SUBORDINATION, INEQUALITIES AND RADIUS CONSTANTS OF CERTAIN ANALYTIC FUNCTIONS

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By

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CANDIDATE'S DECLARATION

I, **Surya Giri**, hereby certify that the work which is being presented in the thesis entitled **"Subordination, Inequalities and Radius Constants of Certain Analytic Functions**" in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in Mathematics, submitted in the Department of Applied Mathematics, Delhi Technological University is an authentic record of my own work carried out during the period from 20^{th} Jan 2020 to 5^{th} Aug 2024 under the supervision of Prof. S. Sivaprasad Kumar.

The matter presented in the thesis has not been submitted by me for the award of any other degree of this or any other Institute.

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CERTIFICATE

Certified that **Mr. Surya Giri** (2K19/PHD/AM/503) has carried out their research work presented in this thesis entitled **"Subordination, Inequalities and Radius Constants of Certain Analytic Functions"** for the award of **Doctor of Philosophy** from Department of Applied Mathematics, Delhi Technological University, Delhi, under my supervision. The thesis embodies results of original work, and studies are carried out by the student himself and the contents of the thesis do not form the basis for the award of any other degree to the candidate or to anybody else from this or any other University/Institution.

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December, 2024 **SURYA GIRI**

Place: Delhi, India.

Dedicated

To

My Family

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Abstract

The Bieberbach conjecture, undoubtedly the most famous coefficient problem in univalent function theory, played a significant role in the development of the field. Various subclasses of the class of normalized analytic univalent functions, denoted by $\mathscr{S}$, were introduced, and determining the sharp estimate of n^{th} Taylor coefficients for functions in these subclasses of $\mathscr{S}$ is still a challenging and intriguing problem. For the classes $\mathscr{S}^*(\varphi)$ and $\mathscr{C}(\varphi)$ introduced by Ma and Minda in *'A unified treatment of some special classes of univalent functions. Proceedings of the Conference on Complex Analysis, Tianjin, Conf Proc Lecture Notes Anal., I Int Press, Cambridge, MA. 157-169 (1992)'*, the estimate of $|a_n|$ for $n = 2, 3, 4$ were known. In Chapter 2, we obtain the sharp estimate of $|a_5|$ for these classes. The derived estimates coincide with some already known bounds for other subclasses of starlike and convex functions, while also providing new cases. In continuation of coefficient problems, Chapter 3 gives the sharp bounds of second and third-order Hermitian-Toeplitz determinants for the same classes along with the class of close-to-convex functions. The established bounds directly extend to various subclasses as well, which show the applicability and significance of the results. This thesis also dealt in the radius problems along with the coefficient problems for a class of semigroup generator denoted by $\mathscr{A}_\beta$ in Chapter 4. Using the sharp estimate of n^{th} coefficient for functions in the class $\mathscr{A}_\beta$, we establish the sharp Bohr radius, Bohr-Rogosinski radius and radius of starlikeness of order α. Additionally, Hankel determinants, Toeplitz and Hermitian Toeplitz determinants, Zalcman functional and bounds of successive coefficient difference also investigated for the same class. In the subsequent chapter, we introduce and study the notion of Toeplitz determinants in the case of higher dimensions. The sharp bounds of second and third-order Toeplitz determinants constructed over the Taylor coefficients of biholomorphic mappings are established. Chapters 5 and 6 give the sharp estimates of Toeplitz determinants for the subclasses of starlike mappings and quasi-convex mappings, respectively, defined on the unit ball in a complex Banach space and on the unit poly disk in $\mathbb{C}^n$. These derived bounds extend the already known bounds for univalent functions to higher dimensions.

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$\mathbb{U}$	Open unit disk $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
$\overline{\mathbb{U}}$	The closure of unit disk $\{z \in \mathbb{C} : \lvert z\rvert \leq 1\}$
$\partial\mathbb{U}$	The boundary of unit disk $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$
$\mathscr{A}$	The class of analytic functions in $\mathbb{U}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$
$\mathscr{S}$	The class of univalent functions in $\mathscr{A}$
$\mathscr{S}^*$	The class of starlike functions in $\mathscr{A}$
$\mathscr{C}$	The class of convex functions in $\mathscr{A}$
$\mathscr{S}^*(\alpha)$	The class of starlike functions of order α in $\mathscr{A}$, $\alpha \in [0,1)$
$\mathscr{C}(\alpha)$	The class of convex functions of order α in $\mathscr{A}$, $\alpha \in [0,1)$
$\mathscr{S}\mathscr{S}^*(\gamma)$	The class of strongly starlike functions of order γ in $\mathscr{A}$, $\gamma \in (0,1]$
$\mathscr{C}\mathscr{C}(\gamma)$	The class of strongly convex functions of order γ in $\mathscr{A}$, $\gamma \in (0,1]$
$\prec$	Subordination
$\mathscr{B}_0$	The class of Schwarz functions
$\mathscr{P}$	The class of Carathéodory functions
$\mathscr{S}^*(\varphi)$	The class of Ma-Minda starlike functions
$\mathscr{C}(\varphi)$	The class of Ma-Minda convex functions
$\mathscr{K}$	The class of close-to-convex functions
$T_{m,n}(f)$	Symmetric Toeplitz determinant
$T_m(n)(f)$	Hermitian-Toeplitz determinant
$H_q(n)$	Hankel determinant
$f * g$	Convolution $f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ of two analytic functions f and g in $\mathscr{A}$
$\mathbb{U}^n$	Unit polydisk ($n-$ copies of the unit disk) in $\mathbb{C}^n$
$\mathbb{B}$	Unit ball in a complex Banach space X
$\overline{\mathbb{U}}^n$	The closure of $\mathbb{U}^n$
$\partial\mathbb{U}^n$	The boundary of $\mathbb{U}^n$
$\partial_0\mathbb{U}^n$	The distinguished boundary of $\mathbb{U}^n$
$\mathscr{S}(\mathbb{B})$	The set of all normalized biholomorphic mappings on $\mathbb{B}$
$\mathscr{S}^*(\mathbb{B})$	The class of starlike mappings on $\mathbb{B}$
$\mathscr{S}_\alpha^*(\mathbb{B})$	The class of starlike mappings of order α on $\mathbb{B}$
$\mathscr{S}\mathscr{S}_\gamma^*(\mathbb{B})$	The class of strongly starlike mappings of order γ on $\mathbb{B}$
$\mathscr{C}_\alpha(\mathbb{B})$	The class of quasi convex mappings of type B and order α

Chapter 1

Introduction

This chapter serves as a foundational framework, presenting definitions and results that provide background for the carried out work. The contents of this chapter are required throughout the thesis. Further, at the end of this chapter, we have included the synopsis of the thesis, a brief of each chapter.

Univalent function theory, a fascinating branch of complex analysis, studies functions that possess a one-to-one mapping from one region of the complex plane to another without overlapping or crossing themselves, revealing profound geometric properties. Classified under geometric function theory, it uncovers insights into the behavior of these functions through geometric intuition, exemplified by the Riemann mapping theorem. The theory of univalent functions traces back to major contributions from the early 20th century, including Koebe's 1907 paper [82], Gronwall's [63] 1914 proof of the area theorem, and Bieberbach's 1916 estimate for the second coefficient of a normalized univalent function [18].

This theory has accumulated a significant body of literature. Goodman's [59] comprehensive book serves as a fundamental resource, covering essential concepts and results. Edited volumes like *Current Topics in Analytic Function Theory* by Srivastava and Owa [174] and works by Brannan and Clunie [26] offer collections of research and survey articles on univalent functions. Bernardi's bibliography [17] provides extensive topic coverage. Various textbooks authored by Duren [41], Goluzin [54], Hayman [69], Pommerenke [144], Hummel [71], Jenkins [75], Milin [120], and Hallenbeck & MacGregor [64] offer detailed insights into different subclasses of univalent functions, while Graham and Kohr's book [61] extensively covers univalent functions in both one and higher dimensions. Additionally, the recent book by Thomas et al. [182] presents up-to-date summaries of key properties and results for significant subclasses of univalent functions, collectively providing a comprehensive foundation for studying this theory and its applications.

In the realm of complex-valued functions, akin to their real counterparts, several differential inequalities provide crucial characterizations of these functions. For instance, the Noshiro-Warschawski theorem states that *If* $f(z)$ *is analytic in the open unit disk* $\mathbb{U}$ *with* $\operatorname{Re} f'(z) > 0$ *for* $z \in \mathbb{U}$*, then* f *is univalent in* $\mathbb{U}$. In 1981, Miller and Mocanu [122] introduced the true complex analogue of such real differential inequalities, termed *differential subordination*, which replaces and extends this approach. Their monograph, *Differential Subordination: Theory and Applications,* containing over 400 papers, offers an extensive discussion of this theory and its numerous applications [121]. The technique of differential subordination is widely employed to establish various results in univalent function theory. Bulbocă's book [28] similarly addresses differential subordination as well as superordination, enriching understanding in this domain. Notable contributions to this field include works by Ali et al. [9, 10], Ravichandran et al. [150, 152, 155], Obradović et al. [135], Ponnusamy [145, 146, 147], and Kumar et al. [91, 94].

A single valued function f which is analytic except for at most one simple pole is said to be univalent (Schlicht) in a domain $\mathscr{D} \subset \mathbb{C}$ if never takes the same value twice, that is $f(z_1) \neq f(z_2)$ for all distinct points z_1 and z_2. The function f is said to be locally univalent at a point $z_0 \in \mathscr{D}$ if it is univalent in some neighbourhood of z_0. If a function f is analytic and univalent in a domain $\mathscr{D}$, then $f'(z) \neq 0$, but the converse is not necessarily true. For instance, the function $f(z) = e^{2\pi z}$ is not univalent in $|z| < 1$. However, the condition $f'(z_0) \neq 0$ ensures the local univalence of f at $z = z_0$. We restrict the domain of study to $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ instead of considering a general simply connected domain in $\mathbb{C}$ due to the celebrated Riemann mapping theorem, which states that *Any proper simply connected domain in* $\mathbb{C}$ *is conformally equivalent to the unit disk* $\mathbb{U}$*.*

Let g be an analytic function in the unit disk $\mathbb{U}$. Since the expression $f(z) = (g(z) - g(0))/g'(0)$, $g'(0) \neq 0$, represents the translations and stretches (or shrinks) of the domain $g(\mathbb{U})$ with a rotation, so properties of the function g can be identified from the corresponding function f. Consequently, we opt for the normalization $f(0) = 0$ and $f'(0) = 1$ for facilitate analysis. Let $\mathscr{A}$ be the class of all such normalized analytic functions. The class of all normalized functions that are analytic and univalent in $\mathbb{U}$, is denoted by $\mathscr{S}$. Thus, a function f in the class $\mathscr{S}$ has the following series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.0.1}$$

The leading example of the class $\mathscr{S}$ is the Koebe function $k(z) = z/(1-z)^2$, which maps the unit disk onto the entire complex plane except the slit from $-1/4$ to $-\infty$. In 1916, Bieberbach [18] gave the estimate for the second coefficient of functions belonging to the class $\mathscr{S}$. He proved that *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}$*, then* $|a_2| \leq 2$*, with equality if and only if* f *is a rotation of the Koebe function.* Using the fact that $|a_2| \leq 2$ for any f in the class $\mathscr{S}$, numerous other geometric properties including the covering theorem, distortion, and growth theorem, were established. The growth and distortion theorems claim that *If* $f \in \mathscr{S}$ *and* $z = re^{i\theta} \in \mathbb{U}$*, then*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2} \quad \text{and} \quad \frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3},$$

respectively. Equality occurs in both the bounds if and only if f is a rotation of the Koebe function. As early as 1907, Koebe [82] discovered that the ranges of all functions in $\mathcal{S}$ contain a common disk $|w| < \rho$, where ρ is an absolute constant. The Koebe function shows that $|\rho| \leq 1/4$, and Bieberbach [19] later established that *The range of every function of class $\mathcal{S}$ contains the disk $\{w \in \mathbb{C} : |w| < 1/4\}$*. The estimate of second coefficient was also the main basis for the famous Bieberbach conjecture: *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, then $|a_n| \leq n$ for any integer $n \geq 2$. Equality holds if and only if f is rotation of the Koebe function.* This conjecture became one of the most celebrated mathematical problems of the twentieth century. Many methods were developed to tackle this problem. In 1923, Löwner [114] proved the conjecture for $n = 3$ by introducing the Löwner differential equation. Garabedian and Schiffer [50] applied the variational method to establish $|a_4| \leq 4$ in 1955. Pederson [140] and Ozawa [137, 138] used the Grunsky inequality to prove $|a_6| \leq 6$ in 1968 and 1969 respectively. Finally, In 1984, L. de Branges [25] gave a remarkable proof using the operator theory and special functions for all values of n. Many partial results were obtained in the intevening years, including results for special subclasses of $\mathcal{S}$ and for particular coefficients, as well as asymptotic estimates and estimates for general n. For more proofs and history of Bieberbach conjecture, one can refer to the books [12, 41, 54, 56, 69, 144].

1.1 Starlike and Convex Functions

It took around 65 years to prove Bieberbach's conjecture. During this period, several subclasses of $\mathcal{S}$ were introduced and the conjecture was verified for them. Apart from their intrinsic properties, the subclasses serve as test cases for the much more difficult class $\mathcal{S}$. In some cases, there are more restrictive coefficient estimates, growth, covering and distortion theorems for the full class $\mathcal{S}$.

A domain $\mathcal{D} \subset \mathbb{C}$ is said to be convex if the line segment joining any two points w_1 and w_2 in $\mathcal{D}$ entirely lies in $\mathcal{D}$. A function $f \in \mathcal{A}$ is called convex if $f(\mathbb{U})$ is a convex domain. The class of all such functions is denoted by $\mathcal{C}$. A domain $\mathcal{D} \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in \mathcal{D}$ if every line segment connecting w_0 to any $w \in \mathcal{D}$ remains entirely in $\mathcal{D}$. A function $f \in \mathcal{A}$ is called starlike if $f(\mathbb{U})$ is a starlike domain with respect to the origin. The class of all such functions is denoted by $\mathcal{S}^*$. Analytically, a function $f \in \mathcal{C}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{1.1.1}$$

Similarly, a function $f \in \mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{1.1.2}$$

The condition (1.1.1) for convexity was first stated by Study [176] in 1913. The condition (1.1.2) is due to R. Nevanlinna [127] in 1921. The inequalities (1.1.1) and (1.1.2) reveal a surprisingly close analytic connection between convex and starlike functions. This was first observed by Alexander [5] stating that *Let $f \in \mathcal{A}$. Then $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$*. Although the search for the sharp bound of $|a_n|$ when f is in the parent class $\mathcal{S}$ continued, this problem was solved for the subclasses $\mathcal{S}^*$ and $\mathcal{C}$.

The sharp estimates $|a_n| \leq n$ for $f \in \mathscr{S}^*$ and $|a_n| \leq 1$ for $f \in \mathscr{C}$ were proved by Nevanlinna [126] and Löwner [113], respectively.

The analytic characterizations mentioned in (1.1.1) and (1.1.2) when appropriately modified, yields various other classes of $\mathscr{S}$. For instance, Robertson [158] introduced and studied the concepts of functions starlike and convex of order α in 1936. A function $f \in \mathscr{A}$ is said to be starlike of order α in $\mathbb{U}$ if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$ for $0 \leq \alpha < 1$. The class of all such functions is denoted by $\mathscr{S}^*(\alpha)$. A function $f \in \mathscr{A}$ is said to be convex of order α in $\mathbb{U}$ if $\operatorname{Re}(1 + zf''(z)/f'(z)) > \alpha$ for $0 \leq \alpha < 1$. The class of all such functions is denoted by $\mathscr{C}(\alpha)$. If $f \in \mathscr{C}(\alpha)$ and $|z| = r < 1$, then

$$\frac{1}{(1+r)^{2(1-\alpha)}} \leq |f'(z)| \leq \frac{1}{(1-r)^{2(1-\alpha)}}.$$

If $\alpha \neq 1/2$, then

$$\frac{(1+r)^{2\alpha-1}-1}{2\alpha-1} \leq |f(z)| \leq \frac{1-(1-r)^{2\alpha-1}}{2\alpha-1}$$

and if $\alpha = 1/2$, then

$$\log(1+r) \leq |f(z)| \leq -\log(1-r).$$

These estimates are sharp. Equality holds in each of the above relations for

$$f_\alpha(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \alpha \neq \dfrac{1}{2}, \\[2ex] -\log(1-z), & \alpha = \dfrac{1}{2}. \end{cases} \tag{1.1.3}$$

Similar results holds when $f \in \mathscr{S}^*(\alpha)$, for example

$$\frac{r}{(1+r)^{2(1-\alpha)}} \leq |f(z)| \leq \frac{r}{(1-r)^{2(1-\alpha)}}.$$

These inequalities are sharp for the function $zf'_\alpha(z)$.

Another natural extension of the definitions of $\mathscr{S}^*$ and $\mathscr{C}$ are the classes of strongly starlike and strongly convex functions, respectively. A function $f \in \mathscr{A}$ is said to be strongly starlike of order γ if and only if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi\gamma}{2}, \quad \gamma \in (0,1].$$

The class of strongly starlike functions of order γ is denoted by $\mathscr{S}\mathscr{S}^*(\gamma)$. Similarly, a function $f \in \mathscr{A}$ is called strongly convex function of order γ if and only if

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi\gamma}{2}, \quad \gamma \in (0,1].$$

The class of strongly convex functions of order γ is represented by $\mathscr{C}\mathscr{C}(\gamma)$.

1.2 Subordination and Carathéodory Functions

For any two analytic functions f and g, it is said that f is subordinate to g, denoted by $f \prec g$, if there exists a Schwarz function $\omega(z)$ such that $f(z) = g(\omega(z))$. Let us denote the family of all Schwarz functions $\omega(z)$ by $\mathscr{B}_0$ satisfying $\omega(0) = 0$, $|\omega(z)| \leq 1$ for $z \in \mathbb{U}$, and having the series expansion

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n. \tag{1.2.1}$$

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Moreover, *If $f \prec g$ in $\mathbb{U}$, then for each $r \in (0,1)$, $f(\mathbb{U}_r) \subset g(\mathbb{U}_r)$, where $\mathbb{U}_r = \{z \in \mathbb{C} : |z| \leq r\}$. Further, if $f(re^{i\theta})$ is on the boundary of $g(\mathbb{U}_r)$ for one point $z_0 = re^{i\theta_0}$, with $0 < r < 1$, then there is a real α such that $f(z) \equiv g(e^{i\alpha}z)$, and $f(re^{i\theta})$ is on the boundary of $g(\mathbb{U}_r)$ for every point z in $\mathbb{U}$* . This concept is called *The Lindelöf Principle*.

Another important family is the class of Carathéodory functions, denoted by $\mathscr{P}$, which consists of analytic functions in $\mathbb{U}$ having the series expansion of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.2.2}$$

and satisfy $\operatorname{Re} p(z) > 0$ in $\mathbb{U}$. Clearly, the function

$$L(z) = \frac{1+z}{1-z}, \quad z \in \mathbb{U} \tag{1.2.3}$$

is a member of $\mathscr{P}$ as it maps the unit disk onto the right half-plane. The analytic characterization of various subclasses of $\mathscr{S}$ can now be expressed involving Carathéodry class or in terms of subordination. For instance, $f \in \mathscr{S}^*$ if and only if $zf'(z)/f(z) \in \mathscr{P}$ or $zf'(z)/f(z) \prec (1+z)/(1-z)$ and $f \in \mathscr{C}$ if and only if $1 + zf''(z)/f'(z) \in \mathscr{P}$ or $1 + zf''(z)/f'(z) \prec (1+z)/(1-z)$. Carathéodory [29] proved that for each $p \in \mathscr{P}$ of the form (1.2.2), the sharp inequality $|p_n| \leq 2$ holds. Ma and Minda [117] found the sharp estimate of $|p_2 - \nu p_1^2|$ for $p \in \mathscr{P}$ and $\nu \in \mathbb{R}$, which is called the Fekete-Szegö functional. The bounds on the coefficients of Carathéodory functions are crucial in coefficient problems. Kwon et al. [101] provided the formula for the fourth coefficient p_4, which led to the derivation of sharp estimates for the third-order Hankel determinant for various subclasses of $\mathscr{S}$. For further details regarding the estimates on coefficients of $p \in \mathscr{P}$, we suggest referring to the survey article [35].

1.3 Other Subclasses of Univalent Functions

In 1971, Janowski [72] introduced the following classes that give further generalizations of starlike and convex functions:

$$\mathscr{S}^*[A,B] = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\} \text{ and } \mathscr{C}[A,B] = \left\{ f \in \mathscr{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \right\},$$

where $-1 \le B < A \le 1$. It can be easily seen that $(1+Az)/(1+Bz)$ maps the unit disk univalently onto a convex domain in the right half plane having the diameter endpoints $(1-A)/(1-B)$ and $(1+A)/(1+B)$ for the range $-1 \le B < A \le 1$. Choosing $A = 1$ and $B = -1$, the classes $\mathcal{S}^*[A,B]$ and $\mathcal{C}[A,B]$ represent the classes of starlike and convex functions, respectively. For $A = (1-2\alpha)$ and $B = -1$, we get the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively. Among other subclasses of starlike and convex functions, the Janowski classes are one of the most studied and the work is still ongoing.

Other interesting classes are the classes of uniformly starlike and uniformly convex functions introduced by Goodman [57, 58] in 1991. A function $f \in \mathcal{S}^*$ is said to be uniformly starlike, if it maps each circular arc γ contained in $\mathbb{U}$, with centre ζ also in $\mathbb{U}$, onto an arc $f(\gamma)$ which is starlike with respect to $f(\zeta)$. The class of all such functions is denoted by UST. Similarly, a function $f \in \mathcal{C}$ is said to be uniformly convex, if it maps each circular arc γ contained in $\mathbb{U}$, with centre ζ also in $\mathbb{U}$, onto a convex arc $f(\gamma)$. This class of all these functions is denoted by UCV.

In 1992, Ma and Minda [117] unified various subclasses of starlike and convex functions by introducing the following classes:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \quad \text{and} \quad \mathcal{C}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}, \tag{1.3.1}$$

where $\varphi(z)$ is an analytic univalent function in $\mathbb{U}$ that satisfies $\varphi'(0) > 0$ and maps the unit disk onto a domain in the right half plane, which is symmetric about the real axis and starlike with respect to $\varphi(0) = 1$. Suppose φ has the following series expansion

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad B_1 > 0. \tag{1.3.2}$$

Since $\varphi(\mathbb{U})$ is symmetric with respect to the real axis and $\varphi(0) = 1$, we have $\overline{\varphi(\bar{z})} = \varphi(z)$, which yields that all B_i's are real. For the family $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ sharp growth, distortion theorems and estimates for the Fekete-Szegö functional $|a_3 - \nu a_2^2|$ are known, where $\nu \in \mathbb{R}$ [117]. Consider, the analytic functions $k_{\varphi,n} : \mathbb{U} \to \mathbb{C}$ defined by

$$k_{\varphi,n} = z \exp \int_0^z \frac{\varphi(t^{n-1}) - 1}{t} dt, \quad n = 2, 3, \cdots. \tag{1.3.3}$$

Clearly $k_{\varphi,n} \in \mathcal{S}^*(\varphi)$ and it plays the role of Koebe function for the class $\mathcal{S}^*(\varphi)$. We simply denote $k_{\varphi,2}$ by k_φ. Similarly in case of the family $\mathcal{C}(\varphi)$, the analytic function $h_{\varphi,n} : \mathbb{U} \to \mathbb{C}$ satisfying

$$1 + \frac{z h''_{\varphi,n}(z)}{h'_{\varphi,n}(z)} = \varphi(z^{n-1}), \quad n = 2, 3, \cdots, \tag{1.3.4}$$

is a member of $\mathcal{C}(\varphi)$ and plays the role of extremal function. We simply take $h_{\varphi,2} =: h_\varphi$.

It is obvious that $\mathcal{S}^*((1+z)/(1-z)) = \mathcal{S}^*$ and $\mathcal{C}((1+z)/(1-z)) = \mathcal{C}$ are the classes of starlike and convex functions, respectively. If $0 \le \alpha < 1$, then $\mathcal{S}^*((1+(1-2\alpha)z)/(1-z)) = \mathcal{S}^*(\alpha)$ and $\mathcal{C}((1+(1-2\alpha)z)/(1-z)) = \mathcal{C}(\alpha)$ become the classes of starlike and convex functions of order α, respectively. Suppose

$$\phi(z) := 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \tag{1.3.5}$$

then $\phi(\mathbb{U}) = \{w : \operatorname{Re} w > |w-1|\}$ and is clearly a Ma-Minda function. It is proved in [116] and [130] that $\mathscr{C}(\phi)$ reduces to the class UCV. Rønning [129, 130] studied the corresponding class of parabolic starlike functions, denoted by $\mathscr{S}_p = \mathscr{S}^*(\phi)$. If we define $\varphi(z) = \sqrt{1+z}$, the class $\mathscr{S}^*(\varphi)$ coincides with the class $\mathscr{S}_L^*$ introduced by Sokół and Stankiewicz [173]. Geometrically, a function $f \in \mathscr{S}_L^*$ if and only if $zf'(z)/f(z)$ lies in the region bounded by the right lemniscate of Bernoulli given by $\{w \in \mathbb{C} : |w^2-1| < 1\}$. Therefore $\mathscr{S}_L^* = \{f \in \mathscr{A} : |(zf'(z)/f(z))^2 - 1| < 1\}$. Mendiratta et al. [118] considered the class $\mathscr{S}_{RL}^*$ of functions f such that the quantity $zf'(z)/f(z)$ lies in the interior of the left half of the shifted lemniscate of Bernoulli given by $\Omega_{RL} = \{w \in \mathbb{C} : \operatorname{Re} w > 0, |(w-\sqrt{2})^2 - 1| < 1$. Note that, the function

$$\varphi(z) = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$$

maps the unit disk onto Ω_{RL}. Thus

$$\mathscr{S}_{RL}^* = \left\{f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}\right\}.$$

Using this approach, various interesting subclasses of starlike functions by confining the values of $zf'(z)/f(z)$ to a defined region within the right half-plane were introduced and studied. Some of them are listed in Table 1.1 along with their respective class notations.

Class	$\varphi(z)$	**Reference**
$\mathscr{S}_{\sin}^*$	$1+\sin z$	Cho et al. [34]
$\mathscr{S}_{SG}^*$	$2/(1+e^{-z})$	Goel and Kumar [52]
$\mathscr{S}_{\wp}^*$	$1+ze^z$	Kumar and Gangania [92]
Δ^*	$z+\sqrt{1+z^2}$	Raina and Sokół [149]
$\mathscr{S}_e^*$	e^z	Mendiratta et al. [119]
$\mathscr{S}_{q_b}^*$	$\sqrt{1+bz}, \quad b \in (0,1]$	Sokół [172]
$\mathscr{S}_{Ne}^*$	$1+z-z^3/3$	Wani and Swaminathan [184]
$\mathscr{S}_B^*$	$e^{e^z}-1$	Kumar et al. [100]

Table 1.1: Subclasses of starlike functions

Starlike and Convex Functions with Respect to Symmetric Points

A regular function f in $\mathbb{U}$ is said to be starlike with respect to symmetrical points if for every r less than and sufficiently close to 1 and every z_0 on $|z| = r$, the angular velocity of $f(z)$ about the point $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction. Sakaguchi [165] showed that a function $f \in \mathscr{A}$ is starlike with respect to symmetrical points if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)-f(-z)} > 0 \quad (z \in \mathbb{U}).$$

The class of all such functions is denoted by $\mathscr{S}_s^*$. It is noted that the class of functions univalent and starlike with respect to symmetrical points includes the classes of convex functions and odd functions starlike with respect to the origin [165]. Afterwards, Das and Singh [40] introduced the class $\mathscr{C}_s$ of $f \in \mathscr{A}$, known as the class of convex functions with respect to symmetrical points, which satisfy

$$\operatorname{Re} \frac{(2zf'(z))'}{(f(z)-f(-z))'} > 0 \quad (z \in \mathbb{U}).$$

The functions in the class $\mathscr{C}_s$ are convex, and Das and Singh proved that the n^{th} coefficient of functions in $\mathscr{C}_s$ is bounded by $1/n$ for $n \geq 2$.

Incorporating the notion of subordination, Ravichandran [151] generalized these classes as

$$\mathscr{S}_s^*(\varphi) = \left\{ f \in \mathscr{A} : \frac{2zf'(z)}{f(z)-f(-z)} \prec \varphi(z) \right\} \tag{1.3.6}$$

and

$$\mathscr{C}_s(\varphi) = \left\{ f \in \mathscr{A} : \frac{(2zf'(z))'}{(f(z)-f(-z))'} \prec \varphi(z) \right\}, \tag{1.3.7}$$

where $\varphi(z)$ is given by (1.3.2) and satisfies all the constraints as considered by Ma and Minda [117].

Clearly, for $\varphi(z) = (1+z)/(1-z)$, the classes $\mathscr{S}_s^*(\varphi)$ and $\mathscr{C}_s(\varphi)$ reduce to the classes $\mathscr{S}_s^*$ and $\mathscr{C}_s$, respectively. For $0 \leq \alpha < 1$, if $\varphi(z) = (1+(1-2\alpha)z)/(1-z)$, then we get the classes of starlike and convex functions with respect to symmetric points of order α, denoted by $\mathscr{S}_s^*(\alpha)$ and $\mathscr{C}_s(\alpha)$, respectively. For more information about these classes, we refer [31, 53, 181].

Close-to-Convex Functions

Another well known subclass of univalent functions in the unit disk is the class of close-to-convex functions introduced by Kaplan [76] in 1952. A function $f \in \mathscr{A}$ is said to be close-to-convex if there exists a convex function $h \in \mathscr{C}$ such that

$$\operatorname{Re}\left(\frac{f'(z)}{h'(z)}\right) > 0.$$

The class of all such functions is denoted by $\mathscr{K}$. By Alexander's theorem, if $h(z)$ is convex, then $g(z) = zh'(z)$ is starlike. Hence, an equivalent form of Kaplan's definition is that there exists a function $g \in \mathscr{S}^*$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{1.3.8}$$

Various choices of function g in (1.3.8) yield some well known subclasses of $\mathscr{K}$. For instance,

$$\mathscr{F}_1 = \left\{ f \in \mathscr{S} : \operatorname{Re}(1-z)f'(z) > 0 \right\}, \quad \mathscr{F}_2 = \left\{ f \in \mathscr{S} : \operatorname{Re}(1-z^2)f'(z) > 0 \right\},$$
$$\mathscr{F}_3 = \left\{ f \in \mathscr{S} : \operatorname{Re}(1-z)^2 f'(z) > 0 \right\}, \quad \mathscr{F}_4 = \left\{ f \in \mathscr{S} : \operatorname{Re}(1-z+z^2)f'(z) > 0 \right\}$$

and

$$\mathscr{R} = \left\{ f \in \mathscr{S} : \operatorname{Re} f'(z) > 0 \right\}.$$

Ozaki [136] proved that the condition (1.3.8) is sufficient for a function f to be univalent. Functions in the classes $\mathscr{F}_2$ and $\mathscr{F}_4$ exhibit nice geometric properties. Functions in the class $\mathscr{F}_2$ map $\mathbb{U}$ univalently onto a domain convex in the direction of the imaginary axis whereas functions in the class $\mathscr{F}_4$ map $\mathbb{U}$ univalently onto a domain convex in the direction of the real axis. By $g(z) = f(z)$, it is evident that $\mathscr{S}^* \subset \mathscr{K}$ and hence $\mathscr{C} \subset \mathscr{S}^* \subset \mathscr{K}$. As in the case of convex and starlike functions, close-to-convex functions also exhibit nice geometric characterization. Let $f(z)$ be analytic function in $\mathbb{U}$ and $f(C_r)$ is the image of circle $|z| = r$, where $0 < r < 1$. Function f is said to be close-to-convex if and only if the image curve $f(C_r)$ has no 'large hairpin turns'; that is there are no sections of the curve $f(C_r)$ in which the tangent vector turns backward through an angle greater than or equal to π. Analytically, Kaplan [76] showed that *If f is analytic and locally univalent in $\mathbb{U}$, then f is close-to-convex if and only if*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta > -\pi, \quad z = re^{i\theta}$$

for each $r \in (0,1)$ and for each pair of real numbers θ_1 and θ_2 with $0 \le \theta_2 - \theta_1 \le 2\pi$. In 1955, Reade [156] proved that the coefficients of close-to-convex functions satisfy the Bieberbach conjecture i.e. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{K}$, then $|a_n| \le n$ for $n = 2, 3, 4, \cdots$. Since the Koebe function is in $\mathscr{K}$, the result is sharp for each n.*

1.4 Geometric Function Theory in Higher Dimensions

There are numerous results in univalent function theory in one complex variable that cannot be extended to higher dimensions, at least without restrictions. One of the most basic results in the theory of univalent functions in one variable is the Riemann mapping theorem. Its failure in several variables, discovered by Poincaré [142], is one of the key difference between geometric function theory in one variable and in higher dimensions. Cartan [30] stated that the most celebrated problem, the Bieberbach conjecture, does not hold in the case of several complex variables. Counterexamples show that many results in the geometric function theory of one complex variable are not applicable for several complex variables [55, 61].

Let $\mathbb{C}^n$ denote the space of $n-$complex variables $z = (z_1, z_2, \cdots, z_n)$, where $z_j \in \mathbb{C}$, $1 \le j \le n$ and $\Omega \subset \mathbb{C}^n$ be a domain. There are several definitions of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$. The book by Graham and Kohr [61] provides a systematic treatment of classical results in univalent function theory and their generalization to higher dimensions.

Holomorphic Mappings in Complex Banach Spaces

Let X and Y be two complex Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. For simplicity, we denote both the norms by $\|\cdot\|$, when there is no possibility of confusion. Let $L(X,Y)$ denote the

Banach space of all continuous linear operators from X into Y with the standard operator norm

$$\|A\| = \sup\{\|A(z)\| : \|z\| = 1\}, \quad A \in L(X,Y).$$

Let I denote the identity in $L(X,X)$. A mapping $f : \Omega \subseteq X \to Y$ is called holomorphic if for any $z \in \Omega$, there is a mapping $Df(z) \in L(X,Y)$, called the Fréchet derivative of f at z, such that

$$\lim_{h \to 0} \frac{\|f(z+h) - f(z) - Df(z)h\|}{\|h\|} = 0,$$

i.e.

$$f(z+h) = f(z) + Df(z)(h) + o(\|h\|).$$

Let $\mathscr{H}(\Omega,\Omega')$ represent the set of mappings from $\Omega \subseteq X$ into a domain $\Omega' \subseteq Y$ and $\mathscr{H}(\Omega,X) =: \mathscr{H}(\Omega)$. If $f \in \mathscr{H}(\Omega,Y)$ and $z \in \Omega$, then for each $k = 1,2,\cdots$, there is a bounded symmetric linear mapping

$$D^k f(z) : \prod_{j=1}^{k} X \to X,$$

called the k^{th}-order Fréchet derivative of f at z, such that

$$f(w) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(z)((w-z)^k)$$

for all w in some neighborhood of z. It is understood that $D^0 f(z)((w-z)^0) = f(z)$ and for $k \geq 1$,

$$D^k f(z)((w-z)^k) = D^k f(z) \underbrace{(w-z, w-z, \cdots, w-z)}_{k \text{ -times}}.$$

Moreover, if $f \in \mathscr{H}(\Omega,Y)$ and the closed segment $[a,a+h]$ is contained in Ω, then the Taylor formula with remainder

$$f(a+h) = f(a) + Df(a)(h) + \cdots + \frac{1}{k!} D^k f(a)(h^k) + \int_0^1 \frac{(1-t)^k}{k!} D^{k+1} f(a+th)(h^{k+1}) dt$$

holds for all $k \in \mathbb{N}$.

On a bounded circular domain $\Omega \subset \mathbb{C}^n$, the first and the m^{th} Fréchet derivative of a holomorphic mapping $f : \Omega \to X$ are written by $Df(z)$ and $D^m f(z)(a^{m-1}, \cdot)$, respectively. The matrix representations are

$$Df(z) = \left(\frac{\partial f_j}{\partial z_k} \right)_{1 \leq j,k \leq n},$$

$$D^m f(z)(a^{m-1}, \cdot) = \left(\sum_{p_1,p_2,\cdots,p_{m-1}=1}^{n} \frac{\partial^m f_j(z)}{\partial z_k \partial z_{p_1} \cdots \partial z_{p_{m-1}}} a_{p_1} \cdots a_{p_{m-1}} \right)_{1 \leq j,k \leq n},$$

where $f(z) = (f_1(z), f_2(z), \cdots f_n(z))'$, $a = (a_1, a_2, \cdots a_n)' \in \mathbb{C}^n$.

If $0 \in \Omega$, mapping $f \in \mathscr{H}(\Omega)$ is said to be normalized if $f(0) = 0$ and $D(f(0)) = I$. A mapping

$f \in \mathcal{H}(\Omega, Y)$ is said to be biholomorphic on the domain Ω if $f(\Omega, Y)$ is a domain in Y, and the inverse f^{-1} exists and is holomorphic on $f(\Omega)$. Let $\mathbb{B} = \{z \in X : \|z\| < 1\}$ be the unit ball in X. By $\mathbb{U}^n$, we denote the unit polydisk ($n-$copies of unit disk) in $\mathbb{C}^n$. Further, let $\partial\mathbb{U}^n$ denote the boundary and $\partial_0\mathbb{U}^n$ be the distinguished boundary of $\mathbb{U}^n$. Keeping the notation used in one variable, let $\mathcal{S}(\mathbb{B})$ represent the set of all normalized biholomorphic mappings from $\mathbb{B}$ into X i.e.

$$\mathcal{S}(\mathbb{B}) = \{f \in \mathcal{H}(\mathbb{B}) : f \text{ is biholomorphic on } \mathbb{B},\ f(0) = 0,\ Df(0) = I\}.$$

The study of the class $\mathcal{S}(\mathbb{B})$ was comparatively slow to develop, although it was suggested by Cartan [30] in 1933.

Starlike and Convex Mappings

It can be seen that the class $\mathcal{S}(\mathbb{B})$ is not normal when the dimension is greater than one, hence there can be no growth or covering theorems or coefficient bounds for the full class. Fitzgerald's [48] counterexample shows that the modulus of any combination of the coefficients in the Taylor expansion of a biholomorphic mapping in any domain in $\mathbb{C}^n$ is unbounded. This counterexample as well as failure of Bieberbach conjecture in higher dimensions strongly suggested that to extend certain results of geometric function theory from one complex variable to several complex variables and expect to obtain some positive results, it is worth adding some other conditions, such as convexity or starlikeness, on the biholomorphic mappings.

Let X and Y be two complex Banach spaces and Ω be a domain in X. A mapping $f \in \mathcal{H}(\Omega, Y)$ is said to be starlike with respect to $z_0 \in \Omega$ if it is biholomorphic on Ω and $f(\Omega)$ is a starlike domain with respect to $f(z_0)$ i.e. $tf(z) + (1-t)f(z_0) \in f(\Omega)$ for all $z \in \Omega$ and $t \in [0,1]$. We use the term starlike to refer starlike with respect to the origin. If $\mathbb{B}$ is the unit ball in X, let $\mathcal{S}^*(\mathbb{B})$ denote the subclass of $\mathcal{S}(\mathbb{B})$ consisting of normalized starlike mappings from $\mathbb{B}$ into X.

For each $z \in X \setminus \{0\}$, consider the set

$$T_z = \{l_z \in L(X, \mathbb{C}) : l_z(z) = \|z\|, \|l_z\| = 1\},$$

This set is non-empty according to the Hahn-Banach theorem. In 1973, Suffridge [178] derived the following necessary and sufficient condition:

Theorem 1.4.1. [178] Let $f : \mathbb{B} \to X$ be a locally biholomorphic mapping such that $f(0) = 0$. Then f is starlike if and only if

$$\mathrm{Re}(l_z([Df(z)]^{-1}f(z))) > 0, \quad z \in \mathbb{B} \setminus \{0\},\ l_z \in T_z.$$

In case of unit polydisc i.e $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, the above condition reduces to

$$\mathrm{Re}\, \frac{q_k(z)}{z_k} > 0, \quad \forall\, z \in \mathbb{U}^n \setminus \{0\},$$

where $q(z) = (q_1(z), q_2(z), \cdots, q_n(z))' = (D(f(z)))^{-1} f(z)$ is a column vector in $\mathbb{C}^n$ and k satisfies $|z_k| = \|z\| = \max_{1\le j\le n}\{|z_j|\}$.

Hamada et al. [68] defined the following:

Definition 1.4.2. [68] Let $f : \mathbb{B} \to X$ be a normalized locally biholomorphic mapping and $\alpha \in (0,1)$. Then f is starlike of order α if

$$\left|\frac{1}{\|z\|} l_z([Df(z)]^{-1} f(z)) - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \quad \forall\, z \in \mathbb{B}\setminus\{0\},\ l_z \in T(z).$$

In case of $X = \mathbb{C}^n$ and $\mathbb{B} = \mathbb{U}^n$, the above condition is equivalent to

$$\left|\frac{q_k(z)}{z_k} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \quad \forall\, z \in \mathbb{U}^n\setminus\{0\},$$

where $q(z) = (q_1(z), q_2(z), \cdots, q_n(z))' = (D(f(z)))^{-1} f(z)$ is a column vector in $\mathbb{C}^n$ and k satisfies $|z_k| = \|z\| = \max_{1\le j\le n}\{|z_j|\}$. For $\mathbb{B} = \mathbb{U}$ and $X = \mathbb{C}$, the relation is equivalent to

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{U}.$$

Let $\mathscr{S}^*_\alpha(\mathbb{B})$ denote the class of starlike mappings of order α on $\mathbb{B}$.

Definition 1.4.3. [84] Let $f : \mathbb{B} \to X$ be a normalized locally biholomorphic mapping and $\gamma \in (0,1]$. Then f is strongly starlike mapping of order γ if

$$\left|\arg l_z([Df(z)]^{-1} f(z))\right| < \frac{\pi}{2}\gamma, \quad \forall\, z \in \mathbb{B}\setminus\{0\},\ l_z \in T(z).$$

In case of $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, the above condition is equivalent to

$$\left|\arg\frac{q_k(z)}{z_k}\right| < \frac{\pi}{2}\gamma, \quad z \in \mathbb{U}^n\setminus\{0\}.$$

where $q(z) = (q_1(z), q_2(z), \cdots, q_n(z))' = (D(f(z)))^{-1} f(z)$ is a column vector in $\mathbb{C}^n$ and k satisfies $|z_k| = \|z\| = \max_{1\le j\le n}\{|z_j|\}$. In case of $\mathbb{B} = \mathbb{U}$ and $X = \mathbb{C}$, the relation is equivalent to

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\gamma, \quad z \in \mathbb{U}.$$

Let $\mathscr{S}\mathscr{S}^*_\gamma(\mathbb{B})$ denote the class of starlike mappings of order γ on $\mathbb{B}$.

Similarly, a biholomorphic mapping $f : \Omega \subseteq X \to Y$ is said to be convex on Ω if $f(\Omega)$ is a convex domain in Y. Let $\mathscr{C}(\mathbb{B})$ denote the subclass of $\mathscr{S}(\mathbb{B})$ consisting of normalized convex mappings from the unit ball $\mathbb{B}$ into the complex Banach space X. Numerous necessary and sufficient criteria for convexity on the different domains were established, for instance, see [81, 179, 178]. In case of holomorphic function defined on $\mathbb{U}$, Sheil-Small [167] and Suffridge [177] proved that *If $f \in \mathscr{A}$, then $f \in \mathscr{C}$ if and only if*

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z)-f(\zeta)} - \frac{z+\zeta}{z-\zeta}\right) \ge 0, \quad z, \zeta \in \mathbb{U}. \tag{1.4.1}$$

Trying to extend this idea in several variables, Roper and Suffridge [161] introduced the classes of quasi-convex mappings. Liu and Liu [112] further generalized these classes by defining the following:

Definition 1.4.4. [112] Suppose $\alpha \in [0,1)$ and $f : \mathbb{B} \to X$ is a normalized locally biholomorphic mapping. If

$$\operatorname{Re}\left\{l_z([Df(z)]^{-1}(D^2 f(z)(z^2) + Df(z)(z)))\right\} \geq \alpha \|z\|, \quad l_z \in T_z,\ z \in \mathbb{B} \setminus \{0\},$$

then f is called a quasi convex mapping of type B and order α on $\mathbb{B}$.

If $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, then the above condition reduces to

$$\left| \frac{q_k(z)}{z_k} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad \forall\, z \in \mathbb{U}^n \setminus \{0\},$$

where $q(z) = (q_1(z), q_2(z), \cdots, q_n(z))' = (Df(z))^{-1}(D^2 f(z)(z^2) + Df(z)(z))$ is a column vector in $\mathbb{C}^n$ and k satisfies $|z_k| = \|z\| = \max_{1 \leq j \leq n}\{|z_j|\}$. For $\mathbb{B} = \mathbb{U}$ and $X = \mathbb{C}$, the relation is equivalent to

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{U}.$$

Let $Q_\alpha(\mathbb{B})$ denote the class of quasi-convex mappings of type B and order α. When $\alpha = 0$, Definition 1.4.4 becomes the definition of quasi-convex mapping of type B, denoted by $Q(\mathbb{B})$, introduced by Roper and Suffridge [161].

1.5 Coefficient and Radius Problems

Finding the estimate of Taylor coefficients for functions belonging to some particular class falls under the category of coefficient problems. It also includes finding bounds for Fekete-Szegö functional ($|a_3 - \mu a_2^2|$, $\mu \in \mathbb{C}$), Zalcman functionals ($|a_n a_m - a_{n+m-1}|$, $m, n \in \mathbb{N}$), Hankel determinants, Toeplitz and Hermitian-Toeplitz determinants formed over the coefficients of functions. Many authors have worked in this direction for various subclasses of $\mathcal{S}$ [13, 107, 154, 166]. In 2018, Ali et al. [6] introduced the symmetric Toeplitz determinant $T_{m,n}(f)$ for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, defined as

$$T_{m,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \cdots & a_n \end{vmatrix} \quad (m \in \mathbb{N}). \tag{1.5.1}$$

They derived the sharp estimates for the determinants $T_{2,n}(f)$, $T_{3,1}(f)$, $T_{3,2}(f)$ and $T_{2,3}(f)$ when f belongs to the classes $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and other subclasses of $\mathcal{A}$. The study of best upper and lower estimates of Toeplitz and Hermitian-Toeplitz determinants for initial values of m and n is now in trend in GFT, which was initiated with the papers [39, 87, 86, 104].

Apart from the growth and distortion theorems established for the class $\mathcal{S}$ using the second coefficient estimates, the importance of coefficient bounds can be seen in the concept of Bohr's radius problems, see [20, 49, 123]. In radius problems, we try to find the disk of maximal radius $0 < r < 1$ so that every function in a class satisfies a certain property P. For instance, every function in the class $\mathscr{S}^*$ is convex in the disk of radius $r = 2 - \sqrt{3}$. For more on the coefficient and radius results, one can refer [59, 182].

1.6 Synopsis of the Thesis

Univalent functions exhibit deep geometric properties, that is why their study is an integral part of geometric function theory. Within this field, coefficient and radius problems are fundamental topics, with significant applications extending recently to areas such as digital image processing and fluid dynamics. This thesis primarily addresses coefficient and radius problems for certain subclasses of $\mathscr{S}$. It begins with coefficient-related problems associated with the Ma-Minda and Sakaguchi classes. Subsequently, it examines geometric properties, including growth estimates and the Bohr and Bohr-Rogosinski phenomena, for subclasses of analytic functions involving semigroup generators. Finally, the thesis explores the extension of the Toeplitz determinants for starlike mappings and quasi-convex mappings of type B in higher dimensions. The thesis is organized into six chapters. The first is an introduction chapter, which contains the essential definitions, terminologies, and foundational results necessary for the subsequent chapters.

Chapter two deals with the coefficient problems for the classes $\mathscr{S}^*(\varphi)$, $\mathscr{C}(\varphi)$, $\mathscr{S}^*_s(\varphi)$ and $\mathscr{C}_s(\varphi)$. For functions belonging to these classes the sharp estimates of second, third and fourth coefficients are already known. The sharp bound of the fifth coefficients were derived only for their particular subclasses, for instance when $\varphi(z) = 1 + ze^z$, $\varphi(z) = e^z$ and $\varphi(z) = \sqrt{1+z}$. This chapter is devoted to establishing the sharp bound of $|a_5|$ for these generalized classes when the initial Taylor coefficients of the function $\varphi(z)$ satisfy certain conditions. The established bounds provide new results and include some already proven bounds as special cases for the subclasses of $\mathscr{S}^*$, $\mathscr{C}$, $\mathscr{S}^*_s$ and $\mathscr{C}_s$ depending on the function $\varphi(z)$. We now furnish below a result of this chapter, obtained for the class $\mathscr{S}^*(\varphi)$:

1. Let $f \in \mathscr{S}^*(\varphi)$ and $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If

$$\textbf{C1}: \quad |B_1^2 + 2B_2| < 4B_1^2,$$

$$\textbf{C2}: |B_1^3 - B_1^2 B_2 + 18B_2^2 - 18B_1 B_3| < 3|(B_1^2 + 2B_1 + 2B_2)(2B_1^2 - 3B_1 + 3B_2)|,$$

$$\begin{aligned}\textbf{C3}: &|30B_1^7 - 9B_1^8 - B_1^6(66B_2 - 5) - 648B_2^3 + 324B_2^4 + B_1^5(170B_2 - 126) - 648B_2B_3^2 + B_1^3(-180B_2 \\ &+ 220B_2^2 + 108B_3 - 360B_2B_3) + B_1(1296B_2B_3 - 720B_2^2B_3) + 648B_2^2B_4 + B_1^4(108 + 10B_2 - 175B_2^2 \\ &+ 90B_3 + 162B_4) + B_1^2(-144B_2^2 + 4B_2^3 + 180B_2B_3 - 324B_3^2 - 648B_4 + 648B_2B_4)| < 8|9B_1^6 + 9B_1^7 \\ &+ B_1^4(-27 + 32B_2) + B_1^5(-52 + 63B_2) + 162B_2^2B_3 + B_1^3(81 - 189B_2 + 164B_2^2 + 9B_3) + B_1^2(18B_2^2 \\ &- 9B_2B_3) + B_1(-162B_2^2 + 198B_2^3 - 81B_3^2)|,\end{aligned}$$

$$\textbf{C4}: \quad 0 < (4B_1^2 + 6(B_2 - B_1)/((3B_1^2 + 6(B_2 - B_1)) < 1.$$

hold, then

$$|a_5| \leq \frac{B_1}{4}.$$

The inequality is sharp.

The conditions **C1**, **C2**, **C3** and **C4** hold true for various choices of the function $\varphi(z)$ including $1+\sin z$, $\sqrt{1+z}$ and $2/(1+e^{-z})$. Consequently, the bounds for the corresponding subclasses of starlike functions can be directly derived from our results, as given below:

1. If $f \in \mathscr{S}^*_{\sin}$, then $|a_5| \leq \frac{1}{4}$. The bound is sharp.

2. If $f \in \mathscr{S}^*_L$, then $|a_5| \leq 1/8$. The bound is sharp.

3. If $f \in \mathscr{S}^*_{SG}$, then $|a_5| \leq 1/8$. The bound is sharp.

In the **third chapter**, we study certain Hermitian-Toeplitz determinants for classes $\mathscr{S}^*(\varphi)$, $\mathscr{C}(\varphi)$, $\mathscr{S}^*_s(\varphi)$, $\mathscr{C}_s(\varphi)$. Additionally, we consider the subclasses of $\mathscr{K}$ depending on the function $g \in \mathscr{S}^*$, denoted by $\mathscr{K}(g)$. The sharp upper and lower bounds of second and third order Hermitian-Toeplitz determinants constructed over the coefficients of functions belonging to these classes are obtained. Since these classes generalize various subclasses of $\mathscr{S}^*$, $\mathscr{C}$, $\mathscr{S}^*_s$, $\mathscr{C}_s$ and $\mathscr{K}$, the bounds for these subclasses also follow directly. We list below a few results of this chapter related to class $\mathscr{S}^*(\varphi)$:

1. Let $f \in \mathscr{S}^*(\varphi)$ and $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. Then the following sharp bounds hold:

   (a) $1 - B_1^2 \leq T_2(1)(f) \leq 1$.

   (b) If $3B_1^4 - 8B_1^2 + 2B_1^2 B_2 - B_2^2 < 0$ and $B_1 \leq |B_2 + B_1^2|$, then
   $$T_3(1)(f) \leq 1.$$

   (c) If $3B_1^4 - 8B_1^2 + 2B_1^2 B_2 - B_2^2 \geq 0$ and $B_1 \leq |B_2 + B_1^2|$, then
   $$T_3(1)(f) \leq B_1^2(B_1^2 + B_2) - \frac{1}{4}\left(B_1^2 + B_2\right)^2 - 2B_1^2 + 1.$$

2. Let $f \in \mathscr{S}^*(\varphi)$ and $B_1^2 \geq B_2$.

   (a) If $\nu \notin [0,4]$, then
   $$T_3(1)(f) \geq \min\left\{1 - \frac{B_1^2}{4}, 1 - 2B_1^2 + \frac{3B_1^4}{4} + \frac{B_1^2 B_2}{2} - \frac{B_2^2}{4}\right\}.$$

   (b) If $\nu = 4$, then
   $$T_3(1)(f) \geq 1 - 2B_1^2 + \frac{3B_1^4}{4} + \frac{B_1^2 B_2}{2} - \frac{B_2^2}{4}.$$

   (c) If $\nu \in (0,4)$, then
   $$T_3(1)(f) \geq 1 - \frac{B_1^2}{4} - \frac{B_1^2(B_1^2 + 3B_1 - B_2)^2}{4\left(B_1(2B_1^2 - B_1 - 2B_2) + (3B_1^2 - B_2)(B_1^2 + B_2)\right)},$$

where
$$v = \frac{4B_1(B_1^2+3B_1-B_2)}{(3B_1^2-B_2)(B_1^2+B_2)+B_1(2B_1^2-2B_2-B_1)}.$$

The first two inequalities are sharp.

The sharp bounds for various other subclasses of $\mathscr{S}^*$ follow directly from these results. Some of them are as listed here:

1. If $f \in \mathscr{S}^*_{sin}$, then $0 \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$.
2. If $f \in \mathscr{S}^*_{\wp}$, then $0 \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$.
3. If $f \in \mathscr{S}_p$, then $1-(64/\pi^4) \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$.
4. If $f \in \mathscr{S}^*_{SG}$, then $3/4 \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$.
5. If $f \in \mathscr{S}^*_{B}$, then $0 \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$.
6. If $f \in \mathscr{S}^*_{\wp}$, then $T_3(1)(f) \geq 0$.
7. If $f \in \mathscr{S}^*_{\sin}$, then $T_3(1)(f) \geq -1/4$.
8. If $f \in \mathscr{S}_p$, then $T_3(1)(f) \geq 1-64\left(19\pi^4-24\pi^2-432\right)/(9\pi^8)$.
9. If $f \in \mathscr{S}^*_{RL}$, then $T_3(1)(f) \geq -9(4130\sqrt{2}-5861)/256$.

In the **fourth chapter**, we consider the class
$$\mathscr{A}_\beta = \left\{ f \in \mathscr{A} : \mathrm{Re}\left(\beta\frac{f(z)}{z}+(1-\beta)f'(z)\right) > 0,\ \ \beta \in [0,1] \right\}.$$

The class $\mathscr{A}_\beta$ were recently studied in the framework of so called filtration theory of semigroup generators [23]. Elin et al. [47] studied certain subclasses of $\mathscr{S}^*$ and their embedding in the class $\mathscr{A}_\beta$. Moreover, they found the radius of starlikeness for $f \in \mathscr{A}_\beta$. Generalizing their work, we obtained the radius of starlike of order α for the class $\mathscr{A}_\beta$. Additionally, we study some coefficient problems for $f \in \mathscr{A}_\beta$. We determine the sharp bounds of coefficient functional such as second order Hankel determinant, certain Zalcman functionals, third order Toeplitz and Hermitian-Toeplitz determinants. Furthermore, the sharp estimates of the n^{th} coefficients and the bounds for difference of successive coefficients are established along with growth and distortion theorems, which further apply to determine the Bohr and the Bohr-Rogosinski radii for the mentioned class. We say that the class $\mathscr{A}_\beta$ satisfies the Bohr-Rogosinski phenomenon if there exist r_N such that
$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(\Omega)), \quad m,N \in \mathbb{N}$$

holds in $|z| = r \leq r_N$.

Some of the results obtained in this chapter are as under:

1. If $f \in \mathscr{A}_\beta$, then for $|z| \leq r$, the following hold:

(i) $-\dfrac{\tilde{f}(-r)}{r} \leq \text{Re}\left(\dfrac{f(z)}{z}\right) \leq \dfrac{\tilde{f}(r)}{r}$,

(ii) $-\tilde{f}(-r) \leq |f(z)| \leq \tilde{f}(r)$,

where

$$\tilde{f}(z) = z\Big(-1+2\Big({}_2F_1\Big[1, \frac{1}{1-\beta}, \frac{2-\beta}{1-\beta}, z\Big]\Big)\Big).$$

All these estimations are sharp.

2. If $f \in \mathscr{A}_\beta$ is of the form (1.0.1), then

$$|f(z^m)| + \sum_{k=N}^{\infty} |a_k z^k| \leq d(0, \partial f(\mathbb{U}))$$

hold for $|z| = r \leq r_N$, where r_N is the root of the equation

$$\tilde{f}(r^m) + \tilde{f}(r) - \hat{f}(r) + \tilde{f}(-1) = 0,$$

with

$$\hat{f}(r) = \begin{cases} 0 & N = 1, \\ r & N = 2, \\ r + \sum_{n=2}^{N-1} \frac{2}{(n-(n-1)\beta)} r^n & N \geq 3. \end{cases}$$

The radius is sharp.

The **fifth chapter** is devoted to introduce the Toeplitz determinants in case of functions of several variables that extends the notion of Toeplitz determinants from one dimension to higher dimensions. We begin with finding the sharp bounds of Toeplitz determinants for a class of functions defined on $\mathbb{U}$ and then extend these bounds to a class of holomorphic mappings defined on the unit ball in a complex Banach space and on the unit polydisc in $\mathbb{C}^n$. The results obtained in higher dimensions yield Toeplitz determinant bounds for a class of starlike mappings defined on the unit ball in a complex Banach space, among other subclasses of normalised univalent mappings. The derived sharp estimates for the classes $\mathscr{S}^*(\mathbb{B})$ and $\mathscr{S}\mathscr{S}^*_\gamma(\mathbb{B})$ are as follows:

1. Let $f \in \mathscr{H}(\mathbb{B}, \mathbb{C})$ with $f(0) = 1$ and $F(z) = zf(z) \in \mathscr{S}^*(\mathbb{B})$. Then the following holds:

$$\Big|\Big(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\Big)^2 - \Big(\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\Big)^2\Big| \leq 13, \quad z \in \mathbb{B}\setminus\{0\}, \ l_z \in T_z.$$

If $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, then

$$\Big\|\frac{1}{3!}D^3F(0)\Big(z^2, \frac{D^3F(0)(z^3)}{3!}\Big) - \frac{1}{2!}D^2F(0)\Big(z, \frac{D^2F(0)(z^2)}{2!}\Big)\Big\| \leq 9\|z\|^5 + 4\|z\|^3, \quad z \in \mathbb{U}^n.$$

All the estimates are sharp.

2. Let $f \in \mathscr{H}(\mathbb{B}, \mathbb{C})$ with $f(0) = 1$ and $F(z) = zf(z) \in \mathscr{S}\mathscr{S}^*_\gamma(\mathbb{B})$. Then for $\gamma \in [1/3, 1]$, the following

holds:

$$\Big|\Big(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\Big)^2-\Big(\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\Big)^2\Big|\le 9\gamma^4+4\gamma^2,\quad z\in\mathbb{B}\setminus\{0\}, l_z\in T_z.$$

If $\mathbb{B}=\mathbb{U}^n$ and $X=\mathbb{C}^n$, then

$$\Big\|\frac{1}{3!}D^3F(0)\Big(z^2,\frac{D^3F(0)(z^3)}{3!}\Big)-\frac{1}{2!}D^2F(0)\Big(z,\frac{D^2F(0)(z^2)}{2!}\Big)\Big\|\le 9\|z\|^5\gamma^4+4\|z\|^3\gamma^2,\quad z\in\mathbb{U}^n.$$

All the estimates are sharp.

The **sixth chapter** is devoted to derive the sharp bounds of certain Toeplitz determinants for a class of holomorphic mappings defined on the unit ball of a complex Banach space. The derived bounds provide certain new results for the subclasses of normalized univalent mappings, including the class of quasi-convex mappings of type B. Additionally, we determine the sharp bounds of Toeplitz determinants for the class of convex function $\mathscr{C}$ such that $f\in\mathscr{C}$ has a zero of order $k+1$ at $z=0$, $k\in\mathbb{N}$. These results are extended to higher dimensions by determining the bounds of Toeplitz determinants for the subclass of quasi-convex mappings of type B. We list below some results belonging to this chapter:

1. Let $f\in\mathscr{H}(\mathbb{B},\mathbb{C})$ and $F(z)=zf(z)\in Q_\alpha(\mathbb{B})$. Then the following inequality holds:

$$\left|\left(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\right)^2-\left(\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\right)^2\right|\le\frac{2(1-\alpha)^2(2\alpha^2-6\alpha+9)}{9},\quad l_z\in T_z,\ z\in\mathbb{B}\setminus\{0\}.$$

If $\mathbb{B}=\mathbb{U}^n$ and $X=\mathbb{C}^n$, then

$$\left\|\frac{1}{3!}D^3F(0)\left(z^2,\frac{D^3F(0)(z^3)}{3!}\right)-\frac{1}{2!}D^2F(0)\left(z,\frac{D^2F(0)(z^2)}{2!}\right)\right\|\le(1-\alpha)^2\|z\|^3+\frac{(2\alpha^2-5\alpha+3)^2\|z\|^5}{9}.$$

All these bounds are sharp.

2. Let $f:\mathbb{B}\to\mathbb{C}$ and $f(z)\neq 0$ for $z\in\mathbb{B}$. If $F(z)=zf(z)\in Q(\mathbb{B})$ and $z=0$ is a zero of order $k+1$ $(k\in\mathbb{N})$ of $F(z)-z$, then

$$\left|\left(\frac{l_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!\|z\|^{2k+1}}\right)^2-\left(\frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!\|z\|^{k+1}}\right)^2\right|\le\frac{(k+2)^2}{k^4(2k+1)^2}+\frac{4}{k^2(k+1)^2}$$

for $z\in\mathbb{B}\setminus\{0\}$. The bound is sharp.

Chapter 2

Coefficient Estimates of Ma-Minda and Sakaguchi Classes

In this chapter, we estimate the sharp bound of the fifth coefficient of functions belonging to the Ma-Minda classes: $\mathscr{S}^*(\varphi)$ and $\mathscr{C}(\varphi)$, as well as the Sakaguchi classes: $\mathscr{S}_s^*(\varphi)$ and $\mathscr{C}_s(\varphi)$, whenever, the coefficients of $\varphi(z)$ satisfy certain conditions. The results obtained yield several new special cases including some already known bounds.

2.1 Introduction

Obtaining sharp estimates for each Taylor coefficients of a function remains a formidable challenge in geometric function theory. These estimates not only reveal valuable information about other geometric properties of functions but also find applications in diverse fields such as image processing and Borel distribution [1, 128, 139]. While the sharp estimate of $|a_n|$, where $n \in \mathbb{N}$, is established for functions in the classes $\mathscr{S}^*$ and $\mathscr{C}$, determining the bound for functions in their subclasses proves to be a challenging endeavor. For instance, for functions in the classes $\mathscr{S}^*(\varphi)$, $\mathscr{C}(\varphi)$, $\mathscr{S}_s^*(\varphi)$, and $\mathscr{C}_s(\varphi)$, only the sharp bounds for $|a_n|$, where $n = 2,3,4$, are currently known. For $n \geq 5$, finding the bound of $|a_n|$ for functions belonging to these classes is still an open problem.

The sharp estimates for the second and third coefficients of functions in $\mathscr{S}^*(\varphi)$ and $\mathscr{C}(\varphi)$ were obtained from the Fekete-Szegö functional estimate for the class $\mathscr{C}(\varphi)$, established in [117], while the sharp bound of the fourth coefficient was derived by Ali et al. [7]. For different subclasses of $\mathscr{S}^*$ depending on the different choices of $\varphi(z)$ in $\mathscr{S}^*(\varphi)$, the bound for $|a_5|$ is known, which immediately provide the bound for the corresponding subclass of $\mathscr{C}$. Goel and Kumar [52] determined the sharp bound of $|a_5|$ for functions in the class $\mathscr{S}^*_{SG}$, whereas Kumar and Gangania [92] addressed the same issue for the class $\mathcal{S}^*_{\wp}$. For functions belonging to the classes $\mathcal{S}^*_L$ and $\mathcal{S}^*_{RL}$, Sokół [171] and Mendiratta et al. [118] proposed the following conjectures, respectively,

$$|a_n| \leq \frac{1}{2(n-1)} \quad \text{and} \quad |a_n| \leq \frac{5-3\sqrt{2}}{2(n-1)} \quad \text{for} \quad n \geq 5.$$

Ravichandran and Verma [153] settled these both conjectures for $n = 5$ by establishing the sharp estimate of the fifth coefficient for functions belonging to the classes $\mathcal{S}^*_L$ and $\mathcal{S}^*_{RL}$. For more work in this direction, we refer [14, 36, 95] and the references cited therein.

This chapter is devoted to finding the sharp estimate of $|a_5|$ for a general choice of $\varphi(z)$, provided that the coefficients of φ satisfy certain stipulated conditions, covering many of the previously known bounds and presents some new examples as well. We require the following lemmas to prove our results.

Lemma 2.1.1. [125, Lemma I] If the functions $1+\sum_{n=1}^{\infty} b_n z^n$ and $1+\sum_{n=1}^{\infty} c_n z^n$ are in $\mathcal{P}$, then the same holds for the function

$$1+\frac{1}{2}\sum_{n=1}^{\infty} b_n c_n z^n.$$

Lemma 2.1.2. [125, Lemma II] Let $h(z) = 1+u_1z+u_2z^2+\cdots$ and $1+G(z) = 1+d_1z+d_2z^2+\cdots$ be functions in $\mathcal{P}$, and set

$$\gamma_n = \frac{1}{2^n}\left[1+\frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}u_k\right], \quad \gamma_0 = 1.$$

If A_n is defined by

$$\sum_{n=1}^{\infty}(-1)^{n+1}\gamma_{n-1}G^n(z) = \sum_{n=1}^{\infty} A_n z^n,$$

then $|A_n| \leq 2$.

It is worth recalling the Möbius function Ψ_ζ, which maps the unit disk onto the unit disk and is given by

$$\Psi_\zeta(z) = \frac{z-\zeta}{1-\bar{\zeta}z}, \quad \zeta \in \mathbb{U}. \tag{2.1.1}$$

Lemma 2.1.3. [32, Lemma 2.4] If $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$, then for some $\zeta_i \in \overline{\mathbb{U}}$, $i \in \{1,2,3\}$,

$$p_1 = 2\zeta_1, \tag{2.1.2}$$

$$p_2 = 2\zeta_1^2+2(1-|\zeta_1|^2)\zeta_2, \tag{2.1.3}$$

$$p_3 = 2\zeta_1^3+4(1-|\zeta_1|^2)\zeta_1\zeta_2-2(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2+2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3. \tag{2.1.4}$$

For $\zeta_1, \zeta_2 \in \mathbb{U}$ and $\zeta_3 \in \partial\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$, there is a unique function $p = L \circ \omega \in \mathcal{P}$ with p_1, p_2 and p_3 as in (2.1.2)-(2.1.4), where

$$\omega(z) = z\Psi_{-\zeta_1}(z\Psi_{-\zeta_2}(\zeta_3 z)), \tag{2.1.5}$$

and the function L is given by (1.2). That is

$$p(z)=\frac{1+(\overline{\zeta_2}\zeta_3+\overline{\zeta_1}\zeta_2+\zeta_1)z+(\overline{\zeta_1}\zeta_3+\zeta_1\overline{\zeta_2}\zeta_3+\zeta_2)z^2+\zeta_3z^3}{1+(\overline{\zeta_2}\zeta_3+\overline{\zeta_1}\zeta_2-\zeta_1)z+(\overline{\zeta_1}\zeta_3-\zeta_1\overline{\zeta_2}\zeta_3-\zeta_2)z^2-\zeta_3z^3},\quad z\in\mathbb{U}.$$

Conversely, if $\zeta_1,\zeta_2\in\mathbb{U}$ and $\zeta_3\in\overline{\mathbb{U}}:=\{z\in\mathbb{C}:|z|\le 1\}$ are given, then we can construct a (unique) function $p\in\mathscr{P}$ of the form (1.2.2) such that p_i satisfy the identities in (2.1.2)-(2.1.4). For this, we define

$$\omega(z)=\omega_{\zeta_1,\zeta_2,\zeta_3}(z)=z\Psi_{-\zeta_1}(z\Psi_{-\zeta_2}(\zeta_3 z)),\quad z\in\mathbb{U}, \tag{2.1.6}$$

where Ψ_ζ is the function given as in (2.1.1). Then $\omega\in\mathscr{B}_0$. Moreover, if we define $p(z)=(1+\omega(z))/(1-\omega(z))$, $z\in\mathbb{U}$, then p is represented by (1.2.2), where p_1, p_2 and p_3 satisfy the identities in (2.1.2)-(2.1.4) (see the proof of [32, Lemma 2.4]).

2.2 Ma-Minda Classes

Recall the Ma-Minda class $\mathscr{S}^*(\varphi)$ and $\mathscr{C}(\varphi)$ given in (1.3.1). We begin with the following lemma:

Lemma 2.2.1. If $-1<\sigma<1$, then $F(z)=(1+2\sigma z+z^2)/(1-z^2)$ belongs to $\mathscr{P}$.

Proof. Let us consider

$$\omega(z)=\frac{F(z)-1}{F(z)+1}=\frac{z(z+\sigma)}{1+\sigma z}.$$

From (2.1.1), we have

$$\omega(z)=z\Psi_{-\sigma}(z),\quad z\in\mathbb{U}.$$

Since $\Psi_{-\sigma}(z)$ is a conformal automorphism of $\mathbb{U}$, which gives $|\omega(z)|<1$ and $\omega(0)=0$. Therefore ω is a Schwarz function and $F\in\mathscr{P}$. □

Theorem 2.2.1. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{S}^*(\varphi)$ and $\varphi(z)$ be as given in (1.3.2). If the following conditions hold:

C1 : $|B_1^2+2B_2|<4B_1^2,$

C2 : $|B_1^3-B_1^2B_2+18B_2^2-18B_1B_3|<3|(B_1^2+2B_1+2B_2)(2B_1^2-3B_1+3B_2)|,$

C3 : $|30B_1^7-9B_1^8-B_1^6(66B_2-5)-648B_2^3+324B_2^4+B_1^5(170B_2-126)-648B_2B_3^2+B_1^3(-180B_2+220B_2^2+108B_3-360B_2B_3)+B_1(1296B_2B_3-720B_2^2B_3)+648B_2^2B_4+B_1^4(108+10B_2-175B_2^2+90B_3+162B_4)+B_1^2(-144B_2^2+4B_2^3+180B_2B_3-324B_3^2-648B_4+648B_2B_4)|<8|9B_1^6+9B_1^7+B_1^4(-27+32B_2)+B_1^5(-52+63B_2)+162B_2^2B_3+B_1^3(81-189B_2+164B_2^2+9B_3)+B_1^2(18B_2^2-9B_2B_3)+B_1(-162B_2^2+198B_2^3-81B_3^2)|,$

C4 : $0<(4B_1^2+6(B_2-B_1))/(3B_1^2+6(B_2-B_1))<1,$

then

$$|a_5|\le\frac{B_1}{4}.$$

The bound is sharp.

Proof. Suppose $f \in \mathcal{S}^*(\varphi)$, then

$$\frac{zf'(z)}{f(z)} = \varphi(\omega(z)),$$

where ω is a Schwarz function. Corresponding to this ω, let there exists a function $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ such that $\omega(z) = (p(z)-1)/(p(z)+1)$. Then by comparing the coefficients of the same powers of z obtained by the series expansion of $f(z)$ together with $p(z)$ and $\varphi(z)$, we obtain

$$a_2 = \frac{B_1 p_1}{2}, \quad a_3 = \frac{1}{8}\left(B_1^2 p_1^2 - B_1 p_1^2 + 2B_1 p_2 + B_2 p_1^2\right),$$
$$a_4 = \frac{1}{48}\left(p_1 p_2 (6B_1^2 - 8B_1 + 8B_2) + p_1^3 (B_1^3 - 3B_1^2 + 3B_1 B_2 + 2B_1 - 4B_2 + 2B_3) + 8B_1 p_3\right)$$

and

$$a_5 = \frac{B_1}{8} I, \tag{2.2.1}$$

where

$$I = p_4 + I_1 p_1^4 + I_2 p_1^2 p_2 + I_3 p_1 p_3 + I_4 p_2^2 \tag{2.2.2}$$

with

$$\left.\begin{aligned} I_1 &= \frac{B_1^4 - 6B_1^3 + 11B_1^2 + 6B_1^2 B_2 - 6B_1 + 3B_2^2 - 22B_1 B_2 + 18B_2 - 18B_3 + 8B_1 B_3 + 6B_4}{48B_1}, \\ I_2 &= \frac{3B_1^3 - 11B_1^2 + 9B_1 - 18B_2 + 11B_1 B_2 + 9B_3}{12B_1}, \quad I_3 = \frac{2B_1^2 - 3B_1 + 3B_2}{3B_1} \end{aligned}\right\} \tag{2.2.3}$$

and

$$I_4 = \frac{B_1^2 - 2B_1 + 2B_2}{4B_1}. \tag{2.2.4}$$

Let $q(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be in $\mathcal{P}$, then by Lemma 2.1.1, we have

$$1 + \frac{1}{2}(p(z)-1) * (q(z)-1) = 1 + \frac{1}{2}\sum_{n=1}^{\infty} b_n p_n z^n \in \mathcal{P}.$$

If we assume $h(z) = 1 + \sum_{n=1}^{\infty} u_n z^n \in \mathcal{P}$ and take $1 + G(z) := 1 + \frac{1}{2}\sum_{n=1}^{\infty} b_n p_n z^n$, then Lemma 2.1.2 gives

$$|A_4| \leq 2,$$

where

$$A_4 = \frac{1}{2}\gamma_0 b_4 p_4 - \frac{1}{4}\gamma_1 b_2^2 p_2^2 - \frac{1}{2}\gamma_1 b_1 b_3 p_1 p_3 + \frac{3}{8}\gamma_2 b_1^2 b_2 p_1^2 p_2 - \frac{1}{16}\gamma_3 b_1^4 p_1^4 \tag{2.2.5}$$

with

$$\gamma_0 = 1, \ \gamma_1 = \frac{1}{2}\left(1 + \frac{1}{2}u_1\right), \ \gamma_2 = \frac{1}{4}\left(1 + u_1 + \frac{1}{2}u_2\right) \text{ and } \gamma_3 = \frac{1}{8}\left(1 + \frac{3}{2}u_1 + \frac{3}{2}u_2 + \frac{1}{2}u_3\right). \tag{2.2.6}$$

So, from (2.2.2) and (2.2.5), we can observe that if there exist $q,\ h \in \mathcal{P}$ such that

$$b_4 = 2, \quad I_1 = -\frac{1}{128}\left(1 + \frac{3}{2}u_1 + \frac{3}{2}u_2 + \frac{1}{2}u_3\right) b_1^4, \quad I_2 = \frac{3}{32}\left(1 + u_1 + \frac{u_2}{2}\right) b_1^2 b_2,$$

$$I_3 = -\frac{1}{4}(1+\frac{u_1}{2})b_1b_3 \quad \text{and} \quad I_4 = -\frac{1}{8}\left(1+\frac{u_1}{2}\right)b_2^2,$$

then we have

$$I = A_4. \tag{2.2.7}$$

The bound for $|A_4|$ can be obtained from Lemma 2.1.2. Consequently, we can estimate the bound for $|I|$ and thus we arrive at the desired bound by using (2.2.1). To prove the theorem, we construct the functions q and h in such a way that we obtain (2.2.7).

From Lemma 2.1.3, suppose that the functions q and h are constructed by taking $\zeta_1, \zeta_2 \in \mathbb{U}$, $\zeta_3 \in \overline{\mathbb{U}}$ and $\xi_1, \xi_2 \in \mathbb{U}$, $\xi_3 \in \overline{\mathbb{U}}$, respectively, as follows:

$$q = L \circ \omega_1 \quad \text{and} \quad h = L \circ \omega_2, \tag{2.2.8}$$

where

$$\omega_1(z) = z\Psi_{-\zeta_1}(z\Psi_{-\zeta_2}(\zeta_3 z)), \quad \omega_2(z) = z\Psi_{-\xi_1}(z\Psi_{-\xi_2}(\xi_3 z)) \tag{2.2.9}$$

and $L(z)$ is given by (1.2.3). So, again from Lemma 2.1.3, the b_i's and u_i's, $i \in \{1,2,3\}$ are given by

$$\begin{aligned} &b_1 = 2\zeta_1, \quad b_2 = 2\zeta_1^2 + 2(1-|\zeta_1|^2)\zeta_2, \\ &b_3 = 2\zeta_1^3 + 4(1-|\zeta_1|^2)\zeta_1\zeta_2 - 2(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3 \end{aligned}$$

and

$$\begin{aligned} &u_1 = 2\xi_1, \quad u_2 = 2\xi_1^2 + 2(1-|\xi_1|^2)\xi_2, \\ &u_3 = 2\xi_1^3 + 4(1-|\xi_1|^2)\xi_1\xi_2 - 2(1-|\xi_1|^2)\overline{\xi_1}\xi_2^2 + 2(1-|\xi_1|^2)(1-|\xi_2|^2)\xi_3. \end{aligned}$$

There may be many solutions for the above set of equations. For our purpose, we impose some restrictions on the parameters. We take all $\xi_i \in \mathbb{R}$, then

$$\begin{aligned} &u_1 = 2\xi_1, \quad u_2 = 2\xi_1^2 + 2(1-\xi_1^2)\xi_2, \\ &u_3 = 2\xi_1^3 + 4(1-\xi_1^2)\xi_1\xi_2 - 2(1-\xi_1^2)\xi_1\xi_2^2 + 2(1-\xi_1^2)(1-\xi_2^2)\xi_3. \end{aligned} \tag{2.2.10}$$

Further, if we define

$$\xi_1 = -\frac{B_1^2+2B_2}{2B_1}, \quad \xi_2 = \frac{B_1^3 - B_1^2B_2 + 18B_2^2 - 18B_1B_3}{3(B_1^2+2B_1+2B_2)(2B_1^2-3B_1+3B_2)},$$

and

$$\begin{aligned} \xi_3 = \Big(& -9B_1^8 + 30B_1^7 - B_1^6(66B_2-5) + 2B_1^5(85B_2-63) + 4B_1^3(5B_2(11B_2-18B_3-9)+27B_3) \\ &+4B_1^2\left(B_2^3 - 36B_2^2 - 81B_3^2 + 45B_2B_3 + 162(B_2-1)B_4\right) - 144B_1(5B_2-9)B_2B_3 + 324B_2(-2B_3^2 \\ &+B_2((B_2-2)B_2+2B_4)) + 18B_1^4(9B_4+5B_3+6) - 5B_1^4B_2(35B_2-2)\Big)\Big/\Big(8(3B_1^4+2B_1^3+18B_2^2 \\ &+B_1^2(10B_2-9) - 9B_1B_3)(B_1(3B_1^2+B_1+11B_2-9)+9B_3)\Big), \end{aligned}$$

then the conditions **C1**, **C2** and **C3** on the coefficients B_1, B_2, B_3 and B_4, respectively, yield

$$|\xi_1| < 1, \quad |\xi_2| < 1 \quad \text{and} \quad |\xi_3| < 1.$$

Using these ξ_i's in (2.2.10), we can obtain u_i's, which in turn by using (2.2.6) gives

$$\left.\begin{aligned}
\gamma_1 &= \frac{1}{4}\left(2 - B_1 - \frac{2B_2}{B_1}\right),\\
\gamma_2 &= \frac{\left(B_1^2+2B_2-2B_1\right)\left(3B_1^3-11B_1^2+B_1\left(11B_2+9\right)+9\left(B_3-2B_2\right)\right)}{24B_1\left(2B_1^2+3B_2-3B_1\right)},\\
\gamma_3 &= -\frac{1}{64B_1\left(2B_1^2+3B_2-3B_1\right)^2}\Big(3\left(B_1^2+2B_2-2B_1\right)^2\left(B_1^4-6B_1^3+B_1^2(6B_2+11)\right.\\
&\quad \left.+B_1\left(8B_3-22B_2-6\right)+3\left(B_2^2+6B_2-6B_3+2B_4\right)\right)\Big).
\end{aligned}\right\} \qquad (2.2.11)$$

Let us consider

$$q(z) = \frac{1+2\sigma z+z^2}{1-z^2},$$

with $\sigma = \sqrt{(4B_1^2+6(B_2-B_1)/((3B_1^2+6(B_2-B_1))}$, then

$$b_1 = b_3 = 2\sigma \quad \text{and} \quad b_2 = b_4 = 2. \qquad (2.2.12)$$

If we choose B_1 and B_2 such that $0<\sigma<1$, which is equivalent to condition **C4**. Then by Lemma 2.2.1, we have $q \in \mathscr{P}$. On putting the values of $b_i's$ and $\gamma_i's$ obtained from (2.2.11) and (2.2.12), respectively, in (2.2.5), we get (2.2.7), which together with (2.2.1) gives the desired bound for $|a_5|$.

The sharpness of the bound follows from the function $k_{\varphi,5}(z) = z+\sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}^*(\varphi)$ defined by (1.3.3), as for this function, $a_2 = a_3 = a_4 = 0$ and $a_5 = B_1/4$. □

It can be easily seen that for $\varphi(z) = 1+\sin z$, all conditions **C1**, **C2**, **C3** and **C4** are valid. Hence, Theorem 2.2.1 provide the following result for the class $\mathscr{S}^*_{\sin}$ immediately.

Corollary 2.2.2. If $f \in \mathscr{S}^*_{\sin}$, then $|a_5| \leq \frac{1}{4}$. The bound is sharp.

For $\varphi(z) = (1+Az)/(1+Bz)$, where $-1 \leq B < A \leq 1$, Theorem 2.2.1 gives the following result directly for the class $\mathscr{S}^*[A,B]$:

Corollary 2.2.3. If $f \in \mathscr{S}^*[A,B]$ such that A and B satisfy the following conditions

i : $|A^2-4BA+3B^2| < 4(A-B)^2$,

ii : $|(A-B)^3(B+1)| < 3|(A^2+(2-4B)A+B(3B-2))(2A^2-(7B+3)A+B(5B+3))|$,

iii : $|(A-B)^4(9A^4-6(17B+5)A^3+(427B^2+260B-5)A^2+(-778B^3-740B^2+20B+126)A$
$+3(172B^4+230B^3+7B^2-102B-36))| < 8|(A-B)^3(9A^4+(9-99B)A^3+(407B^2-59B$
$-52)A^2+(-742B^3+118B^2+293B-27)A+506B^4-68B^3-403B^2+27B+81)|$,

iv : $0 < (10B-4A+6)/(9B-3A+6) < 1$, then

$$|a_5| \le \frac{A-B}{4}.$$

The bound is sharp.

Example 2.2.4. If we take $A=0$ and $B=-1/2$, all the conditions in Corollary 2.2.3 hold true. Therefore, if $f(z)=z+\sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*[0,-1/2]$, then $|a_5| \le 1/8$.

In case of the classes $\mathcal{S}^*_{SG}$, $\mathcal{S}^*_{L}$, $\mathcal{S}^*_{q_b}$ and $\mathcal{S}^*_{RL}$, the coefficients of the corresponding function φ satisfy the conditions **C1**, **C2**, **C3** and **C4**. Consequently, the bounds of $|a_5|$ for these classes can be derived from Theorem 2.2.1 as special cases.

Remark 2.2.1.
1. If $f \in \mathcal{S}^*_{SG}$, then $|a_5| \le 1/8$ [52, Theorem 4.1].
2. If $f \in \mathcal{S}^*_{L}$, then $|a_5| \le 1/8$ [153, Theorem 3.1(a)].
3. If $f \in \mathcal{S}^*_{q_b}$, then $|a_5| \le b/8$, where $b \in (0,1]$ [36, Theorem 3.1].
4. If $f \in \mathcal{S}^*_{RL}$, then $|a_5| \le \left(5-3\sqrt{2}\right)/8$ [153, Theorem 3.1(b)].
5. If $\varphi(z)=((1+z)/(1-z))^{\gamma}$ $(0<\gamma\le 1)$, then the conditions of Theorem 2.2.1 are satisfied only for $0<\gamma\le\gamma_0\approx 0.350162$. Therefore, $|a_5|\le \gamma/2$ for $f \in \mathcal{S}\mathcal{S}^*(\gamma)$ whenever $0<\gamma\le\gamma_0$ [8].

Theorem 2.2.5. Let $\varphi(z)$ be as defined in (1.3.2), whose coefficients satisfy the conditions **C1**, **C2**, **C3** and **C4**. If $f(z)=z+\sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}(\varphi)$, then

$$|a_5| \le \frac{B_1}{20}.$$

The inequality is sharp.

Proof. Since $f \in \mathcal{C}(\varphi)$, we have

$$1+\frac{zf''(z)}{f'(z)} = \varphi\left(\frac{p(z)-1}{p(z)+1}\right), \tag{2.2.13}$$

where $p \in \mathcal{P}$ is given by (1.2.2). By comparison of the coefficients of z, z^2, z^3 and z^4 in (2.2.13) with the series expansion of f, φ and p, we get

$$\begin{aligned}
a_2 &= \frac{B_1 p_1}{4}, \quad a_3 = \frac{1}{24}\left(B_1^2 p_1^2 - B_1 p_1^2 + 2B_1 p_2 + B_2 p_1^2\right),\\
a_4 &= \frac{1}{192}\left(p_1 p_2 (6B_1^2 - 8B_1 + 8B_2) + p_1^3 (B_1^3 - 3B_1^2 + 3B_1 B_2 + 2B_1 - 4B_2 + 2B_3) + 8B_1 p_3\right)
\end{aligned}$$

and

$$a_5 = \frac{B_1}{40} I, \tag{2.2.14}$$

where

$$I = p_4 + I_1 p_1^4 + I_2 p_1^2 p_2 + I_3 p_1 p_3 + I_4 p_2^2$$

with I_1, I_2, I_3 and I_4 given as in (2.2.3) and (2.2.4). Using the same method as in Theorem 2.2.1, we obtain

$$|I| \le 2,$$

when B_1, B_2, B_3 and B_4 satisfy all the conditions **C1**, **C2**, **C3** and **C4**. Thus bound of $|a_5|$ follows from (2.2.14).

It is evident that for the function $h_{\varphi,5}(z) \in \mathscr{C}(\varphi)$ defined by (1.3.4), $a_2 = a_3 = a_4 = 0$ and $a_5 = B_1/20$, showing the sharpness of the bound. □

2.3 Sakaguchi Classes

Recall the Sakaguchi classes $\mathscr{S}_s^*(\varphi)$ and $\mathscr{C}_s(\varphi)$ given by (1.3.6) and (1.3.7), respectively. Ravichandran [151] derived some sufficient conditions in terms of convolution and provided growth and distortion estimates for functions belonging to the classes $\mathscr{S}_s^*(\varphi)$ and $\mathscr{C}_s(\varphi)$. Later, Shanmugam et al. [166] determined the sharp bound of Fekete-Szegö functional for the classes $\mathscr{S}_s^*(\varphi)$ and $\mathscr{C}_s(\varphi)$, which easily provides the bounds for the initial coefficients $|a_2|$ and $|a_3|$. Further, the sharp bound of $|a_4|$ for functions belonging to these classes was established by Khatter et al. [80] and for certain significant choices of φ such as

$$\left.\begin{aligned} \mathscr{S}_{s,e}^* &:= \mathscr{S}_s^*(e^z), \quad \mathscr{S}_{s,L}^* := \mathscr{S}_s^*(\sqrt{1+z}) \quad \text{and} \\ \mathscr{S}_{s,RL}^* &:= \mathscr{S}_s^*\left(\sqrt{2}-(\sqrt{2}-1)\sqrt{(1-z)/(1+2(\sqrt{2}-1)z)}\right), \end{aligned}\right\} \tag{2.3.1}$$

they obtained the sharp estimate of $|a_5|$. This section is devoted to determining the sharp estimates of $|a_5|$ for functions belonging to the classes $\mathscr{S}_s^*(\varphi)$ and $\mathscr{C}_s(\varphi)$. The extremal functions in various problems for $\mathscr{S}_s^*(\varphi)$ and $\mathscr{C}_s(\varphi)$ are $k_{s,\varphi,n}$ and $h_{s,\varphi,n}$, respectively, defined by

$$\frac{2zk'_{s,\varphi,n}(z)}{k_{s,\varphi,n}(z)-k_{s,\varphi,n}(-z)} = \varphi(z^{n-1}) \ \text{ and } \ \frac{(2zh'_{s,\varphi,n}(z))'}{(h_{s,\varphi,n}(z)-h_{s,\varphi,n}(-z))'} = \varphi(z^{n-1}). \tag{2.3.2}$$

We denote $k_{s,\varphi,1}$ and $h_{s,\varphi,1}$ simply as $k_{s,\varphi}$ and $h_{s,\varphi}$, respectively.

Theorem 2.3.1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_s^*(\varphi)$ and $\varphi(z)$ be as given by (1.3.2). If the following conditions hold:

P1 : $|B_1^3 - 2B_1B_2 + 2B_2^2| < |2B_1^2 - B_1^3 - 2B_1B_2|$,

P2 : $|B_1^3 - B_1^2B_2 + 3B_2^2 - 3B_1B_3| < 3|B_1^3 - B_1^2 + B_2^2|$,

P3 : $|B_1^7 - B_1^6(8B_2+3) - 6B_1^4(B_2(3B_2+2B_3+2) - 6B_3 + 9B_4) + B_1^5(7B_2(B_2+4) - 24B_3 + 18B_4) + 6B_1^3(B_2^3 - 2B_2^2 + 8B_2B_3 - 3B_3^2 + 6(B_2+1)B_4) - 6B_1B_2(3B_2^3 - 6B_3^2 + B_2^2(4B_3-6) + 6B_2(B_4 - 2B_3)) + 18B_2^2(-2B_3^2 + B_2((B_2-2)B_2 + 2B_4)) + B_1^2B_2(B_2(B_2(5B_2+6) - 24B_3 + 18B_4) - 36(B_4 + 2B_3))| < 2|((B_1-2)B_1 + 2B_2)(B_1(2B_1+B_2-3) + 3B_3)(4B_1^3 + 6B_2^2 - B_1^2(B_2+3) - 3B_1B_3)|$,

P4 : $0 < (2B_1 - B_1^2 - 2B_2)/(2(B_1 - B_2)) < 1$,

then

$$|a_5| \leq \frac{B_1}{4}.$$

The bound is sharp.

Proof. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{S}_s^*(\varphi)$, then there exist a Schwarz function $\omega(z)$ such that

$$\frac{2zf'(z)}{f(z)-f(-z)}=\varphi(\omega(z)).$$

By the one-to-one correspondence between the class of Schwarz functions and the class $\mathscr{P}$, we obtain

$$\frac{2zf'(z)}{f(z)-f(-z)}=\varphi\left(\frac{p(z)-1}{p(z)+1}\right) \tag{2.3.3}$$

for some $p(z)=1+\sum_{n=1}^{\infty}p_nz^n\in\mathscr{P}$. On the comparison of the same powers of z with the series expansions of functions $f(z)$, $\varphi(z)$ and $p(z)$, the above equation yields

$$a_2=\frac{B_1p_1}{4},\quad a_3=\frac{1}{8}\Big(B_2p_1^2-B_1p_1^2+2B_1p_2\Big),$$
$$a_4=\frac{1}{64}\Big(p_1^3(-B_1^2+B_1B_2+2B_1-4B_2+2B_3)+p_1p_2(2B_1^2-8B_1+8B_2)+8B_1p_3\Big)$$

and

$$a_5=\frac{B_1}{8}(\Upsilon_1p_1^4+\Upsilon_2p_1^2p_2+\Upsilon_3p_1p_3+\Upsilon_4p_2^2+p_4), \tag{2.3.4}$$

where

$$\left.\begin{aligned}\Upsilon_1&=\frac{B_1^2-2B_1+6B_2-2B_1B_2+B_2^2-6B_3+2B_4}{16B_1},\\ \Upsilon_2&=\frac{3B_1-B_1^2-6B_2+B_1B_2+3B_3}{4B_1},\quad \Upsilon_3=\frac{B_2-B_1}{B_1},\\ \Upsilon_4&=\frac{B_1^2-2B_1+2B_2}{4B_1}.\end{aligned}\right\} \tag{2.3.5}$$

Let us consider that $q(z)=1+\sum_{n=1}^{\infty}\kappa_nz^n$ and $h(z)=1+\sum_{n=1}^{\infty}\nu_nz^n$ are the members of $\mathscr{P}$, then by Lemma 2.1.1 for $p\in\mathscr{P}$, we have

$$1+H(z):=1+\sum_{n=1}^{\infty}\frac{p_n\kappa_n}{2}z^n\in\mathscr{P}. \tag{2.3.6}$$

For $h\in\mathscr{P}$ and the function $1+H(z)$ given in (2.3.6), Lemma 2.1.2 gives

$$A_4=\frac{1}{2}\gamma_0\kappa_4p_4-\frac{1}{4}\gamma_1\kappa_2^2p_2^2-\frac{1}{2}\gamma_1\kappa_1\kappa_3p_1p_3+\frac{3}{8}\gamma_2\kappa_1^2\kappa_2p_1^2p_2-\frac{1}{16}\gamma_3\kappa_1^4p_1^4, \tag{2.3.7}$$

where $\gamma_0=1$,

$$\gamma_1=\frac{1}{2}\left(1+\frac{1}{2}\nu_1\right),\ \gamma_2=\frac{1}{4}\left(1+\nu_1+\frac{1}{2}\nu_2\right),\ \gamma_3=\frac{1}{8}\left(1+\frac{3}{2}\nu_1+\frac{3}{2}\nu_2+\frac{1}{2}\nu_3\right) \tag{2.3.8}$$

and

$$|A_4|\le 2. \tag{2.3.9}$$

Now, in order to establish the required bound, we construct the functions $h(z)$ and $q(z)$ such that

$$A_4=\Upsilon_1p_1^4+\Upsilon_2p_1^2p_2+\Upsilon_3p_1p_3+\Upsilon_4p_2^2+p_4, \tag{2.3.10}$$

where Υ's and A_4 are given in (2.3.5) and (2.3.7), respectively. For $0<\tau<1$, define

$$q(z)=\frac{1+2\tau z+2\tau^2z^2+2\tau z^3+z^4}{1-z^4},$$

which yields

$$\kappa_1=\kappa_3=2\tau,\quad \kappa_2=2\tau^2 \quad\text{and}\quad \kappa_4=2. \tag{2.3.11}$$

From [15, Theorem 1], we have $q\in\mathscr{P}$. To construct function $h(z)$, using Lemma 2.1.3, let

$$h(z)=\frac{1+\omega_1(z)}{1-\omega_1(z)}$$

such that

$$\omega_1(z)=z\Psi_{-\varepsilon_1}(z\Psi_{-\varepsilon_2}(\varepsilon_3 z)), \tag{2.3.12}$$

where $\varepsilon_1,\varepsilon_2\in\mathbb{U}$ and $\varepsilon_3\in\overline{\mathbb{U}}$. Thus, we have

$$\nu_1=2\varepsilon_1,\quad \nu_2=2\varepsilon_1^2+2(1-|\varepsilon_1|^2)\varepsilon_2$$

and

$$\nu_3=2\varepsilon_1^3+4(1-|\varepsilon_1|^2)\varepsilon_1\varepsilon_2-2(1-|\varepsilon_1|^2)\overline{\varepsilon_1}\varepsilon_2^2+2(1-|\varepsilon_1|^2)(1-|\varepsilon_2|^2)\varepsilon_3.$$

The above set of equations may be satisfied by many ε's. For our purpose, we impose some restriction on ε's and take all ε's as real numbers. Therefore,

$$\left.\begin{array}{l}\nu_1=2\varepsilon_1,\quad \nu_2=2\varepsilon_1^2+2(1-\varepsilon_1^2)\varepsilon_2,\\ \nu_3=2\varepsilon_1^3+4(1-\varepsilon_1^2)\varepsilon_1\varepsilon_2-2(1-\varepsilon_1^2)\varepsilon_1\varepsilon_2^2+2(1-\varepsilon_1^2)(1-\varepsilon_2^2)\varepsilon_3.\end{array}\right\} \tag{2.3.13}$$

In addition, if we define

$$\begin{aligned}\varepsilon_1=&\frac{B_1^3-2B_1B_2+2B_2^2}{2B_1^2-B_1^3-2B_1B_2},\quad \varepsilon_2=\frac{B_1^3-B_1^2B_2+3B_2^2-3B_1B_3}{3(-B_1^2+B_1^3+B_2^2)},\\ \varepsilon_3=&\Big(B_1^7-B_1^6(8B_2+3)-6B_1^4(B_2(3B_2+2B_3+2)-6B_3+9B_4)+B_1^5(7B_2(B_2\\ &+4)-24B_3+18B_4)+6B_1^3(B_2^3-2B_2^2+8B_2B_3-3B_3^2+6(B_2+1)B_4)\\ &-6B_1B_2(3B_2^3-6B_3^2+B_2^2(4B_3-6)-6B_2(2B_3-B_4))+18B_2^2(-2B_3^2\\ &+B_2((B_2-2)B_2+2B_4))+B_1^2B_2(-36(2B_3+B_4)+B_2(B_2(6+5B_2)-24B_3\\ &+18B_4))\Big)\Big/\Big(2((B_1-2)B_1+2B_2)(B_1(2B_1+B_2-3)+3B_3)(4B_1^3+6B_2^2\\ &-B_1^2(3+B_2)-3B_1B_3)\Big)\end{aligned}$$

and

$$\tau=\sqrt{\frac{2B_1-B_1^2-2B_2}{2(B_1-B_2)}},$$

then by the hypothesis **P1**, **P2**, **P3** and **P4**, we have $|\varepsilon_1|<1$, $|\varepsilon_2|<1$, $|\varepsilon_3|<1$ and $0<\tau<1$, respectively. Putting these defined ε's in (2.3.13), we obtain ν_i's, which in turn together with (2.3.8) yields

$$\left.\begin{aligned}
\gamma_1 &= -\frac{(B_1-B_2)^2}{B_1(B_1^2-2B_1+2B_2)},\\
\gamma_2 &= -\frac{(B_1-B_2)^2(B_1^2+6B_2-B_1(3+B_2)-3B_3)}{3B_1(B_1^2-2B_1+2B_2)^2},\\
\gamma_3 &= -\frac{(B_1-B_2)^2(B_1^2+6B_2+B_2^2-2B_1(1+B_2)-6B_3+2B_4)}{4B_1(B_1^2-2B_1+2B_2)^2}.
\end{aligned}\right\} \tag{2.3.14}$$

On putting the values of $\kappa_i' s$ and $\gamma_i' s$ from (2.3.11) and (2.3.14), respectively, in (2.3.7), we get (2.3.10). Using the bound $|A_4| \leq 2$ in (2.3.10), we get

$$|\Upsilon_1 p_1^4 + \Upsilon_2 p_1^2 p_2 + \Upsilon_3 p_1 p_3 + \Upsilon_4 p_2^2 + p_4| \leq 2,$$

which together with (2.3.4) gives the desired bound of $|a_5|$.

The function $k_{s,\varphi,5} = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_s^*(\varphi)$, given by (2.3.2), serves as the extremal function since, for this function, $a_2 = a_3 = a_4 = 0$ and $a_5 = B_1/4$. □

For $-1 \leq B < A \leq 1$, consider the classes $\mathscr{S}_s^*[A,B] := \mathscr{S}_s^*((1+Az)/(1+Bz))$ and $\mathscr{S}_{s,SG}^* := \mathscr{S}_s^*(2/(1+e^{-z}))$. These classes are analogues to the corresponding classes of starlike functions introduced and studied in [52, 73]. Theorem 2.3.1 directly gives the following result for these classes.

Corollary 2.3.2. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_s^*[A,B]$ such that A and B satisfy the following conditions:

$$\begin{aligned}
&\textbf{i}: && |(A-B)^2(A+B+2B^2)| < |(A-3B-2)(A-B)^2|,\\
&\textbf{ii}: && |(A-B)^3(B+1)| < 3|(A-B)^2(A-1+(B-1)B)|,\\
&\textbf{iii}: && |(A-B)^5(B+1)(A^2(7B+1)+B(B(38+(12-17B)B)+15)+A(B(B(5B-31)-27)\\
& && -3))| < 2|(A-B)^4(A-3B-2)(A(B-2)-4B^2+2B+3)(A(B+4)+2B(B-2)-3)|,\\
&\textbf{iv}: && 0 < (3B-A+2)/(2B+2) < 1,
\end{aligned}$$

then

$$|a_5| \leq \frac{A-B}{4}.$$

The bound is sharp.

Example 2.3.3. For $A = 0$ and $B = -1/2$, all conditions in Corollary 2.3.2 are satisfied. Thus, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_s^*[0,-1/2]$, then $|a_5| \leq 1/8$.

Corollary 2.3.4. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_{s,SG}^*$, then $|a_5| \leq 1/8$ and the bound is sharp.

For the classes $\mathscr{S}_{s,L}^*$ and $\mathscr{S}_{s,RL}^*$, the coefficients of the corresponding function φ satisfy the conditions **P1**, **P2**, **P3** and **P4**. Consequently, Theorem 2.3.1 yields the following result for these classes:

Remark 2.3.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_{s,L}^*$, then $|a_5| \leq 1/8$ [80, Theorem 5(a)].

Remark 2.3.2. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_{s,RL}^*$, then $|a_5| \leq (5-3\sqrt{2})/8$ [80, Theorem 5(b)].

Theorem 2.3.5. If $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathscr{C}_s(\varphi)$ and coefficients of $\varphi(z)$ satisfy the conditions **P1**, **P2**, **P3** and **P4**, then

$$|a_5|\leq\frac{B_1}{20}.$$

The bound is sharp.

Proof. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{C}_s(\varphi)$, then there exists a Schwarz function $\omega(z)$ such that

$$\frac{(2zf'(z))'}{(f(z)-f(-z))'}=\varphi(\omega(z)).$$

Corresponding to the Schwarz function $\omega(z)$, let there is a function $p(z)=1+\sum_{n=1}^{\infty}p_nz^n\in\mathscr{P}$ satisfying $p(z)=(1+\omega(z))/(1-\omega(z))$. Thus, we obtain

$$\frac{(2zf'(z))'}{(f(z)-f(-z))'}=\varphi\left(\frac{p(z)-1}{p(z)+1}\right). \tag{2.3.15}$$

Comparing the coefficients of the same powers of z after applying the series expansion of $f(z)$, $\varphi(z)$ and $p(z)$ leads to

$$a_2=\frac{B_1p_1}{8},\quad a_3=\frac{1}{24}\left(B_2p_1^2-B_1p_1^2+2B_1p_2\right),$$
$$a_4=\frac{1}{256}\left(p_1^3(-B_1^2+B_1B_2+2B_1-4B_2+2B_3)+p_1p_2(2B_1^2-8B_1+8B_2)+8B_1p_3\right)$$

and

$$a_5=\frac{B_1}{20}\left(\Upsilon_1p_1^4+\Upsilon_2p_1^2p_2+\Upsilon_3p_1p_3+\Upsilon_4p_2^2+p_4\right),$$

where Υ_i's are given in (2.3.5). Since, Υ_i's are the same as in the case of $\mathscr{S}_s^*(\varphi)$, therefore following the same methodology as in Theorem 2.3.1, we get the bound of $|a_5|$. The equality case holds for the function $h_{s,\varphi,4}=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{C}_s(\varphi)$ given by (2.3.2). □

We can define the classes $\mathscr{C}_s[A,B]$, $\mathscr{C}_{s,e}$, $\mathscr{C}_{s,SG}$, $\mathscr{C}_{s,L}$ and $\mathscr{C}_{s,RL}$ in a similar manner as $\mathscr{S}_s^*[A,B]$, $\mathscr{S}_{s,e}^*$, $\mathscr{S}_{s,SG}^*$, $\mathscr{S}_{s,L}^*$ and $\mathscr{S}_{s,RL}^*$, respectively. For these classes, Theorem 2.3.5 yields the following sharp bounds directly:

Corollary 2.3.6. If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{C}_s[A,B]$ such that A and B satisfy the conditions given in Corollary 2.3.2, then

$$|a_5|\leq\frac{A-B}{20}.$$

Corollary 2.3.7. If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{C}_{s,RL}$, then

$$|a_5|\leq\frac{5-3\sqrt{2}}{40}.$$

Corollary 2.3.8. If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{C}_{s,e}$, then $|a_5|\leq 1/20$.

Corollary 2.3.9. If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{C}_{s,L}$, then $|a_5|\leq 1/40$.

Corollary 2.3.10. If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{C}_{s,SG}$, then $|a_5|\leq 1/40$.

By focusing on the coefficients of functions, we delve into establishing the sharp bounds of specific Hermitian-Toeplitz determinants in the following chapter. These determinants are constructed over the coefficients of functions belonging to the classes discussed in this chapter, allowing us to explore their properties in more detail.

Highlights of the chapter

Determining sharp coefficient bounds for functions belonging to well-known classes is a significant problem in geometric function theory. Consequently, there is an extensive body of literature addressing the sharp bounds of initial coefficients up to the fourth for functions in the classes $\mathscr{S}^*(\varphi)$, $\mathscr{C}(\varphi)$, $\mathscr{S}^*_s(\varphi)$, and $\mathscr{C}_s(\varphi)$. However, the bounds for the fifth and higher coefficients remain unknown. This chapter addresses this gap by successfully finding the sharp fifth coefficient bound for functions in the aforementioned classes, given certain constraints on the coefficients of $\varphi(z)$. These constraints are commonly satisfied by well-known Ma-Minda functions such as $2/(1+e^{-z})$, $\sqrt{1+z}$, $1+\sin z$ and $\sqrt{2}-(\sqrt{2}-1)\sqrt{(1-z)/(1+2(\sqrt{2}-1)z)}$. Our findings include special cases for other subclasses as well.

The work of this chapter is covered in the following research papers:

S. Giri, S. S. Kumar, Sharp bounds of fifth coefficient and Hermitian-Toeplitz determinants for Sakaguchi classes. Bull. Korean Math. Soc. 2024; 61:317-333. https://doi.org/10.4134/BKMS.b230018

S. Giri, and S. Sivaprasad Kumar, Fifth Coefficient Estimate for Certain Starlike Functions, arXiv preprint arXiv:2201.05803 (Communicated).

Chapter 3

Hermitian-Toeplitz Determinants for Certain Univalent Functions

Sharp upper and lower bounds for the second and third order Hermitian-Toeplitz determinants are obtained for certain subclasses of normalized univalent functions defined on the unit disk. Applications of these results are also discussed for several widely known classes.

3.1 Introduction

Toeplitz matrices and determinants have many applications in pure and in applied mathematics. Toeplitz determinants are closely related to Hankel determinants. Hankel matrices have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. Ye and Lim [194] showed that any $n \times n$ matrix over $\mathbb{C}$ generically can be written as the product of some Toeplitz matrices or Hankel matrices. For a summary of applications of Toeplitz matrices to a wide range of mathematics, we refer to [194].

Extending the notion of Toeplitz determinants defined in (1.5.1), Cudna et al. [39] introduced and studied the Hermitian-Toeplitz determinants $T_m(n)(f)$ for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, defined as

$$T_m(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ \bar{a}_{n+1} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{a}_{n+m-1} & \bar{a}_{n+m-2} & \cdots & a_n \end{vmatrix} \qquad (m \in \mathbb{N}), \tag{3.1.1}$$

where $\bar{a}_k = \overline{a_k}$. It is also noted that the determinant $T_m(1)(f)$ is rotationally invariant that is determinants $T_m(1)(f)$ and $T_m(1)(f_\theta)$ are same, where $f_\theta(z) = e^{-i\theta} f(e^{i\theta} z)$ and $\theta \in \mathbb{R}$. Since $T_m(1)(f)$ for $f \in \mathscr{A}$ is a determinant of the Hermitian matrix, it is a real number. For $f \in \mathscr{A}$, we have $a_1 = 1$, thus the second order Hermitian-Toeplitz determinant is

$$T_2(1)(f) = \begin{vmatrix} 1 & a_2 \\ \bar{a}_2 & 1 \end{vmatrix} = 1 - |a_2|^2$$

and the third order Hermitian-Toeplitz determinant is given by

$$T_3(1)(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ \bar{a}_2 & 1 & a_2 \\ \bar{a}_3 & \bar{a}_2 & 1 \end{vmatrix} = 2\operatorname{Re}\left(a_2^2 \bar{a}_3\right) - 2|a_2|^2 - |a_3|^2 + 1. \tag{3.1.2}$$

Firstly, Cudna et al. [39] established the sharp bounds of $T_2(1)(f)$ and $T_3(1)(f)$ for the classes $\mathscr{S}^*(\alpha)$ and $\mathscr{C}(\alpha)$. Generalizing these results, Kumar et al. [99] established the bounds of certain Hermitian-Toeplitz determinants for the classes $\mathscr{S}^*[A,B]$ and $\mathscr{C}[A,B]$. The bounds of $T_2(1)(f)$ and $T_3(1)(f)$ for functions belonging to the class $\mathscr{S}$ and some of its subclasses were obtained by Obradović and Tuneski [134].

Recently, Kumar [96] obtained the sharp lower and upper bounds of $T_2(1)(f)$ and $T_3(1)(f)$ for the classes $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$ and $\mathscr{F}_4$. Kowalczyk et al. [87] also attained the sharp bounds for the classes $\mathscr{F}_2$ and $\mathscr{F}_3$. For more work in this direction, one can see [4, 6, 33, 104, 105].

For particular choices of φ in $\mathscr{S}^*(\varphi)$, $\mathscr{C}(\varphi)$, $\mathscr{S}_s^*(\varphi)$ and $\mathscr{C}_s(\varphi)$, many authors obtained upper as well as lower bounds of Hermitian-Toeplitz determinants. Here, we consider this problem for these general classes and establish sharp estimates of the second and third-order Hermitian-Toeplitz determinants. Furthermore, we find the bounds of the same for the class $\mathscr{K}$ with a fixed starlike function g. Since the bounds depend on the coefficients of g, we write $\mathscr{K}(g)$ to denote such subclasses of $\mathscr{K}$. Clearly,

$$\mathscr{K} = \bigcup_{g \in \mathscr{S}^*} \mathscr{K}(g).$$

We use the following lemma in deriving our results.

Lemma 3.1.1. [107][143, p. 166] If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathscr{P}$, then

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)\zeta,$$

for some $\zeta \in \overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$.

3.2 Hermitian-Toeplitz determinants for $\mathscr{S}^*(\varphi)$ and $\mathscr{C}(\varphi)$

In the forthcoming results, we obtain the sharp bounds of $T_2(1)(f)$ and $T_3(1)(f)$ for the classes $\mathscr{S}^*(\varphi)$ and $\mathscr{C}(\varphi)$.

Theorem 3.2.1. Let $f \in \mathscr{S}^*(\varphi)$ and $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. Then the following holds:

(i) $1 - B_1^2 \leq T_2(1)(f) \leq 1$.

(ii) If $3B_1^4 - 8B_1^2 + 2B_1^2 B_2 - B_2^2 < 0$ and $B_1 \leq |B_2 + B_1^2|$, then

$$T_3(1)(f) \leq 1. \tag{3.2.1}$$

(iii) If $3B_1^4 - 8B_1^2 + 2B_1^2 B_2 - B_2^2 \geq 0$ and $B_1 \leq |B_2 + B_1^2|$, then

$$T_3(1)(f) \leq B_1^2(B_1^2 + B_2) - \frac{1}{4}\left(B_1^2 + B_2\right)^2 - 2B_1^2 + 1. \tag{3.2.2}$$

All these inequalities are sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}^*(\varphi)$, then $|a_2| \leq B_1$ and whenever $B_1 \leq |B_2 + B_1^2|$, we have

$$|a_3| \leq \frac{1}{2}|B_1^2 + B_2|$$

(see [4]). It is clear that the lower and the upper estimates of $T_2(1)(f)$ will be deduced directly from the bound of $|a_2|$, which are sharp for the functions k_φ and $k_{\varphi,3}$, respectively, given by.

$$k_\varphi(z) = z \exp \int_0^z \frac{\varphi(t) - 1}{t} dt \quad \text{and} \quad k_{\varphi,3}(z) = z \exp \int_0^z \frac{\varphi(t^2) - 1}{t} dt. \tag{3.2.3}$$

Now we proceed to estimate $T_3(1)(f)$. Using the inequality $\operatorname{Re}(a_2^2 \bar{a}_3) \leq |a_2|^2 |a_3|$ in (3.1.2), we have

$$T_3(1)(f) \leq 2|a_2|^2|a_3| - 2|a_2|^2 - |a_3|^2 + 1 =: u(|a_3|),$$

where $u(x) = 2|a_2|^2 x - 2|a_2|^2 - x^2 + 1$. Clearly, maximum of $u(x)$ is the upper bound for $T_3(1)(f)$. Since $B_1 \in [0,2]$ and $B_2 \in [-2,2]$ (see [143, Corollary 2.3]), we have $x = |a_3| \in [0,3]$. Since,

$$u'(x) = 2\left(|a_2|^2 - x\right) \quad \text{and} \quad u''(x) = -2,$$

$u(x)$ attains its maximum value at $x_0 = |a_2|^2$. Now, there arise two cases:
Case 1: When $|a_2|^2$ lies in the range of x that is $|a_2|^2 < |a_3|$, then

$$\begin{aligned} \max u(x) = u(|a_2|^2) &= \left(|a_2|^2 - 1\right)^2 \\ &\leq \begin{cases} 1 & |a_2|^2 \leq 2, \\ \left(\frac{1}{2}\left(B_1^2 + B_2\right) - 1\right)^2 & 2 \leq |a_2|^2 \leq \frac{1}{2}|B_1^2 + B_2|. \end{cases} \\ &= \begin{cases} 1 & |B_1^2 + B_2| \leq 4, \\ \left(\frac{1}{2}\left(B_1^2 + B_2\right) - 1\right)^2 & 4 \leq |B_1^2 + B_2|. \end{cases} \end{aligned}$$

Case 2: If $|a_3| \leq |a_2|^2$, then

$$\begin{aligned}\max u(x) &= u(|a_3|)\\ &= u\left(\frac{1}{2}|B_1^2+B_2|\right) \leq B_1^2(B_1^2+B_2) - \frac{1}{4}\left(B_1^2+B_2\right)^2 - 2B_1^2 + 1.\end{aligned}$$

It can be easily seen that for $|B_1^2+B_2| \geq 4$,

$$\begin{aligned}\max\left\{\left(\frac{1}{2}\left(B_1^2+B_2\right)-1\right)^2, B_1^2(B_1^2+B_2) - \frac{1}{4}\left(B_1^2+B_2\right)^2 - 2B_1^2+1\right\}\\ = B_1^2(B_1^2+B_2) - \frac{1}{4}\left(B_1^2+B_2\right)^2 - 2B_1^2+1.\end{aligned}$$

Also,

$$B_1^2(B_1^2+B_2) - \frac{1}{4}\left(B_1^2+B_2\right)^2 - 2B_1^2+1 \geq 1$$

whenever

$$3B_1^4 - 8B_1^2 + 2B_1^2B_2 - B_2^2 \geq 0.$$

Combining all these facts, we obtain the upper bound of $T_3(1)(f)$.

Sharpness: When $3B_1^4 - 8B_1^2 + 2B_1^2B_2 - B_2^2 < 0$, the inequality (3.2.1) is sharp for $f(z) = z$ and when $3B_1^4 - 8B_1^2 + 2B_1^2B_2 - B_2^2 \geq 0$, the equality sign in (3.2.2) holds for the function k_φ given by (3.2.3). □

Note that, when $\varphi(z)$ is $1+\sin z$ and $1+ze^z$, the class $\mathcal{S}^*(\varphi)$ reduces to the classes $\mathcal{S}^*_{sin}$ and $\mathcal{S}^*_{\wp}$ introduced in [34] and [92], respectively. From Theorem 3.2.1, we easily obtain the following results.

Corollary 3.2.2. (i) If $f \in \mathcal{S}^*_{sin}$, then $0 \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$.

(ii) If $f \in \mathcal{S}^*_{\wp}$, then $0 \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$.

(iii) If $f \in \mathcal{S}_p$, then $1-(64/\pi^4) \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$.

Remark 3.2.1. Theorem 3.2.1 yields some already known results for different subclasses of $\mathcal{S}^*$, obtained by an appropriate choice of φ.

(i) If $f \in \mathcal{S}^*_{SG}$, then $3/4 \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$ [33, Theorem 2.1].

(ii) If $f \in \mathcal{S}^*_{B}$, then $0 \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$ [33, Theorem 2.2].

(iii) If $f \in \Delta^*$, then $0 \leq T_2(1)(f) \leq 1$ and $T_3(1)(f) \leq 1$ [98, Theorem 2].

(iv) If $f \in \mathcal{S}^*$, then $T_3(1)(f) \leq 8$ [39, Corollary 3], [87, Corollary 2].

(v) If $f \in \mathcal{S}^*(1/2)$, then $T_3(1)(f) \leq 1$ [39, Corollary 4].

(vi) If $f \in \mathcal{S}^*_{Ne}$, then $T_3(1)(f) \leq 1$ [90, Theorem 4.2].

(vii) If we take $\varphi(z) = (1+(1-2\alpha)z)/(1-z), \alpha \in (0,1]$, then we obtain the upper bound of $T_3(1)(f)$ for $f \in \mathcal{S}^*(\alpha)$ [39, Theorem 3].

(viii) For $\varphi(z) = ((1+z)/(1-z))^\beta, \beta \in [1/3,1]$, we get the bound of $T_3(1)(f)$ for function f belonging to the class of strongly starlike function $\mathcal{SS}^*(\beta)$ [98, Theorem 1], [87, Theorem 3].

(ix) For $\varphi(z) = (1+Az)/(1+Bz)$, where $-1 \leq B < A \leq 1$ with $A-B \leq |A^2-3AB+2B^2|$, we get the bound of $T_3(1)(f)$ for the class $\mathcal{S}^*[A,B]$ [99, Theorem 2].

In a similar fashion, the bounds of $T_2(1)(f)$ can also be found for all the above mentioned classes.

Theorem 3.2.3. Let $f \in \mathcal{C}(\varphi)$ and $\varphi(z) = 1+B_1z+B_2z^2+B_3z^3+\cdots$. Then

(i) $1-(B_1^2)/4 \leq T_2(1)(f) \leq 1$.

(ii) $T_3(1)(f) \leq 1$, provided $B_1 \leq |B_2+B_1^2|$.

All these estimates are sharp.

Proof. Let $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}(\varphi)$, then $|a_2| \leq B_1/2$ and whenever $B_1 \leq |B_2+B_1^2|$ hold, we have

$$|a_3| \leq \frac{1}{6}|B_1^2+B_2|$$

(see [4]). The bounds of $T_2(1)(f)$ can be obtained using the bound of $|a_2|$. Further, the functions h_φ and $h_{\varphi,3}$ satisfying

$$1+\frac{zh''_\varphi(z)}{h'_\varphi(z)} = \varphi(z) \text{ and } 1+\frac{zh''_{\varphi,3}(z)}{h'_{\varphi,3}(z)} = \varphi(z^2), \tag{3.2.4}$$

respectively, act as extremal functions for lower and upper estimates of $T_2(1)(f)$. Now using the technique of Theorem 3.2.1, in context of the class $\mathcal{C}(\varphi)$, we get

$$T_3(1)(f) \leq u(x) = 2|a_2|^2x-2|a_2|^2-x^2+1,$$

where $x = |a_3|$. Clearly, $u(x)$ attain its maximum value at $x = |a_2|^2$. Therefore,

$$\max u(x) = u(|a_2|^2) = \left(|a_2|^2-1\right)^2 \leq 1.$$

Thus, we get $T_3(1)(f) \leq 1$. The bound is sharp for the identity function $f(z) = z$. □

For different choices of φ, we obtain several known results as listed below, for different subclasses of $\mathcal{C}$ as a special case of Theorem 3.2.3.

Remark 3.2.2. (i) If $\varphi(z) = (1+z)/(1-z)$, then $f \in \mathcal{C}$ and $T_3(1)(f) \leq 1$ [104, Theorem 1].

(ii) If $\varphi(z) = (1+(1-2\alpha)z)/(1-z)$, then $f \in \mathcal{C}(\alpha)$ and $T_3(1)(f) \leq 1$ [39, Theorem 5].

(iii) If $\varphi(z) = ((1+z)/(1-z))^\beta$, $\beta \in [1/3,1]$, then $f \in \mathcal{CC}(\beta)$ and $T_3(1)(f) \leq 1$ [87, Theorem 5].

(iv) If $f \in \mathcal{C}[A,B]$, where $-1 \leq B < A \leq 1$ and $A-B \leq |A^2-3AB+2B^2|$, then $T_3(1)(f) \leq 1$ [99, Theorem 4].

Theorem 3.2.4. Let $f \in \mathcal{S}^*(\varphi)$ and $B_1^2 \geq B_2$.

(i) If $\nu \notin [0,4]$, then

$$T_3(1)(f) \geq \min\left\{1-\frac{B_1^2}{4}, 1-2B_1^2+\frac{3B_1^4}{4}+\frac{B_1^2B_2}{2}-\frac{B_2^2}{4}\right\}. \tag{3.2.5}$$

(ii) If $\nu = 4$, then

$$T_3(1)(f) \geq 1 - 2B_1^2 + \frac{3B_1^4}{4} + \frac{B_1^2 B_2}{2} - \frac{B_2^2}{4}. \tag{3.2.6}$$

(iii) If $\nu \in (0,4)$, then

$$T_3(1)(f) \geq 1 - \frac{B_1^2}{4} - \frac{B_1^2(B_1^2 + 3B_1 - B_2)^2}{4\left(B_1(2B_1^2 - B_1 - 2B_2) + (3B_1^2 - B_2)(B_1^2 + B_2)\right)}, \tag{3.2.7}$$

where

$$\nu = \frac{4B_1(B_1^2 + 3B_1 - B_2)}{(3B_1^2 - B_2)(B_1^2 + B_2) + B_1(2B_1^2 - 2B_2 - B_1)}.$$

The first two inequalities are sharp.

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}^*(\varphi)$, therefore

$$\frac{zf'(z)}{f(z)} = \varphi(\omega(z)), \quad z \in \mathbb{U}, \tag{3.2.8}$$

where ω is a Schwarz function satisfying $|\omega(z)| \leq |z|$ and $\omega(0) = 0$. Corresponding to the function ω, there is some Carathéodory function $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathscr{P}$, which satisfy $\omega(z) = (p(z) - 1)/(p(z)+1)$. On comparing the coefficients of z and z^2 in (3.2.8) with the series expansion of f, ω and φ, we obtain

$$a_2 = \frac{B_1 p_1}{2} \quad \text{and} \quad a_3 = \frac{1}{8}\left((B_1^2 - B_1 + B_2)p_1^2 + 2B_1 p_2\right). \tag{3.2.9}$$

Since the class $\mathscr{S}^*(\varphi)$ and the class of Carathéodory functions $\mathscr{P}$ are invariant under rotation and $|p_1| \leq 2$, without lose of generality, we can take $p_1 \in [0,2]$. Using the values of a_2 and a_3 from (3.2.9) together with Lemma 3.1.1, we get

$$2\operatorname{Re}\left(a_2^2 \bar{a}_3\right) = \frac{B_1^2 p_1^2}{16}\left((B_1^2 - B_1 + B_2)p_1^2 + B_1\left(p_1^2 + (4 - p_1^2)\operatorname{Re}\bar{\zeta}\right)\right)$$

and

$$\begin{aligned} -|a_3|^2 = -\frac{1}{64}\Bigg(& (B_1^2 - B_1 + B_2)^2 p_1^4 + B_1^2\left(p_1^4 + (4 - p_1^2)^2|\zeta|^2 + 2p_1^2(4 - p_1^2)\operatorname{Re}\bar{\zeta}\right) \\ & + 2B_1(B_1^2 - B_1 + B_2)p_1^2\left(p_1^2 + (4 - p_1^2)\operatorname{Re}\bar{\zeta}\right)\Bigg). \end{aligned}$$

Now, by (3.1.2)

$$\begin{aligned} T_3(1)(f) = -\frac{1}{64}\Bigg(& (B_1^2 - B_1 + B_2)^2 p_1^4 + B_1^2\left(p_1^4 + (4 - p_1^2)^2|\zeta|^2 + 2p_1^2(4 - p_1^2)\operatorname{Re}\bar{\zeta}\right) \\ & + 2B_1(B_1^2 - B_1 + B_2)p_1^2\left(p_1^2 + (4 - p_1^2)\operatorname{Re}\bar{\zeta}\right)\Bigg) - \frac{B_1^2 p_1^2}{2} + 1 \\ & + \frac{B_1^2 p_1^2}{16}\left((B_1^2 - B_1 + B_2)p_1^2 + B_1\left(p_1^2 + (4 - p_1^2)\operatorname{Re}\bar{\zeta}\right)\right). \end{aligned}$$

We can rewrite the above equation as

$$\begin{aligned} T_3(1)(f) = & \left(\frac{3B_1^4+2B_1^2B_2-B_2^2}{64}\right)p_1^4-\frac{B_1^2}{2}p_1^2-\frac{B_1^2}{64}(4-p_1^2)^2|\zeta|^2 \\ & +\left(\frac{B_1^3-B_1B_2}{32}\right)p_1^2(4-p_1^2)\operatorname{Re}\bar{\zeta}+1=: F(p_1^2,|\zeta|,\operatorname{Re}\bar{\zeta}). \end{aligned}$$

Note that $F(p_1^2,|\zeta|,\operatorname{Re}\bar{\zeta}) \geq F(p_1^2,|\zeta|,-|\zeta|)$, therefore

$$T_3(1)(f) \geq \left(\frac{3B_1^4+2B_1^2B_2-B_2^2}{64}\right)p_1^4-\frac{B_1^2}{2}p_1^2-\frac{B_1^2}{64}\left(4-p_1^2\right)^2|\zeta|^2-\left(\frac{B_1^3-B_1B_2}{32}\right)p_1^2(4-p_1^2)|\zeta|+1.$$

Let $x=p_1^2 \in [0,4]$ and $y=|\zeta| \in [0,1]$, then $T_3(1)(f) \geq F(x,y)$, where

$$F(x,y)=\left(\frac{3B_1^4+2B_1^2B_2-B_2^2}{64}\right)x^2-\frac{B_1^2}{2}x-\frac{B_1(B_1^2-B_2)}{32}x(4-x)y-\frac{B_1^2}{64}(4-x)^2y^2+1.$$

If $B_1^2 \geq B_2$, then for any fixed x and $y \in [0,1]$, we have

$$\frac{\partial F}{\partial y}=-\frac{B_1(B_1^2-B_2)}{32}x(4-x)-\frac{B_1^2}{32}(4-x)^2y \leq 0,$$

which means that $F(x,y)$ is decreasing function of y and

$$F(x,y) \geq F(x,1)=G(x)$$

with

$$G(x)=\frac{1}{64}\Big((3B_1^2-B_2)(B_1^2+B_2)+B_1(2B_1^2-2B_2-B_1)\Big)x^2-\frac{B_1^2}{4}-\frac{1}{8}B_1(B_1^2+3B_1-B_2)x+1.$$

A computation shows that $G'(x)=0$ at

$$x_0=\frac{4B_1(B_1^2+3B_1-B_2)}{(3B_1^2-B_2)(B_1^2+B_2)+B_1(2B_1^2-2B_2-B_1)}$$

and

$$G''(x_0)=\frac{1}{32}\left((3B_1^2-B_2)(B_1^2+B_2)+B_1(2B_1^2-2B_2-B_1)\right).$$

Since $B_1>0$ and $B_1^2 \geq B_2$, numerator of x_0 is always positive. Note that, denominator of x_0 is same as $32G''(x_0)$, which gives $x_0<0$ (or $x_0>0$) iff $G''(x_0)<0$ (or $G''(x_0)>0$). Now, there arise two cases:
Case 1: If $x_0<0$ or $x_0>4$, which means $G(x)$ does not have any critical point and

$$\begin{aligned} T_3(1)(f) & \geq \min\{G(0),G(4)\} \\ & = \min\left\{1-\frac{B_1^2}{4}, 1-2B_1^2+\frac{3B_1^4}{4}+\frac{B_1^2B_2}{2}-\frac{B_2^2}{4}\right\}, \end{aligned}$$

which yields (3.2.5). Both $G(0)$ and $G(4)$ are sharp for the extremal functions $k_{\varphi,3}$ and k_{φ}, respectively,

given by (3.2.3). The case $x_0 < 0$ also exhaust the possibility of $G''(x_0) < 0$.
Case 2: If $x_0 \in (0,4]$, then $G''(x_0) > 0$ and the function G attains its minimum value at x_0. Here, we discuss two possibilities for x_0, which are $x_0 = 4$ and $x_0 \in (0,4)$. For $x_0 = 4$, we have

$$\begin{aligned} T_3(1)(f) &\geq G(4) \\ &= 1 - 2B_1^2 + \frac{3B_1^4}{4} + \frac{B_1^2 B_2}{2} - \frac{B_2^2}{4}, \end{aligned} \tag{3.2.10}$$

which establishes (3.2.6). If $x_0 \in (0,4)$, then

$$T_3(1)(f) \geq G(x_0),$$

which proves (3.2.7). □

We get lower bound of $T_3(1)(f)$ for different classes with corresponding alternatives of φ.

Corollary 3.2.5. Let $f \in \mathscr{S}^*[A,B]$, where $-1 \leq B < A \leq 1$.

(i) If $\nu \notin [0,4]$, then

$$T_3(1)(f) \geq \min\left\{1 - \frac{(A-B)^2}{4}, 1 + \frac{(A-B)^2(3A^2 - 8AB + 4B^2 - 8)}{4}\right\}.$$

(ii) If $\nu = 4$, then

$$T_3(1)(f) \geq 1 + \frac{(A-B)^2(3A^2 - 8AB + 4B^2 - 8)}{4}.$$

(iii) If $\nu \in (0,4)$, then

$$T_3(1)(f) \geq 1 - \frac{(A-B)^2(A^2 - 2AB + B^2 + 2A + 2)}{(3A^2 - 8AB + 4B^2 + 2A - 1)},$$

where $\nu = 4(3+A)/(3A^2 + 4B^2 - 8AB + 2A - 1)$. The first two inequalities are sharp.

In the following corollary, bounds are given for certain classes when coefficients B_1 and B_2 of $\varphi(z)$ satisfy the condition (3.2.5) or (3.2.6).

Corollary 3.2.6. (i) If $f \in \mathscr{S}^*_{\wp}$, then $T_3(1)(f) \geq 0$.

(ii) If $f \in \mathscr{S}^*_{\sin}$, then $T_3(1)(f) \geq -1/4$.

(iii) If $f \in \mathscr{S}_p$, then $T_3(1)(f) \geq 1 - 64\left(19\pi^4 - 24\pi^2 - 432\right)/(9\pi^8)$.

(iv) If $f \in \mathscr{S}^*_{RL}$, then $T_3(1)(f) \geq -9(4130\sqrt{2} - 5861)/256$.

Remark 3.2.3. Some of the already known results are established as special cases of Theorem 3.2.4, which are given below.

(i) If $f \in \mathscr{S}^*_{SG}$, then $T_3(1)(f) \geq 35/64$ [33, Theorem 2.1].

(ii) If $f \in \mathscr{S}^*_{B}$, then $T_3(1)(f) \geq 0$ [33, Theorem 2.2].

(iii) If $f \in \mathcal{S}^*_{Ne}$, then $T_3(1)(f) \geq -1/4$ [90, Theorem 4.2].

(iv) If $f \in \mathcal{S}^*_L$, then $T_3(1)(f) \geq 135/256$ [98, Theorem 3].

(v) If $f \in \mathcal{S}^*(1/2)$, then $T_3(1)(f) \geq 0$ [39, Corollary 4].

Remark 3.2.4. Special cases of Theorem 3.2.4 for the classes $\mathcal{S}^*$ and Δ^*, which are reducing to the known results, are listed below particularly when the choices of φ satisfy the condition (3.2.7).

(i) If $f \in \mathcal{S}^*$, then $T_3(1)(f) \geq -1$ [39, Corollary 3],[87, Corollary 2].

(ii) If $f \in \Delta^*$, then $T_3(1)(f) \geq -1/15$ [98, Theorem 2].

Theorem 3.2.7. If $f \in \mathcal{C}(\varphi)$ such that $B_1^2 \geq 2B_2$, then

$$T_3(1)(f) \geq \begin{cases} \min\left\{1-\dfrac{B_1^2}{36}, 1-\dfrac{B_1^2}{2}+\dfrac{B_1^4}{18}+\dfrac{B_1^2B_2}{36}-\dfrac{B_2^2}{36}\right\}, & \sigma \notin [0,4], \\ 1-\dfrac{B_1^2}{2}+\dfrac{B_1^4}{18}+\dfrac{B_1^2B_2}{36}-\dfrac{B_2^2}{36}, & \sigma = 4, \\ 1-\dfrac{B_1^3(B_1^3+4B_1^2+28B_1-8B_2)}{16(2B_1^4+B_1^3-B_1^2-2B_1B_2+B_1^2B_2-B_2^2)}, & \sigma \in (0,4), \end{cases}$$

where

$$\sigma = \frac{2B_1(B_1^2+16B_1-2B_2)}{2B_1^4+B_1^3-B_1^2-2B_1B_2+B_1^2B_2-B_2^2}.$$

The first two inequalities are sharp.

Proof. Let $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}(\varphi)$, then we have

$$1+\frac{zf''(z)}{f'(z)} = \varphi(\omega(z)), \quad z \in \mathbb{U},$$

where ω is a Schwarz function. Corresponding to the function ω, there is $p(z) = 1+\sum_{n=2}^{\infty} p_n z^n \in \mathcal{P}$ such that $\omega(z) = (p(z)-1)/(p(z)+1)$. Comparison of the same powers of z in the above equation with the series expansions of functions f, φ and p gives

$$a_2 = \frac{B_1 p_1}{4} \quad \text{and} \quad a_3 = \frac{(B_1^2-B_1+B_2)p_1^2+2B_1p_2}{24}.$$

Following the same procedure as in Theorem 3.2.4 with Lemma 3.1.1, we obtain $T_3(1)(f) \geq F(x,y)$, where

$$F(x,y) = 1-\frac{B_1^2 x}{8}+\frac{1}{576}\Big((2B_1^4+B_1^2B_2-B_2^2)x^2-(B_1^3-2B_1B_2)x(4-x)y-B_1^2(4-x)^2y^2\Big)$$

for $x = p_1^2 \in [0,4]$ and $y = |\zeta| \in [0,1]$. Partial derivative with respect to y shows that

$$\frac{\partial F}{\partial y} = -\frac{(B_1^3-2B_1B_2)(4-x)x}{576}-\frac{B_1^2(4-x)^2y}{288} \leq 0$$

whenever $B_1^2 \geq 2B_2$. Therefore F is a decreasing function of y and $F(x,y) \geq F(x,1) =: G(x)$, where

$$G(x) = 1 - \frac{B_1^2}{36} - \frac{B_1(B_1^2+16B_1-2B_2)x}{144} + \frac{(2B_1^4+B_1^3-B_1^2-2B_1B_2+B_1^2B_2-B_2^2)x^2}{576}.$$

A simple computation reveals that $G'(x) = 0$ at

$$x_0 = \frac{2B_1(B_1^2+16B_1-2B_2)}{2B_1^4+B_1^3-B_1^2-2B_1B_2+B_1^2B_2-B_2^2}$$

and

$$G''(x) = \frac{2B_1^4+B_1^3-B_1^2-2B_1B_2+B_1^2B_2-B_2^2}{288}.$$

Since $B_1^2 \geq B_2$ and $B_1 > 0$, therefore numerator of x_0 is always positive. Also note that, the denominator of x_0 and numerator of $G''(x)$ are same. Thus

Case I: When $0 < x_0 < 4$, $G''(x_0) > 0$ and hence minimum will attain at x_0. In this case

$$\begin{aligned} T_3(1)(f) &\geq \min G(x) = G(x_0) \\ &= 1 - \frac{B_1^3(B_1^3+4B_1^2+28B_1-8B_2)}{16(2B_1^4+B_1^3-B_1^2-2B_1B_2+B_1^2B_2-B_2^2)}. \end{aligned}$$

Case II: If $x_0 < 0$ or $x_0 > 4$, which indicates that the critical point does not lie in the domain, then

$$\begin{aligned} T_3(1)(f) &\geq \min\{G(0), G(4)\} \\ &= \min\left\{1 - \frac{B_1^2}{36}, 1 - \frac{B_1^2}{2} + \frac{B_1^4}{18} + \frac{B_1^2B_2}{36} - \frac{B_2^2}{36}\right\}. \end{aligned}$$

Further, for $x_0 = 4$, $T_3(1)(f) \geq G(4)$.

For the functions h_φ and $h_{\varphi,3}$ given in (3.2.4), we have

$$T_3(1)(h_\varphi) = 1 - \frac{B_1^2}{2} + \frac{B_1^4}{18} + \frac{B_1^2B_2}{36} - \frac{B_2^2}{36} \quad \text{and} \quad T_3(1)(h_{\varphi,3}) = 1 - \frac{B_1^2}{36},$$

which shows the sharpness of the bounds. □

If we take $\varphi(z)$ as $(1+Az)/(1+Bz)$ and $(1+z)/(1-z)$ in Theorem 3.2.7, we obtain the following corollaries, respectively.

Corollary 3.2.8. Let $f \in \mathscr{C}[A,B]$ such that $-1 \leq B < A \leq 1$ and $(A-B)^2 \geq 2(B^2-AB)$. Then

$$T_3(1)(f) \geq \begin{cases} \min\left\{1 - \frac{(A-B)^2}{36}, 1 + \frac{(A-B)^2(2A^2+2B^2-5AB-18)}{36}\right\}, & \sigma \notin [0,4] \\ 1 + \frac{(A-B)^2(2A^2+2B^2-5AB-18)}{36}, & \sigma = 4 \\ 1 - \frac{(A-B)^2(A^2+B^2+4A+4B-2AB+28)}{16(2A^2+2B^2+A+B-5AB-1)}, & \sigma \in (0,4) \end{cases}$$

where $\sigma = 2(A+B+16)/(2A^2+2B^2+A+B-5AB-1)$. The first two inequalities are sharp.

Corollary 3.2.9. If $f \in \mathscr{C}$, then $T_3(1)(f) \geq 0$. The bound is sharp.

3.3 Hermitian-Toeplitz for Close-to-Convex functions

Recently, the lower and upper bounds of $T_2(1)(f)$ and $T_3(1)(f)$ for functions in the classes $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$ and $\mathscr{F}_4$ were obtained [87, 96, 105]. Generalizing these works, we obtain the bounds of $T_2(1)(f)$ and $T_3(1)(f)$ when $f \in \mathscr{K}(g)$ for certain $g \in \mathscr{S}^*$.

Theorem 3.3.1. Let $g \in \mathscr{S}^*$ be of the form $g(z) = z + b_2z^2 + b_3z^3 + \cdots$, such that

$$6|b_2|^3 - 4|b_2|(|b_3|-1) - 4(|b_3|-1)^2 + |b_2|^2(3|b_3|+5) \geq 0 \tag{3.3.1}$$

and let $f \in \mathscr{K}(g)$. Then

(i) $1-(1+|b_2|/2)^2 \leq T_2(1)(f) \leq 1$.

(ii) If $6|b_2|^3 + |b_2|^2(3|b_3|+13) + 4|b_2|(|b_3|-1) - 2(1-|b_3|)^2 - 18 \leq 0$, then

$$T_3(1)(f) \leq 1. \tag{3.3.2}$$

(iii) If $6|b_2|^3 + |b_2|^2(3|b_3|+13) + 4|b_2|(|b_3|-1) - 2(1-|b_3|)^2 - 18 > 0$, then

$$T_3(1)(f) \leq \frac{1}{18}\left(6|b_2|^3 + |b_2|^2(3|b_3|+13) + 4|b_2|(|b_3|-1) - 2(1-|b_3|)^2\right). \tag{3.3.3}$$

All these bounds are sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{K}(g)$, then we have

$$zf'(z) = g(z)p(z),$$

where $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathscr{P}$. By comparing the coefficients of like powers on either side, we have

$$a_2 = \frac{b_2 + p_1}{2} \quad \text{and} \quad a_3 = \frac{1}{3}(b_3 + b_2p_1 + p_2).$$

Using $|p_n| \leq 2$, we obtain

$$|a_2| \leq \frac{2+|b_2|}{2} \quad \text{and} \quad |a_3| \leq \frac{1}{3}(|b_3| + 2|b_2| + 2). \tag{3.3.4}$$

The bounds in (3.3.4) are sharp for the function f_g, given by

$$f_g(z) = \int_0^z \frac{(1+t)g(t)}{t(1-t)}dt. \tag{3.3.5}$$

From (3.3.4), we can easily obtain the lower and upper estimates of $T_2(1)(f)$. The lower bound of $T_2(1)(f)$ is sharp for the function f_g given by (3.3.5) whereas the equality in the upper bound is attained for

$$f_1(z) = \int_0^z \frac{(1+t^3)}{(1-t^3)} \frac{1}{(1-t^2)} dt = z + \frac{z^3}{3} + \cdots.$$

Using the inequality $\mathrm{Re}(a_2^2 \bar{a}_3) \leq |a_2|^2 |a_3|$ in (3.1.2), we get

$$T_3(1)(f) \leq \max u(x),$$

where $u(x) = 2|a_2|^2 x - 2|a_2|^2 - x^2 + 1$ and $x = |a_3|$. Note that $u(x)$ attains its maximum value at $x = |a_2|^2$. When $|a_2|^2$ lies in the range of $x = |a_3|$, that is $|a_2|^2 \leq \frac{1}{3}(|b_3| + 2|b_2| + 2)$, then

$$\begin{aligned}
\max u(x) &= u(|a_2|^2) \\
&= (|a_2|^2 - 1)^2 \\
&\leq \begin{cases} 1, & |a_2|^2 \leq 2, \\ \left(\frac{1}{3}(|b_3| + 2|b_2| + 2) - 1\right)^2, & 2 \leq |a_2|^2 \leq \frac{1}{3}(2 + 2|b_2| + |b_3|) \end{cases} \\
&= \begin{cases} 1, & \frac{1}{3}(|b_3| + 2|b_2| + 2) \leq 2, \\ \left(\frac{1}{3}(|b_3| + 2|b_2| + 2) - 1\right)^2, & 2 \leq \frac{1}{3}(2 + 2|b_2| + |b_3|). \end{cases}
\end{aligned}$$

For $x = |a_3| \leq |a_2|^2$,

$$\begin{aligned}
\max u(x) &= u\left(\frac{1}{3}(|b_3| + 2|b_2| + 2)\right) \\
&\leq \frac{1}{18}\left(6|b_2|^3 + |b_2|^2(3|b_3| + 13) + 4|b_2|(|b_3| - 1) - 2(1 - |b_3|)^2\right).
\end{aligned} \tag{3.3.6}$$

Using (3.3.1), we observe that the maximum value of $u(x)$ at $x = |a_3|$, given in (3.3.6), is greater than the value at $x = |a_2|^2$, that is $\left(\frac{1}{3}(|b_3| + 2|b_2| + 2) - 1\right)^2$. Moreover,

$$\frac{1}{18}\left(6|b_2|^3 + |b_2|^2(3|b_3| + 13) + 4|b_2|(|b_3| - 1) - 2(1 - |b_3|)^2\right) > 1,$$

whenever

$$6|b_2|^3 + |b_2|^2(3|b_3| + 13) + 4|b_2|(|b_3| - 1) - 2(1 - |b_3|)^2 - 18 > 0. \tag{3.3.7}$$

Upper bound of $T_3(1)(f)$ can be obtained from (3.3.6) and (3.3.7).

Bound in (3.3.3) is sharp for the function f_g given by (3.3.5) and for $f(z) = z$ equality holds in (3.3.2). □

Remark 3.3.1. Note that, when $g(z) = z/(1-z)$, the class $\mathscr{K}(g)$ reduces to the class $\mathscr{F}_1$. The series expansion of $z/(1-z)$ shows that $b_2 = b_3 = 1$, which clearly satisfy the condition (3.3.1) and since

$$6|b_2|^3 + |b_2|^2(3|b_3| + 13) + 4|b_2|(|b_3| - 1) - 2(1 - |b_3|)^2 - 18 = 4,$$

the condition (3.3.3) is also true. Similarly, when $g(z) = z/(1-z)^2$, it can be easily verified that the conditions (3.3.1) and (3.3.3) hold and the class $\mathscr{K}(g)$ becomes the class $\mathscr{F}_3$. Consequently, we obtain the following known sharp bounds for the classes $\mathscr{F}_1$ and $\mathscr{F}_3$ as a special case of Theorem 3.3.1.

(i) If $f \in \mathscr{F}_1$, then $-5/4 \leq T_2(1)(f) \leq 1$ [96, Theorem 2.1], [105, Theorem 4].

(ii) If $f \in \mathscr{F}_1$, then $T_3(1)(f) \leq 11/9$ [96, Theorem 2.2], [105, Theorem 5].

(iii) If $f \in \mathscr{F}_3$, then $-3 \leq T_2(1)(f) \leq 1$ [96, Theorem 2.5],[87, Theorem 5].

(iv) If $f \in \mathscr{F}_3$, then $T_3(1)(f) \leq 8$ [96, Theorem 2.6],[87, Theorem 6].

Remark 3.3.2. Taking $g(z) = z/(1-z^2)$ in $\mathscr{K}(g)$, we obtain the class $\mathscr{F}_2$. The series expansion of $z/(1-z)^2$ reveals that $b_2 = 0$ and $b_3 = 1$, which satisfy (3.3.1) and since

$$6|b_2|^3 + |b_2|^2(3|b_3| + 13) + 4|b_2|(|b_3| - 1) - 2(1 - |b_3|)^2 - 18 = -18,$$

the condition (3.3.2) also hold. Similarly, in case of $g(z) = z/(1-z+z^2)$, the conditions (3.3.1) and (3.3.2) are true, and the class $\mathscr{K}(g)$ reduces to the class $\mathscr{F}_4$. Thus, Theorem 3.3.1 directly gives the following sharp bounds for the classes $\mathscr{F}_2$ and $\mathscr{F}_4$.

(i) If $f \in \mathscr{F}_2$, then $0 \leq T_2(1)(f) \leq 1$ [96, Theorem 2.3],[87, Theorem 2].

(ii) If $f \in \mathscr{F}_2$, then $T_3(1)(f) \leq 1$ [96, Theorem 2.4], [87, Theorem 3].

(iii) If $f \in \mathscr{F}_4$, then $-5/4 \leq T_2(1)(f) \leq 1$ [96, Theorem 2.7], [105, Theorem 2].

(iv) If $f \in \mathscr{F}_4$, then $T_3(1)(f) \leq 1$ [96, Theorem 2.8], [105, Theorem 3].

Theorem 3.3.2. Let $f \in \mathscr{K}(g)$ such that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathscr{S}^*$. If $\tilde{g}(z) = z + \sum_{n=2}^{\infty} i^{n-1} b_n z^n \in \mathscr{S}^*$, then

$$|T_2(2)(f)| \leq \frac{1}{4}(2 + |b_2|)^2 + \frac{1}{9}(|b_3| + 2|b_2| + 2)^2.$$

The bound is sharp.

Proof. From (3.1.1), we have

$$|T_2(2)(f)| = |a_2^2 - |a_3|^2| \leq |a_3|^2 + |a_2|^2.$$

Using the estimate of $|a_2|$ and $|a_3|$ for $f \in \mathscr{K}(g)$, given in (3.3.4), we obtain the required bound of $|T_2(2)(f)|$.

The bound is sharp for the function f given by

$$\frac{zf'(z)}{\tilde{g}(z)} = \frac{1+iz}{1-iz}$$

as f has the power series representation $f(z) = z + i(2+b_2)z^2/2 - (2+b_3+2b_2)z^3/3 + \cdots$. □

By changing the function g in Theorem 3.3.2, we can obtain results for other classes.

Corollary 3.3.3. (i) If $f \in \mathscr{F}_1$, then $|T_2(2)(f)| \leq 181/36$ and the bound is sharp for

$$f(z) = \int_0^z \left(\frac{1+it}{1-it}\right) \frac{\tilde{g}(t)}{t} dt,$$

where $\tilde{g}(z) = z/(1-iz) \in \mathcal{S}^*$.

(ii) If $f \in \mathcal{F}_2$, then $|T_2(2)(f)| \leq 2$ and the bound is sharp for

$$f(z) = \int_0^z \left(\frac{1+it}{1-it}\right) \frac{\tilde{g}(t)}{t} dt,$$

where $\tilde{g}(z) = z/(1+z^2) \in \mathcal{S}^*$.

(iii) If $f \in \mathcal{F}_3$, then $|T_2(2)(f)| \leq 13$ and the bound is sharp for

$$f(z) = \int_0^z \left(\frac{1+it}{1-it}\right) \frac{\tilde{g}(t)}{t} dt,$$

where $\tilde{g}(z) = z/(1-iz)^2 \in \mathcal{S}^*$.

(iv) If $f \in \mathcal{F}_4$, then $|T_2(2)(f)| \leq 145/36$ and the bound is sharp for

$$f(z) = \int_0^z \left(\frac{1+it}{1-it}\right) \frac{\tilde{g}(t)}{t} dt,$$

where $\tilde{g}(z) = z/(1-iz+z^2) \in \mathcal{S}^*$.

(v) If $f \in \mathcal{R}$, then $|T_2(2)(f)| \leq 13/9$ and the bound is sharp for

$$f(z) = \int_0^z \left(\frac{1+it}{1-it}\right) dt.$$

3.4 Hermitian-Toeplitz for Sakaguchi Classes

In the past, various subclasses of $\mathcal{S}_s^*$ and $\mathcal{C}_s$ were considered and studied such as $\mathcal{S}_{s,e}^*$, $\mathcal{S}_{s,L}^*$ and $\mathcal{S}_{s,RL}^*$ defined in (2.3.1). Bounds of initial coefficients for functions belonging to these classes were established in [80]. Kumar and Kumar [89] studied second and third order Hermitian-Toeplitz determinants for functions belonging to the classes $\mathcal{S}_s^*$, $\mathcal{C}_s$ and the classes defined in (2.3.1). Sun and Wang [180] considered the same problem for the classes $\mathcal{S}_s^*(\alpha)$ and $\mathcal{C}_s(\alpha)$ and established the sharp bound of $T_3(1)(f)$.

We need the following lemmas to prove our results. Shanmugam et al. [166] obtained the following bounds of $|a_3 - \mu a_2^2|$ for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belonging to the classes $\mathcal{S}_s^*(\varphi)$ and $\mathcal{C}_s(\varphi)$.

Lemma 3.4.1. [166, Theorem 2.1] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_s^*(\varphi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{1}{2}\left(B_2 - \dfrac{\mu}{2}B_1^2\right) & \text{if } \mu \leq \nu_1, \\[2ex] \dfrac{B_1}{2} & \text{if } \nu_1 \leq \mu \leq \nu_2, \\[2ex] -\dfrac{1}{2}\left(B_2 - \dfrac{\mu}{2}B_1^2\right) & \text{if } \mu \geq \nu_2, \end{cases}$$

where $\nu_1 = (2(B_2 - B_1))/B_1^2$ and $\nu_2 = (2(B_2 + B_1))/B_1^2$. The bound is sharp.

Lemma 3.4.2. [166, Corollary 2.4] If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{C}_s(\varphi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{1}{6}\left(B_2 - \dfrac{3}{8}\mu B_1^2\right) & \text{if } \mu \leq \nu_1, \\ \dfrac{B_1}{6} & \text{if } \nu_1 \leq \mu \leq \nu_2, \\ -\dfrac{1}{6}\left(B_2 - \dfrac{3}{8}\mu B_1^2\right) & \text{if } \mu \geq \nu_2, \end{cases}$$

where $\nu_1 = (8(B_2 - B_1))/(3B_1^2)$ and $\nu_2 = (8(B_2 + B_1))/(3B_1^2)$. The bound is sharp.

For $\mu = 0$, the following bounds for $|a_3|$ directly follow from Lemma 3.4.1 and Lemma 3.4.2, respectively, helping us to prove the results:

Lemma A. *If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathscr{S}_s^*(\varphi)$ and $B_1 \leq |B_2|$, then*

$$|a_3| \leq \frac{|B_2|}{2}.$$

Lemma B. *If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathscr{C}_s(\varphi)$ and $B_1 \leq |B_2|$, then*

$$|a_3| \leq \frac{|B_2|}{6}.$$

We begin with the following result for the class $\mathscr{S}^*(\varphi)$.

Theorem 3.4.1. If $f \in \mathscr{S}_s^*(\varphi)$ and $B_1 \leq |B_2|$, then

$$T_3(1)(f) \leq 1.$$

The bound is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_s^*(\varphi)$, then

$$T_3(1)(f) = 1 - 2|a_2|^2 - |a_3|^2 + 2\operatorname{Re}(a_2^2 \bar{a}_3). \tag{3.4.1}$$

Applying the inequality $2\operatorname{Re}(a_2^2 \bar{a}_3) \leq 2|a_2^2||a_3|$ in the last equation, we obtain

$$T_{3,1}(f) \leq 1 - 2|a_2|^2 - |a_3|^2 + 2|a_2^2||a_3| =: g(x),$$

where $g(x) = 1 - 2|a_2|^2 - x^2 + 2|a_2^2|x$ with $x = |a_3|$. For $f \in \mathscr{S}_s^*(\varphi)$, we have $|a_2| \leq B_1/2$ and from Lemma A, $|a_3| \leq |B_2|/2$. Thus $|a_2| \in [0,1]$ and $x = |a_3| \in [0,1]$. As $g'(x) = 0$ at $x = |a_2|^2$ and $g''(x) < 0$ for all $x \in [0,1]$. Consequently, we have

$$\begin{aligned} T_3(1)(f) &\leq \max g(x) \\ &= g(|a_2|^2) = (|a_2|^2 - 1)^2 \leq 1. \end{aligned}$$

Since the identity function $f(z)=z$ is a member of the class $\mathscr{S}_s^*(\varphi)$ and for this function, we have $a_2=0$, $a_3=0$ and $T_{3,1}(f)=1$, which shows that the bound is sharp. □

Theorem 3.4.2. If $f\in\mathscr{C}_s(\varphi)$ and $B_1\leq|B_2|$, then

$$T_3(1)(f)\leq 1.$$

The result is sharp.

Proof. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{C}_s(\varphi)$, then using the inequality $\operatorname{Re}(a_2^2\bar{a}_3)\leq|a_2|^2|a_3|$ in (3.1.2) for $f\in\mathscr{C}_s(\varphi)$, we obtain

$$T_3(1)(f)\leq 1-2|a_2|^2-|a_3|^2+2|a_2|^2|a_3|=:g(x),$$

where $g(x)=1-2|a_2|^2-x^2+2|a_2|^2x$. Since $|a_2|\leq B_1/4$ and from Lemma B, we have $|a_3|\leq|B_2|/6$, therefore $|a_2|\in[0,1/2]$ and $|a_3|\in[0,1/3]$. Also, note that $g(x)$ attains its maximum value at $x=|a_2|^2$. Hence

$$\begin{aligned}T_3(1)(f)&\leq\max g(x)\\&=g(|a_2|^2)=(|a_2|^2-1)^2\leq 1.\end{aligned}$$

The equality case holds for $f(z)=z$. □

Theorem 3.4.3. If $f\in\mathscr{S}_s^*(\varphi)$ such that $B_1^2>2B_2$, then the following estimates hold:

$$T_3(1)(f)\geq\begin{cases}\min\left\{1-\dfrac{B_1^2}{4},1-\dfrac{B_1^2}{2}+\dfrac{B_1^2B_2}{4}-\dfrac{B_2^2}{4}\right\}, & \sigma_1\notin[0,4],\\[2ex] 1-\dfrac{B_1^2}{2}+\dfrac{B_1^2B_2}{4}-\dfrac{B_2^2}{4}, & \sigma_1=4,\\[2ex] 1-\dfrac{B_1^3(B_1^3+4B_1^2-4B_1-8B_2)}{16(B_1^3+B_1^2(B_2-1)-2B_1B_2-B_2^2)}, & \sigma_1\in(0,4),\end{cases}$$

where

$$\sigma_1=\frac{2B_1(B_1^2-2B_2)}{(B_1^2-B_1-B_2)(B_1+B_2)}.$$

The first two inequalities are sharp.

Proof. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{S}_s^*(\varphi)$, then there exist a Schwarz function $\omega(z)$ such that

$$\frac{2zf'(z)}{f(z)-f(-z)}=\varphi(\omega(z)).$$

By the one-to-one correspondence between the class of Schwarz functions and the class $\mathscr{P}$, we obtain

$$\frac{2zf'(z)}{f(z)-f(-z)}=\varphi\left(\frac{p(z)-1}{p(z)+1}\right)\tag{3.4.2}$$

for some $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathscr{P}$. On the comparison of the same powers of z with the series expansions of functions $f(z)$, $\varphi(z)$ and $p(z)$, the above equation yields

$$a_2 = \frac{B_1 p_1}{4} \quad \text{and} \quad a_3 = \frac{1}{8}(-B_1 p_1^2 + B_2 p_1^2 + 2B_1 p_2). \tag{3.4.3}$$

Using Lemma 3.1.1 in (3.4.3), we get

$$-|a_3|^2 = -\frac{1}{64}\left(B_2^2 p_1^4 + B_1^2(4-p_1^2)^2|\zeta|^2 + 2B_1B_2p_1^2(4-p_1^2)\operatorname{Re}\bar{\zeta}\right)$$

and

$$2\operatorname{Re}(a_2^2\bar{a}_3) = \frac{1}{64}B_1^2p_1^2\left((B_2-B_1)p_1^2 + B_1(p_1^2 + (4-p_1^2)\operatorname{Re}\bar{\zeta})\right).$$

Taking these into account in (3.1.2), we get

$$\begin{aligned} T_3(1)(f) = &\frac{1}{64}\left((B_1^2 - B_2)B_2p_1^4 - B_1^2(4-p_1^2)^2|\zeta|^2 + B_1(B_1^2 - 2B_2)p_1^2(4-p_1^2)\operatorname{Re}\bar{\zeta}\right) \\ &- \frac{B_1^2p_1^2}{8} + 1 =: F(p_1, |\zeta|, \operatorname{Re}\bar{\zeta}). \end{aligned}$$

It can be seen that $F(p_1,|\zeta|,\operatorname{Re}\bar{\zeta}) \geq F(p_1,|\zeta|,-|\zeta|) =: G(x,y)$ by considering $p_1^2 = x$ and $|\zeta| = y$, where

$$G(x,y) = \frac{1}{64}\left((B_1^2 - B_2)B_2x^2 - B_1^2(4-x)^2y^2 - B_1(B_1^2 - 2B_2)x(4-x)y\right) - \frac{B_1^2x}{8} + 1.$$

Whenever $B_1^2 > 2B_2$, we have

$$\frac{\partial G}{\partial y} = \frac{1}{64}(-2B_1^2(4-x)^2y - B_1(B_1^2 - 2B_2)x(4-x)) \leq 0$$

for $x \in [0,4]$ and $y \in [0,1]$, which means that $G(x,y)$ is a decreasing function of y and $G(x,y) \geq G(x,1) =: I(x)$ with

$$I(x) = \frac{1}{64}(B_1^3 + B_1^2(B_2 - 1) - 2B_1B_2 - B_2^2)x^2 + \frac{B_1}{16}(2B_2 - B_1^2)x - \frac{B_1^2}{4} + 1.$$

An easy computation yields that $I'(x) = 0$ at

$$x_0 = \frac{2B_1(B_1^2 - 2B_2)}{(B_1^2 - B_1 - B_2)(B_1 + B_2)}$$

and

$$I''(x_0) = \frac{1}{32}(B_1^2 - B_1 - B_2)(B_1 + B_2).$$

Since $B_1^2 > 2B_2$, therefore numerator of x_0 is always positive. Moreover, denominator of x_0 and numerator of $I''(x_0)$ are same, therefore $x_0 < 0$ (or $x_0 > 0$) iff $I''(x_0) < 0$ (or $I''(x_0) > 0$). Here we discuss the following cases:

Case I: Whenever $x_0 \in (0,4)$, then $I''(x_0) > 0$. Thus $I(x)$ attains its minimum value at x_0, which gives

$$\begin{aligned} T_3(1)(f) &\geq I(x_0) \\ &= 1 - \frac{B_1^3(B_1^3 + 4B_1^2 - 4B_1 - 8B_2)}{16(B_1^3 + B_1^2(B_2 - 1) - 2B_1B_2 - B_2^2)}. \end{aligned}$$

Case II: When $x_0 < 0$ or $x_0 > 4$, which indicates that $I(x)$ does not have any critical point, therefore

$$\begin{aligned} T_3(1)(f) &\geq \min\{I(0), I(4)\} \\ &= \min\left\{1 - \frac{B_1^2}{4}, 1 - \frac{B_1^2}{2} + \frac{B_1^2 B_2}{4} - \frac{B_2^2}{4}\right\}. \end{aligned}$$

For $x_0 = 4$, $T_{3,1}(f) \geq I(4)$.

Function $k_{s,\varphi} \in \mathscr{S}_s^*(\varphi)$ and $k_{s,\varphi,3} \in \mathscr{S}_s^*(\varphi)$ given by (2.3.2) shows that these bounds are sharp as

$$T_3(1)(k_{s,\varphi}) = 1 - \frac{B_1^2}{2} + \frac{B_1^2 B_2}{4} - \frac{B_2^2}{4} \quad \text{and} \quad T_3(1)(k_{s,\varphi,3}) = 1 - \frac{B_1^2}{4},$$

which completes the proof. □

Theorem 3.4.4. If $f \in \mathscr{C}_s(\varphi)$ and $3B_1^2 \geq 8B_2$, then the following estimates hold:

$$T_3(1)(f) \geq \begin{cases} \min\left\{1 - \dfrac{B_1^2}{36}, 1 - \dfrac{B_1^2}{8} + \dfrac{B_1^2 B_2}{48} - \dfrac{B_2^2}{36}\right\}, & \sigma_2 \notin [0,4], \\[2ex] 1 - \dfrac{B_1^2}{8} + \dfrac{B_1^2 B_2}{48} - \dfrac{B_2^2}{36}, & \sigma_2 = 4, \\[2ex] 1 - \dfrac{B_1^3(B_1^3 + 12B_1^2 + 4B_1 - 32B_2)}{64(3B_1^3 + B_1^2(3B_2 - 4) - 8B_1B_2 - 4B_2^2)}, & \sigma_2 \in (0,4), \end{cases}$$

where

$$\sigma_2 = \frac{2B_1(3B_1^2 + 10B_1 - 8B_2)}{3B_1^3 + 3B_1^2 B_2 - 4B_1^2 - 8B_1B_2 - 4B_2^2}.$$

First two inequalities are sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{C}_s(\varphi)$, then there exists a Schwarz function $\omega(z)$ such that

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} = \varphi(\omega(z)).$$

Corresponding to the Schwarz function $\omega(z)$, let there is a function $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathscr{P}$ satisfying $p(z) = (1 + \omega(z))/(1 - \omega(z))$. Thus, we obtain

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right). \tag{3.4.4}$$

Comparing the coefficients of the same powers of z after applying the series expansion of $f(z)$, $\varphi(z)$

and $p(z)$ leads to

$$a_2 = \frac{B_1 p_1}{8}, \quad a_3 = \frac{1}{24}((B_2 - B_1)p_1^2 + 2B_1 p_2). \tag{3.4.5}$$

The rotationally invariant property of the classes $\mathscr{C}_s(\varphi)$ and $\mathscr{P}$ allows to take $p_1 \in [0,2]$. Using Lemma 3.1.1 in (3.4.5), we get

$$-|a_3|^2 = -\frac{1}{576}(B_2^2 p_1^4 + B_1^2(4-p_1^2)^2|\zeta|^2 + 2B_1B_2p_1^2(4-p_1^2)\operatorname{Re}\bar{\zeta})$$

and

$$2\operatorname{Re}(a_2^2\bar{a}_3) = \frac{B_1^2p_1^2}{768}(-B_1p_1^2 + B_2p_1^2 + B_1(p_1^2 + (4-p_1^2)\operatorname{Re}\bar{\zeta})).$$

These above values together with (3.1.2) leads to

$$\begin{aligned} T_3(1)(f) = &\left(\frac{B_1^2B_2}{768} - \frac{B_2^2}{576}\right)p_1^4 - \frac{1}{576}B_1^2(4-p_1^2)^2|\zeta|^2 + \left(\frac{3B_1^3 - 8B_1B_2}{2304}\right)p_1^2(4-p_1^2)\operatorname{Re}\bar{\zeta} \\ &- \frac{B_1^2p_1^2}{32} + 1 =: F(p_1, |\zeta|, \operatorname{Re}\bar{\zeta}). \end{aligned}$$

As $\operatorname{Re}\bar{\zeta} \geq -|\zeta|$, hence $F(p_1,|\zeta|,\operatorname{Re}\bar{\zeta}) \geq F(p_1,|\zeta|,-|\zeta|) := G(x,y)$, where

$$G(x,y) = \left(\frac{B_1^2B_2}{768} - \frac{B_2^2}{576}\right)x^2 - \frac{1}{576}B_1^2(4-x)^2y^2 - \left(\frac{3B_1^3 - 8B_1B_2}{2304}\right)x(4-x)y - \frac{B_1^2}{32}x + 1$$

for $x = p_1^2 \in [0.4]$ and $y = |\zeta| \in [0,1]$. Whenever $3B_1^2 \geq 8B_1B_2$, we have

$$\frac{\partial G(x,y)}{\partial y} = -\frac{1}{288}B_1^2(4-x)^2y - \left(\frac{3B_1^3 - 8B_1B_2}{2304}\right)x(4-x) \leq 0.$$

Therefore, $G(x,y)$ is decreasing function of y and $G(x,y) \geq G(x,1) =: I(x)$, where

$$I(x) = 1 - \frac{B_1(3B_1^2 + 10B_1 - 8B_2)}{576}x - \frac{B_1^2}{36} + \frac{x^2(3B_1^3 + 3B_1^2B_2 - 4B_1^2 - 8B_1B_2 - 4B_2^2)}{2304}.$$

An elementary calculation reveals that $I'(x) = 0$ at

$$x_0 = \frac{2B_1(3B_1^2 + 10B_1 - 8B_2)}{3B_1^3 + 3B_1^2B_2 - 4B_1^2 - 8B_1B_2 - 4B_2^2}$$

and

$$I''(x) = \frac{3B_1^3 + 3B_1^2B_2 - 4B_1^2 - 8B_1B_2 - 4B_2^2}{1152}.$$

Since $3B_1^2 \geq 8B_2$ and $B_1 > 0$, therefore numerator of x_0 is always positive. Also, note that, the denominator of x_0 and numerator of $I''(x)$ are the same, therefore the sign of x_0 and $I''(x)$ changes simultaneously. Here, two cases arise:

Case I: When $0 < x_0 < 4$. In this case $I''(x) > 0$, so the minimum of $I(x)$ attains at x_0, which gives

$$\begin{aligned} T_3(1)(f) &\geq I(x_0) \\ &= 1 - \frac{B_1^3(B_1^3 + 12B_1^2 + 4B_1 - 32B_2)}{64(3B_1^3 + B_1^2(3B_2 - 4) - 8B_1B_2 - 4B_2^2)}. \end{aligned}$$

Case II: When $x_0 < 0$ or $x_0 > 4$, that means $I(x)$ has no critical point. Thus

$$\begin{aligned} T_3(1)(f) &\geq \min\{I(0), I(4)\} \\ &= \min\left\{1 - \frac{B_1^2}{36}, 1 - \frac{B_1^2}{8} + \frac{B_1^2 B_2}{48} - \frac{B_2^2}{36}\right\}. \end{aligned}$$

For the case $x_0 = 4$, we have $T_3(1)(f) \geq I(4)$.

The sharpness of these bounds follows from the functions $h_{s,\varphi}(z)$ and $h_{s,\varphi,3}(z)$ defined by (2.3.2). Since

$$T_3(1)(h_{s,\varphi}) = 1 - \frac{B_1^2}{8} + \frac{B_1^2 B_2}{48} - \frac{B_2^2}{36} \quad \text{and} \quad T_3(1)(h_{s,\varphi,3}) = 1 - \frac{B_1^2}{36},$$

which completes the proof. □

We now discuss some special cases of Theorem 3.4.1 and 3.4.2. If $\varphi(z) = (1 + Az)/(1 + Bz)$, the class $\mathcal{S}_s^*(\varphi)$ and $\mathcal{C}_s(\varphi)$ reduces to the classes $\mathcal{S}_s^*[A,B]$ and $\mathcal{C}_s[A,B]$, respectively. Theorem 3.4.1 and 3.4.2 immediately give the following sharp bound for the class $\mathcal{S}_s^*[A,B]$ and $\mathcal{C}_s[A,B]$.

Corollary 3.4.5. (i) If $f \in \mathcal{S}_s^*[A,B]$ and $A - B \leq |B^2 - AB|$, then $T_3(1)(f) \leq 1$.

(ii) If $f \in \mathcal{C}_s[A,B]$ and $A - B \leq |B^2 - AB|$, then $T_3(1)(f) \leq 1$.

Theorem 3.4.3 and 3.4.4 yield the following lower bound of $T_3(1)(f)$ for these classes:

Corollary 3.4.6. If $f \in \mathcal{S}_s^*[A,B]$ such that $A^2 - B^2 > 0$, then the following estimates hold:

$$T_3(1)(f) \geq \begin{cases} \min\left\{1 - \frac{1}{4}(A-B)^2, 1 - \frac{1}{4}(A-B)^2(AB+2)\right\}, & \sigma_1 \notin [0,4], \\ 1 - \frac{1}{4}(A-B)^2(AB+2), & \sigma_1 = 4, \\ 1 + \frac{(A^2 - 2A(B-2) + B^2 + 4B - 4)(A-B)^2}{16(1-A)(1-B)}, & \sigma_1 \in (0,4), \end{cases}$$

where

$$\sigma_1 = -\frac{2(A+B)}{(1-A)(1-B)}.$$

The first two inequalities are sharp.

Corollary 3.4.7. If $f \in \mathscr{C}_s[A,B]$ and $3A^2+2AB-5B^2 \geq 0$, then the following estimates hold:

$$T_3(1)(f) \geq \begin{cases} \min\left\{1-\dfrac{(A-B)^2}{36}, 1-\dfrac{(A-B)^2(B^2+3AB+18)}{144}\right\}, & \sigma_2 \notin [0,4], \\ 1-\dfrac{(A-B)^2(B^2+3AB+18)}{144}, & \sigma_2 = 4, \\ 1-\dfrac{(A^2-2A(B-6)+B^2+20B+4)(A-B)^2}{64(1-B)(3A+B-4)}, & \sigma_2 \in (0,4), \end{cases}$$

where

$$\sigma_2 = \frac{2(3A+5(B+2))}{(1-B)(3A+B-4)}.$$

The first two inequalities are sharp.

For $\varphi(z) = (1+(1-2\alpha)z)/(1-z)$ and $(1+z)/(1-z)$ in $\mathscr{S}_s^*(\varphi)$, we obtain the classes $\mathscr{S}_s^*(\alpha)$ and $\mathscr{S}_s^*$, respectively, where $\alpha \in [0,1)$.

Theorem 3.4.1 and 3.4.3 yield the following sharp lower and upper bounds of $T_{3,1}(f)$ for these classes, proved by Kumar and Kumar [89].

Remark 3.4.1. (i) If $f \in \mathscr{S}_s^*(\alpha)$, then $(3-2\alpha)\alpha^2 \leq T_3(1)(f) \leq 1$ [89, Theorem 2.2].

(ii) If $f \in \mathscr{S}_s^*$, then $0 \leq T_3(1)(f) \leq 1$ [89, Corollary 2.3].

For $\varphi(z) = 2/(1+e^{-z})$, we get the classes $\mathscr{S}_{s,SG}^*$ and $\mathscr{C}_{s,SG}$, which are analogues to the corresponding classes studied in [52]. Theorem 3.4.3 provides the following bound for the class $\mathscr{S}_{s,SG}^*$.

Corollary 3.4.8. If $f \in \mathscr{S}_{s,SG}^*$, then $T_{3,1}(f) \geq 2009/2304$.

For other subclasses of $\mathscr{S}_s^*$, the following already known sharp bounds follow from Theorem 3.4.3.

Remark 3.4.2. (i) If $f \in \mathscr{S}_{s,L}^*$, then $T_{3,1}(f) \geq 221/256$ [89, Theorem 3.1].

(ii) If $f \in \mathscr{S}_{s,RL}^*$, then $T_{3,1}(f) \geq (863-444\sqrt{2})/256$ [89, Theorem 3.3].

Theorem 3.4.2 and 3.4.4 give the following corollaries for different subclasses of $\mathscr{C}_s$.

Corollary 3.4.9. (i) If $f \in \mathscr{C}_s[A,B]$ and $A-B \leq |B^2-AB|$, then $T_{3,1}(f) \leq 1$.

(ii) If $f \in \mathscr{C}_s(\alpha)$, then $T_3(1)(f) \leq 1$.

(iii) If $f \in \mathscr{C}_s$, then $T_3(1)(f) \leq 1$.
All these bounds are sharp.

Corollary 3.4.10. (i) If $f \in \mathscr{C}_{s,SG}$, then $T_3(1)(f) \geq 31/32$.

(ii) If $f \in \mathscr{C}_{s,L}$, then $T_3(1)(f) \geq 4459/4608$.

(iii) If $f \in \mathscr{C}_{s,RL}$, then $T_3(1)(f) \geq (-3731+5835\sqrt{2})/4608$.

All these bounds are sharp.

Up to this point, our focus has been on investigating coefficient problems within diverse Ma-Minda and Sakaguchi classes. However, in the forthcoming chapter, we broaden our scope to encompass not only coefficient problems but also radius problems within a class of functions linked to semigroup generators.

Highlights of the chapter

We determined the sharp bounds for the second and third order Hermitian-Toeplitz determinants for the classes $\mathcal{S}^*(\varphi)$, $\mathcal{C}(\varphi)$, $\mathcal{K}(g)$, $\mathcal{S}_s^*(\varphi)$, and $\mathcal{C}_s(\varphi)$. These results hold significant weight as these classes serve as generalizations for various other subclasses of $\mathcal{S}$. Consequently, our established bounds seamlessly extend to cover these broader classes, showcasing the applicability and importance of our findings.

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S. Giri, and S. S. Kumar, Hermitian-Toeplitz determinants for certain univalent functions, Analysis and Mathematical Physics 13, no. 2 (2023): 37. https://doi.org/10.1007/s13324-023-00800-2
S. Giri, S. S. Kumar, Sharp bounds of fifth coefficient and Hermitian-Toeplitz determinants for Sakaguchi classes. Bull. Korean Math. Soc. 2024; 61:317-333. https://doi.org/10.4134/BKMS.b230018

Chapter 4

A Class of Analytic Functions Involving Semigroup Generators

This chapter explores coefficient problems within the class of semigroup generators, establishing sharp bounds for Hankel, Toeplitz, and Hermitian-Toeplitz determinants. It also provides sharp growth estimates and membership criteria for this class. Finally, some radius results are also established for the class.

4.1 Introduction

Since the early 20th century, complex analysis and geometric function theory are foundational in mathematics, paving the way for various applications. Elin et al. [47] discovered new connections between complex dynamics and geometric function theory. Particularly, they studied certain subclasses of starlike functions and their embedding in the classes of semigroup generators. Complex dynamics refers to the study of dynamical systems defined on complex numbers or complex spaces. Dynamic systems, often expressed as equations of motion, are extensively researched, particularly focusing on monotone or nonlinear operators. Concurrently, the theory of one-parameter semigroups of holomorphic functions evolved, finding applications in diverse fields such as complex analysis [16, 38], manifold theory [2, 3], and optimization [70].

A family $\{u(t,z)\}_{t\geq 0} \subset \mathscr{B}$ is called a one parameter continuous semigroup if (i) $\lim_{t\to 0} u(t,z) = z$, (ii)$u(t+s,z) = u(t,z)\circ u(s,z)$, and (iii) $\lim_{t\to s} u(t,z) = u(s,z)$ for each $z \in \mathbb{U}$ hold, where $\mathscr{B}$ denote the set of holomorphic self mappings of the unit disk $\mathbb{U}$. Berkson and Porta [16] showed that each one parameter semigroup is locally differentiable in parameter $t \geq 0$ and moreover, if

$$\lim_{t \to 0} \frac{z - u(t,z)}{t} = f(z),$$

which is a holomorphic function, then $u(t,z)$ is the solution of the Cauchy problem

$$\frac{\partial u(t,z)}{\partial t} + f(u(t,(z)) = 0, \quad u(0,z) = z.$$

The function f is called the holomorphic generator of semigroup $\{u(t,z)\}_{t\geq 0} \subset \mathscr{B}$. The class of all holomorphic generators is denoted by $\mathscr{G}$. Also, note that each element of $\{u(t,z)\}$ generated by $f \in \mathscr{G}$ is a univalent function while f is not necessarily univalent [44]. Various properties of generators and semigroup generated by them are discussed in [16, 21, 43, 44, 45, 169]. Berkson and Porta [16] proved:

Theorem 4.1.1. [16] The following assertions are equivalent:

(a) $f \in \mathscr{G}$;

(b) $f(z) = (z - \sigma)(1 - z\bar{\sigma})p(z)$ with some $\sigma \in \overline{\mathbb{U}}$ and $p \in \mathscr{H}$, $\operatorname{Re}(p(z)) \geq 0$,

where $\mathscr{H}$ represents the class of analytic functions in $\mathbb{U}$.

The point $\sigma \in \overline{\mathbb{U}} := \{z \in \mathbb{C} : |z| \leq 1\}$ is called the Denjoy–Wolff point of the semigroup generated by f. According to the Denjoy-Wolff theorem [44, 157, 169] for continuous semigroup, if for at least one $t \in [0,\infty)$, any element of the semigroup generated by f is neither the identity and nor an elliptic automorphism of $\mathbb{U}$, then there is a unique point $\sigma \in \overline{\mathbb{U}}$ such that $\lim_{t\to\infty} u(t,z) = \sigma$ uniformly for each $z \in \mathbb{U}$. Let the class of all semigroup generators with Denjoy-Wolff point σ be denoted by $\mathscr{G}[\sigma]$. The class of primary interest is $\mathscr{G}[0] \subset \mathscr{G}$, which can be represented as

$$\mathscr{G}[0] = \{f \in \mathscr{G} : f(z) = zp(z), \ \operatorname{Re} p(z) > 0\}.$$

Note that, the semigroup $\{u(t,\cdot)\}_{t\geq 0}$ generated by $f \in \mathscr{G}[0]$ is real analytic with respect to its parameter, it does not always allow for an analytic extension to a domain in $\mathbb{C}$. In addition, the rate of convergence of the semigroup $\{u(t,\cdot)\}_{t\geq 0}$ generated by $f \in \mathscr{G}[0]$ to zero can be estimated as follows (see [44])

$$|u(t,z)| \leq |z| e^{-t \operatorname{Re} f'(0) \frac{1-|z|}{1+|z|}}, \quad z \in \mathbb{U}.$$

This estimate is not uniform on $\mathbb{U}$ and indeed not all semigroups converge uniformly. Elin et al. [46] proved the following:

Proposition 4.1.2. (i) The semigroup generated by $f \in \mathscr{G}[0]$ can be analytically extended to the sector $\{t : |\arg t| < \pi\alpha/2\}$ if and only if $|\arg(f(z)/z)| < \pi(1-\alpha)/2$, for all $z \in \mathbb{U}$ (see [46]).
(ii) The semigroup $\{u(t,\cdot)\}_{t\geq 0}$ generated by $f(z) = zp(z)$ has a uniform exponential rate of convergence: $|u(t,z)| \leq |z|e^{-t\delta}$ if and only if $\operatorname{Re} p(z) \geq \delta > 0$ for all $z \in \mathbb{U}$ (see [23]).

Bracci et al. [23] considered the class

$$\mathscr{G}_0 := \mathscr{G}[0] \cap \mathscr{A}$$

and studied the asymptotic behaviour of the semigroup generated by $f \in \mathscr{G}_0$. In the study of non-autonomous problems such as Loewner theory, the class $\mathscr{G}_0$ plays a significant role [22, 41]. Criteria for membership to the class $\mathscr{G}_0$ were investigated by many authors [16, 23, 44]. Bracci et al. [23] showed that a sufficient condition for $f \in \mathscr{A}$ to be a semigroup generator is that $\operatorname{Re} f'(z) > 0$ for $z \in \mathbb{U}$. The class of all such functions is called the class of bounded turning functions and is denoted by

$$\mathscr{R} = \{f \in \mathscr{A} : \operatorname{Re} f'(z) > 0\}.$$

According to the Noshiro-Warschawski condition, every $f \in \mathscr{R}$ is univalent but not every semigroup generator needs to be univalent. Therefore, the condition $\operatorname{Re} f'(z) > 0$ is far from being a necessary condition for a function to be a semigroup generator.

Various subclasses of $\mathscr{G}_0$ with certain parameters, such that $\mathscr{R}$ is the smallest one, were recently studied, which is also called filtration (see [23, 42, 45, 168]). In particular, for $\beta \in [0,1]$, the class

$$\mathscr{A}_\beta = \left\{ f \in \mathscr{A} : \operatorname{Re}\left(\beta \frac{f(z)}{z} + (1-\beta) f'(z) \right) > 0 \right\} \tag{4.1.1}$$

is a subclass of $\mathscr{G}_0$. Clearly, when $\beta = 0$, the class $\mathscr{A}_\beta$ reduces to the class $\mathscr{R}$ and for $\beta = 1$, $\mathscr{A}_1 = \mathscr{G}_0$. In [23], the authors proved that

Lemma 4.1.1. [23] If $f \in \mathscr{A}_\beta$, then the following hold:

(i) $\mathscr{A}_{\beta_1} \subsetneq \mathscr{A}_{\beta_2} \subsetneq \mathscr{G}_0$ for $0 \leq \beta_1 < \beta_2 < 1$, and

(ii) $\operatorname{Re}(f(z)/z) \geq \delta(\beta) > 0$ for $z \in \mathbb{U}$, where

$$\delta(\beta) = \int_0^1 \frac{1-s^{1-\beta}}{1+s^{1-\beta}} ds.$$

Note that, $\delta(\beta)$ is a decreasing function of β satisfying $\delta(0) = 2\log 2 - 1$ and $\delta(1) = 0$. It follows from Proposition 4.1.2 that the bound in Lemma 4.1.1 is equivalent to the following uniform exponential rate of convergence to the origin of the semigroup generated by $f \in \mathscr{A}_\beta$,

$$|u(t,z)| \leq e^{-t\delta(\beta)}|z|.$$

A counterexample was given by Elin et al. [47], which shows that the condition in assertion (ii) of Lemma 4.1.1 does not imply the inclusion $f \in \mathscr{A}_\beta$ for any $\beta \in [0,1)$. They also observe the embedding of various subclasses of $\mathscr{S}$ in the class $\mathscr{G}_0$.

Recently, Kumar and Gangania [93] introduced and studied the class

$$\mathscr{F}(\Psi) = \left\{ f \in \mathscr{A} : \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \Psi(z), \ z \in \mathbb{U} \right\},$$

where Ψ is an analytic univalent function in $\mathbb{U}$, $\Psi(0) = 0$ and $\Psi(\mathbb{U})$ is a starlike domain with respect to 0. It should be noted that this class also contains non-univalent functions. For $\Psi(z) = z/(1-\alpha z^2)$,

$\alpha \in [0,1)$, the class $\mathscr{F}(\Psi)$ reduces to the class

$$\mathscr{BS}(\alpha)=\left\{f\in\mathscr{A}:\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{z}{1-\alpha z^2},\ z\in\mathbb{U}\right\},$$

introduced by Karger et al. [77]. The geometric properties of $f \in \mathscr{BS}(\alpha)$ including radii problems for starlike functions of order α were studied by Karger et al. [78]. Another interesting class is

$$\mathscr{U}(\lambda)=\left\{f\in\mathscr{A}:\left|f'(z)\left(\frac{z}{f(z)}\right)^2-1\right|<\lambda,\ \lambda\in(0,1]\right\}, \tag{4.1.2}$$

introduced by Obradović and Ponnusamy [131]. It is well known that $\mathscr{U}(\lambda) \subset \mathscr{S}$ for $\lambda \in (0,1]$. For more works on this class, one can refer to [132, 133] and the references cited therein.

The functions $f(z) = z/(1-z+z^2)$ and $g(z) = z(1+z)/(1-z)$ reveals that neither $\mathscr{S}^* \subset \mathscr{G}_0$ nor $\mathscr{G}_0 \subset \mathscr{S}^*$. Here, the radius problem arises. Elin et al. [47] solved this problem for the class $\mathscr{A}_\beta$, which immediately provides the radius of starlikeness for the class $\mathscr{G}_0$, when $\beta = 1$. They proved that the radius of starlikeness is $r = 2-\sqrt{2}$ for the class $\mathscr{G}_0$.

Generalizing this work, we obtained the radius of starlikeness of order α for the class $\mathscr{A}_\beta$. Further, we see the inclusion of the classes $\mathscr{S}^*(\varphi)$, $\mathscr{F}(\Psi)$ and $\mathscr{U}(\lambda)$ in the class of semigroup generators and find the uniform exponential rate of convergence of semigroup generated by the members of these classes. Additionally, we prove that the convolution of $f \in \mathscr{A}_\beta$ with $g \in \mathscr{C}$ is again in $\mathscr{A}_\beta$ and the class $\mathscr{A}_\beta$ is preserved under some integral operators, where the Hadamard product or convolution of two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as $(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

In 1914, Bohr [20] proved that, if $\omega(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathscr{B}_0$, then $\sum_{n=0}^{\infty} |c_n| r^n \leq 1$ for all $z \in \mathbb{U}$ with $|z| = r \leq 1/3$. The constant $1/3$ is known as the Bohr radius and it can not be improved. Different generalizations of the Bohr inequality are taken into consideration [112, 185]. We say that, the class $\mathscr{A}_\beta$ satisfies the Bohr phenomenon if there exists r_b such that

$$|z|+\sum_{n=2}^{\infty}|a_n||z|^n \leq d(f(0),\partial f(\mathbb{U}))$$

holds in $|z| = r \leq r_b$, where $\partial f(\mathbb{U})$ is the boundary of image domain of $\mathbb{U}$ under $f \in \mathscr{A}_\beta$ and d denotes the Euclidean distance between $f(0)$ and $\partial f(\mathbb{U})$.

Muhanna [123] showed that the Bohr phenomenon holds for the class of univalent functions and the class of convex functions when $|z| = r \leq 3-2\sqrt{2}$ and $|z| = r \leq 1/3$, respectively. We refer to the survey article [124] for further details on this topic. There is also the concept of the Rogosinski radius along with the Bohr radius, although a little is known about Rogosinski radius in comparison to the Bohr radius [102, 160]. It says that, if $\omega(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathscr{B}_0$, then

$$\sum_{n=0}^{N-1}|c_n||z|^n \leq 1, \quad (N \in \mathbb{N})$$

in the disk $|z| = r \leq 1/2$. The radius $1/2$ is called the Rogosinski radius. Kayumov et al. [79] considered the following expression, called Bohr-Rogosinski sum,

$$R_N^f(z) := |f(z)| + \sum_{n=N}^{\infty} |a_n||z|^n$$

and found the radius r_N such that $R_N^f(z) \leq 1$ in $|z| = r \leq r_N$ for the Cesáro operators on the space of bounded analytic functions. The largest such r_N is called the Bohr-Rogosinski radius. Here, we say that:

Definition 4.1.3. Let $f \in \mathscr{A}_\beta$. We say that the class $\mathscr{A}_\beta$ satisfies the Bohr-Rogosinski phenomenon if there exist r_N such that

$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(\Omega)), \quad m, N \in \mathbb{N}$$

holds in $|z| = r \leq r_N$.

Elin et al. [47] established the embedding of various subclasses of starlike functions in the class of semigroup generators and also found the radius of starlikeness for the same class. However, problems related to coefficients, growth estimates, and others were still open. Focusing on these problems, in this chapter, we derive the sharp bounds of coefficient functionals for the class of semigroup generators such as second-order Hankel determinants, Zalcman functional, differences of successive coefficients, and third-order Toeplitz and Hermitian-Toeplitz determinants. Additionally, we determine sharp growth estimates, which are used to prove the Bohr and Bohr-Rogosinski phenomena for this class. The chapter also establishes membership criteria, utilizing the Hadamard product, for normalized analytic functions to belong to the class of semigroup generators. Furthermore, we examine the embedding of various subclasses of $\mathscr{S}$ in this class and address radius problems specific to semigroup generators, thereby generalizing existing results.

4.2 Hankel Determinant and Zalcman Functional

In this section, we obtain the sharp bounds of n^{th} Taylor's coefficients, Hankel determinants and Zalcman functional for functions in the class $\mathscr{A}_\beta$. For $m \geq 1$, the m^{th} Hankel determinant of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}$ is defined by

$$H_{m,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2} \end{vmatrix}. \tag{4.2.1}$$

In particular, $H_{2,2}(f) = a_2 a_4 - a_3^2$ and $H_{2,n}(f) = a_n a_{n+2} - a_{n+1}^2$. Finding sharp estimates for Hankel determinants is a longstanding challenge within the realm of coefficient problems, with a notable recent surge in literature dedicated to addressing it. Janteng et al. [74] established the sharp estimates of $|H_{2,2}(f)|$ for the classes $\mathscr{S}^*$ and $\mathscr{C}$. Krishna and Ramreddy [88] solved the same problem for the classes $\mathscr{S}^*(\alpha)$ and $\mathscr{C}(\alpha)$. A summary of results based on the estimates of Hankel determinants for certain subclasses of $\mathscr{S}$ is provided in [182].

In 1999, Ma [115] proposed a conjecture for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}$ that

$$|J_{m,n}(f)| := |a_n a_m - a_{n+m-1}| \leq (m-1)(n-1).$$

He proved this conjecture for the class of starlike functions and univalent functions with real coefficients. It is also called generalized Zalcman conjecture as it generalizes the Zalcman conjecture $|a_n^2 - a_{2n-1}| \leq (2n-1)^2$ for $f \in \mathscr{S}$. Recently, bounds of $|J_{2,3}(f)|$ are obtained for various subclasses of $\mathscr{A}$ [11, 37]. We now begin to obtain the sharp bounds of $|H_{2,n}(f)|$ and $|J_{2,3}(f)|$ for $f \in \mathscr{A}_\beta$.

Theorem 4.2.1. If $f \in \mathscr{A}_\beta$ is of the form (1.0.1), then

$$|a_n| \leq \frac{2}{n - \beta(n-1)}. \tag{4.2.2}$$

Further, this inequality is sharp for each n.

Proof. Let $f \in \mathscr{A}_\beta$ is given by (1.0.1), then we have

$$\beta \frac{f(z)}{z} + (1-\beta) z f'(z) = p(z) \qquad (z \in \mathbb{U}),$$

where $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ is a member of the Carathéodory class $\mathscr{P}$. Upon comparing the coefficients of the same powers on either side with the series expansion of f and p yields

$$(n - (n-1)\beta) a_n = p_{n-1} \tag{4.2.3}$$

for $n = 2, 3, 4, \cdots$, which gives the needed bound of $|a_n|$ using the Carathéodory coefficient bounds $|p_n| \leq 2$ (see [41]). The function $\tilde{f} : \mathbb{U} \to \mathbb{C}$ defined by

$$\tilde{f}(z) = z\left(-1 + 2\left({}_2F_1\left[1, \frac{1}{1-\beta}, \frac{2-\beta}{1-\beta}, z\right]\right)\right) = z + \sum_{n=2}^{\infty} \frac{2}{n-(n-1)\beta} z^n \tag{4.2.4}$$

satisfies the condition $\operatorname{Re}\left(\beta \tilde{f}(z)/z + (1-\beta)\tilde{f}'(z)\right) > 0$, hence $\tilde{f}$ is a member of $\mathscr{A}_\beta$, where ${}_2F_1$ denotes the Gauss hypergeometric function. Equality in (4.2.2) occurs for $\tilde{f}$, which proves the sharpness of the bound. □

Corollary 4.2.2. If $f \in \mathscr{A}_\beta$, then

$$|a_n a_{n+2} - \mu a_{n+1}^2| \leq \frac{4}{(n-(n-1)\beta)(n+2-(n+1)\beta)} + \frac{4\mu}{(n+1-n\beta)^2}, \quad \mu \geq 0.$$

The bound is sharp.

Proof. Since $|a_n a_{n+2} - \mu a_{n+1}^2| \leq |a_n||a_{n+2}| + \mu |a_{n+1}|^2$. The bound simply follows from (4.2.2). To see the sharpness, consider

$$\tilde{f}_1(z) = z\left(-1+2\left({}_2F_1\left[1, \frac{1}{1-\beta}, \frac{2-\beta}{1-\beta}, iz\right]\right)\right) = z + \sum_{n=2}^{\infty} \frac{2i^{n-1}}{(n-(n-1)\beta)} z^n. \tag{4.2.5}$$

It can be easily seen that $\tilde{f}_1(z)$ satisfy (4.1.1), thus $\tilde{f}_1 \in \mathscr{A}_\beta$. □

For $\mu = 1$, Corollary 4.2.2 gives the following sharp bound:

Corollary 4.2.3. If $f \in \mathscr{A}_\beta$ is of the form (1.0.1), then

$$|H_2(n)(f)| \leq \frac{4\left((2n^2-1)\beta^2-(4n^2+4n-2)\beta+2n^2+4n+1\right)}{(n-(n-1)\beta)(n+2-(n+1)\beta)(n+1-n\beta)^2}.$$

For $n = 2$ and 3, the following sharp bounds of second order Hankel determinant follows:

Corollary 4.2.4. If $f \in \mathscr{A}_\beta$ is of the form (1.0.1), then

$$|H_2(2)(f)| \leq \frac{4(7\beta^2-22\beta+17)}{(4-3\beta)(3-2\beta)^2(2-\beta)}, \quad |H_2(3)(f)| \leq \frac{4(17\beta^2-46\beta+31)}{(5-4\beta)(4-3\beta)^2(3-2\beta)}.$$

Theorem 4.2.5. If $f \in \mathscr{A}_\beta$ is of the form (1.0.1), then

$$|J_{2,3}(f)| \leq \frac{2}{4-3\beta}.$$

The bound is sharp.

Proof. Let $f \in \mathscr{A}_\beta$ is given by (1.0.1), then from (4.2.3), we have

$$|J_{2,3}(f)| = |a_2a_3 - a_4| = \left|\frac{p_1p_2}{(3-2\beta)(2-\beta)} - \frac{p_3}{4-3\beta}\right|. \tag{4.2.6}$$

For $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n \in \mathscr{P}$, Libera et al. [108] proved that

$$\left.\begin{aligned} 2p_2 &= p_1^2 + \zeta_1(4-p_1^2), \\ 4p_3 &= p_1^3 + 2\zeta_1 p_1(4-p_1^2) - \zeta_1^2 p_1(4-p_1^2) + 2\zeta_2(1-|\zeta_1|^2)(4-p_1^2), \end{aligned}\right\} \tag{4.2.7}$$

where $|\zeta_1| \leq 1$ and $|\zeta_2| \leq 1$. Substituting these values of p_2 and p_3 in (4.2.6), we obtain

$$\begin{aligned} |J_{2,3}(f)| = \Bigg| &\frac{p_1^3}{4}\left(\frac{2}{2\beta^2-7\beta+6} + \frac{1}{3\beta-4}\right) - \frac{p_1(4-p_1^2)(1-\beta)^2\zeta_1}{(2-\beta)(3-2\beta)(4-3\beta)} \\ &+ \frac{p_1(4-p_1^2)\zeta_1^2}{4(4-3\beta)} - \frac{(4-p^2)(1-|\zeta_1|^2)\zeta_2}{2(4-3\beta)}\Bigg|. \end{aligned}$$

Since the class $\mathscr{P}$ is rotationally invariant and it is an easy exercise to check that the class $\mathscr{A}_\beta$ is also rotationally invariant, therefore, without losing generality, we can take $p_1 = p \in [0,2]$. Now, applying the triangle inequality with $|\zeta_1| = x$, we obtain

$$\begin{aligned}|J_{2,3}(f)| \leq & \frac{p^3}{4}\left(\frac{2}{2\beta^2-7\beta+6}+\frac{1}{3\beta-4}\right)+\frac{p(4-p^2)(1-\beta)^2x}{(2-\beta)(3-2\beta)(4-3\beta)}+\frac{4-p^2}{2(4-3\beta)} \\ & +x^2\left(\frac{p(4-p^2)}{4(4-3\beta)}-\frac{(4-p^2)}{2(4-3\beta)}\right) =: F(p,x).\end{aligned}$$

To determine the maximum value of $F(p,x)$, first we find out the stationary points, given by the roots of $\partial F/\partial p = 0$ and $\partial F/\partial x = 0$, where

$$\begin{aligned}\frac{\partial F(p,x)}{\partial p} = & \frac{3p^2(x^2(2\beta^2-7\beta+6)+4x(1-\beta)^2+2\beta^2-\beta-2)}{4(-2+\beta)(-3+2\beta)(-4+3\beta)}+p\left(\frac{x^2}{4-3\beta}-\frac{1}{4-3\beta}\right) \\ & +\frac{x^2}{4-3\beta}+\frac{4x(1-\beta)^2}{(4-3\beta)(3-2\beta)(2-\beta)}, \\ \frac{\partial F(p,x)}{\partial x} = & 2x\left(\frac{p(4-p^2)}{4(4-3\beta)}-\frac{4-p^2}{2(4-3\beta)}\right)+\frac{p(4-p^2)(1-\beta)^2}{(4-3\beta)(3-2\beta)(2-\beta)}.\end{aligned}$$

A simple calculation shows that for $p \in [0,2]$ and $x \in [0,1]$, the stationary point is $(0,0)$ and

$$\left(\frac{\partial^2 F}{\partial p^2}\frac{\partial^2 F}{\partial x^2}-\frac{\partial^2 F}{\partial x \partial p}\right)_{(p,x)=(0,0)} = \frac{4(8-11\beta+4\beta^2)}{(3-2\beta)^2(2-\beta)^2(4-3\beta))} > 0 \quad \text{for all} \quad \beta \in [0,1].$$

Thus $F(p,x)$ attains either maximum or minimum at $(p,x) = (0,0)$. Since, we have

$$\left(\frac{\partial^2 F}{\partial p^2}\right)_{(0,0)} = \frac{-1}{4-3\beta} < 0, \quad \left(\frac{\partial^2 F}{\partial x^2}\right)_{(0,0)} = \frac{-4}{4-3\beta} < 0 \quad \text{for all} \quad \beta \in [0,1].$$

Therefore, $F(p,x)$ attains its maximum value at $(p,x) = (0,0)$, which is $2/(4-3\beta)$.

Now, to prove the sharpness of the bound, consider the function $\tilde{f}_2 : \mathbb{U} \to \mathbb{C}$ given by

$$\beta\frac{\tilde{f}_2(z)}{z}+(1-\beta)\tilde{f}_2'(z) = \frac{1+z^3}{1-z^3}. \tag{4.2.8}$$

If $\tilde{f}_2(z) = z+\sum_{n=2}^{\infty} a_n z^n$, then $a_2 = a_3 = 0$ and $a_4 = 2/(4-3\beta)$, thus $|J_{2,3}(f)| = 2/(4-3\beta)$. □

4.3 Toeplitz and Hermitian-Toeplitz Determinant

This section provides the sharp bounds of certain Toeplitz and Hermitian-Toeplitz determinants formed over the coefficients of functions in the class $\mathscr{A}_\beta$.

Theorem 4.3.1. If $f \in \mathscr{A}_\beta$ is of the form (1.0.1), then

(i) $|T_{2,n}(f)| \leq 4\left(\dfrac{1}{(n-\beta(n-1))^2}+\dfrac{1}{(n+1-n\beta)^2}\right)$,

(ii) $|T_{3,1}(f)| \leq \dfrac{4\beta^4-28\beta^3+101\beta^2-196\beta+140}{(3-2\beta)^2(\beta-2)^2}$.

The bounds are sharp.

Proof. From (1.5.1), it follows that

$$|T_{2,n}(f)| = |a_n^2 - a_{n+1}^2| \leq |a_n|^2 + |a_{n+1}|^2.$$

Using the bound of $|a_n|$ from (4.2.2), required bound of $|T_{2,n}(f)|$ follows directly and equality case holds for the function $\tilde{f}_1$ given by (4.2.5).

Now we proceed for $|T_{3,1}(f)|$. Again from (1.5.1), we have

$$|T_{3,1}(f)| = |1 - 2a_2^2 + 2a_2^2a_3 - a_3^2| \leq 1 + 2|a_2|^2 + |a_3||a_3 - 2a_2^2|. \tag{4.3.1}$$

By (4.2.3),

$$|a_3 - 2a_2^2| = \frac{1}{3-2\beta}\left|p_2 - \frac{2(3-2\beta)}{(2-\beta)^2}p_1^2\right|.$$

Applying the well known result $|p_2 - \nu p_1^2| \leq 4\nu - 2$ for $\nu > 1$ (see [117]), we obtain

$$|a_3 - 2a_2^2| \leq \frac{8}{(2-\beta)^2} - \frac{2}{3-2\beta}.$$

Using this bound of $|a_3 - 2a_2^2|$ and the bounds of $|a_2|$, $|a_3|$ from (4.2.2) in (4.3.1), required bound of $|T_{3,1}(f)|$ follows. Sharpness of the bound of $|T_{3,1}(f)|$ follows from the function $\tilde{f}_1$. □

Remark 4.3.1. The bounds of $|T_{2,n}(f)|$ and $|T_{3,1}(f)|$ for the class $\mathscr{R}$ follow from Theorem 4.3.1, when $\beta = 0$ [6, Theorem 2.12].

Theorem 4.3.2. If $f \in \mathscr{A}_\beta$ is of the form (1.0.1), then

$$T_3(1)(f) \leq \begin{cases} \dfrac{4\beta^4 - 28\beta^3 + 37\beta^2 - 4\beta - 4}{(3-2\beta)^2(2-\beta)^2}; & \frac{10-\sqrt{10}}{9} \leq \beta \leq 1, \\ 1; & 0 \leq \beta \leq \frac{10-\sqrt{10}}{9}. \end{cases} \tag{4.3.2}$$

The bounds are sharp.

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}_\beta$, Theorem 4.2.1 yields

$$|a_2| \leq \frac{2}{2-\beta} \quad \text{and} \quad |a_3| \leq \frac{2}{3-2\beta}.$$

Hence $|a_2| \in [0,2]$ and $|a_3| \in [0,2]$ for $\beta \in [0,1]$. From (3.1.1), we have

$$\begin{aligned} T_3(1)(f) &= 1 + 2\text{Re}(a_2^2\bar{a}_3) - 2|a_2|^2 - |a_3|^2 \\ &\leq 1 + 2|a_2|^2|a_3| - 2|a_2|^2 - |a_3|^2 =: G(|a_3|), \end{aligned}$$

where $G(x) = 1 + 2|a_2|^2x - 2|a_2|^2 - x^2$ with $x = |a_3| \in [0,2]$. Since $G'(x) = 0$ at $x_0 := |a_2|^2$ and $G''(x_0) < 0$, therefore $G(x)$ attains its maximum value at $x = x_0$, whenever $|a_2|^2$ belongs to the range of x, that means $|a_2|^2 \leq 2$. Thus

$$\begin{aligned} T_3(1)(f) \leq G(|a_2|^2) &= (|a_2|^2-1)^2 \\ &\leq 1 \quad \text{when} \quad |a_2|^2 \leq 2, \\ &= 1 \quad \text{when} \quad 0 \leq \beta \leq 2-\sqrt{2}. \end{aligned}$$

Now, the other case, when $|a_2|^2$ does not lie in the range of x, that is $|a_2|^2 > 2$ or $2-\sqrt{2} \leq \beta \leq 1$, then

$$\begin{aligned} T_3(1)(f) \leq \max G(x) &= G\left(\frac{2}{3-2\beta}\right) \\ &= 1-2a_2^2-\frac{4}{(3-2\beta)^2}+\frac{4a_2^2}{3-2\beta} \\ &\leq \frac{4\beta^4-28\beta^3+37\beta^2-4\beta-4}{(3-2\beta)^2(2-\beta)^2}. \end{aligned}$$

Using all these above arguments, we obtain

$$T_3(1)(f) \leq \begin{cases} 1, & 0 \leq \beta \leq \beta_0; \\ \dfrac{4\beta^4-28\beta^3+37\beta^2-4\beta-4}{(3-2\beta)^2(2-\beta)^2}, & \beta_0 \leq \beta \leq 1, \end{cases}$$

where $\beta_0 = (10-\sqrt{10})/9$ is the root of the equation $9\beta^2-20\beta+10=0$.

The sharpness of the bound follows from $f(z)=z$ when $0 \leq \beta \leq (10-\sqrt{10})/9$. However, for $(10-\sqrt{10})/9 \leq \beta \leq 1$, equality in (4.3.2) holds for the function $\tilde{f}$ given in (4.2.4). □

Remark 4.3.2. For $\beta = 0$ in Theorem 4.3.2, we obtain $T_3(1)(f) \leq 1$ for $f \in \mathscr{R}$ [97, Example 2.4].

Theorem 4.3.3. If $f \in \mathscr{A}_\beta$ is of the form (1.0.1), then

$$T_3(1)(f) \geq 1-\frac{4\beta-9}{\beta^4-4\beta^3+2\beta^2+8\beta-8}.$$

The bound is sharp.

Proof. Let $f \in \mathscr{A}_\beta$, then from (4.2.3), we have

$$a_2 = \frac{p_1}{2-\beta} \quad \text{and} \quad a_3 = \frac{p_2}{3-2\beta}.$$

Now, by replacing p_2 in terms of p_1 using (4.2.7), we get

$$2\operatorname{Re}(a_2^2\bar{a}_3) = \frac{p_1^4+p_1^2(4-p_1^2)\operatorname{Re}(\zeta_1)}{(3-2\beta)(2-\beta)^2}, \quad -|a_2|^2 = \frac{|p_1|^2}{(2-\beta)^2}$$

and

$$-|a_3|^2 = -\frac{p_1^4+(4-p_1^2)^2|\zeta_1|^2+2p_1^2(4-p_1^2)\operatorname{Re}(\zeta_1)}{4(3-2\beta)^2}.$$

A simple computation yields that

$$T_3(1)(f) = 1 + \frac{1}{4(3-2\beta)^2(2-\beta)^2}\Big(p_1^4(8-4\beta-\beta^2) - 8p_1^2(3-2\beta)^2 - (4-p_1^2)^2(2-\beta)^2|\zeta_1|^2$$
$$+2p_1^2(4-p_1^2)(2-\beta^2)\,\mathrm{Re}\,(\zeta_1)\Big)$$
$$=: g(p_1,\zeta_1,\mathrm{Re}(\zeta_1)).$$

Since the classes $\mathcal{A}_\beta$ and $\mathcal{P}$ are rotationally invariant, we can take $p = p_1 \in [0,2]$. Using $\mathrm{Re}(\zeta_1) \geq -|\zeta_1|$ with notation $|\zeta_1| = x$, we have $g(p_1,|\zeta_1|,\mathrm{Re}\,(\zeta_1)) \geq g_1(p,x)$, where

$$g_1(p,x) = 1 + \frac{1}{4(3-2\beta)^2(2-\beta)^2}\Big(p^4(8-4\beta-\beta^2) - 8p^2(3-2\beta)^2$$
$$-(4-p^2)^2(2-\beta)^2x^2 - 2p^2(4-p^2)(2-\beta^2)x\Big).$$

Also, note that

$$\frac{\partial g_1(p,x)}{\partial x} = -\frac{2(4-p^2)^2x(2-\beta)^2 + 2p^2(4-p^2)(2-\beta^2)}{4(3-2\beta)^2(2-\beta)^2} < 0$$

for all $p \in [0,2]$ and $\beta \in [0,1]$. Hence $g_1(p,x)$ is a decreasing function of x with $g_1(p,x) \geq g_1(p,1) =: g_2(p)$. Minimum of $g_2(p)$ is the lower bound of $T_3(1)(f)$. The equation $g_2'(p) = 0$ gives the following critical points

$$p^{(1)} = 0, \quad p^{(2)} = \pm\sqrt{\frac{(2\beta^2-8\beta+7)}{(2-\beta^2)}}.$$

Using the basic calculus rule, it can be easily observed that the function $g_2(p)$ attains its minimum value at $p^{(2)}$ as $g''(p^{(2)}) > 0$ for all $\beta \in [0,1]$. Thus

$$T_3(1)(f) \geq g_2(p^{(2)}) = 1 - \frac{(4\beta-9)}{(\beta^4-4\beta^3+2\beta^2+8\beta-8)}.$$

To show the sharpness consider the function $\tilde{f}_3 \in \mathcal{A}$ given by

$$\beta\frac{\tilde{f}_3(z)}{z} + (1-\beta)\tilde{f}_3'(z) = \frac{1-z^2}{1-z\sqrt{(2\beta^2-8\beta+7)/(2-\beta^2)}+z^2}.$$

For $\tilde{f}_3(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$a_2 = \frac{1}{2-\beta}\sqrt{\frac{2\beta^2-8\beta+7}{2-\beta^2}}, \quad a_3 = \frac{1-2\beta}{2-\beta^2}$$

and $T_3(1)(\tilde{f}_3) = 1 - (4\beta-9)/(\beta^4-4\beta^3+2\beta^2+8\beta-8)$. $\square$

Remark 4.3.3. For $\beta = 0$ in Theorem 4.3.2, we obtain $T_3(1)(f) \geq -1/8$ for $f \in \mathcal{R}$ [97, Example 2.4].

4.4 Successive Coefficient Difference

Robertson [159] proved that $3|a_{n+1}-a_n| \le (2n+1)|a_2-1|$ for the class of convex functions. Recently, Li and Sugawa [106] obtained the bound of $|a_{n+1}-a_n|$ for particular choices of n for the class of convex functions with fixed second coefficient. In this section, we find the the bound of $|a_{n+1}^N - a_n^N|$ $(N \in \mathbb{N})$ for $f \in \mathscr{A}_\beta$ with fixed second coefficient. In fact, it is more convenient to express our result in terms of $p := p_1$, applying the correspondence

$$(2-\beta)a_2 = p_1 = p.$$

We define the class $\mathscr{A}_\beta(p)$ for $p \in [-2,2]$ as follows

$$\mathscr{A}_\beta(p) = \{f \in \mathscr{A}_\beta : f''(0) = p\}.$$

Clearly,

$$\bigcup_{-2\le p\le 2} \mathscr{A}_\beta(p) \subset \mathscr{A}_\beta \quad \text{and} \quad \bigcup_{-2\le p\le 2} \mathscr{A}_\beta(p) \neq \mathscr{A}_\beta.$$

The following lemmas are used to establish our main results.

Lemma 4.4.1. [27] If $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n \in \mathscr{P}$, then the following estimate holds:

$$|p_{n+1}^N - p_n^N| \le 2^N \sqrt{2-2^{1-N}\operatorname{Re}(p_1^N)} \quad (N \in \mathbb{N}).$$

Equality holds for the function $(1+e^{i\alpha}z)/(1-e^{i\alpha}z)$, where $\alpha = \cos^{-1}(b/2)$ and $\operatorname{Re} p_1 = 2b$.

Lemma 4.4.2. [103] Fix $\xi \in \overline{\mathbb{U}}$. If $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n \in \mathscr{P}$, then

$$|\xi p_{n+1} - p_n| \le \frac{2(1-|\xi|^n)\left(1+|\xi|^2-\operatorname{Re}(\xi p_1)\right)}{1-|\xi|} + |2-\xi p_1||\xi|^n \ \text{ for } \ |\xi|<1.$$

The bounds are sharp for $p(z) = (1+z)/(1-z)$.

According to Komatu [85], if $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n$ and $q(z) = 1+\sum_{n=1}^{\infty} q_n z^n$ both are the members of $\mathscr{P}$, then the weighted Hadamard product, $f*g$, also belongs to $\mathscr{P}$, where

$$f*g = 1+\sum_{n=1}^{\infty} \frac{p_n q_n}{2} z^n.$$

Let us define $F_j(z) = F_{j-1} * p(z)$ for $j \in \mathbb{N}$ with $F_0(z) = p(z)$, then using the above result, we have $F_j \in \mathscr{P}$. Particulary, for $N \in \mathbb{N}$, the function

$$F_{N-1}(z) = 1+\sum_{n=1}^{\infty} \frac{p_n^N}{2^{N-1}} z^n \in \mathscr{P}.$$

Replacing $p(z)$ in Lemma 4.4.2 by F_{N-1}, the result is as follows:

Lemma 4.4.3. Fix $\xi \in \overline{\mathbb{U}}$ and $N \in \mathbb{N}$. If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then

$$|\xi p_{n+1}^N - p_n^N| \leq \frac{2(1-|\xi|^n)\left(2^{N-1} + 2^{N-1}|\xi|^2 - \mathrm{Re}(\xi p_1^N)\right)}{1-|\xi|} + |2^N - \xi p_1^N| \cdot |\xi|^n \quad \text{for } |\xi| < 1.$$

Equality holds for the function $(1+z)/(1-z)$.

Theorem 4.4.1. If $f \in \mathscr{A}_\beta(p)$, then the following inequalities hold:

$$|a_{n+1}^N - a_n^N| \leq \begin{cases} \dfrac{2(\sigma^n - \mu^n)(2^{N-1}\sigma^2 + 2^{N-1}\mu^2 - \sigma\mu p^N)}{(\sigma-\mu)\sigma\mu^{n+1}} + \dfrac{\sigma^n |2^N \mu - \sigma p^N|}{\sigma\mu^{n+1}}; & \beta \in [0,1), \\ \dfrac{2^N \sqrt{2 - 2^{1-N} p^N}}{\sigma}; \quad \beta = 1, \end{cases} \tag{4.4.1}$$

where $\sigma = (n-(n-1)\beta)^N$ and $\mu = (n+1-n\beta)^N$. Bounds for $\beta \in [0,1)$ is sharp for $p = 2$ whereas for $\beta = 1$, bound is sharp for odd N and $p = -2$.

Proof. For $f \in \mathscr{A}_\beta(p)$, from (4.2.3), we have

$$(n-(n-1)\beta)^N |a_{n+1}^N - a_n^N| = \left| \left(\frac{n-(n-1)\beta}{(n+1)-n\beta} \right)^N p_n^N - p_{n-1}^N \right|.$$

From Lemma 4.4.3 with $((n-(n-1)\beta)/((n+1)-n\beta))^N = \xi$, bound in (4.4.1) for $\beta \in [0,1)$ follows as $\xi \in (0,1)$ whenever $\beta \in (0,1)$. For $\beta = 1$, we have $\xi = 1$. Bounds for $\beta = 1$ are obtained using Lemma 4.4.1.

To show the sharpness for $\beta \in [0,1)$, consider the function $\tilde{f}(z)$ given in (4.2.4). As for $\tilde{f}$, we have

$$|a_{n+1} - a_n| = \frac{2^N}{(n-(n-1)\beta)^N} \left| \frac{(n-(n-1)\beta)^N}{(n+1-n\beta)^N} - 1 \right|,$$

which is same as in (4.4.1) for $p = 2$. In case of $\beta = 1$, for the function $\tilde{f}(-z)$, we have

$$|a_{n+1} - a_n| = \frac{2^{N+1}}{(n-(n-1)\beta)^N},$$

which coincides with the bounds in (4.4.1) for odd N and $p = -2$. □

For $N = 1$, Theorem 4.4.1 yields the following bounds:

Corollary 4.4.2. If $f \in \mathscr{A}_\beta(p)$ is of the form (1.0.1), then

$$|a_{n+1} - a_n| \leq \begin{cases} \dfrac{2(\sigma^n - \mu^n)(\sigma^2 + \mu^2 - \sigma\mu p)}{(\sigma-\mu)\sigma\mu^{n+1}} + \dfrac{\sigma^n |2\mu - \sigma p|}{\sigma\mu^{n+1}}; & \beta \in [0,1), \\ \dfrac{2\sqrt{2-p}}{\sigma}; & \beta = 1, \end{cases}$$

The class $\mathscr{A}_\beta$ reduces to the class $\mathscr{R}$ for $\beta = 0$. Let us take the corresponding class $\mathscr{R}(p) = \{f \in \mathscr{R} : f''(0) = p\}$. Theorem 4.4.1 gives the following result for the class $\mathscr{R}(p)$ when $\beta = 0$.

Corollary 4.4.3. If $f \in \mathscr{R}(p)$ is of the form (1.0.1), then the following sharp bounds hold:

$$|a_{n+1}^N - a_n^N| \le \frac{2(\sigma^n - \mu^n)(2^{N-1}\sigma^2 + 2^{N-1}\mu^2 - \sigma\mu p^N)}{(\sigma-\mu)\sigma\mu^{n+1}} + \frac{\sigma^n|2^N\mu - \sigma p^N|}{\sigma\mu^{n+1}}.$$

4.5 Growth Theorem and Bohr Phenomenon

In this section, the sharp Bohr and Bohr-Rogosinski radii are established for the class $\mathscr{A}_\beta$.

Theorem 4.5.1. If $f \in \mathscr{A}_\beta$, then for $|z| \le r$, the following hold:

(i) $-\dfrac{\tilde{f}(-r)}{r} \le \operatorname{Re}\left(\dfrac{f(z)}{z}\right) \le \dfrac{\tilde{f}(r)}{r}$,

(ii) $-\tilde{f}(-r) \le |f(z)| \le \tilde{f}(r)$,

where $\tilde{f}(z)$ is given by (4.2.4). All these estimations are sharp.

Proof. (i) Let $f \in \mathscr{A}_\beta$. Consider $p(z) = f(z)/z$, then we have

$$\operatorname{Re}(p(z) + (1-\beta)zp'(z)) > 0.$$

It can be viewed as $p(z) + (1-\beta)zp'(z) \prec (1+z)/(1-z)$. Further, by Hallenbeck and Rusheweyeh [121, Theorem 3.1b], it follows that

$$p(z) \prec q(z) \prec \frac{1+z}{1-z},$$

where $q(z)$ is convex and best dominant, given by

$$\begin{aligned} q(z) &= \frac{1}{(1-\beta)z^{(\frac{1}{1-\beta})}} \int_0^z \left(\frac{1+t}{1-t}\right) t^{(\frac{1}{1-\beta}-1)} dt \\ &= \frac{\tilde{f}(z)}{z}, \end{aligned}$$

where $\tilde{f}(z)$ is defined in (4.2.4). Since $q(z)$ is convex and all coefficients are real for $\beta \in [0,1]$, therefore image domain of $\mathbb{U}$ under the function $q(z)$ is symmetric with respect to real axis and

$$q(-r) \le \operatorname{Re}(q(z)) \le q(r), \quad |z| = r < 1.$$

As $p(z) = f(z)/z \prec q(z)$, so required bound of $\operatorname{Re}(f(z)/z)$ follows. This completes the first part. The sharpness of the bounds follows as $q(z)$ is the best dominant.
(ii) From [23, Lemma 4.10], $f \in \mathscr{A}_\beta$ if and only if

$$f(z) = z\int_0^1 p(t^{1-\beta}z)dt, \tag{4.5.1}$$

where $p \in \mathscr{P}$. Using the well known bound for Carathéodory functions, $|p(z)| \leq (1+r)/(1-r)$ for $|z| = r < 1$, we have

$$|f(z)| \leq r \int_0^1 \frac{1+rt^{1-\beta}}{1-rt^{1-\beta}} dt = \tilde{f}(r).$$

Now, we proceed for the lower bound of $|f(z)|$. After solving the integration in (4.5.1) for $p(z) = (1+z)/(1-z)$, we get

$$f(z) = z(-1+2H(z)),$$

where

$$H(z) = {}_2F_1\left[1, \frac{1}{1-\beta}, \frac{2-\beta}{1-\beta}, z\right].$$

Thus for $z = re^{i\theta}$,

$$|f(z)| = |z(-1+2H(z))| \geq \min_{\theta \in [0,2\pi]} g(\theta), \tag{4.5.2}$$

where

$$g(\theta) = \sqrt{\mathrm{Re}(re^{i\theta}(-1+2H(re^{i\theta})))^2 + \mathrm{Im}(re^{i\theta}(-1+2H(re^{i\theta})))^2}.$$

Since for different choices of β in $[0,1)$, $H(z)$ reduces to different functions. For instance, when $\beta = 0$, it becomes $-2\log(1-z)/z$ and for $\beta = 1/2$, it reduces to $-4(z+\log(1-z))/z^2$. By a simple calculation, we find that the function $g(\theta)$ is decreasing from $[0,\pi]$ and increasing from $[\pi, 2\pi]$ for $r \in (0,1)$ and $\beta \in [0,1)$. Hence $g(\theta)$ attains its minimum value at $\theta = \pi$. Thus from (4.5.2), we get

$$\begin{aligned} |f(z)| &\geq |-r(-1+2H(-r))| \\ &= r(-1+2H(-r)) = -r\tilde{f}(-r), \end{aligned}$$

which completes the proof. The bounds are sharp for the function $\tilde{f}(z)$. □

Theorem 4.5.2. If $f \in \mathscr{A}_\beta$ is of the form (1.0.1), then

$$|\omega(z^m)| + \sum_{n=2}^{\infty} |a_n z^n| \leq d(0, \partial f(\mathbb{U})), \quad m \in \mathbb{N},$$

in $|z| \leq r^*$, where r^* is the smallest positive root of

$$r^m + \tilde{f}(r) - r + \tilde{f}(-1) = 0. \tag{4.5.3}$$

The radius r^* is sharp.

Proof. Let $f \in \mathscr{A}_\beta$, then by Theorem 4.5.1, the Euclidean distance between $f(0) = 0$ and the boundary of $f(\mathbb{U})$ satisfies

$$d(0, \partial f(\mathbb{U})) \geq \lim_{r \to 1} |f(z)| = -\tilde{f}(-1).$$

Let $|z| \le r$. Now using (4.2.2) with the above inequality, we have

$$\begin{aligned}|\omega(z^m)| + \sum_{n=2}^{\infty}|a_n z^n| &\le r^m + \sum_{n=2}^{\infty}\left(\frac{2}{n-\beta(n-1)}\right) r^n,\\ &= r^m + \tilde{f}(r) - r\\ &\le -\tilde{f}(-1) \le d(0, \partial f(\mathbb{U})),\end{aligned}$$

which is true in $|z| = r \le r^*$, where r^* is the root of $H_1(r) = r(r^{m-1} - 1) + \tilde{f}(r) + \tilde{f}(-1)$. Note that, $H_1(0) = \tilde{f}(-1) < 0$ and $H_1(1) = \tilde{f}(1) + \tilde{f}(-1) > 0$ for all $\beta \in [0,1]$, therefore by the Intermediate value property for continuous functions there must exist a $r^* \in (0,1)$ such that $H_1(r^*) = 0$.

Sharpness holds for the functions $\tilde{f}(z)$ and $\omega(z) = z$. Since at $z = r^*$,

$$\begin{aligned}|\omega(z^m)| + \sum_{n=2}^{\infty}|a_n z^n| &= (r^*)^m + \sum_{n=2}^{\infty}\frac{2}{n-(n-1)\beta}(r^*)^n\\ &= (r^*)^m + \tilde{f}(r^*) - r^* = -\tilde{f}(-1).\end{aligned}$$

Hence the radius is sharp. □

For $\omega(z) = z$ and $m = 1$, Theorem 4.5.2 gives the following Bohr-radius for the class $\mathscr{A}_\beta$.

Corollary 4.5.3. If $f \in \mathscr{A}_\beta$, then $|z| + \sum_{n=2}^{\infty}|a_n z^n| \le d(0, \partial f(\mathbb{U}))$ in $|z| \le r_b$, where r_b is root of $\tilde{f}(r) + \tilde{f}(-1) = 0$. The radius r_b is sharp.

For various values of $\beta \in [0,1]$, the corresponding roots r_b are shown in Figure 4.1 and Table 4.1.

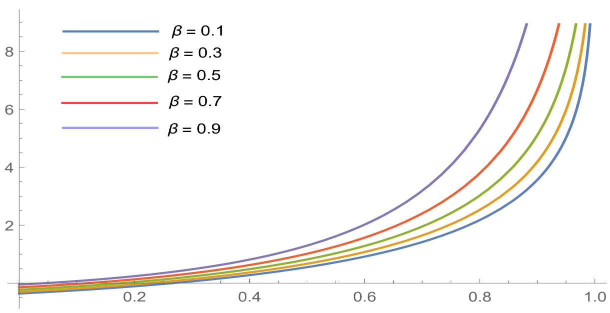

Figure 4.1: Root r_b for different values of β

β	0.1	0.2	0.3	0.5	0.7	0.8	0.9
r_b	0.267139	0.24766	0.22655	0.178366	0.119726	0.085113	0.0457777

Table 4.1: Radius r_b for various choices of β

Theorem 4.5.4. If $f \in \mathscr{A}_\beta$ is of the form (1.0.1), then

$$|f(z^m)| + \sum_{k=N}^{\infty}|a_k z^k| \le d(0, \partial f(\mathbb{U})) \tag{4.5.4}$$

hold for $|z| = r \leq r_N$, where r_N is the root of the equation

$$\tilde{f}(r^m) + \tilde{f}(r) - \hat{f}(r) + \tilde{f}(-1) = 0,$$

with

$$\hat{f}(r) = \begin{cases} 0 & N = 1, \\ r & N = 2, \\ r + \sum_{n=2}^{N-1} \frac{2}{(n-(n-1)\beta)} r^n & N \geq 3. \end{cases}$$

The radius is sharp.

Proof. Suppose $f \in \mathscr{A}_\beta$, then from (4.2.2) and Theorem 4.5.1, we have

$$\begin{aligned} |f(z^m)| + \sum_{k=N}^{\infty} |a_k z^k| &\leq \tilde{f}(r^m) + \sum_{n=N}^{\infty} \frac{2}{n-(n-1)\beta} r^n \\ &= \tilde{f}(r^m) - \hat{f}(r) + \tilde{f}(r) \\ &\leq -\tilde{f}(-1) \\ &\leq d(0, \partial f(\mathbb{U})) \end{aligned}$$

holds in $|z| = r_N$, where r_N is the root of

$$H_2(r) := \tilde{f}(r^m) - \hat{f}(r) + \tilde{f}(r) + \tilde{f}(-1) = 0.$$

Since $H_2(0) = \tilde{f}(-1) < 0$ and $H_2(1) = (\tilde{f}(1) - \hat{f}(1)) + (\tilde{f}(1) + \tilde{f}(-1)) > 0$, therefore there exist a $r_N \in (0,1)$ such that (4.5.4) holds. Note that, for the function $\tilde{f}(z)$ at $|z| = r_N$,

$$\begin{aligned} |f(z^m)| + \sum_{k=N}^{\infty} |a_k z^k| &= \tilde{f}((r_N)^m) + \sum_{n=N}^{\infty} \frac{2}{n-(n-1)\beta} (r_N)^n \\ &= -\tilde{f}(-1), \end{aligned}$$

which proves the sharpness of the radius. □

4.6 Convolution Properties

The following lemmas help us in proving our results.

Lemma 4.6.1. [170] If $g(z)$ is analytic in $\mathbb{U}$, $g(0) = 1$ and $\operatorname{Re} g(z) > 1/2$, then for any function f, analytic in $\mathbb{U}$, the function $g * f$ takes values in the convex hull of the image of $\mathbb{U}$ under f.

Lemma 4.6.2. [164] If $g \in \mathscr{C}$ and $h \in \mathscr{S}^*$, then for each function F, analytic in $\mathbb{U}$, the image of $\mathbb{U}$ under $(g * Fh)/(g * h)$ is a subset of the convex hull of $F(\mathbb{U})$.

Theorem 4.6.1. Let $f \in \mathscr{A}$. Then $f \in \mathscr{A}_\beta$ if and only if

$$f(z) * z\left(\frac{1-z\beta}{(1-z)^2} - \frac{1+\zeta}{1-\zeta}\right) \neq 0, \quad |\zeta| = 1.$$

Proof. From (4.1.1), $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}_\beta$ if and only if

$$\beta \frac{f(z)}{z} + (1-\beta) f'(z) \prec \frac{1+z}{1-z},$$

which ensures the existence of a Schwarz function ω such that

$$\beta \frac{f(z)}{z} + (1-\beta) f'(z) = \frac{1+\omega(z)}{1-\omega(z)}, \quad z \in \mathbb{U}.$$

By the property of subordination, we have

$$\beta \frac{f(z)}{z} + (1-\beta) f'(z) \neq \frac{1+\zeta}{1-\zeta}, \tag{4.6.1}$$

where $|\zeta| = 1$. Using the following basic convolution properties

$$z = f(z) * z, \;\; f(z) = f(z) * \frac{z}{(1-z)} \;\text{ and }\; zf'(z) = f(z) * \frac{z}{(1-z)^2}$$

in (4.6.1), we obtain

$$f(z) * \left(\frac{\beta z}{1-z} + \frac{(1-\beta)z}{(1-z)^2} - \frac{z(1+\zeta)}{1-\zeta} \right) \neq 0,$$

which completes the result. □

Theorem 4.6.2. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}$. If $\sum_{n=2}^{\infty} (\beta + n(1-\beta))|a_n| < 1$, then $f \in \mathscr{A}_\beta$.

Proof. To prove $f \in \mathscr{A}_\beta$, it is sufficient to show that

$$\left| \frac{\beta f(z)/z + (1-\beta) f'(z) - 1}{\beta f(z)/z + (1-\beta) f'(z) + 1} \right| < 1, \quad z \in \mathbb{U}.$$

We have

$$\begin{aligned} \left| \frac{\beta f(z)/z + (1-\beta) f'(z) - 1}{\beta f(z)/z + (1-\beta) f'(z) + 1} \right| &= \left| \frac{\sum_{n=2}^{\infty} (\beta + n(1-\beta)) a_n z^n}{2 + \sum_{n=2}^{\infty} (\beta + n(1-\beta)) a_n z^n} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} (\beta + n(1-\beta))|a_n||z|^n}{2 - \sum_{n=2}^{\infty} (\beta + n(1-\beta))|a_n||z|^n} \\ &\leq \frac{\sum_{n=2}^{\infty} (\beta + n(1-\beta))|a_n|}{2 - \sum_{n=2}^{\infty} (\beta + n(1-\beta))|a_n|}. \end{aligned}$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty} (\beta + n(1-\beta))|a_n| < 2 - \sum_{n=2}^{\infty} (\beta + n(1-\beta))|a_n|,$$

which is equivalent to

$$\sum_{n=2}^{\infty} (\beta + n(1-\beta))|a_n| < 1. \tag{4.6.2}$$

But (4.6.2) is true by hypothesis. Therefore

$$\left|\frac{\beta f(z)/z+(1-\beta)f'(z)-1}{\beta f(z)/z+(1-\beta)f'(z)+1}\right|<1, \quad z\in\mathbb{U}$$

and the proof is complete. □

If we put $\beta=1$ in Theorem 4.6.2, we obtain a sufficient condition for $\mathscr{G}_0$.

Corollary 4.6.3. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{A}$. If $\sum_{n=2}^{\infty}|a_n|<1$, then $f\in\mathscr{G}_0$.

Example 4.6.4. Polynomial $f(z)=z+a_nz^n\in\mathscr{G}_0$, $(n\geq 2)$ whenever $|a_n|<1$.

Theorem 4.6.5. If $f\in\mathscr{A}_\beta$ and $g\in\mathscr{A}$ such that $\operatorname{Re} g(z)>1/2$, then $f*g\in\mathscr{A}_\beta$.

Proof. For $F(z)=(f*g)(z)$, we have

$$zF'(z)=zf'(z)*g(z).$$

Thus,

$$\begin{aligned}\operatorname{Re}\left(\frac{\beta F(z)+(1-\beta)zF'(z)}{z}\right)&=\operatorname{Re}\left(\frac{\beta(f(z)*g(z))+(1-\beta)zf'(z)*g(z)}{z}\right)\\&=\operatorname{Re}\left(\left(\frac{\beta f(z)+(1-\beta)zf'(z)}{z}\right)*g(z)\right).\end{aligned}$$

Since $\operatorname{Re}\left((\beta f(z)+(1-\beta)zf'(z))/z\right)>0$ and by the hypothesis $\operatorname{Re} g(z)>1/2$, therefore by Lemma 4.6.1,

$$\operatorname{Re}\left(\frac{\beta F(z)+(1-\beta)zF'(z)}{z}\right)>0.$$

Consequently, $F(z)=f*g\in\mathscr{A}_\beta$, which completes the proof. □

Theorem 4.6.6. If $f\in\mathscr{A}_\beta$ and $g\in\mathscr{C}$, then $f*g\in\mathscr{A}_\beta$.

Proof. Let $f\in\mathscr{A}_\beta$, then we have $\operatorname{Re} F(z)>0$, where

$$F(z):=\frac{\beta f(z)+(1-\beta)zf'(z)}{z}.$$

For $g\in\mathscr{C}$, we have

$$\frac{g*zF}{g*z}=\left(\frac{\beta(g*f)+(1-\beta)z(g*f)'}{z}\right),$$

which together with Lemma 4.6.2 yields that $\operatorname{Re}\left((\beta(g*f)+(1-\beta)z(g*f)'/z)\right)>0$, hence $f*g\in\mathscr{A}_\beta$. □

If we take

$$g(z)=\sum_{n=1}^{\infty}\left(\frac{1+\gamma}{n+\gamma}\right)z^n,\quad \operatorname{Re}\gamma\geq-\frac{1}{2},$$

which is a convex function in $\mathbb{U}$ (see [162]), in Theorem 4.6.6, the following result follows:

Corollary 4.6.7. If $f \in \mathscr{A}_\beta$, then so is

$$\frac{1+\gamma}{z^\gamma}\int_0^z t^{\gamma-1} f(t)dt, \quad \operatorname{Re}\gamma \geq -\frac{1}{2}.$$

For $\gamma = 0$ and $\gamma = 1$, the function g reduces to $-\log(1-z)$ and $-2(z+\log(1-z))/z$, respectively, and from Theorem 4.6.6, we obtain the following:

Corollary 4.6.8. (i) If $f \in \mathscr{A}_\beta$, then

$$\int_0^z \frac{f(t)-f(0)}{t}dt = \int_0^z \frac{f(t)}{t}dt \in \mathscr{A}_\beta.$$

(ii) If $f \in \mathscr{A}_\beta$, then

$$\frac{2}{z}\int_0^z f(t)dt \in \mathscr{A}_\beta.$$

Consider the function

$$g(z) = \frac{1}{1-x}\log\left(\frac{1-xz}{1-z}\right), \quad |x| \leq 1, \quad x \neq 1.$$

Since $g(z)$ is a convex function, for $f \in \mathscr{A}_\beta$, Theorem 4.6.6 yields:

Corollary 4.6.9. If $f \in \mathscr{A}_\beta$, then

$$\int_0^z \frac{f(t)-f(xt)}{t-xt}dt \in \mathscr{A}_\beta.$$

4.7 Inclusion and Radius Problems

In this section, we see the inclusion of various subclasses of $\mathscr{A}$ in the class of semigroup generators.

Theorem 4.7.1. If Ψ is convex, then $\mathscr{F}(\Psi) \subset \mathscr{G}_0$ whenever

$$\operatorname{Re}\left(\exp\int_0^z \frac{\Psi(t)}{t}\right) \geq \eta > 0. \tag{4.7.1}$$

Moreover, the semigroup $\{u(t,\cdot)\}_{t\geq 0}$ generated by $f \in \mathscr{F}(\Psi)$ satisfies $|u(t,\cdot)| \leq e^{-t\eta}|z|$ for all $z \in \mathbb{U}$.

Proof. Let $f \in \mathscr{F}(\Psi)$, then

$$\frac{zf'(z)}{f(z)} - 1 \prec \Psi(z). \tag{4.7.2}$$

It can be easily seen that

$$\log\left(\frac{1}{1-z}\right) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

is a convex univalent function. Using the result [163], which states that: for any convex univalent functions F and G in $\mathbb{U}$, if $f \prec F$ and $g \prec G$, then $f * g \prec F * G$, in (4.7.2), we get

$$\left(\frac{zf'(z)}{f(z)} - 1\right) * \log\left(\frac{1}{1-z}\right) \prec \Psi(z) * \log\left(\frac{1}{1-z}\right). \tag{4.7.3}$$

Now, applying the following convolution properties

$$\left(\frac{zf'(z)}{f(z)}-1\right)*\log\left(\frac{1}{1-z}\right)=\int_0^z\frac{1}{t}\left(\frac{tf'(t)}{f(t)}-1\right)dt,$$
$$\Psi(z)*\log\left(\frac{1}{1-z}\right)=\int_0^z\frac{\Psi(t)}{t}dt,$$

in (4.7.3), we obtain

$$\int_0^z\frac{1}{t}\left(\frac{tf'(t)}{f(t)}-1\right)dt\prec\int_0^z\frac{\Psi(t)}{t}dt.$$

Consequently,

$$\frac{f(z)}{z}=\exp\int_0^z\frac{1}{t}\left(\frac{tf'(t)}{f(t)}-1\right)dt\prec\exp\int_0^z\frac{\Psi(t)}{t}dt.$$

By subordination principle for $|z|\leq r<1$ (see [59]), we have

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)\geq\operatorname{Re}\left(\exp\int_0^z\frac{\Psi(t)}{t}dt\right).$$

Thus, whenever (4.7.1) holds, we have $\mathscr{F}(\Psi)\subset\mathscr{G}_0$ and the result follows at once from Proposition 4.1.2. □

Example 4.7.2. Let us take $\Psi(z)=-2z/(1-z^2)$, then it can be easily seen that $\Psi(z)$ is analytic, univalent and starlike function with respect to 0 in $\mathbb{U}$ and $\Psi(z)$ is convex in the disk of radius $|z|\leq r_0\approx 0.414214$, where r_0 is the root of the equation

$$r^4-6r^2+1=0.$$

For this Ψ, we can consider

$$\mathscr{F}_1(\Psi)=\left\{f\in\mathscr{A}:\frac{zf'(z)}{f(z)}-1\prec\frac{-2z}{1-z^2}\right\}$$

and a simple calculation yields that

$$\operatorname{Re}\left(\exp\int_0^z\frac{\Psi(t)}{t}\right)=\operatorname{Re}\left(\frac{1-z}{1+z}\right)>0.$$

Therefore, by Theorem 4.7.1, $\mathscr{F}_1(\Psi)\subset\mathscr{G}_0$ in $|z|\leq r_0$.

For $\Psi(z)=z/(1-\alpha z^2)$, the class $\mathscr{F}(\Psi)$ reduces to the class $\mathscr{BS}(\alpha)$. By Theorem 4.7.1, we obtain the following:

Corollary 4.7.3. Let $0<\alpha\leq 3-2\sqrt{2}$. Then $\mathscr{BS}(\alpha)\subset\mathscr{G}_0$ and the semigroup $\{u(t,\cdot)\}_{t\geq 0}$ generated by $f\in\mathscr{BS}(\alpha)$ satisfies

$$|u(t,z)|\leq e^{-t\left(\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}}}|z|,\quad z\in\mathbb{U}.$$

Proof. Since $\Psi(z) = z/(1-\alpha z^2)$ is convex for $0 < \alpha \leq 3-2\sqrt{2}$ [141, Lemma 3.1] and

$$\begin{aligned}\operatorname{Re}\left(\exp\int_0^z \frac{\Psi(t)}{t}\right) &= \operatorname{Re}\left(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{\frac{1}{2\sqrt{\alpha}}} \\ &=: \operatorname{Re} g(z),\end{aligned}$$

where $g(z) = ((1+\sqrt{\alpha}z)/(1-\sqrt{\alpha}z))^{1/(2\sqrt{\alpha})}$. By [78, Theorem 2.4], function g is convex univalent in $\mathbb{U}$ and $g(z)$ is real for real z, therefore it maps the unit disk onto a convex set symmetric with respect to the real axis lying between $g(-1)$ to $g(1)$. Thus $\operatorname{Re} g(z) \geq ((1-\sqrt{\alpha})/(1+\sqrt{\alpha}))^{1/(2\sqrt{\alpha})} > 0$ for $0 < \alpha \leq 3-2\sqrt{2}$, which together with Theorem 4.7.1 and Proposition 4.1.2 establish the result. □

Theorem 4.7.4. The inclusion $\mathcal{S}^*(\varphi) \subset \mathcal{G}_0$ holds, whenever

$$\operatorname{Re}\left\{\exp\int_0^z \frac{\varphi(t)-1}{t}dt\right\} \geq \eta > 0. \tag{4.7.4}$$

Moreover, $f \in \mathcal{S}^*(\varphi)$ generates the semigroup $\{u(t,\cdot)\}_{t\geq 0}$, which satisfies

$$|u(t,z)| \leq e^{-t\eta}|z|, \quad z \in \mathbb{U}.$$

Proof. Let $f \in \mathcal{S}^*(\varphi)$, then from [117, Theorem 1], we have

$$\frac{f(z)}{z} \prec \frac{k_\varphi(z)}{z},$$

where $k_\phi = k_{\phi,2}$ is defined in (1.3.3). By subordination principle for $|z| \leq r < 1$,

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) \geq \operatorname{Re}\left(\frac{k_\varphi(z)}{z}\right) = \operatorname{Re}\left(\exp\left(\int_0^z \frac{\varphi(t)-1}{t}dt\right)\right) > \gamma. \tag{4.7.5}$$

Hence, $\mathcal{S}^*(\varphi) \subset \mathcal{G}_0$ and the result follows at once with Proposition 4.1.2. □

Remark 4.7.1. For $\varphi(z) = 1/(1-z)$, the class $\mathcal{S}^*(\varphi)$ reduces to the class $\mathcal{S}^*(1/2)$, which means $\operatorname{Re}(zf'(z)/f(z)) > 1/2$. From (4.7.5), we have $\operatorname{Re}(f(z)/z) > 1/2$ for $f \in \mathcal{S}^*(1/2)$, which was proved by Marx-Strohäcker [175] using a different technique.

Corollary 4.7.5. If $-1 \leq B < A \leq 0$, then $\mathcal{S}^*[A,B] \subset \mathcal{G}_0$ and the semigroup $\{u(t,\cdot)\}_{t\geq 0}$ generated by $f \in \mathcal{S}^*[A,B]$ satisfies

$$|u(t,z)| \leq e^{-t(1-B)^{\frac{A-B}{B}}}|z|, \quad z \in \mathbb{U}.$$

Proof. For $f \in \mathcal{S}^*[A,B]$,

$$\operatorname{Re}\left\{\exp\int_0^z \frac{\varphi(t)-1}{t}dt\right\} = \operatorname{Re}\left\{(1+Bz)^{\frac{A-B}{B}}\right\}.$$

Now, by taking $z = e^{i\theta}$ for $\theta \in (0,2\pi)$, we have

$$\operatorname{Re}\left\{(1+Be^{i\theta})^{\frac{A-B}{B}}\right\} = \left((1+B\cos\theta)^2 + B^2\sin^2\theta\right)^{\frac{A-B}{2B}}\cos\Theta, \tag{4.7.6}$$

where

$$\Theta := \frac{A-B}{B}\tan^{-1}\left(\frac{\sin\theta}{1+B\cos\theta}\right).$$

For $-1 \le B < A \le 0$ and $\theta \in (0,2\pi)$,

$$\frac{A-B}{B} \in [-1,0) \quad \text{and} \quad -\frac{\pi}{2} < \tan^{-1}\left(\frac{\sin\theta}{1+B\cos\theta}\right) < \frac{\pi}{2}.$$

It is evident from the above that $\Theta \in (-\pi/2,\pi/2)$. Therefore, $\cos\Theta > 0$. Consequently, we conclude from (4.7.6) that condition (4.7.4) holds for the class $\mathscr{S}^*[A,B]$ whenever $-1 \le B < A \le 0$ and hence $\mathscr{S}^*[A,B] \subset \mathscr{G}_0$.

From (4.7.6), for $-1 \le B < A \le 0$ and $\theta \in (0,2\pi)$, we have

$$\begin{aligned}\operatorname{Re}\left\{(1+Be^{i\theta})^{\frac{A-B}{B}}\right\} &\ge \inf_{\theta\in(0,2\pi)} \operatorname{Re}(1+Be^{i\theta})^{\frac{A-B}{B}}\\ &= (1-B)^{\frac{A-B}{B}}.\end{aligned}$$

The result now follows at once from Proposition 4.1.2. □

Remark 4.7.2. Taking $A = 1-2\alpha$ and $B = -1$, we see that the class $\mathscr{S}^*[A,B]$ reduces to the class $\mathscr{S}^*(\alpha)$, $\alpha \in [0,1]$. From Corollary 4.7.5, we obtain $\mathscr{S}^*(\alpha) \subset \mathscr{G}_0$, whenever $\alpha \ge 1/2$ and $|u(t,z)| \le 2^{-(2-2\alpha)}$, proved by Elin et al. [47, Thereom 5].

For $A = 0$ and $B = -1$, Corollary 4.7.5 yields the following result:

Corollary 4.7.6. $\mathscr{S}^*(1/2) \subset \mathscr{G}_0$ and the semigroup $\{u(t,\cdot)\}_{t\ge 0}$ generated by $f \in \mathscr{S}^*(1/2)$ satisfies $|u(t,z)| \le e^{-t/2}|z|$ for all $z \in \mathbb{U}$.

Theorem 4.7.7. If $\lambda \in [0,1/3]$, then $\mathscr{U}(\lambda) \subset \mathscr{G}_0$ and the semigroup $\{u(t,\cdot)\}_{t\ge 0}$ generated by $f \in \mathscr{U}(\lambda)$ satisfies

$$|u(t,z)| \le e^{\left(\frac{t(3\lambda-1)}{2\lambda^2-4\lambda+2}\right)}|z|, \quad z \in \mathbb{U}.$$

The range of λ is best possible.

Proof. For $f \in \mathscr{U}(\lambda)$,

$$\frac{f(z)}{z} \prec \frac{1}{(1+z)(1+\lambda z)}, \quad z \in \mathbb{U} \tag{4.7.7}$$

and

$$\begin{aligned}\operatorname{Re}\frac{1}{(1+z)(1+\lambda z)} &\ge \min_{\theta\in(0,2\pi)} \operatorname{Re}\frac{1}{(1+e^{i\theta})(1+\lambda e^{i\theta})}\\ &= \min_{\theta\in(0,2\pi)} \frac{1-\lambda+2\lambda\cos\theta}{2\lambda^2+4\lambda\cos\theta+2}\\ &= \min_{x\in(-1,1)} \frac{2\lambda x-\lambda+1}{2\lambda^2+4\lambda x+2} =: g(x),\end{aligned}$$

where $x = \cos\theta$. Clearly, $g'(x) = \lambda^2(\lambda+1)/(\lambda^2+2x\lambda+1)^2 > 0$ for all $x \in (-1,1)$ and $\lambda \in [0,1]$. Thus, $g(x)$ is increasing function and the infimum of its range set is attained at $x = -1$, which together with (4.7.7) yields

$$\operatorname{Re}\frac{f(z)}{z} > \frac{1-3\lambda}{2\lambda^2-4\lambda+2}$$

and the required inclusion follows for $\lambda \in [0,1/3]$. By (4.1.2) and (4.7.7), we note that

$$f(z) = \frac{z}{(1+z)(1+\lambda z)} \in \mathscr{U}(\lambda),$$

and $\operatorname{Re}(f(z)/z)$ may be negative when $\lambda > 1/3$, showing that the range of λ is best possible. Now, the exponential rate of convergence of $\{u(t,z)\}_{t\geq 0}$ generated by $f \in \mathscr{U}(\lambda)$ follows by Proposition 4.1.2. □

The following lemma is obtained by Tuan and Anh [183] as a particular case of Theorem 3 of [73] for the class of Carathéodory functions $\mathscr{P}$.

Lemma 4.7.1. Let $p \in \mathscr{P}$. Then for $0 \leq \alpha < 1$,

$$\operatorname{Re}\left(\frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)}\right) \geq \begin{cases} -\dfrac{2(1-\alpha)r}{(1+(2\alpha-1)r)(1+r)} & \text{for } R_1 \leq R_2, \\ -\dfrac{\alpha}{(1-\alpha)} + \dfrac{1}{(1-\alpha)}(2R_1-a) & \text{for } R_2 \leq R_1, \end{cases}$$

where

$$R_1 = \left(\frac{\alpha-\alpha(2\alpha-1)r^2}{1-r^2}\right)^{1/2} \quad \text{and} \quad R_2 = \frac{1+(2\alpha-1)r}{1+r}.$$

The results are sharp and the extremal functions are given by

(i) for $R_1 \leq R_2$, $p(z) = (1-z)/(1+z)$;

(ii) for $R_2 \leq R_1$,

$$p(z) = \frac{1}{2}\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + \frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right),$$

where $\cos\theta$ satisfies the equation

$$\begin{aligned}&(2\alpha-1)r^4 - 2\cos t((2R_1-b-\alpha)(3\alpha-1)+(1-\alpha)^2)r^3 + (2\alpha(2R_1-b-\alpha)(1+2\cos^2\theta)\\ &\quad +4(1-\alpha)^2)r^2 - 2\cos\theta((2R_1-b-\alpha)(1+\alpha)+(1-\alpha)^2)r + (2R_1-b-\alpha) = 0\end{aligned} \tag{4.7.8}$$

with $b = (1-(2\alpha-1)r^2)/(1-r^2)$.

For a given $r \in (0,1)$, the transition from the first case to second case takes place when $\alpha = \alpha_0 \in (0,1)$, where α_0 is determined from the equation $R_1 = R_2$.

Tuan and Anh [183] found the radius of starlikeness for the subclass of $\mathscr{S}$, which satisfies the condition $\operatorname{Re}(f(z)/z) > a$, $0 \leq a < 1$. In the following, we extend their result by considering the class $\mathscr{S}^*(\alpha)$ in place of $\mathscr{S}^*$.

Theorem 4.7.8. Let $f \in \mathscr{A}$ and $\operatorname{Re}(f(z)/z) > a$, $0 \leq a < 1$. Then $f \in \mathscr{S}^*(\alpha)$ in $|z| \leq r_\alpha \in (0,1)$, where r_α is given by

$$r_\alpha = \begin{cases} \dfrac{2a - a\alpha - 1 + \sqrt{(a-1)(a(\alpha-2)^2 - (\alpha-2)\alpha - 2)}}{(1-2a)(1-\alpha)} & \text{for } a \in [0, a_0] \setminus \{a_1\} \\[2ex] \sqrt{\dfrac{\alpha^2(1-a) - \alpha(2-6a) - 4a + 4\sqrt{(\alpha-1)(a-1)a}}{\alpha^2(1-a) - \alpha(4-8a) - 8a + 4}} & \text{for } a \in (a_0, 1) \setminus \{a_1\} \\[2ex] \sqrt{\dfrac{\alpha-1}{\alpha-2}} & \text{for } a = a_1 \end{cases} \tag{4.7.9}$$

with

$$a_0 = \frac{2\alpha^3 - 6\alpha^2 + 9\alpha - 6 + 2\sqrt{(\alpha-1)^4(\alpha(\alpha-2)+4)}}{4\alpha^3 - 21\alpha^2 + 36\alpha - 20} \text{ and } a_1 = \frac{\alpha^2 - 4\alpha + 4}{\alpha^2 - 8\alpha + 8}.$$

The radius is sharp.

Proof. Since $\operatorname{Re}(f(z)/z) > a$, we can write

$$\frac{f(z)}{z} = a + (1-a)p(z),$$

where $p \in \mathscr{P}$. A computation shows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(1 + \frac{(1-a)zp'(z)}{a + (1-a)p(z)}\right). \tag{4.7.10}$$

By Lemma 4.7.1, for $R_1 \leq R_2$, we have

$$\begin{aligned} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &\geq 1 - \frac{2(1-a)r}{(1+(2a-1)r)(1+r)} \\ &=: \xi_1(a, r). \end{aligned}$$

Clearly, $f \in \mathscr{S}^*(\alpha)$ provided $\xi_1(a, r) > \alpha$ or

$$\xi_1(a, \alpha, r) := r^2(2a - \alpha(-1+2a) - 1) + r(4a - \alpha(2a-1) - \alpha - 2) + (1-\alpha) > 0.$$

For the case $R_1 \leq R_2$, r_{α_1} is the smallest positive root of $\xi_1(a, \alpha, r) = 0$ and it is given by

$$r_{\alpha_1} = \frac{2a - a\alpha - 1 + \sqrt{(a-1)(a(\alpha-2)^2 - (\alpha-2)\alpha - 2)}}{(1-2a)(1-\alpha)}.$$

Now, $r_{\alpha_1} < 1$ provided

$$\frac{(6-4\alpha)a - 2}{(\alpha-1)(2a-1)} < 0,$$

which holds when $a < 1/(3-2\alpha)$. Evidently, $\xi_1(a, \alpha, 0) = 1 - \alpha \geq 0$ for all $\alpha \in [0, 1]$ and $\xi_1(a, \alpha, 1) = -2 + (6-4\alpha)a < 0$ for $a < 1/(3-2\alpha)$, which ensures the existence of root $r_{\alpha_1} \in (0, 1)$.

For the case $R_2 \leq R_1$, r_{α_2} is the smallest positive root of $\xi_2(a,\alpha,r)=0$, where

$$\xi_2(a,\alpha,r)=1-\frac{a}{(1-a)}+\frac{1}{(1-a)}\left(2\left(\frac{a-a(2a-1)r^2}{1-r^2}\right)^{1/2}-\frac{1-(2a-1)r^2}{(1-r^2)}\right)-\alpha \qquad (4.7.11)$$

and the root is

$$r_{\alpha_2}=\sqrt{\frac{\alpha^2(1-a)-\alpha(2-6a)-4a+4\sqrt{(\alpha-1)(a-1)a}}{\alpha^2(1-a)-\alpha(4-8a)-8a+4}},$$

where $a \neq (4-4\alpha+\alpha^2)/(8-8\alpha+\alpha^2)$. For $a=(4-4\alpha+\alpha^2)/(8-8\alpha+\alpha^2)$, $\xi_2(a,\alpha,r)=0$ yields the root

$$r_{\alpha_3}=\sqrt{\frac{\alpha-1}{\alpha-2}}.$$

For fixed $\alpha\in[0,1]$, $r_{\alpha_2}\in(0,1)$ whenever $a>\alpha^2/(\alpha-2)^2$. Clearly

$$r_{\alpha_1}=r_{\alpha_2}$$

when

$$a_0=\frac{2\alpha^3-6\alpha^2+9\alpha-6+2\sqrt{(\alpha-1)^4(\alpha(\alpha-2)+4)}}{4\alpha^3-21\alpha^2+36\alpha-20}.$$

For $\alpha\in[0,1]$, a_0 lies in $[1/10,1]$.

Sharpness: For $a\in[0,a_0]\setminus\{a_1\}$, sharpness follows for

$$f(z)=z\left(\frac{1+(2a-1)z}{1+z}\right)$$

as $zf'(z)/f(z)=\alpha$ when $z=r_{\alpha_1}$. For the second inequality, extremal function is given by

$$f(z)=az+\frac{(1-a)}{2}z\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}+\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right),$$

where $\cos\theta$ satisfy (4.7.8) with $r=r_{\alpha_2}$. □

In view of Theorem 4.7.8 and Lemma 4.1.1, we get the radius of starlikeness of order α for the class $\mathscr{A}_\beta$.

Theorem 4.7.9. If $f\in\mathscr{A}_\beta$, $0\leq\beta\leq1$, then $f\in\mathscr{S}^*(\alpha)$ in $\mathbb{U}_{r_\alpha}=\{z\in\mathbb{C}:|z|\leq r_\alpha\}$, where

$$r_\alpha=\begin{cases}\sqrt{\dfrac{\kappa(\beta)(\alpha^2-6\alpha+4)-\alpha^2+2\alpha-4\sqrt{\kappa(\beta)(1-\kappa(\beta))(1-\alpha)}}{k(\alpha^2-8\alpha+8)-(\alpha-2)^2}} & \text{for } \beta\in(0,\beta^*] \\[2ex] \dfrac{2\kappa(\beta)(2-\alpha)-2-\sqrt{(\kappa(\beta)(\alpha-2)^2+\alpha(\alpha-2)+2)(1-\kappa(\beta))}}{2(1-2\kappa(\beta))(1-\alpha)} & \text{for } \beta\in[\beta^*,1]\end{cases}$$

and β^* is the root of

$$\int_0^1 \frac{1-t^{1-\beta^*}}{1+t^{1-\beta^*}}dt = \frac{1}{6-9\alpha+6\alpha^2-2\alpha^3-2\sqrt{(\alpha-1)^4(\alpha^2-2\alpha+4)}}.$$

Remark 4.7.3. For $\alpha = 0$, Theorem 4.7.9 gives the radii of starlikeness for $f \in \mathscr{A}_\beta$ [47, Theorem 8].

So far, we were dealing with coefficient and radius problems associated with different subclasses of $\mathscr{A}$. Nevertheless, as we progress into the upcoming chapters, we are embarking on a fresh trajectory, venturing into uncharted territory by examining functions in higher dimensions. This shift marks an exciting expansion of our research scope, promising to unveil new insights and complexities in the realm of analytic functions.

Highlights of the chapter

We tackled unresolved challenges within the subclass of semigroup generators. By focusing on coefficient problems and growth estimates, we established bounds for the n^{th} Taylor series coefficients and explored specific coefficient functionals. Additionally, we explored the Bohr and Bohr-Rogosinski phenomena, alongside growth estimates, and derived the radius of starlikeness of order α. Our findings also reveal the embedding of well-known subclasses of $\mathscr{S}$. This propels our comprehension of semigroup generator theory to greater depths.

The contents of this chapter are drawn from the following articles:
S. Giri, S. S. Kumar, Coefficient Functional and Bohr-Rogosinski Phenomenon for Analytic functions involving Semigroup Generators, Rocky mountain journal of mathematics (Accepted).
S. Giri, and S. S. Kumar, Radius and Convolution problems of analytic functions involving Semigroup Generators, arXiv preprint arXiv:2205.10777. (Communicated).

Chapter 5

Toeplitz Determinants for Starlike Mappings in Higher Dimensions

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In this chapter, we derive the sharp bounds of certain Toeplitz determinants whose entries are the coefficients of holomorphic functions belonging to a known class defined on the unit disk $\mathbb{U}$. Further, these results are extended to a class of holomorphic functions on the unit ball in a complex Banach space and on the unit polydisc in $\mathbb{C}^n$. The obtained results provide the bounds of Toeplitz determinants in higher dimensions for various subclasses of normalized univalent functions.

5.1 Introduction

The failure of Bieberbach conjecture in several variables, even in its simplest form, is a widely recognized fact. Furthermore, it is noteworthy that the constraint on the modulus of the second coefficients, as observed in Taylor expansions of normalized univalent functions on the unit disc $\mathbb{U}$, does not hold true in the realm of several complex variables. Kohr [83] established the estimations of the second coefficients of Taylor expansions for various classes including the classes of starlike and convex mappings on the Euclidean unit ball of $\mathbb{C}^n$. Gong [56] derived the bounds for the second and third coefficients of starlike mappings on unit polydisc in $\mathbb{C}^n$. Closely related to the Bieberbach conjecture, Xu and Liu [193] obtained the bound of Fekete-Szegö type functional for a subclass of normalized starlike mappings on the unit ball of a complex Banach space. Xu et al. [191] solved the same for a subclass of normalized quasi-convex mappings of type B on the unit ball of complex Banach space. Generalizing this work, Hamada et al. [67] also determined the bound of Fekete-Szegö inequality. Contrary to Fekete-Szegö inequality for various subclasses of $\mathscr{S}(\mathbb{B})$, very few results are known for the inequalities of homogeneous expansions for subclasses of biholomorphic mappings in several complex variables [60, 65, 66, 190]. Numerous best-possible results concerning the coefficient estimates for subclasses of holomorphic mappings in higher dimensions are obtained in [24, 62, 112, 187, 192].

For a biholomorphic function $\Phi : \mathbb{U} \to \mathbb{C}$, which satisfies $\Phi(0) = 1$ and $\operatorname{Re}\Phi(z) > 0$, Kohr [83] introduced the class $\mathscr{M}_\Phi$ containing the functions $h \in \mathscr{H}(\mathbb{B})$ such that $D(h(0)) = I$ and $\|z\|/l_z(h(z)) \in \Phi(\mathbb{U})$. Here, we additionally take $\Phi'(0) > 0$, $\Phi''(0) \in \mathbb{R}$ and define the following:

Definition 5.1.1. Let $\Phi : \mathbb{U} \to \mathbb{C}$ be a biholomorphic function such that $\Phi(0) = 1$, $\operatorname{Re}\Phi(z) > 0$, $\Phi'(0) > 0$ and $\Phi''(0) \in \mathbb{R}$. We define $\mathscr{M}_\Phi$ to be the class of mappings given by

$$\mathscr{M}_\Phi = \left\{ h \in \mathscr{H}(\mathbb{B}) : h(0) = 0, D(h(0)) = I, \frac{\|z\|}{l_z(h(z))} \in \Phi(\mathbb{U}), z \in \mathbb{B} \setminus \{0\},\ l_z \in T(z) \right\}.$$

For $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, the above relation is equivalent to

$$\mathscr{M}_\Phi = \left\{ h \in \mathscr{H}(\mathbb{U}^n) : h(0) = 0, D(h(0)) = I, \frac{z_k}{h_k(z)} \in \Phi(\mathbb{U}), z \in \mathbb{U}^n \setminus \{0\} \right\},$$

where $h(z) = (h_1(z), h_2(z), \cdots, h_n(z))'$ is a column vector in $\mathbb{C}^n$ and k satisfies $|z_k| = \|z\| = \max_{1 \le j \le n}\{|z_j|\}$. For $\mathbb{B} = \mathbb{U}$ and $X = \mathbb{C}$, the relation is equivalent to

$$\mathscr{M}_\Phi = \left\{ h \in \mathscr{H}(\mathbb{U}) : h(0) = 0, h'(0) = 1, \frac{z}{h(z)} \in \Phi(\mathbb{U}), z \in \mathbb{U} \right\}.$$

Also, note that, if $f \in \mathscr{H}(\mathbb{B})$ and $(Df(z))^{-1}f(z) \in \mathscr{M}_\Phi$, then suitable choices of Φ in Definition 5.1.1 provide different subclasses of holomorphic mappings. For instance, when $\Phi(z) = (1+z)/(1-z)$, $\Phi(z) = (1+(1-2\alpha)z)/(1-z)$ and $\Phi(z) = ((1+z)/(1-z))^\gamma$, we easily obtain that $f \in \mathscr{S}^*(\mathbb{B})$, $f \in \mathscr{S}^*_\alpha(\mathbb{B})$ and $f \in \mathscr{S}\mathscr{S}^*_\gamma(\mathbb{B})$, respectively.

In 2018, Ali et al. [6] determined the bound of $|T_{2,2}(f)|$ and $|T_{3,1}(f)|$, when entries of $T_{m,n}(f)$ are the coefficients of starlike functions. Recently, Ahuja et al. [4] obtained the bounds for various subclasses of starlike functions.

Theorem A. *[6] If $f \in \mathscr{S}^*$, then the following sharp bounds hold:*

$$|T_{2,2}(f)| \le 13 \ \ \textit{and} \ \ |T_{3,1}(f)| \le 24.$$

Theorem B. *[4] If $f \in \mathscr{S}^*(\alpha)$, then*

$$|T_{2,2}(f)| \le (1-\alpha)^2(4\alpha^2 - 12\alpha + 13).$$

For $\alpha \in [0, 2/3]$, the following inequality holds:

$$|T_{3,1}(f)| \le 12\alpha^4 - 52\alpha^3 + 91\alpha^2 - 74\alpha + 24.$$

All these estimations are sharp.

Theorem C. *[4] Let $f \in \mathscr{S}^*[A, B]$, $-1 \le B < A \le 1$. Then the following sharp estimations hold:*

(i) If $|A-2B|\geq 1$*, then*

$$|T_{2,2}(f)|\leq \frac{(A-B)^2(A^2+4B^2-4AB+4)}{4}.$$

(ii) If $B\leq \min\{(A-1)/2,(3A-1)/2\}$*, then*

$$|T_{3,1}(f)|\leq 1+2(A-B)^2+\frac{(3A^2-5AB+2B^2)(A^2-3AB+2B^2)}{4}.$$

In the subsequent section, we obtain the sharp bounds of certain Toeplitz determinants for the class of holomorphic functions satisfying $f(z)/f'(z)\in\mathscr{M}_\Phi$, which contain the above results as special cases. Further, these results are generalized in higher dimensions.

5.2 Toeplitz Determinants for Certain Holomorphic Functions

We begin with the following results:

Theorem 5.2.1. Let $f(z)=z+b_2z^2+b_3z^3+\cdots\in\Phi(\mathbb{U})$, where Φ is same as given in Definition 5.1.1 and satisfy

$$|\Phi''(0)+2(\Phi'(0))^2|\geq 2\Phi'(0)>0.$$

If $f(z)/f'(z)\in\mathscr{M}_\Phi$, then

$$|T_{2,2}(f)|\leq \frac{(\Phi'(0))^2}{4}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2+(\Phi'(0))^2.$$

The bound is sharp.

Proof. Since $f(z)/f'(z)\in\mathscr{M}_\Phi$, therefore we have

$$F(z):=\frac{zf'(z)}{f(z)}\in\Phi(\mathbb{U}),$$

which yields $F\prec\Phi$. For $f(z)=z+b_2z^2+b_3z^3+\cdots$, the Taylor series expansion of $F(z)$ is given by

$$F(z)=1+b_2z+(2b_3-b_2^2)z^2+\cdots.$$

Xu et al. [188] proved that

$$|b_3-\lambda b_2^2|\leq \frac{|\Phi'(0)|}{2}\max\left\{1,\left|\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+(1-2\lambda)\Phi'(0)\right|\right\},\quad \lambda\in\mathbb{C}. \tag{5.2.1}$$

Thus, whenever $|\Phi''(0)+2(\Phi'(0))^2|\geq 2\Phi'(0)$, the equation (5.2.1) yields

$$|b_3|\leq \frac{\Phi'(0)}{2}\left|\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right|. \tag{5.2.2}$$

Further, using the bound $|F'(0)| \leq \Phi'(0)$, we obtain

$$|b_2| \leq \Phi'(0). \tag{5.2.3}$$

From (1.5.1), we have

$$|T_{2,2}(f)| = |b_3^2 - b_2^2| \leq |b_3|^2 + |b_2|^2.$$

Clearly, the required bound follows directly from the above relation together with the bounds of $|b_3|$ and $|b_2|$ given in (5.2.2) and (5.2.3), respectively.

To see the sharpness of the bound, consider the function $f_\Phi : \mathbb{U} \to \mathbb{C}$ given by

$$f_\Phi(z) = z \exp \int_0^z \frac{(\Phi(it) - 1)}{t} dt = 1 + i\Phi'(0)z - \frac{1}{2}\left((\Phi'(0))^2 + \frac{\Phi''(0)}{2}\right)z^2 + \cdots. \tag{5.2.4}$$

It can be easily seen that $f_\Phi(z)/f'_\Phi(z) \in \mathscr{M}_\Phi$ and

$$|T_{2,2}(f_\Phi)| = \frac{1}{4}\left((\Phi'(0))^2 + \frac{\Phi''(0)}{2}\right)^2 + (\Phi'(0))^2,$$

which shows that the bound is sharp and completes the proof. □

Theorem 5.2.2. Let $f(z) = z + b_2 z^2 + b_3 z^3 + \cdots \in \Phi(\mathbb{U})$, where Φ is same as given in Definition 5.1.1 and satisfy

$$2\Phi'(0) - 2(\Phi'(0))^2 \leq \Phi''(0) \leq 6(\Phi'(0))^2 - 2\Phi'(0).$$

If $f(z)/f'(z) \in \mathscr{M}_\Phi$, then

$$|T_{3,1}(f)| \leq 1 + 2(\Phi'(0))^2 + \frac{(\Phi'(0))^2}{4}\left(3\Phi'(0) - \frac{\Phi''(0)}{2\Phi'(0)}\right)\left(\frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0)\right).$$

The bound is sharp.

Proof. Since $\Phi''(0) + 2(\Phi'(0))^2 \geq 2\Phi'(0)$, by (5.2.2), we obtain

$$|b_3| \leq \frac{\Phi'(0)}{2}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right). \tag{5.2.5}$$

Also, $6(\Phi'(0))^2 - \Phi''(0) \geq 2\Phi'(0)$ holds, hence (5.2.1) gives

$$|b_3 - 2b_2^2| \leq \frac{\Phi'(0)}{2}\left(3\Phi'(0) - \frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}\right). \tag{5.2.6}$$

From (1.5.1), we have

$$\begin{aligned} |T_{3,1}(f)| &= |2b_2^2 b_3 - 2b_2^2 - b_3^2 + 1| \\ &\leq 1 + 2|b_2|^2 + |b_3||b_3 - 2b_2^2|. \end{aligned}$$

Using the estimates for the second and third coefficients given in (5.2.5) and (5.2.2) together with the bound of $|b_3 - 2b_2^2|$ given in (5.2.6), required bound follows.

The estimate is sharp for the function $f_\Phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$ given by (5.2.4). For this function, we have

$$1 - 2b_2^2 - b_3(b_3 - 2b_2^2) = 1 + 2(\Phi'(0))^2 + \frac{(\Phi'(0))^2}{4}\left(3\Phi'(0) - \frac{\Phi''(0)}{2\Phi'(0)}\right)\left(\frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0)\right),$$

which proves the sharpness of the bound. □

Remark 5.2.1. By taking $\Phi(z) = (1+z)/(1-z)$, $\Phi(z) = (1+(1-2\alpha)z)/(1-z)$ and $\Phi(z) = (1+Az)/(1+Bz)$, Theorem 5.2.1 and 5.2.2 can be deduced to Theorem A, Theorem B and Theorem C, respectively.

The bounds for other classes can also be obtained by changing the corresponding function Φ. For $\Phi(z) = ((1+z)/(1-z))^\gamma$, the following result follows for the class $\mathscr{S}\mathscr{S}^*(\gamma)$.

Corollary 5.2.3. If $f \in \mathscr{S}\mathscr{S}^*(\gamma)$, then for $\gamma \in [1/3, 1]$, the followings sharp inequalities hold:

$$|T_{2,2}(f)| \leq 9\gamma^4 + 4\gamma^2 \text{ and } |T_{3,1}(f)| \leq 15\gamma^4 + 8\gamma^2 + 1.$$

5.3 Bounds in Higher Dimensions

Now, we extend the above results on the unit ball $\mathbb{B}$ in a complex Banach space and on the unit polydisc $\mathbb{U}^n$.

Theorem 5.3.1. Let $f \in \mathscr{H}(\mathbb{B}, \mathbb{C})$ with $f(0) = 1$ and suppose that $F(z) = zf(z)$. If $(DF(z))^{-1}F(z) \in \mathscr{M}_\Phi$ such that Φ satisfy

$$|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0),$$

then

$$\left|\left(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\right)^2 - \left(\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\right)^2\right| \leq \frac{(\Phi'(0))^2}{4}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2 + (\Phi'(0))^2.$$

The bound is sharp.

Proof. Xu et al. [188, Theorem 3.2] proved that

$$\left.\begin{aligned} &\left|\frac{l_z(D^3F(0)(z^3))}{3!||z||^3} - \lambda\left(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\right)^2\right| \\ &\qquad \leq \frac{|\Phi'(0)|}{2}\max\left\{1, \left|\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + (1-2\lambda)\Phi'(0)\right|\right\}, \quad \lambda \in \mathbb{C}, \ z \in \mathbb{B}\setminus\{0\}. \end{aligned}\right\} \tag{5.3.1}$$

Since $|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0)$, the above inequality gives

$$\left|\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\right| \leq \frac{\Phi'(0)}{2}\left|\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right|. \tag{5.3.2}$$

On the other hand, applying a similar method as in [61, Theorem 7.1.14] (also see [188, Theorem 3.2]), we obtain

$$(DF(z))^{-1} = \frac{1}{f(z)}\left(I - \frac{\frac{zDf(z)}{f(z)}}{1+\frac{Df(z)z}{f(z)}}\right).$$

Therefore

$$(DF(z))^{-1}F(z) = z\left(\frac{zf(z)}{f(z)+Df(z)z}\right),$$

which directly gives

$$\frac{\|z\|}{l_z((DF(z))^{-1}F(z))} = 1+\frac{Df(z)z}{f(z)}. \tag{5.3.3}$$

For fix $z \in X\setminus\{0\}$ and $z_0 = \frac{z}{\|z\|}$, define the function $g : \mathbb{U} \to \mathbb{C}$ such that

$$g(\zeta) = \begin{cases} \dfrac{\zeta}{l_z((DF(\zeta z_0))^{-1}F(\zeta z_0))}, & \zeta \neq 0, \\ 1, & \zeta = 0. \end{cases}$$

Then $g \in \mathscr{H}(\mathbb{U})$ and $g(0) = 1 = \Phi(0)$. Further, since $(DF(z))^{-1}F(z) \in \mathscr{M}_\Phi$, we find that

$$\begin{aligned} g(\zeta) =& \frac{\zeta}{l_z((DF(\zeta z_0))^{-1}F(\zeta z_0))} = \frac{\zeta}{l_{z_0}((DF(\zeta z_0))^{-1}F(\zeta z_0))} \\ =& \frac{\|\zeta z_0\|}{l_{\zeta z_0}((DF(\zeta z_0))^{-1}F(\zeta z_0))} \in \Phi(\mathbb{U}), \quad \zeta \in \mathbb{U}. \end{aligned}$$

Taking (5.3.3) into consideration, we obtain

$$g(\zeta) = \frac{\|\zeta z_0\|}{l_{\zeta z_0}((Df(\zeta z_0))^{-1}f(\zeta z_0))} = 1+\frac{Df(\zeta z_0)\zeta z_0}{f(\zeta z_0)}. \tag{5.3.4}$$

In view of the Taylor series expansions of $g(\zeta)$ and $f(\zeta z_0)$, the above equation gives

$$\begin{aligned} &\left(1+g'(0)\zeta+\frac{g''(0)}{2}\zeta^2+\cdots\right)\left(1+Df(0)(z_0)\zeta+\frac{D^2f(0)(z_0^2)}{2}\zeta^2+\cdots\right) \\ &\quad = \left(1+Df(0)(z_0)\zeta+\frac{D^2f(0)(z_0^2)}{2}\zeta^2+\cdots\right)\left(Df(0)(z_0)\zeta+D^2f(0)(z_0^2)\zeta^2+\cdots\right). \end{aligned}$$

Comparison of homogeneous expansions yield that $g'(0) = Df(0)(z_0)$. That is

$$g'(0)\|z\| = Df(0)(z). \tag{5.3.5}$$

Since $F(z) = zf(z)$, therefore, we have

$$\frac{D^2F(0)(z^2)}{2!} = Df(0)(z)z. \tag{5.3.6}$$

Moreover, from (5.3.6), we conclude that

$$\frac{l_z(D^2F(0)(z^2))}{2!} = Df(0)(z)\|z\|. \tag{5.3.7}$$

Thus, equation (5.3.7) together with (5.3.5) gives

$$\left|\frac{l_z(D^2F(0)(z^2))}{2!}\right| = |Df(0)(z)\|z\|| = |g'(0)\|z\|^2|.$$

Since $g \prec \Phi$, therefore $|g'(0)| \leq \Phi'(0)$. Consequently, we obtain

$$\left|\frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2}\right| \leq \Phi'(0). \tag{5.3.8}$$

Using the bounds given in (5.3.8) and (5.3.2) together with

$$\left|\left(\frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2}\right)^2 - \left(\frac{l_z(D^3F(0)(z^3))}{3!\|z\|^3}\right)^2\right| \leq \left|\frac{l_z(D^3F(0)(z^3))}{3!\|z\|^3}\right|^2 + \left|\frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2}\right|^2,$$

the required bound follows.

To see the sharpness, consider the mapping F given by

$$F(z) = z\exp\int_0^{l_u(z)} \frac{(\Phi(it)-1)}{t}dt, \quad z \in \mathbb{B}, \quad \|u\| = 1. \tag{5.3.9}$$

It is a simple exercise to see that $(DF(z))^{-1}F(z) \in \mathscr{M}_\Phi$ and a quick calculation reveals that

$$\frac{D^2F(0)(z^2)}{2!} = i\Phi'(0)l_u(z)z \text{ and } \frac{D^3F(0)(z^3)}{3!} = -\frac{1}{2}\left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2\right)(l_u(z))^2 z.$$

In view of the above equations, we have

$$\frac{l_z(D^2F(0)(z^2))}{2!} = i\Phi'(0)l_u(z)\|z\|$$

and

$$\frac{l_z(D^3F(0)(z^3))\|z\|}{3!} = -\frac{1}{2}\left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2\right)(l_u(z))^2\|z\|^2.$$

Setting $z = ru$ $(0 < r < 1)$, we get

$$\frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2} = i\Phi'(0) \text{ and } \frac{l_z(D^3F(0)(z^3))}{3!\|z\|^3} = -\frac{1}{2}\left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2\right). \tag{5.3.10}$$

Thus for the mapping F, we have

$$\left|\left(\frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2}\right)^2 - \left(\frac{l_z(D^3F(0)(z^3))}{3!\|z\|^3}\right)^2\right| = \frac{(\Phi'(0))^2}{4}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2 + |\Phi'(0)|^2,$$

which proves the sharpness of the bound. □

Theorem 5.3.2. Let $f \in \mathscr{H}(\mathbb{B},\mathbb{C})$ with $f(0)=1$ and suppose that $F(z)=zf(z)$. If $(DF(z))^{-1}F(z) \in \mathscr{M}_\Phi$ such that Φ satisfy

$$2\Phi'(0)-2(\Phi'(0))^2 \leq \Phi''(0) \leq 6(\Phi'(0))^2-2\Phi'(0),$$

then

$$|2b_2^2b_3-b_3^2-2b_2^2+1| \leq 1+2(\Phi'(0))^2+\frac{(\Phi'(0))^2}{4}\left(3\Phi'(0)-\frac{\Phi''(0)}{2\Phi'(0)}\right)\left(\frac{\Phi''(0)}{2\Phi'(0)}+\Phi'(0)\right),$$

where

$$b_3=\frac{l_z(D^3F(0)(z^3))}{3!||z||^3} \quad \text{and} \quad b_2=\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}.$$

The bound is sharp.

Proof. Since $2\Phi'(0)<\Phi''(0)+2(\Phi'(0))^2$, therefore from (5.3.1), we have

$$\left|\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\right| \leq \frac{\Phi'(0)}{2}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right). \tag{5.3.11}$$

Again, since $2\Phi'(0)+\Phi''(0) \leq 6(\Phi'(0))^2$, the inequality (5.3.1) directly gives

$$\left|\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}-2\left(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\right)^2\right| \leq \frac{\Phi'(0)}{2}\left(3\Phi'(0)-\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}\right). \tag{5.3.12}$$

Also, we have

$$|2b_2^2b_3-b_3^2-2b_2^2+1| \leq 1+2|b_2|^2+|b_3||b_3-2b_2^2|. \tag{5.3.13}$$

The required bound is derived by using the estimates given in (5.3.8) and (5.3.11), and the bound given by (5.3.12) in the above inequality.

The equality case holds for the mapping $F(z)$ defined by (5.3.9). It follows from (5.3.10) that for this mapping, we have $b_2=i\Phi'(0)$, $b_3=-(\Phi''(0)+2(\Phi'(0))^2)/4$ and hence

$$1-b_3(b_3-2b_2^2)-2b_2^2=1+2(\Phi'(0))^2+\frac{(\Phi'(0))^2}{4}\left(3\Phi'(0)-\frac{\Phi''(0)}{2\Phi'(0)}\right)\left(\frac{\Phi''(0)}{2\Phi'(0)}+\Phi'(0)\right),$$

which shows the sharpness of the bound. □

Theorem 5.3.3. Let $f \in \mathscr{H}(\mathbb{U}^n,\mathbb{C})$ with $f(0)=1$ and suppose that $F(z)=zf(z)$. If $(DF(z))^{-1}F(z) \in \mathscr{M}_\Phi$ such that Φ satisfies

$$|\Phi''(0)+2(\Phi'(0))^2| \geq 2\Phi'(0),$$

then

$$\left.\begin{aligned} &\left\|\frac{1}{3!}D^3F(0)\left(z^2,\frac{D^3F(0)(z^3)}{3!}\right)-\frac{1}{2!}D^2F(0)\left(z,\frac{D^2F(0)(z^2)}{2!}\right)\right\| \\ &\quad \leq \frac{(\Phi'(0))^2||z||^5}{4}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2+(\Phi'(0))^2\,||z||^3, \quad z \in \mathbb{U}^n. \end{aligned}\right\} \tag{5.3.14}$$

The bound is sharp.

Proof. For $z \in \mathbb{U}^n \setminus \{0\}$, let $z_0 = \frac{z}{\|z\|}$. Define $g_k : \mathbb{U} \to \mathbb{C}$ such that

$$g_k(\zeta) = \begin{cases} \dfrac{\zeta z_k}{p_k(\zeta z_0)\|z_0\|}, & \zeta \neq 0, \\ 1, & \zeta = 0, \end{cases} \tag{5.3.15}$$

where $p(z) = (DF(z))^{-1}F(z)$ and k satisfies $|z_k| = \|z\| = \max_{1\le j\le n}\{|z_j|\}$. Since $(D(F(z)))^{-1}F(z) \in \mathscr{M}_{\Phi}$, we have $g_k(\zeta) \in \Phi(\mathbb{U})$. Further, using (5.3.4), we have

$$g_k(\zeta) = 1 + \frac{Df(\zeta z_0)\zeta z_0}{f(\zeta z_0)}$$

or equivalently,

$$g_k(\zeta)f(\zeta z_0) = f(\zeta z_0) + Df(\zeta z_0)\zeta z_0.$$

A comparison of homogeneous expansions obtained by the Taylor series expansions of f and g_k about ζ gives

$$g_k'(0) = Df(0)(z_0), \quad \frac{g_k''(0)}{2} = D^2f(0)(z_0^2) - (Df(0)(z_0))^2. \tag{5.3.16}$$

Also, using $F(z_0) = z_0 f(z_0)$, we have

$$\frac{D^3F_k(0)(z_0^3)}{3!} = \frac{D^2f(0)(z_0^2)}{2!}\frac{z_k}{\|z\|} \quad \text{and} \quad \frac{D^2F_k(0)(z_0^2)}{2!} = Df(0)(z_0)\frac{z_k}{\|z\|}. \tag{5.3.17}$$

In view of (5.3.16) and (5.3.17), we get

$$\left.\begin{aligned} \left|\frac{1}{2!}D^2F_k(0)\left(z_0, \frac{D^2F(0)(z_0^2)}{2!}\right)\frac{\|z\|}{z_k}\right| &= \left|\frac{1}{2!}D^2F_k(0)(z_0, Df(0)(z_0)z_0)\frac{\|z\|}{z_k}\right| \\ &= \left|Df(0)(z_0)\frac{1}{2!}D^2F_k(0)(z_0, z_0)\frac{\|z\|}{z_k}\right| \\ &= \left|Df(0)(z_0)\frac{1}{2!}D^2F_k(0)(z_0^2)\frac{\|z\|}{z_k}\right| \\ &= |(Df(0)(z_0))^2|. \end{aligned}\right\} \tag{5.3.18}$$

Since $Df(0)(z_0) = g_k'(0)$ and $|g_k'(0)| \le \Phi'(0)$, therefore

$$\left|\frac{1}{2!}D^2F_k(0)\left(z_0, \frac{D^2F(0)(z_0^2)}{2!}\right)\frac{\|z\|}{z_k}\right| = |Df(0)(z_0)|^2 \le (\Phi'(0))^2. \tag{5.3.19}$$

If $z_0 \in \partial_0\mathbb{U}^n$, then

$$\left|\frac{1}{2!}D^2F_k(0)\left(z_0, \frac{D^2F(0)(z_0^2)}{2!}\right)\right| \le (\Phi'(0))^2.$$

Further, since

$$\frac{1}{2!}D^2F_k(0)\left(z_0, \frac{D^2F(0)(z_0^2)}{2!}\right), \quad k = 1, 2, \cdots, n$$

are holomorphic functions on $\overline{\mathbb{U}}^n$, by virtue of the maximum modulus theorem of holomorphic functions on the unit polydisc, we obtain

$$\left|\frac{1}{2!}D^2F_k(0)\left(z,\frac{D^2F(0)(z^2)}{2!}\right)\right|\le(\Phi'(0))^2\|z\|^3,\quad k=1,2,\cdots. \tag{5.3.20}$$

Using the same arguments as in (5.3.18), we get

$$\left|\frac{1}{3!}D^3F_k(0)\left(z_0^2,\frac{D^3F(0)(z_0^3)}{3!}\right)\frac{\|z\|}{z_k}\right|=\left|\left(\frac{D^2f(0)(z_0^2)}{2!}\right)\right|^2. \tag{5.3.21}$$

According to the result established by Xu et al. [188, Theorem 3.3], we have

$$\left.\begin{aligned}&\left|\frac{D^3F_k(0)(z_0^3)}{3!}\frac{\|z\|}{z_k}-\lambda\frac{1}{2}D^2F_k(0)\left(z_0,\frac{D^2F(0)(z_0^2)}{2!}\frac{\|z\|}{z_k}\right)\right|\\&\qquad\le\frac{|\Phi'(0)|}{2}\max\left\{1,\left|\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+(1-2\lambda)\Phi'(0)\right|\right\},\quad\lambda\in\mathbb{C},\ z\in\mathbb{U}^n.\end{aligned}\right\} \tag{5.3.22}$$

Since $|\Phi''(0)+2(\Phi'(0))^2|\ge 2\Phi'(0)$, therefore from (5.3.17) and (5.3.22), it follows that

$$\left|\frac{D^3F_k(0)(z_0^3)}{3!}\frac{\|z\|}{z_k}\right|=\left|\frac{D^2f(0)(z_0^2)}{2!}\right|\le\frac{|\Phi'(0)|}{2}\left|\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right|. \tag{5.3.23}$$

Thus, from (5.3.21) and (5.3.23), we obtain

$$\left|\frac{1}{3!}D^3F_k(0)\left(z_0^2,\frac{D^3F(0)(z_0^3)}{3!}\right)\frac{\|z\|}{z_k}\right|\le\frac{(\Phi'(0))^2}{4}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2.$$

For $z_0\in\partial_0\mathbb{U}^n$, we get

$$\left|\frac{1}{3!}D^3F_k(0)\left(z_0^2,\frac{D^3F(0)(z_0^3)}{3!}\right)\right|\le\frac{(\Phi'(0))^2}{4}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2.$$

Again, since

$$\frac{1}{3!}D^3F_k(0)\left(z_0^2,\frac{D^3F(0)(z_0^3)}{3!}\right),\quad k=1,2,\cdots,n$$

are holomorphic functions on $\overline{\mathbb{U}}^n$, therefore the maximum modulus principle on the unit polydisc yields

$$\left|\frac{1}{3!}D^3F_k(0)\left(z^2,\frac{D^3F(0)(z^3)}{3!}\right)\right|\le\frac{(\Phi'(0))^2\|z\|^5}{4}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2. \tag{5.3.24}$$

Now, using the bounds given in (5.3.20) and (5.3.24), we obtain

$$\begin{aligned}&\left|\frac{1}{3!}D^3F_k(0)\left(z^2,\frac{D^3F(0)(z^3)}{3!}\right)-\frac{1}{2!}D^2F_k(0)\left(z,\frac{D^2F(0)(z^2)}{2!}\right)\right|\\&\qquad\le(\Phi'(0))^2\|z\|^3+\frac{(\Phi'(0))^2\|z\|^5}{4}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2\end{aligned}$$

for $k=1,2,\cdots n$. Therefore,

$$\left\|\frac{1}{3!}D^3F_k(0)\left(z^2,\frac{D^3F(0)(z^3)}{3!}\right)-\frac{1}{2!}D^2F_k(0)\left(z,\frac{D^2F(0)(z^2)}{2!}\right)\right\|$$
$$\leq (\Phi'(0))^2\|z\|^3+\frac{(\Phi'(0))^2\|z\|^5}{4}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2.$$

which is the required bound.

To prove the sharpness, consider the mapping

$$F(z)=z\exp\int_0^{z_1}\frac{\Phi(it)-1}{t}dt,\quad z\in\mathbb{U}^n. \tag{5.3.25}$$

It is a simple exercise to check that $D(F(z))^{-1}F(z)\in\mathscr{M}_\Phi$. From the above relation, we deduce that

$$\frac{D^2F(0)(z^2)}{2!}=i\Phi'(0)z_1z,\quad \frac{D^3F(0)(z^3)}{3!}=-\frac{1}{2}\left(\frac{\Phi''(0)}{2}+(\Phi'(0))^2\right)(z_1)^2z.$$

By taking $z=(r,0,\cdots,0)$, the equality in (5.3.14) holds. □

Remark 5.3.1. (i) When $\mathbb{B}=\mathbb{U}$ and $X=\mathbb{C}$, Theorem 5.3.1 and Theorem 5.3.3 are equivalent to Theorem 5.2.1.

(ii) In case of $\mathbb{B}=\mathbb{U}$ and $X=\mathbb{C}$, Theorem 5.3.2 is equivalent to Theorem 5.2.2.

5.4 Special Cases

If we take $\Phi(z)=(1+z)/(1-z)$, $\Phi(z)=(1+(1-2\alpha)z)/(1-z)$ and $\Phi(z)=((1+z)/(1-z))^\gamma$, Theorem 5.3.1 - 5.3.3 give the following bounds (the branch of the power function is taken such that $((1+z)/(1-z))^\gamma$ =1 at $z=0$).

Corollary 5.4.1. Let $f\in\mathscr{H}(\mathbb{B},\mathbb{C})$ with $f(0)=1$ and $F(z)=zf(z)\in\mathscr{S}^*(\mathbb{B})$. Then the following holds:

$$\left|\left(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\right)^2-\left(\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\right)^2\right|\leq 13,\quad z\in\mathbb{B}\setminus\{0\},l_z\in T_z.$$

If $\mathbb{B}=\mathbb{U}^n$ and $X=\mathbb{C}^n$, then

$$\left\|\frac{1}{3!}D^3F(0)\left(z^2,\frac{D^3F(0)(z^3)}{3!}\right)-\frac{1}{2!}D^2F(0)\left(z,\frac{D^2F(0)(z^2)}{2!}\right)\right\|\leq 9\|z\|^5+4\|z\|^3,\quad z\in\mathbb{U}^n.$$

All these estimates are sharp.

Corollary 5.4.2. Let $f\in\mathscr{H}(\mathbb{B},\mathbb{C})$ with $f(0)=1$ and $F(z)=zf(z)\in\mathscr{S}^*_\alpha(\mathbb{B})$. Then the following holds:

$$\left|\left(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\right)^2-\left(\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\right)^2\right|\leq (1-\alpha)^2(4\alpha^2-12\alpha+13),\ z\in\mathbb{B}\setminus\{0\}.$$

If $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, then

$$\left\| \frac{1}{3!}D^3F(0)\Big(z^2, \frac{D^3F(0)(z^3)}{3!}\Big) - \frac{1}{2!}D^2F(0)\Big(z, \frac{D^2F(0)(z^2)}{2!}\Big) \right\| \le (1-\alpha)^2((3-2\alpha)^2\|z\|^5 + 4\|z\|^3),\ z \in \mathbb{U}^n.$$

All these estimates are sharp.

Corollary 5.4.3. Let $f \in \mathscr{H}(\mathbb{B}, \mathbb{C})$ with $f(0) = 1$ and $F(z) = zf(z) \in \mathscr{S}\mathscr{S}^*_\gamma(\mathbb{B})$. Then for $\gamma \in [1/3, 1]$, the following holds:

$$\left| \Big(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2} \Big)^2 - \Big(\frac{l_z(D^3F(0)(z^3))}{3!||z||^3} \Big)^2 \right| \le 9\gamma^4 + 4\gamma^2, \quad z \in \mathbb{B} \setminus \{0\}, l_z \in T_z.$$

If $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, then

$$\left\| \frac{1}{3!}D^3F(0)\Big(z^2, \frac{D^3F(0)(z^3)}{3!}\Big) - \frac{1}{2!}D^2F(0)\Big(z, \frac{D^2F(0)(z^2)}{2!}\Big) \right\| \le 9\|z\|^5\gamma^4 + 4\|z\|^3\gamma^2, \quad z \in \mathbb{U}^n.$$

All these estimates are sharp.

Corollary 5.4.4. Let $f \in \mathscr{H}(\mathbb{B}, \mathbb{C})$ with $f(0) = 1$ and $F(z) = zf(z) \in \mathscr{S}^*(\mathbb{B})$. Then the following holds:

$$|2b_2^2 b_3 - b_3^2 - 2b_2^2 + 1| \le 24,$$

where

$$b_3 = \frac{l_z(D^3F(0)(z^3))}{3!||z||^3}, \quad b_2 = \frac{l_z(D^2F(0)(z^2))}{2!||z||^2}, \quad l_z \in T_z.$$

The estimate is sharp.

Corollary 5.4.5. Let $f \in \mathscr{H}(\mathbb{B}, \mathbb{C})$ with $f(0) = 1$ and $F(z) = zf(z) \in \mathscr{S}^*_\alpha(\mathbb{B})$. Then for $\alpha \in [0, 2/3]$, the following holds:

$$|2b_2^2 b_3 - b_3^2 - 2b_2^2 + 1| \le 12\alpha^4 - 52\alpha^3 + 91\alpha^2 - 74\alpha + 24,$$

where

$$b_3 = \frac{l_z(D^3F(0)(z^3))}{3!||z||^3} \text{ and } b_2 = \frac{l_z(D^2F(0)(z^2))}{2!||z||^2}, \quad l_z \in T_z.$$

The estimate is sharp.

Corollary 5.4.6. Let $f \in \mathscr{H}(\mathbb{B}, \mathbb{C})$ with $f(0) = 1$ and $F(z) = zf(z) \in \mathscr{S}\mathscr{S}^*_\gamma(\mathbb{B})$. Then for $\gamma \in [1/3, 1]$, the following holds:

$$|2b_2^2 b_3 - b_3^2 - 2b_2^2 + 1| \le 15\gamma^4 + 8\gamma^2 + 1,$$

where

$$b_3 = \frac{l_z(D^3F(0)(z^3))}{3!||z||^3} \text{ and } b_2 = \frac{l_z(D^2F(0)(z^2))}{2!||z||^2}, \quad l_z \in T_z.$$

The estimate is sharp.

In the next chapter, we extend our investigation to the class of quasi-convex mappings in higher dimensions. We will address similar problems of this chapter, adapting our methods and techniques to accommodate the increased complexity of higher-dimensional spaces. This progression allows us to build on the foundational concepts established in this chapter and explore the broader implications and applications of our results in more complex scenarios.

Highlights of the chapter

We established the sharp bounds of $|T_{2,2}(f)|$ and $|T_{3,1}(f)|$ for a specific class of holomorphic functions in the unit disk. Notably, these findings contain Theorems A, B, and C as special cases. Furthermore, we have extended these results to higher dimensions by deriving sharp bounds for certain Toeplitz determinants. These determinants formed over the corresponding terms of a homogeneous expansion of a class of holomorphic mappings defined on the unit ball of complex Banach space or on the unit polydisk in $\mathbb{C}^n$.

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Chapter 6

Toeplitz Determinants for Quasi Convex Mappings

In this chapter, we establish the sharp bounds of certain Toeplitz determinants formed over the coefficients of holomorphic mappings from a class defined on the unit ball of a complex Banach space and on the unit polydisc in $\mathbb{C}^n$. The derived bounds, provide certain new results for the subclasses of normalized univalent functions and extend some known results in higher dimensions. Additionally, we establish the sharp bounds of certain Toeplitz determinants for the class $\mathscr{C}$ such that $z=0$ is a zero of order $k+1$ of $g(z)-z$ $(k\in\mathbb{N})$. Furthermore, these results are extended to higher dimensions by determining the bounds of Toeplitz determinants for a subclass of quasi convex mappings of type B.

It is important to explore new types of univalent mappings in several variables that are not just similar to known subclasses of $\mathscr{S}$ but also due to their importance. Convex mappings of the unit ball in $\mathbb{C}^n$ exhibit a very rigid structure under certain norms. However, even with the Euclidean norm, it can be difficult to determine if a given mapping is convex. Building on the idea presented in (1.4.1), Roper and Suffridge [161] introduced the class of quasi-convex mappings. This new class generalizes the concept, allowing for a broader exploration of univalent mappings beyond the constraints of traditional convexity.

Numerous articles have recently focused on finding sharp estimates for the Toeplitz and Hermitian-Toeplitz determinants for various classes, but in one dimensional complex plane. Ahuja et al. [4] established the sharp bounds of $|\det T_{2,2}(f)|$ and $|\det T_{3,1}(f)|$ for the class $\mathscr{C}$ and its subclasses. The problem of estimating determinants of Hermitian Toeplitz matrices was initiated with the papers [39, 86, 87, 104]. For the class $\mathscr{C}$ and $\mathscr{C}(\alpha)$, the following bounds are proved in [4].

Theorem A. *[4] If $f\in\mathscr{C}$, then $|\det T_{2,2}(f)|\leq 2$. The bound is sharp.*

Theorem B. *[4] If $f \in \mathscr{C}$, then $|\det T_{3,1}(f)| \leq 4$. The bound is sharp.*

Theorem C. *[4] If $f \in \mathscr{C}(\alpha)$, then the following sharp inequality hold:*

$$|\det T_{2,2}(f)| \leq \frac{2(1-\alpha)^2(2\alpha^2-6\alpha+9)}{9}.$$

Theorem D. *[4] If $f \in \mathscr{C}(\alpha)$ and $\alpha \in [0,1/2]$, then the following sharp inequality hold:*

$$|\det T_{3,1}(f)| \leq \frac{8\alpha^4-34\alpha^3+71\alpha^2-72\alpha+36}{9}.$$

In this chapter, we generalize the above results in higher dimensions for a class of holomorphic mappings defined on the unit ball in a complex Banach space and on the unit polydisc in $\mathbb{C}^n$. In 1999, Roper and Suffridge [161] gave a sufficient condition for a normalized biholomorphic convex mapping on the Euclidean unit ball in $\mathbb{C}^n$. Later, Zhu [195] provided a brief proof of this theorem. Xu et al. [189] obtained the sharp bounds of Fekete-Szegö inequality for the class of quasi-convex mappings of type B and order α defined on the unit ball $\mathbb{B}$ and on the unit polydisc in $\mathbb{C}^n$. Liu and Liu [110] derived the sharp estimates of all homogenous expansions for a subclass of holomorphic mappings of quasi-convex mappings of type B and order α in higher dimensions. Various results regarding coefficient estimates for subclasses of holomorphic mappings in higher dimensions were obtained in [60, 62, 65, 67, 83, 191].

In case of one complex variable, many coefficient problems are studied for the class $\mathscr{C}$ such as Theorems A-D. A natural question arises that how to retain these results in higher dimensions. Providing an answer to this question is the aim of this chapter.

6.1 Certain Holomorphic Mappings

Let us recall the class $\mathscr{M}_\Phi$ given in the Definition 5.1.1. We begin with the following results.

Theorem 6.1.1. Let $f \in \mathscr{H}(\mathbb{B},\mathbb{C})$ with $f(0)=1$, $f(z)\neq 0$, $z \in \mathbb{B}$ and suppose that $F(z)=zf(z)$. If $(DF(z))^{-1}(D^2F(z)(z^2)+DF(z)(z)) \in \mathscr{M}_\Phi$ such that Φ satisfies

$$|\Phi''(0)+2(\Phi'(0))^2| \geq 2\Phi'(0) > 0,$$

then

$$\left|\left(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\right)^2-\left(\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\right)^2\right| \leq \frac{(\Phi'(0))^2}{4}+\frac{(\Phi'(0))^2}{36}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2.$$

The bound is sharp.

Proof. Fix $z \in X\setminus\{0\}$ and let $g:\mathbb{U}\to\mathbb{C}$ be defined by

$$g(\zeta)=\begin{cases}\dfrac{l_z((DF(\zeta z_0))^{-1}(D^2F(\zeta z_0)((\zeta z_0)^2)+DF(\zeta z_0)\zeta z_0))}{\zeta}, & \zeta\neq 0,\\ 1, & \zeta=0,\end{cases}$$

where $z_0 = \frac{z}{\|z\|}$. Then $g \in \mathscr{H}(\mathbb{U})$ and $g(0) = \Phi(0) = 1$. Since $(DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z)) \in \mathscr{M}_\Phi$, therefore, we have

$$\begin{aligned} g(\zeta) &= \frac{l_z((DF(\zeta z_0))^{-1}(D^2F(\zeta z_0)((\zeta z_0)^2) + DF(\zeta z_0)\zeta z_0))}{\zeta} \\ &= \frac{l_{z_0}((DF(\zeta z_0))^{-1}(D^2F(\zeta z_0)((\zeta z_0)^2) + DF(\zeta z_0)\zeta z_0))}{\zeta} \\ &= \frac{l_{\zeta z_0}((DF(\zeta z_0))^{-1}(D^2F(\zeta z_0)((\zeta z_0)^2) + DF(\zeta z_0)\zeta z_0))}{\|\zeta z_0\|} \in \Phi(\mathbb{U}), \quad \zeta \in \mathbb{U}. \end{aligned}$$

Applying a similar method used in [61, Theorem 7.1.14], we get

$$(DF(z))^{-1} = \frac{1}{f(z)}\left(I - \frac{\frac{zDf(z)}{f(z)}}{1 + \frac{Df(z)z}{f(z)}}\right). \tag{6.1.1}$$

A simple computation using the fact $F(z) = zf(z)$ yields

$$D^2F(z)(z^2) + DF(z)(z) = (D^2f(z)(z^2) + 3Df(z)(z) + f(z))z. \tag{6.1.2}$$

By using (6.1.1) and (6.1.2), it follows

$$(DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z)) = \frac{D^2f(z)(z^2) + 3Df(z)(z) + f(z)}{f(z) + Df(z)(z)}z. \tag{6.1.3}$$

Consequently

$$l_z((DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z))) = \frac{D^2f(z)(z^2) + 3Df(z)(z) + f(z)}{f(z) + Df(z)(z)}\|z\|. \tag{6.1.4}$$

Using (6.1.4), we obtain

$$\begin{aligned} g(\zeta) &= \frac{l_{\zeta z_0}((DF(\zeta z_0))^{-1}(D^2F(\zeta z_0)((\zeta z_0)^2) + DF(\zeta z_0)\zeta z_0))}{\|\zeta z_0\|} \\ &= \frac{D^2f(\zeta z_0)((\zeta z_0)^2) + 3Df(\zeta z_0)(\zeta z_0) + f(\zeta z_0)}{f(\zeta z_0) + Df(\zeta z_0)(\zeta z_0)}. \end{aligned}$$

Equivalently, we can write

$$g(\zeta)(f(\zeta z_0) + Df(\zeta z_0)(\zeta z_0)) = D^2f(\zeta z_0)((\zeta z_0)^2) + 3Df(\zeta z_0)(\zeta z_0) + f(\zeta z_0).$$

The series expansion in terms of ζ gives

$$\begin{aligned} &\left(1 + g'(0)\zeta + \frac{g''(0)}{2}\zeta^2 + \cdots\right)\left(1 + 2Df(0)(z_0)\zeta + \frac{3Df(0)(z_0^2)}{2}\zeta^2 + \cdots\right) \\ &\qquad = 1 + 4Df(0)(z_0)\zeta + \frac{9Df(0)(z_0^2)}{2}\zeta^2 + \cdots. \end{aligned}$$

Comparison of the homogenous expansions of either sides of the above equality provide $g'(0) =$

$2Df(0)(z_0)$. That is

$$g'(0)\|z\| = 2Df(0)(z). \tag{6.1.5}$$

Also, we have

$$\frac{D^2F(0)(z^2)}{2!} = Df(0)(z)z,$$

which gives

$$\frac{l_z(D^2F(0)(z^2))}{2!} = Df(0)(z)\|z\|.$$

Now, using $|g'(0)| \leq \Phi'(0)$ with (6.1.5), we obtain

$$\left|\frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2}\right| \leq \frac{\Phi'(0)}{2}. \tag{6.1.6}$$

Moreover, for $\lambda \in \mathbb{C}$, Xu et al. [189, Theorem 3.1] proved that

$$\left.\begin{aligned} &\left|\frac{l_z(D^3F(0)(z^3))}{3!\|z\|^3} - \lambda\left(\frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2}\right)^2\right| \\ &\qquad \leq \frac{|\Phi'(0)|}{6}\max\left\{1, \left|\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \left(1-\frac{3}{2}\lambda\right)\Phi'(0)\right|\right\}, \quad z \in \mathbb{B}\setminus\{0\}. \end{aligned}\right\} \tag{6.1.7}$$

Since $|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0)$, therefore the above inequality gives

$$\left|\frac{l_z(D^3F(0)(z^3))}{3!\|z\|^3}\right| \leq \frac{\Phi'(0)}{6}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right). \tag{6.1.8}$$

Also, note that

$$\left|\left(\frac{l_z(D^3F(0)(z^3))}{3!\|z\|^3}\right)^2 - \left(\frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2}\right)^2\right| \leq \left|\frac{l_z(D^3F(0)(z^3))}{3!\|z\|^3}\right|^2 + \left|\frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2}\right|^2.$$

The required bound follows from the above inequality together with the bounds given in (6.1.6) and (6.1.8).

The result is sharp for the mapping F given by

$$DF(z) = I\exp\int_0^{T_u(z)}\frac{\Phi(it)-1}{t}dt, \quad z \in \mathbb{B}, \ \|u\| = 1. \tag{6.1.9}$$

Clearly, $(DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z)) \in \mathscr{M}_\Phi$ and

$$\frac{D^3F(0)(z^3)}{3!} = -\frac{1}{6}\left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2\right)(l_u(z))^2 z \ \text{and} \ \frac{D^2F(0)(z^2)}{2!} = \frac{i\Phi'(0)}{2}l_u(z)z,$$

which immediately gives

$$\frac{l_z(D^2F(0)(z^2))}{2!} = \frac{i\Phi'(0)}{2}l_u(z)\|z\|$$

and

$$\frac{l_z(D^3F(0)(z^3))\|z\|}{3!} = -\frac{1}{6}\left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2\right)(l_u(z))^2\|z\|^2.$$

Taking $z = ru$ $(0 < r < 1)$, we get

$$\frac{l_z(D^3F(0)(z^3))}{3!\|z\|^3} = -\frac{1}{6}\left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2\right) \text{ and } \frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2} = \frac{i\Phi'(0)}{2}. \tag{6.1.10}$$

According to the above equations, we have

$$\left|\left(\frac{l_z(D^3F(0)(z^3))}{3!\|z\|^3}\right)^2 - \left(\frac{l_z(D^2F(0)(z^2))}{2!\|z\|^2}\right)^2\right| = \frac{(\Phi'(0))^2}{4} + \frac{1}{36}\left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2\right)^2,$$

which establishes the sharpness of the bound and completes the proof. □

Theorem 6.1.2. Let $f \in \mathscr{H}(\mathbb{B},\mathbb{C})$ with $f(0) = 1$, $f(z) \neq 0$, $z \in \mathbb{B}$ and suppose that $F(z) = zf(z)$. If $(DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z)) \in \mathscr{M}_\Phi$ such that Φ satisfy

$$2\Phi'(0) - 2(\Phi'(0))^2 \leq \Phi''(0) \leq 4(\Phi'(0))^2 - 2\Phi'(0),$$

then

$$|2b_2^2b_3 - b_3^2 - 2b_2^2 + 1| \leq 1 + 2(\Phi'(0))^2 + \frac{(\Phi'(0))^2}{4}\left(\frac{\Phi''(0)}{2\Phi'(0)} - 3\Phi'(0)\right)\left(\frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0)\right),$$

where

$$b_3 = \frac{l_z(D^3F(0)(z^3))}{3!||z||^3} \text{ and } b_2 = \frac{l_z(D^2F(0)(z^2))}{2!||z||^2}.$$

The bound is sharp.

Proof. Since $2\Phi'(0) \leq 4(\Phi'(0))^2 - \Phi''(0)$, inequality (6.1.7) gives

$$\left|\frac{l_z(D^3F(0)(z^3))}{3!||z||^3} - 2\left(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\right)^2\right| \leq \frac{\Phi'(0)}{6}\left(2\Phi'(0) - \frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}\right) \tag{6.1.11}$$

for $z \in \mathbb{B}\setminus\{0\}$. Also, since Φ satisfy $\Phi''(0) + 2(\Phi'(0))^2 \geq 2\Phi'(0)$, therefore by (6.1.7), we have

$$\left|\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\right| \leq \frac{\Phi'(0)}{6}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right), \quad z \in \mathbb{B}\setminus\{0\}. \tag{6.1.12}$$

Also, we have

$$|2b_2^2b_3 - 2b_2^2 - b_3^2 + 1| \leq 1 + 2|b_2|^2 + |b_3||b_3 - 2b_2^2|.$$

The required bound follows directly from the above inequality along with the bounds given in (6.1.6) and (6.1.12), and the bound of $|b_3 - 2b_2^2|$ given by (6.1.11).

Equality case holds for the mapping $F(z)$ defined in (6.1.9) as for this mapping, we have $b_2 = \frac{i\Phi'(0)}{2}$,

$b_3 = -\frac{1}{6}\left(\frac{\Phi''(0)}{2} + (\Phi'(0))^2\right)$ and hence

$$2b_2^2 b_3 - 2b_2^2 - b_3^2 + 1 = 2(\Phi'(0))^2 + \frac{(\Phi'(0))^2}{12}\left(\frac{\Phi''(0)}{2\Phi'(0)} - 3\Phi'(0)\right)\left(\frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0)\right) + 1,$$

which establish the sharpness of the result. □

Theorem 6.1.3. Let $f \in \mathscr{H}(\mathbb{U}^n, \mathbb{C})$ with $f(0) = 1$, $f(z) \neq 0$, $z \in \mathbb{U}^n$ and suppose that $F(z) = zf(z)$. If $(DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z)) \in \mathscr{M}_\Phi$ such that Φ satisfies

$$|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0),$$

then

$$\left.\begin{aligned} &\left\| \frac{1}{3!}D^3F(0)\left(z^2, \frac{D^3F(0)(z^3)}{3!}\right) - \frac{1}{2!}D^2F(0)\left(z, \frac{D^2F(0)(z^2)}{2!}\right) \right\| \\ &\qquad \leq \frac{(\Phi'(0))^2\|z\|^5}{36}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2 + \frac{(\Phi'(0))^2\|z\|^3}{4}, \quad z \in \mathbb{U}^n. \end{aligned}\right\} \tag{6.1.13}$$

The bound is sharp.

Proof. For $z \in \mathbb{U}^n \setminus \{0\}$ and $z_0 = \frac{z}{\|z\|}$, define $g_k : \mathbb{U} \to \mathbb{C}$ such that

$$g_k(\zeta) = \begin{cases} \dfrac{\zeta z_k}{p_k(\zeta z_0)\|z_0\|}, & \zeta \neq 0, \\ 1, & \zeta = 0, \end{cases} \tag{6.1.14}$$

where $p(z) = (DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z))$ and k satisfies $|z_k| = \|z\| = \max_{1\leq j\leq n}\{z_j\}$. Since by the hypothesis $(DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z)) \in \mathscr{M}_\Phi$, therefore $g_k(\zeta) \in \Phi(\mathbb{U})$. Also, by (6.1.3), we have

$$g_k(\zeta) = \frac{D^2f(\zeta z_0)((\zeta z_0)^2) + 3Df(\zeta z_0)(\zeta z_0) + f(\zeta z_0)}{f(\zeta z_0) + Df(\zeta z_0)(\zeta z_0)},$$

or, equivalently

$$g_k(\zeta)(f(\zeta z_0) + Df(\zeta z_0)(\zeta z_0)) = D^2f(\zeta z_0)((\zeta z_0)^2) + 3Df(\zeta z_0)(\zeta z_0) + f(\zeta z_0).$$

Comparison of same homogeneous expansions in the Taylor series expansions in terms of ζ yield

$$g_k'(0) = 2Df(0)(z_0) \quad \text{and} \quad \frac{g_k''(0)}{2} = 3D^2f(0)(z_0^2) - 4(Df(0)(z_0))^2. \tag{6.1.15}$$

Furthermore, from $F(z_0) = z_0 f(z_0)$, we have

$$\frac{D^2F_k(0)(z_0^2)}{2!} = Df(0)(z_0)\frac{z_k}{\|z\|} \quad \text{and} \quad \frac{D^3F_k(0)(z_0^3)}{3!} = \frac{D^2f(0)(z_0^2)}{2!}\frac{z_k}{\|z\|}. \tag{6.1.16}$$

Using the same argument as in (5.3.18), we get

$$\left|\frac{1}{2!}D^2F_k(0)\left(z_0, \frac{D^2F(0)(z_0^2)}{2!}\right)\frac{\|z\|}{z_k}\right| = |(Df(0)(z_0))^2| \tag{6.1.17}$$

and

$$\left|\frac{1}{3!}D^3F_k(0)\left(z_0^2, \frac{D^3F(0)(z_0^3)}{3!}\right)\frac{\|z\|}{z_k}\right| = \left|\left(\frac{D^2f(0)(z_0^2)}{2!}\right)\right|^2. \tag{6.1.18}$$

Combining (6.1.15) and (6.1.17) with the fact $|g_k'(0)| \leq \Phi'(0)$ gives

$$\left|\frac{1}{2!}D^2F_k(0)\left(z_0, \frac{D^2F(0)(z_0^2)}{2!}\right)\frac{\|z\|}{z_k}\right| \leq \frac{(\Phi'(0))^2}{4}.$$

If $z_0 \in \partial_0\mathbb{U}^n$, then we get

$$\left|\frac{1}{2!}D^2F_k(0)\left(z_0, \frac{D^2F(0)(z_0^2)}{2!}\right)\right| \leq \frac{(\Phi'(0))^2}{4}.$$

Since

$$\frac{1}{2!}D^2F_k(0)\left(z_0, \frac{D^2F(0)(z_0^2)}{2!}\right), \quad k = 1, 2, \cdots, n$$

are holomorphic functions on $\overline{\mathbb{U}}^n$, therefore by the maximum modulus theorem of holomorphic functions on the unit polydisc, we have

$$\left|\frac{1}{2!}D^2F_k(0)\left(z, \frac{D^2F(0)(z^2)}{2!}\right)\right| \leq \frac{(\Phi'(0))^2\|z\|^3}{4}. \tag{6.1.19}$$

For $\lambda \in \mathbb{C}$, Xu et al. [189, Theorem 3.2] established that

$$\left.\begin{aligned} &\left|\frac{D^3F_k(0)(z_0^3)}{3!}\frac{\|z\|}{z_k} \quad -\lambda\frac{1}{2}D^2F_k(0)\left(z_0, \frac{D^2F(0)(z_0^2)}{2!}\frac{\|z\|}{z_k}\right)\right| \\ &\qquad \leq \frac{|\Phi'(0)|}{6}\max\left\{1, \left|\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \left(1 - \frac{3}{2}\lambda\right)\Phi'(0)\right|\right\}. \end{aligned}\right\} \tag{6.1.20}$$

Since $|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0)$, therefore, from (6.1.20) and (6.1.16), we get

$$\left|\frac{D^3F_k(0)(z_0^3)}{3!}\frac{\|z\|}{z_k}\right| = \left|\frac{D^2f(0)(z_0^2)}{2!}\right| \leq \frac{\Phi'(0)}{6}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right). \tag{6.1.21}$$

Thus from (6.1.18) and (6.1.21), we deduce that

$$\left|\frac{1}{3!}D^3F_k(0)\left(z_0^2, \frac{D^3F(0)(z_0^3)}{3!}\right)\frac{\|z\|}{z_k}\right| \leq \frac{(\Phi'(0))^2}{36}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2.$$

Further, if $z_0 \in \partial\mathbb{U}^n$, then

$$\left|\frac{1}{3!}D^3F_k(0)\left(z_0^2, \frac{D^3F(0)(z_0^3)}{3!}\right)\right| \leq \frac{(\Phi'(0))^2}{36}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0)\right)^2.$$

Again, since

$$\frac{1}{3!}D^3F_k(0)\left(z_0^2, \frac{D^3F(0)(z_0^3)}{3!}\right), \quad k = 1,2,\cdots,n$$

are holomorphic functions on $\overline{\mathbb{U}}^n$, therefore by the maximum modulus theorem of holomorphic functions on the unit polydisc, we have

$$\left|\frac{1}{3!}D^3F_k(0)\left(z_0^2, \frac{D^3F(0)(z_0^3)}{3!}\right)\right| \leq \frac{(\Phi'(0))^2\|z\|^5}{36}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2. \tag{6.1.22}$$

Now, using the bounds from (6.1.19) and (6.1.22), we have

$$\left|\frac{1}{3!}D^3F_k(0)\left(z^2, \frac{D^3F(0)(z^3)}{3!}\right) - \frac{1}{2!}D^2F_k(0)\left(z, \frac{D^2F(0)(z^2)}{2!}\right)\right|$$
$$\leq \frac{(\Phi'(0))^2\|z\|^5}{36}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2 + \frac{(\Phi'(0))^2\|z\|^3}{4}$$

for $z \in \mathbb{U}^n$ and $k = 1,2,\cdots n$. Therefore,

$$\left\|\frac{1}{3!}D^3F(0)\left(z^2, \frac{D^3F(0)(z^3)}{3!}\right) - \frac{1}{2!}D^2F(0)\left(z, \frac{D^2F(0)(z^2)}{2!}\right)\right\|$$
$$\leq \frac{(\Phi'(0))^2\|z\|^5}{36}\left(\frac{1}{2}\frac{\Phi''(0)}{\Phi'(0)}+\Phi'(0)\right)^2 + \frac{(\Phi'(0))^2\|z\|^3}{4},$$

which is the required bound.

To prove the sharpness of the bound, consider the mapping F given by

$$DF(z) = I\exp\int_0^{z_1}\frac{\Phi(it)-1}{t}dt. \tag{6.1.23}$$

It can be showed that $(DF(z))^{-1}(D^2F(z)(z^2)+DF(z)(z)) \in \mathscr{M}_\Phi$ and for $z = (r,0,\cdots,0)'$ in (6.1.23), the equality case holds in (6.1.13). □

6.2 Special Cases

Note that if $f \in \mathscr{H}(\mathbb{B})$ and $(Df(z))^{-1}(D^2f(z)(z^2)+Df(z)(z)) \in \mathscr{M}_\Phi$, then various choices of Φ give different subclasses of holomorphic mappings. For instance, when $\Phi(z) = (1+(1-2\alpha)z)/(1-z)$ and $\Phi(z) = (1+z)/(1-z)$, we easily obtain $f \in Q_\alpha(\mathbb{B})$ and $f \in Q(\mathbb{B})$, respectively. For these classes, Theorem 6.1.1 to Theorem 6.1.3 yield the following results.

Corollary 6.2.1. Let $f \in \mathscr{H}(\mathbb{B},\mathbb{C})$ and $F(z) = zf(z) \in Q_\alpha(\mathbb{B})$. Then the following inequality holds:

$$\left|\left(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\right)^2 - \left(\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\right)^2\right| \leq \frac{2(1-\alpha)^2(2\alpha^2-6\alpha+9)}{9}, \quad l_z \in T_z,\ z \in \mathbb{B}\setminus\{0\}.$$

If $\mathbb{B}=\mathbb{U}^n$ and $X=\mathbb{C}^n$, then

$$\left.\begin{aligned} \left\|\frac{1}{3!}D^3F(0)\left(z^2,\frac{D^3F(0)(z^3)}{3!}\right)\right. & \left.-\frac{1}{2!}D^2F(0)\left(z,\frac{D^2F(0)(z^2)}{2!}\right)\right\| \\ & \leq (1-\alpha)^2\|z\|^3+\frac{(2\alpha^2-5\alpha+3)^2\|z\|^5}{9},\ z\in\mathbb{U}^n. \end{aligned}\right\} \tag{6.2.1}$$

All these bounds are sharp.

Remark 6.2.1. In case of $n=1$, eq. (6.2.1) reduces to the following:

$$\left|\left(\frac{F^{(3)}(0)}{3!}\right)^2-\left(\frac{F''(0)}{2!}\right)^2\right|\leq\frac{2(1-\alpha)^2(2\alpha^2-6\alpha+9)}{9},$$

which is equivalent to Theorem C.

Corollary 6.2.2. Let $f\in\mathscr{H}(\mathbb{B},\mathbb{C})$ and $F(z)=zf(z)\in Q_\alpha(\mathbb{B})$. Then for $\alpha\in[0,1/2]$, the following sharp bound holds:

$$|2b_2^2b_3-b_3^2-2b_2^2+1|\leq\frac{8\alpha^4-34\alpha^3+71\alpha^2-72\alpha+36}{9},\quad z\in\mathbb{B}\setminus\{0\},$$

where

$$b_3=\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\quad\text{and}\quad b_2=\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}.$$

In particular, for $\alpha=0$, we obtain the following results for the class $Q(\mathbb{B})$ in higher dimensions.

Corollary 6.2.3. Let $f\in\mathscr{H}(\mathbb{B},\mathbb{C})$ and $F(z)=zf(z)\in Q(\mathbb{B})$. Then the following holds:

$$\left|\left(\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}\right)^2-\left(\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\right)^2\right|\leq 2,\quad l_z\in T_z,\ z\in\mathbb{B}\setminus\{0\}.$$

If $\mathbb{B}=\mathbb{U}^n$ and $X=\mathbb{C}^n$, then

$$\left\|\frac{1}{3!}D^3F(0)\left(z^2,\frac{D^3F(0)(z^3)}{3!}\right)-\frac{1}{2!}D^2F(0)\left(z,\frac{D^2F(0)(z^2)}{2!}\right)\right\|\leq\|z\|^3+\|z\|^5,\quad z\in\mathbb{U}^n. \tag{6.2.2}$$

All these bounds are sharp.

Remark 6.2.2. For $n=1$, (6.2.2) is equivalent to Theorem A.

Corollary 6.2.4. Let $f\in\mathscr{H}(\mathbb{B},\mathbb{C})$ and $F(z)=zf(z)\in Q(\mathbb{B})$. Then the following sharp bound holds:

$$|2a_2^2a_3-a_3^2-2a_2^2+1|\leq 4,\quad z\in\mathbb{B}\setminus\{0\},$$

where

$$a_3=\frac{l_z(D^3F(0)(z^3))}{3!||z||^3}\quad\text{and}\quad a_2=\frac{l_z(D^2F(0)(z^2))}{2!||z||^2}.$$

6.3 Generalized Toeplitz Determinants

In this section, we generalize Theorem A for $f \in \mathscr{C}$ such that $f(z) - z$ has a zero of order $k+1$ at $z = 0$ $(k \in \mathbb{N})$. Further, we obtain the bound of second order Toeplitz determinant for a subclass of quasi-convex mappings defined on the unit ball in a complex Banach space, which extend Theorem A to higher dimensions.

Definition 6.3.1. [109] Suppose that Ω is a domain in X which contains 0 and $f : \Omega \to X$ is a holomorphic mapping. We say that $z = 0$ is the zero of order k of $f(z)$ if $f(0) = 0, \cdots, D^{k-1}f(0) = 0$, but $D^k f(0) \neq 0$, where $k \in \mathbb{N}$.

The following Lemma is due to Liu and Liu [111].

Lemma 6.3.1. [111] Suppose that $f \in \mathscr{S}$. Then F defined by $F(z) = \frac{f(l_u(z))}{l_u(z)} z$, where $z \in \mathbb{B}$, $u \in \partial\mathbb{B}$, belongs to $Q_B(\mathbb{B})$ if and only if $f \in \mathscr{C}$.

We begin with deriving the bound of second order Toeplitz determinant for $f \in \mathscr{C}$ and then extend this bound to higher dimensions.

Theorem 6.3.2. If $f(z) = z + \sum_{n=k+1}^{\infty} b_n z^n \in \mathscr{C}$, then the following sharp bounds hold:

$$|b_{2k+1}^2 - b_{k+1}^2| \leq \frac{(k+2)^2}{k^4(2k+1)^2} + \frac{4}{k^2(k+1)^2}, \quad k \in \mathbb{N}. \tag{6.3.1}$$

Proof. Since $f \in \mathscr{C}$, therefore

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{U}.$$

By the definition of subordination, there exist a Schwarz function ω satisfying

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+\omega(z)}{1-\omega(z)}.$$

Corresponding to the function ω, define $p : \mathbb{U} \to \mathbb{C}$ such that

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + \cdots. \tag{6.3.2}$$

The Taylor series expansion of $(1 + zf''(z)/f'(z))$ is given by

$$1 + \frac{zf''(z)}{f'(z)} = 1 + k(k+1)b_{k+1}z^k + \cdots + (2k(2k+1)b_{2k+1} - k(k+1)^2 b_{k+1}^2)z^{2k} + \cdots. \tag{6.3.3}$$

By (6.3.2) and (6.3.3), the coefficients b_{k+1} can be expressed as follows

$$b_{k+1} = \frac{p_k}{k(k+1)}.$$

Using $|p_k| \leq 2$ [61, Theorem 2.1.5], we obtain

$$|b_{k+1}| \leq \frac{2}{k(k+1)}. \tag{6.3.4}$$

Further, Xu and Lai [186] established that

$$|b_{2k+1}-\lambda b_{k+1}^2| \le \frac{1}{k(2k+1)} \max\left\{1, \frac{|(k+2)(k+1)^2-4(2k+1)\lambda|}{k(k+1)^2}\right\}, \; k\in\mathbb{N},\; \lambda\in\mathbb{C},$$

which immediately provides the following bound

$$|b_{2k+1}| \le \frac{(k+2)}{k^2(2k+1)}, \quad (k\in\mathbb{N}). \tag{6.3.5}$$

Applying the bounds from (6.3.4) and (6.3.5) in the following inequality

$$|b_{2k+1}^2-b_{k+1}^2| \le |b_{2k+1}|^2+|b_{k+1}|^2,$$

the required estimate follows. For sharpness of the bound consider the function

$$f(z)=\int_0^z \frac{1}{(1-it^k)^{2/k}}dt = z+\frac{2i}{k(k+1)}z^{k+1}-\frac{k+2}{k^2(2k+1)}z^{2k+1}+\cdots.$$

Clearly, for this function, we have

$$|b_{2k+1}^2-b_{k+1}^2| = \frac{(k+2)^2}{k^4(2k+1)^2}+\frac{4}{k^2(k+1)^2}$$

proving the sharpness. □

Remark 6.3.1. If we take $k=1$, then Theorem 6.3.2 reduces to Theorem A.

Theorem 6.3.3. Let $f:\mathbb{B}\to\mathbb{C}$ and $f(z)\ne 0$ for $z\in\mathbb{B}$. If $F(z)=zf(z)\in Q(\mathbb{B})$ and $z=0$ is a zero of order $k+1$ $(k\in\mathbb{N})$ of $F(z)-z$, then

$$\left|\left(\frac{l_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!\|z\|^{2k+1}}\right)^2-\left(\frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!\|z\|^{k+1}}\right)^2\right| \le \frac{(k+2)^2}{k^4(2k+1)^2}+\frac{4}{k^2(k+1)^2}$$

for $z\in\mathbb{B}\setminus\{0\}$. The bound is sharp.

Proof. For fix $z\in X\setminus\{0\}$, define $h:\mathbb{U}\to\mathbb{C}$ such that

$$h(\zeta)=\begin{cases} \dfrac{l_z((DF(\zeta z_0))^{-1}(D^2F(\zeta z_0)(\zeta z_0)^2+DF(\zeta z_0)(\zeta z_0)))}{\zeta}, & \zeta\ne 0,\\ 1, & \zeta=0,\end{cases}$$

where $z_0=\frac{z}{\|z\|}$. Then $h\in\mathscr{H}(\mathbb{U})$, $h(0)=1$ and

$$\begin{aligned} h(\zeta) &= \frac{l_z((DF(\zeta z_0))^{-1}(D^2F(\zeta z_0)(\zeta z_0)^2+DF(\zeta z_0)(\zeta z_0)))}{\zeta}\\ &= \frac{l_{z_0}((DF(\zeta z_0))^{-1}(D^2F(\zeta z_0)(\zeta z_0)^2+DF(\zeta z_0)(\zeta z_0)))}{\zeta}\\ &= \frac{l_{\zeta z_0}((DF(\zeta z_0))^{-1}(D^2F(\zeta z_0)(\zeta z_0)^2+DF(\zeta z_0)(\zeta z_0)))}{\|\zeta z_0\|}.\end{aligned}$$

Since $F \in Q_B(\mathbb{B})$, therefore $\operatorname{Re} h(\zeta) > 0$ for $\zeta \in \mathbb{U}$. Applying a similar method used in [61, Theorem 7.1.14], we get

$$(DF(z))^{-1} = \frac{1}{f(z)}\left(I - \frac{\frac{zDf(z)}{f(z)}}{1+\frac{Df(z)z}{f(z)}}\right). \tag{6.3.6}$$

Also, note that $F(z) - z$ has zero of order $k+1$ at $z = 0$, therefore, we have

$$f(z) = 1 + \frac{D^k f(0)(z^k)}{k!} + \cdots + \frac{D^{2k} f(0)(z^{2k})}{(2k)!} + \cdots, \tag{6.3.7}$$

which gives

$$Df(z)z = k\frac{D^k f(0)(z^k)}{k!} + \cdots + 2k\frac{D^{2k} f(0)(z^{2k})}{(2k)!} + \cdots. \tag{6.3.8}$$

By (6.3.7) and (6.3.8), we get

$$f(z) + Df(z)z = 1 + (k+1)\frac{D^k f(0)(z^k)}{k!} + \cdots + (2k+1)\frac{D^{2k} f(0)(z^{2k})}{(2k)!} + \cdots, \tag{6.3.9}$$

$$D^2 f(z)(z^2) + Df(z)(z) = k^2\frac{D^k f(0)(z^k)}{k!} + \cdots + (2k)^2\frac{D^{2k} f(0)(z^{2k})}{(2k)!} + \cdots,$$

and

$$D^2 f(z)(z^2) + 3Df(z)z + f(z) = 1 + (k+1)^2\frac{D^k f(0)(z^k)}{k!} + \cdots + (2k+1)^2\frac{D^{2k} f(0)(z^{2k})}{(2k)!} + \cdots. \tag{6.3.10}$$

Using the facts from (6.3.6), (6.3.9) and (6.3.10), we deduce that

$$(DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z)) = \frac{D^2 f(z)(z^2) + 3Df(z)(z) + f(z)}{f(z) + Df(z)(z)} z, \tag{6.3.11}$$

which gives

$$l_z((DF(z))^{-1}(D^2F(z)(z^2) + DF(z)(z))) = \frac{D^2 f(z)(z^2) + 3Df(z)(z) + f(z)}{f(z) + Df(z)(z)} \|z\|. \tag{6.3.12}$$

Consequently, we obtain

$$\begin{aligned} h(\zeta) &= \frac{l_{\zeta z_0}((DF(\zeta z_0))^{-1}(D^2F(\zeta z_0)((\zeta z_0)^2) + DF(\zeta z_0)\zeta z_0))}{\|\zeta z_0\|} \\ &= \frac{D^2 f(\zeta z_0)((\zeta z_0)^2) + 3Df(\zeta z_0)(\zeta z_0) + f(\zeta z_0)}{f(\zeta z_0) + Df(\zeta z_0)(\zeta z_0)}, \end{aligned}$$

which can be rewritten as

$$D^2 f(\zeta z_0)((\zeta z_0)^2) + 3Df(\zeta z_0)(\zeta z_0) + f(\zeta z_0) = h(\zeta)(f(\zeta z_0) + Df(\zeta z_0)(\zeta z_0)).$$

The Taylor series expansion in terms of ζ gives

$$1+(k+1)^2\frac{D^kf(0)(z_0^k)}{k!}\zeta^k+\cdots+(2k+1)^2\frac{D^{2k}f(0)(z_0^{2k})}{(2k)!}\zeta^{2k}+\cdots=\Big(1+h'(0)\zeta$$
$$+\frac{h''(0)}{2}\zeta^2+\cdots\Big)\Big(1+(k+1)\frac{D^kf(0)(z_0^k)}{k!}\zeta^k+\cdots+(2k+1)\frac{D^{2k}f(0)(z_0^{2k})}{(2k)!}\zeta^{2k}+\cdots\Big).$$

A comparison of homogenous expansions in the above equation leads to

$$h'(0)=h''(0)=\cdots=h^{(k-1)}(0)=0$$

and

$$\frac{h^{(k)}(0)}{k!}=k(k+1)\frac{D^kf(0)(z_0^k)}{k!},$$

which is equivalent to

$$\frac{D^kf(0)(z^k)}{k!}=\frac{1}{k(k+1)}\frac{h^{(k)}(0)\|z\|^k}{k!}. \tag{6.3.13}$$

Since $F(z)=zf(z)$, therefore, we have

$$\frac{D^{k+1}F(0)(z^{k+1})}{(k+1)!}=\frac{D^kf(0)(z^k)}{k!}z. \tag{6.3.14}$$

From (6.3.14), we obtain

$$\frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!}=\frac{D^kf(0)(z^k)}{k!}\|z\|. \tag{6.3.15}$$

The above equation together with (6.3.13) gives

$$\left|\frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!\|z\|^{k+1}}\right|=\left|\frac{h^{(k)}(0)}{k!k(k+1)}\right|.$$

Also, $|h^{(k)}(0)|\le 2(k!)$ [59] holds. Using this fact, we obtain

$$\left|\frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!\|z\|^{k+1}}\right|\le\frac{2}{k(k+1)}. \tag{6.3.16}$$

On the other hand, for $\lambda\in\mathbb{C}$, Xu and Lai [186, Theorem 3.3] proved that

$$\left.\begin{aligned}&\left|\frac{l_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!||z||^{2k+1}}-\lambda\left(\frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!||z||^{k+1}}\right)^2\right|\\&\qquad\le\frac{1}{k(2k+1)}\max\left\{1,\frac{|(k+2)(k+1)^2-4\lambda(2k+1)|}{k(k+1)^2}\right\}\end{aligned}\right\} \tag{6.3.17}$$

for $z\in\mathbb{B}\setminus\{0\}$. The equation (6.3.17) directly yields

$$\left|\frac{l_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!||z||^{2k+1}}\right|\le\frac{k+2}{k^2(2k+1)}. \tag{6.3.18}$$

Using the estimates in (6.3.16) and (6.3.18), we obtain

$$\begin{aligned}
&\left| \left(\frac{l_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!||z||^{2k+1}} \right)^2 - \left(\frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!||z||^{k+1}} \right)^2 \right| \\
&\qquad\qquad \le \left| \frac{l_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!||z||^{2k+1}} \right|^2 + \left| \frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!||z||^{k+1}} \right|^2 \\
&\qquad\qquad \le \left(\frac{k+2}{k^2(2k+1)} \right)^2 + \left(\frac{2}{k(k+1)} \right)^2.
\end{aligned}$$

Sharpness of the bound follows from the mapping

$$F(z) = \frac{\int_0^{l_u(z)} \frac{1}{(1-it^k)^{2/k}} dt}{l_u(z)} z, \quad z \in \mathbb{B}, \quad \|u\| = 1. \tag{6.3.19}$$

From Lemma 6.3.1, the mapping F defined in (6.3.19), is in the $Q(\mathbb{B})$ and $F(z) - z$ has a zero of order $k+1$ at $z = 0$. A simple computation reveals that

$$\frac{D^{2k+1}F(0)(z^{2k+1})}{(2k+1)!} = \frac{-(k+2)}{(2k+1)k^2} (l_u(z))^{2k} z$$

and

$$\frac{D^{k+1}F(0)(z^{k+1})}{(k+1)!} = \frac{2i}{k(k+1)} (l_u(z))^k z.$$

In view of the above equations, we have

$$\frac{l_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!} = -\frac{(k+2)}{(2k+1)k^2} (l_u(z))^{2k} \|z\|$$

and

$$\frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!} = \frac{2i}{k(k+1)} (l_u(z))^k \|z\|.$$

Setting $z = ru$ $(0 < r < 1)$, we obtain

$$\frac{l_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!\|z\|^{2k+1}} = -\frac{(k+2)}{(2k+1)k^2}$$

and

$$\frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!\|z\|^{k+1}} = \frac{2i}{k(k+1)}.$$

Thus, for the mapping F, we have

$$\left| \left(\frac{l_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!\|z\|^{2k+1}} \right)^2 - \left(\frac{l_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!\|z\|^{k+1}} \right)^2 \right| = \frac{(k+2)^2}{k^4(2k+1)^2} + \frac{4}{k^2(k+1)^2},$$

which shows that the bound is sharp. □

Remark 6.3.2. Theorem 6.3.3 is equivalent to Theorem A when $X = \mathbb{C}$, $\mathbb{B} = \mathbb{U}$ and $k = 1$.

Remark 6.3.3. For $k = 1$, Theorem 6.3.3 reduces to [51, Corollary 2.8].

Highlights of the chapter

In this concluding chapter, we have established the sharp bounds of Toeplitz determinants for the class of quasi-convex mappings of Type B and order α defined on the unit ball in a complex Banach space or on the unit polydisk in $\mathbb{C}^n$. Furthermore, we proceeded to establish the sharp bounds of second-order Toeplitz determinants formed over the coefficients of functions $f \in \mathscr{C}$ such that $f(z) - z$ has a zero of order $k+1$ at $z = 0$. Later, these bounds are extended in the case of higher dimensions for the class of quasi-convex mappings defined on the unit ball of a complex Banach space.

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Conclusion, Future Scope & Social Impact

In geometric function theory, the primary focus often revolves around coefficient and radius problems. These problems are crucial because they provide insights into the behavior and properties of analytic and univalent functions. In this work, we explore these problems for various subclasses of $\mathcal{S}$. Specifically, Chapters 2, 3, and 4 deal with coefficient and radius problems for the classes $\mathcal{S}^*(\varphi)$, $\mathcal{C}(\varphi)$, $\mathcal{S}_s^*(\varphi)$, $\mathcal{C}_s(\varphi)$, $\mathcal{K}(g)$, and $\mathcal{A}_\beta$. In the final two chapters, we extend our investigation to geometric function theory in higher dimensions, examining coefficient problems for starlike and quasi-convex mappings on the unit ball in a complex Banach space and on the unit polydisk in $\mathbb{C}^n$.

We have derived the sharp bound for the fifth coefficient for functions in the classes $\mathcal{S}^*(\varphi)$, $\mathcal{C}(\varphi)$, $\mathcal{S}_s^*(\varphi)$, and $\mathcal{C}_s(\varphi)$. However, determining the sharp bound of $|a_n|$ for $n \geq 6$ remains an open problem. Additionally, we have obtained the sharp bounds for $T_2(1)(f)$ and $T_3(1)(f)$ for these classes. The sharp bound of $T_4(1)(f)$ is known only for specific cases, such as when $\varphi(z) = (1+z)/(1-z)$. Extending the bound of $T_4(1)(f)$ to more generalized classes is a future scope for further study.

We have also examined certain coefficient and radius problems for the class $\mathcal{A}_\beta$. For this class, we determined the radius of starlikeness of order α. One can try for radii of other subclasses of starlike functions for the class $\mathcal{A}_\beta$. Another open problem is finding the sharp estimate of the third-order Hankel determinant for $\mathcal{A}_\beta$. In the final chapters, we delve into Toeplitz determinants in higher dimensions, motivated by the Fekete-Szegö problem. Other coefficient functionals, such as the Zalcman functional and Hankel determinants, can also be studied within this framework.

In the theory of univalent functions, studying coefficient and radius problems provides not only profound insights into the geometric properties of analytic functions but also valuable applications across diverse fields. Recently, the estimation of Taylor series coefficients for analytic functions has gained significant attention in digital image processing (DIP). A key technique in DIP for enhancing both the visual appeal and diagnostic utility of images is power law transformation. In recent years, there has been a notable increase in the use of power law transformation and Taylor series coefficient estimates, particularly for functions belonging to the Sakaguchi classes [128, 148]. Integrating power law transformations with coefficient estimates has proven effective for addressing complex image processing tasks such as contrast enhancement, spatial filtering, and image segmentation. These techniques are applicable to a wide range of image sources, including digital cameras, scanners, and medical imaging equipment.

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List of Publications

1. Surya Giri, and S. Sivaprasad Kumar, Hermitian-Toeplitz determinants for certain univalent functions.Anal. Math. Phys.13 (2023), no.2, Paper No. 37, 19 pp. https://doi.org/10.1007/s13324-023-00800-2 (**SCIE, I.F. 1.4**).

2. Surya Giri, and S. Sivaprasad Kumar, Toeplitz Determinants for a Class of Holomorphic Mappings in Higher Dimensions, Complex Anal. Oper. Theory 17 (2023), no. 6, Paper No. 86, 16 pp. https://doi.org/10.1007/s11785-023-01394-0 (**SCIE, I.F. 0.7**).

3. Surya Giri, and S. Sivaprasad Kumar, Toeplitz determinants for a subclass of quasi convex mappings in higher dimensions, J. Anal. 32 (2024), no. 4, 2099-2112. https://doi.org/10.1007/s41478-023-00625-z (**ESCI, I.F. 0.7**).

4. Surya Giri, and S. Sivaprasad Kumar, Sharp bounds of fifth coefficient and Hermitian-Toeplitz determinants for Sakaguchi classes. Bull. Korean Math. Soc. 61 (2024), no. 2, 317-333. https://doi.org/10.4134/BKMS.b230018 (**SCIE, I.F. 0.5**).

5. Surya Giri, and S. Sivaprasad Kumar, Coefficient Functional and Bohr-Rogosinski Phenomenon for Analytic functions involving Semigroup Generators, Rocky Mountain journal of mathematics (Accepted). (**SCIE, I.F. 0.7**).

6. Surya Giri, and S. Sivaprasad Kumar, Toeplitz-determinants in one and higher dimensions, Acta Mathematica Scientia (Accepted) (**SCIE, I.F. 1.1**).

7. Surya Giri, and S. Sivaprasad Kumar, Fifth Coefficient Estimate for Certain Starlike Functions, arXiv preprint arXiv:2201.05803 (Communicated).

8. Surya Giri, and S. Sivaprasad Kumar, Radius and Convolution problems of analytic functions involving Semigroup Generators, arXiv preprint arXiv:2205.10777 (Communicated).

9. Surya Giri, and S. Sivaprasad Kumar, Toeplitz determinants on bounded starlike circular domain in $\mathbb{C}^n$, arXiv preprint arXiv:2211.14532 (Communicated).

10. Surya Giri, and S. Sivaprasad Kumar, Toeplitz determinants of Logarithmic coefficients for Starlike and Convex functions, arXiv preprint arXiv:2303.14712 (Communicated).

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Hermitian–Toeplitz determinants for certain univalent functions

Surya Giri[1] · S. Sivaprasad Kumar[1]

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Abstract
Sharp upper and lower bounds for the second and third order Hermitian–Toeplitz determinants are obtained for some general subclasses of starlike and convex functions. Applications of these results are also discussed for several widely known classes.

Keywords Univalent functions · Starlike functions · Close-to-convex functions · Hermitian–Toeplitz determinant

Mathematics Subject Classification 30C45 · 30C50 · 30C80

1 Introduction

Let $\mathcal{A}$ be the class of functions of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, $(a_2 \neq 0)$, which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}$ be the subclass of $\mathcal{A}$, consisting of univalent functions. The subclasses of $\mathcal{S}$ of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{C}$ respectively. A function $f \in \mathcal{S}^*$ if and only if $\operatorname{Re}\big(zf'(z)/f(z)\big) > 0$, $z \in \mathbb{D}$. Also, a function $f \in \mathcal{C}$ if and only if $\operatorname{Re}\big(1 + zf''(z)/f'(z)\big) > 0$, $z \in \mathbb{D}$. A function $f \in \mathcal{A}$ is said to be close-to-convex [11] if there is a function $g \in \mathcal{S}^*$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{1}$$

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Complex Analysis and Operator Theory

Toeplitz Determinants for a Class of Holomorphic Mappings in Higher Dimensions

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Abstract

In this paper, we establish the sharp bounds of certain Toeplitz determinants formed over the coefficients of holomorphic mappings from a class defined on the unit ball of a complex Banach space and on the unit polydisc in $\mathbb{C}^n$. Derived bounds provide certain new results for the subclasses of normalized univalent functions and extend some known results in higher dimensions.

Keywords Quasi-convex mappings · Toeplitz determinants · Coefficient inequalities

Mathematics Subject Classification 32H02 · 30C45

1 Introduction

Let $\mathcal{S}$ be the class of analytic univalent functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ having the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $\mathcal{K}(\alpha) \subset \mathcal{S}$ denote the class of convex functions of order α, $0 \leq \alpha < 1$. A function $g \in \mathcal{K}(\alpha)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > \alpha, \quad z \in \mathbb{U}.$$

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ORIGINAL RESEARCH PAPER

Toeplitz determinants for a subclass of quasi convex mappings in higher dimensions

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Abstract
In this paper, we establish the sharp bounds of certain Toeplitz determinants formed over the coefficients of a normalized convex function $g(z)$ defined on the unit disk $\mathbb{U}$ such that $z = 0$ is a zero of order $k+1$ of $g(z) - z$. Furthermore, these results are extended to higher dimensions by determining the bounds of Toeplitz determinants for a subclass of quasi convex mappings of type B.

Keywords Holomorphic mapping · Toeplitz determinant · Coefficient inequality

Mathematics Subject Classification 32H02 · 30C45

1 Introduction

Let $\mathcal{S}$ be the class of analytic univalent functions g defined on the unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ with Taylor series

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{1}$$

The classes of starlike and convex functions, denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively, are the well known subclasses of $\mathcal{S}$.

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SHARP BOUNDS OF FIFTH COEFFICIENT AND HERMITIAN-TOEPLITZ DETERMINANTS FOR SAKAGUCHI CLASSES

Surya Giri and S. Sivaprasad Kumar

Abstract. For the classes of analytic functions f defined on the unit disk satisfying

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \varphi(z) \quad \text{and} \quad \frac{(2zf'(z))'}{(f(z)-f(-z))'} \prec \varphi(z),$$

denoted by $\mathcal{S}_s^*(\varphi)$ and $\mathcal{C}_s(\varphi)$, respectively, the sharp bound of the n^{th} Taylor coefficients are known for $n = 2$, 3 and 4. In this paper, we obtain the sharp bound of the fifth coefficient. Additionally, the sharp lower and upper estimates of the third order Hermitian Toeplitz determinant for the functions belonging to these classes are determined. The applications of our results lead to the establishment of certain new and previously known results.

1. Introduction

Let $\mathcal{H}$ be the class of holomorphic functions in the unit disk $\mathbb{D}$ and $A \subset \mathcal{H}$ represent the class of functions f satisfying $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S} \subset \mathcal{A}$ be the class of univalent functions. A function $f \in \mathcal{H}$ is said to be starlike with respect to symmetric point if for r less than and sufficiently close to 1 and every z_0 on $|z| = r$, the angular velocity of $f(z)$ about the point $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction. Sakaguchi [20] showed that a function $f \in \mathcal{A}$ is starlike with respect to symmetrical point if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)-f(-z)} > 0.$$

The class of all such functions is denoted by $\mathcal{S}_s^*$. It is noted that the class of functions univalent and starlike with respect to symmetric points includes the classes of convex functions and odd functions starlike with respect to the origin

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Dear Prof. S. Sivaprasad Kumar,

I am pleased to inform you that your article

Coefficient Functionals and Bohr-Rogosinski Phenomenon for Analytic functions involving Semigroup Generators
by Surya Giri and S. Sivaprasad Kumar

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Objective & Research Area

I aspire to a challenging and rewarding career in teaching and research, where I can nurture and enhance students and my own skills to their fullest potential. My research topic is Geometric function theory in Complex Analysis.

Education

July 2024 — **Ph.D.** ***(Thesis submitted)***
Delhi Technological University, New Delhi
Thesis: *Subordination, Inequalities and Radius Constants of Certain Analytic Functions*

Mar 2023 — **Upgraded JRF to SRF**

Jan 2020 — **Admitted in Ph.D.**
Delhi Technological University, New Delhi

Aug 2019 — **CSIR-UGC JRF (NET)**
AIR 75

Mar 2019 — **GATE**
AIR 108

Aug 2015 - Jul 2017 — **M.Sc.**
Gurukula Kangri Vishwavidyalaya, Haridwar
CGPA: 8.65 *out of* 10.0, Subject: *Mathematics*

Jun 2012 - Aug 2015 — **B.Sc.**
Choudhary Charan Singh University, Meerut
Subject: *Physics, Chemistry, Mathematics*

Research and Research Experience

Jan 2020 – July 2024 — **Delhi Technological University, New Delhi**

- Published **Six** peer-reviewed articles including **five SCIE** and **one ESCI** journal.
- Served as a reviewer for certain journals.
- Participated and Presented papers in international conferences showcasing the significance of the research to a broader audience.
- Taught **Mathematics II and III** courses as part of the B.Tech. undergraduate program.
- Taught and engaged the labs of programming in C/C++, Mathematics & Computing using MATLAB and Probability & Statistics using SPSS.

Software Skills

Latex, Programming in C/C++, Wolfram Mathematica, MATLAB, SPSS, MS Office.

Research Papers (Published & Communicated)

1. **Surya Giri** and S. Sivaprasad Kumar, Hermitian–Toeplitz determinants for certain univalent functions, Analysis and Mathematical Physics 13, no. 2 (2023): 37. https://doi.org/10.1007/s13324-023-00800-2 **(SCIE, I.F. 1.4) (shortlisted for 'Commendable Research Award' by DTU)**

2. **Surya Giri** and S. Sivaprasad Kumar, Toeplitz Determinants for a Class of Holomorphic Mappings in Higher Dimensions, Complex Analysis and Operator Theory 17, no. 6 (2023): 86. https://doi.org/10.1007/s11785-023-01394-0 **(SCIE, I.F. 0.7)**
3. **Surya Giri** and S. Sivaprasad Kumar, Toeplitz determinants for a subclass of quasi convex mappings in higher dimensions, The Journal of Analysis (2023): 1-14. https://doi.org/10.1007/s41478-023-00625-z **(ESCI, I.F. 0.7)**
4. **Surya Giri** and S. Sivaprasad Kumar, Coefficient Functional and Bohr-Rogosinski Phenomenon for Analytic functions involving Semigroup Generators, Rocky Mountain Journal of Mathematics **(Accepted, SCIE, I.F. 0.8)**
5. **Surya Giri** and S. Sivaprasad Kumar, Sharp bounds of fifth coefficient and Hermitian-Toeplitz determinants for Sakaguchi classes, Bulletin of Korean Mathematical Society **(Accepted, SCIE, I.F. 0.5)**
6. **Surya Giri** and S. Sivaprasad Kumar, Toeplitz determinants in one and higher dimensions, Acta Mathematics Scientia, 2024 **(Accepted, SCIE, I.F. 1.1)**.
7. **Surya Giri** and S. Sivaprasad Kumar, Toeplitz determinants on bounded starlike circular domain in $\mathbb{C}^n$, ArXiv preprint arXiv:2211.14532 (2022).
8. **Surya Giri** and S. Sivaprasad Kumar, Radius and Convolution problems of analytic functions involving Semigroup Generators, ArXiv preprint arXiv:2205.10777 (2022).
9. **Surya Giri** and S. Sivaprasad Kumar, Toeplitz determinants of Logarithmic coefficients for Starlike and Convex functions, ArXiv preprint arXiv:2303.14712 (2023).
10. **Surya Giri** and S. Sivaprasad Kumar, Fifth Coefficient Estimate for Certain Starlike Functions, ArXiv preprint arXiv:2201.05803 (2022).

International Conferences & Workshops

1. **Presented** a paper entitled 'Certain Geometrical Properties of Semigroup Generators' in International Virtual Conference on 'Recent Trends in Applied Mathematics (ICRTAM – 2022)' organized by Department of Science and Humanities (Mathematics), Sri Ramakrishna Institute of Technology, Pachapalayam, Coimbatore held on 7th October, 2022.
2. **Presented** a paper entitled 'Hermitian-Toeplitz determinants for certain univalent functions' in the International Conference on 'Evolution in Pure and Applied Mathematics' organized by Department of Mathematics, Akal University, Talwandi Sabo, Bathinda, Punjab (India) during November 16-18, 2022.
3. **Presented** a paper entitled 'Toeplitz Determinants for a Subclass of Quasi Convex Mappings in Higher Dimensions' in the International Conference on 'Mathematical Analysis and Applications' organized by Department of Mathematics, National Institute of Technology, Tiruchirappalli, Tamil Nadu, INDIA during December 15-17, 2022.
4. **Participated** in International Conference on 'Emerging Trends in Pure and Applied Mathematics' organized by Department of Applied Science, School of Engineering in association with Department of Mathematical Sciences, School of Sciences, Tezpur University, Napaam, Sonitpur, Assam held on March 12-13, 2022.
5. **Participated** in the **2 days hands-on workshop on 'Maple 2022 & its applications**' organized by Binary Semantics Ltd. at the Department of Applied Mathematics, DTU on May 26-27, 2022.

6. **Participated** in the **One-week Training Programme on 'Maple, SPSS and Statcraft'** conducted from 12th to 16th February 2024 organized at Delhi Technological University.

Awards

The article *'**Surya Giri** and S. Sivaprasad Kumar, Hermitian–Toeplitz determinants for certain univalent functions, Analysis and Mathematical Physics 13, no. 2 (2023): 37'* has been shortlisted for **'Commendable Research Award'** by DTU, Delhi.

Websites

- https://www.researchgate.net/profile/Surya-Giri-3
- https://scholar.google.com/citations?user=ZmzsU4gAAAAJ&hl=en

Other Information

Nationality: Indian
DOB: 08/09/1995
Languages: English (professional proficiency), and Hindi (native proficiency)
Marital Status: Single
Hobbies: Reading Books, Badminton, Volleyball, Table Tennis

References

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