A Study on Generalizations of UN and VNL Rings and Group Rings

A Thesis
Submitted for the award of degree of
Doctor of Philosophy
in Mathematics
by

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Dedicated to

My Parents

Certificate

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This is to certify that the research work embodied in the thesis entitled “A Study on Generalizations of UN and VNL Rings and Group Rings” submitted by Kanchan Jangra (2K18/Ph.D/AM/502) to the Department of Applied Mathematics, Delhi Technological University, Delhi, for the award of **Doctor of Philosophy**, is a record of bonafide research work carried out by her under supervision and guidance of **Dr. Dinesh Udar**.

In my knowledge, the work reported in this thesis has not been submitted in any form to any other institution or university for the award of a degree or diploma.

Date: July, 2024
Place: Delhi , India

Dr. Dinesh Udar
(Supervisor)

Prof. R. Srivastava
(Head of Department)

Declaration

I declare that the research work in this thesis entitled "**A Study on Generalizations of UN and VNL Rings and Group Rings**" for the award of the degree of ***Doctor of Philosophy in Mathematics*** has been carried out by me under the supervision of ***Dr. Dinesh Udar***, Department of Applied Mathematics, Delhi Technological University, Delhi, India, and has not been submitted by me earlier in part or full to any other university or institute for the award of any degree or diploma.

It is declared that if I have included other people's ideas or words, I have adequately cited and referred to the original sources. This is my own thesis and I have written it in my own way. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission.

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Abstract

The main aim of this thesis is to study generalizations of UN and VNL rings, as well as group rings. As an introduction to this thesis, the introductory chapter collects literature and definitions relevant to each concept that are used throughout this thesis.

Călugăreanu in [7] introduced and investigated UN rings. A ring R is called *UN ring* if every non unit of it can be written as product of a unit and a nilpotent element. We carry his study of UN rings further. We have focused on UN group rings. In our study of UN group rings, necessary and sufficient conditions for group ring RG to be UN have been obtained.

We have studied a generalization of the class of UN rings, called UQ rings. A ring R is called *UQ ring* if every non unit element of R can be represented as a product of a unit and a quasiregular element. Various properties of these rings along with its characterizations are obtained and examples are provided to show that the class of UQ rings properly contains classes of UN rings, J-UN rings and 2-good rings. In study of UQ group rings, necessary and sufficient conditions for commutative group ring RG to be UQ have been established.

An element a of R is called SWR if $a \in aRa^2R$. A ring R is called an *almost SWR* if for any $a \in R$, either a or $1 - a$ is SWR. The class of almost SWR rings properly contains the classes of SWR and abelian VNL rings. Various properties of almost SWR are obtained. We provide characterizations of almost SWR rings. Further, we study SWR group rings and almost SWR group rings.

If a ring, R, satisfies the condition that its every proper homomorphic image has a certain property P, then the ring R is called restricted P ring. This has motivated us to introduce and investigate a new class of rings called *semiboolean neat rings*. The ring R is semiboolean neat provided that every proper homomorphic image of R is semiboolean. The class of semiboolean neat rings lies strictly between the classes of nil neat and neat rings. We obtain characterizations of semiboolean neat rings. Moreover, commutative semiboolean neat group rings have also been studied.

A ring R is said to be *weakly $g(x)$-invo clean* if each element of R is either a sum or difference of an involution and a root of $g(x)$. This class is a proper subclass of weakly $g(x)$-clean rings and a generalization of $g(x)$-invo clean rings. Various properties of weakly $g(x)$-invo clean rings are given. We determine necessary and sufficient conditions for skew Hurwitz series ring (HR, α) to be weakly $g(x)$-invo clean, where α is an endomorphism of R.

Finally, the last chapter summarizes the thesis with a brief conclusion and discusses some future prospects.

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List of Symbols

Set theory

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
P	Set of prime integer
$x \in X$	x is a member of X
$x \notin X$	x is not a member of X
$A \subset X$	A is a proper subset of X
$A \subseteq X$	A is a subset of X

Group theory

(X)	Subgroup generated by X
$\|G\|$	Order of the group G
C_n	Cyclic group of order n
$\|G : H\|$	Index of subgroup H in a group G
gH	Right coset of a subgroup H in G
G/H	Quotient group of G by its normal subgroup H

Ring theory

$\mathbb{Z}_n$	Ring of integers modulo n
$Id(R)$	Set of idempotent elements of R
$Inv(R)$	Set of involution elements of R
$N(R)$	Set of nilpotent elements of R
$Q(R)$	Set of quasiregular elements of R
$J(R)$	Jacobson radical of R
$P(R)$	Prime radical of R
$C(R)$	Center of R

eRe	Corner ring of R
$\bigoplus_{i\in I} R_i$	Direct sum
$\prod_{i\in I} R_i$	Direct product
$Aut(R)$	Group of automorphisms of R
$\mathbb{Z}_{(p)}$	Localization of the integers at prime ideal generated by p
$U(R)$	Group of units of the ring R
$M_n(R)$	Ring of all $n\times n$ matrix over a ring R
$T_n(R)$	Ring of all $n\times n$ upper triangular matrix over a ring R
$R[\mathrm{X}]$	Polynomial ring with coefficients from R
$R[[\mathrm{X}]]$	Formal power series ring in variable x of R
$Char(R)$	Characteristic of the ring R
$\ker(f)$	Kernel of the map f
$\overline{R}$	Factor ring $R/J(R)$
$End_R(M)$	Endomorphism ring of a module M
RG	Group ring of G over R
ωG	Augmentation ideal of RG

Chapter 1

Introduction

Group rings are very interesting algebraic structures. It is a meeting point of various algebraic theories. Group ring is denoted by RG. In the group ring RG, R can be regarded as a subring of RG and G a subgroup of $U(RG)$, the group of units of RG. Thus group rings is a generalization of both, rings and groups. Besides the obvious relationship with group theory and ring theory, the study of group rings involves the theory of fields, linear algebra and algebraic number theory. Group rings exhibits applications in algebraic coding theory. Thus, the theory of group rings provides a subject where many branches of algebra come to a rich interplay. The study of group rings for group representation and the structure of group algebras was started in 1930's. The subject gained importance of its own after the inclusion of questions on group rings in I. Kaplansky's famous list of problems in the theory of rings ([30; 31]). Other important facts to stimulate the area were the paper by I.G. Connell [17], the inclusion of chapters on group rings in the books on ring theory by Lambek [40] and Ribenboim [56], and the two self contained books by Passman ([53; 54]). Since then, many survey articles have appeared and many books on the subject have been published.

1.1 Chapter-wise overview of the thesis

In this section, we give a brief overview and organization of the thesis. In **chapter 2**, we discuss basic definitions, some known results and other preliminaries which are necessary for the development of the work done in the succeeding chapters.

In **chapter 3**, we study the structure of UN rings and group rings. We obtain certain properties of UN rings. We discuss lifting properties of UN rings modulo an ideal I. We discuss the question raised by Călugăreanu [7] that "is $M_n(R)$ over a UN ring R, also UN?". We obtain that if R is commutative, then $M_n(R)$ is UN if and only if R is UN.

Further, we investigate the structure of UN group rings. We first take up the case of group algebra KG of a group G over a field K. It is obtained that if $charK = 0$, then KG can be a UN ring if and only if G is trivial. If $charK = p$, then KG is a UN ring implies that the group G must be a p-group and the converse holds if G is locally finite. We investigate the structure of the group ring RG of a group G over an arbitrary ring R (which may not necessarily be a field) and obtain the result that if RG is a UN ring then R is a UN ring, G is a p-group and $p \in J(R)$; and the converse holds if G is locally finite.

In **chapter 4**, we introduce a new class of rings called *UQ* rings. Various properties of these rings are obtained and examples are given to show that the class of UQ rings properly contains the classes of UN rings, J-UN rings and 2-good rings. We obtain a new characterization of 2-good rings, and it turns out that UQ rings with 2-good identity are equivalent to 2-good rings. We discuss extensions of matrix rings. It is proved that the formal matrix ring $M_n(R; s)$ over R is a UQ ring with quasiregular identity if and only if R is a UQ ring with quasiregular identity and $s \in J(R)$. We determine necessary and sufficient conditions for a commutative group ring RG to be UQ. Let G be an abelian p-group with $p \in J(R)$ and R be a commutative ring. Then RG is a UQ ring if and only if R is UQ. We also characterize UQ group ring if ring R is artinian.

In **chapter 5**, we introduce a new class of rings called almost SWR rings. This class of rings is a generalization of SWR and abelian VNL rings. Various basic properties of these rings are obtained and examples are given to show that the class of almost SWR rings properly contains the classes of SWR, abelian VNL and weakly tripotent. It is proved that a ring R is almost SWR if and only if, for any SWR ideal I of R, R/I is almost SWR. We characterize abelian almost SWR rings. It is proved that if e is an idempotent in an abelian almost SWR ring R, then either eRe or $(1 - e)R(1 - e)$ is SWR, but the converse holds if R is an exchange ring. We prove that if RH is almost SWR for every finitely generated subgroup H of G, then RG is almost SWR, but the converse of this result partially holds. It is proved that if $G = H \rtimes K$ is a semidirect product of finite subgroup H by a subgroup K, then almost s-weakly regularity of RG implies almost s-weakly regularity of RK.

In **chapter 6**, we have investigated that, what happens if every proper homomorphic image of R is semiboolean. So in this chapter, we introduce the concept of semiboolean neat rings. Various properties of semiboolean rings have been investigated. It is proved that a semiboolean neat ring which is not semiboolean is reduced. We prove that matrix ring $M_n(A)$ is semiboolean neat if and only if A is a radical ring. We determine the necessary and sufficient conditions for a commutative group ring RG to be semiboolean neat.

In **chapter 7**, we introduce a new class of rings called weakly g(x)-invo clean rings. This class of rings is a proper subclass of weakly $g(x)$-clean rings and a generalization of $g(x)$-invo clean rings. We obtain various properties of weakly $g(x)$-invo clean rings. Let R be a commutative ring with $2N = 0$. Then it is proved that $R \odot N$ is weakly $g(x)$-invo clean if and only if R is weakly $g(x)$-invo clean. We characterize weakly invo-clean rings as weakly $g(x)$-invo clean rings where $g(x) = x(x-a), a \in C(R) \cap Inv(R)$. It is proved that the ring of skew Hurwitz series (HR, α) is weakly $g(x)$-invo clean ring if and only if R is weakly $g(x)$-invo clean ring. If we take identity endomorphism in skew Hurwitz series ring, then we obtain ring of Hurwitz series to be weakly $g(x)$-invo clean ring.

The last chapter, **chapter 8**, summarises the thesis and then offers some insight into the author's views regarding the future direction of our research.

Chapter 2

Preliminaries

Throughout the thesis, R will denote an associative ring with identity $1 \neq 0$, $J(R)$ the Jacobson radical of R and G a non-trivial group. Also all modules are unitary left R-modules unless otherwise indicated. In this chapter, we provide some prerequisite material which is required to understand the rest of the text. We use the notation and terminology of Lam's book [39], which we refer for noncommutative rings. And for group rings, we refer to Connell [17] and Passman [54].

2.1 Ring Theory

The **Jacobson Radical**, $J(R)$, of a ring R is the intersection of all maximal left (or right) ideals of R. It is in fact a two sided ideal of R. Some of the properties of the Jacobson radical are as follows.

- For $x \in R$, the following statements are equivalent:

  (i) $x \in J(R)$,

  (ii) $1 - xyz \in U(R)$, the group of units of R, for any $x, z \in R$.

- An element $x \in R$ is a unit of R if and only if $x + J$ is a unit of R/J.

- Let $f : R \to S$ be a surjective ring homomorphism, then

$$f(J(R)) \subseteq J(S),$$

  with equality if $ker f \subseteq J(R)$.

- Let I be an ideal of R and $I \subseteq J(R)$, then $J(R/I) = J(R)/I$. In particular, $J(R/J(R)) = 0$.

- For any direct product of rings $\prod R_i$,

$$J\Big(\prod R_i\Big) = \prod J(R_i).$$

- For any ring R and any $n \in \mathbb{N}$,

$$J(M_n(R)) = M_n(J(R)).$$

For proof of the above properties, an interested reader should see Lam [39]. We now mention different types of nilpotency of an ideal I of a ring R.

Definition 2.1.1. *An ideal I, of a ring R is called*

(1) ***nilpotent**, if $I^n = 0$ for some $n \in \mathbb{N}$.*

(2) ***locally nilpotent**, if every finitely generated subring of I is nilpotent.*

(3) ***nil**, if every element of I is nilpotent.*

The chain of containments is as follows:

$$\text{nilpotent} \subseteq \text{locally nilpotent} \subseteq \text{nil} \subseteq J(R).$$

Definition 2.1.2. *A ring R is said to be*

(1) ***left (right) artinian**, if every descending chain of left (right) ideals in R has a minimal element.*

(2) ***semiprimary**, if $R/J(R)$ is artinian and $J(R)$ is nilpotent.*

(3) ***semilocal**, if $R/J(R)$ is artinian.*

The chain of inclusions is as follows:

$$\text{left (right) artinian} \subseteq \text{semiprimary} \subseteq \text{semilocal}.$$

Definition 2.1.3. *An ideal I of a ring R is said to be*

(1) ***prime ideal**, if $I \neq R$ and for any ideals I_1, I_2 of R,*

$$I_1 I_2 \subseteq I \textit{ implies that either } I_1 \subseteq I \textit{ or } I_2 \subseteq I.$$

(2) ***semiprime ideal**, if for any ideal I_1 of R,*

$$I_1{}^2 \subseteq I \textit{ implies that } I_1 \subseteq I.$$

A prime ideal is always semiprime.

Definition 2.1.4. *A ring R is said to be*

(1) ***prime**, if $I_1I_2 = 0$ implies that either $I_1 = 0$ or $I_2 = 0$, for any ideals I_1 and I_2 of R. The above definition is equivalent to the definition of prime ring given by Lam as: A ring R is prime if the* (0) *ideal is prime (see Lam [39, § 10]).*

(2) ***semiprime**, if R has no non-zero nilpotent ideal, or equivalently, if* (0) *is a semiprime ideal (see Lam [39, § 10]).*

Every prime ring is semiprime. The **prime radical**, **P(R)**, of a ring R is the intersection of all prime ideals of R. An element $a \in R$ is called **strongly nilpotent**, if every sequence $a_1, a_2, a_3, ...$ such that $a_1 = a$ and $a_{i+1} \in a_iRa_i$ (for all i) is ultimately zero.

Proposition 2.1.5. *The prime radical, $P(R)$, of a ring R is precisely the set of all strongly nilpotent elements of R.*

Proof. See Lambek [40, Proposition 1, page 56]. □

So $P(R)$ is a nil ideal, and hence $P(R) \subseteq J(R)$.

2.2 UN Rings

An element $e \in R$ is said to be an **idempotent**, if $e^2 = e$. A ring R always has two trivial idempotents namely 0 and 1. A ring R is called **boolean**, if every element of it is an idempotent. Let I be an ideal of R, then we say that **idempotents lift modulo** I, if for an idempotent $\bar{r} \in R/I$ there exists an idempotent $e \in R$ such that $\bar{r} = \bar{e}$.

Proposition 2.2.1. *Let I be a nil ideal in R, then idempotents can be lifted modulo I.*

Proof. See Lam [39, Theorem 21.28] □

One of the active areas of research have been the rings whose elements can be written as a sum/product of units/ idempotents/ nilpotent elements. Nicholson [48] developed the idea of clean rings while studing the lifting of idempotents. An element $a \in R$ is called **clean**, if it can be represented as a sum of an idempotent and a unit. A ring R is said to be **clean ring**, if every element of it is clean. The homomorphic image of a clean ring is clean and direct product of clean rings is clean. If R is clean, then the matrix ring $M_n(R)$ is also clean (see [29, Corollary 1]). Some subclasses of clean rings are as follows.

Definition 2.2.2. *A ring R is said to be*

(1) ***uniquely clean****, if every element of it can be written uniquely as the sum of an idempotent and a unit.*

(2) ***strongly clean****, if every element of it can be written as a sum of an idempotent and a unit that commute.*

(3) ***semiboolean****, if $R/J(R)$ is boolean and idempotents lift modulo $J(R)$.*

(4) ***nil clean****, if its every element can be represented as a sum of an idempotent and a nilpotent element.*

(5) ***strongly nil clean****, if its every element can be represented as a sum of an idempotent and a nilpotent element that commute.*

$$\text{uniquely clean} \subseteq \text{semiboolean} \subseteq \text{clean}.$$

$$\text{strongly nil clean} \subseteq \text{nil clean} \subseteq \text{semiboolean} \subseteq \text{clean}.$$

$$\text{strongly nil clean} \subseteq \text{strongly clean} \subseteq \text{clean}.$$

In clean ring, if we take the multiplication in place of addition, i.e., if every element of a ring R can be represented as product of a unit and an idempotent, then we obtain the well known class of **unit regular rings**. One can think of a multiplicatively analogue for nil-clean rings, that is, rings in which every element is a product of an idempotent and a nilpotent element. If we restrict ourselves to rings with identity, this class has no interest: indeed, it is readily seen that such a ring cannot have identity (unless zero). Taking into consideration the unit and nilpotent elements, Calugareanu and Lam in [8] defined a ring as **fine ring**, if every non zero element of a ring R can be written as a sum of a unit and a nilpotent element. They proved that the class of fine rings is a proper subclass of simple rings. Now, if in place of addition, the multiplication of unit and nilpotent elements is taken into consideration, then, Calugareanu in [7] defined **UN ring**. The simple artinian rings are UN rings.

Definition 2.2.3. *A ring R is called* **UN** *if every non-unit element of R is product of a unit and a nilpotent element.*

Any homomorphic image of a clean ring is again clean leads to the definition of a neat ring. McGovern in [44] defined **neat ring**.

Definition 2.2.4. *A ring R is said to be* ***neat****, if every non-trivial homomorphic image of it is clean.*

Proposition 2.2.5. *The following are equivalent for a ring R.*

(1) R is neat.

(2) R/aR is clean for every nonzero $a \in R$.

(3) For any collection of nonzero prime ideals $\{P_j\}_{j\in J}$ of R with $I = \cap_{j\in J} P_j$ different than 0 we have R/I is clean.

(4) R/aR is neat for every $a \in R$.

(5) R/I is clean for every nonzero semiprime ideal.

Moreover, a homomorphic image of a neat ring is neat.

Proof. See [44, Proposition 2.1]. □

2.3 VNR Rings

Around 1935, John von Neumann in connection with his work on continuous geometry and operator algebras, discovered von Neumann regular rings. For general background and detailed information of von Neumann regular rings, one can see [26] and [32]. The class of von Neumann regular rings is closed under homomorphic images, direct products, and direct limits.

Definition 2.3.1. *A ring R is said to be*

*(1) **von Neumann regular (VNR)** if for any $a \in R$, there exists $x \in R$ such that $a = axa$;*

*(2) **unit-regular** if for any $a \in R$, there exists an invertible $u \in U(R)$ such that $a = aua$;*

*(3) **semiregular** if for any $a \in R$, there exist a regular element $b \in R$ such that $a - b \in J(R)$ or equivalently, if $R/J(R)$ is regular and idempotents lift modulo $J(R)$;*

*(4) **right (left) weakly regular** if for any $a \in R$, there exist $x, y \in R$ such that $a = axay(xaya)$;*

*(5) **s-weakly regular (SWR)** if for any $a \in R$, there exist $x, y \in R$ such that $a = axa^2y$.*

A ring R is said to be **semiprimitive** if $J(R) = 0$; and R is **semisimple** if R is artinian and semiprimitive. We list below the well known inclusions:

$$\begin{array}{c} \text{semisimple} \\ \Downarrow \\ \text{unit-regular} \Longrightarrow \text{VNR} \Longrightarrow \text{semiregular.} \\ \Downarrow \\ \text{semiprimitive} \end{array}$$

$$\text{strongly regular} \Longrightarrow \text{VNR} \Longrightarrow \text{weakly regular.}$$

Theorem 2.3.2. *For any ring R, the following are equivalent:*

(1) For any $a \in R$, there exists $x \in R$ such that $x = xax$.

(2) Every principal left ideal is generated by an idempotent.

(3) Every principal left ideal is a direct summand of $_RR$.

(4) Every finitely generated left ideal is generated by an idempotent.

(5) Every finitely generated left ideal is a direct summand of $_RR$.

Proof. See [39, Theorem 4.23]. □

Since the condition given in Defintion 2.3.1(1) is left-right symmetric, the last four conditions are still valid if we replace the word 'left' by 'right'.

Definition 2.3.3. *A ring R is called* ***local*** *if for any $a \in R$, either a or $1 - a$ is invertible. or equivalently, it has a unique maximal left (right) ideal.*

A commutative ring with identity in which every prime ideal is contained in a unique maximal ideal is called pm-ring. While studying pm-rings, Contessa discovered that a commutative ring R with the property that for every $a \in R$, either a or $1 - a$ is VNR, is a pm-ring. She called the commutative rings with this property as von Neumann local rings. Later noncommutative rings with this property were also called von Neumann local rings. After Contessa, von Neumann local rings were studied by many authors, one can see [11; 15; 27] and [51].

Definition 2.3.4. *A ring R is said to be:*

(1) ***Von Neumann local (VNL)*** *if for any $a \in R$, either a or $1 - a$ is VNR.*

(2) ***almost unit regular*** *if for any $a \in R$, either a or $1 - a$ is unit regular.*

(3) ***feckly semiregular****, if for any $a \in R$, either a or $1 - a$ is semiregular.*

$$\text{abelian VNL} \Longrightarrow \text{almost unit regular} \Longrightarrow \text{VNL}.$$

$$\begin{array}{ccc} \text{semiregular} & \Longrightarrow & \text{feckly semiregular} \\ & & \Uparrow \\ & & \text{VNL.} \end{array}$$

2.4 Hurwitz Series and Skew Hurwitz Series Rings

Following F. keigher [33], Hurwitz series ring over R is denoted by HR and the elements of HR are functions $f : \mathbb{N} \to R$, where $\mathbb{N}$ is the set of natural numbers and f is a sequence of the form (a_n), with componentwise addition and the multiplication for each $f = (a_n)$, $g = (b_n) \in HR$ is defined as $(a_n)(b_n) = (c_n)$ where $c_n = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} a_m b_{n-m}$ for all $n \in \mathbb{N}$. Clearly, HR is a ring with identity $1 = (1, 0, 0, \dots)$. The concept of Hurwitz series rings were extended by K. Paykan (see [55]). The skew Hurwitz series ring $A = (HR, \alpha)$ over R and $\alpha \in End(R)$ is defined as follows: the elements of $A = (HR, \alpha)$ are the ordinary function $f : \mathbb{N} \to R$ with componentwise addition and the operation of multiplication for each $f, h \in A$ is defined as

$$(fh)(n) = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} f(m)\alpha^m(h(n-m))$$

for all $n \in \mathbb{N}$, where $\begin{pmatrix} n \\ m \end{pmatrix}$ is the binomial coefficient.
Define the mappings $l_n : \mathbb{N} \to R$ and $l_r^{'} : \mathbb{N} \to R$ by

$$l_n(x) = \begin{cases} 1 & x = n-1 \\ 0 & x \neq n-1 \end{cases}, \quad l_r^{'}(x) = \begin{cases} r & x = 0 \\ 0 & x \neq 0 \end{cases}$$

respectively. It can be easily shown that $A = (HR, \alpha)$ is a ring with identity $l_1 : \mathbb{N} \to R$ defined as $l_1(0) = 1$ and $l_1(n) = 0$ for all $n \geq 1$.
Let $\mathcal{G}_R : R \to A$ is a ring homomorphism defined as $\mathcal{G}_R(r) = l_r^{'}$ for any $r \in R$. The ring R is then canonically embedded as a subring of A via $r \in A \mapsto l_r^{'} \in A$. Also, let $\mathcal{E}_R : A \to R$ is a ring homomorphism defined as $\mathcal{E}_R(f) = f(0)$ for any $f \in A$. Note that $\mathcal{E}_R \circ \mathcal{G}_R = 0$.

2.5 Group Theory

In this section, we will take a look at some preliminary aspects of group theory.

Definition 2.5.1. *A group G is called*

(1) ***periodic or torsion****, if every element of it is of finite order.*

(2) ***locally finite****, if every finitely generated subgroup of it is finite.*

Obviously, every locally finite group is periodic; whether the converse holds is the famous Burnside's problem. A more stronger condition than local finiteness is locally normal. A group G is called **locally normal**, if every finite subset of it is contained in a finite normal subgroup of G. A group G in which all the elements, except the identity element, have infinite order is called **torsion free**, e.g., infinite cyclic group, C_∞, is a torsion free group.

Definition 2.5.2. *A group G is called a* ***p-group****, if the order of every element of G is a power of the fixed prime p.*

Note that every p-group is a torsion group.

Theorem 2.5.3. *[57] If G is an abelian torsion group, then G is finitely generated.*

Definition 2.5.4. *The* ***FC-center or FC-subgroup*** *of a group G is the set of all elements of G that have finitely many conjugates in G.*

We denote the FC-center of G by $\Delta(G)$. It is easy to see that $\Delta(G) = \{x \in G \| G : C_G(x)| < \infty\}$. If $G = \Delta(G)$, then G is said to be an FC-group. We denote the torsion FC-group by $\Delta^+(G)$, so $\Delta^+(G) = \{x \in G \| G : C_G(x)| < \infty$ and $o(x) < \infty\}$. If $G = \Delta^+(G)$, then G is locally normal.

2.6 Group Rings

The **group ring**, RG, of a group G over a ring R is the set of all formal linear combinations of the form

$$\alpha = \sum_{g \in G} a_g g$$

where $a_g \in R$ and $a_g = 0$ for all but finitely many, that is , only a finite number of coefficients are different from 0 in each of these sums. Sum of two elements in RG is defined componentwise

$$\alpha + \beta = \sum_{g \in G} (a_g + b_g) g.$$

And product is given by

$$\alpha\beta = \sum_{g,h \in G} a_g b_h gh.$$

We can write the product $\alpha\beta$ also as:

$$\alpha\beta = \sum_{q\in G} c_q q$$

where

$$c_q = \sum_{gh=q} a_g b_h.$$

It is easy to verify that, with the operations defined above RG is a ring with identity; namely the element

$$1 = \sum_{g=G} u_g g,$$

where the coefficient corresponding to the identity element of the group is equal to 1 and $u_g = 0$ for every other element $g \in G$.

A product of elements in RG by an element $\lambda \in R$ is defined as:

$$\lambda\Big(\sum_{g\in G} a_g g\Big) = \sum_{g\in G} (\lambda a_g) g.$$

Again, with the operations defined above RG is an R-module. If R is commutative, then RG is called the **group algebra** of G over R.

(1) Given an element $\alpha = \sum_{g\in G} a_g g$ in RG, **support** of α, denoted by $supp(\alpha)$, is the subset of elements in G that have nonzero coefficient in the expression of α, i.e., $supp(\alpha) = \{g \in G : a_g \neq 0\}$.

(2) The homomorphism $\omega : RG \to R$ given by

$$\omega\Big(\sum_{g\in G} a_g g\Big) = \sum_{g\in G} a_g$$

is called the **augmentation map** and its kernel, denoted by ωG, is called the **augmentation ideal** of RG. So we have $RG/\omega G \cong R$.

(3) Notice that if an element $\alpha = \sum_{g\in G} a_g g$ belongs to ωG then $\omega\Big(\sum_{g\in G} a_g g\Big) = \sum_{g\in G} a_g = 0$. So, we can write α in the form

$$\alpha = \sum_{g\in G} a_g g - \sum_{g\in G} a_g = \sum_{g\in G} a_g(g-1).$$

Thus the ideal ωG is generated by the set $\{g - 1 : g \in G, g \neq 1\}$.

If H is a subgroup of G, then we define ωH as the left ideal of RG generated by $\{1 - h : h \in H\}$. In particular, if H is a normal subgroup of G, then ωH is a two sided ideal of RG. For more results related to group rings, one can see [17; 45] and [54].

Lemma 2.6.1. *Let H be a normal subgroup of G, then $RG/\omega H \cong R(G/H)$.*

Proof. Let us define the mapping $\bar{\xi} : RG \to R(G/H)$, as follows:

$$\bar{\xi}(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g (gH).$$

Clearly it is an epimorphism. First we show that $\omega H \subseteq ker(\bar{\xi})$. We have $\bar{\xi}[g(1-h)] = gH - ghH = 0$. So $g(1-h) \in ker(\bar{\xi})$ for all $g \in G, h \in H$. Thus, $\omega H \subseteq ker(\bar{\xi})$.

Conversely, let $\alpha = \Sigma_{g\in G} a_g g \in ker(\bar{\xi})$. Let T be a transversal of H in G. We denote by tH the image of $t \in T$ in the quotient group G/H. So we have

$$0 = \bar{\xi}(\alpha) = \sum_{g\in G} a_g gH = \sum_{t\in T} \Big[\sum_{h\in H} a_{th}\Big] tH.$$

So, for all $t \in T$ we get $\Sigma_{h\in H} a_{th} = 0$, and hence $\Sigma_{h\in H} a_{th} t = 0$. Thus we have

$$\alpha = \sum_{g\in G} a_g g = \sum_{t\in T}\sum_{h\in H} a_{th} th = \sum_{t\in T}\sum_{h\in H} a_{th} th - \sum_{h\in H} a_{th} t = \sum_{t\in T}\sum_{h\in H} a_{th} t(h-1) \in \omega H.$$

From above we get $ker(\bar{\xi}) \subseteq \omega H$. Therefore, $ker(\bar{\xi}) \subseteq \omega H$, and hence $RG/\omega H \cong R(G/H)$.

□

Lemma 2.6.2. *Let I be a two sided ideal of a ring R. Then $IG = \{\sum_{g\in G} a_g g \in RG : a_g \in I\}$ is a two sided ideal of RG and $RG/IG \cong (R/I)G$.*

Proof. Any element in $(IG)(RG)$ is of the form $\alpha\beta$, for some $\alpha = \sum_{g\in G} a_g g \in IG$ and $\beta = \sum_{h\in G} r_h h \in RG$, where $a_g \in I$ and $r_h \in R$. Now

$$\alpha\beta = \sum_{q\in G} c_q q.$$

where, $c_q = \sum_{gh=q} a_g r_h \in I$, because I is an ideal of R. So $\alpha\beta \in IG$, and hence $(IG)(RG) \subseteq IG$. Similarly, we can show that $(RG)(IG) \subseteq IG$. Therefore, IG is a two sided ideal of RG.

The mapping $\theta : RG \to (R/I)G$, induced from the natural map $\phi : R \to R/I$ gives us that

$$\theta(\sum_{g\in G} a_g g) = \sum_{g\in G} (a_g + I) g = \sum_{g\in G} a_g g + IG.$$

So the $ker(\theta) = IG$, and hence $RG/IG \cong (R/I)G$. □

Proposition 2.6.3. *The group ring RG is VNR if and only if*

(1) R is VNR,

(2) G is locally finite, and

(3) the order of every element of G is invertible in R.

Proof. See Connell [17, Theorem 3]. □

Chapter 3

UN Rings and Group Rings

A ring R is called UN ring if every non-unit of it can be written as product of a unit and a nilpotent element. We obtain results about lifting of conjugate idempotents and unit regular elements modulo an ideal I of a UN ring R. Matrix rings over UN rings are discussed and it is obtained that for a commutative ring R, a matrix ring $M_n(R)$ is UN if and only if R is UN. Lastly, UN group rings are investigated and we obtain the conditions on a group G and a field K for the group algebra KG to be UN. Then we extend the results obtained for KG to the group ring RG over a ring R (which may not necessarily be a field).

3.1 Introduction

In last two decades one of the active areas of research have been the rings whose elements can be written as a sum/product of units/ idempotents/ nilpotent elements. For example, *clean rings* are those in which every element of the ring R can be written as sum of a unit and an idempotent. If in place of addition, we take the multiplication, i.e., if every element of a ring R can be written as product of a unit and an idempotent, then we obtain the well known class of *unit regular* rings. Taking into consideration the unit and nilpotent elements, Călugăreanu and Lam in [8] defined a ring as *fine ring*, if every non zero element of a ring R can be written as a sum of a unit and a nilpotent element. They proved that the class of fine rings is a proper subclass of that of simple rings. Now, if in place of addition, the multiplication of unit and nilpotent elements is taken into consideration, then, Călugăreanu in [7] defined a ring R to be *UN ring* if every non-unit of R can be written as product of a unit and a nilpotent element. A non unit element $x \in R$ is called *Strongly UN* if in its UN-decomposition, the unit and nilpotent commute.

In section 3.2, we discuss certain properties of UN rings. Lifting of various types of elements modulo an ideal I have been studied by Khurana, Lam and Nielsen in [35]. We discuss lifting properties of UN rings modulo an ideal I. It is pertinent to mention here that the lifting properties like conjugate idempotent lifting and unit regular elements lifting are considered modulo a two sided ideal in contrast to the exchange rings, where idempotents lift modulo each left (right) ideal. Then we discuss the question raised by Călugăreanu [7] that "is $M_n(R)$ over a UN ring R, also UN?". We obtain that if R is commutative, then $M_n(R)$ is UN if and only if R is UN.

In section 3.3, we focus on UN group rings. The group rings involving units and idempotents to represent every element as sum (clean ring)/product (unit regular ring) of these elements have been studied by many authors. So, our focus is on the group rings involving units and nilpotent elements, i.e., fine rings and UN rings. A group ring RG can never be a fine ring, because a fine ring is a simple ring and RG always has $\omega(G)$ as its proper ideal. So, we investigate the structure of UN group rings. We first take up the case of group algebra KG of a group G over a field K. We obtain the result that if $charK = 0$, then KG can be a UN ring if and only if G is trivial. If $charK = p$, then KG is a UN ring implies that the group G must be a p-group and the converse holds if G is locally finite. Next we investigate that what could be the characteristic of a UN ring R. We arrive at the conclusion that the $charR$ of a UN ring can be either 0 or p^{α} and particularly in case of group ring RG, the characteristic of R can not be 0. Then we investigate the structure of the group ring RG of a group G over an arbitrary ring R (which may not necessarily be a field) and obtain the result that if RG is a UN ring then R is a UN ring, G is a p-group and $p \in J(R)$; and the converse holds if G is locally finite.

3.2 UN Rings

We list below some of the properties of UN rings in the form of the Lemma, which we will require in the following sections.

Lemma 3.2.1. *Let R be a ring, then the following statements hold:*

(1) A UN ring is left-right symmetric ([7], Proposition 1(3)).

(2) Homomorphic image of a UN ring is UN ([7], Proposition 3(1)).

(3) Let I be a nil ideal of R, then R is UN if and only if R/I is UN ([67], Proposition 0(a)).

(4) A UN ring has no nontrivial central idempotents ([67], Proposition 0(c)).

(5) Every left or right regular element of a UN ring R is invertible, i.e., R is its own classical ring of quotients ([67], Proposition 0(f)).

The classes of local rings and UN rings are separate, some examples to this effect can be found in [7]. We prove a theorem below and using it we get a result somewhat in line with local rings.

Theorem 3.2.2. *Let $I \lhd R$, then the following are equivalent:*

(1) R/I is UN.

(2) R/I^n is UN for all $n \in \mathbb{N}$.

(3) R/I^n is UN for some $n \in \mathbb{N}$.

Proof. (1)⇒(2) Let $I \lhd R$ and R/I be UN. It can be seen that

$$R/I \cong (R/I^n)/(I/I^n).$$

Since (I/I^n) is nilpotent in (R/I^n), the result follows by Lemma 3.2.1(c).
(2)⇒(3) is evident.
(3)⇒(1) Let R/I^n be UN for some $n \in \mathbb{N}$. As homomorphic image of a UN ring is UN and

$$R/I \cong (R/I^n)/(I/I^n).$$

So we get that R/I is UN. □

By using the above Theorem we get a result for UN rings similar to the local rings ([39], Ex. 19.5).

Corollary 3.2.3. *Let $I \lhd R$ such that I is maximal as a left ideal, then R/I^n is a UN ring for all $n \in \mathbb{N}$.*

3.2.1 Lifting Properties

In this subsection we discuss about lifting properties of UN rings. We start with the definition of isomorphic and conjugate idempotents in R.

Definition 3.2.4. *Two idempotents $e \in R$ and $f \in R$ are called*

- conjugate *(written as $e \sim f$), if $f = u^{-1}eu$ for some $u \in U(R)$.*
- isomorphic *(written as $e \cong f$), if $eR \cong fR$ as right R-modules.*

The well known results about isomorphic and conjugate idempotents are mentioned below in the form of Lemmas.

Lemma 3.2.5. *Let e and f be idempotents in a ring R, then following are equivalent*

(1) $e \sim f$.

(2) $e \cong f$ *and* $(1-e) \cong (1-f)$.

Lemma 3.2.6. *Let e and f be idempotents in a ring R, then following are equivalent*

(1) $e \cong f$.

(2) $eR \cong fR$ *as right R-modules.*

(3) $Re \cong Rf$ *as left R-modules.*

(4) $e = ab$ *and* $f = ba$ *for some* $a, b \in R$.

We observe that if e be a non trivial idempotent in a UN ring R with $e = ut$ for some $u \in U(R)$ and $t \in N(R)$, then t is unit regular with $t = tut$. And also $f = tu$ is an idempotent isomorphic as well as conjugate to e.

Let $I \lhd R$, we say that idempotents lift modulo I if for any idempotent $\bar{e} \in R/I$ there exists an idempotent $x \in R$ such that $\bar{x} = \bar{e}$. And conjugate idempotents $\bar{e}, \bar{f} \in R/I$ are said to lift modulo I if there exit conjugate idempotents x and y in R such that $\bar{x} = \bar{e}$ and $\bar{y} = \bar{f}$.

Theorem 3.2.7. *In a UN ring if idempotents lift modulo an ideal I, then conjugate idempotents lift modulo I.*

Proof. Let R be a UN ring and $I \lhd R$ such that idempotents lift modulo I. Let $\bar{e}, \bar{f} \in R/I$ be conjugate idempotents in R/I such that $\bar{f} = \overline{u^{-1}}\bar{e}\bar{u}$ for some unit $\bar{u} \in U(R/I)$. Since idempotents lift modulo I, there exit idempotents x and $y \in R$ such that $\bar{x} = \bar{e}$ and $\bar{y} = \bar{f}$. Let pre image of $\bar{u}$ in R be v. As R is UN, so if $v \notin U(R)$, then $v = wt$ for some $w \in U(R)$ and $t \in N(R)$. So,

$$\bar{u} = \bar{v} = \bar{w}\bar{t} \Longrightarrow \bar{t} = \overline{w^{-1}}\bar{v} \in U(R/I)$$

which is not possible. Thus, $v \in U(R)$ and hence pre-image of $\overline{u^{-1}}\bar{e}\bar{u}$ is $v^{-1}xv = z$ (say), which is an idempotent conjugate to x and $\bar{z} = \bar{f}$. □

Theorem 3.2.8. *Let R be a UN ring and $I \lhd R$. Then unit regular elements lift modulo I if and only if idempotents lift modulo I.*

Proof. Let R be UN and unit regular elements lift modulo I. Let $\bar{e}$ be an idempotent in R/I. As idempotents are unit regular elements, so $\bar{e}$ lifts to a unit regular element, say $x \in R$, such that $x = xvx$ for some $v \in U(R)$. By using the fact that $x = xvx$, it is a routine calculation to check that

$$(x - v(x^2 - x))^2 = x - v(x^2 - x).$$

So $y := x - v(x^2 - x)$ is an idempotent in R. Also as $\bar{x} = \bar{e}$, it can be easily seen that

$$\overline{x^2} = \overline{x.1.x} = \bar{x}\bar{1}\bar{x} = \bar{e}\bar{1}\bar{e} = \bar{e}.$$

Thus, we get $(x^2 - x) \in I$, which implies that $\bar{y} = \bar{x} = \bar{e}$. Hence $\bar{e}$ lifts to an idempotent in R.

Conversely, let idempotents lift modulo I. Let $a \in R/I$ be unit regular. It is well known that a unit regular element is a multiple of a unit and an idempotent. So $\bar{a} = \bar{u}\bar{e}$ for some $\bar{u} \in U(R/I)$ and $\bar{e}^2 = \bar{e}$. By following the proof of above Theorem 3.2.7, we get that $\bar{u}$ lifts to some $v \in U(R)$ and by given hypothesis $\bar{e}$ lifts to some idempotent $z \in R$. Thus, $\bar{a}$ lifts to a unit regular element vz. □

3.2.2 UN Matrix Rings

In this subsection we discuss the question raised by Călugăreanu in [7] that "is $M_n(R)$ over a UN ring R, also UN?". As introduced in [63], a ring R is called a *US-ring* if every non unit element of it can be written as product of a unit and a strongly nilpotent element. An element $x \in R$ is called *strongly nilpotent*, if every sequence $x = x_0, x_1, x_2, \cdots$ such that $x_{i+1} \in x_i R x_i$ converges to zero. It is evident that every strongly nilpotent element is nilpotent but the converse may not hold good ([63, Example 1]). In case of a commutative ring R, an element is strongly nilpotent element iff it is nilpotent.

Theorem 3.2.9. *Let R be ring.*

(1) If R is a US-ring, then $M_n(R)$ is UN.

(2) If $M_n(R)$ is UN, then $C(R)$ is a US-ring.

Proof. (1) It is well known that $M_n(R)/J(M_n(R)) \cong M_n(R/J(R))$. Since R is a US-ring, by [63, Theorem 1] we get that $R/P(R)$ is a division ring, and hence $J(R) = P(R)$. Then by [7, Corollary 7], $M_n(R/J(R)) \cong M_n(R)/J(M_n(R))$ is UN. By using the fact that $P(M_n(R)) = M_n(P(R))$, we obtain that

$$J(M_n(R)) = M_n(J(R)) = M_n(P(R)) = P(M_n(R))$$

is nil. Thus, we have obtained that $M_n(R)/J(M_n(R))$ is UN and $J(M_n(R))$ is nil. By Lemma 3.2.1(c), we get that $M_n(R)$ is UN.

(2) Let $M_n(R)$ be UN. The center of $M_n(R)$ is $C(M_n(R)) = \{aI_n : a \in C(R)\}$, i.e., the scalar matrices of the form aI_n for $a \in C(R)$. We have $C(M_n(R)) \cong C(R)$ by the mapping $f : C(R) \to C(M_n(R))$ defined by $f(a) = aI_n, a \in C(R)$. Now the result follows from [67, Proposition 0(b)] and the fact that in a commutative ring, the nilpotent and strongly nilpotent elements coincide.

□

Corollary 3.2.10. *Let R be a commutative ring, then $M_n(R)$ is UN if and only if R is UN.*

Corollary 3.2.11. *[72] Let R be a commutative ring, then $M_n(R)$ is UN if and only if R is a local ring with $J(R)$ nil.*

A ring R is called 2-primal if R/I is a domain for every minimal prime ideal I of R. It is well known in literature that for 2-primal rings $P(R) = N(R)$.

Corollary 3.2.12. *Let R be a 2-primal UN ring, then $M_n(R)$ is UN.*

Since a reduced UN ring is a division ring, we get the result obtained by Călugăreanu in [7] as a corollary of the above Theorem.

Corollary 3.2.13. *A simple Artinian ring is UN.*

3.3 UN Group Rings

3.3.1 Group Algebra

First we take up the case of group algebra of a group G over a field K. If G is a finite group, then let us denote by $\hat{G}$, the following element of KG, $\hat{G} = \sum_{g \in G} g$.

Theorem 3.3.1. *Let K be a field and G be a group.*

(1) If $\mathrm{char}K = 0$, then KG is UN if and only if $G = (1)$.

(2) If $\mathrm{char}K = p$, then KG is UN implies that G is a p-group; the converse holds if G is locally finite.

Proof. First of all we see that if KG is UN, then G is a torsion group irrespective of whether characteristic of field K is 0 or p. Let $g \in G$, then $1 - g \notin U(KG)$. So $1 - g = ut$ for some $u \in U(KG)$ and $t \in N(KG)$. If $g \neq 1$, then $t \neq 0$ and we can choose a positive integer k such that $t^k = 0$ but $t^{(k-1)} \neq 0$. Thus, we have $(1-g)t^{(k-1)} = ut^k = 0$. So $1-g$ is a zero divisor in KG. Hence, the order of g is finite ([17], Proposition 6).

(1) Let $charK = 0$. If G is a finite group, then $|G|^{-1} \in K$. So there would exist a central idempotent $\frac{1}{|G|}\hat{G}$ in KG, which is a contradiction to Lemma 3.2.1(d). Hence, KG can not be UN for a nontrivial finite group G. Now, let us consider the case of infinite group. We observe that, again in light of Lemma 3.2.1(d), G can not be an abelian group. So the only case left out is that G be a non abelian group. In view of Lemma 3.2.1(e) and [[54], Theorem 3.13, page 54] G must be a locally finite group. As G is locally finite, so KG is VNR ring and in particular $J(KG) = 0$. So, $\omega(G)$ is not a quasi regular ideal. Thus, there must exist an $\alpha \in \omega(G)$ such that $1-\alpha \notin U(KG)$. Since KG is UN, we get $1-\alpha = ut$ for some $u \in U(KG)$ and $t \in N(KG)$. Applying augmentation map we get $\omega(1-\alpha) = \omega(ut) \Longrightarrow 1 = \omega(u)\omega(t) \Longrightarrow \omega(t) = \omega(u)^{-1}$, which is absurd. Thus in all the above cases, for KG to be a UN ring, the group G must be trivial.

The converse part is straight forward, since every field is a UN ring.

(2) Let $charK = p$. If G is a finite group, then $|G| \neq p'$, because if it is so then there would exist a central idempotent $\frac{1}{|G|}\hat{G}$ and hence contradicting Lemma 3.2.1(d). So, let $|G| = p^k m$ with $(p,m) = 1$. By Cauchy's Theorem there exists an element $g \in G$ of order p' such that $p'|m$. As $(1+g+g^2+\dots.+g^{(p'-1)})(1-g) = 0$, so we get $(1+g+g^2+\dots.+g^{(p'-1)}) = ut$ for some $u \in U(KG)$ and $t \in N(KG)$. Applying augmentation map we get $\omega(1+g+g^2+\dots.+g^{(p'-1)}) = \omega(ut) \Longrightarrow p' = \omega(u)\omega(t) \Longrightarrow \omega(t) = p'\omega(u)^{-1}$, which is a contradiction, since $p' \in U(K)$. Thus, G must be a p-group. Now let us consider G to be an infinite group. If G is abelian, then G should be p-group, because otherwise there would exist non trivial central idempotents in KG. If G is non abelian, then following the method adopted for finite group, it can be shown that G is a p-group, as desired.

Conversely, let G be a locally finite p-group and K be a field of characteristic p. By [17, Proposition 16(ii)], $\omega(G)$ is a nil ideal. As is well known that $KG/\omega(G) \cong K$, so following Lemma 3.2.1(c) we get that KG is a UN ring. □

3.3.2 Group Ring

Before taking up the group ring case, we discuss the characteristic of a UN ring and obtain the following results below.

Lemma 3.3.2. *Let R be a UN ring and $n \in \mathbb{Z}$, the set of integers. Then for an element $n \in R$, either $n \in U(R)$ or $n \in N(R)$.*

Proof. If $n \notin U(R)$, then $n = ut$ for some $u \in U(R)$ and $t \in N(R)$. This amounts to $u^{-1}nu = tu \Longrightarrow u^{-1}(\underbrace{1 + 1 + ... + 1}_{n\ times})u = tu \Longrightarrow n = tu$. Thus, n is strongly UN and in particular $n \in N(R)$. □

Lemma 3.3.3. *Let R be a UN ring and charR = n, then either n = 0 or $n = p^{\alpha}$, for some prime p and in this case $p \in J(R)$.*

Proof. If $n \neq 0$, then we can write $n = p_1^{\alpha_1} p_2^{\alpha_2} .... p_k^{\alpha_k}$, where p_i's are primes. Since $n = 0$ in R, at least one of the p_i's is nilpotent in R. It can be easily seen that this is possible only when $k = 1$. Thus, $charR = p^{\alpha}$. Now, if $charR = p^{\alpha}$, then $p \in N(R)$. Since, $xp = px$ for all $x \in R$, we get $xp \in N(R)$ for all $x \in R$. So, $1 - xp \in U(R)$ for all $x \in R$. Thus, $p \in J(R)$ ([39], Lemma 4.1). □

Now let us consider the group ring of a group G over an arbitrary ring R (which may not necessarily be a field).

Theorem 3.3.4. *Let R be a ring and G be a non trivial group. If RG is UN, then R is UN of characteristic p^{α}, G is a p-group and $p \in J(R)$; the converse holds if G is locally finite.*

Proof. Let RG be UN, then by augmentation map $\omega : RG \to R$, we obtain that R is a homomorphic image of RG. So, by Lemma 3.2.1(b), R is UN. Going by the proof of Theorem 3.3.1, it can be seen that G is a torsion group. Now let if possible $charR = 0$, then by Lemma 3.3.2 all $n(\neq 0) \in \mathbb{Z}$ are invertible in R. Thus $|g|^{-1} \in R$ for all $g \in G$. Following the proof of Theorem 3.3.1, G can neither be a finite group nor an infinite abelian group. Now, let G be an infinite non abelian group and let the order of an element $g(\neq 1) \in G$ be m, for some positive integer m. So, we have that $(1 + g + g^2 + ... + g^{(m-1)})(1 - g) = 0 \Longrightarrow 1 + g + g^2 + ... + g^{(m-1)} = ut$ for some $u \in U(RG)$ and $t \in N(RG)$ $\Longrightarrow \omega(1 + g + g^2 + ... + g^{(m-1)}) = \omega(ut) \Longrightarrow m = \omega(u)\omega(t) \Longrightarrow \omega(t) = m\omega(u)^{-1}$, which is not possible, since $m \in U(R)$. Thus, $charR \neq 0$. By Lemma 3.3.3, if $charR \neq 0$, then $charR = p^{\alpha}$. In this case also following the proof of Theorem 3.3.1, we can arrive at the result that G can not have p' elements and hence, G is a p-group. By Lemma 3.3.3, we get $p \in J(R)$.

Conversely, let R be a UN ring of characteristic p^{α}, G a locally finite p-group and $p \in J(R)$. By [17, Proposition 16(ii)], $\omega(G)$ is a nil ideal. By augmentation map $\omega : RG \to R$, we observe that $RG/\omega(G) \cong R$. Because R is UN, we get RG is UN (by Lemma 3.2.1(c)). □

Since an abelian torsion group is locally finite, for the commutative group rings we get:

Corollary 3.3.5. *Let R be a commutative ring and G be a non trivial abelian group. Then, RG is UN if and only if R is UN of characteristic p^α, G is a p-group and $p \in J(R)$.*

The above results resemble to the result obtained for local group rings by Nicholson in [47]. But we give below group ring specific examples which show that a UN group ring may not be local and a local group ring may not be UN.

Example 3.3.6. *Let us consider the group ring RG, where $R = M_2(\mathbb{Z}_2)$ ($\mathbb{Z}_2$ be the ring of integers modulo 2) and $G = C_2$ be a cyclic group of order 2. By the mapping ϕ : $M_2(\mathbb{Z}_2)C_2 \to M_2(\mathbb{Z}_2C_2)$ defined by*

$$\phi(\Sigma_{i=1}^{k}(A_i g_i)) = (c_{ij}),$$

where $c_{ij} = \Sigma_{t=1}^{k} a_{ij}^{(m)} g_m$ and $a_{ij}^{(l)}$ is the i-th row and j-th column entry of A_t; it can be seen that $M_2(\mathbb{Z}_2)C_2 \cong M_2(\mathbb{Z}_2C_2)$.

Now $e = \begin{pmatrix} 1 & 1+g \\ 0 & 0 \end{pmatrix}$ is a non zero idempotent in $M_2(\mathbb{Z}_2C_2)$. And hence, $M_2(\mathbb{Z}_2C_2) \cong M_2(\mathbb{Z}_2)C_2 = RG$ is not local. By [7, Corollary 7], $M_2(\mathbb{Z}_2)$ is UN and thus by Theorem 3.3.4 it follows that $RG = M_2(\mathbb{Z}_2)C_2$ is a UN ring.

Example 3.3.7. *Let $R = Z_{(p)}$, i.e., the localization of the ring of integers at a prime ideal generated by p and $G = C_p$ be a cyclic group of order p, where p is a prime. We consider the group ring RG. By [47, Theorem], RG is a local ring.*

It is well known that R is a domain and hence a reduced ring; but R is not a field. Since a reduced UN ring is a division ring, we get that R is not UN. Thus, by Theorem 3.3.4, RG is not UN.

Chapter 4

On UQ rings

In this chapter, we introduce UQ rings. A ring R is called UQ if every non-unit element of R can be represented as a product of a unit and a quasiregular element. We provide various properties of UQ rings along with its characterizations. We give a new characterization of 2-good rings, and it turns out that 2-good rings are precisely the rings in which every element is a product of a unit and a quasiregular element. We discuss various extensions of UQ rings such as Morita contexts, generalized matrix rings, formal matrix rings, group rings etc.

4.1 Introduction

The class of UN rings has investigated in depth in [7] and [67]. In [66], 2-good rings were introduced by Vámos. A ring in which every element is a sum of two units is called 2-*good*. In [7], it has been observed that UN rings with 2-good identity are 2-good, and asked a question to refine the inclusion {UN rings with 2-good identity}⊂{2-good rings}, i.e., find the classes C of rings such that {UN rings with 2-good identity}⊂ C ⊂{2-good identity}. Responding to this question, Zhou in [72] gave an example that refines the above inclusion by taking C = {rings: $R/J(R)$ is UN}. For convenience, we call such a class of rings a J-UN ring.

Motivated by papers [7; 72], here, we introduce a new class of rings called UQ rings. An element $a \in R$ is called quasiregular if $1 + a$ is a unit in R.

Definition 4.1.1. *A ring R is called UQ if every non-unit element of R can be represented as a product of a unit and a quasiregular element.*

In section 4.2, we obtain various properties of UQ rings and examples are provided to show that the class of UQ rings properly contains classes of UN rings, J-UN rings and 2-good rings. The subring of a UQ ring may not be UQ. It is shown that if R is UQ, then the corner ring eRe is UQ for every idempotent $e \in R$. But if corner rings eRe and $(1 - e)R(1 - e)$ are UQ, then R need not be UQ. We provide various characterizations of UQ rings. We obtain a new characterization of 2-good rings, and it turns out that UQ rings with 2-good identity are equivalent to 2-good rings. In section 4.3, we discuss extensions of matrix rings. For a ring R and $s \in C(R)$, we characterize the generalized matrix ring $K_s(R)$ over R to be UQ ring with quasiregular identity. Moreover, we discuss that the formal matrix ring $M_n(R; s)$ over R is a UQ ring with quasiregular identity if and only if R is a UQ ring with quasiregular identity and $s \in J(R)$. If $s = 1$, then the matrix ring $M_n(R)$ is a special case of the formal matrix ring $M_n(R; s)$. Section 4.4 is devoted for the discussion of group ring RG to be UQ ring. We determine necessary and sufficient conditions for a commutative group ring RG to be UQ. It is proved that for a commutative ring R and an abelian group G, if RG is UQ, then R is UQ and G is torsion. The converse holds if G is locally finite p-group where $p \in J(R)$ is a prime number. We prove that for a ring R with $Q(R)$ identity and a finite abelian group G of exponent 2, the group ring RG is UQ if and only if RC_2 is UQ. Also, we determine when the group ring is UQ for an artinian ring.

4.2 On UQ Rings

We first recall some definitions. A ring R whose units are sums $1+n$ for a suitable nilpotent element n, is called *UU ring*. A ring R is UU if and only if every quasiregular element is nilpotent. Recall that $J(R)$ is the unique maximal left quasiregular ideal of R.

Example 4.2.1. *(1) Every element in $J(R)$ is UQ.*

(2) Every element in $N(R)$ is UQ.

(3) A local ring is a UQ ring.

(4) If R is UU ring, then R is UQ if and only if R is UN.

(5) Simple artinian ring is a UQ ring.

(6) A semilocal ring R with $Q(R)$ identity is UQ.

If $a \in R$ is a quasiregular element, then $a = (1 + a) - 1$ is 2-good. Since units can be quasiregular, a quasiregular element may not be UQ:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = I_2 + \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Conversely, UQ elements may not be quasiregular:

$$\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

is UQ but not quasiregular since

$$I_2 + \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

is not a unit.

If R is a UQ ring, then for any non-unit element $a \in R$ we have $a = uq$ where $u \in U(R)$ and $q \in Q(R)$. Since $qu = u(u^{-1}qu)$ where $u \in U(R)$ and $u^{-1}qu \in Q(R)$, so the left-right symmetric definition is equivalent in a UQ ring.

Proposition 4.2.2. *If $a \in R$ is a UQ element, then uav is UQ for any $u, v \in U(R)$.*

Proof. Since a is UQ, $a = wq$ where $w \in U(R)$ and $q \in Q(R)$. Then $uav = (uwv)(v^{-1}qv)$ with $uwv \in U(R)$ and $v^{-1}qv \in Q(R)$. □

Lemma 4.2.3. *(1) Any homomorphic image of a UQ ring is UQ.*

(2) Any homomorphic image of a UQ ring with quasiregular identity is a UQ ring with quasiregular identity.

Proof. (1) The result follows from the fact that every homomorphic image of a unit and a quasiregular element is again a unit and a quasiregular element, respectively.

(2) It is similar to the proof of (1). □

Lemma 4.2.4. *Let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a family of rings and $R = \prod_{\lambda\in\Lambda} R_\lambda$.*

(1) The direct product R is a UQ ring with quasiregular identity if and only if R_λ, for all $\lambda \in \Lambda$, is a UQ ring with quasiregular identity.

(2) The direct product R is UQ if each R_λ, $\lambda \in \Lambda$, is a UQ ring with quasiregular identity.

Proof. (1) ($\Rightarrow$) It follows from Lemma 4.2.3.
($\Leftarrow$) Suppose that for all $\lambda \in \Lambda$, R_λ is a UQ ring with quasiregular identity. Then for a non-unit element $a_\lambda \in R_\lambda$, we have $a_\lambda = u_\lambda q_\lambda$ for some $u_\lambda \in U(R_\lambda)$ and $q_\lambda \in Q(R_\lambda)$. Let $(a_1, a_2, \ldots, a_\lambda, \ldots)$ be an element in R. If each a_λ is a non-unit, then $(a_1, a_2, \ldots, a_\lambda, \ldots)$ is a non-unit element in R. Thus we have $(a_1, a_2, \ldots, a_\lambda, \ldots) = (u_1q_1, u_2q_2, \ldots, u_\lambda q_\lambda, \ldots) = (u_1, u_2, \ldots, u_\lambda, \ldots)(q_1, q_2, \ldots, q_\lambda, \ldots)$ where $(u_1, u_2, \ldots, u_\lambda, \ldots) \in U(R)$ and $(q_1, q_2, \ldots, q_\lambda, \ldots) \in Q(R)$. If some a_λ is a unit, then $(a_1, a_2, \ldots, a_\lambda, \ldots)$ is a non-unit in R. Since each R_λ is a UQ ring with quasiregular identity, so unit element $a_\lambda = a_\lambda 1$ where $a_\lambda \in U(R_\lambda)$ and 1 is quasiregular identity. Thus we obtain that $(a_1, a_2, \ldots, a_\lambda, \ldots) = (u_1, u_2, \ldots, a_\lambda, \ldots)(q_1, q_2, \ldots, 1, \ldots)$ where $(u_1, u_2, \ldots, a_\lambda, \ldots) \in U(R)$. As $(1, 1, \ldots, 1, \ldots) + (q_1, q_2, \ldots, 1, \ldots) = (1 + q_1, 1 + q_2, \ldots, 2, \ldots) \in U(R)$, $(q_1, q_2, \ldots, 1, \ldots) \in Q(R)$. Hence, R is a UQ ring with quasiregular identity.
(2) It follows from (1) and by the fact that a UQ ring with quasiregular identity is a UQ ring. □

If ring R_λ in Lemma 4.2.4 is a UQ ring without quasiregular identity, then the following example shows that Lemma 4.2.4 need not be true.

Example 4.2.5. *Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. Here, $\mathbb{Z}_4$ is a UQ ring without quasiregular identity. Then R is not UQ because non-unit element $(\overline{2}, \overline{3})$ cannot be written as the product of a unit and a quasiregular element in R.*

Remark 4.2.6. *(1) From above Example* 4.2.5, *we also conclude that if corner rings eRe and $(1-e)R(1-e)$ are UQ, then R may not be UQ.*

(2) The triangular matrix ring is not UQ. For example, $T_2(\mathbb{Z}_2)$.

(3) The subring of a UQ ring need not be UQ. For example, $M_2(\mathbb{Z}_2)$ is a UQ ring by Example 4.2.1. *However, the subring $T_2(\mathbb{Z}_2)$ is not UQ.*

A ring R is called *strongly π-regular* if for every element $a \in R$ there exists a positive integer n (depending on a) and an element $x \in R$ such that $a^n = a^{n+1}x$.

Proposition 4.2.7. *Let R be a strongly π-regular ring with trivial idempotents. Then R is a UQ ring.*

Proof. Suppose that R is a strongly π-regular ring. Then by [6, Proposition 2.6], R is a strongly clean ring. So R being strongly clean with trivial idempotents implies that R is local. Thus R is a UQ ring. □

In general, the converse of above Proposition 4.2.7 is false which is shown in the following example.

Example 4.2.8. *Let $R = \{\frac{m}{n} \in \mathbb{Q} \mid n \text{ is odd}\}$. Then R is local. It follows that R is UQ. However, since $J(R)$ is not nil, R is not strongly π-regular.*

Remark 4.2.9. *If R is strongly π-regular ring R with $Q(R)$ identity, then by [25, Theorem 3], every element of R can be written as a sum of two units. Thus R is a 2-good ring. Hence, R is a UQ ring.*

Theorem 4.2.10. *Let R be a ring. Then R is a UQ ring if and only if $R/J(R)$ is UQ.*

Proof. ($\Rightarrow$) By Lemma 4.2.3, it is obvious.

($\Leftarrow$) Let $\overline{R} = R/J(R)$ be a UQ ring and $\overline{a} \in \overline{R}\backslash U(\overline{R})$. Then $\overline{a} = \overline{uq}$ where $\overline{u} \in U(\overline{R})$ and $\overline{q} \in Q(\overline{R})$. We can write $a = uq + j$ for some $j \in J(R)$. Then $a = u(q + u^{-1}j)$ where $u \in U(R)$ and $q + u^{-1}j \in Q(R)$ since $1 + q + u^{-1}j \in U(R) + J(R) = U(R)$. □

Let I be an ideal of a ring R. If R is UQ, then the factor ring R/I is also UQ. In general, however, the converse is not true. For example, let $R = \mathbb{Z}$. Then for any prime p, $R/I = \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p$ is UQ but $\mathbb{Z}$ is not UQ. The following corollary shows that this result is true if I is contained in $J(R)$.

Corollary 4.2.11. *Let I be an ideal of a ring R with $I \subseteq J(R)$. Then R is a UQ ring if and only if R/I is UQ.*

Let $P(R)$ represents prime radical of R. The ideal $P(R)$ is a nil ideal in R and so $P(R) \subseteq J(R)$. As a result, we may immediately come to the following corollary.

Corollary 4.2.12. *Let $P(R)$ be a prime radical of R. Then R is UQ if and only if $R/P(R)$ is UQ.*

Corollary 4.2.13. *Let R be a ring. Then $R[[x]]$ is a UQ ring if and only if R is UQ.*

Proof. ($\Rightarrow$) Suppose that $R[[x]]$ is a UQ ring. It is evident that $R[[x]]/(x) \cong R$. By Lemma 4.2.3, it follows that R is UQ.

($\Leftarrow$) It is easy to show that $R[[x]]/J(R[[x]]) \cong R/J(R)$. Since R is a UQ ring, $R/J(R)$ is UQ by Theorem 4.2.10. Then we obtain that $R[[x]]/J(R[[x]])$ is UQ. Hence, by Theorem 4.2.10, $R[[x]]$ is UQ. □

Theorem 4.2.14. *Let R be a ring. Then R is a UQ ring with quasiregular identity if and only if so is $R/J(R)$.*

Proof. It is similar to the proof of Theorem 4.2.10. □

Proposition 4.2.15. *Let $e^2 = e \in R$. If R is a UQ ring, then eRe is UQ.*

Proof. Note that $eRe/J(eRe) \cong \overline{e}\overline{R}\overline{e} \subseteq \overline{R}$ where $\overline{R} = R/J(R)$. Since R is UQ, $R/J(R)$ is UQ by Theorem 4.2.10. Then we get that $eRe/J(eRe)$ is UQ. Thus by Theorem 4.2.10 again, eRe is UQ. □

Remark 4.2.16. *The converse of Proposition* 4.2.15 *is not true since by Remark* 4.2.6(1)*, if corner rings eRe and* $(1-e)R(1-e)$ *is UQ, then R may not be UQ.*

If R is a UQ ring with quasiregular identity, then it is easy to see that R is a UQ ring with a 2-good identity but the converse is not true. For example, consider $M_2(\mathbb{Z}_2)$. Then the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_2)$ is the sum of two units, i.e. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ but $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is not a unit in $M_2(\mathbb{Z}_2)$. Thus $M_2(\mathbb{Z}_2)$ is a UQ ring with a 2-good identity but not a UQ ring with quasiregular identity. The following theorem provides a new characterization of 2-good rings.

Theorem 4.2.17. *Let R be a ring. Then the following statements are equivalent:*

(1) R is a UQ ring with a 2*-good identity.*

(2) R is a 2*-good ring.*

(3) Every element in R can be represented as a product of a unit and a quasiregular element.

Proof. (1) ⇒ (2) Let $a \in R$ be a non-unit element. Then $a = uq$ where $u \in U(R)$ and $q \in Q(R)$. So $a = u(1+q) - u = uv - u$ where $v := 1 + q \in U(R)$. By hypothesis, we have that the identity $1 = u + v$. So $u = u1 = u^2 + uv$ where $u^2, uv \in U(R)$. Hence, R is a 2-good ring.
(2) ⇒ (1) Suppose that R is a 2-good ring. Then for a non-unit element $a \in R$, we have $a = u + v$ where $u, v \in U(R)$. Then we can write $a = -v(-v^{-1}u - 1) = -vx$ where $x := -v^{-1}u - 1$. Now we only need to show that x is quasiregular in R. Since $u, v \in U(R)$, $1 + x = -v^{-1}u \in U(R)$. Thus R is a UQ ring.
(2) ⇒ (3) It follows from the proof (2) ⇒ (1) by taking an arbitrary element $a \in R$.
(3) ⇒ (2) If $a \in R$ is an arbitrary element in the proof (1) ⇒ (2), then the result follows. □

Now we prove that UQ rings are the generalization of J-UN rings.

Theorem 4.2.18. *Let R be a ring. If R is a J-UN ring, then R is UQ.*

Proof. For any $a \in R$, write $\overline{a} = a + J(R) \in \overline{R}$. Suppose that $\overline{a}$ is a non-unit in $\overline{R}$, then $\overline{a} = \overline{u}\overline{t}$ where $\overline{u} \in U(\overline{R})$ and $\overline{t} \in N(\overline{R})$. So $a = ut + j$ for some $j \in J(R)$. Thus $a = u(t + u^{-1}j)$ with $u \in U(R)$ and $t \in N(R)$. Note that $1 + t + u^{-1}j \in U(R) + J(R) = U(R)$. It implies that $t + u^{-1}j \in Q(R)$. Hence, R is a UQ ring. □

Remark 4.2.19. *The converse of Theorem* 4.2.18 *is not true which is shown below in Example* 4.2.20(2).

We now obtain the following relation between rings.

UN ring	$\Longrightarrow$	J-UN ring	$\Longrightarrow$	UQ ring ($\cong \frac{R}{J(R)}$ UQ)
$\Uparrow$		$\Uparrow$		$\Uparrow$
UN ring with 2-good identity	$\Longrightarrow$	J-UN ring with 2-good identity	$\Longrightarrow$	UQ ring with 2-good identity (2-good ring)

The following examples illustrate that the reverse implications of the above need not be true.

Example 4.2.20. *(1) Let $R = \mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal generated by p. Then $R/J(R) \cong \mathbb{Z}_p$ is UN. Thus, R is J-UN. But R is not UN.*

(2) Let $R = \mathbb{Z}_5 \times \mathbb{Z}_5$. Since $\mathbb{Z}_5$ is a UQ ring with a quasiregular identity, so its direct product is a UQ ring with quasiregular identity by Lemma 4.2.4. *Thus R is UQ. But $\mathbb{Z}_5 \times \mathbb{Z}_5/J(\mathbb{Z}_5 \times \mathbb{Z}_5) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ is not UN. Hence, R is not J-UN.*

(3) Let $R = \mathbb{Z}_4$. Then R is UQ (J-UN, UN). But since the identity 1 *cannot be written as the sum of two units in R, R is a UQ (J-UN, UN) ring without* 2*-good identity.*

(4) [72, Example 1*] Let $R = \mathbb{Q}[[x]]$. Then R is a J-UN ring with* 2*-good identity but R is not UN with* 2*-good identity.*

(5) [72, Example 1*] Let $R = M_n(\mathbb{Z})(n \geq 2)$ is a UQ ring with* 2*-good identity (*2*-good ring) but not J-UN ring with* 2*-good identity.*

Remark 4.2.21. *We now determine whether or not a UQ ring is Morita invariant, i.e. whether the property of being a UQ ring is preserved by the Morita equivalence of rings. Consider $R = \mathbb{Z}_4$ and $S := M_2(\mathbb{Z}_4)$. Then R is a UQ ring. By the Pierce decomposition, $S = eSe \oplus eS(1-e) \oplus (1-e)Se \oplus (1-e)S(1-e)$. If $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $1 - e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $eSe \cong \mathbb{Z}_4 \cong eS(1-e) = (1-e)Se = (1-e)S(1-e)$. So by Lemma* 4.2.4*, the matrix ring S is not a UQ ring. Thus, UQ rings are not Morita invariant.*

4.3 Extensions of Matrix Rings

Recall that a Morita context is a 4-tuple $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ where R and S are rings, ${}_RM_S$ and ${}_SN_R$ are bimodules, and there exist context products $M \times N \to R$ and $N \times M \to S$ written multiplicatively as $(m, n) = mn$ and $(n, m) = nm$ such that $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is an associative ring with the obvious matrix operations. Morita contexts were introduced in 1958 by Morita [46]. The readers are referred to [41; 59; 60] as well as the references there for detailed information on the study on Morita contexts. A Morita context $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is called trivial if the context products are trivial, i.e., $MN = 0$ and $NM = 0$ (see[43]). A trivial Morita context is also called the ring of a Morita context with zero pairings. The class of rings of Morita contexts includes all 2×2 matrix rings and all triangular matrix rings. The following example shows that the UQ property for Morita context with $MN \subseteq J(R)$ and $NM \subseteq J(S)$ is not true.

Example 4.3.1. *Let* $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$. *Then* $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. *Here,* $\mathbb{Z}_2$ *is a UQ ring but its direct product* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *is not UQ. It follows that* $R/J(R)$ *is not UQ. Thus* R *is not UQ.*

Theorem 4.3.2. *Let* $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *be a Morita context such that* $MN \subseteq J(R)$ *and* $NM \subseteq J(S)$. *Then* T *is a UQ ring with quasiregular identity if and only if both* R *and* S *are UQ rings with quasiregular identity.*

Proof. ($\Rightarrow$) Suppose that T is a UQ ring with quasiregular identity. Then by Theorem 4.2.14, $T/J(T)$ is a UQ ring with quasiregular identity. By [60, Lemma 3.1], we have that $T/J(T) \cong R/J(R) \times S/J(S)$. Thus $R/J(R) \times S/J(S)$ is a UQ ring with quasiregular identity. Then by Lemma 4.2.4, both $R/J(R)$ and $S/J(S)$ are UQ rings with quasiregular identity and so are R and S by Theorem 4.2.14.

($\Leftarrow$) If R and S are UQ rings with quasiregular identity, then using the similar theorems, T is a UQ ring with quasiregular identity. □

Corollary 4.3.3. *Let* $E = T(R, S, M) = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *be a formal triangular matrix ring. Then* E *is a UQ ring with quasiregular identity if and only if both* R *and* S *are UQ rings with quasiregular identity.*

Corollary 4.3.4. *For* $n \geq 2$, $T_n(R)$ *is a UQ ring with quasiregular identity if and only if* R *is a UQ ring with quasiregular identity.*

Proof. (⇐) Suppose that R is a UQ ring with quasiregular identity. Then by Theorem 4.2.14, $R/J(R)$ is a UQ ring with quasiregular identity. It is easy to see that

$$T_n(R)/J(T_n(R)) \cong \underbrace{R/J(R) \oplus R/J(R) \oplus \cdots \oplus R/J(R)}_{n}. \qquad (\star)$$

By Lemma 4.2.4, the right hand side of $(\star)$ is UQ with quasiregular identity. Then $T_n(R)/J(T_n(R))$ is UQ with quasiregular identity. Thus by Theorem 4.2.14, $T_n(R)$ is a UQ ring with quasiregular identity.
(⇒) Suppose that $T_n(R)$ is UQ with quasiregular identity. Then by similar theorems, R is UQ with quasiregular identity. □

Consider R to be a ring and M to be a bimodule over R. Write $T(R, M) = \left\{\begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \mid a \in R, m \in M\right\}$, then $T(R, M)$ is a subring of $T(R, R, M)$. The trivial extension of R and M is $R \propto M = \{(a, m) \mid a \in R, m \in M\}$ with addition defined componentwise and multiplication defined by $(a, m)(b, n) = (ab, an + mb)$. Then $T(R, M) \cong R \propto M$ and $T(R, R) \cong R[x]/(x^2)$ where (x^2) is an ideal generated by x^2 in $R[x]$. Note that $J(R \propto M) = \{(a, m) \mid a \in J(R), m \in M\}$ and $U(R \propto M) = \{(a, m) \mid a \in U(R), m \in M\}$.

Corollary 4.3.5. *Let R be a ring and M a bimodule over R. Then the following statements are equivalent:*

(1) R is a UQ ring.

(2) $R \propto M$ is a UQ ring.

(3) $T(R, M)$ is a UQ ring.

(4) $R \propto R$ is a UQ ring.

(5) $T(R, R)$ is a UQ ring.

(6) $R[x]/(x^2)$ is a UQ ring.

Proof. (1) ⇒ (2) Suppose that R is a UQ ring. Then in view of Theorem 4.2.10, $R/J(R)$ is UQ. Since $R \propto M/J(R \propto M) = R \propto M/J(R) \propto M \cong R/J(R)$, we get that $R \propto M/J(R \propto M)$ is UQ. Thus by Theorem 4.2.10 again, $R \propto M$ is UQ.
(2) ⇒ (1) Note that $R \cong R \propto M/(0 \propto M)$. Since $R \propto M$ is UQ, R is UQ by Lemma 4.2.3.
(1) ⇔ (4) It follows from (1) ⇔ (2).
(2) ⇔ (3) Note that $T(R, M) \cong R \propto M$.
(4) ⇔ (5) ⇔ (6) Note that $R \propto R \cong T(R, R) \cong R[x]/(x^2)$. □

Consider R to be a ring and M to be a bimodule over R. Let $R \bowtie M = \{(a, m, b, n) \mid a, b \in R, m, n \in M\}$ with addition defined componentwise and multiplication defined by $(a_1, m_1, b_1, n_1)(a_2, m_2, b_2, n_2) = (a_1a_2, a_1m_2 + m_1a_2, a_1b_2 + b_1a_2, a_1n_2 + m_1b_2 + b_1m_2 + n_1a_2)$. Then $R \bowtie M$ is a ring which is isomorphic to $(R \propto M) \propto (R \propto M)$. Let

$$BT(R, M) = \left\{ \begin{pmatrix} a & m & b & n \\ 0 & a & 0 & b \\ 0 & 0 & a & m \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b \in R, m, n \in M \right\}.$$

Then $BT(R, M) \cong T(T(R, M), T(R, M))$, and we have the following isomorphism as rings: $R[x, y]/(x^2, y^2) \to BT(R, R)$ defined by

$$a + bx + cy + dxy \longmapsto \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

Corollary 4.3.6. *Let R be a ring and M a bimodule over R. Then the following statements are equivalent:*

(1) R is a UQ ring.

(2) $R \bowtie M$ is a UQ ring.

(3) $BT(R, M)$ is a UQ ring.

(4) $BT(R, R)$ is a UQ ring.

(5) $R[x, y]/(x^2, y^2)$ is a UQ ring.

(6) $R \bowtie R$ is a UQ ring.

Given a ring R and an element $s \in C(R)$, the 4-tuple $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ becomes a ring with addition defined componentwise and with multiplication defined by $\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{pmatrix}$. The ring is denoted by $K_s(R)$. The element s is called the multiplier of $K_s(R)$. The ring $K_s(R)$ can be described as a special kind of Morita context. A Morita context $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ with $R = M = N = S$ is called a generalized matrix ring over R. As observed by Krylov [38], the generalized matrix rings over R are determined by their multipliers, i.e., the central elements of the ring R. We prove that $K_s(R)$ is a UQ ring with quasiregular identity if and only if R is a UQ ring with quasiregular identity. If $s = 1$, then $K_1(R)$ is exactly the matrix ring $M_2(R)$ but $K_s(R)$ can be significantly different from $M_2(R)$.

Lemma 4.3.7. (1) *[61, Lemma 2] Let R be a ring with $s \in C(R)$. Then $J(K_s(R)) = \begin{pmatrix} J(R) & M \\ M & J(R) \end{pmatrix}$ where $M = \{x \in R \mid sx \in J(R)\}$.*
(2) *[61, Lemma 14] Let R be a commutative ring with $s \in R$ and let $A \in K_s(R)$. Then A is a unit of $K_s(R)$ iff $det_s(A)$ is a unit of R.*

Theorem 4.3.8. *Let R be a commutative ring with $s \in J(R)$. Then the following statements are equivalent:*

(1) R is a UQ ring with quasiregular identity.

(2) $K_s(R)$ is a UQ ring with quasiregular identity.

Proof. (2) $\Rightarrow$ (1) For $a \in R$, let $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ be a non-unit in $K_s(R)$. Suppose that $K_s(R)$ is a UQ ring with quasiregular identity. Then there exist a unit $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ and a quasiregular element $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $K_s(R)$ such that $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$. By Lemma 4.3.7, $u \in U(R)$ and $1 + q \in U(R)$. Thus $a = uq$ where $u \in U(R)$ and $q \in Q(R)$. Hence, R is a UQ ring with quasiregular identity.
(1) $\Rightarrow$ (2) Consider a map $\phi : K_s(R) \to R/J(R) \times R/J(R)$ defined by

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix} \longmapsto (\overline{a}, \overline{b}).$$

Since $s \in J(R)$, we can easily verify that ϕ is a ring epimorphism. Thus $K_s(R)/J(K_s(R)) \cong R/J(R) \times R/J(R)$. In view of Theorem 4.2.14 and Lemma 4.2.4, $R/J(R) \times R/J(R)$ is a UQ ring with quasiregular identity, so is $K_s(R)/J(K_s(R))$. Thus by Theorem 4.2.14 again, $K_s(R)$ is a UQ ring with quasiregular identity. □

According to Tang and Zhou [62], for $n \geq 2$ and $s \in C(R)$, the $n \times n$ formal matrix ring over R defined by s, denoted as $M_n(R; s)$, is the set of all $n \times n$ matrices over R with usual addition of matrices and with multiplication defined as follows: for (a_{ij}) and (b_{ij}) in $M_n(R; s)$,

$$(a_{ij})(b_{ij}) = (c_{ij}), \text{ where } c_{ij} = \sum_{k=1}^{n} s^{\delta_{ikj}} a_{ik} b_{kj}.$$

Here $\delta_{ijk} = 1 + \delta_{ik} - \delta_{ij} - \delta_{jk}$ with $\delta_{ik}, \delta_{ij}, \delta_{jk}$ the Kronecker delta symbols. If $s = 1$, $M_n(R; s)$ is exactly the matrix ring, although generally it can be significantly different from $M_n(R)$. We obtain necessary and sufficient conditions for $M_n(R; s)$ to be UQ ring with quasiregular identity.

Lemma 4.3.9. *[62, Proposition 32] Let $A \in M_n(R; s)$. Then A is a unit in $M_n(R; s)$ if and only if $det_s A \in U(R)$.*

Theorem 4.3.10. *Let R be a ring with $s \in C(R) \cap J(R)$. Then the following statements are equivalent:*

(1) R is a UQ ring with quasiregular identity.

(2) $M_n(R; s)$ is a UQ ring with quasiregular identity.

Proof. (2) $\Rightarrow$ (1) For $a \in R$, let $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ be a non-unit in $M_n(R; s)$. Suppose that $M_n(R; s)$ is a UQ ring with quasiregular identity. Then there exist a unit $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ and a quasiregular element $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_n(R; s)$ such that $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$. Following Lemma 4.3.9 we obtain that $u \in U(R)$ and $1 + q \in U(R)$. Then $a = uq$ where $u \in U(R)$ and $q \in Q(R)$. Hence, R is a UQ ring with quasiregular identity.
(1) $\Rightarrow$ (2) If $n = 1$, then $M_n(R; s) = R$. So in this case, there is nothing to prove. Suppose that $n > 1$, and the result holds for $M_{n-1}(R; s)$. Let $A = M_{n-1}(R; s)$. Then $M_n(R; s) = \begin{pmatrix} A & M \\ N & R \end{pmatrix}$ is a Morita context where $M = \begin{pmatrix} M_{1n} \\ \vdots \\ M_{n-1,n} \end{pmatrix}$ and $N = \begin{pmatrix} M_{n1} \dots M_{n,n-1} \end{pmatrix}$ with $M_{in} = M_{ni}$ for all $i = 1, 2, \dots, n-1$. For $x = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{n-1,n} \end{pmatrix} \in M$ and $y = \begin{pmatrix} y_{n1} \dots y_{n,n-1} \end{pmatrix} \in N$, we have

$$xy = \begin{pmatrix} s^2 x_{1n} y_{n1} & s x_{1n} y_{n2} & \cdots & s x_{1n} y_{n,n-1} \\ s x_{2n} y_{n1} & s^2 x_{2n} y_{n2} & \cdots & s x_{2n} y_{n,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ s x_{n-1,n} y_{n1} & s x_{n-1,n} y_{n2} & \cdots & s^2 x_{n-1,n} y_{n,n-1} \end{pmatrix} \in J(A)$$

and

$$yx = s^2 y_{n1} x_{1n} + s^2 y_{n2} x_{2n} + \cdots + s^2 y_{n,n-1} x_{n-1,n} \in s^2 R \in J(R).$$

Thus we get that $MN \subseteq J(A)$ and $NM \subseteq J(R)$. Then we obtain that $M_n(R; s)/J(M_n(R; s)) \cong A/J(A) \times R/J(R)$. Since A and R are UQ rings with quasiregular identity, by Theorem 4.2.14 and Lemma 4.2.4, $A/J(A) \times R/J(R)$ is a UQ ring with quasiregular identity. Thus $M_n(R; s)/J(M_n(R; s))$ is a UQ ring with quasiregular identity, and so is $M_n(R; s)$ by Theorem 4.2.14. □

Corollary 4.3.11. *Let R be a ring. Then the following statements are equivalent:*

(1) R is a UQ ring with quasiregular identity.

(2) $M_n(R[[x]]/(x^m); x)$ *is a UQ ring with quasiregular identity.*

Proof. (1) ⇒ (2) Note that $(R[[x]]/(x^m))/J(R[[x]]/(x^m)) \cong R/J(R)$. Since R is a UQ ring with quasiregular identity, $R/J(R)$ is UQ with quasiregular identity by Theorem 4.2.14. It follows that $(R[[x]]/(x^m))/J(R[[x]]/(x^m))$ is UQ with quasiregular identity, and so is $R[[x]]/(x^m)$ by Theorem 4.2.14. Thus by Theorem 4.3.10, (2) is proved.
(2) ⇒ (1) Consider a map $\psi : R[[x]]/(x^m) \to R$ defined by $\psi(\overline{f}) = f(0)$. It can be easily shown that ψ is a ring epimorphism. So R is a homomorphic image of $R[[x]]/(x^m)$. Since $M_n(R[[x]]/(x^m); x)$ is UQ with quasiregular identity, $R[[x]]/(x^m)$ is UQ with quasiregular identity by Theorem 4.3.10. Hence, R is a UQ ring with quasiregular identity. □

4.4 Group Rings

In this section, we discuss group rings to be UQ rings. For a commutative ring and an abelian group, we obtain the necessary conditions for group rings to be UQ.

Theorem 4.4.1. *If RG is a UQ ring, then R is a UQ ring and G a torsion group.*

Proof. First, we show that for a field K and a torsion-free group H, if KH is UQ, then H is the trivial group. We prove this by contradiction, suppose that H is non-trivial and let h be a non-identity element. So $1, h$, and h^2 are the distinct elements in H. It is known that KH has only trivial units and are of the form kh' where $k \in K, k \neq 0, h' \in H$, so quasiregular elements of KH will be of form $kh' - 1$. The element $1 + h + h^2 \in KH$ is a non-unit. Then there exist $k_1h_1' \in U(KG)$ where $k_1 \in K$, $h_1' \in H$ and $k_2h_2' - 1 \in Q(KG)$ where $k_2 \in K$, $h_2' \in H$ such that $1 + h + h^2 = k_1h_1'(k_2h_2' - 1) = k_3h_3' - k_1h_1'$ where $k_3 = k_1k_2 \in K$ and $h_3' = h_1'h_2' \in H$ but it is not possible since the K-linear sum of three distinct elements of H can not be written as the K-linear sum of two elements of H. Thus, it follows that KH is not UQ.

Now for proving the theorem, suppose that RG is UQ. Since R is a homomorphic image of RG, R is UQ by Lemma 4.2.3. Let M be a maximal ideal of R and let $\tau(G)$

denotes the torsion subgroup of G. So R/M is a field and $G/\tau(G)$ is a torsion-free group. Since $(R/M)(G/\tau(G))$ is a homomorphic image of RG, so it is UQ by Lemma 4.2.3. Then by the above claim, we have that $G/\tau(G)$ is the trivial group. So $G = \tau(G)$ is a torsion group. □

The converse of Theorem 4.4.1 is not true.

Example 4.4.2. *Let* $RG = \mathbb{Z}_2C_3 = \{0, 1, x, x^2, 1 + x, 1 + x^2, x + x^2, 1 + x + x^2\}$. *Here,* $R = \mathbb{Z}_2$ *is a UQ ring and* $G = C_3$ *is a torsion group. However,* RG *is not a UQ ring since the non-unit element* $1 + x + x^2 \in RG$ *can not be written as a product of a unit and a quasiregular element in* RG.

If G is a locally finite p-group with $p \in J(R)$, then the converse of the above Theorem 4.4.1 holds.

Theorem 4.4.3. *Let* $p \in J(R)$ *be a prime number. If* R *is a UQ ring and* G *a locally finite* p*-group, then* RG *is UQ.*

Proof. Suppose that G is a locally finite p-group with $p \in J(R)$. Then by [71, Lemma 2] we have that $\omega G \subseteq J(RG)$. Since R is UQ, and it is well known that $RG/\omega G \cong R$, $RG/\omega G$ is UQ. Then in view of Corollary 4.2.11, RG is UQ. □

Combining the above Theorem 4.4.1 and Theorem 4.4.3, we obtain the following characterization for a commutative group ring RG to be UQ.

Theorem 4.4.4. *Let* G *be an abelian* p*-group with* $p \in J(R)$ *and* R *be a commutative ring. Then* RG *is a UQ ring if and only if* R *is UQ.*

Remark 4.4.5. *For any non-trivial finite group* G*, the group ring* $\mathbb{Z}G$ *is not UQ.*

If G is a finite abelian group with exponent 2, then the following theorem states that RG is UQ if and only if RC_2 is UQ. To show this statement, however, we must first prove a lemma.

Lemma 4.4.6. *Let* R *be a ring with* $Q(R)$ *identity. Then* RC_2 *is a UQ ring if and only if* R *is UQ.*

Proof. Since R is a ring with $Q(R)$ identity, $2 \in U(R)$. Then $RC_2 \cong R \times R$. Following Lemma 4.2.4 we get that RC_2 is a UQ ring with quasiregular identity. Hence, RC_2 is a UQ ring. □

Theorem 4.4.7. *Let* G *be a finite abelian group of exponent* 2 *and* R *be a ring with* $Q(R)$ *identity. Then* RG *is UQ if and only if* RC_2 *is UQ.*

Proof. ($\Rightarrow$) Since RC_2 is a homomorphic image of RG, by Lemma 4.2.3, RC_2 is UQ.

($\Leftarrow$) Suppose that RC_2 is a UQ ring. If $n > 1$, then $RC_{2^n} \cong (RC_{2^{n-1}})C_2$ and $RC_{2^{n-1}} \cong (RC_{2^{n-2}})C_2$. Thus by induction and Lemma 4.4.6, RC_{2^n} is UQ for all $n > 0$. Since RG is a homomorphic image of RC_{2^n} for some n. Thus RG is UQ. □

We now obtain another characterization for a group ring to be UQ as follows:

Theorem 4.4.8. *Let R be an artinian ring. Then RG is UQ if and only if $(R/J(R))G$ is UQ.*

Proof. ($\Rightarrow$) Since $(R/J(R))G$ is a homomorphic image of RG, by Lemma 4.2.3, $(R/J(R))G$ is a UQ ring.

($\Leftarrow$) Let $(R/J(R))G$ be a UQ ring. Then $RG/J(R)G \cong (R/J(R))G$ is UQ. Since R is an artinian ring, following [17, Proposition 3] we have $J(R)G \subseteq J(RG)$. We consider a ring epimorphism $\psi : RG/J(R)G \to RG/J(RG)$ defined as:

$$\psi(\alpha + J(R)G) = \alpha + J(RG), \alpha \in RG.$$

Then by Lemma 4.2.3, $RG/J(RG)$ is UQ. Thus by Theorem 4.2.10, RG is UQ. □

If G is a finite p-group and K is a field of $char(K) = p > 0$. Then in view of [39], group algebra KG is local. Thus KG is UQ. The following proposition shows that group algebra KG for a finite abelian group such that $char(K) \nmid |G|$ is a UQ ring.

Proposition 4.4.9. *Let G be a finite abelian group. If K is an algebraic closed field with quasiregular identity and $char(K) \nmid |G|$, then KG is UQ.*

Proof. Following [54] we have

$$KG \simeq \underbrace{K \oplus K \oplus \cdots \oplus K}_{|G|}.$$

Since K is a UQ ring with quasiregular identity, and by Lemma 4.2.4, the direct product of UQ rings with quasiregular identity is UQ, we obtain that $\underbrace{K \oplus K \oplus \cdots \oplus K}_{|G|}$ is UQ. It follows that KG is UQ. □

If, in the above Proposition 4.4.9, K is without quasiregular identity, then group algebra KG may not be UQ.

Example 4.4.10. *By Example 4.4.2, consider $RG = \mathbb{Z}_2C_3$. Then $\mathbb{Z}_2$ is UQ without quasiregular identity and RG is not UQ.*

Chapter 5

On Almost s-Weakly Regular Rings

We introduce almost s-weakly regular (SWR) rings. An element a of R is called SWR if $a \in aRa^2R$*. A ring R is called an almost SWR if for any* $a \in R$*, either a or* $1 - a$ *is SWR. We introduce almost SWR rings as the generalization of abelian VNL rings and SWR rings. We provide various properties and characterizations of almost SWR rings. We discuss various extension rings to be almost SWR. Further, we discuss SWR group rings and almost SWR group rings.*

5.1 Introduction

An element $a \in R$ is *(strongly) VNR* if there exists an element $b \in R$ such that $a = aba$ ($ab = ba$). A ring R is called *(strongly) VNR* if every element of R is (strongly) VNR. Camillo and Xiao [9] investigated weakly regular rings. A ring R is *weakly regular* if it is both right and left weakly regular. As a generalization of strongly VNR rings, in [28], Gupta introduced SWR rings. The class of SWR rings lies strictly between the class of right (or left) weakly regular rings and strongly VNR rings. Contessa in [18], as a common generalization of regular rings and local rings, introduced VNL rings for commutative rings. VNL rings for noncommutative rings were studied by Chen and Tong [15]. Moreover, Grover and Khurana [27] characterized VNL rings in the sense of relating them to some other familiar classes of rings. For more information about VNL rings and their related rings, one can see [13; 15; 18; 51]and [69].

The concept of SWR rings together with the notion of local rings gives motivation for this chapter. In the present chapter, we discuss those elements where either a or $1 - a$ is SWR. We introduce a new class of rings called almost SWR rings. The class of almost

SWR rings is a proper generalization of the class of abelian VNL rings and SWR rings. Our investigation is motivated by papers [15; 28].

Definition 5.1.1. *A ring R is said to be an almost SWR ring if, for any $a \in R$, either a or $1 - a$ is SWR.*

In section 5.2, we prove various properties of almost SWR rings, and some examples are provided to show that the class of almost SWR rings properly contains the classes of SWR, abelian VNL, and weakly tripotent rings. A two sided ideal I in a ring R is said to be SWR ideal if each of its elements is SWR. We prove that a ring R is almost SWR if and only if, for any SWR ideal I of R, R/I is almost SWR. We characterize abelian almost SWR rings. It is proved that if e is an idempotent in an abelian almost SWR ring R, then either eRe or $(1 - e)R(1 - e)$ is SWR, but the converse holds if R is an exchange ring. In section 5.3, we consider extensions of almost SWR rings such as triangular matrix rings, trivial extensions, and so on. In section 5.4, we study semiperfect almost SWR rings. In section 5.5, we prove that if RG is a commutative ring, then RG is SWR if and only if R is SWR, G is locally finite and $n \in o(G)$ is a unit in R where $o(G)$ is the set of orders of all finite subgroups of G. Let KG be a group algebra over a field K satisfying a nontrivial polynomial identity. If KG is SWR, then K is SWR and G is locally finite. It is proved that if RH is almost SWR for every finitely generated subgroup H of G, then RG is almost SWR, but the converse of this result partially holds. We prove that if $G = H \rtimes K$ is a semidirect product of finite subgroup H by a subgroup K, then almost s-weakly regularity of RG implies almost s-weakly regularity of RK. We show that for a finite group G, the group ring RG need not be almost SWR. It is also proved that if R is a commutative local ring and G an abelian p-group with $p \in J(R)$, then RG is almost SWR.

5.2 Basic Properties and Examples

We first recall some definitions. An element a of R is called *tripotent* if $a^3 = a$ and a ring R is tripotent if all elements in R are tripotent. In [23], Danchev introduced weakly tripotent rings. A ring R is *weakly tripotent* if any of its element $a \in R$ satisfies the equations $a^3 = a$ or $a^3 = -a$. Recall that a ring R is called abelian if each idempotent in R is central.

Remark 5.2.1. *(1) Clearly, SWR and local rings are almost SWR rings.*

(2) Every abelian VNL ring is an almost SWR ring.

(3) Every tripotent ring and weakly tripotent ring is an almost SWR ring.

(4) For a commutative ring, $R[[x]]$ is almost SWR if and only if R is local.

(5) *For* $n \geq 2$ *and* $n = \prod_{i=1}^{m} p_i^{k_i}$ *is a prime power decomposition, the ring* $\mathbb{Z}_n$ *of integers mod* n *is almost SWR if and only if* $(pq)^2$ *does not divide* n*, where* p *and* q *are distinct primes.*

(6) *If* $R = \{(q_1, q_2, \ldots, q_n, a, a, \ldots) \mid n \geq 1, q_i \in \mathbb{Q}, a \in \mathbb{Z}_{(2)}\}$*, where* $\mathbb{Z}_{(2)}$ *is the localization of* $\mathbb{Z}$ *at prime ideal generated by* 2*, then* R *is an abelian VNL-ring with* $J(R) = 0$ *but not regular. Thus,* R *is almost SWR but not semiregular.*

Thus, the class of almost SWR rings contains the classes of SWR, abelian VNL, and weakly tripotent rings. Then, we have

$$\begin{array}{c} \text{Abelian VNL} \\ \Downarrow \\ \text{SWR} \Longrightarrow \text{Almost SWR} \\ \Uparrow \\ \text{Weakly Tripotent} \end{array}$$

However, the following examples show that the reverse implication is not true.

Example 5.2.2. *(1) Let* $R = \mathbb{Z}_4$ *be the ring of intergers modulo* 4*. Then,* R *is an almost SWR ring but not SWR.*

(2) *Let* $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$*. Then,*

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

If $r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$*, we can not find* x, y *in* R *such that* $r = rxr^2y$ *but we can easily verify that* $1 - r = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *is SWR. Thus,* R *is an almost SWR ring but not SWR.*

(3) *Let* $R = T_2(\mathbb{Z}_2)$*. Then,* R *is an almost SWR ring but not an abelian VNL because idempotents are not central in* R*.*

(4) *Consider* $R = \mathbb{Z}_4$*. Then,* R *is an almost SWR ring but not weakly tripotent.*

Example 5.2.3. *Let* ${}_R M_S$ *be a bimodule. If* R *is SWR and* S *is local, then* $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *is an almost SWR ring.*

Proof. Let $\beta = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in T$. Since S is local, b or $1_S - b$ is invertible. Assume that b is invertible. By hypothesis, a is SWR in R. So, we have $a = axa^2y$ for some $x, y \in R$. Thus,

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} x & -x(am+mb)b^{-2} \\ 0 & b^{-2} \end{pmatrix}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}^2\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}.$$

It implies that β is SWR.

Assume that $1_S - b$ is invertible. Since $1_R - a$ is SWR, we have $1-a = (1-a)z(1-a)^2w$ for some z, w in R. Similarly, $1_T - \beta = \begin{pmatrix} 1-a & -m \\ 0 & 1-b \end{pmatrix}$ is SWR in T. □

Now we elaborate some properties of almost SWR rings.

Proposition 5.2.4. *The following statements are true for an almost SWR ring R.*

(1) Every homomorphic image of R is almost SWR.

(2) The center of R is a VNL-ring.

(3) The corner ring eRe is almost SWR for every $e^2 = e \in R$.

Proof. (1) It is straightforward.

(2) Let $C(R)$ be the center of R and $x \in C(R)$. Since R is an almost SWR ring, either x or $1-x$ is an SWR element. If x is SWR, we have $x \in xRx^2R = x^3R$ implies immediately that $x = x(x^ky)x$ with $y \in R$ and $k \geq 1$. Moreover, for every $a \in R$, $a(x^ky) = x^{k-1}(xa)y = x^{k-1}(x^{k+1}yx)ay = x^{k-1}ya(x^{k+1}yx) = x^{k-1}y(ax) = (x^ky)a$. Hence, $x^ky \in C(R)$. Similarly, if $1 - x \in C(R)$ is an SWR element in R, then $1 - x$ is regular in C(R).

(3) Let $a \in eRe$. Since R is an almost SWR ring, either a or $1 - a$ is SWR. If a is SWR, we have $a = axa^2y$ for some $x, y \in R$. Thus, $a = eae = eaxa^2ye = aexea^2eye$. It follows that a is SWR in eRe. Similarly, if $1 - a$ is SWR in R, then $e - a$ is SWR in eRe. Hence, eRe is an almost SWR ring. □

The following result follows immediately from Proposition 7.3.2(2).

Corollary 5.2.5. *Let R be an almost SWR ring. Then, R is indecomposable as a ring if and only if its center is local.*

Remark 5.2.6. *In [4], r-clean rings were studied by Ashrafi. A ring R is called r-clean if for any element* $a \in R$*, we have* $a = e + r$ *where e is an idempotent and r is a regular element in R. If R is an r-clean ring with no zero divisor, then by [4, Corollary* 2.10*], R is local. Thus, R is an almost SWR ring.*

It can be easily verified that direct product of SWR rings is SWR if and only if all factors are SWR. But we observe that the direct product of almost SWR rings may not be an almost SWR ring.

Example 5.2.7. *The ring $\mathbb{Z}_4$ of integers modulo 4 is an almost SWR ring. But $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not an almost SWR ring. By choosing $a = (\bar{2}, \bar{3})$, we can easily show that neither a nor $1 - a$ is SWR, and we are done.*

For the direct product of rings to be almost SWR, we prove the following theorem.

Theorem 5.2.8. *Let $R = \prod_{\lambda \in \Lambda} R_\lambda$. Then, R is an almost SWR ring if and only if there exists $\lambda_0 \in \Lambda$ such that R_{λ_0} is an almost SWR ring and for each $\lambda \in \Lambda \backslash \lambda_0$, R_λ is an SWR ring.*

Proof. Let $x = (x_\lambda) \in R$, $\lambda \in \Lambda$. By hypothesis, x_{λ_0} or $1_{R_{\lambda_0}} - x_{\lambda_0}$ is SWR in R_{λ_0}. Assume that x_{λ_0} is SWR in R_{λ_0}, then x is SWR. If $1_{R_{\lambda_0}} - x_{\lambda_0}$ is SWR in R_{λ_0}, then $1 - x$ is SWR in R.

Conversely, suppose that R is an almost SWR ring. Then, R_λ is also an almost SWR ring for every $\lambda \in \Lambda$ by Proposition 7.3.2(1). Write $R = R_{\lambda_0} \times S$, where $S = \prod R_\lambda$, $\lambda \in I \backslash \lambda_0$. If neither R_{λ_0} nor S is SWR, then there exist non SWR elements $a \in R_{\lambda_0}$ and $b \in S$. Now choose $r = (1_{R_{\lambda_0}} - a, b)$. Then, neither r nor $1 - r = (a, 1_S - b)$ is SWR in R, a contradiction. Thus, either R_{λ_0} or S is SWR. If S is an SWR ring, we are done. If S is an almost SWR ring, the iteration of the previous technique completes the proof. □

Lemma 5.2.9. *Let R be an abelian almost SWR ring. Then for every idempotent $e \in R$, either eRe or $(1 - e)R(1 - e)$ is SWR.*

Proof. Consider the Pierce decomposition

$$R \cong \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

Suppose that $a \in eRe$ and $b \in (1-e)R(1-e)$ are not SWR. Then neither $r := \begin{pmatrix} a & 0 \\ 0 & 1-b \end{pmatrix}$ nor $(1 - r) = \begin{pmatrix} 1-a & 0 \\ 0 & b \end{pmatrix}$ is an SWR element in R, which is a contradiction. □

The example given below reveals that the converse of the Lemma 5.2.9 is false.

Example 5.2.10. *Let $R = \{(q_1, q_2, \ldots, q_n, z, z \ldots) \mid q_i \in \mathbb{Q}, z \in \mathbb{Z}, n \geq 1\}$. Clearly, either eRe or $(1 - e)R(1 - e)$ is SWR for every $e^2 = e \in R$. But R is not an almost SWR ring because the homomorphic image $\mathbb{Z}$ of R is not almost SWR.*

An element a of R is said to be an *exchange* [48] if there exists an idempotent $e \in R$ such that $e \in Ra$ and $1 - e \in R(1 - a)$. A ring R is an *exchange ring* if and only if each element of R is exchange. It is easy to show that a commutative almost SWR ring is an exchange ring. The next result shows that the converse of Lemma 5.2.9 is true for an exchange ring.

Theorem 5.2.11. *Let R be an abelian exchange ring. Then R is almost SWR if and only if either eRe or $(1 - e)R(1 - e)$ is SWR for every $e^2 = e \in R$.*

Proof. The 'only if' part follows by Lemma 5.2.9.

Conversely, suppose that R is an exchange ring. Then, for any $a \in R$, we have an idempotent $e \in R$ such that $e \in Ra$ and $1 - e \in R(1 - a)$. Then $Ra + R(1 - e) = R$ and $Re + R(1 - a) = R$. Thus, $Rae = Re$ and $R(1 - a)(1 - e) = R(1 - e)$. So, both ae and $(1 - a)(1 - e)$ are SWR. Since R is an abelian, $eRe = Re$. By hypothesis, if eRe is SWR, then $(1 - a)e$ is SWR. Therefore, $1 - a = (1 - a)e + (1 - a)(1 - e)$ is SWR. Similarly, if $(1 - e)R(1 - e)$ is SWR, then we can prove that a is SWR. □

Proposition 5.2.12. *Let R be a commutative ring. Then $R[x]$ is not an almost SWR ring.*

Proof. Assume that $R[x]$ is an almost SWR ring. Then $R[x]$ being a commutative almost SWR ring implies that $R[x]$ is a VNL-ring, which contradicts [51, Corollary 4.8]. □

Lemma 5.2.13. *Let R be a ring. If $a - aza^2w$ is SWR for some $z, w \in R$, then a is SWR.*

Proof. If $a - aza^2w$ is SWR, then there exist $s, t \in R$ such that

$$(a - aza^2w)s(a - aza^2w)^2t = a - aza^2w.$$

If we set $x = saz-s+z$ and $y = t-wsa^2t+waza^2wt-wsaza^2waza^2wt-za^2wt+wsa^2za^2wt+w$, then it can be verified that $axa^2y = a$. Thus, a is SWR. □

Let R be an almost SWR ring and I an ideal of R. Then, clearly, R/I is almost SWR. But in general, the converse of this result is not true (for example, let $R = \mathbb{Z}_p$ where p is a prime number, then R is almost SWR but $\mathbb{Z}$ is not almost SWR). The following theorem gives another characterization of almost SWR rings.

Theorem 5.2.14. *Let I be an SWR ideal of a ring R. Then, R is an almost SWR ring if and only if R/I is almost SWR.*

Proof. Suppose that R is an almost SWR ring. Then, by Proposition 7.3.2(1), R/I is almost SWR.

Conversely, suppose that R/I is almost SWR. Then, either $a+I$ or $1-a+I$ is SWR. Thus, there exist $x, y, z, w \in R$ such that either $a-axa^2y \in I$ or $(1-a)-(1-a)z(1-a)^2w \in I$. Since I is an SWR ideal, either $a-axa^2y$ or $(1-a)-(1-a)z(1-a)^2w$ is an SWR element of R. If $a-axa^2y$ is SWR, then we have $(a-axa^2y) = (a-axa^2y)t(a-axa^2y)^2s$ for some $t, s \in R$. By Lemma 5.2.13, it follows that $a = aga^2h$ for some $g, h \in R$. Similarly, if $(1-a)-(1-a)z(1-a)^2w$ is SWR, then we can show that $1-a$ is SWR. □

In [28], Gupta introduced $S(R) = \{a \in R \mid (a) \text{ is a SWR ideal in } R\}$, which is the unique maximal two sided SWR ideal of R, where (a) is the principal ideal of R generated by $a \in R$ and proved that $S(R/S(R)) = 0$. Following [5], $M(R) = \{a \in R \mid (a) \text{ is a regular ideal in } R\}$ is the unique maximal two sided regular ideal of R. In [15], Chen and Tong gave a characterization of abelian VNL rings through local rings. Analogously, we characterize commutative almost SWR rings through local rings.

Proposition 5.2.15. *Let R be a commutative ring. Then, R is an almost SWR ring if and only if $R/S(R)$ is a local ring.*

Proof. Suppose that $R/S(R)$ is a local ring. Then $R/S(R)$ is an almost SWR ring. Thus, by Theorem 5.2.14, R is an almost SWR ring.

Conversely, it is easy to see that a commutative almost SWR ring R is a VNL-ring. Let I be a SWR ideal in R. Then, we have

$$\begin{aligned} S(R) &= \{a \in R \mid ar \in I, r \in R\} \\ &= \{a \in R \mid ar = (ar)x(ar)^2y, x, y \in I\} \\ &= \{a \in R \mid ar = (ar)z(ar), z = x(ar)y \in I\} \\ &= M(R). \end{aligned}$$

Then, in view of [15, Lemma 2.7], $R/S(R)$ is local. □

The necessary conditions of Theorem 5.2.15 is not true for arbitrary rings, as shown in the following example.

Example 5.2.16. *Let $R = T_2(\mathbb{Z}_2)$. Then R is an almost SWR ring but $R/S(R)$ is not local. Since in view of [28, Theorem 10(4)], $S(T_2(\mathbb{Z}_2)) = 0$. Then, $R/S(R) = T_2(\mathbb{Z}_2)$ is not local.*

Proposition 5.2.17. *Let L be some nonempty subset of R and $(L)_r$ be a right ideal generated by L. Then, for a commutative ring R, the following are equivalent:*

(1) R is a almost SWR ring;

(2) At least one of the element in L is SWR, whenever $(L)_r = R$.

Proof. For any $a \in R$, let $L = \{a, 1 - a\}$. Since $1 = a + 1 - a \in (L)_r$, $(L)_r = R$. Thus, either a or $1 - a$ is SWR.

Conversely, if R is SWR, the result follows. Otherwise, suppose that R is an almost SWR ring which is not SWR. Now there exist $l_1, l_2, \ldots, l_t$ in any nonempty subset L of R with $(L)_r = R$ such that $l_1R + l_2R + \cdots + l_tR = R$. Then, there exist $r_1, r_2, \ldots, r_t \in R$ satisfying $l_1r_1 + l_2r_2 + \cdots + l_tr_t = 1$, and thus $\bar{l}_1\bar{r}_1 + \bar{l}_2\bar{r}_2 + \cdots + \bar{l}_t\bar{r}_t = \bar{1}$ in $\overline{R} = R/S(R)$. And by Proposition 5.2.15, $\overline{R}$ is a local ring. It follows that there exists an $\bar{l}_k$ such that $\bar{l}_k \in U(\overline{R})$; thus, $\bar{l}_k$ is SWR in $\overline{R}$. So, $\bar{l}_k = \bar{l}_k\bar{x}_k(\bar{l}_k)^2\bar{y}_k$ for some $\bar{x}_k, \bar{y}_k \in \overline{R}$. Then $l_k - l_kx_k(l_k)^2y_k \in S(R)$, it implies that $l_k - l_kx_k(l_k)^2y_k = (l_k - l_kx_k(l_k)^2y_k)a_k(l_k - l_kx_k(l_k)^2y_k)^2b_k$ for some $a_k, b_k \in R$. Thus, l_k is an SWR element by Lemma 5.2.13. □

The following proposition shows that an almost SWR ring R is the direct summand of either $r(a)$ or $r(1 - a)$ for all $a \in R$.

Proposition 5.2.18. *If $r(a) = r(b)$ and $r(1 - a) = r(1 - b)$, for each $a \in R$ and $b \in Ra^2R$. Then R is almost SWR if and only if either $r(a)$ or $r(1 - a)$ is direct summand.*

Proof. Let $a \in R$ and $r(a)$ be the direct summand. Then, we have an ideal $I \subset R$ such that $R = r(a) \oplus I$. So, there exist $d \in r(a)$ and $b \in I$ such that $d + b = 1$ and hence, $a = ad + ab$. Thus, $a = ab$. Since Ra^2R is a two sided ideal of R, $b \in Ra^2R$. Thus, a is SWR. If $r(1 - a)$ is direct summand. There exists an ideal $J \subset R$ such that $R = r(1 - a) \oplus J$, then we can prove that $1 - a$ is SWR.

Conversely, let for any $a \in R$, either a or $1 - a$ is SWR. If a is SWR, then there exists $b = ta^2s \in Ra^2R$ such that $a = ata^2s$ for some $t, s \in R$. Then, $a(1 - ta^2s) = 0$, so $(1 - ta^2s) \in r(a)$. Thus, $1 = (1 - ta^2s) + ta^2s$. Hence, $R = r(a) + Ra^2R$. Now suppose that $x \in r(a) \cap Ra^2R$, then $ax = 0$ and $x = ta^2s$ for some $t, s \in R$. Thus, $ta^2s \in r(a) = r(b)$, so $bta^2s = 0$. Then, $bx = 0$ and so, $x = 0$. Therefore, $r(a) \cap Ra^2R = 0$. Hence, $R = r(a) \oplus Ra^2R$. Similarly, if $1 - a$ is SWR, then we can deduce that $R = r(1 - a) + R(1 - a)^2R$ and $r(1 - a) \cap R(1 - a)^2R = 0$. Hence, $R = r(1 - a) \oplus R(1 - a)^2R$. □

5.3 Extension Rings

We start this section with the necessary conditions for an upper triangular matrix ring to be almost SWR.

The proof of the following lemma is easy.

Lemma 5.3.1. *Let diag$(a_1, a_2, \ldots, a_n)$ be the $n \times n$ diagonal matrix with a_i in each entry on the main diagonal. Then, diag$(a_1, a_2, \ldots, a_n)$ is SWR in $T_n(R)$ if and only if $a_1, a_2, \ldots, a_n$ are all SWR in R.*

Theorem 5.3.2. *If $T_n(R)$ is an almost SWR ring for some $n \geq 2$, then R is an SWR ring.*

Proof. Let $A = \text{diag}(a, 1-a, 1, \ldots, 1) \in T_n(R)$. Then $I_n - A = \text{diag}(1-a, a, 0, \ldots, 0)$. Since $T_n(R)$ is an almost SWR ring, either A or $I_n - A$ is SWR. For any case, by Lemma 5.3.1, a is SWR. Thus, R is SWR. □

The example given below shows that the converse of above Theorem 5.3.2 may not be true.

Example 5.3.3. *The ring $T_2(\mathbb{Z}_6)$ is not almost SWR because neither $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ nor $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ is SWR although $\mathbb{Z}_6$ is an SWR ring.*

Proposition 5.3.4. *For any ring R and $n \geq 4$, $T_n(R)$ is not an almost SWR ring.*

Proof. By applying Proposition 7.3.2(3), we may assume that $n = 4$. Let $C = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, then neither $\text{diag}(C, I_2 - C) = \left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\right)$ nor $diag(I_2 - C, C) = \left(\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right)$ is SWR. Hence, $T_n(R)$ is not an almost SWR ring for any $n \geq 4$. □

Let A be a ring and B a subring of ring A with $1_A \in B$. We set

$$R[A, B] = \{(c_1, c_2 \ldots, c_n, d, d \ldots) | c_i \in A, d \in B, n \geq 1\}$$

with addition and multiplication defined componentwise.

Theorem 5.3.5. *The following statements are equivalent:*

(1) $R[A, B]$ is an almost SWR ring.

(2) A is an SWR ring and B is an almost SWR ring.

Proof. Construct a homomorphism $f : R[A, B] \to B$ defined by $f(c_1, c_2, \ldots, c_n, d, d \ldots) = d$. Then, $R[A, B]/kerf \cong B$. Thus, B is an almost SWR ring by using Proposition 7.3.2(1). If A is not an SWR ring, we have a non SWR element $\alpha \in A$. Let $x = (\alpha, 1-\alpha, 1, 1, \ldots) \in R[A, B]$. So, either x or $1 - x = (1-\alpha, \alpha, 0, 0, \ldots) \in R[A, B]$ is SWR. If x is SWR, so is $\alpha \in A$, a contradiction. Hence, we conclude that A is an SWR ring.

Conversely, for any $(c_1, c_2, \ldots, c_n, d, d, \ldots) \in R[A, B]$ with each $c_i \in A$ and $d \in B$. Since A is an SWR ring, we have $c_i = c_i t_i {c_i}^2 s_i$ for some t_i, s_i in A

and B is an almost SWR ring, then either d or $1-d$ is SWR. If d is SWR, we can find some g, h in B such that $d = dgd^2h$. Thus, $(c_1, c_2, \ldots, c_n, d, d\ldots) = (c_1, c_2, \ldots, c_n, d, d, \ldots)(t_1, t_2, \ldots, t_n, g, g, \ldots)(c_1, c_2, \ldots, c_n, d, d, \ldots)^2(s_1, s_2, \ldots, s_n, h, h, \ldots)$. This implies that $(c_1, c_2, \ldots, c_n, d, d, \ldots) \in R[A, B]$ is SWR. If $1-d$ is SWR, we have $1-d = (1-d)y(1-d)^2z$ for some y, z in B. Thus, we get $(1, 1, \ldots, 1, 1, 1, \ldots)-(c_1, c_2, \ldots, c_n, d, d, \ldots)=(1-c_1, 1-c_2, \ldots, 1-c_n, 1-d, 1-d, \ldots) \in R[A, B]$ is SWR. Therefore, $R[A, B]$ is an almost SWR ring. □

Corollary 5.3.6. *$R[A, A]$ is an almost SWR ring if and only if A is an SWR ring.*

Let R be a ring, then the trivial extension of R over R is

$$R\Theta R = \{(s, n) \mid s \in R, n \in R\}$$

with componentwise addition and multiplication defined by $(s_1, n_1)(s_2, n_2) = (s_1s_2, n_1s_2 + s_1n_2)$. Then, $R\Theta R$ is isomorphic to subring $\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R\right\}$ of $T_2(R)$.

Theorem 5.3.7. *Let R be a ring. If $R\Theta R$ is an almost SWR ring, then R is almost SWR.*

Proof. Let $\theta : R\Theta R \to R$ be a canonical epimorphism. Then, we have $R\Theta R/0\Theta R \cong R$. Hence, R is an almost SWR ring by Proposition 7.3.2(1). □

Proposition 5.3.8. *For a ring S and $n \geq 2$, $R = T_n(S)$. Then, $R\Theta R$ is not an almost SWR ring.*

Proof. Assume that $n = 2$. Let $A = (C, I_2) \in R\Theta R$, where $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Suppose that A is SWR, then there exist (X, Y), $(V, W) \in R\Theta R$ such that $(C, I_2) = (C, I_2)(X, Y)(C, I_2)^2(V, W)$. Thus $(X + CY)C^2V + 2CXCV + CXC^2W = I_2$. Write $X = \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}, Y = \begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix}, V = \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix}$ and $W = \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}$. Then we obtain $\left[\begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix}\right]\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^2\begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^2\begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we get a contradiction by comparing the $(2, 2)$ entry of matrices on both side. Similarly, we can also show that $(I_2, 0) - A$ is not SWR. Hence, $T_2(S)\Theta T_2(S)$ is not an almost SWR ring.

Suppose that $n \geq 3$. Let $C = \begin{pmatrix} C_1 & \alpha \\ 0 & C_2 \end{pmatrix}, D = \begin{pmatrix} D_1 & \beta \\ 0 & D_2 \end{pmatrix} \in R$, where $C_1, D_1 \in T_2(S)$. If (C, D) is SWR ring in $R\Theta R$, then (C_1, D_1) is SWR in $T_2(S)\Theta T_2(S)$. As $T_2(S)\Theta T_2(S)$ is not almost SWR, neither is $R\Theta R$. □

The converse of Theorem 5.3.7 does not hold, which is shown in the following corollary.

Corollary 5.3.9. *Let $R = T_2(\mathbb{Z}_2)$ be an almost SWR ring. Then, $R\Theta R$ is not almost SWR.*

Proof. From Proposition 5.3.8, $R\Theta R = T_2(\mathbb{Z}_2)\Theta T_2(\mathbb{Z}_2)$ is not almost SWR. □

5.4 Semiperfect Almost SWR Rings

In this section, we consider the structure of semiperfect (see [10]) almost SWR rings. Recall that a ring R is called reduced if R has no nonzero nilpotent elements.

Lemma 5.4.1. *[27, Lemma 4.2]. Let e_1 and e_2 be two local idempotents of a ring R. Then, either $e_1R \cong e_2R$, or $e_1Re_2 \subseteq J(R)$ and $e_2Re_1 \subseteq J(R)$.*

Proposition 5.4.2. *Let R be a semiperfect ring with $1 = e_1 + e_2$, where e_1, e_2 are orthogonal primitive idempotents. If R is almost SWR, then R is isomorphic to either of the following:*

(1) $M_2(C)$ for some reduced ring C.

(2) $\begin{pmatrix} A & X \\ Y & B \end{pmatrix}$ where A is a reduced ring, B is a local ring and $XY \subseteq J(A)$, $YX \subseteq J(B)$.

In particular, if $J(R) = 0$. Then, R is isomorphic to either $M_2(C)$ or $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ where A_1, C are reduced and A_2 is a local ring.

Proof. Consider the Pierce decomposition

$$R \cong \begin{pmatrix} e_1Re_1 & e_1Re_2 \\ e_2Re_1 & e_2Re_2 \end{pmatrix}$$

If $e_1R \cong e_2R$, then $R \cong M_2(e_1Re_1)$, where e_1Re_1 is a local ring. By using Lemma 5.2.9, e_1Re_1 is an SWR ring. Then in view of [28, Theorem 5], e_1Re_1 is reduced. If $e_1R \ncong e_2R$, then e_1Re_2 and e_2Re_1 are contained in $J(R)$ by Lemma 5.4.1. Again by Lemma 5.2.9, either e_1Re_1 or e_2Re_2 is SWR. It follows that either e_1Re_1 or e_2Re_2 is a reduced ring. We assume that e_1Re_1 is a reduced ring. Note that $e_1Re_2Re_1 \subseteq J(R) \cap e_1Re_1 = J(e_1Re_1)$ and

$e_2Re_1Re_2 \subseteq J(R) \cap e_2Re_2 = J(e_2Re_2)$. So take $A = e_1Re_1$, $B = e_2Re_2$, $X = e_1Re_2$ and $Y = e_2Re_1$, we obtain that $R \cong \begin{pmatrix} A & X \\ Y & B \end{pmatrix}$. □

Proposition 5.4.3. *Let R be a semiperfect ring with $1 = e_1 + e_2 + e_3$, where $\{e_1, e_2, e_3\}$ is a orthogonal set of primitive idempotents. If R is almost SWR, then R is isomorphic to one of the followings:*

(1) $M_3(C)$ for some reduced ring C.

(2) $\begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix}$ where R_1 is a reduced ring, R_2 is a local ring and $XY \subseteq J(R_1)$, $YX \subseteq J(R_2)$.

(3) $\begin{pmatrix} R_1 & X \\ Y & R_2 \end{pmatrix}$ where R_1 is semiprime, R_2 is a local ring and $XY \subseteq J(R_1)$, $YX \subseteq J(R_2)$.

(4) $\begin{pmatrix} A & X \\ Y & C \end{pmatrix}$ with $A \cong \begin{pmatrix} R_1 & X_1 \\ Y_1 & R_2 \end{pmatrix}$ and R_1, R_2, C are reduced rings, $X_1Y_1 \subseteq J(R_1)$, $Y_1X_1 \subseteq J(R_2)$.

Proof. **Case 1**. If $e_iR \cong e_jR$ for all i, j, then $R \cong M_3(e_1Re_1)$ where e_1Re_1 is a local ring. By Lemma 5.2.9, e_1Re_1 is a reduced ring.

Now we consider Pierce decomposition

$$R \cong \begin{pmatrix} (1-e_1)R(1-e_1) & (1-e_1)Re_1 \\ e_1R(1-e_1) & e_1Re_1 \end{pmatrix}$$

Case 2. Assume that e_1Re_1 is local but not a reduced ring, then $(1-e_1)R(1-e_1)$ is a reduced ring by [28, Theorem 5]. Thus, e_2Re_2 and e_3Re_3 are reduced rings. So by Lemma 5.4.1, $e_1Re_2, e_2Re_1, e_1Re_3$ and e_3Re_1 are all contained in $J(R)$. Thus $(1-e_1)Re_1R(1-e_1) \subseteq J(R) \cap (1-e_1)R(1-e_1) = J((1-e_1)R(1-e_1))$ and $e_1R(1-e_1)Re_1 \subseteq J(R) \cap e_1Re_1 = J(e_1Re_1)$. Hence, R is as in (2) above.

Case 3. Suppose that all e_iRe_i are reduced rings. If $e_2R \cong e_3R$ but $e_1R \ncong e_2R$, then $(1-e_1)R(1-e_1) \cong M_2(C)$ for some reduced ring C, and so C is a semiprime ring. Then $M_2(C)$ is semiprime by [39, Proposition 10.20]. Hence, $(1-e_1)R(1-e_1)$ is semiprime. By Lemma 5.4.1, $(1-e_1)Re_1R(1-e_1) \subseteq J((1-e_1)R(1-e_1))$ and $e_1R(1-e_1)Re_1 \subseteq J(e_1Re_1)$. Thus, R is as in (3).

Case 4. Suppose that e_iRe_i is a reduced ring for all i=1,2,3 and $e_1R \ncong e_2R \ncong e_3R$. Then

$$(1-e_1)R(1-e_1) \cong \begin{pmatrix} e_2Re_2 & e_2Re_3 \\ e_3Re_2 & e_3Re_3 \end{pmatrix},$$

where $e_2Re_3Re_2 \subseteq J(e_2Re_2)$ and $e_3Re_2Re_3 \subseteq J(e_3Re_3)$. Now note that $(1-e_1)Re_1R(1-e_1) \subseteq J(R) \cap (1-e_1)R(1-e_1) = J((1-e_1)R(1-e_1))$. So by taking $e_2Re_2 = R_1, e_3Re_3 = R_2, e_3Re_2 = Y_1, e_2Re_3 = X_1, e_1Re_1 = C, (1-e_1)Re_1 = Y, e_1R(1-e_1) = X$. Thus (3) follows. □

5.5 Almost SWR Group Rings

We start this section with the necessary conditions for RG to be SWR.

Theorem 5.5.1. *If RG is an SWR ring. Then, R is an SWR ring and G is a torsion group.*

Proof. By the augmentation map, R is an image of RG. Since homomorphic image of an SWR ring is SWR, R is SWR. Let $g(\neq 1) \in G$. Since RG is SWR, $1 - g = (1 - g)x$ where $x \in ((1 - g)^2)$. Then, $(1 - g)(1 - x) = 0$. This implies that $1 = x \in \omega G$, which is a contradiction. Thus, $1 - g$ is a zero divisor, and hence g is of finite order by [17, Proposition 6]. Thus, G is a torsion group. □

Recall that an abelian torsion group is locally finite.

Corollary 5.5.2. *Let G be an abelian group. If RG is SWR, then R is SWR, and G is locally finite.*

Theorem 5.5.3. *If RG is an SWR ring. Then for each $n \in o(G)$, n is a unit in R, where $o(G)$ denotes the set of orders of all finite subgroups of G.*

Proof. Let n be the order of $g \in G$. We will show that n is a unit in R. Since RG is SWR, there exist $x, y \in RG$ such that $(1 - g)(1 - x(1 - g)^2y) = 0$. By using [17, Proposition 6], $(1 - x(1 - g)^2y) = (1 + g + g^2 + \cdots + g^{n-1})r$ for some $r \in RG$ and by applying augmentation map $\omega : RG \to R$ on above equation, we get $1 = n\omega(r)$, where $\omega(r) \in R$. □

The following example shows that the converse of Theorem 5.5.1 is not true.

Example 5.5.4. *Let $R = \mathbb{Z}_2C_2$. Then, $\mathbb{Z}_2$ is SWR and C_2 is torsion but R is not SWR.*

If RG is commutative, then we have necessary and sufficient conditions for RG to be SWR.

Theorem 5.5.5. *Let RG be a commutative ring. Then RG is SWR if and only if*

(1) R is SWR.

(2) G is locally finite.

(3) for each $n \in o(G)$, n is a unit in R.

Proof. The necessity follows from Theorem 5.5.1 and Theorem 5.5.3, and the sufficiency follows from the fact that commutative SWR rings are VNR and by [17, Theorem 3]. □

The next result gives necessary conditions for group algebra KG over a field K satisfying a nontrivial polynomial identity to be SWR.

Theorem 5.5.6. *Let KG be a group algebra over a field K satisfying a nontrivial polynomial identity. If KG is SWR, then K is SWR, and G is locally finite.*

Proof. Suppose that KG is SWR. Then, since homomorphic image of an SWR ring is SWR, K is SWR. In view of Theorem 5.5.1, G is a torsion group, and by [53, Theorem 5.5], we have $|G : \Delta(G)| < \infty$. Let H be a finitely generated subgroup of G. Then $|H : H \cap \Delta(G)| < \infty$, and in view of [53, Lemma 6.1], $H \cap \Delta(G)$ is a finitely generated subgroup of $\Delta(G)$. Since by [53, Lemma 2.2], the center $C(H \cap \Delta(G))$ of $H \cap \Delta(G)$ is a subgroup of finite index, $|H : C(H \cap \Delta(G))| < \infty$. Thus, again, by [53, Lemma 6.1], $C(H \cap \Delta(G))$ is a finitely generated torsion group. So, $C(H \cap \Delta(G))$ is finite. Hence, H is finite. □

Remark 5.5.7. *The condition in Theorem* 5.5.3 *is not necessary for RG to be almost SWR since $R = \mathbb{Z}_4C_2$ is almost SWR, but* 2 *is not a unit in* $\mathbb{Z}_4$.

Theorem 5.5.8. *Let R be a commutative local ring and G an abelian p-group with $p \in J(R)$. Then, RG is an almost SWR ring.*

Proof. Suppose that R is a commutative ring and G an abelian p-group with $p \in J(R)$. Following [68, Lemma 2.1] we get that $\omega G \subseteq J(RG)$. Then, R being local implies that RG is local by [47]. Hence, RG is an almost SWR ring. □

Example 5.5.9. *Let $R = \mathbb{Z}_{(p)} = \left\{\frac{b}{a} \mid b, a \in \mathbb{Z}, gcd(a, p) = 1\right\}$ and $G = C_p$. The group ring RG is almost SWR.*

Lemma 5.5.10. *Let G be a group. If RH is almost SWR for every finitely generated subgroup H of G, then RG is almost SWR.*

Proof. Let $\alpha \in RG$ and H be a subgroup generated by the support of α. Then H is a finitely generated subgroup of G. Thus, either α or $1 - \alpha$ is SWR in RH. Assume that α is SWR, then we have $\alpha \in \alpha RH\alpha^2 RH \subseteq \alpha RG\alpha^2 RG$. It follows that α is SWR in RG. Similarly, if $1-\alpha$ is SWR in RH, then $1-\alpha \in (1-\alpha)RH(1-\alpha)^2 RH \subseteq (1-\alpha)RG(1-\alpha)^2 RG$. Thus, $1 - \alpha$ is an SWR element in RG. Hence, RG is an almost SWR ring. □

If H and K are subgroups of G such that: $H \triangleleft G, H \cap K = \{1\}$ and $HK = G$, then G is called a semidirect product of H by K, denoted by $G = H \rtimes K$. The following result shows that the converse of Lemma 5.5.10 partially holds.

Theorem 5.5.11. *Let $G = H \rtimes K$, $|H| < \infty$. If RG is an almost SWR ring, then RK is an almost SWR ring.*

Proof. For any $\alpha \in RK$, either α or $1-\alpha$ is SWR in RG. Assume that α is SWR, then we have $\alpha = \alpha a \alpha^2 b$ for some $a, b \in RG$. Let $a = \sum a_i k_i$ and $b = \sum b_i k_i$, where $a_i, b_i \in RH$, $k_i \in K$ and let $\alpha = \sum \alpha_j k_j$, where $\alpha_j \in R$. Denote $x = \sum \omega(a_i)k_i, y = \sum \omega(b_i)k_i$, so $x, y \in RK$. We will show that $\alpha = \alpha x \alpha^2 y$ for some $x, y \in RK$.

Let $\xi : G \to G/H$ stand for the natural group homomorphism and then extend ξ to a ring homomorphism $\overline{\xi} : RG \to R(G/H)$, defined by $\overline{\xi}(\sum \alpha_i g_i) = \sum \alpha_i \overline{\xi}(g_i)$. Obviously, $Ker(\overline{\xi}) \cap RK = 0$ and $\overline{\xi}(z) = \omega(z)$ for all $z \in RH$.

Since $0 = \alpha - \alpha a \alpha^2 b$, we have

$$\begin{aligned}
0 &= \overline{\xi}(\alpha) - \overline{\xi}(\alpha)\overline{\xi}(a)\overline{\xi}(\alpha^2)\overline{\xi}(b) \\
&= \overline{\xi}(\alpha) - \overline{\xi}(\alpha)\overline{\xi}(\sum a_i k_i)\overline{\xi}(\alpha^2)\overline{\xi}(\sum b_i k_i) \\
&= \overline{\xi}(\alpha) - \overline{\xi}(\alpha)\sum \omega(a_i)\overline{\xi}(k_i)\overline{\xi}(\alpha^2)\sum \omega(b_i)\overline{\xi}(k_i) \\
&= \overline{\xi}(\alpha) - \overline{\xi}(\alpha)\overline{\xi}(\sum \omega(a_i)k_i)\overline{\xi}(\alpha^2)\overline{\xi}(\sum \omega(b_i)k_i) \\
&= \overline{\xi}(\alpha) - \overline{\xi}(\alpha)\overline{\xi}(x)\overline{\xi}(\alpha^2)\overline{\xi}(y) \\
&= \overline{\xi}(\alpha - \alpha x \alpha^2 y).
\end{aligned}$$

Then, $\alpha - \alpha x \alpha^2 y \in Ker(\overline{\xi}) \cap RK = 0$, so we have $\alpha = \alpha x \alpha^2 y$. Similarly, if $1-\alpha$ is SWR, then we can find $t = \sum \omega(t_i)k_i$ and $s = \sum \omega(s_i)k_i$ in RK such that $1 - \alpha = (1-\alpha)t(1-\alpha)^2 s$. □

Remark 5.5.12. *An artinian ring RG may not be an almost SWR ring.*

Example 5.5.13. *The group ring $(\mathbb{Z}_4 \times \mathbb{Z}_4)C_2$ is artinian but not almost SWR.*

For any nontrivial finite group G, group ring RG may or may not be an almost SWR ring.

Example 5.5.14. *$\mathbb{Z}G$ is not almost SWR for any nontrivial finite group G.*

Example 5.5.15. *Let $\mathbb{Z}_2 = \{0, 1\}$ and $G = (g|g^2 = 1)$. An element $1 + g \in \mathbb{Z}_2 G$ is not SWR but $1 - (1 + g)$ is SWR. So, $\mathbb{Z}_2 G$ is an almost SWR ring.*

Proposition 5.5.16. *Let K be a field of $char(K) = p > 0$ and G a finite p-group. Then group algebra KG is almost SWR.*

Proof. Suppose that K be a field of $char(K) = p > 0$ and G a finite p-group. Then by [39, Corollary 8.8], jacobson radical of group algebra $J(KG)$ is equal to augmentation ideal ωG with $J(KG)^{|G|} = 0$. It follows that $KG/J(KG) \cong K$. Since K is a division ring, KG is local. Thus, KG is an almost SWR ring. □

Chapter 6

On Semiboolean Neat Rings

In this chapter, we introduce a new class of ring called semiboolean neat ring. A ring R is said to be semiboolean neat if every proper homomorphic image of R is semiboolean. Semiboolean neat rings are the proper subclass of neat rings and proper generalization of semiboolean and nil neat rings. A commutative semiboolean neat ring which is not semiboolean is reduced. Further, we discuss commutative group ring RG to be semiboolean neat.

6.1 Introduction

Nicholson and Zhou [50] introduced and investigated semiboolean rings. The class of semiboolean rings is strictly between the classes nil clean and clean rings. On the other hand, the class of commutative neat rings was defined and investigated by McGovern [44]. The ring of integers, $\mathbb{Z}$, and any nonlocal PID are example of a neat ring that is not clean. The class of commutative nil neat rings was investigated by Samiei [58]. A ring R is called *nil neat* if every proper homomorphic image of R is nil clean. The class of nil clean rings is nil neat, but the converse containment is not valid, as shown in [58, Example 2.10]. In this chapter, we introduced semiboolean neat rings. The ring R is *semiboolean neat* provided that every proper homomorphic image of R is semiboolean. The class of semiboolean neat rings is a proper subclass of the class of neat rings because every semiboolean ring is clean. And since every nil clean ring is semiboolean, the class of semiboolean neat rings is a proper generalization of class of nil neat rings. Thus, the class of semiboolean neat rings is strictly between the classes nil neat and neat rings.

So, here, our motivation is to refine the class of neat rings. We provide various properties of semiboolean neat rings. It is proved that a semiboolean neat ring which is not semiboolean is reduced. If ring R is semiboolean neat, then the matrix ring $M_n(R)$ may not be semiboolean neat see remark 6.2.13. Recall that a ring A is called radical if $J(A) = A$. We prove that matrix ring $M_n(A)$ is semiboolean neat if and only if A is a radical ring. It is proved that if $J(R) \neq 0$ is a nil ideal, then R is semiboolean neat if and only if R is semiboolean. We determine the necessary and sufficient conditions for a commutative group ring RG to be semiboolean neat.

6.2 Semiboolean Neat Rings

We start this section with some observations about semiboolean neat rings.

Remark 6.2.1. *(1) Every semiboolean ring is semiboolean neat ring.*

(2) Every nil neat ring is semiboolean neat ring.

Nil clean ring	$\Longrightarrow$	Semiboolean ring	$\Longrightarrow$	Clean ring
$\Downarrow$		$\Downarrow$		$\Downarrow$
Nil neat ring	$\Longrightarrow$	Semiboolean neat ring	$\Longrightarrow$	Neat ring

The following examples demonstrate that the reverse implication of aforementioned is not true.

Example 6.2.2. *(1) If $R = \mathbb{Z}_2$ and G is a universal locally finite group, then G is a simple group, $\Delta(G) = \{1\}$ and RG is prime ([54, Theorem 9.4.9]). According to Passman [54, Corollary 9.4.10], ωG is the unique proper ideal of RG. Because $RG/\omega G \cong R$, R is the only proper homomorphic image of RG. As a result, RG is a semiboolean neat ring but not semiboolean. Assume that $\mathbb{Z}_2 G$ is semiboolean. Then, according to [65, Theorem 3.1], G is a 2-group but a universal locally finite group does not have to be a 2-group ([54, Theorem 9.4.8]).*

(2) Let $R = \mathbb{Z}_{(2)} \times \mathbb{Z}_4$ or $R = \mathbb{Z}_2[[x]] \times \mathbb{Z}_4$. Since every proper homomorphic image of R is semiboolean, R is a semiboolean neat ring. But homomorphic image $\mathbb{Z}_{(2)}$ or $\mathbb{Z}_2[[x]]$ respectively of R is not nil clean, so R is not a nil neat ring.

(3) Let $R = \mathbb{Z}$ or $R = \mathbb{Z}_2 \times \mathbb{Z}_3$ or $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then, R is a neat ring but not semiboolean neat since homomorphic image $\mathbb{Z}_3$ of R is not semiboolean.

Proposition 6.2.3. *(1) A homomorphic image of a semiboolean neat ring is a semiboolean neat ring.*

(2) The direct product of semiboolean neat rings does not have to be semiboolean neat.

(3) Polynomial ring is not semiboolean neat.

Proof. (1) It follows from the fact that homomorphic image of a semiboolean ring is semiboolean.

(2) Let $R = \mathbb{Z}_2G \times \mathbb{Z}_4$, where G is a universal locally finite group. Then, both $\mathbb{Z}_2G$ and $\mathbb{Z}_4$ are semiboolean neat since every proper homomorphic image of $\mathbb{Z}_2G$ and $\mathbb{Z}_4$ is semiboolean. But proper homomorphic image $\mathbb{Z}_2G$ of $\mathbb{Z}_2G \times \mathbb{Z}_4$ is not semiboolean, so R is not semiboolean neat.

(3) Consider F is a field and $R = F[x, y]$. Then, R is not semiboolean neat since $R/yR \cong F[x]$ is not clean. It follows that $F[x]$ is not semiboolean. Thus R is not semiboolean neat. □

Proposition 6.2.4. *Let R be a decomposable ring. Then, R is a semiboolean neat ring if and only if R is semiboolean.*

Proof. Suppose that R is a decomposable ring. Then, there are ideals I and J such that $R = I \oplus J$. Now, if R is semiboolean neat, then $J \cong R/I$ (respectively $I \cong R/J$) is also semiboolean. Thus, R being a direct product of semiboolean rings is semiboolean by [50, Example 25(3)]. □

Lemma 6.2.5. *[50, Theorem 29] Let A be a general ring and let $I \lhd A$. Then, A is semiboolean if and only if the following conditions hold:*

(1) I and A/I are semiboolean.

(2) Every idempotent of A/I can be lifted to an idempotent of A.

(3) $J(A/I) = (I + J(A))/I$.

Theorem 6.2.6. *The following statements are equivalent for any commutative ring R:*

(1) R is semiboolean neat.

(2) R/aR is semiboolean for every nonzero $a \in R$.

(3) For any collection of nonzero prime ideals $\{P_j\}_{j\in J}$ of R with $I = \cap_{j\in J} P_j$ different than 0 we have R/I is semiboolean.

(4) R/aR is semiboolean neat for every $a \in R$.

(5) R/I is semiboolean for every nonzero semiprime ideal.

Proof. (1) $\Rightarrow$ (2) It follows from the fact that homomorphic image of a semiboolean ring is semiboolean.

(2) $\Rightarrow$ (1) Let J be an ideal of R and $a \in J$. Now, consider $I = aR$, clearly $I \subseteq J$. Then, J/I is an ideal of R/I. In view of third ring isomorphism theorem $(R/I)/(J/I) \cong R/J$. Since R/I is semiboolean, R/J is semiboolean by [50, Example 25(2)]. Thus, R is a semiboolean neat ring.

(3) $\Leftrightarrow$ (5) It is straightforward.

(1) $\Rightarrow$ (4) Since R is semiboolean neat, R/aR is a semiboolean ring, where a is a nonzero element of R. Thus, R/aR is semiboolean neat for every $a \in R$.

(4) $\Rightarrow$ (1) It is true by using $a = 0$.

(1) $\Rightarrow$ (5) It is obvious.

(5) $\Rightarrow$ (1) It is well known that $P(R)$ is a semiprime ideal.

Case 1: Suppose that $P(R) \neq 0$. As $P(R) \subseteq J(R)$, by [39, Example 10.17(d)] there exist a surjective homomorphism $\phi : R/P(R) \to R/J(R)$. By hypothesis, we have $R/P(R)$ is semiboolean, then $R/P(R)$ is clean. Thus, R is clean by the fact that R is clean if and only if $R/P(R)$ is clean. Since homomorphic image of a semiboolean ring is semiboolean, $R/J(R)$ is semiboolean. It follows that $R/J(R)$ is boolean. In view of [65, Lemma 3.2], R is semiboolean. Hence R is semiboolean neat.

Case 2: Suppose that $P(R) = 0$. Let S be a proper homomorphic image of R, i.e., $\varphi : R \to S$. Then, φ induces a surjection of semiprime rings $\overline{\varphi} : R/P(R) \to S/P(S)$. Since $P(R) = 0$, $\overline{\varphi} : R \to S/P(S)$. By hypothesis, $S/P(S)$ is semiboolean, then $S/P(S)$ is clean. Thus, S is clean by the fact that S is clean if and only if $S/P(S)$ is clean. Since $P(S) \subseteq J(S)$, by [39, Example 10.17(d)] we have $f : S/P(S) \to S/J(S)$. It follows that $S/J(S)$ is semiboolean. So $S/J(S)$ is boolean. In view of [65, Lemma 3.2], S is semiboolean. Thus, R is semiboolean neat. □

We use $N(R)$ to represent the set of all nilpotent elements in R.

Theorem 6.2.7. *Let R be a commutative ring. If R is a semiboolean neat ring but not semiboolean, then R is reduced.*

Proof. Suppose that $N(R) \neq 0$. Then, $R/N(R)$ is semiboolean. Note that $N(R)$ is semiboolean. Since $N(R)$ is a nil ideal, the idempotents of $R/N(R)$ can be lifted to R by [39, Theorem 21.28]. Note that since $N(R) \subseteq J(R)$,

$$\frac{J(R/N(R))}{(J(R)+N(R))/N(R)} = J\left(\frac{R/N(R)}{(J(R)+N(R))/N(R)}\right) = J\left(\frac{R}{J(R)+N(R)}\right) = J\left(\frac{R}{J(R)}\right) = 0.$$

This means that $J(R/N(R)) = (J(R)+N(R))/N(R)$. Thus, by Lemma 6.2.5, R is semiboolean, which gives a contradiction. □

Proposition 6.2.8. *Let R be a ring with $0 \neq J(R)$ nil. Then, R is semiboolean neat if and only if R is semiboolean.*

Proof. (⇒) Suppose that R is a semiboolean neat ring. Then, $R/J(R)$ is semiboolean. It follows that $R/J(R)$ is boolean. Since $0 \neq J(R)$ is nil, idempotents in $R/J(R)$ can be lifted to modulo $J(R)$. Thus, R is semiboolean.

(⇐) It is trivial. □

Remark 6.2.9. *If $J(R) = 0$, then above Proposition 6.2.8 may not be true. In Example 6.2.2, Jacobson radical is 0 and RG is semiboolean neat but not semiboolean since G need not to be a 2-group.*

Corollary 6.2.10. *Let $0 \neq J(R)$ be a nil ideal of a ring R. Then, R is clean UJ ring if and only if R is semiboolean neat ring.*

Proof. (⇒) It follows from [36, Theorem 4.2] and by the fact that semiboolean rings are semiboolean neat.

(⇐) Suppose that R is a semiboolean neat ring. Then, by Proposition 6.2.8, R is semiboolean. Thus, in view of [36, Theorem 4.2], R is a clean UJ ring. □

Proposition 6.2.11. *Let R be a commutative semiboolean neat ring. Then, $R/M \cong \mathbb{Z}_2$ for every nonzero maximal ideal.*

Proof. Suppose that R is a semiboolean neat ring. Then, R/M is semiboolean. Since M is a maximal ideal, R/M is field. A semiboolean field is isomorphic to $\mathbb{Z}_2$. □

Corollary 6.2.12. *Let R be a commutative ring. If R is semiboolean neat, then R is a field or $R/J(R)$ is isomorphic to a subring of a product of copies of $\mathbb{Z}_2$.*

Proof. Suppose that R is a semiboolean neat ring which is not a field. Then, $R/J(R)$ is embeddable inside of $\prod_{m \in Max(R)}(R/M)$; which is isomorphic to a product of copies of $\mathbb{Z}_2$ by Proposition 6.2.11. It follows that $R/J(R)$ is also isomorphic to a subring of product of copies of $\mathbb{Z}_2$. □

Remark 6.2.13. *If R is semiboolean neat ring, then the matrix ring $M_n(R)$ need not be semiboolean neat. For example: $M_2(\mathbb{Z}_4)$. Since $\mathbb{Z}_4 \neq J(\mathbb{Z}_4)$ and $\mathbb{Z}_2 \neq J(\mathbb{Z}_2)$, [50, Example 25(5)] implies that $M_2(\mathbb{Z}_4)$ and $M_2(\mathbb{Z}_4)/J(M_2(\mathbb{Z}_4)) \cong M_2(\mathbb{Z}_2)$ are not semiboolean. Thus, $M_2(\mathbb{Z}_4)$ is not semiboolean neat.*

Alternatively, it is obvious that if $J(R) \neq 0$ and R is semiboolean neat, then $R/J(R)$ is boolean. At the same time $M_2(\mathbb{Z}_4)/J(M_2(\mathbb{Z}_4)) \cong M_2(\mathbb{Z}_2)$ is not boolean. It follows that $M_2(\mathbb{Z}_4)$ is not semiboolean neat. Hence, semiboolean neat property is not Morita invariant.

Theorem 6.2.14. *Let $0 \neq I \subset J(A)$ be an ideal of A. Then, $M_n(A), n \geq 2$, is semiboolean neat if and only if A is a radical ring.*

Proof. ($\Rightarrow$) It follows from [50, Example 25(5)] and Remark 6.2.1.

($\Leftarrow$) Suppose that $M_n(A)$ is a semiboolean neat ring. It is well known that the map $I \mapsto M_n(I)$ is a bijection between the sets of ideals of A and $M_n(A)$. Then, $M_n(A/I) \cong M_n(A)/M_n(I)$ is semiboolean. Then, by using [50, Example 25(5)], we have that A/I is a radical ring. So $A/I = J(A/I)$. Since $I \subset J(A)$, $J(A/I) = J(A)/I$. Hence, $A = J(A)$. □

Theorem 6.2.15. *Let R be a commutative ring and G an abelian group. If group ring RG is semiboolean neat, then the following one condition holds*

(1) G is trivial and R is semiboolean neat.

(2) G is non-trivial 2-group and R is semiboolean.

(3) G is a non-trivial torsion free locally cyclic group and $R \cong \mathbb{Z}_2$.
Moreover, if either condition (1) *or* (2) *holds, then the converse implication is true.*

Proof. If G is trivial, then R is semiboolean neat since $RG \cong R$. So we shall assume hereafter that G is non-trivial.

Claim: If RG is clean, then RG is semiboolean neat if and only if RG is semiboolean if and only if R is semiboolean and G is a 2-group.

Proof. If R is semiboolean and G is a 2-group, then RG is semiboolean by [65, Corollary 3.11]. Thus, RG is semiboolean neat. So we will be concentrated on the inverse implication. Assume on the contrary that RG is not semiboolean. Since RG is clean, indecomposable clean ring is local. So $J(RG) \neq 0$. Since $RG/J(RG)$ is semiboolean, $RG/J(RG)$ is boolean. Then, by [65, Lemma 3.2], RG is semiboolean, which is a contradiction. Thus RG must be semiboolean and so by [65, Corollary 3.2], R is semibooloean and G is a 2-group. □

If RG is not clean, then RG is not semiboolean. By hypothesis, RG is semiboolean neat. The augmentation map $\omega : RG \to R$ is an onto homomorphism and $RG/\omega G \cong R$. Hence, R is a semiboolean ring. Let $I \neq 0$ be a proper ideal of R, then IG be a proper ideal of RG. The map $\theta : RG \to (R/I)G$ defined by

$$\theta(\sum_{g\in G} a_g g) = \sum_{g\in G}(a_g + I)g$$

maps naturally onto $(R/I)G$. Then, $(R/I)G$ is semiboolean. In view of [65, Theorem 3.1], G is a 2-group. Thus G is a torsion group. Since G is abelian, G is a locally finite 2-group.

Then, by [65, Theorem 2.3], *RG* is semiboolean, a contradiction. It follows that *I* is not a proper ideal of *R*. Thus, *R* being a commutative ring with only improper ideals implies that *R* is a field. Hence, *R* is isomorphic to $\mathbb{Z}_2$. As *RG* is semiboolean neat, it follows that *RG* is neat. Then, by [64, Theorem 2.7], *G* is a torsion free locally cyclic group. □

Chapter 7

On Weakly $g(x)$-Invo Clean Rings

In this chapter, we introduce weakly g(x)-invo clean rings. Let $C(R)$ be the center of a ring R and $g(x)$ be a fixed polynomial in $C(R)[x]$. A ring R is said to be weakly $g(x)$-invo clean if each element of R is either a sum or difference of an involution and a root of $g(x)$. This sort of class is a proper subclass of weakly $g(x)$-clean rings and a generalization of $g(x)$-invo clean rings. We provide various properties of weakly $g(x)$-invo clean rings. We characterize weakly invo-clean rings as weakly $g(x)$-invo clean rings where $g(x) \in x(x - a)C(R)[x], a \in C(R) \cap Inv(R)$. We determine necessary and sufficient conditions for skew Hurwitz series ring (HR, α) to be weakly $g(x)$-invo clean, where α is an endomorphism of R. Also, we prove that the ring of skew Hurwitz series $A = (HR, \alpha)$ is weakly invo-clean ring if and only if R is weakly invo-clean ring.

7.1 Introduction

The class of clean rings was introduced by Niholson in [48]. A ring R is called clean if for any $r \in R$, we have $r = u + e$ where $u \in U(R)$ and $e \in Id(R)$. A ring R is strongly clean if $ue = eu$. Following this, some stronger and special concepts of clean rings have been considered (see [12; 49; 70]) and for weaker ones, see [2; 16]. The concept of invo-clean rings were firstly introduced by Danchev [22]. A ring R called an invo-clean if for each $r \in R$, there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $r = v + e$. If the existing idempotent e is unique, R is classified as a uniquely invo-clean ring. An invo-clean ring R is strongly invo-clean if $ve = ev$. In [21], Danchev defined weakly invo-clean rings.

Definition 7.1.1. *A ring R is called weakly invo-clean if for any $r \in R$, either $r = v + e$ or $r = v - e$ where $v \in Inv(R)$ and $e \in Id(R)$. If $ve = ev$, R is a weakly invo-clean ring with strong property.*

Let $C(R)$ signifies the center of a ring R and $g(x)$ be a fixed polynomial such that $g(x) \in C(R)[x]$. In [24], $g(x)$-clean rings were introduced by Fan and Yang. $g(x)$-invo clean rings are those in which every element is the sum of a unit and a root of a polynomial $g(x)$.

Further, weakly $g(x)$-clean rings were studied by Ashrafi and Ahmadi [3].

Definition 7.1.2. *A ring R is weakly $g(x)$-clean if for any $a \in R$, either $a = u + s$ or $a = u - s$ where $u \in U(R)$ and $g(s) = 0$.*

Recently, Maalmi and Mouanis [42] introduced and studied $g(x)$-invo clean rings as a generalization of invo-clean rings. A ring R is $g(x)$-invo clean if for each $r \in R$, we have $r = v + s$ where $v \in Inv(R)$ and $g(s) = 0$. They investigated various basic properties and examples of $g(x)$-invo clean. In their paper, they characterized invo-clean rings as $g(x)$-invo clean rings where $g(x) = (x - a)(x - b), a, b \in C(R)$ and $b - a \in Inv(R)$. Motivated by papers [16] and [24], we introduce weakly $g(x)$-invo clean rings.

Definition 7.1.3. *A ring R is called weakly $g(x)$-invo clean if for any $r \in R$, either $r = v + s$ or $r = v - s$ where $v \in Inv(R)$ and $g(s) = 0$. If $vs = sv$, R is called weakly $g(x)$-invo clean with strong property.*

In section 7.2, we investigate weakly $g(x)$-invo clean rings as a proper subclass of weakly $g(x)$-clean rings and a generalization of $g(x)$-invo clean rings. We obtain various properties of weakly $g(x)$-invo clean rings. We show that a $g(x)$-invo clean ring is a weakly $g(x)$-invo clean ring but its reverse implication is not true (see Example 7.3.1). Consider M to be a R-module. We discuss when trivial extension $R(M)$ is weakly $g(x)$-invo clean. In section 7.3, we characterize weakly invo-clean rings as weakly $g(x)$-invo clean rings where $g(x) = x(x-a), a \in C(R) \cap Inv(R)$. It is proved that skew Hurwitz series ring (HR, α) is weakly invo-clean if and only if R is weakly invo-clean. We determine necessary and sufficient conditions for the ring of skew Hurwitz series (HR, α) to be weakly invo-clean with strong property, where α is an endomorphism of R. We prove that the ring of skew Hurwitz series (HR, α) is weakly $g(x)$-invo clean ring if and only if R is weakly $g(x)$-invo clean ring. If we take identity endomorphism in skew Hurwitz series ring, then we obtain ring of Hurwitz series to be weakly $g(x)$-invo clean ring.

7.2 Necessary and Sufficient Conditions for Weakly Invo-Clean Rings

In this Section, we further explore the concept of weakly invo-clean rings [21].

Let A be a ring and B a subring of ring A with $1_A \in B$. We set

$$R[A, B] = \{(c_1, c_2 \ldots, c_n, d, d \ldots) | c_i \in A, d \in B, n \geq 1\}$$

with addition and multiplication defined componentwise.

Theorem 7.2.1. *Let B be a subring of A, then $R[A, B]$ is weakly invo-clean if and only if A is invo-clean and B is weakly invo-clean.*

Proof. Let $(p_1, p_2, \ldots, p_n, d, d, \ldots) \in R[A, B]$ with each $c_i \in A$ and $d \in B$. Since B is weakly invo-clean and A is invo-clean, so $d = v \pm e$ for some $v \in Inv(R)$ and idempotent $e \in Id(R)$. If $d = v + e$, write $c_i = v_i + e_i$ then $(c_1, c_2, \ldots, c_n, d, d, \ldots) = (v_1, v_2, \ldots, v_n, v, v, \ldots) + (e_1, e_2, \ldots, e_n, e, e, \ldots)$ where $(v_1, v_2, \ldots, v_n, v, v, \ldots) \in Inv(R[A, B])$ and $(e_1, e_2, \ldots, e_n, e, e, \ldots)^2 = (e_1, e_2, \ldots, e_n, e, e, \ldots) \in Id(R[A, B])$. Hence, $R[A, B]$ is weakly invo-clean.

Conversely, since homomorphic image of weakly invo-clean is weakly invo-clean by [21, Lemma 4.1], B is weakly invo-clean. As $R[A, B] = A \oplus A$ so in view of [21, Proposition 4.15], A is invo-clean. □

Theorem 7.2.2. *Let $\alpha \in End(R)$. Then skew Hurwitz series ring $A = (HR, \alpha)$ is a weakly invo-clean ring if and only if so is R.*

Proof. Let $f \in A$ and since R is weakly invo-clean ring, $f(0) = v \pm e$ where $v \in Inv(R)$ and $e^2 = e \in R$. Define an element $h \in A$ by

$$h(n) = \begin{cases} v & n = 0 \\ -v \sum_{m=1}^{n} h(m)\alpha^m(h(n-m)) & n > 0. \end{cases}$$

Since $h(0)$ is an involution in R, h is an involution in A by Lemma 7.4.13, so we have $h \in Inv(A)$. Thus $f = h \pm l'_e$ where $(l'_e)^2 = l'_e \in A$.

Conversely, let $I = \{f \in A | f(0) = 0\}$ is an ideal in A. Then, we have a map $\sigma : R \to A/I$ by $\sigma(r) = l'_r + I$. It is easy to compute that σ is a ring isomorphism. So $R \cong A/I$. By hypothesis, A is weakly invo-clean ring. Then R is weakly invo-clean ring by [21, Lemma 4.1]. □

If we take $\alpha = id_R$, then we have following corollary.

Corollary 7.2.3. *Let R be a ring. Then Hurwitz series ring HR is weakly invo-clean if and only if R is a weakly invo-clean.*

Theorem 7.2.4. *Let $\alpha \in End(R)$ and $e \in A$ is a central idempotent with $\alpha(e) = e$. Then $A = (HR, \alpha)$ is weakly invo-clean ring with strong property if and only if R is weakly invo-clean ring with strong property.*

Proof. Suppose that R is weakly invo-clean ring. Then A is weakly invo-clean ring by Theorem 7.2.2. Thus, each $f \in A$ can be written as $f = (f - l'_e) + l'_e$ or $f = (f + l'_e) - l'_e$. So, all that remains is to demonstrate that $fl'_e = l'_e f$. As e is central in R, $(fl'_e)(n) = f(n)\alpha^n(l'_e(0)) = f(n)\alpha^n(e) = f(n)e = ef(n) = l'_e(0)f(n) = (l'_e f)(n)$ for all $n \in supp(fl'_e)$. Hence, $fl'_e = l'_e f$.

Conversely, assume that A is weakly invo-clean ring, then R is weakly invo-clean by Theorem 7.2.2. Thus, each $r \in R$ can be written as $r = f(0) \pm h(0)$ where f is an involution of A and h is an idempotent element of A. Now, its only remains to demonstrate that $f(0)h(0) = h(0)f(0)$. As f, h commute in A, $f(0)h(0) = f(0)\alpha^0(h(0)) = (fh)(0) = (hf)(0) = h(0)\alpha^0(f(0)) = h(0)f(0)$. □

Corollary 7.2.5. *Let R be a reduced ring, $\alpha \in End(R)$ and $\alpha(e) = e$. Then $A = (HR, \alpha)$ is a weakly invo-clean if and only if $A = (HR, \alpha)$ is a weakly invo-clean ring with strong property.*

Proof. It follows by Theorem 7.2.4 and the fact that a reduced ring R is abelian[1]. □

Proposition 7.2.6. *Let $e \in R$ be a central idempotent and R be a weakly invo-clean ring. Then corner ring eRe is weakly invo-clean.*

Proof. Consider $\varphi : R \to eRe$ defined by $\varphi(r) = er$. Since e be a central idempotent, $\varphi(r) = er = re = ere$. So, eRe is homomorphic image of R. Hence, by [21, Lemma 4.1], the result follows. □

In [29], Han and Nicholson proved that if e is an idempotent element in R such that eRe and $(1 - e)R(1 - e)$ are both clean rings, then R is clean. So $n \times n$ matrix ring $M_n(R)$ is clean. The analogous result for weakly invo-clean rings hold if e is central idempotent in R.

Theorem 7.2.7. *The following statements hold for a ring R.*

(1) If corner ring eRe and $(1 - e)R(1 - e)$ are both weakly invo-clean where e is central idempotent in R, then R is weakly invo-clean.

[1] A ring R is called abelian if all idempotents of R are central.

(2) Let $\{e_1, e_2, \ldots, e_n\}$ be a family of orthogonal idempotents such that $e_1+e_2+\cdots+e_n = 1$. Then e_iRe_i is weakly invo-clean if and only if so is R.

(3) If R is weakly invo-clean, then so is the matrix ring $M_n(R)$ for every $n \geq 1$.

(4) If $N = N_1 \oplus N_2 \oplus \cdots \oplus N_n$ are modules and $End(N)$ is weakly invo-clean for each i, then $End(N)$ is weakly invo-clean.

Proof. (1) Write $\bar{e} = (1 - e)$. By Pierce decomposition

$$R = eRe \oplus eR\bar{e} \oplus \bar{e}Re \oplus \bar{e}R\bar{e}.$$

Since e and $\bar{e}$ are central idempotent, we have

$$R = eRe \oplus \bar{e}R\bar{e} = \begin{pmatrix} eRe & 0 \\ 0 & \bar{e}R\bar{e} \end{pmatrix}$$

Supoose $A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R$, where $x \in eRe$ and $y \in \bar{e}R\bar{e}$. By hypothesis, x and y are weakly invo-clean. Then there exist $v_1, v_2 \in Inv(R)$ and idempotent e_1 and e_2 in R such that $x = v_1 \pm e_1$, $y = v_2 \pm e_2$. Thus,

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} v_1 \pm e_1 & 0 \\ 0 & v_2 \pm e_2 \end{pmatrix} = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \pm \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}.$$

Note that $\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}^2 = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$. Thus R is weakly invo-clean. Hence, R is weakly invo-clean.

(2) The 'if part' follows from (1) by induction and 'only if part' by using the fact that factor ring of weakly invo clean ring is weakly invo clean.

(3) Let E_{ij} denotes that the entry in i^{th} row and j^{th}th column is 1 and other are 0 and $\{E_{ii}\}_{i=0}^{n}$ is a finite set of mutually orthogonal idempotents in the matrix ring $M_n(R)$. The sum of $E_{11} + E_{22} + \cdots + E_{nn}$ is equal to $n \times n$ identity matrix. Thus, result follows from (2).

(4) It follows from the fact that $End(R^n) \cong M_n(R)$ and by (3). □

A ring R is called weakly exchange if, for each $x \in R$, there exists $e \in Id(R)$ such that $e \in xR$ and either $1 - e \in (1 - x)R$ or $1 - e \in (1 + x)R$. The following theorem shows the relation between weakly invo-clean ring and weakly exchange ring.

Theorem 7.2.8. *Let R be a ring. If R is weakly invo-clean ring, then R is weakly exchange.*

Proof. By hypothesis, for any $r \in R$, we have $r = v \pm e$ where $v \in Inv(R)$ and $e \in Id(R)$. If $r = v + e$, then by the inclusion $Inv(R) \subseteq U(R)$, so $v \in U(R)$. Thus, r is clean element and by [48], it satisfies the exchange property.

Suppose $r = v - e$ where $v \in Inv(R)$ and $e \in Id(R)$. Consider $f = v(1 - e)v$, then $f^2 = f$. Note that $v(x + f) = v(v - e + v(1 - e)v) = (v - e)^2 + (v - e) = x^2 + x$. So $x + f \in R(x^2 + x)$. Thus by [20, Lemma 2.1], x satisfies weakly exchange property. □

The converse of above Theorem 7.2.8 is not true, for which we have following example.

Example 7.2.9. *The ring $\mathbb{Z}_7$ is clean ring. This implies that $\mathbb{Z}_7$ is exchange ring by [48, Proposition 1.8]. So, $\mathbb{Z}_7$ is weakly exchange but not weakly invo-clean ring.*

7.3 General Properties of Weakly $g(x)$-Invo Clean Rings

In this section, the general properties of weakly $g(x)$-invo clean rings are discussed. We start with the following observations:

(1) Every weakly $g(x)$-invo clean ring with strong property is weakly $g(x)$-invo clean. Also, in the commutative case, the two definitions coincide. Although, the reverse implication is not true for non commutative rings. For example, the upper triangular matrix ring $T_2(\mathbb{Z}_2)$ is weakly $(x^4 + x^2)$-invo clean but not weakly $(x^4 + x^2)$-invo clean with strong property.

(2) Every weakly $g(x)$-invo clean ring is weakly $g(x)$-clean but its converse is not true. For example, consider $g(x) = (x^2 + x + 2)$, then $\mathbb{Z}_7$ is weakly $g(x)$-clean but not weakly $g(x)$-invo clean.

(3) A $g(x)$-invo clean ring is a weakly $g(x)$-invo clean ring. But the following example shows that its converse is not true.

Example 7.3.1. *Consider $R = \mathbb{Z}_5$ and $g(x) = x^2 - x \in C(R)[x]$. Then R is a weakly $g(x)$-invo clean ring but not $g(x)$-invo clean ring. And non trivial example, it can be easily compute that for a fixed polynomial $g(x) = x^4 + x$, the ring $\mathbb{Z}_5$ is weakly $g(x)$-invo clean but not $g(x)$-invo clean.*

Let R_1 and R_2 be two rings, with a ring homomorphism $\phi : C(R_1) \to C(R_2)$ such that $\phi(1_{R_1}) = 1_{R_2}$. If $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R_1)[x]$, we consider $g_\phi(x) = \sum_{i=0}^{n} \phi(a_i) x^i \in C(R_2)[x]$. We note that if $g(x) \in \mathbb{Z}[x]$, then $g_\phi(x) = g(x)$.

Theorem 7.3.2. *Let $\phi : R_1 \to R_2$ be a ring epimorphism and $g(x) \in C(R_1)[x]$. If R_1 is weakly $g(x)$-invo clean, then R_2 is weakly $g_\phi(x)$-invo clean.*

Proof. Consider $g(x) = \sum_{i=0}^{n} a_i x^i$ and consider $g_\phi(x) = \sum_{i=0}^{n} \phi(a_i)x^i \in C(R_2)[x]$ For any $r_2 \in R_2$, we have an element $r_1 \in R_1$ such that $\phi(r_1) = r_2$. As R_1 is weakly $g(x)$-invo clean, there exist $s \in R_1$ and $v \in Inv(R_1)$ such that $r_1 = v \pm s$ and $g(s) = 0$. So $r_2 = \phi(r_1) = \phi(v \pm s) = \phi(v) \pm \phi(s)$ with $\phi(v) \in Inv(R_2)$ and

$$\begin{aligned} g_\phi(\phi(s)) &= \phi(a_0) + \phi(a_1)\phi(s) + \cdots + \phi(a_n)(\phi(s))^n \\ &= \phi(a_0 + a_1 s + \cdots + a_n s^n) \\ &= a_0 + a_1 s + \cdots + a_n s^n \\ &= 0. \end{aligned}$$

Hence, R_2 is weakly $g_\phi(x)$-invo clean. □

The converse of above Theorem 7.3.2 is false, which is shown by the following example.

Example 7.3.3. *Consider $g(x) = x^2 - x$, the ring $\mathbb{Z}_5 \cong \mathbb{Z}/(5)$ is weakly $g(x)$-invo clean but $\mathbb{Z}$ is not weakly $g(x)$-invo clean.*

Corollary 7.3.4. *If I is any ideal of a weakly $g(x)$-invo clean ring R, then the factor ring R/I is weakly $\overline{g}(x)$-invo clean where $\overline{g}(x) \in C(R/I)[x]$.*

Proof. Consider $\theta : R \to R/I$ be a canonical epimorphism. So, the result follows by Theorem 7.3.2. □

Considering I be an ideal of a ring R, we say that a root $\overline{a}$ of $\overline{g}(x) \in C(R/I)[x]$ can be lifted to $g(x) \in C(R)[x]$, if there exists $b \in R$ such that $g(b) = 0$ and $b - a \in I$. For $g(x) = x^2 - x$, this is the generalization of lifting idempotents modulo I. The next theorem shows that the reverse of Corollary 7.3.4 is true if roots of $\overline{g}(x)$ can be lifted to $g(x)$.

Theorem 7.3.5. *Let $I \subseteq J(R)$ be an ideal of R and $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ with $\overline{g}(x) = \sum_{i=0}^{n} \overline{a}_i x^i \in C(R/I)[x]$. If R/I is weakly $\overline{g}(x)$-invo clean and roots of $\overline{g}(x)$ can be lifted to $g(x)$, then R is weakly $g(x)$-invo clean.*

Proof. For $r \in R$, let $\overline{r} = r + I \in R/I$. Since R/I is weakly $\overline{g}(x)$-invo clean, then there exist $\overline{v} \in Inv(R/I)$ and $\overline{s}$ with $\overline{g}(\overline{s}) = 0$ such that $\overline{r} = \overline{v} \pm \overline{s}$. Since roots of $\overline{g}(x)$ can be lifted to $g(x)$, so we have $t \in R$ such that $g(t) = 0$ and $\overline{s} = \overline{t}$. Thus $\overline{r} = \overline{v} \pm \overline{t}$. Then for some $i \in I$, $r - (v \pm t) = i$. So $a = (v + i) \pm t$ with $v + i \in Inv(R)$. Hence, R is a weakly $g(x)$-invo clean ring. □

The direct product of weakly $g(x)$-invo clean rings does not have a weakly $g(x)$-invo clean. For $g(x) = x^2 - x$, $\mathbb{Z}_5$ is weakly $g(x)$-invo clean but their direct product $\mathbb{Z}_5 \times \mathbb{Z}_5$ is not. Since, in $\mathbb{Z}_5 \times \mathbb{Z}_5$, element $(2, 3)$ and $(3, 2)$ cannot be expressed as sum or difference of involution and a root of $g(x)$.

Theorem 7.3.6. *Let $\{R_i\}_{i=1}^n$ be a family of rings and $g(x) \in \mathbb{Z}[x]$. Then the direct product $R = \prod_{i=1}^{n} R_i$ is weakly $g(x)$-invo clean if and only if there exist $k \in \{1, 2, \ldots, n\}$ such that R_k is weakly $g(x)$-invo clean ring and R_j is $g(x)$-invo clean for all $j \neq k$.*

Proof. Under the projection homomorphism, for each $i \in \{1, 2, \ldots, n\}$, R_i is a homomorphic image of R. Thus R_i is weakly $g(x)$-invo clean by Theorem 7.3.2. Assume that neither R_1 nor R_2 is $g(x)$-invo clean. So, there are $r_1 \in R_1$ and $r_2 \in R_2$ such that $r_1 \neq v_1 + s_1$ where $v_1 \in Inv(R_1)$ and $g(s_1) = 0$ and $r_2 \neq v_2 - s_2$ where $v_2 \in Inv(R_2)$ and $g(s_2) = 0$. Thus, (r_1, r_2) is not weakly $g(x)$-invo clean in $R_1 \times R_2$, which contradicts.

Conversely, let $r = (r_i) \in R$. For a fixed $k \in \{1, 2, \ldots, n\}$, suppose R_k is weakly $g(x)$-invo clean ring. So we have either $r_k = v_k + s_k$ or $r_k = v_k - s_k$ for some $v_k \in Inv(R_k)$ and root s_k of $g(x)$. If $r_k = v_k + s_k$, then write $r_i = v_i + s_i$ for each $i \neq k$ where $v_i \in Inv(R_i)$ and $g(s_i) = 0$. Thus $r = (v_i) + (s_i)$ is the sum of an involution and a root of $g(x)$. If $r_k = v_k - s_k$, then write $r_i = v_i - s_i$ for $i \neq k$ where $v_i \in Inv(R_i)$ and $g(s_i) = 0$. So $r = (v_i) - (s_i)$ is the difference of an involution and a root of $g(x)$. Hence, R is weakly $g(x)$-invo clean ring. Hence, as we required. □

Consider R to be a ring and N be a bimodule. The ideal extension $I(R, N)$ of R by N is defined as the additive abelian group $I(R, N) = R \oplus N$ with multiplication $(r_1, n_1)(r_2, n_2) = (r_1 r_2, r_1 n_2 + n_1 r_2 + n_1 n_2)$. If $g(x) = \sum_{i=0}^{n} (r_i, n_i) x^i \in C(I(R, N))[x]$, then $g_R(x) = \sum_{i=0}^{n} r_i x^i \in C(R)[x]$.

Theorem 7.3.7. *Let R be a ring and N be a bimodule. If $I(R, N)$ is a weakly $g(x)$-invo clean ring, then R is a weakly $g(x)$-invo clean ring.*

Proof. Consider $\vartheta_R : I(R, N) \to R$ defined by $\vartheta_R(r, n) = r$. It can be easily verified that ϑ_R is a ring epimorphism. Thus, by *Theorem* 7.3.2, R is a weakly $g(x)$-invo clean ring. □

The ring of skew power series in x with cofficients from R is denoted by $R[[x, \alpha]]$, where α is a ring epimorphism, with multiplication $xr = \alpha(r)x$ for all $r \in R$. In particular, if we take identity endomorphism, then $R[[x]] = R[[x, 1_R]]$ denotes the ring of formal power series over R. In an analogous way, we can define skew polynomial ring $R[x, \alpha]$. If (x) is the ideal generated by x, then it can be proved that $R[[x, \alpha]] \simeq I(R, (x)$.

Corollary 7.3.8. *Let $\alpha \in End(R)$. If $R[[x,\alpha]](R[[x]])$ is a weakly $g(x)$-invo clean ring, then R is weakly $g(x)$-invo clean.*

Proposition 7.3.9. *Let R be a commutative ring. Then the polynomial ring $R[x]$ is not weakly $g(x)$-invo clean.*

Proof. Suppose $R[x]$ is weakly $g(x)$-invo clean. Then for any $x \in R[x]$, we have $x = v \pm s$ where $v \in Inv(R)$ and $g(s) = 0$. So $x \pm s$ is an involution. Thus by [42, Lemma 3.6], 1 is nilpotent element, which is a contradiction. □

We call $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ a Morita context ring. If $g(x) = \begin{pmatrix} c_0 & n_0 \\ m_0 & d_0 \end{pmatrix} + \begin{pmatrix} c_1 & n_1 \\ m_1 & d_1 \end{pmatrix} x + \cdots + \begin{pmatrix} c_n & n_n \\ m_n & d_n \end{pmatrix} x^n \in C(T)[x]$, then we get $g_R(x) = \sum_{i=0}^{n} c_i x^i \in C(R)[x]$ and $g_S(x) = \sum_{i=0}^{n} d_i x^i \in C(S)[x]$.

Theorem 7.3.10. *Let R, S be two rings and M, N be bimodule and let $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a Morita Context with zero pairings. If T is weakly $g(x)$-invo clean, then R is weakly $g_R(x)$-invo clean and S is weakly $g_S(x)$-invo clean.*

Proof. Suppose T is weakly $g(x)$-invo clean with zero pairings. Consider $I = \begin{pmatrix} 0 & M \\ N & S \end{pmatrix}$ and $J = \begin{pmatrix} R & M \\ N & 0 \end{pmatrix}$ are ideals of R, then $T/I \cong R$ and $T/J \cong S$. In view of Theorem 7.3.2, R is weakly $g_R(x)$-invo clean and S is weakly $g_S(x)$-invo clean. □

Corollary 7.3.11. *Let R, S be two rings and M be a bimodule. If the formal triangular matrix ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is weakly $g(x)$-invo clean, then R is weakly $g_S(x)$-invo clean and S is weakly $g_S(x)$-invo clean.*

The next theorem is the particular case of formal triangular matrix rings. The trivial extension of a commutative ring R and an R-module M is the (commutative) ring

$$R(M) = \left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R, m \in M \right\}$$

with the usual matrix addition and multiplication. If $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$ is an involution of $R(M)$, then r is an involution of R. Naturally, ring R embeds into $R(M)$ via $r \to \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$. So any polynomial $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ can be written as $g(x) = \sum_{i=0}^{n} \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} x^i \in R(M)[x]$ and conversely.

Theorem 7.3.12. *Let R be a commutative ring and M an R-module such that $2M = 0$. Then the trivial extension $R(M)$ of R and M is weakly $g(x)$-invo clean if and only if R is weakly $g(x)$-invo clean.*

Proof. For $\widetilde{M} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} | m \in M \right\}$, we have $R(M)/\widetilde{M} \simeq R$. Hence, by Theorem 7.3.2, R is a weakly $g(x)$-invo clean ring.

Conversely, let $g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ and $r \in R$. Suppose R is weakly $g(x)$-invo clean, then we have $r = v \pm s$ for some $v \in Inv(R)$ and root s of $g(x)$. Then for $m \in M$, $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} = \begin{bmatrix} v & m \\ 0 & v \end{bmatrix} \pm \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$ for some $\begin{pmatrix} v & m \\ 0 & v \end{pmatrix} \in Inv(R(M))$. Also, note that

$$\begin{aligned} g(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}) &= a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} + a_2 \begin{pmatrix} s^2 & 0 \\ 0 & s^2 \end{pmatrix} + \cdots + a_n \begin{pmatrix} s^n & 0 \\ 0 & s^n \end{pmatrix} \\ &= \begin{pmatrix} a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n & 0 \\ 0 & a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence, $R(M)$ is a weakly $g(x)$-invo clean ring. □

7.4 Weakly $g(x)$-Invo Clean Rings

The weakly $(x^2 - x)$-invo clean rings are precisely weakly invo clean rings. However, weakly invo-clean rings are not weakly $g(x)$-invo clean rings.

Example 7.4.1. *Let R be a boolean ring with $|R| > 2$ elements and $c \in R$ with $c \notin \{0, 1\}$. Consider $g(x) = (x + 1)(x + c)$. Since $e = (2e - 1) + (1 - e)$ with $(2e - 1)^2 = 1$ and $(1 - e)^2 = (1 - e)$, then R is weakly invo clean but not weakly $g(x)$-invo clean. Because, if $c = v \pm s$ with $v \in Inv(R)$ and $g(s) = 0$, then it must be $v = 1$ and $s = \pm(c - 1)$. But, clearly, $g(c - 1) \neq 0$.*

However, for some kind of polynomials, weakly invo-clean and weakly $g(x)$-invo clean rings are equivalent.

Theorem 7.4.2. *Let R be a ring and $g(x) \in x(x - a)C(R)[x]$ where $a \in C(R) \cap Inv(R)$. Then, we have following statement.*

(1) R is weakly invo-clean if and only if R is weakly $x(x-a)$-invo clean.

(2) If R is weakly invo-clean, then R is weakly $g(x)$-invo clean.

Proof. (1) Consider $r \in R$. Suppose that R is weakly $x(x-a)$-invo clean, then $ra = v + s$ or $ra = v - s$ for some $v \in Inv(R)$ and root s of $g(x)$. Since $a \in Inv(R)$, $r = va \pm sa$ then $va \in Inv(R)$ and as $g(s) = s(s-a) = 0$, $(sa)^2 = s(s-a-a)a^2 = s(s-a) + sa = sa$. Therefore, R is a weakly invo-clean ring.

Conversely, suppose R is weakly invo-clean. Then for any $r \in R$, either $ra = v + e$ or $ra = v - e$ where $v \in Inv(R)$ and $e^2 = e \in R$. Thus $r = va \pm ea$. Since $a \in C(R)$, $g(ea) = ea(ea - a) = e(e-1)a^2 = 0$. Thus ea is a root of $g(x)$ and as $a \in Inv(R)$, then $va \in Inv(R)$. Hence, R is a weakly $g(x)$-invo clean ring.

(2) It is similar to the proof of (1). □

Corollary 7.4.3. *Let R be a ring and $g(x) = x(x-a) \in C(R)[x]$ where $a \in C(R) \cap Inv(R)$. If R is weakly $g(x)$-invo clean, then $30 \in Nil(R)$.*

Proof. Since R is weakly $g(x)$-invo clean. Then by Theorem 7.4.2, R is weakly invo-clean. Thus by [21, Lemma 4.2], the result follows. □

Remark 7.4.4. *Although, every weakly invo clean ring is weakly $(x^2 + x)$-invo clean but this situation need not be true for elements. For example, let $2 \in \mathbb{Z}$, then $2 = 1 + 1$ is weakly invo clean but not weakly $(x^2 + x)$-invo clean because 1 is not a root of $x^2 + x$.*

Theorem 7.4.5. *Let $n, k \in \mathbb{N}$ and m be a fixed integer > 0. Then, for a ring R, the following statements are equivalent:*

(1) R is weakly $(x^2 - m^n x)$-invo clean.

(2) R is weakly $(x^2 + m^k x)$-invo clean.

(3) R is weakly $(x^2 - mx)$-invo clean.

(4) R is weakly $(x^2 + mx)$-invo clean.

(5) R is weakly invo-clean and $m \in Inv(R)$.

Proof. (1) ⇒ (5). We will prove that $m \in Inv(A)$. Suppose $m \notin Inv(R)$. Then $\overline{R} = R/(m^n R) \neq 0$. Let $m^n = v \pm s$ where $v \in Inv(R)$ and $s^2 - m^n s = 0$. As $\overline{0} = \overline{m^n} = \overline{v} + \overline{s}$, then $\overline{s} = \pm\overline{v} \in Inv(\overline{R})$. But $\overline{s}^2 = \overline{s^2} = \overline{m^n s} = \overline{0}$, which is a contradiction. Thus $m \in Inv(R)$.

(5) ⇒ (1). By Theorem 7.4.2, R is weakly $(x^2 - m^n x)$-invo clean.

Similarly, it can be proved that (2) ⇔ (5), (3) ⇔ (5) and (4) ⇔ (5). □

Proposition 7.4.6. *Let R be a ring. Then we have the following equivalent statements:*

(1) R is weakly $(x^2 - 1)$-invo clean.

(2) Every element of R is the sum or difference of an involution and a square root of 1.

Proof. (1) $\Rightarrow$ (2). Suppose R is weakly $(x^2 - 1)$-invo clean, then we have $v, s \in Inv(R)$ such that $r = v \pm s$ with $s^2 = 1$.

(2) $\Rightarrow$ (1). Let $r \in R$. Then $r = v \pm s$ where $v \in Inv(R)$ and $s^2 = 1$. Then s is the root of $(x^2 - 1)$. Therefore, R is weakly $(x^2 - 1)$-invo clean. □

Khashan and Handam in [34], defined weakly $g(x)$-nil clean rings. A ring R is called weakly $g(x)$-nil clean if for each $r \in R$, we have $r = n \pm s$ where $n \in Nil(R)$ and $g(s) = 0$. Then, we have following proposition.

Proposition 7.4.7. *Let R be a ring,* $2 \in Nil(R)$ *and* $g(x) \in C(R)[x]$*. If R is weakly* $g(x)$*-invo clean, then R is weakly* $g(1 - x)$*-nil clean with bounded index of nilpotence.*

Proof. Suppose R is weakly $g(x)$-invo clean. Then $r = v \pm s$, where $v \in Inv(R)$ and $g(s) = 0$. Write $r = (v + 1) - (1 - s)$, we have $(v + 1)^2 = 2v + 2 = 2(v + 1)$ and $(v + 1)^3 = 2(v + 1)^2 = 2^2(v + 1)$. Then by induction, we get that $(v + 1)^{n+1} = 2^n(v + 1)$ for all $n \in \mathbb{N}$. Since $2 \in Nil(R)$, then $(v + 1)^t = 0$ for some $t \in \mathbb{N}$. Thus $(v + 1) \in Nil(R)$. Similarly, write $r = (v-1)+(1-s)$, we can derive that $(v-1)^{n+1} = (-1)^n 2^n(v-1)$ and since $2 \in Inv(R), (v-1)^m = 0$ for some $m \in \mathbb{N}$. So $(v-1) \in Nil(R)$ and $g(1-(1-s)) = g(s) = 0$. Hence, as we required. □

Proposition 7.4.8. *Let R be a ring with* $char(R) = 2$*. If R is weakly* $(x^k - 1)$*-invo clean, then R is weakly* $(x^k - 1)$*-nil clean.*

Proof. For any $r \in R$, since R is weakly $(x^k - 1)$-invo clean, write $r - 1 = v \pm s$, where $v \in Inv(R)$ and $g(s) = 0$. So $r = (v + 1) \pm s$. Since $char(R) = 2$, $(v + 1)^2 = 2(1 + v) = 0$. This implies that $(v + 1) \in Nil(R)$. Hence, R is weakly $(x^k - 1)$-nil clean. □

Proposition 7.4.9. *Let R be a ring and* $a, b \in R$*,* $n \in \mathbb{N}$*. Then R is a weakly* $(ax^{2n} - bx)$*-invo clean ring if and only if R is a weakly* $(ax^{2n} + bx)$*-invo clean ring.*

Proof. Let $r \in R$. Assume that R is a weakly $(ax^{2n} - bx)$-invo clean ring. Then, $-r = v \pm s$ where $v \in Inv(R)$ and $(as^{2n} - bs) = 0$. Thus $r = (-v) \pm (-s)$ where $(-v) \in Inv(R)$ and $a(-s)^{2n} + b(-s) = 0$. So, r is weakly $(ax^{2n} + bx)$-invo clean. Hence, R is a weakly $(ax^{2n} + bx)$-invo clean ring. Similarly, we can prove for converse. □

The following example shows that Proposition 7.4.9 is not true for odd powers.

Example 7.4.10. *The ring $\mathbb{Z}_4$ is weakly $(x^3 - x)$-invo clean but not a weakly $(x^3 + x)$-invo clean ring.*

Let R be a commutative ring and N be an R-module, then trivial extension of R by N is defined by $R\Theta N = \{(r, n) : r \in R, n \in N\}$ with usual addition and multiplication given by $(r_1, n_1)(r_2, n_2) = (r_1r_2, r_1n_2 + n_1r_2)$.

Theorem 7.4.11. *Let R be a commutative ring and N an R-module such that $2N = 0$. Then $R\Theta N$ is weakly $g(x)$-invo clean if and only if R is weakly $g(x)$-invo clean.*

Proof. Let $n \in N$ and $g(x) = \sum_{i=0}^{n} a_ix^i$. Suppose R is weakly $g(x)$-invo clean, then for any $r \in R$, we have $r = v \pm s$ for some $v \in Inv(R)$ and a root s of $g(x)$. Thus $(r, n) = (v, n) \pm (s, 0)$, where $(v, n) \in Inv(R\Theta N)$ (since $2N = 0$) and $g((s, 0)) = \sum_{i=0}^{n} a_i(s, 0)^i = \sum_{i=0}^{n} a_i(s^i, 0) = \sum_{i=0}^{n}(a_is^i, 0) = (0, 0)$. Hence, $R\Theta N$ is weakly $g(x)$-invo clean.

Conversely, since $R\Theta N$ is weakly $g(x)$-invo clean, then $R \cong (A\Theta N)/(0\Theta N)$ is weakly $g(x)$-invo clean. □

Let R be a commutative ring, let J be an ideal of R and $\psi : R \to R$ be a ring homomorphism. The amalgamated duplication of the ring R along an ideal J is defined as $R \bowtie^{\psi} J = \{(r, r + j) | r \in R, j \in J\}$. This construction is a subring, with identity $(1, 1)$, of $R \times R$ (with the usual componentwise operations). For more information of $R \bowtie^{\psi} J$, one can see [19]. The following theorem gives characterization of $R \bowtie^{\psi} J$ to be weakly $g(x)$-invo clean.

Theorem 7.4.12. *Let R be a commutative ring with $2 = 0$ and $g(x) = \sum_{i=0}^{n} a_ix^i \in R[x]$. If J be a nilpotent ideal of R with nilpotency index 2, then $R \bowtie^{\psi} J$ is weakly $g(x)$-invo clean if and only if R is weakly $g(x)$-invo clean.*

Proof. Let $(r, r + j) \in R \bowtie^{\psi} J$. Suppose R is weakly $g(x)$-invo clean, then $r = v \pm s$ where $v \in Inv(R)$ and $g(s) = 0$. Thus $(r, r + j) = (v \pm s, v \pm s + j) = (v, v + j) \pm (s, s)$. By hypothesis, for each $j \in J$, we have $j^2 = 0$. So, $(v, v + j)^2 = (v^2, v^2 + 2vj + j^2) = (1, 1)$. Hence, $(v, v + j) \in Inv(R \bowtie^{\psi} J)$ and $g((s, s)) = \sum_{i=0}^{n} a_i(s, s)^i = \sum_{i=0}^{n} a_i(s^i, s^i) = (\sum_{i=0}^{n} a_is^i, \sum_{i=0}^{n} a_is^i) = (0, 0)$. Therefore, $R \bowtie^{\psi} J$ is weakly $g(x)$-invo clean.

Conversely, let $0 \bowtie^{\psi} J = \{(0, j) | j \in J\}$ is an ideal of $R \bowtie^{\psi} J$. Note that $R \bowtie^{\psi} J/0 \bowtie^{\psi} J \cong R$. Thus, R is weakly $g(x)$-invo clean by Theorem 7.3.2. □

Lemma 7.4.13. *Let f be an element in $A = (HR, \alpha)$ defined by $f(0) = a$ and $f(n) = -a\sum_{m=1}^{n}\binom{n}{m}f(m)\alpha^m(f(n-m))$ for all $n \geq 1$. Then f is an involution in A if and only if $f(0)$ is an involution in R.*

Proof. Suppose f is an involution in A, then one can easily show that $f(0)$ is an involution in R. Conversely, assume $f \in A$ such that $f(0)$ is an involution in R. It can be computed that for $n = 0$, $f^2 = 1$ and for all $n \geq 1$,

$$\begin{aligned} f^2(n) &= \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} f(m)\alpha^m(f(n-m)) \\ &= f(0)f(n) + \sum_{m=1}^{n} \begin{pmatrix} n \\ m \end{pmatrix} f(m)\alpha^m(f(n-m)) \\ &= 0. \end{aligned}$$

So $f^2 = l_1$. Hence, f is an involution in A. □

Theorem 7.4.14. *Let $\alpha \in End(R)$ and $g(x) = a_0 + a_1x + \cdots + a_kx^k \in C(R)[x]$. Then $A = (HR, \alpha)$ is weakly $g_L(x)$-invo clean where $g_L(x) = l'_{a_0} + l'_{a_1}x + \cdots + l'_{a_k}x^k \in C(A)[x]$ if and only if the ring R is weakly $g(x)$-invo clean.*

Proof. Let $f \in A$ and since R is a weakly $g(x)$-invo clean ring, $f(0) = v \pm s$ where $v \in Inv(R)$ and $g(s) = 0$. Hence, $f = v' \pm l'_s$ where $v' \in A$ defined as $v'(0) = v$ and $v'(n) = -v\sum_{m=1}^{n} \begin{pmatrix} n \\ m \end{pmatrix} v'(m)\alpha^m(v'(n-m))$ for all $n \geq 1$. Since $v'(0)$ is an involution of R, v' is an involution of A by Lemma 7.4.13. Now, $g_L(l'_s) = l'_{a_0} + l'_{a_1}l'_s + \cdots + l'_{a_k}(l'_s)^k = 0$. It follows that l'_s is a root of polynomial $g_L(x)$. Thus, f is weakly $g_L(x)$-invo clean in A.

Conversely, assume that $A = (HR, \alpha)$ is a weakly $g_L(x)$-invo clean ring and $r \in R$, then $\mathcal{G}_R(r) \in A$. Thus $\mathcal{G}_R(r) = f \pm p$ where $f \in Inv(A)$ and $g_L(p) = 0$. So $\mathcal{E}_R(f) \in Inv(R)$ by Lemma 7.4.13 and $g(\mathcal{E}_R(p)) = 0$. Thus, $r = \mathcal{E}_R(f) \pm \mathcal{E}_R(p)$. Hence, R is a weakly $g(x)$-invo clean ring. □

The proof of the next lemma is similar to that of Lemma 7.4.13.

Lemma 7.4.15. *Let sequence (a_n) in HR is defined by $a_0 = v$ and $a_n = -v\sum_{m=1}^{n} \begin{pmatrix} n \\ m \end{pmatrix} a_m a_{n-m}$. Then (a_n) is an involution in HR if and only if a_0 is an involution in R.*

Corollary 7.4.16. *Let R be a ring and $g(x) \in C(R)[x]$. Then Hurwitz series ring HR is weakly $g_L(x)$-invo clean if and only if R is a weakly $g(x)$-invo clean.*

Proof. We take identity endomorphism, i.e., $\alpha = id_R$ in Theorem 7.4.14 and by using Lemma 7.4.15, the result follows. □

Chapter 8

Conclusion and Future Research

This chapter concludes our thesis and shows some of the prospects that define our current and future research endeavors in scientific research. This thesis is mainly a study of generalizations of UN and VNL rings and group rings. The introductory chapter consists of definitions and literature survey of concepts used throughout this thesis. In the third chapter, we study UN rings and group rings. We have studied the question raised by Călugăreanu [7] that "is $M_n(R)$ over a UN ring R, also UN?". We have obtained that if R is commutative, then $M_n(R)$ is UN if and only if R is UN. We have focused on structure of UN group rings. We have found necessary and sufficient conditions for RG to be UN. We have obtained that if RG is a UN ring then R is a UN ring, G is a p-group and $p \in J(R)$; and the converse holds if G is locally finite. As a future scope, we have

Problem 8.0.1. *Find necessary and sufficient conditions for a skew group ring $R *_\theta G$ to be UN.*

In the fourth chapter, We have introduced and investigated a new class of rings which is called UQ rings and establish their relation with already known rings. Various properties of UQ rings have been obtained. We have provided a new characterizations of 2-good rings and discussed extensions of UQ rings such as Morita contexts, generalized matrix rings, formal matrix rings, group rings etc. Further, necessary and sufficient conditions for commutative RG to be UQ have been attained. Let G be an abelian p-group with $p \in J(R)$ and R be a commutative ring. Then RG is a UQ ring if and only if R is UQ. As a future scope, we have

Problem 8.0.2. *Find necessary and sufficient conditions*

(1) for noncommutative group ring RG to be UQ.

(2) for a skew group ring $R *_\theta G$ to be UQ.

In the fifth chapter, we have introduced and investigated a new class of rings which is called almost SWR rings and establish their relation with already known rings. Various properties of almost SWR rings have been obtained. We have characterized almost SWR rings. Also, we have discussed almost SWR group rings. It has been proved that if RH is almost SWR for every finitely generated subgroup H of G, then RG is almost SWR, but the converse of this result partially holds. Thus we have

Problem 8.0.3. *Find the necessary conditions for RG to be almost SWR.*

In the sixth chapter, we have introduced and investigated a new class of rings which is called semiboolean neat rings and established their relation with already known rings. Various properties of semiboolean neat rings have been obtained. It has been proved that commutative semiboolean neat rings, which are not semiboolean, is reduced. We have determined a characterization of group ring RG satisfying semiboolean neat property if R is commutative and G is abelian. As a future scope, we have

Problem 8.0.4. *(1) What is the structure of noncommutative semiboolean neat rings?*

(2) Obtain a complete characterization of noncommutative group ring RG to be semiboolean neat.

Kosan et al. [37], studied rings whose elements are sum of a tripotent and an element from the jacobson radical. An element a of a ring R is called a tripotent if $a^3 = a$. Chen and Sheibani [14], introduced a ring in which every element is the sum of two tripotents and a nilpotent that commute. In future, we intend to study a class of rings for which every element is a sum of an element from $J(R)$ and two tripotents. We shall attempt to determine necessary and sufficient conditions for group ring RG to be sum of a tripotent and an element from the jacobson radical.

Abdolyousefi et al. [1], describe the structure of unit nil-clean rings. A ring R is unit nil-clean if, for any $a \in R$, there exists a unit $u \in R$, such that ua is the sum of an idempotent and a nilpotent. In future, we plan to investigate the ring in which square of each unit is a sum of an idempotent and a nilpotent element. We intend to obtain necessary and sufficient conditions for group ring RG to be square of each unit is a sum of an idempotent and a nilpotent element.

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List of Publications

1. Kanchan Jangra, Dinesh Udar. On almost s-weakly s-weakly regular rings. *Turkish Journal of Mathematics, Scientific and Technological Research Council of Turkey*, 46(5), 1897–1910 (2022). https://doi.org/10.55730/1300-0098.3240 **(SCIE)**

2. Kanchan Jangra, Dinesh Udar. On UQ rings. *Journal of Algebra and its Applications, World Scientific*. 22(08), 2350175 (2023). https://doi.org/10.1142/S021949882350175X **(SCIE)**

3. Kanchan Jangra, Dinesh Udar. UN rings and group rings. *Bulletin of the Korean Mathematical Society*. 60(1), 83–91 (2023). https://doi.org/10.4134/BKMS.b210917 **(SCIE)**

4. Dinesh Udar, Kanchan Jangra. On Semiboolean neat rings. *Lobacheveskii Journal of Mathematics*. 44, 2956–2960 (2023). https://doi.org/10.1134/S1995080223070429 **(ESCI, SCOPUS)**

5. Kanchan Jangra, Dinesh Udar. On weakly g(x)-invo clean rings. *Emerging Advancements in Mathematical Sciences, Nova Science Publishers*. 171-181 (2022). https://doi.org/10.52305/RRIS1719.

6. Kanchan Jangra, Dinesh Udar. A note on weakly g(x)-invo clean rings (Communicated).

Papers presented in International Conferences

1. On weakly g(x)-invo clean rings; *5th International Conference on Recent Advances in Mathematical Sciences and its Applications (RAMSA-2021)*, Jaypee Institute of Information Technology, Noida, India. Dec 02-04, 2021

2. A note on s-weakly regular rings; *Emerging trends in Pure and Applied Mathematics*, Tezpur University, Assam. March 12-13, 2022

3. Further results on weakly g(x)-invo clean rings; *5th International Conference on Frontiers in Industrial and Applied Mathematics (FIAM-2022)*, Central University of Haryana, Mahendragrah, Haryana. Dec 22-23, 2022

4. On UQ rings; *International Conference on Graphs, Networks and Combinatorics (ICGNC-2023)*, Ramanujan College, Delhi. January 10-12, 2023