class $\mathscr{S}^*(\beta)$ of starlike functions of order $\beta$. We define $\mathscr{S}_{\lambda,\beta}$ to be the class of normalised analytic functions $f$ satisfying

$$\operatorname{Re}(e^{i\lambda}(1-z)^{1+\beta} f(z)/z) > 0, \quad z \in \mathbb{D},$$

and introduce a product function $G(z) := (1-z)^{1+\beta} g_1(z) g_2(z)/z$, with $g_1, g_2 \in \mathscr{S}_{\lambda,\beta}$, to find radius constants for $G(z)$ to be in certain desired classes. Notably, earlier known results are identified herein as special cases of our findings and all the results obtained are sharp.

---

## 4.1 Introduction

Radius problems hold significant importance in GFT and are widely studied due to their fundamental role in understanding the behavior of analytic functions. Determining sharp radius estimates has become a common pursuit among researchers in this field, as it offers insights into various geometric properties of functions in different classes. In the past, many authors (see [11, 24, 65, 90, 153]) have contributed to this area, exploring radius-related results through new and innovative approaches. Numerous advancements continue to emerge as this topic remains an active area of investigation. In this chapter, we draw inspiration from the works of Ravichandran et al., see [24, 153], utilizing their ideas to derive sharp and more refined findings in context of some subclasses of starlike functions.

In 1992, Ma-Minda dealt with Growth, distortion, covering and coefficient problems for the class $\mathscr{S}^*(\phi)$. The class $\mathscr{S}^*(\phi)$ has been extensively studied by various authors for different choices of $\phi(z)$, see [11, 23, 39, 59, 79, 81, 83, 104, 106, 139, 154, 156] and the references therein. Some of the popularly known choices of $\phi(z)$ that are needed for our study are listed in Table 1.1.

Let $\mathscr{G}$ and $\mathscr{H}$ be two subclasses of $\mathscr{A}$, then the $\mathscr{G}$-radius of $\mathscr{H}$, denoted by $R_{\mathscr{G}}(\mathscr{H})$ or simply $R_{\mathscr{G}}$ is the largest $R \in (0,1)$ such that $\rho^{-1} f(\rho z) \in \mathscr{G}$, whenever $0 < \rho \leq R$, for all $f \in \mathscr{H}$. For example, consider $\mathscr{H} = \mathscr{S}^*(\beta)$, the class of starlike functions of order $\beta$, where $0 \leq \beta < 1$, defined in (1.2.4) and $\mathscr{G} = \mathscr{S}^*(\phi)$, where $\mathscr{S}^*(\phi)$ is defined by (1.2.2). To find the largest $R \in (0,1)$, suppose $f \in \mathscr{S}^*(\beta)$, then we have

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z} =: \phi_\beta(z).$$

Define $f_\rho : \overline{\mathbb{D}} \to \mathbb{C}$ by $f_\rho(z) := f(\rho z)/\rho$. Consequently, we obtain

$$\frac{zf_\rho'(z)}{f_\rho(z)} \prec \frac{1+(1-2\beta)\rho z}{1-\rho z} = \phi_\beta(\rho z). \tag{4.1.1}$$

If $R_{\mathscr{S}^*(\phi)}$ is the largest $R$ such that $\phi_\beta(\rho z) \prec \phi(z)$, whenever $0 < \rho \leq R$, then it follows from (4.1.1) that $f_\rho \in \mathscr{S}^*(\phi)$. Thus, we can determine the desired radius through $R_{\mathscr{S}^*(\phi)} = \min\limits_{|z|=1} |\phi_\beta^{-1} \circ \phi(z)|$. Since $\phi_\beta(z)$ is univalent, we have

$$\phi_\beta^{-1}(w) = \frac{w-1}{w+1-2\beta}.$$

Therefore, the $\mathcal{S}^*(\phi)-$radius of $\mathcal{S}^*(\beta)$ will become

$$\begin{aligned} R_{\mathcal{S}^*(\phi)} &= \min_{|z|=1} \left| \frac{\phi(z)-1}{\phi(z)+1-2\beta} \right| \\ &= \left( \max_{|z|=1} |\Phi(z)| \right)^{-1}, \end{aligned} \tag{4.1.2}$$

where

$$\Phi(z) := 1 + \frac{2(1-\beta)}{\phi(z)-1}.$$

Note that the expression in (4.1.2), aids in determining $R_{\mathcal{S}^*(\phi)} < 1$ such that $\phi_\beta(\rho z) \prec \phi(z)$, for each $0 < \rho \leq R$. Now recall the generalized Koebe function $\tilde{f}(z)$, given by

$$\tilde{f}(z) = \frac{z}{(1-z)^{2(1-\beta)}}, \; 0 \leq \beta < 1. \tag{4.1.3}$$

In fact, $\tilde{f}(z)$ serves as the extremal function for all the sharp results derived in this section.

In Section 4.2, we focus on determining various radius constants for the class $\mathcal{S}^*(\beta)$, which involves the Ma-Minda subclasses of starlike functions. Additionally, we employ a well-established technique that illustrates an effective approach to tackle radius problems through subordination, yielding sharp results. Further, in Section 4.3, we determine certain radius results for a product function belonging to a class of tilted Carathéodory functions $\mathcal{P}_\lambda$, where $\mathcal{P}_\lambda$ is given by (4.3.14). Furthermore, corrected versions of some known results, are pointed out here, as a special case of our main theorem.

## 4.2 Radius Results for $\mathcal{S}^*(\beta)$

We proceed to estimate $R_{\mathcal{S}^*(\phi)}$, the $\mathcal{S}^*(\phi)-$radius of $\mathcal{S}^*(\beta)$, for various choices of $\phi(z)$, as outlined below:

**Theorem 4.2.1.** *If $f \in \mathcal{S}^*(\beta)$, where $0 \leq \beta < 1$, and $0 < s \leq 1/\sqrt{2}$, then the following radius constants are sharp:*

*(i)* $R_{\mathcal{S}^*_{SG}} = \dfrac{1}{1+2(1-\beta)\coth(1/2)}$.

*(ii)* $R_{\mathcal{S}^*_{\leftmoon}} = \dfrac{1}{1+\sqrt{2}(1-\beta)}$.

*(iii)* $R_{\mathcal{S}^*_{L}(s)} = \dfrac{s(s+2)}{2(1-\beta)+s^2+2s}$.

*(iv)* $R_{\mathcal{S}^*_{\sin}} = \dfrac{\sin 1}{\sin 1 + 2(1-\beta)}$.

*(v)* $R_{\mathcal{S}^*_{\sinh}} = \dfrac{\sinh^{-1} 1}{\sinh^{-1} 1 + 2(1-\beta)}$.

*Proof.* Since $f \in \mathcal{S}^*(\beta)$, then in view of (4.1.1), we have

$$\frac{zf'(z)}{f(z)} \prec \phi_\beta(z) = \frac{1+(1-2\beta)z}{1-z}.$$

(i) To find $R_{\mathscr{S}^*_{SG}}$, the $\mathscr{S}^*_{SG}-$ radius of $\mathscr{S}^*(\beta)$, consider $\phi(z)=2/(1+e^{-z})$, then from (4.1.2), we have

$$\Phi(z)=1+\frac{\lambda}{2/(1+e^{-z})-1}, \tag{4.2.4}$$

where $\lambda=2(1-\beta)$, $0<\lambda\leq 2$. Now to find maximum of $|\Phi(z)|$ on the boundary of $\mathbb{D}$, consider

$$\begin{aligned}
\left|\Phi(e^{it})\right|^2 &= \left|\frac{1+\lambda+(\lambda-1)e^{-e^{it}}}{1-e^{-e^{it}}}\right|^2 \\
&= \frac{(1+\lambda)^2+(\lambda-1)^2e^{-2\cos t}+2(\lambda^2-1)e^{-\cos t}\cos(\sin t)}{1+e^{-2\cos t}-2e^{-\cos t}\cos(\sin t)} \\
&= \frac{(1+\lambda)^2+(\lambda-1)^2e^{-2x}+2(\lambda^2-1)e^{-x}\cos(\sqrt{1-x^2})}{1+e^{-2x}-2e^{-x}\cos(\sqrt{1-x^2})} \\
&= \frac{p(x,\lambda)}{p(x,0)} =: q(x),
\end{aligned} \tag{4.2.5}$$

where $x=\cos t$, $-\pi<t\leq\pi$, and

$$p(x,k)=(1+k)^2+(k-1)^2e^{-2x}+2(k^2-1)e^{-x}\cos(\sqrt{1-x^2}). \tag{4.2.6}$$

Now to find the maximum of $q(x)$ in $[-1,1]$, consider the function $\Upsilon(x)$, defined as below, in the given range of $x$ and $\lambda$ :

$$\begin{aligned}
\Upsilon(x) &= p(x,\lambda)p(1,0)-p(x,0)p(1,\lambda) \\
&= 2e^{-2}(2\lambda\left(\lambda-1+e^2(1+\lambda)\right))e^{-x}\cos(\sqrt{1-x^2})-4\lambda(\lambda+e-1)e^{-(2x+1)} \\
&\quad -4e^{-2}\lambda(e(1+\lambda)-1) \\
&= \upsilon(x)-\upsilon(1),
\end{aligned}$$

where $\upsilon(x)=(2e^{-2}(2\lambda\left(\lambda-1+e^2(1+\lambda)\right))e^{-x}\cos(\sqrt{1-x^2})-4\lambda(\lambda+e-1)e^{-(2x+1)}$. A computation reveals that $\upsilon(x)$ is increasing in $[-1,1]$, therefore we have $\upsilon(x)\leq\upsilon(1)$, for $-1\leq x\leq 1$, which implies $\Upsilon(x)\leq 0$ and hence $q(x)\leq q(1)$. Therefore, the maximum of $q(x)$ is attained at $x=1$. Thus, from (4.2.5) we conclude that

$$|\Phi(e^{it})|\leq|\Phi(1)|. \tag{4.2.7}$$

Finally, by the definition of $R_{\mathscr{S}^*(\phi)}$ and $\Phi(z)$, given by (4.1.2) and (4.2.4), respectively and (4.2.7), we have

$$R_{\mathscr{S}^*_{SG}}=\left(\max_{|z|=1}|\Phi(z)|\right)^{-1}=\left|1+\frac{2(1-\beta)}{2/(1+e^{-1})-1}\right|^{-1}.$$

At $z = R_{\mathscr{S}^*_{SG}}$, the function $\tilde{f}(z)$, given by (4.1.3), satisfies the equality

$$\left|\log\left(\frac{z\tilde{f}'(z)/\tilde{f}(z)}{2 - z\tilde{f}'(z)/\tilde{f}(z)}\right)\right| = \left|\log\left(\frac{1+(1-2\beta)z}{1-(3-2\beta)z}\right)\right| = 1$$

and hence the result is sharp. For the diagrammatic validation of sharpness of the result when $\beta = 1/2$, see Figure 4.2(a).

(ii) To find $R_{\mathscr{S}^*_{\mathfrak{D}}}$, take $\phi(z) = z + \sqrt{1+z^2}$ and $\lambda = 2(1-\beta)$ with $0 < \lambda \leq 2$. From (4.1.2), we have

$$\Phi(z) = 1 + \frac{\lambda}{z - 1 + \sqrt{z^2+1}}. \tag{4.2.8}$$

Since maximum of $|\Phi(z)|$ is obtained on the boundary of $\mathbb{D}$, choose $z = e^{it}$ and $x = \cos(t/2)$ $(-\pi < t \leq \pi)$, then (4.2.8) reduces to

$$\begin{aligned}
\left|\Phi(e^{it})\right|^2 &= \left|\frac{e^{it} + \sqrt{1+e^{2it}} + \lambda - 1}{e^{it} + \sqrt{1+e^{2it}} - 1}\right|^2 \\
&= \frac{1 + (\lambda-1)^2 + 2\sqrt{2}\lambda\cos(t/2)\sqrt[4]{\cos^2 t} + 2(\lambda-1)\cos t + 2\sqrt{\cos^2 t}}{2 + 2(\sqrt{\cos^2 t} - \cos t)} \\
&= \frac{1 + (\lambda-1)^2 + 2\sqrt{2}\lambda x\sqrt[4]{(1-2x^2)^2} + 2(\lambda-1)\left(2x^2-1\right) + 2\sqrt{(2x^2-1)^2}}{4 - 4x^2 + 2\sqrt{(2x^2-1)^2}} \\
&= \frac{p(x,\lambda)}{p(x,0)} = q(x),
\end{aligned}$$

where

$$p(x,k) = 1 + (k-1)^2 + 2\sqrt{2}kx\sqrt[4]{(1-2x^2)^2} + 2(k-1)(2x^2-1) + 2\sqrt{(2x^2-1)^2}.$$

To find maximum of $q(x)$, we now define $\Upsilon(x)$ on $[0,1]$ as:

$$\begin{aligned}
\Upsilon(x) &= p(x,\lambda)p(1,0) - p(x,0)p(1,\lambda) \\
&= 4\sqrt{2}\lambda x\sqrt[4]{(1-2x^2)^2} + 4((\lambda+1+2\sqrt{2})(\lambda-1) + 2(\lambda+\sqrt{2}) + 1)x^2 \\
&\quad + 2(2 - ((\lambda-1)(\lambda+2\sqrt{2}+1) + 2\sqrt{2}+3))\sqrt{(2x^2-1)^2} - 2\lambda(\lambda + 4(1+\sqrt{2})) \\
&= \upsilon(x) - \upsilon(1),
\end{aligned}$$

where

$$\begin{aligned}
\upsilon(x) &= 4\sqrt{2}\lambda x\sqrt[4]{(1-2x^2)^2} + 4((\lambda+1+2\sqrt{2})(\lambda-1) + 2(\lambda+\sqrt{2}) + 1)x^2 \\
&\quad + 2(2 - ((\lambda-1)(\lambda+2\sqrt{2}+1) + 2\sqrt{2}+3))\sqrt{(2x^2-1)^2}.
\end{aligned}$$

Figure 4.1 depicts the graph of $\upsilon(x)$, when $\lambda = 0.2$. However, for each $\lambda \in (0,2]$ the graph retains its same character. From the graph of $\upsilon(x)$, we observe that the function $\upsilon(x)$ steadily increases in the intervals $(0, 1/\sqrt{2})$ and $(1/\sqrt{2}, 1)$ and has a vertical cusp at $x = 1/\sqrt{2}$, where the curve of $\upsilon(x)$ slightly dips and peaks in a small neighborhood of $x = 1/\sqrt{2}$ with $\upsilon(1/\sqrt{2}) < \upsilon(1)$ and $\upsilon(x) \leq \upsilon(1)$ in $[0,1]$.

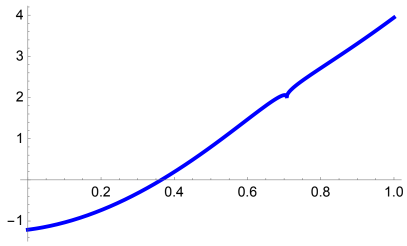


Figure 4.1: Graph of $\upsilon(x)$, with $\lambda = 0.2$

Thus, $\Upsilon(x) \leq 0$, which implies $p(x,\lambda)/p(x,0) \leq p(1,\lambda)/p(1,0)$ and hence $q(x) \leq q(1)$. Consequently, we have

$$|\Phi(e^{it})| \leq |\Phi(1)|. \tag{4.2.9}$$

Now from (4.1.2), (4.2.8) and (4.2.9), we conclude that

$$R_{\mathscr{S}^*_{\mathbb{D}}} = \left( \max_{|z|=1} |\Phi(z)| \right)^{-1} = \frac{1}{1+\sqrt{2}(1-\beta)}.$$

Further, for the function $\tilde{f}(z)$ given by (4.1.3), we have

$$\left| \left( \frac{z\tilde{f}'(z)}{\tilde{f}(z)} \right)^2 - 1 \right| = 2 \left| \frac{z\tilde{f}'(z)}{\tilde{f}(z)} \right|$$

at $z = R_{\mathscr{S}^*_{\mathbb{D}}}$ and hence the result is sharp. Sharpness of this result can be observed in Figure 4.2(c), for $\beta = 1/2$.

(iii) To find $R_{\mathscr{S}^*_L(s)}$, we choose $\phi(z) = (1+sz)^2$ and $\lambda = 2(1-\beta)$ with $0 < \lambda \leq 2$, then (4.1.2) gives $\Phi(z)$ as

$$\Phi(z) = 1 + \frac{\lambda}{(1+sz)^2 - 1}. \tag{4.2.10}$$

Now to find the maximum of $|\Phi(z)|$ on $\partial\mathbb{D}$, we take $z = e^{it}$, $x = \cos t$, where $-\pi < t \leq \pi$, then for each $0 < s \leq 1/\sqrt{2}$, (4.2.10) leads to

$$\begin{aligned} |\Phi(e^{it})|^2 &= \left| \frac{\lambda - 1 + (1+se^{it})^2}{(1+se^{it})^2 - 1} \right|^2 \\ &= \frac{\lambda^2 + s^4 + 2\lambda s^2 \cos 2t + 4s(\lambda + s^2)\cos t + 4s^2}{s^2\,(s^2 + 4s\cos t + 4)} \\ &= \frac{\lambda^2 + s^4 + 4s^3 x + s^2(\lambda\,(4x^2 - 2) + 4) + 4\lambda s x}{s^2(s^2 + 4sx + 4)} \\ &= \frac{p_s(x,\lambda)}{p_s(x,0)} = q_s(x), \end{aligned}$$

where

$$p_s(x,k) = k^2 + s^4 + 4s^3 x + s^2\left(k(4x^2 - 2) + 4\right) + 4ksx.$$

To find optimal value of $q_s(x)$, we now consider a function $\Upsilon_s(x)$ given as:

$$
\begin{aligned}
\Upsilon_s(x) &= p_s(x,\lambda)p_s(1,0) - p_s(1,\lambda)p_s(x,0) \\
&= 4\lambda s^4 x^2 (s+2)^2 - 4\lambda s^3 x\left(\lambda + s^2 - 4\right) + 4\lambda s^3\left(\lambda - s^3 - 3s^2 - 4s - 4\right) \\
&= \upsilon_s(x) - \upsilon_s(1),
\end{aligned} \tag{4.2.11}
$$

where $\upsilon_s(x) = 4\lambda s^4 x^2 (s+2)^2 - 4\lambda s^3 x\left(\lambda + s^2 - 4\right)$, where $-1 \leq x \leq 1$. Observe that $\upsilon_s'(x) = 0$ if and only if $x = (\lambda + s^2 - 4)/(2s(s+2)^2)$. Moreover, $\upsilon_s''(x) = 8\lambda s^4 (s+2)^2 > 0$, which implies $x = (\lambda + s^2 - 4)/2s(s+2)^2$ is the point of local minima, thus maximum exists at $x = \pm 1$. For each $s \in (0, 1/\sqrt{2}]$, one can notice that $\upsilon_s(-1) < \upsilon_s(1)$. Hence

$$
\max_{x \in [-1,1]} \upsilon_s(x) = \upsilon_s(1), \tag{4.2.12}
$$

Finally, from (4.2.11) and (4.2.12), we conclude that $\Upsilon_s(x) \leq 0$ in $[-1,1]$. Thus we have

$$
\frac{p_s(x,\lambda)}{p_s(x,0)} \leq \frac{p_s(1,\lambda)}{p_s(1,0)},
$$

which gives $q_s(x) \leq q_s(1)$ and

$$
|\Phi(e^{it})| \leq |\Phi(1)|. \tag{4.2.13}
$$

Hence (4.1.2), (4.2.10) and (4.2.13), yields,

$$
R_{\mathscr{S}^*_L(s)} = \left(\max_{|z|=1} |\Phi(z)|\right)^{-1} = \frac{s(s+2)}{2(1-\beta) + s^2 + 2s}.
$$

Hence, the function $\tilde{f}(z)$ given by (4.1.3) serves as the extremal function in this result as well. For sharpness, see Figure 4.2(d).

(iv) To find $R_{\mathscr{S}^*_{\sin}}$, we take $\phi(z) = 1 + \sin z$ and $\lambda = 2(1-\beta)$, with $0 < \lambda \leq 2$, then from (4.1.2) we have

$$
\Phi(z) = 1 + \frac{\lambda}{\sin z}.
$$

Since for each $z \in \mathbb{D}$, we have $|\sin z| \geq \sin 1$, then

$$
\left|1 + \frac{\lambda}{\sin z}\right| \leq 1 + \frac{\lambda}{\sin 1}.
$$

Therefore,

$$
R_{\mathscr{S}^*_{\sin}} = \left(\max_{|z|=1} |\Phi(z)|\right)^{-1} = \left|1 + \frac{2(1-\beta)}{\sin 1}\right|^{-1}.
$$

The function $\tilde{f}(z)$, as given in (4.1.3), is the extremal for this result. Infact the sharpness of this result for a specific case: $\beta = 1/2$, can be noticed in Figure 4.2(e). The proof of part (v) is much akin to that of part (iv), therefore it is skipped here. □

Let $-1 \leq B < A \leq 1$, the Janowski class of starlike functions $\mathcal{S}^*[A,B]$ is given by

$$\mathcal{S}^*[A,B] = \{f \in \mathcal{A} : zf'(z)/f(z) \prec (1+Az)/(1+Bz)\}.$$

In fact $\mathcal{S}^*[1-2\beta,-1] = \mathcal{S}^*(\beta)$, the class of starlike functions of order $\beta$. In the past, many radius results were established for the more general class $\mathcal{S}^*[A,B]$, often excluding the cases where $A = 0$ or $\beta = 1/2$, due to limitations of their approach. However, we have overcomed this limitation in our findings for the class $\mathcal{S}^*(\beta)$.

*Remark* 4.2.1. Below, we enumerate earlier results achieved through alternative approaches that align with our findings:

(i) Goel and Kumar [40, Corollary 2.4] established $\mathcal{S}^*_{SG}-$radius for the class $\mathcal{S}^*(\alpha)$, coinciding precisely with the result obtained in Theorem 4.2.1 (i).

(ii) For $A = 1-2\beta$ $(\beta \in [0,1)\setminus\{1/2\})$, $B = -1$ and $n = 1$, the $\mathcal{S}^*_L(s)-$radius, $\mathcal{S}^*_{\sin}-$radius and $\mathcal{S}^*_{\sinh}-$radius, for the class $\mathcal{S}^*(\beta)$ in [14, Theorem 3.6] [23, Theorem 3.7] and [11, Theorem 4.8] matches with the results obtained in Theorem 4.2.1 (ii)-(v), respectively.

(iii) When $s = 1/\sqrt{2}$, Theorem 4.2.1 (iii) reduces to a result of Bano and Raza [14, Theorem 3.6] for $A = 1-2\beta$ and $B = -1$.

(iv) Theorem 4.2.1 (ii) is the corrected version of Gandhi and Ravichandran [37, Theorem 2.2] result, for the case when $A = 1-2\beta$ and $B = -1$.

(v) Theorem 4.2.1 (iv) and (v), addresses the case $\beta = 1/2$, however, it was excluded in the respective findings of the results [23, Theorem 3.7] and [11, Theorem 4.8], when $A = 1-2\beta$ and $B = -1$.

When $\beta = 1/2$, we obtain the following result from Theorem 4.2.1 (see Figure 4.2).

**Corollary 4.2.1.** Let $f \in \mathcal{S}^*(1/2)$, then the following radius constants are sharp:

(i) $R_{\mathcal{S}^*_{SG}} = \dfrac{1}{1+\coth(1/2)} \approx 0.316\cdots.$

(ii) $R_{\mathcal{S}^*_{\mathfrak{D}}} = \dfrac{\sqrt{2}}{1+\sqrt{2}} \approx 0.585\cdots.$

(iii) $R_{\mathcal{S}^*_L(1/\sqrt{2})} = 4\sqrt{2}-5 \approx 0.656\cdots.$

(iv) $R_{\mathcal{S}^*_{\sin}} = \dfrac{\sin 1}{1+\sin 1} \approx 0.456\cdots.$

(v) $R_{\mathcal{S}^*_{\sinh}} = \dfrac{\sinh^{-1} 1}{1+\sinh^{-1} 1} \approx 0.468\cdots.$

The figures presented above depict images of the unit disc $\mathbb{D}$ and $\mathbb{D}_R$ under the mappings $\phi(z)$ and $\phi_\beta(z)$, respectively. Here, $\phi_\beta(z)$ remains fixed while $\phi(z)$ varies among different functions. These figures represent the outcomes outlined in Theorem 4.2.1, specifically

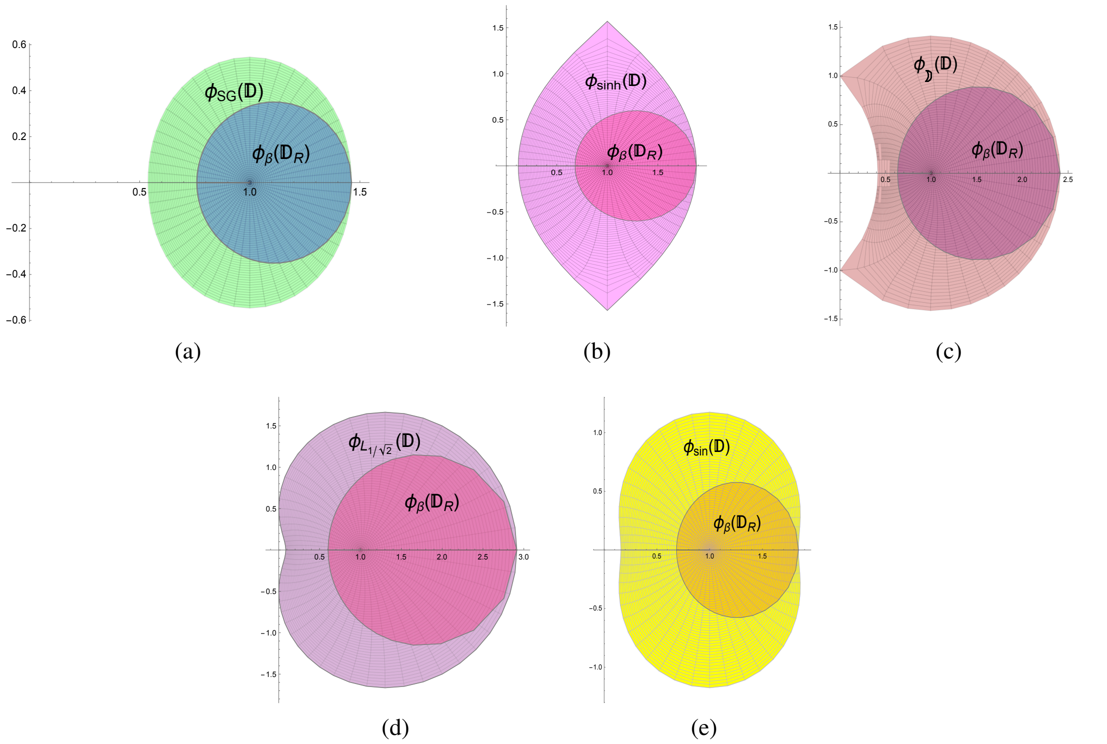


Figure 4.2: (a) $\phi_{SG}(z)=2/(1+e^{-z})$ and $R\approx 0.316\ldots$ (b) $\phi_{\sinh}(z)=1+\sinh^{-1}z$ and $R\approx 0.468\ldots$ (c) $\phi_{\mathfrak{D}}(z)=z+\sqrt{1+z^2}$ and $R\approx 0.585\ldots$ (d) $\phi_{L_{1/\sqrt{2}}}(z)=(1+(1/\sqrt{2})z)^2$ and $R\approx 0.656\ldots$ (e) $\phi_{\sin}(z)=1+\sin z$ and $R\approx 0.456\cdots$.

for the case when $\beta=1/2$. Notably, the boundaries of $\phi(\mathbb{D})$ and $\phi_\beta(\mathbb{D}_R)$ touch in each figure, clearly validating the sharpness of the results.

## 4.3 Radius Estimates for a Product Function Class

In 2010, Wang [171] studied the class of tilted Carathéodory functions $\mathscr{P}_\lambda$, defined as

$$\mathscr{P}_\lambda := \{p\in\mathscr{H}_1 : \mathrm{Re}(e^{i\lambda}p(z))>0\}, \tag{4.3.14}$$

where $-\pi/2<\lambda<\pi/2$. While investigating this class, certain authors limit their focus to the specific case where $\lambda=0$, due to a intrinsic challenge involved in the study of $\mathscr{P}_\lambda$. Note that $\mathscr{P}_0$ is the well known Carathéodory class. For more properties of the class $\mathscr{P}_\lambda$, one may refer to a short survey by Wang in [172]. The function $p_\lambda(z)=(1+e^{-2i\lambda}z)/(1-z)$ maps $\mathbb{D}$ univalently onto the tilted right-half plane $\mathbb{H}_\lambda=\{w\in\mathbb{C}:\mathrm{Re}(e^{i\lambda}w)>0\}$, which serves as the extremal function for several problems. For a function $p\in\mathscr{P}_\lambda$, Wang [171] determined the bound on $n$-th coefficient, sharp bounds on $|zp'(z)/p(z)|$ and $|p'(z)|$. Eventually these bounds are useful in establishing different radius problems. MacGregor [100] studied radius results for functions $f\in\mathscr{A}$ satisfying $\mathrm{Re}\, f(z)/z>0$. Further, Cho et al. [24] studied the class $\mathscr{S}_\lambda$ for $\lambda\in(-\pi/2,\pi/2)$, defined by $\mathscr{S}_\lambda:=\{f\in\mathscr{A}: f(z)/z\in\mathscr{P}_\lambda\}$. Motivated by the aforementioned work, here below we define a class $\mathscr{S}_{\lambda,\beta}$, which generalizes $\mathscr{S}_\lambda$ :

$$\mathscr{S}_{\lambda,\beta} := \left\{ f \in \mathscr{A} : \frac{(1-z)^{1+\beta} f(z)}{z} \in \mathscr{P}_\lambda \right\},$$

where $\beta \in [-1,1]$. We now determine sharp radius of starlikeness in context of various subclasses of starlike functions, for a product function

$$G(z) := \frac{(1-z)^{1+\beta} g_1(z) g_2(z)}{z},$$

where $g_1(z)$ and $g_2(z)$ are chosen from the class $\mathscr{S}_{\lambda,\beta}$. To prove our results, we require the following lemmas:

**Lemma 4.3.1.** [172, Theorem 6, p. 678] Let $p \in \mathscr{P}_\lambda$, $\lambda \in (-\pi/2, \pi/2)$. Then

$$\left| \frac{zp'(z)}{p(z)} \right| \leq M(\lambda, |z|),$$

where

$$M(\lambda, r) := \begin{cases} \dfrac{2r\cos\lambda}{1+r^2-2r|\sin\lambda|}, & r < |\tan(\lambda/2)|, \\[2ex] \dfrac{2r}{1-r^2}, & r \geq |\tan(\lambda/2)|. \end{cases} \tag{4.3.15}$$

The equality holds for some point $z_0 = re^{i\theta}$, $0 < r < 1$, if and only if $p(z) = p_\lambda(xz)$, where $x = e^{i(\tau_0 - \theta)}$ with $\tau_0$ satisfying

$$\begin{cases} \tau_0 = \lambda + \dfrac{\pi}{2}, & r < -\tan(\lambda/2), \\[2ex] \tau_0 = \lambda - \dfrac{\pi}{2}, & r < \tan(\lambda/2), \\[2ex] \sin(\tau_0 - \lambda) = \left(\dfrac{1+r^2}{r^2-1}\right)\sin\lambda, & r \geq |\tan(\lambda/2)|, \end{cases} \tag{4.3.16}$$

where $\mathscr{P}_\lambda$ is given by (4.3.14).

In the following Lemma, we combine all the inclusion results stated in [39, Lemma 2.2], [11, Lemma 2.1], [4, Lemma 2.2] and [79, Lemma 2.2], respectively:

**Lemma 4.3.2.** Let $m_1 < c < m_2$ and $\Delta_c := \{w \in \mathbb{C} : |w-c| < r_c^*\}$, then

$$\Delta_c \subset \begin{cases} \phi_{SG}(\mathbb{D}), & \text{for } m_1 = 2/(1+e),\ m_2 = 2e/(1+e),\ r_c^* = (e-1)/(e+1) - |c-1|, \quad (4.3.17) \\ \phi_{\sinh}(\mathbb{D}), & \text{for } m_1 = 1-\sinh^{-1} 1,\ \ m_2 = 1+\sinh^{-1} 1, \\ & r_c^* = \begin{cases} c-(1-\sinh^{-1} 1), & m_1 < c \leq 1, \\ (1-\sinh^{-1} 1) - c, & 1 \leq c < m_2, \end{cases} \quad (4.3.18) \\ \phi_L(\mathbb{D}), & \text{for } m_1 = 0,\ \ m_2 = \sqrt{2}, \\ & r_c^* = \begin{cases} (\sqrt{1-c^2}-(1-c^2))^{1/2}, & m_1 < c \leq 2\sqrt{2}/3, \\ \sqrt{2}-c, & 2\sqrt{2}/3 \leq c < m_2, \end{cases} \quad (4.3.19) \\ \phi_{\wp}(\mathbb{D}), & \text{for } m_1 = 1-e^{-1},\ m_2 = 1+e, \\ & r_c^* = \begin{cases} (c-1)+e^{-1}, & m_1 < c \leq 1+(e-e^{-1})/2, \\ e-(c-1), & 1+(e-e^{-1})/2 \leq c < m_2, \end{cases} \quad (4.3.20) \\ \phi_R(\mathbb{D}), & \text{for } m_1 = 2(\sqrt{2}-1),\ m_2 = 2, \\ & r_c^* = \begin{cases} c-2(\sqrt{2}-1), & m_1 < c \leq \sqrt{2}, \\ 2-c, & \sqrt{2} \leq c < m_2, \end{cases} \quad (4.3.21) \end{cases}$$

where $\phi_{SG}(z) := 2/(1+e^{-z})$, $\phi_{\sinh}(z) := 1+\sinh^{-1} z$, $\phi_L(z) := \sqrt{1+z}$, $\phi_{\wp}(z) := 1+ze^z$ and $\phi_R(z) := 1+(z/k)((k+z)/(k-z))$, where $k = \sqrt{2}+1$.

We begin with the following radii result:

**Theorem 4.3.1.** *Let $\beta \in [-1,1]$, $\lambda \in (-\pi/2, \pi/2)$ and $G(z) = (1-z)^{1+\beta} g_1(z) g_2(z)/z$ with $g_1, g_2 \in \mathcal{S}_{\lambda,\beta}$. Then $G \in \mathfrak{F}$, whenever $|z| = r < \min\{r_0(\beta), R\}$, where $R$ is given by*

$$R = \begin{cases} r_1(\beta), & r < |\tan(\lambda/2)|, \\ r_2(\beta), & r \geq |\tan(\lambda/2)|, \end{cases} \tag{4.3.22}$$

*with $r_1(\beta)$ being the smallest positive root of the equation $M_{\lambda,\beta}(r) = 0$. The result holds for the following cases:*

*(i) If $\mathfrak{F} = \mathcal{S}_L^*$, we find*

$$M_{\lambda,\beta}(r) = (\sqrt{2}-(\beta+\sqrt{2})r-1)(1+r^2-2r|\sin\lambda|) - 4r(1-r)\cos\lambda, \tag{4.3.23}$$

$$r_0(\beta) = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+\beta}} \quad \text{and} \quad r_2(\beta) = \frac{-(\beta+5)+\sqrt{\beta(\beta+4\sqrt{2}+6)-4\sqrt{2}+33}}{2(\beta+\sqrt{2})}.$$

*(ii) If $\mathfrak{F} = \mathcal{S}^*_{\sinh}$, we find*

$$M_{\lambda,\beta}(r) = \left(\sinh^{-1} 1 - r\left(1+\beta+\sinh^{-1} 1\right)\right)(1+r^2-2r|\sin\lambda|) - 4r(1-r)\cos\lambda, \tag{4.3.24}$$

$$r_0(\beta) = \sqrt{\frac{\sinh^{-1} 1}{1+\beta+\sinh^{-1} 1}}$$

*and*

$$r_2(\beta) = \frac{-(5+\beta)+\sqrt{\beta^2+10\beta+4(1+\beta+\sinh^{-1} 1)\sinh^{-1} 1+25}}{2(1+\beta+\sin^{-1} 1)}.$$

*(iii) If $\mathfrak{F} = \mathscr{S}^*_{SG}$, we find*

$$M_{\lambda,\beta}(r) = (e(1-(\beta+2)r)-1-\beta r)(1+r^2-2r|\sin\lambda|)-4(1+e)(1-r)r\cos\lambda,$$

$$r_0(\beta) = \sqrt{\frac{e-1}{\beta+e(\beta+2)}}$$

*and*

$$r_2(\beta) = \frac{-(1+e)(\beta+5)+\sqrt{(1+e)^2\beta^2+2\left(3+10e+7e^2\right)\beta+33e^2+42e+25}}{2(\beta(e+1)+2e)}.$$

*(iv) If $\mathfrak{F} = \mathscr{S}^*_{p}$, we find*

$$M_{\lambda,\beta}(r) = ((1+r^2)(1-(1+2\beta)r)-2r(1-r(1+2\beta))|\sin\lambda|-8r(1+r)\cos\lambda),$$

$$r_0(\beta) = \sqrt{\frac{1}{2\beta+3}}$$

*and*

$$r_2(\beta) = \begin{cases} \dfrac{5+\beta}{1+2\beta}+\dfrac{\sqrt{\beta^2+8\beta+24}}{|1+2\beta|}, & -1 \le \beta < -1/2 \\[2ex] \dfrac{1}{9}, & \beta = -1/2, \\[2ex] \dfrac{1}{1+2\beta}\left(5+\beta-\sqrt{\beta^2+8\beta+24}\right), & -1/2 < \beta \le 1. \end{cases}$$

*(v) If $\mathfrak{F} = \mathscr{S}^*_{\sin}$, we find*

$$M_{\lambda,\beta}(r) = (\sin 1 - r(1+\beta+\sin 1))(1+r^2-2r|\sin\lambda|)-4r(1-r)\cos\lambda,$$

$$r_0(\beta) = \sqrt{\frac{\sin 1}{1+\beta+\sin 1}}$$

*and*

$$r_2(\beta) = \frac{-(\beta+5)+\sqrt{\beta^2+10\beta+4(1+\beta+\sin 1)\sin 1+25}}{2(1+\beta+\sin 1)}.$$

*The result is sharp.*

*Proof.* Since $g_1, g_2 \in \mathcal{S}_{\lambda,\beta}$, we can find $p_1, p_2 \in \mathcal{P}_\lambda$ such that $p_1(z) = (1-z)^{1+\beta} g_1(z)/z$ and $p_2(z) = (1-z)^{1+\beta} g_2(z)/z$. Thus, we have $G(z) = z(1-z)^{-(1+\beta)} p_1(z) p_2(z)$ and

$$\frac{zG'(z)}{G(z)} = \frac{1+\beta z}{1-z} + \frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)}.$$

Now for $|z| = r < 1$ and from Lemma 4.3.1, we have

$$\left| \frac{zG'(z)}{G(z)} - \frac{1+\beta r^2}{1-r^2} \right| \leq \frac{(1+\beta)r}{1-r^2} + 2M(\lambda, r), \tag{4.3.25}$$

where $M(\lambda, r)$ is given by (4.3.15). Since $-1 \leq \beta \leq 1$, we have

$$\frac{1+\beta r^2}{1-r^2} \geq 1. \tag{4.3.26}$$

(i) From inclusion (4.3.19) of Lemma 4.3.2 and (4.3.26), we have $\Delta_c = \{w : |w-c| < r_c^*\} \subset \{w : |w^2 - 1| < 1\} = \phi_L(\mathbb{D})$, whenever $r_c^* = \sqrt{2} - c$ for $2\sqrt{2}/3 \leq c < \sqrt{2}$. Therefore, if $w := zG'(z)/G(z) \in \Delta_c$, then $zG'(z)/G(z) \prec \phi_L(z)$ iff $G \in \mathcal{S}_L^*$. Now in view of (4.3.25), $w \in \Delta_c$, provided

$$\frac{1+\beta r^2}{1-r^2} \leq \sqrt{2} \tag{4.3.27}$$

and

$$\frac{(1+\beta)r}{1-r^2} + 2M(\lambda, r) \leq \sqrt{2} - \frac{1+\beta r^2}{1-r^2}. \tag{4.3.28}$$

Now on substituting (4.3.15) in (4.3.28), yields the following inequalities:

$$\begin{cases} (\sqrt{2} - (\beta + \sqrt{2})r - 1)(1 + r^2 - 2r|\sin\lambda|) & \text{for } r < |\tan(\lambda/2)|, \\ -4r(1-r)\cos\lambda \geq 0, & \\ & \\ (\beta + \sqrt{2})r^2 + (\beta + 5)r - \sqrt{2} + 1 \leq 0, & \text{for } r \geq |\tan(\lambda/2)|. \end{cases} \tag{4.3.29}$$

Observe that each coefficient of the cubic equation $M_{\lambda,\beta}(r) = 0$, where $M_{\lambda,\beta}(r)$ is given by (4.3.23), are real, therefore it has atleast one real root. Finally, one can conclude that the desired result follows from (4.3.27) and (4.3.29).

(ii) Due to inclusion (4.3.18) of Lemma 4.3.2 and (4.3.26), we have $\Delta_c = \{w : |w-c| < r_c^*\} \subset \{w : |\sinh(w-1)| < 1\} = \phi_\rho(\mathbb{D})$, whenever $r_c^* = 1 + \sinh^{-1} 1 - c$ for $1 \leq c < 1 + \sinh^{-1} 1$. Thus if $w := zG'(z)/G(z) \in \Delta_c$, then $zG'(z)/G(z) \prec \phi_\rho(z)$ iff $G \in \mathcal{S}_{\sinh}^*$. Now in view of (4.3.25), $w \in \Delta_c$, provided

$$\frac{1+\beta r^2}{1-r^2} \leq 1 + \sinh^{-1} 1 \tag{4.3.30}$$

and

$$\frac{(1+\beta)r}{1-r^2}+2M(\lambda,r)\leq 1+\sinh^{-1}1-\frac{1+\beta r^2}{1-r^2}. \tag{4.3.31}$$

Upon using, (4.3.15) in (4.3.31), we have the following inequalities:

$$\begin{cases} \left(\sinh^{-1}1-r\left(1+\beta+\sinh^{-1}1\right)\right)(1+r^2-2r|\sin\lambda|) & \text{for } r<|\tan(\lambda/2)|, \\ \quad -4r(1-r)\cos\lambda\geq 0, & \\ & \\ r^2\left(1+\beta+\sinh^{-1}1\right)+(5+\beta)r-\sinh^{-1}1\leq 0, & \text{for } r\geq|\tan(\lambda/2)|. \end{cases} \tag{4.3.32}$$

Hence the desired result now follows at once from (4.3.30) and (4.3.32).

(iii) In view of inclusion (4.3.17) of Lemma 4.3.2 and (4.3.26), we have $\Delta_c=\{w:|w-c|<r_c^*\}\subset\{w:|\log(w/(2-w))|<1\}=\phi_{SG}(\mathbb{D})$, whenever $r_c^*=(e-1)/(e+1)-|c-1|$ for $1\leq c\leq 2e/(1+e)$. Thus if $w:=zG'(z)/G(z)\in\Delta_c$, then $zG'(z)/G(z)\prec\phi_{SG}(z)$ iff $G\in\mathscr{S}^*_{SG}$. Now in view of (4.3.25), $w\in\Delta_c$, provided

$$\frac{1+\beta r^2}{1-r^2}\leq\frac{2e}{1+e} \tag{4.3.33}$$

and

$$\frac{(1+\beta)r}{1-r^2}+2M(\lambda,r)\leq\frac{e-1}{e+1}-\frac{1+\beta r^2}{1-r^2}+1. \tag{4.3.34}$$

Now in view of (4.3.15), the inequality (4.3.34), simplifies to the following set of inequalities:

$$\begin{cases} (e(1-(\beta+2)r)-1-\beta r)(1+r^2-2r|\sin\lambda|) & \text{for } r<|\tan(\lambda/2)|, \\ \quad -4(1+e)(1-r)r\cos\lambda\geq 0, & \\ & \\ (\beta+e(\beta+2))r^2+(\beta+e(\beta+5)+5)r-e+1\leq 0, & \text{for } r\geq|\tan(\lambda/2)| \end{cases} \tag{4.3.35}$$

Hence, due to a similar argument as in part (i) of this theorem and from the inequalities outlined in (4.3.33) and (4.3.35), we promptly infer the desired result.

The proofs of (iv) and (v) are much akin to the proof of part (i), follows by an application of inclusions in [154, Pg 321] and [23, Lemma 3.3], respectively. Thus we omit the proofs of these parts. For the sake of sharpness of the result, let us consider the function

$$G_0(z)=\frac{(1-z)^{1+\beta}g_1(z)g_2(z)}{z}, \tag{4.3.36}$$

where $g_1(z)$ and $g_2(z)$ are defined as:

$$g_1(z) = g_2(z) = \frac{z(1-z)^{-(1+\beta)}(1+e^{-2i\lambda}xz)}{1-xz}. \tag{4.3.37}$$

We can observe that $(1-z)^{1+\beta}g_1(z)/z = p_1(xz) \in \mathscr{P}_\lambda$ and $(1-z)^{1+\beta}g_2(z)/z = p_2(xz) \in \mathscr{P}_\lambda$. Therefore, according to Lemma 4.3.1, the equality in the result occurs in case for the functions $G_0(z) = (1-z)^{1+\beta}g_1(z)g_2(z)/z$ at some point $z_0 = re^{i\theta}$, where $0 < r < 1$, and $x = e^{i(\tau_0-\theta)}$, with $\tau_0$ as given in (4.3.16). $\square$

In the next theorem, we study another set of radius results in context of these classes: $\mathscr{S}^*_\wp$, $\mathscr{S}^*_R$, $\mathscr{S}^*_c$, $\mathscr{S}^*_q$ and $\mathscr{S}^*_e$.

**Theorem 4.3.2.** *Let $\beta \in [-1,1]$, $\lambda \in (-\pi/2, \pi/2)$ and $G(z) = (1-z)^{1+\beta}g_1(z)g_2(z)/z$ with $g_1, g_2 \in \mathscr{S}_{\lambda,\beta}$. Then $G \in \mathfrak{F}$ whenever $|z| = r < \min\{r_0(\beta), R\}$ where $R$ is given by (4.3.22), with $r_1(\beta)$ being the smallest positive root of the equation $M_{\lambda,\beta}(r) = 0$. The result applies for the following cases:*

*(i) (a) If $\mathfrak{F} = \mathscr{S}^*_\wp$ and $r_0(\beta) = \sqrt{\dfrac{e^2-1}{2e(1+\beta)+e^2-1}}$, we obtain*

$$\begin{aligned} M_{\lambda,\beta}(r) = {} & (r+1)\left(1+r^2-2r|\sin\lambda|\right) - er((1+\beta)\left(1+r^2-2r|\sin\lambda|\right) \\ & +4(1+r)\cos\lambda) \end{aligned}$$

*and*

$$r_2(\beta) = \begin{cases} \dfrac{1}{2}\left(\dfrac{e(\beta+5)}{e(\beta+1)-1} + \dfrac{\sqrt{e^2(\beta+5)^2-4e(\beta+1)+4}}{|e(\beta+1)-1|}\right), & -1 \le \beta < (1-e)/e, \\ \dfrac{1}{1+4e}, & \beta = (1-e)/e, \\ \dfrac{1}{2(e(\beta+1)-1)}\left(e(\beta+5) - \sqrt{e^2(\beta+5)^2-4e(\beta+1)+4}\right), & (1-e)/e < \beta \le 1. \end{cases}$$

*(b) If $\mathfrak{F} = \mathscr{S}^*_\wp$ and $\sqrt{\dfrac{e^2-1}{2e(\beta+1)+e^2-1}} \le r_0(\beta) < \sqrt{\dfrac{e}{\beta+e+1}}$, with $-1 < \beta \le 1$, we obtain*

$$M_{\lambda,\beta}(r) = 4(r-1)r\cos\lambda - ((\beta+1)r + e(r-1))\left(1+r^2-2r|\sin\lambda|\right)$$

*and*

$$r_2(\beta) = \frac{-(5+\beta)+\sqrt{(\beta+5)^2+4e(\beta+1)+4e^2}}{2(1+e+\beta)}.$$

(ii) (a) *If* $\mathfrak{F}=\mathscr{S}^*_R$ *and* $r_0(\beta)=\sqrt{\dfrac{\sqrt{2}-1}{\beta+\sqrt{2}}}$, *we obtain*

$$M_{\lambda,\beta}(r)=(3-(\beta+2\sqrt{2}-2)r-2\sqrt{2})(1+r^2-2r|\sin\lambda|)-4r(1+r)\cos\lambda$$

*and*

$$r_2(\beta)=\begin{cases}\dfrac{1}{2}\left(\dfrac{\beta+5}{\beta+2\sqrt{2}-2}+\dfrac{\sqrt{\beta^2+(8\sqrt{2}-2)\beta-40\sqrt{2}+81}}{|\beta+2\sqrt{2}-2|}\right), & -1\leq\beta<2(1-\sqrt{2}),\\[2ex] \dfrac{3-2\sqrt{2}}{7-2\sqrt{2}}, & \beta=2(1-\sqrt{2}),\\[2ex] \dfrac{1}{2(\beta+2\sqrt{2}-2)}\left(\beta+5-\sqrt{\beta^2+(8\sqrt{2}-2)\beta-40\sqrt{2}+81}\right), & 2(1-\sqrt{2})<\beta\leq 1.\end{cases}$$

(b) *If* $\mathfrak{F}=\mathscr{S}^*_R$ *and* $\sqrt{\dfrac{\sqrt{2}-1}{\beta+\sqrt{2}}}\leq r_0(\beta)<\sqrt{\dfrac{1}{\beta+2}}$, *with* $-1<\beta\leq 1$, *we obtain*

$$M_{\lambda,\beta}(r)=(1-(\beta+2)r)(1+r^2-2r|\sin\lambda|)-4r(1-r)\cos\lambda$$

*and*

$$r_2(\beta)=\frac{-(\beta+5)+\sqrt{\beta^2+14\beta+33}}{2(\beta+2)}.$$

(iii) (a) *If* $\mathfrak{F}=\mathscr{S}^*_c$ *and* $r_0(\beta)=\sqrt{\dfrac{2}{3\beta+5}}$, *we obtain*

$$M_{\lambda,\beta}(r)=(2-(1+3\beta)r)\left(1+r^2-2r|\sin\lambda|\right)-12r(1+r)\cos\lambda$$

*and*

$$r_2(\beta)=\begin{cases}\dfrac{1}{2}\left(\dfrac{3(\beta+5)}{1+3\beta}+\dfrac{\sqrt{9\beta^2+66\beta+217}}{|1+3\beta|}\right), & -1\leq\beta<-1/3,\\[2ex] \dfrac{1}{7}, & \beta=-1/3,\\[2ex] \dfrac{1}{2(1+3\beta)}\left(3(\beta+5)-\sqrt{9\beta^2+66\beta+217}\right), & -1/3<\beta\leq 1.\end{cases}$$

(b) *If* $\mathfrak{F}=\mathscr{S}^*_c$ *and* $\sqrt{\dfrac{2}{3\beta+5}}\leq r_0(\beta)<\sqrt{\dfrac{2}{\beta+3}}$, *with* $-1<\beta\leq 1$, *we obtain*

$$M_{\lambda,\beta}(r)=(2-(3+\beta)r)\left(1+r^2-2r|\sin\lambda|\right)-4r(1-r)\cos\lambda$$

*and*

$$r_2(\beta) = \frac{-(5+\beta)+\sqrt{\beta^2+18\beta+49}}{2(3+\beta)}.$$

*(iv) (a) If* $\mathfrak{F} = \mathcal{S}_q^*$ *and* $r_0(\beta) = \sqrt{\dfrac{\sqrt{2}-1}{\beta+\sqrt{2}}}$, *we obtain*

$$M_{\lambda,\beta}(r) = ((2+r(1-\beta)) - \sqrt{2}(1+r))(1+r^2-2r|\sin\lambda|) - 4r(1+r)\cos\lambda$$

*and*

$$r_2(\beta) = \begin{cases} \dfrac{1}{2}\left(\dfrac{\beta+5}{\beta+\sqrt{2}-1} + \dfrac{\sqrt{\beta^2+2\beta(1+2\sqrt{2})-12\sqrt{2}+41}}{|\beta+\sqrt{2}-1|}\right), & -1 \le \beta < 1-\sqrt{2}, \\ \dfrac{\sqrt{2}-1}{3\sqrt{2}-1}, & \beta = 1-\sqrt{2}, \\ \dfrac{1}{2\left(\beta+\sqrt{2}-1\right)}\left(\beta+5-\sqrt{\beta^2+2\beta(1+2\sqrt{2})-12\sqrt{2}+41}\right), & 1-\sqrt{2} < \beta \le 1. \end{cases}$$

*(b) If* $\mathfrak{F} = \mathcal{S}_q^*$ *and* $\sqrt{\dfrac{\sqrt{2}-1}{\beta+\sqrt{2}}} \le r_0(\beta) < \sqrt{\dfrac{\sqrt{2}}{\beta+\sqrt{2}+1}}$, *with* $-1 < \beta \le 1$, *we obtain*

$$M_{\lambda,\beta}(r) = (\sqrt{2} - (1+\beta+\sqrt{2})r)(1+r^2-2r|\sin\lambda|) - 4r(1-r)\cos\lambda$$

*and*

$$\frac{-(\beta+5)+\sqrt{\beta^2+2(2\sqrt{2}+5)\beta+4\sqrt{2}+33}}{2(\beta+\sqrt{2}+1)}.$$

*(v) (a) If* $\mathfrak{F} = \mathcal{S}_e^*$ *and* $r_0(\beta) = \sqrt{\dfrac{e-1}{2e\beta+e^2+1}}$, *we obtain*

$$M_{\lambda,\beta}(r) = (e(1-Ar) - (1+r))(1+r^2-2r|\sin\lambda|) - 4er(1+r)\cos\lambda)$$

*and*

$$r_2(\beta) = \begin{cases} \dfrac{1}{2}\left(\dfrac{e(\beta+5)}{1+e\beta} + \dfrac{\sqrt{e^2\left(\beta^2+6\beta+25\right)+4e(\beta-1)+4}}{|1+e\beta|}\right), & -1 \le \beta < -1/e, \\ \dfrac{e-1}{5e-1}, & \beta = -1/e, \\ \dfrac{1}{2(1+e\beta)}\left(e(\beta+5)-\sqrt{e^2\left(\beta^2+6\beta+25\right)+4e(\beta-1)+4}\right), & -1/e < \beta \le 1. \end{cases}$$

*(b) If* $\mathfrak{F} = \mathcal{S}_e^*$ *and* $\sqrt{\dfrac{e-1}{2e\beta+e^2+1}} \le r_0(\beta) < \sqrt{\dfrac{e-1}{\beta+e}}$, *with* $-1 < \beta \le 1$, *we obtain*

$$M_{\lambda,\beta}(r) = (e(1-r) - \beta r - 1)(1+r^2-2r|\sin\lambda|) - 4r(1-r)\cos\lambda$$

*and*

$$r_2(\beta) = \frac{-(\beta+5)+\sqrt{\beta^2+(6+4e)\beta+4e^2-4e+25}}{2(\beta+e)}.$$

*The result is sharp.*

*Proof.* (i) From inclusion (4.3.20) of Lemma 4.3.2 and (4.3.26), we have $\Delta_c = \{w : |w-c| < r_c^*\} \subset \phi_{\wp}(\mathbb{D})$, whenever

$$r_c^* = \begin{cases} (c-1)+e^{-1}, & 1-e^{-1} < c \leq 1+(e-e^{-1})/2, \\ e-(c-1), & 1+(e-e^{-1})/2 \leq c < 1+e. \end{cases}$$

Therefore, if $w := zG'(z)/G(z) \in \Delta_c$, then $zG'(z)/G(z) \prec \phi_{\wp}(z)$ iff $G \in \mathscr{S}^*_{\wp}$. Subsequently, Lemma 4.3.1 and inequality (4.3.25) lead to $w \in \Delta_c$, provided

$$\frac{1+\beta r^2}{1-r^2} \leq 1+\frac{e-e^{-1}}{2}$$

and

$$\frac{(1+\beta)r}{1-r^2}+2M(\lambda,r) \leq \frac{1+\beta r^2}{1-r^2}-1+\frac{1}{e} \tag{4.3.38}$$

or

$$1+\frac{e-e^{-1}}{2} \leq \frac{1+\beta r^2}{1-r^2} < 1+e$$

and

$$\frac{(1+\beta)r}{1-r^2}+2M(\lambda,r) \leq 1+e-\frac{1+\beta r^2}{1-r^2}. \tag{4.3.39}$$

On substituting (4.3.15) in (4.3.38) and (4.3.39), we get the following inequalities:

(I). If $1 \leq a = (1+\beta r^2)/(1-r^2) \leq 1+(e-e^{-1})/2$, then

$$\begin{cases} \left(1+r^2-2r|\sin\lambda|\right)(1+r-er(1+\beta)) & \text{for } r < |\tan(\lambda/2)|, \\ \quad -4er(1+r)\cos\lambda \geq 0, & \\ & \\ r(e(\beta+5-(\beta+1)r)+r)-1 \leq 0, & \text{for } r \geq |\tan(\lambda/2)|. \end{cases} \tag{4.3.40}$$

(II). If $1+(e-e^{-1})/2 \leq a = (1+\beta r^2)/(1-r^2) \leq 1+e$, then, we have

$$\begin{cases} -((\beta+1)r^2+e(r-1))\left(1+r^2-2r|\sin\lambda|\right) & \text{for } r < |\tan(\lambda/2)|, \\ \quad 4(r-1)r\cos\lambda \geq 0, & \\ & \\ r(\beta(r+1)+r+5)+er^2-e \leq 0, & \text{for } r \geq |\tan(\lambda/2)|. \end{cases} \tag{4.3.41}$$

Thus, inequalities (4.3.40)-(4.3.41) leads to the required radius and $G_0(z)$ given by (4.3.36), serves as the extremal function.

From inclusion (4.3.21) of Lemma 4.3.2 and (4.3.26), we have $\Delta_c = \{w : |w-c| < r_c^*\} \subset \phi_R(\mathbb{D})$, whenever

$$r_c^* = \begin{cases} c - 2(\sqrt{2}-1), & 2(\sqrt{2}-1) < c \leq \sqrt{2}, \\ 2-c, & \sqrt{2} \leq c < 2. \end{cases}$$

Therefore, if $w := zG'(z)/G(z) \in \Delta_c$, then $zG'(z)/G(z) \prec \phi_R(z)$ iff $G \in \mathcal{S}^*_R$. Consequently, Lemma 4.3.1 and inequality (4.3.25) imply that $w \in \Delta_c$, if

$$\frac{1+\beta r^2}{1-r^2} \leq \sqrt{2} \tag{4.3.42}$$

and

$$\frac{(1+\beta)r}{1-r^2} + 2M(\lambda, r) \leq \frac{1+\beta r^2}{1-r^2} - 2(\sqrt{2}-1) \tag{4.3.43}$$

or

$$\sqrt{2} \leq \frac{1+\beta r^2}{1-r^2} < 2$$

and

$$\frac{(1+\beta)r}{1-r^2} + 2M(\lambda, r) \leq 2 - \frac{1+\beta r^2}{1-r^2} \tag{4.3.44}$$

holds. In view of (4.3.15), inequalities (4.3.43) and (4.3.44) leads us to the following cases:

(I). If $1 \leq a = (1+\beta r^2)/(1-r^2) \leq \sqrt{2}$, then we have

$$\begin{cases} (3-2\sqrt{2}-(\beta+2\sqrt{2}-2)r)\left(1+r^2-2r|\sin\lambda|\right) & \text{for } r < |\tan(\lambda/2)|, \\ \quad -4r(1+r)\cos\lambda \geq 0, & \\ & \\ (\beta+2\sqrt{2}-2)r^2 - (\beta+5)r - 2\sqrt{2}+3 \geq 0, & \text{for } r \geq |\tan(\lambda/2)|. \end{cases} \tag{4.3.45}$$

(II). If $\sqrt{2} \leq a = (1+\beta r^2)/(1-r^2) < 2$, then

$$\begin{cases} (1-(2+\beta)r)\left(1+r^2-2r|\sin\lambda|\right) - 4r(1-r)\cos\lambda \geq 0, & \text{for } r < |\tan(\lambda/2)|, \\ & \\ (\beta+2)r^2 + (\beta+5)r - 1 \leq 0, & \text{for } r \geq |\tan(\lambda/2)|. \end{cases} \tag{4.3.46}$$

Thus the inequalities obtained in (4.3.45) and(4.3.46) yields the required radius. Further, the proof of parts (iii)-(v) are similar to that of part (i) and hence follow by the inclusions given in [156, Lemma 2.5] [37, Lemma 2.1] and [106, Lemma 2.2], respectively, therefore, it is omitted here.

Finally, equality here holds for the functions $g_1(z)$ and $g_2(z)$ given by (4.3.37) such that $G_0(z) = (1-z)^{1+\beta} g_1(z) g_2(z)/z$. □

*Remark* 4.3.1. We establish the corrected versions of the findings presented in [24, Theorem 3, 6, 11, 7, 9, 8 and 10] as special case to Theorem 4.3.1(i), (iv), (v) and Theorem 4.3.2 (ii)-(v), respectively, all articulated in Corollary 4.3.1 and Corollary 4.3.2.

Note that Theorem 4.3.1 and Theorem 4.3.2, holds good for different values of $\beta$ in the range $[-1,1]$. In particular, when $\beta=-1$, center of the disc, given in (4.3.25) becomes $1$, which implies $r_0(-1)=1$. Since $r<1$, the equation $M_{\lambda,-1}(r)=0$ simplifies to a quadratic equation in $r$, whose root is $r_1(-1)=r_1$, and $r_2(-1)=r_2$, resulting in the following corollary:

**Corollary 4.3.1.** Let $\lambda \in (-\pi/2, \pi/2)$ and $G(z) = g_1(z)g_2(z)/z$ with $g_1, g_2 \in \mathscr{S}_\lambda$. Then $G \in \mathfrak{F}$, whenever $|z| = r < R$, where $R$ is given by (4.3.22). The result holds for the following cases:

(i) If $\mathfrak{F} = \mathscr{S}^*_L$, then

$$r_1 = |\sin\lambda| + 2(1+\sqrt{2})\cos\lambda - \sqrt{\cos\lambda\,(4(1+\sqrt{2})\,|\sin\lambda| + (8\sqrt{2}+11)\cos\lambda)}$$

and $r_2 = (1+\sqrt{2})(\sqrt{7-2\sqrt{2}}-2) \approx 0.102\cdots$.

(ii) If $\mathfrak{F} = \mathscr{S}^*_{\sinh}$, then

$$r_1 = |\sin\lambda| + 2(\sinh^{-1}1)^{-1}\cos\lambda - \sqrt{(|\sin\lambda| + 2(\sinh^{-1}1)^{-1}\cos\lambda)^2 - 1}$$

and $r_2 = -2(\sinh^{-1}1)^{-1} + \sqrt{1+4(\sinh^{-1}1)^{-2}} \approx 0.210\cdots$.

(iii) If $\mathfrak{F} = \mathscr{S}^*_{SG}$, then

$$r_1 = |\sin\lambda| + 2(e-1)^{-1}(e+1)\cos\lambda - \sqrt{(|\sin\lambda| + 2(e+1)(e-1)^{-1}\cos\lambda)^2 - 1}$$

and $r_2 = -2(1+e)(e-1)^{-1} + (e-1)^{-1}\sqrt{e(6+5e)+5} \approx 0.114\cdots$.

(iv) If $\mathfrak{F} = \mathscr{S}^*_{\wp}$, then $r_1 = 4\cos\lambda + |\sin\lambda| - \sqrt{15\cos^2\lambda + 8\cos\lambda\,|\sin\lambda|}$ and $r_2 = \sqrt{17} - 4 \approx 0.123\cdots$.

(v) If $\mathfrak{F} = \mathscr{S}^*_{\sin}$, then

$$r_1 = |\sin\lambda| + 2(\sin 1)^{-1}\cos\lambda - \sqrt{\cos^2\lambda\,(4(\sin 1)^{-2} - 1) + 4\cos\lambda\,|\sin\lambda|(\sin 1)^{-1}}$$

and $r_2 = \sqrt{1+4\csc^2 1} - 2\csc 1 \approx 0.201\cdots$.

The result is sharp.

**Corollary 4.3.2.** Let $\lambda \in (-\pi/2, \pi/2)$ and $G(z) = g_1(z)g_2(z)/z$ with $g_1, g_2 \in \mathcal{S}_\lambda$. Then $G \in \mathfrak{F}$, whenever $|z| = r < R$, where $R$ is given by (4.3.22). The result holds for the following cases:

(i) If $\mathfrak{F} = \mathcal{S}^*_{\wp}$, then $r_1 = |\sin\lambda| + 2e\cos\lambda - \sqrt{\cos\lambda(4e|\sin\lambda| + (4e^2-1)\cos\lambda)}$ and $r_2 = -2e + \sqrt{4e^2+1} \approx 0.091\cdots$.

(ii) If $\mathfrak{F} = \mathcal{S}^*_R$, then

$$r_1 = |\sin\lambda| + 2(3+2\sqrt{2})\cos\lambda - \sqrt{\cos\lambda(4(3+2\sqrt{2})|\sin\lambda| + (67+48\sqrt{2})\cos\lambda)}$$

and $r_2 = (2\sqrt{2}+3)\left(\sqrt{3(7-4\sqrt{2})} - 2\right) \approx 0.042\cdots$.

(iii) If $\mathfrak{F} = \mathcal{S}^*_c$, then $r_1 = |\sin\lambda| + 3\cos\lambda - \sqrt{\cos\lambda(6|\sin\lambda| + 8\cos\lambda)}$ and $r_2 = \sqrt{10} - 3 \approx 0.162\cdots$.

(iv) If $\mathfrak{F} = \mathcal{S}^*_q$, then

$$r_1 = |\sin\lambda| + (\sqrt{2}+2)\cos\lambda - \sqrt{((5+4\sqrt{2})\cos\lambda + 2\sqrt{2}(1+\sqrt{2})|\sin\lambda|)\cos\lambda}$$

and $r_2 = \sqrt{4\sqrt{2}+7} - 2 - \sqrt{2} \approx 0.143\cdots$.

(v) If $\mathfrak{F} = \mathcal{S}^*_e$, then $r_1 = |\sin\lambda| + 2e(e-1)^{-1}\cos\lambda - \sqrt{(|\sin\lambda| + 2e(e-1)^{-1}\cos\lambda)^2 - 1}$ and $r_2 = (\sqrt{1+e(5e-2)} - 2e)/(e-1) \approx 0.154\cdots$.

The result is sharp.

**Theorem 4.3.3.** *For* $0 \le \alpha < 1$ *and* $-1 \le \beta \le 1$ *assume*

$$\kappa_{\alpha,\beta}(x) := \frac{\beta+5}{2(\alpha+\beta)} + \frac{x}{2}\sqrt{\frac{4\alpha^2 + 4\alpha\beta - 4\alpha + \beta^2 + 6\beta + 25}{(\alpha+\beta)^2}}, \ \alpha \neq -\beta.$$

*Let* $\lambda \in (-\pi/2, \pi/2)$ *and* $G(z) = (1-z)^{1+\beta} g_1(z)g_2(z)/z$ *with* $g_1, g_2 \in \mathcal{S}_{\lambda,\beta}$. *Then* $G \in \mathcal{S}^*(\alpha)$, *whenever* $|z| = r < R$, *where* $R$ *is given by*

$$R = \begin{cases} r_1(\alpha,\beta), & r < |\tan(\lambda/2)|, \\ r_2(\alpha,\beta), & r \ge |\tan(\lambda/2)|, \end{cases} \tag{4.3.47}$$

*with* $r_1(\alpha,\beta)$ *being the smallest positive root of the equation*

$$((1-\beta r) - \alpha(1+r))(1+r^2 - 2r|\sin\lambda|) - 4r(1+r)\cos\lambda = 0$$

*and*

$$r_2(\alpha,\beta)=\begin{cases}\kappa_{\alpha,-1}(-1), & \textit{if } \beta=-1,\\[2ex] \begin{cases}\kappa_{\alpha,\beta}(1), & 0\le\alpha<-\beta,\\ (1-\alpha)(\beta+5)^{-1}, & \alpha=-\beta,\\ \kappa_{\alpha,\beta}(-1), & -\beta<\alpha<1,\end{cases} & \textit{if } -1<\beta\le 0,\\[2ex] \kappa_{\alpha,\beta}(-1), & \textit{if } 0<\beta\le 1.\end{cases}$$

*The result is sharp.*

*Proof.* From Lemma 4.3.1 and inequality (4.3.25), we have

$$\operatorname{Re}\left(\frac{zG'(z)}{G(z)}\right)>\frac{1-\beta r}{1+r}-2M(\lambda,r).$$

Now $G\in\mathcal{S}^*(\alpha)$ provided

$$M(\lambda,r)\le\frac{1}{2}\left(\frac{1-\beta r}{1+r}-\alpha\right),$$

holds, for $|z|=r<1$. Now by application of Lemma 4.3.1, we arrive at the following two cases:

$$\begin{cases}((1-\beta r)-\alpha(1+r))(1+r^2-2r|\sin\lambda|)-4r(1+r)\cos\lambda\ge 0, & \text{for } r<|\tan(\lambda/2)|,\\ r^2(\alpha+\beta)-(\beta+5)r+1-\alpha\ge 0, & \text{for } r\ge|\tan(\lambda/2)|.\end{cases} \tag{4.3.48}$$

Finally, the result follows at once due to inequalities in (4.3.48). The sharpness of the result is confirmed by the function $G_0(z)$ given by (4.3.36), where $g_1(z)$ and $g_2(z)$ are given by (4.3.37). □

The corrected version of the result [24, Theorem 5], obtained by substituting $\beta=-1$ in Theorem 4.3.3, is stated in the following corollary:

**Corollary 4.3.3.** Let $\alpha\in[0,1)$, $\lambda\in(-\pi/2,\pi/2)$ and $G(z)=g_1(z)g_2(z)/z$ with $g_1,g_2\in\mathcal{S}_\lambda$. Then $G\in\mathcal{S}^*(\alpha)$, whenever $|z|=r<R$, where $R$ is given by

$$R=\begin{cases}r_1(\alpha), & r<|\tan(\lambda/2)|,\\ r_2(\alpha), & r\ge|\tan(\lambda/2)|,\end{cases} \tag{4.3.49}$$

with

$$r_1(\alpha)=\frac{2\cos\lambda+(1-\alpha)|\sin\lambda|-\sqrt{(3-\alpha^2+2\alpha)\cos^2\lambda+4(1-\alpha)\cos\lambda\,|\sin\lambda|}}{1-\alpha}$$

and

$$r_2(\alpha) = \frac{\sqrt{\alpha^2 - 2\alpha + 5} - 2}{1 - \alpha}.$$

The result is sharp.

Here we have determined sharp radius constants for various subclasses of $\mathscr{A}$. Building upon these findings, the subsequent chapter studies a Ma–Minda class of functions associated with the Hyperbolic cosine function. We focus on deriving radius constants and establishing inclusion results for this class, thereby extending our study for various starlike subclasses.

## Highlights of the Chapter

In this chapter, we present a comprehensive derivation of various radius results, emphasizing the intricate calculations involved. To enhance comprehension, key findings are supplemented with diagrammatic representations, providing visual clarity. Additionally, we extend previously known results, showcasing them as special cases within our broader framework. Employing differential subordination techniques, we introduce an elegant and systematic approach to deriving radius results. Through procedural enhancements, we develop new methodologies that yield simplified and sharp radius estimations.

The contents of this chapter is based on the findings presented in the paper:
*S. Sivaprasad Kumar and Mridula Mundalia: On Sharp Radius estimates for $\mathscr{S}^*(\beta)$ and a product function, Mathematica Slovaca,* **75***(2), 281-300 (2025).*

# Chapter 5

# On a Class of Starlike Functions Associated with Hyperbolic Cosine Function

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In this Chapter, we introduce and study a new Ma-Minda subclass of starlike functions $\mathscr{S}^*_\rho$, defined as

$$\mathscr{S}^*_\rho := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \cosh\sqrt{z} =: \rho(z), z \in \mathbb{D} \right\},$$

associated with an analytic univalent function $\cosh\sqrt{z}$, where we choose the branch of the square root function so that $\cosh\sqrt{z} = 1 + z/2! + z^2/4! + \cdots$ . We establish certain inclusion relations for $\mathscr{S}^*_\rho$ and deduce sharp $\mathscr{S}^*_\rho-$radii for certain subclasses of analytic functions.

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## 5.1 Introduction

Ma and Minda's pioneering work established a unified framework for the starlike class $\mathcal{S}^*$ and its subclasses, marking a significant advancement in GFT. They introduced the class $\mathcal{S}^*(\phi)$ given by (1.2.2). This generalization encompassed many previously studied subclasses, fostering extensive research into specific choices of $\phi(z)$ and deepening the understanding of the geometric properties of analytic and univalent functions.

Motivated by the works of [10, 56, 104, 106, 150] (see also Table 1.1), we introduce and study a new Ma-Minda subclass of starlike functions associated with the function $\cosh b\sqrt{z}$. In this chapter, we investigate the properties of $\cosh b\sqrt{z}$, focusing on its geometry when restricted to the principal branch of the square root function. This function can be expressed as:

$$\cosh b\sqrt{z} = \sum_{k=0}^{\infty} \frac{(b^2 z)^k}{(2k)!} = 1 + \frac{b^2 z}{2!} + \frac{b^4 z^2}{4!} + \cdots, \quad \text{where } b \in [-\pi/2, \pi/2] \setminus \{0\}. \tag{5.1.1}$$

This formulation sets the stage for our investigation into the associated starlike subclass and its geometric properties.

Assume $\rho_b(z) = \cosh b\sqrt{z}$, then the conformal mapping $\rho_b : \mathbb{D} \to \mathbb{C}$, maps the unit disc $\mathbb{D}$ onto the region

$$\Omega_{\rho_b} := \{w \in \mathbb{C} : |\log(w + \sqrt{w^2 - 1})|^2 < b^2\} \quad (b \in [-\pi/2, \pi/2] - \{0\}),$$

defined on the principle branch of logarithmic and square root functions. When $b_1 \leq b_2$, we observe that $\rho_{b_1}(\mathbb{D}) \subset \rho_{b_2}(\mathbb{D})$. Moreover, for each circle $|z| = r < 1$,

$$\begin{cases} \min\limits_{|z|=r} \operatorname{Re} \rho_b(z) = \min\limits_{|z|=r} |\rho_b(z)| = \rho_b(\sqrt{-r}) \\ \max\limits_{|z|=r} \operatorname{Re} \rho_b(z) = \max\limits_{|z|=r} |\rho_b(z)| = \rho_b(\sqrt{r}). \end{cases} \tag{5.1.2}$$

Assume $\rho_1(z) =: \rho(z)$. Observe that $\rho_b(z)$ is an analytic and univalent function, satisfying $\operatorname{Re}(\rho_b(z)) > 0$, and it maps the unit disc $\mathbb{D}$ onto a convex region. Additionally, it is symmetric with respect to the real axis, since $\overline{\rho_b(z)} = \rho_b(\bar{z})$, and it is typically real as it satisfies $\rho_b'(0) = b^2/2 > 0$. Therefore, $\rho_b(z)$ is a Ma-Minda function.

In the recent past, Raza and Hussain [9, 13] have investigated the cosine and cosine hyperbolic functions. However, these functions are non-univalent and therefore, they are no more Ma-Minda functions. Several Ma-Minda classes have been studied in the past, including $\mathcal{S}_e^*, \mathcal{S}_L^*(s)$, $\mathcal{S}^*(q_\kappa)$, $\mathcal{S}^*[A,B]$ and $\mathcal{S}\mathcal{S}^*(\beta)$ [10,56,104,106,150]. The geometric and analytical properties of these classes have motivated us to introduce the following new class.

**Definition 5.1.1.** Let $\mathcal{S}^*_{\rho_b}$ be the class of normalized starlike functions, given by

$$\mathcal{S}^*_{\rho_b} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \rho_b(z) := \cosh b\sqrt{z},\ z \in \mathbb{D},\ b \in [-\pi/2, \pi/2] - \{0\} \right\},$$

where we choose the branch of the square root function so that

$$\cosh b\sqrt{z} = 1 + \frac{b^2 z}{2!} + \frac{b^4 z^2}{4!} + \frac{b^6 z^3}{6!} + \cdots .$$

In addition to $\mathcal{S}^*_{\rho_b}$, we also study the class $\mathcal{S}^*_{\rho_1} =: \mathcal{S}^*_{\rho}$, to derive radius constants along with several inclusion relations. The integral representation for $f \in \mathcal{S}^*_{\rho}$ is given by

$$f(z) = z\exp\left(\int_0^z \frac{\hat{\rho}(t) - 1}{t} dt\right), \tag{5.1.3}$$

where $\hat{\rho}(z) \prec \rho(z)$. Note that if $\psi_{\hat{\rho}}(z) = 1 + z/3 + z^2/18$ and $\phi_{\hat{\rho}}(z) = 1 + \sin(z/3)$, then evidently $\psi_{\hat{\rho}}(z)$ and $\phi_{\hat{\rho}}(z)$ are subordinate to $\rho(z)$, so the corresponding functions

$$f_1(z) = z\exp\left(\frac{z}{3} + \frac{z^2}{36}\right) \quad \text{and} \quad f_2(z) = ze^{Si(z)}, \text{ where } Si(z) = \int_0^z \frac{\sin t}{t} dt$$

lie in $\mathcal{S}^*_{\rho}$. Now using the representation in (5.1.3), we obtain different functions, which work as extremal functions for various results. For instance, $\varphi_{\rho_n} \in \mathcal{A}$ $(n = 2, 3, 4, \ldots)$, defined as

$$\varphi_{\rho_n}(z) = z\exp\left(\int_0^z \frac{\rho(t^{n-1}) - 1}{t} dt\right) = z + \frac{z^n}{2(n-1)} + \frac{z^{2n-1}}{48(n-1)} + \cdots, \tag{5.1.4}$$

belongs to $\mathcal{S}^*_{\rho}$. We denote $\varphi_\rho := \varphi_{\rho_2}$. For completeness of our class $\mathcal{S}^*_{\rho}$, we give below a remark.

*Remark* 5.1.1. For $f \in \mathcal{S}^*_{\rho}$ and $\varphi_\rho(z)$ be as defined in (5.1.4), then for $|z| = r_0 < 1$, we have

(i) $-\varphi_\rho(-r_0) \leq |f(z)| \leq \varphi_\rho(r_0)$ (Growth Theorem).

(ii) $\varphi'_\rho(-r_0) \leq |f'(z)| \leq \varphi'_\rho(r_0)$ (Distortion Theorem).

(iii) $|\arg(f(z)/z)| \leq \max_{|z|=r_0} \arg\left(\varphi_\rho(z)/z\right)$ (Rotation Theorem) .

Equality for (i)-(iii) holds for some $z_0 \neq 0$ if and only if $f(z)$ is a rotation of $\varphi_\rho(z)$. Infact if $f \in \mathcal{S}^*_{\rho}$ then either $f$ is a rotation of $\varphi_\rho(z)$ or $f(\mathbb{D}) \supset \{w : |w| \leq -\varphi_\rho(-1) \approx 0.619\ldots\}$.

*Remark* 5.1.2. For $f \in \mathcal{S}^*_{\rho}$, the sharp bounds on initial coefficients, $a_2, a_3, a_4$ and Fekete-Szegö functional, obtained using Theorems 2.2.3 and 2.2.10, are listed below:

(i) $|a_2| \leq 1/2$, (ii) $|a_3| \leq 1/4$, (iii) $|a_4| \leq 1/6$,

(iv) For any complex constant $\mu$, $|a_3 - \mu a_2^2| \leq \frac{1}{4}\max\left\{1, \left|\mu - \frac{7}{12}\right|\right\}$.

Equality in (i) holds for the function $\varphi_\rho(z)$, given by (5.1.4), and $\tilde{f}(z) = z + z^3/4$ is an extremal function for (ii) and (iv).

In section 5.2, we discuss geometric properties of $\cosh\sqrt{z}$, thereby deducing various inclusion results involving, $\mathscr{S}^*(\beta)$, $\mathscr{M}(\alpha)$ and others. Diagrammatic interpretation of the inclusions are also provided, depicting the sharpness of our findings. In section 5.3, sharp radius results are derived in connection to various other Ma-Minda classes such as: $\mathscr{S}^*(\beta)$, $\mathscr{C}(\beta)$, $\mathscr{S}^*[A,B]$ and other classes defined through ratio of analytic expression $f(z)/g(z)$ or $g(z)/z$, where $f$ and $g$ lie in some suitable class of analytic functions.

## 5.2 Properties of Hyperbolic Cosine Function

Motivated by the works of Aouf et al. [10], Janowski [56], Masih and Kanas [104] and others [106, 150], we study inclusion results for the class $\mathscr{S}^*_\rho$. We begin with a Lemma which demonstrates a maximal disc centered at point $(c,0)$ on the real line, that can be subscribed within $\rho_b(\mathbb{D})$.

**Lemma 5.2.1.** Suppose $b \neq 0$, then $\rho_b(z)$ satisfies the following inclusion

$$\{w \in \mathbb{C} : |w - c| < r_{bc}\} \subset \rho_b(\mathbb{D}) =: \Omega_{\rho_b} \quad (-\pi/2 \leq b \leq \pi/2),$$

where

$$r_{bc} = \begin{cases} c - \cos b, & \cos b < c \leq (\cosh b + \cos b)/2, \\ \cosh b - c, & (\cosh b + \cos b)/2 \leq c < \cosh b. \end{cases}$$

*Proof.* Let $\Gamma := \rho_b(e^{it})$, $-\pi \leq t \leq \pi$ be the boundary curve of the function $\rho_b(z)$. Due to symmetricity of the curve $\Gamma$ about real-axis, it is enough to consider $0 \leq t \leq \pi$. Define a function $G_c(\tau)$ as follows:

$$G_c(\tau) := (c - \cosh(b(\cos\tau))\cos(b(\sin\tau)))^2 + \sinh^2(b(\cos\tau))\sin^2(b(\sin\tau)),$$

where $\tau = t/2$. Observe that $G_c(\tau)$ (see Fig. 5.1 for different values of $c$) is the square of the distance from point $(c,0)$ to $\Gamma$. Now we study the following cases:

**Case 1:** For $\cos b < c \leq 1$, $G_c(\tau)$ is monotonically decreasing on $[0,\pi/2]$, then

$$r_{bc} = \min_{\tau\in[0,\pi/2]} \sqrt{G_c(\tau)} = \sqrt{G_c(\pi/2)} = c - \cos b.$$

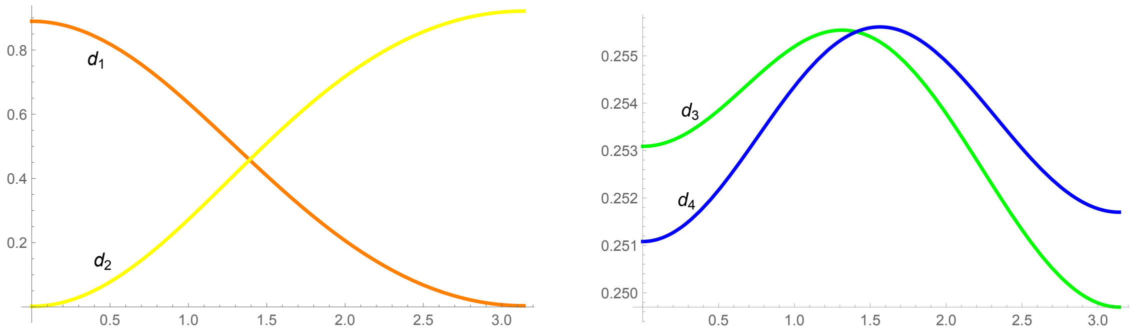


Figure 5.1: Graphs of $G_{0.6}(\tau) = d_1$, $G_{1.5}(\tau) = d_2$, $G_{1.04}(\tau) = d_3$, $G_{1.042}(\tau) = d_4$, (with $b = 1$)

**Case 2:** When $1 \leq c \leq b_0$, where $b_0 < (\cosh b + \cos b)/2$ is a point at which $G_c(\tau)$ changes its character i.e. $G_c(\tau)$ is monotonically decreasing for $1 \leq c \leq b_0$ and has three critical points $\{0, \tau_{\tilde{c}}, \pi/2\}$ for $b_0 < c \leq (\cosh b + \cos b)/2$, where $\tau_{\tilde{c}} \in (0, \pi/2)$ is the only root of the equation

$$\begin{aligned} &2c \tan\tau \cos(b \sin\tau) \sinh(b \cos\tau) + 2c \sin(b \sin\tau) \cosh(b \cos\tau) \\ &= \sin(2b \sin\tau) + \tan\tau \sinh(2b \cos\tau). \end{aligned} \tag{5.2.5}$$

Note that $\tau_c < \tau_{\tilde{c}}$ whenever $c < \tilde{c}$. Further,

$$G_c(0) - G_c(\pi/2) = (\cosh b - \cos b)(\cos b + \cosh b - 2c) \geq 0,$$

yields

$$r_{bc} = \min_{\tau \in [0,\pi/2]} \left\{ \sqrt{G_c(0)}, \sqrt{G_c(\tau_{\tilde{c}})}, \sqrt{G_c(\pi/2)} \right\} = \sqrt{G_c(\pi/2)} = c - \cos b.$$

**Case 3:** For $(\cosh b + \cos b)/2 \leq c \leq b_1$, where $b_1 < \cosh b$ is a point at which $G_c(\tau)$ changes its character i.e. $G_c(\tau)$ has three critical points $\{0, \tau_{\hat{c}}, \pi/2\}$, where $\tau_{\hat{c}} \in (0, \pi/2)$ is the only root of equation (5.2.5) and $G_c(\tau)$ is an increasing function for $b_1 < b < \cosh b$. Infact $G_c(0) \leq G_c(\pi/2)$. Therefore

$$r_{bc} = \min_{\tau \in [0,\pi/2]} \left\{ \sqrt{G_c(0)}, \sqrt{G_c(\tau_{\hat{c}})}, \sqrt{G_c(\pi/2)} \right\} = \sqrt{G_c(0)} = \cosh b - c.$$

Hence the result follows. □

Inclusion results in Lemma 5.2.2, follows from equation (5.1.2) and Lemma 5.2.1.

**Lemma 5.2.2.** For the region $\Omega_{\rho_b} := \rho_b(\mathbb{D})$, following inclusion relations hold:

(i) $\{w : |w - (\cosh b + \cos b)/2| < (\cosh b - \cos b)/2\} \subset \Omega_{\rho_b}$.

(ii) $\Omega_{\rho_b} \subset \{w : \cos b < \operatorname{Re} w < \cosh b\}$ and $\Omega_{\rho_b} \subset \{w : \cos b < |w| < \cosh b\}$.

(iii) $\Omega_{\rho_b} \subset \{w : |\operatorname{Im} w| < l\}$ and $\Omega_{\rho_b} \subset \{w : |w - (\cosh b + \cos b)/2| < l\}$, where $l = |\mathrm{Im}(\cosh(be^{it_0/2}))|$, and $t_0$ is the root of the equation

$$\cos b + \cosh b - 2\cos(b\sin(t/2))\cosh(b\cos(t/2)) = 0.$$

For $b = 1$, Lemma 5.2.1 leads to the following result for the region $\Omega_{\rho_1} =: \Omega_\rho$.

**Theorem 5.2.1.** *Let region* $\rho(\mathbb{D}) = \Omega_\rho := \{w \in \mathbb{C} : |\log(w + \sqrt{w^2-1})|^2 < 1\}$, *then*

$$\Omega_\rho \supset \{w \in \mathbb{C} : |w - c| < r_c\}$$

*where*

$$r_c = \begin{cases} c - \cos 1, & \cos 1 < c \leq (\cosh 1 + \cos 1)/2 \\ \cosh 1 - c, & (\cosh 1 + \cos 1)/2 \leq c < \cosh 1. \end{cases} \tag{5.2.6}$$

*Remark* 5.2.1. Theorem 5.2.1 ensures that $D_c := |w - c| < r_c$, is the maximal disc subscribed in $\rho(\mathbb{D})$, when $c = (\cosh 1 + \cos 1)/2$ and $r_c = (\cosh 1 - \cos 1)/2$. Thus $D_c \subset \rho(\mathbb{D})$.

For all the subsequent results, we shall assume $l_0 := \cos 1$ and $l_1 := \cosh 1$.

**Lemma 5.2.3.** For the region $\Omega_\rho := \rho(\mathbb{D})$, we have the following inclusion relations:

(i) $\{w : |w - (l_0 + l_1)/2| < (l_1 - l_0)/2\} \subset \Omega_\rho$.

(ii) $\Omega_\rho \subset \{w : |\arg w| < m\}$, where $m \approx 0.506053 \approx (0.322163)\,\pi/2 \approx 28.9947°$.

(iii) $\Omega_\rho \subset \{w : l_0 < \mathrm{Re}\, w < l_1\}$ and $\Omega_\rho \subset \{w : l_0 < |w| < l_1\}$.

(iv) $\Omega_\rho \subset \left\{w : |\mathrm{Im}\, w| < l_\rho\right\}$ and $\Omega_\rho \subset \left\{w : |w - (l_0 + l_1)/2| < l_\rho\right\}$, where $l_\rho = |\mathrm{Im}(\cosh(e^{it_0/2}))|$ and $t_0$ is the solution of the equation

$$l_0 + l_1 - 2\cos(\sin(t/2))\cosh(\cos(t/2)) = 0.$$

*Proof.* We can obtain (i), (iii)-(iv) from equations in (5.1.2), Remark 5.2.1 and Lemma 5.2.2 (for $b = 1$). For part (ii) let $\Gamma := \partial(\rho(z)) = \rho(e^{it})$, $-\pi \leq t \leq \pi$, represents the boundary curve of $\rho(z)$. Assume that

$$\mathrm{Re}\,\rho(e^{it}) = \cos(\sin(t/2))\cosh(\cos(t/2)) =: X(t)$$

and

$$\mathrm{Im}\,\rho(e^{it}) = \sin(\sin(t/2))\sinh(\cos(t/2)) =: Y(t).$$

Now consider

$$|\arg\rho(z)| < \max_{|z|=1} |\arg\rho(z)| = \max_{t\in[-\pi,\pi]} |\arg\rho(e^{it})| = \max_{t\in[-\pi,\pi]} \tan^{-1}(Y(t)/X(t))$$
$$= \max_{t\in[-\pi,\pi]} \tan^{-1}(\tan(\sin(t/2))\tanh(\cos(t/2))) =: m(t).$$

Observe that $\tan^{-1} x$ is a monotonically increasing real valued function, therefore, it is enough to obtain the maximum of $m(t)$. The roots of

$$m'(t) = 0.5(\cos(t/2)\tanh(\cos(t/2))\sec^2(\sin(t/2))$$
$$-\sin(t/2)\tan(\sin(t/2))\,\mathrm{sech}^2(\cos(t/2))) = 0$$

are $t_1 \approx -1.91672$ and $t_2 \approx 1.91672$. As $t_1 < t_2$, therefore maximum of $m(t)$ is attained at $t = t_2$. Hence the inclusion in $(ii)$ follows. □

In Theorem 5.2.2 and Corollary 5.2.1, we derive inclusion results pertaining to the class $\mathcal{S}^*_\rho$ involving various other classes such as $\mathcal{S}\mathcal{T}_p(\hat{\gamma})$, $\mathcal{S}^*_{hpl}(\tilde{s})$ and $k-\mathcal{S}\mathcal{T}$ [7,59,62], which are given by:

$$\mathcal{S}\mathcal{T}_p(\hat{\gamma}) := \left\{ f \in \mathcal{A} : \mathrm{Re}\,\frac{zf'(z)}{f(z)} + \hat{\gamma} > \left| \frac{zf'(z)}{f(z)} - \hat{\gamma} \right|, \hat{\gamma} > 0 \right\},$$

$$\mathcal{S}^*_{hpl}(\tilde{s}) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1}{(1-z)^{\tilde{s}}}, 0 < \tilde{s} \leq 1 \right\},$$

$$k-\mathcal{S}\mathcal{T} := \left\{ f \in \mathcal{A} : \mathrm{Re}\,\frac{zf'(z)}{f(z)} > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, k \geq 0 \right\}.$$

**Theorem 5.2.2.** *Let $f \in \mathcal{S}^*_{\rho_b}$, then for each $b \in [-\pi/2, \pi/2] - \{0\}$, following inclusions hold:*

*(i)* $\mathcal{S}^*_{\rho_b} \subset \mathcal{S}^*(\beta)$, *where* $\beta = \cos b$.

*(ii)* $\mathcal{S}^*_{\rho_b} \subset \mathcal{M}(\alpha)$, *where* $\alpha = \cosh b$.

*(iii)* $\mathcal{S}^*_{q_\kappa} \subset \mathcal{S}^*_{\rho_b}$, *whenever* $\kappa \leq 1 - \cos^2 b$.

*(iv)* $k-\mathcal{S}\mathcal{T} \subset \mathcal{S}^*_{\rho_b}$, *whenever* $k \geq \cosh b/(\cosh b - 1)$.

*(v)* $\mathcal{S}^*_{\rho_b} \subset \mathcal{S}^*_{hpl}(\tilde{s})$, *whenever* $\log(\sec b)/\log 2 \leq \tilde{s} \leq 1$, $b \in [-\pi/3, \pi/3] - \{0\}$.

*(vi)* $\mathcal{S}^*_{\rho_b} \subset \mathcal{S}^*_L(s)$, *whenever* $1 - \sqrt{\cos b} \leq s \leq \frac{1}{\sqrt{2}}$.

*Proof.* Observe that, in equation (5.1.2), when $r$ tends to $1^-$, sharp bounds on real part and modulus of $\rho_b(z)$ are obtained. Consequently, due to Lemma 5.2.2 the inclusions in $(i)$ and $(ii)$ are true for the class $\mathcal{S}^*_{\rho_b}$. We know that $q_\kappa(z) = \sqrt{1+\kappa z}$ where $0 < \kappa \leq 1$, is associated with the region $|w^2 - 1| < \kappa$. Therefore part $(iii)$ can be easily established as $q_\kappa(\mathbb{D})$ lies in $\Omega_{\rho_b}$, if and only if,

$\sqrt{1-\kappa} \geq \cos b$, which implies $\kappa \leq 1-\cos^2 b$. For part $(iv)$, let $\Gamma_k = \{w \in \mathbb{C} : \operatorname{Re} w > k|w-1|\}$, then for $k>1$, the set $\Gamma_k$ represents the interior of an ellipse,

$$\gamma_k := \left\{ (x,y) \in \mathbb{R}\times\mathbb{R} : \frac{(x-x_1)^2}{a_0^2} + \frac{y^2}{b_0^2} = 1 \right\},$$

where $x_1 = k^2/(k^2-1)$, $a_0 = k/(k^2-1)$ and $b_0 = 1/\sqrt{k^2-1}$. For $\gamma_k$ to lie in $\Omega_{\rho_b}$ we must have $x_1 + a_0 \leq \cosh b$, which gives a sufficient condition for $\gamma_k$ to lie in $\Omega_{\rho_b}$, this leads us to the required condition. From [59] we know that $\operatorname{Re}(1-z)^{-\tilde{s}} > 2^{-\tilde{s}}$. Therefore for $(v)$ to hold true $2^{-\tilde{s}} \leq \cos b$, which gives $\log(\sec b)/\log 2 \leq \tilde{s} \leq 1$, provided $-\pi/3 \leq b \leq \pi/3$. Furthermore, it was demonstrated in [104], that

$$\begin{aligned}\Omega_s = \phi_s(\mathbb{D}) &= \{x+iy \in \mathbb{C} : ((x-1)^2+y^2-s^4)^2 < 4s^2((x-1+s^2)^2+y^2)\} \\ &\supset \{w : |w-1| < 1-(1-s)^2\},\end{aligned}$$

where $0 < s \leq 1/\sqrt{2}$. Thus for $(vi)$ to hold true we must have $1-(1-s)^2 \geq 1-\cos b$. Thus $\mathcal{S}^*_{\rho_b} \subset \mathcal{S}^*_L(s)$ for each $s \geq 1-\sqrt{\cos b}$. □

In the following Corollary we prove inclusion results for the class $\mathcal{S}^*_\rho$.

**Corollary 5.2.1.** For each function $f \in \mathcal{S}^*_\rho$ the following inclusions hold:

(i) $\mathcal{S}^*_\rho \subset \mathcal{S}^*(\beta)$, where $\beta = l_0$.

(ii) $\mathcal{S}^*_\rho \subset \mathcal{M}(\alpha)$, where $\alpha = l_1$.

(iii) $\mathcal{S}^*_\rho \subset \mathcal{S}\mathcal{S}^*(\beta)$, where $\beta \approx 0.3222163$.

(iv) $\mathcal{S}^*_{q_\kappa} \subset \mathcal{S}^*_\rho$, whenever $\kappa \leq 1-l_0^2$.

(v) $k-\mathcal{S}\mathcal{T} \subset \mathcal{S}^*_\rho$, whenever $k \geq l_1/(l_1-1)$.

(vi) $\mathcal{S}^*_\rho \subset \mathcal{S}^*_{hpl}(\tilde{s})$, whenever $-\log l_0/\log 2 \leq \tilde{s} \leq 1$.

(vii) $\mathcal{S}^*_\rho \subset \mathcal{S}^*_L(s)$, whenever $1-\sqrt{l_0} \leq s \leq \frac{1}{\sqrt{2}}$.

(viii) $\mathcal{S}^*_\rho \subset \mathcal{S}\mathcal{T}_p(\hat{\gamma})$, whenever $\hat{\gamma} \geq \hat{\gamma}_0 \approx 0.0654238$.

*Proof.* Clearly parts $(i)-(ii)$ and $(iv)-(vii)$ can be obtained as a result of Theorem 5.2.2 for $b=1$. Part $(iii)$ is true due to Lemma 5.2.3, for the class $\mathcal{S}^*_\rho$ (see Fig. 5.2). For $(viii)$ in order to show $\mathcal{S}^*_\rho \subset \mathcal{S}\mathcal{T}_p(\hat{\gamma})$, we must have $|w-\hat{\gamma}| - \operatorname{Re} u < \hat{\gamma}$, where $u(z) = \cosh\sqrt{z}$. For $z = e^{it}$ we have

$$H(\tau) := \frac{\sin^2(\sin\tau)\sinh^2(\cos\tau)}{4\cos(\sin\tau)\cosh(\cos\tau)} < \hat{\gamma},$$

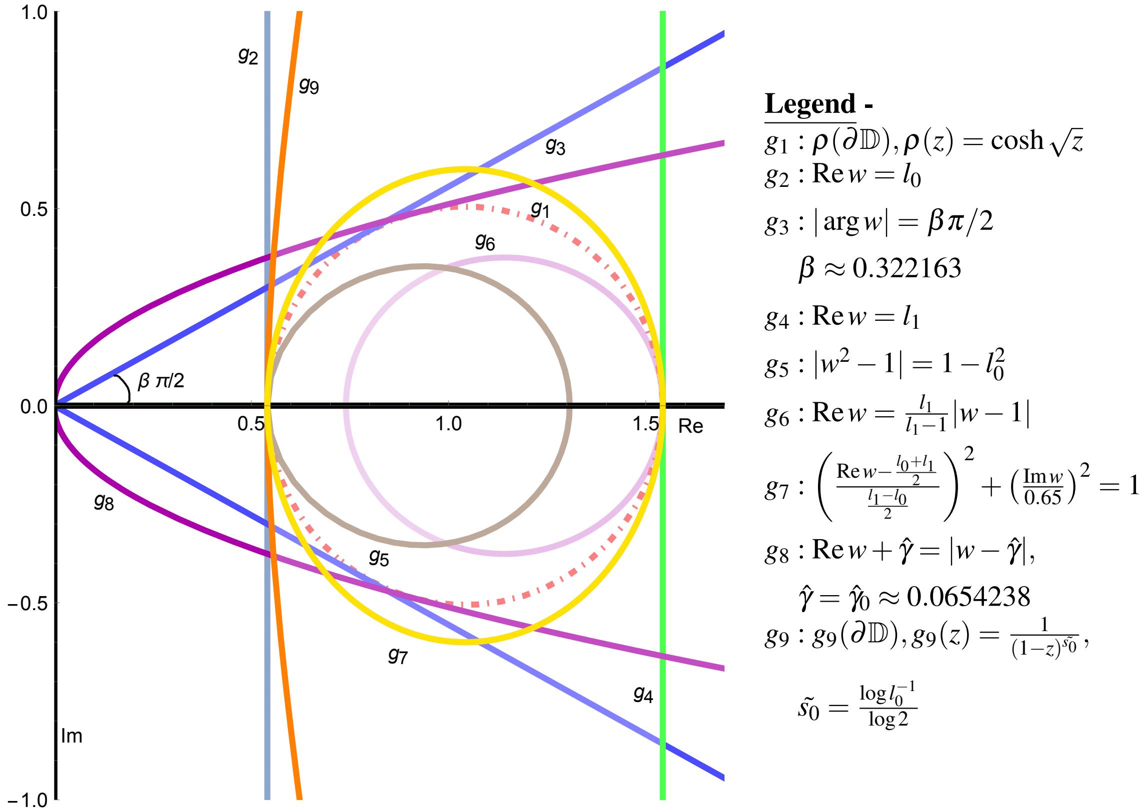


Figure 5.2: Graph depicting the boundary curves of various dominants and subordinants of $\rho(\partial\mathbb{D})$ as deduced in Corollary 5.2.1.

where $\tau = t/2$. Clearly, $H'(\tau)$ vanishes on $\{0, \tilde{\tau}, \pi/2\}$, with $\tau = \tilde{\tau} \approx 0.832934$ as the only root of the equation

$$\begin{aligned} &\tan(\sin\tau)\tanh(\cos\tau)((\cos\tau(\cos(2\sin\tau)+3)\sinh(\cos\tau)\sec(\sin\tau)) \\ &\quad - \sin\tau\sin(\sin\tau)(\cosh(2\cos\tau)+3)\operatorname{sech}(\cos\tau)) = 0 \end{aligned}$$

in $(0, \pi/2)$. Therefore,

$$\max_{\tau\in[0,\pi/2]} H(\tau) = H(\tilde{\tau}) \approx 0.0654238.$$

Observe that $\mathscr{S}\mathscr{T}_p(\hat{\gamma}_1) \subset \mathscr{S}\mathscr{T}_p(\hat{\gamma}_2)$, whenever $\hat{\gamma}_1 < \hat{\gamma}_2$. This leads to the required inclusion relation. □

*Remark* 5.2.2. Fig. 5.2 displays various inclusion relations related to the region $\Omega_\rho := \Omega_{\rho_1}$. A vertical ellipse enclosing the region $\Omega_\rho$ is

$$\frac{\left(x - \dfrac{l_0 + l_1}{2}\right)^2}{\left(\dfrac{l_1 - l_0}{2}\right)^2} + \frac{y^2}{b_2^2} = 1,$$

where $b_2 \geq \max \operatorname{Im} \rho(z)$. For visual purposes we illustrate this ellipse ($g_7$) for $b_2 = 0.65$. Fig. 5.2

depicts the sharpness of inclusion results given in Corollary 5.2.1.

We now state a Lemma needed to prove the next result.

**Lemma 5.2.4.** [152] If $p \in \mathscr{P}_n(\beta)$, then for $|z| = r$

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2(1-\beta)nr^n}{(1-r^n)(1+(1-2\beta)r^n)}.$$

**Theorem 5.2.3.** *Let $p(z) = (1+Az)/(1+Bz)$, where $-1 < B < A \leq 1$, then $p(z) \prec \cosh\sqrt{z}$, if and only if*

$$A \leq \begin{cases} 1-(1-B)l_0, & \text{if } 2(1-AB) \leq (l_0+l_1)(1-B^2), \\ (1+B)l_1 - 1, & \text{if } 2(1-AB) \geq (l_0+l_1)(1-B^2). \end{cases} \tag{5.2.7}$$

*Proof.* Lemma 3.2.3 shows that the $p(z) = (1+Az)/(1+Bz)$, maps $\mathbb{D}$ onto the disc

$$\left|p(z) - \frac{1-AB}{1-B^2}\right| \leq \frac{A-B}{1-B^2}, \quad -1 < B < A \leq 1.$$

By Theorem 5.2.1, $p(z) \prec \cosh\sqrt{z}$ if and only if the above disc lies within $\Omega_\rho$. Conditions in (5.2.7) gives $(1+A) \leq (1+B)l_1$, provided $2(1-AB) \geq (l_0+l_1)(1-B^2)$ holds. Infact $(A-B)/(1-B^2) \leq l_1 - (1-AB)/(1-B^2)$ leads to $(A-B)/(1-B^2) \leq l_1 - c$ provided $2c \geq l_0 + l_1$ where $c = (1-AB)/(1-B^2)$. Also from (5.2.7), $(1-A) \geq (1-B)l_0$ whenever $2(1-AB) \leq (l_0+l_1)(1-B^2)$. Equivalently, $(A-B)/(1-B^2) \leq c - l_0$ whenever $2c \leq l_0+l_1$. Thus $p(z)$ lies in $|w-c| < r_c$, where $r_c$ is given by (5.2.6). □

Theorem 5.2.3 leads to the following result:

**Corollary 5.2.2.** Let conditions on $A$ and $B$ be as given in Theorem 5.2.3, then $\mathscr{S}^*[A,B] \subset \mathscr{S}^*_\rho$.

## 5.3 Radius Problems and Certain Estimates for the Class $\mathscr{S}^*_\rho$

Radius problems have been an active area of research in GFT. Some of the pioneering works in this direction have been discussed by several authors, see [10, 106, 135, 150]. Motivated by these work, we derive radius results for $\mathscr{S}^*_\rho$ involving the following classes:

$$\mathscr{S}^*_n(\rho) = \left\{ f \in \mathscr{A}_n : \frac{zf'(z)}{f(z)} \prec \cosh\sqrt{z} =: \rho(z) \right\},$$

$$\mathscr{S}^*_n[A,B] = \left\{ f \in \mathscr{A}_n : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

and

$$\mathscr{M}_n(\alpha) = \left\{ f \in \mathscr{A}_n : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z}, \alpha > 1 \right\}.$$

We apply lemmas stated in Section 5.2, to obtain sharp $\mathscr{S}_n^*(\rho)-$radius, $\mathscr{S}_n^*[A,B]-$ radius and $\mathscr{M}_n(\alpha)-$radius for the class $\mathscr{S}_\rho^*$. The proof of the following theorem can be obtained from Lemma 5.2.3 and equations in (5.1.2), so it is skipped here.

**Theorem 5.3.1.** *Let $f \in \mathscr{S}_\rho^*$, then $f \in \mathscr{M}(\alpha)$ for $|z| < r_\alpha$, where*

$$r_\alpha = \begin{cases} r(\alpha), & 1 < \alpha < l_1, \\ 1, & \alpha \geq l_1. \end{cases}$$

*and $r(\alpha) \in (0,1)$ is the smallest root of the equation: $\cosh\sqrt{r} = \alpha$. Equality holds when $f(z) = \varphi_\rho(z)$.*

**Theorem 5.3.2.** *Let $f \in \mathscr{S}_\rho^*$, then $f \in \mathscr{S}^*(\beta)$ for $|z| < r_\beta$, where $\beta \in [0,1)$ and $r_\beta < 1$ is the least positive root of the equation: $\cos\sqrt{r} = \beta$. This radius result is sharp.*

*Proof.* As $f \in \mathscr{S}_\rho^*$, then we have $zf'(z) = f(z)\cosh\sqrt{\omega(z)}$, where $\omega(z)$ is a Schwarz function with $\omega(0) = 0$ such that for $-\pi \leq t \leq \pi$, $\omega(z) = Re^{it}$. For each $R = |\omega(z)| \leq |z| = r < 1$, we have $\cos\sqrt{R} \geq \cos\sqrt{r}$, and as a result of equations in (5.1.2)

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \min_{|z|=r} \operatorname{Re}\rho(\omega(z)) = \cos\sqrt{r} \geq \beta.$$

If $s(r,\beta) := \cos\sqrt{r} - \beta$, then there exist $r_{\beta_0} < r_{\beta_1}$ such that $s(r_{\beta_0},\beta) > 0$ and $s(r_{\beta_1},\beta) < 0$, holds. Thus a least positive root $r_\beta$ for the equation $s(r,\beta) = 0$, will serve the purpose. In particular, at $z_0 = -r$, we have $\operatorname{Re}(z_0\tilde{f}'(z_0)/\tilde{f}(z_0)) = \cos\sqrt{r} = \beta$, then function $\tilde{f}(z) = \varphi_\rho(z)$ is the extremal function. □

In Theorem 5.3.3, we establish radius of convexity of order $\beta$ for the class $\mathscr{S}_\rho^*$.

**Theorem 5.3.3.** *Let $f \in \mathscr{S}_\rho^*$, then $f \in \mathscr{C}(\beta)$ for $|z| \leq r_\beta$, where $\beta \in [0,1)$ and $r_\beta \in (0,1)$ is the least positive root of the equation*

$$2(1-r^2)\cos\sqrt{r} - \sqrt{r}\tan\sqrt{r} = \beta.$$

*Proof.* As $f \in \mathscr{S}_\rho^*$, there exists a Schwarz function $\omega(z)$ such that $\omega(0) = 0$ and

$$\frac{zf'(z)}{f(z)} = \cosh\sqrt{\omega(z)}. \tag{5.3.8}$$

On logarithmically differentiating (5.3.8) and applying triangle inequality, we deduce

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &= \operatorname{Re}\frac{zf'(z)}{f(z)}+\operatorname{Re}\left(\frac{z\omega'(z)\tanh\sqrt{\omega(z)}}{2\sqrt{\omega(z)}}\right)\\ &\geq \cos\sqrt{r}-|z||\omega'(z)|\left|\frac{\tanh\sqrt{\omega(z)}}{2\sqrt{\omega(z)}}\right| \quad (|z|=r<1). \end{aligned} \tag{5.3.9}$$

Further as $\omega(z)$ is a Schwarz function, then due to Schwarz Pick Lemma, we have

$$|\omega'(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2}. \tag{5.3.10}$$

Now in view of (5.3.10), we have

$$-|z||\omega'(z)|\left|\frac{\tanh\sqrt{\omega(z)}}{\sqrt{\omega(z)}}\right| \geq -|z|\frac{1-|\omega(z)|^2}{1-|z|^2}\left|\frac{\tanh\sqrt{\omega(z)}}{\sqrt{\omega(z)}}\right|. \tag{5.3.11}$$

Assume $\omega(z)=Re^{it}$, $t\in[-\pi,\pi]$, where $R\leq r<1$, then inequality (5.3.11) yields

$$\operatorname{Re}\left(\frac{z\omega'(z)\tanh\sqrt{\omega(z)}}{2\sqrt{\omega(z)}}\right) \leq \frac{\sqrt{r}\tan\sqrt{r}}{2(1-r^2)}. \tag{5.3.12}$$

Thus from inequalities (5.3.9) and (5.3.12) we conclude that

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f(z)}\right) \geq \cos\sqrt{r}-\frac{\sqrt{r}\tan\sqrt{r}}{2(1-r^2)}.$$

Hence the least positive root of the equation of $2(1-r^2)\cos\sqrt{r}-\sqrt{r}\tan\sqrt{r}=\beta$ will serve the purpose. □

**Theorem 5.3.4.** *For $-1\leq B<A\leq 1$, suppose $f\in\mathscr{S}_n^*[A,B]$, then the sharp $\mathscr{S}_n^*(\rho)-$radius is given by*

*(i)* $\mathscr{R}_{\mathscr{S}_n^*(\rho)}(\mathscr{S}_n^*[A,B])=\min\{1;((1-l_0)/(A-Bl_0))^{1/n}\}=:\mathscr{R}_0$, *where* $0\leq B<A\leq 1$.

*(ii)* $\mathscr{R}_{\mathscr{S}_n^*(\rho)}(\mathscr{S}_n^*[A,B])=\begin{cases}\mathscr{R}_0, & \mathscr{R}_0\leq\mathscr{R}_1,\\ \mathscr{R}_2, & \mathscr{R}_0>\mathscr{R}_1,\end{cases}$ *where* $-1\leq B<0<A\leq 1$.

*where*

$$\mathscr{R}_1:=\left(\frac{l_0-2}{B(l_0B-2A)}\right)^{1/2n}, \quad \mathscr{R}_2:=\min\left\{1;\left(\frac{l_1-1}{A-Bl_1}\right)^{1/n}\right\}.$$

*Proof.* As $f\in\mathscr{S}_n^*[A,B]$, then $p(z)=zf'(z)/f(z)$ lies in the disc $|p(z)-c|<R$, where

$$c=\frac{1-ABr^{2n}}{1-B^2r^{2n}} \quad \text{and} \quad R=\frac{(A-B)r^n}{1-B^2r^{2n}}.$$

If $B \geq 0$, then $c \leq 1$. For $f(z)$ to lie in $\mathcal{S}_n^*(\rho)$, Theorem 5.2.1 and Lemma 3.2.3 yields

$$\frac{(A-B)r^n}{1-B^2r^{2n}} \leq \frac{1-ABr^{2n}}{1-B^2r^{2n}} - l_0.$$

The above inequality gives $r \leq R_0$. Equality here holds for $\tilde{f}(z)$ of the form

$$\tilde{f}(z) = \begin{cases} z(1+Bz^n)^{(A-B)/nB}, & B \neq 0 \\ z\exp(Az^n/n), & B = 0. \end{cases} \tag{5.3.13}$$

Further, if $-1 \leq B < 0 < A \leq 1$ and $\mathcal{R}_0 \leq \mathcal{R}_1$, then $c \leq (l_0+l_1)/2$ if and only if $r \leq \mathcal{R}_1$. Therefore, for $0 < r \leq \mathcal{R}_0$, we deduce that $c \leq (l_0+l_1)/2$. Infact due to Theorem 5.2.1 for each $f \in \mathcal{S}_n^*(\rho)$, we have $(A-B)r^n/(1-B^2r^{2n}) \leq c - l_0$, equivalently $r \leq \mathcal{R}_0$. Furthermore assume that $\mathcal{R}_0 > \mathcal{R}_1$. Then $c \geq (l_0+l_1)/2$ if and only if $r \geq \mathcal{R}_1$. In particular for $r \geq \mathcal{R}_0$, we have $c \geq (l_0+l_1)/2$. Thus by Theorem 5.2.1, for each $f \in \mathcal{S}_n^*(\rho)$, the inequality $(A-B)r^n/(1-B^2r^{2n}) \geq l_1 - c$ is equivalent to $r \leq \mathcal{R}_2$. The function $\tilde{f}(z)$ given in (5.3.13) works as the extremal function. □

**Theorem 5.3.5.** *Let $\alpha > 1$, then the sharp $\mathcal{S}_n^*(\rho)-$radius for the class $\mathcal{M}_n(\alpha)$, is given by*

$$\mathcal{R}_{\mathcal{S}_n^*(\rho)}(\mathcal{M}_n(\alpha)) = \left(\frac{1-l_0}{2\alpha-(1+l_0)}\right)^{1/n}.$$

*Proof.* As $f \in \mathcal{M}_n(\alpha)$, then $zf'(z)/f(z) \prec (1+(1-2\alpha)z)/(1-z)$. Clearly, for each $\alpha > 1$, $(1+(1-2\alpha)r^{2n})/(1-r^{2n}) \leq 1$. Further by Lemma 3.2.3, we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}}\right| \leq \frac{2(\alpha-1)r^n}{1-r^{2n}}.$$

On applying Theorem 5.2.1, we have

$$\frac{2(\alpha-1)r^n}{1-r^{2n}} \leq \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}} - l_0$$

or equivalently $r^{2n}((1-2\alpha)+l_0) - 2(\alpha-1)r^n + 1 - l_0 \geq 0$, which gives $r \leq \mathcal{R}_{\mathcal{S}_n^*(\rho)}(\mathcal{M}_n(\alpha))$. The required extremal function is $\tilde{f}(z) = z/(1-z^n)^{2(1-\alpha)/n}$. □

In view of certain properties of the class $\mathcal{F}_\varphi$, where $\mathcal{F}_\varphi$ is as defined in (3.1.2), with $\varphi$ given by (3.1.1), we establish the following theorem.

**Theorem 5.3.6.** *Suppose $0 \leq \alpha < 1$, then for $f \in \mathcal{A}$, the sharp $\mathcal{F}_\varphi-$radii for the class $\mathcal{S}_\rho^*$ is given by*

$$\mathcal{R}_{\mathcal{F}_\varphi}(\mathcal{S}_\rho^*) = \left(\cosh^{-1}\left(\frac{3}{2}\right)\right)^2.$$

*Proof.* As $f \in \mathcal{S}^*_\rho$, then $f$ lies in $\mathcal{F}_\varphi$, if we have

$$\max_{|z|=r} \operatorname{Re} \cosh\sqrt{z} = \cosh\sqrt{r} \leq \max_{|z|<1} \operatorname{Re}\varphi(z) = \frac{3}{2},$$

provided $r \leq \mathcal{R}_{\mathcal{F}_\varphi}(\mathcal{S}^*_\rho)$. Sharpness holds for the function $f_\rho \in \mathcal{A}$ defined as $zf'_\rho(z)/f_\rho(z) = \cosh\sqrt{z}$. □

As a consequence of Theorem 5.3.6, $\mathcal{S}^*(1+\alpha_0 z)-$ radii for the class $\mathcal{S}^*_\rho$, is stated in Corollary 5.3.1.

**Corollary 5.3.1.** Let $f \in \mathcal{F}_\varphi$, then the sharp $\mathcal{S}^*_\rho-$radius for the class $\mathcal{F}_\varphi$, is given by $\tanh^2(\pi\lambda/2)$, where $\lambda = \sin(1/2)$.

Recently, Lecko et al. [88] investigated the expressions $\operatorname{Re}(1-z^2)f(z)/z > 0$ and $\operatorname{Re}(1-z)^2 f(z)/z > 0$, involving the starlike functions $z/(1-z^2)$ and $z/(1-z)^2$. In 2019, Cho et al. [23] estimated radii constants for classes characterized by the ratio of two analytic functions $f(z)$ and $g(z)$ with certain conditions on $g(z)$, namely $\operatorname{Re} g(z)/z > \alpha$ for $\alpha = 0$ or $1/2$, such that $\operatorname{Re} f(z)/g(z) > 0$. Motivated by these classes, here below we define some subclasses of $\mathcal{A}_n$,

$$\mathfrak{F}_1(\beta) := \left\{ f \in \mathcal{A}_n : \left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ and } \operatorname{Re}\frac{g(z)}{z} > \beta, g \in \mathcal{A}_n \right\} \quad (\beta \in \{0, 1/2\})$$

and

$$\mathfrak{F}_2 := \left\{ f \in \mathcal{A}_n : \left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ and } g \in \mathcal{A}_n \text{ is convex} \right\}.$$

**Definition 5.3.1.** Let $-1 \leq A \leq 1$ and $g \in \mathcal{A}_n$, then for each $n = 1, 2, \ldots$, $\mathfrak{F}_3 \subset \mathcal{A}_n$, be defined as:

$$\mathfrak{F}_3 := \left\{ f \in \mathcal{A}_n : \operatorname{Re}\frac{f(z)}{g(z)} > 0 \text{ and } \operatorname{Re}\frac{(1-z^n)^{(1+A)/n} g(z)}{z} > 0 \right\}.$$

*Remark* 5.3.1. The functions $\tilde{f}(z) = z(1+(1-2\beta)z^n)$ and $\tilde{g}(z) = z(1+(1-2\hat{\beta})z^n)/(1-z^n)$ defined on $\mathbb{D}$ satisfy $|(\tilde{f}(z)/\tilde{g}(z)) - 1| = |z|^n < 1$ and $\operatorname{Re}\tilde{g}(z)/z = \operatorname{Re}(1+(1-2\beta)z^n)/(1-z^n) > \beta$. Therefore $\tilde{f} \in \mathfrak{F}_1(\beta)$, where $\beta \in \{0, 1/2\}$. If $\tilde{f}(z) = z(1+z^n)/(1-z^n)^{1/n}$ and $\tilde{g}(z) = z/(1-z^n)^{1/n}$, then $\tilde{f} \in \mathfrak{F}_2$. Similarly, when $\tilde{f}(z) = z(1+z^n)^2/(1-z^n)^{2+(1+A)/n}$ and $\tilde{g}(z) = z(1+z^n)/(1-z^n)^{1+(1+A)/n}$, then $\tilde{f} \in \mathfrak{F}_3$. Therefore the class $\mathfrak{F}_3$ is non-empty.

**Theorem 5.3.7.** *The sharp $\mathcal{S}^*_n(\rho)-$ radii for the classes $\mathfrak{F}_1(0), \mathfrak{F}_1(1/2)$ and $\mathfrak{F}_2$, are respectively given by*

*(i)* $$\mathcal{R}_{\mathcal{S}^*_n(\rho)}(\mathfrak{F}_1(0)) = \left(\frac{\sqrt{9n^2 - 4(l_0-1)(1+n-l_0)} - 3n}{2(1+n-l_0)}\right)^{1/n}.$$

*(ii)* $$\mathcal{R}_{\mathcal{S}^*_n(\rho)}(\mathfrak{F}_1(1/2)) = \left(\frac{1-l_0}{2n-(l_0-1)}\right)^{1/n}.$$

$$(iii)\ \mathscr{R}_{\mathscr{S}_n^*(\rho)}(\mathfrak{F}_2) = \left(\frac{\sqrt{1+n(n+6)+4l_0(l_0-(1+n))}-(1+n)}{2(n-l_0)}\right)^{1/n}.$$

*Proof.* Assume $f(z)/g(z) = p_1(z)$ and $g(z)/z = p_2(z)$, where $f(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$.

(i) As $f \in \mathfrak{F}_1(0)$, then $p_2 \in \mathscr{P}_n(0)$. We know that $|p_1(z)-1| < 1$ holds if $\operatorname{Re}(1/p_1(z)) > 1/2$ and vice-versa. Assume $f(z) = zp_1(z)p_2(z)$. Now using the expressions of $p_1(z)$, $p_2(z)$ and by applying Theorem 5.2.1 and Lemma 5.2.4 we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{zp_2'(z)}{p_2(z)} - \frac{zp_1'(z)}{p_1(z)}\right| \leq \frac{(3+r^n)nr^n}{1-r^{2n}} \leq 1-l_0.$$

The above inequality leads to $r^{2n}(n+1-l_0)+3nr^n-1+l_0 \leq 0$, provided $r \leq \mathscr{R}_{\mathscr{S}_n^*(\rho)}(\mathfrak{F}_1(0))$. The functions $\tilde{f}(z) = z(1+z^n)/(1-z^n)^2$ and $\tilde{g}(z) = z(1+z^n)/(1-z^n)$ at $z_0 = \mathscr{R}_{\mathscr{S}_n^*(\rho}(\mathfrak{F}_1(0))e^{i\pi/n}$ gives

$$\frac{z_0\tilde{f}'(z_0)}{\tilde{f}(z_0)} - 1 = \frac{(3+z_0^n)nz_0^n}{1-z_0^{2n}} = 1-l_0.$$

Thus $\tilde{f}$ is the extremal function.

(ii) As $f \in \mathscr{R}_{\mathscr{S}_n^*(\rho)}(\mathfrak{F}_1(1/2))$, then $1/p_1, p_2 \in \mathscr{P}_n(1/2)$. Proceeding as in (i), on applying Theorem 5.2.1 and Lemma 5.2.4 we get

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{2nr^n}{1-r^n} \leq 1-l_0.$$

This holds true whenever $r \leq \mathscr{R}_{\mathscr{S}_n^*(\rho)}(\mathfrak{F}_1(1/2))$. For sharpness, consider $\tilde{f}(z) = z$ and $\tilde{g}(z) = z/(1-z^n)$, then at $z_0 = \mathscr{R}_{\mathscr{S}_n^*(\rho)}(\mathfrak{F}_1(1/2))e^{i\pi/n}$, we get

$$\frac{z_0\tilde{f}'(z_0)}{\tilde{f}(z_0)} - 1 = \frac{2nz_0^n}{1-z_0^n} = 1-l_0.$$

(iii) Let $f(z)/g(z) = p(z)$ be a function defined in $\mathbb{D}$. As $f \in \mathfrak{F}_2$, then $|1/p(z)-1| < 1$ if and only if $\operatorname{Re} p(z) > 1/2$. As $g \in \mathscr{A}_n$ is convex, then due to Marx-Strohhäcker theorem, $g \in \mathscr{S}_n^*(1/2)$, $(\mathscr{S}_n^*(1/2) = \{f \in \mathscr{A}_n : \operatorname{Re} zf'(z)/f(z) > 1/2\})$. Therefore, due to Lemma 3.2.3,

$$\left|\frac{zg'(z)}{g(z)} - \frac{1}{1-r^{2n}}\right| \leq \frac{r^n}{1-r^{2n}}.$$

On logarithmically differentiating $f(z)$ and applying Theorem 5.2.1, we get

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-r^{2n}}\right| &= \left|\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)} - \frac{1}{1-r^{2n}}\right| \\ &\leq \frac{nr^{2n}+(1+n)r^n}{1-r^{2n}} \leq \frac{1}{1-r^{2n}} - l_0,\end{aligned}$$

which leads to $r^{2n}(n-l_0)+r^n(1+n)-1+l_0\leq 0$, provided $r\leq \mathscr{R}_{\mathscr{S}_n^*(\rho)}(\mathfrak{F}_2)$. The functions $\tilde{f}(z)=z(1+z^n)/(1-z^n)^{1/n}$ and $\tilde{g}(z)=z/(1-z^n)^{1/n}$ at $z_0=\mathscr{R}_{\mathscr{S}_n^*(\rho)}(\mathfrak{F}_2)e^{i\pi/n}$ gives $|z_0\tilde{f}'(z_0)/\tilde{f}(z_0)|=l_0$.

Hence the result is sharp. □

**Theorem 5.3.8.** *Let $-1\leq A\leq 1$ and $r\in[0,1)$, then for $n=1,2,\ldots$, the sharp $\mathscr{S}_n^*(\rho)-$radius for the class $\mathfrak{F}_3$ is given by*

$$\mathscr{R}_{\mathscr{S}_n^*(\rho)}(\mathfrak{F}_3)=\begin{cases}\mathscr{R}_0, & r\leq \mathscr{R}_0,\\ \mathscr{R}_1, & r\geq \mathscr{R}_0,\end{cases}$$

*where*

$$\mathscr{R}_0=\begin{cases}\left(\dfrac{1+A+4n+\sqrt{(1+A+4n)^2-4(1-l_0)(A+l_0)}}{2(A+l_0)}\right)^{1/n}, & \textit{if } -1\leq A<-l_0,\\[2ex] \left(\dfrac{1-l_0}{1+4n-l_0}\right)^{1/n}, & \textit{if } A=-l_0,\\[2ex] \left(\dfrac{1+A+4n-\sqrt{(1+A+4n)^2-4(1-l_0)(A+l_0)}}{2(A+l_0)}\right)^{1/n}, & \textit{if } -l_0<A\leq 1\end{cases}$$

*and*

$$\mathscr{R}_1=\left(\frac{\sqrt{(1+A+4n)^2+4(A+l_1)(l_1-1)}-(1+A+4n)}{2(A+l_1)}\right)^{1/n}.$$

*Proof.* Let $f\in\mathfrak{F}_3$, then $\operatorname{Re} f(z)/g(z)>0$ and $\operatorname{Re}((1-z^n)^{(1+A)/n}g(z)/z)>0$, where $g\in\mathscr{A}_n$. Define $g(z)/f(z)=p_1(z)$ and $(1-z^n)^{(1+A)/n}g(z)/z=p_2(z)$, where $p_1(z)$ and $p_2(z)$ are analytic in $\mathbb{D}$. Since $A<1$, then for $|z|=r<1$, the inequality $(1+Ar^{2n})\geq 1-r^{2n}$, holds true. Further on logarithmically differentiating $zp_1(z)p_2(z)(1-z^n)^{-(1+A)/n}=f(z)$, we get

$$\frac{zf'(z)}{f(z)}=\frac{1+Az^n}{1-z^n}+\frac{zp_1'(z)}{p_1(z)}+\frac{zp_2'(z)}{p_2(z)}.$$

Due to Lemma 3.2.3 and Lemma 5.2.4, for $|z|=r$, we infer

$$\left|\frac{zf'(z)}{f(z)}-\frac{1+Ar^{2n}}{1-r^{2n}}\right|\leq\frac{4nr^n}{1-r^{2n}}+\frac{(1+A)r^n}{1-r^{2n}}. \tag{5.3.14}$$

Assume $c=(1+Ar^{2n})/(1-r^{2n})$. Then $c\leq(l_0+l_1)/2$ leads to $r\leq\mathscr{R}$ and vice-versa, where $\mathscr{R}=((l_0-2)/(2A+l_0))^{1/2n}$. Algebraically, for each $n=1,2,3,\ldots$, it can be observed that, for the given range of $A$, we have $\mathscr{R}_0<\mathscr{R}_1<\mathscr{R}$. In particular, if $r\leq\mathscr{R}_0$, then $c\leq(l_0+l_1)/2$. Further due to Theorem 5.2.1, inequality (5.3.14) gives

$$\frac{4nr^n}{1-r^{2n}} + \frac{(1+A)r^n}{1-r^{2n}} \leq \frac{1+Ar^{2n}}{1-r^{2n}} - l_0,$$

whenever $r \leq \mathscr{R}_0$. Moreover if $c \geq (l_0+l_1)/2$, then $r \geq \mathscr{R}_0$. Infact, when $r \geq \mathscr{R}_0$, then we have $c \geq (l_0+l_1)/2$. Now inequality (5.3.14) together with Theorem 5.2.1 yields

$$\frac{4nr^n}{1-r^{2n}} + \frac{(1+A)r^n}{1-r^{2n}} \leq l_1 - \frac{1+Ar^{2n}}{1-r^{2n}},$$

provided $r \leq \mathscr{R}_1$. Thus the following functions, mentioned in Remark 5.3.1

$$\tilde{f}(z) = \frac{z(1+z^n)^2}{(1-z^n)^{2+(1+A)/n}} \quad \text{and} \quad \tilde{g}(z) = \frac{z(1+z^n)}{(1-z^n)^{1+(1+A)/n}},$$

serve as the extremal function for both the cases. □

We now determine certain sufficient conditions for the class $\mathscr{S}^*_\rho$.

**Theorem 5.3.9.** *Let $f \in \mathscr{A}$ , then $f \in \mathscr{S}^*_\rho$ if and only if*

$$\frac{1}{z}\left(f(z) * \frac{z-kz^2}{(1-z)^2}\right) \neq 0 \tag{5.3.15}$$

*where $k = \cosh e^{it/2}/(\cosh e^{it/2}-1)$ for $t \in [-\pi,\pi]$. Moreover, $f \in \mathscr{S}^*_\rho$ if and only if*

$$1 - \sum_{n=2}^{\infty} \frac{(n-\cosh e^{it/2})a_n}{\cosh e^{it/2}-1} z^{n-1} \neq 0. \tag{5.3.16}$$

*Proof.* Since $f \in \mathscr{S}^*_\rho$, then $zf'(z)/f(z) = \cosh\sqrt{\omega(z)}$, where $\omega(z)$ is a Schwarz function with $\omega(0) = 0$. Equivalently for $\omega(z) = e^{it}$, $-\pi \leq t \leq \pi$, we have

$$\frac{zf'(z)}{f(z)} \neq \cosh e^{it/2} \Leftrightarrow zf'(z) - (\cosh e^{it/2})f(z) \neq 0 \quad \text{for} \quad t \in [-\pi,\pi],$$

Eventually it leads to $zf'(z) - k(zf'(z) - f(z)) \neq 0$. Thus through simple computations (5.3.15) can be established, and condition in (5.3.16) can be deduced using (5.3.15). □

**Corollary 5.3.2.** Let $f \in \mathscr{A}$ satisfy the following:

$$\sum_{n=2}^{\infty} \left|\frac{n-\cosh e^{it/2}}{\cosh e^{it/2}-1}\right| |a_n| < 1, \tag{5.3.17}$$

then $f \in \mathscr{S}^*_\rho$.

*Proof.* Consider the following inequality with $k = \cosh e^{it/2}/(\cosh e^{it/2} - 1)$,

$$\left|1 - \sum_{n=2}^{\infty}(n(k-1)-k)a_n z^{n-1}\right| \geq 1 - \sum_{n=2}^{\infty}|n(k-1)-k||a_n|.$$

Thus from (5.3.17) we establish

$$\left|1 - \sum_{n=2}^{\infty}(n(k-1)-k)a_n z^{n-1}\right| > 0,$$

Hence due to Theorem 5.3.9, we conclude that $f \in \mathcal{S}_{\rho}^{*}$. □

**Theorem 5.3.10.** *Let $f \in \mathcal{S}_{\rho}^{*}$, then the following inequality holds:*

$$l_1^2 - 1 \geq \sum_{k=2}^{\infty}(k^2 - l_1{}^2)|a_k|^2.$$

*Proof.* Since $f \in \mathcal{S}_{\rho}^{*}$, then $zf'(z) = \cosh(\sqrt{\omega(z)})f(z)$, for a Schwarz function $\omega(z)$ with $\omega(0) = 0$. For $0 < |z| = r < 1$, we get the following

$$\begin{aligned}
2\pi \sum_{k=1}^{\infty} k^2|a_k|^2 r^{2k} &= \int_0^{2\pi} \left|re^{i\theta} f'(re^{i\theta})\right|^2 d\theta \\
&= \int_0^{2\pi} \left|\cosh\left(\sqrt{\omega(re^{i\theta})}\right) f(re^{i\theta})\right|^2 d\theta \\
&\leq \int_0^{2\pi} \cosh^2\left(\sqrt{|\omega(re^{i\theta})|}\right) |f(re^{i\theta})|^2 d\theta \\
&\leq \int_0^{2\pi} (\cosh^2 r)|f(re^{i\theta})|^2 d\theta \\
&= 2\pi(\cosh^2 r)\sum_{k=1}^{\infty}|a_k|^2 r^{2k}. \qquad (5.3.18)
\end{aligned}$$

Thus when $r$ tends to $1^{-}$, we at once obtain the required inequality. □

*Example* 1. Let $f \in \mathcal{A}$, then following functions are members of $\mathcal{S}_{\rho}^{*}$ :

(i) $f(z) = z + a_n z^n \in \mathcal{S}_{\rho}^{*}$, provided $|a_n| \leq (1 - l_0)/(n - l_0)$, $n \in \mathbb{N} - \{1\}$.

(ii) $f(z) = z/(1 - Az)^2 \in \mathcal{S}_{\rho}^{*}$, provided $|A| \leq (l_1 - 1)/(l_1 + 1)$.

(iii) $f(z) = z/(1 - Az) \in \mathcal{S}_{\rho}^{*}$, provided $|A| \leq (l_1 - 1)/l_1$.

(iv) $f(z) = ze^{Az} \in \mathcal{S}_{\rho}^{*}$, provided $|A| \leq 1 - l_0$.

*Proof.* For part (i) we require that $zf'(z)/f(z) = (1 + na_n z^{n-1})/(1 + a_n z^{n-1})$ must lie in the disc $\{w : |w - c| < r_c\} \subset \rho(\mathbb{D})$, centered at $c$, where $r_c$ is defined in (5.2.6). It is a known fact that the function $f(z) = z + a_n z^n$ is univalent if and only if $|a_n| \leq 1/n$. Thus $c = (1 - n|a_n|^2)/(1 - |a_n|^2) \leq 1$.

If $w=(1+na_nz^{n-1})/(1+a_nz^{n-1})$ and $r_c=(1-n|a_n|^2)/(1-|a_n|^2)-l_0$, then due to Theorem 5.2.1,

$$\frac{(n-1)|a_n|}{1-|a_n|^2}\leq\frac{1-n|a_n|^2}{1-|a_n|^2}-l_0.$$

The proofs of (ii)-(iv) are much akin to (i), therefore it is skipped. □

So far, we have thoroughly examined the radius and inclusion properties of the class $\mathcal{S}^*_\rho$, providing significant insights into its geometric characteristics. Building upon these findings, the subsequent chapter investigates into the differential subordination aspects of $\mathcal{S}^*_\rho$, offering a detailed exploration of its analytical properties.

## Highlights of the Chapter

In this chapter, we introduce a new class of Ma-Minda starlike functions defined using the Hyperbolic cosine function. We derive a structural formula for the class, denoted as $\mathcal{S}^*_\rho$, and provide illustrative examples. Utilizing this formula, we establish the Growth, Distortion, and Rotation theorems pertinent to this class. Furthermore, we explore inclusion relations between $\mathcal{S}^*_\rho$ and existing classes such as $\mathcal{S}^*(\beta)$, $\mathcal{M}(\alpha)$, $\mathcal{S}\mathcal{S}^*(\beta)$, $\mathcal{S}^*_{q_k}$, $k$-$\mathcal{S}\mathcal{T}$, $\mathcal{S}^*_{hpl}(\tilde{s})$, $\mathcal{S}^*_L(s)$ and $\mathcal{S}\mathcal{T}_p(\gamma)$. We also provide diagrammatic representations for the inclusion relations for the class under study, depicting sharpness of our findings. Additionally, we determine the sharp $\mathcal{S}^*_\rho-$radius for classes such as $\mathcal{S}^*_n[A,B]$, $\mathcal{M}_n(\alpha)$, and various subclasses characterized by the ratio of analytic functions. This comprehensive study not only enhances the understanding of starlike classes but also offers valuable insights for future research in GFT.

The contents of this chapter is based on the findings presented in the paper:

*Mridula Mundalia and S. Sivaprasad Kumar: On a subfamily of starlike functions related to hyperbolic cosine function, The Journal of Analysis,* **31***(3), 2043-2062, (2023).* https://doi.org/10.1007/s41478-023-00550-1

## Chapter 6

# Sufficient Conditions for Functions to be in $\mathscr{S}^*_{\cosh\sqrt{z}}$

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In this chapter, we use Briot-Bouquet differential subordination and similar techniques to establish sufficient conditions for functions to belong to the class $\mathscr{S}^*_{\rho}$, which consists of starlike functions associated with $\cosh\sqrt{z}$. Additionally, by applying admissibility conditions, we derive several differential subordination results for the class $\mathscr{S}^*_{\rho}$.

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## 6.1 Introduction

Differential subordination has long been a classical method in complex analysis. The foundational work by Miller and Mocanu [110] has significantly advanced this theory, along with contributions from others [16, 34, 39, 69, 110, 137, 155, 157]. Numerous authors have established conditions on $\eta$, ensuring that the subordination $1+\eta zp'(z)/p^n(z) \prec \phi_1(z)$ $(n = 0,1,2)$ implies $p(z) \prec \phi_2(z)$. These conditions vary based on the selection of $\phi_1(z)$ and $\phi_2(z)$, which include: $2/(1+e^{-z})$, $\sqrt{1+z}$, $(1+Az)/(1+Bz)$ $(-1 \leq B < A \leq 1)$ and $e^z$, as noted by [39, 56, 81, 106]. Further, numerous studies have extensively explored the Briot-Bouquet differential subordination, given by

$$p(z)+\frac{zp'(z)}{\eta p(z)+\gamma} \prec h(z), \tag{6.1.1}$$

with contributions from various authors over time. For comprehensive insights, refer to [16, 29, 69, 110]. Specifically, one can refer to the works of Ravichandran et al. [155] and Singh et al. [157] for notable contributions to Briot-Bouquet differential subordination results. This unique form of differential subordination holds significant importance in UFT and has a wide range of applications. Additionally, it is common to analyze the implication results related to (6.1.1) with the assumption that $h(z)$ is a convex function and $\mathrm{Re}(\eta h(z)+\gamma) > 0$. Previously, sufficient conditions for the classes $\mathcal{S}^*_L$, $\mathcal{S}^*[A,B]$, $\mathcal{S}^*_s$, $\mathcal{S}^*_e$ and $\mathcal{S}^*_{\mathfrak{D}}$ (see Table 1.1) are studied.

In this chapter, we obtain sufficient conditions for functions to be in $\mathcal{S}^*_\rho$, the class of starlike functions associated with $\cosh\sqrt{z}$, given by Definition 5.1.1, using Briot-Bouquet differential subordination and similar differential subordination techniques. We list below the image regions of $\mathbb{D}$, corresponding to the functions $e^z$, $z+\sqrt{1+z^2}$, $(1+Az)/(1+Bz)$ and $(1+sz)^2$, respectively, required for our further investigations

$$\begin{aligned}
\Omega_e &:= \{w \in \mathbb{C} : |\log w| < 1\},\\
\Omega_{\mathfrak{D}} &:= \{w \in \mathbb{C} : |w^2-1| < 2|w|\}\\
&= \{w \in \mathbb{C} : |w-1| < \sqrt{2}\} \cap \{w \in \mathbb{C} : |w+1| > \sqrt{2}\} = \Delta_1 \cap \Delta_2, \qquad (6.1.2)\\
\Omega_{A,B} &:= \{w \in \mathbb{C} : |w-1| < |A-Bw|\}\\
\Omega_s &:= \{x+iy : ((x-1)^2+y^2-s^4)^2 < 4s^2((x-1+s^2)^2+y^2)\}\\
&\subset \{w \in \mathbb{C} : |w-1| < |s|(|s|+2)\}. \qquad (6.1.3)
\end{aligned}$$

From Figure 6.1, we see that the crescent domain in (6.1.2) is given by $\Omega_{\mathfrak{D}} = \Delta_1 \cap \Delta_2$, where $\Delta_1 = \{w : |w-1| < \sqrt{2}\}$ and $\Delta_2 = \{w : |w+1| > \sqrt{2}\}$. Generally, while dealing with the crescent domain, we conclude that if $x \notin \Delta_i$, then $x \notin \Omega_{\mathfrak{D}}$, as $\Omega_{\mathfrak{D}} \subset \Delta_i$, where $i = 1, 2$.

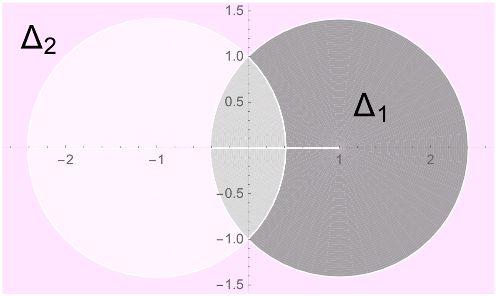


Figure 6.1: $\Omega_{\mathfrak{D}} = \Delta_1 \cap \Delta_2$.

In section 6.2, we derive sufficient conditions for functions to be in $\mathcal{S}^*_{\rho}$, by establishing Briot-Bouquet differential subordination implications with dominants such as $e^z$, $(1+Az)/(1+Bz)$, $z+\sqrt{1+z^2}$ and $(1+sz)^2$. Additionally, in section 6.3, we study some first-order differential subordination results for $\mathcal{S}^*_{\rho}$ and diagrammatically validate the sharpness of our findings. Finally, we deduce certain admissibility results for $\mathcal{S}^*_{\rho}$, accompanied by some applications and illustrations of our findings.

## 6.2 Briot-Bouquet Differential Subordination Results

To proceed, we require the following lemma, which will be instrumental in establishing some of our main results in this section.

**Lemma 6.2.1.** [148, Lemma 1.3, p.28] Let $\omega$ be a meromorphic function in $\mathbb{D}$, $\omega(0) = 0$. If for some $z_0 \in \mathbb{D}$, $\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)|$, then it follows that $z_0\omega'(z_0)/\omega(z_0) \geq 1$.

Now we begin with the following theorem:

**Theorem 6.2.1.** *Let $\eta, \gamma \in \mathbb{R}$ such that $\gamma \neq -\eta$, satisfy any of the following conditions:*

*(i) For $\phi(z) = z + \sqrt{1+z^2}$, we have*

$$\eta_2 \leq \eta \leq \eta_1 \text{ and } \eta_3 \leq \eta \leq \eta_4, \tag{6.2.4}$$

*where*

$$\eta_1 = -\frac{\gamma}{\cosh 1} + \frac{\sinh 1}{(2\cosh 1(1+\sqrt{2}-\cosh 1))}, \ \eta_2 = -\frac{\gamma}{\cosh 1},$$
$$\eta_3 = -\frac{\gamma}{\cos 1} - \frac{\sin 1}{(2\cos 1(1+\sqrt{2}-\cos 1))}, \ \eta_4 = -\frac{\gamma}{\cos 1}.$$

*(ii) For $\phi(z)=(1+sz)^2$ and $0<s\leq 1/\sqrt{2}$, we have*

$$\left.\eta_4+\eta_3\leq\eta\leq\eta_3 \text{ and } \begin{array}{ll}\eta\geq\eta_1 & \text{if } 0<s\leq -1+\sqrt{\cosh 1},\\ \eta_1\leq\eta\leq\eta_1+\eta_2 & \text{if } -1+\sqrt{\cosh 1}<s\leq 1/\sqrt{2},\end{array}\right\} \tag{6.2.5}$$

*where*

$$\begin{aligned}\eta_1&=-\frac{\gamma}{\cosh 1},\ \eta_2=\frac{\sinh 1}{2\cosh 1((1+s)^2-\cosh 1)},\\ \eta_3&=-\frac{\gamma}{\cos 1},\ \eta_4=-\frac{\sin 1}{2\cos 1((1+s)^2-\cos 1)}.\end{aligned}$$

*(iii) For $\phi(z)=e^z$, we have*

$$\eta_2<\eta\leq\eta_1 \text{ and } \eta_3\leq\eta<\eta_4, \tag{6.2.6}$$

*where*

$$\eta_1=-\frac{\gamma}{\cosh 1}+\frac{\sinh 1}{2\cosh 1(e-\cosh 1)},\ \eta_3=-\frac{\gamma}{\cos 1}-\frac{\sin 1}{2\cos 1(e-\cos 1)},$$

*$\eta_2$ and $\eta_4$ are as given in (i).*

*If $p(z)$ is an analytic function, such that $p(0)=1$ and satisfies*

$$p(z)+\frac{zp'(z)}{\eta p(z)+\gamma}\prec\phi(z), \tag{6.2.7}$$

*then $p(z)\prec\cosh\sqrt{z}$.*

*Proof.* Let $\mathfrak{B}(z)$ and $\omega(z)$ be as given below:

$$\mathfrak{B}(z):=p(z)+\frac{zp'(z)}{\eta p(z)+\gamma} \text{ and } \omega(z)=(\cosh^{-1}p(z))^2, \tag{6.2.8}$$

then we have $p(z)=\cosh\sqrt{\omega(z)}$. It is evident that $\omega(z)$ is a well-defined analytic function, with $\omega(0)=0$. Now to prove $p(z)\prec\cosh\sqrt{z}$, we need to show that $|\omega(z)|<1$ in $\mathbb{D}$. For if, there exists $z_0\in\mathbb{D}$ such that $\max_{|z|\leq|z_0|}|\omega(z)|=|\omega(z_0)|=1$, then by Lemma 6.2.1, we have $z_0\omega'(z_0)=k\omega(z_0)$, where $k\geq 1$. Let $\omega(z_0)=e^{2it}$, where $-\pi/2\leq t\leq\pi/2$.

(i) Let $\phi(z)=z+\sqrt{1+z^2}$, then from (6.1.2), we have $\phi(\mathbb{D})\subset\{w:|w-1|<\sqrt{2}\}$. Now we deduce a contradiction by arriving at $|\mathfrak{B}(z_0)-1|^2\geq 2$. To do this, we expand the following expression using (6.2.8):

$$|\mathfrak{B}(z_0)-1|^2 = \left|\cosh\sqrt{\omega(z_0)}-1+\frac{z_0\omega'(z_0)\sinh\sqrt{\omega(z_0)}}{2\sqrt{\omega(z_0)}(\eta\cosh\sqrt{\omega(z_0)}+\gamma)}\right|^2$$

$$= \left|\cosh e^{it}-1+\frac{ke^{it}\sinh e^{it}}{2(\eta\cosh e^{it}+\gamma)}\right|^2 \tag{6.2.9}$$

$$=: \frac{N(t)}{D(t)}, \tag{6.2.10}$$

where

$$\begin{aligned} N(t) = {} & (\sinh(\cos t)(2\gamma k\sin t\cos(\sin t)+2(2\gamma^2-\eta^2)\sin(\sin t)+\eta^2\sin(3\sin t)) \\ & +2\gamma\sin(\sin t)\cosh(\cos t)(k\cos t+4\eta\cos(\sin t)\sinh(\cos t))+\eta k(\sin(2\sin t)\cos t \\ & +\sin t\sinh(2\cos t))+\eta^2\sin(\sin t)\sinh^3(\cos t)+3\eta^2\sin(\sin t)\sinh(\cos t)\cosh^2(\cos t))^2 \\ & +4(\cosh(\cos t)(\eta k\cos t\sinh(\cos t)-\gamma k\sin t\sin(\sin t)+2\gamma(\gamma-2\eta)\cos(\sin t) \\ & +2\eta^2\sin^2(\sin t)\cos(\sin t)\sinh^2(\cos t))-\eta\cos(\sin t)\cosh^2(\cos t)(k\sin t\sin(\sin t) \\ & +2(\eta-2\gamma)\cos(\sin t))+\gamma k\cos t\cos(\sin t)\sinh(\cos t)+2\eta^2\cos^3(\sin t)\cosh^3(\cos t) \\ & +\eta\sin(\sin t)\sinh^2(\cos t)(k\sin t\cos(\sin t)-2\eta\sin(\sin t))-2\gamma^2)^2 \end{aligned} \tag{6.2.11}$$

and

$$D(t) = 16((\gamma+\eta\cos(\sin t)\cosh(\cos t))^2+\eta^2\sin^2(\sin t)\sinh^2(\cos t))^2. \tag{6.2.12}$$

Define a function $F(t)$ on the interval $[-\pi/2,\pi/2]$, as

$$F(t) = N(t)-2D(t).$$

Since $F(t)$ is an even function, it is sufficient to show that $F(t)$ is non-negative in $[0,\pi/2]$ or minimum of $F(t)$ is non-negative in $[0,\pi/2]$. A computation reveals that the minimum of $F(t)$ is obtained either at $t=0$ or $t=\pi/2$. Now we see that

$$F(0) = (k(\eta\sinh 2+2\gamma\sinh 1)-4(1-\cosh 1)(\gamma+\eta\cosh 1)^2)^2-32(\gamma+\eta\cosh 1)^4$$

and

$$F\left(\frac{\pi}{2}\right) = (k(\eta\sin 2+2\gamma\sin 1)-4(\cos 1-1)(\gamma+\eta\cos 1)^2)^2-32(\gamma+\eta\cos 1)^4.$$

Since $k\geq 1$, we have $F(0)\geq(4(\cosh 1-1)(\gamma+\eta\cosh 1)^2+(\eta\sinh 2+2\gamma\sinh 1)^2)^2-32(\gamma+$

$\eta\cosh 1)^4 =: X(\eta)$. But $X(\eta)\geq 0$, whenever $\eta_2\leq\eta\leq\eta_1$, which implies $F(0)\geq 0$. Infact, for $k\geq 1$, we have $F(\pi/2)\geq((\eta\sin 2+2\gamma\sin 1)-4(\cos 1-1)(\gamma+\eta\cos 1)^2)^2-32(\gamma+\eta\cos 1)^4 =: Y(\eta)$. Since,

$$(\eta\sin 2+2\gamma\sin 1)-4(\cos 1-1)(\gamma+\eta\cos 1)^2\geq 4\sqrt{2}(\gamma+\eta\cos 1)^2,$$

whenever $\eta_3\leq\eta\leq\eta_4$, therefore, $Y(\eta)\geq 0$, which implies $F(\pi/2)\geq 0$. Thus $|\mathfrak{B}(z_0)-1|^2\geq 2$, which contradicts (6.2.7), hence the result follows at once.

(ii) Let $\phi(z)=(1+sz)^2$, then from (6.1.3), we have $\phi(\mathbb{D})\subset\{w:|w-1|<s(s+2)\}$. Now we shall show that $|\mathfrak{B}(z_0)-1|\geq s(s+2)$, which leads to the desired contradiction. To achieve this, we use the expansion of $|\mathfrak{B}(z_0)-1|^2$, as given in (6.2.9), with $N(t)$ and $D(t)$ given by (6.2.11) and (6.2.12), respectively. Now for each $0<s\leq 1/\sqrt{2}$, define

$$F_s(t)=N(t)-s^2(s+2)^2D(t),\ \text{where}\ -\pi/2\leq t\leq\pi/2.$$

We observe that $F(t)$ is an even function, consequently, it is suffices to prove that $F(t)\geq 0$ for $t\in[0,\pi/2]$. Furthermore, it is observed that $F(t)$ attains its minimum at either $t=0$ or $\pi/2$. Now for $k\geq 1$, we have

$$\begin{aligned}F_s(0)=&(4(\cosh 1-1)(\gamma+\eta\cosh 1)^2+k(\eta\sinh 2+2\gamma\sinh 1))^2\\&-16s^2(s+2)^2(\gamma+\eta\cosh 1)^4\end{aligned}$$

and

$$\begin{aligned}F_s\left(\frac{\pi}{2}\right)=&(4(\cos 1-1)(\gamma+\eta\cos 1)^2-k(\eta\sin 2+2\gamma\sin 1))^2\\&-16s^2(s+2)^2(\gamma+\eta\cos 1)^4.\end{aligned}$$

For each $k\geq 1$, it can be easily verified that $F_s(0)$ is a monotonically increasing function of $k$, provided $\eta\geq\eta_1$, this implies $F_s(0)\geq(4(\cosh 1-1)(\gamma+\eta\cosh 1)^2+(\eta\sinh 2+2\gamma\sinh 1))^2-16s^2(s+2)^2(\gamma+\eta\cosh 1)^4 =: X_s(\eta)$. Now $X_s(\eta)$ can be written as

$$X_s(\eta)=(X_{s_1}(\eta)-X_{s_2}(\eta))(X_{s_1}(\eta)+X_{s_2}(\eta)),$$

where $X_{s_1}(\eta):=4(\cosh 1-1)(\gamma+\eta\cosh 1)^2+(\eta\sinh 2+2\gamma\sinh 1)-4s(s+2)(\gamma+\eta\cosh 1)^2$ and $X_{s_2}(\eta):=4(\cosh 1-1)(\gamma+\eta\cosh 1)^2+(\eta\sinh 2+2\gamma\sinh 1)+4s(s+2)(\gamma+\eta\cosh 1)^2$. Now, for each $0<s\leq 1/\sqrt{2}$, we need to show that $X_s(\eta)\geq 0$. Observe that, for each $0<s\leq\sqrt{\cosh 1}-1$, $X_s(\eta)\geq 0$ if and only if $X_{s_1}(\eta)\geq X_{s_2}(\eta)$, i.e. $\eta\sinh 2+2\gamma\sinh 1+4(\cosh 1-(s+1)^2)(\gamma+$

$\eta\cosh 1)^2 \geq 0$, which happens, whenever $\eta \geq \eta_1$, thus $X_s(\eta) \geq 0$. Infact, for each $\sqrt{\cosh 1}-1 < s \leq 1/\sqrt{2}$, $X_s(\eta) \geq 0$ if and only if the inequality $4((s+1)^2-\cosh 1)(\eta\cosh 1+\gamma)^2 \leq \eta\sinh 2 + 2\gamma\sinh 1$ holds, which is possible whenever $\eta_1 \leq \eta \leq \eta_1+\eta_2$, which means $X_s(\eta) \geq 0$. Eventually, for each $k \geq 1$ and $0 < s \leq 1/\sqrt{2}$, we have $F_s(0) \geq 0$. Next, for $\eta \leq \eta_3$, it can be seen that $F_s(\pi/2)$ is a monotonically increasing function of $k$ and $F_s(\pi/2) \geq (4(\cos 1-1)(\gamma+\eta\cos 1)^2-(\eta\sin 2+2\gamma\sin 1))^2 - 16s^2(s+2)^2(\gamma+\eta\cos 1)^4 =: Y_s(\eta)$. In addition, for each $0 < s \leq 1/\sqrt{2}$, we have $Y_s(\eta) \geq 0$, whenever $\eta_3+\eta_4 \leq \eta \leq \eta_3$. Therefore, for each $k \geq 1$ and $0 < s \leq 1/\sqrt{2}$, we deduce that $F_s(\pi/2) \geq 0$. Thus $|\mathfrak{B}(z_0)-1|^2 \geq s^2(s+2)^2$. Hence, we get a contradiction to the hypothesis, given in (6.2.7), which completes the proof.

(iii) Choose $\phi(z) = e^z$. We prove this result by the method of contradiction, similar to (i) and (ii). To proceed, it is suffices to show that

$$|\log\mathfrak{B}(z_0)|^2 \geq 1, \tag{6.2.13}$$

where log denotes the principle branch of logarithmic function. Consider,

$$\mathfrak{B}(z_0) = \cosh e^{it} + \frac{ke^{it}\sinh e^{it}}{2(\eta\cosh e^{it}+\gamma)} =: U(t)+iV(t), \tag{6.2.14}$$

where

$$\begin{aligned} U(t) := \kappa_t^{-1}(&\cosh(\cos t)(\gamma(2\gamma\cos(\sin t)-k\sin t\sin(\sin t))+\eta k\cos t\sinh(\cos t) \\ &+2\eta^2\sin^2(\sin t)\cos(\sin t)\sinh^2(\cos t))+k\cos(\sin t)\sinh(\cos t)(\gamma\cos t \\ &+\eta\sin t\sin(\sin t)\sinh(\cos t))+\eta\cos(\sin t)\cosh^2(\cos t)(4\gamma\cos(\sin t) \\ &+2\eta^2\cos^3(\sin t)\cosh^3(\cos t)-k\sin t\sin(\sin t))) \end{aligned}$$

and

$$\begin{aligned} V(t) := \kappa_t^{-1}(&\sinh(\cos t)(2\gamma k\sin t\cos(\sin t)+2(2\gamma^2-\eta^2)\sin(\sin t)+\eta^2\sin(3\sin t)) \\ &+\eta k(\sin(2\sin t)\cos t+\sin t\sinh(2\cos t))+\eta^2\sin(\sin t)\sinh^3(\cos t) \\ &+3\eta^2\sin(\sin t)\sinh(\cos t)\cosh^2(\cos t))+2\gamma\sin(\sin t)\cosh(\cos t)(k\cos t \\ &+4\eta\cos(\sin t)\sinh(\cos t)), \end{aligned}$$

with $\kappa_t := 2((\gamma+\eta\cos(\sin t)\cosh(\cos t))^2+\eta^2\sin^2(\sin t)\sinh^2(\cos t))$.
Assume $F(t) = 4|\log\mathfrak{B}(z_0)|^2-4$. To prove (6.2.13), in view of (6.2.14), it is enough to show that

$$F(t) = \log^2(U^2(t)+V^2(t))+4\left(\tan^{-1}\frac{V(t)}{U(t)}\right)^2-4 \geq 0.$$

Since $F(-t)=F(t)$, for each $t\in[-\pi/2,\pi/2]$, so we confine our findings to the interval $[0,\pi/2]$. It can be easily verified that $F(t)$ attains its minimum at $t=0$ or $\pi/2$. Now for $k\geq 1$, we have

$$F(0)=\left(\log\left(\cosh 1+\frac{k(\eta\sinh 2+2\gamma\sinh 1)}{4(\gamma+\eta\cosh 1)^2}\right)^2\right)^2-4$$

and

$$F\left(\frac{\pi}{2}\right)=\left(\log\left(\cos 1-\frac{k(\eta\sin 2+2\gamma\sin 1)}{4(\gamma+\eta\cos 1)^2}\right)^2\right)^2-4.$$

As $\log x$ is a monotonically increasing function, it is suffices to determine the minimum of $h_1(k):=\cosh 1+k(\eta\sinh 2+2\gamma\sinh 1)/4(\gamma+\eta\cosh 1)^2$. Observe that $h_1'(k)=(2\gamma\sinh 1+\eta\sinh 2)/4(\gamma+\eta\cosh 1)^2>0$, whenever $\eta>\eta_2$, i.e. $h_1(k)$ is an increasing function of $k$, whenever $\eta>\eta_2$. Further, as a consequence of the inequality: $\eta_2<\eta\leq\eta_1$, we have $h_1(k)\geq\cosh 1+(2\gamma\sinh 1+\eta\sinh 2)/4(\gamma+\eta\cosh 1)^2=h_1(1)\geq e$, which gives $(\log(h_1^2(k)))^2\geq(\log e^2)^2=4$, thus $F(0)\geq 0$. Moreover, for each $k\geq 1$, we have $h_2(k):=\cos 1-k(\eta\sin 2+2\gamma\sin 1)/(4(\gamma+\eta\cos 1)^2)\geq\cos 1-(\eta\sin 2+2\gamma\sin 1)/(4(\gamma+\eta\cos 1)^2)=h_2(1)\geq e$, whenever $\eta_3\leq\eta<\eta_4$, which implies $(\log(h_2^2(k)))^2\geq(\log e^2)^2=4$, thus $F(\pi/2)\geq 0$. Hence $|\log\mathfrak{B}(z_0)|^2\geq 1$, which contradicts the hypothesis given in (6.2.7), this completes the proof. □

Below, we derive some special cases of Theorem (6.2.1) by appropriately choosing the value of $\eta$ so that the conditions of the hypothesis are not violated. Next result is obtained by substituting $p(z)=zf'(z)/f(z)$, and take $\eta=1/2$, in Theorem 6.2.1(i) and (iii).

**Corollary 6.2.1.** Let $\gamma\in\mathbb{R}\setminus\{-1/2\}$, and if $f\in\mathscr{A}$ satisfies the following

$$\frac{zf'(z)}{f(z)}\left(1+\frac{1+2\left(\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}\right)}{\dfrac{zf'(z)}{f(z)}+2\gamma}\right)\prec\phi(z),$$

(i) for $\phi(z)=z+\sqrt{1+z^2}$, with

$$-\frac{\cos 1}{2}\left(1+\frac{\tan 1}{\sqrt{2}+1-\cos 1}\right)\leq\gamma\leq-\frac{\cos 1}{2},$$

(ii) for $\phi(z)=e^z$, with

$$-\frac{\cos 1}{2}\left(1+\frac{\tan 1}{e-\cos 1}\right)\leq\gamma\leq-\frac{\cosh 1}{2}\left(1-\frac{\tanh 1}{e-\cosh 1}\right),$$

then $f\in\mathscr{S}^*_\rho$.

On substituting $p(z)=zf'(z)/f(z)$ with $s=0.2$ and $\eta=1$ in Theorem 6.2.1(ii), we obtain the next corollary:

**Corollary 6.2.2.** Let $\gamma \in \mathbb{R}\setminus\{-1\}$ such that $-(\sin 1)/(2((1.2)^2-\cos 1)) \leq \gamma \leq -\cos 1$. If $f \in \mathscr{A}$ satisfies the following

$$\frac{zf'(z)}{f(z)}\left(1+\frac{1+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}}{\dfrac{zf'(z)}{f(z)}+\gamma}\right) \prec (1+(0.2)z)^2,$$

then $f \in \mathscr{S}^*_\rho$.

We obtain the following examples as a byproduct of Lemma [110, Theorem 3.2d, p.86] and Theorem 6.2.1, for a suitable selection of different parameters. Choose $\gamma = -3/5$ and $s = 1/2$, then from Theorem 6.2.1(ii) we have

$$\frac{3}{5\cos 1}-\frac{2\sin 1}{\cos 1\,(9-4\cos 1)} \leq \eta \leq \frac{3}{5\cosh 1}+\frac{2\sinh 1}{\cosh 1\,(9-4\cosh 1)}. \tag{6.2.15}$$

Substitute $a=1=n$, $\beta=\eta$, $\gamma=-3/5$ and $h(z)=(1+(1/2)z)^2$ in [110, Theorem 3.2d, p.86], then the open door function, which is univalent in $\mathbb{D}$, reduces to

$$R_{\eta-3/5,1}(z) = (\eta-3/5)(1+z)/(1-z)+2z/(1-z^2). \tag{6.2.16}$$

*Example* 2. Let $\eta$ be given by (6.2.15), $\mathrm{Re}(\eta) > 3/5$ and $R_{\eta-3/5,1}(z)$ be given by (6.2.16). If

$$\eta(1+(1/2)z)^2 \prec R_{\eta-3/5,1}(z)+3/5,$$

then

$$p(z) = \left(\eta\int_0^1 t^{\eta-8/5} e^{(t-1)(z\eta(z(1+t)+8))/8}dt\right)^{-1}+\frac{3}{5\eta}$$

is analytic in $\mathbb{D}$, and it is a solution of the differential equation $p(z)+zp'(z)/(\eta p(z)-3/5) = (1+(1/2)z)^2$ and satisfies $\mathrm{Re}(\eta p(z)) > 3/5$. Furthermore, $p(z) \prec \cosh\sqrt{z}$.

By taking $\eta = 1/2$, in Theorem 6.2.1(iii), we deduce that

$$-\frac{1}{2}\left(\cos 1+\frac{\sin 1}{e-\cos 1}\right) \leq \gamma \leq -\frac{1}{2}\left(\cosh 1-\frac{\sinh 1}{e-\cosh 1}\right). \tag{6.2.17}$$

Choose $n=a=1, \beta=\eta=1/2$ and $h(z)=e^z$ in [110, Theorem 3.2d, p.86], then the open door function, which is univalent in $\mathbb{D}$, becomes

$$R_{\gamma+1/2,1}(z) = (\gamma+1/2)(1+z)/(1-z)+2z/(1-z^2). \tag{6.2.18}$$

*Example* 3. Let $\gamma$ be given by (6.2.17), $\operatorname{Re}\gamma > -1/2$ and $R_{\gamma+1/2,1}(z)$ be given by (6.2.18). If

$$\gamma + e^z/2 \prec R_{\gamma+1/2,1}(z),$$

then

$$p(z) = \left(\frac{1}{2}\int_0^1 t^{\gamma-1}\left(e^{Chi(tz)-Chi(z)+Shi(tz)-Shi(z)}\right)^{1/2} dt\right)^{-1} - 2\gamma,$$

is an analytic solution of the differential equation $p(z)+2zp'(z)/(p(z)+2\gamma) = e^z$, where $Chi(z) := \xi + \log z + \int_0^z (\cosh t - 1)/t\, dt$ and $Shi(z) = \int_0^z \sinh t/t\, dt$, with $\xi \approx 0.577216$ (the Euler–Mascheroni constant), and satisfies $\operatorname{Re} p(z) > -2\gamma$. Then $p(z) \prec \cosh\sqrt{z}$.

**Theorem 6.2.2.** *Let $-1 \leq B < A \leq 1$ and $\eta, \gamma \in \mathbb{R}$ such that $\gamma \neq -\eta$, satisfy the following conditions:*

*(i)* $(1-B^2)\sinh 1 + 2(\gamma + \eta\cosh 1)(\cosh 1 - 1 + B(A - B\cosh 1)) \geq 0,$

*(ii)* $(\sinh 1 + 2(\cosh 1 - 1)(\gamma + \eta\cosh 1))^2 \geq (B\sinh 1 - 2(A - B\cosh 1)(\gamma + \eta\cosh 1))^2,$

*(iii)* $(1-B^2)\sin 1 + 2(\gamma + \eta\cos 1)(1 - \cos 1 - B(A - B\cos 1)) \geq 0,$

*(iv)* $(\sin 1 + 2(\cos 1 - 1)(\gamma + \eta\cos 1))^2 \geq (B\sin 1 + 2(A - B\cos 1)(\gamma + \eta\cos 1))^2.$

*Let $p(z)$ be an analytic function, such that $p(0) = 1$ and satisfies*

$$p(z) + \frac{zp'(z)}{\eta p(z) + \gamma} \prec \frac{1+Az}{1+Bz},$$

*then $p(z) \prec \cosh\sqrt{z}$.*

The proof of Theorem 6.2.2 is much akin to the previous results, so it is omitted.

The following corollaries illustrate specific outcomes of Theorem 6.2.2, derived by substituting $p(z) = zf'(z)/f(z)$ and setting the parameters as follows: $A = 1$, $B = 0$, $\gamma = 0$; and $A = 0$, $B = -1/2$, $\eta = 1$, respectively.

**Corollary 6.2.3.** Let $\eta \in \mathbb{R}\setminus\{0\}$ such that $-(\tanh 1 \operatorname{sech} 1)/2 \leq \eta \leq (\tanh 1)/(4 - 2\cosh 1)$. If $f \in \mathscr{A}$ satisfies

$$1 + \frac{zf''(z)}{(1-\eta)f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{\eta z}{1-\eta}, \tag{6.2.19}$$

then $f \in \mathscr{S}^*_\rho$.

**Corollary 6.2.4.** Let $\gamma \in \mathbb{R}\setminus\{-1\}$ such that $(\sin 1 + 4\cos 1 - \cos 2 - 1)/(2\cos 1 - 4) \leq \gamma \leq (4\cosh 1 - \sinh 1 - 2\cosh^2 1)(2\cosh 1 - 4)$. If $f \in \mathscr{A}$ satisfies

$$\frac{zf'(z)}{f(z)}\left(1 + \frac{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}}{\frac{zf'(z)}{f(z)} + \gamma}\right) \prec \frac{2}{2-z},$$

then $f \in \mathcal{S}^*_{\rho}$.

Note that (6.2.19) in Corollary 6.2.3 is equivalent to:

$$\left|1+\frac{zf''(z)}{(1-\eta)f'(z)}-\frac{zf'(z)}{f(z)}\right| < \left|\frac{\eta}{1-\eta}\right|.$$

It is observed that for $g \in \mathcal{A}$, the Briot-Bouquet differential equation is closely related to the Bernardi integral operator [110], given by

$$G(z) := \frac{\eta+\gamma}{z^{\gamma}} \int_0^z g^{\eta}(t) t^{\gamma-1} dt. \tag{6.2.20}$$

If $p(z) = zG'(z)/G(z)$, then we have the following Briot-Bouquet differential equation:

$$p(z)+\frac{zp'(z)}{\eta p(z)+\gamma} = \frac{zg'(z)}{g(z)}. \tag{6.2.21}$$

Using this fact, we now derive the following corollary, as a consequence of Theorem 6.2.1 and Theorem 6.2.2:

**Corollary 6.2.5.** Let $-1 \leq B < A \leq 1$ and $\eta, \gamma \in \mathbb{R}$ such that $\gamma \neq -\eta$. Assume the conditions as outlined in Theorem 6.2.1 (6.2.4)-(6.2.6) and Theorem 6.2.2(i)-(iv) holds. If $g \in \mathcal{S}^*(\phi)$, then $G \in \mathcal{S}^*_{\rho}$ for the choices of $\phi(z)$ : $z+\sqrt{1+z^2}$, $(1+sz)^2$, $e^z$ and $(1+Az)/(1+Bz)$ respectively.

*Proof.* Let us first prove the result for the case when $\phi(z) = z+\sqrt{1+z^2}$, and other cases will follow in the similar fashion. Assume that $p(z) = zG'(z)/G(z)$, where $G(z)$ is given by (6.2.20). Since $g \in \mathcal{S}^*(\phi)$, then from (6.2.21), we have

$$p(z)+\frac{zp'(z)}{\eta p(z)+\gamma} = \frac{zg'(z)}{g(z)} \prec z+\sqrt{1+z^2}.$$

Now the result follows at once by an application of Theorem 6.2.1(i). □

## 6.3 First Order Differential Subordination Results

In this section, we study certain differential subordination implication results involving the expressions: $1+\eta zp'(z)/p(z)$, $1+\eta zp'(z)$ and $\eta p(z)+zp'(z)/p(z)$. We require the following Lemma by Miller and Mocanu to derive our results.

**Lemma B.** *[110] Let $q$ be analytic in $\mathbb{D}$ and let $\varpi$ be analytic in domain $\mathcal{D}$ containing $q(\mathbb{D})$ with $\varpi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $\mathcal{Q}(z) := zq'(z)\varpi(q(z))$ and $h(z) := \nu(q(z))+\mathcal{Q}(z)$. Suppose*

*(i) either $h$ is convex, or $\mathcal{Q}$ is starlike univalent in $\mathbb{D}$ and*

*(ii) $\operatorname{Re}(zh'(z)/\mathcal{Q}(z)) > 0$ for $z \in \mathbb{D}$.*

*If $p$ is analytic in $\mathbb{D}$, with $p(0)=q(0)$, $p(\mathbb{D})\subset\mathscr{D}$ and*

$$\nu(p(z))+zp'(z)\varpi(p(z))\prec\nu(q(z))+zq'(z)\varpi(q(z)),$$

*then $p(z)\prec q(z)$, and $q$ is the best dominant.*

We begin with the following result.

**Theorem 6.3.1.** *Suppose $A,B\in\mathbb{C}$, with $A\neq B$ and $|B|<1$, $\eta$ be such that*

$$|\eta|\geq\frac{2|A-B|}{(1-|B|)\tanh 1}.$$

*Let $p(z)$ be analytic in $\mathbb{D}$ with $p(0)=1$, satisfying the subordination*

$$1+\eta\frac{zp'(z)}{p(z)}\prec\frac{1+Az}{1+Bz},$$

*then $p(z)\prec\cosh\sqrt{z}$ and $\cosh\sqrt{z}$ is the best dominant.*

*Proof.* Let $q(z)=\cosh\sqrt{z}$, $\nu(w)=1$ and $\varpi(w)=\eta/w$, then $\mathscr{Q}(z)=\eta zq'(z)/q(z)=\eta(\sqrt{z}\tanh\sqrt{z})/2$, which implies $\operatorname{Re} z\mathscr{Q}'(z)/\mathscr{Q}(z)>0$ for each $z\in\mathbb{D}$. Hence $\mathscr{Q}(z)$ is starlike in $\mathbb{D}$. Since $h(z)=\nu(\rho(z))+\mathscr{Q}(z)$, we have

$$\operatorname{Re}\left(\frac{zh'(z)}{\mathscr{Q}(z)}\right)=\frac{1}{2}+\operatorname{Re}\sqrt{z}\operatorname{csch}2\sqrt{z}>\frac{1}{2}+\operatorname{csch}2>0.$$

Assume $\phi_{A,B}(z)=(1+Az)/(1+Bz)$, then from the representation of $\phi_{A,B}(z)$, we can write $\phi_{A,B}^{-1}(w)=(w-1)/(A-Bw)$. If we choose

$$T(z)=1+\frac{\eta}{2}\sqrt{z}\tanh(\sqrt{z}),$$

then it suffices to show that $\phi_{A,B}(z)\prec T(z)$, for each $z\in\mathbb{D}$ or $\mathbb{D}\subset\phi_{A,B}^{-1}(T(\mathbb{D}))$ or equivalently,

$$|\phi_{A,B}^{-1}(T(e^{it}))|\geq 1\quad(-\pi\leq t\leq\pi).$$

As $|\tanh(e^{it/2})|$ attains its minimum at $t=0$, for $|\eta|\geq 2|A-B|/(1-|B|)\tanh 1$ on $\partial\mathbb{D}$, we have

$$\begin{aligned}|\phi_{A,B}^{-1}(T(e^{it}))|&\geq\frac{|\eta||\tanh(e^{it/2})|}{2|A-B|+|\eta B\tanh(e^{it/2})|}\\&\geq\frac{|\eta|\tanh 1}{2|A-B|+|\eta B|\tanh 1}\\&\geq 1.\end{aligned}$$

Now the result follows at once by an application of Lemma B. ∎

By taking $p(z) = zf'(z)/f(z)$ in Theorem 6.3.1, we obtain the following corollary.

**Corollary 6.3.1.** Assume $\eta$, $A$ and $B$ as given in Theorem 6.3.1. Let $f \in \mathscr{A}$ such that

$$1+\eta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz},$$

then $f \in \mathscr{S}^*_\rho$.

In the next result, we establish sharp lower bound on $\eta$, for which the following implication holds:

$$1+\eta zp'(z) \prec \cosh\sqrt{z} \Rightarrow p(z) \prec \phi(z),$$

where we choose $\phi(z)$ to be any of $\phi_e(z), \phi_L(z), \phi_{\mathfrak{D}}(z), \phi_{A,B}(z)$ and $\phi_s(z)$, given in Table 1.1.

**Theorem 6.3.2.** *Let $\xi \approx 0.577216$, the Euler–Mascheroni constant and assume that*

$$\kappa_0 := \frac{Chi(1)+Ci(1)-2\xi}{Chi(1)-Ci(1)}, \tag{6.3.22}$$

*with*

$$Chi(1) = \xi + \int_0^1 (\cosh\sqrt{t}-1)/t \;\; dt \quad \text{and} \quad Ci(1) = \xi + \int_0^1 (\cos\sqrt{t}-1)/t \;\; dt. \tag{6.3.23}$$

*Let $\eta \in \mathbb{R}$ and $p(z)$ be analytic in $\mathbb{D}$ satisfying $1+\eta zp'(z) \prec \cosh\sqrt{z}$, then*

*(i)* $p(z) \prec e^z$ *and* $(e-1)\eta \geq 2e(\xi - Ci(1))$.

*(ii)* $p(z) \prec \sqrt{1+z}$ *and* $(\sqrt{2}-1)\eta \geq 2(Ci(1)-\xi)$.

*(iii)* $p(z) \prec z+\sqrt{1+z^2}$ *and* $(\sqrt{2}-1)\eta \geq \sqrt{2}(\xi - Ci(1))$.

*(iv)* $p(z) \prec \dfrac{1+Az}{1+Bz}$ *and* $\eta \geq \eta_{A,B}$, *where* $-1 < B < A \leq 1$ *and*

$$\eta_{A,B} = \begin{cases} \dfrac{2}{A-B}(1-B)(\xi - Ci(1)), & \text{if } -1 < B \leq -\kappa_0, \\ \\ \dfrac{-2}{A-B}(1+B)(\xi - Chi(1)), & \text{if } -\kappa_0 < B < 1. \end{cases} \tag{6.3.24}$$

*(v)* $p(z) \prec (1+sz)^2$ *and* $\eta \geq \eta_s$, *where*

$$\eta_s = \begin{cases} \dfrac{2}{s(s+2)}(Chi(1)-\xi), & \text{if } 0 < s \leq 2\kappa_0, \\ \\ \dfrac{2}{s(s-2)}(Ci(1)-\xi), & \text{if } 2\kappa_0 < s \leq 1/\sqrt{2}. \end{cases} \tag{6.3.25}$$

*All bounds on $\eta$ are sharp.*

*Proof.* The differential equation $1+\eta z\phi_\eta'(z)=\cosh\sqrt{z}$, has a solution $\phi_\eta(z):\overline{\mathbb{D}}\to\mathbb{C}$, given by

$$\phi_\eta(z)=1+\frac{1}{\eta}\left(2Chi\sqrt{z}-\log z-2\xi\right),$$

where $\xi\approx 0.577216$ is the Euler–Mascheroni constant. Let $\nu(w)=1$ and $\varpi(w)=\eta$. Define an analytic function $\mathscr{Q}:\overline{\mathbb{D}}\to\mathbb{C}$ by

$$\mathscr{Q}(z)=z\phi_\eta'(z)\varpi(\phi_\eta(z))=-1+\cosh\sqrt{z}.$$

Since $\operatorname{Re} z\mathscr{Q}'(z)/\mathscr{Q}(z)>(1/2)\cot(1/2)>0$, thus $\mathscr{Q}(z)$ is starlike in $\mathbb{D}$. Let $h(z)=\nu(\phi_\eta(z))+\mathscr{Q}(z)$, then we have

$$\operatorname{Re}\left(\frac{zh'(z)}{\mathscr{Q}(z)}\right)=\operatorname{Re}\left(\frac{1}{2}\sqrt{z}\coth\left(\frac{\sqrt{z}}{2}\right)\right)>0.$$

By an application of Lemma B, we obtain $1+\eta zp'(z)\prec 1+\eta z\phi_\eta'(z)$, which implies $p(z)\prec\phi_\eta(z)$. Now we need to show that $\phi_\eta(z)\prec\phi(z)$, where $\phi(z)$ is any of these functions: $\phi_e(z)$, $\phi_L(z)$,$\phi_{\mathfrak{D}}(z)$, $\phi_{A,B}(z)$ and $\phi_s(z)$. If $\phi_\eta(z)\prec\phi(z)$, then

$$\phi(-1)\leq\phi_\eta(-1)<\phi_\eta(1)\leq\phi(1). \tag{6.3.26}$$

Now, for each choice of $\phi(z)$, by solving equation (6.3.26), we obtain sharp bounds on $\eta$. The graphical observations presented in Figure 6.2 demonstrate that the condition in (6.3.26) is not only necessary but also sufficient for the chosen choice of $\phi(z)$.

(i) When $\phi(z)=\phi_e(z)$, then (6.3.26) reduces to the following

$$e^{-1}\leq 1+\frac{2(Ci(1)-\xi)}{\eta}<1+\frac{2(Chi(1)-\xi)}{\eta}\leq e. \tag{6.3.27}$$

Now from (6.3.27) we get

$$e^{-1}\leq 1+\frac{2(Ci(1)-\xi)}{\eta},\text{ which implies }\eta\geq\frac{2e(\xi-Ci(1))}{e-1}=\eta_e\approx 0.758753$$

and

$$1+\frac{2(Chi(1)-\xi)}{\eta}\leq e,\text{ which implies }\eta\geq\frac{2(Chi(1)-\xi)}{e-1}=x_e\approx 0.303386.$$

It can be observed that $\eta\geq\max\{\eta_e,x_e\}$. Therefore, $\phi_\eta(z)\prec\phi(z)$ for each $\eta\geq\eta_e$, consequently, due to transitivity the result follows at once.

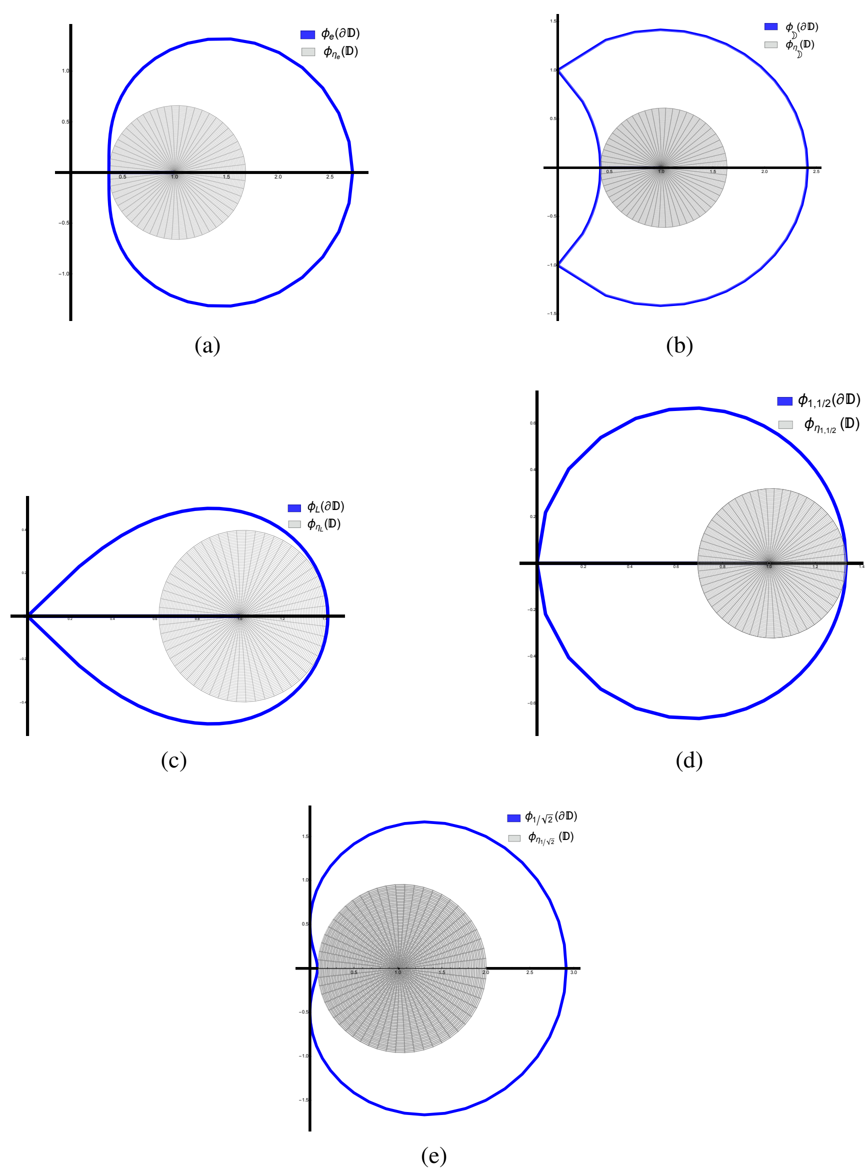


Figure 6.2: Images of $\partial\mathbb{D}$ and $\mathbb{D}$ under the mappings, (a) $\phi_e(z)$ and $\phi_{\eta_e}(z)$, (b) $\phi_{\mathfrak{D}}(z)$ and $\phi_{\eta_{\mathfrak{D}}}(z)$, (c) $\phi_L(z)$ and $\phi_{\eta_L}(z)$, (d) $\phi_{1,1/2}(z)$ and $\phi_{\eta_{1,1/2}}(z)$ & (e) $\phi_{1/\sqrt{2}}(z)$ and $\phi_{\eta_{1/\sqrt{2}}}(z)$, respectively.

(ii) If $\phi(z) = \phi_L(z)$, then inequalities (6.3.26) leads to

$$0 \leq \frac{2(Ci(1)-\xi)}{\eta} + 1 < \frac{2(Chi(1)-\xi)}{\eta} + 1 \leq \sqrt{2},$$

which is true, provided $\eta \geq \eta_L$, where

$$\eta_L = \frac{\sqrt{2}(\xi - Ci(1))}{\sqrt{2}-1} \approx 1.25854.$$

Thus for each $\eta \geq \eta_L$, we have $p(z) \prec \phi_L(z)$.

(iii) For $\phi(z) = \phi_{\mathfrak{D}}(z)$, the expression in (6.3.26), gives

$$\sqrt{2}-1 \leq 1+\frac{2(Ci(1)-\xi)}{\eta} < 1+\frac{2(Chi(1)-\xi)}{\eta} \leq \sqrt{2}+1, \tag{6.3.28}$$

which holds for each $\eta \geq \eta_{\mathfrak{D}}$, where

$$\eta_{\mathfrak{D}} = \frac{\sqrt{2}(\xi - Ci(1))}{\sqrt{2}-1} \approx 0.818769.$$

Therefore, for each $\eta \geq \eta_{\mathfrak{D}}$, we have $\phi_\eta(z) \prec \phi_{\mathfrak{D}}(z)$. Accordingly we conclude that $p(z) \prec \phi_{\mathfrak{D}}(z)$.

The cases (iv) and (v) will follow similar to the above cases. Sharpness of the result is depicted in Figure 6.2, for the above choices of $\phi(z)$. We choose $A=1, B=1/2$ and $s=1/\sqrt{2}$, in parts (iv) and (v), respectively for graphical representation of the case, consequently, we obtain $\eta \geq \eta_{1/\sqrt{2}} \approx 0.52463$ and $\eta \geq \eta_{1,1/2} \approx 1.56391$, respectively, from (6.3.24) and (6.3.25). □

**Corollary 6.3.2.** Let $\kappa_0$, $Chi(1)$ and $Ci(1)$ be as given in Theorem (6.3.2). For $f \in \mathscr{A}$, if

$$\Phi_\eta(z) := 1+\eta\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right),$$

then $f \in \mathscr{S}^*_\rho$ implies the following:

(i) $\Phi_\eta(z) \prec e^z$, whenever $(e-1)\eta \geq 2e(\xi - Ci(1))$.

(ii) $\Phi_\eta(z) \prec \sqrt{1+z}$, whenever $(\sqrt{2}-1)\eta \geq 2(Ci(1)-\xi)$.

(iii) $\Phi_\eta(z) \prec z+\sqrt{1+z^2}$, whenever $(\sqrt{2}-1)\eta \geq \sqrt{2}(\xi - Ci(1))$.

(iv) $\Phi_\eta(z) \prec \dfrac{1+Az}{1+Bz}$, whenever $\eta \geq \eta_{A,B}$, where$\eta_{A,B}$ is given in Theorem 6.3.2 and $-1<B<A\leq 1$.

(v) $\Phi_\eta(z) \prec (1+sz)^2$, whenever $\eta \geq \eta_s$, where $\eta_s$ is given in Theorem 6.3.2.

All the bounds obtained on $\eta$ are sharp.

**Theorem 6.3.3.** *Let $-1<B<A\leq 1$, $\mu = (\cosh 1 - 1)/(1-\cos 1)$, and $B_0$ be the root of the equation $(1+B)(1-B)^\mu = 1$. If $p(z)$ is an analytic function in $\mathbb{D}$ that satisfies*

$$1+\eta z p'(z) \prec \frac{1+Az}{1+Bz},$$

*then* $p(z) \prec \cosh\sqrt{z}$*, provided* $\eta$ *satisfies the following sharp inequalities:*

$$\left.\begin{array}{ll} \eta B(\cosh 1 - 1) \leq (A-B)\log(1+B), & \text{if } -1 < B \leq B_0, \\ \eta B(\cos 1 - 1) \geq (A-B)\log(1-B), & \text{if } B_0 < B < 0, \\ \eta B(\cos 1 - 1) \leq (A-B)\log(1-B), & \text{if } 0 < B < 1, \\ 2\eta \geq A\csc^2(1/2), & \text{if } B = 0. \end{array}\right\} \tag{6.3.29}$$

*Proof.* The analytic function $\phi_\eta : \overline{\mathbb{D}} \to \mathbb{C}$ defined as :

$$\phi_\eta(z) = 1 + \frac{A-B}{\eta B}\log(1+Bz),$$

is a solution of the differential equation $1+\eta z\phi_\eta'(z) = (1+Az)/(1+Bz)$. We shall prove the result by using Lemma B, accordingly we assume $\nu(u) = 1$, $\varpi(u) = \eta$ and define an analytic function $\mathcal{Q} : \overline{\mathbb{D}} \to \mathbb{C}$ as

$$\mathcal{Q}(z) = z\phi_\eta'(z)\varpi(\phi_\eta(z)) = \frac{(A-B)z}{1+Bz}.$$

Clearly, for the given choice of $A$ and $B$, $\mathcal{Q}(z)$ is a starlike function in $\mathbb{D}$. Note that if $h(z) = \nu(\phi_\eta(z)) + \mathcal{Q}(z)$, then $zh'(z)/\mathcal{Q}(z) = 1/(1+Bz)$, thus $\operatorname{Re} zh'(z)/\mathcal{Q}(z) > 1/(1+|B|) > 0$. Further, in view of Lemma B, we conclude that $p(z) \prec \phi_\eta(z)$. Now we need to show that $\phi_\eta(z) \prec \cosh\sqrt{z} =: \rho(z)$. We know that the following condition is necessary:

$$\rho(-1) \leq \phi_\eta(-1) < \phi_\eta(1) \leq \rho(1), \tag{6.3.30}$$

for $\phi_\eta(z) \prec \rho(z)$. However, a graphical observation presented in Figure 6.3 for the value of $\eta$ satisfying (6.3.30) shows that (6.3.30) is not only necessary but also sufficient. For $B \neq 0$, we have from (6.3.30):

$$\cos 1 \leq 1 + \frac{A-B}{\eta B}\log(1-B) < 1 + \frac{A-B}{\eta B}\log(1+B) \leq \cosh 1, \tag{6.3.31}$$

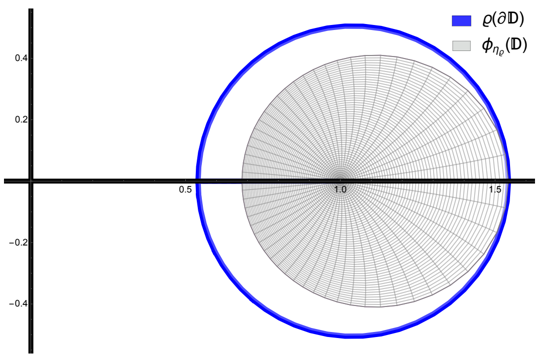


Figure 6.3: Images of $\partial\mathbb{D}$ and $\mathbb{D}$ under the mappings $\rho(z) = \cosh\sqrt{z}$ and $\phi_{\eta_\rho}(z)$ respectively.

then from (6.3.31), we conclude that $\phi_\eta(z) \prec \rho(z)$, provided $\eta$ satisfies (6.3.29). Furthermore, if $B=0$, the function

$$m_\eta(z) := 1 + \frac{A}{\eta} z$$

is a solution of the differential equation: $1+\eta z m_\eta'(z) = 1 + Az$. Now in view of (6.3.30), we have the following inequality

$$\cos 1 \leq m_\eta(-1) < m_\eta(1) \leq \cosh 1, \tag{6.3.32}$$

which holds, whenever $A \leq 2\eta \sin^2(1/2)$. Therefore, for $B=0$ we have $\phi_\eta(z) \prec \rho(z)$, provided $\eta$ satisfies (6.3.32). Hence the result follows at once. □

Note that Figure 6.3, illustrates sharpness on $\eta$ for the case when $A=1/2$, $B=-1/2$ and $\eta = \eta_\rho := (\log 4)/(\cosh 1 - 1) \approx 2.5526$ in Theorem 6.3.3, where the extremal function is $\phi_{\eta_\rho}(z) := 1 - (2/\eta_\rho)\log(1-(z/2))$.

In Theorem 6.3.3, choose $p(z) = zf'(z)/f(z)$, then we deduce the next result:

**Corollary 6.3.3.** Assume $A, B$ and $\mu$ as given in Theorem 6.3.3 and for $f \in \mathscr{A}$,

$$\Phi_\eta(z) := 1 + \eta \frac{zf'(z)}{f(z)} \left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right).$$

If $\Phi_\eta(z) \prec (1+Az)/(1+Bz)$, then $f \in \mathscr{S}^*_\rho$, provided $\eta$ satisfies (6.3.29). All the bounds attained on $\eta$ are sharp.

In the next result, we apply Lemma B and derive its corresponding corollaries.

**Theorem 6.3.4.** *Let $p(z)$ be a non-vanishing analytic function in $\mathbb{D}$ with $p(0)=1$, such that*

$$\eta p(z) + \frac{zp'(z)}{p(z)} \prec \eta \cosh\sqrt{z} + \frac{\sqrt{z}}{2}\tanh\sqrt{z},$$

*where $\eta \geq \eta_1$ with*

$$\eta_1 = -\left(\frac{1}{2} + \operatorname{csch} 2\right)\operatorname{sech} 1 \approx -0.502\cdots. \tag{6.3.33}$$

*Then $p(z) \prec \cosh\sqrt{z}$, and $\cosh\sqrt{z}$ is the best dominant.*

*Proof.* Let $q(z) = \cosh\sqrt{z}$ and $\varpi(w) = 1/w$. Clearly, $q(z)$ is a convex univalent function with $q(0)=1$ and

$$\mathscr{Q}(z) = zq'(z)\varpi(q(z)) = \frac{zq'(z)}{q(z)} = \frac{\sqrt{z}}{2}\tanh\sqrt{z}.$$

Now, it can be easily verified that

$$\operatorname{Re}\frac{z\mathscr{Q}'(z)}{\mathscr{Q}(z)} = \operatorname{Re}\left(\frac{1}{2} + \sqrt{z}\operatorname{csch} 2\sqrt{z}\right) > \frac{1}{2} + \operatorname{csch} 2 > 0,$$

then $\mathscr{Q}(z)$ is starlike in $\mathbb{D}$. Further, $\varpi(z)$ is analytic in $\mathbb{C}\setminus\{0\}$ containing $q(\mathbb{D})$ with $\varpi(w)\neq 0$, where $w\in q(\mathbb{D})$. Now set

$$\nu(w)=\eta w \quad\text{and}\quad h(z)=\nu(q(z))+\mathscr{Q}(z)=\eta q(z)+\frac{zq'(z)}{q(z)}=\eta\cosh\sqrt{z}+\frac{\sqrt{z}}{2}\tanh\sqrt{z}.$$

For $\eta\geq\eta_1$, where $\eta_1$ is given by (6.3.33), it can be verified that

$$\operatorname{Re}\left(\eta\cosh\sqrt{z}+\sqrt{z}\operatorname{csch}2\sqrt{z}\right)>-\frac{1}{2},$$

therefore, $\operatorname{Re}(zh'(z)/\mathscr{Q}(z))>0$ for each $\eta\geq\eta_1$. Finally, by applying Lemma B we conclude that $p(z)\prec\cosh\sqrt{z}$. □

On taking $\eta=0$ in Theorem 6.3.4, we deduce the following corollary.

**Corollary 6.3.4.** Suppose $p(z)$ is analytic in $\mathbb{D}$ with $p(z)\neq 0$ in $\mathbb{D}$ and $p(0)=1$ such that

$$\frac{zp'(z)}{p(z)}\prec\frac{\sqrt{z}}{2}\tanh\sqrt{z},$$

then $p(z)\prec\cosh\sqrt{z}$, and $\cosh\sqrt{z}$ is the best dominant.

On substituting $p(z)=zf'(z)/f(z)$ in Theorem 6.3.4, we state the next corollary.

**Corollary 6.3.5.** Let $f\in\mathscr{A}$ and $\eta\geq\eta_1$, where $\eta_1$ is given by (6.3.33), such that

$$1+\frac{zf''(z)}{f'(z)}-(1-\eta)\frac{zf'(z)}{f(z)}\prec\eta\cosh\sqrt{z}+\frac{\sqrt{z}}{2}\tanh\sqrt{z},$$

then $f\in\mathscr{S}^*_\rho$.

## Some More Results Using Admissibility Conditions

We begin by giving the following definition given by Miller and Mocanu [109]:

**Definition 6.3.1.** Let $\mathscr{Q}$ be the set of functions $q$ that are analytic and injective on $\overline{\mathbb{D}}\backslash\mathbb{E}(q)$, where

$$\mathbb{E}(q)=\left\{\varepsilon\in\partial\mathbb{D}:\lim_{z\to\varepsilon}q(z)=\infty\right\},$$

such that $q'(\varepsilon)\neq 0$ for $\varepsilon\in\partial\mathbb{D}\backslash\mathbb{E}(q)$.

For $\Omega\subset\mathbb{C}$, $q\in\mathscr{Q}$ and $n\in\mathbb{N}$, let the class $\Psi_n[\Omega,q]$ consist of functions $\vartheta:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$

that meet the admissibility conditions:

$$\vartheta(r,s,t;z) \notin \Omega, \text{ whenever } (r,s,t;z) \in \mathbb{C}^3 \times \mathbb{D},$$
$$r = q(\varepsilon),\ s = m\varepsilon q'(\varepsilon),\ \operatorname{Re}\left(1+\frac{t}{s}\right) \geq m \operatorname{Re}\left(1+\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)}\right),$$

for $z \in \mathbb{D}, \varepsilon \in \partial\mathbb{D} \backslash \mathbb{E}(q)$ and $m \geq n$. Denote $\Psi_1[\Omega, q]$ by $\Psi[\Omega, q]$.

**Theorem 6.3.5.** *[110, Theorem 2.3b] Let $\vartheta \in \Psi_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathscr{H}[a,n]$ satisfies*

$$\vartheta(p(z), zp'(z), z^2p''(z); z) \in \Omega,$$

*then $p(z) \prec q(z)$.*

If $\Omega \subsetneq \mathbb{C}$ is a simply connected domain, then there exists a conformal mapping $h(z)$ from $\mathbb{D}$ onto $\Omega = h(\mathbb{D})$. Symbolically, let $\Psi_n[h(\mathbb{D}), q]$ represent $\Psi_n[\Omega, q]$. Further, if the function $\vartheta(p(z), zp'(z), z^2p''(z); z)$ is analytic in $\mathbb{D}$, then $\vartheta(p(z), zp'(z), z^2p''(z); z) \in \Omega$ can be rewritten in terms of subordination:

$$\vartheta(p(z), zp'(z), z^2p''(z); z) \prec h(z).$$

Consider $\Omega \subsetneq \mathbb{C}$ and $q \in \mathscr{H}_1$ be given by $q(z) := \cosh\sqrt{z}$. As $q(z)$ is univalent in $\overline{\mathbb{D}} \backslash \mathbb{E}(q)$, where $\mathbb{E}(q) = \emptyset$, also $q(0) = 1$ and $q(\mathbb{D}) = \Omega_\rho$, where $\Omega_\rho = \{w \in \mathbb{C} : |\log(w + \sqrt{w^2-1})|^2 < 1\}$. Below we study the class of admissible functions $\Psi_n[\Omega, q]$.
Note that for $|\varepsilon| = 1$,

$$q(\varepsilon) \in q(\partial\mathbb{D}) = \partial\Omega_\rho = \left\{\omega \in \mathbb{C} : |\log(\omega + \sqrt{\omega^2-1})|^2 = 1\right\}.$$

Infact if $\varepsilon = e^{i\theta}$, $-\pi < \theta \leq \pi$, then

$$\varepsilon q'(\varepsilon) = \frac{\sqrt{\varepsilon}}{2}\sinh\sqrt{\varepsilon},\ q''(\varepsilon) = \frac{1}{4\varepsilon}\left(\cosh\sqrt{\varepsilon} - \frac{\sinh\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)$$

and

$$1 + \frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)} = \frac{1}{2}\left(1 + \sqrt{\varepsilon}\coth\sqrt{\varepsilon}\right).$$

Further, it can be verified that the minimum value of $\operatorname{Re}(\sqrt{\varepsilon}\coth\sqrt{\varepsilon})$ is attained at $\varepsilon = -1$. Now in view of the above observation, in the following definition, we give the admissibility conditions for $q(z) = \cosh\sqrt{z}$.

**Definition 6.3.2.** Let $\Omega \subsetneq \mathbb{C}$ and $n \geq 1$, then for $q(z) = \cosh\sqrt{z}$, the admissibility conditions are given as follows:

$$\vartheta(r,s,t;z) \notin \Omega \text{ whenever } \begin{cases} r = q(\varepsilon) = \cosh\sqrt{\varepsilon}, \\ s = m\varepsilon q'(\varepsilon) = \dfrac{m}{2}\sqrt{\varepsilon}\sinh\sqrt{\varepsilon}, \\ \operatorname{Re}\left(1+\dfrac{t}{s}\right) \geq \dfrac{m}{2}(1+\cot 1), \end{cases} \tag{6.3.34}$$

where $z \in \mathbb{D}, \varepsilon \in \partial\mathbb{D}\backslash\mathbb{E}(q)$ and $m \geq 1$. We denote this class of admissible functions by $\Psi(q_\rho)$.

In view of Theorem 6.3.5 and Definition 6.3.2, we directly establish the next result:

**Theorem 6.3.6.** *Let* $p \in \mathscr{H}_1$.

*(i) If* $\vartheta \in \Psi(q_\rho)$, *then* $\vartheta(p(z), zp'(z), z^2p''(z); z) \in \Omega \Rightarrow p(z) \prec \cosh\sqrt{z}$.

*(ii) If* $\vartheta \in \Psi(q_\rho)$, *with* $\Omega = \Omega_\rho$, *then* $\vartheta(p(z), zp'(z), z^2p''(z); z) \prec \cosh\sqrt{z} \Rightarrow p(z) \prec \cosh\sqrt{z}$.

Recently insightful work is carried out in establishing several first and second order differential subordination implication results, using the concept of admissibility. For instance, many authors have studied the class of admissible functions associated with different analytic functions, such as: modified sigmoid function, lemniscate of Bernoulli, exponential function, petal shaped function, see [82, 102, 116, 170]. Further, Kumar and Goel [82], modified the existing third order differential subordination results of Antonino and Miller [12], in context of some special type of classes of starlike functions. In the following results, we present a few applications to Theorem 6.3.6.

**Theorem 6.3.7.** *Let* $p \in \mathscr{H}_1$, *such that*

$$|zp'(z) - 1| < \frac{\sin 1}{2} \approx 0.420\ldots,$$

*then* $p(z) \prec \cosh\sqrt{z}$.

*Proof.* Suppose $\Omega = \{w : |w-1| < (\sin 1)/2\}$. Let $\vartheta(p(z), zp'(z), z^2p''(z); z)$ be a function defined on $\mathbb{C}^3 \times \mathbb{D}$, given by $\vartheta(r,s,t;z) = 1+s$. We need to show that for $(r,s,t) \in \mathbb{C}^3$ satisfies admissibility conditions given in (6.3.34). For $m \geq 1$, consider

$$|\vartheta(r,s,t;z) - 1| = |s| = \left|\frac{m}{2}\sqrt{\varepsilon}\sinh\sqrt{\varepsilon}\right|,$$

then for $\varepsilon = e^{i\theta}$, where $-\pi < \theta \leq \pi$, we have

$$|\vartheta(r,s,t;z) - 1| = \frac{m}{2}\left|\sinh e^{i\theta/2}\right| \geq \frac{1}{2}\sin 1.$$

This means $\vartheta(r,s,t;z) \notin \Omega$ for each $r,s,t$ satisfying (6.3.34) and therefore, $\vartheta \in \Psi[\Omega,q]$. Finally, Theorem 6.3.6 leads to the required conclusion. □

**Theorem 6.3.8.** *Let $p \in \mathscr{H}_1$, such that*

$$\left|\frac{zp'(z)}{p(z)}\right| < \frac{\tanh 1}{2} \approx 0.380\ldots,$$

*then $p(z) \prec \cosh\sqrt{z}$.*

*Proof.* Let $\Omega = \{w : |w| < (\tanh 1)/2\}$ and $\vartheta(p(z), zp'(z), z^2p''(z); z)$ be a function defined on $\mathbb{C}^3 \times \mathbb{D}$, given by $\vartheta(r,s,t;z) = s/r$. We need to show that for $(r,s,t) \in \mathbb{C}^3$ satisfying conditions (6.3.34) leads to $\vartheta(r,s,t;z) \notin \Omega$. Consider,

$$|\vartheta(r,s,t;z)| = \left|\frac{s}{r}\right| = \left|\frac{m}{2}\sqrt{\varepsilon}\tanh\sqrt{\varepsilon}\right| \geq \frac{m}{2}\tanh 1.$$

Thus for $m \geq 1$, we conclude that $\vartheta(r,s,t;z) \notin \Omega$ for each $r,s$ and $t$ satisfying (6.3.34). Thus $\vartheta \in \Psi(\Omega,q)$ and Theorem 6.3.6 gives that $p(z) \prec \cosh\sqrt{z}$. □

**Theorem 6.3.9.** *Let $p \in \mathscr{H}_1$, such that*

$$\left|\frac{zp'(z)}{p^2(z)} - 1\right| < \frac{1}{2}\operatorname{sech} 1 \tanh 1 \approx 0.246\ldots,$$

*then $p(z) \prec \cosh\sqrt{z}$.*

*Proof.* Suppose $\Omega = \{w : |w-1| < (\operatorname{sech} 1 \tanh 1)/2\}$. Let $\vartheta(p(z), zp'(z), z^2p''(z); z)$ be a function defined on $\mathbb{C}^3 \times \mathbb{D}$, given by $\vartheta(r,s,t;z) = 1 + s/r^2$, then for $m \geq 1$, we have

$$|\vartheta(r,s,t;z) - 1| = \left|\frac{s}{r^2}\right| = \frac{m}{2}\left|\frac{\sinh\sqrt{\varepsilon}}{\cosh^2\sqrt{\varepsilon}}\right| \geq \frac{\sinh 1}{2\cosh^2 1}.$$

This gives that $\vartheta(r,s,t;z) \notin \Omega$ for each $r,s$ and $t$ satisfying (6.3.34), therefore, $\vartheta \in \Psi(\Omega,q)$. Thus Theorem 6.3.6 leads to the required conclusion. □

On substituting $p(z) = zf'(z)/f(z)$ in Theorem 6.3.7-Theorem 6.3.9, we deduce the following:

**Corollary 6.3.6.** If $f \in \mathscr{A}$ satisfies any of the following inequalities:

(i) $$\left|\frac{z^2f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} - z^2\left(\frac{f'(z)}{f(z)}\right)^2 - 1\right| < \frac{\sin 1}{2} \text{ or}$$

(ii) $$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \frac{\tanh 1}{2} \text{ or}$$

(iii) $$\left|\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\left(\frac{zf'(z)}{f(z)}\right)^{-1}-1\right|<\frac{1}{2}\operatorname{sech}1\tanh 1,$$

then $f \in \mathscr{S}^*_\rho$.

## Highlights of the Chapter

In this chapter, we employ Briot–Bouquet differential subordination techniques to establish sufficient conditions for functions to belong to the class $\mathscr{S}^*_\rho$. By applying admissibility conditions, we will derive several differential subordination results, thereby expanding the understanding of the geometric and analytical properties of the class $\mathscr{S}^*_\rho$. To enhance clarity, diagrammatic validations of our sharp findings are also provided.

The contents of this chapter is mostly based on the findings presented in the paper:
*Mridula Mundalia and S. Sivaprasad Kumar: Sufficient conditions for starlikeness related to a Hyperbolic Cosine function, Ukrainian Mathematical Journal,* **77***(2), (2025).*

# Conclusion, Future Scope & Social Impact

This thesis introduces the class $\mathscr{S}^k_{\gamma,\delta}(\phi)$, unifying and extending existing analytic function classes, including $\mathscr{S}^*_\rho$, $\mathscr{F}_\varphi$, $\mathscr{S}^*(\beta)$, $\mathscr{S}^*_e$, and $\mathscr{S}^*_{\sin}$. It rigorously analyzes geometric properties such as radius problems and inclusion relations, establishing sharp results. A distinctive feature is the diagrammatic illustration of sharpness of results, offering deeper insights into geometric behaviors. This work advances geometric function theory, introduces innovative methodologies, and provides a strong foundation for future research.

The current work establishes radius results for the classes $\mathscr{S}^*_\rho$, $\mathscr{F}_\varphi$, and $\mathscr{S}^*(\beta)$ using well-established techniques. Future research could focus on extending these findings by deriving coefficient bounds such as $|a_n|$ for $n \geq 5$ and exploring additional radius results for related and generalized classes. Further, one may explore second and higher-order Hankel and Toeplitz determinants for $\mathscr{S}^k_{\gamma,\delta}(\phi)$, $\mathscr{F}_\varphi$, and $\mathscr{S}^*_\rho$, offering deeper insights into their geometric properties. There is also scope to extend first-order differential subordination results for $\mathscr{S}^*_\rho$ to higher orders using advanced admissibility conditions. These directions, along with multidimensional settings and alternative convolution operators, hold promise for advancing the understanding and applications of analytic function classes in complex analysis.

The practical relevance of this work is evident in its connections to physical models and applications in engineering and mathematical physics. For instance, hyperbolic cosine functions, central to this research, are widely used in signal processing, structural mechanics, vibration analysis, and differential equations. Moreover, concepts from geometric function theory (GFT) have applications in digital image processing. As demonstrated in studies like [1,112,119,120], convolution techniques enhance image quality by improving clarity and efficiency, showcasing the broader societal benefits of advancements in this field.

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# List of Publications

1. **Mridula Mundalia** and Shanmugam Sivaprasad Kumar, *Coefficient bounds for a unified class of holomorphic functions*, In Mathematical Analysis I: Approximation Theory, pp. 197–210, Springer Proceedings in Mathematics & Statistics **306**, Springer, Singapore (2020) (**Scopus**).

2. **Mridula Mundalia** and S. Sivaprasad Kumar, *On a subfamily of starlike functions related to hyperbolic cosine function,* Journal of Analysis, **31**(3), 2043–2062 (2023) (**ESCI, Impact Factor: 0.8**).

3. **Mridula Mundalia** and S. Sivaprasad Kumar, *Coefficient Problems for Certain Close-to-Convex Functions*, Bulletin of Iranian Mathematical Society, **49**(1), Article 5 (2023) (**SCIE, Impact Factor: 0.7**).

4. S. Sivaprasad Kumar and **Mridula Mundalia,** *Sufficient conditions for starlikeness related to a Hyperbolic Cosine function*, Ukrainian Mathematical Journal, **77**(2) (2025) (**SCIE, Impact Factor: 0.5**).

5. S. Sivaprasad Kumar and **Mridula Mundalia,** *On Sharp Radius estimates for* $\mathscr{S}^*(\beta)$ *and a product function*, Mathematica Slovaca, **75**(2), 281-300 (2025) (**SCIE, Impact Factor: 0.9**).

6. **Mridula Mundalia** and S. Sivaprasad Kumar, *On a Class of Non-Univalent functions Associated with a Parabolic Region.* (Under Review).

# Coefficient Bounds for a Unified Class of Holomorphic Functions

**Mridula Mundalia and Sivaprasad Kumar Shanmugam**

**Abstract** In the present paper, sharp initial coefficient bounds have been estimated for functions in the newly defined classes $\mathcal{S}^k_{\gamma,\delta}(\Phi)$ and $\mathcal{S}^k_{\gamma,\delta,h}(\Phi)$, which in fact, unifies many earlier known classes. Further, sharp bounds of the Fekete–Szegö coefficient functional for functions in the classes introduced here are obtained and special cases of our results are also pointed out.

**Keywords** Univalent functions · Starlike functions · Convex functions · Fekete–Szegö coefficient functional · Subordination

**2010 Mathematics Subject Classification** 30C45 · 30C80

## 1 Introduction and Preliminaries

Let $\mathcal{A}$ be the class all of functions $f$ that are holomorphic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, possessing the series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $h$ and $g$ be holomorphic functions defined in $\mathbb{D}$, $h$ is said to be *subordinate* to $g$, denoted by $h \prec g$, if there exists a Schwarz function $v : \mathbb{D} \to \mathbb{D}$ with $v(0) = 0$ such that $h(z) =$

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ORIGINAL RESEARCH PAPER

# On a subfamily of starlike functions related to hyperbolic cosine function

**Mridula Mundalia**[1] · **S. Sivaprasad Kumar**[1]

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**Abstract**
We introduce and study a new Ma–Minda subclass of starlike functions $\mathcal{S}^*_{\varrho}$, defined as

$$\mathcal{S}^*_{\varrho} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \cosh\sqrt{z} =: \varrho(z), z \in \mathbb{D} \right\},$$

associated with an analytic univalent function $\cosh\sqrt{z}$, where we choose the branch of the square root function so that $\cosh\sqrt{z} = 1 + z/2! + z^2/4! + \cdots$. We establish certain inclusion relations for $\mathcal{S}^*_{\varrho}$ and deduce sharp $\mathcal{S}^*_{\varrho}$-radii for certain subclasses of analytic functions.

**Keywords** Univalent functions · Starlike functions · Radius problems · Hyperbolic Cosine function · Subordination

**Mathematics Subject Classification** 30C45 · 30C80

## 1 Introduction

Let $\mathcal{A}_n$ be the class of all analytic functions defined on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, with Taylor series representation of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$. Let $\mathcal{A} := \mathcal{A}_1$. Assume $\mathcal{S} \subset \mathcal{A}$ as the class of univalent functions. If $f(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$, then $f(z)$ is said to be subordinate to $g(z)$ ($f \prec g$), if there exists a self-map $w(z)$ on $\mathbb{D}$ such that $w(0) = 0$ and $f(z) = g(w(z))$. For instance, if $g(z)$ is a univalent function in $\mathbb{D}$, then $f \prec g$ if

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S. Sivaprasad Kumar contributed equally to this work.

Extended author information available on the last page of the article

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ORIGINAL PAPER

# Coefficient Problems for Certain Close-to-Convex Functions

Mridula Mundalia[1] · Shanmugam Sivaprasad Kumar[1]

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**Abstract**
In this paper, bounds are established for the second Hankel determinant of logarithmic coefficients for normalised analytic functions satisfying certain differential inequality.

**Keywords** Univalent functions · Close-to-convex functions · Hankel determinant · Logarithmic coefficient · Starlike with respect to symmetric points

**Mathematics Subject Classification** 30C45 · 30C50

## 1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions defined on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the Taylor series expansion of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Assume $\mathcal{S} \subset \mathcal{A}$ to be the class of univalent functions defined on $\mathbb{D}$. A function $f \in \mathcal{S}$, lies in $\mathcal{S}^*$ if the domain $f(\mathbb{D})$ is starlike w.r.t origin. A function $f \in \mathcal{A}$ belongs to the class of close-to-convex functions $\mathcal{K}$ [14], if there exists $g \in \mathcal{S}^*$ such that $\mathrm{Re}(zf'(z)/g(z)) > 0$ for $z \in \mathbb{D}$. Note that $\mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$. Moreover, for specific choices of $g(z)$, namely $g(z) = 1/(1-z)$, $1/(1-z^2)$, $1/(1-z+z^2)$ and $1/(1-z)^2$, we obtain some special subclasses of close-to-convex functions $\mathcal{F}_i$ $(i = 1, \ldots, 4)$, defined as

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Mridula Mundalia and Shanmugam Sivaprasad Kumar contributed equally to this work.

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# [UMZh] Editor Decision

From: Oleksandra Vinnichenko via (noreply-umzh@imath.kiev.ua)

To: mridulamundalia@yahoo.co.in; spkumar@dce.ac.in

Date: Monday 25 November, 2024 at 06:31 pm IST

Dear Prof. Dr. Mridula Mundalia, Sivaprasad,

The Editorial Board of the Ukrainian Mathematical Journal has reached the following decision regarding your submission "Sufficient conditions for starlikeness related to a Hyperbolic Cosine function".

Editorial Board Decision is to: Accept your manuscript for publication.

The referee's report: In the article Sufficient conditions for starlikeness related to a Hyperbolic Cosine function the authors determine sufficient conditions for functions to be in the class S^{*}_{\rho} , which consists of starlike functions related to cosh sqrt{z}, using Briot-Bouquet and other differential subordination techniques.

The results of the article are new and interesting and it seems to me that the article fits into the scope of the journal. The article has high scientific level and there are also no issues regarding the scientific soundness and it is prepared respecting the standard requirements.

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# ON SHARP RADIUS ESTIMATES FOR $\mathcal{S}^*(\beta)$ AND A PRODUCT FUNCTION

S. Sivaprasad Kumar — Mridula Mundalia [c]

(*Communicated by Stanisława Kanas*)

ABSTRACT. In the present investigation, we determine various radius constants for the class $\mathcal{S}^*(\beta)$ of starlike functions of order $\beta$. We define $\mathcal{S}_{\lambda,\beta}$ to be the class of normalised analytic functions $f$ satisfying $\operatorname{Re}(e^{i\lambda}(1-z)^{1+\beta}f(z)/z) > 0$ and introduce a product function $G(z) := (1-z)^{1+\beta}g_1(z)g_2(z)/z$ with $g_1, g_2 \in \mathcal{S}_{\lambda,\beta}$, to find radius constants for $G(z)$ to be in certain desired classes. Notably, earlier known results are identified herein as special cases of our findings and all the results obtained are sharp.



## 1. Introduction

Let $\mathbb{D}_r$ be the open disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \le 1$ and $\mathbb{D} := \mathbb{D}_1$. Let $\mathcal{A}$ be the class of all analytic functions defined on $\mathbb{D}$ with the normalisation $f(0) = 0$ and $f'(0) = 1$ and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{A}_0$ be the class of analytic functions $f(z)$ defined on $\mathbb{D}$ with the normalisation $f(0) = 1$. Let $f(z)$ and $g(z)$ be analytic functions in $\mathbb{D}$, if there exists a Schwarz function $w : \mathbb{D} \to \mathbb{D}$ satisfying $|w(z)| \le |z|$, such that $f(z) = g(w(z))$, then $f(z)$ is subordinate to $g(z)$, denoted by $f \prec g$. Further, if $g(z)$ is a univalent function in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. The class of Carathéodory functions denoted by $\mathcal{P}$, consists of $p \in \mathcal{A}_0$ such that $\operatorname{Re} p(z) > 0$ and $\mathcal{S}^*$ denote the class of starlike functions satisfying $zf'(z)/f(z) \in \mathcal{P}$. Ma and Minda [15] unified all subclasses of starlike functions by defining the following class:

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z),\ z \in \mathbb{D} \right\}, \tag{1.1}$$

where $\phi \in \mathcal{P}$ is a univalent function, which is symmetric about the real line and starlike with respect to $\phi(0) = 1$ with $\phi'(0) > 0$. Further they dealt with growth, distortion, covering and coefficient problems for the class $\mathcal{S}^*(\phi)$. The class $\mathcal{S}^*(\phi)$ has been extensively studied by various authors for different choices of $\phi(z)$, see [2, 4, 8, 9, 11–13, 17–20, 22, 23] and the references therein. Some of the popularly known choices of $\phi(z)$ are: $\cosh\sqrt{z}$, $2/(1+e^{-z})$, $1+\sinh^{-1} z$, $1+\sin z$, $z+\sqrt{1+z^2}$, $1+ze^z$, $e^z$, $1+(4/3)z+(2/3)z^2$, $1+(2/\pi^2)(\log((1+\sqrt{z})/(1-\sqrt{z})))^2$, $(1+sz)^2$, where $s \in [-1/\sqrt{2}, 1/\sqrt{2}] \setminus \{0\}$ and $1+(z/k)((k+z)/(k-z))$, where $k = 1+\sqrt{2}$, the corresponding classes are denoted by $\mathcal{S}^*_{\varrho}$, $\mathcal{S}^*_{SG}$, $\mathcal{S}^*_{s}$, $\mathcal{S}^*_{\sin}$, $\mathcal{S}^*_{\mathbb{D}}$, $\mathcal{S}^*_{\wp}$, $\mathcal{S}^*_{e}$, $\mathcal{S}^*_{c}$, $\mathcal{S}^*_{p}$, $\mathcal{S}^*_{L}(s)$ and $\mathcal{S}^*_{\mathcal{R}}$, respectively. Further,

2020 Mathematics Subject Classification: Primary 30C45; Secondary 30C80.
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## SUMMARY

Mathematics researcher and educator with expertise in Complex Analysis and Engineering Mathematics. Over 8 years of experience teaching undergraduate engineering students while conducting research in Geometric Function Theory. Published multiple papers in peer-reviewed journals and proficient in mathematical computing tools.

## EDUCATION

### Ph.D. in Mathematics

Delhi Technological University

- Specialization: Complex Analysis, Geometric Function Theory
- Thesis: "*Coefficient Estimates, Radius Constants and Subordination of Certain Analytic Functions*"

### M.A. Mathematics (Honours)

Lady Shri Ram College, Delhi University | 2013 | Score: 81.7%

### B.A. Mathematics (Honours)

Jesus and Mary College, Delhi University | 2011 | Score: 66.8%

## PROFESSIONAL EXPERIENCE

### Guest Faculty

Delhi Technological University (DTU)
2023 - Present

- Teach Engineering Mathematics to B.Tech. students.
- Lead laboratory sessions using MATLAB, Python, and Mathematica for computational mathematics.

### Guest Faculty

Bhagwan Parshuram Institute of Technology (BPIT)

April - June 2022

- Delivered Engineering Mathematics lectures to B.Tech. students.
- Developed and implemented practical coursework using computational tools.

## RESEARCH, PUBLICATIONS & CONFERENCES

- Published 3 papers in SCIE-indexed journals (Bulletin of Iranian Mathematical Society, Mathematica Slovaca, Ukrainian Mathematical Journal, etc.)
- Research focuses on coefficient estimates and geometric properties of analytic functions.

### Publications

1. **On Sharp Radius Estimates and a Product Function**
   - *Mathematica Slovaca*, 2025, Springer (SCIE, Impact Factor: 0.9)
   - DOI: 10.1515/ms-2025-0022
2. **Sufficient Conditions for Starlikeness Related to a Hyperbolic Cosine Function**
   - *Ukrainian Mathematical Journal*, 2025, Springer (SCIE, Impact Factor: 0.5)
   - DOI: 10.3842/umzh.v77i2.8382
3. **Coefficient Problems for Certain Close-to-Convex Functions**
   - *Bulletin of the Iranian Mathematical Society*, Springer (SCIE, Impact Factor: 0.7).
   - DOI: 10.1007/s41980-023-00751-1.
4. **On a Subfamily of Starlike Functions Related to Hyperbolic Cosine Function**
   - *The Journal of Analysis*, Springer (ESCI, Impact Factor: 0.8).
   - DOI: 10.1007/s41478-023-00550-1.
5. **Coefficient Bounds for a Unified Class of Holomorphic Functions**
   - *Mathematical Analysis I: Approximation Theory*, Springer Proceedings (Scopus).
   - DOI: 10.1007/978-981-15-1153-0_17.
6. **On a Class of Non-Univalent Functions Associated with a Parabolic Region**
   - DOI: 10.48550/arXiv.2304.02608 (Under Review).

### Conferences/Workshops

- Attended 'National Conference on Algebra, Analysis, Coding and Cryptography' at Department of Mathematics, Delhi University, on October 14- 15, 2016.
- Participated in TEQIP Short Term course on Complex Analysis, Fourier Analysis and Special Functions, on March 06 – 10, 2017, at Department of Mathematics, IIT Roorkee.
- Participated in a one-day Workshop on 'Raising Awareness on Plagiarism and Copyrights,' on March 20, 2018.
- Presented a paper titled, 'Coefficient Estimates for a unified class of analytic functions' at the '33rd Annual Conference of the Ramanujan Mathematical Society,' at Delhi University, on June 1-3, 2018.
- Presented a paper entitled, 'Strongly Starlikeness Criteria for Certain Analytic Functions,' at 'International Conference on Recent Advances in Pure and Applied

Mathematics (ICRAPAM 2018),’ at DTU held on October 23-25, 2018.
- Contributed as an organizing member in ‘ICRAPAM 2018.’
- Presented a paper entitled, ‘Inclusion Relations of Strongly Starlike Functions,’ at ‘National Conference on Advances in Mathematical Analysis and its Applications, (NCAMAA - 2019),’ on November 8-10, 2019.
- Participated as Technical Assistant in the DTU sponsored one-day M.Sc. (Mathematics) Curriculum Workshop-2019, on November 14, 2019.
- Presented a Research article entitled: ‘On Coefficient Bounds for certain classes of univalent functions’ at ‘International Conference on Emerging Trends in Pure and Applied Mathematics,’ on March 12-13, 2022 at Tezpur University.
- Presented a Research article entitled, ‘Coefficient functionals for a class associated with Hyperbolic Cosine function,’ at ‘International Conference on Mathematical Analysis and Applications (ICMAA),’ at NIT Trichy, held during December 15-17, 2022.
- Presented a paper titled: ‘On Differential Subordination for Starlike functions,’ at the 29th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications (ICFIDCAA), Ramanujan School of Mathematical Sciences, Department of mathematics, Pondicherry University, held during August 21 - 25, 2023.

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## TECHNICAL SKILLS

- Programming: Python, C++, MATLAB
- Mathematical Software: Mathematica, LATEX
- Statistical Analysis: IBM SPSS

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## ACHIEVEMENTS

- Multiple paper presentations at international mathematical conferences and attended workshops.
- Active contributor to undergraduate and postgraduate research supervision in DTU.
- Organized and participated in collegiate mathematics competitions at graduation level.

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## URL LINKS:

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Linkedin URL: www.linkedin.com/in/dr-mridula-mundalia-625581275
Orcid id: https://orcid.org/0000-0002-1007-6100

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Prof. R. Srivastava, Delhi Technological University. Email id: rsrivastava@dce.ac.in