Spectral Properties of Differential and Integral Operators

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Candidate's Declaration

We, Neeharika Singh and Lisa Dema Kamduk of Msc Mathematics hereby declare that the Dissertation titled "Spectral Properties of Differential and Integral Operator" which is submitted by us to the department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree Masters of Mathematics, is original and not copied from any source without project citation. This work has not previously formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or other similar title or recognition.

Place: Delhi
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Certificate

I hereby certify that the project dissertation titled "Spectral Properties of Differential and Integral Operators" which is submitted by Neeharika Singh and Lisa Dema Kamduk to the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the Degree of Master's of Mathematics, is record of the project work carried out by the students under my supervision. To the best of my knowledge this work has not been submitted in part or full for any other Degree or Diploma of this university or elsewhere.

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Abstract

This dissertation explores the spectral properties of differential and integral operators in infinite-dimensional vector spaces, especially Hilbert spaces. It starts by introducing fundamental concepts to understand how these operators behave in infinite dimensions. An overview of spectral theory is provided, focusing on the importance of eigenvalues, eigenvectors, and spectral decompositions, with a discussion of the spectral theorem and its significance.
Subsequently, the spectral properties of linear differential and integral operators are explored, with particular emphasis on their role in solving differential equations in Hilbert spaces. Various differential operators, including Sturm-Liouville operators, are analyzed to understand their spectral behavior and implications.
The practical importance of these theoretical results is illustrated through numerous applications in physics, engineering, and other scientific fields. Instances include the application of spectral properties in quantum mechanics, signal processing, and structural analysis, demonstrating the wide-ranging real life applications of these ideas.

Contents

Chapter 1

Introduction

Examining the spectral properties of differential and integral operators is a key topic in functional analysis and engineering and science. These operators, which appear in many fields like quantum mechanics, signal processing, and differential equations, are crucial for understanding physical systems and solving mathematical problems. Understanding their spectral properties, like eigenvalues and eigenfunctions, offers valuable insights into how complex systems behave and how different equations that arise in engineering and sciences can be solved. Before we proceed further in our text, we will give a brief review of some standard definitions which is necessary to have an uninterrupted understanding for the rest of the content.

Preliminaries

Riemann and Lebesgue Integration

This section explores the distinctions between Riemann and Lebesgue Integration. In the context of Riemann integration, we start by dividing the interval (a, b) into smaller subintervals denoted as $(t_i, t_i + \Delta t_i)$ for $i = 1, \ldots, n$. This partitioning leads to the construction of the upper and lower Darboux sums, D_{u}^n and D_1^n, respectively. The upper bound f_i^{u} of $f(t)$ in each subinterval $(t_i, t_i + \Delta t_i)$ is defined as $f_i^{\mathrm{u}} = \max_t f(t)$.

Similarly, the lower bound is defined as $f_i' = \min_t f(t)$. The upper Darboux sum, which represents the total area of rectangles with height f_i^{u} and width Δt_i, is expressed as:

$$D_{\mathrm{u}}^n = \sum_{i=1}^{n} f_i^{\mathrm{u}} \Delta t_i$$

The lower Darboux sum, representing the total area of rectangles with height f_i^1 and width Δt_i, is given by:

$$D_1^n = \sum_{i=1}^{n} f_i^1 \Delta t_i$$

The Riemann integral is said to exist and be equal to the common limit of the Darboux sums as the partitioning becomes finer, i.e., as $n \to \infty$ and $\Delta t_i \to 0$, if:

$$\int_a^b f(t)\, dt = \lim_{n\to\infty} D_{\mathrm{u}}^n = \lim_{n\to\infty} D_1^n$$

In Lebesgue's integration theory, functions $f(t)$ are approximated almost everywhere by a sequence of step functions $\psi_n(t)$. This approximation holds everywhere except for a countable set of isolated points. The step functions are defined by:

$$\psi_n(t) \equiv \bar{f}_i^n, \quad t_i < t < t_i + \Delta t_i$$

Here, $\bar{f}_i^n$ represents the value of $f(t)$ within the interval i, excluding isolated points. The integral of these step functions is:

$$\int_a^b \psi_n(t)\, dt = \sum_{i=1}^{n} \bar{f}_i^n \Delta t_i$$

For the Lebesgue integral of $f(t)$ to exist, it must satisfy:

$$\left| \int_a^b \psi_n(t)\, dt \right| < A \neq \infty$$

and

$$f(t) = \lim_{n\to\infty} \psi_n(t)$$

almost everywhere. Thus, the Lebesgue integral is defined as:

$$\int_a^b f(t)\, dt = \lim_{n\to\infty} \int_a^b \psi_n(t)\, dt$$

It is evident that if a Riemann integral exists, then the Lebesgue integral will also exist.
An important concept, Linear Independence is defined below,

Linear Independence

A set of p n-dimensional vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ is said to be linearly independent if there is no non-zero set of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ (where not all α_i are zero) such that

$$\sum_{i=1}^{p} \alpha_i \mathbf{x}_i = \mathbf{0}. \tag{1.1}$$

Norm

Consider a vector $\mathbf{x}$ in a vector space W. The norm of $\mathbf{x}$, represented as $\|\mathbf{x}\|$, is a function $\|\cdot\| : W \to \mathbb{R}$ adhering to the given conditions $\forall\ \mathbf{y}, \mathbf{x} \in W$ and any scalar $\beta \in \mathbb{R}$:

1. **Non-negativity:** $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
2. **Homogeneity:** $\|\beta\mathbf{x}\| = |\beta|\|\mathbf{x}\|$.
3. **Subadditivity:** $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

These conditions ensure that the norm reflects an intuitive sense of magnitude.

Inner Product

An inner product on a vector space W over $\mathbb{R}$ or $\mathbb{C}$) is a function $(\cdot, \cdot) : W \times W \to \mathbb{K}$ that meets the following criteria for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in W$ and $\beta \in \mathbb{K}$:

1. **Conjugate Symmetry:** $(\mathbf{a}, \mathbf{b}) = \overline{(\mathbf{b}, \mathbf{a})}$, where $\overline{(\mathbf{b}, \mathbf{a})}$: complex conjugate of $(\mathbf{b}, \mathbf{a})$.
2. **Linearity in the first argument:** $(\beta\mathbf{a} + \mathbf{b}, \mathbf{c}) = \beta(\mathbf{a}, \mathbf{c}) + (\mathbf{b}, \mathbf{c})$.
3. **Positive-Definiteness:** $(\mathbf{b}, \mathbf{b}) \geq 0$, and $(\mathbf{b}, \mathbf{b}) = 0$ if and only if $\mathbf{b} = \mathbf{0}$.

These criteria ensure that the inner product generalizes the concept of the dot product in Euclidean space.

Normed Linear Space and Inner Product Space

A linear space with a norm as the operation is referred to as a normed linear space, while a linear space with an inner product as the operation is known as an inner product space..

Hilbert Space and Banach Space

A Hilbert Space is characterized as a complete space equipped with an inner product, whereas a Banach Space is defined as a complete space with a norm. Recall, a linear space is termed complete if every Cauchy sequence within it converges.

Linear Operators

We define a linear operator $\mathbf{L}$ in a Hilbert space $\mathcal{H}$ as follows:
(1) if $\mathbf{x}, \mathbf{y} \in \mathcal{H}$ and $\mathbf{Lx}, \mathbf{Ly} \in \mathcal{H}$ and
(2) if

$$\mathbf{L}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{Lx} + \beta\mathbf{Ly}$$

where α and β are complex numbers, then we say $\mathbf{L}$ is a linear operator in $\mathcal{H}$.
Definition- We say that a vector $\mathbf{x}$ in $\mathcal{H}$ belongs to the domain $\mathcal{D}$ of the linear operator $\mathbf{L}$ in $\mathcal{H}$ if $\mathbf{Lx}$ belongs to $\mathcal{H}$.
Theorem- If $\mathbf{L} = \mathbf{I} + \mathbf{K}$ and $\|\mathbf{K}\| < \gamma, 0 < \gamma < 1$, then the inverse of $(\mathrm{I} + \mathrm{K})$ exists and can be represented as

$$(\mathbf{I} + \mathbf{K})^{-1} = \mathbf{I} + \sum_{i=1}^{\infty} (-\mathbf{K})^i$$

This also leads to the conclusion that the solution of

$$(\mathbf{I} + \mathbf{K})\mathbf{u} = \mathbf{f}$$

always exists and is unique. It could be computed from the series

$$\mathbf{u} = \mathbf{f} + \sum_{i=1}^{\infty} (-1)^i \mathbf{K}^i \mathbf{f}$$

Infinite-Dimensional Linear Spaces

Function spaces form linear vector spaces which are, in fact, infinite dimensional.
The dimension of a space is the no. of linearly independent vectors that the space contains. Then, in the space $C^0(0, 1)$, the functions $f_n(t) = t^n, n = 0, 1, 2, ...$, are linearly independent vectors and so $C^(0, 1)$ is an infinite- dimensional vector space.
Hence, many such function spaces can be defined:

1. $C^n(a,b)$: the space of functions $f(t), g(t), \ldots$ in the interval (a,b), whose first n derivatives are continuous.

2. $R(c,d)$: $\mathbf{g} \in R(c,d)$ is the space of functions whose Riemann integral exists
$$\int_c^d f(s)ds$$

3. $\mathcal{L}_1(c,d)$ is the space of functions whose Lebesgue integral exists.

Another Definition of Hilbert Spaces

In previous sections, we established that norms are definable independently of inner products. However, in the subsequent discussion, we will focus solely on spaces whose norm is defined by an inner product, namely, $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$.If an inner product space, is a Banach space w.r.t the norm defined before, then it is called a Hibert Space.

Perfect Operators

An operator in the Hilbert space $\mathcal{H}$ is termed perfect if its eigenvectors constitute a complete basis set. Consider, for instance, the operator $\mathbf{L}$ in the space $\mathcal{L}_2(0,1)$, defined by the differential operator

$$Lv(t) = -\frac{d^2v}{dt^2}$$

subject to the boundary conditions

$$v(0) = v(1)$$

This operator is considered perfect because its eigenvectors

$$v_n = \sqrt{2}\sin(n\pi t), \quad n = 1, 2, \ldots$$

with the their eigenvalues $\lambda_n = (n\pi)^2$ in the space $\mathcal{L}_2(0,1)$, forms its basis.

Chapter 2

An overview of Spectral Theory

2.1 Review of Eigenvalues and Eigenvectors of a Matrix

Consider a m-rowed square matrix $M = (\lambda_{ij})$,

$$Mx = \alpha x$$

We know that an **eigenvalue** of a square matrix M is a number ' α ' such that $Mx = \alpha x$ has a solution $x \neq 0$ and x is known as the **eigenvector**.

Eigenspace: The set of eigenvectors associated with the eigenvalue α (along with the zero vector) constitutes a vector subspace of X. This subspace is known as the eigenspace of M corresponding to the eigenvalue α.

Spectrum: The collection $\sigma(M)$, which includes all the eigenvalues of M.

Resolvent set: The resolvent set is the complement of the spectrum, represented as $\rho(M) = \mathbb{C} \setminus \sigma(M)$.

Theorem 1 (Result from Fundamental Theorem of Algebra) *The eigenvalues of a m-rowed matrix $M = (\lambda_{ij})$ are given by the solution of the characteristic eq" of M. Consequently, M has at least one eigenvalue (and atmost m different eigenvalues).*

2.2 From Matrix Theory to Operator Theory

Consider, $L : X \to X$ be a linear operator, with X as the finite dimensional normed space.

Then, wthe operator L can be represented as matrices. Infact, the spectral theory of T is essentially the matrix eigenvalue theory.

If the normed space X has dimension n, and
(1) Let $e = \{e_1, \ldots, e_n\}$ be any basis for X.
(2) Let $L_e = (\lambda_{ij})$

Then, the eigenvalues of L_e correspond to the eigenvalues of the operator L. The spectrum and resolvent of L_e are the same as the spectrum set and resolvent set of the operator L.

Some Important Results

Theorem 2 *All matrices that represent a linear operator $L : X \to X$ on a finite-dimensional normed space X, when expressed with respect to different bases of X, share the same eigenvalues.*

Theorem 3 *For a linear operator L on a complex normed space $X \neq \{0\}$ (if X is finite-dimensional), there is at least one eigenvalue.*

2.3 Spectrum

Let $L : \mathbb{D}(L) \to X$ be a linear operator on X, where $X \neq \{0\}$ is a complex normed space. Then for the operator defined as:

$$L_\alpha = L - \alpha I$$

where α is a complex number and I is an identity operator on $D(L)$
an inverse operator $R_\alpha L$ is defined as:

$$R_\alpha(L) = L_\alpha^{-1} = (L - \alpha I)^{-1}$$

and is called **Resolvent Operator** of L

Definition 4 *Let $X \neq 0$ be a complex normed space, and let $L : D(L) \to X$ be a linear operator where $D(L) \subset X$. A **regular value** α of L is a complex number that satisfies the following conditions:*

5.1 $R_\alpha(L)$ *exists,*

5.2 $R_\alpha(L)$ *is bounded,*

5.3 $R_\alpha(L)$ *is defined on a subset that is dense in* X*.*

From this, we can define the following important concepts:

Resolvent set $\rho(L)$: The set of all regular values of L. When we gather all such α values, we form the resolvent set $\rho(L)$ of L.

Spectrum $\sigma(L)$: The set of all spectral values of L. This set is the complement of the resolvent set.

Types of Spectrum

Point Spectrum: This is the collection of α values for which $R_\alpha(L)$ does not exist. It is symbolized by $\sigma_p(L)$, and $\alpha \in \sigma_p(L)$ is referred to as an eigenvalue of L. This set is also known as the Discrete Spectrum.

Continuous Spectrum: This consists of α values for which $R_\alpha(L)$ exists and satisfies condition 5.3 but not condition 5.2, meaning $R_\alpha(L)$ is unbounded. It is represented by $\sigma_c(L)$.

Residual Spectrum: This set contains α values for which $R_\alpha(L)$ exists and may or may not be bounded, but the domain of $R_\alpha(L)$ is not dense in X. It is denoted by $\sigma_r(L)$.

It is crucial to observe that the sets $\rho(L)$, $\sigma_p(L)$, $\sigma_c(L)$, and $\sigma_r(L)$ are mutually exclusive, and their union covers the entire complex plane: $\mathbb{C} = \rho(L) \cup \sigma(L) = \rho(L) \cup \sigma_p(L) \cup \sigma_c(L) \cup \sigma_r(L)$.

2.3.1 Some Important Properties

1. If the resolvent operator $R_\alpha(L)$ exists, then it is linear.
2. $R_\alpha(L) : R(L_\alpha) \to \mathbb{D}(L_\alpha)$ exists if and only if null space of $L_\alpha = \{0\}$ i.e. $L_\alpha x = 0 \implies x = 0$.
3. If $L_\alpha x = (L - \alpha I)x = 0$ for some $x \neq 0$, then $\alpha \in \sigma_p(L)$. By definition, α is an eigenvalue of L. If x is the eigenvector corresponding to α then all such vectors form the eigenspace of L corresponding to that α.

Theorem 5 *Let* $L : X \to X$ *be a bounded linear operator, and let* X *be a complete space. If there is some* α *such that the resolvent* $R_\alpha(L)$ *exists and is defined over the entire space* X*, then for this* α*, the resolvent is bounded.*

Lemma: Let $L : X \to X$ be a linear operator and let $\alpha \in \rho(L)$, where X is a complex Banach space. Assume that either (a) L is a closed operator, or (b) L is a bounded operator. Then $R_\alpha(L)$ is defined on the entire space X and is bounded.

2.4 Spectral Decomposition

Spectral decomposition (or spectral theorem) for matrices and operators is a concept from linear algebra and deals with expressing a matrix in terms of its eigenvalues and eigenvectors.

Spectral Theorem

The spectral theorem says that all diagonalizable square matrix M can be decomposed. For any real symmetric matrix M, there exists an orthogonal matrix P and a diagonal matrix Λ such that:

$$M = P\Lambda P^T$$

In this context, P is a matrix with columns that are the orthonormal eigenvectors of M, while Λ is a diagonal matrix with the eigenvalues of M on its diagonal.

Spectral Decomposition Theorem

For a diagonalizable matrix M, the spectral decomposition is as follows:

$$M = U\Lambda U^{-1}$$

where:
U is a matrix composed of the linearly independent eigenvectors of M as its columns,
Λ is a diagonal matrix with the eigenvalues of M as its diagonal entries.

Chapter 3

Linear Integral Operators in Hilbert Space

3.1 Integral Operators

Definition 6 *Let $\mathcal{F}$ denote a space of functions. Consider two functions g and L where g is the function to be transformed and L is known as the kernel function. The integral operator $\mathcal{J}$ modifies g by integrating it with respect to L.*
The operator $\mathcal{J}$ acting on a function g is given by:

$$(\mathcal{J}g)(y) = \int_c^d L(y,s)g(s)\,ds, \tag{3.1}$$

where $L(y,s)$ represents the kernel function, determining how $g(s)$ contributes to the outcome at each point y and $g(s)$ is the function being transformed by the integral operator.

3.1.1 Completely Continuous Operator

A completely continuous operator $\mathcal{J}$ is represented as:

$$\mathcal{J}_m = \sum_{j=1}^{m} \mathbf{u}_j \mathbf{v}_j^*$$

Here, $\mathbf{u}_j$ and $\mathbf{v}_j$ are vectors, and $\mathbf{v}_j^*$ denotes the conjugate transpose of $\mathbf{v}_j$. i.e, it is uniformly approximable by a sequence of finite-dimensional dyadic operators. This implies:

$$\|(\mathcal{J} - \mathcal{J}_m)\mathbf{z}\| < \delta\|\mathbf{z}\|$$

for all $m > m(\delta)$. The parameter δ is a small positive number that determines the degree of approximation, and $m(\delta)$ is the corresponding number of terms needed to achieve this level of accuracy.

3.1.2 Hilbert-Schmidt Operator

An operator $\mathcal{T}$ is classified as a Hilbert-Schmidt operator if it satisfies the following conditions: $\mathcal{T}$ is bounded, and

$$\sum_{k=1}^{\infty} \|\mathcal{T}\varphi_k\|^2 < \infty,$$

where $\{\varphi_k\}$, for $k = 1, 2, \ldots$, is an orthonormal set in the Hilbert space $\mathcal{X}$.

Theorem 7 *A Hilbert-Schmidt operator is completely continuous.*

Proof. Since, $\mathbf{I} = \sum_{m=1}^{\infty} \boldsymbol{\psi}_m \boldsymbol{\psi}_m^{\dagger}$ and use the identity $\mathbf{x} = \mathbf{I}\mathbf{x}$, or

$$\mathbf{x} = \sum_{m=1}^{\infty} \boldsymbol{\psi}_m \boldsymbol{\psi}_m^{\dagger} \mathbf{x}$$

which gives

$$\mathbf{K}\mathbf{x} = \sum_{m=1}^{\infty} (\mathbf{K}\boldsymbol{\psi}_m)\, \boldsymbol{\psi}_m^{\dagger} \mathbf{x}$$

As, $\|\mathbf{K}\| < M$, where $M < \infty$), hence the series converges to $\mathbf{K}\mathbf{x}$. Since $\mathbf{K}\boldsymbol{\psi}_m$ is a vector in $\mathcal{H}$, the operator

$$\mathbf{K}_n = \sum_{m=1}^{n} (\mathbf{K}\psi_m)\, \psi_m^{\dagger}$$

is an n-term dyadic operator and

$$(\mathbf{K} - \mathbf{K}_n)\, \mathbf{x} = \sum_{m=n+1}^{\infty} \mathbf{K}\psi_m \boldsymbol{\psi}_m^{\dagger} \mathbf{x}$$

By the **Triangle and Schwarz Inequalities**, it follows that

$$\begin{aligned}
\left\| \sum_{m=n+1}^{\infty} \mathbf{K}\psi_m \left(\boldsymbol{\psi}_m^{\dagger} \mathbf{x}\right) \right\| &\leq \sum_{m=n+1}^{\infty} \|\mathbf{K}\boldsymbol{\psi}_m\| \left|\boldsymbol{\psi}_m^{\dagger} \mathbf{x}\right| \\
&\leq \left(\sum_{m=n+1}^{\infty} \|\mathbf{K}\boldsymbol{\psi}_m\|^2 \sum_{m=n+1}^{\infty} \left|\boldsymbol{\psi}_m^{\dagger} \mathbf{x}\right|^2 \right)^{1/2} \\
&\leq \left(\sum_{m=n+1}^{\infty} \|\mathbf{K}\boldsymbol{\psi}_m\|^2 \right)^{1/2} \|\mathbf{x}\|
\end{aligned}$$

which implies:

$$\sum_{m=n+1}^{\infty} \|\mathbf{K}\psi_m\|^2 < \epsilon^2$$

for $n >$ for some positive integer. Thus, we conclude that

$$\|(\mathbf{K} - \mathbf{K}_n)\,\mathbf{x}\| < \epsilon\|\mathbf{x}\|$$

for $n > n(\epsilon)$. This proves the theorem.

Theorem 8 *A Lebesgue square integrable kernel $l(\vec{p}, \vec{q})$ of an integral operator $\mathbf{J}$, is completely continuous.*

Proof. Let $\boldsymbol{\psi}_m$ be an orthonormal basis set. Recall, $\mathbf{I} = \sum_{m=1}^{\infty} \psi_m \psi_m^{\dagger}$, is the identity integral operator whose kernel $m(\vec{r}, \vec{s}) = \sum_{m=1}^{\infty} \psi_m(\vec{r})\psi_m^*(\vec{s}) = \delta(\vec{r} - \vec{s})$, the Dirac delta function.

Therefore,

$$\begin{aligned}
\sum_{m=1}^{\infty} \|\mathbf{K}\psi_m\|^2 &= \sum_{m=1}^{\infty} \int d^D r \int k^*(\vec{r}, \vec{s})\psi_m^*(\vec{s}) d^D s \int k(\vec{r}, \vec{s}')\psi_m(\vec{s}') d^D s' \\
&= \int d^D r d^D s d^D s' k^*(\vec{r}, \vec{s}) k(\vec{r}, \vec{s}') \sum_{m=1}^{\infty} \psi_m(\vec{s}')\psi_m^*(\vec{s}) \\
&= \int d^D r d^D s d^D s' k^*(\vec{r}, \vec{s}) k(\vec{r}, \vec{s}')\delta(\vec{s}' - \vec{s}) \\
&= \int d^D r d^D s \, |k(\vec{r}, \vec{s})|^2
\end{aligned}$$

Since $\int d^D r d^D s |k(\vec{r}, \vec{s})|^2 < \infty$ by previous theorem, it follows $\mathbb{J}$ is completely continuous.

It is important to note that Completely Continuous operators are important because if they are self-adjoint or normal, then they obey the sepectral resolution theorem (which reduces the complexity of solving various problems in engineering and sciences). Another reason why they are of special interest is because they obey the following:

Theorem 9 (Fredholm alternative theorem) ***Theorem.*** *Suppose J is a compact operator. If $M = I + J$ or $M = J$, then*

$$Mv = g \quad (1)$$

has a solution if and only if

$$\langle w, g \rangle = 0 \quad (2)$$

for every solution to the adjoint homogeneous equation

$$M^* w = 0. \quad (3)$$

The equation (1) and the homogeneous equation

$$Mv = 0 \quad (4)$$

have the same number of solutions. If equation (2) has no solution, then a unique solution to equation (1) exists for any vector g in the Hilbert space $\mathcal{H}$ in which J is defined. When equations (3) and (4) have nontrivial solutions, the general solution to equation (1) is given by

$$v = v^p + \sum_{i=1}^{m} \beta_i v^i, \quad (5)$$

where v^p is a particular solution to the inhomogeneous equation (1) and v^i, $i = 1, \ldots, m$, are linearly independent solutions to (4).

3.1.3 Volterra Equations

First Kind

An integral of the form:

$$\int_{\Omega_D} k(\vec{r}, \vec{s}) x(\vec{s}) d^D s = f(\vec{r})$$

where

$$k(\vec{r}, \vec{s}) = 0 \quad \text{if any } s_m > r_m$$

and s_m and r_m are the independent components of $\vec{s}$ and $\vec{r}$.

Second Kind

An integral of the form:

$$x(t) + \int_0^t k(t, s) x(s) ds = f(t)$$

Theorem 10 *The Volterra equation of the second kind has only one continuous solution $x(t)$, when $f(t)$ and the kernel $k(t, s)$ is continuous where, t and $s \in [a, b]$*

Proof. Consider the sequence of functions,

$$\begin{aligned} x_1(t) &= f(t) \\ x_2(t) &= f(t) - \int_a^t k(t,s)x_1(s)ds \\ &\vdots \end{aligned}$$

After successive substitution,

$$\begin{aligned} x_n(t) =& f(t) - \int_a^t k(t,v_1)f(v_1)ds_1 \\ &+ \int_a^t k(t,v_1)\int_a^{v_1} k(s_1,s_2)f(s_2)ds_1ds_2 + \cdots \\ &+ (-1)^{n-1}\int_a^t k(t,s_1)\int_a^{s_1} k(s_1,s_2)\cdots\int_a^{s_{n-2}} k(s_{n-2},s_{n-1}) \\ &\times f(s_{n-1})ds_1ds_2\cdots ds_{n-1} \end{aligned}$$

or, in operator form,

$$\begin{aligned} \mathbf{x}_n &= \mathbf{f} - \mathbf{K}\mathbf{f} + \mathbf{K}^2\mathbf{f} + \cdots + (-1)^{n-1}\mathbf{K}^{n-1}\mathbf{f} \\ &= \mathbf{S}_n\mathbf{f} \end{aligned}$$

where

$$\mathbf{S}_n = \mathbf{I} + \sum_{i=1}^{n-1}(-1)^i\mathbf{K}^i$$

Since k and f are continuous, it follows that:

$$\left|\int_a^t k(t,s_1)f(s_1)ds_1\right| \leq AB\int_a^t ds_1 = AB(t-a)$$

where $\mathbf{v}_m$ is a solution to the homogeneous adjoint equation. In infinite-dimensional vector spaces, there can be an infinite number $\mathbf{m}$ of solutions to the homogeneous equation (as are the function spaces associated with integral equations).

3.2 Spectral Theory of Integral Operators

3.2.1 Bessel's Inequality

Theorem 11 *Bessel's Inequality is given as follows:*

$$\sum_m |\langle\phi_m, \mathbf{f}\rangle|^2 \leq \|\mathbf{f}\|^2$$

holds in a hilbert space when $\phi_1, \phi_2, \ldots$ is an orthonormal set.

Proof. To prove the theorem, we define

$$\alpha_m = \langle \phi_m, \mathbf{f} \rangle$$

and note that

$$\left\| \mathbf{f} - \sum_m \alpha_m \phi_m \right\|^2 \geq 0$$

Using an inner product property, we get:

$$\|\mathbf{f}\|^2 - \sum_m \alpha_m^* \langle \phi_m, \mathbf{f} \rangle - \sum_m \alpha_m \langle \mathbf{f}, \phi_m \rangle + \sum_m |\alpha_m|^2 \geq 0$$

where, $\|\phi_m\|^2 = 1$. Since $\alpha_m = \langle \phi_m, \mathbf{f} \rangle$ and $\alpha_m^* = \langle \mathbf{f}, \phi_m \rangle$, the above inequality can be rearranged to yield Bessel's inequality:

$$\|\mathbf{f}\|^2 - 2\sum_m |\alpha_m|^2 + \sum_m |\alpha_m|^2 \geq 0$$

Simplifying this, we get:

$$\|\mathbf{f}\|^2 \geq \sum_m |\alpha_m|^2$$

Hence, we have:

$$\sum_m |\langle \phi_m, \mathbf{f} \rangle|^2 \leq \|\mathbf{f}\|^2$$

This completes the proof of Bessel's inequality.

3.2.2 Properties of Eigenvalue Multiplicity

Consider,

$$\mathbf{K}\psi_m = \lambda \psi_m, \quad m = 1, \ldots, h$$

where h denotes the degeneracy (or multiplicity) of λ.
We have the follwing:

1. An operator $\mathbf{J}$ with a square integrable kernel in $\mathcal{H}$, has eigenvectors with finite number h.

2. For a completely continuous operator, the number of eigenvectors corresponding to non-zero λ is finite.

3.2.3 Eigenvectors of $f(J)$

Analogous to matrix theory, the eigenvector ϕ of the integral operator $\mathbf{J}$ is also an eigenvector of $f(\mathbf{J})$ i.e. the following is obeyed:

$$f(\mathbf{J})\phi_m = f(\lambda_m)\phi_m$$

3.2.4 Some Special Properties of Spectral Operators

From the definition of an integral operator,

$$\langle \mathbf{y}, \mathbf{J}\mathbf{x} \rangle = \left\langle \mathbf{J}^\dagger \mathbf{y}, \mathbf{x} \right\rangle \quad \forall \mathbf{y}, \mathbf{x} \in \mathcal{H}$$

or that

$$\int d^D r y^*(\vec{r}) \int d^D s k(\vec{r}, \vec{s}) x(\vec{s}) = \int d^D s \left[\int d^D r k^*(\vec{s}, \vec{r}) y(\vec{r}) \right]^* x(\vec{s})$$

The following text gives some special properties of $\mathbf{J}$:

1. A self-adjoint $\mathbf{J}$ has real eigenvalues (λ_m) and the set of their eigenvectors (ϕ_m) are orthogonal.

2. The eigenvector of $\mathbf{J}^\dagger$ with eigenvalue λ_m^*, is same as eigenvector of $\mathbf{J}$, provided $\mathbf{J}$ is a normal operator.

3. If $\mathbf{J}$ is a normal operator and if $\lambda_m \neq \lambda_n$, then from

   $$\langle \boldsymbol{\phi}_m, \mathbf{J}\boldsymbol{\phi}_m \rangle = \left\langle \mathbf{J}^\dagger \boldsymbol{\phi}_m, \boldsymbol{\phi}_m \right\rangle$$

   it follows that

   $$(\lambda_n - \lambda_m) \langle \phi_m, \phi_n \rangle = 0,$$

   and so $\langle \phi_m, \phi_n \rangle = 0$, hence if $\lambda_m \neq \lambda_n$, we get orthogonal eigenvectors.

3.3 Spectral Resolution Theorem

For any normal, completely continuous operator $\mathcal{J}$, we have,

$$\mathcal{J} = \sum_j \mu_j \psi_j \psi_j^\dagger, \quad \langle \psi_j, \psi_k \rangle = \delta_{jk}$$

Consequently, the function $h(\mathcal{J})$ of $\mathcal{J}$ also follows,

$$h(\mathcal{J}) = \sum_j h(\mu_j)\psi_j\psi_j^\dagger$$

Proof. Let $\mathbf{v}$ is an arbitrary vector in $\mathcal{G}$, then we get a set of coefficients $\beta_1, \beta_2, \ldots$ such that $\mathbf{v} = \sum_j \beta_j\psi_j$, where

$$\psi_j, \quad j = 1, 2, \ldots$$

are the eigenvectors of $\mathcal{J}$. Given that

$$\langle\psi_j, \psi_k\rangle = \delta_{jk},$$

it follows that $\beta_j = \langle\psi_j^\dagger, \mathbf{v}\rangle = \psi_j^\dagger\mathbf{v}$, hence

$$\mathbf{v} = \sum_j \psi_j(\psi_j^\dagger\mathbf{v}) = \left(\sum_j \psi_j\psi_j^\dagger\right)\mathbf{v}$$

for any $\mathbf{v} \in \mathcal{G}$. Hence,

$$\mathbf{I} = \sum_j \psi_j\psi_j^\dagger$$

which gives the spectral resolution theorem.
This has an important implication in the following theorem:

Theorem 12 *A self-adjoint, completely continuous operator is perfect i.e. it has a complete set of orthonormal eigenvectors.*

Proof. Let $\mathbf{A}$ be a linear self adjoint operator. Let $\mathbf{u}$ be a normalized vector. If:

$$\max_{\mathbf{u}} \langle\mathbf{u}, \mathbf{A}\mathbf{u}\rangle, \quad \|\mathbf{u}\|^2 = 1$$

then, $\mathbf{u}$ is an eigenvector of $\mathbf{A}$. Expanding $\mathbf{u}$ in terms of an arbitrary orthonormal basis $\phi_1, \phi_2, \ldots$, we get:

$$\mathbf{u} = \sum_i \beta_i\phi_i$$

for g, hence we get:

$$g = \sum_{i,j} a_{ij}\beta_i^*\beta_j - \mu\sum_i \beta_i^*\beta_i$$

where $a_{ij} = \langle\phi_i, \mathbf{A}\phi_j\rangle$. This results in the eigenproblem:

$$\sum_j a_{ij}\beta_j = \mu\beta_i, \quad i = 1, 2, \ldots$$

Equivalently:

$$\mathbf{A}\mathbf{u} = \mu\mathbf{u}$$

If $\mathbf{A}$ has no positive eigenvalues, we consider $-\mathbf{A}$ to find a positive maximum, as $\mathbf{A}$ and $-\mathbf{A}$ share the same eigenvectors. We know a completely continuous operator is bounded, i.e., $\|\mathbf{A}\|^2 = \max_{\mathbf{v}\neq 0}\langle \mathbf{A}\mathbf{v}, \mathbf{A}\mathbf{v}\rangle / \langle \mathbf{v}, \mathbf{v}\rangle = N^2 < \infty$. For any normalized vector $\mathbf{u}$ in $\mathcal{H}$:

$$\langle \mathbf{u}, \mathbf{A}\mathbf{u}\rangle \leq \|\mathbf{u}\|\|\mathbf{A}\mathbf{u}\| \leq N\|\mathbf{u}\|^2 = N$$

We get a sequence of normalized vectors, $\mathbf{u}_1, \mathbf{u}_2, \ldots$:

$$\lim_{n\to\infty} \langle \mathbf{u}_n, \mathbf{A}\mathbf{u}_n\rangle = N$$

From the property of completely continuous, self-adjoint operators that $\mathbf{u}_n \to \mathbf{u}$, we have:

$$\langle \mathbf{u}, \mathbf{A}\mathbf{u}\rangle = N$$

Thus, $\mathbf{u}$ is an eigenvector (ψ_1) of $\mathbf{A}$, with N as its eigenvalue (μ_1). Consider the self-adjoint operator:

$$\tilde{\mathbf{A}} = \mathbf{A} - \sum_i \mu_i \psi_i \psi_i^\dagger$$

Let $\tilde{\mathbf{A}}$ is nonzero. Then, there exists a nonzero eigenvalue ν and its eigenvector φ:

$$\tilde{\mathbf{A}}\varphi = \nu\varphi$$

However, the eigenvectors $\psi_1, \psi_2, \ldots$ of $\mathbf{A}$ are also eigenvectors of $\tilde{\mathbf{A}}$ with zero eigenvalue. Thus, $\langle \psi_i, \varphi\rangle = 0$, implying:

$$\tilde{\mathbf{A}}\varphi = \mathbf{A}\varphi = \nu\varphi$$

Therefore, φ must be an eigenvector of $\mathbf{A}$ with a nonzero eigenvalue, which contradicts the generation process of all eigenvectors of $\mathbf{A}$ with nonzero eigenvalues. Hence, $\tilde{\mathbf{A}} = \mathbf{0}$, or:

$$\mathbf{A} = \sum_i \mu_i \psi_i \psi_i^\dagger$$

This establishes the theorem.

3.4 Completely Continuous Operators

Theorem 13 *Any completely continuous operator in $\mathcal{H}$, can be expressed as*

$$\mathbf{T} = \sum \tau_k \mathbf{u}_k \mathbf{v}_k^\dagger$$

where the τ_k are positive real numbers, and the vectors $\mathbf{u}_k$ and $\mathbf{v}_k$ for $k = 1, 2, \ldots$ form orthonormal sets in $\mathcal{H}$. These vectors satisfy the following equations:

$$\mathbf{T}\mathbf{u}_k = \tau_k \mathbf{v}_k$$

and

$$\mathbf{T}^\dagger \mathbf{v}_k = \tau_k \mathbf{u}_k$$

Proof. Given that $\mathbf{T}$ is a completely continuous operator in$\mathcal{H}$, by the spectral theorem, that $\mathbf{T}$ has a discrete spectrum with eigenvalues accumulating only at zero.

Let $\{\tau_k\}$ denote the non-zero singular values of $\mathbf{T}$. These τ_k are positive real numbers. Let $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ be the corresponding orthonormal sets of left and right singular vectors of $\mathbf{T}$, respectively.
By SVD,

$$\mathbf{T} = \sum_k \tau_k \mathbf{u}_k \mathbf{v}_k^\dagger$$

where $\tau_k > 0$, and $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ form orthonormal sets in $\mathcal{H}$.

The vectors $\mathbf{u}_k$ and $\mathbf{v}_k$ fulfill the following equations due to the properties of singular value decomposition:

$$\mathbf{T}\mathbf{u}_k = \tau_k \mathbf{v}_k$$

and

$$\mathbf{T}^\dagger \mathbf{v}_k = \tau_k \mathbf{u}_k$$

To verify the orthonormality of $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$, we must show:

$$\langle \mathbf{u}_k, \mathbf{u}_j \rangle = \delta_{kj} \quad \text{and} \quad \langle \mathbf{v}_k, \mathbf{v}_j \rangle = \delta_{kj}$$

where δ_{kj} is the Kronecker delta.

Since $\mathbf{T}$ is a completely continuous operator, it transforms orthonormal sets into orthonormal sets under the transformation involving its singular values and vectors. Therefore, the orthonormality conditions hold:

$$\langle \mathbf{u}_k, \mathbf{u}_j \rangle = \delta_{kj}$$

and

$$\langle \mathbf{v}_k, \mathbf{v}_j \rangle = \delta_{kj}$$

Therefore, we have represented $\mathbf{T}$ as

$$\mathbf{T} = \sum_k \tau_k \mathbf{u}_k \mathbf{v}_k^\dagger$$

where τ_k are the positive singular values, and $\mathbf{u}_k$ and $\mathbf{v}_k$ are orthonormal sets satisfying the given eigenvector equations.

Theorem 14 *The eigenvectors of a completely continuous and normal integral operator has a complete orthonormal basis set.*

Proof Given,$\mathbf{A}$ is a completely continuous integral operator in $\mathcal{H}$ and,

$$\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger \mathbf{A},$$

we need to demonstrate that the eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots$ of $\mathbf{A}$ form a complete orthonormal basis set in $\mathcal{H}$. Since $\mathbf{A}$ is a normal operator, there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of $\mathbf{A}$. Let $\{\mathbf{u}_k\}$ be the set of eigenvectors of $\mathbf{A}$ corresponding to eigenvalues $\{\lambda_k\}$. First, we show that the eigenvectors $\{\mathbf{u}_k\}$ are orthonormal. For $k \neq j$, consider the eigenvalue equations:

$$\mathbf{A}\mathbf{u}_k = \lambda_k \mathbf{u}_k$$

and

$$\mathbf{A}\mathbf{u}_j = \lambda_j \mathbf{u}_j.$$

Taking inner product in the following way, we obtain:

$$\langle \mathbf{A}\mathbf{u}_k, \mathbf{u}_j \rangle = \lambda_k \langle \mathbf{u}_k, \mathbf{u}_j \rangle$$

and

$$\langle \mathbf{A}\mathbf{u}_j, \mathbf{u}_k \rangle = \lambda_j \langle \mathbf{u}_j, \mathbf{u}_k \rangle.$$

Since $\mathbf{A}$ is normal, we have:

$$\langle \mathbf{A}\mathbf{u}_k, \mathbf{u}_j \rangle = \langle \mathbf{u}_k, \mathbf{A}^\dagger \mathbf{u}_j \rangle = \overline{\lambda_j} \langle \mathbf{u}_k, \mathbf{u}_j \rangle.$$

Equating the expressions for $\langle \mathbf{A}\mathbf{u}_k, \mathbf{u}_j \rangle$, we get:

$$\lambda_k \langle \mathbf{u}_k, \mathbf{u}_j \rangle = \overline{\lambda_j} \langle \mathbf{u}_k, \mathbf{u}_j \rangle.$$

Since $\lambda_k \neq \overline{\lambda_j}$ for $k \neq j$, it follows that:

$$\langle \mathbf{u}_k, \mathbf{u}_j \rangle = 0 \quad \text{for} \quad k \neq j.$$

This shows that the eigenvectors $\{\mathbf{u}_k\}$ are orthogonal. Normalizing them, we obtain an orthonormal set. To demonstrate completeness, we need to show that the set $\{\mathbf{u}_k\}$ spans $\mathcal{H}$. On the contrary, let there exists a vector $\mathbf{w} \in \mathcal{H}$ orthogonal to all $\mathbf{u}_k$:

$$\langle \mathbf{w}, \mathbf{u}_k \rangle = 0 \quad \text{for all} \quad k.$$

Since $\mathbf{A}$ is a completely continuous operator, the spectral theorem for compact operators ensures that the eigenvectors corresponding to nonzero eigenvalues form a complete basis in the closure of the range of $\mathbf{A}$. Additionally, for the zero eigenvalue, the eigenvectors corresponding to zero eigenvalue also contribute to the basis for the null space of $\mathbf{A}$. Since $\mathbf{A}$ is normal, its range and null space are orthogonal complements in $\mathcal{H}$. Thus, the union of the eigenvectors associated with nonzero and zero eigenvalues spans $\mathcal{H}$. Therefore, any vector orthogonal to all eigenvectors must be the zero vector:

$$\mathbf{w} = 0.$$

This proves that the set $\{\mathbf{u}_k\}$ is complete. Consequently, the eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots$ form a complete orthonormal basis set in $\mathcal{H}$.

Chapter 4

Linear Differential Operators in Hilbert Space

4.1 The Differential Operator

Definition 15 *A differential operator is a linear transformation D on H, of the form*

$$D(u) = u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1u' + a_0u$$

where $a_0, a_1, \ldots, a_{n-1}$ are (can be complex) constants. The order of D is n. H is hilbert space.

Linearity. A linear operator D is a linear transformation D on H such that,
1. $D(x + v) = D(x) + D(v)$
2. $D(cx) = cD(x)$

Eigenfunctions. A non-zero function u is an eigenfunction for a differential operator D, such that,

$$Du = \lambda u$$

where the constant λ is the eigenvalue of u.

Consider a 3-Dimensional wave equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\implies \frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \implies \frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \tag{4.1}$$

$$(\frac{\partial^2}{\partial t^2} - c^2\nabla^2)u = 0$$

$$Du = 0$$

Comparing with a generic matrix equation, $D \equiv A$, $u \equiv x$ and $b = 0$ (in this case homogenous), D is the Differential Operator and u is the solution.

$$\begin{aligned} D(\alpha u + \beta v) &= (\frac{\partial^2}{\partial t^2} - c^2\nabla^2)(\alpha u + \beta v) \\ &= \frac{\partial^2(\alpha u + \beta v)}{\partial t^2} - c^2\nabla^2(\alpha u + \beta v) \\ &= \alpha(\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u) + \beta(\frac{\partial^2 v}{\partial t^2} - c^2\nabla^2 v) \\ &= \alpha D(u) + \beta D(v) \end{aligned}$$

$\implies$ D is a *Linear* Differential Operator.
Note that Differential Operator requires specific boundary conditions.
Consider the inhomogeneous equation,

$$Du(x) = g(x), \quad x \in [a, b],$$

where D is a p th-order differential expression given by,

$$\begin{aligned} Du(x) =& a_p(x)\frac{d^p u(x)}{dx_p} + a_{p-1}(x)\frac{d^{p-1}u(x)}{dx^{p-1}} + \cdots \\ &+ a_1(x)\frac{du(x)}{dx} + a_0(x)u(x) \end{aligned}$$

Then boundary functions are given by:

$$\begin{aligned} B_1 u &\equiv \sum_{j=1}^{p} \lambda_{1j} u^{(j-1)}(x) + \sum_{j=1}^{p} \lambda_{1,p+j} u^{(j-1)}(y) \\ &\vdots \\ B_m u &\equiv \sum_{j=1}^{p} \lambda_{mj} u^{(j-1)}(x) + \sum_{j=1}^{p} \lambda_{m,p+j} u^{(j-1)}(y), \end{aligned}$$

where $u^{(i)}$ is the i th derivative of u ($d^i u/dc^i$, at $c = x$ and y). For the matrix

$$\mathbf{B} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1,2p} \\ \vdots & & \\ \alpha_{m1} & \cdots & \alpha_{m,2p} \end{bmatrix}$$

the row vectors

$$\boldsymbol{\alpha}_i^{\mathrm{T}} = [\lambda_{i1}, \ldots, \lambda_{i,2p}], \quad i = 1, \ldots, m$$

are linearly independent (rank of the matrix is m).

The above differential equation is called a "balanced problem". We can find a **unique** solution for a balanced problem whereas sometimes it can fail to have any solution at all. An unbalanced problem can also have a solution.

4.1.1 Unbounded Differential Operators

Definition 16 *An operator, O on normed spaces is said to be bounded when $\|Ox\| \leq c\|x\|$ where c is a real number.*

While some classes of integral operators are bounded, differential operators in Hilbert space are unbounded. Let us illustrate this with a simple example,

Let $D : C^1_{[a,b]} \to C^0_{[a,b]}$ with $a < b$ be the differential operator defined as $Dx = x'$.

Taking, $[a, b] = [0, 1]$, let $n \in \mathbb{N}$ be arbitrary. Now, consider the sequence $f_n(t) = t^n$. Then, $Tf_n(t) = nt^{n-1}$. Equipping C^1 with the infinity norm (**sup** norm), then $\|Tf_n\|_\infty = n$, but $\|f_n\|_\infty = 1$. So,

$$\frac{\|Tf_n\|_\infty}{\|f_n\|_\infty} = n$$

By assumption, this is true for any n. This means there is no c for which $\|Tf_n\|_\infty \leq c\,\|f_n\|_\infty$.

$\implies$ Differential operator is unbounded.

4.1.2 The Adjoint of $\mathbb{D}$

Definition 17 *A differential operator $\mathcal{D}$ has an adjoint operator $\mathcal{D}^*$ such that*

$$\langle g, \mathcal{L}f \rangle = \langle f, \mathcal{L}^* g \rangle,$$

where $f(t)$ and $g(t)$ are arbitrary functions satisfying homogeneous boundary conditions.

The adjoint of a differential operator is similar to the transpose of a matrix. Consider an $m \times m$ matrix $\mathbf{B}$ and arbitrary $m \times 1$ vectors $\mathbf{x}$ and $\mathbf{y}$. Then,

$$\langle \mathbf{y}, \mathbf{B}\mathbf{x} \rangle = \mathbf{y}^T \mathbf{B}\mathbf{x},$$

$$\implies \langle \mathbf{y}, \mathbf{B}\mathbf{x} \rangle = \left(\mathbf{B}^T \mathbf{y}\right)^T \mathbf{x}.$$

In terms of the inner product, this means that

$$\langle \mathbf{y}, \mathbf{B}\mathbf{x} \rangle = \left\langle \mathbf{x}, \mathbf{B}^T \mathbf{y} \right\rangle.$$

Determining the Adjoint of $\mathbb{D}$

Consider the general second-order linear differential equation with variable coefficients

$$\mathcal{D}u = \frac{1}{w(x)} \left[a_0(x)u''(x) + a_1(x)u'(x) + a_2(x)u(x) \right] = 0, \quad a \leq x \leq b, \quad (4.2)$$

Consider an arbitrary function $v(x)$, and take the inner product with $\mathcal{D}u$,

$$\begin{aligned} \langle v, \mathcal{D}u \rangle = \int_a^b w(x)v(x) \left\{ \tfrac{1}{w(x)} \left[a_0(x)u''(x) + a_1(x)u'(x) + a_2(x)u(x) \right] \right\} dx \\ = \int_a^b \left\{ a_0 v u'' + a_1 v u' + a_2 v u \right\} dx, \end{aligned} \quad (4.3)$$

the inner product is taken w.r.t the weight function.

Integrating the second term by parts,

$$\int_a^b a_1 v u' dx = a_1 v u \big|_a^b - \int_a^b u \left(a_1 v\right)' dx$$

where

$$\int g dh = gh - \int h dg$$

with

$$\begin{array}{ll} g = a_1 v, & h = u, \\ dg = \left(a_1 v\right)' dx, & dg = u' dx \end{array}$$

Integrating gives,

$$\int_a^b a_0 v u'' dx = a_0 v u' \big|_a^b - \int_a^b u' \left(a_0 v\right)' dx = \left[a_0 v u' - \left(a_0 v\right)' u \right]_a^b + \int_a^b u \left(a_0 v\right)'' dx$$

where,

$$\begin{array}{ll} g = a_0 v, & h = u', \\ dg = \left(a_0 v\right)' dx, & dh = u'' dx, \end{array}$$

and,

$$\begin{aligned} g &= (a_0 v)', & h &= u, \\ dg &= (a_0 v)'' \, dx, & dh &= u' dx \end{aligned}$$

Substituting into (4.3),

$$\begin{aligned} \langle v, \mathcal{D}u \rangle =& \left[a_0 v u' - (a_0 v)' u + a_1 v u \right]_a^b \\ &+ \int_a^b w(x) u(x) \left\{ \frac{1}{w(x)} \left[(a_0 v)'' - (a_1 v)' + a_2 v \right] \right\} dx, \end{aligned}$$

For homogeneous boundary conditions, we have,

$$\langle v, \mathcal{D}u \rangle = \langle u, \mathcal{D}^* v \rangle ,$$

where the adjoint operator $\mathcal{D}^*$ is:

$$\mathcal{D}^* w = \frac{1}{v(x)} \left\{ [a_0(x) w]'' - [a_1(x) w]' + a_2(x) w \right\} . \tag{4.4}$$

We can use the same approach for finding the adjoint of higher order derivatives.

4.2 Self-Adjoint Differential Operator

A matrix is *symmetric* if it is equivalent to its transpose and *hermitian* if equal to its conjugate transpose and if its eigenvalues are distinct then it has mutually orthogonal eigenvectors.

Analogously, if $\mathcal{D} = \mathcal{D}*$, then the differential operator is **self-adjoint (Hermitian)**.

Proof. Consider, $\mathcal{D} = \mathcal{D}*$ and let $u(x)$ and $v(x)$ be two eigenfunctions of the differential operator, i.e, $\mathcal{D}u = \lambda_1 u$ and $\mathcal{D}v = \lambda_1 v$. Then, by definition of adjoint of a differential operator,

$$\begin{aligned} \langle u, \mathcal{D}v \rangle &= \langle v, \mathcal{D}*u \rangle \\ \langle u, \mathcal{D}v \rangle &= \langle v, \mathcal{D}u \rangle \\ \langle u, \lambda_2 v \rangle &= \langle v, \lambda_1 u \rangle \\ (\lambda_1 - \lambda_2) \langle u, v \rangle &= 0 \end{aligned}$$

$\implies$ Thus, if$\lambda_1 \neq \lambda_2$then$\langle u, v \rangle = 0$, i.e, the corresponding eigenfunctions are orthogonal. (4.5)

This implies that distinct eigenvalues of a self-adjoint differential operator has orthogonal eigenvectors.

4.3 Spectral Representation

Given a differential operator D with a set of eigenfunctions $\{u_n\}$ and corresponding eigenvalues $\{\lambda_n\}$, any function f can be expanded as:

$$f(x) = \sum_n c_n u_n(x)$$

where c_n are coefficients determined by projecting f onto the eigenfunctions:

$$c_n = \int f(x) u_n(x)\, dx$$

4.4 Sturm-Liouville Differential Operator

Recall eq.(4.2). Not all differential expressions give rise to self-adjoint differential operators. Let us find the self-adjoint subset of these equations.

The adjoint of a linear differential operator of second order with variable coefficients is:

$$\mathcal{D}^* v = \frac{1}{w(x)} \left\{ [a_0 v]'' - [a_1 v]' + a_2 v \right\}.$$

Expanding the differentiation terms via the product rule and collecting terms as follows,

$$\mathcal{D}^* v = \frac{1}{w(x)} \left\{ a_0 v'' + [2a_0' - a_1]\, v' + [a_0'' - a_1' + a_2]\, v \right\} \tag{4.6}$$

For $\mathcal{D}$ to be self-adjoint, the operators $\mathcal{D}$ and $\mathcal{D}^*$ must be the same.
Comparing (4.2) and (4.6),
First term is already identical, hence for equivalence of the second term,

$$a_1(y) = 2a_0'(y) - a_1(y),$$

$$a_1(y) = a_0'(y). \tag{4.7}$$

For the equivalence of the third term,

$$a_2(x) = a_0''(x) - a_1'(x) + a_2(x),$$

which is true using (4.7).
Therefore,

$$\mathcal{D}u = \frac{1}{w(x)} \left\{ a_0 u'' + a_0' u' + a_2 u \right\} = 0$$

This can be written as,

$$\mathcal{D} = \frac{1}{w(x)} \left\{ \frac{d}{dx} \left[a_0(x) \frac{d}{dx} \right] + a_2(x) \right\}$$

This is the **Sturm-Liouville differential operator**.

Therefore, the eigenfunctions associated with the Sturm-Liouville differential operator are orthogonal. Any second-order linear ordinary differential equation can be transformed into Sturm-Liouville form. This approach can also be generalized to higher-order differential equations.

4.5 Some General Spectral Properties of Differential Operators

Theorem 18 *If λ_1 and u are eigen value and eigenfuntion of $\mathcal{D}$ and λ_2 and v are eigen value and eigenfuntion of $\mathcal{D}$ with $\lambda_1 \neq \lambda_2$, then u is orthogonal to v.*

Proof. The proof is along the same lines as shown in (4.5).

Theorem 19 *The eigenvalues of a self-adjoint $\mathbf{D}$ are real.*

Proof. Since $\mathbf{D}$ is self-adjoint, clearly, $\langle u, \mathbf{D}u \rangle = \langle \mathbf{D}u, u \rangle = \lambda_i \langle u, u \rangle = \lambda_i^* \langle u, u \rangle$.

Theorem 20 *Consider $\mathbf{D}$ of pth-order.If $\exists$ a number β not equal to an eigenvalue of $\mathbf{D}$ and the coefficient $a_p(x) \neq 0$ and $a_i(x)$ is continuous in the finite interval, then the number of eigenvectors λ_i is finite.*

Proof. Let β not be an eigenvalue of a regular self-adjoint $\mathcal{D}$. We know, the eigenvalues of $\mathcal{D}$ will always be real.

Define the operator $\mathbf{D}$ as,

$$Dv = \sum_{m=0}^{q} a_m(x) \frac{d^m v}{dx^m}, \quad B_n v = 0, n = 1, \ldots, q,$$

where, $\mathbf{v}, D\mathbf{v} \in \mathcal{H}$.

We define a new operator, $\mathbf{D\prime} \equiv \mathbf{D} + \beta \mathbf{I}$ as,

$$\mathcal{D}\prime v = \sum_{m=0}^{q} a_m(x) \frac{d^m v}{dx^m} + \alpha v, \quad B_n v = 0, n = 1, \ldots, q,$$

where, $\mathbf{v}, \mathcal{D}\prime\mathbf{v} \in \mathcal{H}$.

Since the equation,

$$\mathbf{D}\prime u = \lambda\prime u$$

can be rearranged as,

$$\mathbf{D}u = (\lambda\prime - \beta)u$$

both $\mathcal{D}$ and $\mathcal{D}\prime$ have the same eigenvectors.

This implies, $\lambda = \lambda\prime - \beta$ are eigenvalues of $\mathcal{D}$. Now β is not an eigenvalue of $\mathcal{D}$ so the homogenous equation $\mathcal{D}\prime u = 0$ only has trivial solution and so $\mathcal{D}\prime$ exists. $\mathcal{D}\prime^{-1}$ is the integral $\mathbf{G}\prime$ which has Green's function for $\mathbf{L}\prime$ as its kernel, $\tilde{h}(x, y)$.

Again, since the equation,

$$\mathbf{G}\prime u = \alpha\prime u$$

can be rearranged as,

$$\mathbf{D}\prime u = \alpha\prime^{-1} u$$

the eigenvectors of $\mathbf{G}\prime$ and $\mathbf{D}\prime$ are same. Consequently, the eigenvectors of $\mathbf{G}\prime$ and $\mathbf{D}$ are same.

Now, the kernel $\tilde{h}(x, y)$ is continuous, so on a finite interval it is square integrable in a bounded rectangle.This implies, $\mathbf{G}\prime$ has a finite number of eigenvectors for a non-zero eigenvalue.

Theorem 21 *If D is a differential operator of order p, the maximum number of eigenvectors associated with any specific eigenvalue cannot exceed p.*

Proof. Consider the eigenvalue problem:

$$\mathcal{D}u - \lambda u = 0,$$

where $\mathcal{D}$ is a differential operator of order p, subject to boundary conditions:

$$B_i u = 0, \quad i = 1, \ldots, p.$$

Assume λ is an eigenvalue of $\mathcal{D}$. This leads to the homogeneous equation:

$$\mathcal{D}\prime u = a_p(x)\frac{d^p u}{dx^p} + \cdots + a_1(x)\frac{du}{dx} + (a_0(x) - \lambda)u = 0,$$

where $\mathcal{D}\prime$ incorporates λ.

A fundamental system of solutions to $\mathcal{D}\prime u = 0$ consists of $\{u_1, u_2, \ldots, u_p\}$. Any solution u can thus be expressed as a linear combination:

$$u = \sum_{i=1}^{p} \alpha_i u_i.$$

The boundary conditions then determine the coefficients α_i. By substituting this general solution into the boundary conditions, we obtain:

$$\begin{bmatrix} B_1 u_1 & \cdots & B_1 u_p \\ \vdots & & \vdots \\ B_p u_1 & \cdots & B_p u_p \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} = 0.$$

The rank r of the matrix $[B_i u_j]$ determines the number of linearly independent solutions $\{\alpha_1, \ldots, \alpha_p\}$ for this system. Since λ is assumed to be an eigenvalue, the rank r must be between 0 and $p-1$. The number linearly independent solutions is $p - r$. This is also the number of eigenfuntions.

Definition 22 (Normal Differential Operator) **D** *of order p and its adjoint, with the following:*

$$B_j u = B_j^\dagger u = 0, \quad j = 1, \ldots, q.$$

and satisfying:

$$DD^\dagger v = D^\dagger D v.$$

is called a normal differential operator.

Theorem 23 *If D is a normal differential operator and nonsingular, then its eigenvectors $v_i, i = 1, 2, \ldots$, form a complete orthonormal set.*

Proof. Consider **D** of order p and its adjoint, with:

$$B_j u = B_j^\dagger u = 0, \quad j = 1, \ldots, q.$$

and satisfying:

$$DD^\dagger v = D^\dagger D v.$$

D is called a normal differential operator.
We assume that the only solution to $\mathbf{Du} = \mathbf{0}$ is trivial, $\mathbf{u} = \mathbf{0}$.
We denote **G** as the inverse of **D** and $\mathbf{G}^\dagger$ as the inverse of $\mathbf{L}^\dagger$. Then,

$$\mathbf{D}\mathbf{D}^\dagger = \mathbf{D}^\dagger\mathbf{D} \quad \text{or} \quad \left(\mathbf{D}^\dagger\right)^{-1}\mathbf{D}^{-1} = \mathbf{D}^{-1}\left(\mathbf{D}^\dagger\right)^{-1},$$

which implies:

$$\mathbf{G}^\dagger\mathbf{G} = \mathbf{G}\mathbf{G}^\dagger.$$

Since, the kernel of **G** is also square integrable. **G** is a normal, completely continuous operator. And we already know, such operators have a complete set of orthonormal eigenvectors. Hence the theorem is proved.

4.6 Spectral properties of Sturm-Liouville Operators

4.6.1 Regular Sturm-Liouville Operators

Consider the Regular Sturm-Liouville Operator:

$$\mathcal{D}y = -\frac{1}{r(x)}\frac{d}{dx}\left(m(x)\frac{dy}{dx}\right) + \frac{n(x)}{r(x)}y, \quad c < x < d$$

where $[c, d]$ is a bounded interval; $r(y), m(y), m'(y)$, and $n(y)$ are real and continuous functions; and $r(y)$ and $m(y)$ are positive within the interval $[c, d]$. For the sake of convenience, we will be defining $\mathcal{D}$ in $\mathcal{L}_2(a, b; s)$.
The inner product here is defined as,

$$\langle y, z\rangle = \int_c^d r(x)y^*(x)z(x)\,dx$$

The formal adjoint differential expression $\mathcal{D}^\dagger$ of $\mathcal{D}$ is given by:

$$\int_c^d r(y)\left[x^*(y)\mathcal{D}z(y) - (\mathcal{D}^\dagger x(y))^* z(y)\right]\,dy = m(y)\left[x^*(y)z'(y) - (x'(y))^* z(y)\right]\Big|_c^d.$$

Since, $\mathcal{D}^\dagger = \mathcal{D}$, the Sturm-Liouville operator is formally self-adjoint.
The boundary conditions of $\mathcal{D}$ is given by:

$$B_1 y = \beta_{11}y(c) + \beta_{12}y'(c) + \beta_{13}y(d) + \beta_{14}y'(d) = 0$$
$$B_2 y = \beta_{21}y(c) + \beta_{22}y'(c) + \beta_{23}y(d) + \beta_{24}y'(d) = 0$$

where β_{ij} are reals.
$\mathbf{D}^\dagger$ is specified by $\mathcal{D}$ and bc: $B_1^\dagger z = B_2^\dagger z = 0$, derived from the condition:

$$m(x)\left[y^*(x)z'(x) - (y'(x))^* z(x)\right]\Big|_c^d = 0, \quad \text{for all } \mathbf{z} \in \mathcal{D}\wr_{\mathcal{D}}$$

where

$$\mathcal{D}\wr_{\mathcal{D}} = \{\mathbf{z}, \mathcal{D}\mathbf{z} \in \mathcal{L}_2(c, d; r); B_1 y = B_2 y = 0\}$$

Theorem 24 *The Sturm-Liouville operator is self-adjoint if and only if*

$$m(c)\begin{vmatrix} \beta_{13} & \beta_{14} \\ \beta_{23} & \beta_{24} \end{vmatrix} = m(d)\begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix}$$

Proof. Since $D = D'$, self-adjointness requires that $B_1 u = B_1 v$, $i = 1, 2$. This requirement can be summarized by the matrix equation,

$$\begin{pmatrix} B_1 u & B_1 v \\ B_2 u & B_2 v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1}$$

which can be rearranged to give

$$\begin{pmatrix} \beta_{11}u(c) + \beta_{12}u'(c) & \beta_{11}v(c) + \beta_{12}v'(c) \\ \beta_{21}u(c) + \beta_{22}u'(c) & \beta_{21}v(c) + \beta_{22}v'(c) \end{pmatrix} = - \begin{pmatrix} \beta_{13}u(d) + \beta_{14}u'(d) & \beta_{13}v(d) + \beta_{14}v'(d) \\ \beta_{23}u(d) + \beta_{24}u'(d) & \beta_{23}v(d) + \beta_{24}v'(d) \end{pmatrix}. \tag{2}$$

Taking the determinant of each side of Eq. (2) and using the elementary properties of determinants, we find that the expression

$$\begin{vmatrix} \beta_{11} & \beta_{12} & u(c) & v(c) \\ \beta_{21} & \beta_{22} & u'(c) & v'(c) \end{vmatrix} = \begin{vmatrix} \beta_{13} & \beta_{14} & u(d) & v(d) \\ \beta_{23} & \beta_{24} & u'(d) & v'(d) \end{vmatrix} \tag{3}$$

is the condition that the boundary conditions of D and D' are the same. However, by the definition of D', in Eq. (3), u and v satisfy

$$m(c) \begin{pmatrix} v(c) & v'(c) \end{pmatrix} = m(d) \begin{pmatrix} v(d) & v'(d) \end{pmatrix}. \tag{4}$$

There are two possibilities to consider here. The first is that the determinants $\beta_{11}\beta_{22} - \beta_{21}\beta_{12}$ and $\beta_{13}\beta_{24} - \beta_{23}\beta_{14}$ are not zero. The other possibility is that the determinants $\beta_{11}\beta_{22} - \beta_{21}\beta_{12}$ and $\beta_{13}\beta_{24} - \beta_{23}\beta_{14}$ are both 0. If we have one determinant is zero and the other is not then it will not produce a self adjoint operator. Since the rank of the matrix $[\beta_{ij}]$ (for $i = 1, 2; j = 1, \ldots, 4$) is 2, at least one of the determinants

$$R_1 = \begin{vmatrix} \beta_{11} & \beta_{14} \\ \beta_{21} & \beta_{24} \end{vmatrix}, \quad R_2 = \begin{vmatrix} \beta_{12} & \beta_{14} \\ \beta_{22} & \beta_{24} \end{vmatrix}, \tag{5}$$

$$R_3 = \begin{vmatrix} \beta_{11} & \beta_{13} \\ \beta_{21} & \beta_{23} \end{vmatrix}, \quad R_4 = \begin{vmatrix} \beta_{12} & \beta_{13} \\ \beta_{22} & \beta_{23} \end{vmatrix}$$

is nonzero since

$$\begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{vmatrix} \neq 0 \tag{6}$$

for this case. From the linear combinations $\beta_{21}B_1 u - \beta_{11}B_2 u = 0$, $\beta_{23}B_1 u - \beta_{13}B_2 u = 0$, and the relation $\beta_{13}\beta_{24} - \beta_{23}\beta_{14} = 0$, we obtain

$$R_1 u(c) + R_2 u'(c) = 0 R_3 u(c) + R_4 u'(c) = 0 \tag{7}$$

Also,

$$R_1 v(c) + R_2 v'(c) = 0 R_3 v(c) + R_4 v'(c) = 0 \tag{8}$$

Consider $R_1, \ldots, R_4$ to be unknowns of the four equations in Eqs. (7) and (8). At least one of the R_i, $i = 1, \ldots, 4$, is nonzero, hence,

$$\begin{vmatrix} v(c) & v'(c) & 0 & 0 \\ v(c) & v'(c) & 0 & 0 \\ 0 & 0 & v(c) & v'(c) \\ 0 & 0 & v(c) & v'(c) \end{vmatrix} \tag{9}$$

is 0,

$$\begin{vmatrix} v(c) & v'(c) \\ v(c) & v'(c) \end{vmatrix}^2 = 0. \tag{10}$$

Theorem 25 *For a regular Sturm-Liouville operator* $\mathbf{D}$ *in* $\mathcal{L}_2(c, d; s)$*, the operator is bounded below if the bc are one of the following:*

$$u(c) = u(d) = 0$$

$$u(c) = \beta_{11} u(c) \quad \textit{and} \quad u'(d) = -\beta_{21} u(d)$$

where β_{11} *and* β_{21} *are any real numbers.*

Theorem 26 *Consider*

$$\mathcal{D}v = -\frac{1}{r(y)} \frac{d}{dy} \left(m(y) \frac{dv}{dy} \right) + \frac{n(y)}{r(y)} v$$

where $m(y), m'(y), n(y)$*, and* $r(y)$*, real-valued continuous functions on* $[a, b]$*;* $m(y)$ *and* $r(y)$ *being positive on* $[a, b]$*. The boundary conditions are given by*

$$C_1 v \equiv \gamma_{11} v(a) + \gamma_{12} v'(a) = 0$$
$$C_2 v \equiv \gamma_{21} v(b) + \gamma_{22} v'(b) = 0$$

where the coefficients γ_{ij} *are real and satisfy* $\gamma_{11}^2 + \gamma_{12}^2 \neq 0$ *and* $\gamma_{21}^2 + \gamma_{22}^2 \neq 0$*. Under these conditions,* $\mathcal{D}$ *is self-adjoint, and its eigenvalues can be ordered as follows:*

$$\mu_0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots$$

with

$$|\mu_0| < \infty \quad \textit{and} \quad \lim_{n \to \infty} \mu_n = \infty$$

4.6.2 Singular Sturm-Liouville Operators

Now we have already seen Regular Sturm-Liouville Operators. What is the difference between the two? An operator is termed singular if either $|c|$ or $|d|$ on the interval $[c, d]$ is infinite, if $u(x)$ or $v(x)$ has a zero in $[c, d]$, or if $p(x)$ becomes infinite at some point within $[c, d]$. The criterion for self-adjointness matches that of a regular operator, specifically, $\mathcal{D} = \mathcal{D}^*$ and $\mathcal{D}l_{\mathcal{D}} = \mathcal{D}l_{\mathcal{D}^*}$. The domain $\mathcal{D}l_{\mathcal{D}}$ is defined as those vectors w in $\mathcal{H}$ such that $\mathcal{D}^* w \in \mathcal{H}$ and

$$(v, \mathcal{D}u) = (\mathcal{D}^* v, u) \quad \text{for all } u \in \mathcal{D}l_{\mathcal{D}}. \tag{1}$$

Given below are two important examples:
(i) The Legendre equation:

$$\mathcal{D}_1 y = -\frac{d}{dx}\left((1 - x^2)\frac{dy}{dx}\right) = \lambda y, \quad -1 < x < 1. \tag{2}$$

(ii) The Hermite equation:

$$\mathcal{D}_2 y = \frac{d^2 y}{dx^2} + x^2 y = \lambda y, \quad -\infty < x < \infty. \tag{3}$$

The Legendre equation appears in the quantum mechanical analysis of spherically symmetric potentials and angular momentum. The Hermite equation is relevant on the study of energy levels in quantum mechanics. Here, y denotes the wave function and λ represents the energy in appropriate units. The domain of the operator in Eq. (2) is

$$\mathcal{D}l_{\mathcal{D}_1} = \{y, \mathcal{D}_1 y \in L_2(-1, 1),\, y(-1) = y(1),\, y'(-1) = y'(1)\}, \tag{4}$$

and the domain of the operator in Eq. (3) is

$$\mathcal{D}l_{\mathcal{D}_2} = \{y, \mathcal{D}_2 y \in L_2(-\infty, \infty)\}. \tag{5}$$

It can be readily shown that $\mathcal{D}_1 = \mathcal{D}_1^*$ and $\mathcal{D}l_{\mathcal{L}_1} = \mathcal{D}l_{\mathcal{L}_1^*}$, $\mathcal{D}_2 = \mathcal{D}_2^*$, $\mathcal{D}l_{\mathcal{L}_2} = \mathcal{D}l_{\mathcal{L}_2^*}$. Consequently, $\mathcal{D}_1$ and $\mathcal{D}_2$ are self-adjoint. Note, $\mathcal{D}_1$ is singular because the function $u(y) = 1 - y^2$ is zero at $y = -1$ and $y = 1$. Similarly, $\mathcal{D}_2$ is singular since it is defined over the infinite interval $(-\infty, \infty)$.

The eigenfunctions of Eq. (2) are the Legendre polynomials

$$P_k(y) = \frac{1}{2^k k!}\frac{d^k}{dy^k}\left(y^2 - 1\right)^k, \quad k = 0, 1, 2, \ldots, \tag{6}$$

with corresponding eigenvalues $\lambda_k = k(k + 1)$. The eigenfunctions of Eq. (3) are the Hermite functions

$$v_n(y) = H_n(y) \exp\left(-\frac{y^2}{2}\right), \quad n = 0, 1, 2, \ldots, \tag{7}$$

where H_n is the Hermite polynomial defined by

$$H_n(y) = (-1)^n \exp(y^2) \frac{d^n}{dy^n} \left[\exp(-y^2)\right]. \tag{8}$$

The eigenvalue corresponding to v_n is $\mu_n = 2n + \frac{1}{2}$.

Note that the Legendre polynomials form a complete orthogonal set in $L_2(-1, 1)$. While the Hermite functions form a complete orthogonal set in $L_2(-\infty, \infty)$. $P_k(y)$ has exactly k zeros in the interval $(-1, 1)$. While v_n has exactly n zeros in the interval $(-\infty, \infty)$. Moreover, the eigenvalues corresponding to P_k and v_n are ordered in the sequence

$$-\infty < \mu_0 < \mu_1 < \mu_2 < \cdots. \tag{9}$$

Thus, even though the Legendre and Hermite operators are singular self-adjoint Sturm-Liouville operators, they have a complete set of orthogonal eigenvectors, and their eigenfunctions and eigenvalues adhere to the theorems established for regular, self-adjoint Sturm-Liouville operators.These operators are frequently encountered in quantum mechanics, where they result in discrete spectra.

Theorem 27 *If $\mathcal{D}$ is a linear self-adjoint operator within a certain domain $\mathcal{D}_{l_\mathcal{D}}$ in a Hilbert space, the following holds (for I):*

$$j(z, w) = \sum_{m=1}^{m_v} \psi_m(z)\psi_m^*(w) + \int_{p_a}^{p_b} v_p(z)v_p^*(w)\, dp \tag{19.1}$$

where ψ_m are eigenfunctions of $\mathcal{D}$, and v_p are functions that satisfy $\mathcal{D}v_p = \mu_p v_p$. m_v, $|p_m|$, and $|p_a|$ can be finite or infinite.

Equation (19.1), through the relation $g(\mathcal{D}) = g(\mathcal{D})\mathcal{I}$, gives:

$$g(z, w) = \sum_{p=1}^{p_v} g(\mu_p)\psi_p(z)\psi_p^*(w) + \int_{r_a}^{r_b} g(\mu_r)v_r(z)v_r^*(w)\, dr \tag{19.2}$$

if $g(t)$ exists for $t = \mu_p$, and μ_p values. Here, $g(z, w)$ is the spectral decomposition of $g(\mathcal{D})$. In the weighted Hilbert space $\mathcal{L}_2(a, b; t)$ the inner product is $\langle \mathbf{u}, \mathbf{v} \rangle = \int_a^b u^*(x)v(x)t(x)\, dx$, and so

$$\mathcal{I}v = \int_a^b j(z,y)s(y)v(y)\,dy \tag{19.3}$$

and

$$g(\mathcal{D})v = \int_a^b g(z,y)s(y)v(y)\,dy. \tag{19.4}$$

Chapter 5

Applications

5.1 Differential Operators

5.1.1 Applications in Physics

Quantum Mechanics

A particle's wavefunction in a one-dimensional box is described by the equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dz^2} = \mathcal{E}\phi, \quad 0 < z < L,$$

where $\hbar$: Planck's constant divided by 2π.
m: the particle's mass.
L: the length of the box.
$\mathcal{E}$: the particle's energy.

This is the type of one-dimensional problem, $Sv = \mu v$, where S is the self-adjoint Sturm-Liouville operator, specifically the **Schrödinger** operator, that one often comes across in quantum mechanics and finding their eigenvalues is crucial as they represent the allowed energy levels of a quantum system.
The boundary conditions $\phi(0) = \phi(L) = 0$ represent the confinement of the particle within the box.
Quantum mechanics says that the energy of a confined particle can only attain discrete values. This occurs because of the eigenvalue of Sturm-Liouville operator.By introducing the notation

$$\mu = \frac{2m\mathcal{E}L^2}{\hbar^2} \quad \text{and} \quad \eta = \frac{z}{L},$$

the equation becomes

$$-\frac{d^2\phi}{d\eta^2} = \mu\phi, \quad 0 < \eta < 1, \quad \phi(0) = \phi(1) = 0.$$

The solution for this equation is given by $v_1 = \sin(\sqrt{\mu}\eta)$ and $v_2 = \cos(\sqrt{\mu}\eta)$. Taking $\phi = k_1 v_1 + k_2 v_2$ and applying the boundary conditions $\phi(0) = \phi(1) = 0$, we determine that $k_2 = 0$ and the condition

$$\sin(\sqrt{\mu}) = 0$$

This implies that $\sqrt{\mu} = n\pi$ for $n = 1, 2, 3, \ldots$, and hence,

$$\mu = (n\pi)^2.$$

We replace η by z/L and determine k_1 from the normalization criterion, $\int_0^1 \phi_j^2\, dz = 1$, hence we find the wavefunctions as,

$$\phi_j = \sqrt{\frac{2}{L}} \sin\frac{\pi j z}{L}, \quad j = 1, 2, \ldots,$$

The probability amplitude for finding the particle at a given point is represented by these wavefunctions ϕ.
and the particle energies as,

$$\mathcal{E}_j = \frac{\hbar^2}{2m}\frac{\pi^2 j^2}{L^2}, \quad j = 1, 2, \ldots$$

Heat and Mass Transfer

In classical physics, like in the study of heat and mass transfer, we often deal with equations such as:

$$\frac{\partial v}{\partial \tau} = -\kappa \mathcal{M} v, \quad \mathcal{B}_1 v = \mathcal{B}_2 v = 0, \quad \alpha < \xi < \beta,$$

where $v(\xi, \tau)$ is a function of spatial and temporal variable respectively.We consider initial conditions such as $v(\xi, 0) = g(\xi)$. Here, κ is a positive transport coefficient and the operator $\mathcal{M}v = -\frac{d^2 v}{d\xi^2}$. The boundary conditions are such that $\mathcal{M}$ is self-adjoint. Let us solve this equation:

$$\frac{\partial v}{\partial \tau} = \kappa \frac{\partial^2 v}{\partial \xi^2}.$$

with BC:

$$v(\alpha, \tau) = 0, \quad v(\beta, \tau) = 0.$$

and initial condition:

$$v(\xi, 0) = g(\xi).$$

Solving,

$$X(\xi)\frac{dT}{d\tau} = \kappa T(\tau)\frac{d^2X}{d\xi^2}.$$

Dividing both sides by $\kappa X(\xi)T(\tau)$,

$$\frac{1}{\kappa T(\tau)}\frac{dT}{d\tau} = \frac{1}{X(\xi)}\frac{d^2X}{d\xi^2} = -\lambda,$$

where λ is a separation constant. We get two ordinary differential equations (ODEs) as follows:

1. Time-dependent ODE:

$$\frac{dT}{d\tau} + \kappa\lambda T = 0$$

2. Space-dependent ODE:

$$\frac{d^2X}{d\xi^2} + \lambda X = 0$$

We solve the spatial ODE:

$$\frac{d^2X}{d\xi^2} + \lambda X = 0.$$

For $\lambda > 0$:

$$X(\xi) = A\cos(\sqrt{\lambda}\xi) + B\sin(\sqrt{\lambda}\xi).$$

Using BC, $X(\alpha) = 0$ and $X(\beta) = 0$,

$$A\cos(\sqrt{\lambda}\alpha) + B\sin(\sqrt{\lambda}\alpha) = 0,$$
$$A\cos(\sqrt{\lambda}\beta) + B\sin(\sqrt{\lambda}\beta) = 0.$$

since,

$$\sqrt{\lambda} = \frac{n\pi}{\beta - \alpha}, \quad n = 1, 2, 3, \ldots$$

Thus, the eigenvalues are,

$$\lambda_n = \left(\frac{n\pi}{\beta - \alpha}\right)^2,$$

and the corresponding eigenfunctions are,

$$X_n(\xi) = \sin\left(\frac{n\pi(\xi - \alpha)}{\beta - \alpha}\right).$$

Now, we solve the temporal ODE The time-dependent ODE is:

$$\frac{dT}{d\tau} + \kappa\lambda T = 0$$

For each λ_n, we get:

$$\frac{dT_n}{d\tau} + \kappa \left(\frac{n\pi}{\beta - \alpha}\right)^2 T_n = 0.$$

This is a first-order linear ODE with the following solution,

$$T_n(\tau) = C_n \exp\left(-\kappa \left(\frac{n\pi}{\beta - \alpha}\right)^2 \tau\right)$$

Hence, combining the spatial and temporal solutions, we get,

$$v(\xi, \tau) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi(\xi - \alpha)}{\beta - \alpha}\right) \exp\left(-\kappa \left(\frac{n\pi}{\beta - \alpha}\right)^2 \tau\right).$$

Now, to find C_n, we use the initial condition $v(\xi, 0) = g(\xi)$:

$$g(\xi) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi(\xi - \alpha)}{\beta - \alpha}\right)$$

This is a Fourier sine series expansion of $g(\xi)$. Hence, the coefficients C_n are:

$$C_n = \frac{2}{\beta - \alpha} \int_{\alpha}^{\beta} g(\xi) \sin\left(\frac{n\pi(\xi - \alpha)}{\beta - \alpha}\right) d\xi.$$

Substituting C_n, we get the final solution as follows,

$$v(\xi, \tau) = \sum_{n=1}^{\infty} \left(\frac{2}{\beta - \alpha} \int_{\alpha}^{\beta} g(\xi) \sin\left(\frac{n\pi(\xi - \alpha)}{\beta - \alpha}\right) d\xi\right) \sin\left(\frac{n\pi(\xi - \alpha)}{\beta - \alpha}\right) \exp\left(-\kappa \left(\frac{n\pi}{\beta - \alpha}\right)^2 \tau\right).$$

i.e.,

$$v(\xi) = \sum_{n} \exp\left(-\tau\kappa\lambda_n\right) \psi_n(\xi) \int_{\alpha}^{\beta} \psi_n^*(\xi) g(\xi) d\xi,$$

where ψ_n and $\lambda_n (n = 1, 2, \ldots)$ represent the eigenfunctions and eigenvalues of the operator $\mathcal{M}$.
This solution represents the temperature distribution $v(\xi, \tau)$ over time in the given spatial domain.

5.1.2 Applications in Engineering

Fluid Dynamics

Navier-Stokes equation

In fluid dynamics, Navier-Stokes equation is a pde that describes the flow of incompressible fluids. With some initial conditions, it is used to determine the velocity vector applied to the fluid. In real situations, they form a system of non-linear pde but however under certain simplifications, they can be reduced to linear differential equations. Let us see how:
Consider the following Navier-Stokes equation, ignoring external forces and assuming constant viscosity, in its simplest form:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \tag{5.1}$$

where, ρ: density.
$\mathbf{u}$: velocity field.
p: pressure.
ν: kinematic viscosity.
Now we introduce a small perturbation $\mathbf{u}'$ such that $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$. We substitute this in above equation and linearize as follows:

$$\begin{aligned} \frac{\partial \mathbf{u}'}{\partial t} &= -\left(\mathbf{u}_0 \cdot \nabla\right)\mathbf{u}' - \left(\mathbf{u}' \cdot \nabla\right)\mathbf{u}_0 - \frac{1}{\rho}\nabla p' + \nu \nabla^2 \mathbf{u}' \\ \nabla \cdot \mathbf{u}' &= 0 \end{aligned}$$

For simplicity, we ignore the nonlinear term $\left(\mathbf{u}' \cdot \nabla\right)\mathbf{u}_0$ and focus on the Laplacian term $\nu \nabla^2 \mathbf{u}'$. Now we use spectral decomposition, for solving this linearized equation. For a bounded domain, the solution $\mathbf{u}'$ can be expressed as a series of eigenfunctions ϕ_n of the Laplacian as follows:

$$\mathbf{u}'(x,t) = \sum_n a_n(t)\phi_n(x)$$

Substituting $\mathbf{u}'$ into the linearized Navier-Stokes equation, for each term of the series we get:

$$\frac{\mathrm{d}a_n}{\mathrm{d}t}\phi_n = \nu \lambda_n a_n \phi_n$$

where λ_n are the eigenvalues associated with ϕ_n. Hence, this reduces the problem to solving a set of ordinary differential equations for $a_n(t)$:

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = \nu \lambda_n a_n$$

The analysis of fluid flow dynamics becomes easier by the spectral decomposition, as it reduces complicated partial differential equations into solvable ordinary differential equations.In applications ranging from aerospace to maritime engineering, engineers use this technology to forecast flow behaviour under small perturbations, optimise systems, and create more stable configurations.

5.1.3 Applications in Medicine

Diffusion Equation

In the field of medicine, the diffusion equation, is used to describe the rate at which a drug spreads through a biological medium, such as tissue or blood:

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial z^2},$$

where k represents the diffusion coefficient.
This can be reformulated as:

$$\frac{\partial \mathbf{v}}{\partial t} = -k\mathbf{M}\mathbf{v},$$

where $M\varphi = -\frac{d^2\varphi}{dz^2}$.
The initial distribution of the substance (at $t = 0$) is given by:

$$\varphi(z, 0) = \varphi_0(z) \tag{5.2}$$

with boundary conditions:

$$\frac{\partial \varphi}{\partial z}(0, t) = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial z}(A, t) = 0 \tag{5.3}$$

This describes a self-adjoint operation.
Thus, the solution can be expressed as:

$$\mathbf{v} = \exp(-tk\mathbf{M})\mathbf{v}_0 = \sum_i \exp(-tk\mu_i)(\psi_i, \mathbf{v}_0\rangle\psi_i,$$

where μ_i and ψ_i satisfy the eigenrelation $\mathbf{M}\psi_i = \mu_i\psi_i$. Applying the boundary conditions to $\varphi(z) = a_1 \sin\sqrt{\mu}z + a_2\cos\sqrt{\mu}z$, we find $a_1 = 0$ and the eigenvalues must satisfy

$$\sqrt{\mu}\sin(\sqrt{\mu}A) = 0,$$

resulting in $\mu_0 = 0$ and $\sqrt{\mu_i}A = i\pi, i = 1, 2, \ldots$. Hence, the eigenfunctions are

$$\psi_0 = \frac{1}{\sqrt{A}},$$
$$\psi_i = \sqrt{\frac{2}{A}} \cos \frac{\pi i z}{A}, \quad i = 1, 2, \ldots,$$

and the concentration $v(z,t)$ is given by

$$v(z,t) = \frac{1}{A}\int_0^A v_0(z)dz + \sum_{i=1}^{\infty} \exp\left(\frac{-tk\pi^2 i^2}{A^2}\right) \langle \psi_i, \mathbf{v}_0 \rangle \sqrt{\frac{2}{A}} \cos \frac{\pi i z}{A}$$

with,

$$\langle \psi_i, \mathbf{v}_0 \rangle = \sqrt{\frac{2}{A}} \int_0^A v_0(z) \cos \frac{\pi i z}{A} dz.$$

where $\overline{v} = A^{-1}\int_0^A v_0(z)dz$, is the mean value of the initial concentration.
An understanding of drug diffusion through tissues is crucial in optimising drug delivery specially for treatments like chemotherapy, where the effectiveness of the treatment largely depends on the drug reaching all malignant cells in appropriate quantities without overloading healthy tissues.

5.2 Integral Operators

Integral operators that exhibit linear integration are those in which the function is integrated in a linear manner against a kernel function. Because of their adaptability and variety of uses, these operators are essential in many different fields.
The following are some important applications of linear integral operators:

5.2.1 Applications in Mathematical Physics

In Quantum Mechanics: Schrödinger's equation is solved with integral operators; integral kernels are frequently needed to calculate the potential function. In quantum mechanics, integral operators play a role in formulating quantum states and observables.
Schrödinger Equation: The time-independent Schrödinger equation can be formulated in terms of integral operators.
For example, the Green's function approach involves solving-

$$\psi(u) = \int g(u,v)V(v)\psi(v)dv$$

where $g(u, v)$: Green's function, $V(v)$: potential, and $\psi(u)$: wave function.
In Electromagnetics: Electromagnetic scattering problems often involve solving integral equations derived from Maxwell's equations. For instance, the boundary integral equation for scattering by a perfectly conducting object can be expressed using the following:
Electric Field Integral Equation (EFIE):

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\text{inc}}(\mathbf{r}) - \int_S \left[\mathbf{L}\left(\mathbf{r}, \mathbf{r}'\right) \cdot \mathbf{K}\left(\mathbf{r}'\right)\right] dS'$$

where $\mathbf{E}(\mathbf{r})$ is the total electric field, $\mathbf{E}^{\text{inc}}\left(\mathbf{r}\right)$ is the incident electric field, $\mathbf{G}\left(\mathbf{r}, \mathbf{r}'\right)$ is the dyadic Green's function, $\mathbf{J}\left(\mathbf{r}'\right)$ is the surface current density, and the integration is performed over the surface S of the scatterer.
Magnetic Field Integral Equation (MFIE):

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}^{\text{inc}}(\mathbf{r}) + \frac{1}{4\pi} \int_S \left[\mathbf{K}\left(\mathbf{r}'\right) \times \nabla' L\left(\mathbf{r}, \mathbf{r}'\right)\right] dS'$$

where
$\mathbf{H}(\mathbf{r})$: total magnetic field,
$\mathbf{H}^{\text{inc}}(\mathbf{r})$: incident magnetic field,
$G\left(\mathbf{r}, \mathbf{r}'\right)$: scalar Green's function, and
∇' denotes gradient with respect to $\mathbf{r}'$.

5.2.2 Applications in Engineering

Heat Transfer: While addressing heat conduction issues, particularly in non-homogeneous materials, integral operators are applied.
Formulation of Integral Equations for Heat Conduction:
In one-dimensional heat conduction problems, the temperature distribution $t(x, t')$ in a medium described using integral equations derived from the heat equation.
The temperature distribution might be expressed as:

$$t(x, t') = \int_0^t \int_{-\infty}^{\infty} g(x - \sigma, t - \alpha) q(\sigma, \alpha) d\sigma d\alpha$$

where
$g(x - \sigma, t - \alpha)$: Green's function for the heat conduction problem.
$q(\xi, \tau)$: heat source distribution.
Structural Analysis : Integral equations can be used to formulate the

study of stresses and strains in structures, especially in the setting of elasticity theory.

Viscoelastic Stress Analysis

For certain polymers and plastics, traditional Newtonian models of stress and strain fail to capture the time-dependent nature of their responses. These materials often exhibit a viscoelastic behavior, where the stress response is not instantaneous but evolves over time. Boltzmann's viscoelastic model provides a framework for describing this behavior.

In isotropic materials, where the response to deformation is uniform in all directions, the modulus E (analogous to the elastic modulus in purely elastic materials) is the ratio of stress μ to strain β:

$$E = \frac{\mu}{\beta}$$

For perfectly elastic materials, this relationship holds instantaneously. However, viscoelastic materials require a more generalized form to account for time-dependent effects:

$$E(\beta, t) = \frac{\mu(t)}{\beta(t)}$$

This formulation introduces the concept of a relaxation modulus E, which decays over time under a constant strain. This decay can be attributed to the internal molecular rearrangements within the material, leading to a more complex stress-strain relationship.

Memory Functions and Stress Response: To model the time-dependent behavior of viscoelastic materials, we introduce the concept of a memory function $N(t)$, which characterizes how past strains influence the current stress state. The relationship between stress and strain can then be described using an integral operator:

$$\sigma(t) = -\int_{-\infty}^{t} N\left(\epsilon, s - s'\right) \epsilon\left(t'\right) ds'$$

Here, the memory function $N(\epsilon, s)$ is related to the time derivative of the relaxation modulus:

$$N(\epsilon, t) = -\frac{dE(\epsilon, t)}{dt}$$

This integral formulation highlights how the entire history of strain impacts the current stress, encapsulating the viscoelastic nature of the material.

Modeling Relaxation Behavior: One practical approach to model the

relaxation behavior of viscoelastic materials is to consider a series of independent relaxation modes. Each mode contributes to the overall relaxation modulus:

$$E(t) = \sum_{j=1}^{m} E_j \exp\left(\frac{-t}{\beta_j}\right)$$

In this representation, each term E_j and β_j corresponds to a specific mode, with E_j representing the nominal modulus and β_j the relaxation time of the j-th mode. This sum of exponential functions allows for a comprehensive description of the relaxation behavior over time. The memory function $N(t-t')$, integral to the viscoelastic model, can be expressed in terms of these modes, providing a compact operator within, $\mathcal{L}_2(-\infty, c)$, here$c$ is a large finite time. This mathematical framework enables precise modeling and prediction of the viscoelastic response under various loading conditions, essential for structural analysis.

5.2.3 Applications in Signal Processing

In signal processing, integral transforms such as the Fourier and Laplace transforms, which can be viewed as integral operators, are used for analyzing signals.

Fourier Transform: The Fourier transform is an integral operator given by

$$V(g) = \int_{-\infty}^{\infty} v(t) e^{-j2\pi ft} dt$$

$v(t)$: original time-domain signal.
j : imaginary unit.
$V(g)$: Fourier transform of the signal $x(t)$.
g : frequency variable.
The Fourier transform decomposes a signal into its constituent frequencies.

Convolution: Convolution operations, essential in filtering signals, can be expressed using integral operators:

$$i(y) = \int_{-\infty}^{\infty} g(t) h(y-t) dt$$

$i(y)$: output (filtered signal)
$g(y)$: input signal
$h(y)$: impulse response of the filter.
The filter modifies the input signal $g(y)$ based on the characteristics of the filter described by $h(y)$.

5.2.4 Applications in Economics and Finance

Economic Growth Models- Integral operators can describe the accumulation of capital or other resources in growth models, helping to determine optimal investment strategies.
The capital accumulation $K(t)$ over time can be modeled as:

$$K(t) = K(0) + \int_0^t (I(s) - \delta K(s))ds$$

where: $K(0)$: initial capital
$I(s)$: investment at time s
δ : depreciation rate

Aggregation of Demand- Integral equations are used to describe equilibrium states in economics. As in consumer theory, integral operators can be used to model aggregate demand.
If $D_i(p)$ is the demand function of the i-th individual at price p, the aggregate demand $D(p)$ can be written as:

$$D(p) = \int_0^1 D_i(p)di$$

Present Value of Future Cash Flows- Integral operators are used to calculate the present value of a continuous stream of future cash flows, which is crucial for valuing investments, projects, or any financial assets.
The present value PV of a continuous cash flow $C(t)$ over time t from 0 to T is given by:

$$PV = \int_0^T C(t)e^{-rt}dt$$

where- $C(t)$: cash flow at time t.
r : discount rate.

Dynamic Optimization Problems- Many economic models involve optimizing an objective function over time, such as maximizing utility or minimizing cost. These problems can often be expressed and solved using integral operators.
In a consumption-savings model, the objective might be to maximize the total utility U over a lifetime T :

$$\max \int_0^T u(c(t))e^{-\rho t}dt$$

$u(c(t))$: utility of consumption $c(t)$.
ρ : subjective discount rate.

Chapter 6

Conclusion

This dissertation has explored the important spectral characteristics of integral and differential operators in the context of infinite-dimensional vector spaces, with a particularly on Hilbert spaces. The fundamental significance of the spectral theorem on the analysis of these operators was discussed.
Further, we have discussed the spectrum properties of linear differential operators. We examined their crucial function in solving differential equations in Hilbert spaces, delving deeply into specific operators like Sturm-Liouville operators. This allowed to shed a light on the intricate relationship between the spectral features of the operators and their more general mathematical and physical implications.
Although the results presented in this paper are well-established, we have endeavored to present them with our own style in our proofs. Additionally, we have explored the connections of these results to real-life applications, particularly in engineering and science. We hope this paper offers a deeper understanding and an engaging introduction to the rich theory of spectral properties of operators and their practical implications.

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