Some Coefficient Problems for a Class of Starlike Functions

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Abstract

This investigation explores the analytic and geometric properties of complex functions. Specifically, we focus on a novel starlike function that is analytic, denoted as S^*_{nc}, which is uniquely associated with a non-convex domain. The class on which we are going to work is defined as :

$$S^*_{nc} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} = \frac{1+z}{\cos z} \prec \varphi_{nc}(z), \quad z \in \mathbb{D} \right\}.$$

Here, $\mathcal{A}$ represents the set of functions analytic in $\mathbb{D}$ that satisfy $f(0) = 0$ and $f'(0) = 1$. The subordination condition involves a specific non-convex function $\varphi_{nc}(z) = (1+z)/\cos z$, which characterizes the geometric properties of functions belonging to this class.

Our primary objective is to determine the sharp second-order of Hankel and Toeplitz determinats for the logarithmic coefficients of functions f belonging to this newly defined class S^*_{nc}. Furthermore, the study extends to finding these precise bounds for the logarithmic coefficients of z their inverse functions, f^{-1}. The determination of sharp bounds for these determinants and coefficients provides crucial insights into the intricate behavior and structural properties of these analytic functions within the specified non-convex domain, contributing significantly to the understanding of coefficient problems in geometric function theory.

Contents

0.1 List of Symbols

Symbol	Description
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\Re$	Real part of a complex number
$\mathbb{C}$	The set of complex numbers
$\mathbb{D}$	Open unit disk
$\mathbb{D}^*$	Open punctured unit disk
$\mathcal{A}$	The class of functions analytic in $\mathbb{D}$
$\mathcal{S}$	Subclass of A consisting of univalent functions
$\mathcal{M}$	The class of harmonic functions in $\mathbb{D}$
S^*	The class of starlike functions in S
$S^*(\alpha)$	The class of starlike functions of order α in S
CV	The class of convex functions in S
$CV(\alpha)$	The class of convex functions of order α in S
CCV	The class of close-to-convex functions
$CV(\psi)$	The class of Ma-Minda convex functions, where $\psi \in \mathcal{P}$
$S^*(\psi)$	The class of Ma-Minda starlike functions, where $\psi \in \mathcal{P}$
S^*	The class of starlike functions with respect to symmetric points
K	The class of convex functions with respect to symmetric points
$\mathcal{P}(\alpha, \beta)$	Petal-shaped domain
$S_{\mathcal{P}}$	The class of starlike functions under a petal-shaped domain
P	The class of Carathéodory functions
R	The class of bounded turning functions in S
$H_n(f)$	Hankel determinant
$T_n(f)$	Toeplitz determinant
$k(z)$	Koebe function
$f \prec g$	Subordination symbol; $f \prec g$ implies f is subordinate to g
$f * g$	Convolution of analytic functions f and g in A
$M_n(f)$	n-th tail sum $\sum_{k=n}^{\infty} \lvert a_k \rvert^2$ of an analytic function $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$

Σ	Class of all meromorphic functions f of the form $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$
Σ_b	Class of all meromorphic functions f of the form $f(z) = \frac{1}{z} + b + \sum_{k=2}^{\infty} a_k z^k$ $(b \leq 0)$
δ_n	Logarithmic coefficients
Δ_n	Inverse logarithmic coefficients

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Chapter 1

Introduction

This chapter defines classes of analytic functions and introduces some basic terms and concepts that will be used in later chapters. This section introduces the fundamental notations and offers a broad overview of the thesis, emphasizing several significant findings.

Definition 1.0.1 (Univalent Function). [13] A function $f(z)$ is called univalent in a domain D if it is one-to-one on D; that is, whenever

$$f(z_1) = f(z_2) \quad \text{for } z_1, z_2 \in D,$$

it necessarily follows that $z_1 = z_2$ [1].

Definition 1.0.2 (Analytic Function). [13] A complex function $f(z)$ is analytic at a point z_0 within a domain $\mathcal{D}$ if it's differentiable throughout a neighborhood of z_0, meaning its derivative, $f'(z_0)$, exists. If this condition holds for every point in $\mathcal{D}$, then f is considered analytic across the entire domain.
For any point $z \in \mathcal{D}$, an analytic function f can be represented by its Taylor series expansion around z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

where $f^{(n)}(z_0)$ denotes the n-th derivative of f evaluated at z_0.

We define $\mathcal{M}(\mathbb{D})$ as the set of functions that are analytic on the unit disk $\mathbb{D}$ [1]. Within this class, $\mathcal{M}[a,n]$ refers to the subclass consisting of functions whose series expansion begins as:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

We introduce $\mathcal{A}$ be a set of functions f that are analytic in the open unit disk $\mathbb{D}$, which is normalized by $f'(0) = 1$ and $f(0) = 0$ [35]. The Taylor's expansion of a function $f \in \mathcal{A}$ [13] is represented by:

$$f(z) = z + \sum_{n=2}^{\infty} d_n z^n$$

This function operates within the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ [13]. The existence of a solution to the associated coefficient problem, along with its connection to the compactness of a particular function space, highlights the importance of normalization. Since analytic and univalent functions in a domain $\mathcal{D}$ preserve both the magnitude and direction of angles, they are referred to as conformal mappings in $\mathcal{D}$. Assume $\mathcal{S}$ be the subclass of $\mathcal{A}$. Koebe function [1], which maps $\mathbb{D}$ onto the complex plane with the exception of a slit along the half-line $(-\infty, -1/4]$, is the function k given by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n$$

defined on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ [13]. A surprising conclusion known as the Riemann Mapping Theorem [35] was announced by Riemann in 1851.Simply connected domain that isn't the entire complex plane $\mathbb{C}$ can be conformally mapped onto the unit disk $\mathbb{D}$.

Theorem 1.0.1 (**Riemann Mapping Theorem**). [35] Suppose $b \in D$, $D \subset \mathbb{C}$ be a simply connected domain. A distinct analytic function [1] $g : D \to \mathbb{C}$ exists, such that

(A.) $g(b) = 0$ and $g'(b) > 0$;

(B.) g is univalent;

(C.) $g(\mathcal{D}) = \Omega$, where Ω is also a simply connected domain.

The domain under consideration is the unit disk in the complex plane, specified by $|z| < 1$ [13].

Consequently, it is easy to convert the properties of a univalent function defined on the simply connected domain $\mathcal{D}$ into the properties of the original function defined on the open unit disk $\mathbb{D}$ [13]. Studying analytic functions inside the unit disk $\mathbb{D}$ is therefore adequate. Since

$$f_1(z) = \frac{f(z) - f(0)}{f'(0)}, \quad f'(0) \neq 0$$

symbolizes the image domain's $f(\mathbb{D})$ contraction and shifting with rotation and any property of the function $f_1(z)$ is immediately translated into a corresponding property of $f(z)$. Furthermore, the presence of a solution to the coefficient-related problem and its relationship to the compactness of a certain function space demonstrate the significance of normalization.

Theorem 1.0.2 (**Bieberbach's Conjecture**). [17] According to Bieberbach's Conjecture, for any function $f \in \mathcal{S}$, the modulus of the n-th coefficient satisfies the inequality $|a_n| \leq n$ for all $n \geq 2$. This estimate is known to be sharp, with equality attained exclusively by rotations of the Koebe function $k(z)$ [1, 35]. The conjecture saw partial proofs over time: Löwner, Garabedian, and Schiffer confirmed it for $n = 3$ and $n = 4$, respectively. Pederson and Schiffer later proved it for $n = 5$, followed by Pederson and Ozawa independently for $n = 6$. Ultimately, Louis de Branges provided a complete proof for all coefficients n in 1985 [1].

Theorem 1.0.3 (**de Branges' Theorem (formerly Bieberbach's Conjecture)**). [53] If $f \in \mathcal{S}$, then for all $n \geq 2$, the coefficients of f satisfy the bound $|a_n| \leq n$.

The extremal case is attained solely when f coincides with the Koebe function or a rotated form of it. This theorem captures several fundamental properties of univalent functions, including the classical covering result.

Theorem 1.0.4 (**Koebe One-Quarter Theorem**). Let $f \in \mathcal{S}$. Then the image of the open unit disk $\mathbb{D}$ under f contains the disk

$$\left\{ z \in \mathbb{C} : |z| < \frac{1}{4} \right\}.$$

A notable consequence of de Branges' Theorem is the Distortion Theorem, which provides precise bounds on the modulus of the derivative for functions in the class $\mathcal{S}$.

The Distortion Theorem stands as a prominent corollary of de Branges' foundational result which gives sharp upper and lower bounds for the modulus of the derivative of functions in the class $\mathcal{S}$.

Theorem 1.0.5 (**Distortion Theorem**). [35] Let $f \in \mathcal{S}$ and $z = re^{i\theta} \in \mathbb{D}$. Then the derivative $f'(z)$ satisfies the inequality:

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

Equality is attained exclusively when f corresponds to a rotated form of the Koebe function.

The next theorem is a direct implication of the Distortion Theorem, which provides sharp estimates for the modulus $|f(z)|$ when f belongs to the class $\mathcal{S}$ [13].

Theorem 1.0.6 (**Growth Theorem**). [35] Let $f \in \mathcal{S}$. Then for all $z \in \mathbb{D}$ with $|z| = r < 1$, the following inequality holds:

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}.$$

An intellectually striking corollary that emerges from Bieberbach's theorem is the Rotation Theorem, which encapsulates the angular behavior of univalent functions, which provides a bound on the argument of the derivative of functions in $\mathcal{S}$.

Theorem 1.0.7 (**Rotation Theorem**). [35] Let $f \in \mathcal{S}$. Then for $z \in \mathbb{D}$ with $|z| = r < 1$, the argument of the derivative satisfies:

$$|\arg f'(z)| \leq \begin{cases} \dfrac{4\sin^{-1} r}{1-r^2}, & \text{if } r \leq \dfrac{1}{\sqrt{2}}, \\ \log\left(\dfrac{1+r}{1-r}\right), & \text{if } r > \dfrac{1}{\sqrt{2}}. \end{cases}$$

Univalency of analytic functions is also studied using the Fekete-Szegő coefficient functional.

Theorem 1.0.8 (**Fekete–Szegő Theorem**). [35] Let $f \in \mathcal{S}$. Then for any real parameter $\alpha \in (0,1)$, the coefficients of f satisfy the inequality:

$$\left|a_3 - \alpha a_2^2\right| \leq 1 + 2e^{-2\alpha}.$$

1.1 Classes of univalent and starlike functions

The notation $\mathcal{S}$ is used to represent the subclass of $\mathcal{A}$ that contains univalent functions [13]. If $f \in \mathcal{S}$, from this, the Taylor's expansion of the series for f is derived as [1]:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1.1.1}$$

In 1907, Koebe proved that for the class $\mathcal{S}$, there exists an absolute constant $k > 0$ such that the boundary of the image $f(\mathbb{D})$ cannot approach the origin closer than a distance k [1]. Later, in 1916, Bieberbach established the elegant result that $|a_2| \leq 2$ for every function $f \in \mathcal{S}$, and using this result, determined the value of k as $1/4$. This highlights the geometric significance of coefficient bounds in the study of univalent functions. Furthermore, Bieberbach conjectured that $|a_n| \leq n$ for all $n \geq 2$. Although the conjecture remained unproven for a long time, it was verified for several subclasses of $\mathcal{S}$. In 1925, J. E. Littlewood showed that $|a_n| \leq en$, demonstrating that the conjecture held up to a multiplicative constant of $e \approx 2.718$ [1, 35]. In 1985, Louis de Branges finally proved the entire conjecture, utilizing specialized methods.

Definition 1.1.1 (Starlike Function). [1] Assume domain D which is subset of complex domain is said to be starlike to a point $r_0 \in D$ if, for each point $r \in D$, then line segment joining r_0 and r lies entirely within D. That is,

$$(1-p)r_0 + pr \in D \quad \text{for all } p \in [0,1].$$

A function $f(z)$ is designated as a starlike function if it maps the open unit disk $\mathbb{D}$ onto a domain that is stellar with respect to the origin (i.e., $r_0 = 0$) [13].
From an analytical perspective, if $f(z) \in A$ and $\Re(zf'(z)/f(z)) > 0$, then the function $f(z)$ is starlike with respect to the origin. The class of starlike functions is represented as S^* [13]

$$\mathcal{S}^* = \{f \in A : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0\}.$$

Definition 1.1.2 (Starlike Functions of Order α). [1] A function $f \in \mathcal{S}$ is categorized as starlike of order α if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{for all } z \in \mathbb{D}, \quad 0 \leq \alpha < 1.$$

The category encompassing all such functions is denoted by $\mathcal{S}^*(\alpha)$ [13]. In particular, when $\alpha = 0$, we recover the class $\mathcal{S}^*$ of starlike functions, i.e., $\mathcal{S}^*(0) = \mathcal{S}^*$ [1].

Definition 1.1.3 (**Convex Function**). [1] A subset $D \subset \mathbb{C}$ is characterized as convex if, given any two points r_1, r_2 residing in D, their connecting line segment remains entirely subsumed by D. In other words,

$$pr_1 + (1-p)r_2 \in D \quad \forall p \in [0,1].$$

A function f is designated a convex function provided it transforms the open unit disk $\mathbb{D}$ into a convex domain [1].
From an analytical standpoint, A function $f \in \mathcal{A}$ is characterized as convex if it satisfies the prescribed inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{for all } z \in \mathbb{D}.$$

The class of such functions is denoted by $\mathcal{CV}$ (also commonly denoted by $\mathcal{K}$) [1], and is defined as:

$$\mathcal{CV} = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}.$$

The inclusion of convex functions within the class of starlike functions is widely acknowledged, since the image of a Every convex domain is necessarily starlike with respect to any of its interior points. However, the reciprocal assertion typically proves false. As an illustration, consider the function $f(z) = z + \frac{z^2}{2}$, which is starlike yet not convex.
In 1936, Robertson [29] generalized the theory of the classes $\mathcal{S}^*$ (starlike functions) and $\mathcal{CV}$ (convex functions), which led to significant advancements in geometric function theory.

Definition 1.1.4 (**Convex Function of order** α). [1] A function $f \in \mathcal{S}$ is classified as a convex function of order α provided it fulfills the inequality

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha \quad \text{for all } z \in \mathbb{D}, \quad \text{where } 0 \leq \alpha < 1.$$

The ensemble of all such functions is represented by $\mathcal{CV}(\alpha)$ [1]. Notably, in the instance where $\alpha = 0$, the class $\mathcal{CV}(\alpha)$ becomes equivalent to the class of convex functions; that is,

$$\mathcal{CV}(0) = \mathcal{CV}.$$

A fundamental result known as *Alexander's Theorem* establishes a correspondence between the classes $\mathcal{CV}(\alpha)$ and $\mathcal{S}^*(\alpha)$. Specifically,

$$f \in \mathcal{CV}(\alpha) \quad \Longleftrightarrow \quad zf'(z) \in \mathcal{S}^*(\alpha).$$

This relationship provides a powerful tool for translating results between the classes of convex and starlike functions.

Definition 1.1.5 (Bounded Turning [1]). A function $f \in \mathcal{S}$ is said to have *bounded turning* (i.e., $f \in \mathcal{R}$) if its derivative maps the unit disk $\mathbb{D}$ into the right half-plane. That is,

$$\Re\left(f'(z)\right) > 0, \quad \text{for all } z \in \mathbb{D}.$$

Equivalently, this is satisfied if

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < 1, \quad \text{for all} z \in \mathbb{D}.$$

Theorem 1.1.1 (Alexander's Theorem [35]). Discovered in 1915, Alexander's Theorem establishes a fundamental connection between convex and starlike functions. It states:

$$f \in \mathcal{CV} \quad \Leftrightarrow \quad zf'(z) \in \mathcal{S}^*,$$

where $\mathcal{CV}$ and $\mathcal{S}^*$ denote the classes of convex and starlike functions, respectively. This equivalence extends naturally to the generalized classes:

$$f \in \mathcal{CV}(\alpha) \quad \Leftrightarrow \quad zf'(z) \in \mathcal{S}^*(\alpha),$$

as detailed in [1].

Definition 1.1.6 (Close-to-Convex[35]). A function $f \in \mathcal{A}$ is termed close-to-convex in the unit disk $\mathbb{D}$ if, as first stated by Kaplan in 1952, there exists a convex function g and a real parameter $\theta \in (-\pi, \pi)$ such that

$$\Re\left(e^{i\theta}\frac{f'(z)}{g'(z)}\right) > 0, \quad \text{for all } z \in \mathbb{D}.$$

These subclasses of $\mathcal{S}$ follow a chain of inclusions, indicating a hierarchy of geometric properties [1]:

$$\mathcal{CV} \subset \mathcal{S}^* \subset \mathcal{CCV} \subset \mathcal{S}.$$

Theorem 1.1.2 (Noshiro–Warschawski Theorem [1]). Consider a function $g \in \mathcal{A}$ analytic in a convex region R. If there exists a real number γ such that

$$\Re\left(e^{i\gamma}g'(w)\right) > 0, \quad \text{for all } w \in R,$$

then g is univalent in R. Kaplan leveraged this theorem to establish the univalence of all close-to-convex functions [52].

Definition 1.1.7 (Starlike with Respect to Symmetric Points [13]). A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_s^*$, referred to as starlike with respect to symmetric points, as introduced by Sakaguchi in 1959 [22], if the following condition holds for every $z \in \mathbb{D}$:

$$\Re\left(\frac{2zf(z)}{f(z) - f(-z)}\right) > 0.$$

This condition implies that, as z moves counter-clockwise when z traverses the circle $|z| = r$ in the positive (counter-clockwise) direction, the image of z under f rotates around the point $f(-z)$ with a strictly positive angular velocity.

Definition 1.1.8 (Carathéodory Class). A function $p(z)$ is an element of the Carathéodory class $\mathcal{P}$ provided it is holomorphic in the open unit disk $\mathbb{D}$, adheres to the normalization condition

$$p(0) = 1,$$

and maintains a positive real part across $\mathbb{D}$:

$$\Re(p(z)) > 0 \quad \text{for all } z \in \mathbb{D}.$$

Such functions admit a Taylor series expansion of the form:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots .$$

Functions within $\mathcal{P}$ are alternatively referred to as Carathéodory functions or functions exhibiting a positive real part [1, 13].

A fundamental connection exists between the Carathéodory class $\mathcal{P}$ and the class of starlike functions $\mathcal{S}^*$, as expressed by:

$$f \in \mathcal{S}^* \quad \Leftrightarrow \quad \frac{zf'(z)}{f(z)} \in \mathcal{P}. \tag{1.1.4}$$

Lemma 1.1.9. Consider $p(z)$ as an element of the class $\mathcal{P}$ [1], whose representation is provided by the power series

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots .$$

Subsequently, the ensuing estimate is valid:

$$|p_n| \leq 2 \quad \text{for all } n \in \mathbb{N}.$$

For any real number $\alpha \in [0, 1)$, the collection of holomorphic functions $p \in \mathcal{P}$ that fulfill

$$\Re(p(z)) > \alpha \quad \text{for all } z \in \mathbb{D}$$

is called $\mathcal{P}(\alpha)$. Employing the notion of subordination, a function $p(z)$ having a positive real part may be characterized as

$$p(z) \prec \frac{1+z}{1-z} \quad \text{for all } z \in \mathbb{D},$$

owing to the fact that the function $q(z) = \frac{1+z}{1-z}$ maps the unit disk onto the right half-plane [1, 13, 35].

Ma and Minda presented a broader methodology for characterizing various subclasses of starlike and convex functions. Rather than employing the standard function $q(z) = \frac{1+z}{1-z}$, they utilized a more generalized holomorphic function $\varphi(z)$ [1, 35]. This function φ is holomorphic in the unit disk, fulfills the conditions $\varphi(0) = 1, \varphi'(0) > 0$, and yields, as its image of the disk, a domain that exhibits axial symmetry around the real axis and is starlike relative to the point

The class $\mathcal{S}^*(\varphi)$, called Ma-Minda starlike functions, includes $f \in \mathcal{A}$ that is

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad \text{for each } z \in \mathbb{D}.$$

Similarly, the $\mathcal{CV}(\varphi)$, called Ma-Minda convex functions, includes functions $f \in \mathcal{A}$ such that

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \quad \text{for each } z \in \mathbb{D}.$$

Chapter 2

Hankel Determinant and Toeplitz Determinant

This chapter explores the definitions of the Hankel and Toeplitz determinants as well as how they differ across various analytic function sub-classes.

Hankel determinants are instrumental in demonstrating the rationality of functions within the unit disk ($\mathbb{D}$). This is particularly true for functions characterized by specific restrictions, such as those that can be expressed as the ratio of two bounded analytic functions having integral coefficients in their Laurent series. Furthermore, Hankel determinants have provided significant insights when applied to the analysis of meromorphic functions.

One commonly studied form is the Hankel determinant $H_{h,s}(f)$, especially in the framework of the Fekete–Szegö problem. For example, the second-order Hankel determinant $H_{2,1}$ is defined as $H_{2,1} = a_3 a_1 - a_2^2$, which has been generalised to $H_{2,1} = a_3 - \mu a_2^2, \quad \mu \in \mathbb{C}$. Pommerenke, in his foundational work [10], established that for univalent functions, the Hankel determinant satisfies the inequality

$$|H_{h,s}(f)| < K s^{-(h+3)/2+\beta}, \quad \text{with } \beta > \frac{1}{4000},$$

where the constant K depends only on the order h.

Subsequently, Hayman [51] proved that for areally mean univalent functions, the second-order Hankel determinant satisfies the bound $|H_2(s)| < A^2 s, \quad \text{for } s = 1, 2, \ldots,$ where A is an absolute constant.

Further investigations by Pommerenke [10] in 1967 extended the analysis of Hankel determinants to various function classes, including areally mean p-valent functions, univalent functions, and starlike functions.

Additionally, ElHosh derived sharp modulus for determinants of Hankel in the class of injective holomorphic function with positive Hayman index α, as well as for k-fold symmetric and close-to-convex functions [35].

The hth Hankel-determinant for $h, s \in \mathbb{N}$, or $H_{h,s}(f)$, for a function $f \in \mathcal{A}$ is as follows:

$$H_{h,s}(f) = \begin{vmatrix} a_s & a_{s+1} & \cdots & a_{s+h-1} \\ a_{s+1} & a_{s+2} & \cdots & a_{s+h} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s+h-1} & a_{s+h} & \cdots & a_{s+2(h-1)} \end{vmatrix} \tag{2.0.1}$$

The Fekete-Szegö problem is regarded as one of the most significant results concerning univalent functions [3], [7], [55]. It relates to the coefficients of a function's [1] and was introduced by Fekete and Szegő [7]. In Fekete-Szegő problem the optimization of the absolute value of the functional $|a_3 - \mu a_2^2|$ is our goal. Numerous researchers have carefully examined and analyzed this outcome. For the Koebe function, the equality is valid. Keogh and Merkes [39] discovered the sharp upper bound of the Fekete-Szegő $|a_3 - \mu a_2^2|$ in 1969 for a few univalent function sub-classes.

The Fekete Szegő is obtained for $h = 2$ and $s = 1$ in (2.0.1);

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2.$$

Further, sharp bounds for the functional $|a_2 a_4 - a_3^2|$ are obtained in (2.0.1) for $h = 2$ and $s = 2$, the second order Hankel-determinant [10]:

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2$$

In recent years, many researchers have focused on finding sharp upper bounds for the Hankel determinant $|H_{2,2}(f)|$. Exact estimates of $|H_{2,2}(f)|$ have been obtained for important subclasses of univalent functions, namely the classes $\mathcal{R}$ (functions of bounded turning), $\mathcal{S}^*$ (starlike functions), and $\mathcal{K}$ (convex functions).

Recently, Ye and Lim showed that every $n \times n$ matrix over $\mathbb{C}$ can, in general, be written as a product of certain Toeplitz or Hankel matrices. Both Hankel and Toeplitz matrices play a key role in many areas of mathematics due to their wide range of applications [32]. In particular, Toeplitz matrices and their determinants are important tools in both theoretical and applied mathematics. They appear in various fields such as complex analysis, quantum physics, image and signal processing, and the theory of integral equations. A detailed overview of these applications can be found in the related survey literature.

It is worth noting that in a Toeplitz matrix, each diagonal from top-left to bottom-right contains the same element, while in a Hankel matrix, the elements along the anti-diagonals (from top-right to bottom-left) are constant.

For a function $f \in \mathcal{A}$, the h_{th} Toeplitz determinant, $T_{h,s}(f)$ as:

$$T_{h,s}(f) = \begin{vmatrix} a_s & a_{s+1} & \cdots & a_{s+h-1} \\ a_{s+1} & a_s & \cdots & a_{s+h-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{s+h-1} & a_{s+h-2} & \cdots & a_s \end{vmatrix}$$

Bieberbach approximated $H_{2,1}(f)$ for the class $\mathcal{S}$ [9, 10]. For $f \in \mathcal{A}$, the h_{th} Hankel and Toeplitz determinant, $H_{h,s}(F_f/2)$ and $T_{h,s}(F_f/2$ where $h, s \in \mathbb{N}$ are entries of logarithmic coefficients [28](refer Chapter 4), it is expressed as

$$H_{h,s}(F_f/2) = \begin{vmatrix} \delta_s & \delta_{s+1} & \cdots & \delta_{s+h-1} \\ \delta_{s+1} & \delta_{s+2} & \cdots & \delta_{s+h} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{s+h-1} & \delta_{s+h} & \cdots & \delta_{s+2(h-1)} \end{vmatrix}, \quad T_{h,s}(F_f/2) = \begin{vmatrix} \delta_s & \delta_{s+1} & \cdots & \delta_{s+h-1} \\ \delta_{s+1} & \delta_s & \cdots & \delta_{s+h-2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{s+h-1} & \delta_{s+h-2} & \cdots & \delta_s \end{vmatrix}.$$

Kowalczyk et al. [9] studied the Hankel determinant with entries of logarithmic coefficients. In this, we're going to study about $H_{2,1}(F_f)$ which can be found with the help of $H_{2,1}(f) = a_2a_4 - a_3^2$, where $f \in \mathcal{S}$ (following logarithmic function methodology). Given the widespread significance of various coefficients in geometric function theory, Hankel determinant which is formulated from the logarithmic coefficients of functions $f \in \mathcal{S}$, has been recently put forth by Kowalczyk and Lecko [28]. Drawing motivation from this concept and established theories of determinants [9], this work undertakes an analysis of the Hankel determinant $H_{h,s}(F_{f^{-1}}/2)$ and the Toeplitz determinant $T_{h,s}(F_{f^{-1}}/2)$. For these particular determinants, the constituent elements are the logarithmic coefficients of the inverse functions f^{-1}, where $f^{-1} \in \mathcal{S}$ [32]. A more comprehensive discussion of these inverse logarithmic coefficients is provided in Chapter 4.

The determinant $H_{h,s}(F_{f^{-1}}/2)$ is expressed as follows:

$$H_{h,s}(F_{f^{-1}}/2) = \begin{vmatrix} \Delta_s & \Delta_{s+1} & \cdots & \Delta_{s+h-1} \\ \Delta_{s+1} & \Delta_{s+2} & \cdots & \Delta_{s+h} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{s+h1} & \Delta_{s+h} & \cdots & \Delta_{s+2(h-1)} \end{vmatrix}$$

and consequently the determinant $T_{h,s}(F_{f^{-1}}/2)$ is

$$T_{h,s}(F_{f^{-1}}/2) = \begin{vmatrix} \Delta_s & \Delta_{s+1} & \cdots & \Delta_{s+h-1} \\ \Delta_{s+1} & \Delta_s & \cdots & \Delta_{s+h-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{s+h-1} & \Delta_{s+h-2} & \cdots & \Delta_s \end{vmatrix}.$$

Hankel and Toeplitz determinants are a well-researched topic for function classes like starlike, convex, and their subclasses, with sharp bounds established in key studies [9, 23, 33]. Though Hankel determinants involving logarithmic coefficients have been a subject of recent inquiry for specific function subclasses such as close-to-convex, strongly starlike, and strongly convex functions [13], pinning down sharp bounds for Toeplitz determinants with logarithmic coefficients of inverse functions particularly under specific symmetry or domain constraints—is still an open problem.

Progress has been made in related areas. For example, Zaprawa (2021) derived sharp bounds for initial logarithmic coefficients in classes like $\mathcal{S}_S^*$ (starlike with symmetric points) and $\mathcal{K}_S$ (convex with symmetric points) [35, 28]. Recent studies have also addressed 2^{nd} and 3^{rd} order Toeplitz determinants for f^{-1} in Ma-Minda classes, which generalise many classical function families.

These results build on methods using coefficient relationships and growth theorems. For instance, logarithmic coefficients of inverse functions satisfy $\delta_1 = -a_2/2$ and $\delta_2 = -\frac{1}{2}\left(a_3 - \frac{3}{2}a_2^2\right)$, enabling determinant calculations through Taylor coefficients.

Toeplitz matrices (with constant diagonals) and their determinants have wide applications in complex analysis, signal processing, and operator theory. A comprehensive survey by Ye and Lim [21] details their mathematical significance across pure and applied fields. Current challenges involve extending these results to higher-order determinants and non-symmetric function classes.

Chapter 3

Non-Convex Domain

This chapter investigates non-convex domains regions where straight lines between points may cross outside the domain—and their distinct geometric properties, such as irregular boundaries and asymmetric connectivity. Unlike convex shapes, these domains introduce analytical complexities that influence function behavior, particularly in coefficient growth and boundary interactions. A central goal is establishing sharp bounds for Taylor coefficients (e.g., $|a_n|$) of analytic functions defined on such domains, leveraging extremal function methods and subordination theory to generalize classical convex-domain results.

Kumar and Giri recently brought forth a new classification of starlike functions, which are connected to non-convex domains [42]. This classification is characterized as follows:

$$\mathcal{S}^*_{nc} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{\cos z},\ z \in \mathbb{D} \right\}. \tag{3.1}$$

Function f is in $\mathcal{S}^*_{nc}$ class iff it admits the following integral form representation:

$$f(z) = z \exp\left(\int_0^z \frac{q(t)-1}{t}\, dt \right), \tag{3.2}$$

where the auxiliary function $q(z)$ is analytic and subordinate to $\varphi_{nc}(z) = (1+z)/\cos z$, i.e., $q(z) \prec \varphi_{nc}(z)$ [42].

For integers $n \geq 2$, define the family of canonical functions $f_n(z)$ satisfying $f_n(0) = 0$

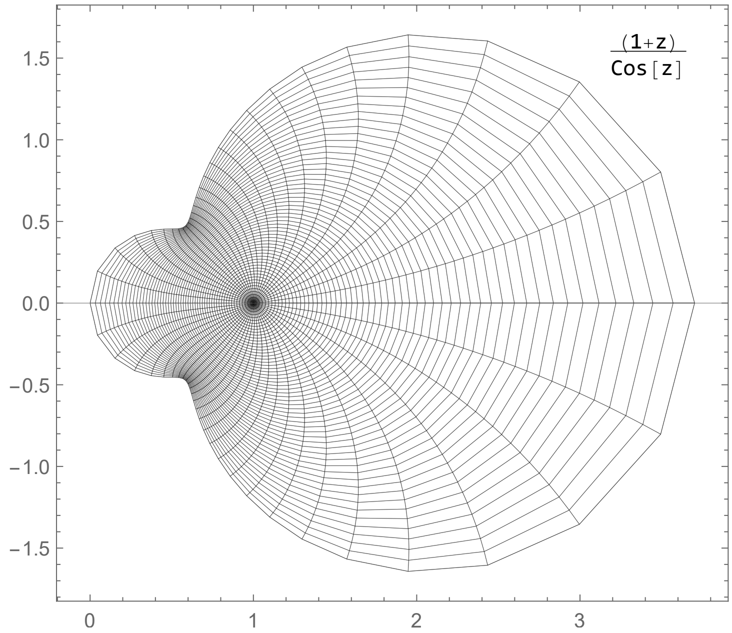

Figure 3.1: Image of the unit disk under the function φ_{nc}.

and $f_n'(0) = 1$, and given by:

$$\frac{zf_n'(z)}{f_n(z)} = \frac{1 + z^{n-1}}{\cos z^{n-1}}. \tag{3.3}$$

Each function f_n is in $\mathcal{S}^*_{nc}$ class. In particular, the case $n = 2$ yields an important extremal function:

$$\tilde{f}(z) = z \exp\left(\int_0^z \frac{1 + t - \cos t}{t \cos t}\, dt\right) = z + z^2 + \frac{3}{4}z^3 + \frac{7}{12}z^4 + \frac{35}{96}z^5 + \cdots, \tag{3.4}$$

which frequently arises in extremal problems within the class $\mathcal{S}^*_{nc}$.
Following the framework established by Ma and Minda [54], for any function $f \in \mathcal{S}^*_{nc}$, the following subordination relationships hold:

$$\frac{f(z)}{z} \prec \frac{\tilde{f}(z)}{z} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \frac{z\tilde{f}'(z)}{\tilde{f}(z)}.$$

These relations confirm that $\tilde{f}$ serves as a dominant function in the class $\mathcal{S}^*_{nc}$ with respect to these key functionals.

Theorem 3.0.1. [42]Let $f \in S^*_{nc}$ and $\tilde{f}$ be the extremal function given by (3.4). Then the following holds:

1. **Growth theorem[42]:** For $|z_0| = r < 1$, we have

$$-\tilde{f}(-r) \leq |f(z_0)| \leq \tilde{f}(r).$$

   Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of $\tilde{f}$.

2. **Rotation theorem:** For $|z_0| = r < 1$, we have

$$\left|\arg\left\{\frac{f(z_0)}{z_0}\right\}\right| \leq \max_{|z|=r} \arg\left\{\frac{\tilde{f}(z)}{z}\right\}.$$

3. **Distortion theorem:** For $|z_0| = r < 1$, we have

$$\tilde{f}'(-r) \leq |f'(z_0)| \leq \tilde{f}'(r).$$

   Equality obtains for some $z_0 \neq 0$ if and only if f constitutes a rotation of $\tilde{f}$.

Theorem 3.0.2 (see [42])**.** Let $f \in S^*_{nc}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} -\mu + \frac{3}{4}, & \mu < \frac{1}{4} \\ \frac{1}{2}, & \frac{1}{4} \leq \mu \leq \frac{5}{4} \\ \mu - \frac{3}{4}, & \mu > \frac{5}{4}. \end{cases}$$

The bound is sharp.

Theorem 3.0.3 (see [42])**.** If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class S^*_{nc}, then

$$|a_2| \leq 1, \quad |a_3| \leq \frac{3}{4}, \quad |a_4| \leq \frac{7}{12}, \quad |a_5| \leq \frac{1}{3}.$$

The bounds of a_n for $n = 2, 3, 4$ are sharp.

Inclusion Relations and Radius Problems

In 1999, Kanas and Wiśniowska [40] introduced the class

$$k\text{-}\mathcal{ST} = \left\{f \in A : \operatorname{Re}\frac{zf'(z)}{f(z)} > k\left|\frac{zf'(z)}{f(z)} - 1\right|\right\},$$

of k-starlike functions. Geometrically, the boundary of the domain

$$\Omega_k = \{w \in \mathbb{C} : \operatorname{Re} w > k|w-1|\}$$

represents an ellipse for $k > 1$, a parabola for $k = 1$, and a hyperbola for $0 < k < 1$. The next result establishes the inclusion relation between the class k-$\mathcal{ST}$ and the following classes (see [1, 3, 35]):

$$\mathcal{S}^*(M) = \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} - M \right| < M \right\}, \quad M > \frac{1}{2},$$
$$\mathcal{ST}_p(a) = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} + a \right) > |w-a| \right\}, \quad a > 0,$$

and

$$\mu(\beta) = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) < \beta \right\}, \quad \beta > 1.$$

A function f in the class $\mathcal{S}^*(M)$ is referred to as an M-starlike function.

Theorem 3.0.4 (see [42]). *k-$\mathcal{ST} \subset \mathcal{S}^*_{nc}$ for $k \geq \frac{4\cos 1}{4\cos 1 - \cos 2 - 1}$.*

Proof. Consider a function $f \in k$-$\mathcal{ST}$, and let Ω_k represent the region defined by the inequality $\operatorname{Re} w > k|w-1|$. For values of $k > 1$, the boundary curve of this region, denoted δ_k, constitutes an ellipse. This ellipse is described by the equation:

$$x^2 = k^2(x-1)^2 + k^2y^2.$$

Rearranging this equation, we can express the ellipse in its standard form:

$$\frac{(x-x_0)^2}{u^2} + \frac{(y-y_0)^2}{v^2} = 1,$$

where the parameters are given by:

$$x_0 = \frac{k^2}{k^2-1}, \quad y_0 = 0, \quad u = \frac{k}{k^2-1}, \quad v = \frac{1}{\sqrt{k^2-1}}.$$

To ensure that the ellipse δ_k is contained within the region Ω_{nc}, a condition must be met. Since $u > v$, the ellipse δ_k will lie inside Ω_{nc} if and only if the range of $x_0 + u$

satisfies

$$x_0 + u \leq \frac{4\cos 1}{1+\cos 2}.$$

A simple calculation confirms that this condition is satisfied whenever $k \geq \frac{4\cos 1}{4\cos 1 - \cos 2 - 1}$.[42]

□

Chapter 4

Logarithmic Coefficient and Inverse Logarithmic Coefficient

The fundamental objective herein is to is to investigate the behavior and properties of logarithmic coefficients and inverse logarithmic coefficients associated with analytic and univalent functions. These coefficients play a crucial role in the geometric function theory, particularly in understanding the distortion, growth, and covering properties of such functions. In this context, we will examine sharp bounds and structural results related to these coefficients. Special emphasis will be placed on deriving estimates for the second-order Hankel determinant, which captures nonlinear interactions among the coefficients. We will present and prove several theorems concerning the bounds of Hankel determinants formulated in terms of logarithmic and inverse logarithmic coefficients, highlighting their relevance in the broader scope of complex analysis and univalent function theory.

Definition 4.0.1 (Logarithmic Coefficients). The logarithmic coefficients δ_n for a function f belonging to the class $\mathcal{S}$ are established through its series expansion. Specifically, for z within the punctured unit disk $\mathbb{D} \setminus \{0\}$, these coefficients arise from the following definition:

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \delta_n z^n. \tag{4.0.1}$$

where δ_n quantify the interaction between f and the logarithmic mapping. These coefficients play a critical role in growth/covering theorems and extremal problems.

Definition 4.0.2 (**Milin's Conjecture**). [35, 13] Milin conjectured that for $f \in \mathcal{S}$,

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\delta_k|^2 - \frac{1}{k}\right) \leq 0,$$

with the sole circumstance under which equality is achieved is when f is a rotation of the Koebe function $k(z) = z/(1-z)^2$. Proved by de Branges in 1984, this conjecture resolved the Bieberbach conjecture. Despite the Koebe function's extremality, logarithmic coefficients $|\delta_n|$ do not universally satisfy $|\delta_n| \leq 1/n$, even asymptotically.

Definition 4.0.3 (**Sharp Bounds for Class $\mathcal{S}$**). For $f \in \mathcal{S}$, the sharp bounds of the first two logarithmic coefficients are:

$$|\delta_1| \leq 1 \quad \text{and} \quad |\delta_2| \leq \frac{1}{2} + \frac{1}{e^2}.$$

No sharp bounds are known for $|\delta_n|$ when $n \geq 3$, highlighting a major open problem in geometric function theory.

Lemma 4.0.4 (**Logarithmic Coefficient Relations**). For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, the logarithmic coefficients satisfy:

$$\begin{aligned}
\delta_1 &= \frac{1}{2}a_2, \\
\delta_2 &= \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \\
\delta_3 &= \frac{1}{4}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \\
\delta_4 &= \frac{1}{4}\left(a_5 - a_2a_4 + a_3^2 - \frac{1}{2}a_2^2a_3 - \frac{1}{4}a_2^4\right), \\
&\cdot
\end{aligned}$$

The 2^{nd} order Hankel determinant for $F_f(z)$ is represented by:

$$H_{2,1}(F_f) = \delta_1\delta_3 - \delta_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \tag{4.0.2}$$

Under rotation $f_\theta(z) = e^{-i\theta} f(e^{i\theta} z)$, the determinant's *magnitude* remains invariant:

$$\left|H_{2,1}(F_{f_\theta})\right| = \left|H_{2,1}(F_f)\right|.$$

Theorem 4.0.1 (Fekete-Szegö Inequality). For $f \in \mathcal{S}$ and $\lambda \in [0, 1)$, the sharp inequality

$$|a_3 - \lambda a_2^2| \le 1 + 2e^{-2\lambda/(1-\lambda)}$$

holds. This result, critical to coefficient estimation, generalises classical bounds for a_3 and a_2^2.

Definition 4.0.5 (Koebe's 1/4 Theorem). Let $f \in \mathcal{S}$ (univalent with $f(0) = 0, f'(0) = 1$). Then the image of $\mathbb{D}$ under f, denoted $f(\mathbb{D})$, encompasses the disk $D(0, 1/4)$, meaning:

$$\{w \in \mathbb{C} : |w| < 1/4\} \subset f(\mathbb{D}),$$

and the constant $1/4$ is sharp. The function that achieves this bound is the Koebe function:

$$k(z) = \frac{z}{(1-z)^2},$$

which maps the unit disk $\mathbb{D}$ to the complex plane excluding the ray from negative infinity to $-1/4$, specifically $\mathbb{C} \setminus (-\infty, -1/4]$. This particular function acts as an extremal example for a variety of questions in geometric function theory [1]. The theorem ensures the existence of a local inverse $F = f^{-1}$ near the origin, enabling the study of inverse coefficients.

Definition 4.0.6 (Inverse Logarithmic Coefficients). Let $f \in \mathcal{S}$ and $f^{-1}(w) = F(w) = w + \sum_{n=2}^{\infty} A_n w^n$ serve as its inverse function, which is holomorphic within the disk $|w| < 1/4$. The inverse logarithmic coefficients Δ_s are defined by:

$$\log\left(\frac{F(w)}{w}\right) = 2\sum_{s=1}^{\infty} \Delta_s w^s, \quad |w| < \frac{1}{4}.$$

Regarding functions $f \in \mathcal{S}$, Ponnusamy [32] established the precise inequality:

$$|\Delta_s| \le \frac{1}{2s}\binom{2s}{s} \quad (s \in \mathbb{N}),$$

where equality is attained only for rotations of the Koebe function. These coefficients connect the geometry of to it's inverse's analytic properties.

4.1 Important lemmas

To reinforce the key findings, we now present several important lemmas that form the foundation for proving theorems related to these determinant bounds. These lemmas play a critical role in establishing sharp estimates and ensuring the mathematical rigor of the analysis presented in this section.
The following lemma is due to Kwon, Lecko, and Sim [34].

Lemma 4.1.1 (see [34]). Let $p \in \mathcal{P}$. Then, for certain $x, \delta, \rho \in D = \{z \in \mathbb{C} : |z| \leq 1\}$, the following relations hold:

$$2c_2 = c_1^2 + (4 - c_1^2)x,$$

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2(1 - |x|^2)(4 - c_1^2)\delta,$$

$$8c_4 = c_1^4 + x\left[c_1^2(x^2 - 3x + 3) + 4x\right](4 - c_1^2) - 4(4 - c_1^2)(1 - |x|^2)\left[c_1(x - 1)\delta + \bar{x}\delta^2 - (1 - |\delta|^2)\rho\right].$$

In this context, x, δ, and ρ represent complex numbers, each with a modulus not exceeding 1. Furthermore, if we set $c_1 = c$, $|x| = m$, and $|\rho| = y$, these real quantities are constrained within the intervals $[0, 2]$, $[0, 1]$, and $[0, 1]$ in their respective order.

The following lemma is from R. J. Libera and E. J. Zlotkiewicz [36, 38].

Lemma 4.1.2. [see [36], [38]] Let $p \in \mathcal{P}$, with $c_1 \geq 0$, then

$$c_1 = 2\zeta_1, \tag{4.1}$$

$$c_2 = 2\zeta_1^2 + 2(1 - \zeta_1^2)\zeta_2, \tag{4.2}$$

$$c_3 = 2\zeta_1^3 + 4(1 - \zeta_1^2)\zeta_1\zeta_2 - 2(1 - \zeta_1^2)\zeta_1\zeta_2^2 + 2(1 - \zeta_1^2)(1 - |\zeta_2|^2)\zeta_3 \tag{4.3}$$

for some $\zeta_1 \in [0, 1]$ and $\zeta_2, \zeta_3 \in \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. For $\zeta_1 \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, we can find a unique function $p \in \mathcal{P}$ with c_1 as in (4.1),

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \quad z \in \mathbb{D}. \tag{4.4}$$

For $\xi_1 \in \mathbb{D}$ and $\xi_2 \in \mathbb{T}$, we can find a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (4.1) and (4.2),

$$p(z) = \frac{1 + (\overline{\xi_1}\xi_2 + \xi_1)z + \xi_2 z^2}{1 + (\overline{\xi_1}\xi_2 - \xi_1)z - \xi_2 z^2}, \quad z \in \mathbb{D}. \tag{4.5}$$

For $\xi_1, \xi_2 \in \mathbb{D}$ and $\xi_3 \in \mathbb{T}$, we can find a unique function $p \in \mathcal{P}$ with c_1, c_2 and c_3 as in (4.1) to (4.3),

$$p(z) = \frac{1 + (\xi_2\xi_3 + \xi_1\xi_2 + \xi_1)z + (\xi_1\xi_3 + \xi_1\xi_2\xi_3 + \xi_2)z^2 + \xi_3 z^3}{1 + (\xi_2\xi_3 + \xi_1\xi_2 - \xi_1)z + (\xi_1\xi_3 - \xi_1\xi_2\xi_3 - \xi_2)z^2 - \xi_3 z^3}, \quad z \in \mathbb{D}. \tag{4.6}$$

The following lemma is due to Choi et al. [16].

Lemma 4.1.3. [16] Let A, B, C be any real numbers and

$$Y(A,B,C) = \max\left\{|A + Bz + Cz^2| + 1 - |z|^2 : z \in \overline{\mathbb{D}}\right\}.$$

(i) If $AC \geq 0$, then

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1-|C|), \\ 1 + |A| + \frac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

(ii) If $AC < 0$, then

$$Y(A,B,C) = \begin{cases} 1 - |A| + \frac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \leq B^2 \wedge |B| < 2(1-|C|), \\ 1 + |A| + \frac{B^2}{4(1+|C|)}, & B^2 < \min\{4(1+|C|)^2, -4AC(C^{-2}-1)\}, \\ R(A,B,C), & \text{otherwise}, \end{cases}$$

where

$$R(A,B,C) = \begin{cases} |A| + |B| - |C|, & |C|(|B| + 4|A|) \leq |AB|, \\ -|A| + |B| + |C|, & |AB| \leq |C|(|B| - 4|A|), \\ (|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}}, & \text{otherwise}. \end{cases}$$

The following lemma is due to [15].

Lemma 4.1.4. [15] Let $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ be a Schwarz function. Then

$$|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2, \quad \text{and} \quad |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}.$$

4.2 Main Results

This section is devoted to deriving bounds for the Hankel and as well as Toeplitz determinants formed from the logarithmic coefficients of starlike functions associated with a non-convex domain, represented by the class $\mathcal{S}^*_{nc}$ (see [20]). These determinants play a significant role in analyzing the geometric characteristics and the coefficient structure of functions belonging to thisclass.

Theorem 4.2.1. Let $f \in \mathcal{S}^*_{nc}$, then

$$|H_{2,1}(F_f/2)| \leq \frac{1}{16}.$$

This is best possible result.

Proof. Consider a function $f \in \mathcal{S}^*_{nc}$. By definition, there will be an analytic Schwarz function $w(z)$, such that

$$\frac{zf'(z)}{f(z)} = \frac{1+w(z)}{\cos w(z)}, \quad z \in \mathbb{D}.$$

Define the function $p(z)$ via the Möbius transformation:

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{4.7}$$

Inverting the transformation, we express $w(z)$ in terms of the coefficients c_n as follows:

$$\begin{aligned} w(z) = \frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}c_1^2\right) z^2 + \frac{1}{2}\left(\frac{1}{4}c_1^3 - c_1c_2 + c_3\right) z^3 \\ + \frac{1}{2}\left(c_4 - c_1c_3 - \frac{1}{2}c_2^2 - \frac{1}{8}c_1^4 + \frac{3}{4}c_1^2c_2\right) z^4 + \cdots . \end{aligned} \tag{4.8}$$

Now, recall the logarithmic derivative expansion of f:

$$\begin{aligned} \frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_2a_3 + 3a_4)z^3 \\ + (4a_5 - a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2)z^4 + \cdots . \end{aligned} \tag{4.9}$$

We also expand the other side of the identity involving $w(z)$ using the Taylor series:

$$\frac{1+w(z)}{\cos w(z)} = 1 + \frac{c_1 z}{2} + \left(\frac{c_2}{2} - \frac{c_1^2}{8}\right) z^2 + \frac{1}{16}\left(c_1^3 - 4c_1c_2 + 8c_3\right) z^3 \\ + \frac{1}{384}\left(-19c_1^4 + 72c_1^2c_2 - 48c_2^2 - 96c_1c_3 + 192c_4\right) z^4 + \cdots . \tag{4.10}$$

Comparing the coefficients of z^n from equations (4.9) and (4.10), we identify the following relationships:

$$a_2 = \frac{c_1}{2}, \quad a_3 = \frac{1}{16}\left(c_1^2 + 4c_2\right), \quad a_4 = \frac{1}{96}\left(c_1^3 + 4c_1c_2 + 16c_3\right). \tag{4.11}$$

Now consider the 2^{nd} order Hankel determinant for the function f. Using the definition and expressions from (4.11), we get:

$$\left|\delta_1\delta_3 - \delta_2^2\right| = \frac{1}{3072}\left|c_1^4 + 8c_1^2c_2 - 48c_2^2 + 32c_1c_3\right|. \tag{4.12}$$

Because $H_{2,1}(F_f/2)$ is invariant under rotation, we may take $c_1 = c \in [0,2]$ without loss of generality. We express the coefficients c_2 and c_3 in terms of c, a real parameter $x \in \mathbb{C}$, and $\delta \in \mathbb{C}$ with $|x| \leq 1$, $|\delta| \leq 1$, using known coefficient bounds for Carathéodory functions:

$$c_2 = \frac{1}{2}(4-c^2)x, \quad c_3 = \frac{1}{4}(4-c^2)(1-|x|^2)\delta.$$

Substituting into equation (4.12) and denoting $m = |x|$, we obtain:

$$\left|\delta_1\delta_3 - \delta_2^2\right| \leq \frac{1}{3072}\Big[c^4 + 8c^2m^2(4-c^2) + 12m^2(4-c^2)^2 + 4c^2m(4-c^2) \\ + 16c(1-m^2)(4-c^2)\Big] = P(c,m).$$

To find the maximum of $P(c,m)$, we differentiate it with respect to m and analyze:

$$\frac{\partial P}{\partial m} = \frac{1}{3072} \cdot 4(4-c^2)\left(c^2 + 6m(4-c^2) + 16c^2m - 32cm\right).$$

As $c \in [0,2]$, it is straightforward to show that $\frac{\partial P}{\partial m} \geq 0$ for all $m \in [0,1]$. Therefore,

$P(c,m)$ attains its maximum at $m=1$, yielding:

$$P(c,1) = \frac{1}{3072}\left[c^4 + 12c^2(4-c^2) + 12(4-c^2)^2\right] = \phi(c).$$

Differentiating $\phi(c)$, we find $\phi'(c) \leq 0$ for $c \in [0,2]$, so $\phi(c)$ is a non increasing function. Hence, the greatest value occurs at $c = 0$, giving:

$$\left|H_{2,1}(F_f/2)\right| \leq \phi(0) = \frac{192}{3072} = \frac{1}{16}.$$

Finally, this bound is sharp. The extremal function achieving equality is given by:

$$f_1(z) = z\exp\left(\int_0^z \frac{(1+t^2) - \cos t^2}{t\cos t^2}\,dt\right) = z + \frac{1}{2}z^3 + \frac{1}{4}z^5 + \cdots.$$

With the help of this function, we can write

$$a_2 = 0, \quad a_3 = \frac{1}{2}, \quad a_4 = 0 \quad \Rightarrow \quad H_{2,1}(F_{f_1}/2) = \frac{1}{16}.$$

With this, the proof is done. □

Theorem 4.2.2. [50] Let $f \in \mathcal{S}^*_{nc}$. Then

$$\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| \leq \frac{17}{192}.$$

This is the best possible result.

Proof. For $f \in \mathcal{S}^*_{nc}$, there will be a Schwarz function $w(z)$ which implies that

$$\frac{zf'(z)}{f(z)} = \frac{1+w(z)}{\cos w(z)}, \quad z \in \mathbb{D}.$$

Using equation (4.9), we have:

$$a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{16}(c_1^2 + 4c_2), \quad a_4 = \frac{1}{96}(16c_3 + 4c_1c_2 + c_1^3). \tag{4.13}$$

Substituting into the Hankel determinant $H_{2,1}(F_{f^{-1}}/2)$, we get:

$$H_{2,1}\left(F_{f^{-1}}/2\right) = \frac{1}{48}\left(13a_2^4 - 12a_2^2a_3 - 12a_3^2 + 12a_2a_4\right)$$
$$= \frac{1}{3072}(41c_1^4 - 56c_1^2c_2 - 48c_2^2 + 64c_1c_3). \tag{4.14}$$

Applying Lemma (4.1.2), we express this in terms of pre-Schwarz function parameters:

$$H_{2,1}\left(F_{f^{-1}}/2\right) = \frac{1}{192}\Big(17\zeta_1^4 - 20(1-\zeta_1^2)\zeta_1^2\zeta_2 - 4(1-\zeta_1^2)(3+\zeta_1^2)\zeta_2^2 + 16\zeta_1(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3\Big). \tag{4.15}$$

We analyze this expression based on the value of $\zeta_1 \in [0,1]$ [50].

Case I: $\zeta_1 = 1$. Then (4.15) yields

$$\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| = \frac{17}{192}.$$

Case II: $\zeta_1 = 0$. Then from (4.15),

$$\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| = \frac{1}{16}|\zeta_2|^2 \leq \frac{1}{16}.$$

Case III: $\zeta_1 \in (0,1)$. Implementing the triangle inequality to (4.15) and taking $|\zeta_3| \leq 1$, we get:

$$\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| \leq \frac{1}{192}\left|17\zeta_1^4 - 20(1-\zeta_1^2)\zeta_1^2\zeta_2 - 4(1-\zeta_1^2)(3+\zeta_1^2)\zeta_2^2 + 16\zeta_1(1-\zeta_1^2)(1-|\zeta_2|^2)\right|$$
$$= \frac{1}{12}\zeta_1(1-\zeta_1^2)\left(|A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2\right),$$

where

$$A = \frac{17\zeta_1^3}{192(1-\zeta_1^2)}, \quad B = -\frac{20\zeta_1}{192}, \quad C = -\frac{4(3+\zeta_1^2)}{192\zeta_1}.$$

Since $AC < 0$, Lemma (4.1.3)(ii) is applicable. We now verify its conditions:

(a) The inequality

$$-4AC\left(\frac{1}{C^2} - 1\right) - B^2 \leq 0$$

is equivalent to

$$\frac{\xi_1^2(-57+9779\xi_1^2+2\xi_1^4)}{576(-3+2\xi_1^2+\xi_1^4)} \le 0,$$

which holds for $\xi_1 \in (0,1)$. However, $|B| < 2(1-|C|)$ fails, so we move to the next condition.

(b) The inequality

$$B^2 < \min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2}-1\right)\right\}$$

is also violated for $\xi_1 \in (0,1)$.

(c) The inequality

$$|C|(|B|+4|A|) - |AB| \le 0$$

reduces to

$$60 - 124\xi_1^2 - 133\xi_1^4 \le 0,$$

which is false in $(0,1)$.

(d) The inequality

$$|AB| - |C|(|B|-4|A|) \le 0$$

reduces to

$$690\xi_1^4 + 1048\xi_1^2 - 240 \le 0,$$

which is satisfied for $0 < \xi_1 \le \xi_1' \approx 0.449569$. In this subinterval, using Lemma (4.1.2), we obtain:

$$|H_{2,1}(F_{f^{-1}}/2)| \le \frac{1}{12}\xi_1(1-\xi_1^2)(-|A|+|B|+|C|),$$

which simplifies to:

$$\varphi(\xi_1) = \frac{1}{2304}(12+24\xi_1^2-41\xi_1^4). \tag{4.16}$$

Define $\varphi(t) = \frac{1}{2304}(12+24t^2-41t^4)$. It attains its maximum in $(0,\xi_1']$ at $t_0 = \sqrt{12/41}$, hence

$$\varphi(\xi_1) \le \varphi(t_0) = \frac{53}{7872} \approx 0.006732 < \frac{17}{192}.$$

(e) For $\xi_1' < \xi_1 < 1$, again using Lemma (4.1.2), we get:

$$\Psi(\xi_1) = \frac{12 - 8\xi_1^2 + 15\xi_1^4}{2304}\sqrt{\frac{76 - 8\xi_1^2}{17(3+\xi_1^2)}}. \tag{4.17}$$

Since

$$\Psi'(t) < 0 \quad \text{for } \xi_1' < t < 1,$$

the function is decreasing, and

$$|H_{2,1}(F_{f^{-1}}/2)| \leq \Psi(\xi_1) \leq \Psi(\xi_1') \approx 0.0055788 < \frac{17}{192}.$$

Combining all cases, the inequality holds for all $f \in \mathcal{S}^*_{nc}$. For sharp bound, we will take the following function

$$f_2(z) = z\exp\left(\int_0^z \frac{1+t-\cos t}{t\cos t}dt\right) = z + z^2 + \frac{3}{4}z^3 + \frac{7}{12}z^4 + \frac{35}{96}z^5 + \cdots$$

A direct computation shows

$$\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| \leq \frac{17}{192},$$

proving that the inequality is sharp. □

Theorem 4.2.3. Let $f \in \mathcal{S}^*_{nc}$. Then

$$\left|T_{2,1}\left(F_{f^{-1}}/2\right)\right| \leq \frac{89}{1024}.$$

This bound is sharp.

Proof. Suppose $f \in \mathcal{S}^*_{nc}$ then, from equations (4.9) and (4.10), we obtain

$$a_2 = \frac{c_1}{2}, \quad a_3 = \frac{1}{16}(4c_2 + c_1^2). \tag{4.18}$$

Substituting these into the formula for the Toeplitz determinant $T_{2,1}(F_{f^{-1}}/2)$, we get

$$T_{2,1}\left(F_{f^{-1}}/2\right) = \frac{1}{16}\left(-9a_2^4 + 4a_2^2 - 4a_3^2 + 12a_2^2a_3\right)$$
$$= \frac{1}{1024}\left(-25c_1^4 + 64c_1^2 + 40c_1^2c_2 - 16c_2^2\right). \tag{4.19}$$

Applying the triangle inequality to (4.19) yields

$$1024\left|T_{2,1}\left(F_{f^{-1}}/2\right)\right| \leq 25|c_1|^4 + 64|c_1|^2 + 40|c_1|^2|c_2| + 16|c_2|^2. \tag{4.20}$$

Letting $x = |c_1|$ and $y = |c_2|$, we define the function

$$Q(x,y) = 25x^4 + 64x^2 + 40x^2y + 16y^2.$$

Then inequality (4.20) becomes

$$1024\left|T_{2,1}\left(\frac{F_{f^{-1}}}{2}\right)\right| \leq Q(x,y). \tag{4.21}$$

According to Lemma 4.1.4, the admissible region for (x,y) is

$$\varrho = \left\{(x,y) \in [0,1]^2 : y \leq 1 - x^2\right\}.$$

We now seek the greatest value of $Q(x,y)$ in this area.
First, compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = 100x^3 + 128x + 80xy, \quad \frac{\partial Q}{\partial y} = 32y + 40x^2.$$

Setting both derivatives to zero leads to no critical points within the interior of ϱ. Therefore, the greatest must occur along the boundary of the region.
We examine three cases: Along $y = 0$, we have

$$Q(x,0) = 25x^4 + 64x^2 \leq Q(1,0) = 89.$$

Along $x = 0$, we get

$$Q(0,y) = 16y^2 \leq 16.$$

Along the curve $y = 1 - x^2$, we compute

$$Q(x, 1 - x^2) = 25x^4 + 72x^2 - 40x^4 + 16.$$

This expression simplifies to

$$Q(x, 1 - x^2) = -15x^4 + 72x^2 + 16.$$

By checking values numerically or through calculus, we verify that

$$Q(x, 1 - x^2) \leq 73 \quad \text{for } x \in [0, 1].$$

Thus, the greatest value of $Q(x, y)$ over ϱ is 89, attained at $(x, y) = (1, 0)$. From (4.21), it follows that

$$\left| T_{2,1}\left(F_{f^{-1}}/2\right) \right| \leq \frac{89}{1024}.$$

To show sharpness, consider the function

$$f_3(z) = z \exp\left(\int_0^z \frac{\sqrt{89}(1 + t^2) - \sqrt{89}\cos(t^2)}{8t\cos(t^2)}\, dt \right) = z + \frac{\sqrt{89}}{16} z^3 + \cdots.$$

A direct calculation shows that this function satisfies

$$\left| T_{2,1}\left(F_{f_3^{-1}}/2\right) \right| = \frac{89}{1024}.$$

Hence, the bound is sharp. □

Theorem 4.2.4. Let $f \in \mathcal{S}^*_{nc}$. Then

$$\left| T_{2,1}\left(F_f/2\right) \right| \leq \frac{65}{1024}. \tag{4.22}$$

This bound is sharp.

Proof. Given that $f \in \mathcal{S}^*_{nc}$, we use the coefficient relationships derived previously:

$$a_2 = \frac{c_1}{2}, \quad a_3 = \frac{1}{16}(4c_2 + c_1^2). \tag{4.23}$$

Substituting these into the expression for $T_{2,1}(F_f/2)$, and after simplify we obtain

$$1024(\delta_1^2 - \delta_2^2) = 64c_1^2 - 16c_2^2 + 8c_1^2c_2 - c_1^4.$$

Now implementing the triangle inequality with Lemma 4.1.4, we estimate each term as follows:

$$\begin{aligned} 1024|\delta_1^2 - \delta_2^2| &\leq |c_1|^4 + 64|c_1|^2 + 8|c_1|^2|c_2| + 16|c_2|^2 \\ &\leq \xi^4 + 64\xi^2 + 8\xi^2(1-\xi^2) + 16(1-\xi^2)^2, \end{aligned} \tag{4.24}$$

where $\xi = |c_1|$, and the bounds $|c_2| \leq 1 - \xi^2$ and $\xi \in [0,1]$ follow from Lemma 4.1.4. Simplifying the (4.24),

$$1024|\delta_1^2 - \delta_2^2| \leq \xi^4 + 64\xi^2 + 8\xi^2(1-\xi^2) + 16(1 - 2\xi^2 + \xi^4) = \xi^4 + 64.$$

Since $\xi \in [0,1]$, we clearly have $\xi^4 + 64 \leq 65$. Thus,

$$\left|T_{2,1}\left(F_f/2\right)\right| \leq \frac{65}{1024}.$$

For the sharp bound, we construct a function for which equality is attained:

$$f_4(z) = z \exp\left(\int_0^z \frac{\sqrt{65}(1+t^2) - \sqrt{65}\cos(t^2)}{8t\cos(t^2)}\, dt\right) = z + \frac{\sqrt{65}}{16}z^3 + \cdots.$$

A direct computation verifies that for this function,

$$\left|T_{2,1}\left(F_{f_4}/2\right)\right| = \frac{65}{1024},$$

which confirms the sharpness of the inequality. □

Chapter 5

Conclusion

This chapter wraps up the research conducted in this dissertation and points out some interesting directions for future work. Our primary goal was to understand the variation in bounds of the Hankel determinant. This specific problem becomes particularly complex when its entries are derived from logarithmic and inverse logarithmic coefficients, especially when dealing with functions defined on a non-convex domain. To tackle this, we relied on the well-established methodology of coefficient problems and ingeniously leveraged known results from the Carathéodory class, a fundamental concept in gft. The detailed findings and the methodologies employed to arrive at these conclusions are thoroughly presented and discussed in Chapter 4 of this dissertation.

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Some Coefficient Problems for A Class of Starlike Functions

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