OPTIMIZATION AND UNCERTAINTY IN FUZZY DECISION MAKING

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for the Degree of

DOCTOR OF PHILOSOPHY

In

Mathematics

by

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CANDIDATE'S DECLARATION

I, Vineet Kumar (2K19/PHDAM/02), a Ph.D. student in the Department of Applied Mathematics, hereby declare that the thesis entitled "Optimization and Uncertainty in Fuzzy Decision Making" which is submitted by me to the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree of "Doctor of Philosophy in Mathematics", is original and not copied from any source without proper citation. This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship, or other similar title or recognition.

Place: Delhi **(Vineet Kumar)**

Date: 2K19/PHDAM/02

<u>CERTIFICATE</u>

Certified that **Mr. Vineet Kumar** (2K19/PHDAM/02), has carried out his research work presented in this thesis entitled **"Optimization and Uncertainty in Fuzzy Decision Making"** for the award of **Doctor of Philosophy in Mathematics** from Delhi Technological University, New Delhi, under our supervision. The thesis embodies the results of the original work and studies carried out by the student himself and the contents of the thesis do not form the basis for the award of any other degree to the candidate or anybody else from this or any other University/Institution.

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Abstract

Decision-making unfolds as a non-deterministic process, generating conflicting situations in various facets of real-life scenarios, thereby giving rise to uncertainty and imprecision. It's commonplace to grapple with vaguely defined data in real-world contexts, necessitating the utilization of mathematical frameworks such as fuzzy sets and fuzzy matrices to navigate through these conflicting events. The imprecision inherent in decision-making problems can manifest in multiple forms, notably fuzziness, which is effectively addressed through the application of fuzzy set theory and fuzzy matrix theory within decision theory. Fuzzy matrices are particularly adept at scrutinizing and resolving real-world issues represented by matrices with varying degrees of vagueness. Conventional or fuzzy quantitative methods often fall short of representing the inherently ambiguous nature of human activities and decisions. Employing a fuzzy approach becomes imperative in such scenarios, engendering the need for computational methods involving uncertainty. Various computation models, stemming from the optimization and uncertainty model, have been introduced in the literature to contend with this approach. Decision-making is intricately linked with strategic information exchange through language, albeit in a manner characterized by rigor and stylization. Despite the efforts towards formalization, uncertainty remains entrenched in the conceptual tools employed in optimization and uncertainty theories, posing a fundamental challenge. Our study delves into addressing linear programming problems and matrices imbued with imprecise information, striving to augment their practical utility. This endeavor contributes to enhancing the process of decision analysis within an uncertain qualitative milieu. The thesis, titled "Optimization and Uncertainty in Fuzzy Decision-Making Problems", comprises six chapters, followed by a summary and delineation of future research directions. Motivated by existing computational models designed to mitigate uncertainty in decision-making, our work extensively scrutinizes the literature. The primary objective is to tackle the prevailing vagueness and imprecision inherent in decision-making problems and matrix quandaries, particularly in the context of fuzzy variables. The thesis concludes with a comprehensive bibliography and a list of publications,

underscoring the breadth and depth of exploration in the field.

The introductory **Chapter 1** presents an overview of the fuzzy sets, fuzzy matrix, imprecise models, and their elementary applications anticipated in the transportation problem (TP) and assignment problem (AP), followed by their implementation in the distinct decision-making problems. Further, this chapter discusses the notion of fuzziness involved in qualitative concept of matrix. Some basic concepts used throughout the thesis have been defined along with the motivation of the research work. Thus, the current chapter creates a background for this thesis's work and motivates the work carried out in this thesis.

The **Chapter 2** entitled, "Transportation problem under interval-valued Pythagorean fuzzy and spherical fuzzy environment" establishes the basis for a theory of TPs. In literature, the TPs with Pythagorean fuzzy and picture fuzzy models are considered and solved. However, the theory of TPs having interval-valued Pythagorean fuzzy sets (IVPyFS) and spherical fuzzy set (SFS) is pristine and yet to be explored. The use of IVPyFS and SFS to represent practical transportation situations has shown to be a powerful approach. The chapter is based on two research papers entitled, "Solution of transportation problem using interval-valued Pythagorean fuzzy approach", published in **Advanced Engineering Optimization Through Intelligent Techniques: Select Proceedings of AEOTIT 2022 (pp. 359-368)**, **Springer, 10 (1), 2199368 (2023)** and "Solution of transportation problem under spherical fuzzy set", published in **2021 IEEE 6th International Conference on Computing, Communication and Automation (ICCCA) (pp. 444-448). , IEEE**.

The chapter mentioned above constitutes a transportation problem with IVPyFS and SFS belonging to an uncertain parameter set where all plausible imprecise descriptors provided by experts have a symmetric and uniform distribution. In practical life decision problems, the experts may prefer another special type of TP model called the "Assignment Problem" model. Several computational models are established in literature to deal with assignment problems with imprecise parameters like cost, condition, road condition, etc. Therefore, in **Chapter 3** entitled, "A novel similarity measure and score function of Pythagorean fuzzy sets and their application in assignment problem," we propose a newly constructed methodology to handle assignment problems with uncertain parameters. To handle the uncertainty in practical applications of assignment problems (AP), a method for solving the Pythagorean fuzzy assignment problem (PyFAP) has been proposed using a similarity measure and a proposed score function. Numerical examples are given to explain the methodology. Hence, this chapter also discusses and solves the decision matrix of AP with uncertain parameters. The chapter is based on the research paper titled, “A novel similarity measure and score function of Pythagorean fuzzy sets and their application in assignment problem”, published in **Economic Computation and Economic Cybernetics Studies and Research**, (SCIE, Impact Factor: 0.9).

Chapter 4 entitled, “Interval-valued picture fuzzy matrix: basic properties and application” proposes a novel concept of the matrix based on interval-valued picture fuzzy sets. Further, based on the defined concept, the several key definitions and theorems for the interval-valued picture fuzzy matrix (IVPFM) and present a procedure for calculating its determinant and adjoint. Using composition functions, a new algorithms to identify the greatest and least eigenvalue for the defined problem is developed. The proposed approach can be perceived as a convenient technique for multiple criteria decision-making (MCDM) problems by the proposed distance measure. The chapter is based on a research paper titled, “Interval-valued picture fuzzy matrix: basic properties and application,” published in **Soft Computing**, Springer (SCIE, Impact Factor: 3.1).

In **Chapter 5** entitled, “Interval-valued spherical fuzzy matrix and its applications in multi-attribute decision-making process”, the concept of matrix with interval-valued spherical fuzzy concept is proposed in which each row of the matrix may correspond to an element, while each column represents a different dimension or attribute of membership, neutrality and non-membership degree in interval number instead to a single point of a real number. The theory of the interval-valued spherical fuzzy matrix (IVSFM) represents more flexibly uncertain and vague information. In this context, we establish significant definitions and theorems about the given matrices. Further, introduces the methodology for determining the determinant and adjoint of IVSFM. Finally, proposes a new score function for the interval-valued spherical fuzzy sets and prove its validity with the help of basic properties and the application of the decision-making problems for a career placement assessment. The chapter is based on a research paper titled, “Interval-valued spherical fuzzy matrix and its applications in the multi-attribute decision-making process,” published in **Maejo International Journal of Science and Technology.**,(SCIE, Impact Factor: 0.8).

Chapter 6 entitled, “Interval-valued fermatean fuzzy matrix and its application” presents an Interval-valued fermatean fuzzy matrix in which the membership and non-membership degrees of the fermatean fuzzy matrix in continuous form (interval number). The methodology presented in this chapter manipulates imprecise and uncertain information in decision-making in interval-valued fermatean fuzzy set theory. The chapter is based on the research paper titled, “Interval-valued fermatean fuzzy matrix and its application” (communicated in

“Cybernetics and Systems").

Chapter 6 is followed by the summary of the research work carried out in this thesis. In addition, the future scope of the thesis has been discussed briefly.

Finally, the thesis ends with the bibliography and list of publications.

List of Publications

1. **Vineet Kumar**, Anjana Gupta, H.C.Taneja; *Interval valued picture fuzzy matrix: basic properties and application*,Soft Computing, Springer, **25**, (2023) (**SCIE, Impact Factor: 3.1**).

2. **Vineet Kumar**, Anjana Gupta, H.C.Taneja; *A novel similarity measure and score function of Pythagorean fuzzy sets and their application in assignment problem*, Economic Computation and Economic Cybernetics Studies and Research, Vol. 3, (2023). DOI: https://doi.org/10.24818/18423264/57.3.23.19.(**SCIE, Impact Factor: 0.9**).

3. **Vineet Kumar**, Anjana Gupta, H.C.Taneja; *Solution of transportation problem using interval-valued Pythagorean fuzzy approach*, In Advanced Engineering Optimization Through Intelligent Techniques: Select Proceedings of AEOTIT 2022 (pp. 359-368), **vol 10 (1)**. Springer, 2199368 (2023), DOI: https://link.springer.com/chapter/10.1007/978-981-19-9285-8-33, (**Conference Proceedings Citation Index (CPCI)**).

4. **Vineet Kumar**, Anjana Gupta, H.C.Taneja; *Solution of Transportation Problem Under Spherical Fuzzy Set*, In 2021 IEEE 6th International Conference on Computing, Communication and Automation (ICCCA)(pp. 444-448), **vol 10 (1)**. IEEE, (2023), DOI: 10.1109/ICCCA52192.2021.9666372, (**Conference Proceedings Citation Index (CPCI)**).

5. **Vineet Kumar**, Anjana Gupta, H.C.Taneja; *Interval-valued spherical fuzzy matrix and its applications in multi-attribute decision-making process*, Maejo International Journal of Science and Technology, **17(3)**, (2023) (**SCIE, Impact Factor: 0.8**).

6. **Vineet Kumar**, Anjana Gupta, H.C.Taneja; *Interval- valued fermatean fuzzy matrix and its application* (**Communicated** in “Cybernetics and Systems, Taylor & Francis").

Papers Presented in International Conferences

- Presented a research paper entitled "**Solution of Transportation Problem using Interval-Valued Pythagorean Fuzzy Approach** " on *11th International Conference on Engineering and Natural Science* held at ISPEC Institute, MUS, TURKEY, during September 18-19, 2021.

- Presented a research paper entitled "**Solution of Pythagorean Fuzzy Linear Programming Problem by using similarity Measure** " in *Recent Advances in Mathematics Computational Optimization (RAMCO-2021)* held online by School of Computational and Integrative Science Jawaharlal Nehru University & International Academy of Physical Science, Delhi during October 26-28, 2021.

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List of Abbreviations

- DM: Decision-maker
- MCDM: Multi- criterion decision-making
- FS: Fuzzy set
- IVFS: Interval-valued fuzzy set
- IFS: Intuitionistic fuzzy set
- IVIFS: Interval-valued intuitionistic fuzzy set
- PyFS: Pythagorean fuzzy set
- IVPyFS: Interval-valued Pythagorean fuzzy set
- FFS: Fermatean fuzzy set
- IVFFS: Interval-valued fermatean fuzzy set
- PFS: Picture fuzzy set
- IVPFS: Interval-valued picture fuzzy set
- SFS: Spherical fuzzy set
- IVSFS: Interval-valued spherical fuzzy set
- FM: Fuzzy matrix
- IVFM: Interval-valued fuzzy matrix
- IFM: Intuitionistic fuzzy matrix

- IVIFM: Interval-valued intuitionistic fuzzy matrix
- PyFM: Pythagorean fuzzy matrix
- FFM: Fermatean fuzzy matrix
- IVFFM: Interval-valued fermatean fuzzy matrix
- PFM: Picture fuzzy matrix
- IVPFM: Interval-valued picture fuzzy matrix
- SFM: Spherical fuzzy matrix
- IVSFM: Interval-valued spherical fuzzy matrix
- μ : Membership degree
- η: Neutrality degree
- ν: Non-membership degree
- π: Hesitancy degree
- GEFS : Greatest eigen fuzzy set
- LEFS: Least eigen fuzzy set
- TP: Transportation problem
- AP: Assignment problem
- MCDA: Multi-criterion decision analysis
- SM: Similarity measure
- MD: Membership degree
- NMD: Non-membership degree
- HD: Hesitancy degree
- FTP: Fuzzy transportation problem
- IFTP: Intuitionistic fuzzy transportation problem
- IVIFTP: Interval-valued intuitionistic fuzzy transportation problem

- PyFTP: Pythagorean fuzzy transportation problem
- IVPyFTP: Interval-valued Pythagorean fuzzy transportation problem
- PFTP: Picture fuzzy transportation problem
- IVPFTP: Interval-valued picture fuzzy transportation problem
- SFTP: Spherical fuzzy transportation problem
- IVSFTP: Interval-valued spherical fuzzy transportation problem
- FTP: Fuzzy assignment problem
- IFTP: Intuitionistic fuzzy assignment problem
- IVIFTP: Interval-valued intuitionistic fuzzy assignment problem
- PyFTP: Pythagorean fuzzy assignment problem
- LPP: Linear programming problem
- PyFN: Pythagorean fuzzy number
- IVPyFN: Interval-valued Pythagorean fuzzy number
- FFN: Fermatean fuzzy number
- IVFFN: Interval-valued fermatean fuzzy number
- PFN: Picture fuzzy number
- IVPFN: Interval-valued picture fuzzy number
- SFN: Spherical fuzzy number
- IVSFN: Interval-valued spherical fuzzy number
- EIVPFS: Eigen interval-valued picture fuzzy set
- GEIVPFS: Greatest eigen interval-valued picture fuzzy set
- LEIVPFS: Least eigen interval-valued picture fuzzy set
- EIVFFS: Eigen interval-valued fermatean fuzzy set
- GEIVFFS: Greatest eigen interval-valued fermatean fuzzy set

- LEIVFFS: Greatest eigen interval-valued fermatean fuzzy set
- $|\tilde{A}|$: Determinant of matrix
- IVPFR: Interval-valued picture fuzzy relation
- IVFFR: Interval-valued fermatean fuzzy relation
- IVMD: Interval-valued membership degree
- IVNMD: Interval-valued non-membership degree
- IVFFPM: Interval-valued fermatean fuzzy permutation matrix

Chapter 1

Introduction

In today's world, decision-making is becoming more and more important, even with the variety of advanced technical tools available to assist with it. Nonetheless, there are situations in which technology cannot make decisions on its own without taking into account the subjective judgment of humans. It is the responsibility of those with clear perceptions to use good decision-making techniques to arrive at compromises. The development of multi-criteria decision-making theory in the early 1970s marked a critical turning point in decision-making methodology. This theory laid the groundwork for more rational and systematic approaches to decision-making, particularly when there are several criteria at play. This decision theory was quickly adopted and recognized, leading to its integration into what is now widely recognized as MCDM. To account for uncertainties, the theory was further refined, incorporating ideas from other theories like fuzzy sets theory and expanding upon it in LPP. This chapter's discussion also looks at how fuzzy matrices are developed and used, which is important when dealing with problems involving decision-making. Therefore, the basic ideas presented in this chapter offer a strong foundation upon which to build and direct the research projects undertaken for the thesis. They provide priceless insights into the complexity of decision-making procedures, emphasizing how important it is to balance human judgment with technical progress to achieve the best results.

Decision-making is a multidisciplinary notion that refers to the process of choosing the best course of action from a variety of options in order to accomplish a particular goal. The first thorough analysis of decision-making theory was given by Edward [1]. A decision matrix is frequently used in decision analysis to evaluate several options in relation to different criteria. Each cell in the matrix indicates how well an alternative performs in relation to a certain criterion, allowing decision-makers to compare and assess possibilities statistically and make better-informed decisions. Everyday life involves making decisions, from small ones like what to eat for breakfast to big ones like choosing a career path. Nevertheless, decision-making isn't always clear; there are frequently ambiguities, imprecisions, and uncertainties in the information that is at hand. In this situation, "fuzzy decision-making" can act as a link between decision-making and imprecise data. Uncertainty and optimization are essential aspects of fuzzy decision-making. Whether it's maximizing gains or reducing expenses, optimization seeks to identify the optimal course of action. However uncertainty is a common result of imprecise or ambiguous information in real-world circumstances. Similar to choosing the best route for a road trip despite unpredictable driving conditions, optimization is making the greatest choice while taking uncertainty into account. For instance, while buying a new phone, one might take pricing, camera quality, and battery life into account. Fuzzy decision-making aids in selecting the phone that best matches these criteria, even if it doesn't fulfill all requirements perfectly.

As we move forward, fuzzy decision-making is likely to become even more powerful in matrix problems. With advancements in the fuzzy matrix, we use fuzzy matrix logic to make smarter choices in various fields, from self-driving cars that navigate unpredictable traffic to medical diagnoses that consider a range of symptoms. Fuzzy matrix theory isn't just a theory; it's a practical tool that empowers us to tackle the complexities of decision-making in an uncertain world. In summary, fuzzy optimization is a problem-solving technique that combines fuzzy logic with optimization methods to deal with decision-making in situations where the available data or criteria are imprecise, uncertain, or vague. It's a powerful approach for handling complex problems in various fields, from engineering and economics to medicine and environmental management. Fuzzy programming is a more flexible and realistic approach to modeling decision-making in situations where precise information is lacking.

1.1 Fuzzy Sets

The theory of fuzzy set (FS) was presented by Zadeh [2] as a means to handle uncertain or ambiguous information more effectively in real-world scenarios. At the heart of fuzzy sets lies the concept of assigning a membership degree to individual elements of a set, representing the degree to which each element pertains to the set. Fuzzy sets are being used in a wide range of fields, including artificial intelligence, decision-making, control systems, and pattern recognition. The idea of Interval-valued fuzzy set (IVFS) was introduced by Zadeh [3], Grattan-Guiness [6], Jahn [5], and Sambuc [4]. The IVFS is more nuanced than traditional fuzzy sets when dealing with uncertainty and imprecision. By employing intervals instead of a single point value as in traditional fuzzy sets, this addition makes it possible to describe a range of possible values. In contrast to fuzzy sets, which only capture membership degrees (MD) and non-membership degrees (NMD), intuitionistic fuzzy set (IFS) introduce an additional dimension known as the hesitancy degree (HD), quantifying the level of uncertainty or ambiguity associated with a specific element. Atanassov [7] extended the concept of fuzzy sets and introduced the novel notion of intuitionistic fuzzy sets. Since their development, intuitionistic fuzzy sets have undergone extensive research and found widespread application across various domains, including decision-making. Atanassov & Gargov [8] further extended the notion of IFS to the interval-valued intuitionistic fuzzy set (IVIFS) in which intervals numbers are used rather than exact numbers to provide flexibility in defining membership degrees to an element. Yager [15] defined the Pythagorean fuzzy set (PyFS). Yager overcomes the situation when the sum of the membership (μ) and non-membership (ν) degree is greater than one i.e. $\mu+\nu>1$. It is a generalization of IFS, which is better at handling problems with incomplete information compared to IFS. PyFS offers more flexibility, and the key rule is that the sum of the squares of the membership and non-membership values for any element in PyFS must be less than or equal to 1 ($\mu^2+\nu^2\leq 1$). Further, Liang et al. [9] defined interval-valued Pythagorean fuzzy set (IVPyFS) by extended membership and non-membership degree in interval number. The efficiency of PyFS was further improved by Senapati and Yager [18] with the introduction of fermatean fuzzy set (FFS). FFS is derived from PyFS but with a relaxed rule - the sum of the cubes of membership and non-membership degree for any element is must be less than or equal to 1 ($\mu^3+\nu^3\leq 1$). In practical situations where real-life problems have incomplete and unclear information, IVFFS are more suitable than IVPyFS. However, providing precise fermatean fuzzy values for all real-life problems with incomplete information may not always be possible. For instance, if a decision-maker defines the MD of an alternative as [0.6, 0.7] and the NMD as [0.65, 0.75], and the sum of the squares of the upper bounds of these values is greater than 1, it doesn't fit traditional categories. In such cases, Rani & Mishra [19] considered as interval-valued fermatean fuzzy sets (IVFFS) since the sum of the upper bounds is less than or equal to 1.

IVFFS is found to be more capable of handling problems with incomplete and imprecise information compared to IVIFS and IVPyFS. As decision-making becomes more complex, traditional FS and IFS theory may not be sufficient. For example, in expert voting, where outcomes can be support, neutrality, opposition, or abstention, Cuong & Kreinovich [44] proposed an extended fuzzy set- picture fuzzy set (PFS). They used three index, membership degree (μ), neutrality degree (η) and non-membership degree (ν) with the condition $0 \leq \mu + \eta + \nu \leq 1$. In multiple criteria decision-making (MCDM) problems, experts may provide interval numbers instead of specific real numbers due to the complexity and ambiguity of the decision-making environment. Hence, Cuong & Kreinovich [44] proposed interval-valued picture fuzzy set (IVPFS) theory to represent membership, neutrality, non-membership, and abstention with interval numbers, enhancing the credibility of decision-making results. The generalization of PFS is the spherical fuzzy set (SFS) which is introduced by Ashraf et al. [121] with condition $0 \leq \mu + \eta + \nu \leq 1$. Consider μ=0.7, η=0.3 and ν=0.5, then 0.7+0.3+0.5 $>$1 but $(0.7)^2 + (0.3)^2 + (0.5)^2 \leq 1$. By this, we can see SFS is more suitable than PFS. The idea behind SFS is to let decision-makers to generalize other extensions of fuzzy sets by defining a membership function on a spherical surface and independently assigning the parameters of that membership function with a larger domain. Gundogdu & Kahraman [116] extended the SFS into the interval-valued spherical fuzzy set (IVSFS), utilizing interval-valued fuzzy sets to incorporate decision makers' opinions about fuzzy set parameters into the model with an interval instead of a single point.

1.2 Eigen Fuzzy Sets and its Extension

In both the realms of theoretical exploration and practical application within the sciences, the mathematical concept of relations stands as a cornerstone for establishing connections among various entities, states, and events. Fuzzy relations, an extension of binary relations, were first introduced by Zadeh [96], who delineated key concepts such as fuzzy equivalency (similarity) relation and fuzzy ordering, shedding light on their essential characteristics.

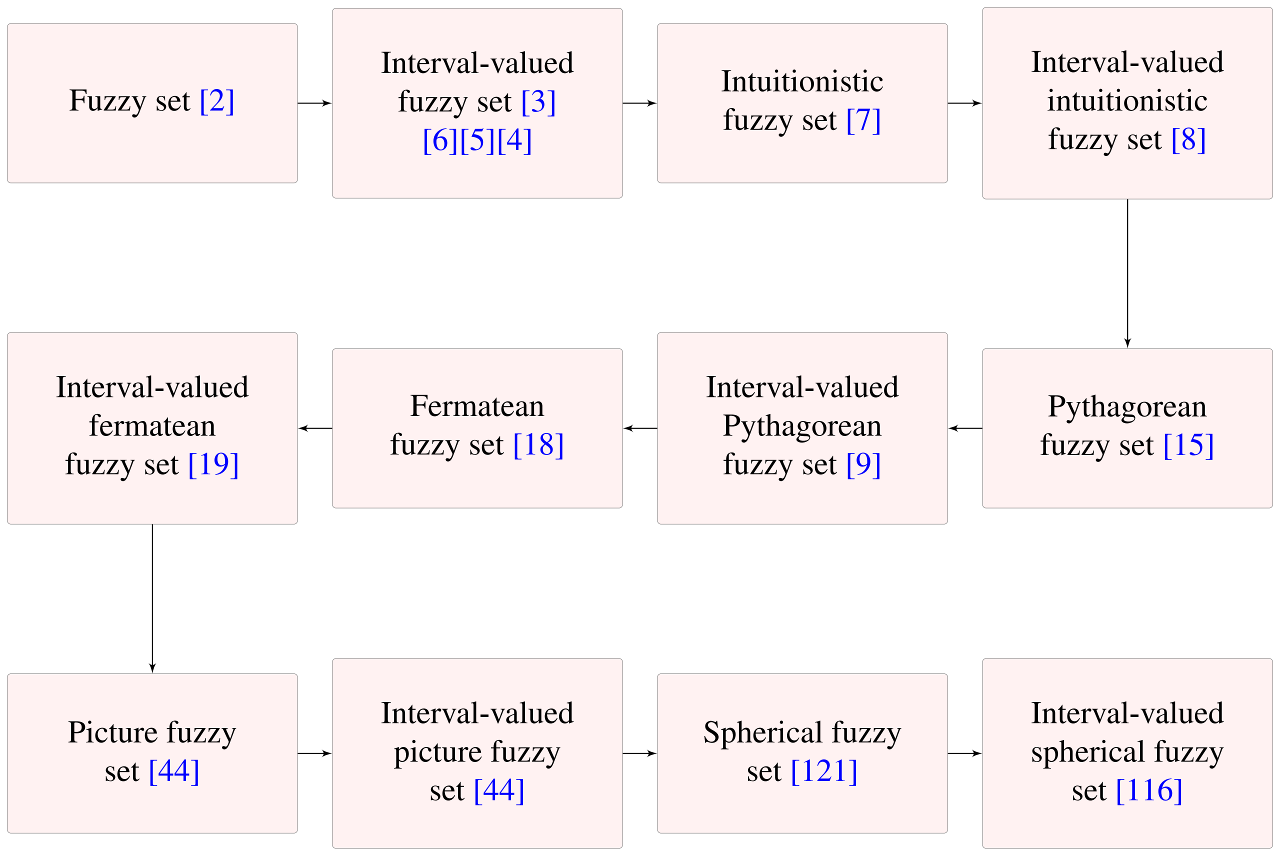

Figure 1.1: Extension of fuzzy sets

In the realm of problem-solving across diverse domains, the significance of eigenvalues and eigenvectors of matrices cannot be overstated. [97] contributed to this area by delving into eigen fuzzy sets, elucidating their importance through the composition of fuzzy relations. Utilizing the max-min composition method, [97] also identified what is known as the greatest eigen fuzzy set (GEFS). Building upon this concept, Martino et al. [48] presented the least eigen fuzzy set, leveraging the min-max composition method. Further advancements came from Goetschel & Voxman [98], who expanded the idea of finding eigen fuzzy sets to eigen fuzzy numbers. Nobuhara & Hirota [99] applied principal component analysis in image processing, defining both the greatest and lowest eigen fuzzy sets. Subsequently, Martino et al. [100] proposed a genetic algorithm for image reconstruction based on fuzzy relations, utilizing both GEFS and the lowest eigen fuzzy set (LEFS) to optimize fitness values. Rakus-Andersson [26] explored the application of fuzzy relations in measuring drug effectiveness levels by establishing connections between potential symptoms, employing both the greatest and lowest eigen fuzzy sets. Additionally, Padde and Murugadas [56] calculated the greatest eigen intuitionistic fuzzy sets, which have found widespread application across various fields. However, it's crucial to consider the concept of neutral membership in refining the notion of greatest eigen intuitionistic fuzzy sets. Addressing this concern, Guleria & Bajajl [62] ventured into the realm of eigen spherical fuzzy sets and their relevance in decision-making scenarios. They proposed two algorithms to determine both the greatest and least eigen spherical fuzzy sets, thereby contributing significantly to the advancement of decision support systems and optimization techniques. Overall, these developments showcase the intricate interplay between mathematical concepts, such as relations, eigenvalues, and fuzzy sets, and their applications across different domains. By bridging theoretical constructs with practical problem-solving methodologies, researchers continue to push the boundaries of knowledge and innovation, enriching our understanding of complex systems and phenomena.

1.3 Fuzzy Matrix

Matrix, a fundamental mathematical concept, hold immense significance across disciplines such as linear algebra, physics, computer science, and economics. A matrix is a rectangular array of numbers, symbols, or expressions arranged in rows and columns. It provides a concise and organized way to represent and manipulate complex data sets or mathematical relationships. Various uncertain and vague data types are involved in real-world situations, which is difficult to express in a classical matrix. To overcome this situation, Thomasom [16] introduced the idea of a fuzzy matrix (FM) in 1977. After that, the concept of an interval-valued fuzzy matrix (IVFM) introduces a nuanced layer of flexibility and expressiveness to the realm of fuzzy matrices. Pal [24] proposed the IVFM theory.The IVFM extends this idea by allowing each matrix element to be represented not as a single value but as an interval, thereby capturing a range of possible degrees of membership IVFM has numerous applications in decision-making like Meenakshi [22] used the IVFM for solving medical diagnosis problems. Mandal & Pal [27] described some methods to find the ranks of IVFM.

In the context of matrices, an intuitionistic fuzzy matrix (IFM) is an arrangement of elements where each entry is associated not only with a degree of membership but also with a degree of non-membership. These parameters capture the essence of uncertainty in a more detailed and expressive way, reflecting the idea that in certain situations. Pal et al. [131] introduced the concept of the IFM. Pal & Khan [132] proposed some important operations on the IFM. Padder [35] worked on max-min operations on the IFM and discussed the convergence of transitive IFM. Further, Khan & Pal [32] extended the notion of the intuitionistic fuzzy matrix to the interval-valued intuitionistic fuzzy matrix (IVIFM) in which MD and NMD are used in interval numbers rather than exact numbers. Silambarasan [40] defined the Hamacher operations of IVIFM and proved some important properties associated with them. Silambarasan and Sriram [30] have introduced the concept of the Pythagorean fuzzy matrix (PyFM) and extended the concept of IFM under the condition that the sum of the square of membership degree and non-membership degree is less than one. They also discussed some basic operations and properties defined on it.

Moreover, Silambarasan [144] introduced the fermatean fuzzy matrices (FFM), which is an extension of PyFM with the condition sum of cube of membership degree and non-membership degree is less than one. Also, proposed some algebraic operations of FFM. The IFMs have been strongly enforced in various areas, yet the concept of neutral membership needs to be considered in IFMs. In this regard, Dogra & Pal [28] proposed the picture fuzzy matrix (PFM) and discussed some of its important aspects. On the theory of PFM, many authors worked on its important concept. The concept of an interval-valued picture fuzzy matrix (IVPFM) is a notable extension that combines the ideas of IVFM and PFM. This hybrid approach introduces a new layer of flexibility and granularity in representing uncertainty and imprecision within a matrix framework. The IVPFM addresses situations where not only is there ambiguity in the membership degrees as in interval-valued fuzzy matrices but the elements themselves are represented in a more visual and descriptive manner through picture fuzzy matrix. Kumar et al. [118] proposed IVPFM and discussed some of its important aspects. The generalization of PFM is the spherical fuzzy matrix (SFM) which is introduced by Silambarasan [139]. Further, the concept of an interval-valued spherical fuzzy matrix (IVSFM) is defined by Kumar et al. [119] and represents a novel and comprehensive approach to handling uncertainty, imprecision, and complexity within a matrix framework. Combining elements from IVFM and SFM, this extension provides a more nuanced representation of relationships, particularly in situations where both interval-based uncertainty and spherical characteristics are pertinent. This is particularly useful in scenarios where relationships exhibit radial patterns or cyclical trends, as seen in various applications such as geographical analysis, circular data modeling, or periodic system dynamics.

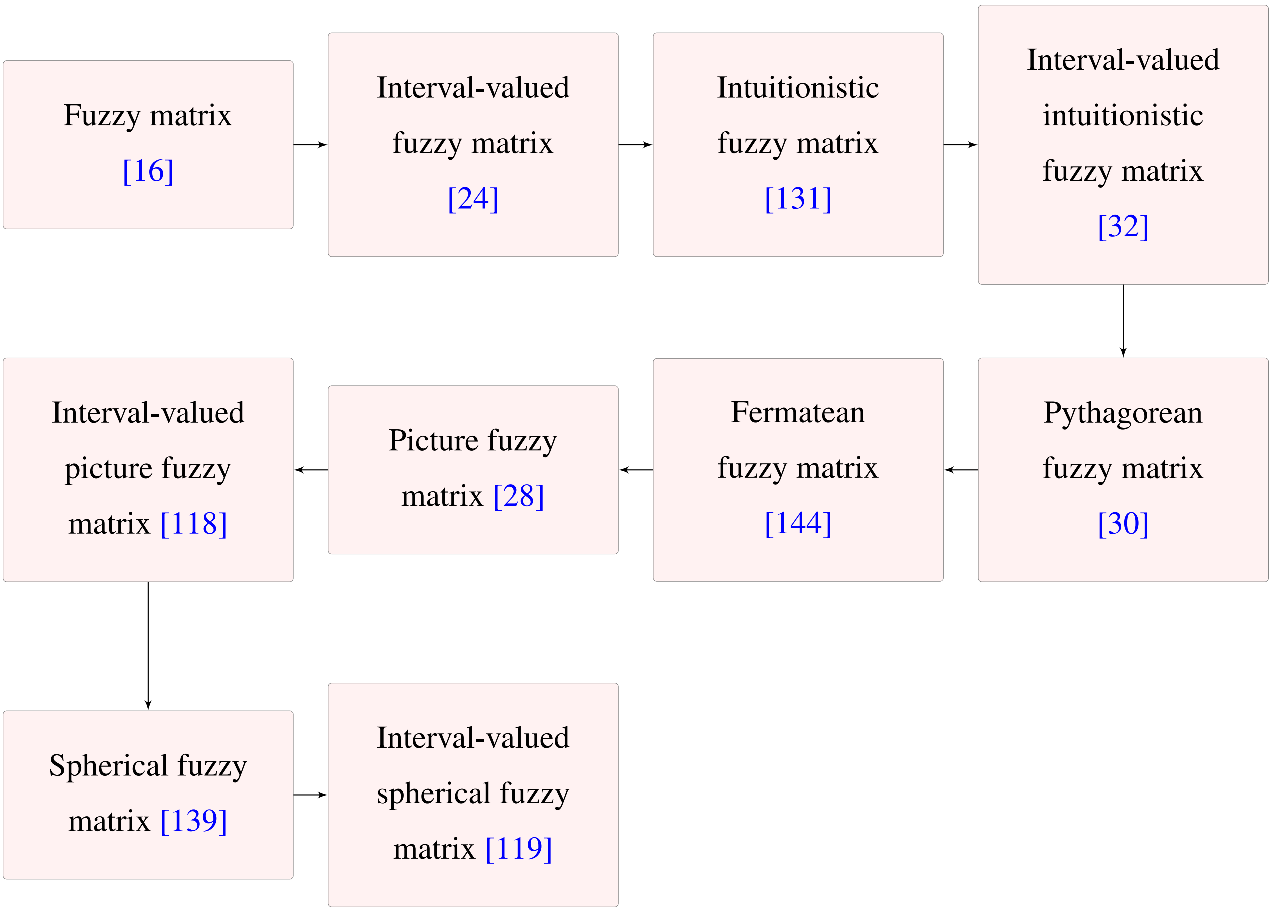

Figure 1.2: Extension of fuzzy matrices

Table 1.1 shows a summary of existing literature based on FM in the chronological order of publication, along with a synopsis of the key contributions followed by their advantages and limitations.

The theories and literature discussed till now are different problems, and depending on the aspects presented by each problem, we can deal with different types of precise numerical values. But in other cases, the problems present complex qualitative aspects to assess utilizing precise values. In the latter case, using the fuzzy matrix approach, Thomasom [16] has provided very good results. Most decision analysis problems involve multiple attributes that exhibit both quantitative and qualitative nature. We can deal with quantitative attributes with different types of precise numerical values. However, it is not proper to represent qualitative aspects that often involve vague, imprecise, and uncertain information similarly. In such cases, experts often use fuzzy matrix descriptors to express their assessments regarding their uncertain knowledge of the problem.

The next section discusses the preliminaries of the existing literature.

Table 1.1: Summary of literature based on fuzzy matrices

Title	Year	Paper	Description	Advantages/Limitations
Convergence of powers of a fuzzy matrix.	1977	[16]	The notion of fuzzy sets is introduced. A heuristical discussion of fuzzy matrices and their relationship to classical matrices is presented.	• The intuitive meaning of fuzzy matrix is defined. • Usual properties of fuzzy matrix are provided. • Limited to choice of membership functions lies in [0,1].
Generalized fuzzy matrices.	1980	[20]	A systematic development of fuzzy matrix theory is discussed.	• Usual basic results of the fuzzy matrix are provided.
The canonical form of a transitive fuzzy matrix	1983	[227]	Discussed the canonical form of a transitive fuzzy matrix and decomposed a transitive matrix into the sum of a nilpotent matrix and a symmetric matrix.	• The intuitive meaning of transitive fuzzy matrix is defined. • Some basic results of transitive fuzzy matrix are provided
The determinant and adjoint of a square fuzzy matrix	1994	[129]	The determinant and adjoint of a square fuzzy matrix are discussed. Also, the circular fuzzy matrices and some properties of a square fuzzy matrix are carried over to the adjoint of the matrix.	• defined the circular fuzzy matrices and showed that some properties of a square fuzzy matrix
On the min-max composition of fuzzy matrices	1995	[21]	Study and prove some properties of the min-max composition of fuzzy matrices.	• How to construct an idempotent fuzzy matrix.
Two new operators on fuzzy matrices	2004	[228]	Two new binary fuzzy operators $\oplus$ and $\odot$ are introduced for fuzzy matrices. Several properties on $\oplus$ are presented here.	• Two new binary fuzzy operators $\oplus$ and $\odot$ are introduced for fuzzy matrices. • Restricted to fuzzy operators $\oplus$ and $\odot$.
The period of powers of a fuzzy matrix	2000	[229]	The period of the powers of a general fuzzy matrix by a graph theoretical viewpoint, and show conditions for convergence under the max–min composition.	• Useful for determining the period of fuzzy matrix. • Restricted to fuzzy matrix.
Note on Convergence of powers of a fuzzy matrix.	2001	[230]	Quoted in Imai et al., Fuzzy Sets and Systems 109 (2000) 405, about the convergence of fuzzy matrices is false.	• The limitations of fuzzy logic, such as the lack of a universally accepted definition or standardization, may also affect the applicability of the convergence concept.
Triangular fuzzy matrices	2007	[17]	Introduced triangular fuzzy matrices (TFMs) and Some elementary operations on triangular fuzzy defined numbers (TFNs) are defined.	• The intuitive meaning of a triangular fuzzy matrix is defined. • Some basic results of a triangular fuzzy matrix are provided. • Limited to a triangular fuzzy matrix.
Intuitionistic fuzzy matrices	2002	[20]	An intuitionistic fuzzy matrix generalizes the concept of an intuitionistic fuzzy set to a matrix format. Instead of dealing with individual elements, an intuitionistic fuzzy matrix contains IFS values organized in rows and columns.	• Limited Adoption and Standardization. • Restricted to the intuitionistic fuzzy matrix.
Some operations on intuitionistic fuzzy matrices	2002	[131]	An operation on intuitionistic fuzzy matrices with fuzzy parameters is introduced and it is shown that it is important for handling the MCDM problems.	• The important operations with fuzzy parameters are introduced. • Improved Handling of Dynamic Systems: • Restricted to the intuitionistic fuzzy matrix.
Interval-valued intuitionistic fuzzy matrices	2005	[32]	This study, explores the interval-valued intuitionistic fuzzy matrices (IVIFMs), introducing essential operators and defining the interval-valued intuitionistic fuzzy determinant (IVIFD). A real-life problem involving IVIFMs is presented, and certain operators are interpreted using this example.	• Enhanced representation of uncertainty • The combination of interval values and intuitionistic fuzzy set parameters offers flexibility in modeling ambiguity and imprecision. • Limited to interval-valued intuitionistic fuzzy matrices.
Some operations on intuitionistic fuzzy matrices.	2006	[132]	Define the Hamacher scalar multiplication and Hamacher exponentiation operations on Intuitionistic fuzzy matrices	• Computation complexity is reduced in fuzzy theory.

Table 1.1: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
Inverse of a fuzzy matrix of fuzzy numbers	2009	[89]	The objective of this study is to broaden the notion of the inverse of a matrix by incorporating fuzzy numbers.	• The advantages of the inverse of a fuzzy matrix lie in its utility for solving systems of fuzzy linear equations and facilitating operations analogous to those in classical linear algebra. • Restricted fuzzy matrix.
An application of interval-valued fuzzy matrices in medical diagnosis	2011	[22]	In this study, we expand upon Sanchez's methodology for medical diagnosis by representing an interval-valued fuzzy matrix as an interval matrix derived from two fuzzy matrices. We introduce the arithmetic mean of an interval-valued fuzzy matrix as the mean between its lower and upper limit matrices. Additionally, we propose a simplified method to examine Sanchez's medical diagnosis approach through the arithmetic mean of an interval-valued fuzzy matrix.	• The use of interval-valued fuzzy matrices is advantageous when dealing with incomplete or imprecise medical information, as it allows for a more flexible and robust representation of uncertain data. • The application may face limitations in terms of practical validation and real-world testing.
Similarity relations, invertibility, and eigenvalues of the intuitionistic fuzzy matrix.	2013	[57]	In this paper, properties of similarity relations, invertibility conditions, and eigenvalues of IFMs. are investigate.	• Restricted to intuitionistic fuzzy matrix
Solving fully fuzzy matrix equations.	2012	[231]	In this paper, a new method is proposed to find the fuzzy optimal solution of fuzzy linear programming problems.	• Usual properties of fuzzy matrix equation are provided. • Limited to fuzzy number.
Some results on the generalized inverse of intuitionistic fuzzy matrices	2014	[232]	In this paper, pseudo-similar intuitionistic fuzzy matrix (IFM) is defined with some properties of pseudo-similar and semi-similar IFMs.	• Limited to some particular concepts.
Interval-valued fuzzy matrices with interval-valued fuzzy rows and columns.	2015	[24]	In this work, a new kind of IVFM has been introduced. In this approach, the rows and columns are in IVFM.	• This matrices can be used to handle images, fuzzy graphs, etc. • limited to IVFM
The rank of interval-valued fuzzy matrices	2016	[27]	In this paper, various techniques are outlined for determining the ranks of interval-valued fuzzy matrices, encompassing three different types of ranks. The interrelationship between these ranks is explored, and a straightforward investigation of these ranks is conducted through the cross-vector approach. Several outcomes are presented, employing the definition of scalar multiplication for interval-valued fuzzy matrices.	• The rank of an interval-valued fuzzy matrix provides a quantitative measure of its significance. • The rank provides information about linear dependencies in a matrix, but it may not capture nonlinear relationships or more intricate patterns in the data. • limited to IVFM
Multiplicative operations of intuitionistic fuzzy matrices	2017	[233]	This paper investigated the algebraic properties of intuitionistic fuzzy matrices under the new operations.	• De Morgan's laws for the operations. • Limited to IFM
Algebraic operations on Pythagorean fuzzy matrices	2018	[30]	The primary aim of this paper is to present a Pythagorean fuzzy matrix and establish several operations on PFMs, subsequently exploring their properties.	• There may be a lack of standardized methods or guidelines for certain operations and applications involving PyFMs. • Restricted to PyFM.
Bipolar fuzzy matrices	2019	[234]	In this article, bipolar fuzzy algebra and bipolar fuzzy relation are defined, and then, the bipolar fuzzy matrix is introduced.	• It formalizes a unified approach to polarity and fuzziness. • It captures the bipolar or double-sided nature of human
Picture fuzzy matrix and its application.	2020	[28]	This paper introduced the concept of picture fuzzy matrix and properties.	• The intuitive meaning of picture fuzzy matrix is defined. • Some basic results of the picture fuzzy matrix are provided. • Limited to picture fuzzy matrix.
Some algebraic structures of picture fuzzy matrices	2020	[38]	Define algebraic operations of Picture fuzzy matrices and their basic properties are proved.	• PFMs need to be explored in the decision-making, risk analysis, and many other uncertain and fuzzy environment.
Some operations over intuitionistic fuzzy matrices based on Hamacher t-norm and t-conorm	2021	[145]	Define the Hamacher scalar multiplication and Hamacher exponentiation operations on Intuitionistic fuzzy matrices.	• Restricted to IFM.
Fermatean fuzzy matrices	2022	[144]	Introduced the concept of FFM and some algebraic operations.	• The intuitive meaning of fermatean fuzzy matrix is defined. • Some basic results of fermatean fuzzy matrix are provided. of giving fuzzy solutions for the fuzzy matrix game. • Limited to FFM.

Table 1.1: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
Eigenvalue and Eigenvector of Picture Fuzzy Matrix	2022	[94]	Find the Eigenvalue and Eigenvector of some particular types of PFMs	• This new algorithm to find Eigenvalue and Eigenvector . • Limited to some particular types of PFM and for some suitable examples.
Spherical fuzzy matrices.	2023	[52]	Introduce the concept of Spherical fuzzy matrices and some algebraic operations.	• The intuitive meaning of the spherical fuzzy matrix is defined. • Some basic results of the spherical fuzzy matrix are provided • Limited to SFM
Interval-valued picture fuzzy matrix: basic properties and application	2023	[118]	Extend the theory of picture fuzzy matrices (PFM) into interval-valued picture fuzzy matrices (IVPFMs) to more flexibly represent uncertain and vague information and discuss some important definitions, theorems and new distance measures. Also calculated the determinant, adjoint, greatest, and least eigenvalue for the same matrix.	• The existing FM, IVFM, IFM, IVIFM, and PFM each have their own shortcomings that prevent them from fully capturing the information. The IVPFM fills the gaps and gives a more flexible opinion. • We can see the drawback in the condition of eigen fuzzy sets and eigen intuitionistic fuzzy sets experts/decision-makers bind their input in a certain area. The generalization feature offered by the proposed eigen interval-valued picture fuzzy set with a significant impact. • A limitation of IVPFM is related to the representation of degrees of membership, neutrality, and non-membership as interval numbers. The limitation arises when the sum of the upper degree of membership, neutrality, and upper degree of non-membership exceeds the interval [0, 1].
Interval-valued spherical fuzzy matrix and its applications in multi-attribute decision-making process	2023	[235]	Extend the theory of spherical fuzzy matrices (SFM) into interval-valued spherical fuzzy matrices (IVSFMs) to more flexibly represent uncertain and vague information and discuss some important definitions, theorems and score function. Also calculated the determinant and adjoint for the same matrix.	• The existing FM, IVFM, IVIFM, and SFM each have their own shortcomings that prevent them from fully capturing the information. The IVSFM fills the gaps and gives a more flexible opinion. • A limitation of IVSFM is related to the representation of degrees of membership, neutrality, and non-membership as interval numbers. The limitation arises when the sum of the upper degree of membership, neutrality, and upper degree of non-membership exceeds the interval [0, 1].

1.4 Preliminaries

This section presents the preliminaries containing a few basic concepts that create the foundations of the work performed in this thesis.

Definition 1.4.1. *[2] Let X be a universal set, then a FS A is defined as*

$$A = \left\{ \left(x, \mu_A(x)\right) \mid x \in X \right\},$$

which is characterized by the membership function

$$\mu_A(x) : X \to [0,1].$$

Here $\mu_A(x)$ is the degree of membership of the element x to the set A.

Definition 1.4.2. [7] *IFS A on X is defined as a set of ordered pairs given by*

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A(x), \nu_A(x) : X \longrightarrow [0,1]$ are respectively, the degree of membership and degree of non-membership of the element x to the set A, with the condition $(\mu_A(x)) + (\nu_A(x)) \leq 1$, the degree of indeterminacy is given by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Definition 1.4.3. [107] *An IVIFS A on a universal set X is defined as*

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A = [\mu_{AL}, \mu_{AU}], \nu_A = [\nu_{AL}, \nu_{AU}]$ and $\mu_A, \nu_A : X \longrightarrow [0,1]$ are respectively. An IVIFS has a condition that the sum of the supremum of the functions must lie in the unit interval and $\mu_A(x)$ and $\nu_A(x)$ are the membership and non-membership degree of the element x to the set A.

Definition 1.4.4. [15] *PyFS A on X is defined as a set of ordered pairs given by*

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A(x), \nu_A(x) : A \longrightarrow [0,1]$ are respectively. the membership and non membership degree of the element x to the set A respectively, with the condition corresponding to its membership function $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$, the degree of indeterminacy is given by $\pi_A(x) = \sqrt{1 - ({\mu_A}^2 + {\nu_A}^2)}$.

Definition 1.4.5. [114] *An IVPyFS A on X is defined as*

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A = [\mu_{AL}, \mu_{AU}], \nu_A = [\nu_{AL}, \nu_{AU}]$ and $\mu_A, \nu_A : X \longrightarrow [0,1]$. An IVPyFS has a condition that the sum of the square of the supremum of functions must lie in the unit interval and $\mu_A(x)$ and $\nu_A(x)$ are the membership and non-membership degree of the element x to the set A

Definition 1.4.6. [18] *FFS A defined on universal set X is given by*

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A(x) : X \longrightarrow [0,1], \nu_A(x) : X \longrightarrow [0,1]$ and $0 \leq \mu_A^3(x) + \nu_A^3(x) \leq 1$.
Here $\mu_A(x)$ and $\nu_A(x)$ are the membership and non-membership degree of the element to the set element x to the set A.

Definition 1.4.7. [19] *An IVFFS A on X is defined as*

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A = [\mu_{AL}, \mu_{AU}], \nu_A = [\nu_{AL}, \nu_{AU}]$ *and* $\mu_A, \nu_A : X \to [0,1]$. *An IVFFS has a condition that the sum of the cube of supremum of functions must lie in the unit interval and* $\mu_A(x)$ *and* $\nu_A(x)$ *are the membership and non-membership degree of the element x to the set A.*

Definition 1.4.8. [127] *PFS A defined on universal set X is given by*

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) | x \in X\},$$

where $\mu_A(x) : X \to [0,1], \eta_A(x) : X \to [0,1], \nu_A(x) : X \to [0,1]$ *and* $0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$. *Here* $\mu_A(x), \eta_A(x), \nu_A(x)$ *are the membership, neutral membership, and non-membership degree of the element x to the set A.*

Definition 1.4.9. [44] *An IVPFS A on a universal set X is defined as*

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A = [\mu_{AL}, \mu_{AU}], \eta_A = [\eta_{AL}, \eta_{AU}], \nu_A = [\nu_{AL}, \nu_{AU}]$ *and* $\mu_A, \eta_A, \nu_A : X \to [0,1]$. *An IVPFS has a condition that the sum of the supremum of all three functions must lie in the unit interval and* $\mu_A(x), \eta_A(x)$ *and* $\nu_A(x)$ *are the membership, neutral membership and non-membership degree of the element to the set element x to the set A.*

Definition 1.4.10. [42] *SFS A defined on universal set X is given by*

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) | x \in X\},$$

where $\mu_A(x) : X \to [0,1], \eta_A(x) : X \to [0,1], \nu_A(x) : X \to [0,1]$ *and* $0 \leq \mu_A^2(x) + \eta_A^2(x) + \nu_A^2(x) \leq 1$. *Here* $\mu_A(x), \eta_A(x)$ *and* $\nu_A(x)$ *are the membership, neutral membership, and non-membership degree of the element x to the set A.*

Definition 1.4.11. [116] *An IVSFS A on a universal set X is defined as*

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A = [\mu_{AL}, \mu_{AU}], \eta_A = [\eta_{AL}, \eta_{AU}], \nu_A = [\nu_{AL}, \nu_{AU}]$ *and* $\mu_A, \eta_A, \nu_A : X \to [0,1]$. *An IVSFS has a condition that the sum of the square of the supremum of all three functions must lie in* *the unit interval and* $\mu_A(x), \eta_A(x)$ *and* $\nu_A(x)$ *are the membership, neutral membership and non-membership degree of the element x to the set A.*

Definition 1.4.12. [16] *Fuzzy matrix (FM)* $\tilde{A} = (\tilde{a}_{ij})$ *of order* $m \times n$ *is defined as*

$$\tilde{A} = (< \tilde{a}_{ij\mu} >),$$

such that $\tilde{a}_{ij\mu} \in [0,1]$ *is the measure of membership degrees of* $\tilde{a}_{ij}$ *respectively where,* $i = 1,2.\dots\dots m$ *and* $j = 1,2.\dots\dots n$ *satisfying* $0 \leq \tilde{a}_{ij\mu} \leq 1$.

Definition 1.4.13. [24] *An IVFM* $\tilde{A}$ *is defined as*

$$\tilde{A} = (\tilde{a}_{ij}) = (< \tilde{a}_{ij\mu} >),\ i = 1,2,\dots,m,\ j = 1,2,\dots,n$$

where $\tilde{a}_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \subseteq [0,1]$ *is the membership degree of* $\tilde{a}_{ij}$.

Definition 1.4.14. [131] *IFM* $\tilde{A} = (\tilde{a}_{ij})$ *of order* $m \times n$ *is defined as*

$$\tilde{A} = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\nu} >),$$

where $\tilde{a}_{ij\mu} \in [0,1]$, $\tilde{a}_{ij\nu} \in [0,1]$ *is the measure of membership and non-membership degrees of* $\tilde{a}_{ij}$ *respectively where,* $i = 1,2.\dots\dots m$ *and* $j = 1,2.\dots\dots n$ *satisfying* $0 \leq \tilde{a}_{ij\mu} + \tilde{a}_{ij\nu} \leq 1$.

Definition 1.4.15. [32] *An IVIFM* $\tilde{A}$ *is defined as*

$$\tilde{A} = (\tilde{a}_{ij}) = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\nu} >),\ i = 1,2,\dots,m,\ j = 1,2,\dots,n$$

where,

$$\tilde{a}_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \subseteq [0,1],$$

$$\tilde{a}_{ij\nu} = [a_{ij\nu L}, a_{ij\nu U}] \subseteq [0,1],$$

with the condition

$$a_{ij\mu U} + a_{ij\nu U} \leq 1.$$

$\tilde{a}_{ij\mu}$ *and* $\tilde{a}_{ij\nu}$ *are the membership and non-membership degree of* $\tilde{a}_{ij}$.

Definition 1.4.16. [30] *PyFM is defined as* $\tilde{A} = (\tilde{a}_{ij})$ *of order* $m \times n$ *is defined as*

$$\tilde{A} = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\nu} >),$$

where $\tilde{a}_{ij\mu} \in$ *[0,1],* $\tilde{a}_{ij\nu} \in$*[0,1] is the measure of membership and non-membership degrees of* $\tilde{a}_{ij}$ *respectively where,* $i = 1,2\ldots\ldots..m$ *and* $j = 1,2\ldots\ldots.n$ *satisfying* $0 \leq (\tilde{a}_{ij\mu})^2 + (\tilde{a}_{ij\nu})^2 \leq 1$.

Definition 1.4.17. *[144] FFM* $\tilde{A} = (\tilde{a}_{ij})$ *of order* $m \times n$ *is defined as*

$$\tilde{A} = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\nu} >),$$

where $\tilde{a}_{ij\mu} \in$ *[0,1],* $\tilde{a}_{ij\nu} \in$*[0,1] is the measure of membership and non-membership degrees of* $\tilde{a}_{ij}$ *respectively where,* $i = 1,2\ldots\ldots..m$ *and* $j = 1,2\ldots\ldots.n$ *satisfying* $0 \leq (\tilde{a}_{ij\mu})^3 + (\tilde{a}_{ij\nu})^3 \leq 1$.

Definition 1.4.18. *[28] PFM* $\tilde{A} = (\tilde{a}_{ij})$ *of order* $m \times n$ *is defined as*

$$\tilde{A} = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu} >),$$

where $\tilde{a}_{ij\mu} \in$ *[0,1],* $\tilde{a}_{ij\eta} \in$*[0,1],* $\tilde{a}_{ij\nu} \in$*[0,1] is the measure of membership, neutral membership and non-membership degrees of* $\tilde{a}_{ij}$ *respectively where,* $i = 1,2\ldots\ldots..m$ *and* $j = 1,2\ldots\ldots.n$ *satisfying* $0 \leq \tilde{a}_{ij\mu} + \tilde{a}_{ij\eta} + \tilde{a}_{ij\nu} \leq 1$.

Definition 1.4.19. *[118] An IVPFM* $\tilde{A}$ *is defined as*

$$\tilde{A} = (\tilde{a}_{ij}) = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu} >),\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n$$

where,

$$\tilde{a}_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \subseteq [0,1],$$

$$\tilde{a}_{ij\eta} = [a_{ij\eta L}, a_{ij\eta U}] \subseteq [0,1],$$

$$\tilde{a}_{ij\nu} = [a_{ij\nu L}, a_{ij\nu U}] \subseteq [0,1],$$

with the condition

$$a_{ij\mu U} + a_{ij\eta U} + a_{ij\nu U} \leq 1.$$

$\tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}$ *and* $\tilde{a}_{ij\nu}$ *are the membership, neutral membership and non-membership degree of* $\tilde{a}_{ij}$.

Definition 1.4.20. *[52] SFM* $\tilde{A} = (\tilde{a}_{ij})$ *of order* $m \times n$ *is defined as*

$$\tilde{A} = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu} >),$$

where $\tilde{a}_{ij\mu} \in$ *[0,1],* $\tilde{a}_{ij\eta} \in$*[0,1],* $\tilde{a}_{ij\nu} \in$*[0,1] is the measure of membership, neutral membership and non-membership degrees of* $\tilde{a}_{ij}$ *respectively where,* $i = 1,2\ldots\ldots..m$ *and* $j = 1,2\ldots\ldots.n$ *satisfying* $0 \leq \tilde{a}^2_{ij\mu} + \tilde{a}^2_{ij\eta} + \tilde{a}^2_{ij\nu} \leq 1$.

Definition 1.4.21. *[119] An IVSFM $\tilde{A}$ is defined as*

$$\tilde{A} = (\tilde{a}_{ij}) = (<\tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu}>),\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n$$

where,

$$\tilde{a}_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \subseteq [0,1],$$

$$\tilde{a}_{ij\eta} = [a_{ij\eta L}, a_{ij\eta U}] \subseteq [0,1],$$

$$\tilde{a}_{ij\nu} = [a_{ij\nu L}, a_{ij\nu U}] \subseteq [0,1],$$

with the condition

$$(a_{ij\mu U})^2 + (a_{ij\eta U})^2 + (a_{ij\nu U})^2 \leq 1.$$

$\tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}$ *and* $\tilde{a}_{ij\nu}$ *are the membership, neutral membership and non-membership degree of* $\tilde{a}_{ij}$.

Definition 1.4.22. *[96] A fuzzy relation, denoted as R, defined on a fuzzy set X can be described as a fuzzy subset of* $X \times X$. *In other words, it can be represented as:*

$$R = \left\{ \left\langle (x_1,x_2), \mu_R(x_1,x_2) | x_1, x_2 \in X \right\rangle \right\}.$$

Here, $\mu_R(x_1,x_2)$ *represents the membership degree of the pair* (x_1,x_2) *in the fuzzy relation R, and it takes values within the interval [0, 1].*

Definition 1.4.23. *[57] An intuitionistic fuzzy relation, denoted as R, defined on a fuzzy set X can be described as a fuzzy subset of* $X \times X$. *In other words, it can be represented as:*

$$R = \left\{ \left\langle (x_1,x_2), \mu_R(x_1,x_2), \nu_R(x_1,x_2) | x_1, x_2 \in X \right\rangle \right\}.$$

Here, $\mu_R(x_1,x_2) and \nu_R(x_1,x_2)$ *represents the membership and non-membership degree of the pair* (x_1,x_2) *with the condition* $\mu_R(x_1,x_2) + \nu_R(x_1,x_2) \leq 1$ *in the fuzzy relation R, and it takes values within the interval [0, 1].*

Definition 1.4.24. *[46] A picture fuzzy relation, denoted as R, defined on a fuzzy set X can be described as a fuzzy subset of* $X \times X$. *In other words, it can be represented as:*

$$R = \left\{ \left\langle (x_1,x_2), \mu_R(x_1,x_2), \eta_R(x_1,x_2), \nu_R(x_1,x_2) | x_1, x_2 \in X \right\rangle \right\}.$$

Here, $\mu_R(x_1,x_2), \eta_R(x_1,x_2) and \nu_R(x_1,x_2)$ *represents the membership, neutrality and non-membership degree of the pair* (x_1,x_2) *with the condition* $\mu_R(x_1,x_2) + \eta_R(x_1,x_2) + \nu_R(x_1,x_2) \leq 1$ *in the fuzzy relation R, and it takes values within the interval [0, 1].*

Definition 1.4.25. *[53] A spherical fuzzy relation, denoted as R, defined on a fuzzy set X can be described as a fuzzy subset of $X \times X$. In other words, it can be represented as:*

$$R = \left\{ \left\langle (x_1, x_2), \mu_R(x_1, x_2), \eta_R(x_1, x_2), \nu_R(x_1, x_2) | x_1, x_2 \in X \right\rangle \right\}.$$

Here, $\mu_R(x_1,x_2), \eta_R(x_1,x_2) and \nu_R(x_1,x_2)$ represents the membership, neutrality and non-membership degree of the pair (x_1,x_2) with the condition $(\mu_R(x_1,x_2))^2 + (\eta_R(x_1,x_2))^2 + (\nu_R(x_1,x_2))^2 \leq 1$ in the fuzzy relation R, and it takes values within the interval [0, 1].

Definition 1.4.26. *[97] Consider a fuzzy relation R on the elements of a fuzzy set $A \subseteq X$, denoted as $R \in FR(X \times X)$. Let T be a subset of X. In this context, T is referred to as an eigen fuzzy set associated with the relation R if it meets the condition $T \circ R = T$, where $\circ$ represents any composition operator.*

Definition 1.4.27. *[56] Consider a intuitionistic fuzzy relation R on the elements of a intuitionistic fuzzy set A, denoted as $R \in FR(X \times X)$. Let T be a subset of X. In this context, T is referred to as an eigen intuitionistic fuzzy set associated with the relation R if it meets the condition $T \circ R = T$, where $\circ$ represents any composition operator.*

Definition 1.4.28. *[62] Consider a spherical fuzzy relation R on the elements of a spherical fuzzy set A, denoted as $R \in FR(X \times X)$. Let T be a subset of X. In this context, T is referred to as an eigen spherical fuzzy set associated with the relation R if it meets the condition $T \circ R = T$, where $\circ$ represents any composition operator.*

1.5 Transportation Problem

The transportation problem (TP) is a classical optimization problem in linear programming that deals with the efficient distribution of goods or resources from multiple suppliers to multiple consumers while minimizing transportation costs. It is often used in logistics, supply chain management, and distribution network planning. The main objective of a transportation problem is to determine how much of a product should be transported from each supplier to each consumer in such a way that the total transportation cost is minimized while respecting supply and demand constraints. Let there be three units, producing scooters, say, A_1, A_2 and A_3 from where the scooters are to be supplied to four depots say B_1, B_2, B_3 and B_4.

Let the number of scooters produced at A_1, A_2 and A_3 be a_1, a_2 and a_3 respectively and the demands at the depots be b_1, b_2, b_3, and b_4 respectively.

We assume the condition

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + +b_4$$

i.e., all scooters produced are supplied to the different depots.

Let the cost of transportation of one scooter from A_1 to B_1 be c_{11}. Similarly, the costs of transportation in other cases.

Let out of a_1 scooters available at A_1, x_{11} be taken at B_1 depot, x_{12} be taken at B_2 depot and to other depots as well.

Total number of scooters to be transported from A_1 to all destinations, i.e., B_1, B_2, B_3 and B_4 must be equal to a_1.

$$x_{11} + x_{12} + x_{13} + x_{14} = a_1$$

Similarly, from A_2 and A_3 the scooters transported be equal to a_2 and a_3 respectively

$$x_{21} + x_{22} + x_{23} + x_{24} = a_2$$

and

$$x_{31} + x_{32} + x_{33} + x_{34} = a_3$$

On the other hand, it should be kept in mind that the total number of scooters delivered to B_1 from all units must be equal to b_1, i.e.

$$x_{11} + x_{21} + x_{31} = b_1$$

Similarly

$$x_{12} + x_{22} + x_{32} = b_2$$

$$x_{13} + x_{23} + x_{33} = b_3$$

$$x_{14} + x_{24} + x_{34} = b_4$$

With the help of the above information we can construct the following table:

The cost of transportation from $A_i(i = 1, 2, 3)$ to $B_j(j = 1, 2, 3, 4)$ will be equal to

$$S = \sum_{ij} c_{ij} x_{ij}$$

where the symbol put before c_{ij}, x_{ij} signifies that the quantities $c_{ij} x_{ij}$ must be summed over all $i = 1,2,3$ and all $j = 1,2,3,4$.

Thus we come across a linear programming problem given by equations and a linear function. We have to find the non-negative solutions of the system such that it minimizes the function.

1.5.1 Fuzzy transportation problem

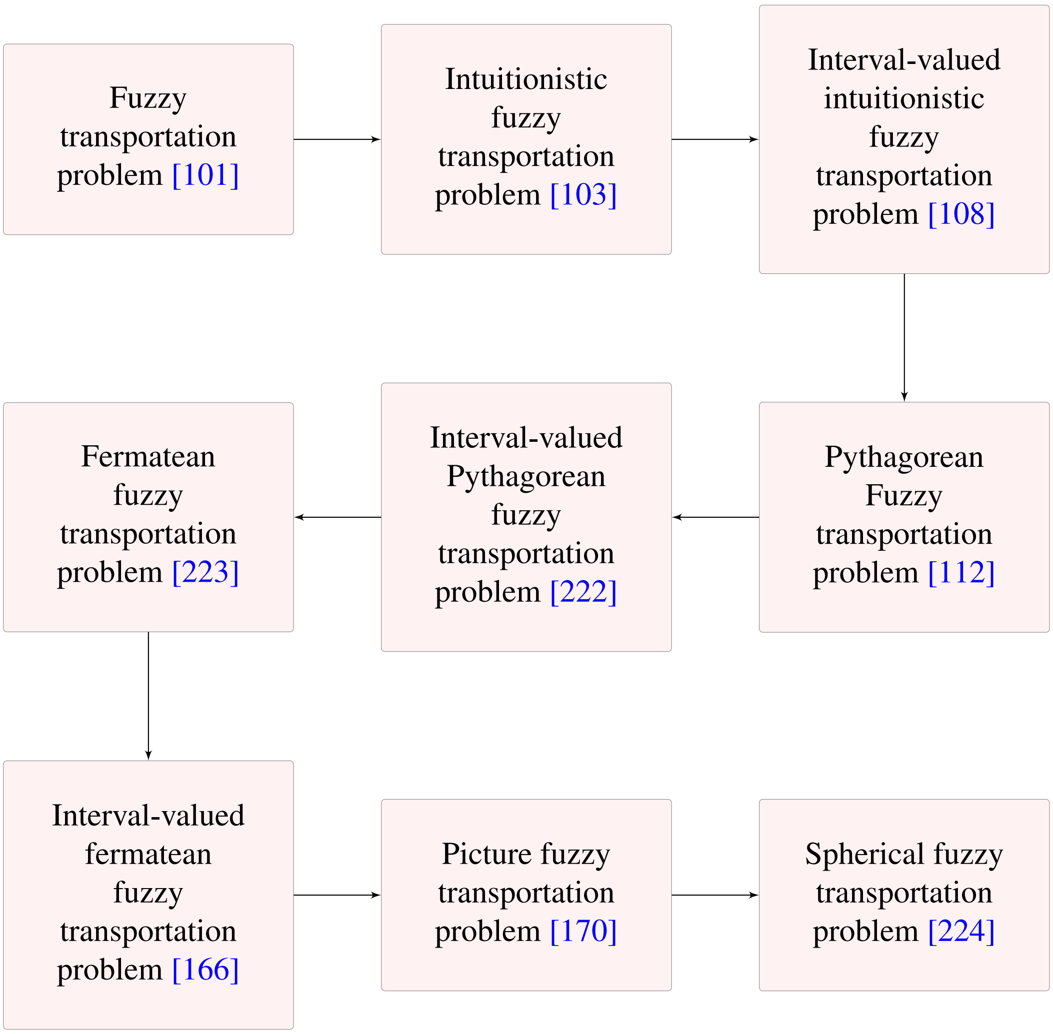

Figure 1.3: Extension of fuzzy transportation problem

In conventional transportation problems it is assumed that the decision maker is sure about the precise values of transportation cost, availability, and demand of the product. In real-world applications, all these parameters of the transportation problems may not be known precisely due to uncontrollable factors. To deal with such situations, fuzzy set theory is applied in the literature to solve transportation problems. Several authors have proposed different methods for solving balanced fuzzy transportation problems by representing the transportation cost, availability, and demand as normal fuzzy numbers. The balanced fuzzy transportation problems, in which a decision maker is uncertain about the precise values of transportation cost, availability, and demand, may be formulated as follows:

$$\text{Minimize} \quad \sum_{i=1}^{p}\sum_{j=1}^{q} c_{ij} \otimes x_{ij}$$

Subject to

$$\sum_{j=1}^{q} x_{ij} = a_i, \quad i = 1,2,3,\ldots,p$$

$$\sum_{i=1}^{p} x_{ij} = b_j, \quad j = 1,2,3,\ldots,q$$

$$\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} b_j, \quad i = 1,2,3,\ldots,p, \quad j = 1,2,3,\ldots,q$$

x_{ij} is a non-negative fuzzy number, where p = total number of sources; q = total number of destination a_i = the fuzzy availability of the product at i^{th} source b_j = the fuzzy demand of the product at j^{th} destination c_{ij} = the fuzzy transportation cost for a unit quantity of the product from i^{th} source to j^{th} destination (or fuzzy decision variables) to minimize the total fuzzy transportation cost x_{ij} = the fuzzy quantity of the product that should be transported from i^{th} source to j^{th} destination (or fuzzy decision variables) to minimize the total fuzzy transportation cost $\sum_{i=1}^{p} a_i$ = total fuzzy availability of the product $\sum_{j=1}^{q} b_j$ = total fuzzy demand of the product

$$\sum_{i=1}^{p}\sum_{j=1}^{q} c_{ij} \otimes x_{ij} = \text{ total fuzzy transportation cost}$$

.

Remark 1. if $\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} b_j$ then the FTP is said to be a balanced fuzzy transportation problem, otherwise it is called an unbalanced fuzzy transportation problem.

Table 1.2 presents a summary of transportation problems having imprecise information with their description as well as advantages and limitations.

1.6 Assignment Problem

The assignment problem (AP) is a specific type of linear programming problem (LPP) that deals with the optimal assignment of a set of tasks to a set of agents or machines, such that the total cost or time required to complete the tasks is minimized. It is characterized by the following features:

- Agents or Machines: There is a set of agents or machines (also referred to as workers or facilities) available to perform tasks.

- Tasks: There is a set of tasks to be assigned to the agents or machines.

- Assignment: Each task must be assigned to exactly one agent, and each agent can be assigned to at most one task. In other words, there is a one-to-one correspondence between tasks and agents.

- Cost or Time: There is a cost or time associated with assigning a particular task to a specific agent. The objective is to minimize the total cost or time required for completing all tasks by making optimal assignments.

Let there be n persons and n jobs. Each job must be done by exactly one person and one person can do at most one job. The problem is to assign the persons to the job so that the total cost of completing all the jobs becomes minimum.

In the problem c_{ij} denotes the cost for assigning the j^{th} job to the i^{th} person. We introduce the $0-1$ variable x_{ij}, where

$$x_{ij} = \begin{cases} 1, & \text{if the person } i \text{ is assigned the job } j; i, j = 1, 2, \ldots\ldots, n \\ 0, & \text{otherwise} \end{cases}$$

Corresponding to the $(i, j)^{th}$ event of assigned person i to job j, the constraint $\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots..n$ means that each job must be done by exactly one person and the constraint $\sum_{j=1}^{n} x_{ij} = 1, j = 1, 2, \ldots..n$ means each person must be assigned at most one job. Thus the model for the crisp assignment problem is given by:

$$\begin{aligned} \text{Minimize} \quad & z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{Subject to} \quad & \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n \\ & x_{ij} = 0 \text{ or } 1, \quad i, j = 1, 2, \ldots, n \end{aligned}$$

1.6.1 Fuzzy assignment problem

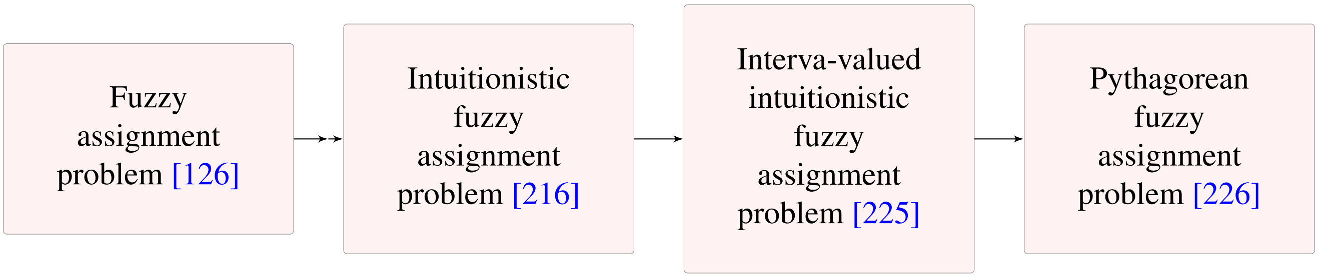

Figure 1.4: Extension of fuzzy assignment problem

An assignment problem is a specific kind of math problem where you have different tasks and an equal number of workers or machines. The goal is to match each task with one worker or machine in a way that makes the total time or cost as minimum as possible. Alternatively, the aim might be to make the total sales, total profit, or the overall happiness of the group as maximum as possible. The fuzzy assignment problem can be mathematically formulated using fuzzy numbers and optimization techniques. Let's consider a scenario where there are 'n' tasks and 'm' agents, and the goal is to find an optimal assignment that minimizes a fuzzy objective function. Here's a general mathematical representation::

$$\text{Minimize or Maximize} \quad \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij} \otimes x_{ij}$$

Subject to

$$\sum_{j=1}^{m} x_{ij} = 1, \quad i = 1, 2, 3, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} \leq 1, \quad j = 1, 2, 3, \ldots, m$$

$$0 \leq x_{ij} \leq 1, \quad i = 1, 2, 3, \ldots, n, \quad j = 1, 2, 3, \ldots, m$$

x_{ij} be the degree of assignment of task i to agent j, where $i = 1, 2, 3, \ldots..., n$, $j = 1, 2, 3, \ldots.., m$. n = total number of task; m = total number of agent. c_{ij}= The fuzzy cost or utility functions involve fuzzy numbers, representing the imprecision or uncertainty associated with the corresponding costs or utilities. These fuzzy numbers can be represented using membership functions.

Table 1.3 presents a summary of fuzzy assignment problems having imprecise information with their description as well as advantages and limitations.

1.7 Score Function

Many useful methods have been developed to enrich the concept of score functions, and valuable applications of score functions have been developed in a variety of fields, especially in MCDA. Chen [146] proposed two similarity measures for measuring the degree of similarity between vague sets on the basis of the score functions. Chen [147], Hong and Kim [148] defined the weighted score of a vague value according to the weighted score functions to determine the weighted similarity measure between the vague values. Liu and Wang [149] presented a weighted score function method for solving multi-criteria fuzzy decision-making problems in an intuitionistic fuzzy environment. By means of intuitionistic fuzzy point operators, they defined a series of new score functions and proposed an evaluation function for the decision-making problems. Chen [150] utilized optimistic and pessimistic point operators to measure the effects of optimism and pessimism and further determined a suitability function in terms of the weighted score functions. Based on the score function and the accuracy function, Xu and Yager [151] introduced an order relation between two intuitionistic fuzzy values. They developed new intuitionistic fuzzy geometric aggregation operators to accommodate the intuitionistic fuzzy environment. Xu [152] provided a method for comparing two intuitionistic fuzzy values using score functions and accuracy functions, and he then developed aggregation operators, such as the intuitionistic fuzzy hybrid aggregation operator, for aggregating intuitionistic fuzzy values. Xu and Chen [153] extended geometric aggregation operators to IVIFSs [151] that were first developed for IFSs and studied their various properties. They applied the proposed operators to solve a multi-attribute decision-making problem with interval-valued intuitionistic fuzzy information using the score function defined on IVIFNs. Xu [155] introduced a score function and an accuracy function to measure an IVIFN and developed a method for making comparisons between two IVIFNs. On the basis of the score function and the accuracy function, Xu and Chen [156] defined the interval-valued intuitionistic judgment matrix, its score matrix, and its accuracy matrix. They developed the ordered weighted aggregation operator and hybrid aggregation operator to aggregate interval-valued intuitionistic preference information and proposed an approach to group decision-making with interval-valued intuitionistic judgment matrices. Building upon the concepts of score and accuracy functions, Xu [157] developed a method based on distance measure for group decision-making with interval-valued intuitionistic fuzzy matrices. Pythagorean fuzzy set seemed to be a more resultful means to depict uncertain information in a greater range point when compared with the intuitionistic fuzzy set. The comparison issue in Pythagorean fuzzy environment is disposed by proposing novel score function Peng et al.[158]. Afterward, Huang et al. [159], a novel score function based on determinacy degree and indeterminacy degree is put forward for approximately representing PyFSs. Then, the original MULTIMOORA method is extended by using the score function and it is used to solve the multicriteria decision-making problems under the PyFS information context. In the context of medical decision-making, Rani et al. [160] defined the new score function for PyFS and used it to solved the case study of the T2D pharmacological therapy selection problem. In view of the situation, combined with the relevant knowledge of interval numbers [161], the concept of interval-valued Pythagorean fuzzy sets (IVPFS) [162] is proposed, which can be used to express membership or non-membership in the form of intervals. The score function [163] can convert the interval number into an exact single value, which is convenient for comparison and decision-making. Senapati & Yager[18] have established some basic mathematical operations over the fermatean fuzzy sets. They have also introduced the concept of score and accuracy function on the class of fermatean fuzzy sets for comparing any two arbitrary FFSs. Jeevaraj [164] has introduced the new score and accuracy function of interval-valued Fermatean fuzzy sets and its applications. Rani et al. [165] also improved the score function for IVFFS. Also Akram et al. [166] defined the new score function for solving the fractional transportation problem under IVFFS. The score function defined on the PFS is a more frequent platform for describing the degree of positive, neutral, and negative membership functions that generalizes the concept of intuitionistic fuzzy score function. In the decision-making process, some researchers like Jaikumar et al. [167] defined the perfect score function in picture fuzzy sets and its applications in decision-making problems. Also Jan et al. [168] proposed the new score function to solve the mathematical analysis of generative adversarial network problems. Wang et al. [169] used the score function on picture fuzzy sets and their application in multiple attribute decision-making. Moreover, Geetha & Selvakumari [170] use three types of ranking functions to find the solution of the picture fuzzy transportation problem. However, in MCDM problems, due to the limitations of experts' understanding of decision-making objects and the ambiguity of the decision-making environment, experts can give an interval number rather than a specific real number when making a decision. Therefore, membership, neutrality, non-membership and abstention can be represented by interval numbers to enhance the credibility of decision-making results. In certain real-world applications, IVPFS theory, an extension of PFS, is more adept at handling and modeling inconsistent, indeterminate, and incomplete data [171, 172, 173]. Therefore, it becomes very important to study the group decision problem under the IVPFS. Ma et al. [174] had used the score function to analysis the MCDM approach for design concept evaluation based on interval-valued picture fuzzy sets. Shanthi & Gayathri [175] worked on the interval-valued picture fuzzy soft sets weighted aggregation operators are used to aggregate the interval-valued picture fuzzy soft sets information corresponding to each alternative. The alternatives are then ranked based on the values of the accuracy function. The idea behind SFS is to let decision-makers to generalize other extensions of fuzzy sets by defining a membership function on a spherical surface and independently assigning the parameters of that membership function with a larger domain. Kutlu & Kahraman, [42] are defined the new score and accuracy functions for SFS. Otay et al.[176] also defined Score and accuracy functions for different types of spherical fuzzy sets. Interval-valued fuzzy sets are used for incorporating the decision-maker's opinions about the parameters of a fuzzy set into the model with an interval instead of a single point. The novel interval-valued spherical fuzzy sets are introduced with their score and accuracy functions by Gundogdu and Kahraman [116] for solving a multiple criteria selection problem among 3D printers to verify the developed approach and to demonstrate its practicality and effectiveness. Lathamaheswari et al. [177] also proposed the new score function for IVSFS and it applied in a decision-making problem to choose the best station which scrutinizes the quality of air.

1.8 Similarity Measure and Distance Measure

In the literature, the idea of the similarity measure has a vital role in identifying the degree of likeliness between two entities . Over the last few decades, researchers have been paying attention to measuring the similarities and distances between two objects and applying them. These measures are useful in various areas like pattern recognition, medical diagnosis, decision-making, and clustering analysis [178, 179, 180, 181, 182, 183, 184]. The commonly used distance measures are Euclidean distance [185], Hamming distance [185], and Hausdorff metric [186]. The research work on the similarity measures has been increased significantly, and many similarity measures for IFS, PFS, and IVIFS have been studied in the literature [187, 188, 189]. An adequate count of similarity and dissimilarity in a distance measure-based similarity measure of IFS is given by szimidt and kacprzyk [187] and further extended to a group of similarity measures and analyzed with the existing models. Peng & Li [188] developed a distance measure-based similarity measure for the IVPyFS. By extending the axioms Xu [190] generalized and proposed some similarity measures from IFS to IVIFS. A similarity measure of IVIFS to solve problems in the domains of medical diagnosis, multi-criteria fuzzy decision-making, and pattern recognition is proposed by Wei [191]. Singh [192] proposed a cosine-based similarity measure of IVIFS for pattern recognition. However, few research works are found in medical diagnosis and pattern recognition for measuring the distance and similarity between two IVPyFS. Mishra et al. [194] proposed two similarity measures to quantify the degree of similarity between two FFSs. Sahoo [195] presented FFS similarity measures with their application in the group decision making. Wei [196] proposed some SM for PFS and applied these SM to mineral field recognition and building material recognition applications. Moreover, Son [77] proposed a generalized picture distance measure and applied it to picture fuzzy clustering. Wei [197] proposed some SM for PFS and applied these to strategy decision-making problems. Wei and Gao [198] proposed generalized dice SM for PFS and applied these to building material recognition. In decision-making problems, the decision-makers have to give their results in the form of different fuzzy frameworks but if there is a large amount of data then it is difficult to cover it in fuzzy framework. For example, in the case of "daily mean temperature of a city", there could be multiple readings taken at different stations within that particular city, all of which are presented in the dataset. To convert this data into one IVPFN or any other fuzzy framework there are some methods discussed in [199, 200, 201]. Some decision-making problems for different tools for uncertainty are discussed in [25, 202, 203, 204, 205, 206, 207, 208]. Some new similarity measures in the framework of spherical fuzzy sets, including cosine similarity measure, grey similarity measure, and set-theoretic similarity measure were proposed by Ullah et al. [25]. These similarity measures were applied to a building material recognition problem. The novel cosine similarity measure under a spherical fuzzy environment was investigated by Ratiq et al. [209]. Some different similarity measures for spherical fuzzy sets based on cosine and cotangent functions were defined by Wei et al. [210].

1.9 Research Gaps

While the literature discussed above effectively addresses the uncertainty associated with extending fuzzy sets and fuzzy matrices in the presence of uncertain information, subsequent studies have identified gaps in the existing research.

1. We have seen that the TP is solved by different approaches for the extension of fuzzy sets. The concept of interval-valued Pythagorean fuzzy sets and spherical fuzzy sets

are useful to explore for the TP. But so far the work has not been done for interval-valued Pythagorean fuzzy and spherical fuzzy sets for TP.

2. In literature, the methodologies have been reported for solving assignment problems. Furthermore, the concept of assignment problem has been extended to fuzzy sets, intuitionistic fuzzy, and interval-valued intuitionistic fuzzy sets and solved by different methods. But, still, work has not been done for Pythagorean fuzzy domain for the assignment problem.

3. Considering picture fuzzy matrix theory, analogous to PFM, a PFM is outlined by a membership degree, neutrality degree, and non-membership degree, respectively. Several authors have significantly solved many decision-making problems in the realm of PFMs. However, it has been noticed that membership, neutrality, and non-membership of PFM are taken to be as a point. If we try to consider the membership, neutrality, and non-membership values as an interval, it is practically useful in the case of real-life problems. Additionally, when there are additional sorts of uncertainty in data, present strategies are ineffective in dealing with them. In these instances, data should be gathered or displayed in the form of an interval-valued picture fuzzy matrix meaning.

4. In order to deal with SFM, several SFMs applications have been introduced in the literature. However, at a point scale set is still used to represent the data; therefore, the information does not retain its literal meaning. If we try to consider the membership, neutrality, and non-membership values as an interval, it is practically useful in the case of real-life problems. Additionally, When there are additional sorts of uncertainty in data, present strategies are ineffective in dealing with them. In these instances, data should be gathered or displayed in the form of an interval-valued spherical fuzzy matrix meaning.

5. In many real-world problems, the concept of a fermatean matrix is crucial. To deal with the ambiguous problem environment, various fermatean matrices applications have been developed at a point scale for the membership and non-membership degrees. If we try to consider the membership and non-membership values as an interval, it is practically useful in the case of real-life problems. Additionally, When there are additional sorts of uncertainty in data, present strategies are ineffective in dealing with them. In these instances, data should be gathered or displayed in the form of an interval-valued fermatean fuzzy matrix meaning.

1.10 Motivation

Human beings are always making decisions in an uncertain environment to tackle ill-structured problems successfully. As our physical world is always changing, it seems to be a common attribute of human intelligence to make decisions with knowledge and solve the pertinent problems that we encounter within our real-life conflicting situations. Decision-making is guided by the cognitive process, which depends on an individual's perception, i.e., in the form of natural language. The process attracts the researchers to model and present the uncertain data using new tools and methodologies. Researchers in fuzzy theory and its allied areas have allowed us to deal with the cognitive process of human behavior concerning incomplete and vague information. Several methods have been developed in the literature for managing such information. Generally, the problems present quantitative aspects easily accessible through precise numeric values; nevertheless, in some cases, they present qualitative aspects that are complicated to handle by precise numeric values. The introduction of various optimization techniques has improved the accuracy of TP, AP, and matrix problems and facilitated the processes by managing uncertain information. The fuzzy sets as well as fuzzy matrices, have delivered remarkable advancement in the uncertainty domain pertaining to the field of TP, AP and matrix theory. Moreover, many works have been done by the researchers on TP and AP under FS, IFS, and so on. Also, matrix problems have been examined by many authors in the literature under FM, IFM, etc., that too in a very straightforward way so that information loss occurs in a huge manner. Further, the interval-valued Pythagorean fuzzy, spherical fuzzy, and Pythagorean fuzzy information prevailing in the TP and AP problems has not yet been taken care of in the existing theories. Apart from this, the representation of information in terms of interval-valued fermatean fuzzy, interval-valued picture fuzzy and interval-valued spherical fuzzy matrix may only eradicate the fuzziness of the problem. This thesis is motivated by the existing literature and the research gaps prevailing in the existing literature. The actual motivation behind this thesis is that no such methodologies are defined to address TP and AP in IVPyFS, SFS and PyFS, and matrix in IVFFM, IVPFM and IVSFM. This extended framework aims to provide a more realistic and flexible model that can effectively handle imprecision, ambiguity, and uncertainty in decision-making processes across diverse domains such as decision support systems, artificial intelligence, risk assessment and many more. Therefore, in this thesis, we have analyzed the uncertainty at a point as well as interval emerging in the TP, AP, and matrix problems due to unavoidable real-life circumstances. We proposed the methodologies for meticulously modeling the uncertainty in intricate TP, AP and matrix problems with broad applications and further validated these methodologies with appropriate data. All these methodologies are capable enough to deal with uncertainty or vague information. Henceforth, the primary purpose of this current study is to develop accurate and innovative techniques to address imprecise concepts in LPP or matrix problems by emphasizing the aforementioned momentous research gaps.

1.11 Organization of the Thesis

This thesis is organized into six chapters. Chapter 1 is introductory. The outline of the proposed research study in Chapter 2 onwards is explained as follows:

Chapter 2 presents a methodology for solution of transportation problem using the interval-valued Pythagorean fuzzy approach and solution of the transportation problem using spherical fuzzy approach. The TP set utilized in the current theory is a balanced TP set having all interval-valued Pythagorean fuzzy and spherical fuzzy terms distributed symmetrically. This study provides the concept of TP, which is considered to be a new study in the literature addressing such types of TP. Even though several methodologies are available in the literature to solve TP under uncertain conditions because in real-world applications, all the parameters of the transportation problems may not be known precisely due to uncontrollable factors. The approach proposed by us is an improvement to offer acceptable results faster. In conventional transportation problems, it is assumed that the decision maker is sure about the precise values of transportation cost, availability, and demand of the product. In real-world applications, all the parameters of the transportation problems may not be known precisely due to uncontrollable factors. This type of imprecise data is not always well represented by random variables selected from a probability distribution. Fuzzy numbers may represent this data. So, fuzzy and its extension decision-making method is needed here. Several pioneering studies have been put forward to support the context of TP. However, no methodology has been yet applied to transportation problems under interval-valued Pythagorean fuzzy and spherical fuzzy sets. Henceforth, in **Chapter 3**, we propose a newly constructed method to handle AP with uncertain parameters. Addressing decision-making issues with AP to apprehend uncertainty has received much interest. However, with rapid societal development, APs have become increasingly complex and challenging. In such a scenario, it is necessary to introduce models that are lucrative for defining the AP and can synthetically explain the notion of the Pythagorean fuzzy assignment problem. Numerous APs are defined to describe the uncertainty with different fuzzy sets. However, the existing research on AP is based on PyFS and is not defined in the literature. For that purpose, we consider that have practical applications for assignment problems. The AP consists of PyFS terms. Hence, in this chapter, we define and solve the Pythagorean fuzzy assignment problem (PFAP) using the proposed similarity measure and a score function. Numerical examples are also given to explain the methodology.

Researchers have been really interested in using matrix theory for decision-making problems. Recently, many researchers have found ways to improve matrices even more. In literature, matrix theory in decision-making based on IVPF terms is not defined. Henceforth, to thoroughly capture the uncertain and vague information in more flexible manner, the work proposed in **Chapter 4** aims to introduce the idea of picture fuzzy matrix (PFM) into interval-valued picture fuzzy matrix (IVPFM).

Further in **Chapter 5**, the work done in Chapter 4 is extended to the Interval-valued spherical fuzzy matrix and its applications in the multi-attribute decision-making process. The limitation arises in the concept of interval-valued picture fuzzy matrix when the sum of the upper degree of membership, neutral membership, and the upper degree of non-membership exceeds the interval [0, 1]. This type of matrix is a useful concept for multiple criteria analysis. Further, to thoroughly capture the uncertainty involved in the matrix, they are converted into the interval-valued spherical fuzzy matrix and describe the uncertainties of real-life problems. This model overcomes the inherent limitation of the existing fuzzy matrix where only the fuzziness of concepts is evaluated.

Chapter 6 introduces the concept of interval-valued fermatean fuzzy matrix and its application. The IVFFM is outlined by an interval membership degree and an interval non-membership degree. By using IVFFM, the DMs can consider an interval hesitancy degree, where they cannot simply convey their perception using one interval term. It is the newest tool for dealing with imprecision. The IVFFM is the extension of FFM and takes it a step further by allowing for a range of possible membership values, addressing additional layers of uncertainty in the modeling process A summary followed by the future scope of the research work is evinced to conclude the thesis after Chapter 6.

Table 1.2: Summary of literature based on fuzzy TP

Title	Year	Paper	Description	Advantages/Limitations
A fuzzy approach to the transportation problem.	1984	[118]	The notion of fuzzy sets is introduced. A heuristical discussion of fuzzy TP and its relationship to classical TP is presented.	• The intuitive meaning of fuzzy TP is defined. • Usual properties of fuzzy sets are provided. • Limited non-negative fuzzy number.
Fuzzy Transportation Problem: A General Analysis	1987	[118]	The transportation problems have a recognized importance. Their range of applications can be enlarged when some fuzziness in its formulation is accepted. This paper is devoted to the study of a resolution method for fuzzy transportation problems.	• The intuitive meaning of fuzzy TP is defined. • Para metric type algorithm is proposed. • More efficient than others existing in the current literature because of the lower dimensionality.
Fuzzy Optimization: An Appraisal	1989	[236]	This paper takes a general look at core ideas that make up the burgeoning body of Fuzzy mathematical programming emphasizing the methodological view. Although Fuzzy mathematical programming has enjoyed a rapidly increasing acceptance within the scientific community, some technical hurdles exist to hinder unanimity. Reasons for this as well as possible ways for improvement are also discussed.	• The intuitive meaning of fuzzy optimization is defined. • Restricted to fuzzy sets.
Transportation route choice model using fuzzy inference technique.	1990	[241]	The model consists of rules that indicate the degree of preference for each route given the approximate travel time of the two routes. Applying the model to each driver and then aggregating the individual preferences, a fuzzy network loading algorithm assigns traffic to each route.	• Limited to fuzzy TP.
Characteristics of the fuzzy LP transportation problem for civil engineering application	1991	[238]	This paper addresses the type of problems which are suited for fuzzy LP transportation modeling.	• useful for the process of determining the suitable level of supply or demand within a known range. • Usual properties of fuzzy sets are provided.
Fuzzy programming approach to multicriteria decision making transportation problem	1992	[239]	This paper presents an application of fuzzy linear programming to the linear multiobjective transportation problem. It gives efficient solutions as well as an optimal compromise solution. For this method, an efficient Fortran program has been developed based on the fuzzy linear programming algorithm, which is an extended version of the simplex algorithm.	• Require lesser computational time and give better compromise solutions • Fuzzy linear programming is a more suitable method for the multiobjective transportation problem.
Interval and fuzzy extensions of classical transportation problems	1993	[240]	Classical, interval, and fuzzy TP, have been introduced and analyzed in this paper.	• The intuitive meaning of interval TP is defined. • More efficient than others existing in the current literature because of the interval number.
Fuzzy sets theory applications in traffic and transportation	1994	[241]	The purpose of this paper is to point out the possibility of using fuzzy set theory in solving complex traffic and transportation problems.	• To develop approximate reasoning algorithms related to the parts of the network with a considerable number of signalized intersections. • The basic elements of fuzzy set theory, fuzzy arithmetic, fuzzy mathematical programming, and especially fuzzy logic are discussed.
Solving bicriteria solid transportation problem with fuzzy numbers by a genetic algorithm	1995	[242]	The proposed Genetic Algorithms (GAs) are incorporated with problem-specific knowledge and the decision maker's degree of optimism, and the criteria space approach for bicriteria linear program conductive to find out the nondominated extreme points in the criteria space.	• The intuitive meaning of the bicriteria solid transportation problem with fuzzy numbers is defined. • New interactive fuzzy satisficing method. Also, the algorithm may be adapted to nonlinear multiobjective solid transportation problems.
An approach for fuzzy linear multicommodity transportation problems and its application	1995	[243]	In this paper, a general model of fuzzy linear multicommodity problem was presented	• Limited to triangular fuzzy number.
An Integer Fuzzy Transportation Problem	1996	[244]	In this paper, the integer fuzzy transportation problem is discussed, in which it is assumed that every value of supply and demand is an integer and that the values of commodities to be transported are all integers.	• The ordinary transportation problem is not useful to solve the case if the total value of integral supplies is less than that of integral demands. It is possible to solve this infeasible problem by relaxing all the supplies and demands as fuzzy supplies and demands. That is, such a case can be formulated and solved as the integer fuzzy transportation problem.
Improved genetic algorithm for solving multi-objective solid transportation problems with fuzzy numbers.	1997	[245]	Using fuzzy numbers, and propose an improved genetic algorithm that incorporates an ordering of fuzzy numbers based on integral values and a processing method for competing objective functions into the conventional genetic algorithm. It becomes possible to directly obtain the optimal solution set, and it also becomes possible to easily provide the best solution that satisfies the decision-maker	• Suitable to the multicriteria optimization problem.

Table 1.2: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
Fuzzy integer transportation problem.	1998	[246]	Proposed an algorithm solving the integer fuzzy transportation problem (with fuzzy supply and demand values as well as with the fuzzy goal) in the sense of maximizing the joint satisfaction of the goal and of the constraints.	• Limited to fuzzy integer concept.
Multi-objective transportation problem with interval cost, source, and destination parameters	1999	[247]	In this paper, we focus on the solution procedure of the multiobjective transportation problem (MOTP) where the cost coefficients of the objective functions, and the source and destination parameters have been expressed as interval values by the decision maker.	• The TP seems to be suitably expressed with interval values, where the cost coefficients of the objective functions, and the source and destination parameters have been expressed in interval values. Whose values are intervals rather than real numbers.
Transportation projects selection process using fuzzy sets theory in fuzzy intervals	2000	[248]	This work presents an efficient technique for the selection of transportation projects using fuzzy sets theory. The selection procedure is a multiple objectives process, and projects are rated both on a quantitative and qualitative basis, using linguistic variables.	• The fuzzy compromise programming approach proposed here is a more suitable method for the multiobjective transportation problem and other multiobjective programming problems.
Fuzzy programming and profit and cost allocation for a production and transportation problem.	2001	[254]	This paper addresses production and transportation challenges in a housing material manufacturer, focusing on optimizing production and transportation planning across multiple factories and regions. It formulates mixed zero–one programming problems to minimize costs while considering factory capacities and regional demands. Additionally, fuzzy programming techniques are integrated to ensure stable production and satisfactory product supply in uncertain environments.	• There may be a lack of standardized methodologies and guidelines for applying fuzzy programming to profit and cost allocation in production and transportation problems.
Transportation of the fuzzy amounts using the fuzzy cost	2002	[250]	In this article, a procedure based on fuzzy one-point method is proposed to solve fuzzy transportation problems where all the parameters are taken as Pentagonal fuzzy numbers.	• Limited to Pentagonal fuzzy numbers.
Fuzzy mathematical programming applied to multi-level programming problems	2003	[251]	An alternative FMP method for a linear/nonlinear MLPP of maximization- and minimization-type objectives which gives better optimal solutions.	• Limited to triangular fuzzy number.
Solving fuzzy transportation problems based on the extension principle	2004	[252]	Develops a procedure to derive the fuzzy objective value of the fuzzy transportation problem, in that the cost coefficients and the supply and demand quantities are fuzzy numbers. The idea is based on the extension principle.	• Objective value is expressed by the membership function rather than by a crisp value, more information is provided for making decisions.
An intuitionistic fuzzy goal programming approach to vector optimization problem	2005	[253]	This paper proposes an intuitionistic fuzzy goal programming (IFGP) approach for solving vector optimization problems under uncertainty.	• Limited to intuitionistic fuzzy number.
Two-stage fuzzy transportation problem.	2006	[118]	Presents a two-stage cost-minimizing fuzzy transportation problem in which supplies and demands are trapezoidal fuzzy numbers. A parametric approach is used to obtain a fuzzy solution.	• The two-stage approach provides a more realistic representation of the transportation process. In many real-world scenarios, the transportation of goods may involve different stages, such as production and distribution. By considering these stages separately, the model can better capture the complexities of the actual system. • Limited to trapezoidal fuzzy numbers.
Fuzzy fixed charge solid transportation problem and algorithm	2007	[255]	Considered a bi-criteria transportation problem on a fuzzy network and developed an algorithm to find non-dominated solutions.	• Adaptability to ambiguous information. • Complexity of implementation.
Fuzzy transportation problem with random transportation costs.	2009	[256]	Considered a bi-criteria transportation problem on a fuzzy network and developed an algorithm to find non-dominated solutions.	• Allowing decision-makers to account for variability and randomness in costs, leading to more robust and adaptable solutions.
Fuzzy programming technique to solve the bi-objective transportation problem.	2010	[257]	In the present paper, fuzzy linear and non-linear programming techniques have been used to find an optimal compromise solution for the two-objective transportation problem.	• Using fuzzy programming to solve a bi-objective transportation problem allows for more flexible and realistic solutions by considering uncertainties and balancing conflicting goals, making it well-suited for complex transportation scenarios.
An approach for solving fuzzy transportation problem	2011	[258]	In this paper, we shall study fuzzy transportation problem, and we introduce an approach for solving a wide range of such problems by using a method that applies it for ranking of the fuzzy numbers.	• This method can be used for all kinds of fuzzy transportation problems, whether triangular and trapezoidal fuzzy numbers with normal or abnormal data. • The new method is a systematic procedure, easy to apply, and can be utilized for all types of transportation problems whether maximize or minimize the objective function.
The transportation problem in an intuitionistic fuzzy environment	2012	[259]	This paper investigates transportation problems in which supplies and demands are intuitionistic fuzzy numbers.	• The advantage of solving the transportation problem in an intuitionistic fuzzy environment lies in its capability to handle uncertainty, vagueness, and hesitation associated with transportation parameters • Limited to triangular intuitionistic fuzzy number.

Table 1.2: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
A new method for solving intuitionistic fuzzy transportation problem	2013	[260]	In this paper, we find the optimal solution for the intuitionistic fuzzy transportation problem using the new method where the supply and demand are triangular intuitionistic fuzzy numbers.	• This new algorithm takes very less iterations to obtain the optimum solution. • The present algorithm will be very helpful for decision-makers who are dealing with logistic and supply chain problems.
The optimal solution of an intuitionistic fuzzy transportation problem.	2014	[261]	This paper deals with the optimal solution of an intuitionistic fuzzy transportation problem whose quantities are intuitionistic triangular fuzzy numbers.	• This method can also be applied for mixed intuitionistic fuzzy transportation problem. • Limited to intuitionistic fuzzy number.
A method for solving unbalanced intuitionistic fuzzy transportation problems	2015	[262]	proposed the algorithm for solving unbalanced intuitionistic fuzzy transportation problems.	• The main advantage of this method is computationally very simple, easy to understand and also the optimum objective value obtained by method is physically meaningful. • The solution obtained by this method is always optimal. • Proved some theorem.
A new approach for solving intuitionistic fuzzy transportation problem of type-2	2016	[105]	IFTP-2 has been proposed. Using a linear ranking function for TIFNs, new methods are proposed to find the starting BFS and optimal solution of IFBTP.	• it can be extended to solve transportation problem having nonsymmetrical data, that is, when the costs as well as availabilities and demands are of different types. • The methodologies are very comfortable. It will be very useful in solving transportation problems having uncertainty as well as hesitation in the prediction of the transportation cost.
Solving intuitionistic fuzzy transportation problem using linear programming	2017	[263]	The present paper focuses on the two methods for solving intuitionistic fuzzy TP. One of the methods uses the intuitionistic fuzzy programming technique together with the three different membership functions—linear, exponential, and hyperbolic and the other method uses crisp linear programming taking intuitionistic fuzzy data in both the methods for the cost objective functions in the TP.	• Intuitionistic fuzzy transportation problems as linear programs, it leverage the computational efficiency of existing solvers. • This advantage is crucial when dealing with intuitionistic fuzzy transportation problems, as it allows for a clear and well-defined representation of the problem.
Transportation problem under interval-valued intuitionistic fuzzy environment	2018	[108]	In the present paper, discussed the TP under the IVIFS and defined the new defuzzified method. Also, stated and proved some theorems/results related to expected values.	• Defined expected interval and expected values of IVIF numbers. • Stated and proved some theorem/results related to expected values.
A Pythagorean fuzzy approach to the transportation problem	2019	[112]	Introduces a simplified presentation of a new computing procedure for solving the fuzzy Pythagorean transportation problem.	• Limited to Pythagorean fuzzy number.
A picture fuzzy approach to solving transportation problem	2020	[170]	This paper presents the introduction of picture fuzzy sets (PFS), designed to better accommodate imprecise, inconsistent, and uncertain data. Three types of ranking are proposed, employing the critical path method, wherein supply, demand, and unit costs are depicted as picture fuzzy numbers. The objective is to minimize transportation costs by comparing solutions obtained through various ranking methods.	• Limited to picture fuzzy number. • This enables decision-makers to consider a wider range of factors and make more informed decisions.
Solution of Transportation Problem Under Spherical Fuzzy Set	2021	[224]	In this study, explores the resolution of the spherical fuzzy transportation problem (SFTP) by introducing three distinct models and also discusses the arithmetic operations.	• SFTP provides a more comprehensive representation of uncertainty compared to traditional transportation problems by incorporating spherical fuzzy sets. • Limited to spherical fuzzy number.
A new score function based fermatean fuzzy transportation problem	2021	[223]	The aim of this paper is to solve the transportation problem where supply, demand, and transportation costs are Fermatean fuzzy numbers (FFNs)	• FFTP provides a more comprehensive representation of uncertainty compared to IFTP by incorporating fermatean fuzzy sets. • Define the new score function based on FFN. • Limited to fermatean fuzzy number.
Fractional transportation problem under interval-valued fermatean fuzzy sets	2022	[166]	The purpose of this paper is to define a triangular interval-valued fermatean fuzzy number (TIVFFN) and its arithmetic operations.	• Limited to TIVFFN
Solution of transportation problem using interval-valued Pythagorean fuzzy approach	2023	[222]	An algorithm has been devised to address the interval-valued Pythagorean fuzzy transportation problem across three distinct cases. Additionally, a novel scoring function has been formulated for IVPyFS and compared against the score function proposed by Peng and Yang [113].	• Defined the new score function for IVPyFS. • Limited to IVPyFS.

Table 1.3: Summary of literature based on fuzzy assignment problem

Title	Year	Paper	Description	Advantages/Limitations
The Hungarian method for the assignment problem	1955	[125]	It is shown that ideas latent in the work of two Hungarian mathematicians may be exploited to yield a new method of solving this problem.	• The intuitive meaning of crisp AP is defined. • Usual Hungarian Method is provided. • The expressions are limited to crisp numbers.
On a fuzzy assignment problem	1985	[126]	The notion of fuzzy AP is introduced. A heuristical discussion of fuzzy AP and their also proved some theorems.	• The intuitive meaning of fuzzy AP is defined. • Usual Hungarian Method is provided. • The expressions are limited to a fuzzy number.
Fuzzy optimal assignment problem	1987	[264]	The notion of fuzzy AP is introduced. A heuristical discussion of fuzzy AP and used the graph theory.	• The intuitive meaning of fuzzy AP is defined • Usual Graph Method is provided. • The expressions are limited to a fuzzy number.
A fuzzy approach to the vehicle assignment problem	1996	[265]	A develops a model based on fuzzy logic that assigns different types of vehicles to planned transportation tasks. The model is tested on a real numerical example. The results of testing the model indicate that it outperforms the work of an experienced dispatcher.	• The intuitive meaning of fuzzy AP is defined. • Used Heuristic Algorithms is provided. • The expressions are limited to a fuzzy number.
Solving an assignment–selection problem with verbal information and using genetic algorithms	1999	[266]	The assignment selection problems deal with finding the best one-to-one match for each of the given number of candidates to positions. Different benefits or costs are involved in each match and the goal is to minimize the total expense.	• The intuitive meaning of fuzzy AP is defined. • Used Genetic algorithms is provided. • The expressions are limited to fuzzy number. • Selection problems.
Fuzzy route choice model for traffic assignment	2000	[267]	A new route choice model taking account of the imprecisions and the uncertainties lying in the dynamic choice process. This model makes possible a more accurate description of the process than those (deterministic or stochastic) used in the AP.	• The intuitive meaning of fuzzy AP is defined. • Used Fuzzy model and LOGIT model (a widely used stochastic discrete choice model) . • The expressions are limited to fuzzy number.
A labeling algorithm for the fuzzy assignment problem	2004	[268]	This paper concentrates on the assignment problem where costs are not deterministic numbers but imprecise ones. Here, the elements of the cost matrix of the assignment problem are subnormal fuzzy intervals with increasing linear membership functions, whereas the membership function of the total cost is a fuzzy interval with decreasing linear membership function.	• The intuitive meaning of fuzzy AP is defined. • Used labeling method. • one may consider the fuzzy AP where the costs are triangular, trapezoidal, or other types of membership functions.
Method for solving fully fuzzy assignment problems using triangular fuzzy numbers.	2009	[277]	A new method is proposed to find the fuzzy optimal solution of fuzzy assignment problems by representing all the parameters as triangular fuzzy numbers.	• The final result are in fuzzy number. • It is easy to apply the proposed method, compare to the existing method,to find the fuzzy optimal solution of fuzzy AP occurring in fuzzy number.
Application of fuzzy ranking method for solving assignment problems with fuzzy costs.	2010	[270]	In this paper cost has been considered to be triangular or trapezoidal fuzzy numbers and Yager's ranking method has been used for ranking the fuzzy numbers. The fuzzy assignment problem has been transformed into crisp assignment problem in the LPP form and solved by using LINGO 9.0.	• The intuitive meaning of triangular or trapezoidal fuzzy numbers AP is defined. • Used Yager's ranking method. • only consider the triangular or trapezoidal fuzzy number.
Solution method for fuzzy assignment problem with restriction of qualification.	2006	[271]	The paper presents a mathematical model for the fuzzy assignment problem with qualification restrictions. It establishes a transforming method to convert this into a traditional assignment problem, facilitating solution assessment. By converting the beneficial matrix into the decision matrix, and subsequently into the solution matrix when feasible, the problem's solvability is determined.	• The parameter such as efficiency, time or cost be fuzzy variables. • Used average method. • It is easy to apply.

Table 1.3: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
Elitist genetic algorithm for assignment problem with imprecise goal	2007	[272]	The objective of this research paper is to solve a generalized assignment problem with imprecise cost(s)/time(s) instead of a precise one by an elitist genetic algorithm (GA). Here, the impreciseness of cost(s)/time(s) has been represented by interval-valued numbers, as interval-valued numbers are the best representation than others like random variable representation with a known probability distribution and fuzzy representation.	• Interval-valued numbers are the best representation than others like random variable representation • Used genetic algorithm.
Methods for solving fuzzy assignment problems and fuzzy traveling salesman problems with different membership functions.	2011	[215]	A fuzzy assignment problem and a fuzzy traveling salesman problem are solved. The proposed methods are method based on fuzzy linear programming formulation and method based on classical methods easy to understand and apply to find optimal solution to them occurring in real-life situations	• The main advantage of both methods is that these can be used for solving both types of fuzzy assignment problems and in addition, both types of fuzzy traveling salesman problems.
Solution of a class of intuitionistic fuzzy assignment problem by using similarity measures.	2012	[216]	A more realistic Intuitionistic Fuzzy Assignment Problem, with and without restrictions on job cost and person-cost based on his/her efficiency or qualification has been introduced.	• The intuitive meaning of intuitionistic fuzzy AP is defined. • Used labeling method. • one may consider the fuzzy AP where the costs are triangular, trapezoidal, or other types of membership functions.
Method for solving fuzzy assignment problem.	2013	[273]	A simple yet effective method was introduced to solve fuzzy assignment problem by using ranking of fuzzy numbers.	• This method can be used for all kinds of fuzzy assignment problem, whether triangular and trapezoidal fuzzy numbers. • Used Ones Assignment Method and Robust's ranking method. • The method is a systematic procedure, easy to apply and can be utilized for all type of assignment problem whether maximize or minimize objective function.
A method for solving balanced intuitionistic fuzzy assignment problem	2014	[274]	A solution of an assignment problem in which cost coefficients are triangular intuitionistic fuzzy numbers.	• This method can be used for all kinds of intuitionistic fuzzy assignment problems, whether triangular and trapezoidal intuitionistic fuzzy numbers. • In solving other types of problems like project schedules, transportation problems, and network flow problems. • The method is a systematic procedure, easy to apply, and can be utilized for all types of assignment problems whether maximize or minimize the objective function
On solution of interval-valued intuitionistic fuzzy assignment problem using similarity measure and score function.	2014	[225]	Proposed two algorithms-one based on the degree of similarity measures and another based on the score function to get the optimal assignment for the interval-valued intuitionistic fuzzy assignment problem.	• This method can be used for all kinds of interval-valued intuitionistic fuzzy assignment problem. • In solving other types of problems like project schedules, transportation problems, and network flow problems. • The method is a systematic procedure, easy to apply and can be utilized for all types of assignment problems whether maximize or minimize objective function. • Restricted to interval-valued intuitionistic fuzzy sets.

Table 1.3: (Continued)

Title	Year	Paper	Description	Advantages/Limitations
A method for solving intuitionistic fuzzy assignment problem using branch and bound method.	2015	[275]	In this paper, we investigate an assignment problem in which cost coefficients are triangular intuitionistic fuzzy numbers. The fuzzy assignment problem has been converted into a crisp assignment problem using the ranking procedure Branch bounded method has been applied to find an optimal solution.	• This method can be used for all kinds of interval-valued intuitionistic fuzzy assignment problems. • In solving other types of problems like project schedules, transportation problems, and network flow problems. • The method is a systematic procedure, easy to apply and can be utilized for all types of intuitionistic fuzzy assignment problems whether maximize or minimize objective function. • Restricted to triangular intuitionistic fuzzy numbers.
An algorithm for solving assignment problems with costs as generalized trapezoidal intuitionistic fuzzy numbers.	2015	[276]	An algorithm has been developed for solving assignment problems with costs as generalized trapezoidal intuitionistic fuzzy numbers by using the given ranking method.	• This method can be used for all kinds of interval-valued intuitionistic fuzzy assignment problems. • In solving other types of problems like project schedules, transportation problems and network flow problems. • The method is a systematic procedure, easy to apply and can be utilized for all types of intuitionistic fuzzy assignment problems whether maximize or minimize objective function. • Restricted to triangular intuitionistic fuzzy numbers.
A simple method for solving fully intuitionistic fuzzy real-life assignment problem	2016	[277]	A new method called PSK (P.Senthil Kumar) method has evolved which provides the opportunity to find the optimal objective value of the fully intuitionistic fuzzy assignment problem (FIFAP)in terms of triangular intuitionistic fuzzy number.	• This method can be used for all kinds of triangular intuitionistic fuzzy assignment problem. • method is computationally very simple, easy to understand • Restricted to triangular intuitionistic fuzzy numbers.
Method for solving fuzzy assignment problem using ones assignment method and robust's ranking technique.	2017	[278]	In this paper first, the proposed fuzzy assignment problem is formulated to the crisp assignment problem in the linear programming problem (LPP) form and solved by using Ones assignment method and using Robust's ranking method for the fuzzy numbers.	• method is computationally very simple, and easy to understand. • Restricted to generalized trapezoidal fuzzy numbers.
Placement of staff in LIC using fuzzy assignment problem.	2018	[214]	In this paper, maximum fuzzy assignment problem for the placement of four candidates for four different posts is solved successfully. The magnitude ranking method is used to convert fuzzy assignment problem into crisp assignment problem and further three methods namely the Hungarian method, Matrix Ones Assignment method and the direct method are used to find out the optimal assignment and solution.	• Limited to FS.
Optimization of fuzzy bi-objective fractional assignment problem.	2019	[279]	A fuzzy bi-objective fractional assignment problem has been formulated. Here the parameters are represented by triangular fuzzy numbers and the fuzzy problem is transformed into standard crisp problem through alpha-cut and then the compromise solution is derived by fuzzy programming.	• fractional assignment problem has been formulated with two conflicting objective functions in a fuzzy environment. • Restricted to fuzzy bi-objective fractional function.
Fuzzy assignment problems.	2020	[280]	the assignment costs are considered as imprecise numbers described by triangular fuzzy numbers. Moreover, the fuzzy assignment problem has been transformed into a crisp assignment problem using ranking function for fuzzy costs matrix and solves it by Hungarian method.	• Restricted to triangular fuzzy numbers. • Method is computationally very simple, easy to understand
Optimal solution of fuzzy assignment problem with centroid methods.	2021	[212]	The optimal solution to the fuzzy assignment problem is introduced in this paper, employing the Fourier elimination method.	• method is computationally very simple, and easy to understand. • Assigning and implementing the persons to work successfully on one–one basis so that it increases the number of potential bidders.
One's fixing method for a distinct symmetric fuzzy assignment model.	2022	[281]	this work, we proposed a new algorithm to compute an optimal solution for assignment problems. This proposed method differs in the procedure of finding the optimal solution with the classical Hungarian method in allocating zero's.	• method is computationally very simple, easy to understand. • The proposed technique provides the optimal solution to distinct symmetric fuzzy assignment problem.
A novel similarity measure and score function of Pythagorean fuzzy sets and their application in assignment problem.	2023	[226]	A method to solve the Pythagorean fuzzy assignment problem (PFAP) using the proposed similarity measure and a score function.	• A new method is defined and it is very simple, and easy to understand. • Pythagorean fuzzy set (PFS) has a larger domain space than the fuzzy sets and intuitionistic fuzzy set to describe the membership grade.

Chapter 2

A Methodology for Solving Transportation Problem under Interval-Valued Pythagorean Fuzzy and Spherical Fuzzy Environment

This chapter [1] [2] is primarily elaborated on the TP under the interval-valued Pythagorean fuzzy and spherical fuzzy environment. This class of TP consists of real-life problems having transportation cost, availability and demand of the product are imprecise and are based on the interval-valued Pythagorean fuzzy set (IVPyFS) and spherical fuzzy set (SFS). In this direction, a novel solution to the transportation problem using interval-valued Pythagorean fuzzy and spherical fuzzy approaches is defined to solve such TP, and a new methodology is proposed to compute the optimal solution. The proposed methodology is sufficient to evaluate such TP and presents three different models of the interval-valued Pythagorean fuzzy transportation problem (IVPyFTP) and spherical fuzzy transportation problem (SFTP). The IVPyFS and SFS arithmetic operations and algorithms are discussed here. Finally, some hypothetical working examples are given to show the practical applicability of the defined methodology and simplified by the proposed score function. Thus, this

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[1] **The content of this chapter is based on the research paper "Solution of transportation problem using the interval-valued Pythagorean fuzzy approach", In: *Advanced Engineering Optimization Through Intelligent Techniques*, Springer, 10 (1), 2199368 (2023). DOI: https://link.springer.com/chapter/10.1007/978-981-19-9285-8-33. (Conference Proceedings Citation Index (CPCI))**

[2] **The content of this chapter is based on the research paper "Solution of Transportation Problem Under Spherical Fuzzy Set.", In: *2021 IEEE 6th International Conference on Computing, Communication, and Automation (ICCCA)*, IEEE, 444-448,(2023). DOI: https://10.1109/ICCCA52192.2021.9666372. (Conference Proceedings Citation Index (CPCI))**

is the new method to tackle the uncertainty in real-life transportation problems. The proposed method is also applied to solve interval-valued picture fuzzy transportation problems (IVPFTP) and interval-valued spherical fuzzy transportation problems (IVSFTP).

2.1 Introduction

TP is a linear programming problem and was originally formulated by Hitchcock [109]. In a classical transportation problem, the decision-maker is certain about the exact value of transportation cost, supply, and demand but in real-life, all these values depend on various factors like fuel costs, condition of whether and customer loyalty, etc. which are not fixed (imprecise). In situations like these, the application of FSs theory becomes important for effectively managing such conditions. Hence, fuzzy transportation methods assume greater significance in such contexts. Zadeh [2] introduced the concept of FSs in 1965, addressing imprecision and vagueness in real-world situations. The tremendous growth and advancement in TP have attracted researchers to work in the field of FTP. In this connection, Chanas et al. [101] used the Zadeh [2] concept to deal with the imprecision of real-world transportation problems. Chanas et al. [101] proposed an approach to the transportation problems. Dinagar and Palanivel [102] solved the transportation problem under a fuzzy environment. Kumar and Amarpreet [111] have given the concept of application of classical transportation method for solving fuzzy transportation problem. Many researchers have worked on fuzzy transportation but they consider only the membership function of the element. Then, Kumar and Hussain [119] consider the TP in an intuitionistic fuzzy environment. Gani and Abbs [103] solved the IFTP by a new ranking method in which supply and demand are triangular intuitionistic fuzzy numbers. Singh & Yadav [105] proposed the transportation problem in which costs are in triangular intuitionistic fuzzy numbers. Dubey & Mehra [104] worked on linear programming with the triangular intuitionistic fuzzy numbers. Bharti [110] solved the trapezoidal intuitionistic fuzzy fractional transportation problem. Instead of a single point of MD and NMD Mishra et al. [107] worked on transportation problems under the interval-valued intuitionistic fuzzy environment. Arora [106] given an algorithm for interval-valued fuzzy fractional transportation problem. Bharati and Singh [108] solved the transportation problem under the interval-valued intuitionistic fuzzy environment.

Building on the foundation of FS theory, Yager [15] introduced the theory of Pythagorean fuzzy set (PyFS) and underscored its critical relevance in TP scenarios. Within PyFS, an element is characterized by its MD and NMD, constrained by their quadratic sum equating to unity. Consequently, PyFS theory has demonstrated its efficacy and adaptability in addressing transportation problems featuring uncertain information. There are various methods in the field of PyFS to solve TP. Kumar et al. [112] extended the IFTP into the

Pythagorean fuzzy transportation problem. Karasan et al. [37] worked on a landfill site selection problem using the novel Pythagorean fuzzy AHP method. Ghosh et al. [122] simplified the multi-objective solid transportation problem with preservation technology using Pythagorean fuzzy sets. Jeyalakshmi et al. [123] also solved the Pythagorean fuzzy transportation problem via Monalisha Technique. Saikia et al. [124] defined the new advanced similarity measure for Pythagorean fuzzy sets and its applications in transportation problems. Additionally, Peng and Yang [16] proposed the concept of interval-valued Pythagorean fuzzy sets (IVPyFS), where the MD and NMD of an element are presented as interval values rather than real numbers. In contrast to PyFS, the MD, and NMD in IVPyFS are represented as intervals rather than precise numbers, enhancing convenience and information richness. Since its introduction, numerous authors have successfully employed IVPyFS to address uncertain information and qualitative data.

Moreover, the idea of membership/belonging (yes), non-membership/non-belonging (no), and indeterminacy/neutral (abstain) has been effectively explained by both the definition of IFSs and PyFS. Let's consider a voting system example where voters fall into four distinct categories: those who vote affirmatively (yes), those who vote negatively (no), those who neither vote for nor against (abstain), and those who refuse to vote (refusal). It's worth noting that the concept of 'refusal' is not considered in any of the aforementioned sets. To address such scenarios and create a concept that aligns closely with human flexibility, Cuong & Kreinovich [44] introduced the concept of picture fuzzy set (PFS). In this framework, all four parameters—degree of membership, degree of indeterminacy (neutral), degree of non-membership, and degree of refusal are taken into consideration. The condition where neutral membership degree is calculated independently in real-life Problem, based on these two sets FS and IFS, the picture fuzzy transportation problem is defined by Geetha and Selvakumari [170] and it solved using the score function. Mehmood and Bashir [124] also worked on the extended transportation models based on PFS.

Further, the spherical fuzzy set (SFS) is introduced by Ashraf et al. [121] with condition $0 \leq \mu + \eta + \nu \leq 1$ which gives additional strength to the idea of picture fuzzy sets by broadening/enlarging the space for the grades of all the four parameters. Consider the example, let μ=0.7, η=0.3 and ν=0.5, in this case 0.7+0.3+0.5 $>$1 but by squaring $(0.7)^2$ $+(0.3)^2+(0.5)^2 \leq 1$. By this, we can see that SFS is more suitable than PFS. Gundogdu and Kahraman [42] worked on spherical fuzzy sets and spherical fuzzy TOPSIS method. Mahmood et al. [53] used the spherical fuzzy sets to solve decision-making and medical diagnosis problems.

The facts of the above literature review on transportation problems, we found that no one worked on transportation problems under the IVPyFS and SFS.

Such that, to overcome the loss of information incurred in the process of TP, the IVPyFTP and SFTP are introduced. However, no concepts of "TP under IVPyFS and SFS" are yet to be accepted broadly. Hence, it is a crucial problem to define the concepts of optimal solutions in TP with imprecise parameters and investigate the characteristics of solutions of such TP. This study aims to introduce a novel concept of "Solution of TP under IVPyFS and SFS using score function". This is done by emphasizing the following research gaps:

- The TP under an interval-valued Pythagorean and spherical fuzzy environment
- The defuzzification methods used to solve fuzzy TP and its extension are restricted to a particular solution procedure. Hence, a more generalized methodology is required to be developed to directly draw the optimal solution.
- In the existing score functions for IVPyFS, they are not given the appropriate results.

To fill the aforementioned research gaps, first, Solution of the Transportation Problem using the interval-valued Pythagorean fuzzy approach is introduced, and then a methodology is defined for solving such problems. The motivation for doing this is to incorporate the ambiguity of human judgments in TP problems having two independent parameters. The primary reason for this is to handle the TP on a continuous scale. For example, Consider a manufacturing company that sources raw materials from multiple suppliers and distributes its finished products to various customers. The company deals with uncertainties in transportation costs due to fluctuating fuel prices, changing road conditions, and other unpredictable factors. In this context, the interval-valued Pythagorean fuzzy transportation problem can be applied to optimize the transportation of goods while considering the imprecise nature of cost estimates. Thus to deal with such applications, we introduce this novel concept. Secondly, solutions of transportation problems under a spherical fuzzy set are introduced, and then a methodology is defined for solving such problems. The motivation for doing this is to incorporate ambiguity of human judgments in TP problems having three independent parameters, in situations where there are more possible responses, such as no, yes, refusal, and abstain, SFSs are important. For example, consider a city's traffic management system where the transportation network is characterized by unpredictable factors, such as varying traffic conditions, road closures, and accidents. The goal is to optimize the traffic flow to minimize congestion while considering the uncertainty in travel times.

This chapter contributes to the theoretical development by proposing a new concept of solutions for TP under the IVPyFS and SFS approach.

This entire chapter is laid out as follows. Section 2.2 introduces the arithmetic operation and score function and accuracy function for IVPyFS and SFS. Two new approaches for solving the transportation problems have been presented in Section 2.3. Section 2.4 discusses the results and comparative study of the proposed methodology in solving TP. Section 2.5 concludes the chapter.

2.2 Arithmetic Operation and Score Function for IVPyFS and SFS

In this section, we'll discuss some arithmetic operations and existing score functions in the literature.

Existing arithmetic operation

Definition 2.2.1. *[113] Let A=* $\{([\mu_{AL},\mu_{AU}],[\nu_{AU},\nu_{AU}])\}$ *and B=*$\{([\mu_{BL},\mu_{BU}],[\nu_{BL},\nu_{BU}])\}$ *are two interval-valued Pythagorean fuzzy sets.*

1. $A\vee B=\{[\max\{\mu_{AL},\mu_{BL}\}, \max\{\mu_{AU},\mu_{BU}\}], [\min\{\nu_{AL},\nu_{BL}\}, \min\{\nu_{AU},\nu_{BU}\}]\}$
2. $A\wedge B=\{[\min\{\mu_{AL},\mu_{BL}\}, \min\{\mu_{AU},\mu_{BU}\}], [\max\{\nu_{AL},\nu_{BL}\}, \max\{\nu_{AU},\nu_{BU}\}]\}$
3. $A\oplus B=[\sqrt{\mu_{AL}^2+\mu_{BL}^2-\mu_{AL}^2.\mu_{BL}^2},\sqrt{(\mu_{AU}^2+\mu_{BU}^2-\mu_{AU}^2.\mu_{BU}^2}],[\nu_{AL}.\nu_{BL},\nu_{AU}.\nu_{BU}]$
4. $A\otimes B=[\mu_{AL}.\mu_{BL},\mu_{AU}.\mu_{BU}],[\sqrt{\mu_{AL}^2+\mu_{BL}^2-\mu_{AL}^2.\mu_{BL}^2},\sqrt{(\mu_{AU}^2+\mu_{BU}^2-\mu_{AU}^2.\mu_{BU}^2}]$

Definition 2.2.2. *[42] Let A & B be two spherical fuzzy sets.*

Then

1.
$$A\bigcup B=\{\max\{\mu_A,\mu_B\},\min\{\nu_A,\nu_B\},$$
$$\min\{(1-((\max\{\mu_A,\mu_B\})^2+(\min\{\nu_A,\nu_B\})^2))^{\frac{1}{2}},\max\{\eta_A,\eta_B\}\}\}$$

2.
$$A\bigcap B=\{\min\{\mu_A,\mu_B\},\max\{\nu_A,\nu_B\},$$
$$\max\{(1-((\min\{\mu_A,\mu_B\})^2+(\max\{\nu_A,\nu_B\})^2))^{\frac{1}{2}},\min\{\eta_A,\eta_B\}\}\}$$

3.
$$A \oplus B = \{(\mu^2{}_A + \mu^2{}_B - \mu^2{}_A.\mu^2{}_B)^{1/2}, \nu_A.\nu_B, ((1-\mu_B{}^2)\eta_A{}^2$$
$$+(1-\mu_A{}^2)\eta_B{}^2 - \eta_A{}^2\eta_B{}^2)^{1/2}\}$$

4.
$$A \otimes B = \{\mu_A.\mu_B, (\nu^2{}_A + \nu^2{}_B - \nu^2{}_A.\nu^2{}_B)^{1/2}, ((1-\nu_B{}^2)\eta_A{}^2$$
$$+(1-\nu_A{}^2)\eta_B{}^2 - \eta_A{}^2\eta_B{}^2)^{1/2}\}$$

Score function:

Definition 2.2.3. *Peng and Yang [113] Consider IVPyFS $A = ([\mu^-, \mu^+], [\nu^-, \nu^+])$. The score function of A is defined as: $S(A) = \frac{1}{2}\left[(\mu^-)^2 + (\mu^+)^2 - (\nu^-)^2 - (\nu^+)^2\right]$, $S(A) \in [-1, 1]$. The accuracy function of A is defined as follow $H(A) = \frac{1}{2}\left[(\mu^-)^2 + (\mu^+)^2 + (\nu^-)^2 + (\nu^+)^2\right]$, $H(A) \in [0, 1]$.*

Comparision for two IVPyFSs A_1 and A_2,

1. if $S(A_1) > S(A_2)$ then $A_1 > A_2$
2. if $S(A_1) < S(A_2)$ then $A_1 < A_2$
3. if $S(A_1) = S(A_2)$ then
   (a) if $H(A_1) > H(A_2)$ then $A_1 > A_2$
   (b) if $H(A_1) < H(A_2)$ then $A_1 < A_2$
   (c) if $H(A_1) = H(A_2)$ then $A_1 = A_2$

Definition 2.2.4. *[42] The Score and accuracy function of SFS are defined by:*

Score $S(A) = (\mu_A - \eta_A)^2 - (\nu_A - \eta_A)^2$, $S(A) \in [-1, 1]$.
Accuracy $H(A)$=$\mu_A{}^2 + \nu_A{}^2 + \eta_A{}^2$, $H(A) \in [0, 1]$.

Suppose any two SFSs A_1 and A_2,

1. if $S(A_1) > S(A_2)$ then $A_1 > A_2$
2. if $S(A_1) < S(A_2)$ then $A_1 < A_2$
3. if $S(A_1) = S(A_2)$ then
   (a) if $H(A_1) > H(A_2)$ then $A_1 > A_2$

(b) if $H(A_1) \prec H(A_2)$ then $A_1 \prec A_2$

(c) if $H(A_1) = H(A_2)$ then $A_1 = A_2$

Suppose, if we consider two IVPyFN A_1 =$\langle[0.1, 0.2], [0.4, 0.5]\rangle$ and A_2 =$\langle[0.1, 0.2], [0.5, 0.5]\rangle$, the score and accuracy value calculated are $S(A_1)$=-0.1800, $S(A_2)$=-0.1800 and $H(A_1)$=0.2300, $H(A_2)$=0.2300.

According to Peng & Yang [113] $A_1 \sim A_2$, but we have seen that $A_1 \neq A_2$. To overcome this limitation, we have proposed a new score function as follows:

Definition 2.2.5. *Let* $A = \left\langle \left[\mu^-, \mu^+\right], \left[\nu^-, \nu^+\right] \right\rangle$*be an IVPyFS.*

Then score and accuracy function A, $\mathrm{S}(A) = \frac{1}{6}\left[1 + (\mu^-)^2 + (\mu^+)^2 - (\nu^-)^2 - (\nu^+)^2\right] \left|\mu^- + \mu^+ - \nu^- - \nu^+\right|$ $\mathrm{S}(A) \in [-1, 1]$.

$$\mathrm{H}(A) = \frac{1}{6}\left[(\mu^-)^2 + (\mu^+)^2 + (\nu^-)^2 + (\nu^+)^2\right] \left|\mu^- + \mu^+ + \nu^- + \nu^+\right|, \mathrm{H}(A) \in [0, 1].$$

Comparision for two IVPyFSs A_1 and A_2,

1. if $S(A_1) \succ S(A_2)$ then $A_1 \succ A_2$
2. if $S(A_1) \prec S(A_2)$ then $A_1 \prec A_2$
3. if $S(A_1) = S(A_2)$ then
   (a) if $H(A_1) \succ H(A_2)$ then $A_1 \succ A_2$
   (b) if $H(A_1) \prec H(A_2)$ then $A_1 \prec A_2$
   (c) if $H(A_1) = H(A_2)$ then $A_1 = A_2$

2.3 Interval Valued Pythagorean Fuzzy and Spherical Fuzzy Transportation Problem

2.3.1 Interval valued Pythagorean fuzzy transportation problem

The IVPyFS Transportation Problem is

$$\text{Min } Z = \sum_i^m \sum_j^n c_{ij}^{IVPyFS} x_{ij}$$

Such that

$$\sum_j^n x_{ij} = a_{ij}^{IVPyFS}, i = 1, 2, ...m$$
$$\sum_i^m x_{ij} = b_{ij}^{IVPyFS}, j = 1, 2, ...n$$
$$x_{ij} \geq 0$$

2.3.2 Spherical fuzzy transportation problem

The SFS Transportation Problem is

$$\text{Min } Z = \sum_i^m \sum_j^n c_{ij}^{SFS} x_{ij}$$

Such that

$$\sum_j^n x_{ij} = a_i^{SFS}, j = 1, 2, ...m$$
$$\sum_i^m x_{ij} = b_j^{SFS}, i = 1, 2, ...n$$
$$x_{ij} \geq 0$$

where c_{ij}^{SFS} is the cost function having SFS data. The two sets of constraints will be consistent i.e., the system will be balanced if $\sum_{i=1}^m a_i = \sum_j^n b_j$.

Above we have taken two cases of transportation problems; the first one is an interval-valued Pythagorean fuzzy transportation problem and the second is a spherical fuzzy transportation problem. In these cases, there are m sources and n destinations. Suppose a_i be the number of supply units available at source $i(i = 1, 2, 3........m)$ and b_j be the number of demand units required at destination $j(j = 1, 2.3..............m)$ in terms of interval-valued Pythagorean fuzzy and spherical fuzzy set. Let c_{ij} represent the unit transportation cost for transporting the units from source i to the destination j in terms of interval-valued Pythagorean fuzzy and spherical fuzzy sets.

2.3.3 Proposed methodology

To solve the transportation problem under interval-valued Pythagorean fuzzy set and spherical fuzzy set, we have taken three different types of transportation problems of interval-valued Pythagorean fuzzy set and spherical fuzzy set.

2.3.4 Illustrative example

- Type 1 (Cost are in IVPyFS)

Table 2.1: TP in Type 1

Destination Sources	**D**$_1$	**D**$_2$	**...**	**D**$_n$	**Supply**
S$_1$	${C_{11}}^{IVPyFN}$	${C_{12}}^{IVPyFN}$	...	${C_{1n}}^{IVPyFN}$	a_1
S$_2$	${C_{21}}^{IVPyFN}$	${C_{22}}^{IVPyFN}$	...	${C_{2n}}^{IVPyFN}$	a_2
S$_m$	${C_{m1}}^{IVPyFN}$	${C_{m2}}^{IVPyFN}$	...	${C_{mn}}^{IVPyFN}$	a_m
Demand	b_1	b_2	...	b_n	

- Type 2 (Supply and Demand are in IVPyFN)

Table 2.2: TP in Type 2

Destination Sources	**D**$_1$	**D**$_2$	**...**	**D**$_n$	**Supply**
S$_1$	C_{11}	C_{12}	...	C_{1n}	${a_1}^{IVPyFN}$
S$_2$	C_{21}	C_{22}	...	C_{2n}	${a_2}^{IVPyFN}$
S$_m$	C_{m1}	C_{m2}	...	C_{mn}	${a_m}^{IVPyFN}$
Demand	${b_1}^{IVPyFN}$	${b_2}^{IVPyFN}$	...	${b_n}^{IVPyFN}$	

- Type 3 (Cost, Supply, and Demand are in IVPyFN)

Table 2.3: TP in Type 3

Destination Sources	**D**$_1$	**D**$_2$	**...**	**D**$_n$	**Supply**
S$_1$	${C_{11}}^{IVPyFN}$	${C_{12}}^{IVPyFN}$	...	${C_{1n}}^{IVPyFN}$	${a_1}^{IVPyFN}$
S$_2$	${C_{21}}^{IVPyFN}$	${C_{22}}^{IVPyFN}$	...	${C_{2n}}^{IVPyFN}$	${a_2}^{IVPyFN}$
S$_m$	${C_{m1}}^{IVPyFN}$	${C_{m2}}^{IVPyFN}$	...	${C_{mn}}^{IVPyFN}$	${a_m}^{IVPyFN}$
Demand	${b_1}^{IVPyFN}$	${b_2}^{IVPyFN}$	...	${b_n}^{IVPyFN}$	

Algorithm

Step1. Write IVPyFTP in matrix form.

Step 2. Use the score function to change the IVPyFTP into a crisp transportation problem
Step 3. Check whether it is balanced or not

Step 4. If it is balanced go to next step 5, otherwise make it balanced to add dummy variables.

Step 5. obtain the initial basic feasible solution.

Step 6. Test optimality of the transportation problem by using any Software MQ, LINGO, and MATLAB.

Step7. Put x_{ij} in the objective function to get the objective function.

Problem based on SFTP

- Type 1 (cost are in SFN)

Table 2.4: SFTP in Type 1

Destination/ Source	**D**$_1$	**D**$_2$	**...**	**D**$_n$	**Supply**
S$_1$	$C_{11}{}^{SFN}$	$C_{12}{}^{SFN}$	...	$C_{1n}{}^{SFN}$	a_1
S$_2$	$C_{21}{}^{SFN}$	$C_{22}{}^{SFN}$	...	$C_{2n}{}^{SFN}$	a_2
S$_m$	$C_{m1}{}^{SFN}$	$C_{m2}{}^{SFN}$	...	$C_{mn}{}^{SFN}$	a_m
Demand	b_1	b_2	...	b_n	

- Type 2 (supply and demand are in SFN)

Table 2.5: SFTP in Type 2

Destination/ Source	**D**$_1$	**D**$_2$	**...**	**D**$_n$	**Supply**
S$_1$	C_{11}	C_{12}	...	C_{1n}	$a_1{}^{SFN}$
S$_2$	C_{21}	C_{22}	...	C_{2n}	$a_2{}^{SFN}$
S$_m$	C_{m1}	C_{m2}	...	C_{mn}	$a_m{}^{SFN}$
Demand	$b_1{}^{SFN}$	$b_2{}^{SFN}$	...	$b_n{}^{SFN}$	

- Type 3 (cost, supply, and demand are in SFN)

Table 2.6: SFTP in Type 3

Destination/ Source	**D**$_1$	**D**$_2$	**…………**	**D**$_n$	**Supply**
S$_1$	${C_{11}}^{SFN}$	${C_{12}}^{SFN}$	……..	${C_{1n}}^{SFN}$	${a_1}^{SFN}$
S$_2$	${C_{21}}^{SFN}$	${C_{22}}^{SFN}$	…………	${C_{2n}}^{SFN}$	${a_2}^{SFN}$
S$_m$	${C_{m1}}^{SFN}$	${C_{m2}}^{SFN}$	………..	${C_{mn}}^{SFN}$	${a_m}^{SFN}$
Demand	${b_1}^{SFN}$	${b_2}^{SFN}$	………..	${b_n}^{SFN}$	

Algorithm

Step 1. Write SFTP in tabular form.

Step 2. Use the score function to change the SFTP into a crisp transportation problem.

Step 3. Check whether it is balanced or not

Step 4. If it is balanced go to next step 5, otherwise make it balanced to add dummy variables.

Step 5. Find the initial basic feasible solution.

Step 6. Test optimality of the transportation problem by using any Software MQ, LINGO, and MATLAB.

Step7. Put x_{ij} in the objective function to get the objective function.

Illustrative Examples

(Type 1): Condition for the proposed transportation problem is shown in Table 2.7.

Table 2.7: IVPyFTP in Type 1

	D$_1$	**D**$_2$	**D**$_3$	**D**$_4$	**Supply**
O$_1$	[.4,.6][.2,.3]	[.4,.5][.1,.2]	[.6,.8][.2,.3]	[.4,.5][.2,.3]	26
O$_2$	[.5,.7][.3,.4]	[.4,.6][.2,.3]	[.5,.7][.2,.3]	[.5,.7][.1,.3]	24
O$_3$	[.4,.5][.1,.2]	[.6,.62][.2,.3]	[.4,.6][.2,.3]	[.4,.6][.2,.3]	30
Demand	17	23	28	12	

(Type 2): The condition for the proposed transportation problem is shown in Table 2.8.

Table 2.8: IVPyFTP in Type 2

	D_1	**D_2**	**D_3**	**D_4**	**Supply**
O_1	13	15	16	17	[.6,.8][.3,.5]
O_2	18	20	7	50	[.5,.6][.3,.4]
O_3	21	19	10	29	[.4,.6][.1,.2]
Demand	[.5,.8][.3,.5]	[.6,.8][.2,.3]	[.8,.9][.2,.3]	[.4,.5][.1,.3]	

(Type 3): Condition for the proposed transportation problem is shown in Table 2.9.

Table 2.9: IVPyFTP in Type 3

	D_1	**D_2**	**D_3**	**D_4**	**Supply**
O_1	[.4,.6][.2,.3]	[.4,.5][.1,.2]	[.6,.8][.2,.3]	[.4,.5][.2,.3]	[.6,.8][.3,.5]
O_2	[.5,.7][.3,.4]	[.4,.6][.2,.3]	[.5,.7][.2,.3]	[.5,.7][.1,.3]	[.5,.6][.3,.4]
O_3	[.4,.5][.1,.2]	[.6,.62][.2,.3]	[.4,.6][.2,.3]	[.4,.6][.2,.3]	[.4,.6][.1,.2]
Demand	[.5,.8][.3,.5]	[.6,.8][.2,.3]	[.8,.9][.2,.3]	[.4,.5][.1,.3]	

2.3.5 Illustrative example

(Type 1): In this SFTP type 1, we have taken the condition for the proposed transportation problem shown in Table 2.10, the costs are in spherical fuzzy numbers, supply and demand are crisp.

Table 2.10: Data for spherical fuzzy transportation problem of Type 1

	D_1	**D_2**	**D_3**	**D_4**	**Supply**
O_1	(0.9,0.1,0.1)	(0.6,0.4,0.4)	(0.91,.03,.02)	(0.91,.03,.02)	26
O_2	(0.89,0.08,0.03)	(0.74,0.16,0.1)	(0.5,0.5,0.5)	(0.7,0.3,0.3)	24
O_3	(0.99,0.05,0.02)	(0.73,0.15,0.08)	(0.73,0.12,0.08)	(0.68,0.26,0.06)	30
Demand	17	23	28	12	80

(Type 2): Condition for the proposed transportation problem is shown in Table 2.11 where the costs are crisp, but the demand and supply are SFN.

Table 2.11: Data for spherical fuzzy transportation problem of Type 2

	D_1	D_2	D_3	D_4	Supply
O_1	0.64	0.04	0.792	0.94	(0.9,0.1,0.1)
O_2	0.73	0.4	0	0.16	(0.89,0.08,0.03)
O_3	0.94	0.41	0.42	0.34	(0.99,0.05,0.02)
Demand	(0.9,0.1,0.1)	(0.6,0.4,0.4)	(0.91,0.03,0.02)	(0.99,0.05,0.02)	

(Type 3): Condition for the proposed transportation problem is shown in Table 2.12. Here all three parameters costs, demands, and supplies are the spherical fuzzy numbers.

Table 2.12: Data for spherical fuzzy transportation problem of Type 3

	D_1	D_2	D_3	D_4	Supply
O_1	(0.61,0.46,0.34)	(0.74,0.27,0.28)	(0.7,0.3,0.3)	(0.62,0.39,0.39)	(0.9,0.1,0.1)
O_2	(0.81,0.2,0.23)	(0.55,0.47,0.43)	(0.5,0.5,0.5)	(0.7,0.3,0.3)	(0.89,0.08,0.03)
O_3	(0.99,.05,.02)	(0.73,0.15,0.08)	(0.73,0.12,0.08)	(0.68,0.26,0.06)	(0.99,0.05,0.02)
Demand	(0.9,0.1,0.1)	(0.6,0.4,0.4)	(0.91,0.03,0.02)	(0.99,0.05,0.02)	

2.4 Result and Comparative Study

We have solved the Transportation problem under interval-valued Pythagorean and spherical fuzzy sets using three different examples. The results for these examples are shown in Table 2.13 and Table 2.14.

Table 2.13: Result of IVPyFTP

IVPyFTP	Optimum value by Peng & Yang [113] score function	Optimum value by proposed score function	Comparison
Type 1	17.4	9.774	Min (Z) by Peng & Yang [113] > Min (Z) of proposed score function
Type 2	8.01	4.577	Min (Z) by Peng & Yang [113] > Min (Z) of proposed score function
Type 3	0.14	0.0445	Min (Z) by Peng & Yang [113] > Min (Z) of proposed score function

Table 2.14: Result of SFTP

SFTP	IBFS	Optimum value	Comparison
Type 1	21.76	21.76	IBFS$\geq$ Optimum value
Type 2	0.7292	0.7052	IBFS$\geq$ Optimum value
Type 3	0.43082	0.4308	IBFS$\geq$ Optimum value

Table 2.15: Analysis of proposed work with others authors work

S.No.	Author name	Work
1	Bharti(2018) [110]	Transportation problem under interval valued intuitionistic fuzzy environment.
2	Kumar et al.(2019) [112]	A Pythagorean fuzzy approach to the transportation problem
3	Geetha and Selvakumari (2020) [170]	A picture fuzzy approach to solving transportation problem
4	Proposed work	Solution of transportation problem under spherical fuzzy sets.
5	Proposed work	Solution of transportation problem under interval-valued Pythagorean fuzzy set.

In Type 1 IVPyFTP, the transportation cost of Peng & Yang [113] is 17.4 and by proposed score function 9.774. In Type 2 IVPyFTP, the transportation cost of Peng & Yang [113] is 8.01 and by the proposed score function 4.577. In Type 3 IVPyFTP, the transportation cost of Peng & Yang [113] is 0.14, and by the proposed score function 0.0445. Thus in all cases, the optimal value of our proposed score function is not more than the optimal value of Peng & Yang [113] score function.

In Type 1 SFTP, we found the IBFS is 21.76 and the optimum transportation cost of 21.76. Also in Type 2 SFTP, we found the IBFS is 0.7292 and the optimum transportation cost of 0.7052. In Type 3 SFTP, we found the IBFS is 0.43082 and optimum transportation cost of 0.4308. So, it is clear from the above observation the optimal solution in all three types is less than or equal to IBFS.

2.5 Conclusion

In first approach, we have developed the algorithm to solve the interval-valued Pythagorean fuzzy transportation problem in all possible three different cases. In the IVPyFTP case, we have used our proposed score function and Peng & Yang [113] score function to convert the IVPyFTP into crisp transportation problem. After that, we applied the computational technique for finding the optimality of the problem. In the final solution, we found the appropriate results by our proposed score function as compared to Peng & Yang [113] score function gives better results. Thus, this is the new method to tackle the uncertainty in real-life transportation problems. In the second approach, we have investigated the solution of the spherical fuzzy transportation problem with three different types. In the present problem, we have given the algorithm for the work, and for a better understanding the problem, we have discussed examples. This is also the new approach to tackle the uncertainty in real-life transportation problems.

Chapter 3

A Novel Similarity Measure and Score Function of Pythagorean Fuzzy Sets and their Application in Assignment Problem

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This chapter [1] is based on Pythagorean fuzzy sets and their application. The objective of this work to handle the uncertainty in practical applications of assignment problems. In real-life problems, things are imprecise because of imprecision/inaccuracy, and the exact value of the measured quantities is impossible to get. Sometimes, due to time pressure/ incomplete knowledge, it is difficult for the decision-makers to provide their opinion. To describe the imprecision, the information in terms of the fuzzy is provided to allow the decision-makers to express their inputs freely. There is a valuable role of the fuzzy set (FS) and intuitionistic fuzzy set (IFS) to describe uncertainty under uncertain situations. In the literature, various models are available for assignment problems under fuzzy sets and intuitionistic fuzzy sets. The Pythagorean fuzzy set has a larger domain space than the IFS to describe the membership grade. To handle the uncertainty in practical applications of assignment problems (AP), we have proposed a method to solve the Pythagorean fuzzy assignment problem (PyFAP) using the proposed similarity measure and a score function. Numerical examples are given to explain the methodology.

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[1] **The content of this chapter is based on the research papers "A novel similarity measure and score function of Pythagorean fuzzy sets and their application in assignment problem",** ***Economic Computation and Economic Cybernetics Studies and Research*****, Issue 3/2023; Vol. 53. https://doi.org/10.24818/18423264/57.3.23.19 (SCIE, Impact Factor : 0.9)**

3.1 Introduction

AP is a linear programming problem and originally formulated by Kuhn [125] that deals with allocation and scheduling. The problem of assignment arises because available resources have varying degrees of efficiency for performing different activities. In the fuzzy assignment problem, instead of having crisp values for costs or benefits associated with each task-agent assignment, the fuzzy values or fuzzy sets representing the degree of membership of each task to each agent.

The tremendous growth and advancement in AP have attracted researchers to work in the field of fuzzy assignment problems (FAP). In this connection Chen [101] used the Zadeh(1965) [2] concept to deal with imprecision of real-world assignment problems. Gurukumaresan et al. [212] used the centroid method for the solution of the fuzzy assignment problem. Tsai et al. [213] worked on the multiobjective fuzzy deployment of manpower. Kumar and Gupta [215] solved the fuzzy assignment problems and fuzzy traveling salesman problems with different membership functions. Thakre et al. [214] worked on the placement of staff in LIC using the fuzzy assignment problems. To consider the vague and imprecise information in the practical problem, the different extensions of the fuzzy assignment problems have been introduced by some authors. Mukherjee and Basu [216] solved the assignment problem under IFS by using similarity measure and score function. Roseline and Amirtharaj [217] solved the intuitionistic fuzzy assignment problem by using the ranking of intuitionistic fuzzy numbers (IFN). Instead of a single point of MD and NMD, Kumar and Bajaj [112] introduced the problem of an interval-valued intuitionistic fuzzy assignment problem and solved it with similarity measure and score function.

The condition where sum of membership degree and non-membership degree is greater than 1 in IFS. Yager [15] overcomes the condition that the square sum of the membership degree and the non-membership degree is less than or equal to 1. The concept of Pythagorean fuzzy sets gives the larger preference domain for decision makers (DM). DMs can define their support and against the degree of membership as μ = 4/5, ν= 2/5. In this case, 4/5+2/5>1 is not valid in IFS but squaring $(4/5)^2+(2/5)^2$<1 implies the PyFS is more suitable than the IFS. Paul Augustine Ejegwa [218] worked on the PyFS and its application in career placement using max-min composition. Fei and Deng [219] solved the problem of the Pythagorean fuzzy multi-criteria problem. Shahzadi et al. [220] proposed the solution of the decision-making approach under the Pythagorean fuzzy Yager weighted operators.

Peng and Yang [221] defined some results for Pythagorean fuzzy sets.

The facts of the above literature review on the assignment problems, we found that no one worked on assignment problems under the Pythagorean fuzzy sets. In this work, we have developed a methodology to solve the assignment problem with Pythagorean fuzzy values. The score function defined by Garg [211] has some limitations. To overcome these limitations, we have proposed a new score function. Additionally, we have defined the new similarity measure to validate our result. So far, there is no literature regarding Pythagorean fuzzy assignment problems using similarity measures and score function. These are the new ways to handle the uncertainty in assignment problem

Further, to overcome the loss of information incurred in the process of AP, the PyFAP is introduced. However, no concepts of AP under PyFS are yet to be accepted broadly. Hence, it is a crucial problem to define the concepts of optimal solutions in AP with imprecise parameters and investigate the characteristics of solutions of such AP. This study aims to introduce a novel concept of “A novel similarity measure and score function of Pythagorean fuzzy sets and their application in assignment problem”. This is done by emphasizing the following research gaps:

- The AP under Pythagorean fuzzy environment
- The defuzzification methods used to solve PyFAP
- In the existing score functions for PyFS, there are some drawbacks of the computation methods.
- Only some similarity measures are defined for PyFS in the literature.

For filling the aforementioned research gaps, A novel similarity measure and score function of Pythagorean fuzzy sets and their application in assignment problems is introduced, and then a methodology is defined for solving such problems. The motivation for doing this is to incorporate the ambiguity of human judgments in AP problems having two independent parameters. The primary reason for this is to handle the AP on PyFS. For example, The Pythagorean fuzzy assignment problem finds application in project management, aiding optimal task allocation to teams considering uncertainty in project complexity and team expertise. It enhances resource utilization and minimizes overall mismatch by incorporating the fuzzy nature of project requirements and team capabilities. This approach is valuable in project environments where precise task complexities and team proficiencies are challenging to quantify, ensuring effective project execution and successful outcomes with adaptable task-team assignments. Thus to deal with such applications, we introduce this novel concept. The motivation for doing this is to incorporate ambiguity of human judgments in AP problems having two independent parameters, in situations where $\mu + \nu$ >1.

This chapter contributes to the theoretical development by proposing a new concept of solutions for AP under the PyFS approach. Here, we have proposed a new score function. Additionally, we have defined the new similarity measure to validate our result. These are the new ways to handle the uncertainty in assignment Problems.

This entire chapter is laid out as follows. In Section 3.2, we have proposed a novel similarity measure and score function. Also, the limitations of previously defined score functions have been pointed out. The methodology to solve PyFAP using similarity measure and score function is given in Section 3.3. Illustrative examples are also given in this section. Section 3.4, presents the comparative study and concluding remarks.

3.2 Similarity Measure and Score Function of Pythagorean Fuzzy Set

In this section, we have defined the novel similarity measure and score function of Pythagorean fuzzy sets.

3.2.1 Similarity measure for Pythagorean fuzzy sets

Definition 3.2.1. *Suppose A and B be two PyFSs. The similarity measure SM: A × B → [0,1] is defined as follow*

$$S(A,B) = \frac{\sum_{j=1}^{m} \mu_A^2(x_j).\mu_B^2(x_j)+\nu_A^2(x_j).\nu_B^2(x_j)}{\sum_{j=1}^{m}\left[\left(\mu_A^6(x_j)\vee\mu_B^6(x_j)\right)+\left(\nu_A^6(x_j)\vee\nu_B^6(x_j)\right)\right]}$$

Theorem 3.2.1. *Similarity measure (SM) between two PyFS A and B, then following are true.*

(S1) $0 \leq S(A,B) \leq 1$

(S2) $S(A,B) = 1$ iff $A=B$

(S3) $S(A,B) = S(B,A)$

(S4) $S(A,C) \leq S(A,B)$ and $S(A,C) \leq S(B,C)$ for all A, B, C are PyFSs such that $A \subseteq B \subseteq C$.

Proof. To prove S is a similarity measure, we have to verify the four conditions of similarity measure □

(S1) Since for all x_j, $1 \leq j \leq m$, we have $\mu_A^2(x_j).\mu_B^2(x_j) \leq \mu_A^4(x_j) \vee \mu_B^4(x_j)$ and $\nu_A^2(x_j).\nu_B^2(x_j) \leq \nu_A^4(x_j) \vee \nu_B^4(x_j)$. Therefore for each x_j, we have

$$\left[\mu_A^2(x_j).\mu_B^2(x_j) + \nu_A^2(x_j).\nu_B^2(x_j)\right] \leq \left[\left\{\mu_A^4(x_j) \vee \mu_B^4(x_j)\right\} + \left\{\nu_A^4(x_j) \vee \nu_B^4(x_j)\right\}\right]$$

Therefore for all x_j, $1 \leq j \leq m$, we have

$$\sum_{j=1}^{m}\left[\mu_A^2(x_j).\mu_B^2(x_j) + \nu_A^2(x_j).\nu_B^2(x_j)\right] \leq \sum_{j=1}^{m}\left[\left\{\mu_A^4(x_j) \vee \mu_B^4(x_j)\right\} + \left\{\nu_A^4(x_j) \vee \nu_B^4(x_j)\right\}\right]$$

$0 \leq S^s(A,B) \leq 1$.

(S2). Suppose $S(A,B) = 1$

$$\frac{\sum_{j=1}^{m}\left[\mu_A^2(x_j).\mu_B^2(x_j)+\nu_A^2(x_j).\nu_B^2(x_j)\right]}{\sum_{j=1}^{m}\left[\left\{\mu_A^4(x_j)\vee\mu_B^4(x_j)\right\}+\left\{\nu_A^4(x_j)\vee\nu_B^4(x_j)\right\}\right]}=1$$

$$\Longrightarrow \sum_{j=1}^{m}\left[\mu_A^2(x_j).\mu_B^2(x_j) + \nu_A^2(x_j).\nu_B^2(x_j)\right] = \sum_{j=1}^{m}\left[\left\{\mu_A^4(x_j) \vee \mu_B^4(x_j)\right\} + \left\{\nu_A^4(x_j) \vee \nu_B^4(x_j)\right\}\right].$$

Now, we claim that $\mu_A^2(x_j).\mu_B^2(x_j) = \mu_A^4(x_j) \vee \mu_B^4(x_j)$ and $\nu_A^2(x_j).\nu_B^2(x_j) = \nu_A^4(x_j) \vee \nu_B^4(x_j)$.
Suppose $\mu_A^2(x_j).\mu_B^2(x_j) \neq \mu_A^4(x_j) \vee \mu_B^4(x_j)$, since $\mu_A^2(x_j).\mu_B^2(x_j) \leq \mu_A^4(x_j) \vee \mu_B^4(x_j)$, there exists $r_1 > 0$ such that $\mu_A^2(x_j).\mu_B^2(x_j) + r_1 = \mu_A^4(x_j) \vee \mu_B^4(x_j)$.
Similarly there exists $r_2 > 0$ such that $\nu_A^2(x_j).\nu_B^2(x_j) + r_2 = \nu_A^4(x_j) \vee \nu_B^4(x_j)$.
By hypothesis it follows that $r_1 + r_2 = 0$. This implies $r_1 = -(r_2)$, which is not possible. This implies that $\mu_A^2(x_j).\mu_B^2(x_j) = \mu_A^4(x_j) \vee \mu_B^4(x_j)$ and $\nu_A^2(x_j).\nu_B^2(x_j) = \nu_A^4(x_j) \vee \nu_B^4(x_j)$. This implies that $\mu_A^2(x_j) = \mu_B^2(x_j)$ and $\nu_A^2(x_j) = \nu_B^2(x_j)$
Hence $A = B$.

(S3) $S(A,B) = S(B,A)$ is trivial .

(S4) For three PyFSs A, B and C in X. The similarity measures between A, B and A, C are given as

$$S(A,B) = \frac{\sum_{j=1}^{m}\left[\mu_A^2(x_j).\mu_B^2(x_j)+\nu_A^2(x_j).\nu_B^2(x_j)\right]}{\sum_{j=1}^{m}\left[\left\{\mu_A^4(x_j)\vee\mu_B^4(x_j)\right\}+\left\{\nu_A^4(x_j)\vee\nu_B^4(x_j)\right\}\right]}$$

$$S(A,C) = \frac{\sum_{j=1}^{m}\left[\mu_A^2(x_j).\mu_C^2(x_j)+\nu_A^2(x_j).\nu_C^2(x_j)\right]}{\sum_{j=1}^{m}\left[\left\{\mu_A^4(x_j)\vee\mu_C^4(x_j)\right\}+\left\{\nu_A^4(x_j)\vee\nu_C^4(x_j)\right\}\right]}$$

Suppose $A \subseteq B \subseteq C$. for all $x_j \in x$, we have $\mu_A^2(x_j) \leq \mu_B^2(x_j) \leq \mu_C^2(x_j)$, $\nu_A^2(x_j) \geq \nu_B^2(x_j) \geq \nu_C^2(x_j)$. This implies that $\mu_A^4(x_j) \leq \mu_B^4(x_j) \leq \mu_C^4(x_j)$, $\nu_A^4(x_j) \geq \nu_B^4(x_j) \geq \nu_C^4(x_j)$. We claim that for all $x_j \in \tilde{x}$, we have

$$\frac{\mu_A^2(x_j).\mu_B^2(x_j)}{\mu_B^4(x_j)+\nu_A^4(x_j)} \leq \frac{\mu_A^2(x_j).\mu_C^2(x_j)}{\mu_C^4(x_j)+\nu_A^4(x_j)}$$

Similarly, we have

$$\frac{\nu_A^2(x_j).\nu_B^2(x_j)}{\mu_B^4(x_j)+\nu_A^4(x_j)} \leq \frac{\nu_A^2(x_j).\nu_C^2(x_j)}{\mu_C^4(x_j)+\nu_A^4(x_j)}$$

by adding all the above equations, we have

$$\frac{\sum_{j=1}^{m}\left[\mu_A^2(x_j).\mu_C^2(x_j)+\nu_A^2(x_j).\nu_C^2(x_j)\right]}{\sum_{j=1}^{m}\left[\left\{\mu_A^4(x_j)\vee\mu_C^4(x_j)\right\}+\left\{\nu_A^4(x_j)\vee\nu_C^4(x_j)\right\}\right]} \leq \frac{\sum_{j=1}^{m}\left[\mu_A^2(x_j).\mu_B^2(x_j)+\nu_A^2(x_j).\nu_B^2(x_j)\right]}{\sum_{j=1}^{m}\left[\left\{\mu_A^4(x_j)\vee\mu_B^4(x_j)\right\}+\left\{\nu_A^4(x_j)\vee\nu_B^4(x_j)\right\}\right]}$$

Hence, $S(A,C) \leq S(A,B)$. Similarly, we can prove $S(A,C) \leq S(B,C)$.

3.2.2 Score function

Here, we discuss the limitations of the previously defined score function and to overcome the limitations, a new score function is proposed in this section. Peng & Yang [113] defined the score function and accuracy function for an interval-valued Pythagorean fuzzy number (IVPyFN) as follows:

Consider IVPyFN $A = \langle[\mu^-,\mu^+],[\nu^-,\nu^+]\rangle$, the score function $S_1(A)$ and accuracy function $H_1(A)$ are defined as follows

$$S_1(A) = \frac{\mu^{-2}+\mu^{+2}-\nu^{-2}-\nu^{+2}}{2}$$

$$H_1(A) = \frac{\mu^{-2}+\mu^{+2}+\nu^{-2}+\nu^{+2}}{2}$$

Suppose, if we consider two IVPyFN $A_1 = \langle[0.1,0.2],$ $[0.4,0.5]\rangle$ and $A_2 = \langle[0.1,0.2],[0.5,0.5]\rangle$, the score and accuracy value calculated are $S_1(A_1)$=-0.1800, $S_1(A_2)$=-0.1800 and $H_1(A_1)$=0.2300, $H_1(A_2)$=0.2300.

According to Peng & Yang [113] $A_1 \sim A_2$, but we have seen that $A_1 \neq A_2$. To overcome this limitation Garg [211] defined the new score function $S_2(A)$ and defined as

$$\frac{(\mu^{-2}-\nu^{-2})(1+\sqrt{1-\mu^{-2}-\nu^{-2}})+(\mu^{+2}-\nu^{+2})(1+\sqrt{1-\mu^{+2}-\nu^{+2}})}{2} \in [-1,1].$$

By this score function, the score value of the above examples are $S_2(A_1)$=−0.3368 and $S_2(A_2)$=−0.3233. Here $S_2(A_2)>S_2(A_1)$, hence $A_2>A_1$.

For IVPyFN $A = \langle[0.81,0.87],[0.11,0.25]\rangle$, the score value $S_2(A)$=1.0022, this is invalid because the score value is greater than 1 i.e. $S_2(A) = 1.0022 \notin [-1,1]$.

We have seen from the above example, that the score function defined by Garg [211] is not giving the appropriate result. To improve this we have proposed a score function as follows.

Definition 3.2.2. *Let $A = \langle[\mu^-,\mu^+],[\nu^-,\nu^+]\rangle$ be an IVPyFN. The score function for IVPyFN is*

$$S(A) = \frac{(\sqrt{3+\mu^- - 3\nu^-}) + (\sqrt{3+\mu^+ - 3\nu^+})}{4} \in [0,1].$$

Suppose, if $\mu^- = \mu^+ = \mu$ and $\nu^- = \nu^+ = \nu$ then, the score function of IVPyFN will become the score function of Pythagorean fuzzy number (PyFN). So, the proposed score function of PyFN is as follows:

$$S(A) = \frac{\sqrt{3+\mu-3\nu}}{2} \in [0,1].$$

3.3 Application of the Pythagorean Fuzzy Assignment Problem

In this section, we introduce the assignment problem with Pythagorean fuzzy number (PyFN) and give two methodologies to solve such problems. One is based on a similarity measure and the other is based on a score function.

Pythagorean fuzzy assignment problem (PyFAP)

$$\text{Min } Z = \sum_i^n \sum_j^n c_{ij}^{PyFN} x_{ij}$$

Subject to

$$\sum_j^n x_{ij} = 1, \quad i = 1,2,...n$$
$$\sum_i^n x_{ij} = 1, \quad j = 1,2,...n$$

$$x_{ij} \in \{0,1\}.$$

In assignment problems the cost is usually deterministic in nature. But in real-life problems, this is very difficult to judge the precise value of the cost. In this unstable condition, we calculate the preference value. Based on preference value we get the preference for the j^{th} work to the i^{th} person in the form of a composite relative degree of similarity with an ideal solution, Thus we replace c_{ij} by composite relative degree.

3.3.1 Proposed algorithm for Pythagorean fuzzy assignment problem using similarity measure

Here we developed an algorithm to solve the Pythagorean fuzzy assignment problem using a similarity measure.

Methodology for Pythagorean fuzzy assignment problem

Step 1: First consider the Pythagorean fuzzy assignment problem decision matrix

$$\text{G}=\{(\text{L}_{ij})\}_{m\times n}$$

$(\text{L}_{ij})=<\mu_{ij}(x), \nu_{ij}(x)>$, *i=1,2....m, j= 1,.... . . ..n*, are Pythagorean fuzzy numbers.

Step 2: Examine whether the problem is balanced or not. If it is not balanced, then add dummy variables so that the problem is converted into a balanced assignment problem.

Step 3: Calculate the similarity measure of each cost value from Pythagorean positive ideal solution (PPIS) $L^+ = \langle 1,0\rangle$ and Pythagorean negative ideal solution (PNIS) $L^- = \langle 0,1\rangle$

$$\text{S}\left(\text{L},\text{L}^+\right) = \frac{\sum_{j=1}^m \mu_L^2\left(x_j\right)\cdot\mu_{L^+}^2\left(x_j\right)+\nu_L^2\left(x_j\right)\cdot\nu_{L^+}^2\left(x_j\right)}{\sum_{j=1}^m\left[\left(\mu_L^4\left(x_j\right)\vee\mu_{L^*}^4\left(x_j\right)\right)+\left(\nu_L^4\left(x_j\right)\vee\nu_{L^+}^4\left(x_j\right)\right)\right]}$$
$$\text{S}\left(\text{L},\text{L}^-\right) = \frac{\sum_{j=1}^m \mu_L^2\left(x_j\right)\cdot\mu_{L^-}^2\left(x_j\right)+\nu_L^2\left(x_j\right)\cdot\nu_{L^-}^2\left(x_j\right)}{\sum_{j=1}^m\left[\left(\mu_L^4\left(x_j\right)\vee\mu_{L^-}^4\left(x_j\right)\right)+\left(\nu_L^4\left(x_j\right)\vee\nu_{L^-}^4\left(x_j\right)\right)\right]}$$

Relative similarity matrix calculated column-wise

$$Q = \frac{S\left(L, L^{+}\right)}{S\left(L, L^{+}\right) + S\left(L, L^{-}\right)}$$

Similarly, the relative similarity matrix calculated row-wise

$$R = \frac{S\left(L, L^{+}\right)}{S\left(L, L^{+}\right) + S\left(L, L^{-}\right)}$$

Step 4: The composite matrix [T] $_{n\times n}$ is evaluated as $T = Q \times R = q_{ij} \times r_{ij}$, the resultant matrix T represents the preference that j^{th} job is chosen by i^{th} person.

3.3.2 Proposed Algorithm for Pythagorean Fuzzy Assignment Problem using Score Function.

Step 1: Write PyFAP in tabular form

Step 2: Convert the Pythagorean fuzzy assignment problem into a crisp assignment problem by using the score function.

Step 3: Examine whether the problem is balanced or not. If it is not balanced then add dummy variables so that the problem is converted into a balanced assignment problem.

Step 4: The greater cell value of the matrix will indicate the preference of j^{th} job to the i^{th} person.

3.3.3 Illustrative Examples

In this section, we consider the following example for PyFAP.

Example 3.3.1. *A manufacturing company decides to make six subassemblies through six contractors. One contractor has to receive only one subassembly. The cost of each subassembly is determined by the bids submitted by each contractor and is shown in Table 3.1 in Pythagorean fuzzy* *number. The problem is how to assign subassemblies to contractors to get the optimal assignment.*

Table 3.1: Assignment problem based on PyFN

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	(0.7,0.6)	(0.7, 0.7)	(0.8, 0.5)	(0.7, 0.6)	(0.6,0.7)	(0.6, 0.5)
$\mathcal{S}_2$	(0.63, 0.67)	(0.9, 0.5)	(0.8, 0.53)	(0.8, 0.3)	(0.9, 0.2)	(0.45, 0.59)
$\mathcal{S}_3$	(0.83, 0.4)	(0.5, 0.7)	(0.6, 0.7)	(0.5, 0.7)	(0.20, 0.81)	(0.5, 0.8)
$\mathcal{S}_4$	(0.63, 0.55)	(0.71, 0.63)	(0.66, 0.35)	(0.9, 0.3)	(0.4, 0.8)	(0.73, 0.4)
$\mathcal{S}_5$	(0.7, 0.5)	(0.65, 0.35)	(0.32, 0.7)	(0.8, 0.5)	(0.4, 0.9)	(0.85, 0.18)
$\mathcal{S}_6$	(0.45, 0.75)	(0.83, 0.3)	(0.35, 0.7)	(0.55, 0.8)	(0.5, 0.6)	(0.3, 0.8)

Table 3.2: S (L, L^+) column wise

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	0.43	0.39	0.60	0.43	0.29	0.33
$\mathcal{S}_2$	0.33	0.76	0.59	0.63	0.8	0.18
$\mathcal{S}_3$	0.67	0.19	0.29	0.20	0.02	0.17
$\mathcal{S}_4$	0.36	0.43	0.42	0.80	0.11	0.41
$\mathcal{S}_5$	0.46	0.41	0.08	0.60	0.09	0.72
$\mathcal{S}_6$	0.15	0.68	0.09	0.21	0.22	0.06

Table 3.3: S (L, L^-) column-wise

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	0.29	0.39	0.17	0.29	0.43	0.22
$\mathcal{S}_2$	0.38	0.23	0.19	0.05	0.02	0.33
$\mathcal{S}_3$	0.10	0.46	0.43	0..46	0.65	0.60
$\mathcal{S}_4$	0.26	0.31	0.10	0.05	0.62	0.12
$\mathcal{S}_5$	0.20	0.10	0.48	0.17	0.15	0.02
$\mathcal{S}_6$	0.54	0.06	0.48	0.58	0.33	0.63

Table 3.4: Relative similarity matrix R (column-wise)

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	0.59	0.5	0.77	0.59	0.40	0.60
$\mathcal{S}_2$	0.46	0.76	0.75	0.92	0.97	0.35
$\mathcal{S}_3$	0.87	0.29	0.40	0.30	0.02	0.22
$\mathcal{S}_4$	0.58	0.58	0.80	0.94	0.15	0.80
$\mathcal{S}_5$	0.69	0.80	0.14	0.77	0.37	0.97
$\mathcal{S}_6$	0.21	0.91	0.15	0.26	0.40	0.08

Table 3.5: Relative similarity matrix S (row-wise)

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	0.34	0.25	0.59	0.34	0.16	0.36
$\mathcal{S}_2$	0.21	0.57	0.56	0.84	0.94	0.12
$\mathcal{S}_3$	0.75	0.08	0.16	0.09	0.00	0.04
$\mathcal{S}_4$	0.33	0.33	0.64	0.88	0.02	0.64
$\mathcal{S}_5$	0.47	0.64	0.01	0.59	0.13	0.94
$\mathcal{S}_6$	0.04	0.82	0.02	0.06	0.16	0.00

Now compute composite matrix T=R×S=$r_{ij} \times s_{ij}$. This matrix T represents the preference the $\mathcal{S}^{th}$ subassembly to $\mathcal{C}^{th}$ contractor

Table 3.6: Composite matrix

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	0.11	0.06	**0.34**	0.11	0.02	0.12
$\mathcal{S}_2$	0.04	0.32	0.31	0.70	**0.88**	0.01
$\mathcal{S}_3$	**0.56**	0.00	0.02	0.00	0.00	0.00
$\mathcal{S}_4$	0.10	0.10	0.4	**0.77**	0.00	0.4
$\mathcal{S}_5$	0.22	0.40	0.00	0.34	0.01	**0.88**
$\mathcal{S}_6$	0.00	**0.67**	0.00	0.00	0.02	0.00

The optimal assignment policy is: subassembly 1→contractor 3, subassembly 2→contractor 5, subassembly 3→contractor 1, subassembly 4→contractor 4, subassembly 5→contractor 6, subassembly 6→contractor 2.

It is to note that both methods are given similar results. It may get change if the problem has none than one solution.

Illustrative Examples

We consider Example 6.4.1 as given above and apply score function for PyFN Defination 3.2.2. We get Table 3.7 corresponding to Table 3.1.

Table 3.7: Decision matrix calculated by score function

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	0.689	0.632	**0.758**	0.689	0.612	0.725
$\mathcal{S}_2$	0.636	0.775	0.743	0.851	**0.908**	0.648
$\mathcal{S}_3$	**0.811**	0.592	0.612	0.592	0.439	0.524
$\mathcal{S}_4$	0.704	0.675	0.808	**0.866**	0.500	0.795
$\mathcal{S}_5$	0.742	0.806	0.552	0.758	0.418	**0.910**
$\mathcal{S}_6$	0.548	**0.856**	0.559	0.536	0.652	0.474

The optimal assignment policy is: subassembly 1$\rightarrow$contractor 3, subassembly 2$\rightarrow$contractor 5, subassembly 3$\rightarrow$contractor 1, subassembly 4$\rightarrow$contractor 4, subassembly 5$\rightarrow$contractor 6, subassembly 6$\rightarrow$contractor 2.

It is note that both the methods are giving similar results. It may get change if the problem has more than one solution.

3.4 Comparative Analysis and Conclusions

The examples mentioned above have vividly demonstrated the proposed similarity measure and score function as a potential tool for solving the assignment problem in PyFS. From the analysis, it is observed that the results obtained by the implementation of the developed similarity measure and score function are more accurate and reliable. Compared to the existing methodologies in the literature, the novel score function proposed in the present paper has the following advantages:

(i) The proposed method has a simple presentation such that it can significantly avoid the information loss that may have previously occurred in the score function defined by Peng & Yang [113] and Garg [211]. It is envisioned that there exist certain values where the Peng & Yang [113] and Garg [211] score function failed to give valid results.

(ii) We have also observed that Example 6.4.1 cannot be solved by using the score function given by Garg [211] as the score values of the cell (2,2) representing S_2Q_2(0.9, 0.5).

(iii) The diversity and fuzziness of the decision maker's assessment information can be well reflected and modeled using the proposed similarity measure and score function.

(iv) The result offered by using the novel similarity measure and score function is consistent with the result obtained in the existing work, Mukherjee and Basu [216], Kumar and Bajaj [112]. Therefore, the proposed method becomes more flexible and convenient for solving the Pythagorean fuzzy assignment problem.

In this chapter, we have proposed a methodology to solve the Pythagorean fuzzy assignment problem and solved the problem using the similarity measure and the score function to test the optimality of the method. It is anticipated that the proposed methodology is capable of managing the uncertainty persisting within the intricate assignment problem. The working of the proposed technique has been illustrated via examples to test its validity. We further provide a comparison with the existing methods in the literature. From the comparative study and analysis, it can be concluded that the proposed method overcomes the limitations present in the existing work.

Table 3.8: Comparative analysis of proposed score function

Problem	**Score Function by Garg**	**Score Function by Peng & Yang**	**Proposed Score Function**
PyFAP	This is not valid for some values	This is not valid when score values & accuracy are same	Solution exist

Table 3.8 provides a comparative analysis of the proposed score function. Additionally, it would be engrossing to explore the application of the developed approach to picture fuzzy sets, spherical fuzzy sets and interval-valued picture fuzzy sets, etc., also to deal with other linear programming problems.

Chapter 4

Interval Valued Picture Fuzzy Matrix: Basic Properties and Application

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This chapter [1] is based on Interval-valued picture fuzzy matrix: basic properties and application. The objective of this work is to handle the uncertainty in practical applications of matrix. In real-life problems, things are imprecise because of imprecision/inaccuracy, and the exact value of the measured quantities is impossible to get. Sometimes, due to time pressure/ incomplete knowledge, it is difficult for the decision-makers to provide their opinion. To describe the imprecision, the information in terms of the fuzzy is provided to allow the decision-makers to express their inputs freely. In the literature, some important matrices are available to tackle uncertain problems. There is a valuable role of the fuzzy matrix (FM), intuitionistic fuzzy matrix (IFM), and picture fuzzy matrix (PFM) to describe uncertainty under uncertain situations. To handle the uncertainty in practical applications of matrices, we have defined, the interval-valued picture fuzzy matrix and its important properties and applications. The IVPFM offers a more robust framework to handle various real-life scenarios and decision-making processes by representing membership, neutral membership, and non-membership degrees as intervals. In this chapter, we define the several key definitions and theorems for the IVPFM and present a procedure for calculating its determinant and adjoint. Using composition functions, we develop algorithms to identify the greatest and least eigenvalue for the defined problem. The work demonstrates this process with a numerical example of a decision-making problem. In addition, we intro-

[1]**The content of this chapter is based on the research papers "Interval-valued picture fuzzy matrix: basic properties and application.", *Soft Computing*, 1-22. https://doi.org/10.1007/s00500-023-09455-4 (SCIE, Impact Factor : 3.1)**

duce a new distance measure for the IVPFSs and prove its validity with the help of basic properties. Further, the application of the proposed concepts has been shown by a real-life numerical example of a computer numerical controlled (CNC) programmer selection problem in a smart manufacturing company.

4.1 Introduction

Decision-making is a key process that profoundly influences our personal and professional lives. However, the real world is often characterized by uncertainty and vagueness, which complicates and challenges the decision-making process [61, 62, 63, 64]. Uncertainty, in this context, arises when the outcomes or consequences of various alternatives are inherently unknown, lacking a definitive level of certainty. Conversely, vagueness encompasses handling imprecise or ambiguous information, characterized by its deficiency in clarity and specificity. Both uncertainty and vagueness introduce risks, trade-offs, and a sense of ambiguity that decision-makers must navigate to make sound and effective choices. Confronting uncertain and vague information adds substantial difficulty to accurately analyzing a given situation, identifying the optimal course of action, and gaining a comprehensive understanding of the potential consequences of a decision [65].

The theory of fuzzy set was proposed by Zadeh [2] to deal with uncertain or vague information more efficiently in practical situations. The core idea behind fuzzy sets is to assign a membership degree to each element of a set, indicating the extent to which the element belongs to the set. The concept of fuzzy sets finds applications in various fields, including artificial intelligence, decision-making, pattern recognition, and control systems, to name a few. Atanassov [7] generalized the idea of FSs and introduced a novel concept called intuitionistic fuzzy set. Unlike fuzzy sets that solely capture MD and non-membership degrees, IFSs introduce an additional dimension called the hesitancy degree, which quantifies the level of uncertainty or ambiguity associated with a particular element. Since its introduction, IFSs have been extensively studied and widely applied in various domains, including decision-making [66, 67], pattern recognition [68], and medical diagnosis [69, 70]. El-Morsy [71] introduced an approach that utilizes Pythagorean fuzzy numbers to assess the rate of risked return, portfolio risk amount, and expected return rate. Atanassov & Gargov [8] further extended the notion of IFS to the interval-valued intuitionistic fuzzy sets in which intervals numbers are used rather than exact numbers to provide flexibility in defining membership degrees to an element. Researchers have successfully applied IVIFS in various domains, demonstrating their flexibility and effectiveness in real-world situations [72, 73, 74, 75].

While IFSs have proven useful in various applications, they encounter limitations when confronted with conflicting information in real-life scenarios. Consider situations such as voting, where the outcomes can be categorized into four distinct groups: vote for, abstain, vote against, and refuse to vote. Cuong & Kreinovich [127] introduced the notion of picture fuzzy sets to address this issue. The PFSs offer a more comprehensive approach by incorporating three functions: membership, neutral membership, and non-membership degree. This expansion of IFS theory provides a powerful tool for handling conflicting and ambiguous information, enabling more robust analysis and decision-making. Cuong & Kreinovich [127] also defined some basic operational laws for PFSs and proved several properties associated with them. Phong et al. [46] worked on the composition of picture fuzzy relations to solve medical diagnosis problems using max-min composition. Singh [76] introduced the idea of a correlation coefficient for PFSs. Son [77] proposed a generalized distance measure between PFSs to solve clustering problems in a picture-fuzzy context. Wei [78] defined cosine similarity measures for PFSs and studied their application in decision-making. Garg [79] proposed weighted average and geometric aggregation operators for picture fuzzy numbers (PFNs) and used them in decision-making. Jana et al. [80] defined Dombi aggregation operators for picture PFNs to solve multiple attribute decision-making problems. Wei et al. [81] developed projection models for solving MADM problems in a picture fuzzy framework. Luo & Zhang [45] defined the new similarity between PFSs and discussed their application in pattern recognition. Joshi [82] formulated a novel picture fuzzy decision-making method based on the R-norm information measure and the VIKOR approach. Jana & Pal [83] used picture fuzzy Hamacher aggregation operators for the performance assessment of an enterprise. Verma & Rohtagi [84] proposed novel similarity measures to resolve pattern recognition and medical diagnosis issues in a picture-fuzzy environment. Ganie [85] defined a new distance measure on PFSs and discussed its application to pattern recognition problems. Hasan et al.[86] defined some picture fuzzy mean operators. Roan et al. [87] worked on the utilization of the picture fuzzy distance measure to manage network power consumption.

The theory of PFS was further generalized by Cuong & Kreinovich [127] by proposing interval-valued picture fuzzy sets. It is worth mentioning that the IVPFSs have various advantages over PFSs and IVIFSs, making them highly suitable for efficiently modeling uncertain and vague information, particularly in complex scenarios. Khalil et al. [39] established some operation laws for IVPFSs and comprehensively analyzed their properties. Liu et al. [128] introduced the similarity measures for IVPFS and studied their applications in decision-making problems. Mahmood et al. [88] defined Frank aggregation operators as designed to solve decision-making problems involving interval-valued picture fuzzy information.

4.1.1 Literature review

Matrix, a fundamental mathematical concept, hold immense significance across disciplines such as linear algebra, physics, computer science, and economics. A matrix is a rectangular array of numbers, symbols, or expressions arranged in rows and columns. It provides a concise and organized way to represent and manipulate complex data sets or mathematical relationships. Various uncertain and vague data types are involved in real-world situations, which is difficult to express in a classical matrix. To overcome this situation, Thomasom [16] introduced the idea of a fuzzy matrix (FM) in 1977. After that, Kim & Roush [20] worked on the FM over boolean algebra. Ragab & Eman [21] gave some results on the max-min composition and studied the construction of an idempotent FM. Ragab & Emam [129] solved the determinant and adjoint of a square FM and discussed some properties defined on it. Shyamal & Pal [17] proposed the triangular fuzzy matrix and gave the methodology to find the corresponding determinant. Dehghan et al. [89] expanded the idea of the inverse of a matrix with fuzzy numbers. Pal [24] proposed the interval-valued fuzzy matrix (IVFM) theory. IVFM has numerous applications in decision-making, medical diagnosis, etc., just like FSs, Meenakshi [130] used the IVFM for solving medical diagnosis problems. Mandal & Pal [133] described some methods to find the ranks of IVFM.

Owing to the advantages of IFSs, Pal et al. [131] introduced the concept of the intuitionistic fuzzy matrix (IFM). Pal & Khan [132] proposed some important operations on the IFM. Padder & Murugadas [35] worked on max-min operations on the IFM and discussed the convergence of transitive IFM. Padder & Murugadas [138] also addressed the determinant of an IFM with basic mathematical properties. Muthuraji et al.[134] studied the decomposition of IFM. Jenita et al.[90] presented a detailed study on ordering in generalized regular IFM. Padder & Murugadas [91] developed an algorithm for controllable and nilpotent IFM. Further, Khan & Pal [32] extended the notion of the intuitionistic fuzzy matrix to the interval-valued intuitionistic fuzzy matrix (IVIFM) in which MD and NMD are used in interval numbers rather than exact numbers. Silambarasan [135] defined the Hamacher operations of IVIFM and proved some important properties associated with them. The IFMs have been strongly enforced in various areas, yet the concept of neutral membership needs to be considered in IFMs. In this regard, Dogra & Pal [28] proposed the picture fuzzy matrix (PFM) and discussed some of its important aspects. On the theory of PFM, many authors worked on its important concept. For instance, Silambarasan [38] defined some algebraic operations and properties of the PFM. Murugadas [92] defined the implication operation on PFM. The picture fuzzy soft matrices were defined by Arikrishnan & Sriram [93]. Further, Kamalakannann & Murugadas [94] studied the eigenvalue and eigenvector of PFM with some examples. Adak et al. [95] explained the concepts of semi-prime ideals of PFS.

Also, the concept of fuzzy relations is an important concept of the matrix. Zadeh [96] initially established the concept of fuzzy relation, including fuzzy equivalency (similarity) relation, and provided the concept of fuzzy ordering along with certain basic features. Moreover, eigenvalues and eigenvectors of the matrix play a very important role in solving many complex problems in different domains. Sanchez [97] worked on eigen fuzzy sets and described the importance of eigen fuzzy sets using the composition of fuzzy relations. Using the max-min composition, Sanchez [97] also determined the greatest eigen fuzzy set (GEFS). Martino et al. [48] presented the least eigen fuzzy set depending on min-max composition. Goetschel & Voxman [98] extended the idea for finding the eigen fuzzy set to the eigen fuzzy number. Using the principal component analysis of images, Nobuhara & Hirota [99] defined the greatest eigen fuzzy set and an adjoint concept of GEFS. Martino et al.[100] proposed a genetic algorithm for image reconstruction based on fuzzy relation, where the GEFS and lowest eigen fuzzy set (LEFS) were used to determine the highest value of fitness. Rakus-Andersson [136] measured the levels of drug effectiveness by establishing fuzzy relations between the potential symptoms and using the greatest and lowest eigen fuzzy sets.

4.1.2 Motivation and contribution

In numerous real-life scenarios, determining precise values for the membership, neutral membership, and non-membership degrees of an element within a given set proves challenging. The IVPFS emerge as a highly effective and prominent tool to address these difficulties, enabling a more versatile and comprehensive depiction of uncertain and vague information. Consequently, it becomes vital to develop a generalized and adaptable matrix theory capable of surpassing the limitations of PFM in representing interval data associated with membership, neutral membership, and non-membership degrees. The contribution of the present work is summarized as follows:

- We generalize the theory of PFM and introduce the novel concept of IVPFM to incorporate the situations where membership, neutral membership, and non-membership degrees are available in terms of interval numbers. We provide comprehensive definitions and theorems related to IVPFS, establishing a solid foundation for further exploration and analysis.

- The procedure to obtain the determinant and adjoint of an IVPFM is established.
- Two Algorithms are developed to evaluate the greatest eigen interval-valued picture fuzzy set (GEIVPFS) and the least eigen interval-valued picture fuzzy set (LEIVPFS) to solve decision-making issues.
- A new distance measure between IVPFSs is also defined with proof of its validity.
- An application of the proposed concepts has been shown in a real-life decision-making problem associated with smart manufacturing.

4.1.3 Organization of the Chapter

The remaining part of the chapter is organized as follows. In Section 4.2, we define the idea of IVPFM with basic definitions, properties, and important theorems. The concepts of determinant, adjoint, and propositions are given in Section 4.3. Section 4.4 proposes the algorithms for finding the GEIVPFS & the LEIVPFS and illustrates them with a numerical example. Section 4.5 introduces a new distance measure for the IVPFSs and discusses its application in decision-making. The comparative study with existing work is conducted in Section 4.6. Finally, Section 4.7 concludes with some future directions.
In the next section, we present the innovative notion of an IVPFM, which serves as an extension of PFM. Additionally, we establish the definitions of fundamental arithmetic operations and demonstrate the proof of essential theorems pertaining to the IVPFM.

4.2 Interval-Valued Picture Fuzzy Matrix

In this section, we define the IVPFM and basic concepts by generalizing the concept of PFM.

Definition 4.2.1. *An IVPFM $\tilde{A}$ is defined as*

$$\tilde{A} = (\tilde{a}_{ij}) = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu} >),\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n$$

where,

$$\tilde{a}_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \subseteq [0,1],$$

$$\tilde{a}_{ij\eta} = [a_{ij\eta L}, a_{ij\eta U}] \subseteq [0,1],$$

$$\tilde{a}_{ij\nu} = [a_{ij\nu L}, a_{ij\nu U}] \subseteq [0,1],$$

with the condition

$$a_{ij\mu U} + a_{ij\eta U} + a_{ij\nu U} \leq 1.$$

$\tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}$ *and* $\tilde{a}_{ij\nu}$ *are the membership, neutral membership and non-membership degree of* $\tilde{a}_{ij}$.

Definition 4.2.2. *An IVPFM is said to be a square interval-valued picture fuzzy matrix (SIVPFM) if the number of rows is equal to the number of columns.*

Definition 4.2.3. *Let* $\tilde{A}$ *and* $\tilde{B}$ *be two IVPFM such that*
$\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])$ *and* $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ij\nu L}, b_{ij\nu U}])$.
Then, we write $\tilde{A} \leq \tilde{B}$ *as follow:*
$a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}; a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}; a_{ij\nu L} \geq b_{ij\nu L}, a_{ij\nu U} \geq b_{ij\nu U}$.

Definition 4.2.4. *An IVPFM* $\tilde{A}$ *is a null matrix if* $\tilde{a}_{ij\mu} = 0$, $\tilde{a}_{ij\eta} = 0$ *and* $\tilde{a}_{ij\nu} = 0 \; \forall \; i = 1, 2, \ldots\ldots m, j = \ldots\ldots n$.

Definition 4.2.5. *An IVPFM* $\tilde{A} = (\tilde{a}_{ij})$ *various kind of matrix have been analogically proposed.*

(I) An IVPFM is called the row matrix if $i = 1$ $(j = 1, 2\ldots\ldots n)$.

(II) An IVPFM is called the column matrix if $j = 1$ $(i = 1, 2\ldots\ldots m)$.

(III) An IVPFM is called the diagonal matrix if all its non-diagonal elements are zero.

(IV) An IVPFM is called the μ universal matrix if $\tilde{\mu}_{ij} = 1, \tilde{\eta}_{ij} = 0, \tilde{\nu}_{ij} = 0 \; \forall \; i = 1, 2, \ldots\ldots m, j = \ldots\ldots n$.

(V) An IVPFM is called the η universal matrix if $\tilde{\mu}_{ij} = 0, \tilde{\eta}_{ij} = 1, \tilde{\nu}_{ij} = 0 \; \forall i = 1, 2, \ldots\ldots m, j = \ldots\ldots n$.

(VI) An IVPFM is called the ν universal matrix if $\tilde{\mu}_{ij} = 0, \tilde{\eta}_{ij} = 0, \tilde{\nu}_{ij} = 1 \; \forall \; i = 1, 2, \ldots\ldots m, j = \ldots\ldots n$.

(VII) An IVPFM is called a symmetric matrix if $\tilde{a}_{ij} = \tilde{a}_{ji}$.

(VIII) An IVPFM is called the skew-symmetric if $\tilde{a}_{ij} = Neg(\tilde{a}_{ij})$

(IX) Two IVPFM are called equal if they have the same order and their corresponding elements are equal.

(X) If $\tilde{A}$ is the SIVPFM, then its trace, denoted by $tr(\tilde{A})$, is the sum of the elements on the main diagonal.

Next, we discuss some operations of IVPFM.

4.2.1 Operations of IVPFM

Before proceeding toward the main theorem of operations of IVPFM, we first define some basic operations.

Definition 4.2.6. *Let* $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$ *and* $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$ *be two IVPFM of same order* $m \times n$ *then, we define some basic operations.*

(i) $\tilde{A}^c = (\left[a_{ijvL}, a_{ijvU}\right] \left[a_{ij\eta L}, a_{ij\eta U}\right], \left[a_{ij\mu L}, a_{ij\mu U}\right])$.

(ii) $\tilde{A} \vee \tilde{B} = (\left[max\left(a_{ij\mu L}, b_{ij\mu L}\right), max\left(a_{ij\mu U}, b_{ij\mu U}\right)\right] \left[min\left(a_{ij\eta L}, b_{ij\eta L}\right), min\left(a_{ij\eta U}, b_{ij\eta U}\right)\right]$ $\left[min\left(a_{ijvL}, b_{ijvL}\right), min\left(a_{ijvU}, b_{ijvU}\right)\right])$.

(iii) $\tilde{A} \wedge \tilde{B} = (\left[min\left(a_{ij\mu L}, b_{ij\mu L}\right), min\left(a_{ij\mu U}, b_{ij\mu U}\right)\right] \left[min\left(a_{ij\eta L}, b_{ij\eta L}\right), min\left(a_{ij\eta u}, b_{ij\eta U}\right)\right]$ $\left[max\left(a_{ijvL}, b_{ijvL}\right), max\left(a_{ijvU}, b_{ijvU}\right)\right])$.

(iv) $\tilde{A}^T = (\left[a_{ji\mu L}, a_{ji\mu U}\right] \left[a_{ji\eta L}, a_{ji\eta U}\right] \left[a_{jivL}, a_{jivU}\right])$.

(v) $\tilde{A} \oplus \tilde{B} = (\left[a_{ij\mu L} + b_{ij\mu L} - a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U} + b_{ij\mu U} - a_{ij\mu U}.b_{ij\mu U}\right] \left[a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U}.b_{ij\eta U}\right]$ $\left[a_{ijvL}.b_{ijvL}, a_{ijvU}.b_{ijvU}\right])$.

(vi) $\tilde{A} \otimes \tilde{B} = (\left[a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U}.b_{ij\mu U}\right], \left[a_{ij\eta L} + b_{ij\eta L} - a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U} + b_{ij\eta U} - a_{ij\eta U}.b_{ij\eta U}\right],$ $\left[a_{ijvL} + b_{ijvL} - a_{ijvL}.b_{ijvL}, a_{ijvU} + b_{ijvU} - a_{ijvU}.b_{ijvU}\right])$.
where $\tilde{A}^c$ *and* $\tilde{A}^T$ *are complement and transpose of* $\tilde{A}$ *respectively.*

Based on the above-defined basic operations, we propose some new properties in the next theorem required throughout the work.

Theorem 4.2.1. *Let* $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$, $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$ *and* $\tilde{C} = ([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ijvL}, c_{ijvU}])$ *be IVPFM of same order* $m \times n$ *then the following properties hold true.*

(i) $\tilde{A} \vee \tilde{B} = \tilde{B} \vee \tilde{A}$.

(ii) $\tilde{A} \wedge \tilde{B} = \tilde{B} \wedge \tilde{A}$.

(iii) $(\tilde{A}^T)^T = \tilde{A}$.

(iv) $(\tilde{A}^c)^T = (\tilde{A}^T)^c$.

(v) $\tilde{A}\vee(\tilde{B}\wedge\tilde{C})=(\tilde{A}\vee\tilde{B})\wedge(\tilde{A}\vee\tilde{C})$.

(vi) $\tilde{A}\wedge(\tilde{B}\vee\tilde{C})=(\tilde{A}\wedge\tilde{B})\vee(\tilde{A}\wedge\tilde{C})$.

(vii) $\tilde{A}\oplus\tilde{B}=\tilde{B}\oplus\tilde{A}$.

(viii) $\tilde{A}\otimes\tilde{B}=\tilde{B}\otimes\tilde{A}$.

(ix) $\tilde{A}\oplus(\tilde{B}\oplus\tilde{C})=(\tilde{A}\oplus\tilde{B})\oplus\tilde{C}$.

(x) $\tilde{A}\otimes(\tilde{B}\otimes\tilde{C})=(\tilde{A}\otimes\tilde{B})\otimes\tilde{C}$.

(xi) *(a)* $\tilde{A}\otimes(\tilde{B}\oplus\tilde{C})\neq(\tilde{A}\otimes\tilde{B})\oplus(\tilde{A}\otimes\tilde{C})$.

(b) $(\tilde{B}\oplus\tilde{C})\otimes\tilde{C}\neq(\tilde{B}\otimes A)\oplus(\tilde{C}\otimes\tilde{A})$.

Proof. The proof of the properties (i) to (vi) is obvious and, therefore, not given here.

(vii)

$$
\begin{aligned}
\tilde{A}\oplus\tilde{B} =& ([a_{ij\mu L}+b_{ij\mu L}-a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U}+b_{ij\mu U}-a_{ij\mu U}.b_{ij\mu U}], [a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U}.b_{ij\eta U}],\\
&\quad [a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U}.b_{ij\nu U}])\\
=& ([b_{ij\mu L}+a_{ij\mu L}-b_{ij\mu L}.a_{ij\mu L}, b_{ij\mu U}+a_{ij\mu U}-b_{ij\mu U}.a_{ij\mu U}], [b_{ij\eta L}.a_{ij\eta L}, b_{ij\eta U}.a_{ij\eta U}],\\
&\quad [b_{ij\nu L}.a_{ij\nu L}, b_{ij\nu U}.a_{ij\nu U}])\\
=& \tilde{B}\oplus\tilde{A}.
\end{aligned}
$$

$(viii)$ Similarly, $\tilde{A}\otimes\tilde{B}=\tilde{B}\otimes\tilde{A}$.

(ix)

$$
\begin{aligned}
A\oplus(\tilde{B}\oplus\tilde{C}) =& ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])\oplus([b_{ij\mu L}+c_{ij\mu L}-b_{ij\mu L}.c_{ij\mu L}, b_{ij\mu U}+c_{ij\mu U}\\
&\quad -b_{ij\mu U}.c_{ij\mu U}], [b_{ij\eta L}.c_{ij\eta L}, b_{ij\eta U}.c_{ij\eta U}], [b_{ij\nu L}.c_{ij\nu L}, b_{ij\nu U}.c_{ij\nu U}])\\
=& ([a_{ij\mu L}+b_{ij\mu L}-a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U}+b_{ij\mu U}-a_{ij\mu U}.b_{ij\mu U}], [a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U}.b_{ij\eta U}],\\
&\quad [a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U}.b_{ij\nu U}])\oplus([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ij\nu L}, c_{ij\nu U}])\\
=& (\tilde{A}\oplus\tilde{B})\oplus\tilde{C}.
\end{aligned}
$$

(x) Similarly, $\tilde{A}\otimes(\tilde{B}\otimes\tilde{C})=(\tilde{A}\otimes\tilde{B})\otimes\tilde{C}$.

(xi)

(a)

$$\tilde{B}\oplus\tilde{C}=([b_{ij\mu L}+c_{ij\mu L}-b_{ij\mu L}.c_{ij\mu L},b_{ij\mu U}+c_{ij\mu U}-b_{ij\mu U}.c_{ij\mu U}],[b_{ij\eta L}.c_{ij\eta L},b_{ij\eta U}.c_{ij\eta U}],$$
$$[b_{ij\nu L}.c_{ij\nu L},b_{ij\nu U}.c_{ij\nu U}])$$
$$\tilde{A}\otimes(\tilde{B}\oplus\tilde{C})=([a_{ij\mu L}.(b_{ij\mu L}+c_{ij\mu L}-b_{ij\mu L}.c_{ij\mu L}),a_{ij\mu U}.(b_{ij\mu U}+c_{ij\mu U}-b_{ij\mu U}.c_{ij\mu U})],$$
$$[a_{ij\eta L}+b_{ij\eta L}.c_{ij\eta L}-a_{ij\eta L}.b_{ij\eta L}.c_{ij\eta L},a_{ij\eta U}+b_{ij\eta U}.c_{ij\eta U}-a_{ij\eta U}.b_{ij\eta U}.c_{ij\eta U}]$$
$$[a_{ij\nu L}+b_{ij\nu L}.c_{ij\nu L}-a_{ij\nu L}.b_{ij\nu L}.c_{ij\nu L},a_{ij\nu U}+b_{ij\nu U}.c_{ij\nu U}-a_{ij\nu U}.b_{ij\nu U}.c_{ij\nu U}]).$$
$$\tilde{A}\otimes\tilde{B}=([a_{ij\mu L}.b_{ij\mu L},a_{ij\mu U}.b_{ij\mu U}],[a_{ij\eta L}+b_{ij\eta L}-a_{ij\eta L}.b_{ij\eta L},a_{ij\eta U}+b_{ij\eta U}-a_{ij\eta U}.b_{ij\eta U}],$$
$$[a_{ij\nu L}+b_{ij\nu L}-a_{ij\nu L}.b_{ij\nu L},a_{ij\nu U}+b_{ij\nu U}-a_{ij\nu U}.b_{ij\nu U}]).$$
$$\tilde{A}\otimes\tilde{C}=([a_{ij\mu L}.c_{ij\mu L},a_{ij\mu U}.c_{ij\mu U}],[a_{ij\eta L}+c_{ij\eta L}-a_{ij\eta L}.c_{ij\eta L},a_{ij\eta U}+c_{ij\eta U}-a_{ij\eta U}.c_{ij\eta U}],$$
$$[a_{ij\nu L}+c_{ij\nu L}-a_{ij\nu L}.c_{ij\nu L},a_{ij\nu U}+c_{ij\nu U}-a_{ij\nu U}.c_{ij\nu U}])$$
$$(\tilde{A}\otimes\tilde{B})\oplus(\tilde{A}\otimes\tilde{C})=([a_{ij\mu L}.(b_{ij\mu U}+c_{ij\mu U})-a^2_{ij\mu U}.b_{ij\mu L}.c_{ij\mu L},a_{ij\mu U}.(b_{ij\mu U}+c_{ij\mu U})-$$
$$a^2_{ij\mu U}.b_{ij\mu U}.c_{ij\mu U}],[(a_{ij\eta L}+b_{ij\eta L}-a_{ij\eta L}.b_{ij\eta L}).(a_{ij\eta L}+c_{ij\eta L}-a_{ij\eta L}-c_{ij\eta L}),$$
$$b_{ij\eta U}.(a_{ij\eta U}+b_{ij\eta U}-a_{ij\eta U}.b_{ij\eta U}).(a_{ij\eta U}+c_{ij\eta U}-a_{ij\eta U}.c_{ij\eta U})],$$
$$[(a_{ij\nu L}+b_{ij\nu L}-a_{ij\nu L}.b_{ij\nu L}).(a_{ij\nu L}+c_{ij\nu L}-a_{ij\nu L}-c_{ij\nu L}),$$
$$b_{ij\nu U}.(a_{ij\nu U}+b_{ij\nu U}-a_{ij\nu U}.b_{ij\nu U}).(a_{ij\nu U}+c_{ij\nu U}-a_{ij\nu U}.c_{ij\nu U})])$$

So, $\tilde{A}\otimes(\tilde{B}\oplus\tilde{C})\neq(\tilde{A}\otimes\tilde{B})\oplus(\tilde{A}\otimes\tilde{C})$.

(b) Similarly, $(\tilde{B}\oplus\tilde{C})\otimes\tilde{C}\neq(\tilde{B}\otimes\tilde{A})\oplus(\tilde{C}\otimes\tilde{A})$.

□

Theorem 4.2.2. *Let $\tilde{A}$, $\tilde{B}$ be two IVPFM of order $m\times n$ then*

(a) $\left(\tilde{A}\vee\tilde{B}\right)^c=\tilde{A}^c\wedge\tilde{B}^c$.

(b) $\left(\tilde{A}\wedge\tilde{B}\right)^c=\tilde{A}^c\vee\tilde{B}^c$.

Proof. (a) Let $\tilde{A}=([a_{ij\mu L},a_{ij\mu U}],[a_{ij\eta L},a_{ij\eta U}],[a_{ij\nu L},a_{ij\nu U}])$ and
$\tilde{B}=([b_{ij\mu L},b_{ij\mu U}],[b_{ij\eta L},b_{ij\eta U}],[b_{ij\nu L},b_{ij\nu U}])$
Then, $\tilde{A}^c=(\left[a_{ij\nu L},a_{ij\nu U}\right],\left[a_{ij\eta L},a_{ij\eta U}\right],\left[a_{ij\mu L},a_{ij\mu U}\right])$ and

$$\tilde{B}^c = \left(\left[b_{ijvL}, b_{ijvU}\right], \left[b_{ij\eta L}, b_{ij\eta U}\right], \left[b_{ij\mu L}, b_{ij\mu U}\right]\right)$$

$$\begin{aligned}
\tilde{A}^c \wedge \tilde{B}^c &= \left(\left[\min\left(a_{ijvL}, b_{ijvL}\right), \min\left(a_{ijvU}, b_{ijvU}\right)\right] \left[\min\left(a_{ij\eta L}, b_{ij\eta L}\right), \min\left(a_{ij\eta U}, b_{ij\eta U}\right)\right]\right. \\
&\quad \left.\left[\max\left(a_{ij\mu L}, b_{ij\mu L}\right), \max\left(a_{ij\mu U}, b_{ij\mu U}\right)\right]\right) \\
\tilde{A} \vee \tilde{B} &= \left(\left[\max\left(a_{ijvL}, b_{ijvL}\right), \max\left(a_{ijvU}, b_{ijvU}\right)\right] \left[\min\left(a_{ij\eta L}, b_{ij\eta L}\right), \min\left(a_{ij\eta U}, b_{ij\eta U}\right)\right]\right. \\
&\quad \left.\left[\min\left(a_{ij\mu L}, b_{ij\mu L}\right), \min\left(a_{ij\mu U}, b_{ij\mu U}\right)\right]\right) \\
\left(\tilde{A} \vee \tilde{B}\right)^c &= \left(\left[\min\left(a_{ijvL}, b_{ijvL}\right), \min\left(a_{ijvU}, b_{ijvU}\right)\right] \left[\min\left(a_{ij\eta L}, b_{ij\eta L}\right), \min\left(a_{ij\eta U}, b_{ij\eta U}\right)\right]\right. \\
&\quad \left.\left[\max\left(a_{ij\mu L}, b_{ij\mu L}\right), \max\left(a_{ij\mu U}, b_{ij\mu U}\right)\right]\right) \\
&= \tilde{A}^c \wedge \tilde{B}^c
\end{aligned}$$

(b) Proof of part (b) can be done on similar lines.

□

Theorem 4.2.3. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ and $\tilde{A} \leq \tilde{C}$ and $\tilde{B} \leq \tilde{C}$, then $\tilde{A} \vee \tilde{B} \leq \tilde{C}$.*

Proof. Let $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$, $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$ and $\tilde{C} = ([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ijvL}, c_{ijvU}])$.
If $\tilde{A} \leq \tilde{C}$ then $a_{ij\mu L} \leq c_{ij\mu L}, a_{ij\mu U} \leq c_{ij\mu U}, a_{ij\eta L} \leq c_{ij\eta L}, a_{ij\eta U} \leq c_{ij\eta U}, a_{ijvL} \geq c_{ijvL}, a_{ijvU} \geq c_{ijvU}$ for all i,j,
and $\tilde{B} \leq \tilde{C}$ then $b_{ij\mu L} \leq c_{ij\mu L}, b_{ij\mu U} \leq c_{ij\mu U}, b_{ij\eta L} \leq c_{ij\eta L}, b_{ij\eta U} \leq c_{ij\eta U}, b_{ijvL} \geq c_{ijvL}, b_{ijvU} \geq c_{ijvU}$ for all i,j.
Now, max $\left(a_{ij\mu L}, b_{ij\mu L}\right) \leq c_{ij\mu L}$, max $\left(a_{ij\mu U}, b_{ij\mu U}\right) \leq c_{ij\mu U}$,
min $\left(a_{ij\eta L}, b_{ij\eta L}\right) \leq c_{ij\eta L}$, min $\left(a_{ij\eta U}, b_{ij\eta U}\right) \leq c_{ij\eta U}$,
min $\left(a_{ijvL}, b_{ijvL}\right) \geq c_{ijvL}$, min $\left(a_{ijvU}, b_{ijvU}\right) \geq c_{ijvU}$.
Thus $\tilde{A} \vee \tilde{B} \leq \tilde{C}$ using Definition 4.2.3. □

Theorem 4.2.4. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ of $\tilde{A} \leq \tilde{B}$ then $\tilde{A} \vee \tilde{C} \leq \tilde{B} \vee \tilde{C}$.*

Proof. Let $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$, $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$ and $\tilde{C} = ([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ijvL}, c_{ijvU}])$ be three IVPFM of same order $m \times n$.
If $\tilde{A} \leq \tilde{B}$ then $a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}, a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}, a_{ijvL} \geq b_{ijvL}, a_{ijvU} \geq b_{ijvU}$.
Now, max $\left(a_{ij\mu L}, c_{ij\mu L}\right) \leq$ max $\left(b_{ij\mu L}, c_{ij\mu L}\right)$, max $\left(a_{ij\mu U}, c_{ij\mu U}\right) \leq$ max $\left(b_{ij\mu U}, c_{ij\mu U}\right)$,
min $\left(a_{ij\eta L}, c_{ij\eta L}\right) \leq$ min $\left(b_{ij\eta L}, c_{ij\eta L}\right)$, min $\left(a_{ij\eta U}, c_{ij\eta U}\right) \leq$ min $\left(b_{ij\eta U}, c_{ij\eta U}\right)$,

min $(a_{ijvL}, c_{ijvL}) \geq$ min (b_{ijvL}, c_{ijvL}) , min $(a_{ijvU}, c_{ijvU}) \geq$ min (b_{ijvU}, c_{ijvU}) for all i,j.
Therefore $\tilde{A} \vee \tilde{C} \leq \tilde{B} \vee \tilde{C}$. □

Theorem 4.2.5. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ and $\tilde{C} \leq \tilde{A}$ and $\tilde{C} \leq \tilde{B}$ then $\tilde{C} \leq \tilde{A} \wedge \tilde{B}$.*

Proof. Proof of the above result directly follows from Theorem 4.2.4. □

Theorem 4.2.6. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ and if $\tilde{A} \leq \tilde{B}$, $\tilde{A} \leq \tilde{C}$ and $\tilde{B} \wedge \tilde{C} = 0$, then $\tilde{A} = 0$.*

Proof. If $\tilde{A} \leq \tilde{B}$ then $a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}; a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}; a_{ijvL} \geq b_{ijvL}, a_{ijvU} \geq b_{ijvU}$. Similarly $\tilde{A} \leq \tilde{C}$ then $a_{ij\mu L} \leq c_{ij\mu L}, a_{ij\mu U} \leq c_{ij\mu U}; a_{ij\eta L} \leq c_{ij\eta L}, a_{ij\eta U} \leq c_{ij\eta U}; a_{ijvL} \geq c_{ijvL}, a_{ijvU} \geq c_{ijvU}$. Thus by Theorem 4.2.5 $\tilde{A} \leq \tilde{B} \wedge \tilde{C}, \tilde{B} \wedge \tilde{C} = 0$ such that $\tilde{A} = 0$. □

Theorem 4.2.7. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ of $\tilde{A} \leq \tilde{B}$ then $\tilde{A} \wedge \tilde{C} \leq \tilde{B} \wedge \tilde{C}$.*

Proof. Proof of the above result directly follows from Definition 4.2.3. □

Theorem 4.2.8. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ and if $\tilde{A} \leq \tilde{B}$, and $\tilde{B} \wedge \tilde{C} = 0$, then $\tilde{A} \wedge \tilde{C} = 0$.*

Proof. By Theorem 4.2.7, the proof is straight forward. □

In the next section, we present a method for determining the determinant and adjoint of the IVPFM. Illustrative examples are provided to showcase the calculation of both the determinant and adjoint of this matrix.

4.3 Determinant and Adjoint of IVPFM

In this section, we define the determinant, and adjoint of the IVPFM and examine some related fundamental observations.

Definition 4.3.1. *Determinant of IVPFM*

Suppose $\tilde{A}$= $([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$ be the IVPFM of order m. Then, the determinant of $\tilde{A}$ is denoted by $|\tilde{A}|$ and defined by

$$|\tilde{A}| = \begin{pmatrix} \vee_{h\in\psi_k}([a_{1h(1)\mu L}, a_{1h(1)\mu U}] \wedge [a_{2h(2)\mu L}, a_{2h(2)\mu U}] \cdots \wedge [a_{kh(k)\mu L}, a_{kh(k)\mu U}]), \\ \wedge_{h\in\psi_k}([a_{1h(1)\eta L}, a_{1h(1)\eta U}] \wedge [a_{2h(2)\eta L}, a_{2h(2)\eta U}] \cdots \wedge [a_{kh(k)\eta L}, a_{kh(k)\eta U}]), \\ \wedge_{h\in\psi_k}([a_{1h(1)\nu L}, a_{1h(1)\nu U}] \vee [a_{2h(2)\nu L}, a_{2h(2)\nu U}] \cdots \vee [a_{kh(k)\nu L}, a_{kh(k)\nu U}]). \end{pmatrix}$$

where ψ_k be the set of permutation on the set {1, 2, 3,. . . , m}.

Example 4.3.1. *Let us consider IVPFM of order 3 as follows*

$$\tilde{A} = \begin{pmatrix} [0.40,0.50][0.12,0.23][0.19,0.23] & [0.71,0.79][0.07,0.09][0.10,0.12] & [0.21,0.30][0.12,0.22][0.39,0.40] \\ [0.27,0.43][0.08,0.28][0.16,0.23] & [0.42,0.51][0.13,0.29][0.07,0.12] & [0.14,0.23][0.21,0.29][0.30,0.40] \\ [0.35,0.39][0.11,0.23][0.06,0.21] & [0.19,0.31][0.04,0.08][0.49,0.59] & [0.48,0.57][0.22,0.3][0,0.07] \end{pmatrix}$$

To find the determinant of $\tilde{A}$, we need to find out all permutations on $\{1,2,3\}$. The permutation on $\{1,2,3\}$

$$\psi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \psi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \psi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

$$\psi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \psi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \psi_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The membership degree of $|\tilde{A}|$ is

$$\begin{aligned} &([a_{1\psi_1(1)\mu L}, a_{1\psi_1(1)\mu U}] \wedge [a_{2\psi_1(2)\mu L}, a_{2\psi_1(2)\mu U}] \wedge [a_{3\psi_1(3)\mu L}, a_{3\psi_1(3)\mu U}]) \\ &\vee([a_{1\psi_2(1)\mu L}, a_{1\psi_2(1)\mu U}] \wedge [a_{2\psi_2(2)\mu L}, a_{2\psi_2(2)\mu U}] \wedge [a_{3\psi_2(3)\mu L}, a_{3\psi_2(3)\mu U}]) \\ &\vee([a_{1\psi_3(1)\mu L}, a_{1\psi_3(1)\mu U}] \wedge [a_{2\psi_3(2)\mu L}, a_{2\psi_3(2)\mu U}] \wedge [a_{3\psi_3(3)\mu L}, a_{3\psi_3(3)\mu U}]) \\ &\vee([a_{1\psi_4(1)\mu L}, a_{1\psi_4(1)\mu U}] \wedge [a_{2\psi_4(2)\mu L}, a_{2\psi_4(2)\mu U}] \wedge [a_{3\psi_4(3)\mu L}, a_{3\psi_4(3)\mu U}]) \\ &\vee([a_{1\psi_5(1)\mu L}, a_{1\psi_5(1)\mu U}] \wedge [a_{2\psi_5(2)\mu L}, a_{2\psi_5(2)\mu U}] \wedge [a_{3\psi_5(3)\mu L}, a_{3\psi_5(3)\mu U}]) \\ &\vee([a_{1\psi_6(1)\mu L}, a_{1\psi_6(1)\mu U}] \wedge [a_{2\psi_6(2)\mu L}, a_{2\psi_6(2)\mu U}] \wedge [a_{3\psi_6(3)\mu L}, a_{3\psi_6(3)\mu U}]) \end{aligned}$$

$$= \begin{aligned} &([a_{11\mu L}, a_{11\mu U}] \wedge [a_{22\mu L}, a_{22\mu U}] \wedge [a_{33\mu L}, a_{33\mu U}]) \\ &\vee([a_{11\mu L}, a_{11\mu U}] \wedge [a_{23\mu L}, a_{23\mu U}] \wedge [a_{32\mu L}, a_{32\mu U}]) \\ &\vee([a_{12\mu L}, a_{12\mu U}] \wedge [a_{21\mu L}, a_{21\mu U}] \wedge [a_{33\mu L}, a_{33\mu U}]) \\ &\vee([a_{12\mu L}, a_{12\mu U}] \wedge [a_{23\mu L}, a_{23\mu U}] \wedge [a_{31\mu L}, a_{31\mu U}]) \\ &\vee([a_{13\mu L}, a_{13\mu U}] \wedge [a_{21\mu L}, a_{21\mu U}] \wedge [a_{32\mu L}, a_{32\mu U}]) \\ &\vee([a_{13\mu L}, a_{13\mu U}] \wedge [a_{22\mu L}, a_{22\mu U}] \wedge [a_{31\mu L}, a_{31\mu U}]) \end{aligned}$$

$$= \begin{array}{l} ([0.40,0.50] \wedge [0.42,0.51] \wedge [0.48,0.57]) \\ \vee([0.40,0.50] \wedge [0.14,0.23] \wedge [0.19,0.31]) \\ \vee([0.71,0.79] \wedge [0.27,0.43] \wedge [0.48,0.57]) \\ \vee([0.71,0.79] \wedge [0.14,0.23] \wedge [0.35,0.39]) \\ \vee([0.21,0.30] \wedge [0.27,0.43] \wedge [0.19,0.31]) \\ \vee([0.21,0.30] \wedge [0.42,0.51] \wedge [0.35,0.39]) \end{array}$$

$$= [0.4,0.50] \vee [0.14,0.23] \vee [0.27,0.43] \vee [0.14,0.23] \vee [0.19,0.30] \vee [0.21,0.30] = [0.4,0.5] .$$

Similarly, the neutral membership degree of $|\tilde{A}|$ is

$$\begin{array}{l} ([a_{1\psi_1(1)\eta L}, a_{1\psi_1(1)\eta U}] \wedge [a_{2\psi_1(2)\eta L}, a_{2\psi_1(2)\eta U}] \wedge [a_{3\psi_1(3)\eta L}, a_{3\psi_1(3)\eta U}]) \\ \wedge([a_{1\psi_2(1)\eta L}, a_{1\psi_2(1)\eta U}] \wedge [a_{2\psi_2(2)\eta L}, a_{2\psi_2(2)\eta U}] \wedge [a_{3\psi_2(3)\eta L}, a_{3\psi_2(3)\eta U}]) \\ \wedge([a_{1\psi_3(1)\eta L}, a_{1\psi_3(1)\eta U}] \wedge [a_{2\psi_3(2)\eta L}, a_{2\psi_3(2)\eta U}] \wedge [a_{3\psi_3(3)\eta L}, a_{3\psi_3(3)\eta U}]) \\ \wedge([a_{1\psi_4(1)\eta L}, a_{1\psi_4(1)\eta U}] \wedge [a_{2\psi_4(2)\eta L}, a_{2\psi_4(2)\eta U}] \wedge [a_{3\psi_4(3)\eta L}, a_{3\psi_4(3)\eta U}]) \\ \wedge([a_{1\psi_5(1)\eta L}, a_{1\psi_5(1)\eta U}] \wedge [a_{2\psi_5(2)\eta L}, a_{2\psi_5(2)\eta U}] \wedge [a_{3\psi_5(3)\eta L}, a_{3\psi_5(3)\eta U}]) \\ \wedge([a_{1\psi_6(1)\eta L}, a_{1\psi_6(1)\eta U}] \wedge [a_{2\psi_6(2)\eta L}, a_{2\psi_6(2)\eta U}] \wedge [a_{3\psi_6(3)\eta L}, a_{3\psi_6(3)\eta U}]) \end{array}$$

$$= \begin{array}{l} ([a_{11\eta L}, a_{11\eta U}] \wedge [a_{22\eta L}, a_{22\eta U}] \wedge [a_{33\eta L}, a_{33\eta U}]) \\ \wedge([a_{11\eta L}, a_{11\eta U}] \wedge [a_{23\eta L}, a_{23\eta Uta}] \wedge [a_{32\eta L}, a_{32\eta U}]) \\ \wedge([a_{12\eta L}, a_{12\eta U}] \wedge [a_{21\eta L}, a_{21\eta U}] \wedge [a_{33\eta L}, a_{33\eta U}]) \\ \wedge([a_{12\eta L}, a_{12\eta U}] \wedge [a_{23\eta L}, a_{23\eta U}] \wedge [a_{31\eta L}, a_{31\eta U}]) \\ \wedge([a_{13\eta L}, a_{13\eta U}] \wedge [a_{21\eta L}, a_{21\eta U}] \wedge [a_{32\eta L}, a_{32\eta U}]) \\ \wedge([a_{13\eta L}, a_{13\eta U}] \wedge [a_{22\eta L}, a_{22\eta U}] \wedge [a_{31\eta L}, a_{31\eta U}]) \end{array}$$

$$= \begin{array}{l} ([0.12,0.23] \wedge [0.13,0.29] \wedge [0.22,0.3]) \\ \wedge([0.12,0.23] \wedge [0.21,0.29] \wedge [0.04,0.08]) \\ \wedge([0.07,0.09] \wedge [0.08,0.28] \wedge [0.22,0.3]) \\ \wedge([0.07,0.09] \wedge [0.21,0.29] \wedge [0.11,0.23]) \\ \wedge([0.12,0.22] \wedge [0.08,0.28] \wedge [0.04,0.08]) \\ \wedge([0.12,0.22] \wedge [0.13,0.29] \wedge [0.11,0.23]) \end{array}$$

$$= [0.12,0.23] \wedge [0.04,0.08] \wedge [0.07,0.09] \wedge [0.07,0.09] \wedge [0.04,0.08] \wedge [0.11,0.22] = [0.04,0.08].$$

Now, the non-membership degree of $|\tilde{A}|$ is

$$\begin{aligned}&([a_{1\psi_1(1)\nu L},a_{1\psi_1(1)\nu U}]\vee[a_{2\psi_1(2)\nu L},a_{2\psi_1(2)\nu U}]\vee[a_{3\psi_1(3)\nu L},a_{3\psi_1(3)\nu U}])\\&\wedge([a_{1\psi_2(1)\nu L},a_{1\psi_2(1)\nu U}]\vee[a_{2\psi_2(2)\nu L},a_{2\psi_2(2)\nu U}]\vee[a_{3\psi_2(3)\nu L},a_{3\psi_2(3)\nu U}])\\&\wedge([a_{1\psi_3(1)\nu L},a_{1\psi_3(1)\nu U}]\vee[a_{2\psi_3(2)\nu L},a_{2\psi_3(2)\nu U}]\vee[a_{3\psi_3(3)\nu L},a_{3\psi_3(3)\nu U}])\\&\wedge([a_{1\psi_4(1)\nu L},a_{1\psi_4(1)\nu U}]\vee[a_{2\psi_4(2)\nu L},a_{2\psi_4(2)\nu U}]\vee[a_{3\psi_4(3)\nu L},a_{3\psi_4(3)\nu U}])\\&\wedge([a_{1\psi_5(1)\nu L},a_{1\psi_5(1)\nu U}]\vee[a_{2\psi_5(2)\nu L},a_{2\psi_5(2)\nu U}]\vee[a_{3\psi_5(3)\nu L},a_{3\psi_5(3)\nu U}])\\&\wedge([a_{1\psi_6(1)\nu L},a_{1\psi_6(1)\nu U}]\vee[a_{2\psi_6(2)\nu L},a_{2\psi_6(2)\nu U}]\vee[a_{3\psi_6(3)\nu L},a_{3\psi_6(3)\nu U}])\end{aligned}$$

$$=\begin{aligned}&([a_{11\nu L},a_{11\nu U}]\vee[a_{22\nu L},a_{22\nu U}]\vee[a_{33\nu L},a_{33\nu U}])\\&\wedge([a_{11\nu L},a_{11\nu U}]\vee[a_{23\nu L},a_{23\nu Uta}]\vee[a_{32\nu L},a_{32\nu U}])\\&\wedge([a_{12\nu L},a_{12\nu U}]\vee[a_{21\nu L},a_{21\nu U}]\vee[a_{33\nu L},a_{33\nu U}])\\&\wedge([a_{12\nu L},a_{12\nu U}]\vee[a_{23\nu L},a_{23\nu U}]\vee[a_{31\nu L},a_{31\nu U}])\\&\wedge([a_{13\nu L},a_{13\nu U}]\vee[a_{21\nu L},a_{21\nu U}]\vee[a_{32\nu L},a_{32\nu U}])\\&\wedge([a_{13\nu L},a_{13\nu U}]\vee[a_{22\nu L},a_{22\nu U}]\vee[a_{31\nu L},a_{31\nu U}])\end{aligned}$$

$$=\begin{aligned}&([0.19,0.23]\vee[0.07,0.12]\vee[0,0.07])\\&\wedge([0.19,0.23]\vee[0.3,0.40]\vee[0.49,0.59)\\&\wedge([0.10,0.12]\vee[0.16,0.23]\vee[0,0.07])\\&\wedge([0.10,0.12]\vee[0.30,0.40]\vee[0.06,0.21])\\&\wedge([0.39,0.44]\vee[0.16,0.23]\vee[0.49,0.59])\\&\wedge([0.39,0.44]\vee[0.07,0.12]\vee[0.06,0.21])\end{aligned}$$

$$=\ [0.19,0.27]\wedge[0.49,0.59]\wedge[0.16,0.23]\wedge[0.30,0.40]\wedge[0.49,0.59]\wedge[0.39,0.44]=[0.16,0.23]\ .$$

$\therefore|\tilde{A}|=([0.4,0.5][0.04,0.08][0.16,0.23])$.

Definition 4.3.2. *Adjoint of IVPFM*

Let $\tilde{A}=(\tilde{a}_{ij})=(<\tilde{a}_{ij\mu},\tilde{a}_{ij\eta},\tilde{a}_{ij\nu}>)$ *be a IVPFM of order m. Then, the adjoint of* $\tilde{A}$ *is denoted by the Adjoint(*$\tilde{A}$*) and defined by*

$$Adjoint(\tilde{A})=\ (<\tilde{q}_{ij\mu},\tilde{q}_{ij\eta},\tilde{q}_{ij\nu}>)$$

where

$$\tilde{q}_{ij\mu}=\vee_{\delta\in S_{m_jm_i}}\wedge_{u\in m_j}\tilde{a}_{u\delta(u)\mu},$$

$$\tilde{q}_{ij\eta}=\wedge_{\delta\in S_{m_jm_i}}\wedge_{u\in m_j}\tilde{a}_{u\delta(u)\eta},$$

$$\tilde{q}_{ij\nu}=\wedge_{\delta\in S_{m_jm_i}}\vee_{u\in m_j}\tilde{a}_{u\delta(u)\nu}.$$

Here $m_j=\{1,2........m\}$-$\{j\}$ *and* $S_{m_jm_i}$ *is the set of all permutation of the set* m_j *over the set* m_i.

Example 4.3.2. *Let us consider IVPFM of order three as follow*

$$\tilde{A} = \begin{pmatrix} [0.40,0.50][0.12,0.23][0.19,0.23] & [0.71,0.79][0.07,0.09][0.10,0.12] & [0.21,0.30][0.12,0.22][0.39,0.40] \\ [0.27,0.43][0.08,0.23][0.16,0.23] & [0.42,0.51][0.13,0.29][0.07,0.12] & [0.14,0.23][0.21,0.29][0.30,0.40] \\ [0.35,0.39][0.11,0.23][0.06,0.21] & [0.19,0.31][0.04,0.08][0.49,0.59] & [0.48,0.57][0.22,0.3][0,0.07] \end{pmatrix}$$

For j=1 and i=1, m_j= $\{1,2,3\}$-$\{1\}$= $\{2,3\}$ and m_i= $\{1,2,3\}$-$\{1\}$= $\{2,3\}$. The permutation of m_i over m_j are

$$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

Now,

$(a_{22\mu} \wedge a_{33\mu}) \vee (a_{23\mu} \wedge a_{32\mu})$

$= ([0.42,0.51] \wedge [0.48,0.57]) \vee ([0.14,0.23] \wedge [0.19,0.31])$

$= [0.42,0.51] \vee [0.14,0.23]$ =[0.42,0.51],

$(a_{22\eta} \wedge a_{33\eta}) \wedge (a_{23\eta} \wedge a_{32\eta})$

$= ([0.13,0.29] \wedge [0.22,0.3]) \wedge ([0.21,0.29] \wedge [0.04,0.08])$

$= [0.13,0.29] \wedge [0.04,0.08]$ =[0.04,0.08],

$(a_{22\nu} \vee a_{33\nu}) \wedge (a_{23\nu} \vee a_{32\nu})$

$= ([0.07,0.12] \vee [0,0.07]) \wedge ([0.3,0.4] \vee [0.49,0.59])$

$= [0.07,0.12] \wedge [0.49,0.59]$ =[0.07,0.12].

For $j = 1$ and $i = 2$, $m_j = \{1,2,3\} - \{1\} = \{2,3\}$ and $m_i = \{1,2,3\} - \{2\} = \{1,3\}$. The permutation of m_i over m_j are

$$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

Now

$(a_{12\mu} \wedge a_{33\mu}) \vee (a_{13\mu} \wedge a_{32\mu})$

$= ([0.71,0.79] \wedge [0.48,0.57]) \vee ([0.21,0.30] \wedge [0.19,0.31])$

$= [0.48,0.57] \vee [0.19,0.30]$ =[0.48,0.57],

$(a_{12\eta} \wedge a_{33\eta}) \wedge (a_{13\eta} \wedge a_{32\eta})$

$= ([0.07,0.09] \wedge [0.22,0.3]) \wedge ([0.12,0.24] \wedge [0.13,0.29])$

$= [0.07,0.09] \wedge [0.12,0.24]$ =[0.07,0.09],

$(a_{12\nu} \vee a_{33\nu}) \wedge (a_{13\nu} \vee a_{32\nu})$

$([0.10,0.12] \vee [0,0.07]) \wedge ([0.39,0.44] \vee [0.49,0.59])$

=$[0.10,0.12] \wedge [0.49,0.59]$ =[0.10,0.12].

For $j=1$ and $i=3$, $m_j=\{1,2,3\}-\{1\}=\{2,3\}$ and $m_i=\{1,2,3\}-\{3\}=\{1,2\}$. The permutation of m_i over m_j are

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Now

$(a_{12\mu}\wedge a_{23\mu})\vee(a_{13\mu}\wedge a_{22\mu})$

$([0.71,0.79]\wedge[0.14,0.23])\vee([0.21,0.30]\wedge[0.42,0.51])$

$=[0.14,0.23]\vee[0.21,0.30]$ =[0.21,0.30],

$(a_{12\eta}\wedge a_{23\eta})\wedge(a_{13\eta}\wedge a_{22\eta})$

$([0.07,0.09]\wedge[0.21,0.29])\wedge([0.12,0.24]\wedge[0.13,0.29])$

$=[0.07,0.09]\wedge[0.12,0.24]$ =[0.07,0.09],

and $(a_{12\nu}\vee a_{23\nu})\wedge(a_{13\nu}\vee a_{22\nu})$

$([0.10,0.12]\vee[0.3,0.4])\wedge([0.39,0.44]\vee[0.07,0.27])$

$=[.3,0.4]\wedge[0.39,0.44]$ =[0.3,0.4].

Calculating in the similar way, Adjoint($\tilde{A}$) is obtained as

$$\text{Adjoint}(\tilde{A})=\begin{pmatrix} [0.49,0.51][0.04,0.08][0.07,0.12] & [0.48,0.57][0.07,0.09][0.10,0.12] & [0.21,0.30][0.07,0.09][0.3,0.4] \\ [0.27,0.43][0.08,0.23][0.06,0.21] & [0.4,0.5][0.11,0.23][0.06,0.21] & [0.21,0.3][0.08,0.23][0.19,0.23] \\ [0.35,0.39][0.04,0.08][0.16,0.23] & [0.35,0.39][0.04,0.08][0.19,0.23] & [0.4,0.5][0.07,0.09][0.10,0.27] \end{pmatrix}$$

Proposition 4.3.1. *If $\tilde{A}$ be a square IVPFM, then $|\tilde{A}|=|\tilde{A}^T|$.*

Proof. Let $\tilde{A}=([a_{ij\mu L},{}_{ij\mu U}],[a_{ij\eta L},a_{ij\eta U}],[a_{ij\nu L},a_{ij\nu U}])$
$\implies \tilde{A}^T=([a_{ji\mu L},{}_{ji\mu U}],[a_{ji\eta L},a_{ji\eta U}],[a_{ji\nu L},a_{ji\nu U}])$. Then

$$|\tilde{A}^T|=\sum_{\sigma\in\delta_n}\begin{pmatrix} [a_{\sigma(1)1\mu L},a_{\sigma(1)1\mu U}],[a_{\sigma(1)1\eta L},a_{\sigma(1)1\eta U}],[a_{\sigma(1)1\nu L},a_{\sigma(1)1\nu U}] \\ [a_{\sigma(2)2\mu L},a_{\sigma(2)2\mu U}],[a_{\sigma(2)2\eta L},a_{\sigma(2)2\eta U}],[a_{\sigma(2)2\nu L},a_{\sigma(2)2\nu U}] \\ \dots \\ [a_{\sigma(n)n\mu L},a_{\sigma(n)n\mu U}],[a_{\sigma(n)n\eta L},a_{\sigma(n)n\eta U}],[a_{\sigma(n)n\nu L},a_{\sigma(n)n\nu U}] \end{pmatrix}$$

□

Let Ψ be the permutation of $\{1,2,...n\}$ such that $\Psi\sigma=I$, the identity permutation. Then $\Psi=\sigma^{-1}$. As σ runs over the whole set of permutation, so does Ψ. Let $\sigma(i)=j, i=\sigma^{-1}(j)=\Psi(j)$.
Therefore $a_{\sigma(i)i\mu L}=a_{j\Psi(j)\mu L}$, $a_{\sigma(i)i\mu U}=a_{j\Psi(j)\mu U}$, $a_{\sigma(i)i\eta L}=a_{j\Psi(j)\mu L}$, $a_{\sigma(i)i\eta U}=a_{j\Psi(j)\eta U}$, $a_{\sigma(i)i\nu L}=a_{j\Psi(j)\nu L}$, $a_{\sigma(i)i\nu U}=a_{j\Psi(j)\nu U}\forall i,j$. As i runs over the set $\{1,2,...n\}$, j so does.

$$\therefore |\tilde{A}^T| = \begin{array}{l} ([a_{\Psi 1(1)\mu L}, a_{\Psi 1(1)\mu U}], [a_{\Psi 1(1)\eta L}, a_{\Psi 1(1)\eta U}], [a_{\Psi 1(1)\nu L}, a_{\Psi 1(1)\nu U}]) \\ ([a_{\Psi 2(2)\mu L}, a_{\Psi 2(2)\mu U}], [a_{\Psi 2(2)\eta L}, a_{\Psi 2(2)\eta U}], [a_{\Psi 2(2)\nu L}, a_{\Psi 2(2)\nu U}]) \dots \\ ([a_{\Psi n(n)\mu L}, a_{\Psi n(n)\mu U}], [a_{\Psi n(n)\eta L}, a_{\Psi n(n)\eta U}], [a_{\Psi n(n)\nu L}, a_{\Psi n(n)\nu U}]) \end{array}$$

$$= \begin{array}{l} \sum_{\Psi \in \delta_n} ([a_{1\Psi(1)\mu L}, a_{1\Psi(1)\mu U}], [a_{1\Psi(1)\eta L}, a_{1\Psi(1)\eta U}], [a_{1\Psi(1)\nu L}, a_{1\Psi(1)\nu U}]) \\ ([a_{2\Psi(2)\mu L}, a_{2\Psi(2)\mu U}], [a_{2\Psi(2)\eta L}, a_{2\Psi(2)\eta U}], [a_{2\Psi(2)\nu L}, a_{2\Psi(2)\nu U}]) \dots \\ ([a_{n\Psi(n)\mu L}, a_{n\Psi(n)\mu U}], [a_{n\Psi(n)\eta L}, a_{n\Psi(n)\eta U}], [a_{n\Psi(n)\nu L}, a_{n\Psi(n)\nu U}]) \end{array}$$

$$= |\tilde{A}|.$$

Proposition 4.3.2. *If $\tilde{A}$ and $\tilde{B}$ be two square IVPFM and $\tilde{A} \leq \tilde{B}$, then Adjoint($\tilde{A}$)$\leq$ Adjoint($\tilde{B}$).*

Proof. $\text{Adjoint}(\tilde{A}) = \Sigma_{\sigma \in S_{n_i} n_j} \Pi_{t \in n_j} ([a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}], [a_{t\sigma(t)\eta L}, a_{t\sigma(t)\eta U}], [a_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu U}])$
and $\text{Adjoint}(\tilde{B}) = \Sigma_{\sigma \in S_{n_i} n_j} \Pi_{t \in n_j} ([b_{t\sigma(t)\mu L}, b_{t\sigma(t)\mu U}], [b_{t\sigma(t)\eta L}, b_{t\sigma(t)\eta U}], [b_{t\sigma(t)\nu L}, b_{t\sigma(t)\nu U}])$.
Using the given hypothesis,
$a_{t\sigma(t)\mu L} \leq b_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U} \leq b_{t\sigma(t)\mu U}, a_{t\sigma(t)\eta L} \leq b_{t\sigma(t)\eta L}, a_{t\sigma(t)\eta U} \leq b_{t\sigma(t)\mu U}, a_{t\sigma(t)\nu L} \geq b_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu U} \geq b_{t\sigma(t)\nu U}, t \neq j, \sigma(t) \neq \sigma(j)$, Therefore Adjoint($\tilde{A}$)$\leq$Adjoint($\tilde{B}$). □

Proposition 4.3.3. *For a square IVPFM A, then Adjoint($\tilde{A}^T$)=(Adjoint$\tilde{A}$)^T.*

Proof. The proof follows using Definition 6.3.2 and Proposition 4.3.1. □

In the next section, first, we introduce the definition of eigen interval-valued picture fuzzy sets and develop the algorithms for identifying the greatest and least eigen interval-valued picture fuzzy sets. Then a numerical example is demonstrated to illustrate the application of the same. Algorithm for the same is provided in fig. 1 and 2.

4.4 Greatest Eigen Interval-Valued Picture Fuzzy Set and Least Eigen Interval-Valued Picture Fuzzy Set

In this section, we introduce the notion of EIVPFS and provide the necessary steps of an appropriate method for finding the GEIVPFS and LEIVPFS with the help of numerical examples.

Definition 4.4.1. *An interval-valued picture fuzzy relation (IVPFR) R between two IVPFS X and Y defined as follows*

$$R = \{\langle (x, y), \mu_R(x, y), \eta_R(x, y), \nu_R(x, y)\rangle | x \in X, y \in Y\},$$

where μ_R=[μ_R^L, μ_R^U], η_R=[η_R^L, η_R^U], ν_R=[ν_R^L, ν_R^U] such that 0$\leq \mu_R^U + \eta_R^U + \nu_R^U \leq$1 for every (x,y)$\in$(X$\times$Y).

Consider $R_1 \in (X \times Y)$ and $R_2 \in (Y \times Z)$ be two IVPFR. The following composition operators for the IVPFR R_1 and R_2 is defined by Cuong [127] as follows:

Max-Min Composition: The max-min composition operator is represented by

$$R_1 \circ R_2 = \{\langle (x_{ij}, z_{ij}), \mu_{R_1 \circ R_2}(x_{ij}, z_{ij}), \eta_{R_1 \circ R_2}(x_{ij}, z_{ij}), \nu_{R_1 \circ R_2}(x_{ij}, z_{ij})\rangle | x_{ij} \in X, z_{ij} \in Z\},$$

where $\mu_{R_1 \circ R_2}(x_{ij}, z_{ij}) = [\mu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \mu^U_{R_1 \circ R_2}(x_{ij}, z_{ij})]$, $\eta_{R_1 \circ R_2}(x_{ij}, z_{ij}) = [\eta^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \eta^U_{R_1 \circ R_2}(x_{ij}, z_{ij})]$, and, $\nu_{R_1 \circ R_2}(x_{ij}, z_{ij})$=$[\nu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \nu^U_{R_1 \circ R_2}(x_{ij}, z_{ij})]$.
Also,

$\mu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \max_{y \in Y}\{\min_{x \in X}(\mu^L_{R_1}(x_{ij}, y_{ij}), \mu^L_{R_2}(y_{ij}, z_{ij})\}$, $\mu^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \max_{y \in Y}\{\min_{x \in X}(\mu^U_{R_1}(x_{ij}, y_{ij}), \mu^U_{R_2}(y_{ij}, z_{ij})\}$,

$\eta^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \min_{y \in Y}\{\min_{x \in X}(\eta^L_{R_1}(x_{ij}, y_{ij}), \eta^L_{R_2}(y_{ij}, z_{ij})\}$, $\eta^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \min_{y \in Y}\{\min_{x \in X}(\eta^U_{R_1}(x_{ij}, y_{ij}), \eta^U_{R_2}(y_{ij}, z_{ij})\}$,

$\nu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \min_{y \in Y}\{\max_{x \in X}(\nu^L_{R_1}(x_{ij}, y_{ij}), \nu^L_{R_2}(y_{ij}, z_{ij})\}$, $\nu^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \min_{y \in Y}\{\max_{x \in X}(\nu^U_{R_1}(x_{ij}, y_{ij}), \nu^U_{R_2}(y_{ij}, z_{ij})\}$.

Definition 4.4.2. *Suppose R is an IVPFR defined on IVPFS of X. An IVPFS N is said to be an eigen interval-valued picture fuzzy set associated with the relation R if $N \odot R = N$, where $\odot$ is any of the above-defined composition operators.*

4.4.1 Greatest eigen interval-valued picture fuzzy set

Here, we apply the max-min composition operator for calculating the GEIVPFS with the IVPFR R. Suppose N_1 be the IVPFS, in which the degree of membership is the greatest of all elements of the column of relation R, the degree of neutral membership and degree of non-membership is the lowest of all the elements of the column of R.

$$\begin{aligned}
\mu_{N_1}(u) &= \max_{x \in X} \mu_R(x, u) \forall u \in Y, \\
\eta_{N_1}(u) &= \min_{x \in X} \eta_R(x, u) \forall u \in Y, \\
\nu_{N_1}(u) &= \min_{x \in X} \nu_R(x, u) \forall u \in Y.
\end{aligned} \tag{4.4.1}$$

It is easy to verify that N_1 is an eigen interval-valued picture fuzzy set, but not the GEIVPFS always. To evaluate GEIVPFS, the following sequences are evaluated using max-min composition.

$$N_1 \circ R = N_2,$$

$$N_2 \circ R = N_1 \circ R^2 = N_3,$$

$$N_3 \circ R = N_1 \circ R^3 = N_4,$$

$$\vdots$$

$$N_n \circ R = N_1 \circ R^n = N_{n+1}.$$

Now, we give an algorithm to evaluate GEIVPFS.

Algorithm 1 (GEIVPFS)

Step 1 Calculate the set N_1 from R using the above Equation 4.4.1.

Step 2 Set the index $n = 1$ and find $N_{n+1} = N_n \circ R$.

Step 3 If $N_{n+1} \neq N_n$ then go to step 2.

Step 4 If $N_{n+1} = N_n$ then N_n is the GEIVPFS associated with R.

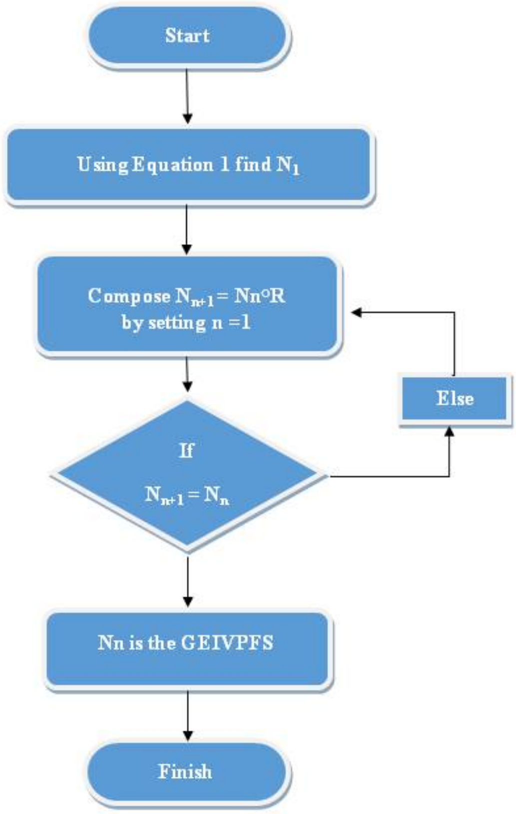

Figure 4.1: Flow Chart for Algorithm I (GEIVPFS)

Example 4.4.1. *Let $\tilde{A} = (\tilde{a}, \tilde{b}, \tilde{c})$ be the IVPFM and R be the interval-valued picture fuzzy relation on $\tilde{A}$ represented as follows.*

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} \qquad a \qquad & \qquad b \qquad & \qquad c \qquad \end{array} \\ \begin{pmatrix} [0.40,0.50][0.12,0.23][0.19,0.23] & [0.71,0.79][0.07,0.09][0.10,0.12] & [0.21,0.30][0.12,0.24][0.39,0.44] \\ [0.27,0.43][0.08,0.28][0.16,0.23] & [0.42,0.51][0.13,0.29][0.07,0.12] & [0.14,0.23][0.21,0.29][0.30,0.40] \\ [0.35,0.39][0.11,0.23][0.06,0.21] & [0.19,0.31][0.04,0.08][0.49,0.59] & [0.48,0.57][0.22,0.3][0,0.07] \end{pmatrix} \end{array}$$

We solve this as follows:

Step 1

$N_1 = ([0.40,0.50][0.08,0.23][0.06,0.21]), ([0.71,0.79][0.04,0.08][0.07,0.12]), ([0.48,0.57][0.12,0.24][0,0.07])$.

Step 2 *For* $n = 1$, $N_2 = N_1 \circ R$,

$$N_2 = ([0.40, 0.50][0.04, 0.08][0.06, 0.21]), ([0.42, 0.51][0.04, 0.08][0.07, 0.12]), ([0.48, 0.57][0.04, 0.08][0, 0.07]).$$

Step 3 *Since* $N_2 \neq N_1$, *we set* $n = 2$ *in step 2 and compose* N_2 *with R, to get* N_3, *i.e,* $N_3 = N_2 \circ R$.

$$N_3 = ([0.40, 0.50][0.04, 0.08][0.06, 0.21]), ([0.42, 0.51][0.04, 0.08][0.07, 0.12]), ([0.48, 0.57][0.04, 0.08][0, 0.07]).$$

Step 4 *Since* N_3=N_2, *thus* N_2 *is the GEIVPFS associated with R.*

4.4.2 Least eigen interval-valued picture fuzzy set

Here, we apply the max-min composition operator for calculating the LEIVPFS with the IVPF relation R. Suppose N_1 be the IVPFS, in which the degree of membership, the degree of neutral membership is the smallest of all elements of the column of relation R, and the degree of non-membership is the greatest of all the elements of the column of R.

$$\begin{aligned}
\mu_{N_1}(u) &= \min_{x\in X} \mu_R(x,u) \forall u \in Y, \\
\eta_{N_1}(u) &= \min_{x\in X} \eta_R(x,u) \forall u \in Y, \\
\nu_{N_1}(u) &= \max_{x\in X} \nu_R(x,u) \forall u \in Y.
\end{aligned} \tag{4.4.2}$$

We can easily find that N_1 is an eigen interval-valued picture fuzzy set, but our focus is to find LEIVPFS. We define the sequence of IVPFS N_n such that

$$\begin{gathered}
N_1 \circ R = N_2, \\
N_2 \circ R = N_1 \circ R^2 = N_3, \\
N_3 \circ R = N_1 \circ R^3 = N_4, \\
\vdots \\
N_n \circ R = N_1 \circ R^n = N_{n+1}.
\end{gathered}$$

For the determination of the LEIVPFS, we now present the following algorithm followed by a numerical example along with real-life application of the defined GEIVPFS and LEIVPFS.

Algorithm II(LEIVPFS)

Step 1 Calculate the set N_1 from R using above Equation 4.4.2.

Step 2 Set the index n=1 and find $N_{n+1} = N_n \circ R$.

Step 3 If $N_{n+1} \neq N_n$ then go to step 2.

Step 4 If $N_{n+1} = N_n$ then N_n is the LEIVPFS associated with R.

We consider the same Example 4.4.1 for the illustration of the computational steps of Algorithm II as below:

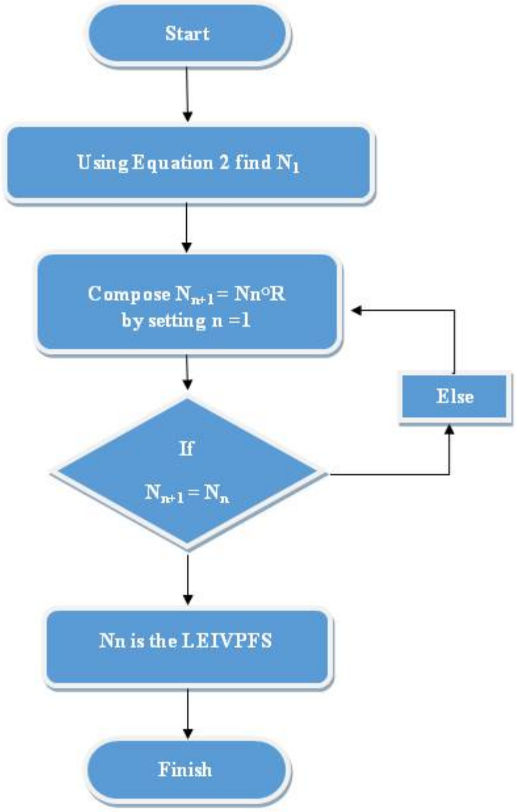

Figure 4.2: Flow Chart for Algorithm II (LEIVPFS)

Step 1

$N_1 = \{([0.27,0.39][0.08,0.23][0.19,0.23]),([0.19,0.31][0.04,0.08][0.49,0.59]),([0.14,0.23][0.12,0.24][0.39,0.44])\}$.

Step 2 For $n = 1$, $N_2 = N_1 \circ R$,

$N_2 = ([0.27,0.39][0.04,0.08][0.19,0.23]),([0.27,0.39][0.04,0.08][0.19,0.23]),([0.21,0.3][0.04,0.08][0.39,0.44])$.

Step 3 Since $N_2 \neq N_1$, we set $n = 2$ in step 2 and compose N_2 with R, to get N_3, i.e, $N_3 = N_2 \circ R$,

$N_3 = ([0.27,0.39][0.04,0.08][0.19,0.23]),([0.27,0.39][0.04,0.08][0.19,0.23]),([0.21,0.3][0.04,0.08][0.3,0.4])$.

Step 4

$N_4 = ([0.27,0.39][0.04,0.08][0.19,0.23]),([0.27,0.39][0.04,0.08][0.19,0.23]),([0.21,0.3][0.04,0.08][0.3,0.4])$.

Step 5 Since N_4=N_3, thus N_3 is the LEIVPFS associated with R.

Next we provide a real-life application of the defined algorithms to validate their applicability. For this purpose, we consider multiple criteria decision-making problems of Health insurance companies where customers' satisfaction /abstain / non-satisfaction levels are taken into account for formulating the multiple criteria decision-making problems.

4.4.3 Application of GEIVPFS and LEIVPFS in Multiple Criteria Decision-Making

Consider a health insurance company interviewing 10 of its most valuable clients or industry professionals to learn about the key aspects of the business. Let the characteristic be listed as follows:

- H_1 : Policies that Value Customers.
- H_2 : Size of the Financial Benefits.
- H_3 : Insurance Post Services.

A survey may be used to determine the customer's feedback. However, we assume a set of data presented below without conducting an exhaustive survey to illustrate the suggested methodology. To evaluate some final observations from the health insurance company's perspective, we assume that each customer's feedback is an interval-valued picture fuzzy information that is relative to all the that are available in Table 4.1, 4.2, and 4.3.

The desire levels can be estimated as satisfaction/abstain/non-satisfaction levels. This is possible by considering the interval-valued picture fuzzy relation. Each pair in the relation $R(H_j, H_k)$ has three values that range from 0 to 1; a membership degree (satisfied), a neutral membership degree (abstain), and the non-membership degree (not satisfied) is given by

$$R_{(H_j,H_k)} = \left(\frac{\sum_{p=1,q=1}^{p=m,q=n} \mu_{pq}}{m}, \frac{\sum_{p=1,q=1}^{p=m,q=n} \eta_{pq}}{m}, \frac{\sum_{p=1,q=1}^{p=m,q=n} \nu_{pq}}{m} \right); \tag{4.4.3}$$

and

$$R_{(H_i,H_i)} = \frac{R_{(H_i,H_j)} + R_{(H_i,H_k)}}{2}; \tag{4.4.4}$$

where $j, k = 1, 2, ..., n$.

From Table 4.1, 4.2 and Table 4.2, 4.3, the membership, neutral membership and non-membership degree for $R_{(H_1,H_1)}$ and $R_{(H_1,H_3)}$ can be computed respectively with the help of Equations 4.4.3, i.e.$R_{(H_1,H_1)}$ = ([0.299,0.406][0.11,0.205][0.209,0.321]) and $R_{(H_1,H_3)}$ =([0.301,0.417][0.148,0.231][0.193,0.292]). Suppose $i = 1, j = 2, k = 3$ in Equation 4.4.4, we find

$$R_{(H_1,H_1)} = \frac{R_{(H_1,H_2)} + R_{(H_1,H_3)}}{2}. \tag{4.4.5}$$

Now, by putting the values of $R_{(H_1,H_1)}$ and $R_{(H_1,H_3)}$ in Equation 4.4.5, we can compute the value of $R_{(H_1,H_2)}$, i.e.

$$R_{(H_1,H_2)} = ([0.297, .395][0.072, 0.179][0.225, 0.35])$$

Similarly, the different pairs of features have been computed as follows:
$R_{(H_2,H_1)}$ = ([0.264,0.366] [0.087.,0.189][0.273,0.343]), $R_{(H_2,H_2)}$ = ([0.27,0.377] [0.132,0.225], [0.227,0.304]),
$R_{(H_2,H_3)}$ =([0.276,0.388][0.177,0.261][0.181,0.265]), $R_{(H_3,H_1)}$ = ([0.284,0.412][0.21,0.311] [0.142,0.299]),
$R_{(H_3,H_2)}$ = ([0.222,0.344][0.176,0.287][0.2,0.287]), $R_{(H_3,H_3)}$ =([0.253,0.378][0.193,0.299][0.171,0.258]).
Next, we construct an IVPFR R using the above-obtained interdependency of the features as follows:

$$\begin{array}{cc} & \begin{array}{ccc} H_1 & H_2 & H_3 \end{array} \\ \begin{array}{c} H_1 \\ H_2 \\ H_3 \end{array} & \begin{pmatrix} R_{(H_1,H_1)} & R_{(H_1,H_2)} & R_{(H_1,H_3)} \\ R_{(H_2,H_1)} & R_{(H_2,H_2)} & R_{(H_2,H_3)} \\ R_{(H_3,H_1)} & R_{(H_3,H_2)} & R_{(H_3,H_3)} \end{pmatrix} \end{array}$$

Setting all the values, we get

$$R = \begin{array}{cc} & \begin{array}{ccc} H_1 & H_2 & H_3 \end{array} \\ \begin{array}{c} H_1 \\ H_2 \\ H_3 \end{array} & \begin{pmatrix} ([0.299,0.406][0.11,0.205][0.209,0.321]) & ([0.297,0.395][0.072,0.179][0.225,0.35]) & ([0.301,0.417][.148,0.231][0.193,0.292]) \\ ([0.264,0.366][0.087.,0.189][0.273,0.343]) & ([0.27,0.377][0.132,0.225],[0.227,0.304]) & ([0.276,0.388][0.177,0.261][0.181,0.265]) \\ ([0.284,0.412][0.21,0.311][0.142,0.299]) & ([0.222,0.344][0.176,0.287][0.2,0.287]) & ([0.253,0.378][0.193,0.299][0.171,0.258]) \end{pmatrix} \end{array}$$

Now we use Algorithm I for finding the GEIVPFS.
N_1 = ([0.299,0.412] [0.087,0.189] [0.142,0.299], [0.297,0.395][0.072, 0.179] [0.2,0.287], [0.301,0.417] [0.148,0.231] [0.171,0.258]),
$N_2 = N_1 \circ R$ =([0.299,0.412] [0.072,0.179] [0.171,0.299], [0.297,0.397][0.072, 0.179] [0.2,0.287], [0.299,0.412][0.072,0.179][0.171,0.258]),
$N_3 = N_2 \circ R$=([0.299,0.412] [0.072,0.179] [0.171,0.299], [0.297,0.395][0.072, 0.179] [0.2,0.287], [0.299,0.412] [0.072,0.179] [0.171,0.299]),
$N_4 = N_3 \circ R$=([0.299,0.412] [0.072,0.179] [0.171,0.299], [0.297,0.395][0.072, 0.179] [0.2,0.287], [0.299,0.412] [0.072,0.179] [0.171,0.299]).
Since $N_4 = N_3$ therefore, we conclude that N_3 is the GEIVPFS.

Further, we use Algorithm II for finding the LEIVPFS.
N_1 = ([0.264,0.366] [0.087,0.189] [0.273,0.343], [0.222,0.344][0.72, 0.179] [0.227,0.35], [0.253,0.378] [.148,0.231] [0.193,0.292]),
$N_2 = N_1 \circ R$ =([0.264,0.378] [0.072,0.179] [0.193,0.35], [0.264,0.366][0.72, 0.179] [0.2,0.292], [0.264,0.378] [0.072,0.179] [0.193,0.292]),
$N_3 = N_2 \circ R$ =([0.264,0.378] [0.072,0.179] [0.193,0.35], [0.264,0.378][0.72, 0.179] [0.2,0.292], [0.264,0.378] [0.072,0.179] [0.193,0.292]),
$N_4 = N_3 \circ R$= ([0.264,0.378] [0.072,0.179] [0.193,0.35], [0.264,0.378][0.72, 0.179] [0.2,0.292], [0.264,0.378] [0.072,0.179] [0.193,0.292]).
Since $N_4 = N_3$ therefore, we conclude that N_3 is the LEIVPFS.

Observations and Results:

Based on calculations, the greatest and least interval-valued picture fuzzy sets are given by ([0.299,0.412]

[0.072,0.179] [0.171,0.299], [0.297,0.395] [0.072, 0.179] [0.2,0.287], [0.299,0.412] [0.072,0.179] [0.171,0.299]) and ([0.264,0.378] [0.072,0.179] [0.193,0.35], [0.264,0.378] [0.72, 0.179] [0.2,0.292], [0.264,0.378] [0.072,0.179] [0.193,0.292]) respectively. The results from these sets show the range of levels of satisfaction/abstain/non-satisfaction for the features that a health insurance company is considering.

- Regarding feature H_1, Customers are between (26.4% to 41.2%) satisfied , abstain (7.2% to 17.9%) and between 19.3% to 29.9% unsatisfied.
- Regarding feature H_2, Customers are between (26.4% to 39.5%) satisfied , abstain (7.2% to 17.9%) and between (2% to 28.7%) unsatisfied.
- Regarding feature H_3, Customers are between (26.4% to 41.2%) satisfied, abstain (7.2% to 17.9%) and between (19.3% to 29.9%) unsatisfied.

It should be noted that the numerical results from the GEIVPFS and LEIVPFS are reasonably close to one another. The proposed algorithms have been illustrated using a specific case with a constrained format and less variety in terms of the dimensions and attributes involved. We might see a sizable fluctuation in the values if we have vast data with higher dimensionality of features. However, the similarity of the results indicates accuracy in the decision-making process.

Table 4.1: Relative feedback with H_1 and H_2

Customers/experts	H_1	H_2
E_1	([0.4,0.5], [0.12,0.23] [0.19,0.23])	([0.42,0.51],[0.13,0.29] [0.07,0.12])
E_2	([0.48,0.57], [0.22,0.3] [0,0.07])	([0.14,0.23],[0.21,0.29] [0.3,0.4])
E_3	([0.35,0.39], [0.11,0.23] [0.06,0.21])	([0.36,.48],[0.03,0.10] [0.33,0.39])
E_4	([0.27,0.43], [0.08,0.28] [0.16,0.23])	([0.22,0.32],[0.13,0.21] [0.11,0.20])
E_5	([0.71,0.79], [0.07,0.09] [0.10,0.11])	([0.15,0.27],[0.09,0.17] [0.42,0.54])
E_6	([0.21,0.30], [0.12,0.24] [0.34,0.44])	([0.15,0.27],[0.09,0.17] [0.42,0.53])
E_7	([0.26,0.35], [0.08,0.18] [0.12,0.20])	([0.35,0.47],[0.11,0.22] [0.18,0.21])
E_8	([0.09,0.19], [0.18,0.32] [0.25,0.49])	([0.7,0.75],[0.06,0.08] [0.09,0.11])
E_9	([0.13,0.25], [0.12,0.22] [0.37,0.49])	([0.2,0.29],[0.11,0.21] [0.38,0.43])
E_{10}	([0.12, 0.22], [0.08, 0.12] [0.25, 0.49])	([0.26,0.42],[0.08,0.27] [0.15,0.22])

Table 4.2: Relative feedback with H_1 and H_3

Customers/experts	H_1	H_3
E_1	([0.1,0.2], [0.1,0.3] [0.4,0.5])	([0.4,0.5], [0.2,0.3] [0.1,0.2])
E_2	([0.6,0.7], [0.1,0.12] [0.15,0.17])	([0.25,0.35],[0.1,0.2] [0.3,0.35])
E_3	([0.1,0.3], [0.1,0.2] [0.3,0.4])	([0.3,0.35], [0.1,0.15] [0.4,0.45])
E_4	([0.4,0.5], [0.1,0.2] [0.2,0.3])	([0.4,0.5],[0.1,0.2] [0.1,0.2])
E_5	([0.7,0.8], [0,0.05] [0.1,0.15])	([0.2,0.3],[.3,.4] [0.1,0.2])
E_6	([0.28,0.37], [0.10,0.21] [0.14,0.39])	([0.1,0.2],[0.2,0.3] [0.4,0.5])
E_7	([0.14,0.22], [0.2,0.28] [0.29,0.45])	([0.1,0.3], [.4,0.5] [.1,.2])
E_8	([0.13,0.33], [0.01,0.22][0.3,0.41])	([0.05,0.15],[0.3,0.4][0.2,0.3])
E_9	([0.41,0.51], [0.12,0.28] [0.07,0.2])	([0.1,0.4], [0.2,0.3] [0.1,0.2])
E_{10}	([0.1, 0.2], [0.2, 0.3] [0.4, 0.5])	([0.1, 0.3], [0.03,0.5] [0.1,0.2])

Table 4.3: Relative feedback with H_2 and H_3

Customers/experts	H_2	H_3
E_1	([0.1,0.4], [0.15,0.35][0.2,0.25])	([0.7,0.75],[0.1,.15][0,0.1])
E_2	([0.2,0.25], [0.3,0.35][0.1,0.2])	([0,0.15],[0.4,0.5][0.1,0.2])
E_3	([0.15,.25], [0.1,.25][0.1,0.2])	([0.1,0.3],[0.2,0.25][0.4,0.45])
E_4	([0.09,.11], [0.06,.08][0.7,0.75])	([0.42,.53],[.3,.4][.15,.27])
E_5	([0.29,.41], [0.2,.28][0.2,0.29])	([0.37,.49],[0.12,0.22][.13,0.25])
E_6	([0.22,.32], [0.13,.21][0.11,0.2])	([0.23,.37],[.2,.3][0.17,0.25])
E_7	([0.42,.54], [0.09,.17][0.15,0.27])	([0.35,.48],[.12,.22][.12,.25])
E_8	([0.18,0.21], [0.11,0.22][0.35,0.47])	([0.1,.2],[.3,.35][0.4,0.45])
E_9	([0.7,.75], [0.06,.08][0.09,0.11])	([0.6,.65],[.1,.2][0.05,.1])
E_{10}	([0.1,0.3], [0.4,0.5][0.1,0.2])	([0.2,.3],[.1,.15][0,.05])

In the next section, firstly, we define the new distance measure for IVPFSs and prove the required properties for the distance measure. Secondly, we use this distance measure to discuss its application in the realm of decision-making using IVPFM.

4.5 Distance Measure and its Application in Decision-Making Problem.

In this section, first we propose distance measure of IVPFS and its properties. Then later we apply this distance measure to find some practical real-life application in the field of smart manufacturing using IVPFM.

4.5.1 Distance Measure of IVPFSs and its properties

Definition 4.5.1. *Suppose*

$$A = \{[a_1(x_i), b_1(x_i)], [c_1(x_i), d_1(x_i)], [e_1(x_i), f_1(x_i)]\}$$

and

$$B = \{[a_2(x_i), b_2(x_i)], [c_2(x_i), d_2(x_i)], [e_2(x_i), f_2(x_i)]\}$$

be two IVPFSs. The distance measure between A and B is defined as follows:

$$D(A,B) = \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_1(x_i) - a_2(x_i)| + |b_1(x_i) - b_2(x_i)| + |c_1(x_i) - c_2(x_i)| + |d_1(x_i) - d_2(x_i)| + |e_1(x_i) - e_2(x_i)|$$
$$+ |f_1(x_i) - f_2(x_i)|\Big)$$

Theorem 4.5.1. *A distance measure between IVPFSs A and B is a mapping $D : A \times B \rightarrow [0,1]$, which satisfy the following properties.* $(D1)$ *$0 \leq D(A,B) \leq 1$.*

$(D2)$ *$D(A,B) = 0$ if and only if A=B.*

$(D3)$ *$D(A,B) = D(B,A)$.*

$(D4)$ *Let A, B, C $\in$ IVPFSs then $D(A,C) \leq D(A,B) + D(B,C)$.*

Proof. Proof of $D1$, $D2$ and $D3$ are trivial as follows:

$(D1)$: As the membership, neutral membership, and non-membership degrees belong to [0,1], it is obvious that the distance measure $D(A,B) \in [0,1]$.

$(D2)$: As $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$, $e_1 = e_2$, $f_1 = f_2$.
Such that $D(A,B) = 0$ if and only if A=B.

$(D3)$:

$$\begin{aligned}
D(A,B) &= \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_1(x_i) - a_2(x_i)| + |b_1(x_i) - b_2(x_i)| + |c_1(x_i) - c_2(x_i)| + |d_1(x_i) - d_2(x_i)| + |e_1(x_i) - e_2(x_i)| + \\
&\quad |f_1(x_i) - f_2(x_i)|\Big) \\
&= \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_2(x_i) - a_1(x_i)| + |b_2(x_i) - b_1(x_i)| + |c_2(x_i) - c_1(x_i)| + |d_2(x_i) - d_1(x_i)| + |e_2(x_i) - e_1(x_i)| + \\
&\quad |f_2(x_i) - f_1(x_i)|\Big) \\
&= D(B,A)
\end{aligned}$$

$(D4)$: Suppose

$$A = (\left[a_1(x_i), b_1(x_i)\right], \left[c_1(x_i), d_1(x_i)\right], \left[e_1(x_i), f_1(x_i)\right]),$$

$$B = (\left[a_2(x_i), b_2(x_i)\right], \left[c_2(x_i), d_2(x_i)\right], \left[e_2(x_i), f_2(x_i)\right])$$

$$C = (\left[a_3(x_i), b_3(x_i)\right], \left[c_3(x_i), d_3(x_i)\right], \left[e_3(x_i), f_3(x_i)\right])$$

Consider

$$\begin{aligned}
D(A,C) &= \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_1(x_i) - a_3(x_i)| + |b_1(x_i) - b_3(x_i)| + |c_1(x_i) - c_3(x_i)| + |d_1(x_i) - d_3(x_i)| + |e_1(x_i) - e_3(x_i)| \\
&\quad + |f_1(x_i) - f_3(x_i)|\Big) \\
&= \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_1(x_i) - a_2(x_i) + a_2(x_i) - a_3(x_i)| + |b_1(x_i) - b_2(x_i) + b_2(x_i) - b_3(x_i)| + |c_1(x_i) - c_2(x_i) + \\
&\quad c_2(x_i) - c_3(x_i)| + |d_1(x_i) - d_2(x_i) + d_2(x_i) - d_3(x_i)| + |e_1(x_i) - e_2(x_i) + e_2(x_i) - e_3(x_i)| + |f_1(x_i) - \\
&\quad f_2(x_i) + f_2(x_i) - f_3(x_i)|\Big) \\
&\leq \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_1(x_i) - a_2(x_i)| + |b_1(x_i) - b_2(x_i)| + |c_1(x_i) - c_2(x_i)| + |d_1(x_i) - d_2(x_i)| + |e_1(x_i) - e_2(x_i)| + \\
&\quad |f_1(x_i) - f_2(x_i)|\Big) + \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_2(x_i) - a_3(x_i)| + |b_2(x_i) - b_3(x_i)| + |c_2(x_i) - c_3(x_i)| + |d_2(x_i) - d_3(x_i)| \\
&\quad + |e_2(x_i) - e_3(x_i)| + |f_2(x_i) - f_3(x_i)|\Big)
\end{aligned}$$

□

Thus, $D(A,C) \leq D(A,B) + D(B,C)$ $\forall$ IVPFSs A, B, C.

4.5.2 Real-life application of the proposed distance measure in smart manufacturing problem

The proposed concept finds practical application in real-life scenarios, particularly in the domain of smart manufacturing problems. In this context, a complex challenge arises due to the existence of l CNC programmers distributed across λ manufacturing companies. The core issue is identifying and selecting k CNC programmers from this pool, aiming to promote and relocate them among the various companies. The selection process revolves around evaluating the CNC programmers' performance within their respective manufacturing companies and the evolving relationships between the companies and CNC hiring agencies. To facilitate the selection process, two distinct IVPFM are provided. The first matrix offers valuable insights into how CNC programmers perceive the support they receive from CNC hiring agencies in each company. Meanwhile, the second matrix delves into the intricate relationships forged between the manufacturing companies and the CNC hiring agencies, particularly during the promotion of CNC machines.

Effectively managing the vast amount of data and preferences involves assessing all IVPFSs from the two IVPFM, primarily focusing on the CNC hiring agencies. These sets' information is then harnessed to compute a distance matrix, a crucial tool in the decision-making process. The distances between each CNC programmer and the CNC hiring agencies are skillfully manipulated to construct the distance matrix. This manipulation is achieved using a specialized distance formula tailored to measure the relationship dynamics between two IVPFM, as defined in Definition 5.4.1. By adopting this comprehensive approach, the selection committee gains deeper insights into the intricate web of interactions and preferences among the CNC programmers, the manufacturing companies, and the CNC hiring agencies. Ultimately, this analysis facilitates the identification of the most deserving CNC programmers for promotion and relocation, thus optimizing the smart manufacturing process.

In this process, the objective is to determine the selected list of CNC programmers for promotion and reassignment among manufacturing companies. To achieve this, we must evaluate the minimum distance between each CNC programmer and the manufacturing companies. This evaluation uses a descending order approach, ranking the distances from the closest to the farthest. To begin the evaluation, the distance of each CNC programmer towards the manufacturing companies is calculated based on the corresponding distance formula, which could be a measure of performance or relationship strength. These distances serve as a key metric in the decision-making process.

As the evaluation continues, the CNC programmers' positions in the list correspond to their respective distances, with the closest ones placed at the top and the farthest ones towards the bottom. This ranking effectively identifies the most suitable candidates for promotion and relocation among the manufacturing companies.

Ultimately, the selected list of CNC programmers is derived from this evaluation, consisting of those with the shortest distances to the companies, ensuring that the most promising and qualified individuals are chosen for the promotion and circulation process. The descending order approach ensures that the best candidates are prioritized based on their close relationships or high-performance levels in the context of manufacturing companies and CNC hiring agencies.

Problem Description:

Imagine a scenario with three CNC stock agencies, namely SA_1, SA_2, SA_3, associated with three smart manufacturing companies, C_1, C_2, C_3. From these companies, five CNC programmers CP_1, CP_2, CP_3, CP_4, and CP_5, are selected for promotion. Let $\tilde{A}$ be an IVPFM, i.e., $\tilde{A} = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu} >)$ which shows the relationship between the CNC programmers and the CNC hiring agencies. In essence, $\tilde{a}_{ij\mu}$ represents the degree of inclination or membership of the CNC programmers towards a particular CNC hiring agency. In contrast, $\tilde{a}_{ij\eta}$ signifies this inclination's degree of neutral membership. Lastly, $\tilde{a}_{ij\nu}$ characterizes the non-membership concerning the CNC hiring agencies,

$$\begin{array}{c} \\ CP_1 \\ CP_2 \\ CP_3 \\ CP_4 \\ CP_5 \end{array} \begin{array}{c} \begin{array}{ccc} \quad SA_1 \quad & \quad SA_2 \quad & \quad SA_3 \quad \end{array} \\ \begin{pmatrix} [0.36,0.48][0.03,0.10][0.33,0.39] & [0.22,0.32][0.13,0.21][0.11,0.20] & [0.15,0.27][0.09,0.17][0.42,0.54] \\ [0.07,0.22][0.11,0.28][0.20,0.32] & [0.13,0.32][0.02,0.13][0.23,0.45] & [0.12,0.25][0.12,0.22][0.35,0.48] \\ [0.26,0.35][0.08,0.18][0.12,0.20] & [0.09,0.19][0.18,0.32][0.22,0.35] & [0.17,0.36][0.04,0.15][0.23,0.37] \\ [0.09,0.42][0.10,0.20][0.14,0.35] & [0.12,0.22][0.08,0.12][0.25,0.49] & [0.13,0.25][0.12,0.22][0.37,0.49] \\ [0.15,0.47][0.14,0.31][0.01,0.10] & [0.13,0.35][0.01,0.23][0.32,0.41] & [0.15,0.27][0.09,0.17][0.42,0.53] \end{pmatrix} \end{array}$$

and $\tilde{B} = (< \tilde{b}_{ij\mu}, \tilde{b}_{ij\eta}, \tilde{b}_{ij\nu} >)$ which represents the relationship with manufacturing companies and CNC hiring agencies during the promotion of CNC.

$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad SA_1 \quad & \quad SA_2 \quad & \quad SA_3 \quad \end{array} \\ \begin{pmatrix} [0.35,0.47][0.11,0.22][0.18,0.21] & [0.7,0.75][0.06,0.08][0.09,0.11] & [0.2,0.29][0.11,0.21][0.38,0.43] \\ [0.26,0.42][0.08,0.27][0.15,0.22] & [0.41,0.51][0.12,0.28][0.07,0.20] & [0.14,0.22][0.2,0.28][0.29,0.41] \\ [0.16,0.47][0.14,0.29][0.01,0.11] & [0.13,0.33][0.01,0.22][0.3,0.41] & [0.28,0.37][0.10,0.21][0.14,0.39] \end{pmatrix} \end{array}$$

From the given matrix $\tilde{A}$, the knowledge about the CNC programmer concerning the CNC hiring agencies (SA_1, SA_2, SA_3) and matrix $\tilde{B}$, the knowledge about the manufacturing companies with respect to the set of CNC hiring agencies (SA_1, SA_2, SA_3). According to the fact, eight IVPFSs are taken out over the set

(SA_1, SA_2, SA_3).

$$CP_1 = [(SA_1, <[0.36, 0.48][0.03, 0.10][0.33, 0.39]>), (SA_2, <[0.22, 0.32][0.13, 0.21][0.11, 0.20]>), (SA_3, <[0.15, 0.27][0.09, 0.17][0.42, 0.54]>)],$$
$$CP_2 = [(SA_1, <[0.07, 0.22][0.11, 0.28][0.20, 0.32]>), (SA_2, <[0.13, 0.32][0.02, 0.13][0.23, 0.45]>), (SA_3, <[0.12, 0.25][0.12, 0.22][0.35, 0.48])],$$
$$CP_3 = [(SA_1, <[0.26, 0.35][0.08, 0.18][0.12, 0.20]>), (SA_2, <[0.09, 0.19][0.18, 0.32][0.22, 0.35]>), (SA_3, <[0.14, 0.22][0.2, 0.28][0.29, 0.41]>)],$$
$$CP_4 = [(SA_1, <[0.09, 0.42][0.10, 0.20][0.14, 0.35]>), (SA_2, <[0.12, 0.22][0.08, 0.12][0.25, 0.49]>), (SA_3, <[0.13, 0.25][0.12, 0.22][0.37, 0.49]>)],$$
$$CP_5 = [(SA_1, <[0.15, 0.47][0.14, 0.31][0.01, 0.10]>), (SA_2, <[0.13, 0.35][0.01, 0.23][0.32, 0.41]>), (SA_3, <[0.28, 0.37][0.10, 0.21][0.14, 0.39]>)],$$
$$C_1 = [(SA_1, <[0.35, 0.47][0.11, 0.22][0.18, 0.21]>), (SA_2, <[0.7, 0.75][0.06, 0.08][0.09, 0.11]>), (SA_3, <[0.2, 0.29][0.11, 0.21][0.38, 0.43]>)],$$
$$C_2 = [(SA_1, <[0.26, 0.42][0.08, 0.27][0.15, 0.22]>), (SA_2, <[0.41, 0.51][0.12, 0.28][0.07, 0.20]>, (SA_3, <[0.14, 0.22][0.2, 0.28][0.29, 0.41]>)],$$
$$C_3 = [(SA_1, <[0.16, 0.47][0.14, 0.29][0.01, 0.11]>), (SA_2, <[0.13, 0.33][0.01, 0.22][0.3, 0.41]>), (SA_3, <[0.28, 0.37][0.10, 0.21][0.14, 0.39]>)].$$

Now get the distance matrix $\delta = (\delta_{ij})$ by using distance measure between two IVPFM $\tilde{A}$ and $\tilde{B}$, here δ_{ij} is the distance between CP_i and C_j where $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3$ given below.

$$\begin{array}{c} \\ CP_1 \\ CP_2 \\ CP_3 \\ CP_4 \\ CP_5 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} 0.1708 & 0.1475 & 0.2042 \\ 0.2092 & 0.1692 & 0.1342 \\ 0.2242 & 0.1508 & 0.1275 \\ 0.2025 & 0.1658 & 0.1392 \\ 0.2142 & 0.1758 & 0.0658 \end{pmatrix} \end{array}$$

Observations and results discussed using distance matrix (δ):

(a) In the first instance of the manufacturing company C_1, the degree of closeness (DOC) between the CNC programmer (CP_1) and the company C_1 is maximum because $DOC(CP_1, C_1) > DOC(CP_4, C_1) > DOC(CP_2, C_1) > DOC(CP_5, C_1) > DOC(CP_3, C_1)$.

(b) In the second instance of the manufacturing company C_2, the degree of closeness (DOC) between the CNC programmer (CP_4) and the company C_2 is maximum because $DOC(CP_1, C_2) > DOC(CP_3, C_2) > DOC(CP_4, C_2) > DOC(CP_2, C_2) > DOC(CP_5, C_2)$.

(c) In the third instance of the manufacturing company C_3, the degree of closeness (DOC) between the

CNC programmer (CP_4) and the manufacturing company C_3 is maximum because $DOC(CP_5,C_3) > DOC(CP_3,C_3) > DOC(CP_2,C_3) > DOC(CP_4,C_3) > DOC(CP_1,C_3)$.

where '$>$' represents the closeness degree. The lesser the value, the higher the closeness. According to the mathematical calculations, we find the final selected list of CNC programmers for a different manufacturing company as follows:

Manufacturing Company	Selected CNC Programmer
C_1	CP_1, CP_4
C_2	CP_1, CP_3
C_3	CP_5, CP_3

The CNC programmer CP_1 and CP_3 are selected (appropriate) for all smart manufacturing companies. Figure 4.3 depicts a flowchart of the procedure to be followed for smart manufacturing companies selection.

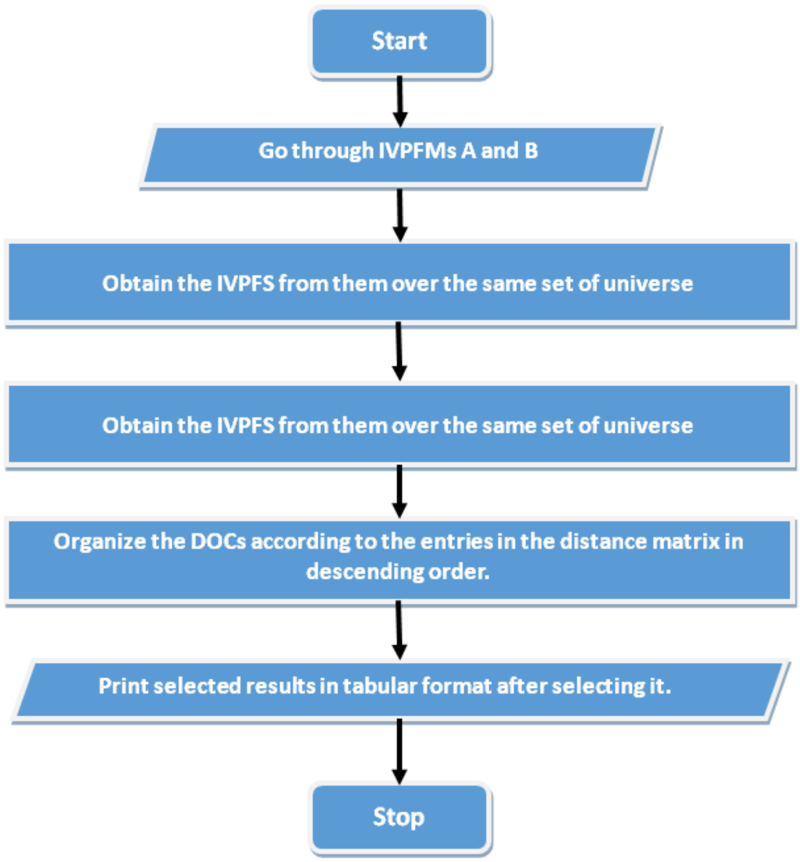

Figure 4.3: Flowchart of the smart manufacturing companies selection procedure

In the next section, we compare the present work with the existing work on FM and its extension. Also, we discuss the advantages of the proposed work in detail in Table 6.8.

4.6 Comparative Study and Results

In the previous studies concerning picture-fuzzy decision-making, information was considered in a picture-fuzzy manner. However, when we face various forms of uncertainty in the information, conventional methods are inadequate for handling such situations. In these scenarios, gathering or representing the information in an interval-valued picture fuzzy sense becomes necessary. In such cases, the currently developed process becomes crucial in arriving at a meaningful and productive conclusion. Dogra & Pal [28] introduced a model for determining a selected list of administrative officers for various governments using the distance formula between two PFM. Their approach considered membership, neutral membership, and non-membership degrees within the PFM framework.

In contrast, our current study extends this work by considering membership, neutral membership, and non-membership degrees as interval numbers, which proves to be more practical for real-life problems in smart manufacturing problems. Additionally, Ejegwa et al. [120] proposed a model for determining students' career paths using the distance formula between two intuitionistic fuzzy sets. In this model, intuitionistic fuzzy sets considered only membership and non-membership aspects. Moreover, Khalaf [117] addressed medical diagnosis problems using IVIFS with max-min-max composition. They formulated these problems as uncertain decision matrices and provided decisions based on fuzzy scores calculated for each attribute. However, our current method takes a different approach with matrices that contain interval-valued picture fuzzy values. We extract interval-valued picture fuzzy sets from these matrices over a defined universe. Using the newly developed distance formula between two IVPFSs, we obtain a distance matrix that leads to a decision. Implementing this method is remarkably straightforward, as it does not require various complex calculations, thereby avoiding any complicacy in its application. Consequently, developing an algorithm and computer programming for this method becomes easy. Furthermore, the data points considered in this method have a remarkable capability to handle a wider range of vagueness in information. Considering that the interval-valued picture fuzzy concept generalizes the picture fuzzy concept, this study can be viewed as a generalization of advanced fuzzy logic.

In summary, the study of IVPFM and EIVPFS yields significant advantages in addressing real-world challenges, particularly in the context of smart manufacturing and health insurance companies. These concepts provide valuable tools and insights for various applications, making them essential components in contemporary research and practical problem-solving scenarios. The following is a detailed list of some substantial advantages of using IVPFM and EIVPFS:

1. From Table 6.8, the existing FM, IVFM, IFM, IVIFM, and PFM each have shortcomings that prevent them from fully capturing the information. The IVPFM effectively fills the gaps left by other matrices and offers a more flexible and versatile approach to expressing opinions and relationships within the data. The IVPFM combines the benefits of both interval-valued and picture-fuzzy concepts, making it a powerful tool for dealing with uncertainties and complexities. The IVPFM offers a more robust framework to handle various real-life scenarios and decision-making processes by representing membership, neutral membership, and non-membership degrees as intervals.

2. We can also see the drawback in the eigen fuzzy sets and eigen intuitionistic fuzzy sets experts/decision-makers bind their input in a certain area. However, the proposed EIVPFS presents a significant impact

due to its ability to offer a generalization feature. This unique characteristic allows for a more comprehensive and versatile representation of uncertain information, empowering decision-makers to make more informed and flexible judgments in various contexts.

3. The implementation of the EIVPFS, IVPFM, and the approach suggested for the problems of health insurance and smart manufacturing in Section 4.4 and Section 4.5 demonstrate how well and consistently the proposed work addresses the extended framework. The observations indicate that the IVPFM is the most generalized structure among all fuzzy matrix models.

The detailed analysis presented in Table 4.4 further compares the proposed work and existing research available in the literature. ⋆ MD-Membership degree, NMD-Non membership degree, IVMD-Interval-valued

Table 4.4: Comparison of the proposed work with existing literature

Characteristics Methods	Whether consider MD	Whether consider MD more flexible, i.e., IVMD	Whether consider MD or NMD	Whether consider MD or NMD more flexible, i.e., IVMD or IVNMD	Whether consider MD, neutral membership and NMD degree	Whether consider MD, neutrality and NMD degree more flexible, i.e., IVMD, interval-valued neutrality degree and IVNMD.
Thomason [16]	✓	×	×	×	×	×
Pal [24]	✓	✓	×	×	×	×
Pal et al. [131]	✓	✓	✓	×	×	×
Silambarasan & Sriram [30]	✓	✓	✓	×	×	×
Silambarasan [144]	✓	✓	✓	×	×	×
Khan & Pal [32]	✓	✓	✓	✓	×	×
Dogra & Pal [28]	✓	✓	✓	✓	✓	×
Proposed work	✓	✓	✓	✓	✓	✓

membership degree, IVNMD-Interval valued non membership degree.

4.7 Concluding Remarks

The exploration of matrix theory has made significant contributions to various applicable fields. In this work, we have introduced the concept of IVPFM along with its essential definitions and theorems. Additionally, we have defined the determinant and adjoint of IVPFM and studied relevant results. The formal definition of an EIVPFS for interval-valued picture fuzzy relations has been presented, and algorithms for determining the GEIVPFS and LEIVPFS using max-min and min-max composition operators have been provided. To illustrate these algorithms, numerical examples have been included. The application of GEIVPFS and LEIVPFS in decision-making problems has been successfully demonstrated. Moreover, we have demonstrated the application of IVPFM in decision-making by introducing a distance formula to solve such problems effectively. The limitation of IVPFM is related to the representation of degrees of membership, neutral membership, and non-membership as interval numbers. The limitation arises when the sum of the upper degree of membership, neutral membership, and upper degree of non-membership exceeds the interval [0, 1]. The current study will help researchers interested in further developing and generalizing our findings in the context of other types of data sets. Also, we can extend in the field of image information retrieval, genetic algorithm for image reconstruction, and outlines to introduce the notion of interval-valued eigen picture fuzzy soft sets/soft matrices have been briefly stated for further research.

Chapter 5

Interval-Valued Spherical Fuzzy Matrix and its Applications in Multi-Attribute Decision-Making Process

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This chapter [1] is based on the interval-valued spherical fuzzy matrix and its applications in multi-attribute decision-making processes. The objective of this work is to handle the uncertainty in practical applications of matrix. In real-life problems, things are imprecise because of imprecision/inaccuracy, and the exact value of the measured quantities is impossible to get. Sometimes, due to time pressure/ incomplete knowledge, it is difficult for the decision-makers to provide their opinion. To describe the imprecision, the information in terms of the fuzzy is provided to allow the decision-makers to express their inputs freely. In the literature, some important matrices are available to tackle uncertain problems. There is a valuable role of the fuzzy matrix (FM), intuitionistic fuzzy matrix (IFM), picture fuzzy matrix (PFM), and spherical fuzzy matrix to describe uncertainty under uncertain situations. To handle the uncertainty in practical applications of matrices, we have defined, the interval-valued spherical fuzzy matrix and its applications in multi-attribute decision-making processes. The IVPFM offers a more robust framework to handle various real-life scenarios and decision-making processes by representing membership, neutral membership, and non-membership degrees as intervals. In this chapter, define several key definitions and theorems for the IVSFM and present a procedure for calculating its determinant and adjoint. The work demonstrates this process with a numerical example of a decision-making problem. For this, propose a new score function for the interval-valued spherical fuzzy sets and prove its validity with the help of basic properties. Further, the application of the proposed concepts is shown by real-life decision-making for a career placement assessment.

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[1] **The content of this chapter is based on the research papers "Interval-valued spherical fuzzy matrix and its applications in multi-attribute decision-making process.",** ***Maejo International Journal of Science and Technology*****, 17(3), 2023. (SCIE, Impact Factor : 0.8)**

5.1 Introduction

Decision-making is the cognitive process of selecting a choice or action among multiple alternatives, a fundamental aspect of human life essential in various contexts, from personal to professional. It involves assessing information, considering consequences, and aligning choices with goals and values. Decisions can range from simple daily choices to complex strategic plans. Effective decision-making necessitates critical thinking, problem-solving skills, and emotional intelligence. It plays a crucial role in shaping our lives, determining success, and mitigating risks. Understanding the decision-making process helps individuals and organizations make well-informed, rational choices that lead to desired outcomes and progress. The application of matrices in decision-making problems plays a pivotal role in various fields. Matrices provide a structured framework that aids in evaluating and comparing different alternatives. They enable decision-makers to quantify and analyze multiple factors or criteria simultaneously, facilitating a systematic and comprehensive approach. Matrices allow for the organization of information and the identification of relationships between variables, providing a visual representation that enhances understanding and aids in making informed decisions [63, 65, 137]. By assigning weights and scores to various elements, matrices help prioritize options and determine the most favorable course of action. Overall, matrices are used in a variety of fields of science and technology to represent data in a meaningful way. However, various sorts of uncertain data are involved in decision-making, making it challenging to solve the issue in a traditional matrix. These problems can stem from data unpredictability, inadequate information, and other factors. To deal with the situation of vague data, fuzzy matrix (FM) plays a fundamental role in dealing with such a situation. Zadeh [2] developed the fuzzy set to deal with uncertainty in practical situations. The FM is defined by Thomason [16] after the fuzzy set is introduced. Kim & Roush [20] worked on the generalization of the FM over boolean algebra. Pal [23] defined the FM with fuzzy rows and columns and presented some properties with the binary operator. Ragab & Eman [21] gave some results on the max-min composition and worked on the construction of an idempotent FM. Ragab & Emam [129] solved the determinant and adjoint of a square FM and discussed some properties defined on it. Pal [24] extended the FM to an interval-valued fuzzy matrix (IVFM) with an interval-valued fuzzy row and column. Meenakshi & Kaliraja [130] used the IVFM for solving medical diagnosis problems. Mandal & Pal [133] described some methods of finding the ranks of IVFM. A number of researchers have worked on FM but they considered only the membership of the element. Atanassov [7] introduced the concept of intuitionistic fuzzy sets with this kind of situation in mind. Khan et al.[131] defined the concept of an intuitionistic fuzzy matrix (IFM). Padder and Murugadas [35] worked on max-min operations on an IFM and discussed the convergence of transitive IFM. Pal and Khan [132] proposed some operations on the IFM. Moreover, Khan & Pal [32] have presented the concept of an interval-valued intuitionistic fuzzy matrix (IVIFM). Silambarasan [135] defined the Hamacher operations of IVIFM and proved some important properties. In the above studies the concept of FM and IFM has been strongly enforced in various areas, yet the concept of neutrality is not considered in FM and IFM. The FM and IFM fail to attain any satisfactory result when the neutral membership degree is calculated independently in real-life problems. After that, Dogra & Pal [28] proposed the picture fuzzy matrix (PFM) using the concept of Cuong & Kreinovich [127] and introduced the method of determinant and adjoint of a PFM. Silambarasan [38] also defined some algebraic operations and properties of the PFM. Khalil et al. [39] established some operation laws for IVPFSs and comprehensively analyzed their properties. Liu et al. [128] introduced the similarity measures for IVPFS and studied their applications in decision-making problems. Further, Silambarasan [139] defined a spherical fuzzy matrix (SFM) using the theory of Gundogdu & Kahraman [42] and proposed some important properties and algebraic operations for the SFM. Muthukumaran et al. [140] defined the n-hyperspherical neutrosophic matrices and compared them with the SFM. Gundogdu and Kahraman [116] extended the spherical fuzzy set into the interval-valued spherical fuzzy set. Menekse and Akdag [141] worked on Seismic risk analysis of hospital buildings: A novel interval-valued spherical fuzzy ARAS. Otay [142] worked on tech-centre location selection by interval-valued spherical fuzzy AHP-based MULTIMOORA methodology. The present work aims to present the notion of interval-valued spherical fuzzy matrix (IVSFM) and its important features.

5.1.1 Reasearch Gap

It has been noted that membership, neutrality, and non-membership degrees for SFM, are frequently precise, despite the fact that this is not the case in reality. As a result, we now face yet another type of uncertainty. Additionally, we've seen that impartiality plays a crucial role in decision-making IVSFMs are significant when there are multiple possible responses, such as no, yes, refusal, and abstain in interval number. Some authors have done excellent work on the concept of FM, IFM, PFMs, and SFM applications in real-life problems, but the idea of IVSFM has not been touched upon.

5.1.2 Organization of the Chapter

The remaining part of the chapter is organized as follows. In Section 5.2, we define the idea of IVSFM with basic definitions, properties, and important theorems. The concepts of determinant, adjoint, and propositions are given in Section 5.3. Section 5.4 introduces a new score function for the IVSFSs and discusses its application in decision-making. The comparative study with existing work is conducted in Section 5.5. Finally, Section 5.6 concludes the paper with some future directions.
In the next section, we present the innovative notion of an IVSFM, which serves as an extension of SFM. Additionally, we establish the definitions of fundamental arithmetic operations and demonstrate the proof of essential theorems pertaining to the IVSFM.

5.2 Interval-Valued Spherical Fuzzy Matrix

In this section, we define the IVSFM and basic concepts by generalizing the concept of SFM.

Definition 5.2.1. *An interval valued spherical fuzzy matrix (IVSFM) $\tilde{A}$ is defined as*

$$\tilde{A} = (\tilde{a}_{ij}) = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu} >),\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n$$

where,

$$\tilde{a}_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \subseteq [0,1],$$

$$\tilde{a}_{ij\eta} = [a_{ij\eta L}, a_{ij\eta U}] \subseteq [0,1],$$

$$\tilde{a}_{ij\nu} = [a_{ij\nu L}, a_{ij\nu U}] \subseteq [0,1],$$

with the condition

$$(a_{ij\mu U})^2 + (a_{ij\eta U})^2 + (a_{ij\nu U})^2 \leq 1,$$

$a_{ij\mu}, a_{ij\eta}$ *and* $a_{ij\nu}$ *are the membership, neutral membership and non-membership degree of* $\tilde{a}_{ij}$.

Definition 5.2.2. *Let* $\tilde{A}$ *and* $\tilde{B}$ *be two IVSFM such that*
$\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])$ *and* $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ij\nu L}, b_{ij\nu U}])$.
Then, we write $\tilde{A} \leq \tilde{B}$ *is following case is true:*
$a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}; a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}; a_{ij\nu L} \geq b_{ij\nu L}, a_{ij\nu U} \geq b_{ij\nu U}$.

Definition 5.2.3. *Let* $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])$ *and* $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ij\nu L}, b_{ij\nu U}])$ *be two IVSFM of same order* $m \times n$ *then, we define some basic operations.*

(i) $\tilde{A}^c = (\left[a_{ij\nu L}, a_{ij\nu U}\right] \left[a_{ij\eta L}, a_{ij\eta U}\right], \left[a_{ij\mu L}, a_{ij\mu U}\right])$.

(ii) $\tilde{A} \vee \tilde{B} = (\left[max\left(a_{ij\mu L}, b_{ij\mu L}\right), max\left(a_{ij\mu U}, b_{ij\mu U}\right)\right] \left[min\left(a_{ij\eta L}, b_{ij\eta L}\right), min\left(a_{ij\eta U}, b_{ij\eta U}\right)\right]$
$\left[min\left(a_{ij\nu L}, b_{ij\nu L}\right), min\left(a_{ij\nu U}, b_{ij\nu U}\right)\right])$.

(iii) $\tilde{A} \wedge \tilde{B} = (\left[min\left(a_{ij\mu L}, b_{ij\mu L}\right), min\left(a_{ij\mu U}, b_{ij\mu U}\right)\right] \left[min\left(a_{ij\eta L}, b_{ij\eta L}\right), min\left(a_{ij\eta u}, b_{ij\eta U}\right)\right]$
$\left[max\left(a_{ij\nu L}, b_{ij\nu L}\right), max\left(a_{ij\nu U}, b_{ij\nu U}\right)\right])$.

(iv) $\tilde{A}^T = (\left[a_{ji\mu L}, a_{ji\mu U}\right] \left[a_{ji\eta L}, a_{ji\eta U}\right] \left[a_{ji\nu L}, a_{ji\nu U}\right])$.

(v) $\tilde{A} \oplus \tilde{B} = (\left[a_{ij\mu L} + b_{ij\mu L} - a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U} + b_{ij\mu U} - a_{ij\mu U}.b_{ij\mu U}\right] \left[a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U}.b_{ij\eta U}\right]$
$\left[a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U}.b_{ij\nu U}\right])$.

(vi) $\tilde{A} \otimes \tilde{B} = (\left[a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U}.b_{ij\mu U}\right], \left[a_{ij\eta L} + b_{ij\eta L} - a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U} + b_{ij\eta U} - a_{ij\eta U}.b_{ij\eta U}\right],$
$\left[a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U}.b_{ij\nu U}\right])$.

(vii)

$$k \cdot A = \left\{ \begin{array}{c} \left[\left(1-\left(1-\left(a_{ij\mu L}\right)^2\right)^k\right)^{1/2}, \left(1-\left(1-\left(a_{ij\mu U}\right)^2\right)^k\right)^{1/2}\right], \left[\left(a_{ij\nu L}\right)^k, \left(a_{ij\nu U}\right)^k\right], \\ \left[\left(\left(1-\left(a_{ij\mu L}\right)^2\right)^k - \left(1-\left(a_{ij\mu L}\right)^2 - \left(a_{ij\eta L}\right)^2\right)^k\right)^{1/2}, \right. \\ \left. \left(\left(1-\left(a_{ij\mu U}\right)^2\right)^k - \left(1-\left(a_{ij\mu U}\right)^2 - \left(a_{ij\eta U}\right)^2\right)^k\right)^{1/2}\right] \end{array} \right\}$$

(viii)

$$A^k = \left\{ \begin{array}{c} \left[\left(a_{ij\mu L}\right)^k, \left(a_{ij\mu U}\right)^k\right], \left[\left(1-\left(1-\left(a_{ij\nu L}\right)^2\right)^k\right)^{1/2}, \left(1-\left(1-\left(a_{ij\nu U}\right)^2\right)^k\right)^{1/2}\right], \\ \left[\left(\left(1-\left(a_{ij\nu L}\right)^2\right)^k - \left(1-\left(a_{ij\nu L}\right)^2 - \left(a_{ij\eta L}\right)^2\right)^k\right)^{1/2}, \right. \\ \left. \left(\left(1-\left(a_{ij\nu U}\right)^2\right)^k - \left(1-\left(a_{ij\nu U}\right)^2 - \left(a_{ij\eta U}\right)^2\right)^k\right)^{1/2}\right] \end{array} \right\},$$

where $\tilde{A}^c$ *and* $\tilde{A}^T$ *are complement and transpose of* $\tilde{A}$ *respectively.*

Theorem 5.2.1. *Let* $\tilde{A}$, $\tilde{B}$ *be two IVSFM of order* $m \times n$ *then*

(a) $\left(\tilde{A} \vee \tilde{B}\right)^c = \tilde{A}^c \wedge \tilde{B}^c$.

(b) $\left(\tilde{A} \wedge \tilde{B}\right)^c = \tilde{A}^c \vee \tilde{B}^c$.

Proof. (*a*) Let $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$ and $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$
Then, $\tilde{A}^c = ([a_{ijvL}, a_{ijvU}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\mu L}, a_{ij\mu U}])$ and $\tilde{B}^c = ([b_{ijvL}, b_{ijvU}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ij\mu L}, b_{ij\mu U}])$

$$\begin{aligned}
\tilde{A}^c \wedge \tilde{B}^c &= ([\min(a_{ijvL}, b_{ijvL}), \min(a_{ijvU}, b_{ijvU})]\, [\min(a_{ij\eta L}, b_{ij\eta L}), \min(a_{ij\eta U}, b_{ij\eta U})] \\
&\quad [\max(a_{ij\mu L}, b_{ij\mu L}), \max(a_{ij\mu U}, b_{ij\mu U})]) \\
\tilde{A} \vee \tilde{B} &= ([\max(a_{ijvL}, b_{ijvL}), \max(a_{ijvU}, b_{ijvU})]\, [\min(a_{ij\eta L}, b_{ij\eta L}), \min(a_{ij\eta U}, b_{ij\eta U})] \\
&\quad [\min(a_{ij\mu L}, b_{ij\mu L}), \min(a_{ij\mu U}, b_{ij\mu U})]) \\
\left(\tilde{A} \vee \tilde{B}\right)^c &= ([\min(a_{ijvL}, b_{ijvL}), \min(a_{ijvU}, b_{ijvU})]\, [\min(a_{ij\eta L}, b_{ij\eta L}), \min(a_{ij\eta U}, b_{ij\eta U})] \\
&\quad [\max(a_{ij\mu L}, b_{ij\mu L}), \max(a_{ij\mu U}, b_{ij\mu U})]) \\
&= \tilde{A}^c \wedge \tilde{B}^c
\end{aligned}$$

(*b*) Proof of part (*b*) can be done on similar lines.

□

Theorem 5.2.2. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVSFM of same order $m \times n$ and $\tilde{A} \leq \tilde{C}$ and $\tilde{B} \leq \tilde{C}$, then $\tilde{A} \vee \tilde{B} \leq \tilde{C}$.*

Proof. Let $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$, $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$ and $\tilde{C} = ([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ijvL}, c_{ijvU}])$.
If $\tilde{A} \leq \tilde{C}$ then $a_{ij\mu L} \leq c_{ij\mu L}, a_{ij\mu U} \leq c_{ij\mu U}, a_{ij\eta L} \leq c_{ij\eta L}, a_{ij\eta U} \leq c_{ij\eta U}, a_{ijvL} \geq c_{ijvL}, a_{ijvU} \geq c_{ijvU}$ for all i, j,
and $\tilde{B} \leq \tilde{C}$ then $b_{ij\mu L} \leq c_{ij\mu L}, b_{ij\mu U} \leq c_{ij\mu U}, b_{ij\eta L} \leq c_{ij\eta L}, b_{ij\eta U} \leq c_{ij\eta U}, b_{ijvL} \geq c_{ijvL}, b_{ijvU} \geq c_{ijvU}$ for all i, j.
Now, max $(a_{ij\mu L}, b_{ij\mu L}) \leq c_{ij\mu L}$, max $(a_{ij\mu U}, b_{ij\mu U}) \leq c_{ij\mu U}$,
min $(a_{ij\eta L}, b_{ij\eta L}) \leq c_{ij\eta L}$, min $(a_{ij\eta U}, b_{ij\eta U}) \leq c_{ij\eta U}$,
min $(a_{ijvL}, b_{ijvL}) \geq c_{ijvL}$, min $(a_{ijvU}, b_{ijvU}) \geq c_{ijvU}$.
Thus $\tilde{A} \vee \tilde{B} \leq \tilde{C}$ using Definition 5.2.2. □

Theorem 5.2.3. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVSFM of same order $m \times n$ of $\tilde{A} \leq \tilde{B}$ then $\tilde{A} \vee \tilde{C} \leq \tilde{B} \vee \tilde{C}$.*

Proof. Let $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$, $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$ and $\tilde{C} = ([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ijvL}, c_{ijvU}])$ be three IVSFM of same order $m \times n$.
If $\tilde{A} \leq \tilde{B}$ then $a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}, a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}, a_{ijvL} \geq b_{ijvL}, a_{ijvU} \geq b_{ijvU}$.
Now, max $(a_{ij\mu L}, c_{ij\mu L}) \leq$ max $(b_{ij\mu L}, c_{ij\mu L})$, max $(a_{ij\mu U}, c_{ij\mu U}) \leq$ max $(b_{ij\mu U}, c_{ij\mu U})$,
min $(a_{ij\eta L}, c_{ij\eta L}) \leq$ min $(b_{ij\eta L}, c_{ij\eta L})$, min $(a_{ij\eta U}, c_{ij\eta U}) \leq$ min $(b_{ij\eta U}, c_{ij\eta U})$,
min $(a_{ijvL}, c_{ijvL}) \geq$ min (b_{ijvL}, c_{ijvL}) , min $(a_{ijvU}, c_{ijvU}) \geq$ min (b_{ijvU}, c_{ijvU}) for all i,j.
Therefore $\tilde{A} \vee \tilde{C} \leq \tilde{B} \vee \tilde{C}$. □

Theorem 5.2.4. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVSFM of same order $m \times n$ and $\tilde{C} \leq \tilde{A}$ and $\tilde{C} \leq \tilde{B}$ then $\tilde{C} \leq \tilde{A} \wedge \tilde{B}$.*

Proof. Proof of the above result directly follows from Theorem 5.2.3. □

Theorem 5.2.5. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVSFM of same order $m \times n$ and if $\tilde{A} \leq \tilde{B}$, $\tilde{A} \leq \tilde{C}$ and $\tilde{B} \wedge \tilde{C} = 0$, then $\tilde{A} = 0$.*

Proof. If $\tilde{A} \leq \tilde{B}$ then $a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}; a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}; a_{ijvL} \geq b_{ijvL}, a_{ijvU} \geq b_{ijvU}$. Similarly $\tilde{A} \leq \tilde{C}$ then $a_{ij\mu L} \leq c_{ij\mu L}, a_{ij\mu U} \leq c_{ij\mu U}; a_{ij\eta L} \leq c_{ij\eta L}, a_{ij\eta U} \leq c_{ij\eta U}; a_{ijvL} \geq c_{ijvL}, a_{ijvU} \geq c_{ijvU}$. Thus by Theorem 5.2.4 $\tilde{A} \leq \tilde{B} \wedge \tilde{C}, \tilde{B} \wedge \tilde{C} = 0$ such that $\tilde{A} = 0$. □

Theorem 5.2.6. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVSFM of order $m \times n$ and if $\tilde{A} \leq \tilde{B}$ then $\tilde{A} \wedge \tilde{C} \leq \tilde{B} \wedge \tilde{C}$.*

Proof. Proof of the above result directly follows from Definition 5.2.2. □

Theorem 5.2.7. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVSFM of order $m \times n$ and if $\tilde{A} \leq \tilde{B}$, and $\tilde{B} \wedge \tilde{C} = 0$, then $\tilde{A} \wedge \tilde{C} = 0$.*

Proof. By Theorem 5.2.6, the proof is straight forward. □

5.3 Determinant and Adjoint of IVSFM

In this section, we define the determinant, and adjoint of the IVSFM.

Definition 5.3.1. *Determinant of IVSFM Suppose $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])$ be the IVSFM of order m. Then, the determinant of $\tilde{A}$ is denoted by $|\tilde{A}|$ and defined by*

$$|\tilde{A}| = \begin{pmatrix} \vee_{h\in\psi_k}([a_{1h(1)\mu L}, a_{1h(1)\mu U}] \wedge [a_{2h(2)\mu L}, a_{2h(2)\mu U}] \cdots \wedge [a_{kh(k)\mu L}, a_{kh(k)\mu U}]), \\ \wedge_{h\in\psi_k}([a_{1h(1)\eta L}, a_{1h(1)\eta U}] \wedge [a_{2h(2)\eta L}, a_{2h(2)\eta U}] \cdots \wedge [a_{kh(k)\eta L}, a_{kh(k)\eta U}]), \\ \wedge_{h\in\psi_k}([a_{1h(1)\nu L}, a_{1h(1)\nu U}] \vee [a_{2h(2)\nu L}, a_{2h(2)\nu U}] \cdots \vee [a_{kh(k)\nu L}, a_{kh(k)\nu U}]). \end{pmatrix}$$

where ψ_k be the set of permutation on the set $\{1, 2, 3, \ldots, m\}$.

Example 5.3.1. *Let us consider IVPFM of order 3 as follows*

$$\tilde{A} = \begin{pmatrix} [0.85,0.95][0.10,0.15][0.05,0.15] & [0.55,0.65][0.25,0.30][0.25,0.30] & [0.13,0.19][0.69,0.79][0.22,0.27] \\ [0.75,0.85][0.15,0.20][0.15,0.20] & [0.47,0.61][0.33,0.41][0.27,0.36] & [0.31,0.42][0.43,0.52][0.30,0.36] \\ [0.56,0.75][0.20,0.25][0.20,0.25] & [0.7,0.88][0.15,0.22][0.10,0.22] & [0.22,0.31][0.53,0.63][0.32,0.38] \end{pmatrix}$$

To find the determinant of $\tilde{A}$, we need to find out all permutations on $\{1, 2, 3\}$. The permutation on $\{1, 2, 3\}$

$$\psi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \psi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \psi_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The membership value of $|\tilde{A}|$ is

$$\begin{aligned} &([a_{1\psi_1(1)\mu L}, a_{1\psi_1(1)\mu U}] \wedge [a_{2\psi_1(2)\mu L}, a_{2\psi_1(2)\mu U}] \wedge [a_{3\psi_1(3)\mu L}, a_{3\psi_1(3)\mu U}]) \\ &\vee([a_{1\psi_2(1)\mu L}, a_{1\psi_2(1)\mu U}] \wedge [a_{2\psi_2(2)\mu L}, a_{2\psi_2(2)\mu U}] \wedge [a_{3\psi_2(3)\mu L}, a_{3\psi_2(3)\mu U}]) \\ &\vee([a_{1\psi_3(1)\mu L}, a_{1\psi_3(1)\mu U}] \wedge [a_{2\psi_3(2)\mu L}, a_{2\psi_3(2)\mu U}] \wedge [a_{3\psi_3(3)\mu L}, a_{3\psi_3(3)\mu U}]) \\ &\vee([a_{1\psi_4(1)\mu L}, a_{1\psi_4(1)\mu U}] \wedge [a_{2\psi_4(2)\mu L}, a_{2\psi_4(2)\mu U}] \wedge [a_{3\psi_4(3)\mu L}, a_{3\psi_4(3)\mu U}]) \\ &\vee([a_{1\psi_5(1)\mu L}, a_{1\psi_5(1)\mu U}] \wedge [a_{2\psi_5(2)\mu L}, a_{2\psi_5(2)\mu U}] \wedge [a_{3\psi_5(3)\mu L}, a_{3\psi_5(3)\mu U}]) \\ &\vee([a_{1\psi_6(1)\mu L}, a_{1\psi_6(1)\mu U}] \wedge [a_{2\psi_6(2)\mu L}, a_{2\psi_6(2)\mu U}] \wedge [a_{3\psi_6(3)\mu L}, a_{3\psi_6(3)\mu U}]) \end{aligned}$$

$$= \begin{array}{l} ([a_{11\mu L}, a_{11\mu U}] \wedge [a_{22\mu L}, a_{22\mu U}] \wedge [a_{33\mu L}, a_{33\mu U}]) \\ \vee([a_{11\mu L}, a_{11\mu U}] \wedge [a_{23\mu L}, a_{23\mu U}] \wedge [a_{32\mu L}, a_{32\mu U}]) \\ \vee([a_{12\mu L}, a_{12\mu U}] \wedge [a_{21\mu L}, a_{21\mu U}] \wedge [a_{33\mu L}, a_{33\mu U}]) \\ \vee([a_{12\mu L}, a_{12\mu U}] \wedge [a_{23\mu L}, a_{23\mu U}] \wedge [a_{31\mu L}, a_{31\mu U}]) \\ \vee([a_{13\mu L}, a_{13\mu U}] \wedge [a_{21\mu L}, a_{21\mu U}] \wedge [a_{32\mu L}, a_{32\mu U}]) \\ \vee([a_{13\mu L}, a_{13\mu U}] \wedge [a_{22\mu L}, a_{22\mu U}] \wedge [a_{31\mu L}, a_{31\mu U}]) \end{array}$$

$$= \begin{array}{l} ([0.85, 0.95] \wedge [0.47, 0.61] \wedge [0.22, 0.31]) \\ \vee([0.85, 0.95] \wedge [0.31, 0.42] \wedge [0.7, 0.88]) \\ \vee([0.55, 0.65] \wedge [0.75, 0.85] \wedge [0.22, 0.31]) \\ \vee([0.55, 0.65] \wedge [0.31, 0.42] \wedge [0.65, 0.75]) \\ \vee([0.13, 0.19] \wedge [0.75, 0.85] \wedge [0.7, 0.88]) \\ \vee([0.13, 0.19] \wedge [0.47, 0.61] \wedge [0.65, 0.75]) \end{array}$$

$= \; [0.22, 0.31] \vee [0.31, 0.42] \vee [0.22, 0.31] \vee [0.31, 0.42] \vee [0.7, 0.19] \vee [0.13, 0.19] =$**[0.31,0.42]**.

Similarly, the neutral membership degree of $|\tilde{A}|$ is

$$= \begin{array}{l} ([a_{11\eta L}, a_{11\eta U}] \wedge [a_{22\eta L}, a_{22\eta U}] \wedge [a_{33\eta L}, a_{33\eta U}]) \\ \wedge([a_{11\eta L}, a_{11\eta U}] \wedge [a_{23\eta L}, a_{23\eta Uta}] \wedge [a_{32\eta L}, a_{32\eta U}]) \\ \wedge([a_{12\eta L}, a_{12\eta U}] \wedge [a_{21\eta L}, a_{21\eta U}] \wedge [a_{33\eta L}, a_{33\eta U}]) \\ \wedge([a_{12\eta L}, a_{12\eta U}] \wedge [a_{23\eta L}, a_{23\eta U}] \wedge [a_{31\eta L}, a_{31\eta U}]) \\ \wedge([a_{13\eta L}, a_{13\eta U}] \wedge [a_{21\eta L}, a_{21\eta U}] \wedge [a_{32\eta L}, a_{32\eta U}]) \\ \wedge([a_{13\eta L}, a_{13\eta U}] \wedge [a_{22\eta L}, a_{22\eta U}] \wedge [a_{31\eta L}, a_{31\eta U}]) \end{array}$$

$$= \begin{array}{l} ([0.10, 0.15] \wedge [0.33, 0.41] \wedge [0.53, 0.63]) \\ \wedge([0.10, 0.15] \wedge [0.43, 0.52] \wedge [0.15, 0.22]) \\ \wedge([0.25, 0.30] \wedge [0.15, 0.20] \wedge [0.53, 0.63]) \\ \wedge([0.25, 0.30] \wedge [0.43, 0.52] \wedge [0.20, 0.25]) \\ \wedge([0.69, 0.79] \wedge [0.15, 0.20] \wedge [0.15, 0.22]) \\ \wedge([0.69, 0.79] \wedge [0.33, 0.41] \wedge [0.20, 0.25]) \end{array}$$

$= \; [0.10, 0.15] \wedge [0.10, 0.15] \wedge [0.15, 0.20] \wedge [0.20, 0.25] \wedge [0.15, 0.22] \wedge [0.20, 0.25] =$**[0.10,0.15]**.

Now, the non-membership degree of $|\tilde{A}|$ is

$$= \begin{array}{l} ([a_{11\nu L},a_{11\nu U}]\vee[a_{22\nu L},a_{22\nu U}]\vee[a_{33\nu L},a_{33\nu U}]) \\ \wedge([a_{11\nu L},a_{11\nu U}]\vee[a_{23\nu L},a_{23\nu Uta}]\vee[a_{32\nu L},a_{32\nu U}]) \\ \wedge([a_{12\nu L},a_{12\nu U}]\vee[a_{21\nu L},a_{21\nu U}]\vee[a_{33\nu L},a_{33\nu U}]) \\ \wedge([a_{12\nu L},a_{12\nu U}]\vee[a_{23\nu L},a_{23\nu U}]\vee[a_{31\nu L},a_{31\nu U}]) \\ \wedge([a_{13\nu L},a_{13\nu U}]\vee[a_{21\nu L},a_{21\nu U}]\vee[a_{32\nu L},a_{32\nu U}]) \\ \wedge([a_{13\nu L},a_{13\nu U}]\vee[a_{22\nu L},a_{22\nu U}]\vee[a_{31\nu L},a_{31\nu U}]) \end{array}$$

$$= \begin{array}{l} ([0.05\wedge 0.15]\vee[0.27\wedge 0.36]\vee[0.32\wedge 0.38]) \\ \wedge([0.05\wedge 0.15]\vee[0.30\wedge 0.35]\vee[0.10\wedge 0.22) \\ \wedge([0.25\wedge 0.30]\vee[0.15\wedge 0.20]\vee[0.32\wedge 0.38]) \\ \wedge([0.25\wedge 0.30]\vee[0.30\wedge 0.36]\vee[0.20\wedge 0.25]) \\ \wedge([0.22\wedge 0.27]\vee[0.15\wedge 0.20]\vee[0.10\wedge 0.22]) \\ \wedge([0.22\wedge 0.27]\vee[0.27\wedge 0.36]\vee[0.20\wedge 0.25]) \end{array}$$

$$= [0.32,0.38]\vee[0.30,0.36]\vee[0.32,0.38]\vee[0.30,0.36]\vee[0.22,0.27]\vee[0.27,0.36] = [0.22,0.27]$$

$\therefore |\tilde{A}| = ([0.31,0.42][0.10,0.15][0.22,0.27])$.

Definition 5.3.2. *Adjoint of IVSFM*
Let $\tilde{A}=(\tilde{a}_{ij})=(<\tilde{a}_{ij\mu},\tilde{a}_{ij\eta},\tilde{a}_{ij\nu}>)$ be a IVSFM of order m. Then, the adjoint of $\tilde{A}$ is denoted by the Adjoint($\tilde{A}$) and defined by

$$Adjoint(\tilde{A}) = (<\tilde{q}_{ij\mu},\tilde{q}_{ij\eta},\tilde{q}_{ij\nu}>)$$

where

$$\tilde{q}_{ij\mu} = \vee_{\delta\in S_{m_j m_i}} \wedge_{u\in m_j} \tilde{a}_{u\delta(u)\mu},$$

$$\tilde{q}_{ij\eta} = \wedge_{\delta\in S_{m_j m_i}} \wedge_{u\in m_j} \tilde{a}_{u\delta(u)\eta},$$

$$\tilde{q}_{ij\nu} = \wedge_{\delta\in S_{m_j m_i}} \vee_{u\in m_j} \tilde{a}_{u\delta(u)\nu}.$$

Here $m_j=\{1,2........m\}$-$\{j\}$ and $S_{m_j m_i}$ is the set of all permutation of the set m_j over the set m_i.

Example 5.3.2. *Let us consider IVSFM of order three as follows*

$$\tilde{A} = \begin{pmatrix} [0.85,0.95][0.10,0.15][0.05,0.15] & [0.55,0.65][0.25,0.30][0.25,0.30] & [0.13,0.19][0.69,0.79][0.22,0.27] \\ [0.75,0.85][0.15,0.20][0.15,0.20] & [0.47,0.61][0.33,0.41][0.27,0.36] & [0.31,0.42][0.43,0.52][0.30,0.36] \\ [0.56,0.75][0.20,0.25][0.20,0.25] & [0.7,0.88][0.15,0.22][0.10,0.22] & [0.22,0.31][0.53,0.63][0.32,0.38] \end{pmatrix}$$

For j=1 and i=1, m_j= $\{1,2,3\}$-$\{1\}$= $\{2,3\}$ and m_i= $\{1,2,3\}$-$\{1\}$= $\{2,3\}$. The permutation of m_i over m_j are

$$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

Now

$(a_{22\mu} \wedge a_{33\mu}) \vee (a_{23\mu} \wedge a_{32\mu})$
$= ([0.47,0.61] \wedge [0.22,0.31]) \vee ([0.31,0.42] \wedge [0.7,0.88])$
$= [0.22,0.31] \vee [0.31,0.42]$ =[0.31,0.42]

$(a_{22\eta} \wedge a_{33\eta}) \wedge (a_{23\eta} \wedge a_{32\eta})$
$= ([0.33,0.41] \wedge [0.53,0.63]) \wedge ([0.43,0.52] \wedge [0.15,0.22])$
$= [0.33,0.41] \wedge [0.15,0.22]$ =[0.15,0.22]

$(a_{22\nu} \vee a_{33\nu}) \wedge (a_{23\nu} \vee a_{32\nu})$
$= ([0.27,0.36] \vee [0.32,0.38]) \wedge ([0.30,0.36] \vee [0.10,0.22])$
$= [0.32,0.38] \wedge [0.30,0.36]$ =[0.30,0.36].

Calculating in the similar way, Adjoint($\tilde{A}$) is obtained as

$$\text{Adjoint}(\tilde{A}) = \begin{pmatrix} [0.31,0.42][0.15,0.22][0.30,0.36] & [0.22,0.31][0.15,0.22][0.22,0.27\ ] & [0.31,0.42][0.25,0.30][0.27,0.36] \\ [0.31,0.42][0.15,0.20][0.30,0.36] & [0.22,0.31][0.10,0.15][0.22,0.27] & [0.31,0.42][0.10,0.15][0.20,0.25] \\ [0.7,0.85][0.15,0.20][0.15,0.22] & [0.55,0.65][0.10,0.15][0.10,0.22] & [0.55,0.65][0.10,0.15][0.25,0.30] \end{pmatrix}$$

5.4 Application of IVSFM in decision-making

In this section, we propose the score function and discuss the application of IVSFM in decision-making.

Definition 5.4.1. *Let $A = \langle [\mu^-,\mu^+],[\nu^-,\nu^+],[\eta^-,\eta^+]\rangle$ be an interval-valued spherical fuzzy number (IVSFN). Then, the score function S(A) and accuracy function H(A) for IVSFN is*

$$S(A) = \frac{1+(\mu^-)^2+(\mu^+)^2-(\nu^-)^2-(\nu^+)^2-(\frac{\eta^-}{2})^2-(\frac{\eta^+}{2})^2}{6}\left|\mu^-+\mu^+-\nu^--\nu^+-\frac{\eta^-}{2}-\frac{\eta^+}{2}\right| \in [-1,1],$$
$$H(A) = \frac{(\mu^-)^2+(\mu^+)^2+(\nu^-)^2+(\nu^+)^2+(\eta^-)^2+(\eta^+)^2}{2} \in [0,1].$$

For any two IVSFN A_1 and A_2,

1. *if $S(A_1) > S(A_2)$, then $A_1 > A_2$*
2. *if $S(A_1) < S(A_2)$, then $A_1 < A_2$*
3. *if $S(A_1) = S(A_2)$, then*
   (a) if $H(A_1) > H(A_2)$, then $A_1 > A_2$

(b) if $H(A_1) < H(A_2)$, then $A_1 < A_2$

(c) if $H(A_1) = H(A_2)$, then $A_1 = A_2$.

One can verify the following properties for the proposed score function.

- Let $A = \langle[1,1],[0,0][0,0]\rangle$ be an IVSFN, then $S(A) = 1$.
- Let $A = \langle[0,0],[0,0][0,0]\rangle$ be an IVSFN, then $S(A) = 0$.

Statement of Problem under Decision-Making

The following is how we localize the interval-valued spherical fuzzy relation concept. Let $V=\{v_1,v_2,v_3,v_4......v_i\}$, $U=\{u_1,u_2,u_3,u_4......u_i\}$ and $W=\{w_1,w_2,w_3,w_4......w_i\}$ are the set of subject related to courses, finite set of courses and finite set of applicant respectively. Suppose, we have the relation $R_1(W \to V)$ and $R_2(V \to U)$ such that

$$R_1 = \{\langle (w,v), \mu_{R_1}(w,v), \nu_{R_1}(w,v), \eta_{R_1}(w,v)\rangle | (w,v) \in (W \times V)\}$$

and

$$R_2 = \{\langle (v,u), \mu_{R_2}(v,u), \nu_{R_2}(v,u), \eta_{R_2}(v,u)\rangle | (v,u) \in (V \times U)\}$$

where $\mu_{R_1}(w,v)$ represents the degree to which the applicant, w, passes the related subject requirement,v; and $\nu_{R_1}(w,v)$ represents the degree to which the applicant, w does not pass the related subject requirements; $\eta_{R_1}(w,v)$ represents the degree to which the applicant may pass may not be the related subject requirement. similarly, $\mu_{R_2}(v,u)$ where represents the degree to which the related subject requirement, v, determines the courses u, $\nu_{R_2}(v,u)$ represents the degree to which the related subject requirement, v, does not determine the course, u and $\eta_{R_2}(v,u)$ represents the degree to which the related subject requirement, v, maybe may not be determine the course.The composition, T , of R_1 and R_2 is given as T $=R_1 \odot R_2$. This describes the state in which the applicants, w_i with respect to the related subjects requirement, v_j fit the courses, u_k. Thus:

$$\mu_T(w_i,u_k) = \bigvee_{v_j \in V} \left\{\mu_{R_1}(w_i,v_j) \wedge \mu_{R_2}(v_j,u_k)\right\},$$

$$\nu_T(w_i,u_k) = \bigwedge_{v_j \in V} \left\{\nu_{R_1}(w_i,v_j) \vee \nu_{R_2}(v_j,u_k)\right\},$$

$$\eta_T(w_i,u_k) = \bigwedge_{v_j \in V} \left\{\eta_{R_1}(w_i,v_j) \wedge \eta_{R_2}(v_j,u_k)\right\}$$ for all $w_i \in W$ and $u_k \in U$ where i,j and k take value from 1,2...n. The values of $\mu_{R_1 \odot R_2}(w_i,u_k)$, $\nu_{R_1 \odot R_2}(w_i,u_k)$ and $\eta_{R_1 \odot R_2}(w_i,u_k)$ of the composition T $=R_1 \odot R_2$ are as follows: If the value of T is maximized, the career placement can be achieved. This value is computed from R_1 and R_2 for the placement of w_i into any u_k relative to v_j, and it is maximized by using the proposed score function 5.4.1.

Example 5.4.1. *Let $W=\{Tom, Ankit, Sahil, Aashima\}$ be the set of applicants for the course placements; $V= \left\{EnglishLanguage, Maths, Biology, Physics, Chemistry\right\}$ be the set of related subjects requirement to the set of courses and $U = \left\{medicine, pharmacy, surgery, anatomy\right\}$ be the set of courses.*

- *A hypothetical relation R_1 ($W \to V$) is given in Table 5.1*

- *A hypothetical relation R_2 ($V \rightarrow U$) is given in Table 5.2*

- *The composition relation R ($W \rightarrow U$) is given in Table 5.3*

- *The degree of affiliation between set of applicants W_i to a set of courses U_i is calculated in Table 5.4.*

Table 5.1 shows us the relation between W and courses placement V in the form of IVSFN. It defines how strongly a set of applicants and a set of related subjects are related. Table 5.2 shows us the relation between the set of related subjects V and a set of courses U.

Table 5.1: Relation between set of applicant W and courses placement V in the form of IVSFN

W/V	**English Lang.**	**Maths**	**Biology**	**Physics**	**Chemistry**
Tom	[.85,.93] [.10,.13] [.05,.15]	[.20,.25] [.65,.75] [.20,.25]	[.6,.7] [.1,.2] [.4,.5]	[.7,.8] [.2,.3] [.3,.4]	[.21,.26] [.64,.74] [.21,.26]
Ankit	[.55,.65] [.25,.30] [.25,.30]	[.85,.9] [.1,.15] [.07,.17]	[.3,.6] [.2,.3] [.3,.4]	[.5,.6] [.1,.2] [.3,.4]	[.76,.88] [.15,.20] [.14,.21]
Sahil	[.65,.75] [.2,.25] [.25,.30]	[.10,.15] [.8,.9] [.08,.15]	[.4,.5] [.2,.3] [.6,.7]	[.6,.8] [.3,.4] [.2,.4]	[.82,.93] [.12,.17] [.09,.17]
Aashima	[.55,.65] [.25,.30] [.25,.30]	[.15,.19] [.7,.8] [.13,.20]	[.8,.9] [.2,.3] [.1,.2]	[.68,.79] [.19,.24] [.19,.24]	[.65,.75] [.20,.25] [.20,.25]

Table 5.2: Relation between the set of related subjects V and a set of courses U

V/U	**Medicine**	**Pharmacy**	**Surgery**	**Anatomy**
English Lang.	[.31,.42] [.15,.22] [.30,.36]	[.55,.65] [.10,.15] [.25,.30]	[.56,.66] [.29,.36] [.24,.30]	[.44,.51] [.43,.52] [.26,.34]
Maths	[.22,.31] [.10,.15] [.10,.15]	[.31,.42] [.15,.20] [.3,.36]	[.59,.68] [.27,.33] [.25,.32]	[.54,.62] [.33,.41] [.26,.33]
Biology	[.22,.31] [.15,.22] [.22,.27]	[.7,.85] [.15,.20] [.15,.22]	[.55,.63] [.32,.31] [.25,.32]	[.48,.57] [.37,.45] [.24,.30]
Physics	[.31,.42] [.25,.30] [.27,.36]	[.58,.67] [.3,.37] [.25,.32]	[.6,.8] [.29,.36] [.24,.30]	[.85,.93] [.10,.13] [.05,.15]
Chemistry	[.7,.85] [.15,.20] [.15,.20]	[.19,.24] [.67,.77] [.19,.24]	[.19,.24] [.67,.77] [.19,.24]	[.65,.75] [.20,.25] [.25,.30]

Table 5.3: The composition relation R (W → U)

W/U	**Medicine**	**Pharmacy**	**Surgery**	**Anatomy**
Tom	[.31,.42] [.10,.13] [.20,.25]	[.7,.85] [.1,.15] [.1,.15]	[.7,.85] [.10,.15] [.10,.15]	[.65,.75] [.10,.15] [.13,.20]
Ankit	[.58,.67] [.1,.13] [.4,.5]	[.55,.65] [.1,.15] [.3,.4]	[.58,.67] [.1,.15] [.19,.24]	[.7,.85] [.10,.15] [.15,.22]
Sahil	[.6,.8] [.1,.13] [.21,.26]	[.59,.68] [.1,.15] [.19,.24]	[.6,.8] [.12,.17] [.19,.24]	[.55,.65] [.19,.24] [.2,.25]
Aashima	[.7,.8] [.1,.13] [.25,.3]	[.65,.75] [.1,.15] [.25,.3]	[.6,.8] [.1,.3] [.2,.4]	[.68,.79] [.2,.25] [.19,.24]

Table 5.4: Final decision matrix

W/U	**Medicine**	**Pharmacy**	**Surgery**	**Anatomy**
Tom	0.0681	**0.425**	**0.425**	0.1335
Ankit	0.1573	0.163	0.226	**0.401**
Sahil	**0.3031**	0.2354	0.2883	0.145
Aashima	**0.3424**	0.2791	0.2788	0.2630

Now to calculate the degree of closeness value (Table 5.4), we use the score function 5.4.1 to find the weights (score value) of Table 5.3 :

$$S(A) = \frac{1+\alpha^2+\beta^2-\gamma^2-\delta^2-\left(\frac{\tau}{2}\right)^2-\left(\frac{\theta}{2}\right)^2}{6}\left|\alpha+\beta-\gamma-\delta-\frac{\tau}{2}-\frac{\theta}{2}\right|.$$

Suppose $A_I = [0.31, 0.42][0.10, 0.13][0.20, 0.25]$; here $\alpha = 0.31, \beta = 0.42, \gamma = 0.10, \delta = 0.13, \tau = 0.20,\ \theta = 0.25$. Then $S(A_I) = 0.0681$. Also, suppose $A_2 = [0.7, 0.85][0.1, 0.15][0.1, 0.15]$; here $\alpha = 0.7,\ \beta = 0.85, \gamma = 0.1, \delta = 0.15, \tau = 0.1, \theta = 0.15$. Then $S(A_2) = 0.425$. Similarly, we can calculate other weights (score values) for Table 5.3.

The analysis of score value i.e., degree of closeness value is given in Table 5.4. Tom is suitable to study either medicine or surgery. Ankit is suitable for studying anatomy. Sahil is suitable for studying medicine. Aashima is suitable to study only medicine.

5.5 Comparative Study and Advantage

Ejegwa [120] worked on the Pythagorean fuzzy set and its application in career placements based on academic performance using max–min–max composition. The information was taken in a Pythagorean fuzzy sense in previous papers on Pythagorean fuzzy decision-making. When there are additional sorts of uncertainty in data, present strategies are ineffective in dealing with them. In these instances, data should be gathered or displayed in the form of an interval-valued spherical fuzzy meaning. In such instances, the currently developed process plays an important role in a successful conclusion. Silambarasan [139] worked on the spherical fuzzy matrix. In the spherical fuzzy matrix, the membership, neutrality, and non-membership degree have a point. In some cases, it is difficult to measure the degree of membership, neutrality, and non-membership values as a point. That way, we consider the membership, neutrality, and non-membership values as an interval, and it is practically useful in the case of real-life problems. So, our study is an extension of the study of Silambarasan [139]. However, in the current technique, the matrices entries that are considered are interval-valued spherical fuzzy values. Over a set of universes, interval-valued spherical fuzzy sets are extracted from them. The final matrix is then calculated using the proposed score function between two interval-valued spherical fuzzy sets, yielding a decision. It is not necessary to perform many different sorts of computations in order to implement the steps of this approach; i.e. the method is not complicated to implement. As a result, developing algorithms and computer programming for this method is relatively simple. Furthermore, the data points used here are capable of tolerating a wider range of information ambiguity. This research can be thought of as a study in an advanced spherical fuzzy sense because the interval-valued spherical fuzzy concept is a generalization of the spherical fuzzy concept. The use of IVSFM has the following comprehensive list of advantages:

1. In the case that information cannot be fully captured by the existing matrices due to each of their limitations, the void is filled by the IVSFM.
2. The limitation of the FM, IFM, PFM, and SFM conditions in the literature at hand is that they prevent experts and decision-makers from assigning membership, neutrality, and non-membership degrees on their personal preferences. The membership, neutrality, and non-membership degrees can all be described broadly as interval numbers in this work.
3. The implementation of the IVSFM and the approach suggested for the problems of career placement assessment demonstrates how well and consistently the proposed work addresses the extended framework. The observations indicate that the IVSFM is the most generalized structure among all fuzzy matrix models.

   The detailed analysis presented in Table 5.5 further compares the proposed work and existing research available in the literature.

5.6 Conclusion

We have defined the interval-valued spherical fuzzy matrix. Important definitions and theorems are defined with their proofs. The procedure of determinant and adjoint of the IVSFM is developed. Such investigations can be seen as an extension of studies on spherical fuzzy matrices. Additionally, we study the application of IVSFM in decision-making processes. A score function is introduced to address decision-making challenges. A limitation of IVSFM is related to the representation of degrees of membership, neutrality, and non-membership as interval numbers. The limitation arises when the sum of the upper degree of membership, neutrality, and upper degree of non-membership exceeds the interval [0, 1]. Exceeding this interval can lead to inconsistencies in calculations and may affect the accuracy and reliability of the results obtained using IVSFM. Therefore, careful consideration and management of this limitation are necessary to ensure the validity of the analysis conducted using IVSFM.

Table 5.5: Analysis of proposed work and existing work in literature

Characteristics Methods	Whether consider MD	Whether consider MD more flexible i.e IVMD	Whether consider MD or NMD	Whether consider MD or NMD more flexible i.e IVMD or IVNMD	Whether consider MD, neutrality and NMD degree	Whether consider MD, neutrality and NMD degree more flexible i.e IVMD, interval-valued neutrality degree and IVNMD.
Thomason [16]	✓	×	×	×	×	×
Pal [24]	✓	✓	×	×	×	×
Pal et al. [131]	✓	✓	✓	×	×	×
Silambarasan and Sriram [30]	✓	✓	✓	×	×	×
Khan and Pal [32]	✓	✓	✓	✓	×	×
Dogra and Pal [28]	✓	✓	✓	✓	✓	×
Silambarasan [139]	✓	✓	✓	✓	✓	×
Proposed work	✓	✓	✓	✓	✓	✓

MD-Membership degree, NMD-Non membership degree, IVMD-Interval-valued membership degree, IVNMD-Interval valued non membership degree.

Chapter 6

Interval Valued Fermatean Fuzzy Matrix and its Application

Interval-valued fermatean fuzzy matrices are outlined by an interval membership and non-membership degree; because of which, the decision-makers are capable of considering an interval hesitancy degree, where they don't simply convey their perception using membership and non-membership degree at a point. This chapter [1] defines interval valued fermatean fuzzy matrix in which each element in the universal set is associated with an interval i.e., expressing the uncertainty range of its membership degree is associated with an interval rather than a precise value, allowing for a more nuanced representation of uncertainty. Some fundamental definition sand theorems have been proposed for IVFFM with their proofs. Also, we have demonstrated the method to find the determinant and adjoint of IVFFM in the paper. Further, we have used the composition functions to introduce the algorithms for the greatest eigen interval-valued fermatean fuzzy set (GEIVFFS) and the least eigen interval-valued fermatean fuzzy set (LEIVFFS). The process is illustrated with a numerical example based on the decision-making problem that has been taken into account for illustrating the proposed methodology. Finally, an application of an interval-valued fermatean fuzzy matrix to a decision-making problem is presented here by using the proposed score function.

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[1] **The work presented in this chapter comprises of the paper entitled, " Interval valued fermatean fuzzy matrix and its application" (Submitted in "Cybernetics and Systems"(Taylor & Francis))**

6.1 Introduction

The fuzzy matrix theory introduced by Thomason [16] has exhibited great zeal and competency in modeling the uncertainty persisting in decision-making to confront the prevailing matrix theories. Since the introduction of the fuzzy matrix, many attempts have been made to generalize the notion of fuzzy matrix. In fuzzy matrix theory, the membership of an element to a set happens to be a value between 0 and 1. Sanchez [97] worked on eigen fuzzy sets and fuzzy relations and used the idea of fuzzy relations composition to explain the significance of eigen fuzzy sets. Sanchez [97] also used the max-min composition to identify the largest eigen fuzzy set. Martino et al. [48] presented the least eigen fuzzy set that depends on min-max composition. Martino and Sessa [154] generalized the method eigen fuzzy sets to evaluate the effectiveness of a drug in making a symptom of the disease disappear in patients. After the notion of FM, Pal [24] introduced the concept of an interval value fuzzy matrix (IVFM) by extending the membership and non-membership degree in interval number. However, in reality, it might not be the case that the degree of non-membership of an element is always equal to 1 minus the membership degree because of the presence of some hesitation degree. Therefore, The intuitionistic fuzzy matrix (IFM) was introduced by Pal et al. [131] with this kind of situation in mind. All the fuzzy matrices are IFM but not conversely. Given the distinct advantages of IFMs, they have become pivotal tools for delineating uncertainty and ambiguity in practical problems [32, 132, 134]. In certain cases, Decision Experts (DEs) express opinions using Membership Degrees (MD) and Non-Membership Degrees (ND), like (0.7, 0.6). Unfortunately, IFMs face limitations when handling this, as 0.7 + 0.6 > 1. Addressing this, Silambarasan and Sriram [30] introduced 'Pythagorean fuzzy matrice (PyFM),' defined by MD and ND with a constraint that the sum of the squares of MD and ND must be ≥ 1. PyFM has proven to be reliable tools for solving complex multiple-criteria decision analysis (MCDA) problems. Venkatesan and Sriram [33] reduced Pythagorean fuzzy matrices to fuzzy matrices, and Gulleria and Bajaj [41] addressed medical diagnosis problems under Pythagorean fuzzy soft matrices. Further, Silambarasan [144] extended Pythagorean fuzzy matrices to FFM with a condition that the cube sum of membership and non-membership degrees must be < 1. A key distinction among IFM, PyFM, and FFM lies in the constraining relationship between MD and ND. FFMs emerge as more potent and operative tools than both IFMs and PyFMs for managing uncertain MCDA problems. Scholars have recently focused on FFMs for diverse applications. However, in practical decision problems under FFMs, DEs face challenges in precisely enumerating decisions due to inadequate information. In such cases, it is beneficial for DEs to express decisions as interval numbers within [0, 1]. Yet, some works have concentrated on FFMs' development without considering extended information. Hence, introducing 'Interval-valued fermatean fuzzy matrix (IVFFM),' certifying MD and ND to assume interval values, becomes essential. This environment aligns with IFMs, leading to the generalization of IFM doctrines to 'IVIFM' [32], indicating that interval values of MD and ND are assigned to a matrix. Inspired by FFM, this work introduces the concept of IVFFM.

6.1.1 Motivation

For decision-makers, the IVFFM has evolved into a useful tool for illustrating uncertainty in issues involving multiple criteria. Another characteristic of IVFFM is the degree of membership and non-membership whose cube sum is less than or equal to one. Compared to the IVFM and IVIFM, the IVFFM has a wider scope. The

IVFFM might be able to fix a problem as compared to IVIFM. In literature, the interval-valued fuzzy matrix introduced by Pal [24] and the concept of an interval-valued intuitionistic fuzzy matrix (IVIFM) presented by Khan and Pal [32] by converting the membership and non-membership degree into interval form. Then, motivated by the concept of IVFM and IVIFM, we have expanded the FFM [144] into IVFFM. On the basis of FFM, we have developed some importants aspects of IVFFM in the present chapter.

6.1.2 Organization of the chapter

The remaining part of the chapter is organized as follows. In Section 6.2, we define the idea of IVFFM with basic definitions, properties, and important theorems. The concepts of determinant, adjoint, and propositions are given in Section 6.3. Section 6.4 proposes the algorithms for finding the GEIVFFS & the LEIVFFS and illustrates them with a numerical example. Section 6.5 introduces a new score function for the IVFFS and discusses its application in decision-making. The comparative study with existing work is conducted in Section 6.6. Finally, Section 6.7 concludes the paper with some future directions.
In the next section, we present the notion of an IVFFM, which serves as an extension of FFM. Additionally, we establish the definitions of fundamental arithmetic operations and demonstrate the proof of essential theorems pertaining to the IVFFM.

6.2 Interval-Valued Fermatean Fuzzy Matrix

In this section, we define the IVFFM and basic concepts by generalizing the concept of FFM.

Definition 6.2.1. *An interval-valued fermatean fuzzy matrix (IVFFM) $\tilde{A}$ is defined as*

$$\tilde{A} = (\tilde{a}_{ij}) = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\nu} >),\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$$

where,

$$\tilde{a}_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \subseteq [0, 1],$$

$$\tilde{a}_{ij\nu} = [a_{ij\nu L}, a_{ij\nu U}] \subseteq [0, 1],$$

with the condition

$$(a_{ij\mu U})^3 + (a_{ij\nu U})^3 \leq 1,$$

$\tilde{a}_{ij\mu}$ and $\tilde{a}_{ij\nu}$ are the membership and non-membership degree of $\tilde{a}_{ij}$.

Definition 6.2.2. *An IVFFM is said to be a square interval-valued fermatean fuzzy matrix (SIVFFM) if the number of rows is equal to the number of columns i.e., $i = 1, 2, \ldots\ldots m, j = \ldots\ldots m$.*

Definition 6.2.3. *Let $\tilde{A}$ and $\tilde{B}$ be two IVFFM such that*
$\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}])$ and $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}])$.
Then, we write $\tilde{A} \leq \tilde{B}$ as follow:
$a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}; a_{ij\nu L} \geq b_{ij\nu L}, a_{ij\nu U} \geq b_{ij\nu U}$.

Definition 6.2.4. *An IVFFM $\tilde{A}$ is a null matrix if $\tilde{a}_{ij\mu} = 0$ and $\tilde{a}_{ij\nu} = 0\ \forall\ i = 1, 2, \ldots\ldots m, j = \ldots\ldots n$.*

Definition 6.2.5. *An IVFFM $\tilde{A} = (\tilde{a}_{ij})$ various kind of matrix have been analogically proposed.*

(i) *An IVFFM is called the row matrix if $i = 1$ ($j = 1, 2.......n$).*

(ii) *An IVFFM is called the column matrix if $j = 1$ ($i = 1, 2.......m$).*

(iii) *An IVFFM is called the diagonal matrix if all its non-diagonal elements are zero.*

(iv) *An IVFFM is called the μ universal matrix if $\tilde{\mu}_{ij} = 1, \tilde{\nu}_{ij} = 0 \;\forall\; i = 1, 2, ...m, j = ...n.$*

(v) *An IVFFM is called the ν universal matrix if $\tilde{\mu}_{ij} = 0, \tilde{\nu}_{ij} = 1 \;\forall\; i = 1, 2, ...m, j = ...n.$*

(vi) *A SIVFFM is called a symmetric matrix if $\tilde{a}_{ij} = \tilde{a}_{ji}$.*

(vii) *A SIVFFM is called the skew-symmetric if $\tilde{a}_{ij} = Neg(\tilde{a}_{ij})$*

(viii) *Two IVFFM are called equal if they have the same order and their corresponding elements are equal.*

(ix) *If $\tilde{A}$ is IVFFM, then its trace, denoted by $tr(\tilde{A})$, is the sum of the elements on the main diagonal.*

Definition 6.2.6. *Let $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}])$ and $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}])$ be two IVFFM of same order $m \times n$ then, we define some basic operations.*

(i) $\tilde{A}^c = (\left[a_{ij\nu L}, a_{ij\nu U}\right], \left[a_{ij\mu L}, a_{ij\mu U}\right]).$

(ii) $\tilde{A} \vee \tilde{B} = (\left[max\left(a_{ij\mu L}, b_{ij\mu L}\right), max\left(a_{ij\mu U}, b_{ij\mu U}\right)\right]\left[min\left(a_{ij\nu L}, b_{ij\nu L}\right), min\left(a_{ij\nu U}, b_{ij\nu U}\right)\right]).$

(iii) $\tilde{A} \wedge \tilde{B} = (\left[min\left(a_{ij\mu L}, b_{ij\mu L}\right), min\left(a_{ij\mu U}, b_{ij\mu U}\right)\right]\left[max\left(a_{ij\nu L}, b_{ij\nu L}\right), max\left(a_{ij\nu U}, b_{ij\nu U}\right)\right]).$

(iv) $\tilde{A}^T = (\left[a_{ji\mu L}, a_{ji\mu U}\right]\left[a_{ji\nu L}, a_{ji\nu U}\right]).$

(v) $\tilde{A} \oplus \tilde{B} = (\left[a_{ij\mu L} + b_{ij\mu L} - a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U} + b_{ij\mu U} - a_{ij\mu U}.b_{ij\mu U}\right], \left[a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U}.b_{ij\nu U}\right]).$

(vi) $\tilde{A} \otimes \tilde{B} = (\left[a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U}.b_{ij\mu U}\right], \left[a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U}.b_{ij\nu U}\right]).$
where $\tilde{A}^c$ and $\tilde{A}^T$ are complement and transpose of $\tilde{A}$ respectively.

Definition 6.2.7. *An interval-valued fermatean fuzzy relation (IVFFR) R between two interval-valued fermatean fuzzy sets (IVFFS) X and Y defined as follows*

$$R = (\langle (x,y), \mu_R(x,y), \nu_R(x,y)\rangle | x \in X, y \in Y)$$

where μ_R=$[\mu_R^L, \mu_R^U]$, ν_R=$[\nu_R^L, \nu_R^U]$ such that $0 \le (\mu_R^U)^3 + (\nu_R^U)^3 \le 1$ for every $(x,y) \in (X \times Y)$. In this paper, we denote the IVFFR$\in$(X×Y) as the set of all the IVFFR between X×Y.

Definition 6.2.8. *Let $R(X,X)$ be an IVFFR on set X. Then the relation $R(X,X)$ is reflexive if the diagonal entries of IVFFM are $\langle [1,1][0,0]\rangle$, i.e.,*

$$\mu_R(x,x) = [1,1] \text{ and } \nu_R(x,x) = [0,0] \forall\, x \in X.$$

Definition 6.2.9. *A relation $R(X,X)$ is symmetric relation, if $\tilde{A}_R = \tilde{A}_R^T$. where $\tilde{A}_R$ be the IVFFM i.e., for*

$$\mu_R(x,y) = [\mu_R^L(x,y), \mu_R^U(x,y)] \text{ and } \nu_R(x,y) = [\nu_R^L(x,y), \nu_R^U(x,y)] \forall\, x,y \in X.$$

$$\mu_R(y,x) = [\mu_R^L(y,x), \mu_R^U(y,x)] \text{ and } \nu_R(x,y) = [\nu_R^L(y,x), \nu_R^U(y,x)] \forall\, x,y \in X.$$

$$\mu_R(x,y) = \mu_R(y,x) \text{ and } \nu_R(x,y) = \nu_R(y,x) \forall\, x,y \in X.$$

Definition 6.2.10. *A relation $R(X,X)$ is said to be transitive relation if $\tilde{A}_R \geq \tilde{A}_R^2$, i.e., $\mu_R(x,z) \geq \max_{y\in X}\left(\min\left\{\mu_R(x,y), \mu_R(y,z)\right\}\right)$ and $\nu_R(x,z) \leq \min_{y\in X}\left(\max\left\{\nu_R(x,y), \nu_R(y,z)\right\}\right)$ for all pair $(x,y),(y,z) \in X\times X$.*

Definition 6.2.11. *The relation is said to be a similarity relation if and only if it is reflexive, symmetric, and transitive.*

The zero matrix O_n and the identity matrix I_n of order $n\times n$ is define as, I_n is the IVFFM whose principle diagonal elements are all $I = \langle[1,1][0,0]\rangle$ and other elements are all $\psi = \langle[0,0][1,1]\rangle$.

Proposition 6.2.1. *An IVFFM $\tilde{A} \in \tilde{A}_n$ ($\tilde{A}_{\mathrm{mn}}$ or simply A_{mn} denotes the set of all IVFFMs of order $m\times n$ over an FFM if $m = n$, then it is denoted by $\tilde{A}_n$) and $\tilde{A} \geq I_n$, then $\tilde{A}$ is reflexive.*

Proof. As $\tilde{A} \geq I_n$ and diagonal entries of $\tilde{A}$ are $\langle[1,1][0,0]\rangle$. Thus by result, $\tilde{A}$ is a reflexive matrix. □

Definition 6.2.12. *Suppose IVFFM $\tilde{A} = (A_{ij}) = \left(\langle a_{ij\mu}, a_{ij\nu}\rangle\right) \in \tilde{A}_n$, we define the following IVFFMs:*

Type of $\tilde{A}$	*Definition*
Idempotent	$\tilde{A} = \tilde{A}^2$.
Reflexive	$A \geq I_n$.
Weakly reflexive	$\tilde{A}\tilde{A}_{ii} \geq \tilde{A}_{ij}$ *for all* $i,j \in \{1,2,3\ldots n\}$
Symmetric	$\tilde{A} = \tilde{A}^T$.
Transitive	$\tilde{A}^2 \leq \tilde{A}$.

Proposition 6.2.2. *If $\tilde{A} \in \tilde{A}_n$ be transitive and reflexive, then $\tilde{A}$ is idempotent.*

Proof. As $\tilde{A}$ is reflexive and $\tilde{A} \geq I_n$. Then

$$\tilde{A}^2 \geq A \geq I_n$$

$\tilde{A}$ is transitive also,

$$\tilde{A}^2 \leq \tilde{A}$$

from above, $\tilde{A}^2 = \tilde{A}$. Thus $\tilde{A}$ is idempotent. □

Proposition 6.2.3. *If $\tilde{A}$ and $\tilde{A}$ are two symmetric IVFFMs such that $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$, then $\tilde{A}\tilde{B}$ is symmetric IVFFM.*

Remark 3.1 If $\tilde{A}$ is a symmetric IVFFM in $\tilde{A}_n$, then $\tilde{A}^n$ is symmetric for any positive integer n.

Proposition 6.2.4. *If $\tilde{A}$ and $\tilde{B}$ are transitive IVFFMs in $\tilde{A}_n$ such that $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$, then $\tilde{A}\tilde{B}$ is transitive.*

Proof. Since $\tilde{A}$ and $\tilde{B}$ are transitive, $\tilde{A}^2 \leq \tilde{A}$ and $\tilde{B}^2 \leq \tilde{B}$. Now

$$\begin{aligned}
(\tilde{A}\tilde{B})^2 = (\tilde{A}\tilde{B})(\tilde{A}\tilde{B}) = \tilde{A}(\tilde{B}\tilde{A})\tilde{B} & \quad \text{(by associative property)}\\
= \tilde{A}(\tilde{A}\tilde{B})\tilde{B} & \quad \text{(Since } \tilde{A}\tilde{B} = \tilde{B}\tilde{A}\text{)}\\
= (\tilde{A}\tilde{A})(\tilde{B}\tilde{B}) & \\
= \tilde{A}^2\tilde{B}^2 \leq \tilde{A}\tilde{B} &
\end{aligned}$$

Hence $\tilde{A}\tilde{B}$ is transitive. □

Definition 6.2.13. *An IVFFM $\tilde{A} \in \tilde{A}_n$ is said to be invertible if and only if there exists $\tilde{B} \in \tilde{A}_m$ such that $\tilde{A}\tilde{B} = \tilde{B}\tilde{A} = I_n$.*

Definition 6.2.14. *If the IVFFM $\tilde{A}$ has exactly one entry $I =< [1,1][0,0] >$ in each row and each column and all the other entries are $\psi = < [0,0[1,1] >$, then IVFFM $\tilde{A} \in A_n$ is called permutation matrix.*

In mathematics, interval-valued fermatean fuzzy permutation matrix (IVFFPM) play a crucial role, particularly in matrix theory. We study whether FFM is invertible if it is a permutation matrix.

Proposition 6.2.5. *Suppose $\tilde{A} \in \tilde{A}_n$ be IVFFPM. Then $\tilde{A}\tilde{A}^T = \tilde{A}^T\tilde{A} = I_n$*

Proof. Let $\tilde{A} = (a_{ij})$=$\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] \rangle$. Then $\tilde{A}^T$= $\langle [a_{ji\mu L}, a_{ji\mu U}], [a_{ji\nu L}, a_{ji\nu U}] \rangle = [q_{ij}]$
Then, $\tilde{A}\tilde{A}^T$ is

$$\sum_{k=1}^{n} a_{ik}q_{kj} = \sum_{k=1}^{n} a_{ik}q_{jk} = \begin{cases} \psi, & \text{if } i \neq j \\ I, & \text{if i=j} \end{cases}$$

□

(As $\tilde{A}$ is an IVFFPM)
Therefore $\tilde{A}\tilde{A}^T$ is an identity matrix of order n.
Also, we can show $\tilde{A}^T\tilde{A} = I_n$. Thus $\tilde{A}\tilde{A}^T = \tilde{A}^T\tilde{A} = I_n$

Proposition 6.2.6. *Suppose $\tilde{A} \in \tilde{A}_n$, $\tilde{A}$ is invertible if and only if $\tilde{A}$ is IVFFPM.*

Proof. Necessary condition. Suppose A be a permutation matrix. Then $\tilde{A}\tilde{A}^T = \tilde{A}^T\tilde{A} = I_n$ (proposition 3.5) Therefore $\tilde{A}$ is invertible and $\tilde{A}^T$ is the inverse of $\tilde{A}$
Sufficient condition. Let $\tilde{A} = (a_{ij})$ is the invertible matrix and $\tilde{B} = (b_{ij})$ inverse of $\tilde{A}$ such that $\tilde{A}\tilde{B} = \tilde{B}\tilde{A} = I_n$ follow that,
$\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} b_{ik}a_{kj} = \psi$ for i $\neq$ j and $\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n} b_{ik}a_{ki} = I$
After that, using max-min algebra,
$a_{ik}b_{kj} = b_{ik}a_{kj} = \psi$ for i $\neq$ j an k $\in$ 1,2,3,.... (1)
and
$a_{ik}b_{ki} = b_{ik}a_{ki} = I$ for at least one k $\in$ 1,2,3,.... and for i $\in$ 1,2,3,.... (2)
by (1)
$a_{ik} = \psi$ or $b_{kj} = \psi$ or both $a_{ik} = b_{kj} = \psi$ for i $\neq$ j , k $\in$ 1,2,3,.... (3)
and
$b_{ik} = \psi$ or $a_{kj} = \psi$ or both $b_{ik} = a_{kj} = \psi$ for i $\neq$ j , k $\in$ 1,2,3,.... (4)
Also (2) implies that
$a_{ik} = b_{ki} = I$ and $a_{ki} = b_{ik} = I$ (5)
for at least one k $\in$ 1,2,3,.... and for each ki $\in$ 1,2,3,....
Let the results of (5) exists for k=r(say), that is $a_{ir} = b_{ri} = I =< 1,0 >$. Then from (3), we get $b_{rj} = \psi =< 0,1 >$ and for i $\neq$ j and $a_{jr} = \psi =< 0,1 >$ for i $\neq$ j . Therefore, the rth row of B has exactly one I and the remaining entries are all ψ and the ath column of a has exactly ine I and the remaining entries are all ψ.

Similarly, by using (4) the ath row of $\tilde{A}$ has exactly one and the remaining entries are all ψ and the rth column of $\tilde{B}$ has exactly one I and the remaining entries are all ψ.
Thus, $\tilde{A}$ and $\tilde{B}$ both are interval-valued fermatean fuzzy permutation matrices. □

Remark 3.2 An interval-valued fermatean fuzzy permutation matrix $\tilde{A} \in \tilde{A}_n$ is invertible and $\tilde{A}^T$ is the inverse of it.

Remark 3.3 The permutation matrices are only the invertible matrices in $\tilde{A}_n$.

Theorem 6.2.1. *Suppose $\tilde{A}$, and $\tilde{B}$ be two IVFFM of order $m \times n$ then*

(a) $\left(\tilde{A} \vee \tilde{B}\right)^c = \tilde{A}^c \wedge \tilde{B}^c$.

(b) $\left(\tilde{A} \wedge \tilde{B}\right)^c = \tilde{A}^c \vee \tilde{B}^c$.

Proof. (a) Let $\tilde{A} = \left\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ijvL}, a_{ijvU}]\right\rangle$, $\tilde{B} = \left\langle [b_{ij\mu L}, b_{ij\mu U}], [b_{ijvL}, b_{ijvU}]\right\rangle$
$\tilde{A}^c = < \left[a_{ijvL}, a_{ijvU}\right], \left[a_{ij\mu L}, a_{ij\mu U}\right] >$, $\tilde{B}^c = < \left[b_{ijvL}, b_{ijvU}\right], \left[b_{ij\mu L}, b_{ij\mu U}\right] >$

$\tilde{A}^c \wedge \tilde{B}^c$=$\langle \left[min\left(a_{ijvL}, b_{ijvL}\right), min\left(a_{ijvU}, b_{ijvU}\right)\right] \left[max\left(a_{ij\mu L}, b_{ij\mu L}\right), max\left(a_{ij\mu U}, b_{ij\mu U}\right)\right]\rangle$
$\tilde{A}^c \vee \tilde{B}^c$=$\langle \left[max\left(a_{ijvL}, b_{ijvL}\right), max\left(a_{ijvU}, b_{ijvU}\right)\right] \left[min\left(a_{ij\mu L}, b_{ij\mu L}\right), min\left(a_{ij\mu U}, b_{ij\mu U}\right)\right]\rangle$

$\left(\tilde{A} \vee \tilde{B}\right)^c = \langle \left[min\left(a_{ijvL}, b_{ijvL}\right), min\left(a_{ijvU}, b_{ijvU}\right)\right] \left[max\left(a_{ij\mu L}, b_{ij\mu L}\right), max\left(a_{ij\mu U}, b_{ij\mu U}\right)\right]\rangle$
$= \tilde{A}^c \wedge \tilde{B}^c$.

(b) Proof of part (b) can be done on similar lines.

□

Theorem 6.2.2. *Suppose $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVFFM of same order. $\tilde{A} \le \tilde{C}$ and $\tilde{B} \le \tilde{C}$, then $\tilde{A} \vee \tilde{B} \le \tilde{C}$.*

Proof. Let $\tilde{A} = \left\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ijvL}, a_{ijvU}]\right\rangle$, $\tilde{B} = \left\langle [b_{ij\mu L}, b_{ij\mu U}], [b_{ijvL}, b_{ijvU}]\right\rangle$
$\tilde{C} = \left\langle [c_{ij\mu L}, c_{ij\mu U}], [c_{ijvL}, c_{ijvU}]\right\rangle$.
If $\tilde{A} \le \tilde{C}$ then $a_{ij\mu L} \le c_{ij\mu L}, a_{ij\mu U} \le c_{ij\mu U}, a_{ijvL} \ge c_{ijvL}, a_{ijvU} \le c_{ijvU}$ for all i,j
and $\tilde{B} \le \tilde{C}$ then $b_{ij\mu L} \le c_{ij\mu L}, b_{ij\mu U} \le c_{ij\mu U}, b_{ijvL} \ge c_{ijvL}, b_{ijvU} \le c_{ijvU}$ for all i,j
Now, max $\left(a_{ij\mu L}, b_{ij\mu L}\right) \le c_{ij\mu L}$, max $\left(a_{ij\mu U}, b_{ij\mu U}\right) \le c_{ij\mu U}$
min $\left(a_{ijvL}, b_{ijvL}\right) \ge c_{ijvL}$, min $\left(a_{ijvU}, b_{ijvU}\right) \ge c_{ijvU}$
Thus $\tilde{A} \vee \tilde{B} \le \tilde{C}$ using Definition 6.2.3. □

Theorem 6.2.3. *Suppose $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVFFM of same order and $\tilde{A} \le \tilde{B}$, then $\tilde{A} \vee \tilde{C} \le \tilde{B} \vee \tilde{C}$.*

Proof. Let $\tilde{A} = \left\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ijvL}, a_{ijvU}]\right\rangle$, $\tilde{B} = \left\langle [b_{ij\mu L}, b_{ij\mu U}], [b_{ijvL}, b_{ijvU}]\right\rangle$
$\tilde{C} = \left\langle [c_{ij\mu L}, c_{ij\mu U}], [c_{ijvL}, c_{ijvU}]\right\rangle$, are three IVFFM of same order $m \times n$.
If $\tilde{A} \le \tilde{B}$ then $a_{ij\mu L} \le b_{ij\mu L}, a_{ij\mu U} \le b_{ij\mu U}, a_{ijvL} \ge b_{ijvL}, a_{ijvU} \le b_{ijvU}$
Now max $\left(a_{ij\mu L}, c_{ij\mu L}\right) \le$ max $\left(b_{ij\mu L}, c_{ij\mu L}\right)$, max $\left(a_{ij\mu U}, c_{ij\mu U}\right) \le$ max $\left(b_{ij\mu U}, c_{ij\mu U}\right)$,
min $\left(a_{ijvL}, c_{ijvL}\right) \ge$ min $\left(b_{ijvL}, c_{ijvL}\right)$, min $\left(a_{ijvU}, c_{ijvU}\right) \ge$ min $\left(b_{ijvU}, c_{ijvU}\right)$ for all i,j
Therefore $\tilde{A} \vee \tilde{C} \le \tilde{B} \vee \tilde{C}$. □

Theorem 6.2.4. *Let $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ be three IVFFM of the same order and $\tilde{C} \leq \tilde{A}$ and $\tilde{C} \leq \tilde{B}$ then $\tilde{C} \leq \tilde{A} \wedge \tilde{B}$.*

Proof. Proof of the above result directly follows from Theorem 6.2.3. □

Theorem 6.2.5. *Suppose $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ be three IVFFM of the same order, and if $\tilde{A} \leq \tilde{B}$, $\tilde{A} \leq \tilde{C}$ and $\tilde{B} \wedge \tilde{C} = 0$, then $\tilde{A}$ = 0.*

Proof. Proof directly follows from Theorem 6.2.3 and above Theorem 6.2.4. □

Theorem 6.2.6. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVFFM of same order of $\tilde{A} \leq \tilde{B}$ then $\tilde{A} \wedge \tilde{C} \leq \tilde{B} \wedge \tilde{C}$.*

Proof. If $\tilde{A} \leq \tilde{B}$ then $a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}, a_{ij\nu L} \geq b_{ij\nu L}, a_{ij\nu U} \leq b_{ij\nu U}$
Now min $\left[a_{ij\mu L}, c_{ij\mu L}\right] \leq$ min $\left[b_{ij\mu L}, c_{ij\mu L}\right]$, min $\left(a_{ij\mu U}, c_{ij\mu U}\right) \leq$ min $\left(b_{ij\mu U}, c_{ij\mu U}\right)$,
max $\left(a_{ij\nu L}, c_{ij\nu L}\right) \geq$ max $\left(b_{ij\nu L}, c_{ij\nu L}\right)$, max $\left(a_{ij\nu U}, c_{ij\nu U}\right) \geq$ max $\left(b_{ij\nu U}, c_{ij\nu U}\right)$ for all i,j
So $\tilde{A} \wedge \tilde{C} \leq \tilde{B} \wedge \tilde{C}$. □

Theorem 6.2.7. *Let $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ be three IVFFM of the same order, and if $\tilde{A} \leq \tilde{B}$, and $\tilde{B} \wedge \tilde{C} = 0$, then $\tilde{A} \wedge \tilde{C} = 0$.*

Proof. By Theorem 6.2.6 the proof is straightforward. □

In the next section, we present a method for determining the determinant and adjoint of the IVFFM. Illustrative examples are provided to showcase the calculation of both the determinant and adjoint of this matrix.

6.3 Determinant and Adjoint of IVFFM

In this section, we define the determinant, and adjoint of the IVFFM and examine some related fundamental observations.

Definition 6.3.1. *Determinant of IVFFM Suppose $\tilde{A}$= $([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}])$ be the IVFFM of order m. Then, the determinant of $\tilde{A}$ is denoted by $|\tilde{A}|$ and defined by*

$$|\tilde{A}| = \begin{pmatrix} \vee_{h\in\psi_k}([a_{1h(1)\mu L}, a_{1h(1)\mu U}] \wedge [a_{2h(2)\mu L}, a_{2h(2)\mu U}] \cdots \wedge [a_{kh(k)\mu L}, a_{kh(k)\mu U}]), \\ \wedge_{h\in\psi_k}([a_{1h(1)\nu L}, a_{1h(1)\nu U}] \vee [a_{2h(2)\nu L}, a_{2h(2)\nu U}] \cdots \vee [a_{kh(k)\nu L}, a_{kh(k)\nu U}]). \end{pmatrix}$$

where ψ_k be the set of permutation on the set $\{1, 2, 3, \ldots, m\}$.

Example 6.3.1. *Let us consider IVFFM of order 3 as follow*

$$\tilde{A} = \begin{pmatrix} [0.45,0.7][0.55,0.75] & [0.60,0.75][0.35,0.60] & [0.40,0.55][0.65,0.80] \\ [0.65,0.70][0.4,0.65] & [0.5,0.6][0.65,0.75] & [0.6,0.65][0.4,0.6] \\ [0.7,0.8][0.3,0.6] & [0.7,0.78][0.3,0.45] & [0.50,0.55][0.5,0.7] \end{pmatrix}$$

To find the determinant of $\tilde{A}$, we need to find out all permutations on $\{1, 2, 3\}$. The permutation on $\{1, 2, 3\}$

$$\psi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \psi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \psi_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The membership degree of $|\tilde{A}|$ *is*

$$
\begin{aligned}
&([a_{1\psi_1(1)\mu L}, a_{1\psi_1(1)\mu U}] \wedge [a_{2\psi_1(2)\mu L}, a_{2\psi_1(2)\mu U}] \wedge [a_{3\psi_1(3)\mu L}, a_{3\psi_1(3)\mu U}])\\
&\vee([a_{1\psi_2(1)\mu L}, a_{1\psi_2(1)\mu U}] \wedge [a_{2\psi_2(2)\mu L}, a_{2\psi_2(2)\mu U}] \wedge [a_{3\psi_2(3)\mu L}, a_{3\psi_2(3)\mu U}])\\
&\vee([a_{1\psi_3(1)\mu L}, a_{1\psi_3(1)\mu U}] \wedge [a_{2\psi_3(2)\mu L}, a_{2\psi_3(2)\mu U}] \wedge [a_{3\psi_3(3)\mu L}, a_{3\psi_3(3)\mu U}])\\
&\vee([a_{1\psi_4(1)\mu L}, a_{1\psi_4(1)\mu U}] \wedge [a_{2\psi_4(2)\mu L}, a_{2\psi_4(2)\mu U}] \wedge [a_{3\psi_4(3)\mu L}, a_{3\psi_4(3)\mu U}])\\
&\vee([a_{1\psi_5(1)\mu L}, a_{1\psi_5(1)\mu U}] \wedge [a_{2\psi_5(2)\mu L}, a_{2\psi_5(2)\mu U}] \wedge [a_{3\psi_5(3)\mu L}, a_{3\psi_5(3)\mu U}])\\
&\vee([a_{1\psi_6(1)\mu L}, a_{1\psi_6(1)\mu U}] \wedge [a_{2\psi_6(2)\mu L}, a_{2\psi_6(2)\mu U}] \wedge [a_{3\psi_6(3)\mu L}, a_{3\psi_6(3)\mu U}])
\end{aligned}
$$

$$
= \begin{aligned}
&([a_{11\mu L}, a_{11\mu U}] \wedge [a_{22\mu L}, a_{22\mu U}] \wedge [a_{33\mu L}, a_{33\mu U}])\\
&\vee([a_{11\mu L}, a_{11\mu U}] \wedge [a_{23\mu L}, a_{23\mu U}] \wedge [a_{32\mu L}, a_{32\mu U}])\\
&\vee([a_{12\mu L}, a_{12\mu U}] \wedge [a_{21\mu L}, a_{21\mu U}] \wedge [a_{33\mu L}, a_{33\mu U}])\\
&\vee([a_{12\mu L}, a_{12\mu U}] \wedge [a_{23\mu L}, a_{23\mu U}] \wedge [a_{31\mu L}, a_{31\mu U}])\\
&\vee([a_{13\mu L}, a_{13\mu U}] \wedge [a_{21\mu L}, a_{21\mu U}] \wedge [a_{32\mu L}, a_{32\mu U}])\\
&\vee([a_{13\mu L}, a_{13\mu U}] \wedge [a_{22\mu L}, a_{22\mu U}] \wedge [a_{31\mu L}, a_{31\mu U}])
\end{aligned}
$$

$$
= \begin{aligned}
&([0.45, 0.7] \wedge [0.5, 0.6] \wedge [0.5, 0.55])\\
&\vee([0.45, 0.7] \wedge [0.6, 0.65] \wedge [0.7, 0.78])\\
&\vee([0.6, 0.75] \wedge [0.65, 0.7] \wedge [0.5, 0.55])\\
&\vee([0.6, 0.75] \wedge [0.6, 0.65] \wedge [0.7, 0.8])\\
&\vee([0.4, 0.55] \wedge [0.65, 0.7] \wedge [0.7, 0.78])\\
&\vee([0.4, 0.55] \wedge [0.5, 0.6] \wedge [0.7, 0.8])
\end{aligned}
$$

$= [0.45, 0.55] \vee [0.45, 0.65] \vee [0.5, 0.55] \vee [0.6, 0.65] \vee [0.4, 0.55] \vee [0.4, 0.55]$ =*[0.6,0.65]*

Now, the non-membership degree of $|\tilde{A}|$ *is*

$$
\begin{aligned}
&([a_{1\psi_1(1)\nu L}, a_{1\psi_1(1)\nu U}] \vee [a_{2\psi_1(2)\nu L}, a_{2\psi_1(2)\nu U}] \vee [a_{3\psi_1(3)\nu L}, a_{3\psi_1(3)\nu U}])\\
&\wedge([a_{1\psi_2(1)\nu L}, a_{1\psi_2(1)\nu U}] \vee [a_{2\psi_2(2)\nu L}, a_{2\psi_2(2)\nu U}] \vee [a_{3\psi_2(3)\nu L}, a_{3\psi_2(3)\nu U}])\\
&\wedge([a_{1\psi_3(1)\nu L}, a_{1\psi_3(1)\nu U}] \vee [a_{2\psi_3(2)\nu L}, a_{2\psi_3(2)\nu U}] \vee [a_{3\psi_3(3)\nu L}, a_{3\psi_3(3)\nu U}])\\
&\wedge([a_{1\psi_4(1)\nu L}, a_{1\psi_4(1)\nu U}] \vee [a_{2\psi_4(2)\nu L}, a_{2\psi_4(2)\nu U}] \vee [a_{3\psi_4(3)\nu L}, a_{3\psi_4(3)\nu U}])\\
&\wedge([a_{1\psi_5(1)\nu L}, a_{1\psi_5(1)\nu U}] \vee [a_{2\psi_5(2)\nu L}, a_{2\psi_5(2)\nu U}] \vee [a_{3\psi_5(3)\nu L}, a_{3\psi_5(3)\nu U}])\\
&\wedge([a_{1\psi_6(1)\nu L}, a_{1\psi_6(1)\nu U}] \vee [a_{2\psi_6(2)\nu L}, a_{2\psi_6(2)\nu U}] \vee [a_{3\psi_6(3)\nu L}, a_{3\psi_6(3)\nu U}])
\end{aligned}
$$

$$= \begin{array}{l} ([a_{11\nu L},a_{11\nu U}] \vee [a_{22\nu L},a_{22\nu U}] \vee [a_{33\nu L},a_{33\nu U}]) \\ \wedge([a_{11\nu L},a_{11\nu U}] \vee [a_{23\nu L},a_{23\nu Uta}] \vee [a_{32\nu L},a_{32\nu U}]) \\ \wedge([a_{12\nu L},a_{12\nu U}] \vee [a_{21\nu L},a_{21\nu U}] \vee [a_{33\nu L},a_{33\nu U}]) \\ \wedge([a_{12\nu L},a_{12\nu U}] \vee [a_{23\nu L},a_{23\nu U}] \vee [a_{31\nu L},a_{31\nu U}]) \\ \wedge([a_{13\nu L},a_{13\nu U}] \vee [a_{21\nu L},a_{21\nu U}] \vee [a_{32\nu L},a_{32\nu U}]) \\ \wedge([a_{13\nu L},a_{13\nu U}] \vee [a_{22\nu L},a_{22\nu U}] \vee [a_{31\nu L},a_{31\nu U}]) \end{array}$$

$$= \begin{array}{l} ([0.55,0.75] \vee [0.65,0.75] \vee [0.5,0.7]) \\ \wedge([0.55,0.75] \vee [0.5,0.6] \vee [0.3,0.45]) \\ \wedge([0.35,0.6] \vee [0.4,0.65] \vee [0.5,0.7]) \\ \wedge([0.35,0.6] \vee [0.4,0.6] \vee [0.3,0.6]) \\ \wedge([0.65,0.80] \vee [0.4,0.65] \vee [0.3,0.45]) \\ \wedge([0.65,0.80] \vee [0.65,0.75] \vee [0.3,0.6]) \end{array}$$

$$= [0.65,0.75] \wedge [0.55,0.75] \wedge [0.5,0.7] \wedge [0.4,0.6] \wedge [0.65,0.8] \wedge [0.65,0.8] = [0.4,0.6]$$

det(A)=([0.6,0.65][0.4,0.6]).

Definition 6.3.2. *Adjoint of IVFFM*
Let $\tilde{A} = (\tilde{a}_{ij}) = (<\tilde{a}_{ij\mu}, \tilde{a}_{ij\nu}>)$ *be the IVFFM of order m. Then, the adjoint of* $\tilde{A}$ *is denoted by the Adjoint*($\tilde{A}$) *and defined by*

$$Adjoint(\tilde{A}) = \quad (<\tilde{q}_{ij\mu}, \tilde{q}_{ij\nu}>)$$

where

$$\tilde{q}_{ij\mu} = \vee_{\delta \in S_{m_j m_i}} \wedge_{u \in m_j} \tilde{a}_{u\delta(u)\mu},$$

$$\tilde{q}_{ij\nu} = \wedge_{\delta \in S_{m_j m_i}} \vee_{u \in m_j} \tilde{a}_{u\delta(u)\nu}.$$

Here $m_j = \{1,2........m\}$-$\{j\}$ *and* $S_{m_j m_i}$ *is the set of all permutation of the set* m_j *over the set* m_i.

Example 6.3.2. *Let us consider IVFFM of order three as follows*

$$\tilde{A} = \begin{pmatrix} [0.45,0.7][0.55,0.75] & [0.60,0.75][0.35,0.60] & [0.40,0.55][0.65,0.80] \\ [0.65,0.70][0.4,0.65] & [0.5,0.6][0.65,0.75] & [0.6,0.65][0.4,0.6] \\ [0.7,0.8][0.3,0.6] & [0.7,0.78][0.3,0.45] & [0.50,0.55][0.5,0.7] \end{pmatrix}$$

For j=1 and i=1, $m_j = \{1,2,3\}$-$\{1\} = \{2,3\}$ *and* $m_i = \{1,2,3\}$-$\{1\} = \{2,3\}$. *The permutation of* m_i *over* m_j *are*

$$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

Now

$(a_{22\mu} \wedge a_{33\mu}) \vee (a_{23\mu} \wedge a_{32\mu})$

$= ([0.5, 0.6] \wedge [0.5, 0.55]) \vee ([0.6, 0.65] \wedge [0.7, 0.78])$
$= [0.5, 0.55] \vee [0.6, 0.65]$ =*[0.6,0.65].*

$(a_{22\nu} \vee a_{33\nu}) \wedge (a_{23\nu} \vee a_{32\nu})$
$= ([0.65, 0.75] \vee [0.5, 0.7]) \wedge ([0.4, 0.6] \vee [0.3, 0.45])$
$= [0.65, 0.75] \wedge [0.4, 0.6]$ =*[0.4,0.6].*

Calculating in the similar way, Adjoint($\tilde{A}$) is obtained as

$$Adjoint(\tilde{A}) = \begin{pmatrix} [0.6,0.65][0.4,0.6] & [0.5,0.55][0.5,0.7] & [0.6,0.65][0.5,0.6] \\ [0.6,0.65][0.5,0.6] & [0.45,0.55][0.55,0.75] & [0.45,0.65][0.55,0.75] \\ [0.65,0.7][0.4,0.65] & [0.6,0.75][0.5,0.6] & [0.6,0.7][0.4,0.65] \end{pmatrix}$$

In the next section, first, we introduce the definition of eigen interval-valued fermatean fuzzy sets and develop the algorithms for identifying the greatest and least eigen interval-valued fermatean fuzzy sets. Then a numerical example is demonstrated to illustrate the application of the same. Algorithm for the same is provided in Fig. 6.1 and 6.2.

6.4 Greatest Eigen Interval-Valued Fermatean Fuzzy Set and Least Eigen Interval-Valued Fermatean Fuzzy Set

In this section, we introduce the notion of EIVFFS and provide the necessary steps of an appropriate method for finding the GEIVFFS and LEIVFFS with the help of numerical example.

Definition 6.4.1. *An interval-valued fermatean fuzzy relation (IVFFR) R between two IVFFS X and Y defined as follows*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle | x \in X, y \in Y\},$$

where $\mu_R=[\mu_R^L, \mu_R^U]$, $\nu_R=[\nu_R^L, \nu_R^U]$ *such that* $0\leq \mu_R^U + \nu_R^U \leq 1$ *for every* $(x,y)\in(X\times Y)$.

Consider $R_1 \in (X \times Y)$ and $R_2 \in (Y \times Z)$ be two IVPFR. The following composition operators for the IVFFR R_1 and R_2 is defined by as follows:

Max-Min Composition The max-min composition operator is represented by

$$R_1 \circ R_2 = \{\langle (x_{ij}, z_{ij}), \mu_{R_1 \circ R_2}(x_{ij}, z_{ij}), \nu_{R_1 \circ R_2}(x_{ij}, z_{ij})\rangle | x_{ij} \in X, z_{ij} \in Z\}$$

$\mu_{R_1\circ R_2}(x_{ij}, z_{ij})=[\mu^L_{R_1\circ R_2}(x_{ij}, z_{ij}), \mu^U_{R_1\circ R_2}(x_{ij}, z_{ij})]$, $\nu_{R_1\circ R_2}(x_{ij}, z_{ij})=[\nu^L_{R_1\circ R_2}(x_{ij}, z_{ij}), \nu^U_{R_1\circ R_2}(x_{ij}, z_{ij})]$
$\mu^L_{R_1\circ R_2}(x_{ij}, z_{ij}) = \max_{y\in Y}\{\min_{x\in X}(\mu^L_{R_1}(x_{ij}, z_{ij}), \mu^L_{R_2}(y_{ij}, z_{ij})\}$, $\mu^U_{R_1\circ R_2}(x_{ij}, z_{ij}) = \max_{y\in Y}\{\min_{x\in X}(\mu^U_{R_1}(x_{ij}, y_{ij}), \mu^U_{R_2}(y_{ij}, z_{ij})\}$,
$\nu^L_{R_1\circ R_2}(x_{ij}, z_{ij}) = \min_{y\in Y}\{\max_{x\in X}(\nu^L_{R_1}(x_{ij}, y_{ij}), \nu^L_{R_2}(y_{ij}, z_{ij})\}$, $\nu^U_{R_1\circ R_2}(x_{ij}, z_{ij}) = \min_{y\in Y}\{\max_{x\in X}(\nu^U_{R_1}(x_{ij}, y_{ij}), \nu^U_{R_2}(y_{ij}, z_{ij})\}$

Min-Max Composition The min-max composition operator is represented by

$$R_1 \circ R_2 = \{\langle (x_{ij}, z_{ij}), \mu_{R_1 \circ R_2}(x_{ij}, z_{ij}), \nu_{R_1 \circ R_2}(x_{ij}, z_{ij}) | x_{ij} \rangle \in X, z_{ij} \in Z\}$$

$\mu_{R_1 \circ R_2}(x_{ij}, z_{ij})=[\mu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \mu^U_{R_1 \circ R_2}(x_{ij}, z_{ij})], \nu_{R_1 \circ R_2}(x_{ij}, z_{ij})=[\nu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \nu^U_{R_1 \circ R_2}(x_{ij}, z_{ij})]$
$\mu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \min_{y \in Y}\{\max_{x \in X}(\mu^L_{R_1}(x_{ij}, y_{ij}), \mu^L_{R_2}(y_{ij}, z_{ij})\}$, $\mu^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \min_{y \in Y}\{\max_{x \in X}(\mu^U_{R_1}(x_{ij}, y_{ij}), \mu^U_{R_2}(y_{ij}, z_{ij})\}$,
$\nu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \max_{y \in Y}\{\min_{x \in X}(\nu^L_{R_1}(x_{ij}, y_{ij}), \nu^L_{R_2}(y_{ij}, z_{ij})\}$, $\nu^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \max_{y \in Y}\{\min_{x \in X}(\nu^U_{R_1}(x_{ij}, y_{ij}), \nu^U_{R_2}(y_{ij}, z_{ij})\}$.

Definition 6.4.2. *Suppose R is an IVFFR defined on IVFFS of X. An IVFFS N is said to be an eigen interval-valued fermatean fuzzy set associated with the relation R if $N \odot R = N$, where $\odot$ is any of the above-defined composition operators.*

6.4.1 Greatest eigen interval-valued fermatean fuzzy set

Here, we apply the max-min composition operator for calculating the GEIVFFS with the IVFFR R. Suppose N_1 be the IVFFS, in which the degree of membership is the greatest of all elements of the column of relation R, and the degree of non-membership is the lowest of all the elements of the column of R.

$$\begin{aligned} \mu_{N_1}(u) &= \max_{x \in X} \mu_R(x, u) \forall u \in Y, \\ \nu_{N_1}(u) &= \min_{x \in X} \nu_R(x, u) \forall u \in Y. \end{aligned} \tag{6.4.1}$$

It is easy to verify that N_1 is an eigen interval-valued fermatean fuzzy set, but not the GEIVFFS always. To evaluate GEIVFFS, the following sequences are evaluated using max-min composition.

$$N_1 \circ R = N_2,$$

$$N_2 \circ R = N_1 \circ R^2 = N_3,$$

$$N_3 \circ R = N_1 \circ R^3 = N_4,$$

$$\vdots$$

$$N_n \circ R = N_1 \circ R^n = N_{n+1}.$$

Now, we give an algorithm to evaluate GEIVFFS.

Algorithm 1 (GEIVFFS)

Step 1 Calculate the set N_1 from R using the above Equation 6.4.1.

Step 2 Set the index $n = 1$ and find $N_{n+1} = N_n \circ R$.

Step 3 If $N_{n+1} \neq N_n$ then go to step 2.

Step 4 If $N_{n+1} = N_n$ then N_n is the GEIVFFS associated with R.

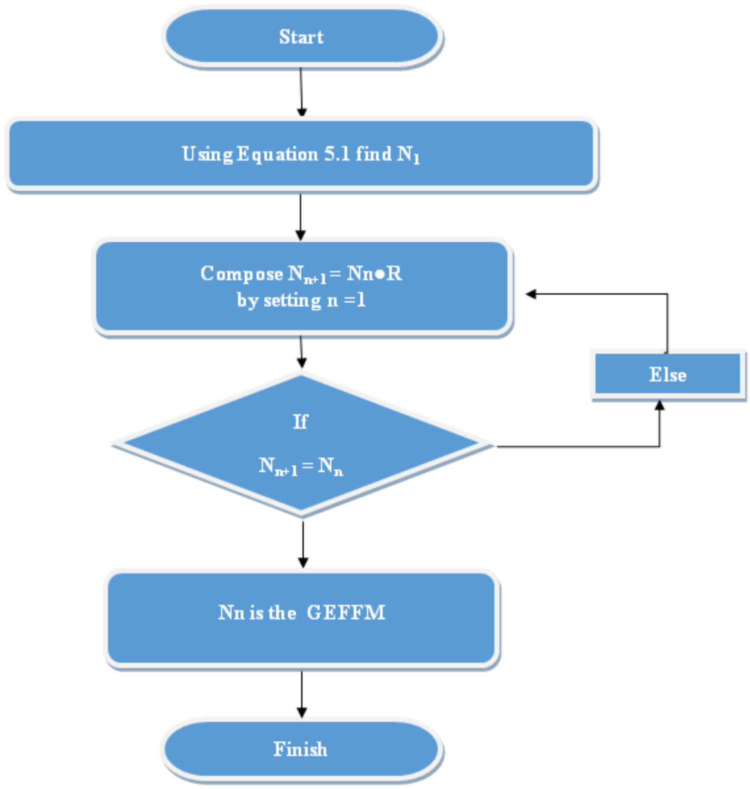

Figure 6.1: Flow Chart for Algorithm I (GEIVFFS)

Example 6.4.1. *Let $\tilde{A}$= $(\tilde{a}, \tilde{b}, \tilde{c})$ be the IVFFM and R be the interval-valued fermatean fuzzy relation on $\tilde{A}$ represented as follows.*

$$R = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} [0.4,0.5][0.19,0.23] & [0.71,0.79][0.10,0.24] & [0.21,0.3][0.34,0.44] \\ [0.27,0.43]\ [0.16,0.23] & [0.42,0.51]\ [0.07,0.12] & [0.14,0.23]\ [0.3,0.4] \\ [0.35,0.39]\ [0.06,0.21] & [0.19,0.31]\ [0.49,0.59] & [0.48,0.57]\ [0,0.07] \end{pmatrix} \end{array}$$

We solve this as follows:

step 1

$$N_1 = \{\langle([0.4,0.5][0.06,0.21]),([0.71,0.79][0.07,0.12]),([0.48,0.57][0,0.07])\rangle\}$$

step 2 for n=1

$$N_2 = N_1 \circ R$$

$$N_2 = \{\langle([0.4,0.5][0.06,0.21]),([0.42,0.51[0.07,0.12]),([0.48,0.57][0,0.07])\rangle\}$$

step 3 Since

$$N_2 \neq N_1$$

n=2 in step 2 and compose N_2 with R, to get N_3 i.e

$$N_3 = N_2 \circ R$$

$$N_3 = \{\langle([0.40,0.50][0.06,0.21]),([0.42,0.51][0.07,0.12]),([0.48,0.57][0,0.07])\rangle\}$$

step 3 Since N_3=N_2, thus N_2 is the greatest eigen interval-valued fermatean fuzzy set.

6.4.2 Least eigen interval-valued fermatean fuzzy set

Here, we apply the max-min composition operator for calculating the LEIVFFS with the IVFF relation R. Suppose N_1 be the IVFFS, in which the degree of membership, and the degree of non-membership is the greatest of all the elements of the column of R.

$$\begin{aligned}\mu_{N_1}(u) &= \min_{x\in X}\mu_R(x,u)\forall u \in Y,\\ \nu_{N_1}(u) &= \max_{x\in X}\nu_R(x,u)\forall u \in Y.\end{aligned} \tag{6.4.2}$$

We can easily find that N_1 is an eigen interval-valued fermatean fuzzy set, but our focus is to find LEIFPFS. We define the sequence of IVFFS N_n such that

$$N_1 \circ R = N_2,$$

$$N_2 \circ R = N_1 \circ R^2 = N_3,$$

$$N_3 \circ R = N_1 \circ R^3 = N_4,$$

$$\vdots$$

$$N_n \circ R = N_1 \circ R^n = N_{n+1}.$$

For the determination of the LEIVFFS, we now present the following algorithm followed by a numerical example along with the real-life application of the defined GEIVFFS and LEIVFFS.

Algorithm II (LEIVFFS)

Step 1 Calculate the set N_1 from R using above Equation 6.4.2.

Step 2 Set the index n=1 and find $N_{n+1} = N_n \circ R$.

Step 3 If $N_{n+1} \neq N_n$ then go to step 2.

Step 4 If $N_{n+1} = N_n$ then N_n is the LEIVFFS associated with R.

We consider the same Example 6.4.1 for the illustration of the computational steps of Algorithm II as below:

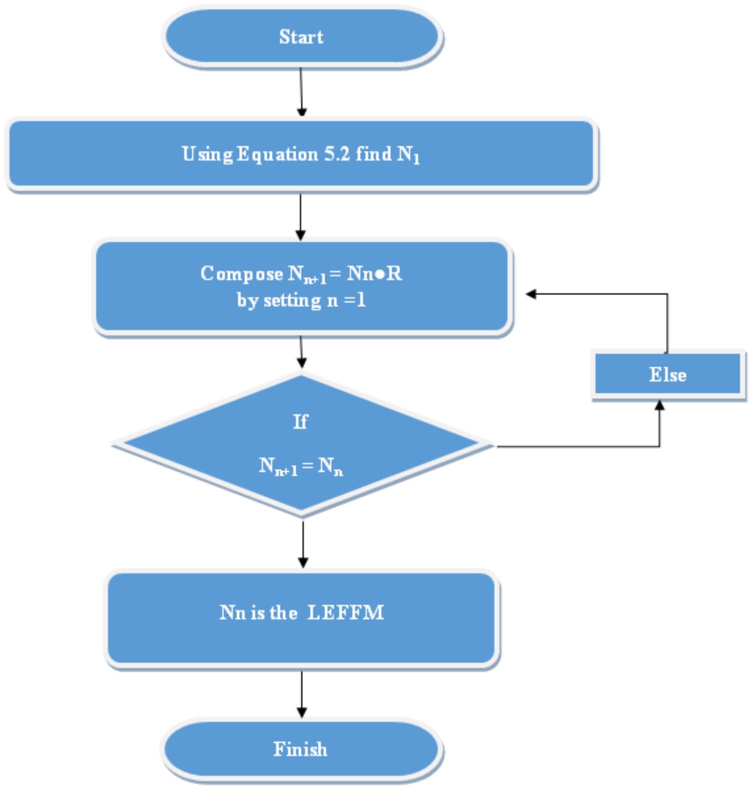

Figure 6.2: Flow Chart for Algorithm II (LEIVFFS)

step 1

$$N_1 = \{\langle([0.27,0.39][0.19,0.23]),([0.19,0.31][0.49,0.59]),([0.14,0.23][0.34,0.44])\rangle\}$$

step 2 for n=1

$$N_2 = N_1 \circ R$$

$$N_2 = \{\langle([0.27,0.39][0.19,0.23]),([0.27,0.39][0.19,0.24]),([0.21,0.3][0.34,0.44])\rangle\}$$

step 3 Since

$$N_2 \neq N_1$$

n=2 in step 2 and compose N_2 with R, to get N_3 i.e

$$N_3 = N_2 \circ R$$

$$N_3 = \{\langle([0.27,0.39][0.19,0.23]),([0.27,0.39][0.19,0.24]),([0.21,0.3][0.3,0.4])\rangle\}$$

step 4 Since

$$N_3 \neq N_2$$

n=3 in step 3 and compose N_3 with R, to get N_4 i.e

$$N_4 = N_3 \circ R$$

$$N_4 = \{\langle([0.27, 0.39][0.19, 0.23]), ([0.27, 0.39][0.19, 0.24]), ([0.21, 0.3][0.3, 0.4])\rangle\}$$

step 5 Since N_4=N_3, thus N_3 is the least eigen interval-valued fermatean fuzzy set associated with R.

6.4.3 MCDM using GEIVFFS and LEIVFFS

Suppose an online grocery company where customer satisfaction/non-satisfaction levels are taken into account for formulating the multiple criteria decision-making problems.

Example 6.4.2. *Consider a grocery company that takes interviews from 10 of its most valuable clients or industry professionals to learn about the key aspects of the business. Let the characteristic be listed as:*

- X_1 : Location
- X_2 : Curated Assortment
- X_3: Prepared Foods

A survey may be used to determine the customer's feedback. However, we assume a set of data presented below without conducting an exhaustive survey in order to illustrate the suggested methodology.

For the purpose of evaluating some final observations in the grocery company's perpective, we assume that each customer's feedback as an interval-valued fermatean fuzzy information that is relative to all are available in Table 6.1, 6.2, and 6.3.

The desire levels can be estimated as satisfaction/non-satisfaction levels possible by considering the interval-valued fermatean fuzzy relation. Each pair in the relation R(X_j, X_k) has two values that range from 0 to 1; a membership value (satisfied) and the non-membership value (not satisfied) is given by

$$R_{(X_j,X_k)} = \left(\frac{\sum_{p=1,q=1}^{p=m,q=n} \mu_{pq}}{m}, \frac{\sum_{p=1,q=1}^{p=m,q=n} \nu_{pq}}{m} \right); \tag{6.4.3}$$

and

$$R_{(X_i,X_i)} = \frac{R_{(X_i,X_j)} + R_{(X_i,X_k)}}{2}; \tag{6.4.4}$$

where $j, k = 1, 2, ..., n$.

Using equations 6.4.3 and 6.4.4 the membership and non-membership values for the different pairs of features have been computed as follows:

$R_{(X_1,X_1)}$ = ([0.52,.65], [0.39,.58]), $R_{(X_1,X_2)}$ = ([0.54,.72],[.43, 0.67]), $R_{(X_1,X_3)}$ = ([.46,.58],[.35,0.49]) ;
$R_{(X_2,X_1)}$ = ([0.57,.62], [0.47,.59]) , $R_{(X_2,X_2)}$ = ([0.48,.58], [0.38,.49]),$R_{(X_2,X_3)}$ =([0.57,.62], [0.47,.59]) ;
$R_{(X_3,X_1)}$ = ([0.4,.55], [0.35,.5]) , $R_{(X_3,X_2)}$ = ([0.48,.59], [.31,.44]), $R_{(X_3,X_3)}$ =([0.44,.57], [0.33,.47]).

We construct the interval-valued fermatean fuzzy relation R using the above obtained interdependency of the features as follows:

$$R = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{pmatrix} R_{(X_1,X_1)} & R_{(X_1,X_2)} & R_{(X_1,X_3)} \\ R_{(X_2,X_1)} & R_{(X_2,X_2)} & R_{(X_2,X_3)} \\ R_{(X_3,X_1)} & R_{(X_3,X_2)} & R_{(X_3,X_3)} \end{pmatrix} \end{array}$$

Setting all the values, we get

$$R = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{pmatrix} ([0.52,.65],[0.39,.59]) & ([0.54,.72],[.43,0.67]) & ([.46,.58],[.35,0.49]) \\ ([0.57,.62],[0.47,.59]) & ([0.48,.58],[0.38,.49]) & ([0.57,.62],[0.47,.59]) \\ ([0.4,.55],[0.35,.5]) & ([0.48,.59],[.31,.44]) & ([0.44,.57],[0.33,.47]) \end{pmatrix} \end{array}$$

Now, we use the first proposed algorithm for finding the greatest eigen interval-valued fermatean fuzzy set, i.e., Algorithm I (GEIVFFS) as ([0.54,.65] [0.35,.5], [0.54,.65][.33, 0.47] , [.57,.62][.33,0.47]).

Further, we use the first proposed algorithm for finding the least eigen interval-valued fermatean fuzzy set, i.e., Algorithm II (LEIVFFS) as ([0.48,.58] [0.47,.59], [0.48,.58][.43, 0.59] , [.48,.58][.47,0.59]).

Observations and Results:

Based on calculations, we discovered that the greatest and least interval-valued fermatean fuzzy set is given by

$$GEIVFFS = ([0.54,.65][0.35,.5],[0.54,.65][.33,0.47],[.57,.62][.33,0.47]),$$

$$LEIVFFS = ([0.48,.58][0.47,.59],[0.48,.58][.43,0.59],[.48,.58][.47,0.59])$$

respectively. The results from these show the range of levels of satisfaction/non-satisfaction for the features that the grocery company is considering.

- Regarding feature X_1, Customers are between (48% to 65%) satisfied and between (47% to 59%) unsatisfied.

- Regarding feature X_2, Customers are between (48% to 65%) satisfied and between (33% to 59%) unsatisfied.

- Regarding feature X_3, Customers are between (48% to 62%) satisfied and between (33% to 59%) unsatisfied.

It should be noticed that the numerical results from the GEIVFFS and LEIVFFS are reasonably close to one another. In actuality, the proposed algorithms have been illustrated using a specific case (Example 5.2) that has a constrained format and less variety in terms of the dimensions and attributes involved. We might see a sizable fluctuation in the values if we have vast data with higher dimensionality of features. However, the similarity of the results indicates accuracy in the decision-making process.

Table 6.1: Relative feedback with X_1 and X_2

Customers/experts	X_1	X_2
P_1	([0.7,.8], [0.4,.65])	([0.6,.75],[.35,.7])
P_2	([0.4,.65], [0.35,.6])	([0.57,.65],[.4,.5])
P_3	([0.5,.55], [0.5,.7])	([0.5,.55],[.6,.65])
P_4	([0.35,.55], [0.45,.65])	([0.5,.65],[.25,.4])
P_5	([0.55,.7], [0.4,.5])	([0.4,.55],[.6,.8])
P_6	([0.6,.65], [0.35,.6])	([0.45,.55],[.6,.7])
P_7	([0.3,.45], [0.55,.7])	([0.55,.6],[.6,.7])
P_8	([0.55,.65], [0.5,.7])	([0.65,.7],[.45,.6])
P_9	([0.1,.5], [0.2,.3])	([0.6,.7],[.3,.5])
P_{10}	([0.7,.9], [0.1,.2])	([0.4,.5],[.2,.3])

Table 6.2: Relative feedback with X_1 and X_3

Customers/experts	X_1	X_3
P_1	([0.5,.55], [0.5,.7])	([0.7,.78],[.3,.45])
P_2	([0.7,.8], [0.3,.6])	([0.45,.7],[.55,.75])
P_3	([0.65,.7], [0.4,.65])	([0.6,.75],[.35,.6])
P_4	([0.4,.55], [0.65,.8])	([0.6,.65],[.4,.6])
P_5	([0.6,.65], [0.4,.6])	([0.7,.78],[.3,.45])
P_6	([0.7,.8], [0.3,.6])	([0.65,.7],[.4,.65])
P_7	([0.45,.65], [0.55,.75])	([0.6,.75],[.35,.5])
P_8	([0.65,.8], [0.4,.55])	([0.68,.75],[.45,.55])
P_9	([0.65,.7], [0.4,.65])	([0.3,.5],[.1,.2])
P_{10}	([0.4,.5], [0.1,.2])	([0.1,.2],[.3,.4])

Comparative Remarks of EIVFFS:

The idea of eigen interval-valued fermatean fuzzy set is a generalization of eigen fuzzy set and eigen intuitionistic fuzzy set. The eigen interval-valued fermatean fuzzy set gives us the advantage to handle the uncertain situation in a wider sense. Below is a detailed summary of some significant comparative points and advantages of EIVFFS.

(1) The existing sets have their own limitations that prevent them from being able to capture all of the information, so the EIVFFS addresses the missing components of fuzzy sets and intuitionistic fuzzy sets.

(2) We use the concept of eigen fuzzy sets in the study of the effect produced by an action performed on entities to safeguard them or improve their performances (for example, a restoration or maintenance intervention on a damaged or degraded building, or a medical treatment prescribed to a patient to eradicate a disease) and also evaluate the effectiveness of a drug in making a symptom of a disease disappear in patients.

(3) The limitations of the criterion in the existing research on eigen fuzzy sets and eigen intuitionistic fuzzy

Table 6.3: Relative feedback with X_2 and X_3

Customers/experts	X_2	X_3
P_1	([0.4,.5], [0.19,.23])	([0.71,.79],[.1,.24])
P_2	([0.21,.3], [0.34,.44])	([0.14,.23],[.3,.4])
P_3	([0.27,.43], [0.16,.23])	([0.42,.51],[.07,.12])
P_4	([0.1,.2], [0.3,.4])	([0.5,.6],[.6,.7])
P_5	([0.8,.9], [0.1,.2])	([0.1,.2],[.3,.5])
P_6	([0.7,.8], [0.3,.4])	([0.3,.5],[.6,.7])
P_7	([0.3,.4], [0.1,.3])	([0.6,.8],[.1,.2])
P_8	([0.7,.85], [0.3,.4])	([0.4,.65],[.7,.8])
P_9	([0.3,.4], [0.2,.3])	([0.1,.2],[.3,.4])
P_{10}	([0.6,.65], [0.75,.8])	([0.3,.5],[.1,.2])

sets experts/decision-makers do not allow the membership and non-membership value as an interval of their own choice. Anyhow, this limits the ability of the decision-makers to offer their opinion in a specific field. However, the defined EIVFFS offers the platform to take the value as an interval form.

(4) The implementation of the EIVFFS and the approach suggested for the problem of the online Grocery business in section 6.4 demonstrate how well and consistently the proposed work addresses the extended framework. In another way, the GEIVFFS and LEIVFFS of the IVFFS have been used to calculate the approximate optimal level.

6.5 Application in Decision-Making

In this section, we propose the score function and discuss the application of IVFFM in decision-making.

Definition 6.5.1. *Let $A = \langle [\mu^-, \mu^+], [\nu^-, \nu^-] \rangle$ be an interval-valued fermatean fuzzy number (IVFFN). The score function S(A) and accuracy function H(A) are*

$$S(A) = \frac{(\mu^-)^3 + (\mu^+)^3 - (\nu^-)^3 - (\nu^+)^3}{4} \left|\mu^- + \mu^+ - \nu^- - \nu^+\right| \in [-1, 1]. \tag{6.5.1}$$

$$H(A) = \frac{(\mu^-)^3 + (\mu^+)^3 + (\nu^-)^3 + (\nu^+)^3}{2} \in [0, 1]. \tag{6.5.2}$$

In this if $\mu^- = \mu^+$ and $\nu^- = \nu^+$ then the score function of IVFFN will become the score function of FFN. So, the proposed score function of FFN is as follows:

$$S(A) = (\mu^3 - \nu^3) \left|\mu - \nu\right| \in [-1, 1].$$

For any two IVFFN A_1 and A_2,

1. if $S(A_1) > S(A_2)$, then $A_1 > A_2$.

2. if $S(A_1) < S(A_2)$, then $A_1 < A_2$.
3. if $S(A_1) = S(A_2)$, then
   (a) if $H(A_1) > H(A_2)$, then $A_1 > A_2$
   (b) if $H(A_1) < H(A_2)$, then $A_1 < A_2$
   (c) if $H(A_1) = H(A_2)$, then $A_1 = A_2$.

One can verify the following properties for the proposed score function.

1. Let $A = \langle [1,1],[0,0] \rangle$ be an IVFFN, then $S(A) = 1$.
2. Let $A = \langle [0,0],[1,1] \rangle$ be an IVFFN, then $S(A) = -1$.
3. Let $A = \langle [0,0],[0,0] \rangle$ be an IVFFN, then $S(A) = 0$.

Statement of problem under decision-making

Here, we have checked how interval-valued fermatean fuzzy relations can be used for medical diagnosis. P denotes a set of varieties of paddy plant, D denotes a set of plant diseases, and S be the common symptoms. We define interval-valued fermatean fuzzy medical knowledge as an interval-valued fermatean fuzzy relation R between a set of plant diseases D, and a set of symptoms S that reveals the degree of a positive and negative association between a set of plant diseases and common symptoms. Let's talk about interval-valued fermatean fuzzy medical diagnosis now. The methodology includes the following three jobs.

(I) Identifying relationships between paddy plants and plant diseases: this relationship expresses the correspondence between plants and diseases.

(II) Creating relationships between diseases and common symptoms: this is a role in the medical knowledge base diagnosis.

(III) Making diagnoses for all patients based on the composition of the relationships: the relationship between plants and common symptoms is composed of relations of the two above relations.

Let $R \in$ IVFFR$(P \times D)$ and $Q \in$ IVFFR$(D \times S)$, clearly the composition T of R and Q i.e $T = R \circ Q$ describes the state of diagnosis. for the sample, the state of plants can be defined as a max-min composition relation T from P to D: $\mu_T(p,s) = \bigvee_{p \in P} \{\mu_Q(P,D) \wedge \mu_R(D,S)\}$

$\nu_T(p,s) = \bigwedge_{p \in S} \{\nu_Q(P,D) \vee \nu_R(D,S)\}$ for all $p_i \in P$ and $s_i \in$ S.

(IV) Ranking of alternatives based on score values of relative closeness value.

Example 6.5.1. *Let the set of paddy plant P=$\{P_1,P_2,P_3,P_4\}$ is to be diagnosis with respect to a set of plant diseases D= $\{D_1,D_2,D_3,D_4\}$ and set of common symptoms S = $\{S_1,S_2,S_3,S_4\}$.*

- A hypothetical relation R_1 $(P \to D)$ is given in Table 6.4

- A hypothetical relation R_2 ($D \rightarrow S$) is given in Table 6.5

- The composition relation R ($P \rightarrow S$) is given in Table 6.6

- The degree of affiliation between set of plants P_i to a set of common symptoms S_i is calculated in Table 6.7.

Table 6.4 shows us the relation between a set of paddy plant P and plant diseases D in the form of IVFFN. It defines how strongly paddy plant and plant diseases are related. Table 6.5 shows us the relation between plant diseases D and common symptoms S. These relationships define how much extent plant diseases D needs the common symptoms S

Table 6.4: Relation between set of paddy plant P and plant diseases D

R_1	D_1	D_2	D_3	D_4
P_1	[0.45, 0.7] [0.55, 0.75]	[0.60, 0.75] [0.35, 0.60]	[0.40, 0.55] [0.65, 0.80]	[0.35, 0.45] [0.75, 0.85]
P_2	[0.65, 0.70] [0.4, 0.65]	[0.3, 0.52] [0.65, 0.75]	[0.6, 0.65] [0.4, 0.6]	[0.40, 0.55] [0.65, 0.80]
P_3	[0.7, 0.8] [0.3, 0.6]	[0.6, 0.68] [0.3, 0.7]	[0.50, 0.8] [0.5, 0.65]	[0.5, 0.65] [0.45, 0.7]
P_4	[0.2, 0.3] [0.8, 0.9]	[0.7, 0.8] [0.3, 0.65]	[0.1, 0.3] [0.5, 0.7]	[0.40, 0.7] [0.65, 0.80]

Table 6.5: Relation between plant diseases D and common symptoms S

R_2	S_1	S_2	S_3	S_4
D_1	[0.1, 0.3] [0.55, 0.97]	[0.40, 0.5] [0.3, 0.45]	[0.7, 0.9] [0.55, 0.65]	[0.15, 0.25] [0.65, 0.80]
D_2	[0.65, 0.75] [0.4, 0.64]	[0.65, 0.75] [0.3, 0.52]	[0.6, 0.8] [0.4, 0.7]	[0.65, 0.75] [0.35, 0.7]
D_3	[0, 0] [0.3, 0.6]	[0.7, 0.8] [0.3, 0.55]	[0.35, 0.45] [0.5, 0.71]	[0.40, 0.55] [0.65, 0.7]
D_4	[0.4, 0.55] [0.3, 0.45]	[0.5, 0.8] [0.6, 0.75]	[0.50, 0.55] [0, 0]	[0.40, 0.65] [0.55, 0.7]

Table 6.6: The composition relation R (P $\rightarrow$ S)

$$R = R_1 \odot R_2$$

R	S_1	S_2	S_3	S_4
P_1	[0.65, 0.75] [0.4, 0.65]	[0.4, 0.55] [0.35, 0.6]	[0..69, 0.75] [0.4, 0.7]	[0.6, 0.75] [0.35, 0.75]
P_2	[0.4, 0.55] [0.4, 0.6]	[0.6, 0.65] [0.3, 0.52]	[0.65, 0.7] [0.5, 0.65]	[0.3, 0.52] [0.65, 0.7]
P_3	[0.6, 0.68] [0.4, 0.64]	[0.5, 0.8] [0.3, 0.52]	[0.7, 0.8] [0.3, 0.6]	[0.6, 0.68] [0.35, 0.7]
P_4	[0.15, 0.68] [0.4, 0.65]	[0.4, 0.7] [0.3, 0.65]	[0.6, 0.8] [0.4, 0.7]	[0.65, 0.75] [0.35, 0.7]

Table 6.7: How much extent set of paddy plant P needs the common symptoms S

R	S_1	S_2	S_3	S_4
P_1	**0.0313**	0	0.0291	0.0280
P_2	0.001620	**0.0347**	0.013	-0.0596
P_3	0.0122	0.0563	**0.0918**	0.0083
P_4	- 0.00114	0.0039	0.024	**0.027**

The analysis of score value i.e. degree of closeness value, is given in Table 6.7. Clearly if the doctor agrees, the plant P_1 suffer from S_1, plant P_2 suffer from S_2, plant P_3 suffer from S_3 and plant P_4 suffer from S_4.

6.6 Comparative Study and Results

The information was taken in a fuzzy sense in previous papers on fuzzy decision-making. When there are additional sorts of uncertainty in data, present strategies are ineffective in dealing with them. In these instances, data should be gathered or displayed in the form of an interval-valued fermatean fuzzy meaning. In such instances, the currently developed process plays an important role in a successful conclusion.

Silambarasan [144] worked on the fermatean fuzzy matrix. In the fermatean fuzzy matrix, the membership and non-membership degrees as a point. In our work, we consider membership and non-membership values as an interval, because, in some cases, it is difficult to measure the degree of membership and non-membership values as a point. That's why, we consider membership and non-membership values as an interval, and it is practically useful in the case of real-life problems. So, our study is an extension of the study of Silambarasan [144]. Khalaf [117] to solve the medical diagnosis problem under interval-valued intuitionistic fuzzy sets by using max-min-max composition. An interval-valued intuitionistic fuzzy score for each attribute was calculated and on the basis of that decision, was provided. The present study is an extension of previous existing work.

However, in the current technique, the matrices entries that are considered are interval-valued fermatean fuzzy values. Over a set of universes, interval-valued fermatean fuzzy sets are extracted from them. The final matrix is then calculated using the proposed score function between two interval-valued fermatean fuzzy sets, yielding a decision. It is not necessary to perform many different sorts of computations in order to implement the steps of this approach; i.e. the method is not complicated to implement. As a result, developing algorithms and computer programming for this method is relatively simple. Furthermore, the data points used here are capable of tolerating a wider range of information ambiguity. This research can be thought of as a study in an advanced fuzzy sense because the interval-valued fermatean fuzzy concept is a generalization of the fuzzy concept.

In summary, the study of IVFFM and EIVFFS yields significant advantages in addressing real-world challenges, particularly in the context of medical diagnosis and online grocery companies. These concepts provide valuable tools and insights for various applications, making them essential components in contemporary research and practical problem-solving scenarios. The following is a detailed list of some substantial advantages of using IVFFM and EIVFFS:

1. From Table 6.8, the existing FM, IVFM, IFM, IVIFM, and FFM each have shortcomings that prevent them from fully capturing the information. The IVFFM effectively fills the gaps left by other matrices and offers a more flexible and versatile approach to expressing opinions and relationships within

the data. The IVFFM combines the benefits of both interval-valued and fermatean fuzzy concepts, making it a powerful tool for dealing with uncertainties and complexities. The IVFFM offers a more robust framework to handle various real-life scenarios and decision-making processes by representing membership, and non-membership degrees as intervals.

2. We can also see the drawback in the eigen fuzzy sets and eigen intuitionistic fuzzy sets experts/decision-makers bind their input in a certain area. However, the proposed EIVFFS presents a significant impact due to its ability to offer a generalization feature. This unique characteristic allows for a more comprehensive and versatile representation of uncertain information, empowering decision-makers to make more informed and flexible judgments in various contexts.

3. The implementation of the EIVFFS, IVFFM, and the approach suggested for the problems of medical diagnosis and online grocery companies in Section 4.4 and Section 4.5 demonstrate how well and consistently the proposed work addresses the extended framework. The observations indicate that the IVFFM is the most generalized structure among all fuzzy matrix models.

The detailed analysis presented in Table 6.8 further compares the proposed work and existing research available in the literature.

Table 6.8: Comparison of the proposed work with existing literature

Characteristics Methods	Whether consider MD	Whether consider MD more flexible, i.e., IVMD	Whether consider MD or NMD	Whether consider MD or NMD more flexible, i.e., IVMD or IVNMD
Thomason [16]	✓	×	×	×
Pal [24]	✓	✓	×	×
Pal et al. [131]	✓	✓	✓	×
Silambarasan & Sriram [30]	✓	✓	✓	×
Silambarasan [144]	✓	✓	✓	✓
Proposed work	✓	✓	✓	✓

⋆ MD-Membership degree, NMD-Non membership degree, IVMD-Interval-valued membership degree, IVNMD-Interval valued non membership degree.

6.7 Concluding Remarks

The exploration of matrix theory has made significant contributions to various applicable fields. In this work, we have introduced the concept of IVFFM along with its essential definitions and theorems. Additionally, we have defined the determinant and adjoint of IVFFM and studied relevant results. The formal definition of an EIVFFS for interval-valued fermatean fuzzy relations has been presented, and algorithms for determining the GEIVFFS and LEIVFFS using max-min and min-max composition operators have been provided. To illustrate these algorithms, numerical examples have been included. The application of GEIVFFS and LEIVFFS in decision-making problems has been successfully demonstrated. Moreover, we have demonstrated the application of IVFFM in decision-making by introducing a score function to solve such problems effectively. The limitation of IVFFM is related to the representation of degrees of membership and non-membership as interval numbers. The limitation arises when the sum of the upper degree of membership, and upper degree of non-membership exceeds the interval [0, 1]. The current study will help researchers interested in further developing and generalizing our findings in the context of other types of data sets. Also, we can extend in the field of image information retrieval, genetic algorithm for image reconstruction, and outlines to introduce the notion of interval-valued eigen fermatean fuzzy soft sets/soft matrices have been briefly stated for further research.

Conclusion, Future Scope and Social Impact

Many practical situations certainly deal with uncertainty and vagueness and further increase the complexity of the problem. Using fuzzy models to deal with such uncertainty has resulted in unsatisfactory results. There are different computational models based on imprecise information that accomplish the process. Further, matrix theory is essentially the study of strategic communication of information through language in a rigorous and stylized way based on the uncertain term set. It is anticipated that within the traditional setup of LPP and decision-making, the payoffs are well-known with certainty. Nevertheless, getting to the practical world, the assumption of certainty is not meaningful on several occasions. The postulations made for the exact payoffs can be considered the stringent ideology in the real-world scenario involving uncertain and ambiguous information. Imprecision and uncertainty have been incorporated into decision theory by using various frameworks like fuzzy, fuzzy matrix, probability, etc. Hence it is necessary to study methodologies that can handle the uncertainty prevailing in optimization and decision-making. The primary purpose of the work done in this thesis is to handle the vagueness and impreciseness prevailing in matrix theory in the form of fuzzy variables. An ample literature survey is performed, and based on this review, different methodologies have been defined to deal with LPP and matrix having imprecise information.

Chapter 2 addresses the transportation problem under interval-valued Pythagorean fuzzy and spherical fuzzy set, for which a theory of LPP with uncertain parameters is established. The use of IVPyFS and SFS to represent LPP problems has shown to be a powerful approach. A methodology to solve such LPP is provided in this thesis, and this methodology is sufficient to evaluate the TP and optimize the solution of transportation problems. However, this work is based on uncertain parameters that are distributed. But in real-life scenarios, this is not always the case that the data is precise.

Hence to describe the assignment problem, two methodologies are designed in Chapter 3 to address AP with costs or utilities that are uncertain or imprecise due to fuzziness.

A fuzzy decision matrix is a tool used in decision-making processes to evaluate alternatives based on multiple criteria, considering uncertainty and imprecision. It consists of rows representing decision alternatives and columns representing decision criteria, with each cell containing fuzzy membership values indicating the degree of satisfaction or preference of each alternative for each criterion. Many researchers are excited about using matrix theory to help solve decision-making problems. Lately, they've been figuring out new ways to make matrices even better. But, in the research we've seen, there hasn't been much about using matrix theory for decision-making with interval-valued picture fuzzy environment.

Then, in Chapter 4, a concept of a matrix to represent imprecise information more accurately. A novel concept of the interval-valued picture fuzzy matrix based on a picture fuzzy matrix is proposed. The innovation of the model overcomes the inherent limitation of the existing matrices and is competent in producing new matrices according to the picture fuzzy theory. Further, based on the defined concept, a generalized spherical fuzzy matrix is introduced to create a mapping between imprecise concepts and numerical intervals. The proposed generalized matrix creates a map between the imprecise data and numerical interval values. Based on the spherical fuzzy matrix, interval-valued spherical fuzzy matrices are transformed, which overcomes the loss of information existing in the spherical fuzzy matrix models where imprecise variables are transformed into a specific numerical value. The proposed approach can be perceived as a convenient technique for MCDM problems.

Chapter 5 presents the concept of the interval-valued spherical fuzzy matrix and its applications in multi-attribute decision-making processes. The limitation arises in the concept of the interval-valued spherical fuzzy matrix when the sum of the upper degree of membership, neutral membership, and upper degree of non-membership exceeds the interval [0, 1]. In many fermatean fuzzy matrices theoretic environments, the precise, crisp values are not always easy to gather as real-life decision-making problems are not inevitably precise and symmetrically distributed in nature. Hence there is a need to explore fermatean matrix having imprecise information. To thoroughly capture the uncertainty involved in the fermatean fuzzy matrix of such decision-making problems, they are converted to the Pythagorean fuzzy matrix that describes the uncertainties of information by continuously implementing the membership degree in interval numbers. Hence this model overcomes the inherent limitation of the existing fermatean fuzzy matrix where only the fuzziness of concepts is evaluated by membership degree as a point. This procedure overcomes the loss of information occurring in the process of FFM. The real-life applications of the defined model are shown to prove its consistency and applicability in the practical world.

Then, Chapter 6 introduces the concept of interval-valued fermatean fuzzy matrix and its application. The IVFFM is outlined by an interval membership degree and an interval non-membership degree. By using IVFFM, the DMs can consider an interval hesitancy degree, where they cannot simply convey their perception using one interval term. It is the newest tools for dealing with imprecision.

Since the inception of uncertain computational models, it is foreseen that a lot of research has been dedicated to these extensions of fuzzy sets and fuzzy matrices in decision-making theory. Our research is also fully dedicated to the development of such LPP problems and fuzzy matrices that have uncertainty prevailing in them. Although this research is advantageous and novel in its own way, there is always a scope for future work. In the near future, we look forward to extending the methodology developed for transportation problems under IVPFS and IVSFS. Moreover, further study can also be focussed on constructing the assignment problems for extension of the Pythagorean fuzzy sets. Fuzzy matrices, a key tool in decision-making, represent uncertain and vague information using fuzzy sets. Decision-making requires solving at every possible system state in every phase. As a result, many fuzzy matrix must be computed over various time phases. Further, The current study will help researchers interested in further developing and generalizing our findings in the context of other types of data sets. Also, we can extend in the field of image information retrieval, genetic algorithm for image reconstruction, and outlines to introduce the notion of interval-valued eigen picture fuzzy soft sets/soft matrices have been briefly stated for further research. Moreover, the efficient resolution of transportation and assignment issues can have a vast array of social effects, including economic efficiency, environmental sustainability, social equity, and disaster response. Organizations and governments can enhance service delivery, improve resource allocation, and ultimately benefit society as a whole by employing optimization techniques to these problems. Also, the social impact of IVPFM, IVSFM, and IVFFM is derived from its capacity to minimize uncertainties, facilitate improved decision-making, and encourage inclusivity and fairness in a variety of sectors. It is probable that the potential to effect positive societal change will increase as research in this discipline grows and applications expand.

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A NOVEL SIMILARITY MEASURE AND SCORE FUNCTION OF PYTHAGOREAN FUZZY SETS AND THEIR APPLICATION IN ASSIGNMENT PROBLEM

Abstract. *In real-life problems, things are imprecise because of imprecision/inaccuracy, and the exact value of the measured quantities is impossible to get. Sometimes, due to time pressure/ incomplete knowledge, it is difficult for the decision-makers to provide their opinion. To describe the imprecision, the information in terms of the fuzzy is provided to allow the decision-makers to express their inputs freely. There is a valuable role of the fuzzy set (FS) and intuitionistic fuzzy set (IFS) to describe uncertainty under the uncertain situations. In the literature, various models are available for assignment problems under fuzzy sets and intuitionistic fuzzy sets. The Pythagorean fuzzy set (PFS) has a larger domain space than the intuitionistic fuzzy set to describe the membership grade. To handle the uncertainty in practical applications of assignment problems (AP), we have proposed a method to solve the Pythagorean fuzzy assignment problem (PFAP) using the proposed similarity measure and a score function. Numerical examples are given to explain the methodology.*

Keywords*: Pythagorean Fuzzy Set, Similarity Measure, Score Function, Assignment Problem*

JEL Classification: C02, C44, C45, C60, C61, C62

1. Introduction

An assignment problem is a linear programming problem (LPP) that deals with allocation and scheduling. The problem of assignment arises because available resources have varying degrees of efficiency for performing different activities. In classical assignment problems, it is assumed that the decision-maker is sure about the precise value of the cost of the assignment problem but, practically, these factors are imprecise.

Zadeh (1965) introduced the fuzzy set (FS) to deal with uncertainty in real-life problems. Gurukumaresan et al. (2020) used the centroid method for the solution of the fuzzy assignment problem. Tsai et al. (1999) worked on the multiobjective fuzzy deployment of manpower. Chanas et al. (1984) took the demand and supply as the fuzzy numbers in the transportation problem and used parametric programming for solving the problem. Verma & Merigó (2019) worked on generalised similarity measures for Pythagorean fuzzy sets and their applications to multiple attribute decision-making. Kumar and Gupta (2011) solved the fuzzy assignment problems and fuzzy travelling salesman problems with different membership functions. Yuen and Ting (2012) performed the textbook selection using the fuzzy PROMETHEE II method. Thakre et al. (2018) worked on the placement of staff in LIC using the fuzzy assignment problems.

To consider the vague and imprecise information in the practical problem, the different extensions of the fuzzy set have been introduced by some authors. The intuitionistic fuzzy set (IFS) proposed by Atanassov (1984) is an extension of the fuzzy set. He considered the membership and nonmembership of the element. Roseline and Amirtharaj (2015) solved the intuitionistic fuzzy assignment problem by using the ranking of intuitionistic fuzzy numbers (IFN). Boran et al. (2012) used the TOPSIS method of the intuitionistic fuzzy set for solving the renewable energy problem. Mukherjee and Basu (2012) solved the assignment problem under IFS by using similarity measure and score function. Kumar and Bajaj [2014] introduced the problem of an interval-valued intuitionistic fuzzy assignment problem and solved it with similarity measure and score function.

Yager (2013) introduced a Pythagorean fuzzy set (PFS). Yager overcomes the situation when membership degree (τ) + nonmembership degree (ζ) >1 in IFS. PFS is an extension of IFS with the condition that the square sum of the membership degree and the nonmembership degree is less than or equal to 1 ($\tau_A(u)^2+\zeta_A(u)^2 \leq 1$). The concept of Pythagorean fuzzy sets (PFS) gives the larger preference domain for decision makers (DM). DMs can define their support and against the degree of membership as $\tau(x)$ = 4/5, $\zeta(x)$= 2/5. In this case, 4/5+2/5>1 is not valid in IFS but squaring $(4/5)^2 + (2/5)^2 <1$ implies the Pythagorean fuzzy set is more suitable than the intuitionistic fuzzy set. Paul Augustine Ejegwa (2019) worked on the Pythagorean fuzzy set and its application in career placement using max-min composition. Fei and Deng [18] solved the problem of the Pythagorean fuzzy multi-criteria problem. Shahzadi et al. (2018)

proposed the solution of the decision-making approach under the Pythagorean fuzzy Yager weighted operators. Peng and Yang (2015) defined some results for Pythagorean fuzzy sets.

Over the period, score function and similarity measure of Pythagorean were introduced by many authors. Agheli et al. (2022) defined the new similarity measure for Pythagorean fuzzy sets and application on multiple-criterion decision- making. Zhang and Xu (2014) worked on TOPSIS for multi-criteria decision- making with PFS. Peng & Yang (2016) defined the score function and distance measure for the interval-valued Pythagorean fuzzy number (IVPFN) to analyse the problem. After that Garg [25] proposed the score function for PFN and IVPFN to overcome some limitations of the score function defined by Peng & Yang (2016).

In this work, we have developed a methodology to solve the assignment problem with Pythagorean fuzzy values. The score function defined by Garg (2017) has some limitations. To overcome these limitations, we have proposed a new score function. Additionally, we have defined the new similarity measure to validate our result. So far, there is no literature regarding Pythagorean fuzzy assignment problems using similarity measure and score function.

The paper is organised as follows: Some basic knowledge of FS, IFS, PFS, and arithmetic operations on Pythagorean fuzzy numbers are discussed in section 2. In section 3, we have proposed a novel similarity measure and score function. Also, the limitations of previously defined score functions have been pointed out. The methodology to solve PFAP using similarity measure and score function is given in section 4. Illustrative examples are also given in this section. Section 5, presents the comparative study and concluding remarks.

2. Preliminaries

In this section, we have discussed some basic definitions and arithmetic operations that are required for our work.

Definition 2.1 (1965) A fuzzy set (FS) $\tilde{A}$ is defined on universal set U as

$$\tilde{A} = \{\langle u, \tau_A(u) | u \in U \rangle\},$$

characterized by the membership function

$$\tau_A(u): U \to [0,1].$$

Here $\tau_A(u)$ is the membership degree of the element u to the set $\tilde{A}$.

Definition 2.2 (1984) An intuitionistic fuzzy set $\tilde{A}$ on U is defined as a set of ordered pair given by

$$\tilde{A} = \{\langle u, \tau_{\tilde{A}}(u), \zeta_{\tilde{A}}(u) \rangle | u \in U\},$$

where $\tau_{\tilde{A}}(u), \zeta_{\tilde{A}}(u): U \to [0,1]$ are the degree of membership and degree of non-membership of the element $u \in U$, with the condition $(\tau_{\tilde{A}}(u)) + (\zeta_{\tilde{A}}(u)) \leq 1$, the degree of indeterminacy is given by $\xi_{\tilde{A}}(u) = 1 - \tau_{\tilde{A}}(u) - \zeta_{\tilde{A}}(u)$.

Definition 2.3 (2013) A Pythagorean fuzzy set $\tilde{A}$ on U is defined as given by

$$\tilde{A} = \{< u, \tau_A(u), \zeta_A(u) > | u \in U\},$$

where $\tau_A(u), \zeta_A(u): U \to [0,1]$ are the degree of membership and degree of non-membership of the element $u \in U$, with the condition $(\tau_A(u))^2 + (\zeta_A(u))^2 \leq 1$, the degree of indeterminacy is given by $\xi_A(u) = \sqrt{1 - (\tau_A^2 + \zeta_A^2)}$.

The domain of a Pythagorean fuzzy set is larger than intuitionistic fuzzy sets. While working in the space of PFS, one may have much more choice of assigning value to member and nonmembership from [0, 1].

Definition 2.4 (2015) The addition, multiplication, and scalar multiplication on two PFNs $\tilde{A}_1 =< \tau_{A_1}(u), \zeta_{A_1}(u) >$ and $\tilde{A}_2 =< \tau_{A_2}(u), \zeta_{A_2}(u) >$ are defined as follows:

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = \left\langle \sqrt{\tau_{A_1}^2 + \tau_{A_2}^2 - \tau_{A_1}^2 \tau_{A_2}^2}, \zeta_{A_1}\zeta_{A_2} \right\rangle,$

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 = \left\langle \tau_{A_1}\tau_{A_2}, \sqrt{\zeta_{A_1}^2 + \zeta_{A_2}^2 - \zeta_{A_1}^2 \zeta_{A_2}^2} \right\rangle,$

(III) $k\tilde{A}_1 = \left\langle \sqrt{1 - (1 - \tau_{A_1}^2)^k}, \zeta_{A_1}^k \right\rangle, k > 0.$

3. Similarity Measure and Score Function of Pythagorean Fuzzy Set

In this section, we have defined the novel similarity measure and score function of Pythagorean fuzzy sets.

Definition 3.1: Suppose $\tilde{A}$ and $\tilde{B}$ be two PFSs. The similarity measure SM: $\tilde{A} \times \tilde{B} \to$ [0, 1] is defined as follows

$$S(\tilde{A}, \tilde{B}) = \frac{\sum_{j=1}^{m} \tau_A^2(u_j).\tau_B^2(u_j) + \zeta_A^2(u_j).\zeta_B^2(u_j)}{\sum_{j=1}^{m} \left[\left(\tau_A^4(u_j) \vee \tau_B^4(u_j) \right) + \left(\zeta_A^4(u_j) \vee \zeta_B^4(u_j) \right) \right]}$$

Theorem 3.1: Similarity measure (SM) between two PFS $\tilde{A}$ and $\tilde{B}$, then the following are true.

(S1) $0 \leq S(\tilde{A}, \tilde{B}) \leq 1$

(S2) $S(\tilde{A}, \tilde{B}) = 1$ iff A=B

(S3) $S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A})$

(S4) $S(\tilde{A},\tilde{C}) \le S(\tilde{A},\tilde{B})$ and $S(\tilde{A},\tilde{C}) \le S(\tilde{B},\tilde{C})$ for all $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ such that $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$.

Proof: **(S1)** Since for all u_j, $1 \le j \le m$, we have $\tau_{\tilde{A}}^2(u_j).\tau_{\tilde{B}}^2(u_j) \le \tau_{\tilde{A}}^4(u_j) \vee \tau_{\tilde{B}}^4(u_j)$ and $\zeta_{\tilde{A}}^2(u_j).\zeta_{\tilde{B}}^2(u_j) \le \zeta_{\tilde{A}}^4(u_j) \vee \zeta_{\tilde{B}}^4(u_j)$.

Therefore, for each u_j, we have

$$\left[\tau_{\tilde{A}}^2(u_j).\tau_{\tilde{B}}^2(u_j) + \zeta_{\tilde{A}}^2(u_j).\zeta_{\tilde{B}}^2(u_j)\right] \le \left[\left\{\tau_{\tilde{A}}^4(u_j) \vee \tau_{\tilde{B}}^4(u_j)\right\} + \left\{\zeta_{\tilde{A}}^4(u_j) \vee \zeta_{\tilde{B}}^4(u_j)\right\}\right]$$

Therefore, for all u_j, $1 \le j \le m$, we have

$$\sum_{j=1}^{m}\left[\tau_{\tilde{A}}^2(u_j).\tau_{\tilde{B}}^2(u_j) + \zeta_{\tilde{A}}^2(u_j).\zeta_{\tilde{B}}^2(u_j)\right] \le$$

$$\sum_{j=1}^{m}\left[\left\{\tau_{\tilde{A}}^4(u_j) \vee \tau_{\tilde{B}}^4(u_j)\right\} + \left\{\zeta_{\tilde{A}}^4(u_j) \vee \zeta_{\tilde{B}}^4(u_j)\right\}\right]$$

$$0 \le S^s(\tilde{A},\tilde{B}) \le 1.$$

(S2). Suppose $S(\tilde{A},\tilde{B}) = 1$, $\dfrac{\sum_{j=1}^{m}\left[\tau_{\tilde{A}}^2(u_j).\tau_{\tilde{B}}^2(u_j) + \zeta_{\tilde{A}}^2(u_j).\zeta_{\tilde{B}}^2(u_j)\right]}{\sum_{j=1}^{m}\left[\left\{\tau_{\tilde{A}}^4(u_j) \vee \tau_{\tilde{B}}^4(u_j)\right\} + \left\{\zeta_{\tilde{A}}^4(u_j) \vee \zeta_{\tilde{B}}^4(u_j)\right\}\right]} = 1$

$$\sum_{j=1}^{m}\left[\tau_{\tilde{A}}^2(u_j).\tau_{\tilde{B}}^2(u_j) + \zeta_{\tilde{A}}^2(u_j).\zeta_{\tilde{B}}^2(u_j)\right] = \sum_{j=1}^{m}\left[\left\{\tau_{\tilde{A}}^4(u_j) \vee \tau_{\tilde{B}}^4(u_j)\right\} + \left\{\zeta_{\tilde{A}}^4(u_j) \vee \zeta_{\tilde{B}}^4(u_j)\right\}\right]$$

Now, we claim that $\tau_{\tilde{A}}^2(u_j).\tau_{\tilde{B}}^2(u_j) = \tau_{\tilde{A}}^4(u_j) \vee \tau_{\tilde{B}}^4(u_j)$ and $\zeta_{\tilde{A}}^2(u_j).\zeta_{\tilde{B}}^2(u_j) = \zeta_{\tilde{A}}^4(u_j) \vee \zeta_{\tilde{B}}^4(u_j)$.

Suppose $\tau_{\tilde{A}}^2(u_j).\tau_{\tilde{B}}^2(u_j) \ne \tau_{\tilde{A}}^4(u_j) \vee \tau_{\tilde{B}}^4(u_j)$, since $\tau_{\tilde{A}}^2(u_j).\tau_{\tilde{B}}^2(u_j) \le \tau_{\tilde{A}}^4(u_j) \vee \tau_{\tilde{B}}^4(u_j)$, there exists $k_1 > 0$ such that $\tau_{\tilde{A}}^2(u_j).\tau_{\tilde{B}}^2(u_j) + k_1 = \tau_{\tilde{A}}^4(u_j) \vee \tau_{\tilde{B}}^4(u_j)$.

Similarly, there exists $k_2 > 0$ such that $\zeta_{\tilde{A}}^2(u_j).\zeta_{\tilde{B}}^2(u_j) + k_2 = \zeta_{\tilde{A}}^4(u_j) \vee \zeta_{\tilde{B}}^4(u_j)$.

By hypothesis it follows that $k_1 + k_2 = 0$. This implies $k_1 = -(k_2)$, which is not possible. This implies that

$\tau_{\tilde{A}}^{2}(u_j).\tau_{\tilde{B}}^{2}(u_j) = \tau_{\tilde{A}}^{4}(u_j) \vee \tau_{\tilde{B}}^{4}(u_j)$ and $\zeta_{\tilde{A}}^{2}(u_j).\zeta_{\tilde{B}}^{2}(u_j) = \zeta_{\tilde{A}}^{4}(u_j) \vee \zeta_{\tilde{B}}^{4}(u_j)$.
This implies that $\tau_{\tilde{A}}^{2}(u_j) = \tau_{\tilde{B}}^{2}(u_j)$ and $\zeta_{\tilde{A}}^{2}(u_j) = \zeta_{\tilde{B}}^{2}(u_j)$.

Hence $\tilde{A} = \tilde{B}$.

(S3) $S(\tilde{A},\tilde{B}) = S(\tilde{B},\tilde{A})$ is trivial.

(S4) For three PFSs $\tilde{A},\tilde{B}$ and $\tilde{C}$ in U. The similarity measures between $\tilde{A},\tilde{B}$ and $\tilde{A},\tilde{C}$ are given as

$$S(\tilde{A},\tilde{B}) = \frac{\sum_{j=1}^{m}\left[\tau_{\tilde{A}}^{2}(u_j).\tau_{\tilde{B}}^{2}(u_j) + \zeta_{\tilde{A}}^{2}(u_j).\zeta_{\tilde{B}}^{2}(u_j)\right]}{\sum_{j=1}^{m}\left[\left\{\tau_{\tilde{A}}^{4}(u_j) \vee \tau_{\tilde{B}}^{4}(u_j)\right\} + \left\{\zeta_{\tilde{A}}^{4}(u_j) \vee \zeta_{\tilde{B}}^{4}(u_j)\right\}\right]}$$

$$S(\tilde{A},\tilde{C}) = \frac{\sum_{j=1}^{m}\left[\tau_{\tilde{A}}^{2}(u_j).\tau_{\tilde{C}}^{2}(u_j) + \zeta_{\tilde{A}}^{2}(u_j).\zeta_{\tilde{C}}^{2}(u_j)\right]}{\sum_{j=1}^{m}\left[\left\{\tau_{\tilde{A}}^{4}(u_j) \vee \tau_{\tilde{C}}^{4}(u_j)\right\} + \left\{\zeta_{\tilde{A}}^{4}(u_j) \vee \zeta_{\tilde{C}}^{4}(u_j)\right\}\right]}.$$

Let $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$ for all $u_j \in \tilde{U}$, we have $\tau_{\tilde{A}}^{2}(u_j) \le \tau_{\tilde{B}}^{2}(u_j) \le \tau_{\tilde{C}}^{2}(u_j)$, $\zeta_{\tilde{A}}^{2}(u_j) \ge \zeta_{\tilde{B}}^{2}(u_j) \ge \zeta_{\tilde{C}}^{2}(u_j)$. This implies $\tau_{\tilde{A}}^{4}(u_j) \le \tau_{\tilde{B}}^{4}(u_j) \le \tau_{\tilde{C}}^{4}(u_j)$, $\zeta_{\tilde{A}}^{4}(u_j) \ge \zeta_{\tilde{B}}^{4}(u_j) \ge \zeta_{\tilde{C}}^{4}(u_j)$.

We claim that for all $u_j \in \tilde{U}$, we have

$$\frac{\tau_{\tilde{A}}^{2}(u_j).\tau_{\tilde{B}}^{2}(u_j)}{\tau_{\tilde{B}}^{4}(u_j) + \zeta_{\tilde{A}}^{4}(u_j)} \le \frac{\tau_{\tilde{A}}^{2}(u_j).\tau_{\tilde{C}}^{2}(u_j)}{\tau_{\tilde{C}}^{4}(u_j) + \zeta_{\tilde{A}}^{4}(u_j)}$$

Similarly, we have

$$\frac{\zeta_{\tilde{A}}^{2}(u_j).\zeta_{\tilde{B}}^{2}(u_j)}{\tau_{\tilde{B}}^{4}(u_j) + \zeta_{\tilde{A}}^{4}(u_j)} \le \frac{\zeta_{\tilde{A}}^{2}(u_j).\zeta_{\tilde{C}}^{2}(u_j)}{\tau_{\tilde{C}}^{4}(u_j) + \zeta_{\tilde{A}}^{4}(u_j)}$$

by adding all above equations, we have

$$\frac{\sum_{j=1}^{m}\left[\tau_{\tilde{A}}^{2}(u_j).\tau_{\tilde{C}}^{2}(u_j)+\zeta_{\tilde{A}}^{2}(u_j).\zeta_{\tilde{C}}^{2}(u_j)\right]}{\sum_{j=1}^{m}\left[\left\{\tau_{\tilde{A}}^{4}(u_j)\vee\tau_{\tilde{C}}^{4}(u_j)\right\}+\left\{\zeta_{\tilde{A}}^{4}(u_j)\vee\zeta_{\tilde{C}}^{4}(u_j)\right\}\right]}\leq$$

$$\frac{\sum_{j=1}^{m}\left[\tau_{\tilde{A}}^{2}(u_j).\tau_{\tilde{B}}^{2}(u_j)+\zeta_{\tilde{A}}^{2}(u_j).\zeta_{\tilde{B}}^{2}(u_j)\right]}{\sum_{j=1}^{m}\left[\left\{\tau_{\tilde{A}}^{4}(u_j)\vee\tau_{\tilde{B}}^{4}(u_j)\right\}+\left\{\zeta_{\tilde{A}}^{4}(u_j)\vee\zeta_{\tilde{B}}^{4}(u_j)\right\}\right]}.$$

Therefore $S(\tilde{A},\tilde{C})\leq S^{s}(\tilde{A},\tilde{B})$. Similarly $S^{s}(\tilde{A},\tilde{C})\leq S(\tilde{B},\tilde{C})$.

Next, we discuss the limitations of the previously defined score function and to overcome the limitations a new score function propose in this section.
Peng & Yang [7] defined score function and accuracy function for interval-valued Pythagorean fuzzy number (IVPFN):

Consider IVPFN $\widetilde{W}=\langle[\alpha,\beta],[\gamma,\delta]\rangle$, the score function $E_1(\widetilde{W})$ and accuracy function $F_1(\widetilde{W})$ are defined as follows

$$E_1(\widetilde{W})=\frac{\alpha^2+\beta^2-\gamma^2-\delta^2}{2}$$

$$F_1(\widetilde{W})=\frac{\alpha^2+\beta^2+\gamma^2+\delta^2}{2}$$

Suppose that if we consider two IVPFN $\widetilde{W}_1=\langle[0.1,0.2],[0.4,0.5]\rangle$ and $\widetilde{W}_2=\langle[0.1,0.2],[0.5,0.5]\rangle$, the score and accuracy value calculated by Peng & Yang (2016) are $E_1(\widetilde{W}_1)$=-0.1800, $E_1(\widetilde{W}_2)$=-0.1800 and $F_1(\widetilde{W}_1)$=0.2300, $F_1(\widetilde{W}_2)$=0.2300.

According to Peng & Yang (2016) $\widetilde{W}_1\sim\widetilde{W}_2$, but we have seen that $\widetilde{W}_1\neq\widetilde{W}_2$. To overcome this limitation Garg (2017) defined the new score function $E_2(\widetilde{W})$ and defined as

$$E_2(\widetilde{W})=\frac{(\alpha^2-\gamma^2)(1+\sqrt{1-\alpha^2-\gamma^2})+(\beta^2-\delta^2)(1+\sqrt{1-\beta^2-\delta^2})}{2}\in[-1,1]$$

By this score function, the score value of above example are $E_2(\widetilde{W}_1)=-0.3368$ and $E_2(\widetilde{W}_2)=-0.3233$. Here $E_2(\widetilde{W}_2)>E_2(\widetilde{W}_1)$, hence $\widetilde{W}_2>\widetilde{W}_1$.

For IVPFN $\widetilde{W}=\langle[0.81,0.87],[0.11,0.25]\rangle$, the score value $E_2(\widetilde{W})$=1.0022, this is invalid because score value is greater than 1 i.e. $E_2(\widetilde{W})=1.0022\notin[-1,1]$.

Proposed Score Function: We have seen from the above example that the score function defined by Garg (2017) is not giving the appropriate result. To improve this, we have proposed a novel score function as follows.

Definition 3.2 Let $\widetilde{W} = \langle[\alpha,\beta],[\gamma,\delta]\rangle$ be an interval valued Pythagorean fuzzy number (IVPFN). The score function for IVPFN is

$$E(\widetilde{W})=\frac{(\sqrt{3+\alpha-3\gamma})+(\sqrt{3+\beta-3\delta})}{4} \in [0,1].$$

In this if $\alpha = \beta = \tau$ and $\gamma = \delta = \zeta$, then the score function of Interval-valued Pythagorean fuzzy set will become score function of Pythagorean fuzzy set. So, the proposed score function of the Pythagorean fuzzy set is as follows:

$$E(\widetilde{W})=\frac{\sqrt{3+\tau-3\zeta}}{2} \in [0,1].$$

For any two IVPFN/PFN $\widetilde{W}_1$ and $\widetilde{W}_2$,

1. if $E(\widetilde{W}_1) > E(\widetilde{W}_2)$, then $\widetilde{W}_1 > \widetilde{W}_2$
2. if $E(\widetilde{W}_1) < E(\widetilde{W}_2)$, then $\widetilde{W}_1 < \widetilde{W}_2$
3. if $E(\widetilde{W}_1) = E(\widetilde{W}_2)$, then $\widetilde{W}_1 = \widetilde{W}_2$.

4. Application of the Pythagorean Fuzzy Assignment Problem

In this section, we introduce the assignment problem with Pythagorean fuzzy number (PFN) and give two methodologies to solve such problems. One is based on similarity measure and the other is based on a score function.

Pythagorean Fuzzy Assignment Problem (PFAP)

Min $\tilde{Y}=\sum_i^n \sum_j^n \tilde{c}_{ij}^{PFN} x_{ij}$

Subject to

$\sum_j^n x_{ij} = 1, i = 1,2,\ldots n$

$\sum_i^n x_{ij} = 1, j = 1,2,\ldots n$

$$x_{ij} \in \{0,1\}$$

In assignment problems, the cost is usually deterministic in nature. But in real-life problem, this is very difficult to judge the precise value of the cost. In this unstable condition, we calculate the preference value. Based on preference value, we get the preference for the j^{th} work to the i^{th} person in the form of a composite relative degree of similarity with an ideal solution, Thus we replace c_{ij} by composite relative degree.

4.1. Methodology for Pythagorean Fuzzy Assignment Problem using similarity measure

Step 1 First consider the Pythagorean fuzzy assignment problem decision matrix G={(L $_{ij}$)} $_{m\times n}$

(L $_{ij}$)=$< \tau_{ij}(x), \zeta_{ij}(x) >$, *i=1, 2....m, j= 1,.....n* are Pythagorean fuzzy numbers.

Step 2 Examine whether the problem is balanced or not. If it is not balanced, then add dummy variables so that the problem is converted into a balanced assignment problem.

Step 3 Calculate the similarity measure of each cost value from Pythagorean positive ideal solution (PPIS) $L^+ = \langle 1,0 \rangle$ and Pythagorean negative ideal solution (PNIS) $L^- = \langle 0,1 \rangle$

$$S(L, L^+) = \frac{\sum_{j=1}^{m} \tau_L^2(x_j).\tau_{L^+}^2(x_j) + \zeta_L^2(x_j).\zeta_{L^+}^2(x_j)}{\sum_{j=1}^{m}\left[\left(\tau_L^4(x_j) \vee \tau_{L^+}^4(x_j)\right) + \left(\zeta_L^4(x_j) \vee \zeta_{L^+}^4(x_j)\right)\right]}$$

$$S(L, L^-) = \frac{\sum_{j=1}^{m} \tau_L^2(x_j).\tau_{L^-}^2(x_j) + \zeta_L^2(x_j).\zeta_{L^-}^2(x_j)}{\sum_{j=1}^{m}\left[\left(\tau_L^4(x_j) \vee \tau_{L^-}^4(x_j)\right) + \left(\zeta_L^4(x_j) \vee \zeta_{L^-}^4(x_j)\right)\right]}$$

Relative similarity matrix calculated column-wise

$$Q = \frac{S(L,L^+)}{S(L,L^+)+S(L,L^-)}$$

Similarly, relative similarity matrix calculated row-wise

$$R = \frac{S(L,L^+)}{S(L,L^+)+S(L,L^-)}.$$

Step 4 The composite matrix $[T]_{n\times n}$ is evaluated as $T = Q \times R = q_{ij} \times r_{ij}$, the resultant matrix T represents the preference that j th job is chosen by i th person.

4.2 Methodology for Pythagorean fuzzy assignment problem using the score function

Step 1 Write PFAP in tabular form

Step 2 Convert the Pythagorean fuzzy assignment problem into a crisp assignment problem by using the score function.

Step 3 Examine whether the problem is balanced or not. If it is not balanced, then add dummy variables so that the problem is converted into a balanced assignment problem.

Step 4 The higher cell value of the matrix will indicate the preference of j th job to the i th person

Illustrative Examples:

Here, we have solved the assignment problem by using similarity measure and score function.

Example 4.1: A manufacturing company decides to make six subassemblies through six contractors. One contractor has to receive only one subassembly. The cost of each subassembly is determined by the bids submitted by each contractor and is shown in Table 1 in Pythagorean fuzzy number. The problem is how to assign subassemblies to contractors to get the optimal assignment.

Table 1. Assignment problem based on PFN

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	(0.7,0.6)	(0.7,0.7)	(0.8,0.5)	(0.7,0.6)	(0.6,0.7)	(0.6,0.5)
$\mathcal{S}_2$	(0.63,0.67)	(0.9,0.5)	(0.8,0.53)	(0.8,0.3)	(0.9,0.2)	(0.45,0.59)
$\mathcal{S}_3$	(0.83,0.4)	(0.5,0.7)	(0.6,0.7)	(0.5,0.7)	(0.20,0.81)	(0.5,0.8)
$\mathcal{S}_4$	(0.63,0.55)	(0.71,0.63)	(0.66,0.35)	(0.9,0.3)	(0.4,0.8)	(0.73,0.4)
$\mathcal{S}_5$	(0.7,0.5)	(0.65,0.35)	(0.32,0.7)	(0.8,0.5)	(0.4,0.9)	(0.85,0.18)
$\mathcal{S}_6$	(0.45,0.75)	(0.83,0.3)	(0.35,0.7)	(0.55,0.8)	(0.5,0.6)	(0.3,0.8)

Solution:

Table 2. S (L, L $^+$) column-wise

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	0.43	0.39	0.60	0.43	0.29	0.33
$\mathcal{S}_2$	0.33	0.76	0.59	0.63	0.8	0.18
$\mathcal{S}_3$	0.67	0.19	0.29	0.20	0.02	0.17
$\mathcal{S}_4$	0.36	0.43	0.42	0.80	0.11	0.51
$\mathcal{S}_5$	0.46	0.41	0.08	0.60	0.09	0.72
$\mathcal{S}_6$	0.15	0.68	0.09	0.21	0.22	0.06

Table 3. S (L, L $^-$) column-wise

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	0.29	0.39	0.17	0.29	0.43	0.22
$\mathcal{S}_2$	0.38	0.23	0.19	0.05	0.02	0.33
$\mathcal{S}_3$	0.10	0.46	0.43	0.46	0.65	0.60
$\mathcal{S}_4$	0.26	0.31	0.10	0.05	0.62	0.12
$\mathcal{S}_5$	0.20	0.10	0.48	0.17	0.15	0.02
$\mathcal{S}_6$	0.54	0.06	0.48	0.58	0.33	0.63

Table 4. Relative similarity matrix R (column-wise)

$\mathcal{S},\mathcal{Q}$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	$\mathcal{Q}_5$	$\mathcal{Q}_6$
$\mathcal{S}_1$	0.59	0.5	0.77	0.59	0.40	0.6
$\mathcal{S}_2$	0.46	0.76	0.75	0.92	0.97	0.35

S, Q	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
S_3	0.87	0.29	0.4	0.3	0.02	0.22
S_4	0.58	0.58	0.80	0.94	0.15	0.80
S_5	0.69	0.8	0.14	0.77	0.37	0.97
S_6	0.21	0.91	0.15	0.26	0.4	0.08

Similarly

Table 5. Relative similarity matrix S (row-wise)

SQ	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
S_1	0.34	0.25	0.59	0.34	0.16	0.36
S_2	0.21	0.57	0.56	0.84	0.94	0.12
S_3	0.75	0.08	0.16	0.09	0.00	0.04
S_4	0.33	0.33	0.64	0.88	0.02	0.64
S_5	0.47	0.64	0.01	0.59	0.13	0.94
S_6	0.04	0.82	0.02	0.06	0.16	0.0

Now compute the composite matrix T=R× S=$r_{ij} \times s_{ij}$. This matrix T represents the preference the S^{th} subassembly to C^{th} contractor

Table 6. the preference the S^{th} subassembly to C^{th} contractor

S, Q	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
S_1	0.11	0.06	0.34	0.11	0.02	0.12
S_2	0.04	0.32	0.31	0.70	0.88	0.01
S_3	0.56	0.0	0.02	0.0	0.0	0.0
S_4	0.10	0.10	0.4	077	0.0	0.4
S_5	0.22	0.40	0.0	0.34	0.01	0.88
S_6	0.0	0.67	0.0	0.0	0.02	0.0

The optimal assignment policy is: subassembly 1→contractor 3, subassembly 2→contractor 5, subassembly 3→contractor 1, subassembly 4→contractor 4, subassembly 5→contractor 6, subassembly 6→contractor 2.

Next, we consider the same Example 4.1 as given above and apply the proposed score function for the Pythagorean fuzzy number (Definition 3.2). We get the Table 7 corresponding to Table 1.

Solution for PFAP:

Table 7. Value calculated by score function

S, Q	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
S_1	0.689	0.632	0.758	0.689	0.612	0.725
S_2	0.636	0.775	0.743	0.851	0.908	0.648
S_3	0.811	0.592	0.612	0.592	0.439	0.524
S_4	0.704	0.675	0.808	0.866	0.500	0.795
S_5	0.742	0.806	0.552	0.758	0.418	0.910
S_6	0.548	0.856	0.559	0.536	0.652	0.474

The optimal assignment policy is: subassembly 1→contractor 3, subassembly 2→contractor 5, subassembly 3→contractor 1, subassembly 4→contractor 4, subassembly 5→contractor 6, subassembly 6→contractor 2.

5. Comparative analysis and conclusions

The examples mentioned above have vividly demonstrated the proposed similarity measure and score function as a potential tool for solving the assignment problem in Pythagorean fuzzy sets. From the analysis, it is observed that the results obtained by the implementation of the developed similarity measure and score function are more accurate and reliable. Compared to the existing methodologies in the literature, the novel score function proposed in the present paper has the following advantages:

(i) The proposed method has a simple presentation such that it can significantly avoid the information loss that may have previously occurred in the score function defined by Peng & Yang (2016) and Garg's (2017). It is envisioned that there exist certain values where Peng & Yang (2016) and Garg (2017) score function failed to give valid results.

(ii) We have also observed that Example 4.1 cannot be solved by using the score function given by Garg (2017) as the score values of the cell (2,2) representing $\mathcal{S}_2\mathcal{Q}_2$ (0.9, 0.5).

(iii) The diversity and fuzziness of the decision maker's assessment information can be well reflected and modelled using the proposed similarity measure and score function.

(iv) The result offered by using the novel similarity measure and score function is consistent with the result obtained in the existing work, Mukherjee and Basu (2012), and Kumar and Bajaj (2014). Therefore, the proposed method becomes more flexible and convenient for solving the Pythagorean fuzzy assignment problem.

In this paper, we have proposed a methodology to solve the Pythagorean fuzzy assignment problem. We have solved the problem using the similarity measure and the score function to test the optimality of the problem. It is anticipated that the proposed methodology is capable of managing the uncertainty persisting within the intricate assignment problem. The working of proposed technique has been illustrated via examples to test the validity. We further provide a comparison with the existing methods in the literature. From the comparative study and analysis, it can be concluded that the proposed method overcomes the limitations present in the existing work. Table 8 provides a comparative analysis of the proposed score function. Additionally, it would be engrossing to explore the application of the developed approach to picture fuzzy sets, spherical fuzzy sets and interval-valued picture fuzzy sets, etc., also to deal with other linear programming problems.

Table 8. Comparative analysis of the present work

Problem	**Score Function by Garg (2017)**	**Score Function by Peng & Yang (2016)**	**Proposed Score Function & Similarity Measure**
PFAP	Failed	This is not valid when score values & accuracy are the same	Solution exists

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FOCUS

Interval valued picture fuzzy matrix: basic properties and application

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Abstract
The use of matrix theory in decision-making problems has been a subject of great interest for researchers. However, recent developments have shown that matrices can be enhanced by incorporating fuzzy, intuitionistic fuzzy, and picture-fuzzy theory. Inspired by the notion of interval-valued picture fuzzy sets (IVPFSs), we extend the idea of picture fuzzy matrix (PFM) into interval-valued picture fuzzy matrix (IVPFM) to represent more flexibly uncertain and vague information. The paper defines several key definitions and theorems for the IVPFM and presents a procedure for calculating its determinant and adjoint. Using composition functions, we develop algorithms to identify the greatest and least eigenvalue interval-valued picture fuzzy sets and use a flow chart to illustrate the procedure. The work demonstrates this process with a numerical example of a decision-making problem. In addition, we introduce a new distance measure for the IVPFSs and prove its validity with the help of basic properties. Further, the application of the proposed concepts has been shown by a real-life numerical example of a computer numerical controlled (CNC) programmer selection problem in a smart manufacturing company.

Keywords Picture fuzzy matrix · Interval-valued picture fuzzy matrix · Greatest eigen interval-valued picture fuzzy set · Least eigen interval-valued picture fuzzy set · Distance measure

1 Introduction

Decision-making is a key process that profoundly influences our personal and professional lives. However, the real world is often characterized by uncertainty and vagueness, which complicates and challenges the decision-making process (Verma and Sharma 2014; Guleria and Bajaj 2021; Verma 2022; Dhumras and Bajaj 2023). Uncertainty, in this context, arises when the outcomes or consequences of various alternatives are inherently unknown, lacking a definitive level of certainty. Conversely, vagueness encompasses handling imprecise or ambiguous information, characterized by its deficiency in clarity and specificity. Both uncertainty and vagueness introduce risks, trade-offs, and a sense of ambiguity that decision-makers must navigate to make sound and effective choices. Confronting uncertain and vague information adds substantial difficulty to accurately analyzing a given situation, identifying the optimal course of action, and gaining a comprehensive understanding of the potential consequences of a decision (Choo 1996).

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The theory of fuzzy sets (FSs) was proposed by Zadeh (1965) to deal with uncertain or vague information more efficiently in practical situations. The core idea behind fuzzy sets is to assign a membership degree (MD) to each element of a set, indicating the extent to which the element belongs to the set. The concept of fuzzy sets finds applications in various fields, including artificial intelligence, decision-making, pattern recognition, and control systems, to name a few. Atanassov (1986) generalized the idea of FSs and introduced a novel concept called intuitionistic fuzzy sets (IFSs). Unlike fuzzy sets that solely capture MD and non-membership degrees (NMD), IFSs introduce an additional dimension called the hesitancy degree (HD), which quantifies the level of uncertainty or ambiguity associated with a particular element. Since its introduction, IFSs have been extensively studied and widely applied in various domains, including decision-making (Verma 2015; Xie et al. 2022), pattern recognition (Zeng et al. 2022), and medical diagnosis (Luo and Zhao 2018; Thao et al. 2019). El-Morsy (2023) introduced an approach that utilizes Pythagorean fuzzy numbers to assess the rate of risked return, portfolio risk amount, and expected return rate. Atanassov and Gargov (1989) further extended the notion of IFS to the interval-valued intuitionistic fuzzy sets (IVIFSs)) in which intervals numbers are used rather than exact numbers to provide flexibility in defining membership degrees to an element. Researchers have successfully applied IVIFSs in various domains, demonstrating their flexibility and effectiveness in real-world situations (Verma and Merigó 2020; Verma and Chandra 2021; Liang et al. 2022; Perçin 2022).

While IFSs have proven useful in various applications, they encounter limitations when confronted with conflicting information in real-life scenarios. Consider situations such as voting, where the outcomes can be categorized into four distinct groups: vote for, abstain, vote against, and refuse to vote. Cuong and Kreinovich (2014) introduced the notion of picture fuzzy sets (PFSs) to address this issue. The PFSs offer a more comprehensive approach by incorporating three functions: membership, neutral membership, and non-membership degree. This expansion of IFS theory provides a powerful tool for handling conflicting and ambiguous information, enabling more robust analysis and decision-making. Cuong and Kreinovich (2014) also defined some basic operational laws for PFSs and proved several properties associated with them. Phong et al. (2014) worked on the composition of picture fuzzy relations to solve medical diagnosis problems using max–min composition. Singh (2015) introduced the idea of a correlation coefficient for PFSs. Son (2016) proposed a generalized distance measure between PFSs to solve clustering problems in a picture-fuzzy context. Wei (2017) defined cosine similarity measures for PFSs and studied their application in decision-making. Garg (2017) proposed weighted average and geometric aggregation operators for picture fuzzy numbers (PFNs) and used them in decision-making. Jana et al. (2019) defined Dombi aggregation operators for picture PFNs to solve multiple attribute decision-making (MADM) problems. Wei et al. (2018) developed projection models for solving MADM problems in picture fuzzy framework. Luo and Zhang (2020) defined the new similarity between PFSs and discussed their application in pattern recognition. Joshi (2020) formulated a novel picture fuzzy decision-making method based on the R-norm information measure and the VIKOR approach. Jana and Pal (2019) used picture fuzzy Hamacher aggregation operators for the performance assessment of an enterprise. Verma and Rohtagi (2022) proposed novel similarity measures to resolve pattern recognition and medical diagnosis issues in a picture-fuzzy environment. Ganie (2023) defined a new distance measure on PFSs and discussed its application to pattern recognition problems. Hasan et al. (2022) defined some picture fuzzy mean operators. Roan et al. (2020) worked on the utilization of the picture fuzzy distance measure to manage network power consumption.

The theory of PFSs was further generalized by Cuong and Kreinovich (2014) by proposing interval-valued picture fuzzy sets. It is worth mentioning that the IVPFSs have various advantages over PFSs and IVIFSs, making them highly suitable for efficiently modeling uncertain and vague information, particularly in complex scenarios. Khalil et al. (2019) established some operation laws for IVPFSs and comprehensively analyzed their properties. Liu et al. (2019) introduced the similarity measures for IVPFS and studied their applications in decision-making problems. Mahmood et al. (2021) defined Frank aggregation operators as designed to solve decision-making problems involving interval-valued picture fuzzy information.

1.1 Literature review

Matrices, a fundamental mathematical concept, hold immense significance across disciplines such as linear algebra, physics, computer science, and economics. A matrix is a rectangular array of numbers, symbols, or expressions arranged in rows and columns. It provides a concise and organized way to represent and manipulate complex data sets or mathematical relationships. Various uncertain and vague data types are involved in real-world situations, which is difficult to express in a classical matrix. To overcome this situation, Thomason (1977) introduced the idea of a fuzzy matrix (FM) in 1977. After that, Kim and Roush (1980) worked on the FM over boolean algebra. Ragab and Emam (1995) gave some results on the max–min composition and studied the construction of an idempotent FM. Ragab and Emam (1994) solved the determinant and adjoint of a square FM and discussed some properties defined on it. Shyamal and Pal (2007) proposed the triangular fuzzy matrix and gave the methodology to find the corresponding determinant. Dehghan et al. (2009) expanded the idea of the inverse of a matrix with fuzzy numbers. Pal (2015) proposed the interval-valued fuzzy matrix (IVFM) theory. IVFM has numerous applications in decision-making, medical diagnosis, etc., just like FSs, Meenakshi and Kaliraja (2011) used the IVFM for solving medical diagnosis problems. Mondal and Pal (2016) described some methods to find the ranks of IVFM.

Owing to the advantages of IFSs, Pal et al. (2002) introduced the concept of the intuitionistic fuzzy matrix (IFM). Pal and Khan (2002) proposed some important operations on the IFM. Padder and Murugadas (2016) worked on max–min operations on the IFM and discussed the convergence of transitive IFM. Padder and Murugadas (2016) addressed the determinant of an IFM with basic mathematical properties. Muthuraji et al. (2016) studied the decomposition of IFM. Jenita et al. (2022) presented a detailed study on ordering in generalized regular IFM. Padder and Murugadas (2022) developed an algorithm for controllable and nilpotent IFM. Further, Khan and Pal (2014) extended the notion of the intuitionistic fuzzy matrix to the interval-valued intuitionistic fuzzy matrix (IVIFM) in which MD and NMD are used in interval numbers rather than exact numbers. Silambarasan (2020) defined the Hamacher operations of IVIFM and proved some important properties associated with them. The IFMs have been strongly enforced in various areas, yet the concept of neutral membership needs to be considered in IFMs. In this regard, Dogra and Pal (2020) proposed the picture fuzzy matrix (PFM) and discussed some of its important aspects. On the theory of PFM, many authors worked on its important concept. For instance, Silambarasan (2020) defined some algebraic operations and properties of the PFM. Murugadas (2021) defined the implication operation on PFM. The picture fuzzy soft matrices were defined by Arikrishnan and Sriram (2020). Further, Kamalakannann and Murugadas (2022) studied the eigenvalue and eigenvector of PFM with some examples. Adak et al. (2023) explained the concepts of semi-prime ideals of PFS.

The concept of fuzzy relations is a generalization of binary relations. Zadeh (1971) initially established the concept of fuzzy relation, including fuzzy equivalency (similarity) relation, and provided the concept of fuzzy ordering along with certain basic features. Eigenvalues and eigenvectors of the matrix play a very important role in solving many complex problems in different domains. Sanchez (1981) worked on eigen fuzzy sets and described the importance of eigen fuzzy sets using the composition of fuzzy relations. Using the max–min composition, Sanchez (1981) also determined the greatest eigen fuzzy set (GEFS). Di-Martino et al. (2004) presented the least eigen fuzzy set depending on min–max composition. Goetschel and Voxman (1985) extended the idea for finding the eigen fuzzy set to the eigen fuzzy number. Using the principal component analysis of images, Nobuhara et al. (2006) defined the greatest eigen fuzzy set and an adjoint concept of GEFS. Di Martino and Sessa (2007) proposed a genetic algorithm for image reconstruction based on fuzzy relation, where the GEFS and lowest eigen fuzzy set (LEFS) were used to determine the highest value of fitness. Rakus-Andersson (2006) measured the levels of drug effectiveness by establishing fuzzy relations between the potential symptoms and using the greatest and lowest eigen fuzzy sets.

1.2 Motivation and contribution

In numerous real-life scenarios, determining precise values for the membership, neutral membership, and non-membership degrees of an element within a given set proves challenging. The IVPFSs emerge as a highly effective and prominent tool to address these difficulties, enabling a more versatile and comprehensive depiction of uncertain and vague information. Consequently, it becomes vital to develop a generalized and adaptable matrix theory capable of surpassing the limitations of PFM in representing interval data associated with membership, neutral membership, and non-membership degrees. The contribution of the present work is summarized as follows:

- We generalize the theory of PFM and introduce the novel concept of IVPFM to incorporate the situations where membership, neutral membership, and non-membership degrees are available in terms of interval numbers. We provide comprehensive definitions and theorems related to IVPFSs, establishing a solid foundation for further exploration and analysis.
- The procedure to obtain the determinant and adjoint of an IVPFM is established.
- Two algorithms are developed to evaluate the greatest eigen interval-valued picture fuzzy set (GEIVPFS) and the least eigen interval-valued picture fuzzy set (LEIVPFS) to solve decision-making issues.
- A new distance measure between IVPFSs is also defined with proof of its validity.
- The application of the proposed concepts has been shown in a real-life decision-making problem associated with smart manufacturing.

1.3 Organization of the paper

The remaining part of this paper is organized as follows. Some preliminaries and definitions are briefly introduced in Sect. 2. In Sect. 3, we define the idea of IVPFM with basic definitions, properties, and important theorems. The concepts of determinant, adjoint, and propositions are given in Sect. 4. Section 5 proposes the algorithms for finding the GEIVPFS and the LEIVPFS and illustrates them with a numerical example. Section 6 introduces a new distance measure for the IVPFSs and discusses its application in decision-making. The comparative study with existing work is conducted in Sect. 7. Finally, Sect. 8 concludes the paper with some future directions.

2 Preliminaries

This section provides a comprehensive overview of fundamental concepts and definitions essential for the subsequent analysis.

Definition 1 (Cuong and Kreinovich 2014) PFSs A defined on universal set X is given by

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) | x \in X\},$$

where $\mu_A(x) : X \to [0, 1], \eta_A(x) : X \to [0, 1], \nu_A(x) : X \to [0, 1]$ and $0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1$.
Here, $\mu_A(x), \eta_A(x), \nu_A(x)$ are the membership, neutral membership, and non-membership degree of the element to the set X.

Definition 2 (Dogra and Pal 2020) PFM $\tilde{A} = (\tilde{a}_{ij})$ of order $m \times n$ is defined as

$$\tilde{A} = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu} >),$$

where $\tilde{a}_{ij\mu} \in [0,1]$, $\tilde{a}_{ij\eta} \in [0,1]$, $\tilde{a}_{ij\nu} \in [0,1]$ is the measure of membership, neutral membership and non-membership degrees of $\tilde{a}_{ij}$ respectively where, $i = 1, 2 \ldots m$ and $j = 1, 2 \ldots n$ satisfying $0 \leq \tilde{a}_{ij\mu} + \tilde{a}_{ij\eta} + \tilde{a}_{ij\nu} \leq 1$.

Definition 3 (Cuong and Kreinovich 2014) An IVPFSs A on a universal set X is defined as

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A = [\mu_{AL}, \mu_{AU}], \eta_A = [\eta_{AL}, \eta_{AU}], \nu_A = [\nu_{AL}, \nu_{AU}]$ and $\mu_A, \eta_A, \nu_A : X \to [0, 1]$. An IVPFSs have a condition that the sum of the supremum of all three functions must lie in the unit interval and $\mu_A(x)$, $\eta_A(x)$ and $\nu_A(x)$ are the membership, neutral membership and non-membership degree of the element to the set X.

Definition 4 (Sanchez 1981) A fuzzy relation, denoted as R, defined on a fuzzy set X can be described as a fuzzy subset of $X \times X$. In other words, it can be represented as

$$R = \{\langle (x_1, x_2), \mu_R(x_1, x_2) | x_1, x_2 \in X \rangle\}.$$

Here, $\mu_R(x_1, x_2)$ represents the membership degree of the pair (x_1, x_2) in the fuzzy relation R, and it takes values within the interval [0, 1].

Definition 5 (Sanchez 1981) Consider a fuzzy relation R on the elements of a fuzzy set A, denoted as $R \in \mathrm{FR}(X \times X)$. Let T be a subset of X. In this context, T is referred to as an eigen fuzzy set associated with the relation R if it meets the condition $T \circ R = T$, where $\circ$ represents any composition operator.

Definition 6 (Khalil et al. 2019) Let $\beta_1 = ([\mu_{1L}, \mu_{1U}], [\eta_{1L}, \eta_{1U}], [\nu_{1L}, \nu_{1U}])$ and $\beta_2 = ([\mu_{2L}, \mu_{2U}], [\eta_{2L}, \eta_{2U}], [\nu_{2L}, \nu_{2U}])$ be the interval-valued picture fuzzy sets (IVPFSs), $\forall \alpha > 0$, then the operations of IVPFSs are defined as follows:

(a) $\beta_1{}^c = ([\nu_{1L}, \nu_{1U}], [\eta_{1L}, \eta_{1U}], [\mu_{1L}, \mu_{1U}])$, where c denotes the compliment of matrix;
(b) $\beta_1 \cap \beta_2 = ([\min(\mu_{1L}, \mu_{2L}), \min(\mu_{1U}, \mu_{2U})], [\min(\eta_{1L}, \eta_{2L}), \min(\eta_{1U}, \eta_{2U})], [\max(\nu_{1L}, \nu_{2L}), \max(\nu_{1U}, \nu_{2U})])$; $\beta_1 \cup \beta_2 = ([\max(\mu_{1L}, \mu_{2L}), \max(\mu_{1U}, \mu_{2U})], [\min(\eta_{1L}, \eta_{2L}), \min(\eta_{1U}, \eta_{2U})], [\min(\nu_{1L}, \nu_{2L}), \min(\nu_{1U}, \nu_{2U})])$;
(c) $\beta_1 \oplus \beta_2 = ([\mu_{1L} + \mu_{2L} - \mu_{1L}\mu_{2L}, \mu_{1U} + \mu_{2U} - \mu_{1U}\mu_{2U}], [\eta_{1L}\eta_{2L}, \eta_{1U}\eta_{2U}], [\nu_{1L}\nu_{2L}, \nu_{1U}\nu_{2U}])$;
(d) $\beta_1 \otimes \beta_2 = ([\mu_{1L}\mu_{2L}, \mu_{1U}\mu_{2U}], [\eta_{1L} + \eta_{2L} - \eta_{1L}\eta_{2L}, \eta_{1U} + \eta_{2U} - \eta_{1U}\eta_{2U}], [\nu_{1L} + \nu_{2L} - \nu_{1L}\nu_{2L}, \nu_{1U} + \nu_{2U} - \nu_{1U}\nu_{2U}])$;
(e) $\alpha\beta_1 = \left(\left[1 - (1 - \mu_{1L})^\alpha, 1 - (1 - \mu_{1U})^\alpha\right], \left[\eta_{1L}^\alpha, \eta_{1U}^\alpha\right], \left[\nu_{1L}^\alpha, \nu_{1U}^\alpha\right]\right)$;
(f) $\beta_1^\alpha = \left(\left[\mu_{1L}^\alpha, \mu_{1U}^\alpha\right], \left[1 - (1 - \eta_{1L})^\alpha, 1 - (1 - \eta_{1U})^\alpha\right], \left[1 - (1 - \nu_{1L})^\alpha, 1 - (1 - \nu_{1U})^\alpha\right]\right)$.

In the next section, we present the innovative notion of an IVPFM, which serves as an extension of PFM. In addition, we establish the definitions of fundamental arithmetic operations and demonstrate the proof of essential theorems pertaining to the IVPFM.

3 Interval-valued picture fuzzy matrix

In this section, we define the IVPFM and basic concepts by generalizing the concept of PFM.

Definition 7 An IVPFM $\tilde{A}$ is defined as

$$\tilde{A} = (\tilde{a}_{ij}) = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu} >),\ i = 1, 2, \ldots, m,\quad j = 1, 2, \ldots, n$$

where,

$$\tilde{a}_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \subseteq [0, 1],$$
$$\tilde{a}_{ij\eta} = [a_{ij\eta L}, a_{ij\eta U}] \subseteq [0, 1],$$
$$\tilde{a}_{ij\nu} = [a_{ij\nu L}, a_{ij\nu U}] \subseteq [0, 1],$$

with the condition

$$a_{ij\mu U} + a_{ij\eta U} + a_{ij\nu U} \leq 1.$$

$\tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}$ and $\tilde{a}_{ij\nu}$ are the membership, neutral membership and non-membership degree of $\tilde{a}_{ij}$.

Definition 8 An IVPFM is said to be a square interval-valued picture fuzzy matrix (SIVPFM) if the number of rows is equal to the number of columns i.e., $i = 1, 2, \ldots m$, $j = \ldots m$.

Definition 9 Let $\tilde{A}$ and $\tilde{B}$ be two IVPFM, such that

$$\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}]) \text{and}$$
$$\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ij\nu L}, b_{ij\nu U}]).$$

Then, we write $\tilde{A} \leq \tilde{B}$ as follows:

$$a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}; a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}; a_{ij\nu L} \geq b_{ij\nu L}, a_{ij\nu U} \geq b_{ij\nu U}.$$

Definition 10 An IVPFM $\tilde{A}$ is a null matrix if $\tilde{a}_{ij\mu} = 0$, $\tilde{a}_{ij\eta} = 0$ and $\tilde{a}_{ij\nu} = 0\ \forall\ i = 1, 2, \ldots m$, $j = \ldots n$.

Definition 11 An IVPFM $\tilde{A} = (\tilde{a}_{ij})$ various kind of matrix have been analogically proposed.

(i) An IVPFM is called the row matrix if $i = 1$ ($j = 1, 2 \ldots n$).
(ii) An IVPFM is called the column matrix if $j = 1$ ($i = 1, 2 \ldots m$).
(iii) An IVPFM is called the diagonal matrix if all its non-diagonal elements are zero.
(iv) An IVPFM is called the μ universal matrix if $\tilde{\mu}_{ij} = 1, \tilde{\eta}_{ij} = 0, \tilde{\nu}_{ij} = 0\ \forall\ i = 1, 2, \ldots m, j = \ldots n$.
(v) An IVPFM is called the η universal matrix if $\tilde{\mu}_{ij} = 0, \tilde{\eta}_{ij} = 1, \tilde{\nu}_{ij} = 0\ \forall i = 1, 2, \ldots m, j = \ldots n$.
(vi) An IVPFM is called the ν universal matrix if $\tilde{\mu}_{ij} = 0, \tilde{\eta}_{ij} = 0, \tilde{\nu}_{ij} = 1\ \forall\ i = 1, 2, \ldots m, j = \ldots n$.
(vii) A SIVPFM is called a symmetric matrix if $\tilde{a}_{ij} = \tilde{a}_{ji}$.
(viii) A SIVPFM is called the skew-symmetric if $\tilde{a}_{ij} = Neg(\tilde{a}_{ij})$
(ix) Two IVPFM are called equal if they have the same order and their corresponding elements are equal.
(x) If $\tilde{A}$ is the SIVPFM, then its trace, denoted by $tr(\tilde{A})$, is the sum of the elements on the main diagonal.

Next, we discuss some operations of IVPFM.

3.1 Operations of IVPFM

Before proceeding toward the main theorem of operations of IVPFM, we first define some basic operations.

Definition 12 Let $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])$ and $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ij\nu L}, b_{ij\nu U}])$ be two IVPFM of same order $m \times n$ then, we define some basic operations.

(i) $\tilde{A}^c = ([a_{ij\nu L}, a_{ij\nu U}] [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\mu L}, a_{ij\mu U}])$.
(ii) $\tilde{A} \vee \tilde{B} = ([max(a_{ij\mu L}, b_{ij\mu L}), max(a_{ij\mu U}, b_{ij\mu U})] [min(a_{ij\eta L}, b_{ij\eta L}), min(a_{ij\eta U}, b_{ij\eta U})] [min(a_{ij\nu L}, b_{ij\nu L}), min(a_{ij\nu U}, b_{ij\nu U})])$.
(iii) $\tilde{A} \wedge \tilde{B} = ([min(a_{ij\mu L}, b_{ij\mu L}), min(a_{ij\mu U}, b_{ij\mu U})] [min(a_{ij\eta L}, b_{ij\eta L}), min(a_{ij\eta u}, b_{ij\eta U})] [max(a_{ij\nu L}, b_{ij\nu L}), max(a_{ij\nu U}, b_{ij\nu U})])$.
(iv) $\tilde{A}^T = ([a_{ji\mu L}, a_{ji\mu U}] [a_{ji\eta L}, a_{ji\eta U}] [a_{ji\nu L}, a_{ji\nu U}])$.
(v) $\tilde{A} \oplus \tilde{B} = ([a_{ij\mu L} + b_{ij\mu L} - a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U} + b_{ij\mu U} - a_{ij\mu U}.b_{ij\mu U}] [a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U}.b_{ij\eta U}] [a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U}.b_{ij\nu U}])$.
(vi) $\tilde{A} \otimes \tilde{B} = ([a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U}.b_{ij\mu U}], [a_{ij\eta L} + b_{ij\eta L} - a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U} + b_{ij\eta U} - a_{ij\eta U}.b_{ij\eta U}], [a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U}.b_{ij\nu U}])$.
where $\tilde{A}^c$ and $\tilde{A}^T$ are complement and transpose of $\tilde{A}$ respectively.

Based on the above-defined basic operations, we propose some new properties in the next theorem required throughout the work.

Theorem 1 *Let $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])$, $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ij\nu L}, b_{ij\nu U}])$ and $\tilde{C} = ([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ij\nu L}, c_{ij\nu U}])$ be IVPFM of same order $m \times n$ then the following properties hold true.*

(i) $\tilde{A} \vee \tilde{B} = \tilde{B} \vee \tilde{A}$.
(ii) $\tilde{A} \wedge \tilde{B} = \tilde{B} \wedge \tilde{A}$.
(iii) $(\tilde{A}^T)^T = \tilde{A}$.
(iv) $(\tilde{A}^c)^T = (\tilde{A}^T)^c$.
(v) $\tilde{A} \vee (\tilde{B} \wedge \tilde{C}) = (\tilde{A} \vee \tilde{B}) \wedge (\tilde{A} \vee \tilde{C})$.
(vi) $\tilde{A} \wedge (\tilde{B} \vee \tilde{C}) = (\tilde{A} \wedge \tilde{B}) \vee (\tilde{A} \wedge \tilde{C})$.
(vii) $\tilde{A} \oplus \tilde{B} = \tilde{B} \oplus \tilde{A}$.
(viii) $\tilde{A} \otimes \tilde{B} = \tilde{B} \otimes \tilde{A}$.
(ix) $\tilde{A} \oplus (\tilde{B} \oplus \tilde{C}) = (\tilde{A} \oplus \tilde{B}) \oplus \tilde{C}$.
(x) $\tilde{A} \otimes (\tilde{B} \otimes \tilde{C}) = (\tilde{A} \otimes \tilde{B}) \otimes \tilde{C}$.
(xi) (a) $\tilde{A} \otimes (\tilde{B} \oplus \tilde{C}) \neq (\tilde{A} \otimes \tilde{B}) \oplus (\tilde{A} \otimes \tilde{C})$.
(b) $(\tilde{B} \oplus \tilde{C}) \otimes \tilde{C} \neq (\tilde{B} \otimes A) \oplus (\tilde{C} \otimes \tilde{A})$.

Proof The proof of the properties (i) to (vi) are obvious, therefore, not given here.

(vii)

$$\begin{aligned}
\tilde{A} \oplus \tilde{B} =& ([a_{ij\mu L} + b_{ij\mu L} - a_{ij\mu L}.b_{ij\mu L},\\
& a_{ij\mu U} + b_{ij\mu U} - a_{ij\mu U}.b_{ij\mu U}],\\
& [a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U}.b_{ij\eta U}],\\
& [a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U}.b_{ij\nu U}])\\
=& ([b_{ij\mu L} + a_{ij\mu L} - b_{ij\mu L}.a_{ij\mu L}, b_{ij\mu U}\\
& + a_{ij\mu U} - b_{ij\mu U}.a_{ij\mu U}],\\
& [b_{ij\eta L}.a_{ij\eta L}, b_{ij\eta U}.a_{ij\eta U}],\\
& [b_{ij\nu L}.a_{ij\nu L}, b_{ij\nu U}.a_{ij\nu U}])\\
=& \tilde{B} \oplus \tilde{A}.
\end{aligned}$$

$(viii)$ Similarly, $\tilde{A} \otimes \tilde{B} = \tilde{B} \otimes \tilde{A}$.
(ix)

$$\begin{aligned}
& A \oplus (\tilde{B} \oplus \tilde{C})\\
&= ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])\\
&\oplus ([b_{ij\mu L} + c_{ij\mu L} - b_{ij\mu L}.c_{ij\mu L}, b_{ij\mu U} + c_{ij\mu U}\\
&- b_{ij\mu U}.c_{ij\mu U}], [b_{ij\eta L}.c_{ij\eta L},\\
& b_{ij\eta U}.c_{ij\eta U}], [b_{ij\nu L}.c_{ij\nu L}, b_{ij\nu U}.c_{ij\nu U}])\\
&= ([a_{ij\mu L} + b_{ij\mu L} - a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U}\\
&+ b_{ij\mu U} - a_{ij\mu U}.b_{ij\mu U}], [a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U}.b_{ij\eta U}],\\
& [a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U}.b_{ij\nu U}])
\end{aligned}$$

$$\oplus\,([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ijvL}, c_{ijvU}])$$
$$= (\tilde{A} \oplus \tilde{B}) \oplus \tilde{C}.$$

(x) Similarly, $\tilde{A}\otimes(\tilde{B} \otimes \tilde{C}) = (\tilde{A} \otimes \tilde{B})\otimes\tilde{C}$.

(xi)
(a)

$$\begin{aligned}
&\tilde{B} \oplus \tilde{C}\\
&= ([b_{ij\mu L} + c_{ij\mu L} - b_{ij\mu L}.c_{ij\mu L}, b_{ij\mu U}\\
&+ c_{ij\mu U} - b_{ij\mu U}.c_{ij\mu U}],\\
&[b_{ij\eta L}.c_{ij\eta L}, b_{ij\eta U}.c_{ij\eta U}],\\
&[b_{ijvL}.c_{ijvL}, b_{ijvU}.c_{ijvU}])\\
&\tilde{A} \otimes (\tilde{B} \oplus \tilde{C})\\
&= ([a_{ij\mu L}.(b_{ij\mu L} + c_{ij\mu L} - b_{ij\mu L}.c_{ij\mu L}),\\
&a_{ij\mu U}.(b_{ij\mu U} + c_{ij\mu U} - b_{ij\mu U}.c_{ij\mu U})],\\
&[a_{ij\eta L} + b_{ij\eta L}.c_{ij\eta L} - a_{ij\eta L}.b_{ij\eta L}.c_{ij\eta L},\\
&a_{ij\eta U} + b_{ij\eta U}.c_{ij\eta U} - a_{ij\eta U}.b_{ij\eta U}.c_{ij\eta U}]\\
&[a_{ijvL} + b_{ijvL}.c_{ijvL} - a_{ijvL}.b_{ijvL}.c_{ijvL},\\
&a_{ijvU} + b_{ijvU}.c_{ijvU} - a_{ijvU}.b_{ijvU}.c_{ijvU}]).\\
&\tilde{A} \otimes \tilde{B}\\
&= ([a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U}.b_{ij\mu U}], [a_{ij\eta L} + b_{ij\eta L}\\
&- a_{ij\eta L}.b_{ij\eta L}, a_{ij\eta U} + b_{ij\eta U} - a_{ij\eta U}.b_{ij\eta U}],\\
&\times [a_{ijvL} + b_{ijvL} - a_{ijvL}.b_{ijvL}, a_{ijvU}\\
&+ b_{ijvU} - a_{ijvU}.b_{ijvU}]).\\
&\tilde{A} \otimes \tilde{C}\\
&= ([a_{ij\mu L}.c_{ij\mu L}, a_{ij\mu U}.c_{ij\mu U}], [a_{ij\eta L} + c_{ij\eta L}\\
&- a_{ij\eta L}.c_{ij\eta L}, a_{ij\eta U} + c_{ij\eta U} - a_{ij\eta U}.c_{ij\eta U}],\\
&\times [a_{ijvL} + c_{ijvL} - a_{ijvL}.c_{ijvL}, a_{ijvU}\\
&+ c_{ijvU} - a_{ijvU}.c_{ijvU}])\\
&(\tilde{A} \otimes \tilde{B}) \oplus (\tilde{A} \otimes \tilde{C})\\
&= ([a_{ij\mu L}.(b_{ij\mu U} + c_{ij\mu U}) - a^2_{ij\mu U}.b_{ij\mu L}.c_{ij\mu L},\\
&a_{ij\mu U}.(b_{ij\mu U} + c_{ij\mu U}) - a^2_{ij\mu U}.b_{ij\mu U}.c_{ij\mu U}],\\
&[(a_{ij\eta L} + b_{ij\eta L} - a_{ij\eta L}.b_{ij\eta L}).\\
&(a_{ij\eta L} + c_{ij\eta L} - a_{ij\eta L} - c_{ij\eta L}), b_{ij\eta U}.(a_{ij\eta U} + b_{ij\eta U}\\
&- a_{ij\eta U}.b_{ij\eta U}).(a_{ij\eta U} + c_{ij\eta U} - a_{ij\eta U}.c_{ij\eta U})],\\
&[(a_{ijvL} + b_{ijvL} - a_{ijvL}.b_{ijvL}).(a_{ijvL} + c_{ijvL}\\
&- a_{ijvL} - c_{ijvL}), b_{ijvU}.(a_{ijvU} + b_{ijvU}\\
&- a_{ijvU}.b_{ijvU}).(a_{ijvU} + c_{ijvU} - a_{ijvU}.c_{ijvU})])
\end{aligned}$$

So, $\tilde{A}\otimes(\tilde{B} \oplus \tilde{C}) \neq (\tilde{A} \otimes \tilde{B})\oplus(\tilde{A} \otimes \tilde{C})$.
(b) Similarly, $(\tilde{B} \oplus \tilde{C})\otimes\tilde{C} \neq (\tilde{B} \otimes \tilde{A})\oplus(\tilde{C} \otimes \tilde{A})$. □

Theorem 2 *Let $\tilde{A}$, $\tilde{B}$ be two IVPFM of order $m \times n$ then*

(a) $\left(\tilde{A} \vee \tilde{B}\right)^c = \tilde{A}^c \wedge \tilde{B}^c$.
(b) $\left(\tilde{A} \wedge \tilde{B}\right)^c = \tilde{A}^c \vee \tilde{B}^c$.

Proof (a) Let $\tilde{A} = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$ and $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$
Then, $\tilde{A}^c = ([a_{ijvL}, a_{ijvU}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\mu L}, a_{ij\mu U}])$ and $\tilde{B}^c = ([b_{ijvL}, b_{ijvU}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ij\mu L}, b_{ij\mu U}])$

$$\begin{aligned}
&\tilde{A}^c \wedge \tilde{B}^c\\
&= ([\min(a_{ijvL}, b_{ijvL}), \min(a_{ijvU}, b_{ijvU})]\\
&[\min(a_{ij\eta L}, b_{ij\eta L}), \min(a_{ij\eta U}, b_{ij\eta U})]\\
&[\max(a_{ij\mu L}, b_{ij\mu L}), \max(a_{ij\mu U}, b_{ij\mu U})])\\
&\tilde{A} \vee \tilde{B}\\
&= ([\max(a_{ijvL}, b_{ijvL}), \max(a_{ijvU}, b_{ijvU})]\\
&[\min(a_{ij\eta L}, b_{ij\eta L}), \min(a_{ij\eta U}, b_{ij\eta U})]\\
&[\min(a_{ij\mu L}, b_{ij\mu L}), \min(a_{ij\mu U}, b_{ij\mu U})])\\
&\left(\tilde{A} \vee \tilde{B}\right)^c\\
&= ([\min(a_{ijvL}, b_{ijvL}), \min(a_{ijvU}, b_{ijvU})]\\
&[\min(a_{ij\eta L}, b_{ij\eta L}), \min(a_{ij\eta U}, b_{ij\eta U})]\\
&[\max(a_{ij\mu L}, b_{ij\mu L}), \max(a_{ij\mu U}, b_{ij\mu U})])\\
&= \tilde{A}^c \wedge \tilde{B}^c
\end{aligned}$$

(b) Proof of part (b) can be done on similar lines. □

Theorem 3 *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ and $\tilde{A} \leq \tilde{C}$ and $\tilde{B} \leq \tilde{C}$, then $\tilde{A} \vee \tilde{B} \leq \tilde{C}$.*

Proof Let $\tilde{A}=([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$, $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$ and $\tilde{C} = ([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ijvL}, c_{ijvU}])$.

If $\tilde{A} \leq \tilde{C}$ then $a_{ij\mu L} \leq c_{ij\mu L}, a_{ij\mu U} \leq c_{ij\mu U}, a_{ij\eta L} \leq c_{ij\eta L}, a_{ij\eta U} \leq c_{ij\eta U}, a_{ijvL} \geq c_{ijvL}, a_{ijvU} \geq c_{ijvU}$ for all i, j, and $\tilde{B} \leq \tilde{C}$ then $b_{ij\mu L} \leq c_{ij\mu L}, b_{ij\mu U} \leq c_{ij\mu U}, b_{ij\eta L} \leq c_{ij\eta L}, b_{ij\eta U} \leq c_{ij\eta U}, b_{ijvL} \geq c_{ijvL}, b_{ijvU} \geq c_{ijvU}$ for all i, j. Now, $\max(a_{ij\mu L}, b_{ij\mu L}) \leq c_{ij\mu L}$, $\max(a_{ij\mu U}, b_{ij\mu U}) \leq c_{ij\mu U}$, $\min(a_{ij\eta L}, b_{ij\eta L}) \leq c_{ij\eta L}$, $\min(a_{ij\eta U}, b_{ij\eta U}) \leq c_{ij\eta U}$, $\min(a_{ijvL}, b_{ijvL}) \geq c_{ijvL}$, $\min(a_{ijvU}, b_{ijvU}) \geq c_{ijvU}$. Thus $\tilde{A} \vee \tilde{B} \leq \tilde{C}$ using Definition 9. □

Theorem 4 *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ of $\tilde{A} \leq \tilde{B}$ then $\tilde{A} \vee \tilde{C} \leq \tilde{B} \vee \tilde{C}$.*

Proof Let $\tilde{A}=([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$, $\tilde{B} = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$ and $\tilde{C} = ([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ijvL}, c_{ijvU}])$ be three IVPFM of same order $m \times n$.

If $\tilde{A} \le \tilde{B}$ then $a_{ij\mu L} \le b_{ij\mu L}, a_{ij\mu U} \le b_{ij\mu U}, a_{ij\eta L} \le b_{ij\eta L}, a_{ij\eta U} \le b_{ij\eta U}, a_{ij\nu L} \ge b_{ij\nu L}, a_{ij\nu U} \ge b_{ij\nu U}$.

Now, max $\left(a_{ij\mu L}, c_{ij\mu L}\right) \le$ max $\left(b_{ij\mu L}, c_{ij\mu L}\right)$, max $\left(a_{ij\mu U}, c_{ij\mu U}\right) \le$ max $\left(b_{ij\mu U}, c_{ij\mu U}\right)$,min $\left(a_{ij\eta L}, c_{ij\eta L}\right) \le$ min $\left(b_{ij\eta L}, c_{ij\eta L}\right)$, min $\left(a_{ij\eta U}, c_{ij\eta U}\right) \le$ min $\left(b_{ij\eta U}, c_{ij\eta U}\right)$, min $\left(a_{ij\nu L}, c_{ij\nu L}\right) \ge$ min $\left(b_{ij\nu L}, c_{ij\nu L}\right)$, min $\left(a_{ij\nu U}, c_{ij\nu U}\right) \ge$ min $\left(b_{ij\nu U}, c_{ij\nu U}\right)$ for all i,j.

Therefore $\tilde{A} \vee \tilde{C} \le \tilde{B} \vee \tilde{C}$. □

Theorem 5 *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ and $\tilde{C} \le \tilde{A}$ and $\tilde{C} \le \tilde{B}$ then $\tilde{C} \le \tilde{A} \wedge \tilde{B}$.*

Proof Proof of the above result directly follows from Theorem 4. □

Theorem 6 *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ and if $\tilde{A} \le \tilde{B}$, $\tilde{A} \le \tilde{C}$ and $\tilde{B} \wedge \tilde{C} = 0$, then $\tilde{A} = 0$.*

Proof If $\tilde{A} \le \tilde{B}$ then $a_{ij\mu L} \le b_{ij\mu L}, a_{ij\mu U} \le b_{ij\mu U}; a_{ij\eta L} \le b_{ij\eta L}, a_{ij\eta U} \le b_{ij\eta U}; a_{ij\nu L} \ge b_{ij\nu L}, a_{ij\nu U} \ge b_{ij\nu U}$. Similarly $\tilde{A} \le \tilde{C}$ then $a_{ij\mu L} \le c_{ij\mu L}, a_{ij\mu U} \le c_{ij\mu U}; a_{ij\eta L} \le c_{ij\eta L}, a_{ij\eta U} \le c_{ij\eta U}; a_{ij\nu L} \ge c_{ij\nu L}, a_{ij\nu U} \ge c_{ij\nu U}$. Thus by Theorem 5 $\tilde{A} \le \tilde{B} \wedge \tilde{C}$, $\tilde{B} \wedge \tilde{C} = 0$, such that $\tilde{A} = 0$. □

Theorem 7 *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ of $\tilde{A} \le \tilde{B}$ then $\tilde{A} \wedge \tilde{C} \le \tilde{B} \wedge \tilde{C}$.*

Proof Proof of the above result directly follows from Definition 9. □

Theorem 8 *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three IVPFM of same order $m \times n$ and if $\tilde{A} \le \tilde{B}$, and $\tilde{B} \wedge \tilde{C} = 0$, then $\tilde{A} \wedge \tilde{C} = 0$.*

Proof By Theorem 7, the proof is straight forward. □

In the next section, we present a method for determining the determinant and adjoint of the IVPFM. Illustrative examples are provided to showcase the calculation of both the determinant and adjoint of this matrix.

4 Determinant and adjoint of IVPFM

In this section, we define the determinant, and adjoint of the IVPFM and examine some related fundamental observations.

Definition 13 Determinant of IVPFM Suppose $\tilde{A}$= $([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])$ be a IVPFM of order m. Then, the determinant of $\tilde{A}$ is denoted by $|\tilde{A}|$ and defined by

$$|\tilde{A}| = \begin{array}{l} (\vee_{h\in H_k}([a_{1h(1)\mu L}, a_{1h(1)\mu U}] \wedge [a_{2h(2)\mu L}, a_{2h(2)\mu U}] \cdots \\ \wedge [a_{kh(k)\mu L}, a_{kh(k)\mu U}]), \\ \wedge_{h\in H_k}([a_{1h(1)\eta L}, a_{1h(1)\eta U}] \wedge [a_{2h(2)\eta L}, a_{2h(2)\eta U}] \cdots \\ \wedge [a_{kh(k)\eta L}, a_{kh(k)\eta U}]), \\ \wedge_{h\in H_k}([a_{1h(1)\nu L}, a_{1h(1)\nu U}] \vee [a_{2h(2)\nu L}, a_{2h(2)\nu U}] \cdots \\ \vee [a_{kh(k)\nu L}, a_{kh(k)\nu U}])). \end{array}$$

where H_k be the set of permutation on the set $\{1, 2, 3, \ldots, m\}$.

Example 1 Let us consider IVPFM of order 3 as follows:

$$\tilde{A} = \begin{pmatrix} [0.40, 0.50][0.12, 0.23][0.19, 0.23] & [0.71, 0.79][0.07, 0.09][0.10, 0.12] & [0.21, 0.30][0.12, 0.22][0.39, 0.40] \\ [0.27, 0.43][0.08, 0.28][0.16, 0.23] & [0.42, 0.51][0.13, 0.29][0.07, 0.12] & [0.14, 0.23][0.21, 0.29][0.30, 0.40] \\ [0.35, 0.39][0.11, 0.23][0.06, 0.21] & [0.19, 0.31][0.04, 0.08][0.49, 0.59] & [0.48, 0.57][0.22, 0.3][0, 0.07] \end{pmatrix}$$

To find the determinant of $\tilde{A}$, we need to find out all permutations on $\{1, 2, 3\}$. The permutation on $\{1, 2, 3\}$

$$\psi_1 = \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix},$$
$$\psi_4 = \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix}, \quad \psi_5 = \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix}, \quad \psi_6 = \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix}$$

The membership degree of $|\tilde{A}|$ is

$$\begin{array}{l} ([a_{1\psi_1(1)\mu L}, a_{1\psi_1(1)\mu U}] \wedge [a_{2\psi_1(2)\mu L}, a_{2\psi_1(2)\mu U}] \\ \wedge [a_{3\psi_1(3)\mu L}, a_{3\psi_1(3)\mu U}]) \vee ([a_{1\psi_2(1)\mu L}, a_{1\psi_2(1)\mu U}] \\ \wedge [a_{2\psi_2(2)\mu L}, a_{2\psi_2(2)\mu U}] \wedge [a_{3\psi_2(3)\mu L}, a_{3\psi_2(3)\mu U}]) \\ \vee ([a_{1\psi_3(1)\mu L}, a_{1\psi_3(1)\mu U}] \wedge [a_{2\psi_3(2)\mu L}, a_{2\psi_3(2)\mu U}] \\ \wedge [a_{3\psi_3(3)\mu L}, a_{3\psi_3(3)\mu U}]) \vee ([a_{1\psi_4(1)\mu L}, a_{1\psi_4(1)\mu U}] \\ \wedge [a_{2\psi_4(2)\mu L}, a_{2\psi_4(2)\mu U}] \wedge [a_{3\psi_4(3)\mu L}, a_{3\psi_4(3)\mu U}]) \\ \vee ([a_{1\psi_5(1)\mu L}, a_{1\psi_5(1)\mu U}] \wedge [a_{2\psi_5(2)\mu L}, a_{2\psi_5(2)\mu U}] \\ \wedge [a_{3\psi_5(3)\mu L}, a_{3\psi_5(3)\mu U}]) \vee ([a_{1\psi_6(1)\mu L}, a_{1\psi_6(1)\mu U}] \\ \wedge [a_{2\psi_6(2)\mu L}, a_{2\psi_6(2)\mu U}] \wedge [a_{3\psi_6(3)\mu L}, a_{3\psi_6(3)\mu U}]) \end{array}$$

$$= \begin{array}{l} ([a_{11\mu L}, a_{11\mu U}] \wedge [a_{22\mu L}, a_{22\mu U}] \wedge [a_{33\mu L}, a_{33\mu U}]) \\ \vee ([a_{11\mu L}, a_{11\mu U}] \wedge [a_{23\mu L}, a_{23\mu U}] \wedge [a_{32\mu L}, a_{32\mu U}]) \\ \vee ([a_{12\mu L}, a_{12\mu U}] \wedge [a_{21\mu L}, a_{21\mu U}] \wedge [a_{33\mu L}, a_{33\mu U}]) \\ \vee ([a_{12\mu L}, a_{12\mu U}] \wedge [a_{23\mu L}, a_{23\mu U}] \wedge [a_{31\mu L}, a_{31\mu U}]) \\ \vee ([a_{13\mu L}, a_{13\mu U}] \wedge [a_{21\mu L}, a_{21\mu U}] \wedge [a_{32\mu L}, a_{32\mu U}]) \\ \vee ([a_{13\mu L}, a_{13\mu U}] \wedge [a_{22\mu L}, a_{22\mu U}] \wedge [a_{31\mu L}, a_{31\mu U}]) \end{array}$$

$$= \begin{array}{l} ([0.40, 0.50] \wedge [0.42, 0.51] \wedge [0.48, 0.57]) \\ \vee ([0.40, 0.50] \wedge [0.14, 0.23] \wedge [0.19, 0.31]) \\ \vee ([0.71, 0.79] \wedge [0.27, 0.43] \wedge [0.48, 0.57]) \\ \vee ([0.71, 0.79] \wedge [0.14, 0.23] \wedge [0.35, 0.39]) \\ \vee ([0.21, 0.30] \wedge [0.27, 0.43] \wedge [0.19, 0.31]) \\ \vee ([0.21, 0.30] \wedge [0.42, 0.51] \wedge [0.35, 0.39]) \end{array}$$

$$= \begin{array}{l}[0.4, 0.50] \vee [0.14, 0.23] \vee [0.27, 0.43] \\ \vee[0.14, 0.23] \vee [0.19, 0.30] \vee [0.21, 0.30] = [0.4, 0.5]\end{array}.$$

Similarly, the neutral membership degree of $|\tilde{A}|$ is

$$\begin{array}{l}([a_{1\psi_1(1)\eta L}, a_{1\psi_1(1)\eta U}] \wedge [a_{2\psi_1(2)\eta L}, a_{2\psi_1(2)\eta U}] \\ \wedge[a_{3\psi_1(3)\eta L}, a_{3\psi_1(3)\eta U}]) \wedge ([a_{1\psi_2(1)\eta L}, a_{1\psi_2(1)\eta U}] \\ \wedge[a_{2\psi_2(2)\eta L}, a_{2\psi_2(2)\eta U}] \wedge [a_{3\psi_2(3)\eta L}, a_{3\psi_2(3)\eta U}]) \\ \wedge([a_{1\psi_3(1)\eta L}, a_{1\psi_3(1)\eta U}] \wedge [a_{2\psi_3(2)\eta L}, a_{2\psi_3(2)\eta U}] \\ \wedge[a_{3\psi_3(3)\eta L}, a_{3\psi_3(3)\eta U}]) \wedge ([a_{1\psi_4(1)\eta L}, a_{1\psi_4(1)\eta U}] \\ \wedge[a_{2\psi_4(2)\eta L}, a_{2\psi_4(2)\eta U}] \wedge [a_{3\psi_4(3)\eta L}, a_{3\psi_4(3)\eta U}]) \\ \wedge([a_{1\psi_5(1)\eta L}, a_{1\psi_5(1)\eta U}] \wedge [a_{2\psi_5(2)\eta L}, a_{2\psi_5(2)\eta U}] \\ \wedge[a_{3\psi_5(3)\eta L}, a_{3\psi_5(3)\eta U}]) \wedge ([a_{1\psi_6(1)\eta L}, a_{1\psi_6(1)\eta U}] \\ \wedge[a_{2\psi_6(2)\eta L}, a_{2\psi_6(2)\eta U}] \wedge [a_{3\psi_6(3)\eta L}, a_{3\psi_6(3)\eta U}])\end{array}$$

$$= \begin{array}{l}([a_{11\eta L}, a_{11\eta U}] \wedge [a_{22\eta L}, a_{22\eta U}] \wedge [a_{33\eta L}, a_{33\eta U}]) \\ \wedge([a_{11\eta L}, a_{11\eta U}] \wedge [a_{23\eta L}, a_{23\eta Uta}] \wedge [a_{32\eta L}, a_{32\eta U}]) \\ \wedge([a_{12\eta L}, a_{12\eta U}] \wedge [a_{21\eta L}, a_{21\eta U}]\ wedge[a_{33\eta L}, a_{33\eta U}]) \\ \wedge([a_{12\eta L}, a_{12\eta U}] \wedge [a_{23\eta L}, a_{23\eta U}] \wedge [a_{31\eta L}, a_{31\eta U}]) \\ \wedge([a_{13\eta L}, a_{13\eta U}] \wedge [a_{21\eta L}, a_{21\eta U}] \wedge [a_{32\eta L}, a_{32\eta U}]) \\ \wedge([a_{13\eta L}, a_{13\eta U}] \wedge [a_{22\eta L}, a_{22\eta U}] \wedge [a_{31\eta L}, a_{31\eta U}])\end{array}$$

$$= \begin{array}{l}([0.12, 0.23] \wedge [0.13, 0.29] \wedge [0.22, 0.3]) \\ \wedge([0.12, 0.23] \wedge [0.21, 0.29] \wedge [0.04, 0.08]) \\ \wedge([0.07, 0.09] \wedge [0.08, 0.28] \wedge [0.22, 0.3]) \\ \wedge([0.07, 0.09] \wedge [0.21, 0.29] \wedge [0.11, 0.23]) \\ \wedge([0.12, 0.22] \wedge [0.08, 0.28] \wedge [0.04, 0.08]) \\ \wedge([0.12, 0.22] \wedge [0.13, 0.29] \wedge [0.11, 0.23])\end{array}$$

$$\begin{array}{l}= [0.12, 0.23] \wedge [0.04, 0.08] \wedge [0.07, 0.09] \\ \quad \wedge[0.07, 0.09] \wedge [0.04, 0.08] \wedge [0.11, 0.22] \\ = [0.04, 0.08].\end{array}$$

Now, the non-membership degree of $|\tilde{A}|$ is

$$\begin{array}{l}([a_{1\psi_1(1)\nu L}, a_{1\psi_1(1)\nu U}] \vee [a_{2\psi_1(2)\nu L}, a_{2\psi_1(2)\nu U}] \\ \vee[a_{3\psi_1(3)\nu L}, a_{3\psi_1(3)\nu U}]) \wedge ([a_{1\psi_2(1)\nu L}, a_{1\psi_2(1)\nu U}] \\ \vee[a_{2\psi_2(2)\nu L}, a_{2\psi_2(2)\nu U}] \vee [a_{3\psi_2(3)\nu L}, a_{3\psi_2(3)\nu U}]) \\ \wedge([a_{1\psi_3(1)\nu L}, a_{1\psi_3(1)\nu U}] \vee [a_{2\psi_3(2)\nu L}, a_{2\psi_3(2)\nu U}] \\ \vee[a_{3\psi_3(3)\nu L}, a_{3\psi_3(3)\nu U}]) \wedge ([a_{1\psi_4(1)\nu L}, a_{1\psi_4(1)\nu U}] \\ \vee[a_{2\psi_4(2)\nu L}, a_{2\psi_4(2)\nu U}] \vee [a_{3\psi_4(3)\nu L}, a_{3\psi_4(3)\nu U}]) \\ \wedge([a_{1\psi_5(1)\nu L}, a_{1\psi_5(1)\nu U}] \vee [a_{2\psi_5(2)\nu L}, a_{2\psi_5(2)\nu U}] \\ \vee[a_{3\psi_5(3)\nu L}, a_{3\psi_5(3)\nu U}]) \wedge ([a_{1\psi_6(1)\nu L}, a_{1\psi_6(1)\nu U}] \\ \vee[a_{2\psi_6(2)\nu L}, a_{2\psi_6(2)\nu U}] \vee [a_{3\psi_6(3)\nu L}, a_{3\psi_6(3)\nu U}])\end{array}$$

$$= \begin{array}{l}([a_{11\nu L}, a_{11\nu U}] \vee [a_{22\nu L}, a_{22\nu U}] \vee [a_{33\nu L}, a_{33\nu U}]) \\ \wedge([a_{11\nu L}, a_{11\nu U}] \vee [a_{23\nu L}, a_{23\nu Uta}] \vee [a_{32\nu L}, a_{32\nu U}]) \\ \wedge([a_{12\nu L}, a_{12\nu U}] \vee [a_{21\nu L}, a_{21\nu U}] \vee [a_{33\nu L}, a_{33\nu U}]) \\ \wedge([a_{12\nu L}, a_{12\nu U}] \vee [a_{23\nu L}, a_{23\nu U}] \vee [a_{31\nu L}, a_{31\nu U}]) \\ \wedge([a_{13\nu L}, a_{13\nu U}] \vee [a_{21\nu L}, a_{21\nu U}] \vee [a_{32\nu L}, a_{32\nu U}]) \\ \wedge([a_{13\nu L}, a_{13\nu U}] \vee [a_{22\nu L}, a_{22\nu U}] \vee [a_{31\nu L}, a_{31\nu U}])\end{array}$$

$$= \begin{array}{l}([0.19, 0.23] \vee [0.07, 0.12] \vee [0, 0.07]) \\ \wedge([0.19, 0.23] \vee [0.3, 0.40] \vee [0.49, 0.59]) \\ \wedge([0.10, 0.12] \vee [0.16, 0.23] \vee [0, 0.07]) \\ \wedge([0.10, 0.12] \vee [0.30, 0.40] \vee [0.06, 0.21]) \\ \wedge([0.39, 0.44] \vee [0.16, 0.23] \vee [0.49, 0.59]) \\ \wedge([0.39, 0.44] \vee [0.07, 0.12] \vee [0.06, 0.21])\end{array}$$

$$\begin{array}{l}= [0.19, 0.27] \wedge [0.49, 0.59] \wedge [0.16, 0.23] \\ \wedge[0.30, 0.40] \wedge [0.49, 0.59] \wedge [0.39, 0.44] \\ = [0.16, 0.23].\end{array}$$

$\therefore |\tilde{A}| = ([0.4, 0.5][0.04, 0.08][0.16, 0.23])$.

Definition 14 Adjoint of IVPFM Let $\tilde{A} = (\tilde{a}_{ij}) = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu} >)$ be a IVPFM of order m. Then, the adjoint of $\tilde{A}$ is denoted by the Adjoint$(\tilde{A})$ and defined by

$$\text{Adjoint}(\tilde{A}) = \tilde{Q} = (< \tilde{q}_{ij\mu}, \tilde{q}_{ij\eta}, \tilde{q}_{ij\nu} >)$$

where

$$\tilde{q}_{ij\mu} = \vee_{\delta \in S_{m_j m_i}} \wedge_{u \in m_j} \tilde{a}_{u\delta(u)\mu},$$
$$\tilde{q}_{ij\eta} = \wedge_{\delta \in S_{m_j m_i}} \wedge_{u \in m_j} \tilde{a}_{u\delta(u)\eta},$$
$$\tilde{q}_{ij\nu} = \wedge_{\delta \in S_{m_j m_i}} \vee_{u \in m_j} \tilde{a}_{u\delta(u)\nu}.$$

Here m_j= {1, 2 ... m}-{j} and $S_{m_j m_i}$ is the set of all permutation of the set m_j over the set m_i.

Example 2 Let us consider IVPFM of order three as follows:

$$\tilde{A} = \begin{pmatrix} [0.40, 0.50][0.12, 0.23][0.19, 0.23] & [0.71, 0.79][0.07, 0.09][0.10, 0.12] & [0.21, 0.30][0.12, 0.22][0.39, 0.40] \\ [0.27, 0.43][0.08, 0.23][0.16, 0.23] & [0.42, 0.51][0.13, 0.29][0.07, 0.12] & [0.14, 0.23][0.21, 0.29][0.30, 0.40] \\ [0.35, 0.39][0.11, 0.23][0.06, 0.21] & [0.19, 0.31][0.04, 0.08][0.49, 0.59] & [0.48, 0.57][0.22, 0.3][0, 0.07] \end{pmatrix}$$

For j=1 and i=1, m_j= {1, 2, 3}-{1}= {2, 3} and m_i= {1, 2, 3}-{1}= {2, 3}. The permutation of m_i over m_j are

$$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

Now,

$$(a_{22\mu} \wedge a_{33\mu}) \vee (a_{23\mu} \wedge a_{32\mu})$$

$$= ([0.42, 0.51] \wedge [0.48, 0.57])$$
$$\vee([0.14, 0.23] \wedge [0.19, 0.31])$$
$$= [0.42, 0.51] \vee [0.14, 0.23] = [0.42, 0.51],$$
$$(a_{22\eta} \wedge a_{33\eta}) \wedge (a_{23\eta} \wedge a_{32\eta})$$
$$= ([0.13, 0.29] \wedge [0.22, 0.3])$$
$$\wedge([0.21, 0.29] \wedge [0.04, 0.08])$$
$$= [0.13, 0.29] \wedge [0.04, 0.08] = [0.04, 0.08],$$
$$(a_{22\nu} \vee a_{33\nu}) \wedge (a_{23\nu} \vee a_{32\nu})$$
$$= ([0.07, 0.12] \vee [0, 0.07]) \wedge ([0.3, 0.4] \vee [0.49, 0.59])$$
$$= [0.07, 0.12] \wedge [0.49, 0.59] = [0.07, 0.12].$$

For $j = 1$ and $i = 2$, $m_j = \{1, 2, 3\} - \{1\} = \{2, 3\}$ and $m_i = \{1, 2, 3\} - \{2\} = \{1, 3\}$. The permutation of m_i over m_j are

$$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

For $j = 1$ and $i = 3$, $m_j = \{1, 2, 3\} - \{1\} = \{2, 3\}$ and $m_i = \{1, 2, 3\} - \{3\} = \{1, 2\}$. The permutation of m_i over m_j are

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Now

$$(a_{12\mu} \wedge a_{23\mu}) \vee (a_{13\mu} \wedge a_{22\mu})$$
$$([0.71, 0.79] \wedge [0.14, 0.23]) \vee ([0.21, 0.30] \wedge [0.42, 0.51])$$
$$= [0.14, 0.23] \vee [0.21, 0.30] = [0.21, 0.30],$$
$$(a_{12\eta} \wedge a_{23\eta}) \wedge (a_{13\eta} \wedge a_{22\eta})$$
$$([0.07, 0.09] \wedge [0.21, 0.29]) \wedge ([0.12, 0.24] \wedge [0.13, 0.29])$$
$$= [0.07, 0.09] \wedge [0.12, 0.24] = [0.07, 0.09],$$
$$\text{and } (a_{12\nu} \vee a_{23\nu}) \wedge (a_{13\nu} \vee a_{22\nu})$$
$$([0.10, 0.12] \vee [0.3, 0.4]) \wedge ([0.39, 0.44] \vee [0.07, 0.27])$$
$$= [.3, 0.4] \wedge [0.39, 0.44] = [0.3, 0.4].$$

Calculating in the similar way, Adjoint$(\tilde{A})$ is obtained as

$$\text{Adjoint}(\tilde{A}) = \begin{pmatrix} [0.49, 0.51][0.04, 0.08][0.07, 0.12] & [0.48, 0.57][0.07, 0.09][0.10, 0.12] & [0.21, 0.30][0.07, 0.09][0.3, 0.4] \\ [0.27, 0.43][0.08, 0.23][0.06, 0.21] & [0.4, 0.5][0.11, 0.23][0.06, 0.21] & [0.21, 0.3][0.08, 0.23][0.19, 0.23] \\ [0.35, 0.39][0.04, 0.08][0.16, 0.23] & [0.35, 0.39][0.04, 0.08][0.19, 0.23] & [0.4, 0.5][0.07, 0.09][0.10, 0.27] \end{pmatrix}$$

Now

$$(a_{12\mu} \wedge a_{33\mu}) \vee (a_{13\mu} \wedge a_{32\mu})$$
$$= ([0.71, 0.79] \wedge [0.48, 0.57])$$
$$\vee([0.21, 0.30] \wedge [0.19, 0.31])$$
$$= [0.48, 0.57] \vee [0.19, 0.30] = [0.48, 0.57],$$
$$(a_{12\eta} \wedge a_{33\eta}) \wedge (a_{13\eta} \wedge a_{32\eta})$$
$$= ([0.07, 0.09] \wedge [0.22, 0.3])$$
$$\wedge([0.12, 0.24] \wedge [0.13, 0.29])$$
$$= [0.07, 0.09] \wedge [0.12, 0.24] = [0.07, 0.09],$$
$$(a_{12\nu} \vee a_{33\nu}) \wedge (a_{13\nu} \vee a_{32\nu})$$
$$([0.10, 0.12] \vee [0, 0.07]) \wedge ([0.39, 0.44] \vee [0.49, 0.59])$$
$$= [0.10, 0.12] \wedge [0.49, 0.59] = [0.10, 0.12].$$

Proposition 1 *If $\tilde{A}$ be a square IVPFM, then $|\tilde{A}| = |\tilde{A}^T|$.*

Proof Let $\tilde{A}$= $([a_{ij\mu L}, {}_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])$ $\implies$ $\tilde{A}^T = ([a_{ji\mu L}, {}_{ji\mu U}], [a_{ji\eta L}, a_{ji\eta U}], [a_{ji\nu L}, a_{ji\nu U}])$. Then

$$|\tilde{A}^T| = \sum_{\sigma \in \delta_n} \begin{pmatrix} [a_{\sigma(1)1\mu L}, a_{\sigma(1)1\mu U}], [a_{\sigma(1)1\eta L}, a_{\sigma(1)1\eta U}], [a_{\sigma(1)1\nu L}, a_{\sigma(1)1\nu U}] \\ [a_{\sigma(2)2\mu L}, a_{\sigma(2)2\mu U}], [a_{\sigma(2)2\eta L}, a_{\sigma(2)2\eta U}], [a_{\sigma(2)2\nu L}, a_{\sigma(2)2\nu U}] \\ \dots \\ [a_{\sigma(n)n\mu L}, a_{\sigma(n)n\mu U}], [a_{\sigma(n)n\eta L}, a_{\sigma(n)n\eta U}], [a_{\sigma(n)n\nu L}, a_{\sigma(n)n\nu U}] \end{pmatrix}$$

Let Ψ be the permutation of $\{1, 2, \dots n\}$, such that $\Psi\sigma = I$ then $\Psi = \sigma^{-1}$ $\sigma(i) = j, i = \sigma^{-1}(j) = \Psi(j)$.
Therefore $a_{\sigma(i)i\mu L} = a_{j\Psi(j)\mu L}$, $a_{\sigma(i)i\mu U} = a_{j\Psi(j)\mu U}$, $a_{\sigma(i)i\eta L} = a_{j\Psi(j)\mu L}$, $a_{\sigma(i)i\eta U} = a_{j\Psi(j)\eta U}$, $a_{\sigma(i)i\nu L} = a_{j\Psi(j)\nu L}$, $a_{\sigma(i)i\nu U} = a_{j\Psi(j)\nu U} \forall i, j$. As i runs over the set $\{1, 2, \dots n\}$, so does j.

$$\therefore |\tilde{A}^T| = \begin{matrix} ([a_{\Psi 1(1)\mu L}, a_{\Psi 1(1)\mu U}], [a_{\Psi 1(1)\eta L}, a_{\Psi 1(1)\eta U}], [a_{\Psi 1(1)\nu L}, a_{\Psi 1(1)\nu U}]) \\ ([a_{\Psi 2(2)\mu L}, a_{\Psi 2(2)\mu U}], [a_{\Psi 2(2)\eta L}, a_{\Psi 2(2)\eta U}], [a_{\Psi 2(2)\nu L}, a_{\Psi 2(2)\nu U}]) \ldots \\ ([a_{\Psi n(n)\mu L}, a_{\Psi n(n)\mu U}], [a_{\Psi n(n)\eta L}, a_{\Psi n(n)\eta U}], [a_{\Psi n(n)\nu L}, a_{\Psi n(n)\nu U}]) \end{matrix}$$

$$= \begin{matrix} \sum_{\Psi \in \delta_n} ([a_{1\Psi(1)\mu L}, a_{1\Psi(1)\mu U}], [a_{1\Psi(1)\eta L}, a_{1\Psi(1)\eta U}], [a_{1\Psi(1)\nu L}, a_{1\Psi(1)\nu U}]) \\ ([a_{2\Psi(2)\mu L}, a_{2\Psi(2)\mu U}], [a_{2\Psi(2)\eta L}, a_{2\Psi(2)\eta U}], [a_{2\Psi(2)\nu L}, a_{2\Psi(2)\nu U}]) \ldots \\ ([a_{n\Psi(n)\mu L}, a_{n\Psi(n)\mu U}], [a_{n\Psi(n)\eta L}, a_{n\Psi(n)\eta U}], [a_{n\Psi(n)\nu L}, a_{n\Psi(n)\nu U}]) \end{matrix}$$

$$= |\tilde{A}|.$$

□

Proposition 2 *If $\tilde{A}$ and $\tilde{B}$ be two square IVPFM and $\tilde{A} \leq \tilde{B}$, then Adjoint($\tilde{A}$)$\leq$ Adjoint($\tilde{B}$).*

Proof Adjoint($\tilde{A}$) $= \Sigma_{\sigma \in S_{n_i} n_j} \Pi_{t \in n_j} ([a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}], [a_{t\sigma(t)\eta L}, a_{t\sigma(t)\eta U}], [a_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu U}])$ and Adjoint($\tilde{B}$) $= \Sigma_{\sigma \in S_{n_i} n_j} \Pi_{t \in n_j} ([b_{t\sigma(t)\mu L}, b_{t\sigma(t)\mu U}], [b_{t\sigma(t)\eta L}, b_{t\sigma(t)\eta U}], [b_{t\sigma(t)\nu L}, b_{t\sigma(t)\nu U}])$. Using the given hypothesis, $a_{t\sigma(t)\mu L} \leq b_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U} \leq b_{t\sigma(t)\mu U}, a_{t\sigma(t)\eta L} \leq b_{t\sigma(t)\eta L}, a_{t\sigma(t)\eta U} \leq b_{t\sigma(t)\mu U}, a_{t\sigma(t)\nu L} \geq b_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu U} \geq b_{t\sigma(t)\nu U}, t \neq j, \sigma(t) \neq \sigma(j)$, Therefore Adjoint($\tilde{A}$)$\leq$Adjoint($\tilde{B}$). □

Proposition 3 *For a square IVPFM A, then Adjoint($\tilde{A}^T$) =(Adjoint$\tilde{A}$)^T.*

Proof The proof follows using Definition 14 and Proposition 1. □

In the next section, first, we introduce the definition of eigen interval-valued picture fuzzy sets and develop the algorithms for identifying the greatest and least eigen interval-valued picture fuzzy sets. Then a numerical example is demonstrated to illustrate the application of the same. Algorithm for the same is provided in Figs. 1 and 2.

5 Greatest Eigen interval-valued picture fuzzy set and least Eigen interval-valued picture fuzzy set

In this section, we introduce the notion of EIVPFS and provide necessary steps of an appropriate method for finding the GEIVPFS and LEIVPFS with the help of numerical example.

Definition 15 An interval-valued picture fuzzy relation (IVPFR) R between two IVPFS X and Y defined as follows:

$$R = \{\langle (x, y), \mu_R(x, y), \eta_R(x, y), \nu_R(x, y)\rangle | x \in X, y \in Y\},$$

where μ_R=$[\mu_R^L, \mu_R^U]$, η_R=$[\eta_R^L, \eta_R^U]$, ν_R=$[\nu_R^L, \nu_R^U]$, such that $0 \leq \mu_R^U + \eta_R^U + \nu_R^U \leq 1$ for every $(x,y) \in (X \times Y)$.

Consider $R_1 \in (X \times Y)$ and $R_2 \in (Y \times Z)$ be two IVPFR. The following composition operators for the IVPFR R_1 and R_2 is defined by Cuong and Kreinovich (2014) as follows:
Max–Min composition: The max–min composition operator is represented by

$$R_1 \circ R_2 = \{\langle (x_{ij}, z_{ij}), \mu_{R_1 \circ R_2}(x_{ij}, z_{ij}), \eta_{R_1 \circ R_2}(x_{ij}, z_{ij}), \nu_{R_1 \circ R_2}(x_{ij}, z_{ij})\rangle | x_{ij} \in X, z_{ij} \in Z\},$$

where $\mu_{R_1 \circ R_2}(x_{ij}, z_{ij}) = [\mu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \mu^U_{R_1 \circ R_2}(x_{ij}, z_{ij})]$, $\eta_{R_1 \circ R_2}(x_{ij}, z_{ij}) = [\eta^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \eta^U_{R_1 \circ R_2}(x_{ij}, z_{ij})]$, and, $\nu_{R_1 \circ R_2}(x_{ij}, z_{ij})$=$[\nu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}), \nu^U_{R_1 \circ R_2}(x_{ij}, z_{ij})]$.
In addition

$$\mu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \max_{y \in Y}\{\min_{x \in X}(\mu^L_{R_1}(x_{ij}, y_{ij}), \mu^L_{R_2}(y_{ij}, z_{ij})\},$$

$$\mu^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \max_{y \in Y}\{\min_{x \in X}(\mu^U_{R_1}(x_{ij}, y_{ij}), \mu^U_{R_2}(y_{ij}, z_{ij})\},$$

$$\eta^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \min_{y \in Y}\{\min_{x \in X}(\eta^L_{R_1}(x_{ij}, y_{ij}), \eta^L_{R_2}(y_{ij}, z_{ij})\},$$

$$\eta^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \min_{y \in Y}\{\min_{x \in X}(\eta^U_{R_1}(x_{ij}, y_{ij}), \eta^U_{R_2}(y_{ij}, z_{ij})\},$$

$$\nu^L_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \min_{y \in Y}\{\max_{x \in X}(\nu^L_{R_1}(x_{ij}, y_{ij}), \nu^L_{R_2}(y_{ij}, z_{ij})\},$$

$$\nu^U_{R_1 \circ R_2}(x_{ij}, z_{ij}) = \min_{y \in Y}\{\max_{x \in X}(\nu^U_{R_1}(x_{ij}, y_{ij}), \nu^U_{R_2}(y_{ij}, z_{ij})\}.$$

Min–max composition: The min–max composition operator is represented by

$$R_1 \bullet R_2 = \{\langle (x_{ij}, z_{ij}), \mu_{R_1 \bullet R_2}(x_{ij}, z_{ij}), \eta_{R_1 \bullet R_2}(x_{ij}, z_{ij}), \nu_{R_1 \bullet R_2}(x_{ij}, z_{ij})\rangle | x_{ij} \in X, z_{ij} \in Z\},$$

where $\mu_{R_1 \bullet R_2}(x_{ij}, z_{ij})$=$[\mu^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}), \mu^U_{R_1 \bullet R_2}(x_{ij}, z_{ij})]$, $\eta_{R_1 \bullet R_2}(x_{ij}, z_{ij})$=$[\eta^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}), \eta^U_{R_1 \bullet R_2}(x_{ij}, z_{ij})]$, and, $\nu_{R_1 \bullet R_2}(x_{ij}, z_{ij})$=$[\nu^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}), \nu^U_{R_1 \bullet R_2}(x_{ij}, z_{ij})]$
Further,

$$
\begin{aligned}
&\mu^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}) \\
&\quad = \min_{y \in Y}\{\max_{x \in X}(\mu^L_{R_1}(x_{ij}, y_{ij}), \mu^L_{R_2}(y_{ij}, z_{ij})\}, \\
&\mu^U_{R_1 \bullet R_2}(x_{ij}, z_{ij}) \\
&\quad = \min_{y \in Y}\{\max_{x \in X}(\mu^U_{R_1}(x_{ij}, y_{ij}), \mu^U_{R_2}(y_{ij}, z_{ij})\}, \\
&\eta^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}) \\
&\quad = \min_{y \in Y}\{\min_{x \in X}(\eta^L_{R_1}(x_{ij}, y_{ij}), \eta^L_{R_2}(y_{ij}, z_{ij})\}, \\
&\eta^U_{R_1 \bullet R_2}(x_{ij}, z_{ij}) \\
&\quad = \min_{y \in Y}\{\min_{x \in X}(\eta^U_{R_1}(x_{ij}, y_{ij}), \eta^U_{R_2}(y_{ij}, z_{ij})\} \\
&\nu^L_{R_1 \bullet R_2}(x_{ij}, z_{ij}) \\
&\quad = \max_{y \in Y}\{\min_{x \in X}(\nu^L_{R_1}(x_{ij}, y_{ij}), \nu^L_{R_2}(y_{ij}, z_{ij})\}, \\
&\nu^U_{R_1 \bullet R_2}(x_{ij}, z_{ij}) \\
&\quad = \max_{y \in Y}\{\min_{x \in X}(\nu^U_{R_1}(x_{ij}, y_{ij}), \nu^U_{R_2}(y_{ij}, z_{ij})\}.
\end{aligned}
$$

It is easy to verify that N_1 is an eigen interval-valued picture fuzzy set, but not the GEIVPFS always. To evaluate GEIVPFS, the following sequences are evaluated using max–min composition:

$$
\begin{aligned}
&N_1 \circ R = N_2, \\
&N_2 \circ R = N_1 \circ R^2 = N_3, \\
&N_3 \circ R = N_1 \circ R^3 = N_4, \\
&\vdots \\
&N_n \circ R = N_1 \circ R^n = N_{n+1}.
\end{aligned}
$$

Now, we give an algorithm to evaluate GEIVPFS.
Algorithm 1 (GEIVPFS)

Step 1 Calculate the set N_1 from R using the above Eq. 1.
Step 2 Set the index $n = 1$ and find $N_{n+1} = N_n \circ R$.
Step 3 If $N_{n+1} \neq N_n$ then go to step 2.
Step 4 If $N_{n+1} = N_n$ then N_n is the GEIVPFS associated with R.

Example 3 Let $\tilde{A}$= $(\tilde{a}, \tilde{b}, \tilde{c})$ be the IVPFM and R be the interval-valued picture fuzzy relation on $\tilde{A}$ represented as follows:

$$
\begin{array}{cc}
 & \begin{array}{ccc} a & b & c \end{array} \\
\begin{array}{c} a \\ b \\ c \end{array} &
\begin{pmatrix}
[0.40, 0.50][0.12, 0.23][0.19, 0.23] & [0.71, 0.79][0.07, 0.09][0.10, 0.12] & [0.21, 0.30][0.12, 0.24][0.39, 0.44] \\
[0.27,0.43][0.08,0.28][0.16,0.23] & [0.42,0.51][0.13,0.29][0.07,0.12] & [0.14,0.23][0.21,0.29][0.30,0.40] \\
[0.35,0.39][0.11,0.23][0.06,0.21] & [0.19,0.31][0.04,0.08][0.49,0.59] & [0.48,0.57][0.22,0.3][0,0.07]
\end{pmatrix}
\end{array}
$$

Definition 16 Suppose R is an IVPFR defined on IVPFS of X. An IVPFS N is said to be an eigen interval-valued picture fuzzy set associated with the relation R if $N \odot R$=N, where $\odot$ is any of the above-defined composition operators.

5.1 Greatest Eigen interval valued picture fuzzy set

Here, we apply the max–min composition operator for calculating the GEIVPFS with the IVPFR R. Suppose N_1 be the IVPFS, in which the degree of membership is the greatest of all elements of the column of relation R, the degree of neutral membership and degree of non-membership is the lowest of all the elements of the column of R:

$$
\begin{aligned}
\mu_{N_1}(u) &= \max_{x \in X} \mu_R(x, u) \forall u \in Y, \\
\eta_{N_1}(u) &= \min_{x \in X} \eta_R(x, u) \forall u \in Y, \\
\nu_{N_1}(u) &= \min_{x \in X} \nu_R(x, u) \forall u \in Y.
\end{aligned}
\tag{1}
$$

We solve this as follows:
Step 1

$$
\begin{aligned}
N_1 = &([0.40, 0.50][0.08, 0.23][0.06, 0.21]), \\
&([0.71, 0.79][0.04, 0.08][0.07, 0.12]), \\
&([0.48, 0.57][0.12, 0.24][0, 0.07]).
\end{aligned}
$$

Step 2 For $n = 1$, $N_2 = N_1 \circ R$,

$$
\begin{aligned}
N_2 = &([0.40, 0.50][0.04, 0.08][0.06, 0.21]), \\
&([0.42, 0.51][0.04, 0.08][0.07, 0.12]), \\
&([0.48, 0.57][0.04, 0.08][0, 0.07]).
\end{aligned}
$$

Step 3 Since $N_2 \neq N_1$, we set $n = 2$ in step 2 and compose N_2 with R, to get N_3, i.e, $N_3 = N_2 \circ R$:

$$
\begin{aligned}
N_3 = &([0.40, 0.50][0.04, 0.08][0.06, 0.21]), \\
&([0.42, 0.51][0.04, 0.08][0.07, 0.12]), \\
&([0.48, 0.57][0.04, 0.08][0, 0.07]).
\end{aligned}
$$

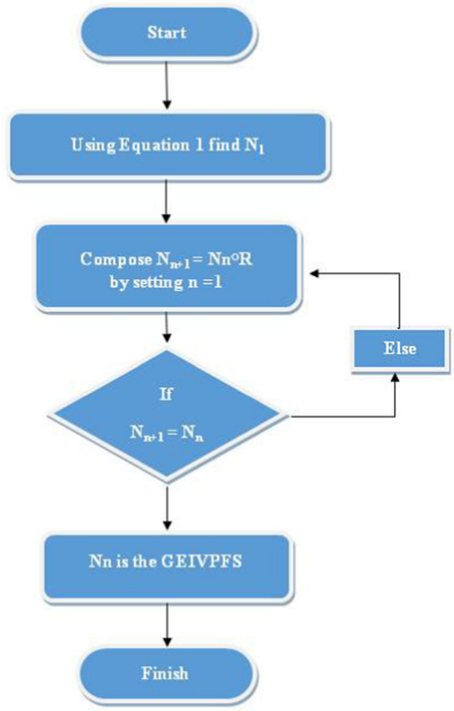

Fig. 1 Flow chart for Algorithm I (GEIVPFS)

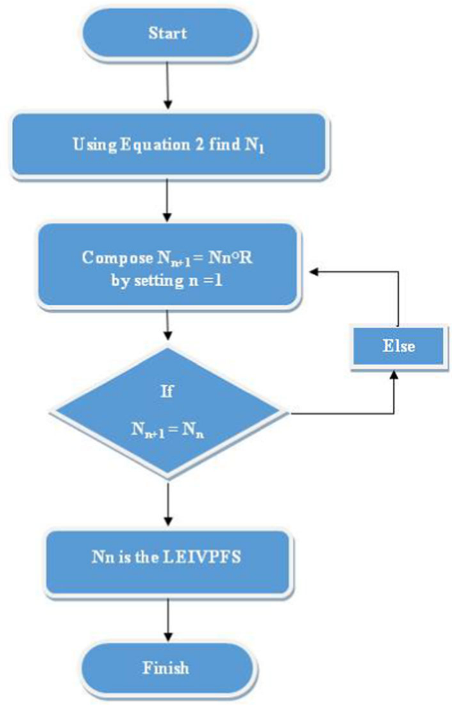

Fig. 2 Flow chart for Algorithm II (LEIVPFS)

Step 4 Since N_3=N_2, thus N_2 is the GEIVPFS associated with R.

5.2 Least Eigen interval-valued picture fuzzy set

Here, we apply the max–min composition operator for calculating the LEIVPFS with the IVPF relation R. Suppose N_1 be the IVPFS, in which the degree of membership, the degree of neutral membership is the smallest of all elements of the column of relation R, and the degree of non-membership is the greatest of all the elements of the column of R:

$$\begin{aligned} \mu_{N_1}(u) &= \min_{x\in X} \mu_R(x,u) \forall u \in Y, \\ \eta_{N_1}(u) &= \min_{x\in X} \eta_R(x,u) \forall u \in Y, \\ \nu_{N_1}(u) &= \max_{x\in X} \nu_R(x,u) \forall u \in Y. \end{aligned} \tag{2}$$

We can easily find that N_1 is an eigen interval-valued picture fuzzy set, but our focus is to find LEIVPFS. We define the sequence of IVPFS N_n, such that

$$\begin{aligned} &N_1 \circ R = N_2, \\ &N_2 \circ R = N_1 \circ R^2 = N_3, \\ &N_3 \circ R = N_1 \circ R^3 = N_4, \\ &\vdots \\ &N_n \circ R = N_1 \circ R^n = N_{n+1}. \end{aligned}$$

For the determination of the LEIVPFS, we now present the following algorithm followed by a numerical example along with real-life application of the defined GEIVPFS and LEIVPFS.

Algorithm II(LEIVPFS)

Step 1 Calculate the set N_1 from R using above Eq. 2.
Step 2 Set the index n=1 and find $N_{n+1} = N_n \circ R$.
Step 3 If $N_{n+1} \neq N_n$ then go to step 2.
Step 4 If $N_{n+1} = N_n$ then N_n is the LEIVPFS associated with R.

We consider the same Example 3 for the illustration of the computational steps of Algorithm II as below:
Step 1

$$\begin{aligned} N_1 = \{&([0.27, 0.39][0.08, 0.23][0.19, 0.23]), \\ &([0.19, 0.31][0.04, 0.08][0.49, 0.59]), \\ &([0.14, 0.23][0.12, 0.24][0.39, 0.44])\}. \end{aligned}$$

Step 2 For $n = 1$, $N_2 = N_1 \circ R$,

$$\begin{aligned} N_2 = &([0.27, 0.39][0.04, 0.08][0.19, 0.23]), \\ &([0.27, 0.39][0.04, 0.08][0.19, 0.23]), \\ &([0.21, 0.3][0.04, 0.08][0.39, 0.44]). \end{aligned}$$

Step 3 Since $N_2 \neq N_1$, we set $n = 2$ in step 2 and compose N_2 with R, to get N_3, i.e, $N_3 = N_2 \circ R$,

$$\begin{aligned} N_3 &= ([0.27, 0.39][0.04, 0.08][0.19, 0.23]),\\ &([0.27, 0.39][0.04, 0.08][0.19, 0.23]),\\ &([0.21, 0.3][0.04, 0.08][0.3, 0.4]). \end{aligned}$$

Step 4

$$\begin{aligned} N_4 = &([0.27, 0.39][0.04, 0.08][0.19, 0.23]),\\ &([0.27, 0.39][0.04, 0.08][0.19, 0.23]),\\ &([0.21, 0.3][0.04, 0.08][0.3, 0.4]). \end{aligned}$$

Step 5 Since N_4=N_3, thus N_3 is the LEIVPFS associated with R.
Next, we provide a real-life application of the defined algorithms to validate their applicability. For this purpose, we consider multiple criteria decision-making problems of Health insurance companies where customers' satisfaction/abstain/non-satisfaction levels are taken into account for formulating the multiple criteria decision-making problems.

5.3 Application of GEIVPFS and LEIVPFS in multiple criteria decision-making

Consider a health insurance company interviewing 10 of its most valuable clients or industry professionals to learn about the key aspects of the business. Let the characteristic be listed as follows:

- H_1: Policies that Value Customers.
- H_2: Size of the Financial Benefits.
- H_3: Insurance Post Services.

A survey may be used to determine the customer's feedback. However, we assume a set of data presented below without conducting an exhaustive survey to illustrate the suggested methodology. To evaluate some final observations from the health insurance company's perspective, we assume that each customer's feedback is an interval-valued picture fuzzy information that is relative to all the that are available in Tables 1, 2, and 3.
The desire levels can be estimated as satisfaction/abstain/non-satisfaction levels. This is possible by considering the interval-valued picture fuzzy relation. Each pair in the relation $R(H_j, H_k)$ has three values that range from 0 to 1; a membership degree (satisfied), a neutral membership degree (abstain), and the non-membership degree (not satisfied) is given by

$$R_{(H_j,H_k)}$$

$$= \left(\frac{\sum_{p=1,q=1}^{p=m,q=n} \mu_{pq}}{m}, \frac{\sum_{p=1,q=1}^{p=m,q=n} \eta_{pq}}{m}, \frac{\sum_{p=1,q=1}^{p=m,q=n} \nu_{pq}}{m} \right); \tag{3}$$

and

$$R_{(H_i,H_i)} = \frac{R_{(H_i,H_j)} + R_{(H_i,H_k)}}{2}; \tag{4}$$

where $j, k = 1, 2, \ldots, n$.
From Tables 1, 2 and Tables 2, 3, the membership, neutral membership and non-membership degree for $R_{(H_1,H_1)}$ and $R_{(H_1,H_3)}$ can be computed respectively with the help of Eqs. 3, i.e. $R_{(H_1,H_1)}$ = ([0.299,0.406][0.11,0.205][0.209,0.321]) and $R_{(H_1,H_3)}$ =([0.301,0.417][0.148,0.231][0.193,0.292]). Suppose $i = 1$, $j = 2$, $k = 3$ in Eq. 4, we find

$$R_{(H_1,H_1)} = \frac{R_{(H_1,H_2)} + R_{(H_1,H_3)}}{2}. \tag{5}$$

Now, by putting the values of $R_{(H_1,H_1)}$ and $R_{(H_1,H_3)}$ in Eq. 5, we can compute the value of $R_{(H_1,H_2)}$, that is

$$R_{(H_1,H_2)} = ([0.297, .395][0.072, 0.179][0.225, 0.35])$$

Similarly, the different pairs of features have been computed as follows:

$$\begin{aligned} R_{(H_2,H_1)} &= ([0.264, 0.366][0.087., 0.189][0.273, 0.343]),\\ R_{(H_2,H_2)} &= ([0.27, 0.377][0.132, 0.225], [0.227, 0.304]),\\ R_{(H_2,H_3)} &= ([0.276, 0.388][0.177, 0.261][0.181, 0.265]),\\ R_{(H_3,H_1)} &= ([0.284, 0.412][0.21, 0.311][0.142, 0.299]),\\ R_{(H_3,H_2)} &= ([0.222, 0.344][0.176, 0.287][0.2, 0.287]),\\ R_{(H_3,H_3)} &= ([0.253, 0.378][0.193, 0.299][0.171, 0.258]). \end{aligned}$$

Next, we construct an IVPFR R using the above-obtained interdependency of the features as follows:

$$\begin{array}{cc} & \begin{array}{ccc} H_1 & H_2 & H_3 \end{array} \\ \begin{array}{c} H_1 \\ H_2 \\ H_3 \end{array} & \begin{pmatrix} R_{(H_1,H_1)} & R_{(H_1,H_2)} & R_{(H_1,H_3)} \\ R_{(H_2,H_1)} & R_{(H_2,H_2)} & R_{(H_2,H_3)} \\ R_{(H_3,H_1)} & R_{(H_3,H_2)} & R_{(H_3,H_3)} \end{pmatrix} \end{array}$$

Setting all the values, we get

$$R = \begin{array}{c} \\ H_1 \\ H_2 \\ H_3 \end{array} \begin{array}{c} \begin{array}{ccc} H_1 & H_2 & H_3 \end{array} \\ \begin{pmatrix} ([0.299, 0.406][0.11, 0.205][0.209, 0.321]) & ([0.297, 0.395][0.072, 0.179][0.225, 0.35]) & ([0.301, 0.417][.148, 0.231][0.193, 0.292]) \\ ([0.264, 0.366][0.087., 0.189][0.273, 0.343]) & ([0.27, 0.377][0.132, 0.225], [0.227, 0.304]) & ([0.276, 0.388][0.177, 0.261][0.181, 0.265]) \\ ([0.284, 0.412][0.21, 0.311][0.142, 0.299]) & ([0.222, 0.344][0.176, 0.287][0.2, 0.287]) & ([0.253, 0.378][0.193, 0.299][0.171, 0.258]) \end{pmatrix} \end{array}$$

Now, we use Algorithm I for finding the GEIVPFS:

$$\begin{aligned} N_1 &= ([0.299, 0.412][0.087, 0.189][0.142, 0.299], \\ &\quad [0.297, 0.395][0.072, 0.179][0.2, 0.287], \\ &\quad [0.301, 0.417][0.148, 0.231][0.171, 0.258]), \\ N_2 &= N_1 \circ R = ([0.299, 0.412][0.072, 0.179][0.171, 0.299], \\ &\quad [0.297, 0.397][0.072, 0.179][0.2, 0.287], \\ &\quad [0.299, 0.412][0.072, 0.179][0.171, 0.258]), \\ N_3 &= N_2 \circ R = ([0.299, 0.412][0.072, 0.179][0.171, 0.299], \\ &\quad [0.297, 0.395][0.072, 0.179][0.2, 0.287], \\ &\quad [0.299, 0.412][0.072, 0.179][0.171, 0.299]), \\ N_4 &= N_3 \circ R = ([0.299, 0.412][0.072, 0.179][0.171, 0.299], \\ &\quad [0.297, 0.395][0.072, 0.179][0.2, 0.287], \\ &\quad [0.299, 0.412][0.072, 0.179][0.171, 0.299]). \end{aligned}$$

Since $N_4 = N_3$ therefore, we conclude that N_3 is the GEIVPFS.
Furthermore, we use Algorithm II for finding the LEIVPFS:

$$\begin{aligned} N_1 &= ([0.264, 0.366][0.087, 0.189][0.273, 0.343], \\ &\quad [0.222, 0.344][0.72, 0.179][0.227, 0.35], \\ &\quad [0.253, 0.378][.148, 0.231][0.193, 0.292]), \\ N_2 &= N_1 \circ R = ([0.264, 0.378][0.072, 0.179][0.193, 0.35], \\ &\quad [0.264, 0.366][0.72, 0.179][0.2, 0.292], \\ &\quad [0.264, 0.378][0.072, 0.179][0.193, 0.292]), \\ N_3 &= N_2 \circ R = ([0.264, 0.378][0.072, 0.179][0.193, 0.35], \\ &\quad [0.264, 0.378][0.72, 0.179][0.2, 0.292], \\ &\quad [0.264, 0.378][0.072, 0.179][0.193, 0.292]), \\ N_4 &= N_3 \circ R = ([0.264, 0.378][0.072, 0.179][0.193, 0.35], \\ &\quad [0.264, 0.378][0.72, 0.179][0.2, 0.292], \\ &\quad [0.264, 0.378][0.072, 0.179][0.193, 0.292]). \end{aligned}$$

Since $N_4 = N_3$ therefore, we conclude that N_3 is the LEIVPFS.
Observations and results:
Based on calculations, the greatest and least interval-valued picture fuzzy set are given by ([0.299,0.412] [0.072,0.179] [0.171,0.299], [0.297,0.395] [0.072, 0.179] [0.2,0.287], [0.299,0.412][0.072,0.179] [0.171,0.299]) and ([0.264,0.378] [0.072,0.179] [0.193,0.35], [0.264,0.378][0.72, 0.179] [0.2,0.292], [0.264,0.378][0.072,0.179][0.193, respectively. The results from these sets show the range of levels of satisfaction/abstain/non-satisfaction for the features that an health insurance company is considering.

- Regarding feature H_1, Customers are between (26.4–41.2%) satisfied, abstain (7.2–17.9%) and between 19.3 and 29.9% unsatisfied.
- Regarding feature H_2, Customers are between (26.4–39.5%) satisfied, abstain (7.2–17.9%) and between (2–28.7%) unsatisfied.
- Regarding feature H_3, Customers are between (26.4–41.2%) satisfied, abstain (7.2% to 17.9%) and between (19.3–29.9%) unsatisfied.

It should be noted that the numerical results from the GEIVPFS and LEIVPFS are reasonably close to one another. The proposed algorithms have been illustrated using a specific case with a constrained format and less variety in terms of the dimensions and attributes involved. We might see a sizable fluctuation in the values if we have vast data with higher dimensionality of features. However, the similarity of the results indicates accuracy in the decision-making process. In the next section, firstly, we define the new distance measure for IVPFSs and prove the required properties for the distance measure. Second, we use this distance measure to discuss its application in the realm of decision-making using IVPFM.

6 Distance measure and its application in decision-making problem

In this section, first, we propose distance measure of IVPFS and its properties. Then later, we apply this distance measure to find some practical real-life application in the field of smart manufacturing using IVPFM.

6.1 Distance measure of IVPFSs and its properties

Definition 17 Suppose

$$A = \{[a_1(x_i), b_1(x_i)], [c_1(x_i), d_1(x_i)], [e_1(x_i), f_1(x_i)]\}$$

Table 1 Relative feedback with H_1 and H_2

Customers/experts	H_1	H_2
E_1	([0.4,0.5], [0.12,0.23][0.19,0.23])	([0.42,0.51],[0.13,0.29][0.07,0.12])
E_2	([0.48,0.57], [0.22,0.3][0,0.07])	([0.14,0.23],[0.21,0.29][0.3,0.4])
E_3	([0.35,0.39], [0.11,0.23][0.06,0.21])	([0.36,.48],[0.03,0.10][0.33,0.39])
E_4	([0.27,0.43], [0.08,0.28][0.16,0.23])	([0.22,0.32],[0.13,0.21][0.11,0.20])
E_5	([0.71,0.79], [0.07,0.09][0.10,0.11])	([0.15,0.27],[0.09,0.17][0.42,0.54])
E_6	([0.21,0.30], [0.12,0.24][0.34,0.44])	([0.15,0.27],[0.09,0.17][0.42,0.53])
E_7	([0.26,0.35], [0.08,0.18][0.12,0.20])	([0.35,0.47],[0.11,0.22][0.18,0.21])
E_8	([0.09,0.19], [0.18,0.32][0.25,0.49])	([0.7,0.75],[0.06,0.08][0.09,0.11])
E_9	([0.13,0.25], [0.12,0.22][0.37,0.49])	([0.2,0.29],[0.11,0.21][0.38,0.43])
E_{10}	([0.12, 0.22], [0.08, 0.12][0.25, 0.49])	([0.26,0.42],[0.08,0.27][0.15,0.22])

Table 2 Relative feedback with H_1 and H_3

Customers/experts	H_1	H_3
E_1	([0.1,0.2], [0.1,0.3][0.4,0.5])	([0.4,0.5],[0.2,0.3][0.1,0.2])
E_2	([0.6,0.7], [0.1,0.12][0.15,0.17])	([0.25,0.35],[0.1,0.2][0.3,0.35])
E_3	([0.1,0.3], [0.1,0.2][0.3,0.4])	([0.3,0.35],[0.1,0.15][0.4,0.45])
E_4	([0.4,0.5], [0.1,0.2][0.2,0.3])	([0.4,0.5],[0.1,0.2][0.1,0.2])
E_5	([0.7,0.8], [0,0.05][0.1,0.15])	([0.2,0.3],[.3,.4][0.1,0.2])
E_6	([0.28,0.37], [0.10,0.21][0.14,0.39])	([0.1,0.2],[0.2,0.3][0.4,0.5])
E_7	([0.14,0.22], [0.2,0.28][0.29,0.45])	([0.1,0.3],[.4,0.5][.1,.2])
E_8	([0.13,0.33], [0.01,0.22][0.3,0.41])	([0.05,0.15],[0.3,0.4][0.2,0.3])
E_9	([0.41,0.51], [0.12,0.28][0.07,0.2])	([0.1,0.4],[0.2,0.3][0.1,0.2])
E_{10}	([0.1, 0.2], [0.2, 0.3][0.4, 0.5])	([0.1, 0.3],[0.03,0.5][0.1,0.2])

Table 3 Relative feedback with H_2 and H_3

Customers/experts	H_2	H_3
E_1	([0.1,0.4], [0.15,0.35][0.2,0.25])	([0.7,0.75],[0.1,.15][0,0.1])
E_2	([0.2,0.25], [0.3,0.35][0.1,0.2])	([0,0.15],[0.4,0.5][0.1,0.2])
E_3	([0.15,.25], [0.1,.25][0.1,0.2])	([0.1,0.3],[0.2,0.25][0.4,0.45])
E_4	([0.09,.11], [0.06,.08][0.7,0.75])	([0.42,.53],[.3,.4][.15,.27])
E_5	([0.29,.41], [0.2,.28][0.2,0.29])	([0.37,.49],[0.12,0.22][.13,0.25])
E_6	([0.22,.32], [0.13,.21][0.11,0.2])	([0.23,.37],[.2,.3][0.17,0.25])
E_7	([0.42,.54], [0.09,.17][0.15,0.27])	([0.35,.48],[.12,.22][.12,.25])
E_8	([0.18,0.21], [0.11,0.22][0.35,0.47])	([0.1,.2],[.3,.35][0.4,0.45])
E_9	([0.7,.75], [0.06,.08][0.09,0.11])	([0.6,.65],[.1,.2][0.05,.1])
E_{10}	([0.1,0.3], [0.4,0.5][0.1,0.2])	([0.2,.3],[.1,.15][0,.05])

and

$$B = \{[a_2(x_i), b_2(x_i)], [c_2(x_i), d_2(x_i)], [e_2(x_i), f_2(x_i)]\}$$

be two IVPFSs. The distance measure between A and B is defined as follows:

$$D(A, B) = \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_1(x_i) - a_2(x_i)| + |b_1(x_i) - b_2(x_i)| + |c_1(x_i) - c_2(x_i)| + |d_1(x_i) - d_2(x_i)| + |e_1(x_i) - e_2(x_i)| + |f_1(x_i) - f_2(x_i)|\Big)$$

Theorem 9 *A distance measure between IVPFSs A and B is a mapping D:IVPFSs × IVPFSs → [0,1], which satisfy the following properties* ($D1$) *$0 \leq D(A, B) \leq 1$* ($D2$) *$D(A, B) = 0$ if and only if A=B* ($D3$) *$D(A, B) = D(B, A)$* ($D4$) *Let $A, B, C \in$ IVPFSs then $D(A,C) \leq D(A,B) + D(B,C)$.*

Proof Proof of $D1$, $D2$ and $D3$ are trivial as follows:

($D1$): As the membership, neutral membership and non-membership degrees belong to [0,1], it is obvious that the distance measure $D(A, B) \in [0, 1]$.

($D2$): As $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$, $e_1 = e_2$, $f_1 = f_2$.
Such that $D(A, B) = 0$ if and only if A=B.

($D3$) :

$$
\begin{aligned}
D(A, B) = \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_1(x_i) - a_2(x_i)| + |b_1(x_i) \\
- b_2(x_i)| + |c_1(x_i) \\
- c_2(x_i)| + |d_1(x_i) - d_2(x_i)| + |e_1(x_i) \\
- e_2(x_i)| + |f_1(x_i) - f_2(x_i)|\Big) \\
= \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_2(x_i) - a_1(x_i)| + |b_2(x_i) \\
- b_1(x_i)| + |c_2(x_i) - c_1(x_i)| \\
+ |d_2(x_i) - d_1(x_i)| + |e_2(x_i) - e_1(x_i)| \\
+ |f_2(x_i) - f_1(x_i)|\Big) = D(B, A)
\end{aligned}
$$

($D4$): Suppose
$A = ([a_1(x_i), b_1(x_i)], [c_1(x_i), d_1(x_i)], [e_1(x_i), f_1(x_i)])$,
$B = ([a_2(x_i), b_2(x_i)], [c_2(x_i), d_2(x_i)], [e_2(x_i), f_2(x_i)])$
$C = ([a_3(x_i), b_3(x_i)], [c_3(x_i), d_3(x_i)], [e_3(x_i), f_3(x_i)])$

Consider

$$
\begin{aligned}
D(A, C) = \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_1(x_i) - a_3(x_i)| \\
+ |b_1(x_i) - b_3(x_i)| + |c_1(x_i) - c_3(x_i)| \\
+ |d_1(x_i) - d_3(x_i)| + |e_1(x_i) - e_3(x_i)| \\
+ |f_1(x_i) - f_3(x_i)|\Big) \\
= \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_1(x_i) - a_2(x_i) + a_2(x_i) - a_3(x_i)| \\
+ |b_1(x_i) - b_2(x_i) + b_2(x_i) - b_3(x_i)| \\
+ |c_1(x_i) - c_2(x_i) + c_2(x_i) - c_3(x_i)| \\
+ |d_1(x_i) - d_2(x_i) + d_2(x_i) - d_3(x_i)| \\
+ |e_1(x_i) - e_2(x_i) + e_2(x_i) - e_3(x_i)| \\
+ |f_1(x_i) - f_2(x_i) + f_2(x_i) - f_3(x_i)|\Big) \\
\le \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_1(x_i) - a_2(x_i)| + |b_1(x_i) - b_2(x_i)| \\
+ |c_1(x_i) - c_2(x_i)| + |d_1(x_i) - d_2(x_i)| \\
+ |e_1(x_i) - e_2(x_i)| \\
+ |f_1(x_i) - f_2(x_i)|\Big) + \frac{1}{4n}\Big(\sum_{i=1}^{n} |a_2(x_i) \\
- a_3(x_i)| + |b_2(x_i) - b_3(x_i)| \\
+ |c_2(x_i) - c_3(x_i)| + |d_2(x_i) - d_3(x_i)| \\
+ |e_2(x_i) - e_3(x_i)| + |f_2(x_i) - f_3(x_i)|\Big)
\end{aligned}
$$

Thus, $D(A, C) \le D(A, B) + D(B, C)\ \forall$ IVPFSs $A,\ B,\ C$.□

6.2 Real-life application of the proposed distance measure in smart manufacturing problem

The proposed concept finds practical application in real-life scenarios, particularly in the domain of smart manufacturing problems. In this context, a complex challenge arises due to the existence of l CNC programmers distributed across λ manufacturing companies. The core issue is identifying and selecting k CNC programmers from this pool, aiming to promote and relocate them among the various companies. The selection process revolves around evaluating the CNC programmers' performance within their respective manufacturing companies and the evolving relationships between the companies and CNC hiring agencies. To facilitate the selection process, two distinct IVPFM are provided. The first matrix offers valuable insights into how CNC programmers perceive the support they receive from CNC hiring agencies in each company. Meanwhile, the second matrix delves into the intricate relationships forged between the manufacturing companies and the CNC hiring agencies, particularly during the promotion of CNC machines.

Effectively managing the vast amount of data and preferences involves assessing all IVPFSs from the two IVPFM, primarily focusing on the CNC hiring agencies. These sets' information is then harnessed to compute a distance matrix, a crucial tool in the decision-making process. The distances between each CNC programmer and the CNC hiring agencies are skillfully manipulated to construct the distance matrix. This manipulation is achieved using a specialized distance formula tailored to measure the relationship dynamics between two IVPFM, as defined in Definition 17. By adopting this comprehensive approach, the selection committee gains deeper insights into the intricate web of interactions and preferences among the CNC programmers, the manufacturing companies, and the CNC hiring agencies. Ultimately, this analysis facilitates the identification of the most deserving CNC programmers for promotion and relocation, thus optimizing the smart manufacturing process.

In this process, the objective is to determine the selected list of CNC programmers for promotion and reassignment among manufacturing companies. To achieve this, we must evaluate the minimum distance between each CNC programmer and the manufacturing companies. This evaluation uses a descending order approach, ranking the distances from the closest to the farthest. To begin the evaluation, the distance of each CNC programmer towards the manufacturing companies is calculated based on the corresponding distance formula, which could be a measure of performance or relationship strength. These distances serve as a key metric in the decision-making process.

As the evaluation continues, the CNC programmers' positions in the list correspond to their respective distances, with the closest ones placed at the top and the farthest ones towards the bottom. This ranking effectively identifies the most suitable candidates for promotion and relocation among the manufacturing companies. Ultimately, the selected list of CNC programmers is derived from this evaluation, consisting of those with the shortest distances to the companies, ensuring that the most promising and qualified individuals are chosen for the promotion and circulation process. The descending order approach ensures that the best candidates are prioritized based on their close relationships or high-performance levels in the context of manufacturing companies and CNC hiring agencies.

Problem description:

Imagine a scenario with three CNC stock agencies, namely SA_1, SA_2, SA_3, associated with three smart manufacturing companies, C_1, C_2, C_3. From these companies, five CNC programmers CP_1, CP_2, CP_3, CP_4, and CP_5, are selected for promotion. Let $\tilde{A}$ be an IVPFM, i.e., $\tilde{A} = (< \tilde{a}_{ij\mu}, \tilde{a}_{ij\eta}, \tilde{a}_{ij\nu} >)$ which shows the relationship between the CNC programmers and the CNC hiring agencies. In essence, $\tilde{a}_{ij\mu}$ represents the degree of inclination or membership of the CNC programmers towards a particular CNC hiring agency. In contrast, $\tilde{a}_{ij\eta}$ signifies this inclination's degree of neutral membership. Lastly, $\tilde{a}_{ij\nu}$ characterizes the non-membership concerning the CNC hiring agencies,

$$
\begin{array}{c} \\ CP_1 \\ CP_2 \\ CP_3 \\ CP_4 \\ CP_5 \end{array}
\begin{array}{c}
\begin{array}{ccc} \quad SA_1 \quad & \quad SA_2 \quad & \quad SA_3 \quad \end{array} \\
\begin{pmatrix}
[0.36, 0.48][0.03, 0.10][0.33, 0.39] & [0.22, 0.32][0.13, 0.21][0.11, 0.20] & [0.15, 0.27][0.09, 0.17][0.42, 0.54] \\
[0.07, 0.22][0.11, 0.28][0.20, 0.32] & [0.13, 0.32][0.02, 0.13][0.23, 0.45] & [0.12, 0.25][0.12, 0.22][0.35, 0.48] \\
[0.26, 0.35][0.08, 0.18][0.12, 0.20] & [0.09, 0.19][0.18, 0.32][0.22, 0.35] & [0.17, 0.36][0.04, 0.15][0.23, 0.37] \\
[0.09, 0.42][0.10, 0.20][0.14, 0.35] & [0.12, 0.22][0.08, 0.12][0.25, 0.49] & [0.13, 0.25][0.12, 0.22][0.37, 0.49] \\
[0.15, 0.47][0.14, 0.31][0.01, 0.10] & [0.13, 0.35][0.01, 0.23][0.32, 0.41] & [0.15, 0.27][0.09, 0.17][0.42, 0.53]
\end{pmatrix}
\end{array}
$$

and $\tilde{B} = (< \tilde{b}_{ij\mu}, \tilde{b}_{ij\eta}, \tilde{b}_{ij\nu} >)$ which represents the relationship with manufacturing companies and CNC hiring agencies during the promotion of CNC:

$$
\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array}
\begin{array}{c}
\begin{array}{ccc} \quad SA_1 \quad & \quad SA_2 \quad & \quad SA_3 \quad \end{array} \\
\begin{pmatrix}
[0.35,0.47][0.11,0.22][0.18,0.21] & [0.7,0.75][0.06,0.08][0.09,0.11] & [0.2,0.29][0.11,0.21][0.38,0.43] \\
[0.26,0.42][0.08,0.27][0.15,0.22] & [0.41,0.51][0.12,0.28][0.07,0.20] & [0.14,0.22][0.2,0.28][0.29,0.41] \\
[0.16,0.47][0.14,0.29][0.01,0.11] & [0.13,0.33][0.01,0.22][0.3,0.41] & [0.28,0.37][0.10,0.21][0.14,0.39]
\end{pmatrix}
\end{array}
$$

From the given matrix $\tilde{A}$, the knowledge about the CNC programmer concerning the CNC hiring agencies (SA_1, SA_2, SA_3) and matrix $\tilde{B}$, the knowledge about the manufacturing companies with respect to the set of CNC hiring agencies (SA_1, SA_2, SA_3). According to the fact, eight IVPFSs are taken out over the set (SA_1, SA_2, SA_3):

$$
\begin{aligned}
CP_1 = [&(SA_1, < [0.36, 0.48][0.03, 0.10][0.33, 0.39] >),\\
&(SA_2, < .[0.22, 0.32][0.13, 0.21][0.11, 0.20] >),\\
&(SA_3, < [0.15, 0.27][0.09, 0.17][0.42, 0.54] >)],\\
CP_2 = [&(SA_1, < [0.07, 0.22][0.11, 0.28][0.20, 0.32] >),\\
&(SA_2, < [0.13, 0.32][0.02, 0.13][0.23, 0.45] >),\\
&(SA_3, < [0.12, 0.25][0.12, 0.22][0.35, 0.48])],\\
CP_3 = [&(SA_1, < [0.26, 0.35][0.08, 0.18][0.12, 0.20] >),\\
&(SA_2, < [0.09, 0.19][0.18, 0.32][0.22, 0.35] >),\\
&(SA_3, < [0.14, 0.22][0.2, 0.28][0.29, 0.41] >)],\\
CP_4 = [&(SA_1, < [0.09, 0.42][0.10, 0.20][0.14, 0.35] >),\\
&(SA_2, < [0.12, 0.22][0.08, 0.12][0.25, 0.49] >),\\
&(SA_3, < [0.13, 0.25][0.12, 0.22][0.37, 0.49] >)],\\
CP_5 = [&(SA_1, < [0.15, 0.47][0.14, 0.31][0.01, 0.10] >),\\
&(SA_2, < [0.13, 0.35][0.01, 0.23][0.32, 0.41] >),\\
&(SA_3, < [0.28, 0.37][0.10, 0.21][0.14, 0.39] >)],\\
C_1 = [&(SA_1, < [0.35, 0.47][0.11, 0.22][0.18, 0.21] >),\\
&(SA_2, < [0.7, 0.75][0.06, 0.08][0.09, 0.11] >),\\
&(SA_3, < [0.2, 0.29][0.11, 0.21][0.38, 0.43] >)],\\
C_2 = [&(SA_1, < [0.26, 0.42][0.08, 0.27][0.15, 0.22] >),\\
&(SA_2, < [0.41, 0.51][0.12, 0.28][0.07, 0.20] >,\\
&(SA_3, < [0.14, 0.22][0.2, 0.28][0.29, 0.41] >)],\\
C_3 = [&(SA_1, < [0.16, 0.47][0.14, 0.29][0.01, 0.11] >),\\
&(SA_2, < [0.13, 0.33][0.01, 0.22][0.3, 0.41] >),\\
&(SA_3, < [0.28, 0.37][0.10, 0.21][0.14, 0.39] >)].
\end{aligned}
$$

Now get the distance matrix $\delta = (\delta_{ij})$ by using distance measure between two IVPFM $\tilde{A}$ and $\tilde{B}$, here δ_{ij} is the distance between CP_i and C_j, where $i = 1, 2, 3, 4, 5$ and

$j = 1, 2, 3$ given below:

$$\begin{array}{c} \\ CP_1 \\ CP_2 \\ CP_3 \\ CP_4 \\ CP_5 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} 0.1708 & 0.1475 & 0.2042 \\ 0.2092 & 0.1692 & 0.1342 \\ 0.2242 & 0.1508 & 0.1275 \\ 0.2025 & 0.1658 & 0.1392 \\ 0.2142 & 0.1758 & 0.0658 \end{pmatrix} \end{array}$$

Observations and results discussed using distance matrix (δ):

(a) In the first instance of the manufacturing company C_1, the degree of closeness (DOC) between the CNC programmer (CP_1) and the company C_1 is maximum because $DOC(CP_1, C_1) > DOC(CP_4, C_1) > DOC(CP_2, C_1) > DOC(CP_5, C_1) > DOC(CP_3, C_1)$.

(b) In the second instance of the manufacturing company C_2, the degree of closeness (DOC) between the CNC programmer (CP_4) and the company C_2 is maximum because $DOC(CP_1, C_2) > DOC(CP_3, C_2) > DOC(CP_4, C_2) > DOC(CP_2, C_2) > DOC(CP_5, C_2)$.

(c) In the third instance of the manufacturing company C_3, the degree of closeness (DOC) between the CNC programmer (CP_4) and the manufacturing company C_3 is maximum because $DOC(CP_5, C_3) > DOC(CP_3, C_3) > DOC(CP_2, C_3) > DOC(CP_4, C_3) > DOC(CP_1, C_3)$.

where ‘ $>'$ represents the closeness degree. The lesser the value, higher is the closeness. According to the mathematical calculations, we find the final selected list of CNC programmer for a different manufacturing company as follows:

Manufacturing Company	Selected CNC Programmer
C_1	CP_1, CP_4
C_2	CP_1, CP_3
C_3	CP_5, CP_3

The CNC programmer CP_1 and CP_3 are selected (appropriate) for all smart manufacturing companies. Figure 3 depicts a flowchart of the procedure to be followed for smart manufacturing companies selection.

In the next section, we compare the present work with the existing work on FM and its extension. Also, we discuss the advantage of the proposed work in detail in Table 4.

7 Comparative study and results

In the previous studies concerning picture-fuzzy decision-making, information was considered in a picture-fuzzy manner. However, when we face various forms of uncertainty

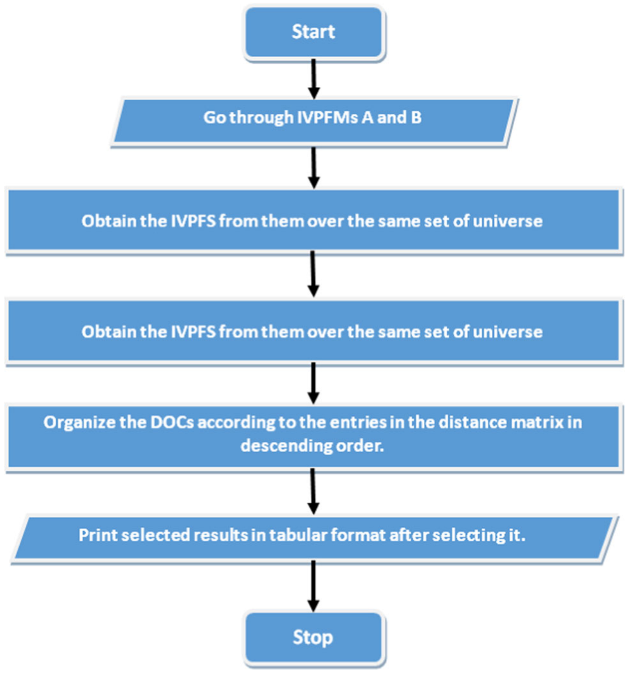

Fig. 3 Flowchart of the smart manufacturing companies selection procedure

in the information, conventional methods are inadequate for handling such situations. In these scenarios, gathering or representing the information in an interval-valued picture fuzzy sense becomes necessary. In such cases, the currently developed process becomes crucial in arriving at a meaningful and productive conclusion. Dogra and Pal (2020) introduced a model for determining a selected list of administrative officers for various governments using the distance formula between two PFM. Their approach considered membership, neutral membership, and non-membership degrees within the PFM framework.

In contrast, our current study extends this work by considering membership, neutral membership, and non-membership degrees as interval numbers, which proves to be more practical for real-life problems in smart manufacturing problems. In addition, Ejegwa et al. (2017) proposed a model for determining students' career paths using the distance formula between two intuitionistic fuzzy sets. In this model, intuitionistic fuzzy sets considered only membership and non-membership aspects. Moreover, Khalaf (2013) addressed medical diagnosis problems using IVIFS with max–min–max composition. They formulated these problems as uncertain decision matrices and provided decisions based on fuzzy scores calculated for each attribute. However, our current method takes a different approach with matrices that contain interval-valued picture fuzzy values. We extract interval-valued picture fuzzy sets from these matrices over a defined universe. Using the newly developed distance formula between two IVPFSs, we obtain a distance matrix that

Table 4 Comparison of the proposed work with existing literature

Characteristics methods	Whether consider MD	Whether consider MD more flexible, i.e., IVMD	Whether consider MD or NMD	Whether consider MD or NMD more flexible, i.e., IVMD or IVNMD	Whether consider MD, neutral membership and NMD degree	Whether consider MD, neutrality and NMD degree more flexible, i.e., IVMD, interval-valued neutrality degree and IVNMD
Thomason (1977)	✓	×	×	×	×	×
Pal (2015)	✓	✓	×	×	×	×
Pal et al. (2002)	✓	✓	✓	×	×	×
Silambarasan and Sriram (2018)	✓	✓	✓	×	×	×
Silambarasan (2020)	✓	✓	✓	×	×	×
Khan and Pal (2014)	✓	✓	✓	✓	×	×
Dogra and Pal (2020)	✓	✓	✓	✓	✓	×
Proposed work	✓	✓	✓	✓	✓	✓

MD membership degree, *NMD* non membership degree, *IVMD* interval-valued membership degree, *IVNMD* interval valued non membership degree

leads to a decision. Implementing this method is remarkably straightforward, as it does not require various complex calculations, thereby avoiding any complicacy in its application. Consequently, developing an algorithm and computer programming for this method becomes easy. Furthermore, the data points considered in this method have a remarkable capability to handle a wider range of vagueness in information. Considering that the interval-valued picture fuzzy concept generalizes the picture fuzzy concept, this study can be viewed as a generalization of advanced fuzzy logic.
In summary, the study of IVPFM and EIVPFS yields significant advantages in addressing real-world challenges, particularly in the context of smart manufacturing and health insurance companies. These concepts provide valuable tools and insights for various applications, making them essential components in contemporary research and practical problem-solving scenarios. The following is a detailed list of some substantial advantages of using IVPFM and EIVPFS:

1. From Table 4, the existing FM, IVFM, IFM, IVIFM, and PFM each have shortcomings that prevent them from fully capturing the information. The IVPFM effectively fills the gaps left by other matrices and offers a more flexible and versatile approach to expressing opinions and relationships within the data. The IVPFM combines the benefits of both interval-valued and picture-fuzzy concepts, making it a powerful tool for dealing with uncertainties and complexities. The IVPFM offers a more robust framework to handle various real-life scenarios and decision-making processes by representing membership, neutral membership, and non-membership degrees as intervals.
2. We can also see the drawback in the eigen fuzzy sets and eigen intuitionistic fuzzy sets experts/decision-makers bind their input in a certain area. However, the proposed EIVPFS presents a significant impact due to its ability to offer a generalization feature. This unique characteristic allows for a more comprehensive and versatile representation of uncertain information, empowering decision-makers to make more informed and flexible judgments in various contexts.
3. The implementation of the EIVPFS, IVPFM, and the approach suggested for the problems of health insurance and smart manufacturing in Sect. 5 and Sect. 6 demonstrate how well and consistently the proposed work addresses the extended framework. The observations indicate that the IVPFM is the most generalized structure among all fuzzy matrix models.

The detailed analysis presented in Table 4 further compares the proposed work and existing research available in the literature.

8 Concluding remarks

The exploration of matrix theory has made significant contributions to various applicable fields. In this work, we have introduced the concept of IVPFM along with its essential definitions and theorems. In addition, we have defined the determinant and adjoint of IVPFM and studied relevant results. The formal definition of an EIVPFS for interval-valued picture fuzzy relations has been presented, and algorithms for determining the GEIVPFS and LEIVPFS using max–min and min–max composition operators have been provided. To illustrate these algorithms, numerical examples have been included. The application of GEIVPFS and LEIVPFS in decision-making problems has been successfully demonstrated. Moreover, we have demonstrated the application of IVPFM in decision-making by introducing a distance formula to solve such problems effectively. The limitation of IVPFM is related to the representation of degrees of membership, neutral membership, and non-membership as interval numbers. The limitation arises when the sum of the upper degree of membership, neutral membership, and upper degree of non-membership exceeds the interval [0, 1]. The current study will help researchers interested in further developing and generalizing our findings in the context of other types of data sets. Also, we can extend in the field of image information retrieval, genetic algorithm for image reconstruction, and outlines to introduce the notion of interval-valued eigen picture fuzzy soft sets/soft matrices have been briefly stated for further research.

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Declarations

Conflict of interest The authors have no conflicts of interest to declare.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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Full Paper

Interval-valued spherical fuzzy matrix and its applications in multi-attribute decision-making process

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Abstract: Despite significant advancement in matrix problem research over the past few decades, there remains a considerable gap in the literature when it comes to addressing real-life problems. The application of matrix is widespread in various real-life scenarios. In the literature researchers have always been more interested in studying matrices in fuzzy settings including intuitionistic fuzzy, picture fuzzy and spherical fuzzy environments. Inspired by the notion of interval-valued spherical fuzzy sets, we extend the theory of spherical fuzzy matrix into interval-valued spherical fuzzy matrix (IVSFM) to represent more flexibly uncertain and vague information. In this context we establish significant definitions and theorems about the given matrices. Further, we introduce the methodology for determining the determinant and adjoint of IVSFM. Finally, we propose a new score function for the interval-valued spherical fuzzy sets and prove its validity with the help of basic properties. Further, the application of the proposed concepts is shown by real-life decision-making for a career placement assessment.

Keywords: decision-making, interval-valued spherical fuzzy matrix, determinant of interval-valued spherical fuzzy matrix, adjoint of interval-valued spherical fuzzy matrix, score function

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INTRODUCTION

Decision-making is the cognitive process of selecting a choice or action among multiple alternatives, a fundamental aspect of human life essential in various contexts, from personal to professional. It involves assessing information, considering consequences and aligning choices with goals and values. Decisions can range from simple daily choices to complex strategic plans. Effective decision-making necessitates critical thinking, problem-solving skills and emotional intelligence. It plays a crucial role in shaping our lives, determining success and mitigating risks.

Understanding the decision-making process helps individuals and organisations make well-informed, rational choices that lead to desired outcomes and progress. The application of matrices in decision-making problems plays a pivotal role in various fields. Matrices provide a structured framework that aids in evaluating and comparing different alternatives. They enable decision-makers to quantify and analyse multiple factors or criteria simultaneously, facilitating a systematic and comprehensive approach. Matrices allow for the organisation of information and the identification of relationships between variables, providing a visual representation that enhances understanding and aids in making informed decisions [1-3]. By assigning weights and scores to various elements, matrices help prioritise options and determine the most favourable course of action. Overall, matrices are used in a variety of fields of science and technology to represent data in a meaningful way. However, various sorts of uncertain data are involved in decision-making, making it challenging to solve the issue in a traditional matrix. These problems can stem from data unpredictability, inadequate information and other factors.

To deal with the situation of vague data, fuzzy matrix (FM) plays a fundamental role in dealing with such a situation. Zadeh [4] developed the fuzzy set to deal with uncertainty in practical situations. The FM is defined by Thomason [5] after the fuzzy set is introduced. Kim and Roush [6] worked on the generalisation of the FM over boolean algebra. Pal [7] defined the FM with fuzzy rows and columns and presented some properties with the binary operator. Ragab and Eman [8] gave some results on the max-min composition and worked on the construction of an idempotent FM. Ragab and Emam [9] solved the determinant and adjoint of a square FM and discussed some properties defined on it. Pal [10] extended the FM to an interval-valued fuzzy matrix (IVFM) with an interval-valued fuzzy row and column. Meenakshi and Kaliraja [11] used the interval-valued FM for solving medical diagnosis problems. Mandal and Pal [12] described some methods of finding the ranks of IVFM. A number of researchers have worked on FM but they considered only the membership of the element.

Atanassov [13] introduced the concept of intuitionistic fuzzy sets with this kind of situation in mind. Khan et al. [14] defined the concept of an intuitionistic fuzzy matrix (IFM). Padder and Murugadas [15] worked on max-min operations on an IFM and discussed the convergence of transitive IFM. Pal and Khan [16] proposed some operations on the IFM. Moreover, Khan and Pal [17] have presented the concept of an interval-valued intuitionistic fuzzy matrix (IVIFM). Silambarasan [18] defined the Hamacher operations of IVIFM and proved some important properties.

In the above studies the concept of FM and IFM has been strongly enforced in various areas, yet the concept of neutrality is not considered in FM and IFM. The FM and IFM fail to attain any satisfactory result when the neutral membership degree is calculated independently in real-life problems. After that, Dogra and Pal [19] proposed the picture fuzzy matrix (PFM) using the concept of Cuong and Kreinovich [20] and introduced the method of determinant and adjoint of a PFM. Silambarasan [21] also defined some algebraic operations and properties of the PFM. Khalil et al. [22] worked on new operations on interval-valued picture fuzzy sets. Liu et al. [23] introduced the similarity measures for interval-valued picture fuzzy sets and their applications in decision-making problems. Further, Silambarasan [24] defined a spherical fuzzy matrix (SFM) using the theory of Gundogdu and Kahraman [25] and proposed some important properties and algebraic operations for the SFM. Muthukumaran et al. [26] defined the n-hyperspherical neutrosophic matrices and compared them with the SFM. Gundogdu and Kahraman [25, 27] extended the spherical fuzzy set into the interval-valued spherical fuzzy set. Menekse and Akdag [28] worked on risk analysis of hospital buildings using an interval-valued picture fuzzy set. Otay [29] worked on tech-centre location selection by interval-valued spherical fuzzy AHP-based MULTIMOORA methodology. The present work aims to present the notion of interval-valued spherical fuzzy matrix (IVSFM) and its important features. The key operations are as follows:

(i) We explore the SFM into an IVSFM.
(ii) Based on PFM and SFM, we introduce some important definitions and theorems.
(iii) The method of finding the determinant and adjoint is proposed.
(iv) We propose the score function for the interval-valued spherical fuzzy set with some basic properties.
(v) The real-life application of decision-making is performed by using the proposed score function.
(vi) Some comparative studies are illustrated.

PRELIMINARIES

In this section we discuss some basic definitions of the related work.

Definition 1 [5]. The fuzzy matrix A of order $m \times n$ is defined as

$$A = (< a_{ij}, a_{ij\mu} >),$$

where $a_{ij\mu}$ is called the membership degree of a_{ij}, satisfying the condition $0 \le a_{ij\mu} \le 1$.

Definition 2 [14]. The intuitionistic fuzzy matrix A of order $m \times n$ is defined as

$$A = (< a_{ij}, (a_{ij\mu}, a_{ij\nu}) >),$$

where $a_{ij\mu}$, $a_{ij\nu}$ are the degree of membership and degree of non-membership value of a_{ij} respectively with condition $0 \le a_{ij\mu} + a_{ij\nu} \le 1$.

Definition 3 [17]. An IVIFM A of order $m \times n$ is defined as

$$A = (< a_{ij}, (a_{ij\mu}, a_{ij\nu}) >),$$

where $a_{ij\mu}$ membership degree and $a_{ij\nu}$ non-membership degree are both the subsets of [0,1], which are denoted respectively by $a_{ij\mu}=[a_{ij\mu L}, a_{ij\mu U}]$ and $a_{ij\nu}=[a_{ij\nu L}, a_{ij\nu U}]$ under the condition $a_{ij\mu U}+a_{ij\nu U} \le 1$ for $i = 1,2, \dots .m$ and $j = 1,2 \dots .n$.

Definition 4 [24]. A spherical fuzzy matrix A of order $m \times n$ is defined as

$$A = (< a_{ij}, (a_{ij\mu}, a_{ij\eta}, a_{ij\nu}) >),$$

where $a_{ij\mu} \in [0,1]$, $a_{ij\eta} \in [0,1]$, $a_{ij\nu} \in [0,1]$ are the measure of membership, neutrality and non-membership degree of a_{ij} respectively, and $i = 1,2 \dots .m$; $j = 1,2 \dots .n$, satisfying $0 \le a_{ij\mu}^2 + a_{ij\eta}^2 + a_{ij\nu}^2 \le 1$.

Definition 5 [27]. An interval-valued spherical fuzzy set A on X is defined as

$$A = \{< x, [\mu_A^L(x), \mu_A^U(x)], [\eta_A^L(x), \eta_A^U(x)], [\nu_A^L(x), \nu_A^U(x)] > / [\mu_A^L(x), \mu_A^U(x)], [\eta_A^L(x), \eta_A^U(x)], [\nu_A^L(x), \nu_A^U(x)] \in D[0,1], (\mu_A^U(x))^2 + (\eta_A^U(x))^2 + (\nu_A^U(x))^2 \le 1, x \in X\},$$

where $[\mu_A^L(x), \mu_A^U(x)], [\eta_A^L(x), \eta_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]$ are the membership, neutrality and non - membership degree of A at x respectively.

The interval of the indeterminacy degree relative to A for each $x \in X$ is defined as

$$\pi_A(x) = [\pi^L{}_A(x), \pi^U{}_A(x)] = [\sqrt{1 - \mu_A{}^L(x))^2 - \eta_A{}^L(x))^2 - (\nu_A{}^L(x))^2}, \sqrt{1 - \mu_A{}^U(x))^2 - \eta_A{}^U(x))^2 - (\nu_A{}^U(x))^2}].$$

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INTERVAL-VALUED SPHERICAL FUZZY MATRIX (IVSFM)

In this section we introduce the IVSFM, along with the important definition and theorem for the proposed matrix.

Definition 6. An interval-valued spherical fuzzy matrix A of order $m \times n$ is defined as
$A = (< a_{ij}, (a_{ij\mu}, a_{ij\eta}, a_{ijv}) >)$,
where $a_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}] \in [0,1]$, $a_{ij\eta} = [a_{ij\eta L}, a_{ij\eta U}] \in [0,1]$,
$a_{ijv} = [a_{ijvL}, a_{ijvU}] \in [0,1]$ with the condition $(a_{ij\mu U})^2 + (a_{ij\eta U})^2 + (a_{ijvU})^2 \le 1$, and $a_{ij\mu}$, $a_{ij\eta}$ and a_{ijv} are the membership, neutrality and non-membership degree of the element a_{ij} respectively.

Definition 7. Let A and B be two IVSFM such that
$A = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$,
$B = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$.
Then we write $A \le B$ if following case is true:
$a_{ij\mu L} \le b_{ij\mu L}, a_{ij\mu U} \le b_{ij\mu U}; a_{ij\eta L} \le b_{ij\eta L}, a_{ij\eta U} \le b_{ij\eta U}; a_{ijvL} \ge b_{ijvL}, a_{ijvU} \ge b_{ijvU}$.

Definition 8. Let $A = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$ and
$B = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$ be two IVSFM of order $m \times n$. Then we define the following operators:

(i) $A^c = ([a_{ijvL}, a_{ijvU}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\mu L}, a_{ij\mu U}])$
(ii) $A \vee B = ([max(a_{ij\mu L}, b_{ij\mu L}), max(a_{ij\mu U}, b_{ij\mu u})][min(a_{ij\eta L}, b_{ij\eta L}), min(a_{ij\eta u}, b_{ij\eta U})]$
$[min(a_{ijvL}, b_{ijvL}), min(a_{ijvU}, b_{ijvU})])$
(iii) $A \wedge B = ([min(a_{ij\mu L}, b_{ij\mu L}), min(a_{ij\mu U}, b_{ij\mu u})][min(a_{ij\eta L}, b_{ij\eta L}), min(a_{ij\eta u}, b_{ij\eta U})]$
$[max(a_{ijvL}, b_{ijvL}), max(a_{ijvU}, b_{ijvU})])$
(iv) $A^T = ([a_{ji\mu L}, a_{ji\mu U}], [a_{ji\eta L}, a_{ji\eta U}], [a_{jivL}, a_{jivU}])$
(v)

$$
\mathrm{A} \oplus B = \left\{\begin{matrix} [((\mu_{aij\mu L})^2 + (\mu_{bij\mu L})^2 - (\mu_{aij\mu L})^2(\mu_{bij\mu L})^2)^{\frac{1}{2}}, ((\mu_{aij\mu U})^2 + (\mu_{bij\mu U})^2 - (\mu_{aij\mu U})^2(\mu_{bij\mu U})^2)^{\frac{1}{2}}], \\ [\mu_{aijvL}\mu_{bijvL}, \mu_{aijvU}\mu_{bijvU}], \\ \begin{bmatrix} ((1-(\mu_{bij\mu L})^2)(\mu_{aij\eta L})^2 + (1-(\mu_{aij\mu L})^2)(\mu_{bij\eta L})^2 - (\mu_{aij\eta L})^2(\mu_{bij\eta L})^2)^{1/2} \\ ((1-(\mu_{bij\mu U})^2)(\mu_{aij\eta U})^2 + (1-(\mu_{aij\mu U})^2)(\mu_{bij\eta U})^2 - (\mu_{aij\eta U})^2(\mu_{bij\eta U})^2)^{1/2} \end{bmatrix} \end{matrix}\right\}
$$

(vi)

$$
A \otimes B = \left\{\begin{matrix} [\mu_{aij\mu L}\mu_{bij\mu L}, \mu_{aij\mu U}\mu_{bij\mu U}], [((\mu_{aijvL})^2 + (\mu_{bijvL})^2 - (\mu_{aijvL})^2(\mu_{bijvL})^2)^{\frac{1}{2}}, \\ ((\mu_{aijvU})^2 + (\mu_{bijvU})^2 - (\mu_{aijvU})^2(\mu_{bijvU})^2)^{1/2}], \\ \begin{bmatrix} ((1-(\mu_{bijvL})^2)(\mu_{aij\eta L})^2 + (1-(\mu_{aijvL})^2)(\mu_{bij\eta L})^2 - (\mu_{aij\eta L})^2(\mu_{bij\eta L})^2)^{1/2} \\ ((1-(\mu_{bijvU})^2)(\mu_{aij\eta U})^2 + (1-(\mu_{aijvU})^2)(\mu_{bij\eta U})^2 - (\mu_{aij\eta U})^2(\mu_{bij\eta U})^2)^{1/2} \end{bmatrix} \end{matrix}\right\}
$$

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(vii)

$$k.A = \begin{Bmatrix} [(1-(1-(\mu_{aij\mu L})^2)^k)^{1/2}, (1-(1-(\mu_{aij\mu U})^2)^k)^{1/2}], [(\mu_{aijvL})^k, (\mu_{aijvU})^k], \\ [((1-(\mu_{aij\mu L})^2)^k - (1-(\mu_{aij\mu L})^2 - (\mu_{aij\eta L})^2)^k)^{1/2}, \\ ((1-(\mu_{aij\mu U})^2)^k - (1-(\mu_{aij\mu U})^2 - (\mu_{aij\eta U})^2)^k)^{1/2}] \end{Bmatrix}$$

(viii)

$$A^k = \begin{Bmatrix} [(\mu_{aij\mu L})^k, (\mu_{aij\mu U})^k], [(1-(1-(\mu_{aijvL})^2)^k)^{1/2}, (1-(1-(\mu_{aijvU})^2)^k)^{1/2}], \\ [((1-(\mu_{aijvL})^2)^k - (1-(\mu_{aijvL})^2 - (\mu_{aij\eta L})^2)^k)^{1/2}, \\ ((1-(\mu_{aijvU})^2)^k - (1-(\mu_{aijvU})^2 - (\mu_{aij\eta U})^2)^k)^{1/2}] \end{Bmatrix},$$

where A^c and A^T are the complement and transpose of A respectively, and k is any scalar value.

Theorem 1. Let A, B be two IVSFM of order $m \times n$. Then
(i) $(A \vee B)^c = A^c \wedge B^c$
(ii) $(A \wedge B)^c = A^c \vee B^c$.

Proof: (i) Let $A = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$,
$B = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$,
$A^c = (< [a_{ijvL}, a_{ijvU}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\mu L}, a_{ij\mu U}] >)$,
$B^c = (< [b_{ijvL}, b_{ijvU}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ij\mu L}, b_{ij\mu U}] >)$. Then
$A^c \wedge B^c = ([min(a_{ijvL}, b_{ijvL}), min(a_{ijvU}, b_{ijvU})][min(a_{ij\eta L}, b_{ij\eta L}), min(a_{ij\eta U}, b_{ij\eta U})]$
$[max(a_{ij\mu L}, b_{ij\mu L}), max(a_{ij\mu U}, b_{ij\mu U})])$,
$A \vee B$=($[max(a_{ijvL}, b_{ijvL}), max(a_{ijvU}, b_{ijvU})][min(a_{ij\eta L}, b_{ij\eta L}), min(a_{ij\eta U}, b_{ij\eta U})]$
$[min(a_{ij\mu L}, b_{ij\mu L}), min(a_{ij\mu U}, b_{ij\mu U})]$),
$(A \vee B)^c$=($[min(a_{ijvL}, b_{ijvL}), min(a_{ijvU}, b_{ijvU})][min(a_{ij\eta L}, b_{ij\eta L}), min(a_{ij\eta U}, b_{ij\eta U})]$
$[max(a_{ij\mu L}, b_{ij\mu L}), max(a_{ij\mu U}, b_{ij\mu U})]$)
$= A^c \wedge B^c$.

The proof of part (ii) can be done on similar lines.

Theorem 2. Suppose A, B and C are three IVSFM and $A \leq C$ and $B \leq C$. Then $A \vee B \leq C$.

Proof: Let $A = ([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}])$,
$B = ([b_{ij\mu L}, b_{ij\mu U}], [b_{ij\eta L}, b_{ij\eta U}], [b_{ijvL}, b_{ijvU}])$,
$C = ([c_{ij\mu L}, c_{ij\mu U}], [c_{ij\eta L}, c_{ij\eta U}], [c_{ijvL}, c_{ijvU}])$.
If $A \leq C$, then $a_{ij\mu L} \leq c_{ij\mu L}, a_{ij\mu U} \leq c_{ij\mu U}, a_{ij\eta L} \leq c_{ij\eta L}, a_{ij\eta U} \leq c_{ij\eta U}, a_{ijvL} \geq c_{ijvL}, a_{ijvU} \geq c_{ijvU}$ for all i,j.
And if $B \leq C$, then $b_{ij\mu L} \leq c_{ij\mu L}, b_{ij\mu U} \leq c_{ij\mu U}, b_{ij\eta L} \leq c_{ij\eta L}, b_{ij\eta U} \leq c_{ij\eta U}, b_{ijvL} \geq c_{ijvL}, b_{ijvU} \geq c_{ijvU}$ for all i,j.
Now max $(a_{ij\mu L}, b_{ij\mu L}) \leq c_{ij\mu L}$, max $(a_{ij\mu U}, b_{ij\mu U}) \leq c_{ij\mu U}$,
min $(a_{ij\eta L}, b_{ij\eta L}) \leq c_{ij\eta L}$, min $(a_{ij\eta U}, b_{ij\eta U}) \leq c_{ij\eta U}$,
min $(a_{ijvL}, b_{ijvL}) \geq c_{ijvL}$, min $(a_{ijvU}, b_{ijvU}) \geq c_{ijvU}$.
Thus, $A \vee B \leq C$ using Definition 7.

Theorem 3. Suppose A, B and C are three IVSFM and $A \leq B$. Then $A \vee C \leq B \vee C$.

Proof: Let $A = \left(\langle a_{ij\mu L}, a_{ij\mu U}\rangle, \langle a_{ij\eta L}, a_{ij\eta U}\rangle, \langle a_{ijvL}, a_{ijvU}\rangle\right)$,
$B = \left(\langle b_{ij\mu L}, b_{ij\mu U}\rangle, \langle b_{ij\eta L}, b_{ij\eta U}\rangle, \langle b_{ijvL}, b_{ijvU}\rangle\right)$ and
$C = \left(\langle c_{ij\mu L}, c_{ij\mu U}\rangle, \langle c_{ij\eta L}, c_{ij\eta U}\rangle, \langle c_{ijvL}, c_{ijvU}\rangle\right)$ be three IVSFM of the same order $m \times n$.
If $A \leq B$, then $a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}, a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}, a_{ijvL} \geq b_{ijvL}, a_{ijvU} \geq b_{ijvU}$. Now $\max\left(a_{ij\mu L}, c_{ij\mu L}\right) \leq \max\left(b_{ij\mu L}, c_{ij\mu L}\right)$, $\max\left(a_{ij\mu U}, c_{ij\mu U}\right) \leq \max\left(b_{ij\mu U}, c_{ij\mu U}\right)$,
$\min\left(a_{ij\eta L}, c_{ij\eta L}\right) \leq \min\left(b_{ij\eta L}, c_{ij\eta L}\right)$, $\min\left(a_{ij\eta U}, c_{ij\eta U}\right) \leq \min\left(b_{ij\eta U}, c_{ij\eta U}\right)$,
$\min\left(a_{ijvL}, c_{ijvL}\right) \geq \min\left(b_{ijvL}, c_{ijvL}\right)$, $\min\left(a_{ijvU}, c_{ijvU}\right) \geq \min\left(b_{ijvU}, c_{ijvU}\right)$ for all *i, j*.
Therefore, $A \vee C \leq B \vee C$.

Theorem 4. Let *A*, *B* and *C* be three IVSFM of the same order, and $C \leq A$ and $C \leq B$. Then $C \leq A \wedge B$.

Proof: Proof of the above result directly follows from Theorem 3.

Theorem 5. Suppose *A*, *B* and *C* are three IVSFM of the same order, and if $A \leq B$, $A \leq C$ and $B \wedge C = 0$, then $A = 0$.

Proof: The proof directly follows from Definition 7 and Theorem 4.

Theorem 6. Let *A*, *B* and *C* be three IVSFM of the same order of $A \leq B$. Then $A \wedge C \leq B \wedge C$.

Proof: If $A \leq B$, then
$a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}, a_{ij\eta L} \leq b_{ij\eta L}, a_{ij\eta U} \leq b_{ij\eta U}, a_{ijvL} \geq b_{ijvL}, a_{ijvU} \geq b_{ijvU}$.
Now $\min\left[a_{ij\mu L}, c_{ij\mu L}\right] \leq \min\left[b_{ij\mu L}, c_{ij\mu L}\right]$, $\min\left(a_{ij\mu U}, c_{ij\mu U}\right) \leq \min\left(b_{ij\mu U}, c_{ij\mu U}\right)$,
$\min\left(a_{ij\eta L}, c_{ij\eta L}\right) \leq \min\left(b_{ij\eta L}, c_{ij\eta L}\right)$, $\min\left(a_{ij\eta U}, c_{ij\eta U}\right) \leq \min\left(b_{ij\eta U}, c_{ij\eta U}\right)$,
$\max\left(a_{ijvL}, c_{ijvL}\right) \geq \max\left(b_{ijvL}, c_{ijvL}\right)$, $\max\left(a_{ijvU}, c_{ijvU}\right) \geq \max\left(b_{ijvU}, c_{ijvU}\right)$ for all *i* , *j*.
So $A \wedge C \leq B \wedge C$.

Theorem 7. Let *A*, *B* and *C* be three IVSFM of the same order, and if $A \leq B$ and $B \wedge C = 0$, then $A \wedge C = 0$.

Proof: By Theorem 6 the proof is straightforward.

DETERMINANT AND ADJOINT OF IVSFM

In this section we define determinant and adjoint of the IVSFM.

Definition 9. Consider the IVSFM $A = \left([a_{ij\mu L}, a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ijvL}, a_{ijvU}]\right)$ of order *k*. Then the determinant of *A* is denoted by $|A|$ and define by

$$|A| = \begin{pmatrix} \vee_{h \in H_k}\left(\left[a_{1h(1)\mu L}, a_{1h(1)\mu U}\right] \wedge \left[a_{2h(2)\mu L}, a_{2h(2)\mu U}\right] \ldots \wedge \left[a_{kh(k)\mu L}, a_{kh(k)\mu U}\right]\right), \\ \wedge_{h \in H_k}\left(\left[a_{1h(1)\eta L}, a_{1h(1)\eta U}\right] \wedge \left[a_{2h(2)\eta L}, a_{2h(2)\eta U}\right] \ldots \wedge \left[a_{kh(k)\eta L}, a_{kh(k)\eta U}\right]\right), \\ \wedge_{h \in H_k}\left(\left[a_{1h(1)vL}, a_{1h(1)vU}\right] \vee \left[a_{2h(2)vL}, a_{2h(2)vU}\right] \ldots \vee \left[a_{kh(k)vL}, a_{kh(k)vU}\right]\right), \end{pmatrix},$$

where H_k denotes the symmetric group of all permutations on the indices $\{1, 2, 3 \ldots\ldots.. k\}$.

Example 1. Consider IVSFM of order 3:

$$A = \begin{pmatrix} [0.85,0.95][0.10,0.15][0.05,0.15] & [0.55,0.65][0.25,0.30][0.25,0.30] & [0.13,0.19][0.69,0.79][0.22,0.27] \\ [0.75,0.85][0.15,0.20][0.15,0.20] & [0.47,0.61][0.33,0.41][0.27,0.36] & [0.31,0.42][0.43,0.52][0.30,0.36] \\ [0.56,0.75][0.20,0.25][0.20,0.25] & [0.7,0.88][0.15,0.22][0.10,0.22] & [0.22,0.31][0.53,0.63][0.32,0.38] \end{pmatrix}.$$

To find the determinant of A, we need to find out all the permutations on $\{1,2,3\}$. The permutations on $\{1,2,3\}$ are

$\psi_1 = \binom{1\ 2\ 3}{1\ 2\ 3}$, $\psi_2 = \binom{1\ 2\ 3}{1\ 3\ 2}$, $\psi_3 = \binom{1\ 2\ 3}{2\ 1\ 3}$ $\psi_4 = \binom{1\ 2\ 3}{2\ 3\ 1}$, $\psi_5 = \binom{1\ 2\ 3}{3\ 1\ 2}$, $\psi_6 = \binom{1\ 2\ 3}{3\ 2\ 1}$.

The membership degree of $|A|$ is

$$
\begin{aligned}
&([a_{1\psi_1(1)\mu L}, a_{1\psi_1(1)\mu U}] \wedge [a_{2\psi_1(2)\mu L}, a_{2\psi_1(2)\mu U}] \wedge [a_{3\psi_1(3)\mu L}, a_{3\psi_1(3)\mu U}])\\
&\vee ([a_{1\psi_2(1)\mu L}, a_{1\psi_2(1)\mu U}] \wedge [a_{2\psi_2(2)\mu L}, a_{2\psi_2(2)\mu U}] \wedge [a_{3\psi_2(3)\mu L}, a_{3\psi_2(3)\mu U}])\\
&\vee ([a_{1\psi_3(1)\mu L}, a_{1\psi_3(1)\mu U}] \wedge [a_{2\psi_3(2)\mu L}, a_{2\psi_3(2)\mu U}] \wedge [a_{3\psi_3(3)\mu L}, a_{3\psi_3(3)\mu U}])\\
&\vee ([a_{1\psi_4(1)\mu L}, a_{1\psi_4(1)\mu U}] \wedge [a_{2\psi_4(2)\mu L}, a_{2\psi_4(2)\mu U}] \wedge [a_{3\psi_4(3)\mu L}, a_{3\psi_4(3)\mu U}])\\
&\vee ([a_{1\psi_5(1)\mu L}, a_{1\psi_5(1)\mu U}] \wedge [a_{2\psi_5(2)\mu L}, a_{2\psi_5(2)\mu U}] \wedge [a_{3\psi_5(3)\mu L}, a_{3\psi_5(3)\mu U}])\\
&\vee \left(\left[a_{1\psi_6(1)\mu L}, a_{1\psi_6(1)\mu U}\right] \wedge \left[a_{2\psi_6(2)\mu L}, a_{2\psi_6(2)\mu U}\right] \wedge \left[a_{3\psi_6(3)\mu L}, a_{3\psi_6(3)\mu U}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= ([a_{11\mu L},\ a_{11\mu U}] \wedge [a_{22\mu L},\ a_{22\mu U}] \wedge [a_{33\mu L},\ a_{33\mu U}])\\
&\vee ([a_{11\mu L},\ a_{11\mu U}] \wedge [a_{23\mu L}, a_{23\mu U}] \wedge [a_{32\mu L},\ a_{32\mu U}])\\
&\vee ([a_{12\mu L},\ a_{12\mu U}] \wedge [a_{21\mu L},\ a_{21\mu U}] \wedge [a_{33\mu L},\ a_{33\mu U}])\\
&\vee ([a_{12\mu L},\ a_{12\mu U}] \wedge [a_{23\mu L},\ a_{23\mu U}] \wedge [a_{31\mu L},\ a_{31\mu U}])\\
&\vee ([a_{13\mu L},\ a_{13\mu U}] \wedge [a_{21\mu L},\ a_{21\mu U}] \wedge [a_{32\mu L},\ a_{32\mu U}])\\
&\vee \left(\left[a_{13\mu L},\ a_{13\mu U}\right] \wedge \left[a_{22\mu L}, a_{22\mu U}\right] \wedge \left[a_{31\mu L}, a_{31\mu U}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= ([0.85, 0.95] \wedge [0.47, 0.61] \wedge [0.22, 0.31])\\
&\vee ([0.85, 0.95] \wedge [0.31, 0.42] \wedge [0.7, 0.88])\\
&\vee ([0.55, 0.65] \wedge [0.75, 0.85] \wedge [0.22, 0.31])\\
&\vee ([0.55, 0.65] \wedge [0.31, 0.42] \wedge [0.65, 0.75])\\
&\vee ([0.13, 0.19] \wedge [0.75, 0.85] \wedge [0.7, 0.88])\\
&\vee ([0.13, 0.19] \wedge [0.47, 0.61] \wedge [0.65, 0.75])
\end{aligned}
$$

$$
\begin{aligned}
&= [0.22, 0.31] \vee [0.31, 0.42] \vee [0.22, 0.31] \vee [0.31, 0.42] \vee [0.7, 0.19] \vee [0.13, 0.19]\\
&= [0.31, 0.42].
\end{aligned}
$$

Similarly, the neutrality degree of $|A|$ is

$$
\begin{aligned}
&([0.10, 0.15] \wedge [0.33, 0.41] \wedge [0.53, 0.63])\\
&\wedge ([0.10, 0.15] \wedge [0.43, 0.52] \wedge [0.15, 0.22])\\
&\wedge ([0.25, 0.30] \wedge [0.15, 0.20] \wedge [0.53, 0.63])\\
&\wedge ([0.25, 0.30] \wedge [0.43, 0.52] \wedge [0.20, 0.25])\\
&\wedge ([0.69, 0.79] \wedge [0.15, 0.20] \wedge [0.15, 0.22])\\
&\wedge ([0.69, 0.79] \wedge [0.33, 0.41] \wedge [0.20, 0.25])
\end{aligned}
$$

$=[0.10, 0.15] \wedge [0.10, 0.15] \wedge [0.15, 0.20] \wedge [0.20, 0.25] \wedge [0.15, 0.22] \wedge [0.20, 0.25]$
$= [0.10, 0.15]$.

Now the non-membership degree of $|A|$ is

$$
\begin{aligned}
&([0.05, 0.15] \vee [0.27, 0.36] \vee [0.32, 0.38])\\
&\wedge ([0.05, 0.15] \vee [0.30, 0.35] \vee [0.10, 0.22)\\
&\wedge ([0.25, 0.30] \vee [0.15, 0.20] \vee [0.32, 0.38])\\
&\wedge ([0.25, 0.30] \vee [0.30, 0.36] \vee [0.20, 0.25])\\
&\wedge ([0.22, 0.27] \vee [0.15, 0.20] \vee [0.10, 0.22])\\
&\wedge ([0.22, 0.27] \vee [0.27, 0.36] \vee [0.20, 0.25]).
\end{aligned}
$$

$= [0.32, 0.38] \vee [0.30, 0.36] \vee [0.32, 0.38] \vee [0.30, 0.36] \vee [0.22, 0.27] \vee [0.27, 0.36]$
$= [0.22, 0.27]$.

$|A|= ([0.31, 0.42]\ [0.10, 0.15]\ [0.22, 0.27])$.

Definition 10. Suppose IVSFM $A = ([a_{ij\mu L} a_{ij\mu U}], [a_{ij\eta L}, a_{ij\eta U}], [a_{ij\nu L}, a_{ij\nu U}])$ of order x. Then the adjoint of A is denoted by $Adj.(A)$ and defined by

Q= $([q_{ij\mu}, q_{ij\eta}, q_{ij\nu}])= Adj(A)$,

where $q_{ij\mu}=\vee_{\delta\in S_{x_j x_i}} \wedge_{u\in x_j} a_{u\delta(u)\mu}$,

$q_{ij\eta}=\wedge_{\delta\in S_{x_j x_i}} \wedge_{u\in x_j} a_{u\delta(u)\eta}$,

$q_{ij\nu}=\wedge_{\delta\in S_{x_j x_i}} \vee_{u\in x_j} a_{u\delta(u)\nu}$.

Here x_j= $\{1,2 \ldots\ldots x\}$-$\{j\}$ and $S_{x_j x_i}$ represents the set of all permutations of set x_j over set x_i.

Example 2. Consider IVSFM of order 3:

$$A = \begin{pmatrix} [0.85,0.95][0.10,0.15][0.05,0.15] & [0.55,0.65][0.25,0.30][0.25,0.30] & [0.13,0.19][0.69,0.79][0.22,0.27] \\ [0.75,0.85][0.15,0.20][0.15,0.20] & [0.47,0.61][0.33,0.41][0.27,0.36] & [0.31,0.42][0.43,0.52][0.30,0.36] \\ [0.56,0.75][0.20,0.25][0.20,0.25] & [0.7,0.88][0.15,0.22][0.10,0.22] & [0.22,0.31][0.53,0.63][0.32,0.38] \end{pmatrix}.$$

For j=1 and i=1, x_j= {1,2,3}-{1}= {2,3} and x_i= {1,2,3}-{1}= {2,3}. The permutations of x_i over x_j are

$$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

Now

$(a_{22\mu} \wedge a_{33\mu}) \vee (a_{23\mu} \wedge a_{32\mu})$

$= ([0.47,0.61] \wedge [0.22,0.31]) \vee ([0.31,0.42] \wedge [0.7,0.88])$

$= [0.22,0.31] \vee [0.31,0.42]$ =[0.31, 0.42],

$(a_{22\eta} \wedge a_{33\eta}) \wedge (a_{23\eta} \wedge a_{32\eta})$

$= ([0.33,0.41] \wedge [0.53,0.63]) \wedge ([0.43,0.52] \wedge [0.15,0.22])$

$= [0.33,0.41] \wedge [0.15,0.22]$ =[0.15, 0.22],

and $(a_{22\nu} \vee a_{33\nu}) \wedge (a_{23\nu} \vee a_{32\nu})$

$= ([0.27,0.36] \vee [0.32,0.38]) \wedge ([0.30,0.36] \vee [0.10,0.22])$

$= [0.32,0.38] \wedge [0.30,0.36]$ = [0.30, 0.36].

Similarly, we can find other values of $Adj.(A)$. Thus,

$$Adj.A = \begin{pmatrix} [0.31,0.42][0.15,0.22][0.30,0.36] & [0.22,0.31][0.15,0.22][0.22,0.27] & [0.31,0.42][0.25,0.30][0.27,0.36] \\ [0.31,0.42][0.15,0.20][0.30,0.36] & [0.22,0.31][0.10,0.15][0.22,0.27] & [0.31,0.42][0.10,0.15][0.20,0.25] \\ [0.7,0.85][0.15,0.20][0.15,0.22] & [0.55,0.65][0.10,0.15][0.10,0.22] & [0.55,0.65][0.10,0.15][0.25,0.30] \end{pmatrix}.$$

APPLICATION OF IVSFM IN DECISION-MAKING

In this section we propose the score function and discuss the application of IVSFM in decision-making.

Definition 11. Let N= $([\alpha,\beta],[\gamma,\delta],[\tau,\theta])$ be an interval-valued spherical fuzzy number. Then the score function for interval-valued spherical fuzzy number is

$$E(N) = \frac{1+\alpha^2+\beta^2-\gamma^2-\delta^2-(\frac{\tau}{2})^2-(\frac{\theta}{2})^2}{6}\left|\alpha+\beta-\gamma-\delta-\frac{\tau}{2}-\frac{\theta}{2}\right| \in [-1,1],$$

with accuracy function $F(N) = \frac{\alpha^2+\beta^2+\gamma^2+\delta^2+\tau^2+\theta^2}{2} \in [0,1]$.

For any two interval-valued spherical fuzzy numbers N_1 and N_2,

i. If $E(N_1) > E(N_2)$, then $N_1 > N_2$

ii. If $E(N_1) < E(N_2)$, then $N_1 < N_2$
iii. If $E(N_1) = E(N_2)$, then
(a) If $F(N_1) > F(N_2)$, then $N_1 > N_2$
(b) If $F(N_1) < F(N_2)$, then $N_1 < N_2$
(c) If $F(N_1) = F(N_2)$, then $N_1 = N_2$.

One can verify the following properties for the proposed score function:

Property 1. Let $N = ([1,1],[0,0][0,0])$ be an interval-valued spherical fuzzy number; then $E(N) = 1$.

Property 2. Let $N = ([0,0],[0,0][0,0])$ be an interval-valued spherical fuzzy number; then $E(N) = 0$.

Problem in Career Placement Assessment

The following is how we localise the interval-valued spherical fuzzy relation concept. Let $V=\{v_1, v_2, v_3, v_4 \ldots\ldots v_i\}$, $U=\{u_1, u_2, u_3, u_4 \ldots\ldots u_i\}$ and $W=\{w_1, w_2, w_3, w_4 \ldots\ldots w_i\}$ be the set of subjects related to courses, finite set of courses and finite set of applicants respectively. Suppose we have the relations $R_1(W \rightarrow V)$ and $R_2(W \rightarrow V)$ such that

$$R_1 = \{((w,v), \mu_{R_1}(w,v), \nu_{R_1}(w,v), \eta_{R_1}(w,v)) | (w,v) \in (W \times V)\},$$

$$R_2 = \{((v,u), \mu_{R_2}(v,u), \nu_{R_2}(v,u), \eta_{R_2}(v,u)) | (v,u) \in (V \times U)\},$$

where $\mu_{R_1}(w,v)$ represents the degree with which the applicant, *w*, passes the related subject requirement, v; $\nu_{R_1}(w,v)$ represents the degree with which the applicant, w, does not pass the related subject requirement; and $\eta_{R_1}(w,v)$ represents the degree with which the applicant may pass or may not pass the related subject requirement. Similarly, $\mu_{R_2}(v,u)$ represents the degree with which the related subject requirement, v, determines the courses u; $\nu_{R_2}(v,u)$ represents the degree with which the related subject requirement, v, does not determine the course, u; and $\eta_{R_2}(v,u)$ represents the degree with which the related subject requirement, v, may or may not determine the course. The composition, *T*, of R_1 and R_2 is given as $T = R_1 \odot R_2$. This describes the state in which the applicants, w_i, with respect to the related subjects requirement, v_j, fit the courses, u_k. Thus,

$$\mu_T(w_i, u_k) = \bigvee_{v_j \in V} \{\mu_{R_1}(w_i, v_j) \wedge \mu_{R_2}(v_j, u_k)\},$$

$$\nu_T(w_i, u_k) = \bigwedge_{v_j \in V} \{\nu_{R_1}(w_i, v_j) \vee \nu_{R_2}(v_j, u_k)\},$$

$$\eta_T(w_i, u_k) = \bigwedge_{v_j \in V} \{\eta_{R_1}(w_i, v_j) \wedge \eta_{R_2}(v_j, u_k)\}$$

for all $w_i \in W$ and $u_k \in U$ where *i, j* and *k* take value from 1,2...n. The values of $\mu_{R_1 \odot R_2}(w_i, u_k)$, $\nu_{R_1 \odot R_2}(w_i, u_k)$ and $\eta_{R_1 \odot R_2}(w_i, u_k)$ of the composition $T = R_1 \odot R_2$ are as follows: If the value of T is maximised, the career placement can be achieved. This value is computed from R_1 and R_2 for the placement of w_i into any u_k relative to v_j, and it is maximised by using the proposed score function (Definition 11).

Example 3. Let $W = \{Tom, Ankit, Sahil, Aashima\}$ be a set of applicants for the course placements; $V = \{English\ Lang., Maths, Biology, Physics, Chemistry\}$ be a set of related subject requirements for the set of courses, and $U = \{medicine, pharmacy, surgery, anatomy\}$ be a set of courses. The following results are then obtained:

• A hypothetical relation R_1 $(W \rightarrow V)$ is given in Table 1;

• A hypothetical relation R_2 ($V \rightarrow U$) is given in Table 2;
• The composition relation T ($W \rightarrow U$) = T =$R_1 \odot R_2$ is given in Table 3;
• The degree of affiliation between the set of applicants W_i to the set of courses U_i is calculated in Table 4 by using the proposed score function (Definition 11).

Table 1 shows the relation between W and course placement V in the form of interval-valued spherical fuzzy number. It defines how strongly the set of applicants and that of related subjects are related. Table 2 shows the relation between the set of related subjects V and that of courses U.

Table 1. Hypothetical relation R_1 ($W \rightarrow V$)

W/V	English Lang.	Mathematics	Biology	Physics	Chemistry
Tom	[0.85,0.93] [0.10,0.13] [0.05,0.15]	[0.20,0.25] [0.65,0.75] [0.20,0.25]	[0.6,0.7] [0.1,0.2][0.4,0.5]	[0.7,0.8] [0.2,0.3][0.3,0.4]	[0.21,0.26] [0.64,0.74] [0.21,0.26]
Ankit	[0.55,0.65] [0.25,0.30] [0.25,0.30]	[0.85,0.9] [0.1,0.15] [0.07,0.17]	[0.3,0.6][0.2,0.3][0.3,0.4]	[0.5,0.6][0.1,0.2][0 .3,0.4]	[0.76,0.88] [0.15,0.20] [0.14,0.21]
Sahil	[0.65,0.75] [0.2,0.25] [0.25,0.30]	[0.10,0.15] [0.8,0.9] [0.08,0.15]	[0.4,0.5][0.8,0.9][0.6,0.7]	[0.6,0.8][0.3,0.4][0 .2,0.4]	[0.82,0.93] [0.12,0.17] [0.09,0.17]
Aashima	[0.55,0.65] [0.25,0.30] [0.25,0.30]	[0.15,0.19] [0.7,0.8] [0.13,0.20]	[0.8,0.9][0.2,0.3][0.1,0.2]	[0.68,0.79] [0.19,0.24] [0.19,0.24]	[0.65,0.75] [0.20,0.25] [0.20,0.25]

Table 2. Hypothetical relation R_2 ($V \rightarrow U$)

V/U	Medicine	Pharmacy	Surgery	Anatomy
English Lang.	[0.31,0.42][0.15,0.22] [0.30,0.36]	[0.55,0.65][0.10,0.15] [0.25,0.30]	[0.56,0.66] [0.29,0.36] [0.24,0.30]	[0.44,0.51][0.43,0.52] [0.26,.034]
Mathematics	[0.22,0.31][0.10,0.15] [0.10,0.15]	[0.31,0.42][0.15,0.20] [0.3,0.36]	[0.59,0.68] [0.27,0.33] [0.25,0.32]	[0.54,0.62][0.33,0.41] [0.26,0.33]
Biology	[0.22,0.31][0.15,0.22] [0.22,0.27]	[0.7,0.85][0.15,0.20] [0.15,0.22]	[0.55,0.63] [0.32,0.31] [0.25,0.32]	[0.48,0.57][0.37,0.45] [0.24,0.30]
Physics	[0.31,0.42] [0.25,0.30] [0.27,0.36]	[0.58,0.67][0.3,0.37] [0.25,0.32]	[0.6,0.8] [0.29,0.36] [0.24,0.30]	[0.85,.93][.10,.13] [0.05,0.15]
Chemistry	[0.7,0.85][0.15,0.20] [0.15,0.20]	[0.19,0.24][0.67,0.77] [0.19,0.24]	[0.19,0.24] [0.67,0.77] [0.19,0.24]	[0.65,0.75][0.20,0.25] [0.25,0.30]

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Table 3. Composition relation T: $(W \rightarrow U) = R_1 \odot R_2$

W/U	Medicine	Pharmacy	Surgery	Anatomy
Tom	[0.31,0.42][0.10,0.13] [0.20,0.25]	[0.7,0.85][0.1,0.15] [0.1,0.15]	[0.7,0.85][0.10,0.15] [0.10,0.15]	[0.65,0.75] [0.10,0.15] [0.13,0.20]
Ankit	[0.58,0.67][0.1,0.13] [0.4,0.5]	[0.55,0.65][0.1,0.15] [0.3,0.4]	[.58,.67][.1,.15] [.19,.24]	[0.7,0.85][0.10,0.15] [0.15,0.22]
Sahil	[0.6,.8][0.1,0.13] [.21,.26]	[0.59,0.68][0.1,0.15] [0.19,0.24]	[0.6,0.8][0.12,0.17] [0.19,0.24]	[0.55,0.65][0.19,0.24] [0.2,0.25]
Aashima	[0.7,0.8] [0.1,0.13] [0.25,0.3]	[0.65,0.75][0.1,0.15] [0.25,0.3]	[0.6,0.8][0.1,0.3] [0.2,0.4]	[0.68,0.79][0.2,0.25] [0.19,0.24]

Now to calculate degree of closeness value (Table 4), we use the score function (Definition 11) for finding the weights (score value) of Table 3:

$$E(N) = \frac{1+\alpha^2+\beta^2-\gamma^2-\delta^2-(\frac{\tau}{2})^2-(\frac{\theta}{2})^2}{6}\left|\alpha+\beta-\gamma-\delta-\frac{\tau}{2}-\frac{\theta}{2}\right|.$$

Suppose N_1= [0.31, 0.42] [0.10,0.13] [0.20,0.25]; here α=0.31, β=0.42, γ=0.10, δ=0.13, τ =0.20, θ=0.25. Then E (N_1) = 0.0681. Also, suppose N_2= [0.7,0.85] [0.1,0.15] [0.1,0.15]; here α=0.7, β=0.85, γ=0.1, δ=0.15, τ =0.1, θ=0.15. Then E (N_2) = 0.425. Similarly, we can calculate other weights (score values) for Table 3.

Table 4. Degree of closeness value

W/U	Medicine	Pharmacy	Surgery	Anatomy
Tom	0.0681	**0.425**	**0.425**	0.1335
Ankit	0.1573	0.163	0.226	**0.401**
Sahil	**0.3031**	0.2354	0.2883	0.145
Aashima	**0.3424**	0.2791	0.2788	0.2630

According to the analysis of score value, i.e. degree of closeness value, given in Table 4, Tom is suited to studying either medicine or surgery and Ankit is suited to studying anatomy. Sahil is suited to studying medicine while Aashima is suited to studying only medicine.

COMPARATIVE STUDY AND ADVANTAGES

Ejegwa [30] worked on the Pythagorean fuzzy set and its application in career placements based on academic performance using max–min–max composition. The information was taken in a Pythagorean fuzzy sense in previous papers on Pythagorean fuzzy decision-making. When there are additional kinds of uncertainty in data, present strategies are ineffective in dealing with them. In these instances data should be gathered or displayed in the form of an interval-valued spherical fuzzy meaning. In such instances the currently developed process plays an important role in a successful conclusion.

Silambarasan [24] worked on the SFM, in which the membership, neutrality and non-membership degree have a point. However, in some cases it is difficult to measure the degree of membership, neutrality and non-membership values as a point. In those cases we consider the membership, neutrality and non-membership values as an interval, and it is practically useful in the case of real-life problems. So our study is an extension of the study of Silambarasan [24].

However, in the current technique the matrix entries that are considered are interval-valued spherical fuzzy values. Over a set of universes, IVSFM is extracted from them. The final matrix is then calculated using the proposed score function between two IVSFM, yielding a decision. It is not necessary to perform many different sorts of computation to implement the steps of this approach. As a result, developing algorithms and computer programming for this method is relatively simple. Furthermore, the data points used here are capable of tolerating a wider range of information ambiguity. This study can be thought of as one in an advanced spherical fuzzy sense because the interval-valued spherical fuzzy concept is a generalisation of the spherical fuzzy concept. The use of IVSFM has the following advantages:

i. In the case that information cannot be fully captured by the existing matrices due to each of their flaws, the void is filled by the IVSFM.

ii. The limitation of the FM, IFM, PFM and SFM conditions in the literature at hand is that they prevent experts and decision-makers from assigning membership, neutrality and non-membership degrees by their personal preferences. The membership, neutrality and non-membership degrees can all be described broadly as interval numbers in this work.

iii. The implementation of the IVSFM and the approach suggested for the problems of career placement assessment demonstrate how well and consistently the proposed work addresses the extended framework. The observations indicate that the IVSFM is the most generalised structure among all fuzzy matrix models.

The detailed analysis presented in Table 5 further compares the proposed work with existing research available in the literature.

Table 5. Analysis of proposed work and existing work in literature

Characteristic method	Whether consider membership degree (MD)	Whether consider MD more flexible, i.e IVMD	Whether consider MD or NMD	Whether consider MD or NMD more flexible, i.e IVMD or IVNMD	Whether consider MD, neutrality and NMD degree	Whether consider MD neutrality and NMD degree more Flexible, i.e IVMD, interval-valued neutrality degree and IVNMD
Thomason [5]	✓	X	X	X	X	X
Pal [10]	✓	✓	X	X	X	X
Pal et al. [14]	✓	✓	✓	X	X	X
Khan and Pal [17]	✓	✓	✓	✓	X	X
Dogra and Pal [19]	✓	✓	✓	✓	✓	X
Silambarasan [24]	✓	✓	✓	✓	✓	X
Proposed work	✓	✓	✓	✓	✓	✓

Note: MD=membership degree, IVMD=interval-valued membership degree, NMD=non-membership degree, IVNMD= interval-valued non-membership degree

CONCLUSIONS

We have defined the interval-valued spherical fuzzy matrix. Important definitions and theorems are defined with their proofs. The procedure of determinant and adjoint of the IVSFM is developed. Such investigations can be seen as an extension of studies on spherical fuzzy matrices.

Additionally, we study the application of IVSFM in decision-making processes. A score function is introduced to address decision-making challenges. A limitation of IVSFM is related to the representation of degrees of membership, neutrality and non-membership as interval numbers. The limitation arises when the sum of the upper degree of membership, neutrality and upper degree of non-membership exceeds the interval [0, 1]. Exceeding this interval can lead to inconsistencies in calculations and may affect the accuracy and reliability of the results obtained using IVSFM. Therefore, careful consideration and management of this limitation are necessary to ensure the validity of the analysis conducted using IVSFM.

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Solution of Transportation Problem Using Interval-Valued Pythagorean Fuzzy Approach

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1 Introduction

The transportation problem is a subset of the linear programming problem (LPP). Transportation issues have a wide range of applications in logistics and supply chain management in terms of cost reduction. It is concerned with transporting a commodity from various sources of supply to various sinks of demand while minimizing the total distribution cost. Transportation problem is originally formulated by Hitchcock [1]. To solve the transportation problem, the decision parameters must be fixed, such as availability, requirement, and transportation cost per unit. The determination of supply, demand, and unit transportation cost may be imprecise due to some uncontrollable circumstances.

To deal with the imprecision of real-world problems, Zadeh [2] introduced the concept of fuzzy set (FS). Chanas et al. [3] proposed a solution to the transportation problem. The optimal solution of the TP with fuzzy cost has also been proposed by Chanas and Kuchta [4]. Palanivel and Natarajan [5] worked on the transportation problem under fuzzy set. Kumar and Amarpreet [6] presented the concept of application of classical transportation method for solving fuzzy TP. Many researchers have worked on fuzzy transportation, but they only consider the element's membership function.

The concept of intuitionistic fuzzy set (IFS) was introduced by Atanassov [7]. They considered the element's membership and non-membership values and defined

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by (μ, ν) with the condition $\mu + \nu \leq 1$. The intuitionistic fuzzy transportation problem (IFTP) was studied by Gani and Abbs [8]. Singh and yadav [9] proposed a new method for solving IFTP. Dubey and Mehra [10] worked on linear programming with triangular intuitionistic fuzzy number. The trapezoidal intuitionistic fuzzy fractional TP was solved by Bharti [11]. The interval-valued intuitionistic fuzzy set (IVIFS) was presented by Atanssov and Gargov [12]. Under the IVIFS, Mishra et al. [13] worked on the transportation problem. Arora [14] presented an algorithm for interval-valued fuzzy fractional TP. The Pythagorean fuzzy set (Pyfs) was introduced by Yager [15]. When $\mu + \nu > 1$, Yager overcomes the situation. The Pythagorean fuzzy set is a generalization of IFS with the condition that "the square sum of membership and non-membership degrees is less than or equal to 1". Zhang and Xu [16] worked on the TOPSIS method to MCDM with Pythagorean fuzzy set methodology. Geng et al. [17] worked on TODIM method with Pythagorean fuzzy uncertain linguistic model and in MCDM problem. Karasan et al. [18] proposed a novel Pythagorean fuzzy AHP method to solve a landfill site selection problem. Kumar et al. [19] worked on a Pythagorean fuzzy approach to the TP.

Liang et al. [20] defined interval-valued Pythagorean fuzzy set (IVPFS). He extended membership and non-membership values in interval number. Peng and Yang [21] worked on IVPFS fundamental properties. Peng and Li [22] proposed a two-measure multiparametric measure and WDBA-based algorithm for IVPFS in emergency decisions. Using the induced IVPFS Einstein aggregation operator, Rahaman et al. [23] developed a methodology for multi-attribute group decision-making.

According to the findings of the above literature review on the transportation problem, no one has worked on the problem of TP using IVPFS. Using three different cases, we proposed a method to solve the interval-valued Pythagorean fuzzy transportation problem (IVPFTP). We used the proposed score function in these cases to convert the IVPFTP into a crisp transportation problem. After that, we have applied the computational technique for finding the optimality of the problem. In the final solution, we found the appropriate results by our proposed score function as compare to Peng and Yang [24] score function result.

We have divided the whole paper in 5 sections. We have discussed preliminaries in Sect. 2. In Sect. 3, we have proposed score function for the interval-valued Pythagorean fuzzy set. In Sect. 4, mathematical structure and algorithm for transportation problem under interval-valued Pythagorean set are shown. Also illustrate examples are discussed for IVPFTP in this section. In Sect. 5, result and conclusion of the defined problem are shown.

2 Preliminaries

Definition 1 Fuzzy set [2]

Fuzzy set G on universal set U is defined as

$$G = \{\langle u, \omega_G(u) | u \in U \rangle\}$$

Such that $\omega_G(u) : U \to [0, 1]$.
and $\omega_G(u)$ is the degree of membership of the element u to the set G.

Definition 2 Pythagorean fuzzy set (Pyfs) [15]
Pythagorean fuzzy set '$\widetilde{G}$' on U is defined as

$$\widetilde{G} = \{\langle u, \omega_G(u), \tau_G(u)\rangle / u \in U\}$$

where $\omega_G(u),\ \tau_G(u) : U \to [0, 1]$ are the membership and non-membership degree of the element $u \in U$ with the condition $(\omega_G(u))^2 + (\tau_G(u))^2 \leq 1$ the degree of indeterminacy is given by $\varsigma_G(u) = \left(\sqrt{1 - \left(\omega_G^2 + \tau_G^2\right)}\right)$.

Definition 3 Interval-valued Pythagorean fuzzy set [21].
The definition of an interval-valued Pythagorean fuzzy set G (IVPFS) on U is defined as

$$\widetilde{G} = \left\{\left\langle u, \left[\omega_G^-(u), \omega_G^+(u)\right], \left[\tau_G^-(u), \tau_G^+(u)\right]\right\rangle / \left[\omega_G^-(u), \omega_G^+(u)\right], \left[\tau_G^-(u), \tau_G^+(u)\right] \in D[0, 1], \left(\omega_G^+(u)\right)^2 + \left(\tau_G^+(u)\right)^2 \leq 1,\ u \in U\right.$$

$[\omega_G^-(u), \omega_G^+(u)]$, $[\tau_G^-(u), \tau_G^+(u)]$ are the degree of membership and non-membership values in interval form. The degree of indeterminacy is defined as follows,

$$\varsigma_G(u) = \left[\varsigma_G^-(u), \varsigma_G^+(u)\right] = \left[\sqrt{1 - \left(\omega_G^+(u)\right)^2 - \left(\tau_G^+(u)\right)^2},\ \sqrt{1 - \left(\omega_G^-(u)\right)^2 - \left(\tau_G^-(u)\right)^2}\right]$$

3 Score Function

3.1 Proposed Score Function for Interval-Valued Pythagorean Fuzzy Set

Let $\widetilde{G} = \langle[\omega^-, \omega^+], [\tau^-, \tau^+]\rangle$ be an IVPFN.

Then, score function $\widetilde{G}$, $\text{Score}(\widetilde{G}) = \frac{1}{6}\left[1 + (\omega^-)^2 + (\omega^+)^2 - (\tau^-)^2 - (\tau^+)^2\right]$ $|\omega^- + \omega^+ - \tau^- - \tau^+|$ $\text{Score}(\widetilde{G}) \in [-1, 1]$.

$\text{Accuracy}(\widetilde{G}) = \frac{1}{6}\left[(\omega^-)^2 + (\omega^+)^2 + (\tau^-)^2 + (\tau^+)^2\right]|\omega^- + \omega^+ + \tau^- + \tau^+|$, $\text{Acc.}(\widetilde{G}) \in [0, 1]$.

For any two IVPFNs $\widetilde{G}_1$ and $\widetilde{G}_2$,

(1) If $\text{Score}(\widetilde{G}_1) \succ \text{Score}(\widetilde{G}_2)$ then $\widetilde{G}_1 \succ \widetilde{G}_2$

(2) If $\text{Score}(\widetilde{G}_1) \prec \text{Score}(\widetilde{G}_2)$ then $\widetilde{G}_1 \prec \widetilde{G}_2$
(3) If $\text{Score}(\widetilde{G}_1) = \text{Score}(\widetilde{G}_2)$ then $\widetilde{G}_1 = \widetilde{G}_2$

(a) If $\text{Acc.}(\widetilde{G}_1) \prec \text{Acc.}(\widetilde{G}_2)$ then $\widetilde{G}_1 \prec \widetilde{G}_2$
(b) If $\text{Acc.}(\widetilde{G}_1) \succ \text{Acc.}(\widetilde{G}_2)$ then $\widetilde{G}_1 \succ \widetilde{G}_2$
(c) If $\text{Acc.}(\widetilde{G}_1) = \text{Acc.}(\widetilde{G}_2)$ then $\widetilde{G}_1 = \widetilde{G}_2$

3.2 Peng and Yang [24] Score Function For Interval-Valued Pythagorean Fuzzy Number

Let $\widetilde{G} = \langle [\omega^-, \omega^+], [\tau^-, \tau^+] \rangle$.

The score of $\widetilde{G}$ is defined as follows

$$\text{Score}(\widetilde{G}) = \frac{1}{2}\left[(\omega^-)^2 + (\omega^+)^2 - (\tau^-)^2 - (\tau^+)^2\right],\ \text{Score}(\widetilde{G}) \in [-1, 1]$$

The accuracy function of $\widetilde{G}$ is defined as follows

$$\text{Acc.}(\widetilde{G}) = \frac{1}{2}\left[(\omega^-)^2 + (\omega^+)^2 + (\tau^-)^2 + (\tau^+)^2\right],\ \text{Acc.}(\widetilde{G}) \in [0, 1]$$

For any two IVPFNs $\widetilde{G}_1$ and $\widetilde{G}_2$,

(4) If $\text{Score}(\widetilde{G}_1) \succ \text{Score}(\widetilde{G}_2)$ then $\widetilde{G}_1 \succ \widetilde{G}_2$
(5) If $\text{Score}(\widetilde{G}_1) \prec \text{Score}(\widetilde{G}_2)$ then $\widetilde{G}_1 \prec \widetilde{G}_2$
(6) If $\text{Score}(\widetilde{G}_1) = \text{Score}(\widetilde{G}_2)$ then $\widetilde{G}_1 = \widetilde{G}_2$

(d) If $\text{Acc.}(\widetilde{G}_1) \prec \text{Acc.}(\widetilde{G}_2)$ then $\widetilde{G}_1 \prec \widetilde{G}_2$
(e) If $\text{Acc.}(\widetilde{G}_1) \succ \text{Acc.}(\widetilde{G}_2)$ then $\widetilde{G}_1 \succ \widetilde{G}_2$
(f) If $\text{Acc.}(\widetilde{G}_1) = \text{Acc.}(\widetilde{G}_2)$ then $\widetilde{G}_1 = \widetilde{G}_2$

4 Mathematical Structure

4.1 Interval-Valued Pythagorean Fuzzy Transportation Problem

The IVPFS transportation problem is

$$\text{Min } \widetilde{Z} = \sum_{i}^{m} \sum_{j}^{n} \tilde{c}_{ij}^{\text{IVPFS}} x_{ij}$$

Such that

$$\sum_{j}^{n} x_{ij} = \tilde{a}_{ij}^{\text{IVPFS}}, \; i = 1, 2, \ldots m$$

$$\sum_{i}^{m} x_{ij} = \tilde{b}_{ij}^{\text{IVPFS}}, \; j = 1, 2, \ldots n$$

$$x_{ij} \geq 0$$

In this case, there are m sources and n destination. Let a_j be the number of supply units available at source i ($i = 1, 2, 3, \ldots m$) and b_j be the number of demand units required at destination j ($j = 1, 2, \ldots m$) in terms of interval-valued Pythagorean fuzzy set. Let c_{ij} represents the unit transportation cost for transporting the units from source i to the destination j in terms of interval-valued Pythagorean fuzzy set.

4.2 Proposed Methodology

Here, we have taken three different types of transportation problem of interval-valued Pythagorean fuzzy set.

Type 1 (cost is in IVPFN)

Destination → Sources ↓	E_1	E_2	…	E_n	Supply
F_1	C_{11}^{IVPFN}	C_{12}^{IVPFN}	…	C_{1n}^{IVPFN}	a_1
F_2	C_{21}^{IVPFN}	C_{22}^{IVPFN}	…	C_{2n}^{IVPFN}	a_2
$\vdots$					
F_m	C_{m1}^{IVPFN}	C_{m2}^{IVPFN}	…	C_{mn}^{IVPFN}	a_m
Demand	b_1	b_2	…	b_n	

Type 2 (supply and demand are in IVPFN)

Destination → Sources ↓	E_1	E_2	…	E_n	Supply
F_1	C_{11}	C_{12}	…	C_{1n}	a_1^{IVPFN}
F_2	C_{21}	C_{22}	…	C_{2n}	a_2^{IVPFN}
$\vdots$					
F_m	C_{m1}	C_{m2}	…	C_{mn}	a_m^{IVPFN}
Demand	b_1^{IVPFN}	b_2^{IVPFN}	…	b_n^{IVPFN}	

Type 3 (cost, supply, and demand are in IVPFN)

Destination → / Sources ↓	E_1	E_2	…	E_n	Supply
$F1$	C_{11}^{IVPFN}	C_{12}^{IVPFN}	…	C_{1n}^{IVPFN}	a_1^{IVPFN}
$F2$	C_{21}^{IVPFN}	C_{22}^{IVPFN}	…	C_{2n}^{IVPFN}	a_2^{IVPFN}
⋮					
Fm	C_{m1}^{IVPFN}	C_{m2}^{IVPFN}	…	C_{mn}^{IVPFN}	a_m^{IVPFN}
Demand	b_1^{IVPFN}	b_2^{IVPFN}	…	b_n^{IVPFN}	

4.3 *Algorithm*

Step 1 In tabular form, write IVPFTP.
Step 2 Using the score function, convert the interval-valued Pythagorean fuzzy transportation problem to a crisp transportation problem.
Step 3 Check to see if it is balanced or not.
Step 4 If it is balanced, proceed to step 5, if not, balance it.
Step 5 Software such as MQ, LINGO, and MATLAB can be used to find the solution of the TP.
Step 6 To get the objective value.

4.4 *Illustrative Examples*

(**Type 1: cost is in IVPFN**): Condition for proposed TP is shown in Table 1.

(**Type 2 supply and demand are in IVPFN**): Condition for proposed TP in Table 2.

Table 1 Conditions for the proposed TP (costs in IVPFN, supply and demand in crisp values)

	E_1	E_2	E_3	E_4	Supply
F_1	[0.4, 0.6][0.2, 0.3]	[0.4, 0.5][0.1, 0.2]	[0.6, 0.8][0.2, 0.3]	[0.4, 0.5][0.2, 0.3]	26
F_2	[0.5, 0.7][0.3, 0.4]	[0.4, 0.6][0.2, 0.3]	[0.5, 0.7][0.2, 0.3]	[0.5, 0.7][0.1, 0.3]	24
F_3	[0.6, 0.7][0.4, 0.5]	[0.7, 0.8][0.5, 0.6]	[0.5, 0.8][0.2, 0.4]	[0.4, 0.5][0.1, 0.2]	30
Demand	17	23	28	12	

Table 2 Conditions for the proposed TP (costs in crisp values, supply and demand in IVPFN)

	E_1	E_2	E_3	E_4	Supply
F_1	13	15	16	17	[0.6, 0.8][0.3, 0.5]
F_2	18	20	7	50	[0.5, 0.6][0.3, 0.4]
F_3	21	19	10	29	[0.4, 0.6][0.1, 0.2]
Demand	[0.5, 0.8][0.3, 0.5]	[0.6, 0.8][0.2, 0.3]	[0.8, 0.9][0.2, 0.3]	[0.4, 0.5][0.1, 0.3]	

Table 3 Conditions for the proposed TP (costs, supply and demand in IVPFN)

	E_1	E_2	E_3	E_4	Supply
F_1	[0.4, 0.6][0.2, 0.3]	[0.4, 0.5][0.1, 0.2]	[0.6, 0.8][0.2, 0.3]	[0.4, 0.5][0.2, 0.3]	[0.6, 0.8][0.3, 0.5]
F_2	[0.5, 0.7][0.3, 0.4]	[0.4, 0.6][0.2, 0.3]	[0.5, 0.7][0.2, 0.3]	[0.5, 0.7][0.1, 0.3]	[0.5, 0.6][0.3, 0.4]
F_3	[0.6, 0.7][0.4, 0.5]	[0.7, 0.8][0.5, 0.6]	[0.5, 0.8][0.2, 0.4]	[0.4, 0.5][0.1, 0.2]	[0.4, 0.6][0.1, 0.2]
Demand	[0.5, 0.8][0.3, 0.5]	[0.6, 0.8][0.2, 0.3]	[0.8, 0.9][0.2, 0.3]	[0.4, 0.5][0.1, 0.3]	

(**Type 3: cost, supply, and demand are in IVPFN**): Condition for proposed TP in Table 3.

5 Results and Conclusion

Table 4 shows the results for these examples.

In Type 1, the transportation cost of Peng and Yang [24] is 17.4 and by proposed score function 9.774. In Type 2, the transportation cost of Peng and Yang [24] is 8.01 and by proposed score function 4.577. In Type 3, the transportation cost of Peng and Yang [24] is 0.14 and by proposed score function 0.0445. Thus in all cases that the optimal value of our proposed score function is not more than the optimal value of Peng and Yang [24] score function.

The previous research was compared in this study

See Table 5.

Table 4 Comparison of results

IVPFTP	Solution by Peng and Yang [24] score function	Solution by proposed score function	Comparison
Type 1	17.4	9.774	Min Z by Peng and Yang [24] > Min Z of proposed score function
Type 2	8.01	4.577	Min Z by Peng and Yang [24] > Min Z of proposed score function solution
Type 3	0.14	0.0445	Min Z by Peng and Yang [24] > Min Z of proposed score function

Table 5 Approaches used in previous research works and the proposed work

	Author name	Work
1	S.K Bharti (2018) [25]	Transportation problem under interval-valued intuitionistic fuzzy environment
2	R.Kumar (2019) [19]	A Pythagorean fuzzy approach to the transportation problem
3	Proposed work	Solution of transportation problem using interval-valued Pythagorean fuzzy approach

6 Conclusion

In proposed work, we have developed the algorithm to solve the interval-valued Pythagorean fuzzy transportation problem using three different cases. In these cases, we have used our proposed score function and Peng and Yang [24] score function to convert the IVPFTP into crisp transportation problem. After that we have applied the computational technique for finding the optimality of the problem. In the final solution, we found the appropriate results by our proposed score function as compare to Peng and Yang [24] score function result. Thus, this is the new method to tackle the uncertainty in real-life transportation problem.

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Solution of Transportation Problem Under Spherical Fuzzy Set

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***Abstract*—Transportation problem (TP) is used to solve real-life problems as agriculture, management and engineering science etc. But in real-life transportation problems, the cost depends on uncontrollable factors like fuel price, weather conditions and consumers behaviour etc. To deal with this type of imprecision, fuzzy numbers and their extension have been introduced by the authors. In literature, many methods are available for finding the solution of TP under fuzzy set (FS), intuitionistic fuzzy set (IFS) and Pythagorean fuzzy set (Pyfs) etc. In the proposed work, we have investigated the solution of the spherical fuzzy transportation problem (SFTP) and presented three different models of the spherical fuzzy transportation problem. Spherical fuzzy set (SFS) arithmetic operations and algorithms are discussed here. To illustrate the proposed work we have taken the numerical problem for these sets. Finally, we have discussed the result and conclusion of the paper.**

***Index Terms*—Transportation problem, Fuzzy set, Spherical fuzzy set, Score function.**

I. Introduction

TP is a linear programming problem and originally formulated by Hitchcock [18]. In a traditional transportation dilemma, the decision-maker knows exactly how much transportation costs, how much supply there is, and how much demand there is, but in real-life transportation problems all these values depend on various factors like fuel costs, condition of whether and customer loyalty, etc. which are not fixed (imprecise).
Zadeh(1965) [4] defined the concept of the fuzzy set to deal with imprecision of real-world problems. Chanas et al. [1] worked on fuzzy TP. Dinagar and Palanivel [3] solved the TP under a fuzzy environment. Kumar and Amarpreet [19] have presented the idea of using a traditional transportation strategy to solve a fuzzy transportation challenges. Many authors have worked on fuzzy transportation, but they only address the element's membership function..

The intuitionistic fuzzy set was introduced by Atanassov [2]. The intuitionistic fuzzy set is defined by membership and non-membership function of the element to the set i.e (δ, λ) with the condition $\delta + \lambda \leq 1$. The IFTP was solved by Gani and Abbs [5] using a new ranking algorithm in which supply and demand are triangular intuitionistic fuzzy numbers. Singh and Yadav [8] introduced the transportation problem in which expenses are expressed as triangular intuitionistic fuzzy numbers. Dubey and Mehra [7] worked on linear programming with the triangular intuitionistic fuzzy numbers. Bharti [14] solved the trapezoidal intuitionistic fuzzy fractional transportation problem. Atanssov and Gargov [6] defined the interval-valued intuitionistic fuzzy set (IVIFS). Mishra et al. [10] worked on transportation problem under IVIF. Arora [9] given an approach for interval-valued fuzzy fractional TP. Bharati and Singh [11] solved the TP under IVIFS.

Yager [16] defined the pythagorean fuzzy set (Pyfs). Yager overcome the situation when $\delta + \lambda > 1$. A pythagorean fuzzy set is an extention of IFS. Zhang and Xu [12] defined the methodology of the TOPSIS method to MCDM with Pyfs. Geng et al. [13] presented an exteended TODIM method and proposed pythagorean fuzzy uncertain linguistic set. Karasan et al. [15] worked on landfill site selection problem using novel pythagorean fuzzy AHP method. Kumar et al. [20] presented the pythagorean fuzzy approach to the transportation problem. Zhang [28] proposed the interval-valued pythagorean fuzzy set (IVPFS). He extended membership and non-membership in interval numbers. Peng and Yang [21] defined the fundamental properties of IVPFS. Peng and Li [22] proposed an algorithm for IVPFS in emergency decisions depend on two measures multiparametric measure and WDBA. Rahaman et al. [23] gave the methodology to multi-attribute group decision-making based on induced IVPFS Einstein aggregation operator.

The condition where neutral membership degree calculated independently in real-life Problem, based on these two sets FS and IFS, the picture fuzzy set (PFS) introduced by Cuong and Kreinovich [29]. They used three index, membership degree (δ), neutral-membership degree $(\eta)$and non-membership degree (λ) in picture fuzzy set with the condition $0 \leq \delta+\eta+\lambda \leq 1$. Dutta and Ganju [24] proposed some characteristics of of PFS. Van and Thao [25] worked on a measure of PFS and their use in multi-attribute decision-making problem. Geetha and Selvakumari [30] solved the picture Fuzzy transportation problem using the score function.

The generalization of PFS is SFS which is introduced by Ashraf et al. [26] with condition $0 \leq \delta + \eta + \lambda \leq 1$.

Consider the example, let δ=0.7, η=0.3 and λ=0.5, in this case 0.7+0.3+0.5 $>$1 but by squaring $(0.7)^2$ +$(0.3)^2$+$(0.5)^2$$\leq$1. By this, we can see SFS is more suitable than PFS. Spherical fuzzy sets and the spherical fuzzy TOPSIS method were studied by Gundogdu and Kahraman [27]. The spherical fuzzy set was utilised by Mahmood et al. [17] to solve decision-making and medical diagnosis challenges.

The findings of the preceding literature analysis on transportation issues, we found that no one worked on transportation problems under the spherical fuzzy sets. In our work, we have proposed the method to solve spherical fuzzy transportation problem using three different cases. In these cases we have used the score function to convert the data into a crisp transportation problems. After that, we have applied the vogel approximation method (VAM) for an initial basic feasible solution (IBFS) and verified the optimality with computational technique. This is the new method to handle the uncertainty in TP.

We have divided the whole paper into 4 Sections. In Section 2 some basic knowledge of FS, Pyfs, PFS, SFS, and arithmetic operations on spherical fuzzy numbers are presented. Methodology to solve spherical fuzzy transportation problem and their illustrative examples are given in Section 3. Section 4, presents the result and concluding remarks.

II. Preliminaries

Definition 2.1 [4] Let G be a universal set, then a fuzzy set (FS) H is defined as

$$H = \{\langle g, \theta_H(g)\,|g \in G\rangle\}$$

where

$$\mu_H(g) : G \to [0,1]$$

and $\theta_H(g)$ is the membership degree of the element g to the set H.

Definition 2.2 [16] Pythagorean fuzzy set '$\tilde{H}$' on G is defined as a set of ordered pair given by

$$\tilde{H} = \{< g, \theta_H(g), \lambda_H(g) > |g \in G\}$$

where $\theta_H(g), \lambda_H(g) : G \to [0,1]$ are the membership and non-membership degree of the element $g \in G$ respectively, with the condition $(\theta_H(g))^2 + (\lambda_H(g))^2 \leq 1$, the degree of indeterminacy is given by $\alpha_H(g) = \left(\sqrt{1-(\theta_H^2+\lambda_A^2)}\right)$

Definition 2.3 [25] Picture fuzzy set $\tilde{H}$ on a universe G is an object in the form of

$$\tilde{H} = \{[g, \theta_{\tilde{H}}(g), \eta_{\tilde{H}}(g), \lambda_{\tilde{H}}(g)]|g \in G\}$$

where $\theta_{\tilde{H}}(g)$$\in$[0, 1] is the degree of membership of g in $\tilde{H}$, $\eta_{\tilde{H}}(g)$$\in$[0,1] is the neutral membership of g in $\tilde{H}$ and $\lambda_{\tilde{H}}(g)$$\in$[0, 1] is the non-membership of g in $\tilde{H}$, where $\lambda_{\tilde{H}}(g), \eta_{\tilde{H}}(g), \lambda_{\tilde{H}}(g)$ satisfy the following condition

$$(g \in G), \theta_{\tilde{H}}(g) + \eta_{\tilde{H}}(g) + \lambda_{\tilde{H}}(g)$$

now (1-($\theta_{\tilde{H}}(g) + \eta_{\tilde{H}}(g) + \lambda_{\tilde{H}}(g)$)) called the refusal membership degree of g in $\tilde{H}$.

Definition 2.4 [27] A spherical fuzzy set (SFS) H̃ defined on universal set G is given by

$$\tilde{H}_s = \{< g, \theta_{H_s}(g), \eta_{H_S}(g), \lambda_{H_S}(g) >, g \in G\}$$

where

$$\theta_{H_s}(g) : G \to [0,1], \eta_{H_S}(g) : G \to [0,1], \lambda_{H_S}(g) : G \to [0,1]$$

and

$$0 \leq \theta_{H_s}(g)^2 + \eta_{H_S}(g)^2 + \lambda_{H_S}(g)^2 \leq 1$$

Definition 2.5 [27] Let $\tilde{A}_s$ & $\tilde{B}_s$ are two SFS. Then

1) Union:

$$\begin{aligned}\tilde{A}_s \cup \tilde{B}_s = \{&\max\{\theta_{A_s}, \theta_{B_s}\}, \min\{\lambda_{A_s}, \lambda_{B_s}\},\\ &\min\{(1-((\max\{\theta_{\tilde{A}_S}, \theta_{\tilde{B}_S}\})^2+\\ &(\min\{\lambda_{\tilde{A}_S}, \lambda_{\tilde{B}_S}\})^2))^{\frac{1}{2}}, \max\{\eta_{\tilde{A}_S}, \eta_{\tilde{B}_S}\}\}\}\end{aligned}$$

2) Intersection:

$$\begin{aligned}\tilde{A}_S \cap \tilde{B}_S = \{&\min\{\theta_{A_s}, \theta_{B_s}\}, \max\{\lambda_{A_s}, \lambda_{B_s}\},\\ &\max\{(1-((\min\{\theta_{\tilde{A}_S}, \theta_{\tilde{B}_S}\})^2+\\ &(\max\{\lambda_{\tilde{A}_S}, \lambda_{\tilde{B}_S}\})^2))^{\frac{1}{2}}, \min\{\eta_{\tilde{A}_S}, \eta_{\tilde{B}_S}\}\}\}\end{aligned}$$

3) Addition:

$$\begin{aligned}\tilde{A}_S \oplus \tilde{B}_S = \Big(&(\theta^2_{\tilde{A}_S} + \theta^2_{\tilde{B}_S} - \theta^2_{\tilde{A}_S}.\theta^2_{\tilde{B}_S})^{\frac{1}{2}}, \lambda_{\tilde{A}_S}.\lambda_{\tilde{B}_S},\\ &((1-\theta^2_{B_S})\eta^2_{A_S} + (1-\theta^2_{A_S})\eta^2_{B_S} - \eta^2_{A_S}\eta^2_{B_S})^{\frac{1}{2}}\Big)\end{aligned}$$

4) Multiplication:

$$\begin{aligned}\tilde{A}_S \otimes \tilde{B}_S = \{&\theta_{A_S}.\theta_{B_S}, (\lambda^2_{\tilde{A}_S} + \lambda^2_{\tilde{B}_S} - \lambda^2_{\tilde{A}_S}.\lambda^2_{\tilde{B}_S})^{\frac{1}{2}},\\ &((1-\lambda^2_{B_S})\eta^2_{A_S} + (1-\lambda^2_{A_S})\eta^2_{B_S} - \eta^2_{A_S}\eta^2_{B_S})^{\frac{1}{2}}\}\end{aligned}$$

Definition 2.6 [27] Let $\tilde{H}_S$ be the SFS, The score function and accuracy function are defined by:

Score $(\tilde{H}_S) = (\theta_{\tilde{H}_S} - \eta_{\tilde{H}_S})^2 - (\lambda_{\tilde{H}_S} - \eta_{\tilde{H}_S})^2$, $S(\tilde{H}_S) \in [-1,1]$
Accuracy $(\tilde{H}_S)$=${\theta_{H_S}}^2 + {\lambda_{H_S}}^2 + {\eta_{H_S}}^2$, $Acc.(\tilde{H}_S) \in [0,1]$
Note that:$\tilde{H}_S < \tilde{Q}_S$ if and only if

1) S($\tilde{H}_S$)<S($\tilde{Q}_S$) or
2) S ($\tilde{H}_S$)=S ($\tilde{Q}_S$) and Acc.($\tilde{H}_S$)< Acc.($\tilde{Q}_S$)

III. Spherical Fuzzy Transportation Problem

A. Spherical Fuzzy Transportation Problem (SFTP)

The SFTP is

$$Min\tilde{Z} = \sum_{i}^{m}\sum_{j}^{n}\tilde{c}_{ij}^{SFS}x_{ij}$$

Such that

$$\sum_{j}^{n} x_{ij} = \tilde{a}_i^{SFS}, i = 1, 2, ...m$$

$$\sum_{i}^{m} x_{ij} = \tilde{b}_j^{SFS}, j = 1, 2, ...n$$

$$x_{ij} \geq 0$$

B. Proposed Methodology

Three distinct models of transportation problems were used to address the transportation problem under the spherical fuzzy set.

- Model 1 (In this paradigm, costs are expressed as spherical fuzzy numbers (SFNs), whereas supply and demand are expressed as crisp numbers.)

Destination	**D**$_1$	**D**$_2$	**...**	**D**$_n$	**Supply**
Sources					
S$_1$	$C_{11}{}^{SFN}$	$C_{12}{}^{SFN}$	...	$C_{1n}{}^{SFN}$	a_1
S$_2$	$C_{21}{}^{SFN}$	$C_{22}{}^{SFN}$	...	$C_{2n}{}^{SFN}$	a_2
S$_m$	$C_{m1}{}^{SFN}$	$C_{m2}{}^{SFN}$	...	$C_{mn}{}^{SFN}$	a_m
$Demand$	b_1	b_2	...	b_n	

- Model 2 In this paradigm, supply and demand are expressed as spherical fuzzy numbers (SFNs), whereas costs are expressed as crisp numbers.)

Destination	**D**$_1$	**D**$_2$	**...**	**D**$_n$	**Supply**
Sources					
S$_1$	C_{11}	C_{12}	...	C_{1n}	$a_1{}^{SFN}$
S$_2$	C_{21}	C_{22}	...	C_{2n}	$a_2{}^{SFN}$
S$_m$	C_{m1}	C_{m2}	...	C_{mn}	$a_m{}^{SFN}$
$Demand$	$b_1{}^{SFN}$	$b_2{}^{SFN}$	...	$b_n{}^{SFN}$	

- Model 3 (The cost, supply, and demand are all represented by a spherical fuzzy number.)

Destination	**D**$_1$	**D**$_2$	**…………**	**D**$_n$	**Supply**
Sources					
S$_1$	$C_{11}{}^{SFN}$	$C_{12}{}^{SFN}$	……..	$C_{1n}{}^{SFN}$	$a_1{}^{SFN}$
S$_2$	$C_{21}{}^{SFN}$	$C_{22}{}^{SFN}$	…………	$C_{2n}{}^{SFN}$	$a_2{}^{SFN}$
S$_m$	$C_{m1}{}^{SFN}$	$C_{m2}{}^{SFN}$	………..	$C_{mn}{}^{SFN}$	$a_m{}^{SFN}$
Demand	$b_1{}^{SFN}$	$b_2{}^{SFN}$	………..	$b_n{}^{SFN}$	

Algorithm

Step1. Write SFTP in tabular form.
Step 2. Use score function to change the SFTP into a crisp transportation problem.
Step 3. Check it is a balance or not.
Step 4. If it's balanced, move to step 5, otherwise make it balanced to add dummy variables.
Step 5. Find the IBFS by VAM.
Step 6. Using any Software MQ, LINGO, and MATLAB, test the optimality of the transportation problem.
Step7. To get the objective function, put x_{ij} in the objective function.

C. Illustrative Example

- (Model 1 SFTP): In this SFTP model 1, we have taken the condition for the proposed transportation problem is shown in Table 1, the cost are in spherical fuzzy numbers, supply and demand are crisp.

TABLE I
Data for spherical fuzzy TP of model 1

	D$_1$	**D**$_2$	**D**$_3$	**D**$_4$	**Supply**
O$_1$	(0.9,0.1,0.1)	(0.6,0.4,0.4)	(0.91,.03,.02)	(0.91,.03,.02)	26
O$_2$	(0.89,0.08,0.03)	(0.74,0.16,0.1)	(0.5,0.5,0.5)	(0.7,0.3,0.3)	24
O$_3$	(0.99,0.05,0.02)	(0.73,0.15,0.08)	(0.73,0.12,0.08)	(0.68,0.26,0.06)	30
Demand	17	23	28	12	80

We obtain the crisp transportation problem by using score function (Definiation 2.6) .

TABLE II
Defuzzified spherical fuzzy transportation problem of model 1

	D$_1$	**D**$_2$	**D**$_3$	**D**$_4$	**Supply**
O$_1$	0.64	0.04	0.792	0.94	26
O$_2$	0.73	0.40	0	0.16	24
O$_3$	0.94	0.41	0.42	0.34	30
Demand	17	23	28	12	80

The IBFS is as follows
Minimum cost = 0.04×23+0×24+0.34×12+0.42×4+0.64×3+0.94×14 =21.76
It has degeneracy solution because the m + n−1= 6. so, we have to proceed for the optimality, we use LINGO software.

Minimum cost
=3×0.64+0.04×23+0×24+0.94×14+0.42×4+0.34×12
=21.76

- (Model 2): Condition for the proposed transportation problem is shown in Table III where the costs are crisp, but the demand and supply are SFN.

We obtain the crisp transportation problem by using score function (Definiation 2.6).

The IBFS is as follows
Minimum cost

TABLE III
DATA FOR SPHERICAL FUZZY TP OF MODEL 2

	D_1	D_2	D_3	D_4	Supply
O_1	0.64	0.04	0.792	0.94	(0.9,0.1,0.1)
O_2	0.73	0.4	0	0.16	(0.89,0.08,0.03)
O_3	0.94	0.41	0.42	0.34	(0.99,0.05,0.02)
Demand	(0.9,0.1,0.1)	(0.6,0.4,0.4)	(0.91,0.03,0.02)	(0.99,0.05,0.02)	

TABLE IV
DEFUZZIFIED SPHERICAL FUZZY TP OF MODEL 2

	D_1	D_2	D_3	D_4	Supply
O_1	0.64	0.04	0.792	0.94	0.64
O_2	0.73	0.4	0	0.16	0.73
O_3	0.94	0.41	0.42	0.34	0.94
Demand	0.64	0.04	0.792	0.94	

$=0.64\times0.64+0\times0.73+0\times0.062+0\times0.04+0.94\times0.34=0.7292$

It has degeneracy solution because the m + n− 1= 6, so, we have to proceed for the optimality, we use LINGO software

The Minimum cost
$=0.64\times0.6+0.04\times0.04+0\times0.73+0\times0.42+0.34\times0.94+0\times0.04+0\times0.062=0.7052$

- (Model 3 SFN): Condition for the proposed transportation problem is shown in Table V. Here all the three paremeters costs, demands and supplies are Spherical fuzzy number.

TABLE V
DATA FOR SPHERICAL FUZZY TP OF MODEL 3

	D_1	D_2	D_3	D_4	Supply
O_1	(0.61,0.46,0.34)	(0.74,0.27,0.28)	(0.7,0.3,0.3)	(0.62,0.39,0.39)	(0.9,0.1,0.1)
O_2	(0.81,0.2,0.23)	(0.55,0.47,0.43)	(0.5,0.5,0.5)	(0.7,0.3,0.3)	(0.89,0.08,0.03)
O_3	(0.99,.05,.02)	(0.73,0.15,0.08)	(0.73,0.12,0.08)	(0.68,0.26,0.06)	(0.99,0.05,0.02)
Demand	(0.9,0.1,0.1)	(0.6,0.4,0.4)	(0.91,0.03,0.02)	(0.99,0.05,0.02)	

We obtain the crisp transportation problem by using score function (Definiation 2.6).

TABLE VI
DEFUZZIFIED SPHERICAL FUZZY TP OF MODEL 3

	D_1	D_2	D_3	D_4	Supply
O_1	0.0585	0.2115	0.16	0.3021	0.64
O_2	0.3355	.0128	0	0.34	0.73
O_3	0.73	0.4	0.41	0.42	0.94
Demand	0.64	0.04	0.792	0.94	

The IBFS is obtained as follows:
Min cost
$=0\times0.102+0.0585\times0.64+0\times0.73+0.42\times0.838+0.41\times0.062$
$+0.4\times0.04=0.43082$

To obtain the optimality, we use LINGO software. Therefore, the optimal solution is as follows:
Minimum cost = 0.4308

IV. RESULT AND CONCLUSION

We have solved TP under spherical fuzzy set using three different examples. The result for these examples are shown in Table VII and compared our work with some previous work as shown in Table VIII.

TABLE VII
COMPARISION OF INTIAL BASIC FEASIBLE SOLUTION WITH OPTIMAL SOLUTION

SFTP	IBFS	Optimum Solution	Comparison
Model 1	21.76	21.76	IBFS$\geq$ optimum solution
Model 2	0.7292	0.7052	IBFS$\geq$ optimum solution
Model3	0.43082	0.4308	IBFS$\geq$ optimum solution

In model 1 SFTP, we found the IBFS is 21.76 and optimum transportation cost of 21.76. Also in model 2 SFTP, we found the IBFS is 0.7292 and optimum transportation cost of 0.7052. In mode 3 SFTP, we found the IBFS is 0.43082 and optimum transportation cost of 0.4308. So, it is clear from the above observation the optimal solution in all three models is less than or equal to IBFS.

TABLE VIII
ANALYSIS OF PROPOSED WORK WITH OTHERS AUTHORS WORK

S.No.	Author name	Work
1	Bharti(2018) [14]	TP under interval valued intuitionistic fuzzy environment.
2	Kumar et al.(2019) [20]	A pythagorean fuzzy approach to the TP
3	Geetha and Selvakumari (2020) [30]	A picture fuzzy approach to solving TP
4	Proposed work	Solution of TP under spherical fuzzy sets.

Finally conclude that we have developed the algorithm to solve the SFTP using three different models. In these models, we have used the score function to convert the data into crisp transportation problems. After that, we have applied the vogel approximation method for an IBFS and verified the optimality with computational technique. This is the new approach to tackle the uncertainty in real-life transportation problem.

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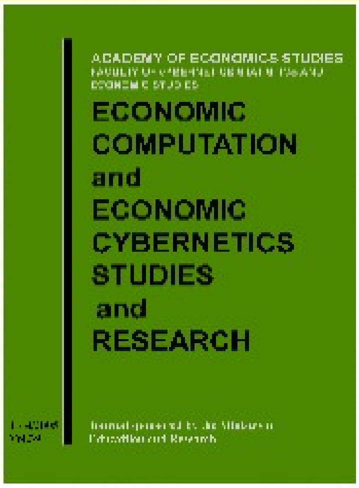

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Department Applied Mathematics

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I pledge and pronounce that all the above information is true to best of my knowledge and behalf.

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