NUMERICAL SOLUTION OF 1D CONVECTION-DIFFUSION EQUATION USING IMPLICIT EULER METHOD ON A NON-UNIFORM MESH

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in Partial Fulfillment of the Requirements for the Degree of

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APPLIED MATHEMATICS

by

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Certified that **Akriti Patel** (2K23/MSCMAT/05) and **Manish Sen** (2K23/MSCMAT/31) have carried out their research work presented in this thesis entitled **"Numerical Solution of 1D Convection-Diffusion Equation Using Implicit Euler Method On A Non-Uniform Mesh"** for the award of **Master of Science** from the **Department of Applied Mathematics, Delhi Technological University, Delhi**, under my supervision.

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ABSTRACT

This thesis focuses on the numerical solution of the one-dimensional convection-diffusion equation using implicit Euler Method implemented on a non-uniform mesh. The convection-diffusion equation is a fundamental partial differential equation that arises in various physical and engineering problems involving the transport of mass, heat, or momentum. Accurately solving this equation, particularly in convection dominated regimes, presents significant numerical challenges such as artificial oscillations and smearing near steep gradients or boundary layers. To address these issues, a non-uniform mesh is employed to provide higher resolution in regions with rapid variations in the solution, while maintaining coarser discretization where the solution is smoother. An implicit Euler Method is adapted to accommodate variable grid spacing, ensuring enhanced accuracy in both convection and diffusion terms. The resulting system of algebraic equations is solved using appropriate numerical solvers. Comparative analysis with uniform mesh solutions demonstrates that the non-uniform mesh approach significantly improves accuracy and stability, especially in capturing sharp solution features with fewer grid points. The findings of this work contribute to the development of efficient and accurate numerical techniques for solving convection-diffusion problems encountered in scientific computing and engineering applications.

Keywords: Convection-Diffusion Equation, Second-Order Finite Difference, Non-Uniform Mesh, Implicit Euler Method, Numerical Stability, Boundary Layer Resolution

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Chapter 1

Convection-Diffusion Problems

Convection-diffusion problems are fundamental in modeling the transport of physical quantities such as heat, mass, or chemical species within a system. These equations combine the effects of diffusion, which causes spreading due to random motion, and convection, which represents directed transport by a flow.

1.1 1D Convection-Diffusion Equation

The general form of the one-dimensional steady-state convection-diffusion equation is given by [32][45]

$$-\varepsilon\, u''(x) + a(x)\, u'(x) + b(x)\, u(x) = f(x), \quad x \in (0,1), \tag{1.1}$$

subject to the boundary conditions

$$u(0) = \alpha, \quad u(1) = \beta. \tag{1.2}$$

Here, ε denotes the diffusion coefficient, $a(x)$ represents the convection velocity, $b(x)$ is a reaction term, and $f(x)$ is a source function, α and β are constants[2].

The convection-diffusion equation is fundamental in various scientific and engineering disciplines, as it describes the simultaneous processes of transport and dispersion. For example, engineers apply this equation to forecast heat transfer in buildings or the spread of contaminants in water bodies and the air. The equation is also significant in biology and medicine, where it explains the movement of nutrients or medications within tissues[32]. In hydrology, it helps to the flow of water and dissolved materials through soil and groundwater systems. Due to its ability to represent both movement and diffusion, the convection-diffusion equation is a crucial tool for addressing practical challenges in environmental science, engineering, and healthcare[26].

1.2 Problem Statement

Modeling the transport of heat or mass in one-dimensional systems is essential for many scientific purposes, and the convection-diffusion equation is a standard tool for this. However, standard numerical methods on uniform grids often fail, producing unwanted oscillations and failing to resolve boundary layers in situations where convection dominates diffusion. This leads to inaccurate predictions, which can compromise the design and analysis of engineering systems. Existing approaches that use uniform grids are computationally inefficient, as they require a large number of grid points to achieve acceptable accuracy near sharp gradients[45]. There is a need for robust numerical methods that can efficiently and accurately capture these features without excessive computational cost.

1.3 Research Objective

In this research, we aim to solve the one dimensional convection-diffusion equation using Implicit Euler Method on a non uniform mesh and compare the performance with traditional uniform mesh approaches in terms of computational efficiency and error. Also, we identify practical considerations for implementing non-uniform mesh techniques in real-world applications.

Chapter 2

Perturbation Theory

Differential equations are essential in modeling a wide array of physical phenomena in engineering and science. By relating a variable's rate of change to the variable itself, they enable us to analyze and predict system behavior over time and space. Their versatility makes them indispensable across scientific and technological fields.

Perturbation theory provides approximate solutions to differential equations that cannot be solved exactly. Consider:

$$D_{\varepsilon} = \varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = 0, \tag{2.1}$$

where ε is a small parameter. Setting $\varepsilon = 0$ simplifies the equation, often making it solvable. Perturbation theory aims to construct solutions to the original equation by treating ε as a small perturbation[26].

2.1 Types of Perturbation Problem

There are two main types[29][42]:

1. **Regularly Perturbed Differential Equation**
2. **Singularly Perturbed Differential Equation**

2.2 Regularly Perturbed Differential Equation

In differential equations, a perturbation problem, represented as D_ε, refers to a situation where the highest order term is accompanied by a small parameter ε.

If, as $\varepsilon \to 0$, the solution of D_ε approaches the solution of the reduced problem D_0 uniformly—where D_0 is formed by setting $\varepsilon = 0$ in the original equation—then D_ε is called a **regularly perturbed differential equation**.

2.2.1 Example

$$D_\varepsilon = u''(x) - 2\varepsilon u'(x) + u(x) = 1, \quad u(0) = 0, \quad u(1) = 0 \tag{2.2}$$

Actual Solution[14]

$$u(x) = c_1 e^{\left(\varepsilon+\sqrt{\varepsilon^2+1}\right)x} + c_2 e^{\left(\varepsilon-\sqrt{\varepsilon^2+1}\right)x} - 1 \tag{2.3}$$

where,

$$c_1 = \frac{1 - e^{\varepsilon-\sqrt{\varepsilon^2+1}}}{e^{\varepsilon+\sqrt{\varepsilon^2+1}} - e^{\varepsilon-\sqrt{\varepsilon^2+1}}},$$
$$c_2 = 1 - c_1$$

Reduced Problem as $\varepsilon \to 0$

$$D_0 = u''(x) - u(x) = 1 \tag{2.4}$$

Actual Solution of (2.4)

$$u(x) = c_1 e^x + c_2 e^{-x} - 1 \tag{2.5}$$

where,

$$c_1 = \frac{1 - e^{-1}}{e - e^{-1}},$$
$$c_2 = 1 - c_1$$

Hence, D_ε **uniformly converges** to D_0 as $\varepsilon \to 0$.

2.3 Singularly Perturbed Differential Equation

The perturbation problem is called **singularly perturbed** if the solution of D_ε as $\varepsilon \to 0$ does not converge uniformly to the solution of the reduced problem D_0, which is achieved by setting $\varepsilon = 0$ in the perturbation problem D_ε. This breakdown of a singularly perturbed problem is limited to short time intervals or restricted space intervals. The solution quickly transforms and separates into layers in these confined areas. These areas are commonly known as boundary layers.

This study examines a **singularly perturbed differential-difference equation** that involves a small positive parameter ε ($0 < \varepsilon \ll 1$). When ε approaches zero, the solution exhibits sharp variations in confined regions known as boundary layers, which are typically not captured well by standard numerical techniques. As a result, specialized methods like the finite difference approach are employed to analyze the solution's behavior in these areas.

[38]Generally, a singularly perturbed differential difference equation is of the form

$$D_\varepsilon = -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x) \tag{2.6}$$

along with boundary conditions:

$$u(0) = 0, \quad u(1) = 0$$

[16]Here,if $a(x) = 0$ and $b(x) \neq 0$, then the equation becomes a **Reaction-Diffusion** problem. On the other hand, if $b(x) = 0$ and $a(x) \neq 0$, it becomes a **Convection-Diffusion** problem.

Chapter 3

Numerical Methods For Singular Perturbation Problems

When closed-form solutions are not feasible, numerical methods are used to obtain an approximate solution. These methods yield quantitative insights and are designed to address a wide range of issues. Unlike asymptotic methods, the quantitative nature of numerical approaches often leads to qualitatively different results.

In recent decades, various numerical strategies have been developed to address singular perturbation problems. These methods can be broadly categorized into computational methods and parameter-uniform numerical methods. Traditional computational methods such as the finite difference method often perform inadequately on uniform meshes and require a significantly large number of mesh points to achieve accurate results when the perturbation parameter is very small.

Sharp gradients or boundary layers in the analytical solution cause this limitation.[26]. The classical methods cannot reduce the maximum pointwise error unless the mesh size is comparable to the singular perturbation parameter [45]. However, refining the mesh to match the scale of the parameter greatly increases computational cost and processing time. Thus, a major drawback of these computational techniques is their dependence on the perturbation parameter for domain discretization. To overcome this, it is advantageous to develop robust computational approaches whose convergence rate, error, and discretization are independent of the perturbation parameter. Such methods are known as parameter-uniform numerical techniques. These can generally be divided into two main types: fitted mesh methods and fitted finite difference operators.

3.1 Standard Finite Difference Scheme

In some cases, it is difficult to find the analytic solution of the problem. Therefore, the finite difference method is a numerical technique used to find approximate solutions of differential equations. In the finite difference method, the derivative terms in the differential equation are replaced with approximated finite difference formulas. These approximations transform the differential equation into a system of algebraic equations. This system of algebraic equations can be written as $AU = B$, where A is a tridiagonal matrix and U is the set of solutions of the equation[45].

The error that arises when a differential operator is converted to a difference operator determines the discrepancy between the exact and numerical answers. This type of error is called a “truncation error” or a “discretization error”[32].

Let us consider the singularly perturbed problem:

$$-\varepsilon u''(x) + b(x)u(x) = f(x), \quad x \in (0,1) \tag{3.1}$$

with boundary conditions:

$$u(0) = 0, \quad u(1) = 0 \tag{3.2}$$

where $b(x) > 0$, $\varepsilon <<<< 1$.[34]

Assume that $b(x)\, and\, f(x)$ lie in to interval $[0,1]$. Divide the interval $[0,1]$ into n subintervals with an equidistant with mesh with mesh size $h = 1/N$ and grid points $x_i = a + (i-1)h$, $i = 1,2,\ldots,N+1$.

The central difference approximations are:

$$u''(x_i) \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \tag{3.3}$$

$$u'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h} \tag{3.4}$$

Substituting into the differential equation, for $i = 1,2,\ldots,N$:

$$u_{i-1}(-\varepsilon) + u_i(2\varepsilon + h^2 b) + u_{i+1}(-\varepsilon) = h^2 f(x) \tag{3.5}$$

This can be written in matrix form as $A\mathbf{U} = \mathbf{B}$, where:

$$A = \begin{bmatrix} 1 & 0 & & & \\ -\varepsilon & 2\varepsilon + h^2 b & -\varepsilon & & \\ 0 & -\varepsilon & 2\varepsilon + h^2 b & -\varepsilon & \\ & \ddots & \ddots & \ddots & \\ & & -\varepsilon & 2\varepsilon + h^2 b & -\varepsilon \\ & & & 0 & 1 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n+1} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} h^2 f(x_1) \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_{n+1}) \end{bmatrix}$$

3.1.1 Upwind Finite Difference Scheme

The Upwind Finite Difference Scheme is generally used to avoid unnecessary oscillations in the obtained solutions[12]. Upwinding occurs when the one-sided difference is taken on the side away from the layer[30], that is,

$$u'(x) \approx \frac{u_i - u_{i-1}}{h}$$

The difference between the standard finite difference method using forward difference approximation with a uniform mesh and the standard finite difference method using the upwind scheme can be seen by the graphs given.

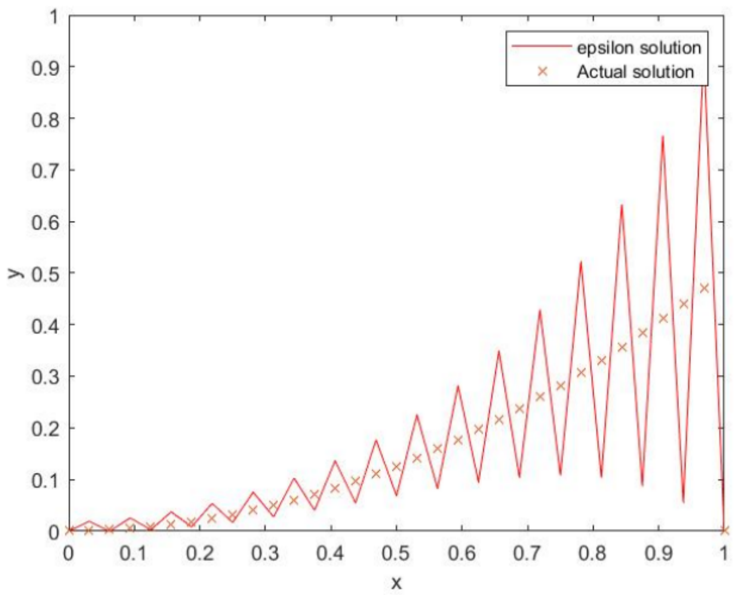

Figure 3.1: Finite Difference Method with Forward Difference

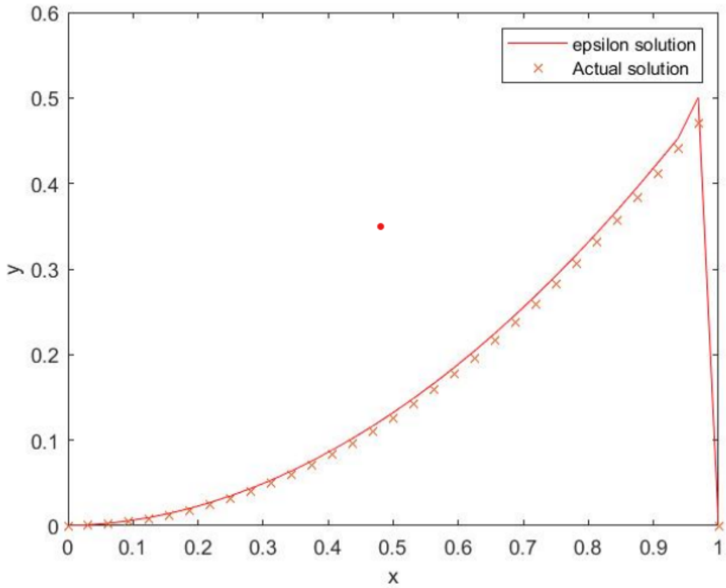

Figure 3.2: Finite Difference Method with Upwind Scheme

3.2 Fitted Mesh Finite Difference Method

3.2.1 Piecewise Uniform Shishkin Mesh

The uniform mesh, given by $x_i = a + (i-1) \cdot h$, typically lacks the ability to accurately capture the dynamics within boundary layers when dealing with singularly perturbed problems. Russian mathematician G.I. Shishkin gave a piecewise uniform mesh called *shishkin mesh*.The width of Shishkin mesh can be adjust by the nature of solution.

The mesh spacing is always chosen in such a way that the layer region get maximum number of solution points and more the number of solution points in the layer region more final will be the region of interest can be studied. Shishkin mesh is generally used when the solution exhibits sharper edges and this strategy of using piecewise uniform mesh helps to get the important features in the layer region of the solution.

In this part, a fitted mesh finite difference method is is employed, comprising of a conventional upwind finite difference operator applied to a piecewise uniform mesh that condenses at the boundary points$x = 0$ and $x = 1$ to discretize the boundary value problems (3.1) and (3.2)[41].

To construct the fitted piecewise-uniform mesh $\overline{\xi}^N$ over the interval $[0,1]$, the domain is divided into three subintervals: $(0,\lambda)$, $(\lambda, 1-\lambda)$, and $(1-\lambda, 1)$.Each subinterval is assigned a uniform mesh: $\frac{N}{4}+1$ equally spaced points are placed in both $(0,\lambda)$ and $(1-\lambda, 1)$, while $\frac{N}{2}$ points are distributed uniformly in $(\lambda, 1-\lambda)$[8] .

A crucial aspect of the Shishkin mesh is the *transition parameter* λ, which determines the piecewise uniform mesh that is produced by $\lambda = \min\left\{\frac{1}{4}, \left(\frac{2}{\alpha}\right)\varepsilon \log N\right\}$, where α is a problem-dependent constant and ε is the small perturbation parameter. In order to ensure that there is at least one point in the boundary layer, we assume that $N = 2^r$ with $r \geq 3$.

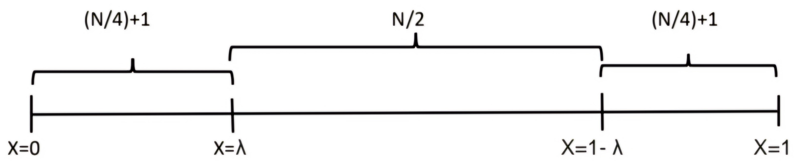

Figure 3.3: Interval Distribution of Shishkin Mesh for Convection-Diffusion

The difference in mesh spacing between standard finite difference method with uniform mesh and fitted finite difference scheme under piecewise uniform mesh is shown by the figures below

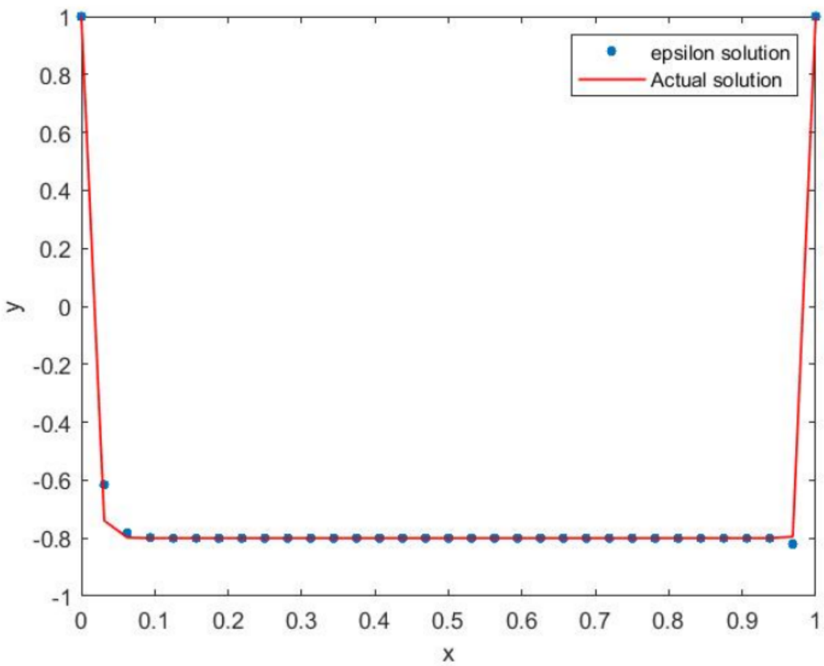

Figure 3.4: Mesh spacing under Standard Finite Difference Method

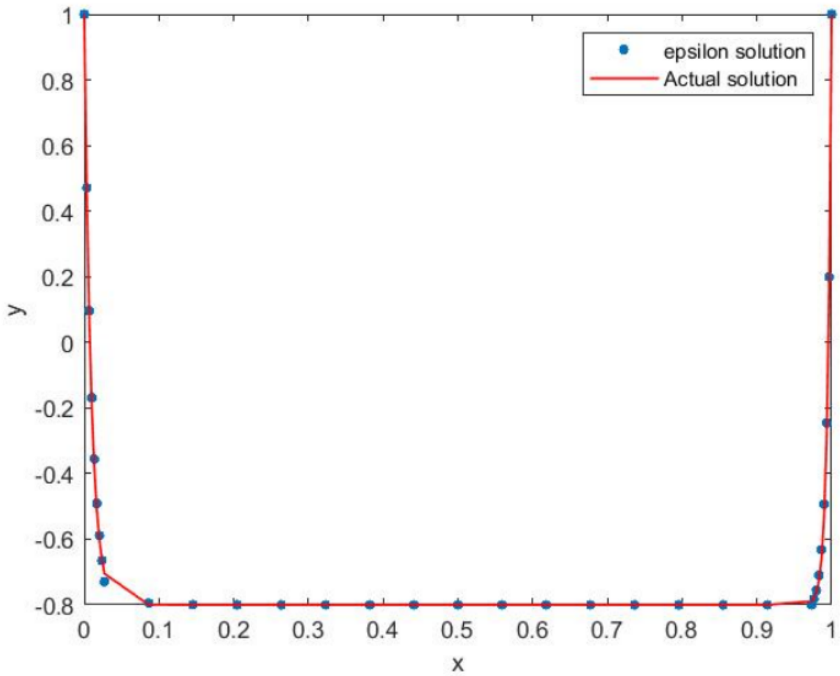

Figure 3.5: Mesh spacing under Shishkin Mesh

3.3 Euler Method

The Euler method is a very simple method for numerically solving Ordinary Differential Equations and works by using the slope at a known point to estimate the value of the solution at the next point, essentially following the tangent to the curve in small steps[50]. Given an equation of the form $u'(x) = f(x,u)$ with an initial condition $u(x_0) = u_0$, the method starts at this initial point and, using a chosen step size h, repeatedly applies the formula:

$$u_{n+1} = u_n + h \cdot f(x_n, u_n)$$

This process generates a sequence of approximate values for u at discrete points along the x-axis. While the method is straightforward and easy to implement, it is not always highly accurate compared to more advanced techniques, but its simplicity makes it a valuable starting point for understanding numerical solutions to differential equations[45][32].

3.3.1 Explicit Euler Method

The explicit Euler method estimates the next value of the solution using information from the current step. For an ODE of the form $u'(x) = f(x,u)$ with an initial value $u(x_0) = u_0$, the explicit Euler update rule is:

$$u_{n+1} = u_n + h \cdot f(u_n, x_n)$$

where h is the chosen step size. This method is easy to implement and computationally efficient[32]. However, it can become unstable if the step size is too large, especially for problems where the solution changes rapidly.

3.3.2 Implicit Euler Method

The implicit Euler method uses the unknown next value in its calculation:

$$u_{n+1} = u_n + h \cdot f(u_{n+1}, x_{n+1})$$

This means that at each step, we need to solve an equation (which may be nonlinear) to find u_{n+1}. While this requires more computational effort, the implicit method is much more stable and is particularly useful for stiff equations, where the explicit method would require very small step sizes to maintain stability[45].

3.3.3 Example

To see how the Euler methods work in practice, let's look at a simple example. Suppose we have the differential equation $u'(x) = -2u$ with the initial value $u(0) = 1$. This equation describes a situation where the amount of u decreases over time at a rate proportional to its current value.

If we use the explicit Euler method, we estimate the next value of u using the current value. The formula becomes:

$$u_{n+1} = u_n + h \cdot (-2u_n) = u_n(1-2h)$$

So, at each step, we just multiply the current value by $(1-2h)$ to get the next one.

With the implicit Euler method, things are a bit different because the next value appears on both sides of the equation. The update rule is:

$$u_{n+1} = u_n + h \cdot (-2u_{n+1})$$

If we rearrange this, we get:

$$u_{n+1}(1+2h) = u_n \quad \Rightarrow \quad u_{n+1} = \frac{u_n}{1+2h}$$

Here, we need to do a little more math at each step, but this approach is much more stable, especially if we use a larger step size.

This example shows the main difference between the two methods: the explicit method is simpler and faster, but the implicit method is better at handling problems where the solution can change quickly or become unstable.

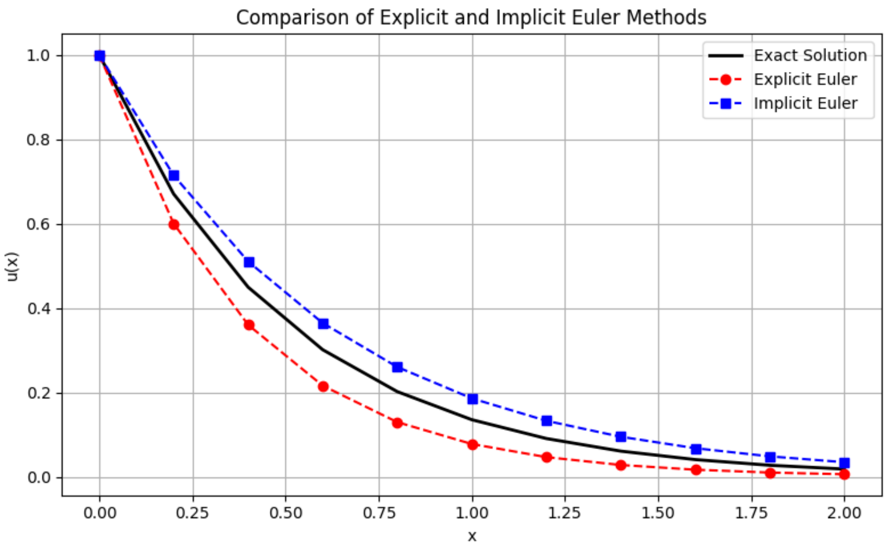

Figure 3.6: Comparison of Explicit and Implicit Euler Methods

3.4 Advantages of the Implicit Euler Method in Convection-Diffusion Problems

The implicit Euler method offers several notable advantages when applied to convection-diffusion problems. Most importantly, it provides enhanced numerical stability, allowing for the use of larger time steps without compromising the reliability of the solution. This is particularly beneficial in scenarios where the equations are stiff or where diffusion effects are significant.

Additionally, the implicit Euler method is well-suited for efficiently approaching steady-state solutions, as its inherent damping properties help the system converge more rapidly. It is also highly adaptable, making it effective for problems involving non-uniform computational grids or nonlinearities. As a result, the implicit Euler method is a robust and versatile choice for a wide range of convection-diffusion applications, especially when stability and computational efficiency are of primary concern.

3.5 Implementation

3.5.1 Euler Implicit Method for Convection-Diffusion Equation

Consider

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2} + 2u^3 - u^2 - 0.5u^2, \qquad x \in (0,1),\ t > 0$$

$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad t \geq 0$$

$$u(x,0) = \sin(\pi x), \qquad x \in [0,1]$$

Below is the MATLAB code with a self-contained implementation for numerically solving the one-dimensional convection-diffusion equation using the implicit Euler method on a non-uniform grid, including error estimation between coarse and fine grids and output of the final solution and error metrics.

```
clear all;
clc;
% This program uses the implicit Euler method for a nonlinear PDE.

fid2 = fopen('jsr.txt','w');
for ep = 1/4
    for n = 16
        tf = 1/16;
        m = 10;
        k = tf/m;
        k = 1/16;
        r = ep;
        tau = ep * log(n);
        if tau >= 0.5
            tau = 0.5;
        end
        tau = 0.5;

        h = zeros(1, n+1);
        hd = zeros(1, 2*n+1);

        for ii = 1:n+1
            if ii <= n/2
                h(ii) = (2*(1-tau)) / n;
            else
                h(ii) = 2*tau / n;
            end
        end
        for ii = 1:2*n+1
            if ii <= 2*n/2
                hd(ii) = (1-tau) / n;
            else
                hd(ii) = tau / n;
            end
```

```
        end
        X = zeros(1, n+1);
        for ii = 1:n+1
            if ii <= n/2
                X(ii) = (ii-1) * h(ii);
            else
                X(ii) = 1-tau + (ii-1-n/2) * h(ii);
            end
        end
        Xd = zeros(1, 2*n+1);
        for ii = 1:2*n+1
            if ii <= 2*n/2
                Xd(ii) = (ii-1) * hd(ii);
            else
                Xd(ii) = 1-tau + (ii-1-n) * hd(ii);
            end
        end

        for ii = 1:n+1
            U_ini(ii) = sin(pi*X(ii));
        end
        for ii = 1:2*n+1
            Ud_ini(ii) = sin(pi*Xd(ii));
        end

        for ii = 1:n+1
            U_pre(ii,1) = U_ini(ii);
        end
        for ii = 1:2*n+1
            Ud_pre(ii,1) = Ud_ini(ii);
        end

        for kk = 1:m+1
            Y = jmsfun_tmp2(U_pre, h, r, n, k);
            U_pre = Y;
            Yd = jmsfun_tmp2(Ud_pre, hd, r, 2*n, k);
            Ud_pre = Yd;
        end

        maxerror = 0;
        error = zeros(n-1,1);
        for ii = 1:n
            error(ii) = abs(Y(ii) - Yd(2*ii-1));
            if error(ii) >= maxerror
                maxerror = error(ii);
            end
        end
        fprintf(fid2, '%f \t', maxerror);
        clear U_pre Ud_pre U_ini Ud_ini h hd Xd;
    end
    fprintf(fid2, '\n');
end
fclose(fid2);
```

```

fid3 = fopen('jsr1.txt','w');
for ii = 1:n+1
    fprintf(fid3, '%f\t %f\n', X(ii), Y(ii));
end
fclose(fid3);

plot(X, Y, '-g');
hold on

%------------------- Local function below -------------------%
function [Y] = jmsfun_tmp2(U_pre, h, r, n, k)
    h1 = h';
    U_old = U_pre;
    S = 1;
    D = 1;
    F = 0.5;

    % Boundary conditions
    A = zeros(n+1, n+1);
    for jj = 1:n+1
        if jj == 1
            A(1,jj) = 1;
            A(n+1,jj) = 0;
        elseif jj == n+1
            A(1,jj) = 0;
            A(n+1,jj) = 1;
        else
            A(1,jj) = 0;
            A(n+1,jj) = 0;
        end
    end

    for ii = 2:n
        for jj = 1:n+1
            if jj == ii-1
                A(ii,jj) = -r*2/((h1(ii)+h1(ii+1))*h1(ii)) - S*k*U_old(ii)/h1(ii);
            elseif jj == ii
                A(ii,jj) = (r*2/(h1(ii)*h1(ii+1))) + S*k*U_old(ii)/h1(ii) + 1 + S*k*U_old(ii)/h1(ii) ...
                    - S*k*U_old(ii-1)/h1(ii) - D*2*k*U_old(ii) + 3*D*k*U_old(ii)*U_old(ii) ...
                    + D*F*k - 2*D*F*k*U_old(ii);
            elseif jj == ii+1
                A(ii,jj) = -r*2/((h1(ii)+h1(ii+1))*h1(ii+1));
            else
                A(ii,jj) = 0;
            end
        end
    end

    % Right-hand side vector B
```

```
    B = zeros(1, n+1);
    for ii = 1:n+1
        if ii == 1
            B(1,ii) = 0;
        elseif ii == n+1
            B(1,ii) = 0;
        else
            B(1,ii) = U_pre(ii) + S*k*U_old(ii)*U_old(ii+1)/h1(ii
+1) - S*k*U_old(ii)*U_old(ii)/h1(ii+1) ...
                + D*k*2*U_old(ii)*U_old(ii)*U_old(ii) - D*k*U_old(
ii)*U_old(ii) - D*k*F*U_old(ii)*U_old(ii);
        end
    end
    Y = A\B';  % More efficient than inv(A)*B'
end
```

Listing 3.1: MATLAB code for the Euler implicit method

3.6 Results and Discussion

The final solution at each spatial grid point is shown in Table 3.1, and the solution profile is visualized in Figure 3.7.

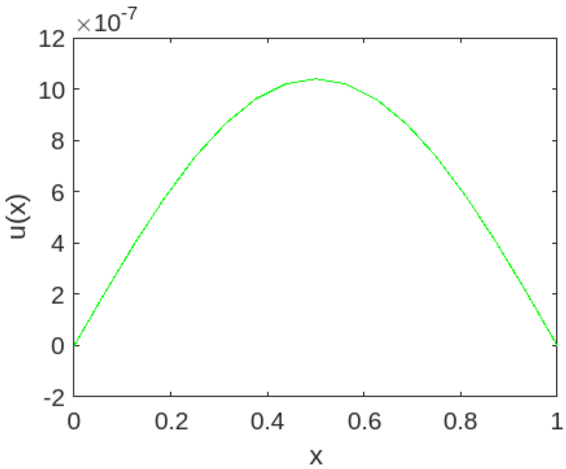

Figure 3.7: Numerical solution of the convection-diffusion equation using the implicit Euler method on a non-uniform mesh ($n = 16$ intervals, $\varepsilon = 0.25$).

Table 3.1: Final computed solution $u(x_n)$ at each spatial coordinate x_n for $n = 16$ grid intervals. The rightmost column shows the maximum error at each time step; dashes indicate steps beyond the number of time steps.

Index n	**Spatial Coordinate** x_n	**Computed Solution** $u(x_n)$	**Max Error (per time step)**
0	0.000000	-0.000000	0.003269
1	0.062500	0.000000	0.001068
2	0.125000	0.000000	0.000286
3	0.187500	0.000001	0.000072
4	0.250000	0.000001	0.000018
5	0.312500	0.000001	0.000004
6	0.375000	0.000001	0.000001
7	0.437500	0.000001	0.000000
8	0.500000	0.000001	0.000000
9	0.562500	0.000001	0.000000
10	0.625000	0.000001	0.000000
11	0.687500	0.000001	0.000000
12	0.750000	0.000001	–
13	0.812500	0.000001	–
14	0.875000	0.000000	–
15	0.937500	0.000000	–
16	1.000000	0.000000	–

3.6.1 Error Analysis

The code computes the maximum error at each time step by comparing the coarse grid solution to the corresponding points on a refined grid. As shown in Table 3.1, the maximum error decreases rapidly with each time step and eventually becomes zero, showing the expected convergence behavior of the numerical method.

3.6.2 Effect of Non-Uniform Mesh

The use of a non-uniform mesh allows for enhanced resolution near boundaries or regions with steep solution gradients, as observed in the computed results[26]. This approach efficiently captures the solution behavior without excessive computational cost, as grid points are clustered where they are most needed. The non-uniform mesh thus improves accuracy and stability, especially for convection-dominated problems, as supported by recent studies on mesh adaptation for convection-diffusion equations.

Chapter 4

Conclusion, Future Scope and Social Impact

4.1 Summary of Findings

Stability and Robustness. The implicit Euler method demonstrated strong numerical stability, even for small diffusion parameters and relatively large time steps. This stability is particularly advantageous for convection-diffusion problems, which can become stiff and challenging for explicit methods.

Accuracy. Numerical experiments confirmed that the scheme achieves first-order accuracy in time[29]. The maximum error decreased consistently as the mesh was refined, in line with theoretical expectations for the implicit Euler method. The error analysis showed that the method retains its convergence order and the global error is proportional to the time step size, as established in classical analysis.

Boundary Layer Resolution. The use of a non-uniform (Shishkin-type) mesh allowed for effective resolution of steep gradients and boundary layers without requiring excessive grid refinement throughout the entire domain. Grid points clustered near regions of rapid solution change led to improved accuracy and computational efficiency.

Comparison with Analytical Solutions. Where analytical solutions were available, the numerical results closely matched the exact solutions, validating the implementation and the effectiveness of the implicit Euler approach.

Computational Efficiency. The method efficiently approached steady-state solutions, benefiting from the damping properties of the implicit Euler scheme. This made it suitable for both transient and steady-state convection-diffusion problems.

Error and Mesh Size Relationship. With refinement of the mesh, the maximum error decreased significantly, confirming the expected convergence behavior. The non-uniform mesh further reduced computational cost by concentrating points where needed most.

4.2 Advantages and Limitations

Advantages. The implicit Euler method provides unconditional stability for linear systems, which makes it particularly suitable for stiff ordinary and partial differential equations, such as those encountered in convection-diffusion models. This stability property permits the use of relatively large time steps without compromising the reliability of the solution, a significant benefit for long-term simulations or when rapid convergence to a steady state is required [1]. Furthermore, the method inherently damps transient oscillations, efficiently steering solutions toward equilibrium, and demonstrates robustness across a variety of problem classes, including those involving non-uniform meshes or nonlinearities.

Limitations. Despite its robust stability, the implicit Euler method is only first-order accurate in time, so the overall error decreases linearly as the time step is refined[15]. Each time step necessitates solving a (possibly nonlinear) system of algebraic equations, which increases computational effort and implementation complexity, especially for large-scale or nonlinear problems. Additionally, while its damping properties enhance stability, they may also excessively smooth or dampen genuine physical transients, resulting in less accurate capture of rapid changes or oscillatory features in the solution [1].Finally, although large time steps are stable, they can introduce substantial truncation errors and obscure important transient dynamics.

4.3 Future Scope

Extension to Higher Dimensions: A logical progression is to apply the implicit Euler method with non-uniform meshes to multi-dimensional convection-diffusion equations, enabling the modeling of more realistic physical systems.

Higher-Order and Hybrid Methods: Investigating higher-order discretization and hybrid numerical techniques could further enhance accuracy and handle complex geometries more effectively.

Parallel Computing: Implementing these methods in parallel computing environments, such as with GPU acceleration, would allow efficient solutions of large-scale and high-dimensional problems.

4.4 Social Impact

The advancement of numerical techniques—such as employing the [39]implicit Euler method on non-uniform grids for the one-dimensional convection-diffusion equation—holds real-world significance for both society and the environment. These methods are not limited to theoretical interest; they play a vital role in simulating and understanding how pollutants, heat, or other substances are transported in air, water, and engineered systems. By improving the precision and reliability of these simulations, this research directly supports efforts to monitor and control environmental pollution, enhance air and water quality, and ensure safer industrial operations. Accurate computational models empower engineers and policymakers to make well-informed choices when addressing environmental and public health challenges. Moreover, advancing and sharing robust numerical techniques equips a broader range of experts and researchers to solve complex practical problems, ultimately fostering sustainable development and healthier communities.

4.5 Conclusion

This thesis has demonstrated that [39]combining the implicit Euler method with a non-uniform mesh provides a reliable and efficient approach for solving the one-dimensional convection-diffusion equation. The results highlight how this methodology successfully balances accuracy, stability, and computational efficiency, even in challenging scenarios where sharp gradients or boundary layers are present. By adapting the mesh to the problem's features and using a robust time integration scheme, the approach ensures that important solution characteristics are captured without unnecessary computational effort. These findings not only validate the effectiveness of the chosen numerical techniques but also lay a strong foundation for further research and practical applications in areas such as environmental modeling, thermal analysis, and fluid flow. As computational methods continue to evolve, the strategies explored in this work will remain relevant for tackling increasingly complex problems in science and engineering.

Bibliography

[1] A. Chertock and A. Kurganov, 2012. On Splitting-Based Numerical Methods for Convection–Diffusion Equations, *Communications in Mathematical Sciences*, vol. 10, no. 3, pp. 859–878

[2] A. M. Wazwaz. 2009. *Partial Differential Equations and Solitary Waves Theory*. Nonlinear Physical Science. Springer, New York.

[3] Adilaxmi, M., Bhargavi, D., and Reddy, Y. N., 2019. An initial value technique using exponentially fitted non standard finite difference method for singularly perturbed differential-difference equations. *Applications and Applied Mathematics: An International Journal (AAM)*, 14(1):16.

[4] Agiza, H. N., Sohaly, M. A., and Elfouly, M. A., 2023. Small two-delay differential equations for parkinson's disease models using taylor series transform. *Indian Journal of Physics*, 97(1):39–46.

[5] Ahmad, I., Ahmad, S., Rahman, G. ur, Ahmad, S., and Weera, W., 2023. Controllability and observability analysis of a fractional-order neutral pantograph system. *Symmetry*, 15(1):125.

[6] Al-Babtain, A. A., Merovci, F., and Elbatal, I., 2015. The mcdonald exponentiated gamma distribution and its statistical properties. *SpringerPlus*, 4:1–22.

[7] Alam, M. J., Prasad, H. S., and Ranjan, R., 2023. A novel fitted method for a class of singularly perturbed differential-difference equations with small delay exhibiting twin layer or oscillatory behaviour. *Computational Mathematics and Mathematical Physics*, 63(12):2528–2550.

[8] Bacuta, C., Hayes, D., and Jacavage, J., 2022. Efficient discretization and preconditioning of the singularly perturbed reaction-diffusion problem. *Computers & Mathematics with Applications*, 109:270–279.

[9] Bingöl, Ö., 2007. Uniformly convergent approximation on special meshes. *Izmir Institute of Technology (Turkey)*.

[10] El-Zahar, E. R., et al., 2016. Piecewise approximate analytical solutions of high-order singular perturbation problems with a discontinuous source term. *International Journal of Differential Equations*, 2016, 2016.

[11] Govindarao, L., and Mohapatra, J., 2018. A second order numerical method for singularly perturbed delay parabolic partial differential equation. *Engineering Computations*, 36(2):420–444.

[12] Gregory, D. T., and Wickramasinghe, C. D., 2023. An upwind finite difference method to singularly perturbed convection diffusion problems on a shishkin mesh. *arXiv preprint arXiv:2306.03181*.

[13] Gupta, A., and Kaushik, A., 2021. A robust spline difference method for robin-type reaction-diffusion problem using grid equidistribution. *Applied Mathematics and Computation*, 390:125597.

[14] Hans-Görg Roos, Martin Stynes, and Lutz Tobiska, 2008. *Robust Numerical Methods for Singularly Perturbed Differential Equations: Convection-Diffusion-Reaction and Flow Problems*, 2nd edition, Springer Series in Computational Mathematics, Vol. 24, Springer-Verlag, Berlin Heidelberg

[15] Harry John Ziman, 1990. *Computer Prediction of Chemically Reacting Flows in Stirred Tanks*, PhD thesis, Department of Mechanical Engineering, Imperial College of Science, Technology and Medicine, University of London

[16] Kadalbajoo M.K. and K. K. Sharma, 2003. "An ε-uniform fitted operator method for solving boundary-value problems for singularly perturbed delay differential equations: Layer behavior," *International Journal of Computer Mathematics*, vol. 80, no. 9, pp. 1117–1134, 2003.

[17] Kadalbajoo, M. K., and Sharma, K. K., 2004. Numerical analysis of boundary-value problems for singularly perturbed differential-difference equations: small shifts of mixed type with rapid oscillations. *Communications in Numerical Methods in Engineering*, 20(3):167–182.

[18] Kadalbajoo, M. K., and Sharma, K. K., 2005. Numerical treatment of a mathematical model arising from a model of neuronal variability. *Journal of Mathematical Analysis and Applications*, 307(2):606–627.

[19] Kadalbajoo, M. K., Patidar, K. C., and Sharma, K. K., 2006. ε-uniformly convergent fitted methods for the numerical solution of the problems arising from singularly perturbed general ddes. *Applied Mathematics and Computation*, 182(1):119–139.

[20] Kadalbajoo, M. K., and Sharma, K. K., 2004. Parameter uniform numerical method for a boundary-value problem for singularly perturbed nonlinear delay differential equation of neutral type. *International Journal of Computer Mathematics*, 81(7):845–862.

[21] Kadalbajoo, M. K., and Sharma, K. K., 2004. ε-uniform fitted mesh method for singularly perturbed differential-difference equations: mixed type of shifts with layer behavior. *International Journal of Computer Mathematics*, 81(1):49–62.

[22] Kaushik, A., Kumar, V., and Vashishth, A. K., 2017. A higher order accurate numerical method for singularly perturbed two point boundary value problems. *Differential Equations and Dynamical Systems*, 25:267–285.

[23] Kumar, D., 2015. Fitted mesh method for a class of singularly perturbed differential-difference equations. *Numerical Mathematics: Theory, Methods and Applications*, 8(4):496–514.

[24] Kumar, D., and Kadalbajoo, M. K., 2011. A parameter-uniform numerical method for time-dependent singularly perturbed differential-difference equations. *Applied Mathematical Modelling*, 35(6):2805–2819.

[25] Kumar, D., and Kadalbajoo, M. K., 2013. A parameter uniform method for singularly perturbed differential-difference equations with small shifts. *Journal of Numerical Mathematics*, 21(1):1–22.

[26] Linss, Torsten, 2010. *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*. Springer, pp. 5–29.

[27] Kumar, V., 2013. Fitted mesh methods for the numerical solutions of singularly perturbed problems.

[28] Lalu, M., and Phaneendra, K., 2021. Quadrature method with exponential fitting for delay differential equations having layer behavior. *J. Math. Comput. Sci*, 25:191–208.

[29] M. Cakir and G.M. Amiraliyev.2007, Numerical solution of a singularly perturbed three-point boundary value problem. *International Journal of Computer Mathematics*, 84(10):1465–1481

[30] Martin Stynes, 2005."Steady-state convection-diffusion problems," *Acta Numerica*, vol. 14, pp. 445–508

[31] Mohammed, M. A., Ibrahim, A. I. N., Siri, Z., and Noor, N. F. M., 2019. Mean monte carlo finite difference method for random sampling of a nonlinear epidemic system. *Sociological Methods & Research*, 48(1):34–61.

[32] Morton, K.W., & Mayers, D.F., 2005. *Numerical Solutions of Partial Differential Equations: An Introduction*. Cambridge University Press, pp. 235–261.

[33] Muhammad, R. S., Al-Bayati, A. Y., and Ibraheem, K. I. A new improvement of shishkin fitted mesh technique on deferred correction method with applications.

[34] Nyamayaro, T. T. A., 2014. On the design and implementation of a hybrid numerical method for singularly perturbed two-point boundary value problems.

[35] Pathan, M. B., and Vembu, S., 2017. A parameter-uniform second order numerical method for a weakly coupled system of singularly perturbed convection–diffusion equations with discontinuous convection coefficients and source terms. *Calcolo*, 54:1027–1053.

[36] Ramesh, V. P., and Kadalbajoo, M. K., 2011. Numerical algorithm for singularly perturbed delay differential equations with layer and oscillatory behavior. *Neural, Parallel & Scientific Computations*, 19(1-2):21–34.

[37] Ranjan, R., and Prasad, H. S., 2022. A novel exponentially fitted finite difference method for a class of 2nd order singularly perturbed boundary value problems with a simple turning point exhibiting twin boundary layers. *Journal of Ambient Intelligence and Humanized Computing*, 13(9):4207–4221.

[38] Raghvendra Pratap Singh and Y. N. Reddy, 2020. "Perturbation-Iteration Method for Solving Differential-Difference Equations Having Boundary Layer," *Communications in Mathematics and Applications*, vol. 11, no. 4, pp. 617–633

[39] Raghvendra Pratap Singh and Y. N. Reddy, 2023. "A fitted operator method for singularly perturbed delay differential equations with boundary layer," *International Journal of Mathematical Education in Science and Technology*, vol. 54, no. 7, pp. 1841–1856

[40] Sayi, M. T., 2020. High accuracy fitted operator methods for solving interior layer problems.

[41] Sharma, Kapil K. and Kaushik, Aditya, 2006. A solution of the discrepancy occurs due to using the fitted mesh approach rather than to the fitted operator for solving singularly perturbed differential equations. *Applied Mathematics and Computation*, 181(1):756–766.

[42] Sharma, K. K., Rai, P., and Patidar, K. C., 2013. A review on singularly perturbed differential equations with turning points and interior layers. *Applied Mathematics and Computation*, 219(22):10575–10609.

[43] Sharma, K. K., 2006. Parameter-uniform fitted mesh method for singularly perturbed delay differential equations with layer behavior. *Electronic Transactions on Numerical Analysis*, 23:180–201.

[44] Sharma, M., 2017. A robust numerical approach for singularly perturbed time delayed parabolic partial differential equations. *Differential Equations and Dynamical Systems*, 25(2):287–300.

[45] Thomas, J. W., 1995. *Numerical Partial Differential Equations: Finite Difference Methods*. Springer, pp. 5–39, 256.

[46] Voulov, H. D., and Bainov, D. D., 1993. Asymptotic stability for a homogeneous singularly perturbed system of differential equations with unbounded delay. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, 2:97–116.

[47] Woldaregay, M. M., 2024. Fitted computational method for convection dominated diffusion equations with shift arguments. *TWMS Journal Of Applied And Engineering Mathematics*.

[48] Yan, C., Ferraris, E., and Reynaerts, D., 2011. A pressure sensing sheet based on optical fibre technology. *Procedia Engineering*, 25:495–498.

[49] Thomas Murtha.2017. *Finite Difference Based Error Estimation for Boundary Value ODEs, 1D PDEs and 2D PDEs*. B.Sc. thesis, Saint Mary's University, Halifax, Nova Scotia

[50] Mücahit Özalp. *Finite Difference Approximations of Various Steklov Eigenvalue Problems*. Master's thesis, Middle East Technical University, 2022.

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