Spectral Theory of Operators and its Applications

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CANDIDATE'S DECLARATION

We, **Ruchita Sareen**, Roll No. 2K22/MSCMAT/34, and **Sumit Dhull**, Roll No. 2K22/MSCMAT/46, hereby certify that the work which is being presented entitled *'Spectral Theory of Operators and its Applications'*, in partial fulfillment of the requirements for the award of Degree of Master of Science, submitted in the Department of Applied Mathematics, Delhi Technological University is an authentic record of our own work carried out during the period from August 2023 to May 2024 under the supervision of Mr. Jamkhongam Touthang.
The matter presented in the thesis has not been submitted by us for the award of any other degree of this or any other Institute.

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CERTIFICATE

Certified that **Ruchita Sareen**(2K22/MSCMAT/34) and **Sumit Dhull** (2K22/MSCMAT/46) have carried out their search work presented in the thesis entitled "**Spectral Theory of Operators and its Applications**" for the award of **Master of Science** from the department of Applied Mathematics , Delhi Technological University , Delhi, under my supervision. The thesis embodies results of original work, and studies are carried out by the students themselves and the contents of the thesis do not form the basis for the award of any other degree to the candidate or to anybody else from this or any other University.

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Abstract

A key idea in both mathematics and physics, spectral theory investigates the characteristics of self-adjoint operators and the eigenvalues and eigenfunctions that go along with them. There are several uses for this theory in partial differential equation (PDE) solving as well as quantum problems. A self-adjoint operators represent observables like momentum and energy, which are crucial for understanding the behavior of quantum systems and the results of measurements. They are crucial for solving the Schrödinger equations, as they reveal the energy sataes of various particles, such as the electrons in a hydrogen atom. However, in the field of PDEs, spectral theory makes it easier to break down complicated equations into more manageable eigenvalue issues, which leads to effective solutions. As illustrated by examples like the heat equation and Laplace's equation in different geometries, spectral theory provides a methodical way to comprehend the behaviour of solutions to PDEs by splitting variables and using eigenfunctions and eigenvalues. All in all, spectrum theory offers a fundamental framework with a wide range of applications in mathematical physics, providing understanding of both quantum and spatial events that are represented by partial differential equations.

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Chapter 1

Introduction

1.1 Overview of the topic

The evolution of spectral theory stands out as one of the most illuminating narratives in contemporary mathematics, connecting pure with applied branches. Its inception can be traced back to the principal axes theorem of analytical geometry established by Fermat and Descartes during the seventeenth century— further expounded by luminaries such as Euler, Lagrange, and Jacobi. It wasn't until the nineteenth century that Sylvester and Cayley propelled the theory forward through a matrix notation: not until 19th-century Sylvester and Cayley introduced this theory in a matrix notation. Fourier's work in the early 1800s was based on representing functions by a sum of sine and cosine waves. It led to spectral theory for infinite dimensional matrices— later refined notably by Hill, Appell, and Poincaré [1].

In today's mathematical education, the spectral theorem — crucial in grasping linear operators on spaces with infinite dimensions — is one of the first things students come across. The evolution of spectral theory through history highlights the dual relationship it shares with theory and application: showing how its significance reaches out to include not only abstract mathematics but also real scientific issues. Quantum mechanics (QM) is a branch of mathematics that uses functional analysis, measure theory, probability, and mathematical logic to combine elegance and a deep grasp of the physical world. The first method used Hilbert spaces but the spectral concept of unbounded operators, and werecreated by von Neumann in 1932. An other, more sophisticated approach, designed to solve problems in quantum field theory, uses abstract algebras (*-algebras and $C*$-algebras) inspired by von Neumann's operator algebras (W*-algebras or von Neumann algebras), which work independently of the Hilbert space structure. These methods demonstrate the mathematical and descriptive versatility of QM [2].

The algebraic characteristics and spectrum theory applications of self-adjoint operators are examined in this work, with a particular focus on quantum mechanics and the solution of PDEs. Self-adjoint and symmetric operators are examined stressing their significance in representing physical observables. Other applications include time independent Schrodinger equation for hydrogen atoms and variable separation methods for solving PDEs in heat and wave equations which add information within a number of scientific and technical context

1.2 Introduction to Operator theory

1.2.1 Normed Space

A vector space that has a norm applied to it is called a normed vector space ;on the other hand, a vector space that has a seminorm applied to it is called an seminormed vector space [3].

Definition 1. *X represent a vector space over $\mathbb{C}$ or $\mathbb{R}$. When a mapping $\rho : X \to \mathbb{R}$ satisfies the following conditions, it is called a norm on X, establishing a normed space (X, ρ).*

1. $\rho 0$. $\rho(x) \geq 0$ *for all* $x \in X$;
2. $\rho 1$. $\rho(\lambda x) = |\lambda|\rho(x)$ *for all* $\lambda \in F$ *and* $x \in X$;
3. $\rho 2$. $\rho(x+y) \leq \rho(x) + \rho(y)$ *for all* $x, y \in X$;
4. $\rho 3$. $\rho(x) = 0$ *implies* $x = 0$ *for all* $x \in X$.

While conditions $\rho 0, \rho 1$, and $\rho 2$ are satisfied, $\rho 3$ may not necessarily hold, thereby distinguishing ρ as a seminorm.

1.2.2 Properties of Normed space

1. $(V, \|\cdot\|)$ is anormed space.

Set of all points in the vector space V that are within a distance r from a specific point $x_0 \in V$ is known as the open ball centered at x_0 with radius $r > 0$.

$$B_r(x_0) := \{x \in V \mid \|x - x_0\| < r\}$$

Set $U \subset V$ termed **bounded** if an open ball $B_r(x_0)$ (with r being a finite radius) such that:

$$B_r(x_0) \supset U$$

This describes the characteristic of a set to be bounded within normed spaces.

2. Within the framework of a normed vector space $(V, |\cdot|)$:

(a) A subset $U \subset V$ is considered open if either U is empty ($U = \emptyset$) or if U can be represented as the union of open balls.

(b) The set of open sets in V is the norm-induced topology over V.

3. A function $g : A \to B$ between normed spaces $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ is considered continuous at a point $a_0 \in A$ if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|g(a) - g(a_0)\|_B < \varepsilon$ whenever $\|a - a_0\|_A < \delta$.

A mapping $g : A \to B$ is defined as continuous if **continuous** at every point within domain A.

4. In normed spaces $(W, \|\cdot\|_W)$ and $(V, \|\cdot\|_V)$ over same field $\mathbb{C}$ or $\mathbb{R}$,linear transformation $T : V \to W$ is defined as isometric or an isometry, if $\|T(v)\|_W = \|(v)\|_V$. Should the isometry $T : V \to W$ be surjective, it qualifies as an isomorphism between the normed spaces. In the context of an isomorphism T between normed spaces, this suggests that the domain and codomain are viewed as isomorphic under T [4].

1.3 Banach spaces

If every Cauchy sequence within a normed space converges to a point within the space, it is termed complete and identified as a Banach space [5].
A Banach space represents normed space $(X, \|\cdot\|)$ where X is a complete vector space. A normed space is represented by the pair $(X, \|\cdot\|)$,Here X a vector space over a scalar field $\mathbb{K}$, usually $\mathbb{R}$ or $\mathbb{C}$, and includes a specific norm $\|\cdot\| : X \to \mathbb{R}$ [4].Like all norms, this one defines the canonical or induced metric, a translation-invariant distance function, which is applicable to all vectors $x, y \in X$.

$$d(x,y) := \|y-x\| = \|x-y\|.$$

This makes X into a metric space (X,d). A sequence $x_1, x_2, \ldots$ is called **Cauchy** in (X,d) or d-Cauchy or $\|\cdot\|$-Cauchy if for every real $r > 0$, there exists some index N such that

$$d(x_n, x_m) = \|x_n - x_m\| < r$$

When m and n surpass N, the normed space $(X, \|\cdot\|)$ earns the title of a Banach space [5]. A complete metric space is formed by (X,d) when the canonical metric d reaches completion. A complete metric space is formed by (X,d) when the canonical metric d reaches completion. This indicates that for every Cauchy sequence $x_1, x_2, \ldots$ in (X,d), there exists a $x \in X$ such that

$$\lim_{n\to\infty} x_n = xin(X,d)$$

where because $\|x_n - x\| = d(x_n, x)$, this **sequence's convergence** to x expressed as:

$$\lim_{n\to\infty} \|x_n - x\| = 0in\mathbb{R}.$$

The norm $\|\cdot\|$ of a normed space $(X, \|\cdot\|)$ is called a **complete norm** if $(X, \|\cdot\|)$ is a Banach space [6] .

Remark: 1 Although it is not invariant over homeomorphisms, the completeness property holds for isomorphisms of normed spaces. This means that continuous maps have continuous inverses which are not always linear.$\mathbb{R}$ and $(0,1)$, which are all normalised with absolute value, provide a counterexample. The completeness on the line contrasts to the interval's incompleteness, even if the pair is homeomorphic.

Theorem 1. ***Completion of Banach spaces:***

(a) A Banach space (Z, ρ) over the field $K = \mathbb{C}$ or $\mathbb{R}$, referred to as the completion of X, contains X as a dense subspace through an isometric embedding achieved by a linear injective mapping $\Phi : X \to Z$.

Put differently, there exists a unique linear mapping $\Phi : X \to Z$ where the image $\Phi(X)$ is dense within Z, and $\rho(\Phi(x)) = \|x\|$ for all $x \in X$.

(b) Given the triplet (Ψ_1, Z_1, ρ_1), where $\Psi_1 : X \to Z_1$ is linear isometry and (Z_1, ρ_1) is Banach space over K, it is known as $(X, \|\cdot\|)$ is isometrically embedded as dense subspace of Z_1 via Ψ_1. As a result, there exists a unique isomorphism $\sigma : Z \to Z_1$ such that Ψ_1 can be written as $\sigma \circ \Phi$ [7].

Normed algebras, Banach algebras

A vector space having a binary operation $\odot$ is an algebra $\mathscr{D}$ over the field $\mathbb{F}$ (which can be either $\mathbb{C}$ or $\mathbb{R}$): $\mathscr{D} \times \mathscr{D} \to \mathscr{D}$, called the product, which is associative:

$$(p \odot q) \odot r = p \odot (q \odot r) forall p,q,r \in \mathscr{D},$$

and follows these distributive laws over vector space operations:

D1. $p \odot (q+r) = p \odot q + p \odot r$ for all $p,q,r \in \mathscr{D}$;

D2. $(q+r) \odot p = q \odot p + r \odot p$ for all $p,q,r \in \mathscr{D}$;

D3. $\lambda(p \odot q) = (\lambda p) \odot q = p \odot (\lambda q)$ for $\lambda \in \mathbb{F}$ and all $p,q \in \mathscr{D}$;

The algebra $(\mathscr{D}, \odot)$ is called **commutative** (or **Abelian**) if:

D4. $p \odot q = q \odot p$ for any $p,q \in \mathscr{D}$;

If there is a value e in $\mathscr{D}$, which is referred to as a unit element, such that:

D5. $e \odot x = x \odot e = x$ for any $x \in \mathscr{D}$;

D6. $\|p \odot q\| \leq \|p\| \cdot \|q\|$ for all $p,q \in \mathscr{D}$;

and if there is a unit element, additionally:

D7. $\|e\| = 1$;

Utilizing an identical norm for both the algebraic and vector space aspects, a Banach algebra, or a Banach algebra with unit, refers to a Banach space $\mathscr{D}$ that functions as both a normed algebra and a normed vector space [8].

1.3.1 Algebra Homomorphisms and Isomorphisms

Definition 2. *A homomorphism $\phi : \mathscr{X}_1 \to \mathscr{X}_2$ between algebras, regardless of the presence of a unit or whether they are normed or Banach, is a linear mapping that preserves the algebraic structure, including products and units, if present. To put it formally:*

$$\phi(x \cdot_1 y) = \phi(x) \cdot_2 \phi(y) \quad forall x,y \in \mathscr{X}_1, and \phi(1_1) = 1_2.$$

An algebra homomorphism transforms into an algebra isomorphism when it is bijective, regardless of whether it operates between normed or Banach algebras. If an isomorphism $\phi : \mathscr{X} \to \mathscr{Y}$ exists, it establishes an isomorphic relationship between the algebras $\mathscr{X}$ and $\mathscr{Y}$, irrespective of their properties such as being normed or Banach, and whether they possess a unit [9].

1.4 Operators, Operator Spaces, and Norms

The operator norm is a real integer that is used in mathematics to quantify the "size" of specific linear operators.It denotes a norm established within a designated pair of normed vector spaces, applied to the set of bounded linear operators.

1.4.1 Linear Operator

Consider vector spaces U and V over a field F, which can be either $\mathbb{R}$ or $\mathbb{C}$.

1. An operator $S : U \to V$ is called linear if it satisfies the following condition:

$$S(au + bv) = aS(u) + bS(v)$$

for every pair $u, v \in U$ and $a, b \in F$. $L(U,V)$ represents a set that includes all linear operators over U to V. The continuous operators make up subset $B(U,V) \subseteq L(U,V)$ where U and V are normed spaces. Specifically, we denote $L(U) := L(U,U)$ and $B(U) := B(U,U)$.

2. A **functional** $S : U \to F$ is termed linear if it adheres to the linearity condition described above.

3. The set U^*, defined as $L(U,F)$, is referred to as the algebraic dual space of U. Conversely, U', denoted as $B(U,F)$, is called the topological dual space or simply the dual of U, where F is normed by its absolute value [10].

Theorem 2. *Consider normed spaces A and B over the same field $F = \mathbb{R}$ or $\mathbb{C}$, and let $S \in L(A,B)$.*

1. *The following statements are equivalent:*

   (a) A real constant M such that $\|Sv\|_B \leq M \|v\|_A$ every all $v \in A$.

   (b) $\sup_{v \in A \setminus \{0\}} \frac{\|Sv\|_B}{\|v\|_A} < +\infty$.

2. *If either condition (i) or (ii) holds, then: The supremum of $\frac{\|Sv\|_B}{\|v\|_A}$ over all non-zero $v \in A$ is equal to infimum of set of real numbers M such that $\|Sv\|_B \leq M \|v\|_A$ for any $v \in A$.*

1.4.2 Bounded Operators and Operator Norms

Consider normed spaces X and Y over either $\mathbb{C}$ or $\mathbb{R}$. The theorem previously mentioned asserts that $T \in L(X,Y)$ is classified as bounded if any of the conditions are satisfied. $\|T\|$, the value in question, is specified as:

$$\|T\| := \sup_{\|u\|_X \neq 0} \frac{\|Tu\|_Y}{\|u\|_X}$$

is referred to as the (operator) norm of T [11].

Theorem 3. *In the context of normed spaces X and Y over either $\mathbb{C}$ or $\mathbb{R}$, the theorem mentioned above asserts operator $T \in L(X,Y)$ is deemed bounded if the given conditions are satisfied. The quantity $\|T\|$ is defined as the following:*
Then, the following assertions hold true and are interchangeable:

1. *The continuity of T at the zero point.*

2. *The uniform continuity of T.*

3. *The boundedness of T.*

Here, T denotes a linear transformation $T : V \to W$.

Conjugate Operator in Normed Spaces

Consider normed spaces A and B, denoted by $\mathbb{C}$ or $\mathbb{R}$. In the realm of normed spaces, the adjoint (or conjugate) operator of S, denoted as S^*, belongs to $B(B', A')$. Its definition is as follows: For any $a \in A$ and $g \in B'$, the action of S^* on g is expressed as $(S^*g)(a) = g(S(a))$.

Extension of Bounded Operators to Dense Subspaces

Suppose B is a Banach space, and both A and B are normed spaces defined over either $\mathbb{C}$ or $\mathbb{R}$ [12]. Let $U : R \to B$ be a bounded linear operator on R, where R acts as a dense subset of A.

1. A singular bounded linear operator $\widetilde{U} : A \to B$ exists, uniquely defined, such that when restricted to R and denoted as $\widetilde{U}\Big|_R$, it precisely matches U.

2. The norm of the operator $\widetilde{U}$ is equivalent to the operator norm of U, represented as $\|\widetilde{U}\| = \|U\|$.

1.4.3 Riesz's theorem for complex measures

Theorem 4. *Suppose $\Gamma : C_0(Y) \to \mathbb{C}$ is a continuous linear functional, and Y is a locally compact Hausdorff space [13]. Then, for any $g \in C_0(Y)$, there exists a unique singular complex Borel measure μ_Γ associated with Γ.*

$$\Gamma(g) = \int_Y g \, d\mu_\Gamma.$$

Moreover, the norms of Γ and μ_Γ are equal, i.e., $\|\Gamma\| = \|\mu_\Gamma\|$.

1.4.4 Riesz's theorem for complex measures on $\mathbb{R}^n$

Theorem 5. *Given a compact set E in either $\mathbb{R}^n$ or $\mathbb{C}$, and a continuous linear functional $\Phi : C_0(E) \to \mathbb{C}$, there exists a unique complex Borel measure μ_Φ associated with E. This measure μ_Φ satisfies $\Phi(g) = \int_E g \, d\mu_\Phi$ for every $g \in C_0(E)$. Additionally, μ_Φ is regular [14].*

1.4.5 Fundamental theorems of Banach spaces

In quantum mechanics, a number of important topological structures are crucial, especially the domain viewed as the Hilbert space that comprises theory. Specific observables are indicated by bounded operators, while the basic characteristics of quantum systems with measurement procedures are orthogonal projectors subgroup.To get through the algebra with observables into the algebra of projectors with continuity, weaker topologies beyond the standard one are needed. These kinds of problems lead to the idea that of the von Neumann algebra [15].

1.5 Hahn-Banach theorem and its direct consequences

The Hahn-Banach theorem stands as a pivotal instrument within the domain of functional analysis. It becomes particularly valuable when a sufficient number of continuous linear functionals are constructed for each normed vector space, rendering the study of the dual space compelling.

This facilitates the expansion of bounded linear functionals from a vector subspace linked with the vector space to encompass entirety of the space, thus finalizing depiction. The Hahn-Banach separation theorem, often referred to as the hyperplane separation theorem, finds diverse applications in convex geometry [16].

1.5.1 Hahn–Banach theorem

The foundation for both geometric and analytical perspectives on the Hahn-Banach theorem lies in a fundamental assertion regarding the extension of linear functionals over a real vector space [17].
Let's posit that U constitutes a subset of V in the normed linear space V on $\mathbb{C}$. Continuous linear functional acting on U represented by $\phi : U \to \mathbb{C}$. The norm of L in V^* then equals the norm of ϕ in U^*, as there is a continuous linear extension $L : V \to \mathbb{C}$ of ϕ.

$$\|L\|_{V^*} = \|\phi\|_{U^*}$$

.
Corollary 1: A linear functional ϕ exists within the dual space U^* where $|\phi| = 1$ and $\phi(u_0) = \|u_0\|$ if u_0 represents a nonzero element within the normed space U on $\mathbb{C}$ or $\mathbb{R}$. This suggests that there exists a linear functional ϕ with a norm of 1 that, if applied to u_0, equals u_0's norm.
Corollary 2. In a normed space U with $\{0\}$ as its only element, defined over the field $F = \mathbb{C}$ or $\mathbb{R}$, for any distinct $u_1, u_2 \in U$, in U^*, there is a functional ψ such that $\psi(u_1) \neq \psi(u_2)$.

Corollary 3. Alinear mapping $T : V \to (V^*)^*$ defined on normed spaces over the fields of complex numbers and real numbers. This transformation is defined such that for any element v in V and any functional g in the dual space V^*, the action of T on v yields $g(v)$. Remarkably, this transformation T possesses the property of being an isometry.

1.5.2 The Banach–Steinhaus theorem, the uniform boundedness principle:

The uniform boundedness principle also called the Banach–Steinhaus theorem, asserts pointwise boundedness given group of continuous linear operators over Banach space entails uniform boundedness. To be more precise, operators are uniformly bounded throughout the space if they are bounded at every point [18].

Theorem 6. ***Banach–Steinhaus theorem:*** *In a Banach space denoted as $\mathscr{V}$ and a normed space designated as $\mathscr{W}$ over either the field $\mathbb{F}$ or $\mathbb{R}$, assuming there exists a collection of operators $\{S_\beta\}_{\beta \in B}$ belonging to the set $\mathfrak{B}(\mathscr{V}, \mathscr{W})$. For any element u in $\mathscr{V}$, if the supremum of $\|S_\beta u\|$ over all $\beta \in B$ is finite, then there exists a non-negative constant denoted as C that uniformly bounds each operator S_β in the aforementioned family [19]. In other words, for all $\beta \in B$, it holds that*

$$\|S_\beta\| \leq C$$

for all $\beta \in B$.

Corollary 1. Banach-Steinhaus theorem, set of operators T_α, for $\alpha \in A$, exhibits uniform continuity. This implies every $\varepsilon > 0$, $\delta > 0$ such that if the distance between x and x', denoted by $|x - x'|$, is less than δ for any $x, x' \in X$,the difference $|T_\alpha x - T_\alpha x'|$ is less than ε every $\alpha \in A$.

Corollary 2. Let Y a normed space on the real and complex field. If $T \subseteq Y$ is weakly bounded, then for every functional g in the dual space Y', there is non-negative constant c_g such that $|g(y)| \leq c_g$ for every $y \in T$. Thus, T is norm-bounded in Y.

Chapter 2

Hilbert spaces and bounded operators

Hilbert spaces, which are named after David Hilbert, take Euclidean vector spaces into infinite dimensions— a development that plays a central role in both mathematics and physics. The defining feature of these spaces is the existence of an inner product that gives rise to a complete metric structure. This concept finds applications in various areas including PDEs and quantum mechanics where geometric intuition provided by such spaces leads us to results like Pythagorean theorem or optimization using projections onto subspaces.Additionally we see how orthonormal bases allow us to have unique representations much like what we achieve with Cartesian coordinates (though this becomes more interesting when dealing with countably infinite bases). Some specific examples include Sobolev spaces which consist of generalised functions, Hardy spaces which consist of holomorphic functions, and spaces which are made up of square-integrable functions (or sequences). These are just a few instances where these different types of spaces appear. Functional analysis entered a fruitful age thanks to Hilbert space approaches, which are crucial for a wide range of applications [20].

Inner product spaces and Hilbert spaces

Vector space V, $F : V \times V \to \mathbb{C}$ is designated as a Hermitian inner product, or Hermitian scalar pairing, if it meets particular requirements. Referring to (V,F) as an inner product space implies that the function F adheres to certain conditions.

1. Positive Definiteness(H_0): $F(x,x) \geq 0$ for any $x \in V$.
2. Linearity in the First Argument** (H_1): $F(x, \alpha y + \beta z) = \alpha F(x,y) + \beta F(x,z)$ for all $\alpha, \beta \in \mathbb{C}$ and $x,y,z \in V$.
3. Conjugate Symmetry** (H_2): $F(x,y) = \overline{F(y,x)}$ for all $x,y \in V$.
4. Non-Degeneracy** (H_3): $F(x,x) = 0 \Rightarrow x = 0$ for all $x \in V$.

If F satisfies H_0, H_1, and H_2 but not H_3, it denotes a Hermitian semi-inner product[21]. In this case, if $F(x,y) = 0$, vectors x and y in V is orthogonal, expressed by $x \perp y$.
We can also express this using different variables. Let's denote the subset as N instead of M. Thus, the orthogonal complement of N, denoted as $N^{\perp}$, is given by:
$N^{\perp} = \{w \in V \mid w \perp y for all y \in N\}$

Remark 2:

1. The prerequisites H_1 and H_2 indicate that T operates antilinearly in its initial argument: $T(\alpha u + \beta v, x) = \alpha T(u,x) + \beta T(v,x)$ for every $\alpha, \beta \in \mathbb{C}$ and $u,v,x \in Y$.

2. By H1, it's clear that $M^{\perp}$ forms a vector subspace in Y, justifying the use of the term "orthogonal space."

3. In a Hermitian inner product space (Y,T), orthogonality entails a useful property: $L \subseteq L_1 L^{\perp} \supseteq L_1^{\perp}$ for $L, L_1 \subseteq Y$.

4. Henceforth, to avoid confusion, "(semi-)inner product" will consistently denote "Hermitian (semi-)inner product."

Propostion 1 (a) According to the Cauchy-Schwarz inequality, the square of the absolute value of the semi-inner product of a vector pair (u, v) and its norms (u, v) is always smaller or equivalent to the product of the norms of u and v for any vectors u and v in vector space V. In other words,

$$|L(u,v)|^2 \leq L(u,u) \cdot L(v,v).$$

(i) When there is a linear dependence between the vectors u and v, the Cauchy-Schwarz inequality applies exactly.

(ii) Equality in the Cauchy-Schwarz inequality occurs when L represents an inner product if and only when the vectors u and v were collinear.

(b) As the square of the root for the inner product of u with itself, or $\sqrt{L(u,u)}$, for u inside V, we define the function $p(u)$. The norm obtained from the inner product L is represented by this function p.
(c) The parallelogram rule can be expressed as the equality: the total of the squares of the norms of the combined and the disparity between two vectors u and v equals twice the sum of the squares of their respective norms.

$$p(u+v)^2 + p(u-v)^2 = 2(p(u)^2 + p(v)^2),$$

where u and v are elements of V.
(d) The polarization formula states that semi-inner product $L(u,v)$ can expressed in one-fourth of sum of four terms involving the norms of u and v, and their combinations under addition and scalar multiplication by i.

$$L(u,v) = \frac{1}{4}(p(u+v)^2 - p(u-v)^2 - ip(u+iv)^2 + ip(u-iv)^2),$$

here u and v are elements of V.

Propostion 2 :

1. The Cauchy-Schwarz inequality shows that, with regard to the product topology, the inner product $L : W \times W \to \mathbb{C}$ is continuous on $W \times W$ where W is endowed with the topology generated by the inner product's norm. This demonstrates that when the inner product is examined as a function on each component independently, it is continuous.

2. Similar definitions hold for the vector space (W) defined on the real field (mathbbR), provided that a real inner product (L: W times W rightarrow mathbbR) is specified. The symmetry property $H2'$, which guarantees $L(a,b) = L(b,a)$ for every $a, b \in W$, is incorporated into this inner product while adhering to characteristics $H0$, $H1$, and $H3$. A true semi-inner product is defined similarly, with the exception of condition $H3$.

3. The previous proposition holds for **real (semi-)inner products**, with the updated polarization formula given by:

$$L(u,v) = \frac{1}{4}\left(p(u+v)^2 - p(u-v)^2\right),$$

in the context of the field $\mathbb{R}$ [22].

A well-known result, attributed to **Fréchet**, **von Neumann**, and **Jordan**, follows from this.
Theorem 1.Completion of Hilbert spaces:

1. Consider W is vector space over $\mathbb{C}$ endowed with in inner product T.

   In Hilbert space H, inner product extends to a completion W via an injective linear map $F : W \to H$, ensuring preservation of the inner product structure. Alternatively, a one-to-one linear transformation exists.

$$F : W \to H$$

   with $F(W) = H$ and $\langle F(a), F(b)\rangle = T(a,b)$ for all $a,b \in W$.

2. If $(G,K,\langle\cdot,\cdot\rangle_K)$ is a triple where $G : W \to K$ is a linear isometry, and $(K,\langle\cdot,\cdot\rangle_K)$ is a Hilbert space, with G extending T to $\langle\cdot,\cdot\rangle_K$ to identify W densely in K, a unique unitary map $\psi : H \to K$ exists such that $G = \psi \circ F$.

2.0.1 Riesz's theorem and its consequences

Frigyes Riesz and Maurice René Fréchet developed the Riesz representation theorem which basically links a Hilbert space to a twin or companion space which we can term as continuous dual space. Particularly it bears the name Riesz-Fréchet representation theorem and it underscores that the two spaces are in effect the same when a set of real numbers is used as the basic field. Conversely, if the base field is comprised of complex numbers then the two spaces are not the same mathematically [23].
Defination 1 A non-empty set M in vector space Y known as convex if:

$$\lambda x + (1-\lambda)y \in M, \quad for any \lambda \in [0,1] and x, y \in M.$$

It is evident that any subspace of Y is convex, although not every convex subset of Y
Theorem 2. Consider $(H,\langle\cdot,\cdot\rangle)$ is a Hilbert space and $M \subseteq H$ a **non-empty subset**.

1. Orthogonal complement $M^{\perp}$ constitutes a subspace of H and possesses closure.
2. $M^{\perp}$ corresponds to the orthogonal complement of the linear subspace spanned by M, denoted as $\langle M\rangle^{\perp}$, and it is also equivalent to the orthogonal complement of the closure of the linear subspace spanned by M, represented as $\overline{\langle M\rangle}^{\perp}$.
3. Suppose M is both closed and convex. Then, for every x in H, there exists a unique element $PM(x)$ in M such that x and $PM(x)$ are at the minimal distance from each other, measured utilizing the norm derived from the inner product $\langle\cdot,\cdot\rangle$.
4. If M is a closed subspace, each x in H expressed as a combination of two vectors: one belonging to M, denoted as z_x, and the other belonging to $M^{\perp}$, denoted as y_x. Thus, H constitutes the direct sum of M and $M^{\perp}$, and z_x corresponds to $PM(x)$ as mentioned in (c).
5. $\left(M^{\perp}\right)^{\perp} = \langle M\rangle$.

2.0.2 Compact operators in normed and Banach spaces

Compact operators throughout normed vector spaces are examined in this section. Initially, it examines several overarching observations regarding compact sets in normed spaces, with particular attention given to infinite-dimensional sets. The subsequent part will explore ideas within Hilbert spaces [24].

2.0.3 Compact sets in Normed Spaces

Definition 2: Consider topological space (X,T) and subset $K \subseteq X$.

(a) An open covering of set K is deemed compact if it possesses a finite subcovering within the topological space (X,T). In other words, there exists a finite subset $J \subseteq I$ such that $\bigcup_{i\in I} A_i \supseteq K$ holds for any collection $\{A_i\}_{i\in I} \subseteq T$, given that $\bigcup_{i\in J} A_i \supseteq K$.

(b) In a topological space (X,T), a set K is considered relatively compact if it exhibits compactness.

(c) In a topological space (X,T), if every point in X possesses a neighborhood that is considered sufficiently compact, then (X,T) is labeled locally compact.

(d) If every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq K$ contains a subsequence $\{x_{n_p}\}_{p\in\mathbb{N}}$ that converges within K, then K is termed sequentially compact in the topological space. This implies that irrespective of how points from K are arranged in a sequence, there always exists a subsequence that converges within K [25].

2.0.4 Compact operators in normed spaces

Definition 3: Given normed spaces X,Y over $\mathbb{R}$ or $\mathbb{C}$, let's explore. If any of the following equivalent assertions are valid, an operator $T \in \mathscr{L}(X,Y)$ is considered compact or fully continuous.

(a) For every bounded subset $M \subseteq X$, the mapping $T(M)$ is contained in a compact subset of Y.

(b) Given a bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ for which the sequence $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ converges in Y.

The set of compact operators acting on X is denoted by $B_\infty(X,Y)$, while those mapping from X to Y are symbolized as $B_\infty(X)$ [26].

Theorem 3: Analyzing Eigenvalues of Compact Operators in Normed Spaces
Suppose T belongs to $B_\infty(X)$, where X is a normed vector space.

1. For every $\delta > 0$, there exists a finite set of eigenspaces of T with associated eigenvalues λ such that $|\lambda| > \delta$.

2. If $\lambda = 0$ is an eigenvalue of T, then the corresponding eigenspace possesses finite dimension.

3. The eigenvalues of T, typically complex numbers, form a bounded, potentially countable collection. They can be ordered by decreasing modulus: $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq 0$, with the chance of 0 being the sole accumulation point.

2.0.5 Compact operators in Hilbert spaces:

The concept of compact operators on a Hilbert space arises from the idea of matrices operating within finite-dimensional vector spaces within the mathematical discipline of functional analysis. Specifically, finite-rank operators in the topology created by the operator norm (represented by finite-dimensional matrices) are closed by compact operators in Hilbert space [27].

2.0.6 General properties and examples

Consider H as a Hilbert space.

1. The collection $B_\infty(H)$ constitutes an element of $B(H)$.
2. The set $B_\infty(H)$ functions as a two-sided *-ideal within $B(H)$, showcasing its dual role as both a subset and an adherent to the condition $TK, KT \in B_\infty(H)$ for any $T \in B_\infty(H)$, $K \in B(H)$, and $T^* \in B_\infty(H)$.
3. The set $B_\infty(H)$ is encompassed by the uniform topology, thereby forming a C*-algebra (without a unit if $dimH = \infty$), essentially serving as a C*-subalgebra embedded within $B(H)$.

2.0.7 Hilbert–Schmidt operators:

Definition 4: An operator T in the space $\mathscr{B}(\mathscr{H})$ earns the label "Hilbert-Schmidt operator" (HS) under a specific condition: there exists a basis V where the sum of squares of the norms of T acting on each basis element converges in the Hilbert space $\mathscr{H}$, equipped with the norm derived from its inner product. The collection of all such operators is denoted by $\mathscr{B}_2(\mathscr{H})$. In simpler terms, an operator T qualifies as a Hilbert-Schmidt operator if this sum converges.

$$\|T\|_2 := \left(\sum_{v \in V} \|Tv\|^2 \right)^{\frac{1}{2}}$$

where V represents the aforementioned basis [28].

Proposition 3: In a Hilbert space $(\mathscr{H}, (\langle \cdot, \cdot \rangle))$, where the norm $\| \cdot \|$ is derived from the inner product, consider two potentially overlapping bases S and T, along with an operator $B \in \mathscr{B}(\mathscr{H})$.
(a) The summation of $\|Bs\|^2$ over all $s \in S$ is finite if and only if the summation of $\|Bt\|^2$ over all $t \in T$ is finite. When this condition holds, it can be expressed as:

$$\sum_{s \in S} \|Bs\|^2 = \sum_{t \in T} \|Bt\|^2.$$

(b) The summation of $\| Bs|^2$ over all $s \in S$ is finite if and only if the summation of $\|B^*t\|^2$ over all $t \in T$ is finite. When this condition is met, it can be denoted as:

$$\sum_{s \in S} \|Bs\|^2 = \sum_{t \in T} \|B^*t\|^2$$

.

2.1 Closed and closable operators

Definition 5: A linear operator $A : D(A) \to H$ is defined as closed if for any sequence $f_n \in D(A)$ where $f_n \to f$ and $Af_n \to g$ as $n \to \infty$, it follows that $f \in D(A)$ and $Af = g$. Bounded linear operators are naturally closed; in fact, if $f_n \to f$, then $Af_n \to Af$ automatically [26].

Definition 6: An operator $A : D(A) \to H$ is deemed closable if there exists a closed operator that extends it.

Propostion:4 Let T be an operator acting on the normed space X. The following statements are interchangeable:

1. T is closable;
2. $G(T)$ excludes elements in the form $(0,z)$ when $z = 0$.
3. T allows closed extensions.

2.2 General properties of the Hermitian adjoint operator

Definition 7: In the context of operators acting on the Hilbert space H, if T is an operator with its domain $D(T) = H$, then the adjoint operator T^* in H is defined as follows: $D(T^*)$ consists of all vectors x in H such that there exists a vector $z_{T,x}$ in H satisfying the inner product property $\langle x, Ty\rangle = \langle z_{T,x}, y\rangle$ for every y in $D(T)$, and T^* maps each vector x to its corresponding $z_{T,x}$.

Theorem 4. Within the category of operators on the Hilbert space H, we consider an operator T whose domain is all of H.

(a) The adjoint operator T^* is closed, and its graph $G(T^*)$ is the orthogonal complement of the closure of T's graph, denoted $\tau(G(T))^{\perp}$.

(b) The operator T^* is closed, and the graph $G(T^*)$ is the orthogonal complement of the closure of T's graph, denoted $\tau(G(T))^{\perp}$.

(c) The null space of T^* is orthogonal to the closure of T's range, $[Ran(T)]^{\perp}$. The null space of T lies within the orthogonal complement of the closure of T^*'s range, $[Ran(T^*)]^{\perp}$. These spaces are equal if T^*'s domain is dense in H and T is closed.

(d) When T is closed, the Hilbert space $H \oplus H$ undergoes an orthogonal decomposition.

$$H \oplus H = \tau(G(T)) \oplus G(T^*)$$

.

Definition 8: Consider $(H, (\cdot))$ as a Hilbert space, and let $T : D(T) \to H$ represent an operator acting on H [29].

(a) T is termed Hermitian if $(u|Tv) = (Tu|v)$ for every $u, v \in D(T)$.

(b) T is considered symmetric if:

(i) T is Hermitian;

(ii) The domain $D(T)$ is dense in H.

Thus, T is symmetric if and only if:

(i)' $D(T) = H$;

(ii)' $T \subseteq T^*$.

(c) T is self-adjoint if:

(i) The domain $D(T)$ is dense in H;

(ii) $T = T^*$.

(d) An operator T achieves the status of essentially self-adjoint when:

(i) The domain $D(T)$ is dense in H;

(ii) The domain $D(T^*)$ is dense in H;

(iii) $T^* = (T^*)^*$ (the adjoint is self-adjoint).

Alternatively, T is essentially self-adjoint if:

(i)' The domain $D(T)$ is dense;

(ii)' T is closable;

(iii)' $T^* = T$.

(e) T is termed normal if $T^*T = TT^*$, where both expressions are defined on their customary domains.

Hilbert spaces are essential to modern analysis, quantum theory, or many other scientific fields because they offer a strong foundation for taking geometric and algebraic ideas into infinite dimensions. Significant insights in functional analysis and its many applications are made possible by the theory for operators on Hilbert spaces, inner product structure, and completeness.

Chapter 3

Spectral Theory

David Hilbert dubbed Hilbert space theory "spectral theory" in his first formulation since it was represented in terms of quadratic forms over an infinite number many variables. As a result, the initial spectral theorem was understood to be an infinite-dimensional application of the theorem on an ellipsoid's principal axes. That spectral theory eventually turned out to have been able to describe certain aspects of atomic spectra within quantum mechanics was therefore fortuitous. This chapter lays the groundwork for spectral theory in normed spaces, focusing on operators and C*-algebras. Topics include spectrum, resolvent sets, and operators, leading to the spectral theorem. The theorem offers a decomposition for bounded normal operators, akin to diagonalization for matrices. It connects to quantum mechanics through projector-valued measures, aligning observables with self-adjoint operators. Sections cover key concepts like Gelfand-Najmark theorem, -homomorphisms, and spectral measures. The final section explores Fuglede's theorem and its implications. We'll delve into the spectral theory's core, setting the stage for deeper quantum mechanics exploration [30].

3.1 Spectrum, resolvent set and resolvent operator

The resolvent set is a notion from operator theory or linear algebra that designates a set of complex numbers over which a linear operator behaves favourably and consistently.The resolvent set is a fundamental element of the resolvent formalism [31].

Defination .1 A function $f : \Omega \to X$ is called analytic in a Banach space in the complex numbers, denoted as $(X, \|\cdot\|)$, given a non-empty open set $\Omega \subset \mathbb{C}$. This function can be expressed as an infinite series centred at z_0 for every point z_0 within Ω, where each term in the series has the following form:

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \cdot a_n \tag{3.1}$$

This equation indicates that the function $f(z)$ can be expressed as an endless sum of terms $(z - z_0)^n$ multiplied with coefficient a_n in X for each z that belongs within the ball $B_\delta(z_0) \subset \Omega$. It is important to remember that for each natural integer n, the coefficients a_n are elements within the Banach space X. Moreover, in the norm $\|\cdot\|$, this series converges uniformly.

Fundamental principles in normed spaces

Norm-governing axioms are fundamental to norm-governed spaces. These include fundamental ideas like the triangle inequality, which states that the norm of a vector sum is always less than or equal to the sum of the individual vector norms, and the Cauchy-Schwarz inequality, which sets a maximum limit in the absolute value of the inner product of two vectors based on their individual norms [32].

Defination 2: Resolvent operator In Banach space X, consider linear operator L defined on a subset $D(L)$ of X, where the identity operator is denoted as id. Let $L_\lambda = L - \lambda id$ for any complex number λ . A complex value λ is termed a regular value when three conditions coincide simultaneously. when a trio of conditions align concurrently.

A complex number λ is identified as a regular value if it fulfills the following requirements:

1. λ is a regular value if L_λ is injective, implying that the restriction of L_λ to its image has an inverse, known as the resolvent $R(\lambda, L)$.
2. λ is regarded as a regular value if the resolvent $R(\lambda, L)$ is a bounded linear operator.
3. λ is classified as a regular value if the resolvent $R(\lambda, L)$ is defined on a dense subset of X, indicating that L_λ has a dense range.

The resolvent set of L is the set of all regular values of L :

$$\rho(L) = \{\lambda \in \mathbb{C} \mid \lambda is a regular value of L\}.$$

When all regular values are removed from a set of all possible complex numbers, what remains is the spectrum.

$$\sigma(L) = \mathbb{C} \backslash \rho(L)$$

The spectrum splits into different groups based on how it's broken down. If the first condition isn't met, you get what's called the "point spectrum." When the second condition isn't fulfilled, you end up with the "continuous spectrum." And if the third condition isn't satisfied, you get what's known as the "residual spectrum."

Every L_λ for a closed operator L is likewise closed, therefore condition 3 can be replaced by making sure that L_λ is surjective. This suggests that there are more features to the resolvent operator L_λ than just being nonempty and compact [33].

Theorem 1: Characteristics of Closed Operators in Banach Spaces

Consider an operator T acting on a nontrivial Banach space X. Here are the key features associated with T:

1. **Bijective Mapping Property:** The resolvent set $\rho(T)$ includes the complex number λ if and only if the operator $T - \lambda I$ establishes a one-to-one correspondence from the domain $D(T)$ to the space X.
2. **Openness and Closedness:**
   (a) The resolvent set $\rho(T)$ is distinguished by its openness.
   (b) Conversely, the spectrum $\sigma(T)$ is characterized by its closedness.
   (c) Furthermore, when $\rho(T)$ contains at least one element, the function mapping $\lambda \in \rho(T)$ to $R_\lambda(T) \in B(X)$, where $B(X)$ denotes the collection of bounded linear operators on X, displays analytic behavior.

3. **Properties with Full Domain:**
   (a) If $D(T) = X$, thus $T \in B(X)$:
   (b) The resolvent set $\rho(T)$ is guaranteed to be nonempty.
   (c) Moreover, the spectrum $\sigma(T)$ is both nonempty and compact.
   (d) For any λ within $\sigma(T)$, the absolute value of λ does not exceed the operator norm $\|T\|$.
4. **Resolvent Equation:**
   - For every pair of complex numbers $\lambda, \mu \in \rho(T)$, the resolvent equation $R_\lambda(T) - R_\mu(T) = (\lambda - \mu)R_\lambda(T)R_\mu(T)$ holds true.

These properties encapsulate the behavior of closed operators within Banach spaces.

Examining the spectrum of specific categories of normal operators within Hilbert spaces.

Normal operators hold significance due to their applicability in the spectral theorem. Their classification as a distinct class of operators is widely recognized.The spectral theorem states that normal operators have specific characteristics. In particular, unitary transformations can be used to diagonalize compact normal operators, especially in a finite-dimensional inner product space [34].

Proposition: 1 Consider $\mathscr{H}$ represent Hilbert space.
(a) In $\mathscr{H}$, let A be a self-adjoint operator that isn't necessarily defined or bounded on the whole of $\mathscr{H}$ as general. . Here are some key properties:

(i) The spectrum $\sigma(A)$ lies entirely within the real numbers.

(ii) The residual spectrum $\sigma_r(A)$ is vacant, indicating the absence of any residual spectrum for A.

(iii) Eigenspace of A corresponding to eigenvalue, denoted by points in $\sigma_p(A)$, is orthogonal to each other.

(b) For a unitary operator $U \in B(\mathscr{H})$, we observe:

(i) Spectrum $\sigma(U)$ forms a compact subset of a unit circle in the complex plane, specifically the set of complex numbers with magnitude 1.

(ii) The residual spectrum $\sigma_r(U)$ is also empty, implying no residual spectrum for U.

(c) When $T \in B(\mathscr{H})$ is a normal operator, the following holds:

(i) Both the residual spectra $\sigma_r(T)$ and $\sigma_r(T^*)$ are empty, indicating the absence of any residual spectrum.

(ii) The point spectra $\sigma_p(T^*)$ match those of T, denoted by $\sigma_p(T)$, signifying the same set of point spectra for T and its adjoint T^*.

(iii) The continuous spectra $\sigma_c(T^*)$ correspond to $\sigma_c(T)$ under complex conjugation, ensuring consistency between the continuous spectra of T and T^*.

3.2 Fundamental concepts in C-algebras: Gelfand-Mazur theorem, spectral norms, Gelfand's identity, Gelfand–Najmark theorem:

A cornerstone theorem in Banach algebra theory, the Gelfand–Mazur Theorem, asserts the following: When dealing with complex numbers, every instance of normed division algebra can be isomorphically matched with the algebraic structures of quaternions $\mathbb{H}$, real numbers $\mathbb{R}$, or complex numbers $\mathbb{C}$. Every complex normed division algebra isometrically isomorphic to $\mathbb{C}$ in the context of real numbers [35].

Theorem 2:Gelfand-Mazur . The complex field $\mathbb{C}$ is intrinsically isomorphic to any non-trivial complex Banach algebra $\mathfrak{B}$ containing a unit element and where every non-zero element possesses an inverse. Remarkably, $\mathfrak{B}$ also exhibits commutativity [36].
Theorem:3 The field $\mathbb{C}$ is naturally isomorphic to any non-trivial complex Banach algebra $\mathfrak{B}$ with a unit element, where every non-zero element is invertible. Furthermore, $\mathfrak{B}$ is commutative.
Theorem:4 In a Banach algebra A with identity I, for an element $a \in A$:

1. The resolvent set $\rho(a)$, representing regular values, includes complex numbers λ for which $(a-\lambda I)^{-1}$ exists in A.

2. The spectrum $\sigma(a)$ consists of complex numbers λ not in $\rho(a)$, signifying those values that render $(a-\lambda I)$ non-invertible within A.

This theorem sheds light on the definitions and implications of the resolvent set and spectrum within Banach algebras.
Defination 3 Consider B a Banach algebra with a unit element. The greatest absolute value among all of the elements in the spectrum of x is the spectral radius for an element x within B, represented as $\rho(x)$. This can be stated as:

$$\rho(x) := \sup\{|\lambda| \mid \lambda \in \sigma(x)\}.$$

This concept applies, for instance, when B is the space of bounded operators $B(X)$, here X denotes Banach space.
Propostion:2 Let A be a Banach algebra with identity element E, and let $b \in A$ and $q = q(z)$ be a complex polynomial function of $z \in \mathbb{C}$.

(a) Substituting b for z in $q(z)$, and considering the powers of b (where $b^0 := E$), results in an element $q(b) \in A$. The spectrum of $q(b)$ is then given by:

$$\sigma(q(b)) = q(\sigma(b)) := \{q(\lambda) \mid \lambda \in \sigma(b)\}.$$

This is particularly relevant when $A = B(X)$, where X is a Banach space.

(b) If A is also a *-algebra, then the spectrum of the adjoint element b^* is:

$$\sigma(b^*) = \sigma(b) := \{\lambda \mid \lambda \in \sigma(b)\}.$$

This relationship specifically applies when $A = B(H)$, where H is a Hilbert space.

Functional Analysis: Characterizing Commutative C-Algebras of Bounded Operators

In order to preserve the foundation of a commutative C * algebra, this section investigates how to represent of limited measurable function by operators on a Hilbert space through a continuous map $\Phi\&T$. Functional calculus is the name given to this representation, which involves a compact set K and a normal operator T. Creating $f(T,T*)$ on H, it bridges the gap between operators and functions. By means of the Gelfand transform, continuous functional calculus is first established abstractly on self-adjoint elements and then extended to normal elements. It characterises positive components and makes *-homomorphisms of C *-algebras easier. Operator algebras are ultimately connected to measurable functions [37].

Functional Analysis: Continuous Functional Calculus for Self-Adjoint Elements in Abstract C-Algebras

Let's delve into abstract C*-algebras, where we denote by C a C*-algebra containing the unit element I. While C can be envisioned as $B(H)$ for some Hilbert space H, the principles we're discussing extend far beyond this specific instance [38].
To define *-homomorphisms, let's first explore polynomial functions of a self-adjoint element, represented as $a^* = a \in C$. The function φ_H operates by substituting a into a polynomial with complex coefficients, treating products accordingly. Moreover, φ_H exhibits several important properties:

1. **Linearity**: $\varphi_H(\alpha p + \beta q) = \alpha\varphi_H(p) + \beta\varphi_H(q)$ for any $\alpha, \beta \in \mathbb{C}$;
2. **Product Transformation**: $\varphi_H(p \cdot q) = \varphi_H(p) \cdot \varphi_H(q)$;
3. **Identity Mapping**: $\varphi_H(1) = I$.

These characteristics establish φ_H as a mapping from the commutative *-algebra of complex polynomials to the C*-algebra C [39]. Further significant attributes encompass:

4. φ_H maps the polynomial $R \times x \to x$ to a, denoted as $\varphi_H(x) = a$;
5. For any polynomial p and its conjugate p^*, $\varphi_H(p)^* = \varphi_H(p^*)$;
6. If $ba = ab$ for some $b \in C$, then $b\varphi_H(p) = \varphi_H(p)b$.

Property (e) confirms φ_H as a *-homomorphism from the *-algebra of polynomials to C. Moreover, within the confined domain of polynomials over the compact set $\sigma(a) \subset \mathbb{R}$, $\varphi_H(p)$ assumes self-adjointness, rendering it normal. Consequently, $\|p(a)\| = r(p(a))$ and $\|\varphi_H(p)\| = \sup\{|p(x)| \mid x \in \sigma(a)\}$, signifying that if the polynomial algebra over $\sigma(a)$ adopts the norm $\|\cdot\|_\infty$, φ_H functions as an isometry. This observation extends beyond polynomials.
Theorem 5: Continuous functional calculus for mappings and self-adjoint elements. Consider a C*-algebra A with a unit element I, and let $a \in A$ be an element that equals its adjoint [40].

1. A unique *-homomorphism is defined on the commutative C*-algebra with unit $C(\sigma(a))$.
$$\Psi_a : C(\sigma(a)) \to A,$$
such that $\Psi_a(x) = a$, where x represents the mapping $\sigma(a) \times x \to x$.

2. The ensuing properties hold:

   (a) Ψ_a is an isometric mapping: for any $g \in C(\sigma(a))$, $\|\Psi_a(g)\| = \|g\|_\infty$;

   (b) If $ba = ab$ for a given $b \in A$, then $bg(a) = g(a)b$ for every $g \in C(\sigma(a))$;

   (c) Ψ_a conserves involutions: $\Psi_a(g) = \Psi_a(g)^*$ for every $g \in C(\sigma(a))$.

3. The spectrum of $g(a)$ equals $g(\sigma(a))$ for every $g \in C(\sigma(a))$.

4. Consider B as a C*-algebra with a unit element, and let $\pi : A \to B$ be a *-homomorphism:
$$\pi(g(a)) = g(\pi(a))$$
for each $g \in C(\sigma(A(a)))$.

Important Characteristics of -Homomorphisms in C-Algebras, Spectral Analysis, and Nonnegative Elements:

Grasp the characteristics of nonnegative elements and spectral analysis requires a grasp of homomorphisms in C*-algebras. Their essential qualities and importance are outlined in this introduction [41].
Theorem 6: A $*$-homomorphism $\psi : X \to Y$ between C-algebras with units is characterized by its continuity, expressed as the inequality $\psi(a)_Y \leq a_X$ for every a in X [42]. Additionally:

1. ψ is injective if and only if it preserves norms, meaning $\psi(a) = a$ for all a in X.

2. The range of X under ψ, denoted $\psi(X)$, constitutes a C*-subalgebra with a unit element in Y.

Theorem 7: Invariance of spectrum
(a) In C^*-algebras X and Y, if X is a subalgebra of Y, then the spectra of elements in X and Y coincide.
(b) For a *-homomorphism $\varphi : X \to Y$, the spectrum of $\varphi(x)$ in Y is contained in the spectrum of x in X. If φ is injective, these spectra are equal. $\sigma_Y(\varphi(x)) = \sigma_{\varphi(X)}(\varphi(x)) \subset \sigma_X(x)$ for any $x \in X$. The last inclusion is an equality if φ is injective.

Commutative Banach algebras and the Gelfand transform

In the framework of the general Banach algebras, we offer a technical solution that makes clear how maximum ideals relate to multiplicative linear functionals. Here all the Banach algebras on a complex field are considered. The commutative Gelfand-Najmark theorem, elucidating how the collection of continuous functions on a compact Hausdorff space, when endowed with the supremum norm, inherently corresponds to a commutative Banach algebra containing a unit element [43].
Defination 4 Let us take a closer look at a Banach algebra Z that has a unit element. In this case, a subset $\mathscr{M} \subseteq Z$ is considered a maximum ideal if and only if it satisfies the subsequent requirements:

1. $\mathscr{M}$ is a subset in Z;

2. For any $z \in \mathscr{M}$ and $c \in Z$, we have $cz, zc \in \mathscr{M}$;

3. $\mathscr{M} \subset Z$;

4. If $\mathscr{M} \subset \mathscr{N}$, where $\mathscr{N}$ satisfies (a) and (b), then either $\mathscr{N} = \mathscr{M}$ or $\mathscr{N} = Z$.

Defination 5:
Consider X as a Banach algebra with a unit element. Suppose $\Phi(xy) = \Phi(x)\Phi(y)$ represents a multiplicative linear functional $\Phi : X \to \mathbb{C}$, referred to as a character of X. When X is commutative, the collection of non-trivial characters forms the spectrum of the algebra, denoted by $\sigma(X)$.

Propostion 3:

1. A character ψ of A is non-zero if and only if $\psi(e) = 1$.

2. Any maximal ideal M in A is a closed set.

3. If A is commutative, then the function $\Delta : \Sigma(A) \to \mathscr{Y}$, where $\Sigma(A)$ is the spectrum of A and $\mathscr{Y}$ is the set of characters, defined by $\Delta(\xi) = Ker(\xi) \subset A$, is a one-to-one correspondence between the spectrum and the set of maximal ideals.

4. In the context of commutative A, $\Sigma(A)$ lies within the dual space A^*, indicating that all characters are indeed continuous linear functionals.

Theorem 8: Let us consider a Banach algebra $\mathfrak{B}$ with a unique unit element $\mathbb{U}$ that is commutative. Consider the Gelfand transform

$$H : \mathfrak{M} \ni y \mapsto \widetilde{y} : \sigma(\mathfrak{S}) \to \mathbb{C},$$

where

$$\widetilde{y}(\varphi) := \varphi(y), \quad y \in \mathfrak{B}, \varphi \in \sigma(\mathfrak{B}).$$

Then:

1. The set $\sigma(\mathfrak{B})$ is a σ^*-weakly compact Hausdorff space. Additionally, for any $\varphi \in \sigma(\mathfrak{B})$, the norm $\|\varphi\| \leq 1$, where $\|\cdot\|$ denotes the norm on the dual space $\mathfrak{B}^*$.

2. For any $y \in \mathfrak{S}$:
$$\sigma(y) = \{\widetilde{y}(\varphi) \mid \varphi \in \sigma(\mathfrak{S})\}.$$

3. The set $\widetilde{\mathfrak{B}}$ is contained within $C(\sigma(\mathfrak{B}))$, and the map $H : \mathfrak{B} \to C(\sigma(\mathfrak{B}))$ is a Banach algebra homomorphism with a unit.

4. The map $H : \mathfrak{B} \to C(\sigma(\mathfrak{B}))$ is continuous, and for any $y \in \mathfrak{S}$, it holds that $\|\widetilde{y}\|_\infty \leq \|y\|$.

Theorem 9: Commutative Gelfand–Najmark theorem Let's reconsider $\mathfrak{A}$ as a commutative C^*-algebra containing a unit element. When we view $C(\sigma(\mathfrak{A}))$ as a distinct commutative C^*-algebra with a unit element, using the $\|\cdot\|_\infty$ norm, we can introduce the Gelfand transform in the following manner:

$$T : \mathfrak{X} \ni y \mapsto \widetilde{y} \in C(\sigma(\mathfrak{A}))$$

where $\widetilde{y}(\varphi) := \varphi(y)$, $y \in \mathfrak{A}$, $\varphi \in \sigma(\mathfrak{A})$, serves as an isometric *-isomorphism [44].

3.3 Spectral theorems

According to the Spectral Theorem, there is an orthonormal basis of eigenvectors for any self-adjoint operator over a finite-dimensional inner product space. Eigenvalues are diagonal entries in the diagonal matrix representation of the operator within this basis. Numerous mathematical as well as practical applications are based on this theorem [45].

3.4 Spectral Theorem

Self-adjoint operator $(A, D(A))$ defined by Hilbert space H, there exists a singular spectral family $\{E_\lambda\}$, referred to as the spectral family of A, which uniquely associates with it. This spectral family satisfies the properties such that domain of A, represented by $D(A)$, is precisely domain of the spectral family, and A itself can be expressed as the integral of λ over the entire real line from $-\infty$ to ∞, weighted by the spectral projections $E(\mathrm{d}\lambda)$: Sure, here's the proper formatting in LaTeX:

$$D(A) = D_{id}\, E(\lambda) \quad \textit{and} \quad A = \int_{-\infty}^{\infty} \lambda\, E(\mathrm{d}\lambda).$$

In conclusion, spectral families and self-adjoint operators have a bijective connection. The statement of any $n \times n$ hermitian matrix was diagonalizable is extended by this theorem. This theorem requires a lengthy and nontrivial proof. We will assume it in the sequel and list the implications of this theorem. [46] .

3.4.1 Spectral decomposition

Spectral decomposition of a normal operator in functional analysis dissects an operator into its constituent eigenvalues and eigenvectors, providing a fundamental framework for understanding its behavior and structure within a Hilbert space. This decomposition offers insight into spectral properties, facilitating analysis and applications in diverse mathematical and scientific contexts.

1. Symmetric Matrices

The spectral decomposition theorem primarily applies to square matrices, especially those that are symmetric or Hermitian. A matrix A is considered symmetric if it is equal to its transpose ($A = A^T$), while matrix A is Hermitian if its conjugate transpose ($A = A^*$).

2. Eigenvalues and Eigenvectors

In spectral decomposition, we utilize the eigenvalues and eigenvectors of the matrix. Eigenvalues are scalars that represent how a transformation (or matrix multiplication) stretches or compresses space in a particular direction, eigenvectors, on the other hand, are the equivalent vectors that, despite potentially scaling, continue in the exact same direction following the transformation. .

3. Decomposition

Any symmetric (or Hermitian) matrix A decomposed in product between an orthogonal matrix Q while diagonal matrix Λ, where Q contains the associated eigenvectors and Λ includes the eigenvalues of A. This is known as the spectral decomposition theorem.
Mathematically, for a symmetric matrix:

$$A = Q\Lambda Q^T$$

And for a Hermitian matrix:

$$A = Q\Lambda Q^*$$

Where:

- Q is an orthogonal (or unitary) matrix, meaning $Q^T = Q^{-1}$ (or $Q^* = Q^{-1}$),its columns are A's eigenvectors.
- The diagonal elements of the diagonal matrix Λ constitute the eigenvalues of A.

4. Properties and Applications

Spectral decomposition provides several insights and applications:

- **Matrix Operations:** It simplifies matrix operations by diagonalizing the matrix, making calculations more straightforward.
- **Spectral Properties:** It displays the matrix's spectral characteristics, including its eigenvalues and eigenvectors, which are crucial in various mathematical and scientific analyses.
- **Signal Processing:** In signal processing, spectral decomposition is used for filtering, feature extraction, and data compression.
- **Quantum Mechanics:** In quantum mechanics, it helps describe observable quantities and their measurements in terms of eigenvalues and eigenvectors of corresponding operators.

5. Generalization

While spectral decomposition is commonly associated with matrices, it also extends to linear operators on vector spaces, especially self-adjoint operators on Hilbert spaces in functional analysis.
In summary, spectral decomposition is a powerful mathematical tool that enables us to analyze and understand the structure and behavior of matrices and operators in various contexts, providing deep insights into their spectral properties through eigenvalues and eigenvectors.

3.5 Spectral Decomposition Theorem:

Let T represent a normal operator over $\mathscr{H}$, which is a Hilbert space. Then, on the Borel subsets in the complex plane $\mathbb{C}$, there is a singular spectral measure E such that:

1. The equivalent spectral projection $E(B)$ for every Borel subset B of $\mathbb{C}$ is a bounded operator over $\mathscr{H}$ satisfies:

   - $E(B)^* = E(\overline{B})$ (where $\overline{B}$ denotes the complex conjugate of B).
   - $E(\emptyset) = 0$, $E(\mathbb{C}) = I$ (where I is the identity operator).
   - If B_1 and B_2 are disjoint Borel subsets of $\mathbb{C}$, then $E(B_1)$ and $E(B_2)$ commute (i.e., $E(B_1)E(B_2) = E(B_2)E(B_1)$).

2. The operator T has the spectral decomposition:

$$T = \int_{\mathbb{C}} \lambda \, dE(\lambda)$$

   here the Bochner integral is used to interpret the integral, and λ ranges over the spectrum of T.

Important Theorems Related to Spectral Decomposition:

1. **Compact Self-Adjoint Operators:**
   It obtains orthonormal basis of $\mathscr{H}$ that includes eigenvectors of T, and the associated eigenvalues constitute a sequence converging to zero, if T represents compact self-adjoint operator over Hilbert space $\mathscr{H}$.

2. **Continuous Functional Calculus:**
   Let f be an bounded Borel observable function on a spectrum of T, and T is normal operator in Hilbert space $\mathscr{H}$.Then, on $\mathscr{H}$, single limited operator $f(T)$ such that

$$f(T) = \int_{\sigma(T)} f(\lambda) \, dE(\lambda)$$

   Here $\sigma(T)$ represents spectrum of T.

3. **Spectral Mapping Theorem:**
   Given a Hilbert space $\mathscr{H}$ with a normal operator T, as well as a continuous function f defined on the spectrum of T, therefore

$$\sigma(f(T)) = f(\sigma(T))$$

   where $\sigma(\cdot)$ denotes spectrum.

Theorem 10 : A self-adjoint operator A in Hilbert space $\mathscr{H}$ possesses distinctive decomposition. This implies that it can be exclusively expressed as
$\langle Ax, y \rangle = \int_{\mathbb{R}} \lambda \, d \langle P_\lambda x, y \rangle_{\mathscr{H}}$,
A x $= \int_{\mathbb{R}} \lambda \, d (P_\lambda x)$
This statement essentially indicates that the result of applying the operator A to vector x and then taking the inner product with another vector y can be expressed as an integral across the real number line. In this integral, each eigenvalue λ of A plays a role based on the inner product of the projection of x onto the eigenspace associated with λ with the vector y. This distinctive representation illustrates how the action of the operator A on vectors x and y can be understood in terms of integrals over the real line, with contributions from each eigenvalue corresponding to their respective eigenspaces. Additionally, the action of A on x can be written

as an integral over the real line, where each eigenvalue λ contributes to the integral by scaling the projection of x onto its corresponding eigenspace [47].
Finally, spectral theory highlights the importance of spectra for resolvent operators by offering a thorough framework for examining operators in Hilbert spaces. Our understanding of the structure and behaviour of self-adjoint and normal operators is greatly enhanced by the spectral decomposition theorem, which opens up a wide range of mathematical or practical applications.

Chapter 4

Self adjoint operators in spectral theory

When an operator in a finite-dimensional inner product space possesses the property of self-adjoint, this termed as self-adjoint in mathematical contexts. A Hermitian matrix exhibits a characteristic where its value equals its own conjugate transpose. These operators are intimately connected to the orthonormal basis that defines the underlying space through a finite-dimensional spectral theorem. In the realm of quantum mechanics, self-adjoint operators acting upon a Hilbert space serve as physical observables like position, momentum, angular momentum, and spin. Within this article, we elucidate and explore the algebraic properties associated with self-adjoint operators [48]. The Hamiltonian operator holds immense significance in this context.

$$H\psi = V\psi - \frac{\hbar^2}{2m}\nabla^2\psi$$

which, in an actual potential field V, is equivalent to the entire energy of a particle with mass m. Differential operators stand out as a vital category among unbounded operators. Spectrum theory, dealing with linear operators in Hilbert space. Indeed, quantum mechanics and linear operators are intrinsically intertwined, each offering insights that complement the other. The emergence of quantum mechanics heralded a profound transformation in physics, prompting the exploration of nontrivial questions inherent in fundamental systems, necessitating a mathematical lens for their resolution. Notably, self-adjoint operators within Hilbert space serve as the mathematical representation of all physical observables in quantum mechanics. This article aims to provide formal elucidations of common quantum concepts, such as the significance of self-adjoint operators over merely Hermitian ones, within a text that emphasizes mathematical rigor [49].

4.1 Hellinger-Toeplitz theorem

The Hellinger-Toeplitz theorem in functional analysis states that a symmetric operator constructed over a Hilbert space is inevitably limited. An operator A is deemed symmetric if, for each pair of vectors x and y within its domain, it satisfies the condition: $\langle Ax, y\rangle = \langle x, Ay\rangle$. Notably, symmetric operators established over the entire Hilbert space also exhibit self-adjointness. Consequently, the essence of the theorem can be succinctly summarized: an everywhere-defined self-adjoint operator inevitably possesses boundedness. This theorem pays tribute to the contributions of scientists Ernst David Hellinger and Otto Toeplitz, who are commemorated through its name.

Since self-adjoint operators have been closed operators, this result can be obtained directly through the closed graph theorem. As an alternative, the uniform boundedness concept can be

utilised to demonstrate it. It is essential to recognize that the importance of the operator A stems from its capacity to define operations throughout the entire Hilbert space, thereby representing the space in its entirety.

The Hellinger-Toeplitz theorem underscores certain complexities inherent in the mathematical framework of quantum mechanics. Yet, it's noteworthy that certain observables, such as the energy operator, do not exhibit boundedness. The Hellinger-Toeplitz theorem states that such operators must be specified over a dense subset of Hilbert space rather than on the entire Hilbert space.

Let's contemplate quantum harmonic oscillator scenario, where Hilbert space under consideration is $L^2(\mathbb{R})$, depicting functions that are square-integrable across the real number line.

$$[H\psi](\xi) = -\frac{1}{2}\frac{d^2}{d\xi^2}\psi(\xi) + \frac{1}{2}\xi^2\psi(\xi)$$

This operator, being self-adjoint and unbounded, exhibits eigenvalues of $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. However, its unbounded nature implies that it lacks a well-defined action over the entirety of $L^2(\mathbb{R})$ [50].

4.2 Symmetric Operator

When an operator A in a Hilbert space H satisfies the condition

$$\langle Ax \mid y\rangle = \langle x \mid Ay\rangle$$

for every x and y within its domain, it is termed symmetric. This notion of symmetry is prevalent in the literature on functional analysis, where partially characterized linear operators derived from the continuous dual space E^* via the topological vector space E are often considered. An operator achieving symmetry across its entire domain is known as self-adjoint, and the Hellinger-Toeplitz theorem imposes bounds on such symmetric operators. Another term for bounded symmetric operators is Hermitian.
While this definition aligns with matrices when considering finite-dimensional Hilbert spaces like $\mathbf{C}^n$, it also extends to more generic contexts encompassing significant infinite-dimensional Hilbert spaces. Interestingly, some bounded symmetric operators exhibit a real spectrum, and even in cases where a symmetric operator lacks eigenvalues, any eigenvalues it does possess are always real. These observations underscore the broad applicability and significance of symmetric operators across various mathematical and physical domains [51].

4.3 Self-Adjoint Operators

On a Hilbert space H, an adjoint A^* of a densely specified linear operator A is defined as follows:

(a) A Hilbert space H, the adjoint operator A^* associated with the densely characterized linear operator A can be defined as follows: Its domain comprises vectors u in H for which the continuous linear functional arising from the densely specified linear mapping $v \to \langle u|Av\rangle$ is attainable. This mapping uniquely extends to form a continuous linear functional across the entirety of H due to the continuity and density of A's domain.

(b) According to the Riesz representations theorem every linear functionals, there exists a unique vector z within the Hilbert space H for every vector u in the adjoint operator A^*'s domain. For any vector v within the domain of the operator A, this vector z satisfies the following condition: $\langle u|Av\rangle = \langle z|v\rangle$.

It can be demonstrated that z has a linear dependence on u. Observe that the adjoint operator's well-definedness is guaranteed by the operator's dense domain and the uniqueness component of the Riesz representation. According to a Hellinger-Toeplitz type result, an operator is bounded if its adjoint is everywhere defined and bounded. Being self-adjoint is a greater requirement for a linear operator on a Hilbert space than being symmetric [38]. Every densely defined operator A in a Hilbert space admits the existence of its adjoint operator A. In the case of a symmetric operator A, the domain of A encompasses that of A, and the boundedness of A on the domain of A mirrors that of A itself. This implies an inclusive relationship, suggesting that A is a subset of A, thereby indicating that A acts as an extension of A. On the other hand, for a self-adjoint operator A, the domain of A precisely corresponds to the domain of A, and notably, A is equal to its own adjoint, denoted as $A = A$. This equivalence underscores the fundamental property of self-adjoint operators, wherein A and A exhibit complete alignment in terms of their operations and characteristics [52].

4.4 Unbounded operators

Unbounded operators in Hilbert spaces are not only manageable but also hold significant importance in quantum mechanics. Operators, denoted as T acting on a Hilbert space H, represent linear mappings with domains $\mathscr{D}(T)$ that constitute linear subspaces within H. Typically, the domain $\mathscr{D}(T)$ is densely situated within H, which characterizes T as a densely defined operator. Unbounded operators that are self-adjoint hold particular significance, as they play a vital role in describing observables within the realm of quantum physics [53].

Within the Hilbert space $L^2(\mathbb{R})$, instances of self-adjoint unbounded operators include extensions of the differential operator and the multiplication-by-x operator. These operators correspond respectively to the momentum and position observables in quantum mechanics.

(a) $(A\psi)(y) = i\frac{d}{dy}\psi(y)$ is a valid extension to the differential operator, where ψ represents a differentiable function with compact support and i is the imaginary unit.

(b) The multiplication-by-y operator: $(B\phi)(y) = y\phi(y)$.

These are the observables for momentum and position, respectively.There are many uses for unbounded linear operators, especially in quantum physics and differential equations. Essentially, the goal of creating a sound mathematical basis for quantum mechanics drove the creation of the theory pertaining to unbounded operators [54] .
Proof: The demonstration that the spectrum of a self-adjoint operator T lies within $\mathbb{R}$ is analogous to the method applied for bounded operators. To establish the non-emptiness of the spectrum, we employ a proof by contradiction. Assuming initially that T has an empty spectrum, we consider the inverse operator T^{-1}, which becomes a bounded

self-adjoint operator with a spectrum confined to $\{0\}$. Specifically, for $\lambda \neq 0$, if λ^{-1} belongs to the resolvent set $\rho(T)$, then the inverse of $T^{-1} - \lambda$ is given by $\lambda^{-1}T\left(T - \lambda^{-1}\right)^{-1}$. However, contrary to the analysis conducted for bounded self-adjoint operators, we deduce that $T^{-1} = 0$, contradicting the property $T \circ T^{-1} = I$. Notably, this assertion does not hold in scenarios where T is not self-adjoint.

4.5 Extensions Of Symmetric Operators

Extensions for symmetric operators acting upon a Hilbert space catch the attention of researchers studying functional analysis. The existence of self-adjoint extensions, and occasionally their explicit formulations, is particularly significant. This issue occurs, for instance, when formal representations of observables in quantum physics require the specification of domains of self-adjointness. Additional uses if solutions to this question can be observed in different moment problems [55].

Theorem 1: Consider a symmetric operator B. A unique, partially defined linear operator $V(B)$ that maps from $ran(B+i)$ to $ran(B-i)$. This operator is defined by the equation:

$$V(B)(By+iy) = By - iy \quad for \quad y \in dom(B).$$

The operator $V(B)$ is isometric on its domain, meaning it preserves the norm of vectors within its domain. Furthermore, the range of $1 - V(B)$ is dense in the Hilbert space K.

Conversely, though, let us consider a partially characterised operator Q that meets the criterion that $1 - Q$ includes a dense range and is isometric over its domain (which may or may not be closed). Next, from $ran(1-Q)$ to $ran(1+Q)$, there is a unique operator $T(Q)$ that is defined as follows:

$$T(Q)(y - Qy) = i(y + Qy) \quad for \quad y \in dom(Q).$$

T(Q) is a symmetric densely defined operator. The mappings denoted as V and T were reciprocal mappings. More specifically, the Cayley transform is denoted by V. It links any symmetric densely specified operator to a partially defined isometry. These mappings were monotone as well. This implies that $V(D)$ extended $V(B)$, and vice versa for T, if D constitutes a symmetric operator which extends another densely characterised symmetric operator B.

Theorem 2:A sufficient and necessary condition for a self-adjoint extension of A can be obtained straight from the reality that its Cayley transform, denoted as $\mathrm{W}(A)$, must be unitary. This condition offers an immediate criterion for determining when A can have a self-adjoint extension.

Theorem 3: It is both required and sufficient for A to possess a self-adjoint extension if $W(A)$ possesses a unitary extension. Additionally, a partially specified operator V in Hilbert space H is known as a partial isometry. This type of operator, with a closed and partially defined domain, uniquely extends isometrically beyond the norm closure of $dom(V)$.

Theorem 4: An Unfinished Isometry If and only if there are identical missing indices, V has a unitary extension. Furthermore, only when both of V's deficiency indices are zero does it possess a unique unitary extension.

4.6 Stone's Theorem

Strongly continuous unitary groups over $\mathscr{K}$ and self-adjoint operators on $\mathscr{K}$ have one-to-one connection. To be more precise, $\left\{\mathrm{e}^{-itB}\right\}_{t\in\mathbb{R}}$ forms a highly continuous unitary group if B is a self-adjoint operator on $\mathscr{K}$. Conversely, if $\{V_t\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group, we define

$$\mathrm{D}(B) := \left\{ g \in \mathscr{K} \;\; \exists s - \lim_{t\to 0} \frac{1}{t}\left[V_t - 1\right] g \right\}$$

and for $g \in \mathrm{D}(B)$, we set $Bg = s - \lim_{t\to 0} \frac{i}{t}\left[V_t - 1\right] g$. The pair $(B, \mathrm{D}(B))$ is then a self-adjoint operator [56].

Proof First, bounded , measurable functions ψ may be extended into continuous and bounded ψ via the map $\psi \mapsto \psi(B)$. By taking into account its Lebesgue-Stieltjes integrals of the weak form, this extension can be achieved. This expansion is specifically required for defining $\psi(B)$ on any Borel set W, whenever ψ is a characteristic function.

Regarding the properties regarding the unitary group $\left\{\mathrm{e}^{-isB}\right\}_{s\in\mathbb{R}}$, the second observation offers an alternate formula for $\psi(B)$. Assuming that (L1(mathbbR)) is the home of the inverse Fourier transform (checkpsi) of (psi), the following equality is true:

$$\psi(B) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{\psi}(s) \mathrm{e}^{-isB}\, \mathrm{d}s$$

Indeed, observe that:

$$\langle g, \psi(B) g \rangle = \int_{\mathbb{R}} \psi(\mu)\, n_g(\mathrm{d}\mu) = \int_{\mathbb{R}} n_g(\mathrm{d}\mu) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-i\mu s} \check{\psi}(s)\, \mathrm{d}s$$

By applying Fubini's theorem,

$$\langle g, \psi(B) g \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}s\, \check{\psi}(s) \int_{\mathbb{R}} \mathrm{e}^{-i\mu s}\, n_g(\mathrm{d}\mu)$$

$= 1 \frac{}{\sqrt{2\pi}\int_{\mathbb{R}} \mathrm{d}s\, \check{\psi}(s) \langle g, \mathrm{e}^{-isB} g\rangle}$
$= \left\langle g, \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}s\, \check{\psi}(s) \mathrm{e}^{-isB} g \right\rangle,$

4.7 Self-Adjoint Extensions In Quantum Mechanics

In quantum physics, observables are associated with self-adjoint operators. For unitary groups, these operators are exact infinitesimal generators that control the evolution of time in accordance with Stone's theorem for unitary groups with one parameter. On the other hand, time-evolution equations using differential operators, wherein the Hamiltonian is symmetric, do not always explain self-adjoint physical issues. There are two possibilities in such cases: either self-adjoint extensions to the Hamiltonian correlate to distinct kinds of conditions at infinity or boundary conditions, or the Hamiltonian is fundamentally self-adjoint, resulting in unique solutions for the physical problem [57].

Example 1. The Schrödinger operator operating in one dimension, with a potential

$$U(z) = -(1 + |z|)^{\delta}$$

initially applies to smoothly bounded functions. It exhibits essential self-adjointness as well within the range $0 < \delta \leq 2$. However, for values of δ exceeding 2, this essential self-adjointness no longer holds.

Example 2. For a particle restricted to motion along a half-line, its force operator P does not show self-adjointness. On the other hand, a Hamiltonian P^2 of a "free" particles has several self-adjoint extensions on a half-line, corresponding to different types of boundary conditions. The particle's reflections near the origin are physically related to these boundary conditions.

This mathematical construct constitutes a space formed by combining a vector space with an inner product, alternatively known as a scalar product. Every physical observable in quantum mechanics may be expressed as an infinite dimensional symmetric matrix $[(\mathrm{T}x, y)] = (x,\ \mathrm{T}y)$, which is a self-adjoint operator. For instance, when specified in the proper domain, $-\Delta$ is a self-adjoint operator on $\mathrm{L}^2\left(\Re^d\right)$. It is comparable to an unbound particle (simply moving in space) in quantum physics. Moreover, a wave function $\phi(x,t) \in \mathrm{L}^2\left(\Re^d\right)$ describes a particle's position. The wavefunction's physical interpretation is that $\int |Q(x,t)|dx$ denotes the chance of finding it in an area Ω at time t. All n-tuples of complex numbers in the space of C^n become a Hilbert space without inner product. An further prerequisite for defining a Hilbert space is the fact that each Cauchy sequence needs to converge, signifying the space's completion. Quantum mechanics provides a mathematical framework to understand the interplay between energy and matter. It posits that the phase space aligns with the Hilbert space H, alongside additional principles that govern the theory.

1. Physical states in quantum mechanics are exclusively represented by vectors with a norm of 1.
2. Vectors that vary only by a phase, specifically by a complex number with a magnitude of 1, denote equivalent physical states. This means that if ψ is a vector in Hilbert space, then $z\psi$, here z is a complex quantity with $|z| = 1$, represents the same state.

Chapter 5

Application of Spectral theory

The spectral theory of self-adjoint operators finds extensive applications in both physics and mathematics. In quantum mechanics, these operators act as representations of observables like energy and momentum, enabling a deeper understanding of quantum systems and measurement outcomes. They facilitate the decomposition of operators into eigenfunctions and eigenvalues, thereby aiding in the resolution of such issues. Spectral features are useful for functional analysis because they make it easier to investigate linear operators on Hilbert spaces and make conclusions like the spectral decomposition theorem possible. In differential geometry, the spectral theory leverages the eigenfunctions and eigenvalues associated with the Laplace-Beltrami operator to examine the geometric properties of Riemannian manifolds. Spectral analysis is a tool used in signal processing to help with filtering and compression by breaking down signals into their component frequency parts. Furthermore, numerical analysis uses spectral methods to approximate eigenvalues and eigenfunctions in order to provide high-accuracy solutions to PDEs. All things considered, the spectrum theory of self-adjoint operators provides a fundamental framework with extensive applications in solving problems in a variety of mathematical and physical fields

5.1 Spectral Theory Application in Quantum Mechanics

The area of mechanics known as quantum mechanics is concerned with with a mathematical representation of motion with respect to subatomic particles. It is concerned with describing the energy level and motion of small particles, including molecules and atoms. Like in any physical theory, the only quantities we need to take into account are those that can be measured. The observables are those physical quantities whose values are discovered through experimentation or measurement. In quantum physics, it is often impossible to forecast the exact result of a measurement.This is a real variable, and the laws of distribution of such variables are studied in quantum mechanics. Out of all the observables, energy is one that matters the most. The analogous operator, represented by H or the Schroperator, is known as the Hamiltonian.

$$i\hbar\frac{d\psi(t)}{dt} = H\psi(t)$$

Where,the Hamiltonian operator represents by H , the planck's constant is resprents by $\hbar$.
In mathematics, if an operator is self-adjoint across a finite-dimensional inner product space by its very nature, it is termed self-adjoint.These operators can be connected to the orthonormal basis describing the fundamental asset space via the finite-dimensional spectral theorem. The diagonal matrix whose members are in the real $\Re$ appears as the operator in this basis. Physical quantities in quantum physics, like spin, are represented as self-adjoint operators in a

Hilbert space, encompassing attributes like position, momentum, and angular momentum. The relevance of the algebra of self-adjoint operators in the mathematical framework of quantum mechanics is examined [58].

5.1.1 The Time-Independent Schrödinger Equation

The spatial and temporal separation required to formulate a time-independent Schrödinger equation of quantum mechanics is made easier by spectral theory. The method begins with a one-dimensional wave equation:

$$\frac{\partial^2 \mu}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 \mu}{\partial t^2}$$

This equation is fundamental for exploring refraction, interference, and other intricate optical phenomena in quantum mechanics. Using separation variable $\mu(x,t) = \psi(x)f(t)$, we get:

$$f(t)\frac{d^2\psi(x)}{dx^2} = \frac{1}{V^2}\psi(x)\frac{d^2 f(t)}{dt^2}$$

Using a standard solution for $f(t)$, such as $e^{i\omega t}$, we obtain:

$$\frac{d^2\psi(x)}{dx^2} = \frac{-\omega^2}{V^2}\psi(x)$$

This results in an ordinary differential equation that shows the matter wave's spatial amplitude as a function of position. The sum of a particle's kinetic and potential energies yields its total energy, expressed as $E = \frac{p^2}{2m} + V(x)$.
Evaluate for the momentum p, we obtain:

$$p = \sqrt{2m[E - V(x)]}$$

Using the de Broglie formula for the wavelength λ, we get:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m[E - V(x)]}}$$

The term $\frac{\omega^2}{V^2}$ from the previous equation this can be written as λ. Given $\omega = 2\pi V$ and $V\lambda = v$, we have:

$$\frac{\omega^2}{V^2} = \frac{4\pi^2 V^2}{V^2} = \frac{4\pi^2}{\lambda^2} = \frac{2m[E - V(x)]}{\hbar^2}$$

We obtain the well-known time-independent Schrödinger equation by substituting the following result back into our previous equation:

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}[E - V(x)]\psi(x) = 0$$

This equation is commonly written in a more familiar form:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

For three-dimensional cases, this extends to:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r) + V(r)\psi(r) = E\psi(r)$$

This equation is central to quantum mechanics, describing the spatial evolution of a physical system's quantum state [59].

5.2 Energy States in the Hydrogen Atom Using Spectral Theory

As mentioned earlier, a hydrogen atom comprises a nucleus carrying a positive charge of $+ze$ and a solitary electron. The potential energy V of a two-particle system with charges q_1 and q_2 is given by:

$$V = \frac{q_1 q_2}{4\pi\varepsilon_0 r}$$

where r is the distance between the particles. For the hydrogen atom, where $q_1 = +e$ and $q_2 = -e$, the potential energy can be expressed as:

$$V(r) = \frac{-e^2}{4\pi\varepsilon_0 r}$$

The hydrogen atom's potential energy function is represented by this. We must solve Schrödinger's equation using the potential energy $V(r)$ from the calculation added to determine the hydrogen atom's energy levels.
We employ Schrödinger's equation in its three-dimensional form since the electron in a hydrogen atom is contained within a three-dimensional potential well

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r) + V(r)\psi(r) = E\psi(r)$$

The energy states associated with the hydrogen atom's quantum states are obtained by solving this equation.

$$E_n = -\frac{me^4}{8\varepsilon_0^2 h^2 n^2}, \quad n = 1,2,3,\ldots$$

The energy that an electron within a hydrogen atom may possess when it is constrained by the nucleus is shown by this equation. For the hydrogen atom, the energy level diagram is well known. The lowest energy level (ground state) has an energy of -13.6 eV, which is typically occupied by the electron. Excitement states are higher energy levels that a hydrogen atom can achieve by taking in the right quantity of energy. The atom becomes ionised if the energy absorbed is high enough to allow the electron to leave the atom. Hydrogen has an ionisation energy of 13.6 eV.

5.3 Discussion

Self-adjoint operators are important because of their basic characteristics. First, eigenvalues belonging to separate eigenvectors are orthogonal since the self-adjoint operator T has real eigenvalues.. Eigenvalues and eigenvectors, serving as solutions to specific operators, are pivotal elements in spectral theory. The complete collection of eigenvalues associated with an operator constitutes its spectrum.Hermitian, and self-adjoint, operators are special in the context of quantum mechanics. These operators are important because the eigenvalues of these operators are the possible measurement outcomes for physical observables like momentum, which are represented by Hermitian operators (i.e., he operators' Hermitian nature ensures that the eigenvalues are real).

Several formulas have been derived for calculating the energy levels of the hydrogen atom, which can be applied to determine the energy states of the atom. For instance, employing the constants, ascertain the energy level of an electron in the ground state of a hydrogen atom [60].

$$\varepsilon_0 = 8.85 \times 10^{-12} C^2/N \cdot m^2, \quad m = 1.9 \times 10^{-31} kg, \quad h = 6.6 \times 10^{-34} Js, \quad e = 1.6 \times 10^{-19} C$$

we use direct substitution into the energy formula with $n = 1$:

$$E_1 = -\frac{me^4}{8\varepsilon_0^2 h^2} = -\frac{(1.6 \times 10^{-19} C)^4 \cdot (1.9 \times 10^{-31} kg)}{8 \cdot (8.85 \times 10^{-12} C^2/N \cdot m^2)^2 \cdot (6.6 \times 10^{-34} Js)^2} = -2.17 \times 10^{-18} J$$

Calculating the energy levels of various elements is also possible. In three dimensions, the energy of an atom in its n-th state can be found using a time-independent Schrödinger equation. The related relationships with a nucleus of charge Ze orbited with one electron are:

$$r_n = (0.0053\, nm) \cdot \frac{n^2}{Z} \quad and \quad E_n = -\frac{13.6\, eV \cdot Z^2}{n^2}$$

The energy levels within an atom are the precise energy values that the electron in that atom can only have according to wave mechanics. These quantized energy levels are fundamental to understanding atomic structure and behavior in quantum mechanics [59].

5.4 Spectral Theory of Self-Adjoint Operators in PDEs

One of the most important mathematical frameworks for solving partial differential equations (PDEs) is spectral theory of self-adjoint operators. This theory, which has its roots in operator theory and functional analysis, offers an organised method for comprehending how PDE solutions behave, especially when applied to engineering and physical problems. Because of their well-defined spectral features, self-adjoint operators make it possible to break down complicated PDEs into smaller, easier-to-manage parts. PDEs can be solved more easily thanks to this decomposition, which converts them into algebraic equations with eigenvalues and eigenfunctions. Spectral theory has many broad and varied applications in solving PDEs; these include wave propagation, quantum mechanics, heat conduction, and stability analysis in a variety of physical systems [61].

5.5 Eigenvalue Problem for the Laplace Operator

Let's examine the Laplace operator Δ within a bounded region Ω contained in $\mathbb{R}^n$, subject to specific boundary conditions[62]. In this context, we can pose the eigenvalue problem as follows:

Differential Equation:

$$-\Delta u = \lambda u \quad in \Omega \tag{5.1}$$

Boundary Conditions: The boundary conditions can vary, but common examples include:

- **Dirichlet Boundary Condition:**

$$u = 0 \quad on \partial\Omega \tag{5.2}$$

- **Neumann Boundary Condition:**

$$\frac{\partial u}{\partial n} = 0 \quad on \partial\Omega \tag{5.3}$$

- **Robin Boundary Condition:**

$$\frac{\partial u}{\partial n} + \alpha u = 0 \quad on \partial\Omega \tag{5.4}$$

where the normal derivative is represented by $\frac{\partial u}{\partial n}$ and α is a particular function or constant.

General Eigenvalue Problem for a Self-Adjoint Operator

In a broader sense, consider A as a self-adjoint operator over H Hilbert space. The eigenvalue problem for A is to find scalars λ (eigenvalues) and non-zero vectors $u \in H$ (eigenfunctions) such that:

$$Au = \lambda u \tag{5.5}$$

Steps to Solve the Eigenvalue Problem

1. Formulate the Operator

Define the differential operator and specify the domain and boundary conditions.

2. Find Eigenfunctions and Eigenvalues

The eigenfunctions u_n with accompanying eigenvalues λ_n can be found by solving the differential equation in conjunction with the boundary conditions.

3. Orthogonality and Completeness

When the problem has been placed in a Hilbert space, confirm that its eigenfunctions constitute an orthonormal basis of the space H. This permits the growth of functions using the eigenfunctions as a basis.

Example: Solving the Eigenvalue Problem for the Laplace Operator

Consider case where Ω is the interval $[0,L]$ in $\mathbb{R}$ and we impose Dirichlet boundary conditions:

$$\{ -\frac{d^2u}{dx^2} = \lambda u, for x \in (0,L) u(0) = 0 u(L) = 0$$

To solve this:

1. Solve the Differential Equation:

$$\frac{d^2u}{dx^2} + \lambda u = 0$$

Assume a solution of the form $u(x) = \sin(kx)$. Then:

$$k^2 = \lambda$$

2. Apply Boundary Conditions:

$$u(0) = 0\sin(0) = 0 \quad (satisfied)$$

$$u(L) = 0\sin(kL) = 0kL = n\pi k = \frac{n\pi}{L}$$

Hence:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad for n = 1,2,3,\ldots$$

3. Eigenfunctions: The corresponding eigenfunctions are:

$$u_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Thus, the eigenvalue problem is solved with eigenvalues λ_n and eigenfunctions $u_n(x)$.
Self-adjoint operator eigenvalue issues simplify complicated partial differential equations (PDEs), supporting reliable solutions in engineering and science. Under certain boundary conditions, spectral features guarantee stability and significant solutions [62].

5.6 Solving PDEs Using Separation of Variables

One effective technique to solve PDEs is variable separation, taking advantage of variable independence to make difficult problems simpler. Using this strategy, PDEs are reduced to manageable eigenvalue issues and ordinary differential equations (ODEs). It offers a strong framework for comprehending the self-adjoint operators that are inherent in PDEs, and it is enhanced by spectral theory. The explanation of eigenvalues and eigenfunctions using spectral theory improves understanding of mathematical structures and solution behaviours. In this brief review, we examine the usefulness of variable separation, enhanced by spectrum theory, in explaining homogeneous PDEs, highlighting its critical function in deciphering the complexities of physical events described by differential equations aleroev2020solving.

Example

Evaluate the one-dimensional heat equation using a simple initial condition and homogeneous Dirichlet boundary conditions:

$$u_t = u_{xx} \quad for \quad x \in [0,\pi], t > 0$$

$$u(0,t) = u(\pi,t) = 0 \quad for \quad t > 0$$
$$u(x,0) = f(x)$$

Our goal is to determine the heat equation solution that, under the provided boundary and initial conditions, solves the partial differential equation (PDE).

Solution:

1. Guessed Separated Solution: Assume a separated solution in the form $u(x,t) = F(t)G(x)$. Plugging this into the PDE, we get:

$$F(t)G''(x) = F(t)G(x)$$

2. Separation and Identification of Eigenvalue Problems:

Separating variables, we obtain:

$$\frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)} = -\lambda$$

This leads to the following ordinary differential equations:

$$F'(t) = -\lambda F(t)$$
$$G''(x) + \lambda G(x) = 0$$

At boundary conditions$u(0,t) = u(\pi,t) = 0$ to $G(x)$, we find that $G(0) = G(\pi) = 0$. Thus, we have the eigenvalue problem:

$$G''(x) + \lambda G(x) = 0, \quad G(0) = G(\pi) = 0$$

Eigenvalue problem solution are $G_n(x) = \sin\left(\frac{n\pi x}{\pi}\right) = \sin(nx)$ with corresponding eigenvalues $\lambda_n = -(n\pi)^2$ for $n = 1,2,3,\ldots$.

3. Determining Temporal Solutions:

For $F(t)$, solving the ODE $F'(t) = -\lambda F(t)$ yields $F_n(t) = b_n e^{-(n\pi)^2 t}$, where b_n are constants.

4. Combining Solutions: Therefore, The superposition of such modes provides the resultant solution for the PDE with boundary conditions:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin(nx)$$

Using the Dirichlet boundary and initial conditions we solved the heat equation. The solution provides insights into the temperature distribution over the one-dimensional domain $[0,\pi]$ at different time instances [62].

The usefulness of variable seperation approach in PDE equations having defined boundary and initial conditions is demonstrated by this problem. It emphasises how crucial it is to comprehend the characteristics of the problem domain then use the relevant solution techniques in order to produce meaningful solutions[63] .

5.7 Wave equation

The equation for waves over the interval $[0,\pi]$ is presented to us together with Neumann starting and boundary conditions. The wave equation is given by:

$$u_{tt} = c^2 u_{xx}, \quad u_x(0,t) = u_x(\pi,t) = 0$$

with initial conditions:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

Solution:

1. **Guessed Separated Solution:** We assume a separated solution of the form $u(x,t) = F(t)G(x)$. Plugging this into the wave equation, we get:

$$\frac{1}{c^2}\frac{F''}{F} = \frac{G''}{G} = -\lambda$$

2. **Separation and Identification of Eigenvalue Problems:** After obtaining the separated solutions, we separate the variables and equate each part to a constant. This leads to ordinary differential equations for $F(t)$ and $G(x)$.

   For $F(t)$, we have:

$$\frac{F''(t)}{F(t)} = -\lambda$$

   where λ is a constant.

   For $G(x)$, we have:

$$\frac{G''(x)}{G(x)} = -\lambda$$

   Minus sign is introduced to match earlier notation, and λ is again a constant.

   We may now describe the original partial differential equation into a set called ordinary differential equations, every centred on a different variable, thanks to this process of separation.

3. **Determining Eigenvalue Problems:** For $F(t)$, the ordinary the differential equation is:

$$F''(t) + c^2\lambda F(t) = 0$$

   The Neumann boundary conditions yield the following boundary conditions for $G(x)$:

$$G'(0) = G'(\pi) = 0$$

   So, For $G(x)$, the eigenvalue problem becomes:

$$G''(x) + \lambda G(x) = 0, \quad G'(0) = G'(\pi) = 0$$

   The condition $F(t)G'(0) = F(t)G'(\pi) = 0$ from the original boundary conditions implies that $G'(0) = G'(\pi) = 0$. This constraint is essential in determining the appropriate eigenvalue problem for $G(x)$.

   Separating the variables, we obtain two ordinary differential equations:

   For $G(x)$:

$$G'' + \lambda G = 0, \quad G'(0) = G'(\pi) = 0$$

   This forms the eigenvalue problem for $G(x)$.

   For $F(t)$:

$$F'' + c^2\lambda F = 0$$

   This ODE determines the temporal behavior of the solution.

4. **Solution for Eigenvalue Problem:** The solutions for $G(x)$ correspond to the eigenfunctions of the system. The eigenvalues λ and corresponding eigenfunctions $G(x)$ are obtained by solving the eigenvalue problem.

5. **Temporal Solution:** Solving the ODE for $F(t)$, we determine the temporal behavior of the solution. The solution for $F(t)$ is specified by $F(t) = A\cos(\omega t) + B\sin(\omega t)$, where $\omega = c\lambda$.
6. **Combining Solutions:** Combining both spatial and temporal solutions yields the generic solution to the waves equation given Neumann boundary conditions with initial conditions:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right) G_n(x)$$

where $G_n(x)$ are the eigenfunctions corresponding to the eigenvalues λ_n and $\omega_n = c\lambda_n$. The eigenvalue problem for the spatial component $G(x)$ is critical to understanding the behavior of the solution in this problem, which applies the spectral theory of self-adjoint operators [64]. The eigenvalue problem is caused by the Neumann boundary conditions, and its solutions yield the system's spatial modes of vibration. The way these spatial modes change over time is dictated by the temporal behavior, which is defined by $F(t)$c[62].

Example

The Spectral Aspect of Laplace's Equation in a Disc: This involves utilizing eigenfunctions and eigenvalues to comprehend the behavior of solutions $u(r,\theta)$. We investigate the spectral features crucial for analyzing physical systems by assuming a separated solution and solving the resulting ordinary differential equations for $R(r)$ and $G(\theta)$.

Problem Statement: Analyse Laplace's equation for $u(r,\theta)$ in a disc with radius a.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad \theta \in [0, 2\pi],\, r < a$$

Using a boundary condition applied to the disc's edge:

$$u(a,\theta) = f(\theta)$$

We assume a separated solution:

$$u(r,\theta) = R(r)G(\theta)$$

Substitution into the PDE: Substitute the separated solution into the partial differential equation (PDE):

$$R''(r)G(\theta) + \frac{1}{r}R'(r)G(\theta) + \frac{1}{r^2}R(r)G''(\theta) = 0$$

Divide and Rearrange: Divide the equation by RG and rearrange to separate the r and θ terms:

$$-r^2\frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} = \frac{G''(\theta)}{G(\theta)} = -\lambda$$

Obtain ODEs for R and G: This yields the ordinary differential equations for R and G:

$$G''(\theta) + \lambda G(\theta) = 0, \quad r^2R''(r) + rR'(r) - \lambda R(r) = 0$$

Boundary Condition for $G(\theta)$: Plugging into the boundary condition $u(a,\theta) = f(\theta)$:

$$R(a)G(\theta) = f(\theta)$$

However, $f(\theta)$ is not directly an eigenfunction of $G(\theta)$, so we'll address this later.
The spectral analysis of Laplace's equation involves separating variables, yielding ODEs for R and G, essential for solving physical systems.

Chapter 6

Future aspects of Spectral theory

The future trajectory of spectral theory in the realm of self-adjoint operators holds immense potential across scientific, technological, and engineering domains. In quantum mechanics, advancements in spectral theory could revolutionize information theory and quantum computing, yielding superior algorithms and more robust communication protocols. Understanding the spectral properties of material Hamiltonians linked to self-adjoint operators may unlock new materials with tailored optical, mechanical, and electrical characteristics, paving the way for innovations in photonics and energy storage technologies. Moreover, progress in quantum chemistry driven by spectral analysis promises accelerated drug discovery processes and enhanced understanding of electronic transport properties in organic materials. Biophysics and medical imaging stand to benefit from spectral analysis, offering insights into disease detection and enabling personalized therapy through precise imaging modalities. Furthermore, advancements in signal processing and communication systems, particularly through spectral analysis of self-adjoint operators, could lead to groundbreaking methods ensuring the reliability and efficiency of communication networks, including satellite communications and quantum communication protocols. As these developments unfold, they have the potential to reshape various industries and drive significant technological advancements.

The search for these novel mathematical structures would contribute to the creation of new opportunities for the advancement of mathematical analysis and physics. Besides, a study of spectral properties in finance has good applications as it helps to create a model of financial systems through the prediction of asset prices — which also involves investment risk management. These milestones are likely to develop better financial models with effective algorithms for pricing derivatives and portfolio optimization: in turn, this will spur innovation leading towards economic expansion.

Chapter 7

Conclusion

This paper explores the algebraic properties and applications of self-adjoint operators in spectral theory, with a focus on quantum mechanics. It talks about self-adjoint and symmetric operators and emphasises how they describe physical observables. Gaining insight into ideas such as deficiency indices and the Cayley transform improves comprehension of self-adjoint extensions. It also shows how important these ideas are for the time-independent Schrödinger equation, which sheds light on the energy states of hydrogen atoms. The paper emphasises the fundamental function of self-adjoint operators in quantum mechanics through rigorous mathematical derivations and examples, successfully linking abstract mathematical ideas with tangible physical occurrences. The spectral theory of self-adjoint operators provides insights into Laplace's equation and other physical processes like heat conduction and wave propagation by decomposing PDEs into eigenvalue problems. The importance of the theory is demonstrated by eigenvalues and eigenfunctions obtained by spectral analysis, which clarify solution behaviour. It facilitates the comprehension and solution of complicated PDEs across disciplines by acting as a vital link between theory and application. This thorough investigation highlights the significant ramifications of spectral theory in numerous scientific and technical fields, deepening our grasp of the complex interplay between mathematical frameworks and the explanation of observable phenomena.

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