NUMERICAL METHODS FOR SINGULARLY PERTURBED DIFFERENTIAL AND DELAY DIFFERENTIAL EQUATIONS

A Thesis Submitted to

DELHI TECHNOLOGICAL UNIVERSITY

In Partial Fulfillment of the Requirements of the Award of the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICS

By

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November, 2022 **Enroll. No. : 2K17/Ph.D/AM/02**

DECLARATION

I declare that the work in this thesis entitled "**Numerical Methods for Singularly Perturbed Differential and Delay Differential Equations**" for the award of the degree of *Doctor of Philosophy in Mathematics* is original and has been carried out by me under the supervision of *Dr. Vivek Kumar*, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

This thesis has not been submitted by me earlier in part or full to any other University or Institute for the award of any degree or diploma. Wherever applicable, proper acknowledgement is given to the other's work.

Date : **Kartikay Khari**

CERTIFICATE

"On the basis of declaration submitted by **Mr. Kartikay Khari**, Ph.D. research scholar, I hereby certify that the thesis titled **"Numerical Methods for Singularly Perturbed Differential and Delay Differential Equations"** submitted to the Department of Applied Mathematics, Delhi Technological University, Delhi, India for the award of the degree of *Doctor of Philosophy in Mathematics*, is a record of bonafide research work carried out by him.

I have read this thesis and in my opinion, it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy.

To the best of my knowledge the work reported in this thesis is original and has not been submitted to any other Institution or University in any form for the award of any Degree or Diploma.

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ACKNOWLEDGEMENT

It gives me great pleasure in expressing my gratitude to all those people who have supported me and had their contributions in making this thesis possible. First and foremost, I would like to express my sincere gratitude to my thesis advisor, Dr. Vivek Kumar, for his guidance and support throughout my doctoral research. He has introduced me to a very interesting area of mathematics, an area that I would like to continue researching in the years ahead. His expertise, skills, and throughout encouragement are major pluses for this research. I definitely consider it an honor to have worked with him. The completion of this thesis would not have been possible without him and I owe a great deal of debt to him. I would like to give a heartfelt thanks to Prof. Mani Mehra for her constant support.

I express my deep gratitude and special thanks to Prof. Sree Hari Rao Vadrevu , Prof S. K. Tomar (Panjab University, Chandigarh) and Prof R. K. Mohanty (South Asian University Delhi) for his unconditional support, research discussions and for the research ideas which he had given to me.

I take an opportunity to thank the Head of the department Prof. S. Shiva Parasad, DRC Chairperson Prof. Neo Kant and Prof. V P Kaushik for providing necessary facilities during my research work.

I am thankful to the Delhi Technological University, Delhi, India for providing me financial assistance during my Ph.D. I take an opportunity to thank all the faculty members of the Department of Applied Mathematics, Delhi Technological University for their support and encouragement during Ph.D. I wish to thank staff members of the Department of Applied Mathematics, Delhi Technological University for providing their help at different times. I gratefully acknowledge the South Asian University, New Delhi for providing necessary facilities.

On this auspicious occasion, I am very happy to take this opportunity of expressing my heartfelt gratitude to my uncle's Kapil, Kuldeep and my aunty Nishi for their constant encouragement and blessings throughout my academic career since my primary education till date.

I am thankful to my friends and Ph.D. colleagues Ankit, Surya, Monu, Abhay, Kirti, Himansi, Akansha, Shweta, Monika at the Delhi Technological University for their concern, support, advice, encouragement and friendly environment. I would like to acknowledge my senior Dr.Ram Pratap, Dr. Sudhakar, Dr. Abhishek and Dr. Rahul Bansal for the research discussions. I also thank my dear Ph.D. friend Saurabh Yadav at LNMIIT, Jaipur for their unconditional support, care and providing pleasant working. They are always there for me, when I need help and moral support.

The love and good wishes of my family have been an invaluable source of inspiration for me in all these years. I am extremely grateful to all the family members for always being there for me. I would like to say a big thanks to my late parents for their blessings, which is beyond any acknowledgement. My deepest gratitude goes to dada and dadi for their blessings, extreme support and continuous encouragement. Finally, a heartfelt thanks to all other friends and relatives for their support and best wishes.

Above all, I acknowledge and thank the Almighty for his blessings, providing me the capability to proceed successfully in life and for everything I am blessed with.

I acknowledge Delhi Technological University, for providing fellowship that made my Ph.D. work possible.

I thank almighty God for showing me the right path.

Date : **KARTIKAY KHARI**

Place : Delhi, India.

Dedicated to my Late Dadi, Maa and Papa

Contents

Preface

“This thesis contributes to the numerical methods for singularly perturbed differential and delay differential equations. The purpose of this research is to propose numerical techniques for solving singularly perturbed differential and delay differential equations. We investigate, develop, and analyse numerical methods, as well as their implementations, for such challenging problems. A chapter-by-chapter structure of the thesis is as follows:

- **Chapter 1** presents introduction to SPPs, a brief survey on numerical analysis of SPDDEs and the need for parameter uniform numerical techniques. Objectives, motivations and a brief summary of the present work is also included in this chapter.

- In **Chapter 2**, we have implemented finite element method for the class of SPP-PDDEs with time delay. The solution of this class of problems exhibits parabolic boundary layers. The domain is discretized with a piecewise uniform mesh (Shishkin mesh) for spatial variable to capture the exponential behaviour of the solution in the boundary layer region and backward-Euler method on equidistant mesh in time direction. The error analysis id carried out in maximum norm and the proposed method is shown to be of order $[O(N^{-1}\ln N)^2+\Delta t]$. The effect of shifts on the boundary layer behaviour of the solution is shown by numerical experiments. *The results of this chapter have been published in the journal “Numerical Method for Partial Differential Equation”.*

- **Chapter 3** is devoted to develop numerical collocation method based on Bernstein polynomial for nonlinear singularly perturbed parabolic reaction-diffusion problems. The existence uniqueness of the proposed problem is carried out. The strategy behind this mesh is to deal with delay term and capture boundary as well as interior layer behaviour of the solution. The performance of the method is corroborated by numerical examples. *The results of this chapter have been accepted for publication in the journal "Journal of Mathematical Chemistry"*.

- The main aim of **Chapter 4** is to provide finite element method with Richardson extrapolation techniques for singularly perturbed parabolic time delay reaction diffusion problem and to improve the order of convergence of the numerical scheme proposed in **Chapter 2**. The solution of this class of problems is polluted by a small positive parameter due to which the solution of the said problem exhibits parabolic boundary layers. The spatial variable domain is evaluated by implementing finite element method along with piecewise uniform mesh (Shishkin mesh) to capture the exponential behaviour of the solution in the boundary layer region and for time variable author has implemented implicit backward-Euler method with Richardson extrapolation on equidistant mesh in time direction to attain a good accuracy along with the higher order convergence. The proposed method is shown to be accurate of order $[O(N^{-1}\ln N)^2 + \Delta t^2]$ in maximum norm. *The results of this chapter have been communicated.*

- The main purpose of **Chapter 5** is to overcome the well-known difficulties associated with numerical methods and to remove restriction on the choice of mesh generation for singularly perturbed problems. In this chapter a closed-form iterative analytic approximation to a class of nonlinear singularly perturbed parabolic partial differential equation is developed and analysed for convergence. We have considered both parabolic reaction diffusion and parabolic convection diffusion type of problems in this chapter. The solution of this class of problem is polluted by a small dissipative parameter, due to which solution often shows boundary and interior layers. A sequence of approximate analytic solution for the above class of problems is constructed using Lagrange multiplier approach. Numerical experiments are provided to illustrate the performance of the method. *The results of this chapter have been communicated.*

- Finally, the **Chapter 6** is devoted to conclusion of the study and discussion on some future directions of the current research work."

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Chapter 1

Introduction

The purpose of this chapter is to give a glimpse into the Singular perturbation theory, particularly the Singularly perturbed differential equations, with or without delay. We shall cover its development from 1900 to 1950, then some crucial contributions from 1980 to 2022, and its evolution from 1900 till today, which further justifies the motivation for this thesis. It also covers some basic notations and the synopsis of the thesis.

1.0.1 Singularly Perturbed Differential Equations

It is undoubtedly not hyperbole to state that the differential equations and numerical analysis are the backbones of present-day human civilization. These are broad topics that are equally important. The mathematical model of differential equations concerning a perturbation parameter offers the most accurate and, in numerous instances, the only realistic simulation of real-life phenomena. Mathematically, the model could be expressed in the form of an algebraic equation, an ordinary differential equation, an integral equation, a partial differential equation, or a system of these equations. The study of the effects of minor disruptions in a mathematical model of a physical system is known as perturbation theory. As differential equations, perturbation problems $\mathscr{P}_\varepsilon$ are ones in which a small positive perturbation parameter ε is attached to the leading derivative of the differential equation. The perturbation problem $\mathscr{P}_\varepsilon$ is called to be regularly perturbed if the solution of $\mathscr{P}_\varepsilon$ as $\varepsilon \to 0$, converges uniformly to the solution of the reduced problem $\mathscr{P}_0$, which is achieved by setting ε equal to zero in the perturbation problem $\mathscr{P}_\varepsilon$. Otherwise, it has referred to be singularly perturbed, with the singular perturbation parameter as the perturbation parameter. Such a breakdown of singular perturbation problems **(SPPs)** occurs only in narrow intervals of space or short intervals of time. The solution changes quickly and creates layers in these confined regions. Boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum physics, and Stroke lines and surfaces in mathematics are all terms used to describe these narrow regions.

The name "singular perturbation" comes from the fact that when the singular perturbation parameter is set to zero, the behavior of the differential equations changes totally in the limit situation. For example, the conservation of momenta and the conservation of energy equations change from being nonlinear parabolic equations to nonlinear hyperbolic equations. The birth of “singular perturbation” occurred at the Third International Congress of Mathematicians in Heidelberg in 1904, where Ludwig Prandtl delivered a lecture on “Fluid motion with small friction”. His seven-page report was published in the proceedings of the conference [246], in which he pointed out that the domain of fluid flow past a body can be divided into two parts, i) a narrow region (boundary layer) adjacent to the body in which the frictional effects are prominent, and ii) the remaining region (outer region) away from the body in which the flow is smooth. However, the terminology "boundary layer" was introduced by Prandtl, but Wasow introduced a substantial generality to the word "boundary layer" in his doctoral dissertation [297]. The term "singular perturbation" was first used by Friedrichs and Wasow in their paper [87].

The singularly perturbed boundary value problems **(SPBVPs)** are used frequently for describing and modeling mathematically many real-life phenomena in engineering, biology, economics, and physics, for example, fluid mechanics, fluid dynamics, aerodynamics, plasma-dynamics, the Michelis-Menten theory for enzyme reactions, the drift-diffusion equations of semiconductor device physics, magneto-hydrodynamics, rarefied-gas dynamics, chemical-reactor theory, elasticity, quantum mechanics, oceanography, plasticity, meteorology and radiating flows. Singular perturbation is now a relatively mature mathematical subject with a reasonably long history. The subject is now commonly a part of graduate study in applied mathematics and many engineering fields.

1.0.2 Singularly Perturbed Delay Differential Equations

In many branches of science and engineering, including control theory, epidemiology, laser optics, and viscoelastic behavior, mathematical models have emerged that account for the current state and the history of a physical system. Delay differential equations **DDEs**, a family of functional differential equations, are commonly used to explain such types of models. When some unknown hidden variables and processes are known to produce a temporal lag but are present in the majority of applications in the life sciences, a delay is added. The delay differential equation can describe various processes mathematically, offering the most accurate simulation of observable phenomena and occasionally the only one-in many application disciplines.

Delay differential equations (DDEs) were first used to describe technical devices, such as control circuits. In that context, the delay is a quantifiable physical property (e.g., the time it takes for the signal to reach the controlled device, for the user to react, and for the signal to return). DDEs are widely used in many biology and control theory fields, including ecology, chemostat systems, epidemiology, immunology, compartmental studies, neural networks, and the navigational control of ships and planes (with varying lag times).

There are many biological phenomena where the delay model is used to model them, for example, in physiological kinetics [9], growth model for cell proliferation [18], mathematical models of the early embryonic cell cycle [47], chemical kinetics [77], periodic oscillations of breathing frequency [205], blood cell production [210], immune responses [211], modeling population dynamics [231], etc. There is numerous collection of books that indicate application areas and theory for DDEs; the list is quite long, but we cite here a few of them: the books by Kuang [174], Asachenkov et al. [13], R. Bank [24], MacDonald [203], Marchuk [211], Bellman and Cooke [34], Driver [71], Gopalsamy [96] and Kolmanovskii, et al. [164].

The primary reasons for studying and using DDE models are that
i) they offer a more robust mathematical foundation for analyzing the dynamics of biological systems than ordinary differential equations,
ii) they exhibit superior congruence with the nature of the underlying processes and prediction outcomes. While DDEs and related functional differential equations represent issues where there is an after-effect impacting at least one of the variables, ordinary differential equations (ODEs) model problems where the variables react to current conditions.

Some modelers ignore the "lag" effect and use an ODE model as a substitute for a DDE model. Kuang ([174], pp. 11) comments under the heading "Small delays can have large effects" on the dangers that researchers risk if they ignore lags which they think are small; see also El' sgol'ts and Norkin ([76], pp. 243). Now, if we restrict the above classes to a class with the characteristics of both the classes, *,i.e.,* delay or advance and singularly perturbed behavior, this class is classified as the singularly perturbed differential-difference equation with delay or advance. In the literature, the expression "negative (or left) shift" and "positive (or right) shift" are used for "delay" and "advance", respectively. Such differential equations arise in modeling various practical phenomena in bioscience, engineering, control theory, etc. For example, the first exit time problem in modeling of activation of neuronal variability [188], in the study of bistable devices [68] and evolutionary biology [299], in a variety of models for physiological processes or diseases [204], to describe the human pupil-light reflex [200], variational problems in control theory [91], and in describing the motion of the sunflower [244], where they provide the best and in many cases the only realistic simulation of the observed phenomena.

1.0.3 Nonlinear Singularly Perturbed Differential and Partial Differential Equation

Whenever a real-life phenomenon is converted into a mathematical model, a nonlinear differential equation, nonlinear partial differential equation, and system of the differential equation play a vital role in modeling natural evolution, we primarily try to obtain what is important, retaining the essential physical quantities and neglecting the negligible ones which involve small positive parameters. These problems occur in oceanography, population dynamic, generic repression, size-dependent cell growth, division modeling, ecology, quantum physics, chemistry, finance(Black-Schole Equation), and material science. The Navier-Stokes equation with a higher Reynolds number is one of the most striking examples of nonlinear singularly perturbed parabolic partial differential equations **(NSPPPDEs)** that arise in fluid dynamics. These problems are essential to the environmental sciences in analyzing pollution from manufacturing sources entering the atmosphere. This type of problem occurs in chemical kinetics in catalytic reaction theory. The NSPPDEs problem models an isothermal reaction catalyzed in a pellet and modeled. In considering these types of problems, it is essential to acknowledge that the diffusion coefficient of the admixture in the material may be sufficiently small, resulting in substantial variations of concentration along with the material depth. Then, the diffusion boundary layers rise. Hence these types of problems exhibit a singularly perturbed character. The mathematical model of such problems has a perturbation parameter, a small coefficient multiplying the differential equation's highest derivatives. Such specific problems rely on a small positive factor so that the solution changes swiftly in some areas of the domain and gradually in other sections. The mathematical model for an adiabatic tubular chemical reactor that processes an irreversible exothermic chemical reaction is also represented by NSPPDEs problems. The concentrations of the various chemical species involved in the reaction can be determined in a simple manner.

1.1 Real World Aspects of Nonlinear Singularly Perturbed Differential Equations

1.1.1 Navier-Stokes equation

Consider the NSPPDES that emerge in fluid dynamics is the Navier-Stokes equation with a greater Reynolds number. Considering the following Navier-Stokes equation that governs unsteady in-compressible viscous fluids flow problem.

$$\frac{\partial \upsilon_\varepsilon}{\partial t} + \upsilon_\varepsilon.\Delta\upsilon_\varepsilon + \Delta\rho = \frac{1}{\Re e}\Delta^2\upsilon_\varepsilon$$

$$\Delta\upsilon_\varepsilon = 0$$

Where ρ signifies pressure and $\upsilon = (\upsilon_1, \upsilon_2)$ denotes the velocity, with velocity components υ_1 and υ_2 along with x and y directions. The parameter $\Re e$ represents Reynolds number which is directly proportional to the velocity scale, length scale and inversely proportional to the kinematic viscosity of the fluid.

1.1.2 Enzyme kinetics

Enzyme kinetics commonly describes the process by which an enzyme transforms a substrate (p) into a product through the formation of a substrate-enzyme complex (q). The pair of equations is a model for this procedure and this is known as the Michaelis-Menton reaction

$$\begin{cases} p^{'}(x) = -p + (p + a - b)q \\ \varepsilon q^{'}(x) = p - (p - a)q \end{cases}$$

The variable $q(x)$ production rate is measured by the parameter ε. Where a and b are positive constants and initial conditions are $p(0) = 1$ and $q(0) = 0$.

1.1.3 The Belousov-Zhabotinskii reaction

Belousov introduced this well-known and well researched phenomenon in 1951. He found that it was feasible for a catalyst's concentration to oscillate steadily between its oxidised and reduced states. This can be demonstrated in a suitable medium as a dramatic change in colour over the course of about a minute, with each colour representing a different condition. For this chemical reaction, a set of model equations are

$$\begin{cases} \varepsilon v^{'} = v + u - vu - \varepsilon\sigma v^2 \\ \varepsilon\mu u^{'} = kw - u - vu \\ w^{'} = v - w \end{cases}$$

where w denotes the concentration of the catalyst. Here σ, μ and k are positive constant independent of ε, measures the rate constant for the production of v and specifies the size of the corresponding constant for u as well as the nonlinearity in the first equation.

1.1.4 Ginzburg-Landau equation

Gingburg and Landau first introduced Ginzburg-Landau functional in the context of superconductivity in 1950.

$$\begin{cases} \varepsilon\Delta V = V^3 - V \quad \text{in} \quad \Omega \\ V(0) = 0 \quad \text{in} \quad \partial\Omega \end{cases}$$

1.2 Real World Aspects of Singularly Perturbed Delay Differential Equations

1.2.1 Respiratory Physiology to Laser-Based Optical Devices

In respiratory physiology and laser-based optical systems, delayed recruitment/renewal equation is defines as:

$$\varepsilon\frac{d\mu(t)}{dt} = -\mu(t) + f(\mu(t-1)),$$

Here, ε is inversely proportional to the product of the time-delay inherent in the physical system and its rate of decay, provides the mathematical model [204] in a diverse spectrum of practical applications.

1.2.2 Population Dynamics

The mathematical model of population dynamic, which is also known as Britton-model modelled as:

$$\frac{\partial v_\varepsilon(x,t)}{\partial t} - \varepsilon \Delta v_\varepsilon(x,t) = v_{ep}(x,t)(g * v_{ep}(x,t)), \tag{1.2.1}$$

where

$$g * v_\varepsilon = \int_{t-\tau}^{t} \int_{\Omega} g(x,-y,t-s) v_\varepsilon(y,s) dyds \tag{1.2.2}$$

Here $0 < \varepsilon << 1$, $v(x,t)$ signifies a density of population that evolves by random migration, which is described by the diffusion term and reproduction term is modelled by the non-linear reaction term. The convolution operator's kernel $g(x,t)$ describes the evolution's distributed age-structure dependence as well as its dependence on local population densities.

1.2.3 Van der Pol Equation

Considers the van der Pol equation defined by Oliveira [67] as

$$\frac{d^2\omega(t)}{dt^2} - \varepsilon \frac{d\omega(t)}{dt} + \varepsilon \omega^2(t-\tau) \frac{d\omega(t-r)}{dt} + \omega(t) = 0,$$

where τ and ε are real parameters, with $\varepsilon > 0$ small, with the delayed time τ, and $0 \le \tau < \pi/2$. The existence and stability of a periodic solution with period near 2π and amplitude near $2/\sqrt{cos\ (\tau)}$ is shown.

1.2.4 Singularly Perturbed Functional Differential Equations Arising in Optimal Control Theory

A controlled singularly-perturbed system with point wise delay in the state variables [91] is:

$$\frac{dx(t)}{dt} = A_1(t)x(t) + A_2(t)y(t) + H_1(t)x(t-\varepsilon h) + H_2(t)y(t-\varepsilon h) + B_1(t)u(t) + f_1(t),$$

$$\varepsilon \frac{dy(t)}{dt} = A_3(t)x(t) + A_4(t)y(t) + H_3(t)x(t-\varepsilon h) + H_4(t)y(t-\varepsilon h) + B_2(t)u(t) + f_2(t),$$

$$x(t) = \phi_x(t), \quad y(t) = \phi_y(t), \quad -\varepsilon h \le t < 0,$$
$$x(0) = \phi_x^0, \quad y(0) = \phi_y^0,$$

where x and y are state variables, u is a control variable, $\varepsilon > 0$ is a small parameter $(\varepsilon \ll 1)$, $h > 0$ is some constant independent of ε, A_i, H_i, B_j, $(i = 1,\dots,4; \quad j = 1,2)$, are time depending matrices of corresponding dimensions.

1.2.5 Neural Reflex Mechanism

$$\varepsilon v'(t) + \alpha(\varepsilon v'(t))y(t) = f(v(t-\tau)).$$

Since neuromuscular reflexes with delayed negative feedback have varying rates depending on the direction of movement, these models are essential [19].

1.2.6 Activation of Neurons

The initial efforts in studying the Stein's model and compared it with the diffusion model was done by Roy and Smith [265], Tuckwell and Cope [285].

In Stein's model, the distribution representing inputs is taken as a Poisson process with exponential decay. If in addition, there are inputs that can be modeled as a Wiener process with variance parameter σ and drift parameter μ, then the problem for expected first-exit time y, given initial membrane potential $x \in (x_1, x_2)$, can be formulated as a general boundary value problem **(BVPs)** for a linear second order differential difference equation(DDE) [187]

$$\frac{\sigma^2}{2} y''(x) + (\mu - x)y'(x) + \lambda_E y(x + a_E) + \lambda_I y(x - a_I) - (\lambda_E + \lambda_I)y(x) = -1, \tag{1.2.3}$$

where the values $x = x_1$ and $x = x_2$ correspond to the inhibitory reversal potential and to the threshold value of membrane potential for action potential generation, respectively. σ and μ are variance and drift parameters, respectively, y is the expected first-exit time and the first order derivative term $-xy'(x)$ corresponds to exponential decay between synaptic inputs. The undifferentiated terms correspond to excitatory and inhibitory synaptic inputs, modeled as Poisson process with mean rates λ_E and λ_I, respectively, and produce jumps in the membrane potential of amounts a_E and a_I, respectively, which are small quantities and could be dependent on voltage. The boundary condition is

$$y(x) \equiv 0, \quad x \notin (x_1, x_2).$$

1.2.7 Mathematical Model of Red Cell System

For describing the production of red blood cells, Wazewska-Czyzewska and Lasota [299] used the equation

$$\varepsilon y'(t,\varepsilon) = -y(t,\varepsilon) + \lambda (y(t-1,\varepsilon))^8 e^{(-y(t-1,\varepsilon))}$$

1.2.8 The Mathematical Model Describing the Motion of the Sunflower

The following mathematical model [244] is used to explain the motion of sunflower.

$$\varepsilon v''(t) + a v'(t) + b \sin(v(t-\tau)) = 0, \quad t \in [-\tau, 0],$$

$\varepsilon > 0$ with $v'(0)$ prescribed. Here a and b are positive parameters which can be obtained experimentally, the $v(t)$ is the angle of the plant with the vertical and the time lag (τ) is geotropic reaction.

1.2.9 Dynamics of a Network of Two Identical Amplifiers

For $\varepsilon > 0$ and $f \in C^m(\Re \times \Re)$, $\lambda \in \Re$, $m \geq 3$, the following system of DDEs

$$\varepsilon \frac{dx(t)}{dt} = -x(t) + f(y(t-1), \lambda),$$

$$\varepsilon \frac{dy(t)}{dt} = -y(t) + f(x(t-1), \lambda),$$

describes the dynamics of a network of two identical amplifiers (or neurons) with delayed outputs [58].

1.3 Numerical Methods

1.3.1 Numerical Analysis

The development and study of algorithms for numerical computations is known as numerical analysis, and it can be approached in one of three ways:
i) It can involve designing methods for solving specific computational problems that originate from mathematical applications.
ii) Providing and analyzing algorithms for fundamental mathematical estimations that are common to numerous applications or performing theoretical research on questions that are critical to the success of algorithms.
iii) Function approximation, data fitting and smoothing, optimization, matrix calculations, ordinary, partial, and functional differential equations, computational aspects of dynamical systems, theory of orthogonal polynomials, and special functions are some areas where it conducts the study.

1.3.2 Numerical Method for Singular Perturbation Problems

The solutions of singularly perturbed differential equations exhibit multi-scale character. The area is referred to as a boundary layer when the solution changes swiftly over a segment of the independent variable. In the outer region, the solution changes gradually. Thus, the "two-time-scale" characteristic of the singular perturbation problems **(SPPs)** arises. The terminology "boundary layer" is borrowed from physics. The problem is "stiff" from the perspective of approximating the answer since both the slow and fast phenomena exist at the same time. Various methods have been proposed so far to obtain the approximate solution of singular perturbation problems, which are classified as;
i) Finite Difference Methods (FDM),
ii) Finite Element Methods (FEM),
iii) Finite Volume Methods (FVM),
iv) Collocation Methods,
v) Operational Matrix Methods (OMM),
vi) Iterative methods and

vii) Asymptotic Approach.

Asymptotic methods are applicable only to the restrictive class of problems. They are not conveniently applicable to two-dimensional problems. For complex one-dimensional nonlinear problems, the asymptotic methods are valid for small values of singular perturbation parameter ε. The asymptotic technique provides qualitative behavior of the solution of a family of problems and semi-quantitative information about the solution of any given member of the family. In contrast, the numerical approach offers quantitative information about the solution to a specific problem. In this section, we give an overview of the numerical approach for tackling involving singular perturbations.

The textbooks provide a thorough summary of SPPs as well as their theoretical and numerical implementation can be found in [42, 78, 108, 111, 214, 217, 220, 221, 234] and the citations therein.

The growth of research activity in the area of numerical treatment for solving singularly perturbed ODEs resulted in the publication of two survey papers [128, 145]. In [128], Kadalbajoo and Reddy presented a survey on the numerical methods for one dimensional SPPs, starting from Pearson's work [242, 243] in the year 1968 up to 1984. In continuation, Kadalbajoo with Patidar [145] published another survey paper on the numerical treatment for singularly perturbed ODEs which deals with the work done after 1984, starting from Ascher's work [14] up to Kopteva's work [170] in 1999. In this section, we focus on the work done after 1999 till date.

In [189], Lengerink considers a one-dimensional convection-diffusion type singularly perturbed BVPs. The author has proposed and discussed a centered difference or finite element discretization and implemented it to solve such types of BVPs. A piecewise equidistant mesh is used for discretization. The author has shown that the proposed scheme is second order convergent concerning the number of nodes concerning the perturbation parameter.

Beckett and Mackenzie [27] consider a model for inhomogeneous second-order SPBVPs on a non-uniform grid and derived a ε-uniform error estimates for first-order upwind. The mesh is discretized by using the equidistribution of a positive monitor function which is a linear combination of an appropriate power of the second derivative of the solution and a positive constant. The authors showed how the constant should be chosen to ensure ε-uniform convergence.

This research article [216] aims to implement a numerical scheme for a singularly perturbed differential equation of reaction-diffusion type with a discontinuous source term. The numerical scheme proposed by the authors consists of a standard finite difference operator and a non-standard Shishkin mesh (piece-wise uniform mesh). The Shishkin mesh is constructed in such a way as to capture the boundary and interior layers that arise inside the solution to the problem. The authors have shown the proposed scheme to be uniformly convergent with concerning singular perturbation parameters. A brief overview is given to show the occurrence of such problems in the context of models of simple semiconductor devices.

In this paper [86], the authors have proposed a defect correction method based on the FDM on the piece-wise uniform mesh known as Shishkin mesh to solve convection-type singularly perturbed BVPs. The convergence analysis is carried out, and the authors obtained bounds for error uniformly in the perturbation parameter ε.

In 2001, Kopteva and Stynes published research articles on the numerical approximation for the solutions of SPDEs. In [171], Kopteva and Stynes deal with the singularly perturbed BVPs of the convection-diffusion type in conservative form. They present an upwind conservative FDM to solve such BVPs. They establish bounds for the errors in approximating the derivative of the exact solution by divided differences of the computed solution on any arbitrary mesh with the weight of the small diffusion coefficient. These bounds are then made more explicit for the specific cases of Shishkin and Bakhvalov meshes.

Kadalbajoo and Patidar [142] present a two-point BVP of singularly perturbed type and discuss the cases when solutions to such BVPs exhibit a most interesting “turning point” phenomena. The authors construct a numerical technique based on the cubic spline technique with non-uniform mesh to solve such BVPs. The proposed numerical scheme is analyzed for convergence and stability. Some test problem is taken into account, and numerical experiments are carried out in support of the predicted theory.

In 2001, Liu and Tang [199] consider a singularly perturbed BVPs and developed a Galerkin-spectral method, which makes use of a class of trial functions that are suitable for coordinate stretching for solving such type of problems. The authors have carried out the error analysis for the proposed spectral method. When solving SSPs using traditional spectral methods, spectral accuracy can be obtained only when $N = O(\varepsilon^{-g})$, where ε is the singular perturbation parameter and g is a positive constant. They obtained similar results for advection-diffusion equations. Two essential features of the proposed method are as follows: i) the coordinate transformation does not involve the singular perturbation parameter ε; ii) machine accuracy can be achieved with N of the order of several hundred, even when ε is very small.

In [207], MacMullen et al. present a self-adjoint singularly perturbed ordinary differential equation and construct a parameter uniform numerical scheme to approximate such type of BVPs. It is demonstrated that an appropriately constructed discrete Schwarz method based on maximum norm converges to the exact solution uniformly concerning the singular perturbation parameter. The suggested scheme has second-order maximum norm convergence.

Aziz and Khan [16] considered singularly-perturbed BVPs and proposed a numerical technique based on a quintic spline method for such types of problems. The scheme leads to a pentadiagonal linear system. The authors have claimed in their article that the proposed approach has fourth-order convergence.

In this article [80], Farrell et al. proposed and analyzed numerical methods based on upwind FDM for solving SSPs. The authors have studied the convergence analysis in maximum norm. The authors employ the technique for computing Reynolds-uniform error bounds in the maximum norm for the numerical solutions obtained by a proposed method applied to Prandtl's problem arising from laminar flow past a thin flat plate.

In 2002, Kadalbajoo again published a survey paper [145] with Patidar on the numerical techniques for solving singularly perturbed ODEs. In this article, the authors present the survey of the work done after 1984, starting from Ascher's work [14] up to 1999. In 2002, Kadalbajoo and Patidar presented several numerical difference schemes based on the cubic spline to solve the BVPs for singularly perturbed linear and nonlinear ODEs. In [143], they consider spline in tension to obtain the numerical solution of SPBVPs. In this paper, the authors consider three types of problems, i) the reaction-diffusion type, ii) the convection-diffusion type with no reaction term, and iii) a more general convection-diffusion type with a reaction term. The authors use the continuity of the spline function's first derivative to resolve these BVPs. The resulting spline produces a tridiagonal system that can be successfully solved using well-established algorithms. The authors have obtained the error estimates for the numerical solution of such types of BVPs. This article [144] deals with the nonlinear SPBVPs. In this article, the authors propose a numerical technique based on cubic splines with non-uniform mesh for nonlinear SPBVPs. The proposed nonlinear problem is first linearized by the quasilinearization technique. The authors drive the difference schemes linearized SPBVPs using a variable-mesh cubic spline. In [146], the authors present a method based on spline in tension for the self-adjoint two-point SPBVPs. The proposed approach is shown to be of almost second order convergence. In [132], Kadalbajoo et al. considered a singular perturbation problem and developed a B-spline collocation method using artificial viscosity for such type of problems.

B. Zhang et al. [308] consider SPBVPs of reaction-diffusion type and derived a posteriori error bounds for nonconforming finite element approximations to such type of problems under special equilibration conditions.

J. Zhao and S. Chen [312] a reaction-diffusion type SPBVPs on anisotropic meshes. The authors have derived robust a posteriori error estimates using the nonconforming finite element method over the anisotropic mesh.

In [206], the author proposes a numerical technique to solve the SPBVPs of convection-diffusion type with a regular boundary layer and construct an iterative method based on the Schwarz alternating procedure over a nonuniform grid (Shishkin mesh) to solve such type of problems. It is demonstrated that the numerical approximations produced by the overlapping Schwarz method with uniform meshes and arbitrary fixed interface positions are not ε-uniform convergent. The approximations produced by a numerical approach using uniform meshes on overlapping meshes and Shishkin interface positions need to converge to exact solutions. Finally, the authors examine a non-overlapping method using Shishkin interface positions, uniform meshes, and artificial Dirichlet interface conditions for a two-dimensional elliptic problem with regular boundary layers.

In [60], C. Clavero et al. present a uniformly convergent alternating direction HODIE FDM for a 2D parabolic convection-diffusion partial differential equation and construct an FDM to solve such types of problems. The authors implement a Peaceman and Rachford technique in the time direction and (HODIE) high-order differences via an identity expansion along with FDM in the space direction. The author discretizes the mesh grid using piece-wise uniform mesh in space direction. In [44], C. Clavero et al. present a 2D parabolic partial differential equation of reaction-diffusion singularly perturbed type. The proposed method consists of implementing the (HODIE) scheme, the finite difference method in space direction, and the Peaceman and Rachford technique in the time direction. The authors have proved that method is second order convergent in time and third-order convergent in space.

Matthews et al. [213] deals with a two-coupled system of singularly perturbed reaction-diffusion ODEs of Dirichlet type. The authors have constructed a numerical technique whose approximate solutions converge point-wise at all domain points independently of the singular perturbation parameter.

In this article [224], Natesan and Ramanujam deal with singularly perturbed second-order ODEs of Robin type. To approximate the multi-scale character of such problems, an asymptotic approximate solution combined with the solution obtained by a numerical method comprising an exponential FDM is obtained suitably. The 'transition point' is chosen inside the interval of integration when the solution of the reduced problem is evaluated, which will be taken as a boundary value for the boundary layer region problem. The authors perform iterations here by slowly moving the transition point towards the right-hand side until the solution stabilizes. In the outer region, the solution of the reduced problem is taken as an approximation to the original problem. Error estimates are established for the approximate solution.

C. Xenophontos and L. Oberbroeckling [303] proposed a FEM for numerically approximating singularly perturbed systems of reaction-diffusion problems. The error analysis is carried out, and the authors claim that the proposed method is convergent with an exponential convergence rate.

In [223], two-point SPBVPs having less severe boundary layers are proposed, and a numerical scheme using the shooting method is constructed. The authors divide the domain into two subintervals of consideration of the differential equation, namely the outer region and the boundary layer or inner region. In the boundary layer region, an IVP obtained from the given BVP is solved by an exponentially fitted FDM. In contrast, the authors have used a classical upwind scheme in the outer region. The authors carry out the convergence and error analysis. The implementation of the method on parallel architectures is discussed.

In this artice [261], the authors deal with implementing numerical integration techniques for solving SPBVPs. The authors replace the original second ODE with an approximate first-order differential equation with a term recurrence relationship. The proposed method is iterative on the deviating argument.

F. Celiker and B. Cockburn [51] present a class of convection-diffusion problems in one space dimension. The author implements discontinuous Galerkin, discontinuous Petrov-Galerkin technique, and hybridized techniques to study the superconvergence of these numerical schemes. The authors have proved the superconvergence of order $(2q+1)$, q denotes the degree of a polynomial. The proposed numerical test's results support the theoretical findings.

Valarmathi and Ramanujam consider an asymptotic numerical fitted mesh method for singularly perturbed third-order ODEs of the reaction-diffusion type [290,291] and fourth-order ODEs of the convection-diffusion type [270]. The authors transform the third-order singularly perturbed BVPs into an equivalent problem of a weakly coupled system of one first-order and one second-order ODE with the parameter ε multiplying the highest order derivative and a fourth-order SPBVPs into the equivalent problem of a weakly coupled system of two-second order ODEs, one with a small parameter and other without the parameter. A computational method based on asymptotic expansion is proposed to solve these systems. In [290], up to the transformation of the third order into the system of ODEs is the same as the authors did in [291]. Then in this paper, the authors divide the domain of the definition of the differential equations into two sub-intervals, namely, the boundary layer region and the outer region. Now the differential equation is solved in these regions separately. The solutions so obtained in these intervals are combined to give the solution in the whole interval. To obtain boundary conditions at the transition points, the authors use mostly the zeroth-order asymptotic expansion of the solution of the BVP or a suitable asymptotic expansion solution. To tackle semi-linearity in the differential equation, Newton's method of quasi-linearization is applied.

Valarmathi and Ramanujam continue their numerical study on the third-order singularly perturbed ODE. In [289], the authors consider singularly perturbed third-order ODEs of the convection-diffusion type. In this article, a numerical method is proposed to solve such problems. This method transforms the given BVPs into a weakly coupled system of two ODEs subject to suitable initial and boundary conditions. The authors reduce the weakly coupled system into a decoupled system. Then, to solve this decoupled system numerically, a boundary value technique is used, in which the domain of the definition of the differential equation is divided into two non-overlapping subintervals called inner and outer regions. Then the decoupled system is solved over these regions as two-point BVPs. An exponential-fitted FDM is used in the inner region, and a classical FDM is in the outer region. The boundary conditions at the transition point are obtained using the zero-order asymptotic expansion approximation of the solution to the problem.

H. Zarin et al. [306] deal with singularly perturbed third-order BVPs on a layer adaptive mesh. The author's design of an interior penalty finite element technique on piece wise uniform mesh (Shishkin mesh). The error estimates are derived in energy norm, and the proposed method is found to be robust.

J.M. Melenk and C. Xenophontos [215] consider a singularly perturbed reaction-diffusion equation that is posed on a two-dimensional domain with an analytical boundary, and the hp-version of the FEM is used to solve such type of problems numerically. The authors have carried out the convergence analysis in the balanced norm, and the method is uniformly convergent concerning the singular perturbation parameter.

In 2003, Heinrichs [107] presented least squares spectral collocation for discontinuous and singular perturbation problems. For the first derivative operator, the author decomposed the domain into subdomains where the jumps are imposed at the discontinuities and used the equal order polynomial on all subdomains. He uses spectral collocation with Chebyshev polynomials for discretization. The collocation and interface conditions lead to an overdetermined system that least squares can efficiently solve. The solution technique involves only symmetric positive definite linear systems. The author extends this approach to singular perturbation problems where least-squares are used for stabilization. A suitable decomposition of the domain well resolves the boundary layer.

S.C.S. Rao and S. Kumar [260] deal with the coupled system of singularly perturbed IVPs and design a second-order global uniformly convergent numerical method for such problems. M. Kumar and S.C.S. Rao [178] proposed a high-order parameter robust numerical technique for 1D singularly perturbed time-dependent reaction-diffusion Dirichlet problem. The proposed method consists of the Crank-Nicolson technique on a uniform grid in the time direction and the fourth-order high-order compact scheme directly on a Shishkin method in the space.

In [147], the authors consider singularly perturbed two-point BVPs and developed a difference scheme based on spline in compression on a non-uniform mesh to solve such types of BVPs. The proposed scheme is second-order accurate. The authors have taken a few numerical examples to support the theoretical results.

Natesan et al. [225] developed and analyzed a numerical technique for SPPs exhibiting weak boundary layers. In this article, the authors have divided the domain $[0,1]$ into two non-overlapping sub-domains, as, $[0,k\varepsilon]$ and $[k\varepsilon,1]$. An exponential FDM is implemented in the layer region $[0,k\varepsilon]$, subject to the transition boundary condition at $x=k\varepsilon$ to solve the proposed problem. A classical FDM is implemented to approximate the SPPs differential equation in the regular region $[k\varepsilon,1]$. In order to obtain the boundary condition at the interior point $x=k\varepsilon$ (called the transition point), the value of the asymptotic approximation is used. The authors have carried out the error estimate, and the proposed technique is found to be convergent with concerning the perturbation parameter ε.

H. Zarin et al. [306] deal with singularly perturbed third-order BVPs on a layer adaptive mesh. The author's design of an interior penalty finite element technique on layer adaptive mesh is known as Shishkin mesh. The error estimates are derived in energy norm, and the proposed method is found to be robust.

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In [288], the authors developed and analyzed a computational technique to approximate non-turning-point SPBVPs for second-order ODEs subject to Dirichlet-type boundary conditions. In this article, the authors proposed a zeroth-order asymptotic expansion for the numerical solution of SPBVPs. Then, the problem is integrated to obtain an equivalent IVP for a first-order ODE. This IVP classical method or a fitted operator method after approximating some of the terms in the differential equations by using the zeroth order asymptotic expansion. The proposed technique is shown to be convergent of order h, where h is the mesh size by deriving the error estimate for the numerical solution.

Z. Du and L. Kong [72] deal with singularly perturbed second-order differential equations and their application to multi-point BVPs. The authors solve the proposed problem by a hybrid method which is a combination of asymptotic solutions and the Liouville-Green transform.

Natividad and Stynes [226] develop a Richarchardson extrapolation technique along with FDM for numerically approximating singularly perturbed convection-diffusion problem over a piece-wise uniform Shishkin mesh. In this paper, the authors show how to construct a Richardson extrapolant of the computed solution and prove that the extrapolation technique minimizes the nodal errors from $O(N^{-1}\ln N)$ to $O(N^{-2}\ln^2 N)$, where $N+1$ points are used in the mesh.

Vivek et al. [183] proposed a numerical technique to solve convection-dominated, convection-diffusion SSPs using second-order central difference methods and a new adaptive mesh strategy. The proposed method employs a novel, entropy-like variable as the convection-diffusion problems adaptation parameter. The proposed method does not require prior knowledge of the position and width of the layers (interior and boundary). In [182], Vivek Kumar has developed and analyzed a high-order compact finite-difference (HOCFD) technique to solve 1D and 2D singularly-perturbed elliptic and parabolic reaction-diffusion problems. The author has claimed parameter uniform convergence. In [184], Vivek et al. consider a singularly perturbed star graph with k + 1 nodes and k edges, leading to a system of k separate partial differential equations along the edges with coupling conditions at the common junction.

C.Y. Jung and R. Temam [127] initiated the study of singularities and boundary layers created by a convection-diffusion problem in a circle of noncompatible data. The circle's boundary has two characteristics that are associated with noncompatible data. The authors have observed a complex singular phenomenon and carefully analyzed it for highly noncompatible data.

C. Clavero et al. [61] consider a parabolic singularly perturbed partial differential equation of convection type with degenerating convective term and discontinuous (discontinuity of the first type) source term. The authors have developed a monotone finite difference method with piece-wise uniform mesh for solving the proposed problem. C. Clavero et al. [62] deal with the 1d parabolic convection reaction partial differential equation of singularly perturbed type. The proposed problem is more complex by multiplying the convection term by the parameter μ, and the source term has a discontinuity. Due to this, interior and boundary layers emerge. The authors have designed a finite difference method over a Shishkin mesh in space direction and an implicit Euler method over the uniform grid in time direction for solving such types of problems. The authors claim that the proposed approach is convergent with first order in time and second order in space.

In [55], The authors have designed and implemented a hybrid difference scheme for a singularly perturbed reaction-diffusion problem with discontinuous (discontinuity of the first type) source term. The author also discusses the case of semiconductor modeling. The authors have proved second order convergence in maximum norm.

R. Lin and M. Stynes [190] consider singularly perturbed reaction-diffusion problems of $d \geq 2$. The authors first proved that the energy norm is too weak to capture the error in the layer part. The authors introduced a balanced norm and proved a uniform convergence for $d \geq 2$.

In [310], J. Zhang et al. derived error bounds of the streamline diffusion finite element method (SDFEM) for 2D convection-dominated BVPs with boundary layers. The authors demonstrate the method's convergence with a point-wise precision of nearly order $7/4$ away from the characteristic layers, independent of the perturbation parameter ε. Numerical experiments support these theoretical findings. [309] J. Zhang and X. Liu proposed SDFEM on hybrid meshes and Shishkin meshes for convection-diffusion BVPs with characteristic layers. In [311], Z. Zhang proposed a finite element and derived the super-convergence for 2D convection-diffusion BVPs over Shishkin mesh.

In this article, [106], A.F. Hegarty and E. O'Riordan dealt with linear SPBVPs convection dominated over a circular domain and constructed a uniform numerical method. The method consists of a monotone FDM over a layer-adapted mesh of the Shishkin type.

N.N. Nefedov et al. [227] take into account a singularly perturbed parabolic periodic BVPs of reaction advection-diffusion type. The author has implemented a modified asymptotic approach by the upper and lower solution method. The authors have derived the existence and asymptotic stability of periodic solutions for the proposed problem.

Madden et al. [208] present a system of two coupled singularly perturbed linear reaction-diffusion two-point BVPs and construct a numerical method to solve such problems. Each equation's leading term is multiplied by a small positive parameter, but the size of these parameters can have different magnitudes. There are boundary layers that overlap and interact in the system solutions. By examining the structure of these layers, the authors were able to develop a piecewise-uniform Shishkin mesh. They also prove that on this mesh, central differencing is almost first-order accurate, uniformly in both small parameters. Numerical results are presented for a test problem in support of the predicted theory.

H.-G. Roos and H. Zarin [264] present a convection-diffusion problem on a unit square with Dirichlet boundary conditions. On a layer-adapted mesh with linear/bilinear components, the problem is discretized using a combination of the standard Galerkin FEM and an h-version of the nonsymmetric discontinuous Galerkin FEM with interior penalties. The authors demonstrate uniform convergence (in the perturbation parameter) in an associated norm using specifically selected penalty parameters for edges from the coarse region of the mesh.

In [305], H. Zarin examines and implements a numerical scheme based on discontinuous FEM for convection-dominated diffusion BVPs. The technique combines a hp version of nonsymmetric discontinuous Galerkin FEM with ordinary Galerkin FEM with bilinear components. The authors have obtained super closeness results. The method is found to be robust and efficient.

In [48], The authors developed a numerical technique combining discontinuous Galerkin FEM and ordinary Galerkin FEM. The author has implemented Galerkin fem in the regular region and discontinuous Galerkin method in the layer region. The convergence and stability of the proposed method are derived from advection-diffusion reaction problems.

Z. Cen et al. [52] developed a reliable method to numerically approximate a nonlinear singularly perturbed IVPs. The authors have also analyzed the behavior of the exact solution and developed a second order method accordingly.

In [236], consider singularly perturbed ordinary differential equations containing two small parameters and develop parameter-uniform numerical methods for solving such types of BVPs. The authors have derived parameter-explicit theoretical bounds on the derivatives of the solutions. A numerical technique consists of an upwind finite difference operator, and an appropriate piecewise uniform mesh is constructed. The proposed method is proved to be convergent, and the parameter-uniform error bounds are derived for the numerical solution of the problem. The obtained numerical results support the theoretical findings.

In [43], M. Brdar and H. Zarin consider a two-parameter singularly perturbed differential equation and constructed a numerical method for solving such a problem. The authors demonstrate uniform convergence of a piecewise linear Galerkin FEM on a Bakhvalov-type mesh.

H.-G. Roos [263], derived error estimate for finite linear elements on Bakhvalov type meshes. Optimal error estimates for finite linear elements on Shishkin-type meshes are for convection-diffusion problems with exponential layers. The author provides the first energy norm optimal convergence result for a Bakhvalov-type mesh.

M. Chandru, T. Prabha, and V. Shanthi [56] consider singularly perturbed two parameters problems with non-smooth data and developed a numerical scheme to numerically approximate such types of problems. The authors have obtained the theoretical bounds on the derivatives. On a Shishkin mesh, authors have designed a hybrid difference scheme.

J.L. Gracia and E. O'Riordan [99] proposed an efficient numerical technique that consists of a finite difference operator on a piecewise-uniform Shishkin mesh to solve a singularly perturbed parabolic partial differential equation.

S. Kumar and B.V.R. Kumar [179] developed and analyzed a reliable numerical method for the numerical approximation of a parabolic singularly perturbed differential equation. The authors have designed an algorithm combining the Domain Decomposition Method (DDM) based on the Schwarz alternating and three-step Taylor Galerkin Finite Element (3TGFE) method. A convergence analysis has been carried out.

In [209], Martin Stynes and Niall Madden develop and analyze numerical techniques to approximate the multi-scale character of singularly perturbed reaction-diffusion type BVPs. The proposed technique is based on the weighted and balanced finite element method on a piecewise uniform grid. The authors have proved a parameter uniform convergence and found that the proposed technique is almost first-order convergent.

S. Franz [85] consider a singularly perturbed convection-diffusion problem and constructed discontinuous Galerkin FEM for such type of problems. In this approach, the author has addressed the lack of stability with ordinary Galerkin FEM.

In [158], the authors have developed a modified graded mesh to capture the boundary layer phenomena of the singularly perturbed reaction-diffusion BVPs. The authors have designed a finite element method and a modified graded mesh, and the convergence is studied in the energy norm.

In [287], the authors have implemented a Bernstein operator method for approximating Volterra integral differential equations of singularly perturbed type. The author has carried out error and convergence analysis. Numerical results show the effectiveness and robustness of the proposed method.

1.3.3 Numerical Methods of Delay Differential Equations

The fact that many phenomena frequently modeled by ODEs can be better modeled by DDEs has not escaped the attention of the numerical analysis community. The main difference between DDEs and ODEs is that the evolution of DDE involves preliminary information on the state variable. The singularly perturbed delay differential equation (SPDDE) is usually the first estimate of the physical model being considered. The solution of the DDEs needs information on not only the current state but also the state at a particular time previously. In such situations, however, a more practical framework would add some of the system's past and future states; therefore, differential equations with lag or progress should be based on a real system. Due to their presence in a broad category of application fields, there has been tremendous growth in the numerical study of **(DDEs)**. Many research publications, technical reports, and textbooks so far have documented a substantial amount of work on DDEs. The most part of the work on DDEs has focused on Initial value problems **(IVPs)**. The main numerical approaches for solving IVPs for ordinary first-order DDEs fall into two classes:

i) single-step methods which use one starting value at each step of the solution.

ii) multistep methods, which are based on several values of the solution. Runge-Kutta methods among single-step methods and the Predictor-corrector method among the multistep methods are of main interest.

While certain DDEs might have solutions that suffer from temporary discontinuities, other DDEs may have solutions that are smooth, and these DDEs may benefit from the employment of customized linear formulae approaches. Contrarily, one-step methods, particularly Runge-Kutta methods, are particularly well suited to DDEs, for which there is no derivative smoothing and so may require repetitive step size modification.

To the best of our knowledge the first mathematician who introduced a delay in a biological model was Hutchinson [114]. To take into consideration hatching and maturation periods, Hutchinson revised Verhulst's traditional model. He brought up the possibility that a discrete time delay in the resource or crowding term could account for the oscillation that has been observed in some types of biological phenomena.

Then, Driver in 1962 published a research article [70] on the *"Existence and stability of solutions of a delay differential system"*. The fundamental theorem and an overview of the Lyapunov method for a generalized difference-differential system are presented by the author in this article. In 1962, Bellman et al. initiated the study of the DDEs and implemented of numerical technique. In [35], the authors have taken into account systems with variable time delays. The authors then converted DDEs into ODEs in order to implement a numerical technique. Subsequent research will focus on situations with multiple lags and lags caused by the solution itself. In [36], the authors considered the first order DDEs $\mu'(t) = \mu(t-1-k\sin(\omega t))+\sin(t)$ and pointed out some interesting properties of the solution via numerical study.

Feldstein and Goodman [82] studied the discretization of propagation of error for discontinuous ODE and DDEs. In [95], the same authors considered a retarded ordinary differential equation (RODE), and for the linear case, the authors estimated the cumulated round-off error, which is a linear combination of the preceding local round-off errors. For a nonlinear retarded ODE, they were obtained by similar estimates.

In [228], Neves investigated the automatic integration of functional differential equations. The author described a method for converting automatic DDEs solvers from ODE solvers. The method kept the essential elements of the first ode solver, like error estimates and step altering.

In order to solve DDEs of the form "$\mu'(t) = \mathscr{F}[(t,\mu(t),\mu(t-\tau))];(\tau>0)$", Bleyer [37] introduced a numerical technique based on spline. He derived a general theorem and the proposed method's convergence.

A finite difference technique for determining parameters in linear DDEs was proposed by Burns and Hirsch in [46]. For a straightforward explicit approach (Euler's), they obtained convergence results and convergence rates.

In [6], Allen and McKee conducted research on fixed-step discretization techniques for DDEs with variable delay. The authors were able to prove global order convergence, which allows the case for the existence of discontinuities. This makes use of a revised formulation of the prerequisite for Dahlquist stability.

Oberle and Pesch [232] deal with the numerical treatment of DDEs using Hermite interpolation. In this research article, the authors constructed a class of numerical techniques based on the RK-Fehlberg methods for treating DDEs. The authors use a multipoint Hermite interpolation to tackle the retarded argument.

In the paper [30], The authors have considered first- and second-order linear delay differential operators in periodic function spaces. Some conditions, in order to ensure that these operators are of 'monotonic type,' that is, isotonic if $Lu \leq Lv$ implies $u \leq v$ and antitonic if $Lu \leq Lv$ implies $u \geq v$, are given. The cases of a variable delay $\tau = \tau(t)$ and that of a constant delay τ are considered. For constant delays, optimal results are obtained.

In 1982, a technical report published by Arndt [11] on *"the influence of interpolation on global error for retarded differential equations"*. In order to obtain a numerical solution of an IVPs for retarded DDEs, generally, one replaces it with an IVP for an ODE along with an appropriate interpolation scheme. In this report, the actual global error was estimated by the author in terms of controllable quantities. In addition, the author demonstrated how the concept of local error, which comes from the theory of ODEs, needs to be generalized for the DDE of the retarded type.

Flower published a research paper [84] on *"Asymptotic analysis of the delayed logistic equation when the delay is large"*. This paper is devoted to a construction of an asymptotic approximation to the delay equation "$\mu'(t) = \rho\mu(1-\mu_1)$," analogous to the asymptotic limit $\rho \to \infty$.

H. T. Banks continued the work on parameter identification problems for delay systems, and in [23], he published a research paper on *"Estimation of delay and other parameters in nonlinear functional differential equations"*. They covered a spline-based approximation method for nonlinear non-autonomous DDEs in this work. They analyzed convergence outcomes in the context of parameter estimation concerns, which included estimating various delays, beginning data, and the typical coefficient-type parameters utilizing estimates of the underlying nonlinear operators of the dissipative type.

Bellen and Zennaro [31], consider a collocation method for the numerical solution of a BVPs of the form

$$u''(t) = f(t, u(t), u(g(t)), u'(t), u'(h(t))),$$

with appropriate boundary conditions. The solutions of such BVPs usually have jump discontinuities in the second derivative even for smooth functions. The given problem is reformulated as a nonlinear integral equation with a compact operator. Then Vainikko's results on projection methods with uniformly bounded projections give a convergence estimate. This result is used for carrying out the approximation by global polynomials and by piecewise polynomials of increasing degree on a fixed subdivision of the given interval ("p-convergence"). The zeros of orthogonal polynomials are used as collocation points. In this case the Erdos-Turan theorem gives the bound for the projections. The convergence rate of this method depends critically on the smoothness of the solution on the interval or its subdivision, respectively. By using h and g sometimes subdivisions can be constructed on which the solution is smooth. Two numerical examples confirm the theory and demonstrate how much can be gained by a suitable subdivision.

In [28], Bellen considered an IVP for the DDEs

$$\begin{aligned} \mu'(t) &= f(t,\mu(t),\mu(t-\alpha(t))), \quad t_0 \leq t \leq x, \quad 0 \leq \alpha(t) \leq r, \\ \mu(t_0) &= \mu_0 \qquad \mu(t) = \phi(t), \quad t_0 - r \leq t \leq t_0, \end{aligned}$$

where f, ϕ and μ are m-vector valued functions and α is a piecewise continuous scalar function. In this paper, author implemented numerical technique that consist of one-step collocation Legendre orthogonal method along with continuous piecewise polynomial functions. The obtained results on convergence and super convergence are shown for appropriate choices of the mesh $\triangle$ and for smooth f.

Feldstein and Neves [83] presented numerical methods to solving state-dependent DDEs of the form

$$\begin{aligned} &\mu'(t) = f(t,\mu(t),\mu(\alpha(t,\mu(t)))), \quad t \in [a,b], \\ &\alpha(t,\mu(t)) \leq t, \\ &\mu(t) = \pi(t), \quad t \in [\overline{a},a], \end{aligned}$$

where $\overline{a} = \min_{t\in[a,b]} \alpha(t,\mu(t))$. There are jump discontinuities in the solution of such types of problems at the initial jump point $t = a$. Thus, in order to find the accurate location of jump discontinuities in lower-order derivatives of the solution $\mu(t)$, a high-order numerical method is implemented. Now one can question whether these unknown jump discontinuities are determined accurately enough to develop high-order methods? The authors explored the special properties of DDEs to reply affirmatively to this question.

In [122], a stability analysis is made of the θ-methods for the solution of DDEs. Such types of equations are relevant to nonlinear problems which arise in a wide variety of practical applications. The approach used in this paper is to investigate whether the asymptotic behavior of solutions of DDEs is inherited by the numerical solution when the θ-method is applied.

In [29], the author considered a BVPs for second-order delay differential systems:

$$\begin{aligned} &\text{“}\mu''(t) = f(t,\mu(t),\mu'(t),\mu(t-\tau(t)),\mu'(t-\sigma(t))) && t_0 \le t \le b \\ &\mu(t) = \phi(t) && t \le t_0, \\ &\mu'(t) = \phi'(t) && t < t_0, \\ &\mu(b) = \mu_b\text{”}, \end{aligned}$$

where $\mu : \Re \to \Re^m$, $f : [t_0,b] \times \Re^{4m} \to \Re$ and $\tau(t),\sigma(t) > 0$. In this article, the author have implemented collocation method in piecewise polynomial spaces by shooting or in the global approach.

Bellen continued his work on the numerical solution of DDEs, and Bellen, with Zennaro, submitted a technical report [32] on *“Numerical solution of delay differential equations”* [33]. In this article, the authors constructed numerical techniques based on a modified version in a predictor-corrector mode of the single-step collocation method at n Gaussian points to approximate the solutions of DDEs.

In continuation of the numerical treatment of IVPs for the retarded DDEs, in 1985, Arndt et al. published a research article [12] on the numerical integration of retarded differential equations. The prime objective of the author’s work is to demonstrate that such linear multistep method’s minimax versions, which were initially developed for an ODE with a periodic solution, are equally appropriate for the integration of retarded DDEs with periodic solutions.

In this article [115], Hwang and Chen deal with parameter identification and problems of analysis of time delay systems using rectangular functions such as block-pulse and Walsh functions. The authors have solved the proposed problems by the continuously shifted Legendre polynomials. The DDEs have been converted into an algebraic form using the operational matrices of integration.

Watanabe and Roth [298] described a geometric technique for the analysis of the DDEs

$\mu' = p\mu(t) + q\mu(t-\tau)$, where the delay $\tau \geq 0$ and the quantities q and p are complex constants.It is demonstrated that there is an equivalent formula for DDEs with similar stability qualities for each A- or $A(\alpha)$-stable linear multistep ODE. However, this result does not extend to implicit Runge-Kutta formulae, and a particular example of the midpoint rule is discussed.

In 1986, Houwen with Sommeijer and Baker published a research paper [292] that deals with parabolic equations with delay, and the authors have proved stability analysis of predictor-corrector. In diffusion problems, when the current state depends on the past history, it gives rise to parabolic equations with lag. The authors have developed an efficient numerical solution of classical parabolic equations via a predictor-corrector type technique with extended real stability intervals. The authors have analyzed the test problem $\mu'(t) = q_1\mu(t) + q_2\mu(t-\omega)$, where, in view of the class of parabolic delay equations which the authors want to consider, their primary interest is in the case $|q_1| \gg |q_2|$.

A single-step subregion method for DDEs was devised in [293]. The proposed technique approximates the solution throughout the entire interval with a piecewise polynomial of fixed degree n. It is shown that for an appropriate choice of the mesh points, the method has uniform convergence of order h^{n+1} and superconvergence of order h^{2n} at nodes.

S. P. Banks [25] published a research article entitled *"Existence of periodic solutions in n-dimensional retarded functional differential equations"*. Here the author proved the existence of periodic orbits of n-dimensional delay differential systems of the form $\mu'(t) = -f(\mu(t-p))$. The result is applied to systems of the form $\mu'(t) = -\mu(t-1)N(\mu(t))$ and to a certain type of Hamiltonian system.

In [198], IVPs for DDEs are studied. It is assumed that such equations are solved numerically by two different methods. If there is no delay, these methods are the one-leg θ-method and the linear θ-method, $0 \leq \theta \leq 1$. The same names are also used when there are delay terms in the equations. The stability properties of these two numerical methods are studied concerning the test problem $\mu'(t) = \lambda\mu(t) + \alpha\mu(t-\tau)$, $\tau \geq 0$, with λ and α complex constants. It is proved that the stability regions of the two numerical methods are different except in the two extreme cases $\theta = 0$ and $\theta = 1$, when the methods have the same stability region. A significant result is that for all $\theta \in (0,1)$, the linear θ-method has a larger stability region than the corresponding one-leg θ-method. Numerical results indicate that for delay equations with constant coefficients, the linear θ-method tends to perform better than the one-leg θ-method.

Will$\acute{e}$ and Baker submitted two technical reports in 1988 [300] and 1990 [2] on the handling of derivative discontinuities in systems of DDEs and presented an overview of DDEs solver *DELSOL*, which were letter published in the form of two consecutive research papers [2, 300].

In [123], Jankowski deals with the numerical approximation of DDEs with parameters and constructs a one-step technique for numerically approximating the proposed problems. The error estimate was derived, as well as the convergence theorem.

Murphy [222] dealt with nonlinear non-autonomous DDE and constructed a parameter estimation technique to evaluate state-dependent delays and other parameters that appear in this type of problem. The linear splines are used to approximate original differential equations as well as variable delays.

In this article [279], Thompson considered ODEs with either time-dependent or state-dependent lags and implemented continuously embedded RK-Sarafyan methods to obtain the solution to such types of problems. The author described a method for finding credible solutions to these challenging problems without taking the impact of local approximation error and local integration error into separate considerations. Additionally, he provided a technique for dealing with derivative discontinuities that occur during the solution of differential equations with delays.

Hout and Spijker [119] consider a linear test problem "$\mu'(t) = \lambda\mu(t) + \alpha\mu(t-\tau)''$ where (i) $\tau > 0$ and (ii) λ and α are complex. This equation is used in the stability analysis of numerical methods for DDEs. In this paper, these results are generalized: the authors prove a theorem that gives a necessary and sufficient condition for stability that contains the results of the four papers mentioned above as special cases. It is also shown that the general results derived by the authors contain interesting cases that are not covered in those works.

Paul submitted a technical report [239] on developing a DDEs solver. The author provided a brief overview of several phenomena that should be analyzed when developing a robust code for DDEs, including the selection of an interpolant, tracking discontinuities, vanishing delays, and issues with floating point arithmetic. Paul continued his study on DDEs and, as a co-author with Baker, submitted a technical report [20] on *"Vomputin stability regions - Runge-Kutta methods for delay differential equations"* which later was published as a research paper [21]. In this article, the authors discussed the application of various fixed step size (RK) methods, along with continuous extensions to such problems, along with the practical determination of stability regions. Consider the linear DDEs,

$$\mu'(t) = \lambda\mu(t) + \alpha\mu(t-\tau) \quad t \geq 0,$$

with fixed delay τ, which is not an integer multiple of the step size. Based on the stability loci obtained in practice, the stability region for Runge-Kutta methods for DDEs may not be accurately mapped by the standard boundary-locus technique. The paper's primary goal is to present a different stability boundary algorithm that avoids the drawbacks of the conventional boundary-locus method. Both explicit and implicit Runge-Kutta methods can make use of the new algorithm.

In this article [49], Cao considered the state-dependent DDEs and generalized the discrete Lyapunov function to such problems. The discrete Lyapunov function quantifies the oscillation of solutions on intervals with lengths equal to the time delay at the zero states. The author established the relationship between the oscillation, exponential decay rate, and first-order estimation of solutions that go to zero as $t \to \infty$. He also observed that a solution decays more quickly the faster it oscillates.

Hout [117] deals with the ODEs with a lagging argument. This article analyzes the stability of the Runge-Kutta approach for DDEs. They emphasize the subclass of collocation techniques that have abscissas in $[0,1]$, and they demonstrate that each of these methods violates a crucial stability requirement pertaining to the category of test problems "$\mu'(t) = \lambda\mu(t) + \alpha\mu(t-\tau)$, where $\lambda, \alpha \in C, \Re(\lambda) < -|\alpha|$, and $\tau > 0$."

In 1990, Hout presented an article on the stability analysis of a class of RK methods for DDEs at the International Conference on Numerical Solution of Volterra and DDEs. Later in 1992 this paper [117] was published. This paper deals with the stability analysis of Runge-Kutta-type methods for DDEs. They focus on the subclass of collocation methods with abscissas in $[0,1)$ and prove that all of these methods violate an important stability condition related to the class of test problems $U'(t) = \lambda U(t) + \mu U(t-\tau)$ with $\lambda, \mu \in \mathbf{C}$, $\operatorname{Re}\lambda < -|\mu|$ and $\tau > 0$.

In this paper [229], the authors deal with systems of functional differential equations which contain delays or lags and are concerned with the solution of this class of problems. The solution of differential equations with state-dependent delays using software is discussed. The methods used in the software which are described provide a natural means of solving such problems. The package uses the continuously embedded Runge-Kutta methods of Sarafyan. These methods are based on C^1 piecewise polynomial interpolants, which are used to handle tasks associated with root finding and interpolation. In addition to providing a means to handle user-defined root-finding requirements, they offer a way to identify derivative discontinuities automatically when they appear during the solution of differential equations with delays.

In [173], the authors deal with a nonlinear system of DDEs. The authors of this article examine the asymptotic behavior of theoretical and numerical approximations of nonlinear DDE systems. When the assumption for the right-hand function is the same as that in a paper, in this article [284], the authors derived the theoretical solutions of the nonlinear systems of DDEs and proved obtained solution asymptotically stable. The analogous behavior of the numerical solutions produced by θ-methods is also shown.

In [110], Higham broadened the scope of the analysis to include a specific class of ODEs with low-order derivative discontinuities and DDEs with constant delays. The author demonstrated that standard error control techniques would be successful if delays are calculated with sufficiently precise interpolants to ensure asymptotic proportionality and discontinuities are crossed with sufficiently small steps.

In [283], the authors construct the numerical methods for pure DDEs and derive the stability properties. The proposed method employs an interpolant and quadrature rule to approximate the retarded part (continuous quadrature rule). The authors consider the test equation as $\mu'(t) = -\sum_{r=1}^{R} b_r(t)\mu(t-r\tau)\ (t>0)$, $\mu(t) = \phi(t)\ (t \leq 0)$ and provide sufficient conditions on the boundedness of the solution. The continuous quadrature rule maintains the same behavior when the parameters are restricted.

In 1994, Paul submitted a twelve-page technical report [240] on the performance and properties of continuous explicit R-K (CERK) approach for ODEs and DDEs.

Karoui and Vaililancourt [151] dealt with state-dependent DDEs and presented a computer solution to solve such problems. In order to develop the numerical technique for approximating the state-dependent DDEs with nonvanishing lag, The authors have implemented an adaptive R-K-Verner $(5,6)$ method. A fifth-degree divided difference Newton interpolation was employed to locate the derivative jump discontinuities of the solution. A three-point Hermite polynomial was used to approximate the solution's value at the delay.

In this article [38], the author deals with delay-differential systems arising in immune response modeling and performs the numerical study of the parameter identification problem. The equations for the models are nonlinear stiff systems of DDEs. When relevant data with significant magnitude variations, the criteria for the best-fit solution are discussed. The fitting procedures are based on a combination of imprecise, globally applicable methods of fitting the models to the data and more precise, locally convergent methods. In order to simplify an optimization problem, an algorithm for sequential parameter identification is based on subdividing the total fitting interval. The authors use short-cut methodologies to enhance several poor initial estimates for some parameters, such as modifying the model with spline functions that approximate the data over the entire observation time interval. They demonstrate an example of the real-life parameter identification problem for the antiviral immune response model in the context of the hepatitis B virus infection modeling and use a modification of the DIFSUB Code to solve the stiff DDEs.

In this article [113], the authors examine the stability analysis of a few prominent numerical approaches for systems of neutral delay-differential equations (NDDEs). The stability regions of linear multistep, explicit RK, and implicit A-stable RK methods are examined when they are implemented to asymptotically stable linear NDDEs after establishing a sufficient condition for asymptotic stability for linear NDDEs. Some comments are made regarding the results' extension in the case of multiple delays.

Karoui et al. [152] present a numerical method for solving vanishing-lag DDEs. For asymptotically vanishing lag as $t \to \infty$, once the lag is sufficiently small, the solution can be determined by solving an ordinary differential equation approximating the original delay equation.

In 1996, Hout published a research article [118] on the adaptation of Runge-Kutta methods to IVPs for systems of DDEs. Presently, three main types of interpolation procedures can be distinguished in the literature for adapting Runge-Kutta methods to DDEs, viz. Hermite interpolation concerning the grid points, interpolation procedures that use continuous extensions of the Runge-Kutta method, and the interpolation procedures that have been introduced by K J In' t Hout (1992). In this paper, the author discusses the stability of the corresponding three types of adaptations of the class of Runge-Kutta methods and presents a survey on them. In [160], Khajah and Ortiz discuss a differential-delay equation arising in number theory. They approximate Buchstab's function, , using the Tau method, which is given by the DDEs "$(\mu w(\mu))' = w(\mu-1)$ for $\mu \geq 2$ and $w(\mu) = 1/\mu$ for $1 \leq \mu \leq 2$".

In [301], the authors consider the second-order neutral differential equation with constant delay

$$\frac{d^2}{dt^2}(\mu(t) - p\mu(t-\tau)) + q(t)f(\mu(t-\sigma)) = 0, \quad t \in [0,\infty),$$

where $f(x)$, $q(t)$ are continuous functions such that $q(t) \geq 0, q(t) \in C[0,\infty)$, $\mu f(\mu) > 0$ if $\mu \neq 0$, and $0 < p < 1$, $\tau > 0$, $\sigma > 0$. In this case, $f(\mu)$ appeases the sub-linear or super-linear constraints, respectively, with the special case $f(\mu) = \mu|\mu|^{\gamma-1}$ for $0 < \gamma < 1$, and $\gamma > 1$ respectively. In this article, they discover the essential and sufficient elements for all continuously resolvable solutions of the aforementioned DDEs. If $\tau = p = \sigma = 0$ in the aforementioned DDEs, the obtained results in this article reduce to the well-established known theorems of Atkinson and Belohorec in the special case when $f(\mu) = \mu|\mu|^{\gamma-1}$, $\gamma \neq 1$. In this paper [22], the author provides detailed information, develops a theoretical framework for fundamental numerical concepts (such as the existence of a close approximation, convergence to the actual solution, and numerical stability), and then presents some research results. Jiang and Wang [125] consider a BVP for the singular second-order functional differential equation,

$$\begin{aligned}
&y'' = -f(x, y(w(x))), \quad 0 < x < 1, \\
&\alpha y(x) - \beta y'(x) = \xi(x), \quad a \leq x \leq 0, \\
&\gamma y(x) + \delta y'(x) = \eta(x), \quad 1 \leq x \leq b,
\end{aligned}$$

where $f(x,y)$ is a function defined on $(0,1) \times (0,\infty)$, which appeases specific prerequisites and may exhibit a singularity at $y = 0$, and $w(x)$ is a continuous function defined on $[0,1]$.

The authors obtain the positive solution of the BVP by applying the Schauder fixed point theorem.

In [241], Paul published a paper on designing efficient software for solving DDEs. The author of this paper describes how to employ numerical software to solve DDEs. Several methodologies for enhancing DDE solver effectiveness that has been developed over the past 25 years are also addressed in this article.

Aykut and Yildiz [15] consider a BVP for a differential equation with a variant retarded argument

$$\begin{aligned} &x''(t)+a(t)x(t-\tau(t))=x(t),\\ &x(t)=\phi(t) \quad \lambda_0\leq t\leq 0,\\ &x(T)=x_T, \end{aligned}$$

where $0\leq t\leq T$ and $a(t),\ f(t),\ \tau(t)\geq 0\ (0\leq t\leq T)$ are known continuous functions. In this paper, the authors apply two approximate methods for the solution of the above BVP.

In [120], the authors present an embedded singly diagonally implicit RK technique to approximate stiff systems of DDEs. The authors use Newton's divided difference interpolation to tackle the delay argument. Initially, the whole system is considered non-stiff and solved by a simple iteration; if stiffness is indicated, the whole system is considered stiff and solved using Newton iteration.

In [75], a pseudo-spectral estimation method is proposed for the class of time-delayed functional differential equation control systems. In this paper, the authors first formulate the problem optimal control problem as a delay-free governed by a system of partial differential equations with boundary conditions of nonlocal type. Next, a Chebyshev spectral method and the cell-averaging Chebyshev integration technique are used to discretize the delay-free optimal control problem. The optimal control problem is thereby transformed into a nonlinear programming problem that can be approximated using well-developed nonlinear programming techniques. As claimed by the authors, the proposed method avoids many of the numerical challenges typically encountered when solving common time-delayed optimal control problems because of its dynamic nature.

In [100], Guofeng discusses the consistency and stability of implicit one-block techniques for the numerical solutions of DDEs systems. The behavior of these techniques, when used to solve the linear test problem mentioned below, is the author's primary concern.

$$\mu'(t) = L\mu(t) + M\mu(t-\tau), \quad t \geq 0,$$
$$\mu(t) = g(t), \qquad -\tau \leq t \leq 0,$$

where $L,\ M \in C^{d\times d}$ are constant complex valued matrices, $\tau > 0$ is the constant delay, and $g(t)$ is a predetermined initial function, $\mu(t)$ signifies a d-dimensional vector-valued function. The author demonstrates how the asymptotic stability property of the system's analytical solutions is preserved by a A-stable implicit one-block method.

DDEs with constant delays can be solved in "Matlab" using the program "dde23," which was written by Shampine and Thompson. In [269], Shampine and Thompson go over some of its features, such as event location, iteration for brief delays, and discontinuity tracking. Convergence, error estimation, and the effects of brief delays on stability are just a few of the theoretical findings that the authors develop and which serve as the foundation for the solver.

Oliveira [67] considers the van der Pol equation

$$\frac{d^2u(t)}{dt^2} - \varepsilon\frac{du(t)}{dt} + \varepsilon u^2(t-r)\frac{du(t-r)}{dt} + u(t) = 0,$$

with the delayed time r, where r and ε are real parameters, with $\varepsilon > 0$ small and $0 \leq r < \pi/2$ and proved the stability and existence of a periodic solution with period near 2π and amplitude near $2/\sqrt{cos\ (r)}$.

In [267], the author considers a functional differential equation of the neutral type and presents a class of numerical methods to obtain the approximate solution to the problem. The methods presented in this paper are based on spline functions. The study of the existence and uniqueness of $3h$-step spline functions of degree $m = 4$ are considered.

This paper [50] is devoted to the construction of a numerical method to solve the BVPs for second-order differential equations with retarded arguments, $i.e.$ of type

$$\mu''(t) + a(t)\mu(t-\tau(t)) = f(t)$$
$$\mu(t) = \phi(t) \ \ \lambda_0 \leq t \leq 0, \quad \mu(T) = \mu_T,$$

where $0 \leq t \leq T$ and $a(t),\ f(t), \tau(t) \geq 0\ (0 \leq t \leq T)$ and $\phi(t),\ \lambda_0 \leq t \leq 0$ are known continuous functions.

In this article [73], the authors study generic oscillation and generic non-oscillation of second order impulsive DDEs. In this article, the authors obtain some essential and sufficient conditions for both phenomena based on the roots of the characteristic equation.

In this article [268], George Seifert considers functional differential equations of the neutral delay type with piecewise constant time dependence. He obtains conditions for the existence and uniqueness of almost periodic solutions for such DDEs.

In [109], Henríquez and Vásquez consider a second-order semilinear functional differentiable equation with unbounded delay and analyze the differentiability of solutions of such type of differential equations. The authors employ their research results to describe the infinitesimal generators of several firmly continuous semigroups of linear operators that appear in the theory of linear abstract retarded functional differential equations with unbounded delay on an axiomatically defined phase space.

In 2003, Jankowski [124] made a study to obtain the approximate solution of the BVPs for differential equations with delayed arguments. The sufficient conditions are established for the existence of a unique solution or extremal ones of the given problem. The author applies a monotone iterative technique [186] for the nonlinear problem.

In [304], the authors deal with a class of second-order DDEs with impulses and obtain sufficient oscillation conditions for all solutions of such differential equations. The authors first calculate the power series of the proposed system and then transform it into Padé (approximates) series form.

The article [202] investigates the existence of periodic solutions for nonautonomous equations with multiple deviating arguments

$$u''(t) + f(u(t))u'(t) + \sum_{j=1}^{n} \beta_j(t) g(u(t - \gamma_j(t))) = p(t)$$

where $f,\ g \in C(R,R)$, $p(t)$, $\beta_j(t)$, $\gamma_j(t)$ $(j = 1,2,\ldots,n)$ are continuous periodic functions with period $T > 0$. In this paper, the authors obtain new results on the existence and nonexistence of periodic solutions of the above differential equations by using the continuation theorem of coincidence degree theory and some analysis techniques.

In 2003, El-Harwary et al. [74] present spline collocation methods for solving DDEs. The authors consider the convergence and stability analysis of seventh C^3-spline collocation methods applied to DDEs.

In [121], Ismail et al. solve DDEs using the embedded Singly Diagonally Implicit RK (SDIRK) technique $(3,4)$ in $(4,5)$. The authors use Newton divided difference interpolation and the interpolation developed by In't Hout to approximate the delay term. The polynomial and the stability regions of the SDIRK method $(2,2)$ using In't Hout interpolation for the delay term are presented.

In the article [17], the authors consider the BVPs on an infinite interval for second-order DDEs:

$$\begin{aligned} &\mu''(t) - p\mu'(t) - q\mu(t) + f(t,\mu(t)) = 0, && t \in [0,\infty), \\ &\alpha\mu(t) - \beta\mu'(t) = \xi(t), && t \in [-\tau,0], \\ &\lim_{t\to\infty} \mu(t) = 0, \end{aligned}$$

and

$$\begin{aligned} &\mu''(t) - p\mu'(t) - q\mu(t) + f(t,\mu(t),\mu'(t)) = 0, && t \in [0,\infty), \\ &\alpha\mu(t) - \beta\mu'(t) = \xi(t), && t \in [-\tau,0], \\ &\lim_{t\to\infty} \mu(t) = 0, \end{aligned}$$

where $p,\ \alpha,\ \beta \geq 0,\ \alpha^2+\beta^2 > 0$, and $q > 0$. They discuss the existence of the positive solution for such BVPs.

In [149], the author extends the quasi-Monte Carlo methods for Runge-Kutta solution techniques to differential equations developed by Stengle, Lécot, Coulibaly and Koudiary for DDEs. Interpolation is used to simulate the retarded argument before the traditional quasi-Monte Carlo Runge-Kutta techniques can be employed. The authors provide general proof for the convergence of this method and its order that is independent of any particular quasi-Monte Carlo Runge-Kutta method.

In article [296], Wang and Li deal with the second-order neutral differential equation with distributed deviating arguments. By introducing parameter functions and employing integral averaging techniques, we can obtain some general oscillatory criteria of solutions for such types of DDEs.

In the paper [271], the authors present a class of second-order neutral functional differential equations. Using an extended Riccati transformation and again incorporating parameter functions and integral averaging approaches, the authors derive some novel conditions that ensure solution oscillation.

In [129], Kadalbajoo and Sharma deals a BVPs for a SPDDE containing both delay and advance arguments in reaction terms. Taylor's series is employed to handle terms with minor shifts. The authors develop a numerical technique based on FDM and a discrete invariant embedding algorithm. Furthermore, the authors constructed a parameter uniform numerical scheme [148] for SPDDEs based on a fitted operator. The solution to the proposed problem exhibits multi-scale character, i.e., the boundary layer appears on either the boundary's left or right side, depending on the sign of the convection term. The authors discussed both the cases and derived parameter uniform error estimates.

In [139], Kadalbajoo and Sharma introduce SPDDEs both with positive and negative shifts and develop a robust numerical scheme based on a fitted mesh finite difference scheme. The authors demonstrate the impact of shifts over the layer behavior of the computed solution and acquire robust error estimates. In [130], the authors continued the study of a similar type of problem and discussed the case when solutions exhibit rapid oscillations. In [140], Kadalbajoo and Sharma proposed a numerical technique for non-linear SPDDEs. The authors have implemented a quasi-linearization approach to handle non-linearity. The numerical scheme is based on a fitted mesh finite difference scheme. In [141], the authors continued the study of non-linear SPDDEs with a negative shift. They discussed two numerical approaches depending on a particular type of mesh with the fitted operator and fitted mesh method. In this article [131], authors studied the model in a more general form. They presented a robust numerical scheme based on Shishkin mesh (piecewise uniform mesh), which is finer in the boundary layer region than that of the outside.

In [148, 238], Patidar et al. dealt with numerical schemes for SPDDEs and developed a new class of fitted operator techniques to approximate the solution of such types of problems. These schemes are based on modeling rules for non-standard FDMs proposed by Mickens.

In [256], Ramos constructed and implemented an exponentially fitted operator technique to find an approximate multi-scale solution of linear SPDDE. The proposed approaches are based on an analytical piecewise solution of advection-reaction-diffusion operators with non-local approximations. For the proposed problem, it is demonstrated that these techniques provide better approximations to solutions than methods based on the analytical solution of the advection-diffusion operator.

In [137], Kadalbajoo et al. considered the BVPs for SPPs to develop and analyze numerical schemes based on upwind, midpoint upwind, and hybrid schemes. In [137], a comparative study is given, and it is shown that the hybrid scheme yields a better approximate solution over a wide range of δ and ε than that of the standard upwind and midpoint upwind scheme.

To approximate the solution of SPDDEs, Devendra and Kadalbajoo developed several numerical techniques to obtain an approximation of the solution of SPDDEs. In article [134], the authors investigated the effectiveness of B-spline collocation approaches with a fitted mesh in designing a parameter uniform numerical scheme for a linear second-order singularly perturbed convection-diffusion-reaction problem with a small delay in the convection term. In [135], a non-linear SPDDE with a negative shift is considered, and non-linearity is handled using a quasi-linearisation process. The mesh is designed to be dense inside the layer region to capture the boundary layer behavior of the solution and coarser in the outer region.

To design numerical techniques for SPPs, one has to rely on highly appropriate non-uniform meshes only if sufficient information like width, location, and presence of boundary layer is known. In 2010, Mohapatra and Natesan [219] proposed an adaptive grid method to evaluate an approximate solution of SPDDEs. The primary characteristic of the proposed technique is that it does not depend upon a priori considerable information about the exact solution and can be used to construct robust schemes for SPPs as well as SPDDEs. The primary approach for creating a reliable method is to evenly distribute the numerical solutions and equidistribute positive monitor function over the domain to establish the grid points and automatically combine them in the boundary layer region. Many factors influence the choice of monitor function, including the problem under consideration, the norm used to calculate error estimates, and the numerical simulation of the proposed problem.

A computational method for solving SPDDE with twin layers or oscillatory behavior was presented by Swamy et al. [185]. First-order DDEs asymptotically equivalent to the original problem is used in their substitute. Numerical integration and linear interpolation are used to find the discrete solution in the remaining parts of the work.

Sirisha and Reddy [53, 274, 275] developed and implemented a numerical technique to obtain approximate solutions for a class of linear second order SPDDEs. The concept underlying their numerical techniques is to replace the original SPDDE with an asymptotically comparable singularly perturbed ordinary differential equation.

In 2011, Sharma et al. initiated the numerical study of a very interesting problem SPDDEs with convection coefficient vanishes or changes sign in the domain, i.e., SPDDEs with turning points, which can result in twin boundary layer or interior layer, depending upon the values of coefficients of various terms involved in the differential-difference equation. In [252], the authors described a numerical method for SPDDEs with turning points exhibiting interior layers. Their method is based on El-Mistikawy-Werle exponential FDM with some modifications. In [253], they analyzed fitted operator finite difference schemes for SPDDEs with mixed shifts and a turning point. In [251], the same authors studied SPDDE with an isolated turning point at $x = 0$. A priori estimates have been established to prove the proposed numerical scheme's convergence. The paper [254] is concerned with SPDDEs, which arise in modeling neuronal variability.

In article [94], Amiraliyev and Erdogan considered SPIVP for linear first-order DDEs having fixed delay. The authors have constructed a numerical method for this problem based on appropriate piecewise-uniform mesh on each time subinterval. Amiraliyev and Erdogan [8] considered the following quasilinear SPDDEs:

$$\varepsilon\mu'(t) + g(t, \mu(t), \mu(t-r)) = 0, \quad t \in (0, T], \tag{1.3.1}$$

$$\mu(t) = \phi(t), \quad t \in [-r, 0], \tag{1.3.2}$$

Here $r > 0$ is a large delay. To construct a robust scheme, the authors used an adaptive grid that involved a piecewise-uniform mesh (Shishkin Mesh) over every time subinterval.

Amiraliyev and Cimen [7] studied the following singularly perturbed second order convection-diffusion type problem with large delay. For $D = D_1 \cup D_2$, $D_1 = (0, k]$, $D_2 = (k, l)$, $\bar{D} = [0, l]$, $D_0 = [-k, 0]$, consider

$$\varepsilon\mu''(x) + a(x)\mu'(x) + b(x)\mu(x-k) = f(x), \quad x \in D, \tag{1.3.3}$$

subject to the boundary conditions:

$$\mu(x) = \phi(x), \quad x \in D_0, \quad \mu(l) = B,$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, $a(x)$, $b(x)$, $f(x)$ and $\phi(x)$ are given sufficiently smooth functions satisfying certain regularity conditions to be specified and k $(l < 2k)$ is a large delay, which is independent of ε, and B is a given constant. For small values of ε, the function $\mu(x)$ has a boundary layer near $x = 0$. The proposed numerical scheme involves exponentially fitted finite difference method accomplished by the method of integral identities.

So far, the authors have considered singularly perturbed differential-difference equations with continuous coefficient and large delay. Subburayan and Ramanujam [278] suggested a numerical technique based on asymptotic initial value for the solution of convection type SPDDEs with discontinuous convection-diffusion coefficient term.

In 2013, Subburayan and Ramanujam [276] considered following reaction-diffusion problem:

$$-\varepsilon\mu''(x) + a(x)\mu(x) + b(x)\mu(x-1) = f(x), \quad x \in (0,1) \cup (1,2), \tag{1.3.4}$$

subject to the boundary conditions defined as

$$\mu(x) = \phi(x), \quad x \in [-1,0], \qquad \mu(2) = l, \tag{1.3.5}$$

where $0 < \varepsilon \ll 1$, $a(x) \geq \alpha_1 > \alpha > 0$ and $\beta_0 \leq b(x) \leq \beta < 0$. $a(x)$, $b(x)$ and $f(x)$ are given sufficiently smooth functions on $[0,2]$, $\phi(x)$ is a smooth function on $[-1,0]$ and l is a given constant independent of ε. The solution of BVP (1.3.4) exhibits boundary layers at $x = 0$, $x = 2$ and an interior layers at $x = 1$ for small values of ε. The authors approximated the solution of (1.3.4) by the second order hybrid finite difference scheme.

Nicaise and Xenophontos [230] considered a second-order singularly perturbed ordinary differential equations with large delay and developed a robust numerical method to solve such types of problems based on the hp-finite element method.

In [307], Helena Zarin proposed a discontinuous Galerkin FEM with interior penalties for SPDDEs. The author demonstrated a robust convergence in the appropriate energy norm using higher-order polynomials on Shishkin-type layer-adapted meshes. Numerical experiments support theoretical conclusions.

In article [181], S. Kumar and M. Kumar deals with the singularly perturbed DDEs and develop a numerical technique to solve such types of problems.

In 2018 Subburayan and Mahendran [277] considered convection type third order SPDDEs with discontinuous convection and source term. The authors have constructed an FDM on Shishkin mesh to approximate such types of problems. Moreover, the existence and uniqueness of the proposed problem have been derived.

1.3.4 Numerical Methods for Singularly Perturbed Parabolic Differential Equations with Time and Space Delay

The SPPDE problem with the time delay model's a more realistic biological and natural phenomenon than the conventional singularly perturbed problems with no delay do. The methodology and dynamics of the SPPDE problem with time delay are utterly different from the conventional partial differential equations without time lag. The solution of (SPPDE) problem with time delay is evaluated by $\psi_b(x,t)$ an initial value function for $t-\tau<0$ rather than by a simple initial value function $\psi_b(x,t)$ as happens in case of singularly perturbed PDEs. The main difficulty in designing the computational algorithms to solve such problems is the simultaneous presence of the singular perturbation parameter and the delay term. The present thesis is mainly divided into two sections. First, we have considered a class of problems involving differential-difference equations in which the highest order derivative is multiplied by a small parameter ε. In the second part, the thesis is concerned with nonlinear SSPs.

Due to its applications in disciplines like neurobiology, [187], optimal control theory [93], in describing the so called human pupil-light reflex [200], in the study of an optically bistable device [68],, in variety of models for physiological processes or diseases [205], there has recently been an increase in interest in the numerical study of such problems.

In [102], Hairer et al. discussed the convergence of Runge-Kutta methods for singularly perturbed ODEs by studying the ε-expansion of the solution. Tian [280] adopted their technique to study Runge-Kutta methods for singularly perturbed DDEss.

Glizer authored numerous research works on singularly perturbed DDEs. In 1998, Glizer [90] took into account a BVP for the set of functional differential equations with partial derivatives of the Riccati type linked to a singularly perturbed linear quadratic optimal control problem with state delay. In addition to being asymptotically solved, the problem is given an expression for a solution that converts it to the explicit singular perturbation form. In [91], the author presents the asymptotic solution of a BVP for linear SPDDEs. The asymptotic solution of the Hamiltonian BVP is constructed and justified assuming boundary layer stabilizability and detectability. The article [92] is concerned with the study of singularly perturbed linear systems with an infinite horizon H_∞ state-feedback control with a small state delay. Concerning this problem, the authors build an asymptotic solution of the hybrid system of Riccati-type algebraic, ODEs, and PDEs with deviating arguments. Based on this asymptotic solution, the authors are able to determine the prerequisites for the existence of a singular perturbation parameter independent solution to the original H_∞ problem. When all sufficiently small values of this parameter are taken into account, they finally arrive at a simplified controller with parameter-independent gain matrices that solves the original problem. In this article [93], the author considers a singularly perturbed system of linear ODEs with a small delay. Estimates of blocks of the fundamental matrix solution to this system uniformly valid for all sufficiently small values of the parameter of singular perturbation are obtained in the cases of time-independent and time-dependent coefficients of the system. In the first case, the author considers the system on an infinite time interval, while in the second case, it is considered on a finite time interval. Finally, the author applies these estimates to justify a uniform asymptotic solution of an IVP for this system in both cases.

In [88], consider the first order singular perturbation problems with delay of the form

$$\begin{aligned}
&\mu'(t) = f(\mu(t), \mu(t-\tau), v(t), v(t-\tau)), && t \in [0,T],\\
&\varepsilon v'(t) = g(\mu(t), \mu(t-\tau), v(t), v(t-\tau)), && 0 < t \ll 1,\\
&\mu(t) = \varphi(t), \quad v(t) = \psi(t), && t \le 0,
\end{aligned}$$

where τ and ε are constants, $\tau > 0$. φ and ψ are continuous functions. $f : R^M \times R^M \times R^N \times R^N \to R^N$ and $g : R^M \times R^M \times R^N \times R^N \to R^N$ are given mappings, which are sufficiently smooth. In this article, the authors deals with the one-parameter stiff SSPs with delay and study the error analysis of linear multi-step methods and RK methods.

In this research article [281], the author discusses the exponential stability of SPDDEs with a bounded lag. The author derives a generalized Halanay inequality and proves a sufficient condition to ensure that any solution of the SPDDEs with a bounded delay is exponentially stable uniformly for sufficiently small singular perturbation parameters.

In [282], Tian presents the asymptotic expansion for singularly perturbed DDEs. In this paper, the author extends singular perturbation theory in ODEs to DDEs with fixed delay and gives an adequate condition so that the solution of a class of SPDDEs can be asymptotically expanded.

A numerical study of BVPs for singularly perturbed second-order differential-difference equation with small shifts was initiated in 2002 [129]. The authors of this article take into account the case where such BVP solutions exhibit boundary layer behavior. A numerical technique scheme based on FDM is devised to evaluate the numerical solution of these BVPs. The presented difference scheme is analyzed for stability and convergence. The authors conduct several numerical experiments to demonstrate the impact of shifts on the behavior of the solution's boundary layer.

In 2007, Ansari, Bakr, and Shishkin [10] studied a time-dependent singularly perturbed BVP, a linear parabolic differential equation with a delay argument in the time variable. The authors developed a parameter uniform-fitted mesh FDM for this type of problem. In 2010, Kaushik et al. [156] developed a numerical technique for numerically approximating non-stationary time delay convection-dominated singularly perturbed parabolic problems. The Authors have implemented FDM with Shishkin mesh to capture the layer phenomena.

Bashier and Patidar [26] studied SPPPDDEs with delay argument in time variable during 2010-2011. The authors proposed parameter uniform numerical schemes consisting of fitted numerical methods with finite difference schemes. They proved the proposed methods to be unconditionally stable and convergent.

In 2012, Kaushik and Sharma [157] considered SPPPDDEs with time delay. The authors constructed and analyzed a parameter-uniform numerical method for this type of problem. The method extends to the case of adaptive meshes, which can be used to improve the solution.

In this article [97], Gowrisankar and Natesan proposed a parameter uniform computational technique to approximate the solution of singularly perturbed delay parabolic initial BVPs, exhibiting parabolic boundary layers. On time and spatial domain, uniform and non-uniform meshes are placed via a monitor function's equidistribution. The proposed technique is proved to be parameter uniform convergent with an optimal error bound $C(N^{-2}+M^{-1})$ in the discrete maximum norm. In this article, Das and Natesan proposed a parameter uniform technique with the hybrid scheme in space direction and backward-Euler method in time direction to numerically approximate the convection-dominated singularly perturbed delay parabolic time delay BVPs. Then, in 2017, Gowrisankar and Natesan [98] studied the convection dominated singularly perturbed delay parabolic time delay BVPs and developed parameter uniform scheme.

In [176], Kamlesh Kumar et al. used a methodology of entropy function and, by this concept, developed an adaptive mesh for numerically approximating the convection type singularly perturbed parabolic problem with the delay. In this adaptive mesh, the authors do not need prior knowledge of mesh location, unlike Bakhvalov and Shishkin.

S. Kumar and M. Kumar [180] deal with singularly perturbed BVPs, a linear parabolic differential equation with delay argument in the time variable. The authors have implemented and developed a hybrid scheme over Shishkin mesh in a spatial direction and the Euler method in a time direction. Moreover, the authors have implemented the Richardson extrapolation technique in the time direction to improve accuracy and convergence. The authors have derived a priori bound on the derivative and exact solution. The proposed method has proved to be of almost four orders in space and order two in time.

In this article [273], Joginder et al. considered a parabolic singularly perturbed reaction-diffusion type time delay problem and designed a numerical technique to approximate the proposed problem. The authors have developed a domain decomposition method for the proposed problem over the piecewisepiecewise uniform mesh. The convergence analysis is carried out, and the proposed method is found to be robust and convergent.

In this article [175], Devendra Kumar deals with the numerical analysis of singularly perturbed parabolic partial differential equation of convection type with time delay. The author has developed a cubic B-spline collocation approach on a piecewise uniform mesh. The author has conducted the convergence analysis, and theoretical bounds have been derived. In [69], Devendra Kumar and Parvin Kumari deals with the SPPPDDEs with time delay and develop a collocation technique based on an extended cubic B-spline approach to solve such types of problems. The author has shown the advantage of extended cubic B-spline over the cubic B-spline technique.

Vikas Gupta et al. [101] proposed a numerical technique to approximate multi-scale solution of singularly perturbed time-dependent differential-difference convection-dominated diffusion equations. The authors have constructed a higher-order Richardson extrapolation technique.

Pratima Rai et al. [247] presented a singularly perturbed DDEs with a turning point. This type of problem exhibits a boundary layer and an interior layer. The authors have implemented the FDM over a Shishkin mesh to approximate the proposed problem.

Komal Bansal et al. [165] deal with the SPPPDDEs with delay arising from modeling neuronal variability. The author constructed a non-standard finite difference method based on interpolation, θ -technique, and Micken's method to approximate the multi-scale solution of the problem. It is proved that the method is unconditionally stable for $0 \leq \theta \leq 1/2$.

Monika Choudhary et al. [59] proposed a defect correction technique to approximate the convection-dominated SPPPDDEs with delay. The authors examine the convergence analysis and discover that the method is convergent in the discrete maximum norm.

1.3.5 Numerical Methods for Nonlinear Singularly Perturbed Differential and Partial Equation

The nonlinear singularly perturbed partial differential equations (NSPPDEs) are essential in converting a real-life phenomenon into a mathematical model. The dynamics of NSPPDEs are utterly different from the conventional nonlinear partial differential equations. These types of problems depend on a small positive parameter, which makes the solution vary rapidly in narrow regions of the domain and change slowly in the rest of the domain. This behavior of the solution in narrow regions is called layer phenomena, and this class of problem is known as SSPs. The designing of computational algorithms for such types of problems is burdened with difficulties because the solution of the proposed problem is contaminated by a small positive parameter ε and nonlinear term simultaneously. Since only a few nonlinear systems can be solved explicitly, we rely on numerical techniques by linearizing the nonlinear problems. Due to the linearization of the nonlinear problem, the approximated solution's accuracy somehow degenerates, which leads to deceptive solutions, and we have to compromise the accuracy of the solution. In this thesis, an attempt has been made to overcome the difficulties associated with nonlinear problems. In this thesis, the authors have presented two numerical techniques to solve the NSPPDEs without linearization or discretization. Very little literature has been reported for nonlinear singularly perturbed differential equations (NSPDEs) and NSPPDEs so far.

In 1952 N. Levinson and E.A. Coddington [64] published a research paper on BVPs for a nonlinear differential equation of convection type with a small parameter. The authors studied the problem in the context of existence and uniqueness.

In 1973, D.S. Cohen [65] analyzed a nonlinear two-point BVPs of singularly perturbed type. The author has carried out the existence and uniqueness of the proposed problem.

In 1988 M.K. Kadalbajoo and Y.N. Reddy [138] proposed a numerical technique for numerically approximating a class of nonlinear singular perturbation two-point BVPs. The author designed an iterative boundary value method. The theoretical analysis is carried out.

In [81], Farrell et al. deal with a semilinear singularly perturbed two-point BVP and develop an FDM on an equidistant mesh with nodal distancing h. Through this article, the authors have shown that the proposed numerical technique with a fixed fitting factor cannot converge ε-uniformly to the solution of the proposed problem as h tends to zero in the maximum norm. They have considered a set of numerical experiments, and the numerical result validates the theoretical findings with variable fitting factors. In [79], Farrell, O'Riordan, Miller, and Shishkin consider quasilinear singularly perturbed BVPs exhibiting boundary layer phenomena. In this article, the authors construct an upwind difference operators technique along with a unique piecewise-uniform mesh known as Shishkin mesh which are fitted to these boundary layers. The convergence analysis is shown to be parameter uniform in a discrete maximum norm concerning the singular perturbation parameter.

In 2000, O'Malley, Jr. [262] published a research article on the asymptotic solution for singularly perturbed BVPs. In this paper, he considered the BVPs for certain differential equations of the form "$\varepsilon\ddot{x}(t) = g(x(t))f(\dot{x}(t))$ on $0 \leq t \leq 1$" whose solution exhibits an interior shock layer. He constructed an asymptotic solution to such BVPs.

Kadalbajoo and Patidar [136] deal with singularly-perturbed nonlinear BVP. The authors design a numerical technique based on cubic spline over Shishkin mesh. First, the original nonlinear BVP is linearized using the quasilinearization technique, and then the proposed numerical method is implemented. The authors have done the convergence analysis, and the method is found to be parameter uniform with a second order convergence rate.

VUlanović, Relja [294] constructed a numerical method for approximating the semilinear singular perturbation BVPs. The author has constructed sixth-order FDM. The theoretical estimates are derived, and the method is found to be parameter uniform.

In this paper [126], the authors extend the study of Stynes and O'Riordan on local exponentially fitted FEM for singularly perturbed two-point BVPs. In the present article, the authors consider a local exponentially fitted FEM in which exponential splines are implemented only in the layer part and away from the layer, the normal continuous piecewise linear function instead of a singularly perturbed two-point BVPs and derive an ε-uniform $h|\ln h|^{1/2}$ order error estimate in the energy norm for this scheme under the assumption that the mesh is quasi-uniform. The authors also consider the two higher-order numerical schemes for approximating the solution of singularly perturbed two-point BVP.

In this article [272], G.I. Shishkin, and L.P. Shishkina published a research article on numerical technique for a quasilinear singularly perturbed elliptic reaction-diffusion equation in a vertical strip. The authors have implemented the Richardson extrapolation technique to increase the accuracy and rate of convergence.

In [313], Zhao, et al. implement a variation of the iteration method to obtain a close-form analytic solution of singularly perturbed IVPs and a system of these. They carried out convergence analysis, and the technique proved convergent concerning the singular perturbation parameter ε. A numerical experiment is carried out, and numerical results support the theoretical finding.

In [286], the author has developed and implemented an iterative analytic technique known as the homotopy perturbation technique for numerically solving a nonlinear two-point singularly perturbed BVPs. The method seems more effective in faster convergence when the optimal convergence control parameters are computed utilizing the absolute residual error concept. The author considers three nonlinear test problems solved in this article by the proposed method.

In [153], article, the author has proposed an iterative analytic method for nonlinear singularly perturbed convection-diffusion problems. The method is based on a Lagrange multiplier's which is evaluated by Liouville-Green transformation and variational theory. The author has also proved the existence and uniqueness of the proposed problem. To analyze the efficiency of the proposed method, two linear and two nonlinear test problems have been considered, and the method is found to be robust.

In the article [172], the authors consider a quasilinear two-point BVP of convection-diffusion type and present a robust adaptive method based on an upwind FDM to approximate such types of BVPs. A straightforward novel approach based on equidistribution of the arc length of the current derived piecewise linear solution is utilized to adaptively shift the nodes of the employed mesh. The mesh has a fixed number $(N+1)$ of nodes and is initially uniform. The authors demonstrate the existence of a mesh that uniformly distributes the arc length throughout the polygonal solution curve, and the method is first-order concerning the singular perturbation parameter ε. In 2007, N. Kopteva [166] deals with a nonlinear two-point BVP singularly perturbed type over a non-uniform mesh. The author obtained a second order of convergence in discrete maximum norm using an equidistribution grid via monitor function. Numerical results are presented by the author and support the theoretical finding. In 2007, [167] N. Kopteva presented a 2D semilinear SSP of reaction-diffusion type derived estimates in the maximum norm. In 2011, [54] N. Kopteva et al. considered a 3d semilinear SSP and derived estimates in maximum norm. In 2009, [169] N. Kopteva et al. considered a semilinear two-point BVP of reaction-diffusion type with multiple solutions and analyze the proposed problem using an overlapping Schwarz method. In [168], N. Kopteva deals with singularly perturbed BVP of reaction diffusion and derives posteriori error estimates in maximum-norm on adaptive anisotropic meshes.

In this research article [194], the authors have considered a quasilinear singularly perturbed two-point BVP and constructed a nonstandard upwind first-order FDM on piecewise uniform meshes. The proposed method is considered parameter uniform in a discrete L_∞ norm.

In [154], the authors considered nonlinear singularly perturbed IVPs. The author has constructed a closed-form iterative analytic method for solving such types of problems. The proposed technique consists of a variation of the iteration approach depending on Lagrange's multiplier. The authors have considered a few problems, and numerical results suggest the theoretical results.

In [159], authors have constructed a closed-form iterative analytic method for one-dimensional nonlinear singularly perturbed BVP of reaction-diffusion type. The method is based on a Lagrange multiplier evaluated by Liouville-Green transformation and variational theory known as the Variation of iteration method (VIM). The authors do a comparative analysis, and the method is found to be robust concerning the singular perturbation parameter ε.

SC Rao and M Kumar, [259] deal with nonlinear singularly perturbed two-point BVPs with Robin boundary conditions and developed a collocation method based on B-spline for solving such type of problem. The authors have first linearized the original problem by the quasilinearization technique, then the proposed method, along with piecewise uniform mesh, is implemented over the linearized problem. The stability and convergence of the proposed technique are derived.

In this article [161], S.A. Khuri et al. implemented an adaptive variational algorithm for self ad-joint singularly perturbed BVP. The method's fundamental idea is to handle this class of problems by using a mixed piecewise domain decomposition and modifying the variational iterative technique. It is shown that the method uniformly converges to the exact solution. The authors have shown the method's convergence, effectiveness, and applicability through test problems and numerical results. In [162] S.A. Khuri et al. develop and implement a patching approach technique, a combination of two different numerical schemes, the cubic spline collocation method and the variational of iteration method (VIM), for numerically approximating singularly perturbed self-adjoint BVP. In the patching approach, the authors have divided the domain into layers and outer regions. The authors implement the VIM method in the layer region and the cubic spline collocation method in the outer region. To demonstrate the method's convergence, efficiency, and applicability, numerical results and computational comparison are done by the authors.

In [255], the authors present an artificial neural network(ANN) method based on a neuro-evolutionary technique for approximating singularly perturbed BVPs of both linear and nonlinear types. The proposed method is a combination of feed-forward artificial neural networks, sequential quadratic programming (SQP) techniques, and genetic algorithms. The authors consider six linear and nonlinear BVPs of SSP to determine how well the proposed design scheme performs, and the method is found to be robust and effective.

In [177], M. Kumar et al. present linear and nonlinear SSPs on a domain $[p,q]$ and prosed an initial value technique to solve such types of problems. The authors have implemented the initial value method directly to linear problems, and for nonlinear problems, the authors have adopted the quasi-linearization technique to linearize the nonlinear problem. Then the reduced problems are solved by the proposed methods.

In [66] article, the author has developed an adaptive mesh, and comparative analysis is carried out for singularly perturbed nonlinear BVPs. The author has done a posteriori error analysis for the proposed problem. A comparative analysis is carried out using existing numerical techniques in the literature.

Q. Zheng et al. [314] proposed a hybrid FDM over a Bakhvalov-Shishkin mesh to a quasilinear singularly perturbed BVPs. The convergence analysis is carried out, and the method is proven uniformly convergent concerning the singular perturbation parameter.

In 2019 Pankaj Mishra et al. [218] considered nonlinear singularly perturbed reaction-diffusion problems and proposed a collocation method based on an orthogonal spline on piece-wise uniform mesh (Shishkin Mesh). The authors have carried out the convergence analysis and used splines of degree ≥ 3. The authors have also proved error estimates in other norms for which analysis has not yet been established, and the results of the numerical experiments confirm the theoretical outcomes of the analysis.

In [212], Mario Amrein and Thomas P Wihler proposed an adaptive numerical known as the Newton-Galerkin method for numerically approximating semilinear parabolic SSP. The proposed technique consists of Newton's technique for linearization, FEM for adaptive discretization in spatial variables, and the backward Euler method in the temporal direction. A posteriori error analysis is derived by the authors.

In [63], Clavero et al. developed and analyzed a parameter uniform numerical method for numerically approximating a semilinear system of one-dimensional parabolic singularly perturbed reaction-diffusion IVP. The method consists of central FDM to discretize in space and the implicit Euler method in time. The proposed method is proved to be second order convergence in space and first-order in time.

In [197], the authors considered a system of nonlinear first-order SSPs and proposed a numerical method based on the backward-Euler approach on an adaptive grid. The proposed method is first-order convergent, according to the convergence analysis results.

The authors [258] have considered a system of coupled semilinear singularly perturbed reaction-diffusion BVPs interior layers and developed a numerical method for such problems. The authors have used Shishkin mesh to capture the boundary layer phenomena. It is demonstrated that the proposed method is convergent and robust with an almost second order convergence rate.

J. Quinn [249] proposed parameter-uniform numerical methods for general nonlinear singularly perturbed reaction-diffusion problems having a stable reduced solution. E. O'Riordan and J. Quinn [237] proposed parameter uniform numerical methods for linear and nonlinear singularly perturbed convection-diffusion boundary turning point problems.

Igor Boglaev in [40] has developed and implemented a discrete monotone iterative technique to approximate the nonlinear parabolic singularly perturbed of reaction-diffusion type. The author first implements upper and lower solution techniques to obtain a nonlinear difference scheme. Then to solve the nonlinear difference scheme author implemented a monotone domain decomposition method based on the Schwartz method. The method solves the linear system at each iteration step. In this article [39], Igor Boglaev deals with semilinear SSPs of elliptic and parabolic types. The author has implemented monotone iterative methods for solving such types of problems, and the proposed method is proven uniformly convergent. In article [193], Igor Boglaev has constructed a numerical method for solving nonlinear parabolic SSPs and constructed a monotone alternating direction technique (ADI) for numerically solving such types of problems. The author has generated a monotone sequence using this technique's upper and lower solution method. With the help of this sequence, a nonlinear difference scheme is developed, by which the nonlinear parabolic problem is approximated. The convergence rate of monotone sequences is found to be quadratic.

In [195], Chein-Shan Liu et al. develop and analyze a numerical method based on a modified asymptotic approach for numerically solving nonlinear singularly perturbed BVPs. The authors have done a few numerical tests, and the method is found to be robust.

In article [3], the authors present a nonlinear singularly perturbed BVPs with integral and multi-point integral boundary conditions. The author developed a multi-resolution Haar wavelet collocation method for approximating nonlinear singularly perturbed BVPs with integral and multi-point integral boundary conditions. To linearize a nonlinear problem quasilinearization technique is used. Numerical results show the effectiveness and robustness of the proposed method. A few test problems are taken into account, and numerical experiments are carried out in support of the predicted theory.

1.3.6 Finite Element Method

In the 1940s, a series of scientific papers were published by some research which founded the finite element method (FEM), which developed into the finite element analysis concept for numerically approximating the ODEs and PDEs. The FEM method is based on a problem domain's subdivision into simpler parts called finite elements. It also uses the calculus of variational methods to minimize an associated error function. The main advantage of the FEM is to handle and approximate the complex elasticity and structural analysis problems. In article [1] **A. Hrennikoff** in 1941 and in [250] in 1943 **R. Courant** first published their work but the method was not recognized at that time. Then in 1950s, the analysis of FEM was began and method was rediscovered by mathematicians.

Let $H^1(\Omega)$ be the Sobolev space, defined as

$$H^1(\Omega) = \{f : f \quad and \quad f^{'} \in L^2(\Omega)\}. \tag{1.3.6}$$

Suppose $H_0^1(\Omega)$ be the subspace of all function of $H^1(\Omega)$. To construct FEM first we are going to convert the problem into its weak formulation as:

$$\begin{cases} find \quad u_\varepsilon \in V, \quad s.t. \quad a(u_\varepsilon, v) = l(v) \quad \forall v \in V. \end{cases} \tag{1.3.7}$$

where V belongs to $H_0^1(\Omega)$ is the solution space, $a(.,.)$ be bilinear functional $V\times V$, and $l(.)$ be the linear functional on V.

After replacing V in (1.3.7) with a finite dimensional subspace $V_h \subset V$ that consists of continuous piecewise polynomial functions of the fixed degree connected with a subdivision of the computational domain, we take into account the following approximation of (1.3.7).

$$\begin{cases} find \quad u_\varepsilon \in V_h, \quad s.t. \quad a(u_\varepsilon, v_h) = l(v) \quad \forall v_h \in V_h. \\ dimV_h = N(h) \quad and \quad V_h = span\{\phi_1, \phi_2, \cdots, \phi_N(h)\} \end{cases} \tag{1.3.8}$$

Each $\phi_i, i = 1, \cdots, N(h)$ are linearly independent basis functions which have small compact support. The approximate solution u_h can be expressed in terms of basis function, ϕ_i, as:

$$u_h(x) = \sum_{i=1}^{N(h)} U_i \phi_i(x),$$

Here, $U_i, i = 1, \cdots, N(h)$, are unknowns and is to be determined. Now we rewrite equation (1.3.8) as follows:

find $\quad (U_1, \cdots, U_N(h)) \in \mathbb{R}^{N(h)}$ such that

$$\sum_{i=1}^{N(h)} a(\phi_i, \phi_j) U_i = l(\phi_j), \quad j = 1, \cdots, N(h). \tag{1.3.9}$$

Hence we obtained a linear system of equation, where $U = (U_1, \cdots, U_N(h))^T$ is to be evaluated, with the matrix of the system $A = (a(\phi_j, \phi_i))$ of the size $N(h) \times N(h)$. As these ϕ_i's have small support, i.e. $a(\phi_i, \phi_j) = 0$ for most pairs of i and j, so the obtained matrix A is a sparse.

Notations and Symbols Through out the thesis C denotes the positive constant independent of parameter ε. We have assumed $\sqrt{\varepsilon} \leq CN^{-1}$ in our analysis. Now $\|v\| = \sup_{x\in[0,1]} |v(x)|$ and $\|v\|_\omega = \max_{i=1,\cdots,N-1} |v(x_i)|$ be the continuous and discrete maximum norm respectively.

Chapter 2

Finite Element Analysis of Singularly Perturbed Parabolic Partial Differential Equation With Retarded Argument

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In this chapter[1], a parameter uniform Galerkin finite element method for solving singularly perturbed parabolic reaction diffusion problems with retarded argument is proposed. The solution of this class of problems exhibits parabolic boundary layers. The domain is discretized with a piecewise uniform mesh (Shishkin mesh) for spatial variable to capture the exponential behaviour of the solution in the boundary layer region and backward-Euler method on equidistant mesh in time direction. The method is proved to be unconditional stable and parameter uniform. The method is shown to be accurate of order $[O(N^{-1}\ln N)^2 + \Delta t]$ in maximum norm using green's function approach. The convergence of the proposed method does not depend on the singular perturbation parameter.

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[1]"Khari K. and Kumar V., Finite Element Analysis of Singularly Perturbed Parabolic Partial Differential Equation With Retarded Argument, Numer. Methods Partial Differ. Equ. **38 8 (2022) , 997-1014, https://doi.org/10.1002/num.22785."**

2.1 Introduction

To transform a real-life phenomenon into a mathematical model, we mainly try to obtain whatever is necessary, preserving the essential physical quantities and neglecting the negligible things. Such specific problems rely on a small positive factor so that the solution changes swiftly in some areas of the domain and gradually in other sections. Therefore, usually, there are thin intermediate layers in which the solution changes quickly or leaps suddenly while the solution behaves gradually and differs slowly away from these layers. These types of problems are called singularly perturbed problems (SPP). The problem is singular in that the order of the reduced problem is less than the order of the original problem. At the same time, the number of boundary conditions stays the same, which makes the problem behave abnormally in a particular domain region to satisfy the boundary conditions.

The main difference between delay differential equations and ordinary differential equations is that the evolution of delay differential equations involves prior information on the state variable. The singularly perturbed delay differential equation is usually the first estimate of the physical model being considered. The solution of the delay differential equations needs information on not only the current state but also the state at a particular time previously. In such situations, however, a more practical framework would add some of the system's past and future states; therefore, differential equations with lag or progress should be based on a real system. The last few decades witnessed the growth of interest in studying this class of problem.

The main difficulty in designing the computational algorithms to solve such types of problems is the presence of the singular perturbation parameter and the delay term simultaneously. Suppose we use the existing classical numerical methods such as finite element and finite difference on the uniform mesh to solve these types of problems. In that case, large oscillations may occur, and the solution can be contaminated due to the presence of a boundary layer in the entire interval, or one has to decrease the step size in similarity with ε, which seems to be unrealistic when the singular perturbation parameter is very small to obtain the stability result.

To the best of our knowledge, the finite element method is not developed to find the approximate solution to this class of problems. The novelty of this chapter is to design an efficient Galerkin finite element method for the SPPRD problem with retarded argument. The energy norm is weak to capture the sharp layers for SPPRD problems of type (4.2.1) as it involves an excessive power of small parameter [190]. Therefore, to capture the sharp layer and the singularities that arise in the problem's solution, the error estimates are carried out in the maximum norm using Green's function approach.

The chapter is organized as follows: The continuous model problem is defined in Section 2.2. In Section 2.3, we provide auxiliary results. In Section 2.4, we have defined the piece-wise uniform mesh (Shishkin mesh) for our continuous problem. The problem is discretized using Galerkin finite element for space component, and the backward-Euler method for the time component in Section 2.5. In Section 2.6, we carried out the stability and error analysis. In Section 2.7, two test problems are taken into account to validate our theoretical results. Finally, Section 2.8 contains a conclusion.

2.2 Statement of Problem

Consider the following class of SPPRD problems with retarded argument on a rectangular domain. Let $Q = \omega \times (0,T]$, $\omega = (0,1)$ and $\Upsilon = \Upsilon_l \cup \Upsilon_b \cup \Upsilon_r$, where $\Upsilon_b = \bar{\omega} \times [-\tau, 0]$, $\Upsilon_l = \{(0,t) : t \in [0,T]\}$ and $\Upsilon_r = \{(1,t) : t \in [0,T]\}$ are the initial boundary condition, left boundary condition and right boundary condition of the rectangular domain Q respectively.

$$\text{“}(v_\varepsilon)_t(x,t) - \varepsilon (v_\varepsilon)_{xx}(x,t) + a(x)v_\varepsilon(x,t) + b(x,t)v_\varepsilon(x,t-\tau) = f(x,t), \tag{2.2.1}$$

where $(x,t) \in Q$, subject to the initial condition and boundary conditions given as:

$$\begin{cases} v_\varepsilon(x,t) = & \psi_b(x,t), \text{ on } \quad (x,t) \in \Upsilon_b, \\ v_\varepsilon(x,t) = & \psi_l(t), \quad \text{on} \quad (x,t) \in \Upsilon_l = \{(0,t) : t \in [0,T]\}, \\ v_\varepsilon(x,t) = & \psi_r(t), \quad \text{on} \quad (x,t) \in \Upsilon_r = \{(1,t) : t \in [0,T]\}\text{”}. \end{cases} \tag{2.2.2}$$

The model problem (4.2.1) can be recast as

$$“(v_\varepsilon)_t(x,t) + L_{\varepsilon,x} v_\varepsilon(x,t) = F(x,t), \tag{2.2.3}$$

$$L_{\varepsilon,x} v_\varepsilon(x,t) = \begin{cases} -\varepsilon(v_\varepsilon)_{xx} + a(x)v_\varepsilon(x,t) & \text{for } x \in (0,1), \\ & t \in (0,\tau], \\ \\ -\varepsilon(v_\varepsilon)_{xx} + a(x)v_\varepsilon(x,t) + b(x,t)v_\varepsilon(x,t-\tau) & \text{for } x \in (0,1), \\ & t \in (\tau,1], \end{cases} \tag{2.2.4}$$

and

$$F(x,t) = \begin{cases} -b(x,t)\psi_b(x,t-\tau) + f(x,t), & \text{for } x \in (0,1), t \in (0,\tau], \\ f(x,t), & \text{for } x \in (0,1), t \in (\tau,1]”. \end{cases} \tag{2.2.5}$$

Here, $0 < \varepsilon \ll 1$ is the singular perturbation parameter and $\tau > 0$ be the delay term. The problem data $\psi_l(t)$, $\psi_r(t)$, $\psi_b(x,t)$, $f(x,t)$, $a(x)$, and $b(x,t)$ are supposed to be sufficiently smooth, bounded and independent of parameter ε.

$$a(x) \geq \alpha > 0, \qquad b(x,t) \geq \beta > 0, \qquad (x,t) \in \bar{Q}. \tag{2.2.6}$$

Where α and β are the positive constants independent of singular perturbation parameter ε. As $\varepsilon \to 0$ the solution of the problem (4.2.1)-(5.1.5) exhibits boundary layers of equal width on both Υ_l and Υ_r boundary points.

2.3 Auxiliary Result

Compatibility Conditions: To ensure the existence and uniqueness of the solution of SPPRD problem with retarded argument (4.2.1)-(5.1.5), it is presumed that given data $f(x,t)$, $a(x)$, and $b(x,t)$ is holder's continuous and the compatibility conditions at $(0,0)$, $(1,0)$, $(0,-\tau)$ and $(1,-\tau)$ corner points are fulfilled. Then compatibility condition's are defined as

$$\begin{cases} \psi_b(0,0) = \psi_l(0), \\ \psi_b(1,0) = \psi_r(0), \\ \frac{\partial \psi_l(0)}{\partial t} - \varepsilon \frac{\partial^2 \psi_b(0,0)}{\partial x^2} + a(0)\psi_b(0,0) = f(0,0) - b(0)\psi_b(0,-\tau), \\ \frac{\partial \psi_r(0)}{\partial t} - \varepsilon \frac{\partial^2 \psi_b(1,0)}{\partial x^2} + a(1)\psi_b(1,0) = f(1,0) - b(1)\psi_b(1,-\tau). \end{cases} \tag{2.3.1}$$

Note that it is presumed that $\psi_l(t)$, $\psi_r(t)$ and $\psi_b(x,t)$ are sufficiently smooth in order to satisfy the compatibility conditions i.e., $\psi(x,t) \in \mathscr{C}^{2,1}(\Upsilon_b)$, $\psi_r(t) \in \mathscr{C}^1(\Upsilon_r)$, $\psi_l(t) \in \mathscr{C}^1(\Upsilon_l)$. Under the above assumption the problem (4.2.1) has a unique solution [163].

Maximum Principle: Let $a,b \in \mathscr{C}^0(\bar{Q})$ and $\Theta(x,t) \in \mathscr{C}^2(\bar{Q}) \cap \mathscr{C}^0(\bar{Q})$ such that $\Theta \geq 0$ on Υ. Then, $\left(L_{\varepsilon,x}\Theta(x,t) + \frac{\partial \Theta(x,t)}{\partial t}\right) \geq 0 \quad \forall \quad (x,t) \in Q$ implies that $\Theta(x,t) \geq 0 \quad \forall \quad (x,t) \in Q$. The subsequent theorem provides the $\varepsilon-$uniform bound for the solution of (4.2.1)-(5.1.5) in maximum norm and the stability of the $(L_{\varepsilon,x}v_\varepsilon(x,t) + (v_\varepsilon)_t(x,t))$ should be.

Theorem 2.3.1. *Suppose v_ε is any function in differential operator's $(\frac{\partial}{\partial t} + L_{\varepsilon,x})$ domain of definition in* (4.2.1)-(5.1.5)*. Then*

$$\|v_\varepsilon\| \leq (1+\mu T)\max\{\| v_\Upsilon \|, \| ((v_\varepsilon)_t + L_{\varepsilon,x}v_\varepsilon) \|\}, \tag{2.3.2}$$

and we have parameter uniform upper bound for any solution v_ε of (4.2.1)-(5.1.5).

$$\|v_\varepsilon\| \leq (1+\mu T)\max\{\|f\|, \|\psi\|_\Upsilon\}. \tag{2.3.3}$$

Where $\mu = \max_{\bar{\omega}}\{0, 1-\gamma\} \leq 1$.

Theorem 2.3.2. *Suppose $a(x) \in \mathscr{C}^{2+\mu}(\bar{\omega})$,$b(x,t), f(x,t) \in \mathscr{C}^{(2+\mu,1+\mu/2)}(\bar{Q})$, ψ_l, $\psi_r \in \mathscr{C}^{2+\mu/2}([0,T])$, $\psi_b \in \mathscr{C}^{(4+\mu,2+\mu/2)}(\Upsilon_b), \mu \in (0,1)$ and higher order compatibility conditions* (2.3.1) *at corner points are satisfied. Then problem* (4.2.1) *has a unique solution v_ε and $v_\varepsilon \in \mathscr{C}^{(4+\mu,2+\mu/2)}(\bar{Q})$. Further, the derivative of the solution v_ε of* (4.2.1) *satisfies the following:*

$$\left\| \frac{\partial^{p+q} v_\varepsilon}{\partial x^p \partial t^q} \right\| \leq C\varepsilon^{-p/2}.$$

Proof. For proof of above theorem we recommend [163]. □

The bounds in the above theorem do not depend explicitly on the boundary layer. So we decompose v_ε into its singular and smooth component to obtain the stronger estimates on its derivatives and of its partial derivative.
We decompose $v_\varepsilon(x,t)$ the solution of (4.2.1) as:

$$v_\varepsilon = r + s.$$

Where s and r are the singular and regular component. We further decompose the smooth component as:

$$r = r_0 + \varepsilon r_1.$$

Further we define r_0 and r_1 as

$$\begin{aligned} \frac{\partial r_0}{\partial t} + a r_0 &= b r_0(x, t-\tau) + f, && (x,t) \in Q, \\ r_0(x,t) &= \psi_b(x,t), && \text{on} \quad \Upsilon, \end{aligned}$$

and

$$\begin{aligned} L_{\varepsilon,x} r_1 + \frac{\partial r_1}{\partial t} &= -b r_1(x, t-\tau) + \frac{\partial^2 r_1}{\partial x^2}, && (x,t) \in Q, \\ r_1(x,t) &= 0, && \text{on} \quad \Upsilon. \end{aligned}$$

By r_0 and r_1, we define the smooth component r as:

$$\begin{aligned} \frac{\partial r}{\partial t} + L_{\varepsilon,x} r &= -b r(x, t-\tau) + f, && (x,t) \in Q, \\ r(x,t) &= \psi_b(x,t), && \text{on} \quad \Upsilon_b, \\ r(0,t) &= r_0(0,t), && \text{on} \quad \Upsilon_l, \\ r(1,t) &= r_0(1,t), && \text{on} \quad \Upsilon_r. \end{aligned}$$

Similarly the singular component s can be defined as

$$\begin{aligned} \frac{\partial s}{\partial t} + L_{\varepsilon,x} s &= -bs(x,t-\tau), && (x,t) \in Q, \\ s(x,t) &= 0, && \text{on} \quad \Upsilon_b, \\ s(0,t) &= \psi_l(t) - r_0(0,t), && \text{on} \quad \Upsilon_l, \\ s(1,t) &= \psi_r(t) - r_0(1,t), && \text{on} \quad \Upsilon_r. \end{aligned}$$

The singular component s is further decomposes into a right and left layer component s_r and s_l respectively as:

$$s(x,t) = s_r(x,t) + s_l(x,t), \tag{2.3.4}$$

where s_l satisfies as

$$\begin{aligned} \frac{\partial s_l}{\partial t} + L_{\varepsilon,x} s_l &= -bs_l(x,t-\tau), && (x,t) \in Q, \\ s_l(x,t) &= 0, && \text{on} \quad \Upsilon_r \cup \Upsilon_b, \\ s_l(0,t) &= \psi_l(t) - r_0(0,t), && \text{on} \quad \Upsilon_l, \end{aligned}$$

and s_r as

$$\begin{aligned} \frac{\partial s_r}{\partial t} + L_{\varepsilon,x} s_r &= -bs_r(x,t-\tau), && (x,t) \in Q, \\ s_r(x,t) &= 0, && \text{on} \quad \Upsilon_l \cup \Upsilon_b, \\ s_r(0,t) &= \psi_r(t) - r_0(1,t), && \text{on} \quad \Upsilon_r. \end{aligned}$$

The subsequent theorem provides the explicit bounds for the singular component s, the smooth component r, and all its partial derivatives, that perform an essential part in the error analysis throughout Section 6.

Theorem 2.3.3. *Suppose* $a(x) \in \mathscr{C}^{4+\mu}(\bar{\omega})$, $b(x,t), f(x,t) \in \mathscr{C}^{(4+\mu,2+\mu/2)}(\bar{Q})$, ψ_l, $\psi_r \in \mathscr{C}^{3+\mu/2}([0,T])$, $\psi_b \in \mathscr{C}^{(6+\mu,3+\mu/2)}(\Upsilon_b)$, $\mu \in (0,1)$ *and compatibility conditions* (2.3.1) *of high order at corner* *points are satisfied. Then the integer p,q such that $0 \leq p+2q \leq 4$ we have the following estimate:*

$$
\begin{aligned}
\left\| \frac{\partial^{p+q} r}{\partial x^p \partial t^q} \right\|_{\bar{Q}} &\leq C(1+\varepsilon^{1-p/2}), \\
\left\| \frac{\partial^{p+q} s_l}{\partial x^p \partial t^q} \right\| &\leq C\varepsilon^{-p/2} e^{\frac{-x}{\sqrt{\varepsilon}}}, \\
\left\| \frac{\partial^{p+q} s_r}{\partial x^p \partial t^q} \right\| &\leq C(1+\varepsilon^{-p/2}) e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}.
\end{aligned}
$$

Proof. For proof of the above theorem, we recommend [163]. □

Theorem 2.3.4. *The partial derivative of $s(x,t)$ satisfies:*

$$
\left\| \frac{\partial^{p+q} s}{\partial x^p \partial t^q} \right\| \leq C\varepsilon^{-p/2} \mathscr{B}_\varepsilon(x), \qquad (x,t) \in \bar{Q},
$$

$\mathscr{B}_\varepsilon(x) = \{e^{\frac{-x}{\sqrt{\varepsilon}}} + e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}\}$ *for integer p,q such that $0 \leq p+2q \leq 4$.*

Proof. Using estimations of theorem (4.3.1) and decomposition (5.3.7) proof of the above theorem is completed. □

2.4 Mesh Discretization

The classical numerical methods, such as finite difference, finite element, etc. for singular perturbation problems are incompetent on the uniform mesh because they require a massive number of mesh points to construct satisfying computational results. Shishkin [133] proposes a non-uniform mesh, which is also known as the piece-wise mesh for construction of the parameter uniform method for singular perturbation problems. The piece-wise mesh is coarse away from the layer region and fine near the layer. The piece-wise mesh is attractive due to the simplicity and appropriate handling of a wide range variety of singular perturbation problems. One of the main drawbacks of piece-wise mesh is that one should know the prior knowledge about the location of the boundary layer. Since our problem (4.2.1) exhibits the strong boundary layer of parabolic type at $x = 0$ and $x = 1$. So we divide our interval into three subinterval $\omega_1 = [0, \sigma]$, $\omega_2 = [\sigma, 1-\sigma]$

and $\omega_3 = [1-\sigma, 1]$, such that

$$\sigma = \min\left\{\frac{1}{4}, \sqrt{\frac{\varepsilon}{\alpha+\beta}}\ln N\right\}$$

where σ is called the mesh transition point. Shishkin mesh is built by partitioning the interval ω_2 into $\frac{N}{2}$ equidistant mesh point and dividing the interval ω_1 and ω_2 into $\frac{N}{4}$ equidistant mesh points.
The step-size h_i is calculated as:

$$h_i = \begin{cases} \dfrac{4\sigma}{N} & \text{for } i = 1, \cdots, \frac{N}{4} \\ \dfrac{2(1-2\sigma)}{N} & \text{for } i = \frac{N}{4}+1, \cdots, \frac{3N}{4} \\ \dfrac{4\sigma}{N} & \text{for } i = \frac{3N}{4}+1 \cdots, N \end{cases} \tag{2.4.1}$$

Where x_i is calculated as:

$$x_i = \begin{cases} (i-1)h_i & \text{for } i = 1, \cdots, \frac{N}{4} \\ \sigma + (i - (\dfrac{N}{4}+1))h_i & \text{for } i = \frac{N}{4}+1, \cdots, \frac{3N}{4} \\ 1-\sigma + (i - (\dfrac{3N}{4}+1))h_i & \text{for } i = \frac{3N}{4}+1 \cdots, N \end{cases} \tag{2.4.2}$$

2.5 Weak Formulation

Consider the Sobolev space $H^1(\omega)$, space of function defined as

$$H^1(\omega) = \{f : f \quad and \quad f^{'} \in L^2(\omega)\}. \tag{2.5.1}$$

Let $H_0^1(\omega)$ be the subspace of all function of $H^1(\omega)$ that vanishes at boundary points $x=0$ and $x=1$. Let $\mathbb{V}(\bar{\omega}^N)$ is the finite dimensional subspace of $H_0^1(\omega)$ of standard piecewise linear polynomials on given Shishkin mesh ω^N condensed at boundary points $x=0$ and $x=1$. We shall consider $\bar{\omega}^N = \{x_0 = 0 < x_1 < \cdots, x_N = 1\}$ to be the set of mesh points x_i, for some positive integer N. We set $h_i = x_i - x_{i-1}$ to be the local step size. The linear basis function of $\bar{V}(\omega)^N$ is $\{\phi_i\}^{N-1}$, ϕ_i are given by:

$$\phi(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i} & \text{for } x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h_{i+1}} & \text{for } x \in [x_i, x_{i+1}], \\ 0 & \text{for otherwise.} \end{cases} \tag{2.5.2}$$

The weak formulation of the problem (4.2.1) can be interpreted as to determine $v_\varepsilon \in H_0^1(\omega)$, such that

$$A(v_\varepsilon, v) = F(v), \qquad \text{for all} \quad v \in H_0^1(\omega), \tag{2.5.3}$$

and condition (4.2.6) provides the uniqueness of weak formulation.

$$A(v_\varepsilon, v) = \int_\omega \{\varepsilon v_\varepsilon^{'} v^{'} + (v_\varepsilon)_t v + a(x) v_\varepsilon v + b(x,t) v_\varepsilon(x, t-\tau) v\} dx, \tag{2.5.4}$$

$$F(v) = \int_\omega f(x,t) v dx. \tag{2.5.5}$$

Now we write

$$v_\varepsilon(x,t) = \sum_{i=1}^{N} V_i(t) \phi_i(x) \quad and \quad v_\varepsilon(x, t-\tau) = \sum_{i=1}^{N} V_i(t-\tau) \phi_i(x), \tag{2.5.6}$$

$$\begin{aligned} A(v_\varepsilon, v) \;\; = \;\; & \int_\omega \sum_{i=1}^{N} \{\varepsilon V_i(t) \phi_i^{'}(x) \phi^{'}(x)_j + V_i^{'}(t) \phi_i(x) \phi_j(x) + a(x) V_i(t) \phi_i(x) \phi_j(x) \\ & + b(x,t) V_i(t-\tau) \phi_i(x) \phi_j(x)\} dx. \qquad \forall \quad j = 1, \cdots, N. \end{aligned}$$

$$F(v) = \int_\omega \sum_{j=1}^{N} f(x,t) \phi_j(x) dx.$$

In matrix notation it can be expressed as:

$$\Big[M V^{'}(t) + (\varepsilon K + a(x) M) V(t) + b(x) M V(t-\tau) = F(t) \Big]. \tag{2.5.7}$$

Where $M = [M_{ij}]_{1 \leq i,j \leq N}$ is the mass matrix with elements $m_{ij} = \int_\omega \phi_i(x) \;\; \phi_j(x) dx$, $K =$

$[K_{ij}]_{1\leq i,j\leq N}$ is the stiffness matrix with entries $k_{ij}=\int_\omega \phi_i^{'}(x)\ \phi_j^{'}(x)dx$ and $F=[F_j]_{1\leq j\leq N}$ is the vector with entries $F_j=\int_\omega f\phi_j^{'}(x)dx$, $V(t)$ is the vector of unknowns $V_i(t)$. Now to solve delay differential equation (2.5.7) we use backward Euler method.

$$K=\begin{bmatrix} (h_i+h_{i+1})/h_ih_{i+1} & -1/h_{i+1} & & & & \\ -1/h_i & (h_i+h_{i+1})/h_ih_{i+1} & -1/h_{i+1} & & 0 & \\ & & & \ddots & & \\ & 0 & & -1/h_i & (h_i+h_{i+1})/h_ih_{i+1} & -1/h_{i+1} \\ & & & & -1/h_i & (h_i+h_{i+1})/h_ih_{i+1} \end{bmatrix},$$

$$M=\begin{bmatrix} (h_i+h_{i+1})/3 & h_{i+1}/6 & & & & \\ h_i/6 & (h_i+h_{i+1})/3 & h_{i+1}/6 & & 0 & \\ & & & \ddots & & \\ & 0 & & h_i/6 & (h_i+h_{i+1})/3 & h_{i+1}/6 \\ & & & & h_i/6 & (h_i+h_{i+1})/6 \end{bmatrix}.$$

Backward Euler method: The time domain $[0,T]$ is descretized by equidistant mesh with constant step size Δt as:

$$\nabla_t^M=\{t_0=0,\Delta t,\cdots,(M-1)\Delta t, M\Delta t=T\}. \tag{2.5.8}$$

Where ∇_t^M is the partition of the time domain with no. of mesh points M in t- direction on $[0,T]$ and Δt satisfies constrains $\Delta t=\tau/k$, where k is a positive integer $t_n=n\Delta t$, $n\geq -k$. Now we define the backward difference as:

$$\delta_t V=\frac{V^n-V^{n-1}}{\Delta t},$$

and we discretize the equation (2.5.7) by backward-Euler method for time derivative as:

$$\left[\frac{M}{\Delta t}+\varepsilon K+a(x)M\right]V^n=\mathscr{G}^n, \tag{2.5.9}$$

where $\mathscr{G}^n$ is as:

$$\mathscr{G}^n \equiv \begin{cases} \frac{M}{\Delta t}V^{n-1} - b^n \psi_b^n + F^n & \text{for } n = 1, \cdots, k, \\ \\ \frac{M}{\Delta t}V^{n-1} - b^n M V^{n-k+1} + F^n & \text{for } n = k+1, \cdots, M. \end{cases} \tag{2.5.10}$$

2.6 Convergence Analysis

Let us consider the steady-state version of the abstract problem (4.2.1)-(5.1.5):

$$\begin{aligned} L_{\varepsilon,x} v_\varepsilon &= -\varepsilon (v_\varepsilon)_{xx} + a v_\varepsilon + b v_\varepsilon = f, \qquad x \in \omega, \\ & v_\varepsilon(0) = v_\varepsilon(1) = 0, \end{aligned} \tag{2.6.1}$$

with $0 < \varepsilon \ll 1$ and $a \geq \alpha$ and $b \geq \beta$ on ω and $\alpha, \beta > 0$.
The corresponding variational formulae is, find $v_\varepsilon \in \mathbb{V}(\bar{\omega}^N)$ such that

$$A(v_\varepsilon, v) = \varepsilon((v_\varepsilon)_x, v_x) + (a v_\varepsilon, v) + (b v_\varepsilon, v) = (f, v),$$

where $(.,.)$ is the standard inner-product in $L_2(\omega)$. Numerical Solution of (2.6.1) is obtain by using piece-wise linear basis. The above weak formulation is equivalent to the finite difference scheme using piece-wise linear basis:

$$\begin{aligned} [\mathscr{L}_\varepsilon V]_i &= \frac{-\varepsilon}{\hbar}\left(\frac{V_{i+1}-V_i}{h_{i+1}} - \frac{V_i - V_{i-1}}{h_i}\right) + \frac{h_i}{\hbar}\frac{(a_{i-1}+b_{i-1})}{6}V_{i-1} + \\ & \frac{2(a_i+b_i)}{3}V_i + \frac{h_{i+1}}{\hbar}\frac{(a_{i+1}+b_{i+1})}{6}V_{i+1} = \frac{h_i}{\hbar}\frac{f_{i-1}}{6} + \frac{2}{3}f_i \\ & + \frac{h_{i+1}}{\hbar}\frac{f_{i+1}}{6}, \\ V_0 &= V_N = 0, \quad \forall \quad i = 1, \cdots, N-1. \end{aligned} \tag{2.6.2}$$

Where $\hbar = \frac{h_i + h_{i+1}}{2}$.
For stability of (2.6.1), Let us assume $a(x), b(x) \in \mathscr{C}^{0,a}[0,1]$ and for arbitrary $\rho \in (0,1)$. Then

$$\|u_\varepsilon\|_\omega \leq \frac{3}{(1-\rho)(\alpha+\beta)} \|L_{\varepsilon,x} u_\varepsilon\|_\omega$$

Proof: Clearly the matrix corresponding to the weak formulation are $M-$ Matrix. So by $M-$criterion $L_{\varepsilon,x}u$ is stable [192, p. 37].

2.6.1 Green's Function

Let $G \in H_0^1(\omega)$ be the Green's function corresponding to the weak formulation $A(.,.)$ with mesh points x_i. It satisfies

$$A(u,G) = u(\varsigma) \quad \forall \quad u \in H_0^1(\omega) \tag{2.6.3}$$

where $G \in \mathscr{C}^2((0,\varsigma)\cup(\varsigma,1))\cap\mathscr{C}[0,1]$ such that

$$\mathscr{L}_\varepsilon G = 0 \quad \text{in} \quad (0,\varsigma)\cup(\varsigma,1) \tag{2.6.4}$$

Equivalently, using standard basis function in $\mathbb{V}(\bar{\omega}^N)$ and $\mathscr{L}_\varepsilon$ from equation (2.6.2). G can be written as

$$\begin{aligned} [\mathscr{L}_\varepsilon G]_i^j &= w_i^j, \\ G_0 = G_N &= 0 \quad \forall \quad i = 1,\cdots,N-1. \end{aligned}$$

Where w_i is the Dirac-delta function defined as:

$$w_i = \begin{cases} \hbar^{-1}, & i = j, \\ 0 & otherwise. \end{cases}$$

Since $L_{\varepsilon,x}u$ is inverse monotone then $G \geq 0$ and G has the following bound in L_1 norm.

Theorem 2.6.1. *Let G be the Green function corresponding with the discrete operator $L_{\varepsilon,x}$ then we have the following bound.*

$$\|(a+b)G_i\|_{1,\omega} \leq 1, \tag{2.6.5}$$

$$\|G_{\zeta,i,.}\|_{1,\omega} \leq \frac{2(1+\varepsilon(\alpha+\beta)^{-1})^2}{\varepsilon(\alpha+\beta)}, \tag{2.6.6}$$

$$\|G_{\zeta\zeta,i}\|_{1,\omega} \leq 2\varepsilon^{-2}, \quad \forall \quad i = 1,\cdots,N-1. \tag{2.6.7}$$

Proof: For proof of the above theorem we refer [41, p. 193]

2.6.2 Interpolation Error

Theorem 2.6.2. *Let* $v_\varepsilon^I(x,t_n)$ *be the* $\mathbb{V}(\bar{\omega}^N)$ *interpolant of the finite element approximation of the solution* $v_\varepsilon(x,t)$ *of* (2.6.1) *on mesh* ω_N*. Then the maximum-norm error satisfies:*

$$\|v_\varepsilon^I - v_\varepsilon\|_\omega \leq CN^{-2}(\ln N)^2 \tag{2.6.8}$$

where C is a constant independent of ε.

Proof. The estimate is obtain separately on each sub interval $\omega_i = (x_{i-1} - x_i)$. Let us assume any function z on ω_i:

$$z^I = z_{i-1}\phi_{i-1} + z_i\phi_i,$$

so it is evident that, on ω_i

$$z^I(x) \leq \max_{\omega_i} z(x)[\phi_{i-1}(x) + \phi_i(x)],$$

taking maximum on both sides

$$\mid z^I(x) \mid \leq \max_{\omega_i} \mid z(x) \mid, \tag{2.6.9}$$

and by applying Taylor's expansion, it is straightforward to see that

$$|z^I(x) - z(x)| \leq Ch_i^2 \max |z''(x)|. \tag{2.6.10}$$

Now from Theorem (2.3.2) and (2.6.10) on ω_i

$$\begin{aligned} |v_\varepsilon{}^I(x) - v_\varepsilon(x)| &\leq Ch_i^2 \max |v_\varepsilon''(x)|, \\ &\leq C\frac{h_i^2}{\varepsilon}. \end{aligned} \tag{2.6.11}$$

Also, from (2.6.9), (2.6.10), Theorem (2.3.4) on $\bar{\omega}_i$, and from [217, p. 80].

$$\begin{aligned}
|v_\varepsilon{}^I(x,t_n) - v_\varepsilon(x,t_n)| &= |r^I(x,t_n) + s_l^I(x,t_n) + s_r^I(x,t_n) - r(x,t_n) \\
&\quad - s_l(x,t_n) - s_r(x,t_n)|, \\
|v_\varepsilon{}^I(x,t_n) - v_\varepsilon(x,t_n)| &\leq |r^I(x,t_n) - r(x,t_n)| + |s_l^I(x,t_n) - s_l(x,t_n)| \\
&\quad + |s_r^I(x,t_n) - s_r(x,t_n)|, \\
|v_\varepsilon{}^I(x,t_n) - v_\varepsilon(x,t_n)| &\leq Ch_i^2 \max_{\omega_i} |r''(x,t_n)| + 2\max_{\omega_i} |s_l(x,t_n)| \\
&\quad + 2\max_{\omega_i} |s_r(x,t_n)|, \\
|v_\varepsilon{}^I(x,t_n) - v_\varepsilon(x,t_n)| &\leq C\left(h_i^2 + e^{\frac{-x_i\sqrt{(\alpha+\beta)}}{\sqrt{\varepsilon}}} + e^{\frac{-\sqrt{(\alpha+\beta)}(1-x_i)}{\sqrt{\varepsilon}}}\right).
\end{aligned}$$

Case-1 When $\sigma \geq \frac{1}{4}$ and $\frac{1}{4} \leq \sqrt{\frac{\varepsilon}{\alpha+\beta}}\ln N$ implies that the mesh is uniform, hence it is clear that $x_i - x_{i-1} = N^{-1}$ and $\varepsilon^{\frac{-1}{2}} \leq C\ln N$ and hence from equation (2.6.11) and [133, p. 48].

$$\|v_\varepsilon^I - v_\varepsilon\|_\omega \leq CN^{-2}(\ln N)^2. \tag{2.6.12}$$

Case-2 When $\sigma < \frac{1}{4}$ implies $\sigma = \sqrt{\frac{\varepsilon}{\alpha+\beta}}\ln N$, then the mesh is piece-wise uniform with mesh spacing $h_i = \frac{2(1-2\sigma)}{N}$ in $[\sigma, 1-\sigma]$ and $h_i = \frac{4\sigma}{N}$ in each sub-interval $[0,\sigma]$ and$[1-\sigma, 1]$.
Now in interval $[0,\sigma]$ and$[1-\sigma,1]$ we have $h_i = 4\sqrt{\frac{\varepsilon}{\alpha+\beta}}\frac{\ln N}{N}$ hence from (2.6.11). We have

$$\|v_\varepsilon^I - v_\varepsilon\|_\omega \leq CN^{-2}(\ln N)^2. \tag{2.6.13}$$

Now in interval $[\sigma, 1-\sigma]$ we have $h_i = \frac{2(1-2\sigma)}{N}$. □

Theorem 2.6.3. *Let $v_\varepsilon(x,t_n)$ be the solution of stationary problem* (2.6.1) *and let $V(x,t)$ be the numerical approximation of* (2.6.1) *then,*

$$\|v_\varepsilon^I - V\|_\omega \leq CN^{-2}(\ln N)^2. \tag{2.6.14}$$

Proof. Let

$$\|v_\varepsilon^I - V\|_\omega = \|v_\varepsilon - v_\varepsilon^I + v_\varepsilon^I - V\|,$$

by triangle inequality

$$\|v_\varepsilon^I - V\|_\omega \leq \|v_\varepsilon - v_\varepsilon^I\| + \|v_\varepsilon^I - V\|, \tag{2.6.15}$$

where $\|v_\varepsilon - v_\varepsilon^I\|_\omega$ is the interpolation error which is bounded in theorem (3.3.1) as

$$\|v_\varepsilon^I - v_\varepsilon\|_\omega \leq CN^{-2}(\ln N)^2, \tag{2.6.16}$$

now to prove the above theorem, it remains to bound the term $e = |v_\varepsilon^I(x,t_n) - V(x,t_n)|$ and let $E = f - (a+b)u$, then

$$e_i = (E^I - E, G) - ((a+b)e)^I, G) + \frac{2}{3}\int_0^1 ((a+b)eG)^I(x)dx. \tag{2.6.17}$$

We use quadrature to evaluate the integral $(a+b)v_\varepsilon$ and f. So $(a+b)v_\varepsilon$ and f are replaced by their interpolants E^I and $E = (a+b)v_\varepsilon - f = \varepsilon(v_\varepsilon)_{xx}$. Hence $(E^I - E, G)$ is again the interpolation error [154] and can be bounded as

$$|(E^I - E, G)| \leq CN^{-2}(\ln N)^2, \tag{2.6.18}$$

and from equation (2.6.2)

$$\begin{aligned} \frac{2}{3}\int_0^1 (((a+b)eG)^I - ((a+b)e)^I, G)(x)dx &= \frac{-1}{3}\sum \frac{h_i}{2}((a_k+b_k)e_kG_k \\ &\quad +(a_{k-1}+b_{k-1})e_{k-1}G_{k-1}). \end{aligned}$$

Suppose ρ be the arbitrary small and from equation (4.6.20), (4.6.21) and theorem (4.6.1).

$$\begin{aligned} \frac{2}{3}\int_0^1 ((a+b)eG)^I - ((a+b)e)^I, G)(x)dx &\leq \left(\frac{1}{2} + \frac{\mathscr{M}h}{2(\alpha^2+\beta^2)}\right)\|e\|_\omega, \\ &\leq \frac{(1+\rho)}{2}\|e\|_\omega. \end{aligned} \tag{2.6.19}$$

Hence from equation (4.6.18), (4.6.20) and (4.6.22)

$$|e_i| \leq CN^{-2}(\ln N)^2 + \frac{(1+\kappa)}{2}\|e\|. \tag{2.6.20}$$

Now we take maximum over $i = 1, \cdots, N-1$ in equation (4.6.23) to get the general error bound.

Using theorem (3.3.1) and from equation (4.6.17), (4.6.20), (4.6.23).

$$\|v_\varepsilon - V\| \le CN^{-2}(\ln N)^2. \tag{2.6.21}$$

□

Theorem 2.6.4. *Let $v_\varepsilon(x,t_n)$ is exact solution of the continuous problem* (4.2.1)-(5.1.5) *and $V(x,t_n)$ is the finite element approximation from the space $\mathbb{V}(\bar{\omega}^N)$ of the exact solution $v_\varepsilon(x,t)$ and compatibility conditions* (2.3.1) *are satisfied at the corner points then the corresponding error is:*

$$\|v_\varepsilon - V\| \le [CN^{-2}(\ln N)^2 + \Delta t]. \tag{2.6.22}$$

Proof. Let the error term is denoted by $\eta = v_\varepsilon - V$, then the truncation error is denoted by:

$$[\delta_t \eta + \mathscr{L}_\varepsilon \eta]_i^n = \chi_{1,i}^n + \chi_{2,i}^n, \tag{2.6.23}$$

where $[\chi_{1,i}^n] = [L_{\varepsilon,x} v_\varepsilon]_i^n - [\mathscr{L}_\varepsilon V]_i^n$ is the truncation error in space and $\chi_{2,i}^n = [\delta_t v_\varepsilon]_i^n - \partial_t v_{\varepsilon i}^n$ is the truncation error in time discretisation.

We prove the above theorem step by step.

First we are going to provide error estimates on interval $[0,\tau]$ i.e in this interval time discretisation parameter n varies from 0 to k. So in interval $[0,\tau]$ our discretisation scheme is as:

$$\delta_t V_i^n + L_{\varepsilon,x} V_i^n = -b_i^n \psi_b(x_i, t_n - k) + f_i^n, \quad i = 1,\cdots,N-1, \quad n = 1,\cdots,k. \tag{2.6.24}$$

As a consequence, the truncation error in equation (2.6.24) can be written as follows, as shown in [191] as:

$$\delta_t \eta_i^n + \mathscr{L}_\varepsilon \eta_i^n = \chi_{1,i}^n + \chi_{2,i}^n, \quad for \quad (x_i,t_n) \in \bar{Q}, \tag{2.6.25}$$

where $\chi_{1,i}^n$ and $\chi_{2,i}^n$ are same as in equation (5.3.14). With the truncation error partition, the error η can decomposed as $\eta = \phi + \psi$, where the function ϕ_i^n is the solution of the corresponding stationary problem (2.6.1) for each fixed $n = 1,\cdots,k$ as:

$$\begin{aligned} L_{\varepsilon,x}\phi_i^n &= \chi_{1,i}^n, \qquad \text{for} \quad i = 1,\cdots,N-1, \\ \phi_0^n &= \phi_N^n = 0, \end{aligned} \tag{2.6.26}$$

and

$$\begin{aligned}[\delta_t\psi + L_{\varepsilon,x}\psi]_i^n &= [\delta_t(\eta-\phi) + L_{\varepsilon,x}(\eta-\phi)]_i^n,\\ &= \delta_t\eta_i^n - \delta_t\phi_i^n + L_{\varepsilon,x}\eta_i^n - L_{\varepsilon,x}\phi_i^n,\\ &= [\delta_t\eta_i^n + L_{\varepsilon,x}\eta_i^n] - [\delta_t\phi_i^n - L_{\varepsilon,x}\phi_i^n],\\ &= \chi_{1,i}^n + \chi_{2,i}^n - \delta_t\phi_i^n - \chi_{1,i}^n,\\ &= \chi_{2,i}^n - \delta_t\phi_i^n.\end{aligned}$$

Now we have

$$\begin{aligned}[\delta_t\psi + L_{\varepsilon,x}\psi]_i^n &= \chi_{2,i}^n - \delta_t\phi_i^n; \quad i = 1,\cdots,N-1,\\ \psi_0^n &= \psi_N^n = 0, \quad n = 1,\cdots,k\\ \psi_i^0 &= -\psi_{bi}^0.\end{aligned} \tag{2.6.27}$$

Now to bound equation (2.6.27), we use theorem (3.3.1), as ϕ_i^n is sequence of the stationary problem (2.6.1) corresponding to the problem (4.2.1), we bound as

$$\|\phi^n\|_\omega \le C(N^{-1}\ln N)^2, \quad for \quad all \quad i,n \le k, \tag{2.6.28}$$

with the assumption that $N^{-1} \gg \sqrt{\varepsilon}$ and our problem exhibits regular boundary layers of the parabolic type, the other error component ψ is bounded by using the theroem (2.3.1) and *Lin* 2007 [155] as

$$\|\psi^n\|_\omega \le \|\phi^0\| + C\max(\|\delta_t u - u_t\| + \|\delta_t\phi\|), \tag{2.6.29}$$

using Taylor's expansion and theorem (3.3.1)

$$\|\psi^n\|_\omega \le C(N^{-1}\ln N)^2 + C(\Delta t + \max\|\delta_t\phi\|).$$

Now it remains to bound the term $\delta_t\phi$

$$[L\delta_t\phi]_i^n = \delta_t\chi_{1,i}^n - \delta_t a^n\phi^{n-1}, \tag{2.6.30}$$

now a is a function of x only. So

$$\begin{aligned} [L_{\varepsilon,x}\delta_t\phi]_i^n &= \delta_t\chi_{1,i}^n, \quad i=1,\cdots,N-1 \\ (\delta_t\phi)_0^n &= (\delta_t\phi)_N^n = 0. \end{aligned}$$

Since ϕ is a sequence of stationary problem. So by theorem (3.3.1)

$$\|\delta_t\phi^n\| \leq C(N^{-1}\ln N)^2 \quad \forall \quad n=1,\cdots,k. \tag{2.6.31}$$

Now from inequalities (2.6.28), (2.6.29) and (2.6.31). We have the error bound for time-delay singularly perturbed problem as:

$$\|v_\varepsilon - V\| \leq [CN^{-2}(\ln N)^2 + \Delta t]. \tag{2.6.32}$$

now for interval $[\tau,T]$, we follow the similar argument as in interval $[0,\tau]$ and have the following error estimates:

$$\|v_\varepsilon - V\| \leq [CN^{-2}(\ln N)^2 + \Delta t]. \tag{2.6.33}$$

□

2.7 Numerical Experiment and Discussion

In this section, two test problems are taken into account, and rigorous comparative analysis is done. The test problems are solved by using the Galerkin finite element method over Shishkin mesh for the spatial variable and backward-Euler method on equidistant mesh for time variable. Let $\eta_\varepsilon^{N,M}$, and $O(N)$ denote the maximum point-wise error and rate of convergence (ROC), respectively.

$$\eta_\varepsilon^{N,M} = \max_{(x_i,t_n)\in Q_\varepsilon^{(N,M)}} |v_\varepsilon(x_i,t_n) - V(x_i,t_n)|, \qquad O(N) = \log_2\left(\frac{\eta_\varepsilon^{N,M}}{\eta_\varepsilon^{2N,4M}}\right),$$

where $V(x_i,t_n)$ and $v_\varepsilon(x_i,t_n)$ denotes the numerical and exact solution of SPPRD problem with retarded argument (4.2.1) respectively with M mesh points in temporal direction with equidistant time-step Δt and N mesh points in spatial direction. The method proposed in this chapter is of order two in space, but only of order one in time. So to accommodate the two error difference, the number of mesh intervals in space direction N is selected with respect to the number of M time step so that $N = 4\sqrt{M}$ with rounded to the nearest integer divisible by 4 to assure adequate construction on Shishkin mesh.

Example 2.7.1. *Consider the following example of SPPRD problem with retarded argument [163].*

$$\begin{cases} \text{“}(v_\varepsilon)_t(x,t) - \varepsilon(v_\varepsilon)_{xx}(x,t) = 2e^{-1}v_\varepsilon(x,t-1), & (x,t) \in \omega \times (0,2], \\ v_\varepsilon(x,t) = e^{\left(-t-\frac{x}{\sqrt{\varepsilon}}\right)}, & (x,t) \in [0,1] \times [-1,0], \\ v_\varepsilon(0,t) = e^{(-t)}, \quad v_\varepsilon(1,t) = e^{\left(-t-\frac{1}{\sqrt{\varepsilon}}\right)}, & t \in [0,2]\text{”} \end{cases} \tag{2.7.1}$$

Since the exact solution of the above test problem (5.5.1) is known to us which is $v_\varepsilon(x,t) = e^{\left(-t-\frac{x}{\sqrt{\varepsilon}}\right)}$. From the exact solution, we can see that there is only one boundary layer at the left side of the domain Γ_l, which is of parabolic type.

As example, (5.5.1) has a non-homogeneous boundary condition, so to apply the finite element method, first the boundary conditions are transformed from non-homogeneous to homogeneous boundary conditions then, we proceed for approximation.

Let us assume the solution of example (5.5.1) $v_\varepsilon(x,t)$ as:

$$\begin{aligned} v_\varepsilon(x,t) &= S(x,t) + U^*(x,t), \\ S(x,t) &= A(t)(1-x) + B(t)x, \end{aligned}$$

Now S has to satisfy the boundary condition and we calculate $A(t)$ and $B(t)$ as:

$$A(t) = e^{(-t)} \qquad B(t) = e^{\left(-t-\frac{1}{\sqrt{\varepsilon}}\right)},$$

Now

$$\begin{aligned} v_\varepsilon(x,t) &= e^{(-t)}(1-x) + e^{\left(-t-\frac{1}{\sqrt{\varepsilon}}\right)}x + U^*(x,t), \\ (v_\varepsilon)_t(x,t) &= -e^{(-t)}(1-x) - e^{\left(-t-\frac{1}{\sqrt{\varepsilon}}\right)}x + U_t^*(x,t), \\ (v_\varepsilon)_{xx}(x,t) &= U_{xx}^*(x,t). \end{aligned}$$

Putting values of $v_\varepsilon(x,t)$, $(v_\varepsilon)_t(x,t)$ and $(v_\varepsilon)_{xx}(x,t)$ in equation (4.8.1). We obtain transformed homogeneous problem as:

$$\begin{cases} U_t^*(x,t) - \varepsilon U_{xx}^*(x,t) = 2e^{-1}U^*(x,t-1) - (1-x)e^{(-t)} - xe^{\left(-t-\frac{1}{\sqrt{\varepsilon}}\right)}, \\ U^*(x,t) = -e^{(-t)}(1-x) - xe^{\left(-t-\frac{1}{\sqrt{\varepsilon}}\right)} + e^{\left(-t-\frac{x}{\sqrt{\varepsilon}}\right)}, (x,t) \in \bar{\omega} \times [-1,0], \\ U^*(0,t) = 0, \quad U^*(1,t) = 0, \qquad\qquad t \in [0,2] \end{cases} \tag{2.7.2}$$

The maximum point-wise error $\eta_\varepsilon^{N,M}$ and the ROC $O(N)$ have been calculated by the proposed scheme for example (5.5.1) and are given in table (2.1) and (4.3) respectively. As we analyse the numerical results given in table (2.1) and (4.3) for example (5.5.1). It is observed that the proposed scheme is parameter-uniform convergent.

Number of N mesh points/Number of M mesh points.			
ε	$32/10$	$64/40$	$128/160$
10^0	$5.424051e-04$	$1.492265e-04$	$3.855375e-05$
10^{-2}	$3.822828e-02$	$1.010200e-02$	$2.562070e-03$
10^{-4}	$5.905033e-02$	$1.610509e-02$	$4.334634e-03$
10^{-6}	$6.624503e-02$	$2.007671e-02$	$6.067123e-03$
10^{-8}	$6.696112e-02$	$2.078518e-02$	$6.626397e-03$

Table 2.1: "Maximum point-wise error $\eta_\varepsilon^{N,M}$ obtained using finite element method for Example (5.5.1)."

Number of N mesh points/Number of M mesh points.			
ε	$32/10$	$64/40$	$128/160$
10^0	1.8618	1.9525	1.9864
10^{-2}	1.9199	1.9792	1.9947
10^{-4}	1.8744	1.8935	1.5030
10^{-6}	1.7223	1.7264	1.3845
10^{-8}	1.6877	1.6492	1.5110

Table 2.2: "ROC for solution of Example (5.5.1)"

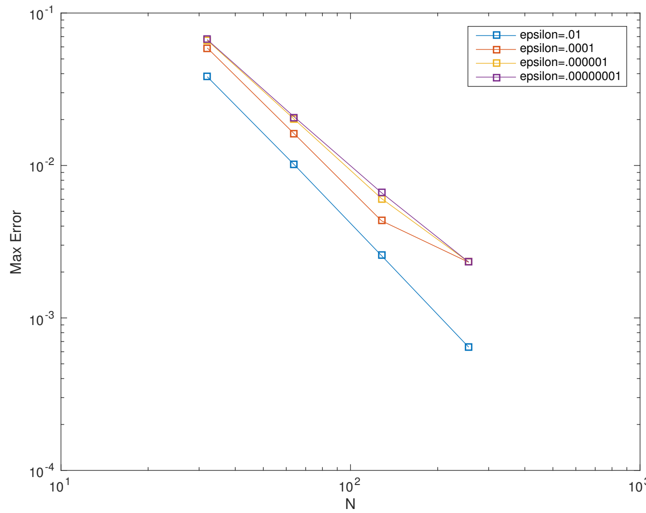


Figure 2.1: "Log-log plot for maximum point-wise error of the solution of Example (5.5.1)."

Example 2.7.2. *Consider the following example of SPPRD problem with retarded argument.*

$$\begin{cases} \text{``}(v_\varepsilon)_t(x,t) - \varepsilon(v_\varepsilon)_{xx}(x,t) + x^2 v_\varepsilon(x,t) + v_\varepsilon(x,t-1) = t^3, \; (x,t) \in \omega \times (0,2] \\ v_\varepsilon(x,t) = 0, \qquad (x,t) \in [0,1] \times [-1,0] \\ v_\varepsilon(0,t) = v_\varepsilon(1,t) = 0, \qquad t \in [0,2]\text{''}. \end{cases} \tag{2.7.3}$$

The exact solution of above test problem (5.5.2) is not available. In order to determine the accuracy and parameter uniform convergence of the proposed numerical scheme, the double mesh principle is used by analysing the numerical solution $V^{N,M}$ to the numerical solution $V^{2N,4M}$, which is calculated on the mesh that is two times fine in space direction and four-time as finer in time direction. The maximum point-wise error $\eta_\varepsilon^{N,M}$ and the ROC $O(N)$ for example (5.5.2) are given in table (2.3) and table (2.4) respectively.

Number of N mesh points/Number of M mesh points.			
ε	$32/10$	$64/40$	$128/160$
10^0	$5.2417e-03$	$1.3268e-03$	$3.4345e-04$
10^{-2}	$1.3926e-01$	$3.6468e-02$	$9.2237e-03$
10^{-4}	$1.4064e-01$	$3.6689e-02$	$9.2574e-03$
10^{-6}	$1.3981e-01$	$3.66118e-02$	$9.2554e-03$
10^{-8}	$1.3864e-01$	$3.6545e-02$	$9.2428e-03$

Table 2.3: "Maximum point-wise error $\eta_\varepsilon^{N,M}$ obtained using finite element method for Example (5.5.2)."

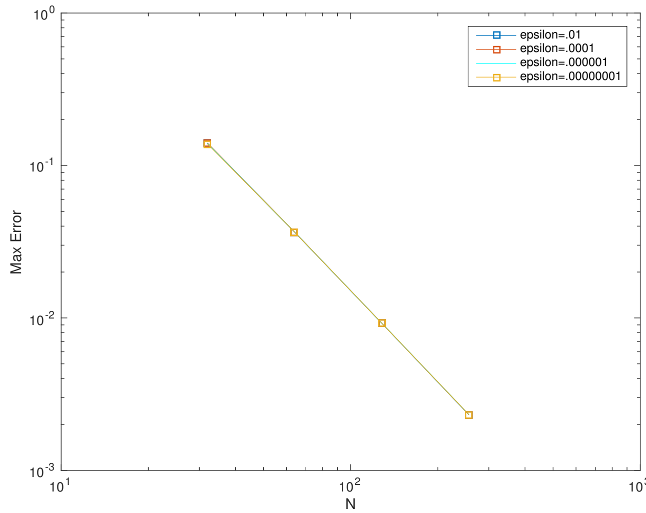


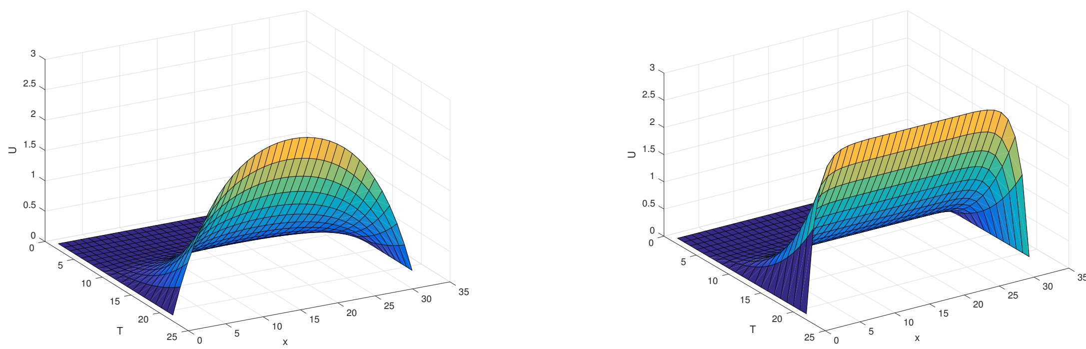
Figure 2.2: “Log-log plot for maximum point-wise error of the solution of Example (5.5.2)”.

Number of N mesh points/Number of M mesh points.			
ε	$32/10$	$64/40$	$128/160$
10^0	1.9820	1.9497	1.9966
10^{-2}	1.9330	1.9832	1.9957
10^{-4}	1.9385	1.9866	1.9977
10^{-6}	1.9330	1.9839	1.9938
10^{-8}	1.9235	1.9832	1.9919

Table 2.4: “ROC of the solution of Example (5.5.2).“

2.8 Conclusion

In this chapter, the Galerkin finite element method has been successfully applied to the SPPRD problems with retarded argument. Parameter uniform convergence is derived in maximum norm using Green’s function approach. Two linear test problems have been solved by using propose method. The proposed method is shown to be accurate of order $(O(N^{-1}\ln N)^2 + \Delta t)$ in maximum norm.



(a) For $\varepsilon = 10^{-1}$

(b) For $\varepsilon = 10^{-10}$

Figure 2.3: "Numerical solution using finite element method of Example (5.5.2) for $\Delta t = 0.1$ and $N = 32$".

Chapter 3

An Efficient Numerical Method for Solving Singularly Perturbed Nonlinear Reaction Diffusion Problems

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In this chapter[1], a numerical method based on Bernstein polynomial for nonlinear singularly perturbed reaction-diffusion problems is proposed. The solution of this type of problem is polluted by a small positive parameter ε along with non-linearity due to which the solution often shows boundary layers, interior layers, and shock waves that arise due to non-linearity. The existence and uniqueness of the solution of the said problems are proved using Nagumo's condition. Moreover, the convergence analysis is carried out of the proposed problem in maximum norm. To illustrate the proposed method's efficiency, two nonlinear test problems have been taken into account, and a comparative analysis has been done with other existing methods. The proposed method's approximated solution seems to be superior or in good agreement with the existing method..

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[1] " Khari, K., Kumar, V. An efficient numerical technique for solving nonlinear singularly perturbed reaction diffusion problem. J Math Chem 60, 1356-1382 (2022). https://doi.org/10.1007/s10910-022-01365-4".

3.1 Introduction

Whenever a real-life phenomenon is converted into a mathematical model, differential equation, partial differential equation and system of differential equation plays a vital role in modelling natural evolution, we primarily try to obtain what is important, retaining the essential physical quantities and neglecting the negligible ones which involve small positive parameters. Due to their occurrence in a wide range of applications, the study of nonlinear singularly perturbed reaction-diffusion (SPNRD) problems has always been the topic of considerable interest for many mathematicians and engineers. These problems seem to be of significance to the environmental sciences in analyzing pollution from manufacturing sources that is entering the atmosphere. These type of problem occurs in chemical kinetics in catalytic reaction theory. The SPNRD problem models an isothermal reaction which is catalyzed in a pellet and modelled by equation (3.3.1) [295]. Where the concentration of reactant is denoted by v and $\frac{1}{\sqrt{\varepsilon}}$ is called the Thiele module defined by $\frac{K}{D}$, K is the reaction rate and D is the diffusion coefficient. In considering these types of problems, it is essential to acknowledge that the diffusion coefficient of the admixture in the material may be sufficiently small, resulting in substantial variations of concentration along with the material depth. Then, the diffusion boundary layers rise. Hence these type of problems exhibit a singularly perturbed character. The mathematical model of such problems has a perturbation parameter, which is a small coefficient multiplying the differential equation's highest derivatives. Such specific problems rely on a small positive factor so that the solution changes swiftly in some areas of the domain and changes gradually in other sections of the domain. The mathematical model for an adiabatic tubular chemical reactor which processes an irreversible exothermic chemical reaction is also represented by SPNRD problems. The concentrations of the various chemical species involved in the reaction can be determined in a simple manner from a knowledge of v.

We rely on the numerical schemes to get the approximate solution of nonlinear systems by linearizing the nonlinear problems as only few nonlinear systems can be solved explicitly. On a uniform mesh, the existing numerical technique, such as finite difference, finite element, spline collocation, etc., gives unsatisfactory results or one has to modify the local mesh that works fine near the layer region and standard away from the layer region by designing a suitable layer adaptive mesh.

The novelty of this chapter is to drive an analytic iterative approximation to nonlinear singularly perturbed reaction diffusion problems using a Bernstein collocation method based on Bernstein polynomial and operational matrix. Bernstein polynomials perform a vital role in numerous mathematics areas, e.g., in approximation theory and computer-aided geometry design [112]. The Bernstein polynomial method's main advantage over the other existing approach is its simplicity of implementation for nonlinear problems. The key feature of this approach is that it reduces such problem to one of solving the system of algebraic equation via operational matrices.

Due to the flexibility and ability, the Bernstein collocation method (BCM) has emerged as a powerful tool to solve linear and nonlinear systems.

The main advantages of this method are i) it provides the approximate solution over the entire domain while other existing numerical method provide the approximate solution on the discrete point of the domain, ii) to solve the nonlinear problem, one often use a quasi-linearization technique to linearize the problem and then solve the linearize problem by numerical or other existing techniques. Due to linearization of nonlinear problem the accuracy of nonlinear problem somehow degenerate, which may lead to deceptive solution some time. In this method, we solve the nonlinear problem without linearization. iii) It is easy to implement.

The chapter is organised as, in Section 3.2 brief sketch of Bernstein collocation method and auxiliary results are presented. Then in Section 3.3 the existence and uniqueness of the said problem is carried out. In Section 3.4 the error analysis is done. In Section 3.5, two nonlinear test problems are taken into account to validate the theoretical finding of the proposed method and a comparative analysis is carried out with the other existing methods. Section 3.6, contains the conclusion.

3.2 Brief Sketch of the Method

In this section we give some brief sketch and auxiliary results corresponding to our proposed method.

3.2.1 Properties of Bernstein polynomial

Generalized form of Bernstein polynomial of m^{th} order on interval $[0,1]$ is defined as

$$\mathbf{B}_{i,m}(x) = \binom{m}{i} x^i (x-1)^{m-i}, \quad 1 \le i \le m. \tag{3.2.1}$$

Using Binomial expansion of $(x-1)^{m-i}$, Bernstein polynomial of m^{th} order reads as:

$$\mathbf{B}_{i,m}(x) = \binom{m}{i} x^i \left(\sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} x^k \right), \tag{3.2.2}$$

$$= \sum_{k=0}^{m-i} (-1)^k \left(\binom{m}{i} \binom{m-i}{k} x^{k+i} \right), \quad i = 0, 1, \cdots, m. \tag{3.2.3}$$

$\mathbf{B}_{i,m}(x)$ has the following properties:

1. $\mathbf{B}_{i,m}(x)$ is continuous over interval $[0,1]$,
2. $\mathbf{B}_{i,m}(x) \ge 0 \ \forall \ x \in [0,1]$,
3. Sum of Bernstein polynomial is 1 (unity) i.e

$$\sum_{i=0}^{m} \mathbf{B}_{i,m}(x) = 1 \quad x \in [0,1]. \tag{3.2.4}$$

4. Bernstein polynomial $\mathbf{B}_{i,m}(x)$ can be written in form of recursive relation as

$$\mathbf{B}_{i,m}(x) = (1-x)\mathbf{B}_{i,m-1}(x) + x\mathbf{B}_{i-1,m-1}(x). \tag{3.2.5}$$

Let $\varphi(x) = [\mathbf{B}_{0,m}(x), \mathbf{B}_{1,m}(x), \cdots, \mathbf{B}_{m,m}(x)]^T$, then we can write $\varphi(x)$ as:

$$\varphi(x) = Q \times T_m(x). \tag{3.2.6}$$

Where vector Q_{i+1} is defined as follows:

$$Q_{i+1} = \left\{ \overbrace{0, 0, \cdots, 0}^{i \ times}, (-1)^0 \binom{m}{i}, (-1)^1 \binom{m}{i} \binom{m-i}{1}, \cdots (-1)^{m-i} \binom{m}{i} \binom{m-i}{m-i} \right\}, \tag{3.2.7}$$

$T_m(x)$ as

$$T_m(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^m \end{bmatrix}, \tag{3.2.8}$$

and Q is an $(m+1)\times(m+1)$ matrix and written as follows:

$$Q_m(x) = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{m+1} \end{bmatrix}, \tag{3.2.9}$$

and

$$\varphi(x) = \begin{bmatrix} \mathbf{B}_{0,m}(x) \\ \mathbf{B}_{1,m}(x) \\ \vdots \\ \mathbf{B}_{m,m}(x) \end{bmatrix}. \tag{3.2.10}$$

From equation (5.5.1), it is concluded that matrix Q is an invertible matrix as it is an upper triangular matrix with non zero diagonal entries and determinant $|Q| = \prod_{i=0}^{i=m} \binom{m}{i}$.

3.2.2 Operational Matrix for Differentiation

In this subsection a Bernstein operational matrix associated with differentiation is derived. From Eq. (3.2.6) we have

$$\varphi(x) = Q \times T_m(x), \tag{3.2.11}$$

and differentiation of $\varphi(x)$ is calculated as:

$$\frac{d\varphi(x)}{dx} = Q_m \begin{bmatrix} 0 \\ 1 \\ 2x \\ \vdots \\ mx^{m-1} \end{bmatrix}, \tag{3.2.12}$$

the above expression can be written as:

$$\frac{d\varphi(x)}{dx} = Q_m \Lambda' X'(x). \tag{3.2.13}$$

Where Λ' and X' are written as:

$$\Lambda' = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m \end{bmatrix}, \tag{3.2.14}$$

and

$$X'(x) = \begin{bmatrix} 0 \\ 1 \\ x \\ \vdots \\ x^{m-1} \end{bmatrix}. \tag{3.2.15}$$

Now vector $X'(x)$ can be expressed in form of Bernstein polynomial basis as $\mathbf{B}_{i,m}$ as $X'(x) = \Delta^* \varphi(x)$, where

$$\Delta^* = \begin{bmatrix} Q_1^{-1} \\ Q_2^{-1} \\ Q_3^{-1} \\ \vdots \\ Q_m^{-1} \end{bmatrix}, \tag{3.2.16}$$

hence,

$$\frac{d\varphi(x)}{dx} = Q_m \Lambda^{'} \Delta^{*} \varphi(x). \tag{3.2.17}$$

Where $\mathscr{O} = Q_m \Lambda^{'} \Delta^{*}$ is called the operational matrix of the derivatives. Let us assume that $v(x)$ is approximated as:

$$v(x) \simeq V^T \varphi(x), \tag{3.2.18}$$

Then the differentiation of $v(x)$ in term of operational matrix is defined as:

$$v^{(n)}(x) \simeq V^T \varphi^{(n)}(x) = V^T \mathscr{O}^n \varphi(x). \tag{3.2.19}$$

3.2.3 Operational Matrix of Product

The main concern of this subsection is to explicitly evaluate the product of operational matrix corresponding to Bernstein polynomial of m^{th} degree operational matrix. Let c be a column vector of $(m+1) \times 1$ and Let $\check{C}$ be a $(m+1) \times (m+1)$ product of operational matrices.

$$c^T \varphi(x) \varphi(x)^T \simeq \varphi(x)^T \check{C}, \tag{3.2.20}$$

where $\varphi(x)$ is defined in (3.2.6) and we have $c^T \varphi(x) = \sum_{i=0}^{i=m} c_i \mathbf{B}_{i,m}$, we rewrite equation (3.2.20) in form of Bernstein basis as

$$\begin{aligned} c^T \varphi(x) \varphi(x)^T &= c^T \varphi(x) T_m^T(x) Q^T, & (3.2.21) \\ &= [c^T \varphi(x), x c^T \varphi(x), x^2 c^T \varphi(x), \cdots, x^m c^T \varphi(x)] Q^T, & (3.2.22) \\ &= \sum_{i=0}^{i=m} [c_i \mathbf{B}_{i,m}, c_i x \mathbf{B}_{i,m}, c_i x^2 \mathbf{B}_{i,m}, \cdots, c_i x^m \mathbf{B}_{i,m}] Q^T. & (3.2.23) \end{aligned}$$

Now we evaluate all $x^k \mathbf{B}_{i,m}$ in term of $\{\mathbf{B}_{i,m}\}$ for all $k, i = 0, 1, \cdots, m$. Let

$$\mathbf{e}_{k,i} = [\mathbf{e}_{k,i}^0, \mathbf{e}_{k,i}^1, \cdots, \mathbf{e}_{k,i}^m]^T. \tag{3.2.24}$$

Let D be a $(m+1) \times (m+1)$ dual matrix of $\varphi(x)$ such that

$$D = \int_0^1 \varphi(x) \varphi(x)^T dx. \tag{3.2.25}$$

Now for $i,k=0,1,\cdots,m$ we have

$$\mathbf{e}_{k,i}^T\varphi(x)\simeq x^k\mathbf{B}_{i,m}. \tag{3.2.26}$$

Now we define

$$\mathbf{e}_{k,i}=D^{-1}\left[\int_0^1 x^k\mathbf{B}_{i,m}\mathbf{B}_{0,m}(x),\int_0^1 x^k\mathbf{B}_{i,m}\mathbf{B}_{1,m}(x),\int_0^1 x^k\mathbf{B}_{i,m}\mathbf{B}_{2,m}(x),\cdots,\int_0^1 x^k\mathbf{B}_{i,m}\mathbf{B}_{m,m}(x)\right]^T, \tag{3.2.27}$$

$$\mathbf{e}_{k,i}=D^{-1}\left(\frac{\binom{m}{i}}{2m+k+1}\right)\left[\frac{\binom{m}{0}}{\binom{2m+k}{i+k}},\frac{\binom{m}{1}}{\binom{2m+k}{i+k+1}},\cdots,\frac{\binom{m}{m}}{\binom{2m+k}{i+k+m}}\right]^T,\quad \text{for}\quad i,k=0,1,\cdots,m. \tag{3.2.28}$$

Let $\check{C}_{m+1}$ be a $(m+1)\times(m+1)$ matrix of columns vectors $[\check{C}_1,\check{C}_2,\cdots,\check{C}_{m+1}]$ and $\check{C}_{k+1}$ is defined as

$$\check{C}_{k+1}=[\mathbf{e}_{k,0},\mathbf{e}_{k,1},\cdots,\mathbf{e}_{k,m}]c \quad \forall \quad k=0,1,\cdots,m. \tag{3.2.29}$$

Then from equation (3.2.23)

$$c^T\varphi(x)\varphi(x)^T=\begin{bmatrix}\sum_{i=0}^{i=m}c_i\mathbf{B}_{i,m}\\ \sum_{i=0}^{i=m}c_ix\mathbf{B}_{i,m}\\ \sum_{i=0}^{i=m}c_ix^2\mathbf{B}_{i,m}\\ \vdots\\ \sum_{i=0}^{i=m}c_ix^m\mathbf{B}_{i,m}\end{bmatrix}Q^T. \tag{3.2.30}$$

$$\begin{aligned} c^T\varphi(x)\varphi(x)^T &\simeq \varphi(x)^T[\check{C}_1,\check{C}_2,\cdots,\check{C}_{m+1}]Q^T, && (3.2.31)\\ &\simeq \varphi(x)^T\check{C}Q^T. && (3.2.32)\end{aligned}$$

Hence, the operational matrix of product is defined as:

$$\check{C}=\check{C}Q^T. \tag{3.2.33}$$

3.3 Existence and Uniqueness

Consider the following class of singularly perturbed non-linear reaction diffusion problem.

$$\begin{cases} \varepsilon v''(x) = g(x, v(x)); \qquad x \in (0,1) = \omega, \\ v(0) = A, \quad v(1) = B, \end{cases} \tag{3.3.1}$$

where ε is singular perturbation parameter with $0 < \varepsilon << 1$ and $g \in C^{\infty}[0,1] \times R$. Let assume that

$$g_u(x, v) > \Im^2 > 0 \qquad \forall (x, v) \in \bar{\omega} \times R. \tag{3.3.2}$$

Let $\alpha(x)$ and $\beta(x)$ are two smooth function such that $\alpha(x) \leq \beta(x)$ and satisfies

$$\begin{cases} -\varepsilon\alpha''(x) + g(x, \alpha(x)) \leq 0 \quad \text{and} \quad -\varepsilon\beta''(x) + g(x, \beta(x)) \geq 0 \\ \alpha(0) \leq A \leq \beta(0), \quad \alpha(1) \leq B \leq \beta(1), \end{cases} \tag{3.3.3}$$

Nagumo condition holds:

$$\begin{cases} g(x, v) = O(|v|)^2, \\ \text{as} \quad |v| \to \infty \quad \forall \quad (x, v) \in (\alpha, \beta) \times [0,1]. \end{cases} \tag{3.3.4}$$

Theorem 3.3.1. *Condition* (5.1.1) *and Nagumo condition* (5.1.4) *provides the existence of solution* $v(x) \in C^2[0,1]$ *of problem* (3.3.1)*, satisfying the condition* $\alpha(x) \leq v(x) \leq \beta(x)$ *for all* $x \in [0,1]$.

Proof. Let us write the problem in operator form as:

$$\pounds v = g(x, v), \tag{3.3.5}$$

where $\pounds = \varepsilon \dfrac{d^2}{dx^2}$

$$\pounds\alpha \geq g(x, \alpha), \qquad \text{and} \qquad \pounds\beta \leq g(x, \beta) \qquad \text{on} \quad [a,b] \times R. \tag{3.3.6}$$

As g is continuous for $(x, v) \in [a,b] \times R$, which ensure the existence of $v(x)$ s.t. $\alpha(x) \leq v(x) \leq \beta(x)$

and satisfing the boundary value problem (3.3.1).

The proof of the above theorem can be done by using maximum principal [235, 248]. □

Theorem 3.3.2. *Let the function g be continuous with respect to* (x,v) *and also g belongs to the class of* C^1 *with respect to v for* (x,v) *in* $(\alpha,\beta)\times[0,1]$ *and there exist a positive constant m such that* $g_v(x,v)\geq m>0$ *for* $[0,1]\times\mathscr{R}$*. Then for each* $\varepsilon>0$*, the problem* (3.3.1) *has a unique solution* $v(x,\varepsilon)\in[0,1]$ *such that* $|v(x,\varepsilon)|\leq\frac{\mathscr{M}}{m}$*. Where* $\mathscr{M}=\max\{\max|g(x,0)|, m|B|, m|A|\}$

Proof. Suppose for $x\in[0,1]$,

$$\alpha(x)=\frac{-\mathscr{M}}{m},\quad \text{and}\quad \beta(x)=\frac{\mathscr{M}}{m}.\quad \text{Then}$$

$$\alpha(x)\leq\beta(x),\quad \alpha(0)\leq A\leq\beta(0),\quad \alpha(1)\leq B\leq\beta(1).$$

Applying Taylor's theorem for some point $\zeta\in(\alpha,0)$, it is obtained as:

$$\begin{aligned} g(x,\alpha,0) &= g(x,0,0)+\alpha g_v(x,\zeta,0),\\ g(x,\alpha,0) &\leq g(x,0,0)+\alpha m\leq\mathscr{M}+m(\frac{-\mathscr{M}}{m})\leq 0=-\varepsilon\alpha. \end{aligned}$$

Similarly for intermediate point $\eta\in(0,\beta)$,

$$\begin{aligned} g(x,\beta,0) &= g(x,0,0)+\beta g_v(x,\eta,0),\\ g(x,\beta,0) &\geq g(x,0,0)+\beta m\geq -\mathscr{M}+m(\frac{\mathscr{M}}{m})\geq 0=-\varepsilon\beta. \end{aligned}$$

Hence, it follows from Theorem (3.3.1) that for each $\varepsilon>0$ the problem (3.3.1) has a solution $v(x,\varepsilon)$ on $[0,1]$ satisfying :

$$\frac{-\mathscr{M}}{m}\leq v(x,\varepsilon)\leq\frac{\mathscr{M}}{m}. \tag{3.3.7}$$

The uniqueness of the solution of problem (3.3.1) follows from maximum principle. □

3.3.1 Stability of Degenerate Solution

In this subsection we are concern with the existence and stability of solution for problem (3.3.1). However we only stick with stable solution of the proposed problem. Let $z(x)\in$

$C^1[a,b]$ be the solution of equation $g(x,v(x)) = g(x,z(x))$ in $\Omega = [a,b]$. Then we define

$$\phi_0(z) = \{(x, v(x) : |v(x) - z(x)| \leq \psi(x), x \in \Omega\}, \tag{3.3.8}$$

where $\psi(x)$ is defined as

$$\psi(x) = \begin{cases} |A - z(a)| + \rho & \text{for } x \in [a, a+\rho/2], \\ \rho & \text{for } x \in [a+\rho, b-\rho], \\ |B - z(b)| + \rho & \text{for} x \in [b-\rho/2, b]. \end{cases} \tag{3.3.9}$$

Where ρ be a small positive constant and suppose if $A \geq z(a)$ and $B \geq z(b)$, then we define

$$\phi_1(z) = \{(x, v(x)) : |v(x) - z(x)| \in [0, \psi(x)], x \in \Omega\}, \tag{3.3.10}$$

IIrly if $A \leq z(a)$ and $B \leq z(b)$ then

$$\phi_2(z) = \{(x, v(x)) : |v(x) - z(x)| \in [-\psi(x), 0], x \in \Omega\}, \tag{3.3.11}$$

Now we discuss and define the stability for the solution of problem (3.3.1). Let us presume that $g(x,v(x))$ has the stated number of continuous partial derivatives w.r.t $v(x)$ in ϕ_i, $i = 0,1$ or 2 and $n \geq 2$, $q \geq 0$ be the integers.

Definition 3.3.3. The function $z = z(x)$ be I_q-stable on Ω if $\exists$ a constant m such that

$$\frac{\partial^j g(x, z(x))}{\partial v^j} = 0 \quad \forall \quad x \in \Omega, 0 \leq j \leq 2q, \tag{3.3.12}$$

and

$$\frac{\partial^q g(x, z(x))}{\partial v^q} \geq m > 0 \quad \text{in} \quad \phi_0(z). \tag{3.3.13}$$

Definition 3.3.4. The function $z = z(x)$ be II_n-stable on Ω and $A \leq z(a)$, $B \leq z(b)$ if $\exists$ a constant $m \geq 0$ such that

$$\frac{\partial^j g(x, z(x))}{\partial v^j} \geq 0 \quad \forall \quad x \in \Omega, 1 \leq j \leq n-1, \tag{3.3.14}$$

and

$$\frac{\partial^n g(x, z(x))}{\partial v^n} \geq m > 0 \quad \text{in} \quad \phi_1(z). \tag{3.3.15}$$

Theorem 3.3.1. *Let $g(x,v(x)) = 0$ satisfies definition (3.3.3) i.e have I_q stable solution $z = z(x) \in$*

$C^2(\Omega)$. *Then* $\exists\ \varepsilon_0 > 0$ *such that* $0 < \varepsilon < \varepsilon_0$. *Then problem* (3.3.1) *has a solution* $v(x) = v(x,\varepsilon)$ *which satisfies the following*

$$|v(x) - z(x)| \leq s_l(x) + s_r(x) + C\varepsilon^{1/(2q+1)}, \tag{3.3.16}$$

where s_l *and* s_r *is defined as*

$$s_l = \begin{cases} |A - z(a)|\exp(-\sqrt{m/\varepsilon}(x-a)) & \textit{if } q = 0, \\ |A - z(a)|[1 + \rho|A - z(a)|^q\varepsilon - 1/2(x-a)^{-1/q} & \textit{if } q \geq 1. \end{cases} \tag{3.3.17}$$

And

$$s_r = \begin{cases} |B - z(b)|\exp(-\sqrt{m/\varepsilon}(b-x)) & \textit{if } q = 0, \\ |B - z(b)|[1 + \rho|B - z(b)|^q\varepsilon - 1/2(b-x)^{-1/q} & \textit{if } q \geq 1. \end{cases} \tag{3.3.18}$$

where $\rho = \sqrt{mq}[(q+1)(2q+1)!]^{-1/2}$.

Proof. For detail proof [57] □

Theorem 3.3.2. *Let* $g(x, v(x)) = 0$ *satisfies definition* (3.3.4) *i.e have* II_n *stable solution* $z = z(x) \in C^2(\Omega)$ *such that* $z(a) \leq A$, $z(b) \leq B$ *and* $z'' \geq 0$ *in* (a,b). *Then* $\exists\ \varepsilon_0 > 0$ *such that* $0 < \varepsilon < \varepsilon_0$. *Then problem* (3.3.1) *has a solution* $v(x) = v(x,\varepsilon)$ *which satisfies the following*

$$0 \leq v(x) - z(x) \leq s_l(x) + s_r(x) + C\varepsilon^{\frac{1}{2}}, \tag{3.3.19}$$

where w_l *and* w_r *is defined as*

$$w_l(x) = (A - z(a))\left[1 + (x-a)(A - z(a))^{\frac{1}{2(n-1)}}\rho_1/\sqrt{\varepsilon}\right]^{\frac{-2}{n-1}}, \tag{3.3.20}$$

and

$$w_r(x) = (B - z(b))\left[1 + (b-x)(B - z(b))^{\frac{1}{2(n-1)}}\rho_1/\sqrt{\varepsilon}\right]^{\frac{-2}{n-1}}, \tag{3.3.21}$$

and $\rho_1 = (n-1)(\frac{m}{2}(m+1)!)^{1/2}$.

Proof. For detail proof [57]. □

3.4 Error Analysis

In this section error analysis is carried out in maximum norm of proposed problem (3.3.1). Let us assume that $\varepsilon \leq Ch$ where C is a positive constant independent. Let us define the collocation points as $x_j = x_0 + \frac{j}{m}$ and $h_j = x_j - x_{j-1}$ $\forall$ $j = 0, 1, \cdots, m$.
First, let us consider the possible cases with $g(x, v(x))$ as

$$g(x,v(x)) = \begin{cases} f(x,v(x)), \\ f(x,v(x)) + p(x)v(x), \\ f(x,v(x)) - p(x)v(x). \end{cases} \tag{3.4.1}$$

There only three possible cases associated with $g(x, v(x))$. In first case $g(x, v(x))$ is non-linear function and other two cases are when, linear part is extracted out from $g(x, v(x))$ with positive and negative signs. Let $p(x) \leq |\wp|$.
Suppose $\chi = C[0,1]$ be the Banach space equipped with norm defined as

$$\|v\| = \max_{x \in [0,1]} |v(x)|. \tag{3.4.2}$$

Theorem 3.4.1. *Let $v(x)$ is the solution of* (3.3.1) *and $g \in C^\infty[0,1] \times R$ then we have the following bound on the derivative of $v(x)$*

$$|v^{(i)}(x)| \leq |C(1 + \varepsilon^{-i} e^{-\Im x/\sqrt{\varepsilon}} + \varepsilon^{-i} e^{\Im(-1+x)/\sqrt{\varepsilon}})|, \tag{3.4.3}$$

for $i = 1, 2, 3$.

Proof. For proof of the above theorem see [166]. □

Theorem 3.4.2. *Suppose $\mathscr{F} \in \chi$ and $\boldsymbol{B}_m(\mathscr{F})$ be a sequence converges to $\mathscr{F}$ uniformly, where $\boldsymbol{B}_m(\mathscr{F})$ is defined in* (3.2.1). *Then for any $\delta \geq 0$ $\exists$ m such that*

$$\|\boldsymbol{B}_m(\mathscr{F}) - \mathscr{F}\| \leq \delta. \tag{3.4.4}$$

Proof. For detailed proof see [245]. □

Theorem 3.4.3. *Let $\mathscr{F}$ be a bounded and continuous function and $\mathscr{F}^{''}$ exist in [0,1], then we have the following error bound*

$$\|\boldsymbol{B}_m(\mathscr{F})-\mathscr{F}\| \leq \frac{1}{2m}x(1-x)\|\mathscr{F}^{''}\|. \tag{3.4.5}$$

Proof. For detail proof see [201]. □

Theorem 3.4.4. *Let v be the exact solution and v_m denotes the approximate solution by BCM. Suppose nonlinear function $g(x,v)$ satisfies the Lipschitz condition*

$$|g(x,v)-g(x,v^{\star})| \leq \mathscr{L}|v-v^{\star}|, \tag{3.4.6}$$

then the error bound for the BCM is given as:

$$\|v-v_m\| \leq \frac{\mathscr{L}\wp}{8m}\|v^{''}\|. \tag{3.4.7}$$

where $\mathscr{L}$ is known as Lipschitz constant.

Proof. Let

$$\|v-v_m\| = \max_{x\in[0,1]}|g(x,v(x))-g(x,v_m(x))|. \tag{3.4.8}$$

Case 1. When $g(x,v(x)) = f(x,v(x))$, then

$$\|v-v_m\| = \max_{x\in[0,1]}|f(x,v(x))-f(x,v_m(x))|, \tag{3.4.9}$$

now using Lipschitz condition (3.4.6)

$$\|v-v_m\| \leq \mathscr{L}\max_{x\in[0,1]}|v(x)-v_m(x)|, \tag{3.4.10}$$

now we have approximated $v(x)$ by BCM then we have

$$\|v-v_m\| \leq \mathscr{L}\max_{x\in[0,1]}|v(x)-\mathbf{B}_m(x)|, \tag{3.4.11}$$

Now from theorem (3.4.3), we have

$$\|v-v_m\| \leq \frac{\mathscr{L}}{2m}\max_{x\in[0,1]}|x(1-x)|\|v^{''}\|, \tag{3.4.12}$$

$$\leq \frac{\mathscr{L}}{8m}\|v^{''}\|, \tag{3.4.13}$$

Case 2. When $g(x,v(x)) = f(x,v(x)) + p(x)v(x)$, then

$$\|v-v_m\| = \max_{x\in[0,1]}|f(x,v(x))+p(x)v(x)-f(x,v_m(x))-p(x)v_m(x)|, \tag{3.4.14}$$

$$= \max_{x\in[0,1]}|f(x,v(x))-f(x,v_m(x))+p(x)(v(x)-v_m(x))|, \tag{3.4.15}$$

$$\leq \max_{x\in[0,1]}|f(x,v(x))-f(x,v_m(x))|+\max_{x\in[0,1]}|p(x)||(v(x)-v_m(x))|, \tag{3.4.16}$$

using Lipschitz condition (3.4.6)

$$\|v-v_m\| \leq \mathscr{L}\left[\max_{x\in[0,1]}|v(x)-v_m(x)|+\max_{x\in[0,1]}|p(x)||v(x)-v_m(x)|\right]. \tag{3.4.17}$$

Now the proof is straight forward. Using conditions (3.4.11), (3.4.12). We obtain the following bound

$$\|v-v_m\| \leq \frac{\mathscr{L}\wp}{8m}\|v^{''}\|, \tag{3.4.18}$$

Case 3. When $g(x,v(x)) = f(x,v(x)) - p(x)v(x)$, The proof is similar and we have

$$\|v-v_m\| \leq \frac{\mathscr{L}\wp}{8m}\|v^{''}\|. \tag{3.4.19}$$

□

3.5 Numerical Results and Discussion

This section analyzes the proposed method's efficiency and implements the BCM to solve two nonlinear singularly perturbed reaction-diffusion problems. The proposed method approximated solution is compared with spline technique [136], B-spline collocation method [259], a patching approach based on novel combination of variation of iteration and cubic spline collocation method [162] and a neuro-evolutionary artificial technique [255].

Example 3.5.1. Consider the non-linear problem singularly perturbed problem used as a model of Michaelis-Menten process, the model takes the form of an equation describing the rate of enzymatic reaction in biology [218].

$$-\varepsilon v''(x) - \frac{v(x)-1}{2-v(x)} + f(x) = 0, \qquad v(0) = v(1) = 0. \tag{3.5.1}$$

The $f(x)$ of the above problem is calculate so that the exact solution of the above problem is $v(x) = 1 - \frac{e^{\frac{-x}{\sqrt{\varepsilon}}} + e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}}{1+e^{\frac{-1}{\sqrt{\varepsilon}}}}$. The approximate solution obtained by BCM and exact solution of example (3.5.1) for different values of ε are given in tables (3.1) and (3.3). The absolute error calculated for example (3.5.1) is given in tables (3.2) and (3.6).

$\varepsilon = 0.1$			
x	$v(x)$	$v_6(x)$	$v_9(x)$
0.0	0	0	0
0.1	0.2449922124062	0.244920738131	0.24499155042781
0.2	0.4138523719037	0.4137925266378	0.41385182864694
0.3	0.5236076811362	0.5235555298431	0.52360718596683
0.4	0.5853249820947	0.585276186368	0.58532498209477
0.5	0.6052290251285	0.6051808878855	0.60522857705331
0.6	0.585324982094	0.585276186368	0.58532498209477
0.7	0.5236076811362	0.5235555298431	0.52360718596683
0.8	0.4138523719037	0.4137925266378	0.41385182864694
0.9	0.2449922124062	0.244920738131	0.24499155042781
1.0	0	0	0

Table 3.1: “Comparison between exact solution and the approximate solution for $M = 6,9$ of Example (3.5.1) for $\varepsilon = 0.1$."

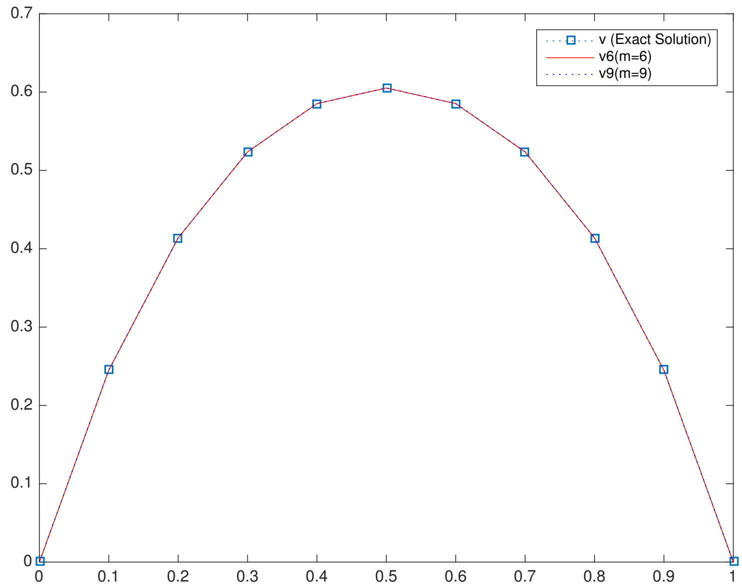

Figure 3.1: “Comparison between exact solution and the numerical solution computed by BCM of Example (3.5.1) for $\varepsilon = 0.1$."

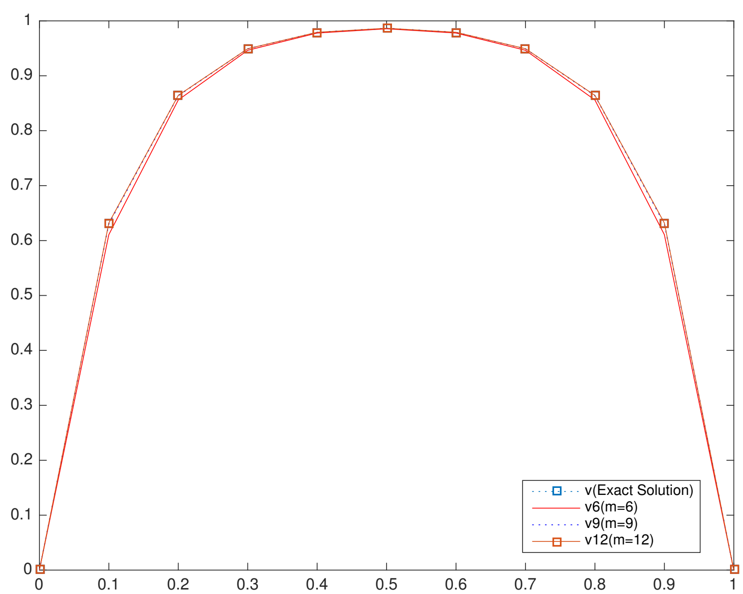

Figure 3.2: “Comparison between exact solution and the numerical solution computed by BCM of Example (3.5.1) for $\varepsilon = 0.01$."

$\varepsilon = 0.1$			
x	Error for $m = 6$	Error for $m = 9$	Error for $m = 12$
0.0	0.0	0.0	0.0
0.1	7.14743×10^{-5}	6.61978×10^{-7}	7.40155×10^{-11}
0.2	5.98453×10^{-5}	5.43257×10^{-7}	6.41531×10^{-11}
0.3	5.21513×10^{-5}	4.95169×10^{-7}	5.74667×10^{-11}
0.4	4.92528×10^{-5}	4.57111×10^{-7}	5.34776×10^{-11}
0.5	4.81372×10^{-5}	4.48075×10^{-7}	5.21325×10^{-11}
0.6	4.92528×10^{-5}	4.57111×10^{-7}	5.34776×10^{-11}
0.7	5.21513×10^{-5}	4.95169×10^{-7}	5.74667×10^{-11}
0.8	5.98453×10^{-5}	5.43257×10^{-7}	6.41531×10^{-11}
0.9	7.14743×10^{-5}	6.61978×10^{-7}	7.40155×10^{-11}
1.0	0.0	0.0	0.0

Table 3.2: “Absolute error at different iterations of Example (3.5.1) for $\varepsilon = 0.1$."

$\varepsilon = 0.01$				
x	$v(x)$	$v_6(x)$	$v_9(x)$	$v_{12}(x)$
0.0	0.0	0.0	0.0	0.0
0.1	0.632014	0.610778	0.630021	0.631995
0.2	0.864335	0.855949	0.8636	0.864328
0.3	0.949303	0.946408	0.94899	0.9493
0.4	0.979207	0.977815	0.979085	0.979205
0.5	0.986525	0.985572	0.986436	0.986524
0.6	0.979207	0.977815	0.979085	0.979205
0.7	0.949303	0.946408	0.94899	0.9493
0.8	0.864335	0.855949	0.8636	0.864328
0.9	0.632014	0.610778	0.630021	0.631995
1.0	0.0	0.0	0.0	0.0

Table 3.3: “Comparison between exact solution and the approximate solution for $M = 6, 9, 12$ of Example (3.5.1) for $\varepsilon = 0.01$."

$\varepsilon = 0.01$			
x	Error for $m = 6$	Error for $m = 9$	Error for $m = 12$
0.0	0.0	0.0	0
0.1	2.12363×10^{-2}	1.99293×10^{-3}	1.85967×10^{-5}
0.2	8.38674×10^{-3}	7.35248×10^{-4}	7.6668×10^{-6}
0.3	2.89526×10^{-3}	3.13402×10^{-4}	3.01472×10^{-6}
0.4	1.39188×10^{-3}	1.21095×10^{-4}	1.27381×10^{-6}
0.5	9.53085×10^{-4}	8.86043×10^{-5}	8.32789×10^{-7}
0.6	1.39188×10^{-3}	1.21095×10^{-4}	1.27381×10^{-6}
0.7	2.89526×10^{-3}	3.13402×10^{-4}	3.01472×10^{-6}
0.8	8.38674×10^{-3}	7.35248×10^{-4}	7.6668×10^{-6}
0.9	2.12363×10^{-2}	1.99293×10^{-3}	1.85967×10^{-5}
1.0	0.0	0.0	0.0

Table 3.4: “Absolute error at different iterations of Example (3.5.1) for $\varepsilon = 0.01$."

$\varepsilon = 0.01$			
x	CPU time for $m = 6$	CPU time for $m = 9$	CPU time for $m = 12$
0.0	–	–	–
0.1	0.000089	0.000065	0.000078
0.2	0.000087	0.00008	0.000092
0.3	0.000109	0.000072	0.000105
0.4	0.000068	0.000067	0.000076
0.5	0.000082	0.000087	0.000086
0.6	0.0001	0.00008	0.000086
0.7	0.000063	0.00007	0.000099
0.8	0.000051	0.000071	0.000085
0.9	0.000083	0.000076	0.000094
1.0	–	–	–

Table 3.5: “CPU time of table (3.6) of Example (3.5.1) for $\varepsilon = 0.01$."

$\varepsilon = 0.001$			
x	Error for $m = 9$	Error for $m = 12$	Error for $m = 20$
0.0	0.0	0.0	0.0
0.1	3.6262×10^{-2}	6.67567×10^{-3}	6.79712×10^{-5}
0.2	1.1625×10^{-3}	3.23463×10^{-4}	2.97539×10^{-6}
0.3	8.158×10^{-4}	2.9402×10^{-6}	1.25976×10^{-7}
0.4	4.31425×10^{-4}	3.59187×10^{-6}	5.29514×10^{-9}
0.5	3.606×10^{-4}	7.67527×10^{-7}	4.26708×10^{-10}
0.6	4.31425×10^{-4}	3.59187×10^{-6}	5.29514×10^{-9}
0.7	8.158×10^{-4}	2.9402×10^{-6}	1.25976×10^{-7}
0.8	1.1625×10^{-3}	3.23463×10^{-4}	2.97539×10^{-6}
0.9	3.6262×10^{-2}	6.67567×10^{-3}	6.79712×10^{-5}
1.0	0.0	0.0	0.0

Table 3.6: “Absolute error at different iterations of Example (3.5.1) for $\varepsilon = 0.001$."

Example 3.5.2. Consider the following non-linear singularly perturbed problem from [136, 162, 255, 259]. This problem can be used to describe a mathematical model of an adiabatic tubular chemical reactor that processes an irreversible exothermic chemical reaction. Where ε represents the dimensionless adiabatic temperature. In fact, the steady state temperature of the reaction is equivalent to a positive solution v.

$$\begin{cases} -\varepsilon v'' + v + v^2 = e^{\frac{-2x}{\sqrt{\varepsilon}}} \\ v(0) = 1, \quad v(1) = e^{\frac{-1}{\sqrt{\varepsilon}}}. \end{cases} \tag{3.5.2}$$

The exact solution of the above problem is given as: $v(x) = e^{\frac{-x}{\sqrt{\varepsilon}}}$. The approximate solution obtained by the BCM and the exact solution of example (3.5.2) for different values of ε are given in tables (3.7) and (3.9). The absolute error calculated for example (3.5.2) is given in tables (3.8) and (3.10). We have compared the error obtained by the BCM for example (3.5.2) with spline technique [136], B-spline collocation method [259], a patching approach based on novel combination of variation of iteration and cubic spline collocation method [162] and a neuro-evolutionary artificial technique [255] in table (3.13), (3.14) , (3.12) and (3.11) respectively.

Figure (3.1) and (3.2) depict the comparison between the exact solution and the approximate solution obtained from the proposed method for example (3.5.1) with $\varepsilon = 0.1$ and 0.01 respectively and Figure (3.3) and (3.4) depict the comparison between the exact solution and the approximate solution obtained from the proposed method for example (3.5.2) with $\varepsilon = 0.1$ and 0.01 respectively. It is observed in all figures as m increases the approximate solution converges to the exact solutions, which demonstrate the convergence of our proposed method.

$\varepsilon = 0.1$				
x	$v(x)$	$v_6(x)$	$v_9(x)$	$v_{12}(x)$
0.0	1	1	1	1
0.1	0.728893	0.69735	0.729038	0.728893
0.2	0.531286	0.48428	0.531366	0.531286
0.3	0.387251	0.337681	0.387294	0.387251
0.4	0.282264	0.238303	0.282291	0.282264
0.5	0.205741	0.170761	0.205746	0.205741
0.6	0.149963	0.123527	0.149947	0.149963
0.7	0.109307	0.0889388	0.109276	0.109307
0.8	0.0796732	0.0631934	0.0796146	0.0796732
0.9	0.0580733	0.04635	0.0579722	0.0580733
1.0	0.0423292	0.0423292	0.0423292	0.0423292

Table 3.7: “Comparison between exact solution and the approximate solution for $M = 6, 9, 12$ of Example (3.5.2) for $\varepsilon = 0.1$"

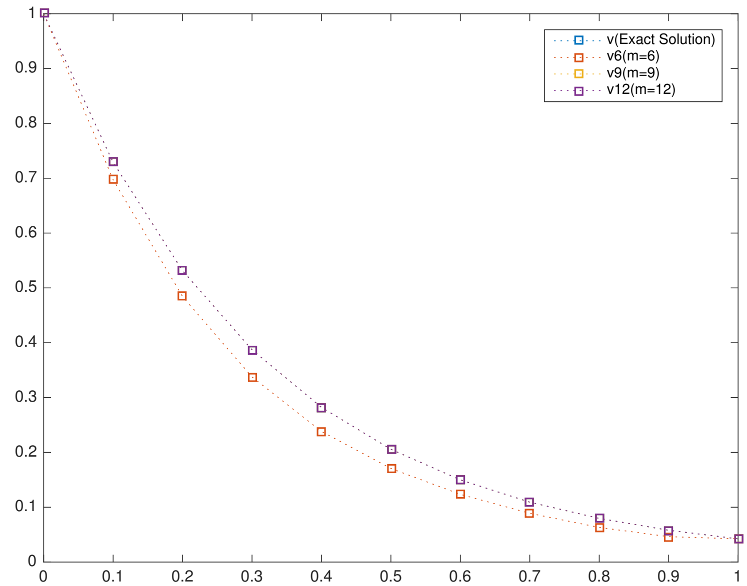

Figure 3.3: “Comparison between exact solution and the numerical solution computed by BCM of Example (3.5.2) for $\varepsilon = 0.1$."

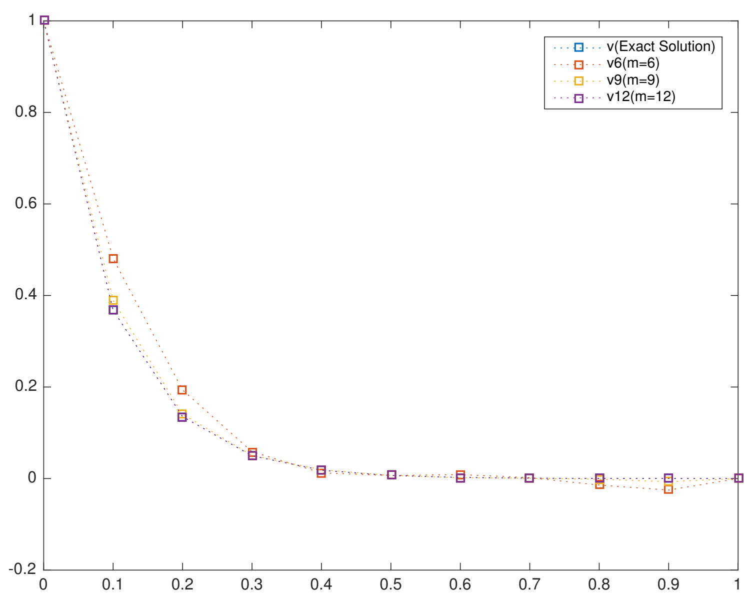

Figure 3.4: “Comparison between exact solution and the numerical solution computed by BCM of Example (3.5.2) for $\varepsilon = 0.01$."

$\varepsilon = 0.1$			
x	Error for $m = 6$	Error for $m = 9$	Error for $m = 12$
0.0	0.0	0.0	0.0
0.1	2.12363×10^{-2}	2.95251×10^{-7}	2.44075×10^{-10}
0.2	8.38674×10^{-3}	1.82323×10^{-7}	1.47531×10^{-10}
0.3	2.89526×10^{-3}	1.37572×10^{-7}	8.51842×10^{-11}
0.4	1.39188×10^{-3}	1.10479×10^{-7}	3.9244×10^{-11}
0.5	9.53085×10^{-4}	1.04473×10^{-7}	6.61762×10^{-13}
0.6	1.39188×10^{-3}	1.10767×10^{-7}	4.05694×10^{-11}
0.7	2.89526×10^{-3}	1.34799×10^{-7}	8.54505×10^{-11}
0.8	8.38674×10^{-3}	1.69239×10^{-7}	1.41512×10^{-10}
0.9	2.12363×10^{-2}	2.46245×10^{-7}	2.16198×10^{-10}
1.0	0.0	0.0	0.0

Table 3.8: “Absolute error at different iterations of Example (3.5.2) for $\varepsilon = 0.1$."

Example 3.5.3. Consider the following non-linear singularly perturbed problem.

$$\begin{cases} -\varepsilon v'' + v + (v+1)^3 = -1 \\ v(0) = 0, \quad v(1) = 0. \end{cases} \tag{3.5.3}$$

The exact solution of the above problem is not known. The approximate solution obtained by the BCM and the absolute error calculated for example (3.5.3) is given in tables (3.15) and (3.16). As the true solution of problem (3.5.3) is not known to us. So to calculate the error we take a reference solution computed using $m = 20$.

$\varepsilon = 0.01$				
x	$v(x)$	$v_6(x)$	$v_9(x)$	$v_{12}(x)$
0.0	1.0	1.0	1.0	1.0
0.1	0.367879	0.480727	0.389305	0.367897
0.2	0.135335	0.193552	0.140804	0.135341
0.3	0.0497871	0.0587971	0.050756	0.0497889
0.4	0.0183156	0.0125445	0.0193477	0.0183163
0.5	0.00673795	0.00663759	0.00692563	0.00673799
0.6	0.00247875	0.00868034	0.00195098	0.00247826
0.7	0.000911882	0.00203716	0.000678469	0.000910368
0.8	0.000335463	-0.0141669	-0.00144098	0.00033117
0.9	0.00012341	-0.0250462	-0.00663395	0.000111523
1.0	0.000045399	0.000045399	0.000045399	0.000045399

Table 3.9: “Comparison between exact solution and the approximate solution for $M = 6, 9, 12$ of Example (3.5.2) for $\varepsilon = 0.01$."

3.6 Conclusion

In this chapter, we have successfully implemented the Bernstein collocation method for solving SPNRD problems. To solve the nonlinear problems, one often uses a quasi-linearization technique to linearize the problem and then solve the linearized problem by numerical or other existing techniques. Due to the linearization of the nonlinear problem, the approximated solution's accuracy somehow degenerates, which may leads to deceptive solutions sometimes. Here we address this issue and solve the nonlinear problem without linearization. The proposed method is easy to implement as it changes complex nonlinear problems to a system of algebraic equation system and can be extended to even a general class of problems. This method yield a higher level of precision just using lower degree polynomials without any limiting assumption.

$\varepsilon = 0.01$			
x	Error for $m = 6$	Error for $m = 9$	Error for $m = 12$
0.0	0.0	0.0	0.0
0.1	2.12363×10^{-2}	8.44402×10^{-4}	1.7746×10^{-5}
0.2	8.38674×10^{-3}	2.06568×10^{-4}	5.48799×10^{-6}
0.3	2.89526×10^{-3}	8.72922×10^{-5}	1.83925×10^{-6}
0.4	1.39188×10^{-3}	2.73424×10^{-5}	6.16373×10^{-7}
0.5	9.53085×10^{-4}	1.98908×10^{-5}	3.88065×10^{-8}
0.6	1.39188×10^{-3}	1.9252×10^{-5}	4.91776×10^{-7}
0.7	2.89526×10^{-3}	5.62131×10^{-5}	1.51435×10^{-6}
0.8	8.38674×10^{-3}	1.25513×10^{-4}	4.29289×10^{-6}
0.9	2.12363×10^{-2}	4.23724×10^{-4}	1.1887×10^{-5}
1.0	0.0	0.0	0.0

Table 3.10: “Absolute error at different iterations of Example (3.5.2)."

$\varepsilon = 0.01$		
x	Our proposed method for $m = 12$	Method in [255]
0.0	0	0
0.1	1.7746×10^{-5}	1.23×10^{-4}
0.2	5.48799×10^{-6}	4.99×10^{-5}
0.3	1.83925×10^{-6}	7.32×10^{-5}
0.4	6.16373×10^{-7}	5.70×10^{-5}
0.5	3.88065×10^{-8}	6.61×10^{-5}
0.6	4.91776×10^{-7}	6.42×10^{-5}
0.7	1.51435×10^{-6}	5.97×10^{-5}
0.8	4.29289×10^{-6}	7.89×10^{-5}
0.9	1.1887×10^{-5}	6.35×10^{-5}
1.0	0	0

Table 3.11: “Absolute error comparison of proposed method for Example (3.5.2) with neuro-evolutionary model technique [255]."

$\varepsilon = 0.1$	
Our proposed method for $m = 9$	Method in [162]
2.95251×10^{-7}	5.0000×10^{-4}

Table 3.12: "Maximum absolute error comparison of proposed method for Example (3.5.2) with a patching approach Method in [162]."

	$\varepsilon = \frac{1}{16}$		$\varepsilon = \frac{1}{64}$		$\varepsilon = \frac{1}{128}$	
N	Method in [136]	BCM $m = 12$	Method in [136]	BCM $m = 12$	Method in [136]	BCM $m = 12$
32	0.80×10^{-3}	3.2057×10^{-9}	0.93×10^{-3}	2.81829×10^{-6}	0.14×10^{-3}	4.4966×10^{-5}
64	0.15×10^{-3}	3.2057×10^{-9}	0.63×10^{-3}	2.81829×10^{-6}	0.76×10^{-3}	4.4966×10^{-5}
128	0.64×10^{-4}	3.2057×10^{-9}	0.59×10^{-4}	2.81829×10^{-6}	0.25×10^{-3}	4.4966×10^{-5}
256	0.13×10^{-6}	3.2057×10^{-9}	0.10×10^{-4}	2.81829×10^{-6}	0.24×10^{-4}	4.4966×10^{-5}

Table 3.13: "Maximum norm error comparison of proposed method for Example (3.5.2) with spline technique [136] on piece-wise uniform mesh."

	$\varepsilon = 2^{-4}$		$\varepsilon = 2^{-6}$		$\varepsilon = 2^{-8}$	
N	Method in [259]	BCM $m = 12$	Method in [259]	BCM $m = 12$	Method in [259]	BCM $m = 12$
32	1.81×10^{-4}	3.2057×10^{-9}	7.31×10^{-4}	2.81829×10^{-6}	2.44×10^{-3}	2.81829×10^{-4}
64	4.50×10^{-5}	3.2057×10^{-9}	1.82×10^{-4}	2.81829×10^{-6}	7.31×10^{-4}	2.81829×10^{-4}
128	1.12×10^{-5}	3.2057×10^{-9}	4.53×10^{-5}	2.81829×10^{-6}	1.81×10^{-4}	2.81829×10^{-4}

Table 3.14: "Maximum norm error comparison of proposed method for Example (3.5.2) with B-spline collocation method [259] on piece-wise uniform mesh".

$\varepsilon = 1$		
x	Error for $m = 9$	Error for $m = 12$
0.0	0.0	0.0
0.1	4.58387×10^{-7}	1.39753×10^{-9}
0.2	3.9138×10^{-7}	1.25563×10^{-9}
0.3	3.69418×10^{-7}	1.16725×10^{-9}
0.4	3.51124×10^{-7}	1.11688×10^{-9}
0.5	3.47409×10^{-7}	1.10017×10^{-9}
0.6	3.51124×10^{-7}	1.11688×10^{-9}
0.7	3.69418×10^{-7}	1.16725×10^{-9}
0.8	3.9138×10^{-7}	1.25563×10^{-9}
0.9	4.58387×10^{-7}	1.39752×10^{-9}
1.0	0.0	0.0

Table 3.15: “Absolute error at different iterations of Example (3.5.3) for $\varepsilon = 1$."

$\varepsilon = 0.1$		
x	Error for $m = 9$	Error for $m = 12$
0.0	0.0	0.0
0.1	5.47695×10^{-4}	3.06129×10^{-5}
0.2	3.53802×10^{-4}	2.10407×10^{-5}
0.3	2.7224×10^{-4}	1.58368×10^{-5}
0.4	2.23452×10^{-4}	1.31449×10^{-5}
0.5	2.10797×10^{-4}	1.2301×10^{-5}
0.6	2.23452×10^{-4}	1.31449×10^{-5}
0.7	2.7224×10^{-4}	1.58368×10^{-5}
0.8	3.53802×10^{-4}	2.10407×10^{-5}
0.9	5.47695×10^{-4}	3.06129×10^{-5}
1.0	0.0	0.0

Table 3.16: “ Absolute error at different iterations of Example (3.5.3) for $\varepsilon = 0.1$."

Chapter 4

Galerkin Finite Element Method with Richardson Extrapolation for Singularly Perturbed Time Delay Parabolic Reaction-Diffusion Problem

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In this chapter[1], a parameter uniform Galerkin finite element method with Richardson extrapolation in time for numerically approximating singularly perturbed parabolic reaction diffusion problems with retarded argument is proposed. The solution of this class of problems is polluted by a small positive parameter, due to which the solution of the said problem exhibits parabolic boundary layers. The spatial variable domain is evaluated by implementing the finite element method along with piecewise uniform mesh (Shishkin mesh) to capture the exponential behavior of the solution in the boundary layer region, and for the time variable author has implemented implicit backward-Euler method with Richardson extrapolation on equidistant mesh in time direction to attain a good accuracy along with the higher order convergence. The proposed method is shown to be robust, efficient, unconditionally stable, and parameter uniform. The error analysis is carried out

[1]"K. Khari and V. Kumar, Galerkin Finite Element Method with Richardson Extrapolation for Singularly Perturbed Time Delay Parabolic Reaction-Diffusion Problem (Communicated) "

in maximum norm using Green's function approach. The proposed method is shown to be accurate of order $[O(N^{-1}\ln N)^2+\Delta t^2]$ in maximum norm. Few numerical examples are taken into account to validate the theoretical findings.

4.1 Introduction

Singularly perturbed parabolic partial differential with a time delay (SPPDE) plays an essential role in converting a natural phenomenon into a mathematical model in which the system's response is delayed or depends on past history. Moreover, in most life science models, a delay term is inducted when the process needs to be better understood or there are some hidden variables that cause a time lag. The (SPPDE) problem with the time delay model's a more realistic biological and natural phenomenon than the conventional singularly perturbed problems with no delay do. The methodology and dynamics of (the SPPDE) problem with time delay are utterly different from the conventional partial differential equations without time lag. The solution of (SPPDE) problem with time delay is evaluated by $\psi_b(x,t)$ an initial value function for $t-\tau<0$ rather than by a simple initial value function $\psi_b(x,t)$ as happens in case of singularly perturbed PDEs. These problems depend on a small positive parameter so that solution varies rapidly in some parts of the domain and changes slowly in other parts. These phenomena are called boundary layer phenomena, and this type of problem is called singularly perturbed. Due to their occurrence in a wide range of applications, the study of singularly perturbed delay differential equations has been an area of research over the last decades. A singularly perturbed delay differential equation is a delay differential equation whose leading derivative has a small positive parameter multiplied by it, which makes the problems much more complex. Finding an approximate solution independent of this singular perturbation parameter ε is very challenging.

The Richardson extrapolation method is used to analyze the numerical solutions of SPP problems in order to improve the proposed scheme's rate of convergence and accuracy. We have shown in this article that in the case of a time-dependent parabolic problem, the extrapolated approximation for the time variable can be examined irrespective of the space variable. Diffusion is a natural occurrence that happens all the time. This happens when molecules move from a high concentration to a low concentration. Contaminant diffusion in porous media, oxygen diffusion in our cells, and other natural processes exist. In many cases, a reaction happens simultaneously with diffusion, and when these two equations are coupled, the diffusion equation becomes the reaction-diffusion equation.

The paper is organized as follows: The continuous model problem is defined in Section 4.2. In Section 4.3, we provide auxiliary results. In Section 4.4, we have defined the piece-wise uniform mesh (Shishkin mesh) for our continuous problem. The problem is discretized using Galerkin finite element for the space component and the backward-Euler method for the time component in Section 4.5. In Section 4.6, we carried out the stability and error analysis. In Section 4.7, Richardson extrapolation is discussed. Section 4.8 test problem is taken into account to validate our theoretical results. Finally, Section 4.9 contains a conclusion.

Notations Through out the paper C denotes the positive constant independent of parameter ε. We have assumed $\sqrt{\varepsilon} \leq CN^{-1}$ in our analysis. Now $\|v\| = \max_{x\in[0,1]}[v]$ and $\|v\|_\omega = \max_{i=1,\cdots,N-1}[v]$ be the discrete and continuous maximum norm respectively.

4.2 Statement of Problem

Consider the following class of SPPRD problems with retarded argument on a rectangular domain. Let $Q = \omega \times (0,T]$, $\omega = (0,1)$ and $\Upsilon = \Upsilon_l \cup \Upsilon_b \cup \Upsilon_r$, where $\Upsilon_b = \bar{\omega} \times [-\tau, 0]$, $\Upsilon_l = \{(0,t) : t \in [0,T]\}$ and $\Upsilon_r = \{(1,t) : t \in [0,T]\}$ are the initial boundary condition, left boundary condition and right boundary condition of the rectangular domain Q respectively.

$$“(v_\varepsilon)_t(x,t) - \varepsilon(v_\varepsilon)_{xx}(x,t) + a(x)v_\varepsilon(x,t) + b(x,t)v_\varepsilon(x,t-\tau) = f(x,t), \tag{4.2.1}$$

where $(x,t) \in Q$, subject to the initial condition and boundary conditions given as:

$$\begin{cases} v_\varepsilon(x,t) = & \psi_b(x,t), \text{ on } \quad (x,t) \in \Upsilon_b, \\ v_\varepsilon(x,t) = & \psi_l(t), \quad \text{ on } \quad (x,t) \in \Upsilon_l = \{(0,t) : t \in [0,T]\}, \\ v_\varepsilon(x,t) = & \psi_r(t), \quad \text{ on } \quad (x,t) \in \Upsilon_r = \{(1,t) : t \in [0,T]\}”. \end{cases} \tag{4.2.2}$$

The model problem (4.2.1) can be recast as

$$“(v_\varepsilon)_t(x,t) + L_{\varepsilon,x}v_\varepsilon(x,t) = F(x,t), \tag{4.2.3}$$

$$L_{\varepsilon,x}v_\varepsilon(x,t)=\begin{cases}-\varepsilon(v_\varepsilon)_{xx}+a(x)v_\varepsilon(x,t) & \text{for } x\in(0,1),\\ & t\in(0,\tau],\\ \\ -\varepsilon(v_\varepsilon)_{xx}+a(x)v_\varepsilon(x,t)+b(x,t)v_\varepsilon(x,t-\tau) & \text{for } x\in(0,1),\\ & t\in(\tau,1],\end{cases} \tag{4.2.4}$$

and

$$F(x,t)=\begin{cases}-b(x,t)\psi_b(x,t-\tau)+f(x,t), & \text{for } x\in(0,1),t\in(0,\tau],\\ f(x,t), & \text{for } x\in(0,1),t\in(\tau,1]".\end{cases} \tag{4.2.5}$$

Here, $0<\varepsilon\ll 1$ is the singular perturbation parameter and $\tau>0$ be the delay term. The problem data $\psi_l(t)$, $\psi_r(t)$, $\psi_b(x,t)$, $f(x,t)$, $a(x)$, and $b(x,t)$ are supposed to be sufficiently smooth, bounded and independent of parameter ε.

$$a(x)\geq\alpha>0, \qquad b(x,t)\geq\beta>0, \qquad (x,t)\in\bar{Q}. \tag{4.2.6}$$

Where α and β are the positive constants independent of singular perturbation parameter ε. As $\varepsilon\to 0$ the solution of the problem (4.2.1)-(5.1.5) exhibits boundary layers of equal width on both Υ_l and Υ_r boundary points.

4.3 Auxiliary Result and Time Discretization

Theorem 4.3.1. *Suppose $a(x)\in\mathscr{C}^{4+\mu}(\bar{\omega}), b(x,t), f(x,t)\in\mathscr{C}^{(4+\mu,2+\mu/2)}(\bar{Q}), \psi_l, \psi_r\in\mathscr{C}^{3+\mu/2}([0,T]), \psi_b\in\mathscr{C}^{(6+\mu,3+\mu/2)}(\Upsilon_b), \mu\in(0,1)$ and compatibility conditions (2.3.1) of high order at corner points are satisfied. Then the integer p,q such that $0\leq p+q\leq$ we have the following estimate:*

$$\begin{aligned}\left\|\frac{\partial^{p+q}v_\varepsilon}{\partial x^p\partial t^q}\right\|_{\bar{Q}} &\leq C(1+\varepsilon^{-p}e^{\frac{-\alpha(1-x)}{\varepsilon}},\\ \left\|\frac{\partial^{p+q}r}{\partial x^p\partial t^q}\right\|_{\bar{Q}} &\leq C(1+\varepsilon^{2-p}),\\ \left\|\frac{\partial^{p+q}s}{\partial x^p\partial t^q}\right\| &\leq C(1+\varepsilon^{-p/2})e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}.\end{aligned}$$

Proof. For proof of the above theorm we recommend [163]. □

Backward-Euler method is implemented for discretization of time derivative and let us assume

$$L_{\varepsilon,x}v_\varepsilon(x,t)=\begin{cases}-\varepsilon\frac{\partial^2}{\partial x^2}+a(x)\mathscr{I} & \text{for } (x,t)\in Q^-\\ -\varepsilon\frac{\partial^2}{\partial x^2}+(a(x)+b(x))\mathscr{I} & \text{for } (x,t)\in Q^+\end{cases} \tag{4.3.1}$$

and

$$F(x,t)=\begin{cases}-b(x,t)\psi_b(x,t-\tau)+f(x,t), & \text{for } (x,t)\in Q^-,\\ f(x,t), & \text{for } (x,t)\in Q^+.\end{cases} \tag{4.3.2}$$

We discretize the time domain $[0,T]$ by equidistant mesh with constant step-size Δt. Let ∇_t^M denotes the partition of the time interval $[0,T]$ with no. of grid points M.

$$\nabla_t^M=\{t_0=0,\Delta t,\cdots,(M-1)\Delta t,M\Delta t=T\}. \tag{4.3.3}$$

Where Δt satisfies constrains $\Delta t=\tau/k$, where k is a positive integer $t_n=n\Delta t$, $n\geq -k$. Now the backward difference is defined as follows:

$$\delta_t v_\varepsilon=\frac{v_\varepsilon^n-v_\varepsilon^{n-1}}{\Delta t},$$

and the discretized form of the equation (4.3.1) by backward-Euler method for time derivative is defined as follows:

$$(\Delta t L_{\varepsilon,x}+I)v_\varepsilon^n=\mathscr{F}^n, \tag{4.3.4}$$

where $\mathscr{F}^n$ is as:

$$\mathscr{F}^n\equiv\begin{cases}\frac{1}{\Delta t}v_\varepsilon^{n-1}-b^n\psi_b^n+f^n & \text{for } n=1,\cdots,k,\\ \\ \frac{1}{\Delta t}v_\varepsilon^{n-1}-b^nv_\varepsilon^{n-k+1}+f^n & \text{for } n=k+1,\cdots,M.\end{cases} \tag{4.3.5}$$

Lemma 4.3.1. Suppose v_ε^n is the semi-discrete approximation of the exact solution $v_\varepsilon(x,t)$ at n^{th} time level $t_n=n\Delta t$ and Eq. (4.3.4) satisfies the maximum principle then.
$(x,t)\in Q^-$

$$\|(\Delta t L_{\varepsilon,x}+I)^{-1}\|_\infty\leq\frac{1}{(1+b\Delta t)} \tag{4.3.6}$$

$(x,t) \in Q^+$

$$\|(\Delta t L_{\varepsilon,x} + I)^{-1}\|_\infty \leq \frac{1}{(1+(b+c)\Delta t)} \tag{4.3.7}$$

Proof. For proof of the above lemma see [196]. □

Hence the lemma (4.3.1) proves the stability of Euler-scheme for semi-discrete form(4.3.4).

So we get a system of ordinary differential equation in spatial variable x for each time step n^{th} time step and v_ε^n denotes the discretized solution at each time level t_n with step size Δt. Now we rewrite the obtained ordinary differential equation in more simplified way as:

When $(x,t) \in Q^-$

$$-\varepsilon \frac{\partial^2 v_\varepsilon^n(x)}{\partial x^2} + a(x) v_\varepsilon^n(x) = \frac{1}{\Delta t} v_\varepsilon^{n-1}(x) - b^n \psi_b^n + f^n. \tag{4.3.8}$$

When $(x,t) \in Q^+$,

$$-\varepsilon \frac{\partial^2 v_\varepsilon^n(x)}{\partial x^2} + a(x) v_\varepsilon^n(x) = \frac{1}{\Delta t} v_\varepsilon^{n-1}(x) - b^n v_\varepsilon^{n-k+1}(x) + f^n, \tag{4.3.9}$$

with boundary conditions:

$$v_\varepsilon^n(0) = v_\varepsilon^n(1) = 0.$$

4.4 Mesh Discretization

Since our problem (4.2.1) exhibits the strong boundary layer of parabolic type at $x=0$ and $x=1$. So we divide our interval into three subinterval $\omega_1 = [0,\sigma]$, $\omega_2 = [\sigma, 1-\sigma]$ and $\omega_3 = [1-\sigma, 1]$, such that

$$\sigma = \min\left\{\frac{1}{4}, \sqrt{\frac{\varepsilon}{\alpha+\beta}} \ln N\right\}$$

where σ is called the mesh transition point. Shishkin mesh is built by partitioning the interval ω_2 into $\frac{N}{2}$ equidistant mesh point and dividing the interval ω_1 and ω_2 into $\frac{N}{4}$ equidistant mesh points.

The step-size h_i is calculated as:

$$h_i = \begin{cases} \dfrac{4\sigma}{N} & \text{for } i = 1, \cdots, \frac{N}{4} \\ \dfrac{2(1-2\sigma)}{N} & \text{for } i = \frac{N}{4}+1, \cdots, \frac{3N}{4} \\ \dfrac{4\sigma}{N} & \text{for } i = \frac{3N}{4}+1 \cdots, N \end{cases} \tag{4.4.1}$$

Where x_i is calculated as:

$$x_i = \begin{cases} (i-1)h(i) & \text{for } i = 1, \cdots, \frac{N}{4} \\ \sigma + (i - (\dfrac{N}{4}+1))h(i) & \text{for } i = \frac{N}{4}+1, \cdots, \frac{3N}{4} \\ 1-\sigma + (i - (\dfrac{3N}{4}+1))h(i) & \text{for } i = \frac{3N}{4}+1 \cdots, N \end{cases} \tag{4.4.2}$$

4.5 Weak Formulation

Let $H_0^1(\omega)$ be the subspace of all function of $H^1(\omega)$ that vanishes at boundary points $x = 0$ and $x = 1$. Let $\mathbb{V}(\bar{\omega}^N)$ is the finite dimensional subspace of $H_0^1(\omega)$ of standard piecewise linear polynomials on given Shishkin mesh ω^N condensed at boundary points $x = 0$ and $x = 1$. We shall consider $\bar{\omega}^N = \{x_0 = 0 < x_1 < \cdots, < x_N = 1\}$ to be the set of mesh points x_i, for some positive integer N. We set $h_i = x_i - x_{i-1}$ to be the local step size and $\hbar = \frac{h_i + h_{i+1}}{2}$. The linear basis function of $\bar{V}(\omega)^N$ is $\{\phi_i\}^{N-1}$, ϕ_i are given by:

$$\phi(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i} & \text{for } x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h_{i+1}} & \text{for } x \in [x_i, x_{i+1}], \\ 0 & \text{for otherwise.} \end{cases} \tag{4.5.1}$$

The Galerkin weak formulation of the problem (4.2.1) can be interpreted as to determine $v_\varepsilon^n \in H_0^1(\omega)$, such that

$$A(v_\varepsilon^n, v) = F(v), \qquad \text{for} \quad \text{all} \quad v \in H_0^1(\omega), \tag{4.5.2}$$

and condition (4.2.6) provides the uniqueness of weak formulation.

For $(x,t) \in Q^-$

$$A^-(v_\varepsilon^n, v) = \int_\omega \{\varepsilon (v_\varepsilon^n)' v' + a(x) v_\varepsilon^n v\} dx, \tag{4.5.3}$$

$$F^-(v) = \int_\omega \left\{ f(x,t_n)v - b(x)\psi_b^n(x)v + \frac{1}{\Delta t} v_\varepsilon^{n-1}(x)v \right\} dx. \tag{4.5.4}$$

For $(x,t) \in Q^+$

$$A^+(v_\varepsilon^n, v) = \int_\omega \{\varepsilon (v_\varepsilon^n)' v' + b(x) v_\varepsilon^n v\} dx, \tag{4.5.5}$$

$$F^+(v) = \int_\omega \left\{ f(x,t_n)v - b(x)\psi_b^n(x)v + \frac{1}{\Delta t} v_\varepsilon^{n+1-k}(x)v \right\} dx. \tag{4.5.6}$$

The above Galerkin weak formulation can be expressed in terms of difference scheme and written as:

$$\mathbf{L}_N(v_\varepsilon)_i = \varepsilon \left[D^+(v_\varepsilon)_i - D^-(v_\varepsilon)_i\right] + \alpha_i D^-(v_\varepsilon)_i + \beta_i D^+(v_\varepsilon)_i + \gamma_i (v_\varepsilon)_i = F(v) \tag{4.5.7}$$

Where $D^+(v_\varepsilon)_i^n$ and $D^-(v_\varepsilon)_i^n$ are defined as

$$D^+(v_\varepsilon)_i^n = \frac{(v_\varepsilon)_{i+1}^n - (v_\varepsilon)_i^n}{h_{i+1}} \qquad D^-(v_\varepsilon)_i^n = \frac{(v_\varepsilon)_i^n - (v_\varepsilon)_{i-1}^n}{h_i} \tag{4.5.8}$$

4.6 Convergence Analysis

4.6.1 Green's Function

Let $\mathscr{G} \in H_0^1(\omega)$ be the green's function corresponding to the weak formulation $A(.,.)$ with mesh points x_i. It satisfies

$$A(u,\mathscr{G}) = u(\varsigma) \quad \forall \quad u \in H_0^1(\omega) \tag{4.6.1}$$

where $\mathscr{G} \in \mathscr{C}^2((0,\varsigma)\cup(\varsigma,1)) \cap \mathscr{C}[0,1]$ such that

$$\mathscr{L}_\varepsilon \mathscr{G} = 0 \quad \text{in} \quad (0,\varsigma)\cup(\varsigma,1) \tag{4.6.2}$$

Equivalently, using standard basis function in $\mathbb{V}(\bar{\omega}^N)$ and $\mathscr{L}_\varepsilon$ from equation (2.6.2). $\mathscr{G}$ can be written as

$$\begin{aligned}
[\mathscr{L}_\varepsilon \mathscr{G}]_i^j &= w_i^j, \\
\mathscr{G}_0 = \mathscr{G}_N &= 0 \quad \forall \quad i = 1, \cdots, N-1.
\end{aligned}$$

Where w_i is the Dirac-delta function defined as:

$$w_i = \begin{cases} \hbar^{-1}, & i = j, \\ 0 & otherwise. \end{cases}$$

Since $L_{\varepsilon,x}u$ is inverse monotone then $\mathscr{G} \geq 0$ and $\mathscr{G}$ has the following bound in L_1 norm.

Theorem 4.6.1. *Let $\mathscr{G}$ be the green function corresponding with the discrete operator $L_{\varepsilon,x}$ then we have the following bound.*

$$\begin{aligned}
\|(a+b)\mathscr{G}_i\|_{1,\omega} &\leq 1, && (4.6.3)\\
\|\mathscr{G}_{\zeta,i,.}\|_{1,\omega} &\leq \frac{2(1+\varepsilon(\alpha+\beta)^{-1})^2}{\varepsilon(\alpha+\beta)}, && (4.6.4)\\
\|\mathscr{G}_{\zeta\zeta,i}\|_{1,\omega} &\leq 2\varepsilon^{-2}, \quad \forall \quad i = 1, \cdots, N-1. && (4.6.5)
\end{aligned}$$

Proof: For proof of the above theorem we refer [41, p. 193]

Let $\Theta = v_\varepsilon(x,t_n) - v_\varepsilon^h(x,t_n)$ be the point-wise error at each time level $t_n \; \forall \; n = 0, 1 \cdots, M$ and $x_i \in \omega$. The point-wise error can be written as:

$$|v_\varepsilon(x,t_n) - v_\varepsilon^h(x,t_n)| = |v_\varepsilon(x,t_n) - v_\varepsilon^I(x,t_n) + v_\varepsilon^I(x,t_n) - v_\varepsilon^h(x,t_n)| \tag{4.6.6}$$

and

$$v_\varepsilon(x,t_n) - v_\varepsilon^h(x,t_n) = \Theta_1 + \Theta_2 \tag{4.6.7}$$

where $\Theta_1 = v_\varepsilon(x,t_n) - v_\varepsilon^I(x,t_n)$ be the interpolation error and $\Theta_2 = v_\varepsilon^I(x,t_n) - v_\varepsilon^h(x,t_n)$ represents the discretization error. So in this section first we are going to prove interpolation error and then discretization error by using Green's function.

Theorem 4.6.2. *Let $v_\varepsilon^I(x,t_n)$ be the $\mathbb{V}(\bar{\omega}^N)$ linear interpolant of solution $v_\varepsilon(x,t)$ of* (2.6.1) *on* *mesh ω_N. Then we have the following bound:*

$$\|v_\varepsilon - v_\varepsilon^h\|_\infty \leq CN^{-2}(\ln N)^2, \tag{4.6.8}$$

where C is a constant independent of ε.

Proof. We first decomposes $v_\varepsilon(x,t)$ into its boundary layer and outer region component. Then, the interpolation error is evaluate for boundary layer and outer layer component separately. The interpolation error can be written as:

$$\begin{aligned} |\Theta_1| &= |v_\varepsilon(x,t_n) - v_\varepsilon^I(x,t_n)|, & (4.6.9)\\ &= |r(x,t_n) - r^I(x,t_n) + s(x,t_n) - s^I(x,t_n)|. & (4.6.10) \end{aligned}$$

First, we prove the error for regular region $r(x,t_n)$. Recalling theorem (2.3.4) on $\bar{\omega}_i$, and from [133, p. 80]

$$\|r - r^I\|_\infty \leq Ch_i^2 \max_{\omega_i} |r^{''}(x,t_n)|, \tag{4.6.11}$$

Here, there are two cases, $1st$ when $\frac{1}{4} \leq \frac{\varepsilon}{\alpha+\beta}\ln N$ and $2nd$ when $\frac{1}{4} \geq \frac{\varepsilon}{\alpha+\beta}\ln N$. So in case $1st$ results are straightforward as in this mesh becomes uniform and $\varepsilon^{-1} \leq C\ln N$. In second case the outer region lies, $0 \leq x_i \leq N/2$ and $\sigma = \frac{\varepsilon}{\alpha+\beta}\ln N$. Then we have the following bound:

$$\|r - r^I\|_\infty = \max_{i=1,\cdots,N} \max_{\omega_i} |r(x,t_n) - r^I(x,t_n)| \leq CN^{-2}(\ln N)^2, \tag{4.6.12}$$

Now it remains to prove the interpolation error for boundary layer (singular) component $s(x,t_n)$.

$$\begin{aligned} |s(x,t_n) - s^I(x,t_n)|_{\omega_i} &\leq \frac{1}{2}\left|\int_{x_{i-1}}^{x_i} s^{''}(x,t_n)(x_i - x)(x - x_{i-1})\mathrm{d}x\right|, & (4.6.13)\\ &\leq C\left|\int_{x_{i-1}}^{x_i} s^{''}(x,t_n)(x - x_{i-1})\mathrm{d}x\right| & (4.6.14)\\ &\leq C\left(\int_{x_{i-1}}^{x_i} |s^{''}(x,t_n)|^{1/2}\mathrm{d}x\right)^2. \\ &\leq C\left(\int_{x_{i-1}}^{x_i} |s^{''}(x,t_n)|^{1/2}\mathrm{d}x\right)^2. \\ &\leq CN^{-2}(\ln N)^2. \end{aligned}$$

Now taking maximum form $i = 1, \cdots, N$ of equation (4.6.15). We have the following bound

$$\|s - s^I\|_\infty = CN^{-2}(\ln N)^2 \tag{4.6.15}$$

Hence from equation (4.6.15) and (4.6.12) results is proved. □

Theorem 4.6.3. *Let $v_\varepsilon(x,t_n)$ be the solution of stationary problem* (2.6.1) *and let $v(x,t)$ be the numerical approximation of* (2.6.1) *then,*

$$\|v_\varepsilon^I - v^h\|_\omega \leq CN^{-2}(\ln N)^2. \tag{4.6.16}$$

Proof. Let

$$|\Theta|_\omega = |v_\varepsilon(x,t_n) - v_\varepsilon^I(x,t_n) + v_\varepsilon^I(x,t_n) - v_\varepsilon^h(x,t_n)|$$

by triangle inequality

$$\|\Theta\|_\omega \leq \|v_\varepsilon - v_\varepsilon^I\| + \|v_\varepsilon^I - v^h\|, \tag{4.6.17}$$

where $\Theta_1 = \|v_\varepsilon - v_\varepsilon^I\|_\omega$ is the interpolation error which is bounded in theorem (3.3.1) as

$$\|\Theta_1\|_\omega \leq CN^{-2}(\ln N)^2, \tag{4.6.18}$$

now to prove the above theorem it is remain to bound the term $\Theta_2 = |v_\varepsilon^I(x,t_n) - v_h(x,t_n)|$ and we define

$$\eta = \mathscr{F}^n - (a+b)v^n \tag{4.6.19}$$

then

$$\Theta_2 = (\eta^I - \eta, \mathscr{G}) - ((a+b)e)^I, \mathscr{G}) + \frac{2}{3}\int_0^1 ((a+b)e\mathscr{G})^I. \tag{4.6.20}$$

We use quadrature to evaluate the integral $(a+b)v_\varepsilon^n$ and $\mathscr{F}^n$. So $(a+b)v_\varepsilon^n$ and $\mathscr{F}^n$ are replaced by their interpolants η^I and $\eta = (a+b)v_\varepsilon^n - \mathscr{F}^n = \varepsilon(v_\varepsilon^n)_{xx}$. Hence $(\eta^I - \eta, \mathscr{G})$ is again the interpolation error [154] and can be bounded as

$$|(\eta^I - \eta, \mathscr{G})| \leq CN^{-2}(\ln N)^2, \tag{4.6.21}$$

and from equation (2.6.2)

$$\frac{2}{3}\int_0^1 (((a+b)e\mathscr{G})^I - ((a+b)e)^I, \mathscr{G}) \;=\; \frac{-1}{3}\sum \frac{h_i}{2}((a_k+b_k)e_k\mathscr{G}_k + (a_{k-1}+b_{k-1})e_{k-1}\mathscr{G}_{k-1}).$$

Suppose ρ be the arbitrary small and from equation (4.6.20), (4.6.21) and theorem (4.6.1).

$$\begin{aligned}\frac{2}{3}\int_0^1 ((a+b)e\mathscr{G})^I - ((a+b)e)^I, \mathscr{G}) &\le \left(\frac{1}{2} + \frac{Mh}{2(\alpha^2+\beta^2)}\right)\|e\|_\omega, \\ &\le \frac{(1+\rho)}{2}\|e\|_\omega. \end{aligned} \tag{4.6.22}$$

Hence from equation (4.6.18), (4.6.20) and (4.6.22)

$$|(\Theta_2)_i| \le CN^{-2}(\ln N)^2 + \frac{(1+\kappa)}{2}\|e\|. \tag{4.6.23}$$

Now we take maximum over $i = 1, \cdots, N-1$ in equation (4.6.23) to get the general error bound. Using theorem (3.3.1) and from equation (4.6.17), (4.6.20), (4.6.23).

$$\|v_\varepsilon - v_h\| \le CN^{-2}(\ln N)^2. \tag{4.6.24}$$

□

Theorem 4.6.4. *Let $v_\varepsilon(x,t)$ is exact solution of the continuous problem* (4.2.1)-(5.1.5) *and $V(x,t)$ is the finite element approximation from the space $\mathbb{V}(\bar{\omega}^N)$ of the exact solution $v_\varepsilon(x,t)$ and compatibility conditions* (2.3.1) *are satisfied at the corner points then the corresponding error is:*

$$\|v_\varepsilon - V\| \le [CN^{-2}(\ln N)^2 + \Delta t]. \tag{4.6.25}$$

4.7 Richardson Extrapolation

In this section we have implemented Richardson extrapolation to improve the rate of convergence of our discrete scheme in time direction. We assume the tensor product of meshes $\bar{Q}^{N,\Delta t}$ and $\bar{Q}^{N,\Delta t/2}$, which are piecewise uniform in spatial direction and uniform step size Δt and $\Delta t/2$ in time direction respectively.

$$\bar{Q}_0^{;\Delta t} = \bar{Q}^{N,\Delta t} \cap \bar{Q}^{N,\Delta t/2} \tag{4.7.1}$$

Let $V^1(x,t) \in \bar{Q}^{N,\Delta t}$ and $V^2(x,t) \in \bar{Q}^{N,\Delta t/2}$ be the solution of scheme (4.2.1). Then we define

$$V^{EXT}(x,t) = 2V^2 - V^1,(x,t) \qquad (x,t) \in \bar{Q}^{N,\Delta t}. \tag{4.7.2}$$

Here V^{EXT} denotes the Richardson extrapolation approximation of V which has an improved rate of convergence in time direction. Let $V^k(x,t)$ for all $(x,t) \in \bar{Q}^{N,\Delta t/k}$, $k = 1,2$ and the expansion of $V^k(x,t)$ is written as:

$$V^k(x,t) = V + 2^{-(k-1)}\Delta t\varsigma(x,t) + \mathbf{R}^k(x,t) \tag{4.7.3}$$

where ς is the residual term which is the solution of the problem:

$$\left(L_{\varepsilon,x} + \frac{\partial}{\partial t}\right)\varsigma(x,t) = 2^{-1}\frac{\partial^2}{\partial t^2}v(x,t) \tag{4.7.4}$$

$\varsigma(x,t) = 0$ for all $(x,t) \in \partial Q$.

Now in order to derive the convergence analysis of the Richardson technique, we have to derive the convergence analysis of the reminder term $\mathbf{R}^k(x,t)$ on $\bar{Q}^{N,\Delta t/k}$ for $k = 1,2$. Now

$$\left(L_{\varepsilon,x} + \frac{\partial}{\partial t}\right)\mathbf{R}^k(x,t) = \left(L_{\varepsilon,x} + \frac{\partial}{\partial t}\right)(V^k - v) - 2^{-(k-1)}\left(L_{\varepsilon,x} + \frac{\partial}{\partial t}\right)\varsigma(x,t) \tag{4.7.5}$$

Hence,

$$\mathbf{R}^k(x,t) \leq [CN^{-2}(\ln N)^2 + \Delta t^2], \tag{4.7.6}$$

the above estimate yield the error bound of Richardson scheme.

Theorem 4.7.1. *Let $v_\varepsilon(x,t_n)$ is exact solution of the continuous problem* (4.2.1)-(5.1.5) *and $V^{EXT}(x,t_n)$ is the finite element approximation from the space $\mathbb{V}(\bar{\omega}^N)$ of the exact solution $v_\varepsilon(x,t)$ and compatibility conditions* (2.3.1) *are satisfied at the corner points then the corresponding error is:*

$$\|v_\varepsilon - V^{EXT}\| \leq [CN^{-2}(\ln N)^2 + \Delta t^2]. \tag{4.7.7}$$

4.8 Numerical Experiment and Discussion

In this section, to examine the efficiency of our proposed method few numerical experiment is carried out. Let $\eta_{\varepsilon}^{N,M}$ and $O(N)$ denote the maximum point-wise error and rate of convergence (ROC), respectively.

$$\eta_{\varepsilon}^{N,M} = \max_{(x_j,t_i)\in Q_{\varepsilon}^{(N,M)}} \|v_{\varepsilon} - V\|, \qquad O(N) = \log_2\left(\frac{\eta_{\varepsilon}^{N,M}}{\eta_{\varepsilon}^{2N,4M}}\right),$$

where $V(x_j,t_i)$ and $v_{\varepsilon}(x_j,t_i)$ denotes the numerical and exact solution of SPPRD problem with retarded argument (4.2.1) respectively with M mesh points in temporal direction with equidistant time-step Δt and N mesh points in spatial direction.

Example 4.8.1. *Consider the following example of SPPRD problem with retarded argument.*

$$\begin{cases} \text{“}(v_{\varepsilon})_t(x,t) - \varepsilon(v_{\varepsilon})_{xx}(x,t) + v_{\varepsilon}(x,t) = v_{\varepsilon}(x,t-1) + f(x,t), & (x,t) \in \omega \times (0,2], \\ v_{\varepsilon}(x,t) = e^{-t}\left(\dfrac{e^{-\frac{x}{\sqrt{\varepsilon}}} + e^{-\frac{1-x}{\sqrt{\varepsilon}}}}{e^{-\frac{1}{\sqrt{\varepsilon}}}+1} - \cos^2(\pi x)\right), & (x,t) \in [0,1]\times[-1,0], \\ v_{\varepsilon}(0,t) = 0, \quad v_{\varepsilon}(1,t) = 0, \quad t \in [0,2]\text{”} \end{cases} \tag{4.8.1}$$

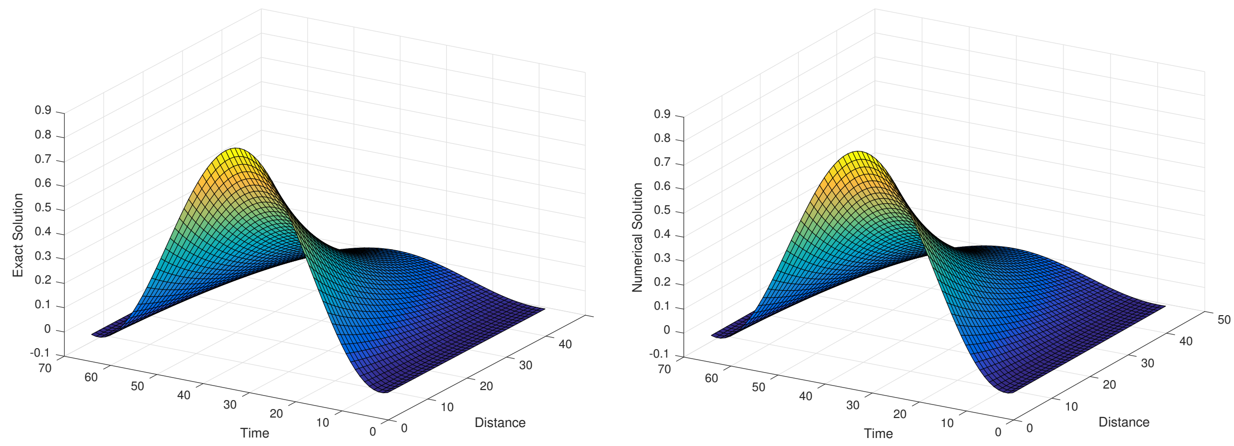
Where the exact solution of the above problem is

$$v_{\varepsilon}(x,t) = e^{-t}\left(\frac{e^{-\frac{x}{\sqrt{\varepsilon}}} + e^{-\frac{1-x}{\sqrt{\varepsilon}}}}{e^{-\frac{1}{\sqrt{\varepsilon}}}+1} - \cos^2(\pi x)\right) \tag{4.8.2}$$

The maximum point-wise error $\eta_{\varepsilon}^{N,M}$ and the ROC $O(N)$ have been calculated by the proposed scheme for example (5.5.1) and are given in table (2.1) and (4.3) respectively. As we analyse the numerical results given in table (4.1) and (4.2) for example (5.5.1). It is observed that the proposed scheme is parameter-uniform convergent.

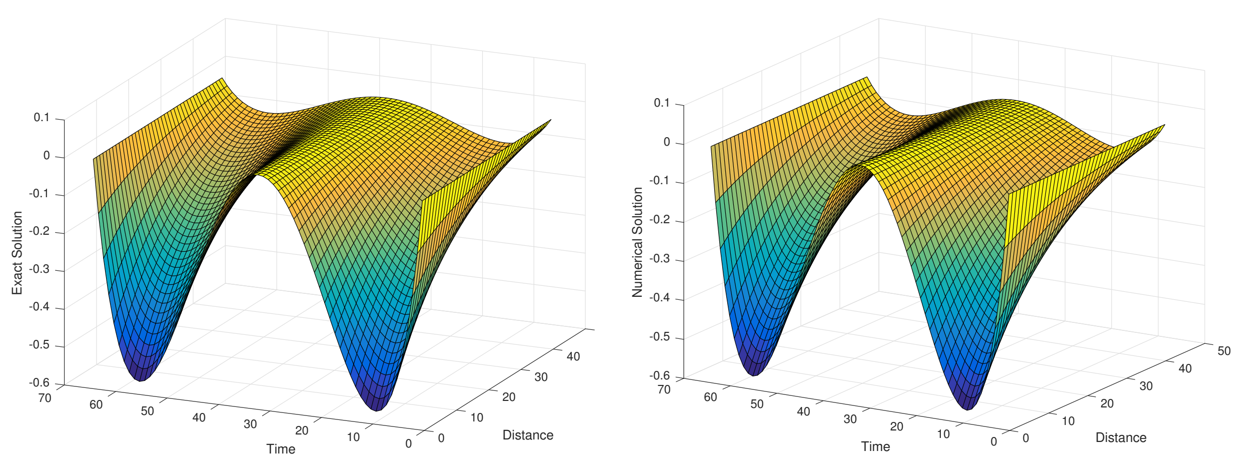
4.9 Conclusion

In this chapter, we have successfully implemented Galerkin FEM with Richardson extrapolation for numerically approximating the SPPRD problem with retarded argument.



(a) For $\varepsilon = 10^{0}$ (b) For $\varepsilon = 10^{0}$

Figure 4.1: "Comparison between exact and numerical solution using finite element method of Example (5.5.1) for $M = 40$ and $N = 64$."



(a) For $\varepsilon = 10^{-2}$ (b) For $\varepsilon = 10^{-2}$

Figure 4.2: "Comparison between exact and numerical solution using finite element method of Example (5.5.1) for $M = 40$ and $N = 64$."

Number of N mesh points/Number of M mesh points.			
ε	$32/20$	$64/40$	$128/80$
2^{0}	$1.449750e-03$	$1.021228e-03$	$5.917450e-04$
2^{-2}	$3.473786e-03$	$1.969669e-03$	$1.046272e-03$
2^{-4}	$1.452867e-03$	$8.433870e-04$	$4.523180e-04$
2^{-6}	$5.964408e-03$	$3.026289e-03$	$1.523524e-03$
2^{-8}	$1.245321e-02$	$6.358727e-03$	$3.207173e-03$
2^{-10}	$1.753827e-02$	$8.963481e-03$	$4.519927e-03$
2^{-12}	$2.906274e-02$	$1.388118e-02$	$6.974805e-03$
2^{-14}	$3.174558e-02$	$1.545738e-02$	$7.579147e-03$
2^{-16}	$3.069788e-02$	$1.490232e-02$	$7.222957e-03$

Table 4.1: "Maximum point-wise error $\eta_{\varepsilon}^{N,M}$ obtained using finite element method for Example (5.5.1) before Richardson extrapolation."

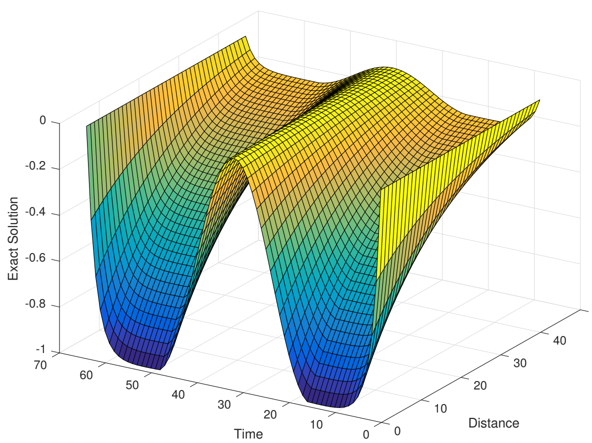
The error analysis is carried out in the discrete maximum norm. The proposed method is shown to be accurate of order $(O(N^{-1}\ln N)^2 + \Delta t^2)$ in maximum norm. The obtained numerical result shows the robustness of the proposed method and validates the theoretical finding.

Number of N mesh points/Number of M mesh points.			
ε	32/20	64/40	128/80
2^0	1.058027$e-03$	2.643077$e-04$	6.642837$e-05$
2^{-2}	6.394781$e-04$	1.538965$e-04$	3.903915$e-05$
2^{-4}	4.672427$e-04$	1.166427$e-04$	2.912724$e-05$
2^{-6}	4.291767$e-04$	1.072185$e-04$	2.676694$e-05$
2^{-8}	1.134271$e-03$	2.883891$e-04$	7.257301$e-05$
2^{-10}	3.898567$e-03$	1.018128$e-03$	2.560283$e-04$
2^{-12}	3.992687$e-02$	1.136948$e-02$	3.120726$e-03$
2^{-14}	4.198006$e-02$	1.170268$e-02$	3.203880$e-03$
2^{-16}	4.245185$e-02$	1.196698$e-02$	3.292116$e-03$

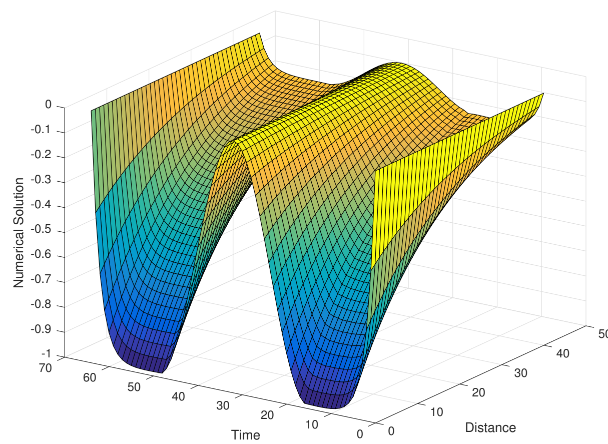
Table 4.2: "Maximum point-wise error $\eta_{\varepsilon}^{N,M}$ obtained using finite element method for Example (5.5.1) after Richardson extrapolation."

Number of N mesh points/Number of M mesh points.			
ε	32/10	64/40	128/160
2^0	2.0011	1.9923	1.9956
2^{-2}	2.0549	1.9790	1.9826
2^{-4}	2.0021	2.0017	2.0012
2^{-6}	2.0010	2.0020	2.0014
2^{-8}	1.9757	1.9905	1.9956
2^{-10}	1.9370	1.9915	1.9881
2^{-12}	1.8122	1.8652	1.8812
2^{-14}	1.8429	1.8689	1.8749
2^{-16}	1.8268	1.8620	1.8631

Table 4.3: "ROC for solution of Example (5.5.1)"



(a) For $\varepsilon = 10^{-5}$



(b) For $\varepsilon = 10^{-5}$

Figure 4.3: “Comparison between exact and numerical solution using finite element method of Example (5.5.1) for $M = 40$ and $N = 64$.”

Chapter 5

An Iterative Analytic Approximation of Non-Linear Singularly Perturbed Parabolic Partial Differential Equations

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[1] *In this chapter a closed-form iterative analytic approximation to a class of nonlinear singularly perturbed parabolic partial differential equation is developed and analysed for convergence. We have considered both parabolic reaction diffusion and parabolic convection diffusion type of problems in this chapter. The solution of this class of problem is polluted by a small dissipative parameter, due to which solution often shows boundary and interior layers. A sequence of approximate analytic solution for the above class of problems is constructed using Lagrange multiplier approach. Within a general frame work, the Lagrange multiplier is optimally obtained using variational theory and Liouville-Green's transformation. The sequence of approximate analytic solutions so obtained is proved to converge the exact solution of the problem.*

---

[1]This work is .. " Kartikay Khari and Vivek Kumar, An Iterative Analytic Approximation for a Class of Non-Linear Singularly Perturbed Parabolic Partial Differential Equations (Communicated)".

The nonlinear singularly perturbed parabolic partial differential equations (NSPPPDEs) plays an essential role in converting a real life phenomenon into a mathematical model. The dynamics of NSPPPDEs are utterly different from the conventional nonlinear partial differential equations. These types of problems depend on a small positive parameter, which makes the solution varies rapidly in narrow regions of the domain and changes slowly in rest parts of the domain. This behaviour of the solution in narrow regions is called layer phenomena and this class of problem is known as singularly perturbed problems. The problem is singular in such a way that when $\varepsilon \to 0$ changes the order of the differential equation, but the number boundary conditions remain the same, causing the problem ill-posed. This type of problems occur in oceanography, population dynamic, generic repression, size dependent cell growth, division modelling, ecology, quantum physics, chemistry, finance(Black-Schole Equation) and material science.

The designing of computational algorithms for such type of problems burden with difficulties because the solution of the proposed problem is contaminated by a small positive parameter ε and nonlinear term simultaneously. Since only few nonlinear systems can be solved explicitly, we rely on numerical techniques by linearizing the nonlinear problems. To best of our knowledge, till date there is no analytic method (exact or approximate) is developed to solve the NSPPPDEs. The novelty of this chapter is the development of an iterative analytic method based on variation of iteration approach to find the approximate solution of the NSPPPDEs. The Variation of iteration method (VIM) was first developed by J. H. He [104] in 1999. The VIM is based on a Lagrange's multiplier developed by Inokuti *et al.* [116] in which they constructed adjoint operators and state that Lagrange multiplier λ is regarded as a Green's function rather then a constant. In [257] J. L. Ramos proves that the claim made by Inokuti *et al.* are indeed correct.

We can drive the VIM method by using Green's function, method of weighted residuals and using integration by parts. The best property of VIM method is to drive a closed form solution of a nonlinear problem without any need of linearization and discretization. Due to the flexibility and ability, the VIM has emerged as a promising tool to solve non-linear or complex systems. Consequently, this method was successfully implimented to find the approximate solution of Hilfer fractional advection diffusion equations with power law initial condition [5], of reaction diffusion system with fast reversible reaction [4], of wave-like and heat-like equations in large domains [89], of delay differential equations [103], of nonlinear singular boundary value problems [150], of the time fractional Fornberg-Whitham equation [266], of Burger's flow with fractional derivatives [302], of non-isothermal reaction diffusion model equations in a spherical catalyst [158], of Optical bright and dark soliton solutions for coupled nonlinear Schrodinger (CNLS) equations [45], to Bratu-like equation arising in electrospinningand [105], to free vibration analysis of a tapered beam mounted on two-degree of freedom subsystems [196].

The chapter is organized as follows: The continuous model problem is defined in Section 5.1. In Section 5.2, a brief sketch of the method is given. Section 5.3, contains the solution methodology and optimal value of Lagrange multiplier is evaluated. In Section 5.4 the convergence analysis is carried out . In Section 5.5, one linear and one nonlinear test problems are taken into account to validate theoretical results obtained in section 5.6 Finally, Section 5.7, contains a conclusion.

5.1 Statement of Problem

5.1.1 Parabolic Reaction Diffusion Problem

Consider the following class of nonlinear singularly perturbed parabolic problem of reaction diffusion type.

$$\text{“}(v_\varepsilon)_t(x,t) - \varepsilon (v_\varepsilon)_{xx}(x,t) + v_\varepsilon(x,t) - f(x,t,v) = 0, \tag{5.1.1}$$

where $(x,t) \in (0,1) \times (0,1)$, subject to the initial condition and boundary conditions prescribed as:

$$\begin{cases} v_\varepsilon(x,0) = & \psi_b(x), \\ v_\varepsilon(0,t) = & \psi_l(t), \\ v_\varepsilon(1,t) = & \psi_r(t)\text{”}. \end{cases} \tag{5.1.2}$$

Where $0 < \varepsilon \ll 1$ is a small positive parameter known as singular perturbation parameter. Let $f(x,t,v)$ be sufficiently smooth and ρ ,$\bar{\rho}$ be some constant satisfies,

$$0 \leq \rho^2 \leq f_v(x,t,v) \leq \bar{\rho}^2 \quad \forall (x,t,v) \in [0,1] \times [0,1] \times \mathbf{R} \tag{5.1.3}$$

5.1.2 Parabolic Convection Diffusion Problem

Consider the following class of singularly perturbed parabolic convection dominated problem.

$$\text{“}(v_\varepsilon)_t(x,t) - \varepsilon(v_\varepsilon)_{xx}(x,t) + (v_\varepsilon)_x(x,t) - f(x,t,v) = 0, \tag{5.1.4}$$

where $(x,t) \in (0,1) \times (0,1)$, subject to the initial condition and boundary conditions prescribed as:

$$\begin{cases} v_\varepsilon(x,0) = & \psi_b(x), \\ v_\varepsilon(0,t) = & \psi_l(t), \\ v_\varepsilon(1,t) = & \psi_r(t)\text{”}. \end{cases} \tag{5.1.5}$$

Where $0 < \varepsilon \ll 1$ is a small positive parameter known as singular perturbation parameter. Let $f(x,t,v)$ be sufficiently smooth and ρ ,$\bar{\rho}$ be some constant satisfies,

$$0 \leq \rho^2 \leq f_v(x,t,v) \leq \bar{\rho}^2 \quad \forall (x,t,v) \in [0,1] \times [0,1] \times \mathbf{R} \tag{5.1.6}$$

5.2 Variation of Iteration Method

In this section, we have given a brief sketch of variation of iteration method. Consider a general class non-linear parabolic partial differential equation.

$$Dv(x,t) = \mathscr{L}(v(x,t)) + \mathscr{N}(v(x,t)) = \mathscr{F}. \tag{5.2.1}$$

Where $\mathscr{L}$ and $\mathscr{N}$ are the linear and non-linear operator respectively and $\mathscr{F}(x,t)$ be the force term or inhomogeneous term. Then the correction functional associated with (5.2.1) is defined as:

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(s)\{\mathscr{L}v_n(x,s) + \mathscr{N}\tilde{v_n}(x,s) - \mathscr{F}(x,s)\}ds \qquad n \geq 0. \tag{5.2.2}$$

Where λ be the general Lagrange's multiplier, which is evaluated optimally [104] by using variational theory and Liouville-Green transformation. Furthermore, v_n denotes the nth iterative approximation of v and $\tilde{v}_n$ denotes the restricted variation i.e $\delta\tilde{v_n} = 0$. After evaluating the Lagrange's multiplier λ and choosing an appropriate initial condition v_0, and using (5.2.2), one obtain the successive approximations v_n of solution $v(x,t)$. Hence the exact solution of the problems (5.2.1) is obtained as

$$v(x,t) = \lim_{n\to\infty} v_n(x,t).$$

5.3 Solution Methodology

5.3.1 Lagrange's Multiplier for Parabolic Reaction Diffusion Problem

Taking the correction functional in $t-$direction. The problem (5.1.1) can be expressed as:

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(s)\left(\frac{\partial v_n(x,s)}{\partial s} - \varepsilon\frac{\partial^2 v_n(x,s)}{\partial x^2} + f_n(x,s,v)\right)ds, n \geq 0. \tag{5.3.1}$$

Now let $\delta\tilde{v_n}$ is the restricted variation and the restricted variation of nonlinear source term is denoted by $\tilde{f}_n(x,s,v)$.

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(s)\left(\frac{\partial v_n(x,s)}{\partial s} - \varepsilon\frac{\partial^2 \tilde{v_n}(x,s)}{\partial x^2} + \tilde{f}_n(x,s,v)\right)ds. \tag{5.3.2}$$

Using integration by part and restricting the restricted variation term,

$$v_{n+1}(x,t) = v_n(x,t) + [\lambda(s)v_n(x,s)]_{s=0}^t - \left[\frac{d\lambda(s)}{ds}v_n(x,s)\right]_{s=0}^t. \tag{5.3.3}$$

We recall the variational theory for evaluating Lagrange's multiplier optimally and applying variation corresponding to v_n and making correction functional stationary i.e. $\delta v_{n+1} = 0$.

$$\delta v_{n+1}(x,t) = \delta v_n(x,t) + \lambda(s)\delta v_n(x,t) - \frac{d\lambda(s)}{ds}\delta v_n(x,t) = 0. \tag{5.3.4}$$

We obtain the following Euler-Lagrange equation and stationary conditions as:

$$\left[\frac{d\lambda(s)}{ds}\right]_{s=t} = 0; \qquad [1-\lambda(s)]_{s=t} = 0. \tag{5.3.5}$$

From stationary condition (5.3.5) we obtain the optimal value of Lagrange's multiplier

$$\lambda(s) = -1 \tag{5.3.6}$$

Now the iteration formulae is written as

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,s)}{\partial s} - \varepsilon \frac{\partial^2 v_n(x,s)}{\partial x^2} + f_n(x,s,v) \right) ds. \tag{5.3.7}$$

5.3.2 Lagrange's Multiplier for Parabolic Convection Diffusion Problem

Taking the correction functional in $t-$direction. The problem (5.1.4) can be expressed as:

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(s) \left(\frac{\partial v_n(x,s)}{\partial s} - \varepsilon \frac{\partial^2 v_n(x,s)}{\partial x^2} + \frac{\partial v_n(x,s)}{\partial x} + f_n(x,s,v) \right) ds, n \geq 0. \tag{5.3.8}$$

Now let $\delta \tilde{v_n}$ is the restricted variation and the restricted variation of nonlinear source term is denoted by $\tilde{f_n}(x,s,v)$.

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(s) \left(\frac{\partial v_n(x,s)}{\partial s} - \varepsilon \frac{\partial^2 \tilde{v_n}(x,s)}{\partial x^2} + \frac{\partial \tilde{v_n}(x,s)}{\partial x} + \tilde{f_n}(x,s,v) \right) ds. \tag{5.3.9}$$

Using integration by part and restricting the restricted variation term,

$$v_{n+1}(x,t) = v_n(x,t) + [\lambda(s) v_n(x,s)]_{s=0}^t - \left[\frac{d\lambda(s)}{ds} v_n(x,s) \right]_{s=0}^t. \tag{5.3.10}$$

We recall the variational theory for evaluating Lagrange's multiplier optimally and applying variation corresponding to v_n and making correction functional stationary i.e. $\delta v_{n+1} = 0$.

$$\delta v_{n+1}(x,t) = \delta v_n(x,t) + \lambda(s) \delta v_n(x,t) - \frac{d\lambda(s)}{ds} \delta v_n(x,t) = 0. \tag{5.3.11}$$

We obtain the following Euler-Lagrange equation and stationary conditions as:

$$\left[\frac{d\lambda(s)}{ds} \right]_{s=t} = 0; \qquad [1 - \lambda(s)]_{s=t} = 0. \tag{5.3.12}$$

From stationary condition (5.3.12) we obtain the optimal value of Lagrange's multiplier

$$\lambda(s) = -1 \tag{5.3.13}$$

Now the iteration formulae is written as

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,s)}{\partial s} - \varepsilon \frac{\partial^2 v_n(x,s)}{\partial x^2} + \frac{\partial v_n(x,s)}{\partial x} + f_n(x,s,v) \right) ds. \tag{5.3.14}$$

5.4 Convergence

Theorem 5.4.1. *Let us suppose that χ be a Banach space such that $\boldsymbol{A} : \chi \to \chi$ be the non-linear mapping and suppose*

$$\|\boldsymbol{A}[v] - \boldsymbol{A}[\bar{v}]\| \leq \rho \|v - \bar{v}\|, \qquad v, \bar{v} \in \chi \tag{5.4.1}$$

then $\boldsymbol{A}$ has a unique fixed point for $\rho < 1$. Moreover

$$v_{n+1} = \boldsymbol{A}[v_n] \tag{5.4.2}$$

Sequence converges to a fixed point $\boldsymbol{A}$ for any arbitrary choice of $v_0 \in \chi$,

$$\|v_k - v_l\| \leq \|v_1 - v_0\| \sum_{j=l-1}^{k-2} \rho^j \tag{5.4.3}$$

Proof. See [233] □

The above iteration formulae (5.3.7) generates a sequence $< v_n(x,t) >$. Now we construct the series from variational iteration formulae (5.3.7) as,

$$v_0(x,t) + [v_1(x,t) - v_0(x,t)] + [v_2(x,t) - v_1(x,t)] + \cdots + [v_n(x,t) - v_{n-1}(x,t)] + \cdots . \tag{5.4.4}$$

Now let

$$S_{n+1}(x,t) = v_0(x,t) + [v_1(x,t) - v_0(x,t)] + [v_2(x,t) - v_1(x,t)] + \cdots + [v_n(x,t) - v_{n-1}(x,t)] = v_n(x,t), \tag{5.4.5}$$

then we have the following bounds.

Theorem 5.4.2. *Let $v_n(x,t) \in [0,1]$ be the sequence generated by* (5.4.4) *with $v_0 = q(x)c$, where $q(x) \in C^2[0,1]$ bounded in interval $[0,1]$ and c is an arbitrary constant. Let the optimal value of Lagrange's multiplier corresponding to the stationary conditions* (5.3.5) *belongs to $C^\infty[0,1]$. Then the sequence v_n convergences to the exact solution to $v(x,t)$ of the problem.*

Proof.

$$
\begin{aligned}
|v_1(x,t) - v_0(x,t)| &= |\int_0^t \lambda(s)\left(\frac{\partial v_0(x,s)}{\partial s} - \varepsilon\frac{\partial^2 v_0(x,s)}{\partial x^2} + \tilde{f}_n(x,s,v_0)\right)ds|, && (5.4.6)\\
&\le \int_0^t |\lambda(s)|\left(\|\frac{\partial v_0(x,s)}{\partial s} - \varepsilon\frac{\partial^2 v_0(x,s)}{\partial x^2}\| + \|\tilde{f}_n(x,s,v_0)\|\right)ds, && (5.4.7)\\
&\le \int_0^t |A_i|\left(\|B_i\|_\infty + \|C_i\|_\infty\right)ds && (5.4.8)\\
&\le M\int_0^t ds, && (5.4.9)\\
&\le Mt. && (5.4.10)
\end{aligned}
$$

Where $|A_i| = \max_{s\in[0,1]}|\lambda(s)|$; $\|B_i\|_\infty \le \max_i |B_i|$ and $\|C_i\|_\infty \le \max_i |C_i|$,

$$
M = \max\{|A_i|\left(\|B_i\|_\infty + \|C_i\|_\infty\right), \|B_i\|_\infty\}. \qquad (5.4.11)
$$

Similarly ,

$$
\begin{aligned}
|v_2(x,t) - v_1(x,t)| &= |\int_0^t \lambda(s)\left(\frac{\partial v_1(x,s)}{\partial s} - \varepsilon\frac{\partial^2 v_1(x,s)}{\partial x^2} + \tilde{f}(x,s,v_1)\right)ds|, && (5.4.12)\\
&\le \int_0^t |\lambda(s)|\left(\|\frac{\partial v_1(x,s)}{\partial s} - \varepsilon\frac{\partial^2 v_1(x,s)}{\partial x^2}\| + \|\tilde{f}(x,s,v_1)\|\right)ds, && (5.4.13)\\
&\le \int_0^t |A_i|\left(\|B_i\|_\infty + \|C_i\|_\infty\right)ds, && (5.4.14)\\
&\le M^2\int_0^t s\,ds, && (5.4.15)\\
&\le \frac{M^2t^2}{2!}. && (5.4.16)
\end{aligned}
$$

By mathematical induction,

$$|v_{n+1}(x,t)-v_n(x,t)| = |\int_0^t \lambda(s)\left(\frac{\partial v_n(x,s)}{\partial s}-\varepsilon\frac{\partial^2 v_n(x,s)}{\partial x^2}+\tilde{f}(x,s,v_n)\right)ds|, \quad (5.4.17)$$

$$\leq \int_0^t |\lambda(s)|\left(\|\frac{\partial v_n(x,s)}{\partial s}-\varepsilon\frac{\partial^2 v_n(x,s)}{\partial x^2}\|+\|\tilde{f}(x,s,v_n)\|\right)ds, \quad (5.4.18)$$

$$\leq \int_0^t |A_i|\left(\|B_i\|_\infty+\|C_i\|_\infty\right)ds, \quad (5.4.19)$$

$$\leq M^{n+1}\int_0^t s^n ds, \quad (5.4.20)$$

$$\leq \frac{M^{n+1}t^{n+1}}{(n+1)!}. \quad (5.4.21)$$

Hence, for $t\in[0,1]$ the series $\frac{M^n t^n}{n!}$ is uniformly convergent and converges to 0 as $n\to\infty$. Therefore we conclude that the sequence $< v_n >$ is uniformly convergent and converges to the exact solution $v(x,t)$. From above convergence analysis it is concluded that the proposed method is parameter uniform convergent i.e independent of singular perturbation parameter ε. □

Theorem 5.4.3. *Let $v_n(x,t)\in[0,1]$ be the sequence generated by* (5.3.14) *with $v_0 = q_1(x)c_1$, where $q_1(x)\in C^2[0,1]$ bounded in interval $[0,1]$ and c_1 is an arbitrary constant. Let the optimal value of Lagrange's multiplier corresponding to the stationary conditions* (5.3.12) *belongs to $C^\infty[0,1]$. Then the sequence v_n convergences to the exact solution to $v(x,t)$ of the problem.*

Proof. The proof is similar to the theorem (5.4.2). □

5.5 Numerical Discussion

In this section, two linear and one non-linear reaction-diffusion test problems are considered, as well as one linear convection reaction diffusion kind of problem to validate the theoretical finding and evaluate the mentioned method's efficiency. The maximum absolute error is calculated by using the following:

$$\eta = \|v-v_n\| \quad (5.5.1)$$

The approximate solution at the nth iteration is symbolized by vn. We define

$$\hat{\eta} = \max_{x=0.1,0.2\cdots,0.9}\|\eta\| \quad (5.5.2)$$

The approximate solution obtained by proposed method for example (5.5.1) and example (5.5.2) with $\varepsilon = 2^{-4}$ and 2^{-6} at 10^{th} and 4^{th} iteration are plotted in Fig. (5.1) and Fig. (5.5) respectively, which coincide with the exact solutions plotted in Fig. (5.2) and Fig. (5.6), respectively. The approximate solution computed by proposed method for example (5.5.1) and example (5.5.2) with $\varepsilon = 2^{-10}$ at 10^{th} and 4^{th} iteration are plotted in Fig. (5.3) and Fig. (5.7) respectively, which coincide with the exact solutions plotted in Fig. (5.4) and Fig. (5.8), respectively. From figure (5.1) and (5.3), it is clear that the solution exhibits the boundary layers at the both boundaries, which corroborate the theoretical property of the considered test problem (5.5.1). The example (5.5.2) is a time depend second order differential equation, therefore the basic theory of singularly perturbed problem says that there should be two boundary layers at both the boundaries of spatial domain but contrary the graphs of the solution of the examples (5.5.2) in figure (5.5) and (5.8) shows one boundary layer at $x = 0$. This is happened because the exact solution of the example (5.5.2) $v(x,t) = e^{-t-\frac{x}{\sqrt{\varepsilon}}}$ satisfies the boundary condition $v(1,t) = e^{\left(-t-\frac{1}{\sqrt{\varepsilon}}\right)}$ at $x = 1$.

The computed results are tabulated in table (5.1) for example (5.5.1) and in table (5.2) for example (5.5.2). Therefore, from table (5.1) and (5.2) we observe that after further increases the number of iteration the error term decreases. Hence it is concluded that the proposed method is convergent and converges to the exact solution.

5.5.1 Reaction-diffusion problem

Example 5.5.1. *"Consider the following linear parabolic singularly perturbed reaction diffusion problem.*

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} - \varepsilon \frac{\partial^2 v(x,t)}{\partial x^2} + v(x,t) = f(x,t), \\ v(x,0) = \left(\dfrac{e^{-\frac{x}{\sqrt{\varepsilon}}} + e^{-\frac{1-x}{\sqrt{\varepsilon}}}}{e^{-\frac{1}{\sqrt{\varepsilon}}} + 1} - \cos^2(\pi x) \right), \quad x \in [0,1], \\ v(0,t) = 0, \quad v(1,t) = 0, \quad t \in [0,1] \end{cases} \tag{5.5.3}$$

with exact solution

$$v(x,t) = e^{-t}\left(\frac{e^{-\frac{x}{\sqrt{\varepsilon}}}+e^{-\frac{1-x}{\sqrt{\varepsilon}}}}{e^{-\frac{1}{\sqrt{\varepsilon}}}+1} - \cos^2(\pi x)\right).” \tag{5.5.4}$$

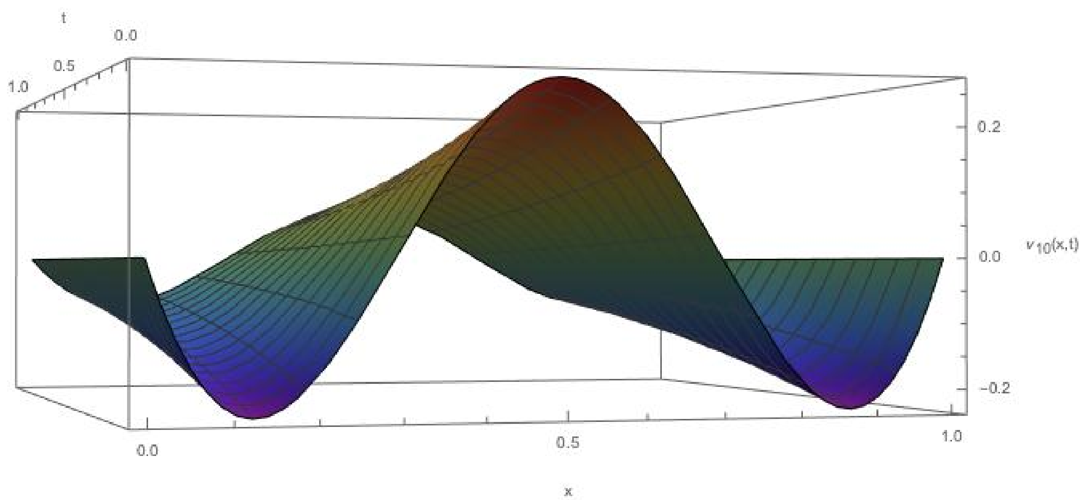

Figure 5.1: “Solution Obtained by VIM Method at 10^{th} iteration for $\varepsilon = 2^{-4}$ for Example (5.5.1)."

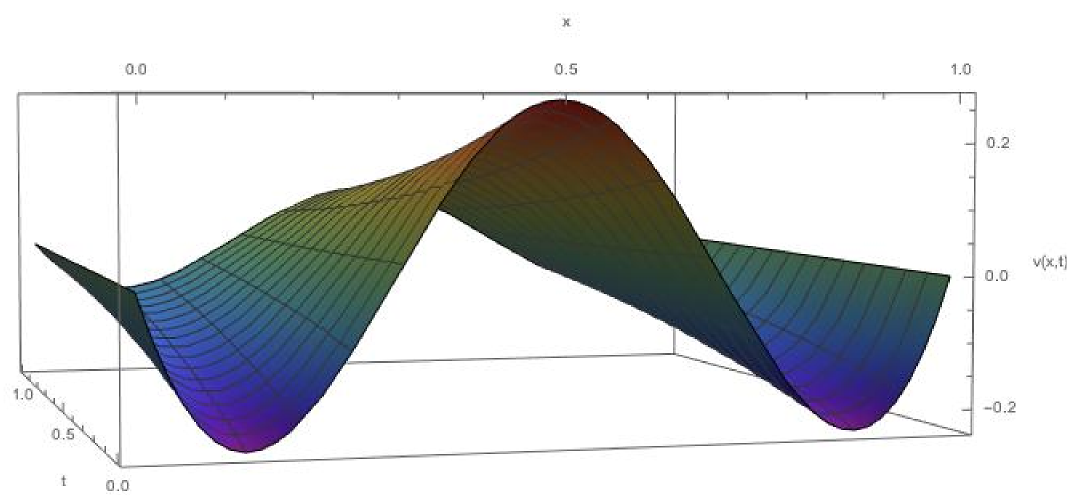

Figure 5.2: “Exact Solution for $\varepsilon = 2^{-4}$ for Example (5.5.1)."

Example 5.5.2. *“Consider the following non-linear parabolic singularly perturbed reaction dif-*

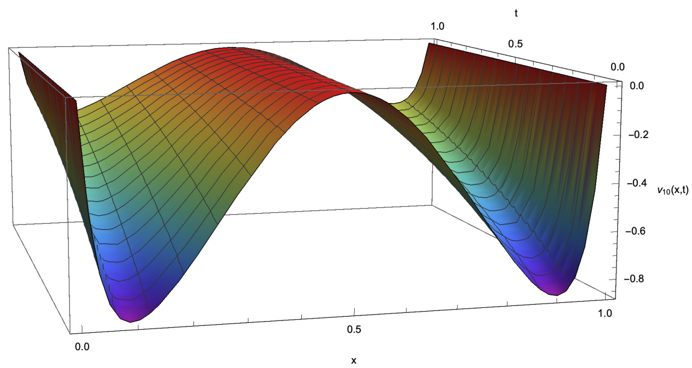

Figure 5.3: “Solution Obtained by VIM Method at 10^{th} iteration for $\varepsilon = 2^{-10}$ for Example (5.5.1)."

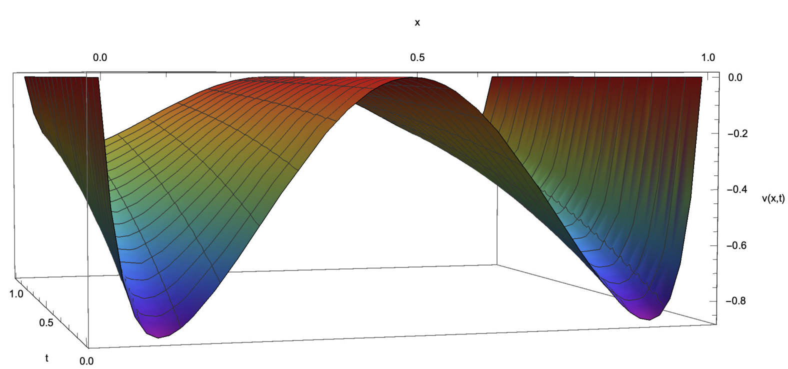

Figure 5.4: “Exact Solution for $\varepsilon = 2^{-10}$ for Example (5.5.1)."

t	$\varepsilon = 2^{-4}$	$\varepsilon = 2^{-6}$	$\varepsilon = 2^{-8}$	$\varepsilon = 2^{-12}$	$\varepsilon = 2^{-16}$
0.2	2.6148×10^{-12}	2.1399×10^{-13}	1.3250×10^{-13}	3.1572×10^{-15}	1.2101×10^{-14}
0.4	6.8904×10^{-10}	4.6791×10^{-10}	2.0960×10^{-10}	8.5420×10^{-13}	9.40137×10^{-13}
0.6	5.9053×10^{-8}	3.9830×10^{-8}	1.7842×10^{-8}	7.2197×10^{-11}	7.8050×10^{-11}
0.8	1.3761×10^{-6}	9.2819×10^{-7}	4.1580×10^{-7}	1.6825×10^{-9}	1.8193×10^{-9}

Table 5.1: “Maximum absolute error $\hat{\eta}$ of proposed method for Example (5.5.1) solved for fifth iteration v_5."

fusion equation.

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} - \varepsilon \frac{\partial^2 v(x,t)}{\partial x^2} + v(x,t) + (v(x,t))^2 = e^{-2t-\frac{2x}{\sqrt{\varepsilon}}} - e^{-t-\frac{x}{\sqrt{\varepsilon}}}, \\ v(x,0) = e^{\left(-\frac{x}{\sqrt{\varepsilon}}\right)}, \qquad x \in [0,1], \\ v(0,t) = e^{(-t)}, \quad v(1,t) = e^{\left(-t-\frac{1}{\sqrt{\varepsilon}}\right)}, \quad t \in [0,1] \end{cases} \tag{5.5.5}$$

with exact solution

$$v(x,t) = e^{-t-\frac{x}{\sqrt{\varepsilon}}}.” \tag{5.5.6}$$

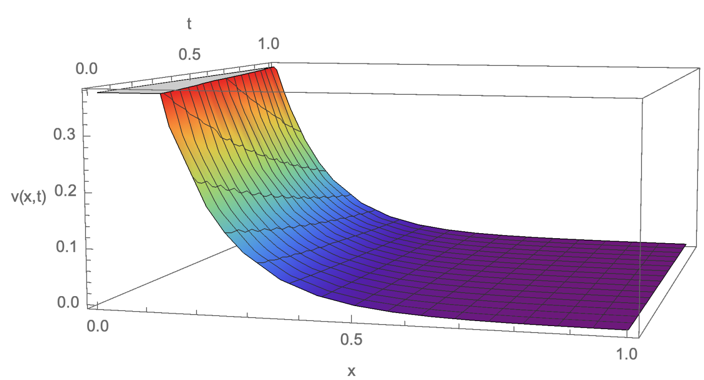

Figure 5.5: “Exact Solution $v(x,t)$ for $\varepsilon = 2^{-6}$ for Example (5.5.2)."

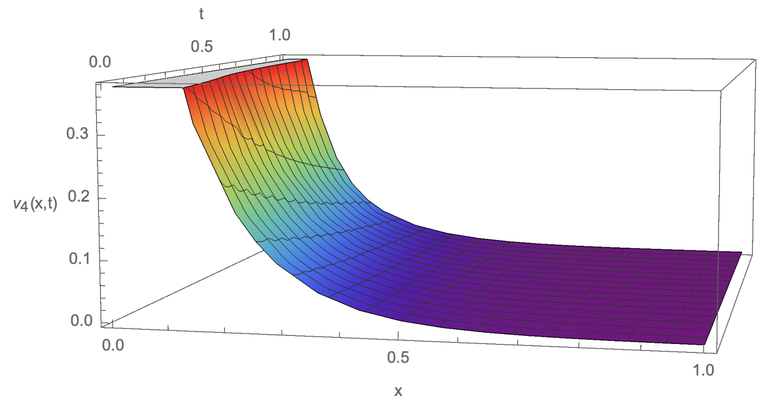

Figure 5.6: “Solution Obtained by VIM Method at 4^{th} iteration for $\varepsilon = 2^{-6}$ for Example (5.5.2)."

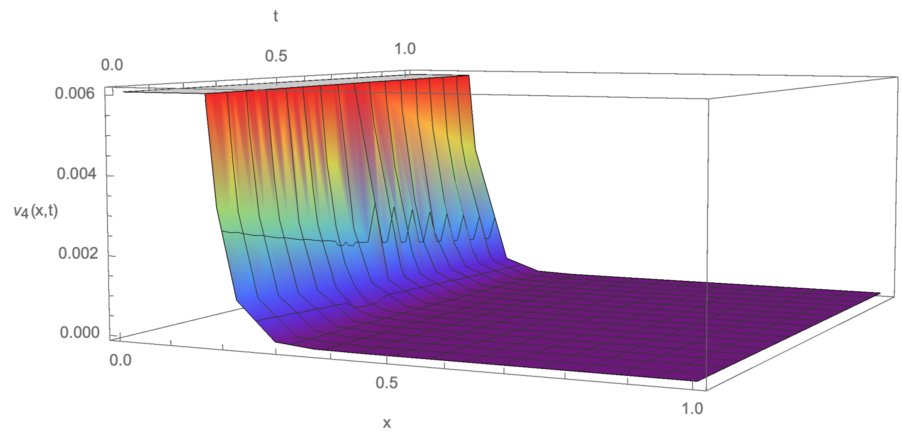

Figure 5.7: “Solution Obtained by VIM Method at 4^{th} iteration for $\varepsilon = 2^{-10}$ for Example (5.5.2)."

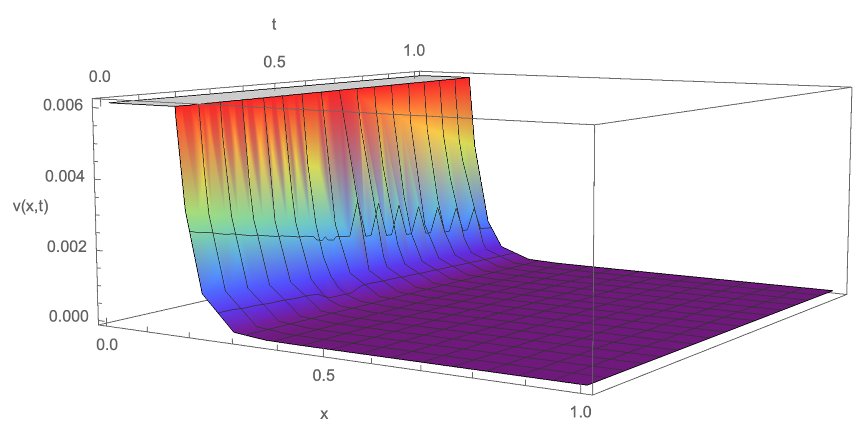

Figure 5.8: “Exact Solution for $\varepsilon = 2^{-10}$ for Example (5.5.2)."

t	$\varepsilon = 2^{-2}$	$\varepsilon = 2^{-4}$	$\varepsilon = 2^{-6}$	$\varepsilon = 2^{-8}$	$\varepsilon = 2^{-10}$
0.2	3.85497×10^{-4}	1.3886×10^{-4}	2.40926×10^{-5}	2.62891×10^{-5}	1.68401×10^{-6}
0.4	3.56725×10^{-3}	8.5865×10^{-4}	7.28079×10^{-4}	4.24775×10^{-4}	2.53068×10^{-5}
0.6	5.11157×10^{-3}	5.11257×10^{-3}	5.11157×10^{-3}	2.15044×10^{-3}	1.20588×10^{-4}
0.8	2.26075×10^{-2}	2.19989×10^{-2}	1.98954×10^{-2}	6.7509×10^{-3}	3.59495×10^{-4}

Table 5.2: “Maximum absolute error $\hat{\eta}$ of proposed method for Example (5.5.2) solved for third iteration v_3."

5.5.2 Convection Reaction-diffusion problem

Example 5.5.3. *“Consider the following non-linear parabolic singularly perturbed convection-reaction diffusion problem.*

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} - \varepsilon \frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial v(x,t)}{\partial x} + v(x,t) + (v(x,t))^2 = f(x,t), \\ v(x,0) = e^{\left(-\frac{x}{\sqrt{\varepsilon}}\right)}, \qquad x \in [0,1], \\ v(0,t) = e^{(-t)}, \quad v(1,t) = e^{\left(-t-\frac{1}{\sqrt{\varepsilon}}\right)}, \quad t \in [0,1] \end{cases} \tag{5.5.7}$$

with exact solution

$$v(x,t) = e^{-t}\left(x\left(1-e^{-1/\varepsilon}\right) - e^{-\frac{1-x}{\varepsilon}} + e^{-1/\varepsilon}\right).\text{”} \tag{5.5.8}$$

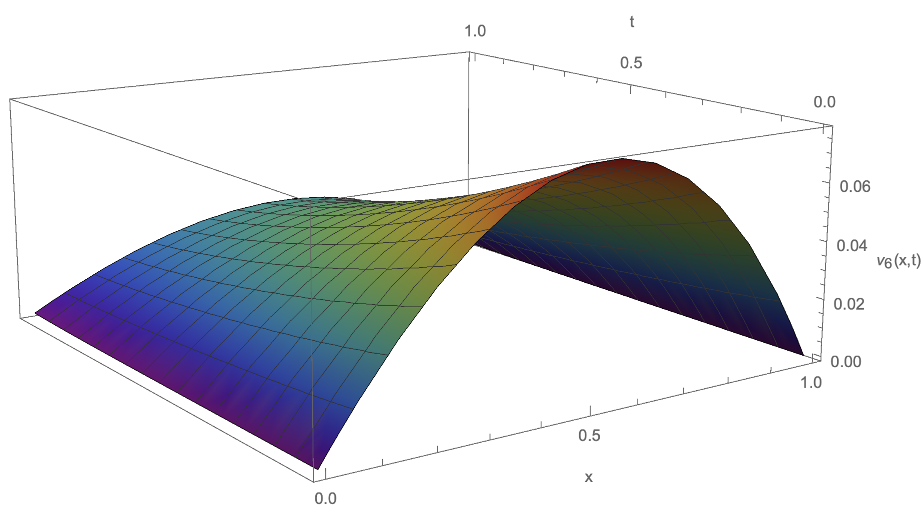

Figure 5.9: “Solution Obtained by VIM Method at 6^{th} iteration for $\varepsilon = 2^0$ for Example (5.5.3)."

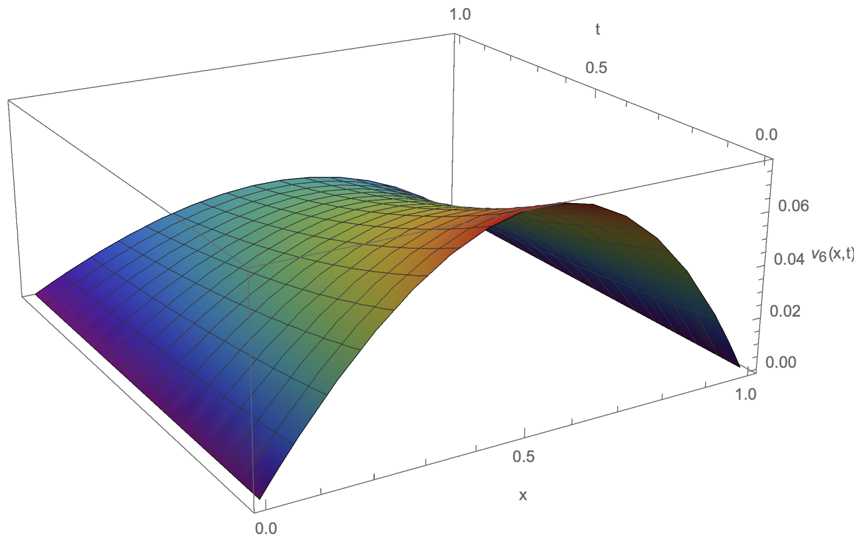

Figure 5.10: "Exact Solution for $\varepsilon = 2^0$ for Example (5.5.3)."

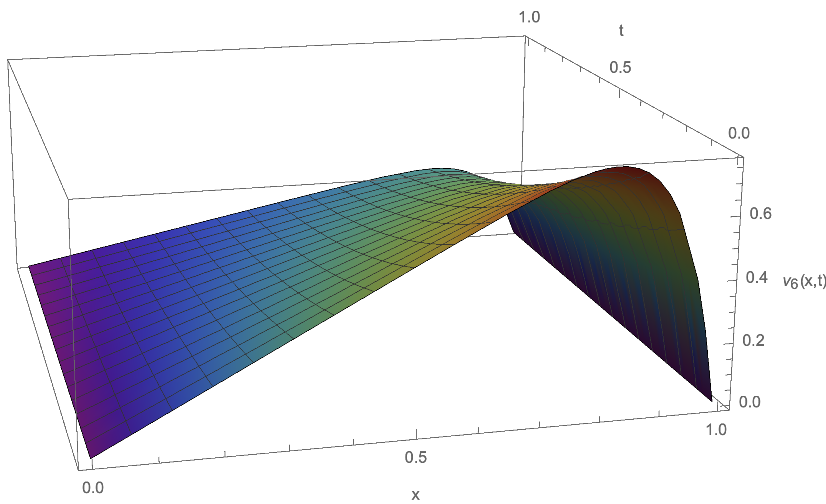

Figure 5.11: "Solution Obtained by VIM Method at 6^{th} iteration for $\varepsilon = 2^{-4}$ for Example (5.5.3)."

5.6 Conclusion

The flow of a fluid through a permeable medium is a well-connected topic in oil recovery, geo-technical engineering, and shallow groundwater hydrology, among several others fields. Ground water pollution is a worry for many environmentalists, scientists, soil and agricultural experts, oceanographers, and biochemists. Several physical problems in oceanography are modelled using mathematical models including such convection models, diffusion models, reaction-convection diffusion models, convection-diffusion models, and so on, which analyse the solute flow in an aquifer / ground water to un-

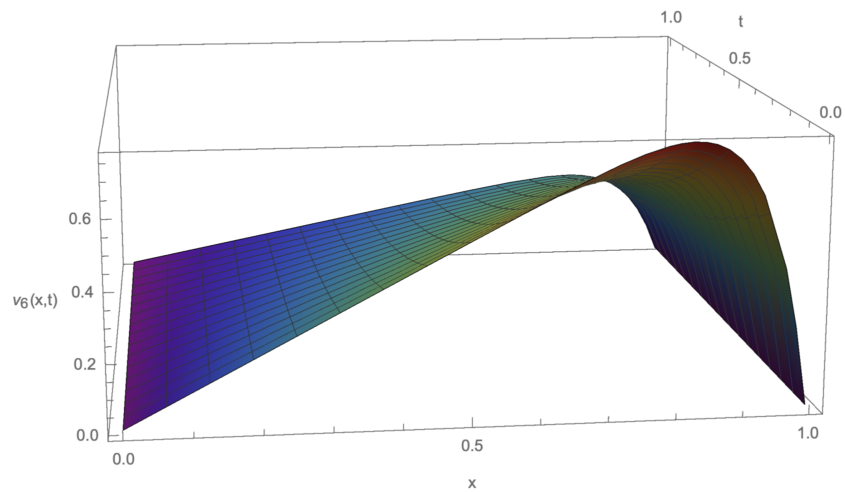

Figure 5.12: “Exact Solution for $\varepsilon = 2^{-4}$ for Example (5.5.3)."

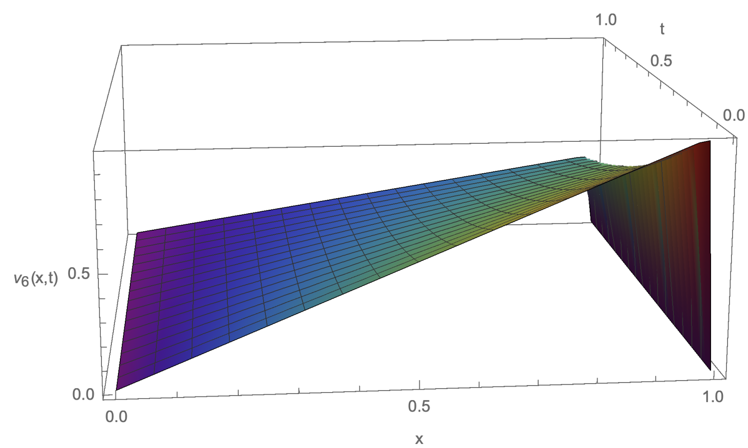

Figure 5.13: “Solution Obtained by VIM Method at 6^{th} iteration for $\varepsilon = 2^{-8}$ for Example (5.5.3)."

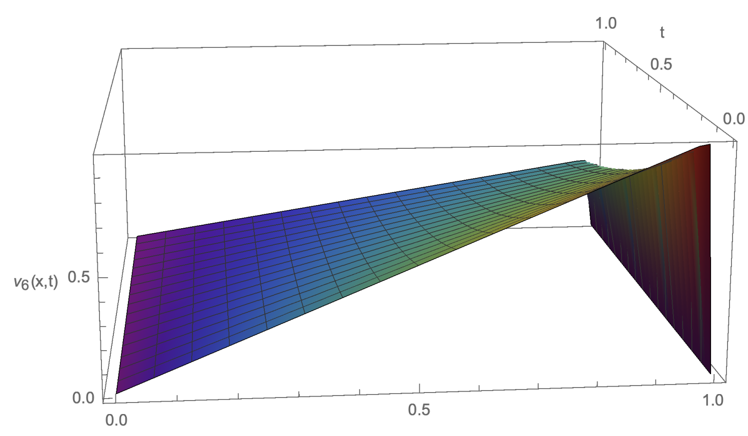

Figure 5.14: “Solution Obtained by VIM Method at 6^{th} iteration for $\varepsilon = 2^{-8}$ for Example (5.5.3)."

derstand the physical behaviour of ground water pollution, how it will affect the environment, and how it will affect human life. The fundamental equations that describe the kinematic and dynamic interactions among flow parameters, fluid, and medium at any location within the flow domain under consideration are ordinary differential equations and partial differential equations. The nonlinear singularly perturbed problems such as nonlinear singularly perturbed reaction diffusion, singularly perturbed convection diffusion, Navier-stokes equations are all crucial in understanding the mechanics of transport phenomena in permeable mediums. Many laboratory studies on 1-D fluid flow have been performed, in which a uniform pressure was applied to the lower edge of the column to evaluate the fluid flow rate in a uniformly length column packed with porous medium. In this chapter, we have successfully developed the variational iteration method for solving NSPPPDEs. In nonlinear problems, one often uses a quasi-linearization technique to linearize the problem and then solve the linearized problem by numerical or other existing techniques. Due to the linearization of the nonlinear problem, the approximated solution's accuracy somehow degenerates, which leads to deceptive solutions. In this chapter, an attempt has been made to address such types of issues associated with NSPPPDEs. The method presented in this chapter seems to be robust, as it maintains high accuracy concerning the perturbation parameter ε. Unlike numerical approaches, the proposed method is simple to implement because it does not require either linear or nonlinear systems of equations. It eliminates the need for time-consuming algebraic equations, prior simplification, discretion, or linearization, allowing it to greatly reduce the size of the calculation while maintaining excellent accuracy. We obtain a higher level of precision just in few iterations without any limiting assumption. The obtained numerical results appear to be promising and justify the efficiency of the proposed method.

Chapter 6

Summary and Future Scope

6.1 Summary of Results

The thesis is divided into two parts; in the first part, we deal with the singularly perturbed delay differential equation. We have developed and implemented finite element methods to solve such types of problems. We have considered a nonlinear singularly perturbed differential equation in the second part of the thesis. We rely on the numerical method through quasilinearization to solve a nonlinear differential equation. In this thesis, we have developed and implemented numerical schemes in which we do not need linearization or discretization of our problem.

6.1.1 Model Problem:1

Singularly Perturbed Parabolic Reaction Diffusion Problem with Time Delay:

$$“(v_\varepsilon)_t(x,t) - \varepsilon(v_\varepsilon)_{xx}(x,t) + a(x)v_\varepsilon(x,t) + b(x,t)v_\varepsilon(x,t-\tau) = f(x,t), \tag{6.1.1}$$

where $(x,t) \in Q$, subject to the initial condition and boundary conditions given as:

$$\begin{cases} v_\varepsilon(x,t) = & \psi_b(x,t), \text{ on } \quad (x,t) \in \Upsilon_b, \\ v_\varepsilon(x,t) = & \psi_l(t), \quad \text{on} \quad (x,t) \in \Upsilon_l = \{(0,t) : t \in [0,T]\}, \\ v_\varepsilon(x,t) = & \psi_r(t), \quad \text{on} \quad (x,t) \in \Upsilon_r = \{(1,t) : t \in [0,T]\}”. \end{cases} \tag{6.1.2}$$

Here, $0 < \varepsilon \ll 1$ is the singular perturbation parameter and $\tau > 0$ be the delay term. The problem data $\psi_l(t)$, $\psi_r(t)$, $\psi_b(x,t)$, $f(x,t)$, $a(x)$, and $b(x,t)$ are supposed to be sufficiently smooth, bounded and independent of parameter ε.

$$a(x) \geq \alpha > 0, \qquad b(x,t) \geq \beta > 0, \qquad (x,t) \in \bar{Q}. \tag{6.1.3}$$

Where α and β are the positive constants independent of singular perturbation parameter ε. As $\varepsilon \to 0$ the solution of the such problem exhibits boundary layers of equal width on both Υ_l and Υ_r boundary points.

6.1.2 Model Problem:2

Singularly Perturbed Non-Linear Reaction Diffusion Problem.

$$\begin{cases} \text{“}\varepsilon v''(x) = g(x, v(x)); \qquad x \in (0,1) = \omega,\text{”} \\ v(0) = A, \quad v(1) = B, \end{cases} \tag{6.1.4}$$

where ε is singular perturbation parameter with $0 < \varepsilon << 1$ and $g \in C^{\infty}[0,1] \times R$. Let assume that

$$g_u(x,v) > \Im^2 > 0 \qquad \forall (x,v) \in \bar{\omega} \times R. \tag{6.1.5}$$

6.1.3 Model Problem:3

Nonlinear Singularly Perturbed Parabolic Reaction Diffusion Problem

$$\text{“}(v_\varepsilon)_t(x,t) - \varepsilon (v_\varepsilon)_{xx}(x,t) + v_\varepsilon(x,t) - f(x,t,v) = 0, \tag{6.1.6}$$

where $(x,t) \in (0,1) \times (0,1)$, subject to the initial condition and boundary conditions prescribed as:

$$\begin{cases} v_\varepsilon(x,0) = & \psi_b(x), \\ v_\varepsilon(0,t) = & \psi_l(t), \\ v_\varepsilon(1,t) = & \psi_r(t).\text{”} \end{cases} \tag{6.1.7}$$

Where $0 < \varepsilon \ll 1$ is a small positive parameter known as singular perturbation parameter. Let $f(x,t,v)$ be sufficiently smooth and ρ ,$\bar{\rho}$ be some constant satisfies,

$$0 \le \rho^2 \le f_v(x,t,v) \le \bar{\rho}^2 \quad \forall (x,t,v) \in [0,1] \times [0,1] \times \mathbf{R} \tag{6.1.8}$$

6.1.4 Model Problem:4

Singularly Perturbed Parabolic Convection Diffusion Problem Consider the following class of singularly perturbed parabolic convection dominated problem.

$$\text{``}(v_\varepsilon)_t(x,t) - \varepsilon (v_\varepsilon)_{xx}(x,t) + (v_\varepsilon)_x(x,t) - f(x,t,v) = 0, \tag{6.1.9}$$

where $(x,t) \in (0,1) \times (0,1)$, subject to the initial condition and boundary conditions prescribed as:

$$\begin{cases} v_\varepsilon(x,0) = & \psi_b(x), \\ v_\varepsilon(0,t) = & \psi_l(t), \\ v_\varepsilon(1,t) = & \psi_r(t).\text{''} \end{cases} \tag{6.1.10}$$

Where $0 < \varepsilon \ll 1$ is a small positive parameter known as singular perturbation parameter. Let $f(x,t,v)$ be sufficiently smooth and ρ ,$\bar{\rho}$ be some constant satisfies,

$$0 \le \rho^2 \le f_v(x,t,v) \le \bar{\rho}^2 \quad \forall (x,t,v) \in [0,1] \times [0,1] \times \mathbf{R} \tag{6.1.11}$$

6.2 Numerical Schemes

Chapter 2 implements a Galerkin finite element method on a singularly perturbed parabolic partial differential equation with time delay. The convergence analysis is carried out, and the method is found to be robust.

In chapter 3, a Bernstein collocation method is implemented on a Nonlinear singularly perturbed reaction-diffusion problem. The existence and uniqueness are derived, and the method is robust.

Chapter 4 implements a Galerkin finite element method with Richardson extrapolation in time on a singularly perturbed parabolic partial differential equation with time delay. The convergence analysis is carried out, and the method is found to be robust.

In chapter 5, an iterative method known as the variation of iteration method based on Lagrange's multiplier is implemented on a nonlinear singularly perturbed parabolic partial differential equation of both convection-diffusion and reaction-diffusion type. The convergence analysis is carried out, and the method is found to be robust.

6.3 Future Scope

During the development and implementation of the numerical schemes to solve singularly perturbed differential and differential-difference equations, we noticed some issues that require further investigation. Some of these points and plans for future work are as follows:

1. Development and analysis of parameter uniform finite element method to 2-D singularly perturbed parabolic delay differential equation.

2. Development and analysis of operational matrix methods on a piece-wise uniform mesh to higher dimension singularly perturbed parabolic differential equation with and without delay.

3. Development and analysis of operational matrix methods on the piece-wise uniform mesh to the system of coupled singularly perturbed parabolic differential equation with and without delay.

4. Development and analysis of numerical methods for singularly perturbed turning point parabolic differential equation with and without delay.

5. With increasing popularity of Caputo types differential equations a future work can be involved the analysis of the following time fractional singularly perturbed initial

value problem.

$$\mathscr{D}_t^{\rho} v_{\varepsilon}(x,t) - \varepsilon \frac{\partial v_{\varepsilon}^2(x,t)}{\partial x^2} + a(x,t)\frac{\partial v_{\varepsilon}(x,t)}{\partial x} + b(x,t)v_{\varepsilon}(x,t-\tau) = f(x,t),$$

where $(x,t) \in Q$, subject to the initial condition and boundary conditions given as:

$$\begin{cases} v_{\varepsilon}(x,t) = & \psi_b(x,t), \text{ on } \quad (x,t) \in \Upsilon_b, \\ v_{\varepsilon}(x,t) = & \psi_l(t), \quad \text{ on } \quad (x,t) \in \Upsilon_l = \{(0,t) : t \in [0,T]\}, \\ v_{\varepsilon}(x,t) = & \psi_r(t), \quad \text{ on } \quad (x,t) \in \Upsilon_r = \{(1,t) : t \in [0,T]\}. \end{cases} \tag{6.3.1}$$

Where $\mathscr{D}_t^{\rho}$ is defined as:

$$\mathscr{D}_t^{\rho} g(x,t) = \frac{1}{\Gamma(1-\rho)} \int_{s=0}^{t} (t-s)^{-\rho} \frac{\partial g_{\varepsilon}(x,s)}{\partial s} ds \tag{6.3.2}$$

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List of Publications

- “**Kartikay Khari** and Vivek Kumar; *Finite element analysis of the singularly perturbed parabolic reaction-diffusion problems with retarded argument, Numer. Methods Partial Differ. Eq. 38(2022), 997-1014. https://doi.org/10.1002/num.22785"*
- “**Kartikay Khari** and Vivek Kumar; *Efficient Numerical Techniques for Nonlinear Singularly Perturbed Reaction Diffusion Problem, J. Math. Chem. (2022). https://doi.org/10.1007/s10910-022-01365-4”*
- “**Kartikay Khari** and Vivek Kumar ; *Iterative analytic approximation to class of nonlinear singularly perturbed parabolic partial differential equation.” (Communicated)*
- “**Kartikay Khari** and Vivek Kumar; *Finite element analysis along with Richarson extrapolation technique singularly perturbed time delay parabolic reaction-diffusion problem". (Communicated)*
