# ANALYSIS OF CERTAIN APPROXIMATION OPERATORS

*A Thesis Submitted to*

**Delhi Technological University**

*for the Award of Degree of*

**Doctor of Philosophy**

*in*

**Mathematics**

*By*


**Chandra Prakash**

**(Enrollment No.: 2K18/PHD/AM/504)**


*Under the Supervision of*


**Prof. Naokant Deo**

*and*

**Dr. Durvesh Kumar Verma**

**Department of Applied Mathematics**

**Delhi Technological University (Formerly DCE)**

**Bawana Road, Delhi-110042, India.**


**APRIL, 2024**

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# DECLARATION

I declare that the research work in this thesis entitled "**Analysis of Certain Approximation Operators**" for the award of the degree of ***Doctor of Philosophy*** in ***Mathematics*** has been carried out by me under the supervision of ***Prof. Naokant deo***, Department of Applied Mathematics, Delhi Technological University, Delhi, and ***Dr. Durvesh Kumar Verma***, Department of Mathematics, Miranda House, University of Delhi, Delhi, India, and has not been submitted by me earlier in part or full to any other university or institute for the award of any degree or diploma.

I declare that this thesis represents my ideas in my own words and where other's ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission.

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## CERTIFICATE

This is to certify that the research work embodied in the thesis entitled **"Analysis of Certain Approximation Operators"** submitted by **Mr. Chandra Prakash** with enrollment number (2K18/PHD/AM/504) is the result of his original research carried out in the Department of Applied Mathematics, Delhi Technological University, Delhi, for the award of **Doctor of Philosophy** under the supervision of **Prof. Naokant Deo**, and **Dr. Durvesh Kumar Verma**.

It is further certified that this work is original and has not been submitted in part or fully to any other University or Institute for the award of any degree or diploma.

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

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**Date :** **(Chandra Prakash)**

**Place : Delhi, India.**

**Dedicated to My Parents**

**Teachers**

**&**

**Wife**

# Contents

# Abstract

The thesis is divided into seven chapters, the contents which are organized as follows:

**Chapter**1 of the thesis covers the literature and historical foundation of certain important approximation operators. We give a brief overview of the chapters that make up this thesis and talk about some of the preliminary tools we'll use to get to the subject depth.

**Chapter** 2 introduces a new sequence of operators involving Apostol-Genocchi polynomials and their integral variants. We estimate some direct convergence results using the second-order modulus of continuity, Voronovskaja type approximation theorem. Moreover, we find weighted approximation results of these operators. Finally, we derive the Kantorovich variant of the given operators involving Apostol-Genocchi polynomials, and their approximation properties are studied.

Next **Chapter** 3 is mainly focused on the Bézier variant of the Bernstein-Durrmeyer type operators. First, we estimate the moments for these operators. Then, we determine the rate of approximation of the operators $\breve{R}_{n,r,s}^{(\rho,\alpha)}(f;x)$ in terms of the Ditzian-Totik modulus of continuity and over the Lipschitz-type spaces. It is addressed how smooth functions with derivatives of bounded variation converge. Finally, graphic depiction of the theoretical findings and the efficiency of these operators are shown.

**Chapter** 4 deals with certain approximation properties of Cheney-Sharma Chlodovsky Durrmeyer operators. Using the moments of these operators Bohman-Korovkin's theorem is validated. After that, the convergence of the CSCD operators is discussed over Lipschitz-type space and in terms of modulus of continuity. In the next section, the weighted approximation result is obtained. Lastly, some estimates on the A-Statistical convergence of these operators are established.

**Chapter** 5 provides the generalization of the family of Bernstein polynomials over a different set of operators proposed by Mache and Zhou [66]. We investigate certain approximation properties for these operators, such as the rate of convergence via second-order modulus of continuity, Lipschitz space, Ditzian-Totik moduli of smoothness, Voronovskaya theorem, Gruss-Voronovskaya theorem, and weighted approximation properties. Finally, we have illustrated the convergence of our operators graphically. In the next section of this chapter, we consider the Durrmeyer variant of modified Bernstein polynomials. First, we provide the auxiliary results and demonstrate the Bohman-Korovkin theorem. Then, we explore some approximation properties such as the rate of convergence using the Ditzian-Totik modulus of continuity, Voronovaskaja type and weighted approximation theorem for these operators. Finally, the convergence behavior have been shown graphically.

The aim of **Chapter** 6 is to introduce and study a new sequence of operators using Appell polynomials of class $A^2$. First, the moments for these operators are established. Then, we study an estimate of error in approximation in terms of modulus of continuity and rate of convergence in weighted space for these operators. Finally, we obtain the rate of convergence for the function having the derivatives of bounded variation.

The thesis is summarised in **Chapter 7**, before providing some insight into the author's thoughts about the future research.

# List of Figures

# List of Symbols and Notations

| | |
|---|---|
| $\mathbb{N}$ | set of natural numbers. |
| $\mathbb{N} \cup \{0\}$ | the set of natural number including zero, |
| $\mathbb{R}$ | set of real numbers, |
| $\mathbb{R}^+$ | the set of positive real numbers, |
| $[a,b]$ | a closed interval, |
| $(a,b)$ | an open interval, |
| $C[a,b]$ | the set of all real-valued and continuous function defined in $[a,b]$ |
| $C^r[a,b]$ | the set of all real-valued, $r$-times continuously differentiable function ($r \in \mathbb{N}$), |
| $C^2[0,\infty)$ | the set of all real-valued, 2-times continuously differentiable function on $[0,\infty)$, |
| $C[0,\infty)$ | the set of all continuous functions defined on $[0,\infty)$, |
| $C_B[0,\infty)$ | the set of all bounded functions on $C[0,\infty)$, |
| $C_B^r[0,\infty)$ | the set of all $r$-times continuously differentiable functions in $C_B[0,\infty)$ ($r \in \mathbb{N}$) |
| | $\lvert f(x)\rvert \leq M(1+x^2)$, $M$ is a positive constant, |
| $C_\tau^*[0,\infty)$ | the subspace of all function $f \in C_B[0,\infty)$ for which $\lim\limits_{x\to\infty} \frac{f(x)}{1+x^2}$ exists and finite. |
| $\lVert f\rVert$ | $\lVert f\rVert = \sup\{\lvert f(x)\rvert : x \in [a,b]\}$, |
| $Lip_\sigma(M)$ | the set of all $C[a;b]$- functions which holds the Lipschitz condition |
| | $\lvert f(t)-f(x)\rvert \leq M\lvert t-x\rvert^\sigma$ for all $t,x \in [a,b]$, $0<\sigma\leq 1$, $M>0$, |
| $e_m$ | the test function with $e_m(x)=x^m$, $m \in \mathbb{N}\cup\{0\}$. |

# List of Publications and Conferences

## Published

1. **Chandra Prakash**, D. K. Verma, Naokant Deo. Approximation by a new sequence of operators involving Apostol-Genocchi polynomials. Mathematica Slovaca, 71(5), 1179–1188 (2021). https://doi.org/10.1515/ms2021–0047 **(SCIE, Impact Factor: 0.996)**

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3. **Chandra prakash**, Naokant Deo. D. K. Verma. Approximation by Apostol-Genocchi Summation integral type operators. Miskolc Mathematical Notes. 24(1), 369–382 (2023). https://doi.org/10.18514/MMN.2023.3848 **(SCIE, Impact Factor: 1.22)**

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2. Naokant Deo, **Chandra Prakash**, D. K. Verma. Approximation by $\alpha-$Bernstein operators based on certain parameters, Communicated

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## Papers presented in International Conferences

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2. Approximation by the operators using Apostol-Genocchi polynomials; $2^{nd}$ International Conference on Nonlinear Applied Analysis and Optimization and NMD (ICNAAO-2022), Department of Mathematical Science, IIT(BHU), Varanasi, Uttar Pradesh. December 19–22, 2022

3. Approximation by Integral Form of modified Bernstein polynomials; International Conference on Analysis and Its Applications (ICAA-2023), Department of Mathematics, Shivaji College, University of Delhi, Delhi. February 27–28, 2023

# Chapter 1

# Introduction

### 1.0.1 Historical Backdrop Review

One of the branches of mathematical analysis, the theory of approximation plays a principal role. The theory of convergence of these types of sequences has been a major field of research in the past couple of decades. The approximation theory contains both theoretical and practical components. Even after not being the best approximation polynomials, linear positive operators have some desirable characteristics, such as the willingness to approximate derivatives, maintain certain function properties (convexity, smoothness), and classify prioritizes of functions according to the degree of approximation that can be accomplished.

One of the most fundamental theorems of approximation theory was stated by Weierstrass in the year 1885 and is also known as the Weierstrass approximation theorem. This theorem states that given any continuous function $f(x)$ on an interval $[a,b]$ and a sufference $\varepsilon > 0$, a polynomial $p_n(x)$ on sufficiently degree $n$ exists, such that $|f(x) - p_n(x)| < \varepsilon$, for $x \in [a,b]$. The sequence of polynomials can be used to approximate any continuous function $f(x)$ uniformly. In the case of trigonometric polynomials, the Weierstrass theorem says that if a continuous function $f$ has period $2\pi$ on the real axis then this function $f$ can be uniformly approximated by a trigonometric polynomials $T_n(x)$ on an interval.

After that, important mathematicians affiliated with the Weierstrass approximation theorem include Runge, Lebesgue, Landau, Vallee-Poussin, Fejer, Jackson, and Bernstein. Lebesgue, Landau, and Sergei Natanovich Bernstein gave the analytical proof of the Weierstrass theorem in respective publications in the year 1898, 1908 and 1912, respectively. Russian mathematician Bernstein [14] developed the Bernstien polynomial and this polynomial sequence uniformly converges to $f$ on $[0,1]$, also its derivative converges to $f$ as $n \to \infty$, providing useful proof of the Weierstrass theorem. Bernstein Operators $B_n : C[0,1] \to \mathbb{R}$ are given by

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where the basis function $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. $B_n(f,x)$ are a convex combination of $f(0)$, $f(\frac{1}{n}), \cdots f(1)$. These operators are linear and positive.

Stancu polynomials [90], Durrmeyer polynomials [27], Szász operators [96], Cheney–Sharma operators [16], modified Berstein polynomials, modified Szász operators, Kantorovich operators [60], Baskakov operators [12], and more are just a few examples of other approximation operators whose construction are proposed by the Bernstein polynomials as a result of there continuous form.

Myriad researchers worked on these finite summation polynomials and achieved approximation results on the finite intervals. Over the last few years several mathematicians have worked on new modifications of such operators and studied the approximation properties like the degree of approximation and asymptotic formula (cf.[21, 22, 42, 44, 46] and [55]).

Addressing the main concepts of approximation theory (rate of convergence and asymptotic behavior) is the goal of a general approximation method. To bring it to the appropriate stage, the accuracy can be established. We are also concerned about the computation that be required to accomplish this accuracy. A direct theorem gives the order of approximation for functions with a certain level of smoothness. In modeling real-world processes, asymptotic analysis is a crucial tool for investigating the ordinary and partial differential equations that occur there. A convergent sequence approaches its limit quickly according to the rate of convergence.

Working continuously in this area, S. Mirakyan [68] and O. Szász [96] proposed the extension of the Bernstein operators for $[0,\infty)$ in 1941 and 1950, defined as:

$$S_n(f,x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in [0,\infty), \tag{1.1}$$

where $f$ is a continuous function on $[0,\infty)$. These operators are called Szász-Mirakyan operators in (1.1). Generalization of Szász operators for the approximation in the infinite interval was presented by Jakimovski et al..

In 1972, one step forward in the field of the industry was given by Bézier, who invented the Renault vehicle and introduced the curve is known as Bézier curves. Symmetry of these curves makes them valuable for computer-aided design. In 1983, Chang [35] discussed the generalized Bernstein-Bézier polynomials and calculated the rate of convergence. Zeng and Piriou [102] studied the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions. The order of approximation of the summation-integral type operators for functions with derivatives of bounded variation is estimated in [7, 100, 103]. Various authors have studied the approximation behaviour of such Bézier type operators (see [1, 39, 48, 85, 89]).

Positive linear operators prompted various researchers interest in the theory of approximation of functions following the work of Bohman and Korovkin. Several examples of new sequences and classes of linear positive operators were created and investigated as well as the previously established cases. Operators are broadly applied in many computer science and mathematics applications. Moments serve a vital significance for comprehending how a sequence of linear positive operators converges. Moments can be obtained using a multitude of methods, involving direct computations, recurrence relations, and the use of hypergeometric series. Degree of approximation and the asymptotic formula are two approximation qualities that many mathematicians have been studying recently while working on novel adaptations of these operators.

In the study of the major theorems of approximation theory, rate of convergence, asymptotic behavior are important aspects. The order of approximation for functions of a particular smoothness is given by a direct theorem. When modeling real-world processes mathematically, asymptotic analysis is a crucial tool for investigating the ordinary and partial differential equations results.

Many connections exist between mathematics and other applied sciences, including machine learning, computer-aided geometric design (CAGD), and combinatorics, due to the generating functions. The purpose of generating functions is to efficiently transfer issues involving sequences into problems involving functions. Many researchers have used Genocchi polynomials and found results from generating functions in their studies. Counting the number of up-down ascent sequences is one of the applications of Genocchi numbers that Jeff Remmel studied. Srivastava et al.[84] proposed a new form of Euler-type polynomials with the aid of their generating function and presented an analog abstraction of the closely related Genocchi-type polynomials.

Approximation of continuous functions by linear positive operators using the statistical convergence or matrix summability approach is one of the most actively researched topics in approximation theory. In mathematics perception, when a sequence is not convergent, we use uniform convergence alongside statistical convergence and utilize gesture approximation for different function spaces. Initial approach to address statistical convergence for linear positive operator sequences was put forward in 2002 by A. D. Gadjiev and C. Orhan. Many researchers have turned to approximation theory as an outcome of this technique's proven fruitfulness. It has been noted in the statistical convergence theory that certain sequences have statistical convergence but not regular convergence. Inspired by above fact, our focus is to derive positive linear operators of integral type and study their statistical approximation effects.

## 1.1 Definitions and well-known operators

### 1.1.1 Definitions

**Definition 1.1.1 (Positive linear operators).** Assume that $X,Y$ are two real function linear spaces. A linear operator is defined as a mapping $L: X \to Y$, if

$$L(af+bg;x) = aL(f;x)+bL(g;x),\ f,g \in X,\ \&\ a,b \in \mathbb{R}.$$

The operators $L$ is a positive linear operator if for all $f \geq 0$, $f \in X$ such that $L(f;x) \geq 0$.

**Proposition 1.1.1.** *Consider $L: X \to Y$ be a of positive linear operator. Then the following inequalities are true:*

- *If $f,g \in X$, with $f \leq g$, then $L(f;x) \leq L(g;x)$,*
- $|L(f;x)| \leq L(|f|;x)$.

**Definition 1.1.2 (Modulus of continuity).** As a long history of the modulus of smoothness (continuity), In 1911, D. Jackson used it in his Ph.D. thesis, this work is known today as Quantitative Approximation Theory. Continuation work in this area, In the year 1987, Ditzian and Totik proposed a natural modulus of smoothness which is a better outfit to give out with the rate of best approximation.
For $k \in \mathbb{N}$, $\delta \in \mathbb{R}^+$ and $f \in C[a,b]$, then the modulus of smoothness of order $k$ is defined by

$$\omega_k(f,\delta) = \sup\{|\Delta_h^k f(x)|\ |0 \leq h \leq \delta,\ x, x+kh \in [a,b]\},$$

where

$$\Delta_\delta^k f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x+j\delta).$$

**Definition 1.1.3 (Peetre-K functional).** The regular modulus of continuity of $f \in C[a,b]$ by

$$\omega(f,t) = \sup_{0<h\leq t}\ \sup_{x,x+h\in[a,b]} |f(x+h)-f(x)|.$$

Let $f \in C[a,b]$, the Peetre's $\mathscr{K}$-functional is defined by

$$\mathscr{K}_2(f;t) = \inf_{g\in C^2[a,b]} \{\|f-g\| + t\|g'\| + t^2\|g''\|\},\ t > 0$$

and $C^2[a,b] = \{g \in C[a,b] : g', g'' \in C[a,b]\}$. By [16], there exists an absolute constant $C > 0$ such that

$$\mathscr{K}_2(f;t) \leq C\omega_2(f;\sqrt{t}). \tag{1.2}$$

For $f \in C_B[0,\infty)$ and $\delta > 0$, Peetre-K functional, a different technique to estimate the smoothness of a functions is given by

$$K_2(f,\delta) = \inf_{g \in W^2} \{\|f-g\| + \delta\|g''\|\},$$

where $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$ and $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$. By ([24] p.177, Thm. 2.4), there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\,\omega_2(f,\sqrt{\delta}), \tag{1.3}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness.

**Definition 1.1.4 (Ditzian-Totik modulus of smoothness).** The Ditzian-Totik modulus of smoothness $\omega_\phi(f,t),\ t \in [0,1]$ (cf. [25]). For $\phi(x) = \sqrt{x(1-x)},\ f \in C[0,1]$, the first order modulus of smoothness is given by

$$\omega_\phi(f;t) = \sup_{0<h\leq t} \left\{ \left| f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right) \right|, x \pm \frac{h\phi(x)}{2} \in [0,1] \right\},$$

and the appropriate Peetre's $K-$functional is defined by

$$\overline{K}_\phi(f;t) = \inf_{g \in W_\phi} \{\|f-g\| + t\|\phi g'\| + t^2\|g'\|\}\ (t > 0),$$

where $W_\phi = \{g : g \in AC_{loc}, \|\phi g'\| < \infty, \|g'\| < \infty\}$ and $\|.\|$ is the uniform norm on $C[0,1]$. It is known from ([25], Thm. 3.1.2) that $\overline{K}_\phi(f;t) \sim \omega_\phi(f;t)$ which means there exists a constant $M > 0$ such that

$$M^{-1}\omega_\phi(f;t) \leq \overline{K}_\phi(f;t) \leq M\omega_\phi(f;t). \tag{1.4}$$

**Definition 1.1.5 (Lipschitz class).** Spaces of the Lipschitz type are defined as:

$$Lip_M(\sigma) = \left\{ f \in C_B[0,\infty) : |f(t) - f(x)| \leq\ M\frac{|t-x|^\sigma}{(t+x)^{\frac{\sigma}{2}}} \right\}.$$

Özarslan and Aktuğlu [71], consider the Lipschitz-type space with two parameters $a,b > 0$, we have

$$Lip_M^{a,b}(\sigma) = \left\{ f \in C_B[0,\infty) : |f(t) - f(x) \leq\ M\frac{|t-x|^\sigma}{(t+ax^2+bx)^{\frac{\sigma}{2}}} \right\},$$

where $0 < x,t < \infty$, $M > 0$ is a constant and $0 < \sigma \leq 1$.

**Definition 1.1.6 (Weighted Approximation).** Consider the space

$$C_\tau^*[0,\infty) = \left\{ f \in C_B[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{\tau(x)} \text{ exists and is finite} \right\}.$$

Yüksel and Ispir defined the weighted modulus of continuity $\Omega^*(f;\delta)$, as

$$\Omega^*(f;\delta) = \sup_{x\in[0,\infty),0<h<\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^2},$$

where $f \in C_\tau^*[0,\infty),\ \tau(x) = 1+x^2$.

**Proposition 1.1.2.** *Let $f \in C_\tau^*[0,\infty)$. Then the following results hold:*

1. $\Omega^*(f;\delta)$ *is a monotonically increasing function of* $\delta$;
2. $\lim_{\delta\to 0^+} \Omega^*(f;\delta) = 0$;
3. *For each* $m \in \mathbb{N}$, $\Omega^*(f;m\delta) \le m\Omega^*(f;\delta)$;
4. *For each* $\lambda \in (0,\infty)$, $\Omega^*(f;\lambda\delta) \le (1+\lambda)\Omega^*(f;\delta)$.

**Definition 1.1.7 (Statistical Convergence).** Steinhaus first laid out the concept of statistical convergence over sixty years ago [94]. Fast [28] explored it further, as well as several authors glanced into it. Various researchers have engaged in the statistical affiliation Korovkin type approximation theorems for positive linear operators after Gadjiev and Orhan [31]. The essential objective underlying the statistical convergence of a sequence is that we just incorporate the majority of the sequence's components and discard the remaining elements. The sequence of numbers $(x_n)$ is called statistical convergent to $l$, if for every $\varepsilon > 0$,

$$\lim_j \frac{|\{n \le j : |x_n - l| \ge \varepsilon\}|}{j} = 0.$$

It is denoted by $st - \lim_n x_n = l$. The hypothesis is basically depends on the non-negative regular summability matrix $A = (a_{jn})$. Then, the $A-$ transformation of $x$, denoted by $Ax = ((Ax)_j)$, where $(Ax)_j = \sum_{n=1}^{\infty} a_{jn}x_n$, on condition that the series converges for each $j$. The summability matrix $A$ is regular if $\lim_j (Ax)_j = l$ whenever $\lim_j x_j = l$.

In 1981, Freedman et.al. defined the A-density of $K \subset \mathbb{N}$ by

$$\delta_A(K) = \lim_j \sum_{n\in K} a_{jn},$$

on condition that the limit exists.

A sequence $x = (x_n)$ is said to be A-statistically convergent to $l$, if for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n-l|\ge\varepsilon} a_{jn} = 0$$

holds, and it is denoted by

$$st_A - \lim_n x_n = l,$$

if for every $\varepsilon > 0, \delta_A\{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\} = 0$. In the particular case, when $A = C_1 = [c_{jn}]$, where the Cesáro matrix is given by

$$c_{jn} = \begin{cases} \frac{1}{j}, & \text{if } 1 \leq n \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

the Cesáro matrix of order one, A-statistical convergence coincides with the statistical convergence [30], *i.e.*, $st_{C_1} - \lim_n x_n = st - \lim_n x_n = l$. When we have take $A = I$, the identity matrix then A-statistical convergence reduces to the ordinary convergence, *i.e.*, $st_I - \lim x = \lim x = l$. Kolk [61] proved that in the case of $\lim_j \max |a_{jn}| = 0$. A-statistical convergence is stronger than ordinary convergence.

**Definition 1.1.8 (Derivative of Bounded Variation (DBV)).** Let DBV$[a,b]$ be the class of all functions in $C_B[a,b]$ having a derivative that is local of bounded variation on $[a,b]$. The function $f \in DBV[a,b]$ is defined as

$$f(x) = \int_0^x g(t)dt + f(0),$$

where $g$ is a function of bounded variation on every finite subintervals of $[a,b]$.

**Definition 1.1.9 (Bohman-Korovkin result).** [62] The essential condition for the feasibility of the function is an approximation approach using linear positive operators. The Bohmn-Korovkin theorem, are a result of this explanatory problem. This explanatory problem gives credence to the Bohman-Korovkin theorem. If following conditions

1. $L_n(e_0;x) = 1 + \alpha_n(x)$.

2. $L_n(e_1;x) = x + \beta_n(x)$

3. $L_n(e_2;x) = x^2 + \gamma_n(x)$

are satisfied by the positive linear operators $L_n : C[a,b] \to C[a,b]$ such that $\alpha_n(x)$, $\beta_n(x)$, and $\gamma_n(x)$ uniformly converges to zero in $[a,b]$ then $L_n(f;x)$ converges uniformly to $f(x)$. T. Popoviciu, H. Bohman, and P. P. Korovkin, three eminent mathematicians, presented this hypothesis in the years 1951, 1952, and 1953, respectively.

**Definition 1.1.10 (Hölder inequality (finite form)).** For $1 < p < \infty$, $1 < q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, the Hölder inequality is defined as

$$\sum_{l=0}^{n} |\xi_l \eta_l| \leq \left( \sum_{l=0}^{n} (\xi_l)^p \right)^{\frac{1}{p}} \left( \sum_{l=0}^{n} (\eta_l)^q \right)^{\frac{1}{q}},$$

where $\xi_l, \eta_l \in \mathbb{R}$.

**Definition 1.1.11** (**Cauchy-Schwarz inequality (finite form)**). The Cauchy-Schwarz inequality is defined as

$$\sum_{l=0}^{n} |\xi_l \eta_l| \leq \left( \sum_{l=0}^{n} (\xi_l)^2 \right)^{\frac{1}{2}} \left( \sum_{l=0}^{n} (\eta_l)^2 \right)^{\frac{1}{2}},$$

where $\xi_l, \eta_l \in \mathbb{R}$.

### 1.1.2 Some well-known operators

This section deals with some well-studied operators.

**(i)** The linear positive operator $K_n : C[0,1] \to C[0,1]$ was proposed and investigated in the year 1930, by L. V. Kantorovich [60], which are defined as:

$$K_n(f;x) = n \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt.$$

**(ii)** With the use of a Lebesgue integrable function on the interval $[0,1]$, Durrmeyer constructed an integral modification of the Bernstein polynomials in 1967 [27], which are as follows: $D_n : C[0,1] \to C[0,1]$

$$D_n(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)dt,$$

where $0 \leq x \leq 1$, $f \in [0,1]$.

**(iii)** In 1985, Mazhar and Totik studied what they called Szász-Durrmeyer operators, which are integral modifications of Szász operators. They are as:

$$S_n(f,x) = n \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^k}{k!} f(t)dt.$$

# Chapter 2

# Approximation by a New Sequence of Operators Involving Apostol-Genocchi Polynomials and its Summation-Integral form

---

The Italian mathematician Angelo Genocchi discovered the unique class of polynomials known as Genocchi polynomials in the $18^{th}$ century. By using these polynomials, he worked out the Genocchi numbers. Layout of this chapter is based on two sections. In the first section, we present a new sequence of operators involving Apostol-Genocchi polynomials, Here, we address the preliminary results and estimate some direct convergence outcomes using the second-order modulus of continuity and the Voronovaskaja-type asymptote. Furthermore, we find the weighted approximation results and the Kantorovich variant using the proposed operators. Next section, we provide the integral variant of the proposed operators given in the first section and study approximation results using the second-order modulus of continuity, Voronovskaja-type asymptotic theorem, Lipschitz-space, and Ditzian-Totik modulus of smoothness. Finally, we present the weighted approximation results using the integral variant.

---

## 2.1 Introduction

A group of polynomials on the real number field that includes numerous classical polynomial systems. P. E. Appell [10] established their Appell polynomials. The description of the Appell polynomial series $A_n(x)$ is as:

$$e^{xt}f(t)=\sum_{n=0}^{\infty}A_n(x)\frac{t^n}{n!},$$

where $f(t)$ be a formal power series in $t$. This polynomial found stunning applications in several field of mathematics, see [49, 86, 87]. One special case known as Genocchi polynomials $G_n(x)$ are defined by

$$\frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\ |t|<\pi.$$

In this case the Genocchi numbers $G_n$ have many applications in number theory, special functions, combinatorics and numerical analysis, where

$$\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!},\ |t|<\pi.$$

The Apostol-Genocchi polynomials $G_k^{\alpha}(x;\beta)$, $\beta\in\mathbb{R}$, of order $\alpha$ (non negative integers) are defined and found some results in [49].

Motivated by such work we proposed a new sequence of the operators by the use of Apostol-Genocchi polynomials defined as:

$$\mathscr{M}_n^{\alpha,\beta}(f;x)=e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\sum_{k=0}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}f\left(\frac{k}{n}\right),\ f\in C[0,\infty), \tag{2.1}$$

where $G_k^{\alpha}(x;\beta)$ is generalized Apostol-Genocchi polynomials, which have the generating function of the form

$$\left(\frac{2t}{1+\beta e^t}\right)^{\alpha}e^{xt}=\sum_{k=0}^{\infty}G_k^{\alpha}(x;\beta)\frac{t^k}{k!},\qquad (\alpha,\beta\in\mathbb{C},|t|<\pi). \tag{2.2}$$

The Apostol-Genocchi polynomials and their properties are studied by many researchers for the detail here we refer (cf. [11, 54, 63, 64, 72, 88]).

In [65], the following explicit formula for the Apostol-Genocchi polynomials $G_k^{\alpha}(x;\beta)$ is given:

$$\begin{aligned}G_k^{\alpha}(x;\beta) &= 2^{\alpha}\alpha!\binom{k}{\alpha}\sum_{i=0}^{k-\alpha}\frac{\beta^i}{(1+\beta)^{\alpha+i}}\binom{k-\alpha}{i}\binom{\alpha+i-1}{i}\\ &\quad\times\sum_{j=0}^{i}(-1)^j\binom{i}{j}j^i(x+j)^{k-i-\alpha}{}_2F_1[\alpha+i-k,i;i+1;j/(x+j)],\end{aligned}$$

where $k,\alpha\in\mathbb{N}\cup\{0\}$, $\beta\in\mathbb{C}\setminus\{-1\}$ and ${}_2F_1[a,b;c;z]$ denotes the Gaussian hypergeometric function defined by

$$ {}_2F_1[a,b;c;z] \quad = \quad \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c} \frac{z^2}{2!} + \cdots, $$

where $(\alpha)_0 = 1$, $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}$, $(n \geq 1)$, (see [93], pp. 37).

**Remark 2.1.1.** *For $\alpha = 1$ and $\beta = \lambda$ our operators (2.1) reduces to following operators*

$$ \mathscr{M}_{n,\lambda}(f;x) \quad = \quad \frac{e\lambda+1}{2e^{nx}} \sum_{k=0}^{\infty} \frac{G_k(nx;\lambda)}{k!} f\left(\frac{k}{n}\right), \tag{2.3} $$

*where $G_k(x;\lambda)$ are the Apostol-Genocchi polynomials generated by the function*

$$ \frac{2te^{xt}}{e^t\lambda+1} = \sum_{k=0}^{\infty} G_k(x;\lambda) \frac{t^k}{k!}, \quad (|t+\log\lambda| < \pi, \lambda \neq 1; x \in \mathbb{R}). $$

*When $\lambda = 1$ the operators (2.3) involves the classical Genocchi polynomials.*

For $\lambda = 1$ and $x = 0$, the Apostol Genocchi polynomials reduce to the Genocchi numbers which have many applications from Combinatorics to numerical analysis and other fields of applied mathematics like number theory. As such, it makes this very appealing to use for applications in Combinatorics. One of the applications of Genocchi numbers that was investigated by Jeff Remmel in [78] is counting the number of up-down ascent sequences. Another application of Genocchi numbers is in Graph Theory. For instance, Boolean numbers of the associated Ferrers Graphs are the Genocchi numbers of the second kind [20].

The aim of this section is first to give moments for given operators (2.1). Then, we estimated the direct theorem and Voronovskaja type asymptotic formula. Next, we have established weighted approximation results for these operators. In the last, we have given the Kantorovich variant of the given operators. Then we discussed the convergence of the operators (2.7) using Korovkin's theorem and the order of approximation in terms of modulus of continuity.

## 2.2 Approximation by the Operators involving Apostol-Genocchi Polynomials

### 2.2.1 Preliminaries

The auxiliary results of the $\mathscr{M}_n^{\alpha,\beta}$ operators are estimated in this section, which will be useful in the main outcome.

**Lemma 2.2.1.** *For* $\mathscr{M}_n^{\alpha,\beta}(t^m;x)$, $m=0,1,2,3$ *and* $4$, *we have*

$$\begin{aligned}
\mathscr{M}_n^{\alpha,\beta}(1;x) &= 1;\\
\mathscr{M}_n^{\alpha,\beta}(t;x) &= x+\frac{\alpha}{n(1+e\beta)};\\
\mathscr{M}_n^{\alpha,\beta}(t^2;x) &= x^2+\frac{(1+2\alpha+e\beta)}{n(1+e\beta)}x+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{n^2(1+e\beta)^2};\\
\mathscr{M}_n^{\alpha,\beta}(t^3;x) &= x^3+\frac{(3+3\alpha+3e\beta)}{n(1+e\beta)}x^2+\frac{\left(3\alpha^2+3\alpha+e^2\beta^2-3\alpha e^2\beta^2-3\alpha e\beta+2e\beta+1\right)}{n^2(1+e\beta)^2}x\\
&\quad+\frac{\left(\alpha^3-6\alpha^2e\beta-3\alpha^2e^2\beta^2-5\alpha e\beta-4\alpha e^2\beta^2-\alpha e^3\beta^3\right)}{n^3(1+e\beta)^3};\\
\mathscr{M}_n^{\alpha,\beta}(t^4;x) &= x^4+\frac{(3+2\alpha+3e\beta)}{n(1+e\beta)}x^3+\frac{\left(6\alpha^2+25e^2\beta^2-50e\beta-6\alpha e^2\beta^2+12\alpha+25\right)}{n^2(1+e\beta)^2}x^2\\
&\quad+\frac{x}{n^3(1+e\beta)^3}(4\alpha^3+6\alpha^2+42e\beta+48e\alpha\beta-18e\alpha^2\beta+42e^2\beta^2+6e^2\alpha\beta^2\\
&\quad-12e^2\alpha^2\beta^2+14e^3\beta^3-10e^3\alpha\beta^2+40\alpha+14)\\
&\quad+\frac{1}{n^4(1+e\beta)^4}(16\alpha^4-1056e\alpha\beta+256e\alpha^2\beta-192e\alpha^3\beta-1888e^2\alpha\beta^2+224e^2\alpha^2\beta^2\\
&\quad-96e^2\alpha^3\beta^2-1312e^3\alpha\beta^3+128e^3\alpha^2\beta^3-304e^4\alpha\beta^4+48e^4\alpha^2\beta^4+288\alpha^2-80\alpha).
\end{aligned}$$

*Proof.* From the generating functions of the classical Euler polynomials given by (2.2), we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}k &= e^{nx}\left(\frac{2}{1+e\beta}\right)^{\alpha}\left[nx+\frac{\alpha}{1+e\beta}\right];\\
\sum_{k=0}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}k^2 &= e^{nx}\left(\frac{2}{1+e\beta}\right)^{\alpha}\left[n^2x^2+\frac{nx(1+2\alpha+e\beta)}{(1+e\beta)}\right.\\
&\quad\left.+\frac{\left(\alpha^2-2\alpha e\beta-\alpha e^2\beta^2\right)}{(1+e\beta)^2}\right];\\
\sum_{k=0}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}k^3 &= e^{nx}\left(\frac{2}{1+e\beta}\right)^{\alpha}\left[n^3x^3+\frac{3n^2x^2(1+\alpha+e\beta)}{(1+e\beta)}\right.\\
&\quad+\frac{nx\left(3\alpha^2+e^2\beta^2-3\alpha e^2\beta^2+3\alpha-3\alpha e\beta+2e\beta+1\right)}{(1+e\beta)^2}\\
&\quad\left.+\frac{\left(\alpha^3-6\alpha^2e\beta-3\alpha^2e^2\beta^2-5\alpha e\beta-4\alpha e^2\beta^2-\alpha e^3\beta^3\right)}{(1+e\beta)^3}\right];
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}k^4 &= e^{nx}\left(\frac{2}{1+e\beta}\right)^{\alpha}\left[n^4x^4+\frac{(3+2\alpha+3e\beta)}{(1+e\beta)}n^3x^3+\frac{n^2x^2}{(1+e\beta)^2}(6\alpha^2+25e^2\beta^2\right.\\
&\quad-50e\beta-6\alpha e^2\beta^2+12\alpha+25)+\frac{nx}{(1+e\beta)^3}(4\alpha^3+6\alpha^2+42e\beta+48e\alpha\beta\\
&\quad-18e\alpha^2\beta+42e^2\beta^2+6e^2\alpha\beta^2-12e^2\alpha^2\beta^2+14e^3\beta^3-10e^3\alpha\beta^2+40\alpha+14)\\
&\quad+\frac{1}{(1+e\beta)^4}(16\alpha^4-1056e\alpha\beta+256e\alpha^2\beta-192e\alpha^3\beta-1888e^2\alpha\beta^2\\
&\quad+224e^2\alpha^2\beta^2-96e^2\alpha^3\beta^2-1312e^3\alpha\beta^3+128e^3\alpha^2\beta^3-304e^4\alpha\beta^4
\end{aligned}$$

$$+48e^4\alpha^2\beta^4+288\alpha^2-80\alpha)\big].$$

In view of these equalities, we get the required result. □

**Remark 2.2.2.** *From Lemma 2.2.1 and simple computation, we have*

$$\begin{aligned}
\mathscr{M}_n^{\alpha,\beta}(t-x;x) =& \frac{\alpha}{n(1+e\beta)};\\
\mathscr{M}_n^{\alpha,\beta}((t-x)^2;x) =& \frac{x}{n}+\frac{\left(\alpha^2-2e\alpha\beta-\alpha e^2\beta^2\right)}{n^2(1+e\beta)^2};\\
\mathscr{M}_n^{\alpha,\beta}((t-x)^3;x) =& \frac{x}{n^2}+\frac{3\alpha x}{n^2(1+e\beta)}+\frac{\left(\alpha^3-6e\alpha^2\beta-3e^2\alpha^2\beta^2-5e\alpha\beta-4e^2\alpha\beta^2-\alpha e^3\beta^3\right)}{n^3(1+e\beta)^3}.
\end{aligned}$$

**Remark 2.2.3.** *The limiting case of the central moments of the given operators* $\mathscr{M}_n^{\alpha,\beta}((t-x)^m;x)$ *are as:*

$$\begin{aligned}
\lim_{n\to\infty} n\mathscr{M}_n^{\alpha,\beta}(t-x;x) &= \frac{\alpha}{(1+e\beta)};\\
\lim_{n\to\infty} n\mathscr{M}_n^{\alpha,\beta}((t-x)^2;x) &= x.
\end{aligned}$$

### 2.2.2 Direct Result and Asymptotic Formula

In this section, we will discuss the direct result and Voronovskaja type asymptotic formula.

**Theorem 2.2.4.** *For* $f\in C_B[0,\infty)$, *we have*

$$|\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)|\leq C\omega_2(f,\sqrt{\delta})+\omega\left(f,\frac{\alpha}{n(1+e\beta)}\right),$$

*where C is a positive constant and* $\delta=\left|\mathscr{M}_n^{\alpha,\beta}((t-x)^2;x)\right|+\left(\frac{\alpha}{n(1+e\beta)}\right)^2$.

*Proof.* We introduce auxiliary operators $H_n^{\alpha,\beta}(t;x)$ as follows:

$$H_n^{\alpha,\beta}(f;x)=\mathscr{M}_n^{\alpha,\beta}(f;x)-f\left(x+\frac{\alpha}{n(1+e\beta)}\right)+f(x).$$

In view of Lemma 2.2.1, linear functions are preserved by $H_n^{\alpha,\beta}$. Let $g\in W^2$. From the Taylor's expansion of $g$, we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)du.$$

Applying the operator $H_n^{\alpha,\beta}$ on above, we get

$$H_n^{\alpha,\beta}(g;x)=g(x)+g'(x)H_n^{\alpha,\beta}((t-x);x)+H_n^{\alpha,\beta}\left(\int_x^t(t-u)g''(u)du;x\right)$$

$$
\begin{aligned}
|H_n^{\alpha,\beta}(g;x)-g(x)| &= \left|H_n^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\
&\leq \left|\mathscr{M}_n^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du;x\right)\right| + \left|\int_x^{x+\frac{\alpha}{n(1+e\beta)}} \left(x+\frac{\alpha}{n(1+e\beta)}-u)g''(u)\right) du\right| \\
&\leq \left[\left|\mathscr{M}_n^{\alpha,\beta}\left(\int_x^t (t-u)du;x\right)\right| + \left|\int_x^{x+\frac{\alpha}{n(1+e\beta)}} \left(x+\frac{\alpha}{n(1+e\beta)}-u\right) du\right|\right] \|g''\| \\
&\leq \left[\left|\mathscr{M}_n^{\alpha,\beta}((t-x)^2;x)\right| + \left(\frac{\alpha}{n(1+e\beta)}\right)^2\right] \|g''\| \\
&= \delta\|g''\|,
\end{aligned}
$$

where $\delta = \left|\mathscr{M}_n^{\alpha,\beta}((t-x)^2;x)\right| + \left(\frac{\alpha}{n(1+e\beta)}\right)^2$.

$$
\begin{aligned}
|\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)| &= |H_n^{\alpha,\beta}(f-g;x)-(f-g)(x)| + |H_n^{\alpha,\beta}(g;x)-g(x)| \\
&\quad + \left|f\left(x+\frac{\alpha}{n(1+e\beta)}\right)-f(x)\right| \\
&\leq 2\|f-g\| + \delta\|g''\| + \omega\left(f, \left|\frac{\alpha}{n(1+e\beta)}\right|\right) \\
&= 2\|f-g\| + \delta\|g''\| + \omega\left(f, \frac{\alpha}{n(1+e\beta)}\right).
\end{aligned}
$$

Taking infimum over all $g \in W^2$, we get

$$
|\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)| \leq K_2(f,\delta) + \omega\left(f, \frac{\alpha}{n(1+e\beta)}\right).
$$

In view of (1.3), we obtain

$$
|\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)| \leq C\omega_2(f,\sqrt{\delta}) + \omega\left(f, \frac{\alpha}{n(1+e\beta)}\right),
$$

which proves the theorem. □

**Theorem 2.2.5.** *(Voronvaskaja type theorem) For any function $f \in C^2[0,\infty)$ such that $f'$, $f'' \in C^2[0,\infty)$, we have*

$$
\lim_{n\to\infty} n[\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)] = \frac{\alpha}{(1+e\beta)} f'(x) + \frac{x}{2} f''(x),
$$

*for every $x \geq 0$.*

*Proof.* Let $f, f', f'' \in C^2[0,\infty)$ and $x \in [0,\infty)$ be fixed. By Taylor expansion we can write

$$
f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!} f''(x) + r(t,x)(t-x)^2,
$$

where $r(t,x)$ is the Peano form of the remainder, $r(t,x) \in C_B[0,\infty)$ and $\lim_{t\to x} r(t,x) = 0$. Applying $\mathscr{M}_n^{\alpha,\beta}$, we get

$$n[\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)] = f'(x)n\mathscr{M}_n^{\alpha,\beta}(t-x;x)+\frac{f''(x)}{2!}n\mathscr{M}_n^{\alpha,\beta}((t-x)^2;x) \\ +n\mathscr{M}_n^{\alpha,\beta}(r(t,x)(t-x)^2;x).$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} n[\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)] &= f'(x)\lim_{n\to\infty} n\mathscr{M}_n^{\alpha,\beta}(t-x;x)+\frac{f''(x)}{2!}\lim_{n\to\infty} n\mathscr{M}_n^{\alpha,\beta}((t-x)^2;x) \\ &\quad +\lim_{n\to\infty} n\mathscr{M}_n^{\alpha,\beta}(r(t,x)(t-x)^2;x) \\ &= \frac{\alpha}{1+e\beta}f'(x)+\frac{x}{2}f''(x)+\lim_{n\to\infty} n\mathscr{M}_n^{\alpha,\beta}\left(r(t,x)(t-x)^2;x\right) \\ &= \frac{\alpha}{1+e\beta}f'(x)+\frac{x}{2}f''(x)+R.\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$R = \lim_{n\to\infty} n\mathscr{M}_n^{\alpha,\beta}(r^2(t,x);x)^{\frac{1}{2}}\mathscr{M}_n^{\alpha,\beta}\left((t-x)^4;x\right)^{\frac{1}{2}}. \tag{2.4}$$

Observe that $r^2(x,x)=0$ and $r^2(x,x)\in C^2[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty} n\mathscr{M}_n^{\alpha,\beta}(r^2(t,x);x)=r^2(x,x)=0 \tag{2.5}$$

uniformly with respect to $x\in[0,\infty)$. Now from (2.4) and (2.5), we obtain

$$\lim_{n\to\infty} n\mathscr{M}_n^{\alpha,\beta}(r(t,x)(t-x)^2;x)=0.$$

Hence $R=0$. Thus, we otained

$$\lim_{n\to\infty} n[\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)]=\frac{\alpha}{1+e\beta}f'(x)+\frac{x}{2}f''(x),$$

which completes the proof. □

### 2.2.3 Weighted Approximation results

**Theorem 2.2.6.** *(Weighted approximation) For each $f\in C_\tau^*[0,\infty)$, we have*

$$\lim_{n\to\infty}\|\mathscr{M}_n^{\alpha,\beta}(f;.)-f\|_\tau=0$$

*Proof.* Using [32], we see that it is sufficient to verify the following conditions

$$\lim_{n\to\infty}\|\mathscr{M}_n^{\alpha,\beta}(t^r;x)-x^r\|_\tau=0,\ \ r=0,1,2. \tag{2.6}$$

Since $\mathscr{M}_n^{\alpha,\beta}(1;x)=1$, therefore for $r=0$ (2.6) holds. By Lemma 2.2.1, we have

$$\begin{aligned}
\|\mathscr{M}_n^{\alpha,\beta}(t;x)-x\|_\tau &= \sup_{x\in[0,\infty)} \frac{|\mathscr{M}_n^{\alpha,\beta}(t;x)-x|}{1+x^2} \\
&\leq \left|\frac{\alpha}{n(1+e\beta)}\right| \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\
&\leq \frac{\alpha}{n(1+e\beta)},
\end{aligned}$$

the condition (2.6) holds for $r=1$ as $n\to\infty$.

Again by Lemma 2.2.1, we have

$$\begin{aligned}
\|\mathscr{M}_n^{\alpha,\beta}(t^2;x)-x^2\|_\tau &= \sup_{x\in[0,\infty)} \frac{|\mathscr{M}_n^{\alpha,\beta}(t^2;x)-x^2|}{1+x^2} \\
&\leq \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left[\left(x^2+\frac{(1+2\alpha+e\beta)}{n(1+e\beta)}x+\frac{(\alpha^2-e\alpha\beta\,(2+e\beta))}{n^2(1+e\beta)^2}\right)-x^2\right] \\
&\leq \left[\frac{(1+2\alpha+e\beta)}{n(1+e\beta)}\right] \sup_{x\in[0,\infty)} \frac{x}{1+x^2}+\frac{(\alpha^2-e\alpha\beta\,(2+e\beta))}{n^2(1+e\beta)^2} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\
&\leq \frac{(1+2\alpha+e\beta)}{n(1+e\beta)}+\frac{(\alpha^2-e\alpha\beta\,(2+e\beta))}{n^2(1+e\beta)^2}
\end{aligned}$$

the condition (2.6) holds for $r=2$ as $n\to\infty$. Hence the theorem proved. □

**Corollary 2.2.7.** *For each $f\in C_\tau^*[0,\infty)$, and $\alpha>0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)|}{(\tau(x))^\alpha}=0.$$

*Proof.* For any fixed $x_0>0$,

$$\begin{aligned}
\sup_{x\in[0,\infty)} \frac{|\mathscr{M}_n^{\alpha,\beta}(f;x,a)-f(x)|}{(1+x^2)^\alpha} &\leq \sup_{x\leq x_0} \frac{|\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^\alpha}+\sup_{x\geq x_0} \frac{|\mathscr{M}_n^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^\alpha} \\
&\leq \|\mathscr{M}_n^{\alpha,\beta}(f;.)-f\|_{C[0,x_0]}+\|f\|_{x^2} \sup_{x\geq x_0} \frac{|\mathscr{M}_n^{\alpha,\beta}(1+t^2;x)|}{(1+x^2)^\alpha} \\
&+ \sup_{x\geq x_0} \frac{|f(x)|}{(1+x^2)^\alpha}.
\end{aligned}$$

The first term of the above inequality tends to zero from Theorem 2.2.5. By Lemma 2.2.1 for any fixed $x_0$ it is easily seen that $\sup_{x\geq x_0} \frac{|\mathscr{M}_n^{\alpha,\beta}(1+t^2;x)|}{(1+x^2)^\alpha}$ tends to zero as $n\to\infty$. We can choose $x_0$ so large that the last part of above inequality can be made small enough. Thus the proof is completed. □

## 2.3 Kantorovich variant of $T_n^{\alpha,\beta}$ operators

In this section, we will discuss the Kantorovich type generalization of the operators $T_n^{\alpha,\beta}$ with the help of the operators (2.1) and we obtain main theorem with the help of Korovkin's theorem and estimate the order of approximation by modulus of continuity. The operators is defined as

$$T_n^{\alpha,\beta}(f(t);x) = \frac{n}{e^{nx}}\left(\frac{1+e\beta}{2}\right)^{\alpha}\sum_{k=0}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt. \tag{2.7}$$

**Lemma 2.3.1.** *For $T_n^{\alpha,\beta}(t^m;x)$, $m=0,1,2$ and 3, we have*

$$\begin{aligned}
T_n^{\alpha,\beta}(1;x) &= 1;\\
T_n^{\alpha,\beta}(t;x) &= x+\frac{(1+2\alpha+e\beta)}{2n(1+e\beta)};\\
T_n^{\alpha,\beta}(t^2;x) &= x^2+\frac{2(1+\alpha+e\beta)}{n(1+e\beta)}x+\frac{1}{n^2(1+e\beta)^2}(3\alpha^2-3e\alpha\beta-3e^2\alpha\beta^2+3\alpha\\
&\quad +e^2\beta^2+2e\beta+1);\\
T_n^{\alpha,\beta}(t^3;x) &= x^3+\frac{(9+6\alpha+9e\beta)}{2n(1+e\beta)}x^2+\frac{6\alpha^2+7(1+e\beta)^2-6\alpha(-2+e^2\beta^2)}{2n^2(1+e\beta)^2}x+\\
&\quad \frac{1}{4n^3(1+e\beta)^3}(4\alpha^3-4e\alpha^4\beta^3+(1+e\beta)^3-6\alpha^2(-1+3e\beta+2e^2\beta^2)\\
&\quad -2\alpha(-2+12e\beta+15e^2\beta^2+3e^3\beta^3)).
\end{aligned}$$

*Proof.* Using Lemma 2.2.1, we get the proof. □

**Remark 2.3.2.** *From Lemma 2.3.1 and simple computation, we have*

$$\begin{aligned}
T_n^{\alpha,\beta}((t-x);x) &= \frac{(1+2\alpha+e\beta)}{2n(1+e\beta)};\\
T_n^{\alpha,\beta}((t-x)^2;x) &= \frac{x}{n}+\frac{(3\alpha^2-3e\alpha\beta-3\alpha e^2\beta^2+3\alpha+e^2\beta^2+2e\beta+1)}{n^2(1+e\beta)^2};\\
T_n^{\alpha,\beta}((t-x)^3;x) &= \frac{(-12\alpha^2+(1+e\beta)^2)+6\alpha(-1+3e\beta+2e^2\beta^2)}{2n^2(1+e\beta)^2}x\\
&\quad +\frac{1}{4n^3(1+e\beta)^3}\bigg(4\alpha^3-4e\alpha^4\beta^3+(1+e\beta)^3-6\alpha^2(-1+3e\beta+2e^2\beta^2)\\
&\quad -2\alpha(-2+12e\beta+15e^2\beta^2+3e^3\beta^3)\bigg).
\end{aligned}$$

**Theorem 2.3.3.** *Let $f\in C[0,\infty)\cap E^*$. Then*

$$\lim_{n\to\infty} T_n^{\alpha,\beta}(f;x) = f(x).$$

*The operators given by (2.7) converge uniform in each compact subset* $[0,\infty)$*, where*

$$E^* = \left\{ f : [0,\infty) \to \mathbb{R}, |F(x)| = \left| \int_0^x f(t)dt \right| \le Ke^{Bx}, B \in \mathbb{R} \text{ and } K \in \mathbb{R}^+ \right\}.$$

*Proof.* From Lemma 2.3.1, we get

$$\lim_{n\to\infty} T_n^{\alpha,\beta}(f;x) = x^i, \quad i = 0,1,2,3.$$

The proof is established by virtue of above uniform convergence in each compact subset of $[0,\infty)$ and Korovkin's theorem. □

**Theorem 2.3.4.** *Let* $f \in C[0,\infty] \cap E^*$*, for the operators* $T_n^{\alpha,\beta}$ *given by (2.7) hold the estimation*

$$\left| T_n^{\alpha,\beta}(f;x) - f(x) \right| \le \left\{ 1 + \left[ x + \frac{3\alpha^2 - 3\alpha e\beta - 3\alpha e^2\beta^2 + 3\alpha + e^2\beta^2 + 2e\beta + 1}{n(1+e\beta)^2} \right]^{\frac{1}{2}} \right\} \omega(f,\delta). \quad (2.8)$$

*Proof.* From Lemma 2.3.1 and the property of modulus of continuity, we have

$$T_n^{\alpha,\beta}(f;x) - f(x) = ne^{-nx} \left( \frac{1+e\beta}{2} \right)^{\alpha} \sum_{k=0}^{\infty} \frac{G_k^{\alpha}(nx;\beta)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (f(t) - f(x))dt,$$

$\Rightarrow$

$$\left| T_n^{\alpha,\beta}(f;x) - f(x) \right| \le ne^{-nx} \left( \frac{1+e\beta}{2} \right)^{\alpha} \sum_{k=0}^{\infty} \frac{G_k^{\alpha}(nx;\beta)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)|\, dt.$$

By using the Cauchy-Schwarz inequality for the integral, we have

$$\left| T_n^{\alpha,\beta}(f;x) - f(x) \right| \le \left\{ 1 + \frac{1}{\delta} e^{-nx} \left( \frac{1+e\beta}{2} \right)^{\alpha} \sum_{k=0}^{\infty} \frac{G_k^{\alpha}(nx;\beta)}{k!} \left( n \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2 dt \right)^{\frac{1}{2}} \right\} \omega(f,\delta). \quad (2.9)$$

Applying the Cauchy-Schwarz inequality in the above sum, (2.9) becomes

$$\begin{aligned}
\left| T_n^{\alpha,\beta}(f;x) - f(x) \right| \quad \le \quad & \left\{ 1 + \frac{1}{\delta} \left( e^{-nx} \left( \frac{1+e\beta}{2} \right)^{\alpha} \sum_{k=0}^{\infty} \frac{G_k^{\alpha}(nx;\beta)}{k!} \right)^{\frac{1}{2}} \right. \\
& \left. \times \left( ne^{-nx} \left( \frac{1+e\beta}{2} \right)^{\alpha} \sum_{k=0}^{\infty} \frac{G_k^{\alpha}(nx;\beta)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2 dt \right)^{\frac{1}{2}} \right\} \omega(f,\delta) \\
& \le 1 + \frac{1}{\delta} \left[ T_n^{\alpha,\beta}((t-x)^2,x) \right]^{\frac{1}{2}} \omega(f,\delta) \\
& \le \left\{ 1 + \frac{1}{\delta} \left[ \frac{x}{n} + \frac{3\alpha^2 - 3\alpha e\beta - 3\alpha e^2\beta^2 + 3\alpha + e^2\beta^2 + 2e\beta + 1}{n^2(1+e\beta)^2} \right]^{\frac{1}{2}} \right\} \omega(f,\delta) \\
& \le \left\{ 1 + \left[ x + \frac{3\alpha^2 - 3\alpha e\beta - 3\alpha e^2\beta^2 + 3\alpha + e^2\beta^2 + 2e\beta + 1}{n(1+e\beta)^2} \right]^{\frac{1}{2}} \right\} \omega(f,\delta),
\end{aligned}$$

if we choose $\delta = \frac{1}{\sqrt{n}}$, which gives (2.8) . □

**Remark 2.3.5.** *We define the Stancu type generalization of the operators $\mathscr{M}_n^{\alpha,\beta}$ based on two parameters $\zeta$ and $\eta$ with the property $0 \leq \zeta < \eta$ as follows:*

$$S_{n,\alpha,\beta}^{\zeta,\eta}(f;x) = e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\sum_{k=0}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}f\left(\frac{k+\zeta}{n+\eta}\right).$$

*One can study the approximating properties for the operators $S_{n,\alpha,\beta}^{\zeta,\eta}$ similar manner as in [99].*

## 2.4 Approximation by integral form of $\mathscr{M}_n^{\alpha,\beta}$ operators

In 2007, Srivastava et al. [84] proposed a family of summation-integral type operators and estimated rate of convergence and function having derivative of bounded variation. Here we refer to some more articles related to summation-integral type operators ([37, 45, 67]) for the readers. Inspired by the work (2.1), for any non-negative integer $\alpha$ and $f \in C[0,\infty)$ we give the integral-type generalization of the operators, which are as:

$$T_{n,\rho}^{\alpha,\beta}(f,x) = e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\left(G_0^{\alpha}(nx;\beta)f(0) + \sum_{k=1}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)f(t)dt\right), \tag{2.10}$$

where $\rho > 0$, and

$$\Theta_{n,k}^{\rho}(t,c) = \begin{cases} \dfrac{n\rho}{\Gamma(k\rho)}e^{-n\rho t}(n\rho t)^{k\rho-1}, & c = 0, \\ \dfrac{\Gamma\left(\frac{n\rho}{c}+k\rho\right)}{\Gamma(k\rho)\Gamma\left(\frac{n\rho}{c}\right)}\dfrac{c^{k\rho}t^{k\rho-1}}{(1+ct)^{\frac{n\rho}{c}+k\rho}}, & c = 1,2,3,... \end{cases}.$$

It can we easily observed by simple computation that:

$$\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)t^r dt = \begin{cases} \frac{\Gamma(k\rho+r)}{\Gamma(k\rho)}\dfrac{1}{\prod_{i=1}^{r}(n\rho - ic)}, & r \neq 0 \\ 1, & r = 0 \end{cases}. \tag{2.11}$$

This section is dedicated to a new sequence of summation-integral type operators involving Apostol-Genocchi polynomials, and estimated their moments and central moments in order to study Voronovskaja type asymptotic theorem, Lipschitz space, Ditzian-Totik modulus of smoothness, and second-order modulus of continuity. Finally, the Weighted approximation is obtained.

### 2.4.1 Preliminaries

This section estimates the auxiliary operator $T_{n,\rho}^{\alpha,\beta}$ outcomes that will be helpful in the main result.

**Lemma 2.4.1.** *The moments of the proposed operator $T_{n,\rho}^{\alpha,\beta}\left(t^i,x\right)$, $i=\overline{0,3}$, we have*

$$\begin{aligned}
T_{n,\rho}^{\alpha,\beta}(1,x) &= 1;\\
T_{n,\rho}^{\alpha,\beta}(t,x) &= \frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)};\\
T_{n,\rho}^{\alpha,\beta}\left(t^2,x\right) &= \frac{n^2\rho^2}{(n\rho-c)(n\rho-2c)}x^2+\frac{n\rho^2+2n\alpha\rho^2+n\rho^2e\beta}{(n\rho-c)(n\rho-2c)(1+e\beta)}x+\frac{\alpha^2\rho^2-2\alpha\rho^2e\beta-\alpha\rho^2e^2\beta^2}{(n\rho-c)(n\rho-2c)(1+e\beta)^2};\\
T_{n,\rho}^{\alpha,\beta}\left(t^3,x\right) &= \frac{n^3\rho^3}{(n\rho-c)(n\rho-2c)(n\rho-3c)}x^3+\frac{3n^2\rho^2+3n^2\rho^3+3\alpha n^2\rho^3+3n^2\rho^2e\beta+3n^2\rho^3e\beta}{(n\rho-c)(n\rho-2c)(n\rho-3c)(1+e\beta)}x^2\\
&\quad+\frac{x}{(n\rho-c)(n\rho-2c)(n\rho-3c)(1+e\beta)^2}\left[n\alpha^2\rho^3+9n\alpha\rho^2+n\rho^2e^2\beta^2(3-2n\rho)\right.\\
&\quad\left.+3n\alpha\rho^2e\beta+n\rho(1+e\beta)^2+8n\rho^2e\beta+3n\rho^2+n\rho\right]\\
&\quad+\frac{1}{(n\rho-c)(n\rho-2c)(n\rho-3c)(1+e\beta)^3}\left[\alpha^3\rho^3+3\alpha^2\rho^2e\beta(1-2\rho-\rho e\beta)+3\alpha^2\rho^2\right.\\
&\quad\left.-5\alpha\rho^3e\beta-\alpha\rho^2e^2\beta^2(4\rho+\rho e\beta+9e\beta+3)-6\alpha\rho^2e\beta+2\alpha\rho+2\alpha\rho e^2\beta^2+4\alpha\rho e\beta\right].
\end{aligned}$$

*Proof.* The proof the above Lemma follows from (2.10), (2.11) and Lemma 2.2.1. □

**Remark 2.4.2.** *Using Lemma 2.4.1 and simple estimation, we have*

$$\begin{aligned}
T_{n,\rho}^{\alpha,\beta}(t-x,x) &= \frac{cx}{(n\rho-c)}+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)};\\
T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right) &= \frac{n\rho c+2c^2}{(n\rho-c)(n\rho-2c)}x^2+\frac{n\rho^2+n\rho^2e\beta+4c\alpha\rho}{(n\rho-c)(n\rho-2c)(1+e\beta)}x\\
&\quad+\frac{\alpha^2\rho^2-2\alpha\rho^2e\beta-\alpha\rho^2e^2\beta^2}{(n\rho-c)(n\rho-2c)(1+e\beta)^2}.
\end{aligned}$$

**Remark 2.4.3.** *For the central moments $T_{n,\rho}^{\alpha,\beta}((t-x)^m;x)$ for $m=1,2$, we have*

$$\begin{aligned}
\lim_{n\to\infty} nT_{n,\rho}^{\alpha,\beta}((t-x);x) &= \frac{xc}{\rho}+\frac{\alpha}{(1+e\beta)};\\
\lim_{n\to\infty} nT_{n,\rho}^{\alpha,\beta}((t-x)^2;x) &= \frac{x(cx+\rho)}{\rho}.
\end{aligned}$$

**Remark 2.4.4.** *For $n\in\mathbb{N}$, the bound for the second moment is as follows:*

$$T_{n,\rho}^{\alpha,\beta}((t-x)^2;x)\le\frac{2\phi^2(x)}{n\rho},\ \textit{where}\ \phi(x)=\sqrt{x(cx+\rho)}.$$

**Lemma 2.4.5.** *Let $f\in C_B[0,\infty)$, $0\le x<\infty$ and $n\in\mathbb{N}$, then*

$$|T_{n,\rho}^{\alpha,\beta}(f;x)|\le\|f\|,$$

*where $\|.\|$ is the uniform norm on $[0,\infty)$.*

*Proof.* We have $T_{n,\rho}^{\alpha,\beta}(e_0;x) = 1$, so

$$|T_{n,\rho}^{\alpha,\beta}(f;x)| \leq T_{n,\rho}^{\alpha,\beta}(e_0;x)\|f\| = \|f\|.$$

□

### 2.4.2 Direct Results

In this section, we establish the uniform convergence of the operators (2.10) using the Bohman-Korovkin theorem, the rate of convergence with the aid of different kind of modulus of smoothness and for functions in Lipschitz type spaces, and asymptotic theorem.

**Theorem 2.4.6.** *(Fundamental convergence theorem) Let $f \in C[0,\infty)$ and adequately large n, then the sequence $\{T_{n,\rho}^{\alpha,\beta}(f,.)\}$ converges uniformly to $f$ in $[a,b]$, where $0 \leq a < b < \infty$.*

*Proof.* From Lemma 2.4.1 we have $T_{n,\rho}^{\alpha,\beta}(1,x) = 1$ for every $n \in \mathbb{N}$,

$$T_{n,\rho}^{\alpha,\beta}(t,x) = \frac{n\rho}{(n\rho - c)}x + \frac{\alpha\rho}{(n\rho - c)(1+e\beta)}$$

tends to $x$ and

$$T_{n,\rho}^{\alpha,\beta}\left(t^2,x\right) = \frac{n^2\rho^2}{(n\rho - c)(n\rho - 2c)}x^2 + \frac{n\rho^2 + 2n\alpha\rho^2 + n\rho^2 e\beta}{(n\rho - c)(n\rho - 2c)(1+e\beta)}x + \frac{\alpha^2\rho^2 - 2\alpha\rho^2 e\beta - \alpha\rho^2 e^2\beta^2}{(n\rho - c)(n\rho - 2c)(1+e\beta)^2}$$

tends to $x^2$ as $n \to \infty$, similarly $T_{n,\rho}^{\alpha,\beta}\left(t^3,x\right)$ tends to $x^3$ uniformly on every compact subset of $[0,\infty)$. Hence, by Bohman-Korovkin theorem the required results hold. □

**Theorem 2.4.7.** *Let $f \in C_B[0,\infty)$ and $x \in [0,\infty)$. Then we have*

$$\left|T_{n,\rho}^{\alpha,\beta}(f,x) - f(x)\right| \leq C\omega_2\left(f,\sqrt{\delta_n}\right) + \omega\left(f,|\alpha_2|\right),$$

*where C is a positive constant, $\delta_n = \frac{1}{2}[\alpha_1 + \alpha_2^2]$, $\alpha_1 = T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)$ and $\alpha_2 = \left(\frac{cx}{n\rho - c} + \frac{\alpha\rho}{(n\rho - c)(1+e\beta)}\right)$.*

*Proof.* Define the auxiliary operators $L_{n,\rho}^{\alpha,\beta} : C_B[0,\infty) \to C_B[0,\infty)$ as follows:

$$L_{n,\rho}^{\alpha,\beta}(f,x) = T_{n,\rho}^{\alpha,\beta}(f,x) - f\left(\frac{n\rho}{(n\rho - c)}x + \frac{\alpha\rho}{(n\rho - c)(1+e\beta)}\right) + f(x). \tag{2.12}$$

These operators are linear and $L_{n,\rho}^{\alpha,\beta}(t-x,x) = 0$.
Let $j \in W^2$ and $x,t \in [0,\infty)$. By Taylor's series expansion

$$j(t) = j(x) + (t-x)j'(x) + \int_x^t (t-u)j''(u)du.$$

Applying the operator $L_{n,\rho}^{\alpha,\beta}$ on above, we obtain

$$L_{n,\rho}^{\alpha,\beta}(j;x)=j(x)+j'(x)L_{n,\rho}^{\alpha,\beta}((t-x);x)+L_{n,\rho}^{\alpha,\beta}\left(\int_x^t(t-u)j''(u)du;x\right)$$

implies that

$$\begin{aligned}\left|L_{n,\rho}^{\alpha,\beta}(j,x)-j(x)\right| &\leq L_{n,\rho}^{\alpha,\beta}\left(\left|\int_x^t(t-u)j''(u)du\right|,x\right)\\ &\leq T_{n,\rho}^{\alpha,\beta}\left(\left|\int_x^t(t-u)j''(u)du\right|,x\right)\\ &+\left|\int_x^{\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}}\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}-u\right)j''(u)du\right|\\ &\leq\frac{1}{2}T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)||j''||\\ &+\left|\int_x^{\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}}\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}-u\right)du\right|||j''||\end{aligned}$$

$$\begin{aligned}\left|L_{n,\rho}^{\alpha,\beta}(j,x)-j(x)\right| &\leq\frac{1}{2}\left[T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)+\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}-x\right)^2\right]||j''||\\ &\leq\frac{1}{2}\left[T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)+\left(\frac{c}{n\rho-c}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\right)^2\right]||j''||\\ &\leq\frac{1}{2}\left[\alpha_1+\alpha_2^2\right]||j''||=\delta_n||j''||.\end{aligned}\tag{2.13}$$

Since

$$\left|T_{n,\rho}^{\alpha,\beta}(f,x)\right|\leq e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\left(G_0^{\alpha}(nx;\beta)|f(0)|+\left|\sum_{k=1}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)f(t)dt\right|\right)\leq||f||.$$

Now by (2.12), we have

$$||L_{n,\rho}^{\alpha,\beta}(f,.)||\leq||T_{n,\rho}^{\alpha,\beta}(f,.)||+2||f||\leq3||f||,\ f\in C_B[0,\infty).\tag{2.14}$$

Using (2.12), (2.13) and (2.14), we have

$$\begin{aligned}\left|T_{n,\rho}^{\alpha,\beta}(f,x)-f(x)\right| &\leq\left|L_{n,\rho}^{\alpha,\beta}(f-j,x)-(f-j)(x)\right|+\left|L_{n,\rho}^{\alpha,\beta}(j,x)-j(x)\right|\\ &+\left|f\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\right)-f(x)\right|\\ &\leq4||f-j||+\delta_n||j''||+\left|f(x)-f\left(\frac{n\rho}{(n\rho-c)}x+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\right)\right|\\ &\leq4||f-j||+\delta_n||j''||+\omega(f,\alpha_2).\end{aligned}$$

Taking infimum over all $j\in W^2$, and using (1.3), we get the required result. □

Our next result is the Voronovskaja-type asymptotic formula.

**Theorem 2.4.8.** *(Voronvaskaja type theorem) Let $f$ be a bounded and integrable function on $[0,\infty)$ such that $f''(x)$ exist at $0 \leq x < \infty$, then*

$$\lim_{n\to\infty} n\left[T_{n,\rho}^{\alpha,\beta}(f,x) - f(x)\right] = \left[\frac{cx}{\rho} + \frac{\alpha}{(1+e\beta)}\right] f'(x) + \left[\frac{c}{\rho}x^2 + x\right]\frac{f''(x)}{2!}.$$

*Proof.* Using the well known Taylor's series expansion

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + \sigma(t,x)(t-x)^2,$$

where $\sigma(t,x) \to 0$ as $t \to x$ and the function $\rho$ is bounded on $[0,\infty)$.

Now,

$$\begin{aligned}
n\left[T_{n,\rho}^{\alpha,\beta}(f,x) - f(x)\right] &= n\left[T_{n,\rho}^{\alpha,\beta}(t-x,x) f'(x) + \frac{T_{n,\rho}^{\alpha,\beta}\left((t-x)^2,x\right)}{2!} f''(x) + T_{n,\rho}^{\alpha,\beta}\left(\sigma(t,x)(t-x)^2,x\right)\right] \\
&= n\left[\left(\frac{cx}{(n\rho-c)} + \frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\right) f'(x) + \frac{n\rho c + 2c^2}{(n\rho-c)(n\rho-2c)}\frac{x^2}{2!} f''(x)\right. \\
&\quad \left. + \frac{n\rho^2 + n\rho^2 e\beta + 4c\alpha\rho}{(n\rho-c)(n\rho-2c)(1+e\beta)}\frac{x}{2!} f''(x) + \frac{\alpha^2\rho^2 - 2\alpha\rho^2 e\beta - \alpha\rho^2 e^2\beta^2}{2!(n\rho-c)(n\rho-2c)(1+e\beta)^2} f''(x)\right] \\
&\quad + \hbar(n,x),
\end{aligned}$$

where

$$\hbar(n,x) = e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\left(G_0^{\alpha}(nx;\beta)\,\sigma(0,x)(-x)^2 + \sum_{k=1}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)\,\sigma(t,x)(t-x)^2 dt\right).$$

Now it is sufficient to show that $\hbar(n,x) \to 0$ as large $n$. Since $\sigma(t,x) \to 0$ as $t \to x$, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\sigma(t,x)| < \varepsilon$, whenever $-\delta < t - x < \delta$,

$$\begin{aligned}
|\hbar(n,x)| \leq e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\sum_{k=1}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}&\left[\int_{|t-x|<\delta}\Theta_{n,k}^{\rho}(t,c)\,|\sigma(t,x)|(t-x)^2 dt\right. \\
&\left. + \int_{|t-x|\geq\delta}\Theta_{n,k}^{\rho}(t,c)\,|\sigma(t,x)|(t-x)^2 dt\right] \\
&+ e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha} G_0^{\alpha}(nx;\beta)\,\sigma(0,x)(-x)^2 \\
&\leq I_1 + I_2.
\end{aligned}$$

By using the Remark 2.4.2, we have $I_1 = \sigma O(1)$ and for $m \geq 2$, we have

$$\begin{aligned}
I_2 &\leq Ce^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\left(G_0^{\alpha}(nx;\beta)\,\sigma(0,x)(-x)^2 + \sum_{k=1}^{\infty}\frac{G_k^{\alpha}(nx;\beta)}{k!}\int_0^{\infty}\Theta_{n,k}^{\rho}(t,c)\,\frac{(t-x)^{2m}}{\delta^{2m-2}}dt\right) \\
&= O(n^{-m+1}),
\end{aligned}$$

where $C = \sup_{t\in[x,\infty)} |\sigma(t,x)|$. Due to arbitrariness of $\varepsilon > 0$, $\hbar(n,x) \to 0$ for sufficiently large $n$. This completes the proof. $\square$

In the following theorem, we provide the rate of convergence of the proposed operators for functions in Lipschitz-type spaces.

**Theorem 2.4.9.** *(Lipschitz class) Let $f \in Lip_M^*(\sigma)$ and $0 < \sigma \leq 1$. Then for each $x \in (0,\infty)$, we have*

$$|T_{n,\rho}^{\alpha,\beta}(f(t);x) - f(x)| \leq M \left( \frac{T_{n,\rho}^{\alpha,\beta}((t-x)^2;x)}{x} \right)^{\frac{\sigma}{2}}.$$

*Proof.* Let $f \in Lip_M^*(\sigma)$ and $x \in (0,\infty)$, $t \in [0,\infty)$, we have

$$\begin{aligned} |T_{n,\rho}^{\alpha,\beta}(f(t);x) - f(x)| &\leq T_{n,\rho}^{\alpha,\beta}(|f(t)-f(x)|;x) \\ &\leq M.T_{n,\rho}^{\alpha,\beta}\left( \frac{|t-x|^{\sigma}}{(t+x)^{\frac{\sigma}{2}}};x \right) \\ &\leq \frac{M}{x^{\frac{\sigma}{2}}} T_{n,\rho}^{\alpha,\beta}(|t-x|^{\sigma};x). \end{aligned} \tag{2.15}$$

Taking $p = \frac{2}{\sigma}$ and $q = \frac{2}{2-\sigma}$ and applying Hölder's inequality, we obtain

$$\begin{aligned} T_{n,\rho}^{\alpha,\beta}(|t-x|^{\sigma};x) &\leq \left\{ T_{n,\rho}^{\alpha,\beta}(|t-x|^2;x) \right\}^{\frac{\sigma}{2}} . \left\{ T_{n,\rho}^{\alpha,\beta}(1^{\frac{2}{2-\sigma}};x) \right\}^{\frac{2-\sigma}{2}} \\ &\leq \left\{ T_{n,\rho}^{\alpha,\beta}(|t-x|^2;x) \right\}^{\frac{\sigma}{2}}. \end{aligned} \tag{2.16}$$

Using (2.15) and (2.16), we get the required result

$$|T_{n,\rho}^{\alpha,\beta}(f(t);x) - f(x)| \leq M \left( \frac{T_{n,\rho}^{\alpha,\beta}((t-x)^2;x)}{x} \right)^{\frac{\sigma}{2}}.$$

$\square$

**Theorem 2.4.10.** *(Ditzian-totik modulus of smoothness) For $f \in C_B[0,\infty)$ and $x \in (0,\infty)$, we have*

$$|T_{n,\rho}^{\alpha,\beta}(f(t);x) - f(x)| \leq C\omega_{\phi^{\gamma}}\left( f; \frac{\phi^{1-\gamma}(x)}{\sqrt{n\rho}} \right),$$

*for sufficiently large $n$ and constant $C > 0$ is independent of $f$ and $n$.*

*Proof.* For $j \in W_{\gamma}$, we get

$$j(t) = j(x) + \int_x^t j'(u)du. \tag{2.17}$$

Applying $T_{n,\rho}^{\alpha,\beta}$ in (2.17) and using Hölder's inequality, we obtain

$$\begin{aligned}|T_{n,\rho}^{\alpha,\beta}(j(t);x)-j(x)| &\leq T_{n,\rho}^{\alpha,\beta}\left(\int_x^t |j'|du;x\right)\\ &\leq \|\phi^{\gamma}j'\|T_{n,\rho}^{\alpha,\beta}\left(\left|\int_x^t \frac{du}{\phi^{\gamma}(u)}\right|;x\right)\\ &\leq \|\phi^{\gamma}j'\|T_{n,\rho}^{\alpha,\beta}\left(|t-x|^{1-\gamma}\left|\int_x^t \frac{du}{\phi(u)}\right|^{\gamma};x\right). \end{aligned} \tag{2.18}$$

Take $A=\left|\int_x^t \frac{du}{\phi(u)}\right|$, we find

$$\begin{aligned}A &\leq \left|\int_x^t \frac{du}{\sqrt{u}}\right|\left|\left(\frac{1}{\sqrt{cx+\rho}}+\frac{1}{\sqrt{ct+\rho}}\right)\right|\\ &\leq 2|\sqrt{t}-\sqrt{x}|\left(\frac{1}{\sqrt{cx+\rho}}+\frac{1}{\sqrt{ct+\rho}}\right)\\ &\leq 2\frac{|t-x|}{\sqrt{x}+\sqrt{t}}\left(\frac{1}{\sqrt{cx+\rho}}+\frac{1}{\sqrt{ct+\rho}}\right)\\ &\leq 2\frac{|t-x|}{\sqrt{x}}\left(\frac{1}{\sqrt{cx+\rho}}+\frac{1}{\sqrt{ct+\rho}}\right). \end{aligned} \tag{2.19}$$

Using the inequality $|a+b|^{\gamma}\leq |a|^{\gamma}+|b|^{\gamma},\ 0\leq\gamma\leq 1$ and (2.19), we get

$$\|\int_x^t \frac{du}{\phi(u)}\|^{\gamma}\leq 2^{\beta}\frac{|t-x|^{\gamma}}{x^{\frac{\gamma}{2}}}\left(\frac{1}{(cx+\rho)^{\frac{\gamma}{2}}}+\frac{1}{(ct+\rho)^{\frac{\gamma}{2}}}\right). \tag{2.20}$$

From (2.18), (2.20) and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}|T_{n,\rho}^{\alpha,\beta}(j(t);x)-j(x)| &\leq \frac{2^{\gamma}\|\phi^{\gamma}j'\|}{x^{\frac{\gamma}{2}}}T_{n,\rho}^{\alpha,\beta}\left(|t-x|\left(\frac{1}{(cx+\rho)^{\frac{\gamma}{2}}}+\frac{1}{(ct+\rho)^{\frac{\gamma}{2}}}\right);x\right)\\ &\leq \frac{2^{\gamma}\|\phi^{\gamma}j'\|}{x^{\frac{\gamma}{2}}}\left(\frac{1}{(ct+\rho)^{\frac{\gamma}{2}}}(T_{n,\rho}^{\alpha,\beta}((t-x)^2;x))^{\frac{1}{2}}\right.\\ &\left.+(T_{n,\rho}^{\alpha,\beta}\left((t-x)^2;x)\right)^{\frac{1}{2}}.(T_{n,\rho}^{\alpha,\beta}\left((ct+\rho)^{-\gamma};x)\right)^{\frac{1}{2}}\right). \end{aligned}$$

If $n$ is adequately large, then we get

$$\left(T_{n,\rho}^{\alpha,\beta}\left((t-x)^2;x\right)\right)^{\frac{1}{2}}\leq\sqrt{\frac{2}{n\rho}}\phi(x), \tag{2.21}$$

where $\phi(x)=\sqrt{x(cx+\rho)}$.
For each $x\in(0,\infty)$, $T_{n,\rho}^{\alpha,\beta}\left((ct+\rho)^{-\gamma};x\right)\to(cx+\rho)^{-\gamma}$ as $n\to\infty$. For $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$T_{n,\rho}^{\alpha,\beta}\left((ct+\rho)^{-\gamma};x\right)\leq(cx+\rho)^{-\gamma}+\varepsilon,\ \forall\, n\geq n_0=n_0(c,x,\rho,\beta).$$

By choosing $\varepsilon = (cx+\rho)^{-\gamma}$, then we obtain

$$T_{n,\rho}^{\alpha,\beta}\left((ct+\rho)^{-\gamma};x\right) \le 2(cx+\rho)^{-\gamma},\ \forall\, n \ge n_0. \tag{2.22}$$

From (2.18) to (2.22), we get

$$\begin{aligned}|T_{n,\rho}^{\alpha,\beta}(j(t);x)-j(x)| &\le 2^{\gamma}\|\phi^{\gamma}j'\|\sqrt{\frac{2}{n\rho}}\phi(x)\left(\phi^{-\gamma}(x)+\sqrt{2}x^{\frac{-\gamma}{2}}(cx+\rho)^{\frac{-\gamma}{2}}\right)\\ &\le 2^{\gamma+\frac{1}{2}}(1+\sqrt{2})\|\phi^{\gamma}j'\|\sqrt{\frac{2}{n\rho}}\phi^{1-\gamma}(x). \end{aligned}\tag{2.23}$$

We may write

$$\begin{aligned}|T_{n,\rho}^{\alpha,\beta}(f(t);x)-f(x)| &\le |T_{n,\rho}^{\alpha,\beta}(f(t)-j(x);x)|\\ &\quad+|T_{n,\rho}^{\alpha,\beta}(j(t);x)-j(x)|+|j(x)-f(x)|\\ &\le 2\|f-j\|+|T_{n,\rho}^{\alpha,\beta}(j(t);x)-j(x)|.\end{aligned}\tag{2.24}$$

From (2.23) to (2.24) and for adequately large $n$, we get

$$\begin{aligned}|T_{n,\rho}^{\alpha,\beta}(f(t);x)-f(x)| &\le 2\|f-j\|+2^{\gamma+\frac{1}{2}}(1+\sqrt{2})\sqrt{\frac{2}{n\rho}}\phi^{1-\gamma}\|\phi^{\beta}j'\|\\ &\le m_1\left\{\|f-j\|+\frac{\phi^{1-\gamma}(x)}{\sqrt{n\rho}}\|\phi^{\gamma}j'\|\right\}\\ &\le CK_{\phi^{\gamma}}\left(f,\frac{\phi^{1-\gamma}(x)}{\sqrt{n\rho}}\right),\end{aligned}\tag{2.25}$$

where $m_1 = \max(2, 2^{\gamma+\frac{1}{2}}(1+\sqrt{2})\sqrt{2})$, from (1.4) and (2.25), we get the appropriate result. □

### 2.4.3 Weighted Approximation

In this section, we will examine the weighted estimation hypothesis of the operators (2.10).

**Theorem 2.4.11.** *(Weighted approximation) For each $f \in C_{\tau}^{*}[0,\infty)$ and $n > 2c$, we have*

$$\lim_{n\to\infty}||T_{n,\rho}^{\alpha,\beta}(f)-f||_{\tau} = 0.$$

*Proof.* Using Lemma 2.4.1, we see that it is sufficient to verify the following conditions

$$\lim_{n\to\infty}||T_{n,\rho}^{\alpha,\beta}(t^r,x)-x^r||_{\tau} = 0,\ r = 0,1,2. \tag{2.26}$$

Since $T_{n,\rho}^{\alpha,\beta}(1,x)=1$, for $r=0$, (2.26) holds. For $n\rho>c$, we have

$$\begin{aligned}
||T_{n,\rho}^{\alpha,\beta}(t,x)-x||_\tau &= \sup_{x\in[0,\infty)} \frac{|T_{n,\rho}^{\alpha,\beta}(t,x)-x|}{1+x^2}\\
&= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{n\rho x}{(n\rho-c)}+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}-x\right|\\
&\leq \left[\frac{n\rho}{(n\rho-c)}-1\right]\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{\alpha\rho}{(n\rho-c)(1+e\beta)}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}
\end{aligned}$$

the condition (2.26) holds for $r=1$ as $n\to\infty$. Again $n\rho>2c$, we have

$$\begin{aligned}
||T_{n,\rho}^{\alpha,\beta}(t^2,x)-x^2||_\tau &= \sup_{x\in[0,\infty)} \frac{|T_{n,\rho}^{\alpha,\beta}(t^2,x)-x^2|}{1+x^2}\\
&= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{n^2\rho^2}{(n\rho-c)(n\rho-2c)}x^2\right.\\
&+\frac{n\rho^2(1+2\alpha+e\beta)}{(n\rho-c)(n\rho-2c)(1+e\beta)}x\\
&\left.+\frac{\rho^2(\alpha^2-2\alpha e\beta-\alpha e^2\beta^2)}{(n\rho-c)(n\rho-2c)(1+e\beta)^2}-x^2\right|\\
&\leq \left[\frac{n^2\rho^2}{(n\rho-c)(n\rho-2c)}-1\right]\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\\
&+\frac{n\rho^2(1+2\alpha+e\beta)}{(n\rho-c)(n\rho-2c)(1+e\beta)}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\
&+\frac{\rho^2(\alpha^2-2\alpha e\beta-\alpha e^2\beta^2)}{(n\rho-c)(n\rho-2c)(1+e\beta)^2}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}.
\end{aligned}$$

The condition (2.26) holds for $r=2$ as $n\to\infty$.

This completes the proof of the theorem. □

**Corollary 2.4.12.** *For each $f\in C_\tau[0,\infty)$, and $\alpha>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|T_{n,\rho}^{\alpha,\beta}(f;x)-f(x)|}{(\tau(x))^\alpha}=0.$$

*Proof.* For any fixed $x_0>0$,

$$\begin{aligned}
\sup_{x\in[0,\infty)}\frac{|T_{n,\rho}^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}} &\leq \sup_{x\leq x_0}\frac{|T_{n,\rho}^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}+\sup_{x\geq x_0}\frac{|T_{n,\rho}^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}\\
&\leq \|T_{n,\rho}^{\alpha,\beta}(f;.)-f\|_{C[0,x_0]}+\|f\|_{x^2}\sup_{x\geq x_0}\frac{|T_{n,\rho}^{\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\alpha}}\\
&+\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}.
\end{aligned}$$

From the Theorem 2.4.7, in the above inequality first term tends to zero and by Lemma 2.4.1 for any fixed $x_0$ it can be easily seen that

$$\sup_{x\geq x_0}\frac{|T_{n,\rho}^{\alpha,\beta}(1+t^2;x)-f(x)|}{(1+x^2)^{1+\alpha}}\leq\frac{\varphi}{(1+x_0^2)^{\alpha}}.$$

Constant $\varphi>0$ is independent of $x$, and choose adequately large $x_0$ the right-hand side of the earlier inequality and last part can be made small, we get the required result. □

# Chapter 3

# Approximation by Bézier Variant of Bernstein-Durrmeyer Blending type Operators

Bézier curves are parametric curves that have been used extensively in image processing, time domain, and computer graphics. These techniques are primarily utilized in curve fitting, approximation, and interpolation. Fierre Bezier introduced this method and design curves for the Renault automobile bodywork in 1960. In the present chapter, we construct the Bézier variant of Bernstein-Durrmeyer blending type operators. We estimate the moments of these operators. Subsequently, we demonstrate the rate of approximation of the given operators in terms of Ditzian-Totik modulus of continuity and Lipschitz–type space. We also obtain the rate of convergence for functions having a bounded variation for these operators. Lastly, we reveal the rate of convergence of these operators to a certain function by graphics.

## 3.1 Introduction

D.D. Stancu has received admiration from all over the world for his thorough study and noteworthy findings. In the year 1998, Stancu [91] proposed the Bernstein-type operators which are as follows:

$$D_{n,r,s}f(x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{\nu=0}^{s} p_{s,\nu}(x) f\left(\frac{\mu+\nu r}{n}\right), \tag{3.1}$$

where parameters $r,s \in \mathbb{N} \cup \{0\}$, and found some approximation results using these operators. For $r,s = 0$, the given operators reduce to classical Bernstein operators. In the literature survey, various authors have studied the approximation behaviour of these mixed type operators [2, 3, 4, 5, 6, 23, 24, 32, 41, 42, 46, 47, 52, 56, 62, 70, 74, 75].

Incitation by above work Kajla and Goyal [58], introduced the Durrmeyer variant of the operators (3.1) depending on three parameters $r,s$ and $\rho$ with $\rho > 0$ as follows:

$$\mathfrak{L}^{(\rho)}_{n,r,s}(f;x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{\nu=0}^{s} p_{s,\nu}(x) \int_0^1 \varphi^{(\rho)}_{n,\mu+\nu r}(t) f(t)dt, \tag{3.2}$$

where $x \in [0,1]$ and

$$\varphi^{(\rho)}_{n,\mu+\nu r}(t) = \frac{t^{(\mu+\nu r)\rho}(1-t)^{(n-(\mu+\nu r))\rho}}{B((\mu+\nu r)\rho+1,(n-(\mu+\nu r))\rho+1)}.$$

They proved the order of convergence and discuss some important results of these operators. In 1983, Chang [35] discussed the generalized Berstein-Bézier polynomials. Zeng and Piriou [102] studied the rate of convergence of two Berstein-Bézier type operators for bounded variation functions. The order of approximation of the summation-integral type operators for functions with derivatives of bounded variation is estimated in [7, 100, 103]. Various authors have studied the approximation behaviour of these Bézier type operators [1, 39, 48, 85, 89].

Motivated by such remarkable work in this area, we introduce the Bézier variant of Bernstein-Durrmeyer blending type operators $\mathfrak{L}^{(\rho)}_{n,r,s}$ is defined as:

$$\breve{R}^{(\rho,\alpha)}_{n,r,s}(f;x) = \sum_{\mu=0}^{n-sr} Q^{(\alpha)}_{n,\mu}(x) \int_0^1 \varphi^{(\rho)}_{n,\mu+\nu r}(t) f(t)dt, \tag{3.3}$$

where $f \in C[0,1]$, $x \in [0,1]$ and $\rho,\ \alpha \geq 1$, $Q^{(\alpha)}_{n,\mu}(x) = [J_{n,\mu}(x)]^{\alpha} - [J_{n,\mu+1}(x)]^{\alpha}$ and

$$J_{n,\mu}(x) = \sum_{j=\mu}^{n-sr} p_{n-sr,j}(x) \sum_{\nu=0}^{s} p_{s,\nu}(x),$$

when $\mu \leq n - sr$ and 0 otherwise.

The operator (3.3) can be rewrite as:

$$\breve{R}^{(\rho,\alpha)}_{n,r,s}(f;x) = \int_0^1 \mathbf{U}^{(\rho,\alpha)}_{n,r,s}(x,t) f(t)dt,\ x \in [0,1],$$

where

$$\mathbf{U}^{(\rho,\alpha)}_{n,r,s}(x,t) = \sum_{\mu=0}^{n-sr} Q^{\alpha}_{n,\mu}(x) \varphi^{(\rho)}_{n,\mu+\nu r}(t)dt.$$

The focus of the present chapter is to discuss some useful results for the generalized operators (3.3). We study rate of convergence using Lipschitz-type space and Ditzian-Totik modulus of continuity. Next, we estimate rate of convergence for functions having a derivative of bounded variation for the given operators. In the last, we show graphical representation of the theoretical results and the effectiveness of the operators. Throughout this chapter $\rho > 0$.

## 3.2 Preliminaries

The results used in our primary findings are discussed in this section.

**Lemma 3.2.1.** *[58] The moments of the operator $\mathfrak{L}_{n,r,s}^{(\rho)}(e_m(x) = x^m);x)$, $m = \overline{0,3}$, we have*

$$\begin{aligned}
\mathfrak{L}_{n,r,s}^{(\rho)}(e_0;x) &= 1;\\
\mathfrak{L}_{n,r,s}^{(\rho)}(e_1;x) &= \frac{n\rho x+1}{n\rho+2};\\
\mathfrak{L}_{n,r,s}^{(\rho)}(e_2;x) &= \frac{x^2\rho^2[n(n-1)-rs(r-1)]+x\rho[n(3+\rho)+rs\rho(r-1)]+2}{(n\rho+3)(n\rho+2)};\\
\mathfrak{L}_{n,r,s}^{(\rho)}(e_3;x) &= \frac{x^3\rho^3[n(n-1)(n-2)+r(r-1)(2-3n+2r)s]}{(n\rho+4)(n\rho+3)(n\rho+2)}\\
&\quad+\frac{3x^2\rho^2[n^2(2+\rho)-rs(r-1)(2+\rho+r\rho)+n(r^2s\rho-(1+2sr)(2+\rho)+rs(4+\rho))]}{(n\rho+4)(n\rho+3)(n\rho+2)}\\
&\quad+\frac{x\rho(rs\rho(r-1)(6+\rho+r\rho)+n(11+\rho(6+\rho)))}{(n\rho+4)(n\rho+3)(n\rho+2)}\\
&\quad+\frac{6}{(n\rho+4)(n\rho+3)(n\rho+2)}.
\end{aligned}$$

**Lemma 3.2.2.** *The central moments of the operators $\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^m;x)$, for $m = 1,2$, we have*

$$\begin{aligned}
\mathfrak{L}_{n,r,s}^{(\rho)}((t-x);x) &= \frac{1-2x}{n\rho+2};\\
\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x) &= \frac{(\rho(n+(n+(r-1)rs)\rho)-6)x(1-x)+2}{(n\rho+2)(n\rho+3)}.
\end{aligned}$$

**Lemma 3.2.3.** *For the operators $\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^m;x)$ for $m = 1,2$, we have*

$$\begin{aligned}
\lim_{n\to\infty} n\mathfrak{L}_{n,r,s}^{(\rho)}((t-x);x) &= \frac{1-2x}{\rho};\\
\lim_{n\to\infty} n\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x) &= \frac{(1+\rho)x(1-x)}{\rho}.
\end{aligned}$$

**Remark 3.2.4.** *For $n \in \mathbb{N}$, we have*

$$\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x) \le \frac{\chi_{r,s}^{\rho}\, x(1-x)}{n\rho}, \tag{3.4}$$

*where $\chi_{r,s}^{\rho}$ is a positive constant depending on $r,s$ and $\rho$.*

**Remark 3.2.5.** *We have*

$$\begin{aligned}\breve{R}_{n,r,s}^{(\rho,\alpha)}(e_0;x) &= \sum_{\mu=0}^{n-sr} Q_{n,\mu}^{(\alpha)}(x) = [J_{n,0}(x)]^{\alpha} \\ &= \sum_{j=0}^{n-sr} p_{n-sr,j}(x) \sum_{\nu=0}^{s} p_{s,\nu}(x) = 1.\end{aligned}$$

**Lemma 3.2.6.** *Let $f$ be a real-valued continuous bounded function on $[0,1]$, then*

$$|\mathfrak{L}_{n,r,s}^{(\rho)}(f)| \leq \|f\|,$$

*where sup-norm $\|.\|$ is defined as $\|f\| = \sup_{x\in[0,1]} |f(x)|$.*

**Lemma 3.2.7.** *Let $f \in C[0,1]$ , then for $x \in [0,1]$ and $\alpha \geq 1$, we have*

$$|\breve{R}_{n,r,s}^{(\rho,\alpha)}(f)| \leq \alpha\|f\|.$$

*Proof.* We know the inequality

$$|a^{\alpha} - b^{\alpha}| \leq \alpha|a-b| \text{ with } 0 \leq a,\ b \leq 1, \text{ and } \alpha \geq 1,$$

using the provided inequality, we obtain

$$0 < [J_{n,\mu}(x)]^{\alpha} - [J_{n,\mu+1}(x)]^{\alpha} \leq \alpha(J_{n,\mu}(x) - J_{n,\mu+1}(x)) = \alpha[p_{n-sr,\mu}(x)].$$

Hence, from the definition of the Bézier variant operators $\breve{R}_{n,r,s}^{(\rho,\alpha)}(f;x)$ and Lemma 3.2.6, we get

$$|\breve{R}_{n,r,s}^{(\rho,\alpha)}(f)| \leq \alpha\|\mathfrak{L}_{n,r,s}^{(\rho)}(f)\| \leq \alpha\|f\|.$$

□

## 3.3 Direct Results

Here, we estimate rate of convergence of the function $f \in Lip_M^*(\sigma)$ by the operators $\breve{R}_{n,r,s}^{(\rho,\alpha)}$.

**Theorem 3.3.1.** *(Lipschitz class) Let $f \in Lip_M^*(\sigma)$ and $\sigma \in (0,1]$. Then for all $x \in (0,1)$, we have*

$$|\breve{R}_{n,r,s}^{(\rho,\alpha)}(f(t);x) - f(x)| \leq \alpha M \left(\frac{\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x)}{x}\right)^{\frac{\sigma}{2}},$$

*where* $\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x) = \dfrac{(\rho(n+(n+(r-1)rs)\rho)-6)x(1-x)+2}{(n\rho+2)(n\rho+3)}$.

*Proof.* From the Remark 3.2.5, we have

$$\begin{aligned} |\breve{R}_{n,r,s}^{(\rho,\alpha)}(f(t);x)-f(x)| &\leq \breve{R}_{n,r,s}^{(\rho,\alpha)}(|f(t)-f(x)|;x) \\ &\leq \alpha\mathfrak{L}_{n,r,s}^{(\rho)}(|f(t)-f(x)|;x) \\ &\leq \alpha M\mathfrak{L}_{n,r,s}^{(\rho)}\left(\frac{|t-x|^{\sigma}}{(t+x)^{\frac{\sigma}{2}}};x\right) \\ &\leq \frac{\alpha M}{x^{\frac{\sigma}{2}}}\mathfrak{L}_{n,r,s}^{(\rho)}(|t-x|^{\sigma};x). \end{aligned} \tag{3.5}$$

Taking $p=\frac{2}{\sigma}$ and $q=\frac{2}{2-\sigma}$ and applying Hölder's inequality, we obtain

$$\begin{aligned} \mathfrak{L}_{n,r,s}^{(\rho)}(|t-x|^{\sigma};x) &\leq \left\{\mathfrak{L}_{n,r,s}^{(\rho)}(|t-x|^{2};x)\right\}^{\frac{\sigma}{2}}.\left\{\mathfrak{L}_{n,r,s}^{(\rho)}(1^{\frac{2}{2-\sigma}};x)\right\}^{\frac{2-\sigma}{2}} \\ &\leq \left\{\mathfrak{L}_{n,r,s}^{(\rho)}(|t-x|^{2};x)\right\}^{\frac{\sigma}{2}}. \end{aligned} \tag{3.6}$$

Combining (3.5, 3.6), we get the required result

$$|\breve{R}_{n,r,s}^{(\rho,\alpha)}(f(t);x)-f(x)|\leq \alpha M\left(\frac{\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^{2};x)}{x}\right)^{\frac{\sigma}{2}}.$$

Hence the proof follows. □

**Theorem 3.3.2.** *Let $f\in C[0,1]$ and $\rho\geq 1$, $\alpha_n\in[0,1]$, we get*

$$|\breve{R}_{n,r,s}^{(\rho,\alpha)}(f(t);x)-f(x)|\leq C\omega_{2}\left(f;\sqrt{\frac{\chi_{r,s}^{\rho}}{16n\rho}}\right),$$

*for sufficient large n and C is a positive constant.*

*Proof.* Let $g\in C^{2}[0,1]$, followed by well-known Taylor series expansion, we have

$$g(t)=g(x)+g'(x)(t-x)+\int_{x}^{t}(t-u)g''(u)du.$$

We have $S_{n}^{(\rho)}(1;x)=1$, applying $S_{n}^{(\rho)}(.;x)$ to both sides of the above equation, we get

$$\breve{R}_{n,r,s}^{(\rho,\alpha)}(g;x)=g(x)+g'(x)\breve{R}_{n,r,s}^{(\rho,\alpha)}(t-x;x)+\breve{R}_{n,r,s}^{(\rho,\alpha)}\left(\int_{x}^{t}(t-u)g''(u)du;x\right).$$

By the Cauchy-Schwarz inequality and using equations of Lemma 3.2.2 and 3.4.1, we have

$$\begin{aligned} |\breve{R}_{n,r,s}^{(\rho,\alpha)}(g;x)-g(x)| &\leq |g'(x)||\breve{R}_{n,r,s}^{(\rho,\alpha)}(|t-x|;x)|+|\breve{R}_{n,r,s}^{(\rho,\alpha)}\left(\int_{x}^{t}(t-u)g''(u)du;x\right)| \\ &\leq \|g'\|\breve{R}_{n,r,s}^{(\rho,\alpha)}(|t-x|;x)+\frac{\|g''\|}{2}\breve{R}_{n,r,s}^{(\rho,\alpha)}((t-x)^{2};x) \\ &\leq \|g'\|\sqrt{\breve{R}_{n,r,s}^{(\rho,\alpha)}((t-x)^{2};x)}+\frac{\|g''\|}{2}\breve{R}_{n,r,s}^{(\rho,\alpha)}((t-x)^{2};x) \end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{\alpha}\|g'\|\sqrt{\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x)} + \frac{\alpha\|g''\|}{2}\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x) \\
&\leq \sqrt{\alpha}\|g'\|\sqrt{\frac{\chi_{r,s}^{\rho}}{4n\rho}} + \frac{\alpha\|g''\|}{2}\frac{\chi_{r,s}^{\rho}}{4n\rho} \\
&\leq \frac{\sqrt{\alpha}\|g'\|}{2}\sqrt{\frac{\chi_{r,s}^{\rho}}{n\rho}} + \frac{\alpha\|g''\|}{8}\frac{\chi_{r,s}^{\rho}}{n\rho}.
\end{aligned}
$$

Then, using the above inequality, we have

$$
\begin{aligned}
|\breve{R}_{n,r,s}^{(\rho,\alpha)}(f;x) - f(x)| &\leq |\breve{R}_{n,r,s}^{(\rho,\alpha)}(f-g;x)| + |(f-g)(x)| + |\breve{R}_{n,r,s}^{(\rho,\alpha)}(g;x) - g(x)| \\
&\leq 2\|f-g\| + \frac{\sqrt{\alpha}}{2}\|g'\|\sqrt{\frac{\chi_{r,s}^{\rho}}{n\rho}} + \frac{\alpha}{8}\|g''\|\frac{\chi_{r,s}^{\rho}}{n\rho} \\
&\leq 2\left(\|f-g\| + \frac{\sqrt{\alpha}}{4}\|g'\|\sqrt{\frac{\chi_{r,s}^{\rho}}{n\rho}} + \frac{\alpha}{16}\|g''\|\frac{\chi_{r,s}^{\rho}}{n\rho}\right) \\
&\leq 2\left(\|f-g\| + \|g'\|\sqrt{\frac{\alpha\chi_{r,s}^{\rho}}{16n\rho}} + \|g''\|\frac{\alpha\chi_{r,s}^{\rho}}{16n\rho}\right).
\end{aligned}
$$

Hence, taking infimum on the right-hand side over all $g \in C^2[0,1]$, we get

$$
|\breve{R}_{n,r,s}^{(\rho,\alpha)}(f;x) - f(x)| \leq 2K_2\left(f;\frac{\chi_{r,s}^{\rho}}{4n\rho}\right)
$$

and using (1.3), we get the desired result

$$
|\breve{R}_{n,r,s}^{(\rho,\alpha)}(f;x) - f(x)| \leq C\omega_2\left(f;\sqrt{\frac{\chi_{r,s}^{\rho}}{16n\rho}}\right).
$$

□

**Theorem 3.3.3.** *(Ditzian-totik modulus of continuity) Let $f \in C[0,1]$ and $\alpha \geq 1$, then for any $x \in (0,1)$, we have*

$$
|\breve{R}_{n,r,s}^{(\rho,\alpha)}(f;x) - f(x)| \leq C\omega_{\phi}\left(f;\frac{\phi(x)}{\sqrt{n\rho}}\right),
$$

*where $\phi(x) = \sqrt{x(1-x)}$ and $C$ is a constant independent of $n$ and $x$.*

*Proof.* By the definition of $\overline{K}_{\phi}(f;t)$, for fixed $n,x$, we can choose $g = g_n(x) \in W_{\phi}$ such that

$$
\begin{aligned}
&\|f-g\| + \frac{1}{\sqrt{n\rho}}\|\phi g'\| + \frac{1}{n\rho}\|g'\| \\
&\qquad \leq 2\overline{K}_{\phi}\left(f;\frac{1}{\sqrt{n\rho}}\right). \qquad (3.7)
\end{aligned}
$$

Using remark (3.2.5), we can write

$$\begin{aligned}|\breve{R}_{n,r,s}^{(\rho,\alpha)}(f;x)-f(x)| &\leq |\breve{R}_{n,r,s}^{(\rho,\alpha)}(f-g;x)|+|f-g|+|\breve{R}_{n,r,s}^{(\rho,\alpha)}(g;x)-g(x)| \\ &\leq 2\|f-g\|+|\breve{R}_{n,r,s}^{(\rho,\alpha)}(g;x)-g(x)|. \end{aligned} \tag{3.8}$$

We only need to compute the second term in the above equation. We will have to split the estimate into two domains, i.e, $x \in F_n^c = [0, \frac{1}{n}]$ and $x \in F_n = (\frac{1}{n}, 1)$.
Using the representation $g(t) = g(x) + \int_x^t g'(u)du$, we may write

$$|\breve{R}_{n,r,s}^{(\rho,\alpha)}(g;x)-g(x)| = \left|\breve{R}_{n,r,s}^{(\rho,\alpha)}\left(\int_x^t g'(u)du;x\right)\right|. \tag{3.9}$$

If $x \in F_n = (\frac{1}{n}, 1)$ then $\mathfrak{L}_{n,r,s}^{(\rho)} \leq \frac{\phi^2(x)}{n\rho}$, we have

$$\left|\int_x^t g'(u)du\right| \leq \|\phi g'\| \left|\int_x^t \frac{1}{\phi(u)}du\right|. \tag{3.10}$$

For any $x, t \in (0,1)$, we find that

$$\begin{aligned}\left|\int_x^t \frac{1}{\phi(u)}du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| \\ &\leq \left|\int_x^t \left(\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{1-u}}\right)du\right| \\ &\leq 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}}+\frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right) \\ &< 2|t-x|\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right) \\ &\leq \frac{2\sqrt{2}|t-x|}{\phi(x)}. \end{aligned} \tag{3.11}$$

Combining (3.9 - 3.11) and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\breve{R}_{n,r,s}^{(\rho,\alpha)}(g;x)-g(x)| &< 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)\breve{R}_{n,r,s}^{(\rho,\alpha)}(|t-x|;x) \\ &\leq 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)(\breve{R}_{n,r,s}^{(\rho,\alpha)}(|t-x|;x))^{\frac{1}{2}} \\ &\leq 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)(\alpha\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x))^{\frac{1}{2}}. \end{aligned}$$

Now, using the relation

$$\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x) \leq \frac{\chi_{r,s}^{\rho}\,\phi(x)}{n\rho},$$

we have

$$|\breve{R}_{n,r,s}^{(\rho,\alpha)}(g;x)-g(x)| < \frac{C\|\phi g'\|}{\sqrt{n\rho}}. \tag{3.12}$$

For $x \in F_n^c = [0, \frac{1}{n}]$, $\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x) \sim \frac{1}{\sqrt{n\rho}}$ and $|\int_x^t g'(u)du \leq \|g'\||t-x|$. Therefore using the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|\breve{R}_{n,r,s}^{(\rho,\alpha)}(g;x) - g(x)| &\leq \|g'\|\breve{R}_{n,r,s}^{(\rho,\alpha)}(|t-x|;x) \\
&\leq \|g'\|\alpha.\mathfrak{L}_{n,r,s}^{(\rho)}(|t-x|;x) \\
&\leq \frac{C\|g'\|}{n\rho}. \qquad (3.13)
\end{aligned}
$$

From (3.12) and (3.13), we have

$$
|\breve{R}_{n,r,s}^{(\rho,\alpha)}(g;x) - g(x)| \leq C\left(\frac{\|\phi g'\|}{\sqrt{n\rho}} + \frac{\|g'\|}{n\rho}\right). \qquad (3.14)
$$

Using (1.2), (3.8) and (3.14), we get the desired result. □

## 3.4 Rate of convergence

**Lemma 3.4.1.** *Let $x \in (0,1]$, then for $\alpha \geq 1$ and sufficiently large n, we have*

$$
\begin{aligned}
\xi_{n,r,s}^{(\rho,\alpha)}(x,y) &= \int_0^y U_{n,r,s}^{(\rho,\alpha)}(x,t)dt \leq \frac{\alpha}{(x-y)^2}\frac{\chi_{r,s}^{(\rho)}\phi^2(x)}{n\rho}, 0 \leq y < x \text{ and} \\
1 - \xi_{n,r,s}^{(\rho,\alpha)}(x,z) &= \int_z^1 U_{n,r,s}^{(\rho,\alpha)}(x,t)dt \leq \frac{\alpha}{(z-x)^2}\frac{\chi_{r,s}^{(\rho)}\phi^2(x)}{n\rho}, x < z < 1.
\end{aligned}
$$

*Proof.* From (3.4) and (3.6), we get

$$
\begin{aligned}
\xi_{n,r,s}^{\rho,\alpha}(x,y) &= \int_0^y U_{n,r,s}^{\rho,\alpha}(x,t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 U_{n,r,s}^{\rho,\alpha}(x,t)dt \\
&= \breve{R}_{n,r,s}^{(\rho,\alpha)}((t-x)^2;x)(x-y)^{-2} \\
&\leq \alpha.\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x)(x-y)^{-2} \\
&\leq \frac{\alpha}{(x-y)^2}\frac{\chi_{r,s}^{(\rho)}\phi^2(x)}{n\rho}.
\end{aligned}
$$

and

$$
\begin{aligned}
1 - \xi_{n,r,s}^{\rho,\alpha}(x,z) &= \int_z^1 U_{n,r,s}^{\rho,\alpha}(x,t)dt \leq \int_z^1 \left(\frac{x-t}{z-x}\right)^2 U_{n,r,s}^{\rho,\alpha}(x,t)dt \\
&= (z-x)^{-2}\int_z^1 (x-t)^2 U_{n,r,s}^{\rho,\alpha}(x,t)dt \\
&= \breve{R}_{n,r,s}^{(\rho,\alpha)}((t-x)^2;x)(z-x)^{-2} \\
&\leq \alpha.\mathfrak{L}_{n,r,s}^{(\rho)}((t-x)^2;x)(z-x)^{-2} \\
&\leq \frac{\alpha}{(z-x)^2}\frac{\chi_{r,s}^{(\rho)}\phi^2(x)}{n\rho}.
\end{aligned}
$$

□

**Theorem 3.4.2.** *(Derivative of bounded variation) Let $f \in DBV(0,1)$, $\alpha \geq 1$ and let $\bigvee_a^b (f'_x)$ be the total variation of $f'_x$ on $[a,b] \subset [0,1]$. Then for every $x \in (0,1)$ and for sufficiently large n, we have*

$$
\begin{aligned}
|\breve{R}^{(\rho,\alpha)}_{n,r,s}(f;x) - f(x)| \quad \leq \quad & \frac{1}{\alpha+1}\left| f'(x+) + \alpha f'(x-) \right| \sqrt{\frac{\alpha \chi^{(\rho)}_{r,s}}{n\rho}\phi(x)} \\
& + \sqrt{\frac{\alpha \chi^{(\rho)}_{r,s}}{n\rho}\phi(x)} . \frac{\alpha}{\alpha+1}\left| f'(x+) + \alpha f'(x-) \right| \\
& + \alpha . \frac{\chi^{(\rho)}_{r,s}(1-x)}{n\rho} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\left(\frac{x}{k}\right)}^{x} (f'_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\left(\frac{x}{\sqrt{n}}\right)}^{x} (f'_x) \\
& + \frac{\alpha \chi^{(\rho)}_{r,s} x}{n\rho} . \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{(1-x)}{k}} (f'_x) + \frac{1-x}{\sqrt{n}} \bigvee_{x}^{x+\frac{(1-x)}{\sqrt{n}}} (f'_x),
\end{aligned}
$$

*where $\chi^{(\rho)}_{r,s} > 0$ and the total variation function $f'_x$ is defined by*

$$
f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+), & x < t \leq 1. \end{cases}
$$

*Proof.* Since $\breve{R}^{(\rho,\alpha)}_{n,r,s}(1;x) = 1$, we have

$$
\begin{aligned}
\breve{R}^{(\rho,\alpha)}_{n,r,s}(f;x) - f(x) &= \int_0^1 |f(t) - f(x)| U^{(\rho)}_{n,r,s}(x,t)dt \\
&= \int_0^1 \left( \int_x^t f'(u)du \right) U^{(\rho)}_{n,r,s}(x,t)dt. \qquad (3.15)
\end{aligned}
$$

From the definition of the function $f'_x$, for any $f \in \mathrm{DBV}(0,1)$, we can write

$$
\begin{aligned}
f'(t) \quad = \quad & \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) + f'_x(t) + \frac{1}{2}(f'(x+) - f'(x-)) \left( sgn(t-x) + \frac{\alpha-1}{\alpha+1} \right) \\
& + \delta_x(t) \left( f'(x) - \frac{1}{2}\left( f'(x+) + f'(x-) \right) \right), \qquad (3.16)
\end{aligned}
$$

where delta function $\delta_x$ is defined as

$$
\delta_x(t) = \begin{cases} 1, & x = t \\ 0, & x \neq t \end{cases} .
$$

It is clear that

$$
\int_0^1 \left( \int_x^t \left( f'(t) - \frac{1}{2}\left( f'(x+) + f'(x-) \right) \right) \delta_x(t)du \right) U^{(\rho)}_{n,r,s}(x,t)dt = 0. \qquad (3.17)
$$

Using Lemma 3.4.1, we get

$$\begin{aligned} A_1 &= \int_0^1 \left( \int_x^t \frac{1}{\alpha+1} \Big( f'(x+) + \alpha f'(x-) \Big) du \right) U_{n,r,s}^{(\rho)}(x,t)dt \\ &= \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) \int_0^1 (t-x) U_{n,r,s}^{(\rho)}(x,t)dt \\ &= \frac{1}{\alpha+1}(f'(x+) + \alpha f'(x-)) \breve{R}_{n,r,s}^{(\rho,\alpha)}((t-x);x) \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} A_2 &= \int_0^1 U_{n,r,s}^{\rho}(x,t) \left( \int_x^t \frac{1}{2}(f'(x+) - f'(x-)) \left( sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) du \right) dt \\ &= \frac{1}{2}(f'(x+) - f'(x-)) \Bigg[ - \int_0^x \left( \int_t^x \left( sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) du \right) U_{n,r,s}^{(\rho)}(x,t)dt \\ &\quad + \int_x^1 \left( \int_x^t \left( sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) du \right) U_{n,r,s}^{(\rho)}(x,t)dt \Bigg] \\ &\leq \frac{\alpha}{\alpha+1}(f'(x+) - f'(x-)) \int_0^1 |t-x| U_{n,r,s}^{(\rho)}(x,t)dt \\ &\leq \frac{\alpha}{\alpha+1}(f'(x+) - f'(x-)) \breve{R}_{n,r,s}^{(\rho,\alpha)}((t-x);x). \end{aligned} \tag{3.19}$$

Using Lemma 3.4.1 and (3.15-3.19) and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\breve{R}_{n,r,s}^{(\rho,\alpha)}(f;x) - f(x)| &\leq \frac{1}{\alpha+1}|f'(x+) + \alpha f'(x-)| \sqrt{\alpha}\delta_{n,\rho}(x) + \frac{\alpha}{\alpha+1}|f'(x+) - \alpha f'(x-)| \sqrt{\alpha}\delta_{n,\rho}(x) \\ &+ \left| \int_0^x \left( \int_x^t f'_x(u)du \right) U_{n,r,s}^{\rho}(x,t)dt \right| + \left| \int_x^1 \left( \int_x^t f'_x(u)du \right) U_{n,r,s}^{(\rho)}(x,t)dt \right|. \end{aligned} \tag{3.20}$$

Now, let

$$A_{n,r,s}^{(\rho,\alpha)}(f'_x,x) = \int_0^x \int_x^t f'_x(u)du U_{n,r,s}^{(\rho)}(x,t)dt,$$

and

$$B_{n,r,s}^{(\rho,\alpha)}(f'_x,x) = \int_x^1 \int_x^t f'_x(u)du U_{n,r,s}^{(\rho)}(x,t)dt.$$

Our claim is that to calculate the estimates of the terms $A_{n,r,s}^{(\rho,\alpha)}(f'_x,x)$ and $B_{n,r,s}^{\rho,\alpha}(f'_x,x)$.
From the definition of $\xi_{n,r,s}^{(\rho,\alpha)}$ given in Lemma 3.4.1, we can write

$$A_{n,r,s}^{(\rho,\alpha)}(f'_x,x) = \int_0^x \left( \int_x^t f'_x(u) \right) \frac{\partial}{\partial t} \xi_{n,r,s}^{(\rho,\alpha)}(x,t)dt.$$

Applying the integration by parts, we get

$$|A_{n,r,s}^{(\rho,\alpha)}(f'_x,x)| \leq \int_0^x |f'_x(t)| \xi_{n,r,s}^{\rho,\alpha}(x,t)dt$$

$$
\begin{aligned}
&\leq \int_{0}^{x-\frac{x}{\sqrt{n}}} |f'_x(t)| \xi^{\rho,\alpha}_{n,r,s}(x,t)dt + \int_{x-\frac{x}{\sqrt{n}}}^{x} |f'_x(t)| x - \frac{x}{\sqrt{n}} \\
&\leq I_1 + I_2.
\end{aligned}
$$

Since $f'_x(x) = 0$ and $\xi^{\rho,\alpha}_{n,r,s}(x,t) \leq 1$, we have

$$
\begin{aligned}
I_2 &= \int_{x-\frac{x}{\sqrt{n}}}^{x} |f'_x(t) - f'_x(x)| \xi^{\rho,\alpha}_{n,r,s}(x,t)dt \leq \int_{x-\frac{x}{\sqrt{n}}}^{x} \left(\bigvee_{t}^{x} f'_x\right) dt \\
&\leq \left(\bigvee_{t}^{x} f'_x\right) \int_{x-\frac{x}{\sqrt{n}}}^{x} dt = \frac{x}{\sqrt{n}} \left(\bigvee_{t}^{x} f'_x\right).
\end{aligned}
$$

By applying Lemma 3.4.1 and considering $t = x - \frac{x}{u}$, we get

$$
\begin{aligned}
I_1 &\leq \alpha \delta^2_{n,\rho}(x) \int_{0}^{x-\frac{x}{\sqrt{n}}} |f'_x(t) - f'_x(x)| \frac{dt}{(x-t)^2} \\
&\leq \alpha \delta^2_{n,\rho}(x) \int_{0}^{x-\frac{x}{\sqrt{n}}} \left(\bigvee_{t}^{x} f'_x\right) \frac{dt}{(x-t)^2} \\
&\leq \frac{\alpha \delta^2_{n,\rho}(x)}{x} \int_{1}^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{u}}^{x} f'_x\right) du \\
&\leq \frac{\alpha \delta^2_{n,\rho}(x)}{x} \sum_{k=1}^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right).
\end{aligned}
$$

Therefore,

$$
|A^{(\rho,\alpha)}_{n,r,s}(f'_x, x)| \leq \frac{\alpha \delta^2_{n,\rho}(x)}{x} \sum_{k=1}^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{k}}^{x} f'_x\right) + \frac{x}{\sqrt{n}} \left(\bigvee_{t}^{x} f'_x\right). \tag{3.21}
$$

Also, using integration by parts in $B^{(\rho,\alpha)}_{n,r,s}(f'_x, x)$ and applying Lemma 3.4.1 with $z = x + \frac{(1-x)}{\sqrt{n}}$, we have

$$
\begin{aligned}
|B^{(\rho,\alpha)}_{n,r,s}(f'_x, x)| &= \left| \int_{x}^{1} \left( \int_{x}^{t} f'_x(u)du \right) U^{(\rho)}_{n,r,s}(x,t)dt \right| \\
&= \left| \int_{x}^{1} \left( \int_{x}^{t} f'_x(u)du \right) \frac{\partial}{\partial t}(1 - \xi^{(\rho,\alpha)}_{n,r,s}(x,t))dt \right. \\
&\quad \left. + \int_{z}^{1} \left( \int_{x}^{t} f'_x(u)du \right) \frac{\partial}{\partial t}(1 - \xi^{(\rho,\alpha)}_{n,r,s}(x,t))dt \right| \\
&= \left| \left[ \int_{x}^{t} (f'_x(u)du)(1 - \xi^{(\rho,\alpha)}_{n,r,s}(x,t)) \right]_{x}^{z} - \int_{x}^{z} f'_x(t)(1 - \xi^{(\rho,\alpha)}_{n,r,s}(x,t))dt \right. \\
&\quad \left. + \int_{z}^{1} \int_{x}^{t} (f'_x(u)du) \frac{\partial}{\partial t}(1 - \xi^{(\rho,\alpha)}_{n,r,s}(x,t))dt \right| \\
&= \left| \int_{x}^{z} (f'_x(u)du)(1 - \xi^{(\rho,\alpha)}_{n,r,s}(x,z)) - \int_{x}^{z} f'_x(t)(1 - \xi^{(\rho,\alpha)}_{n,r,s}(x,t))dt \right. \\
&\quad \left. + \left[ \int_{x}^{t} (f'_x(u)du)(1 - \xi^{(\rho,\alpha)}_{n,r,s}(x,t)) \right]_{z}^{1} - \int_{z}^{1} f'_x(t)(1 - \xi^{(\rho,\alpha)}_{n,r,s}(x,t))dt \right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left| \int_x^z f_x'(t)(1-\xi_{n,r,s}^{(\rho,\alpha)}(x,t))dt + \int_z^1 f_x'(t)(1-\xi_{n,r,s}^{(\rho,\alpha)}(x,t))dt \right| \\
&\leq \alpha\delta_{n,\rho}^2(x) \int_z^1 \left( \bigvee_x^t f_x' \right) (t-x)^{-2} dt + \int_x^z \bigvee_x^t f_x' dt \\
&\leq \alpha\delta_{n,\rho}^2(x) \int_{x+\frac{(1-x)}{\sqrt{n}}}^1 \left( \bigvee_x^t f_x' \right) (t-x)^{-2} dt + \frac{(1-x)}{\sqrt{n}} \left( \bigvee_x^{x+\frac{(1-x)}{\sqrt{n}}} f_x' \right).
\end{aligned}
$$

Substituting $u = \frac{(1-x)}{(t-x)}$ in the above equation, we get

$$
\begin{aligned}
|B_{n,r,s}^{(\rho,\alpha)}(f_x',x)| &\leq \alpha\delta_{n,\rho}^2(x) \int_1^{\sqrt{n}} \left( \bigvee_x^{x+\frac{(1-x)}{u}} f_x' \right) (1-x)^{-1} du + \frac{(1-x)}{\sqrt{n}} \left( \bigvee_x^{x+\frac{(1-x)}{\sqrt{n}}} f_x' \right) \\
&\leq \frac{\alpha\delta_{n,\rho}^2(x)}{(1-x)} \sum_{k=1}^{[\sqrt{n}]} \left( \bigvee_x^{x+\frac{(1-x)}{k}} f_x' \right) + \frac{(1-x)}{\sqrt{n}} \left( \bigvee_x^{x+\frac{(1-x)}{\sqrt{n}}} f_x' \right). \qquad (3.22)
\end{aligned}
$$

Using the equations (3.20- 3.22), we get the required result. □

## 3.5 Numerical Results with Conclusions

Using a function and various parameter values, we show the convergence of the operators (3.2) and (3.3) with the help of Mathematica software in this section.

Let $f(x) = -x^3 + x^2 - 1$ for $r = 1$, $s = 1$, $\rho = 1$ and $n \in \{10, 50, 110, 200\}$. The operators $\mathfrak{L}_{n,r,s}^{(\rho)}$ converges to the function $f(x)$ for large $n$ are shown in figure (a) and proposed Bézier variant operators $\breve{R}_{n,r,s}^{(\rho,\alpha)}$ for $\alpha = 1.05$, $r = 1$, $s = 1$, $\rho = 1$ and $n \in \{10, 50, 110, 200\}$, converges to the function $f(x)$ for large $n$ are shown in figure (b).

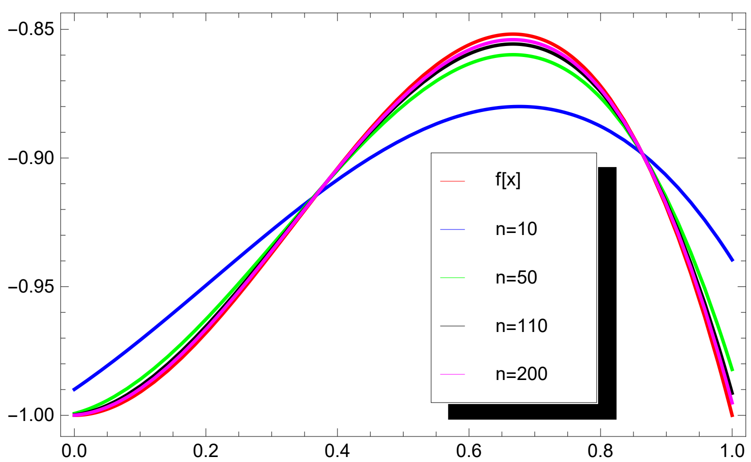


Figure 3.1: Figure:(a) This graph shows how the Blending type Bernstein-Durrmeyer operators uniformly converge.

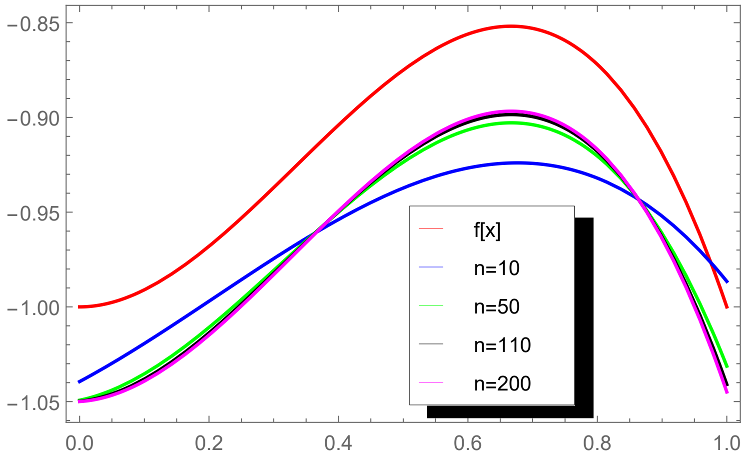


Figure 3.2: Figure:(b) convergence of the Bézier variant of Bernstein-Durrmeyer blending type operators.

The Figure $(a)$, and Figure $(b)$ give out with the uniform convergence of the operators (3.2) and (3.3).

# Chapter 4

# Approximation by a Durrmeyer Variant of Cheney-Sharma Chlodovsky Operators

---

For the sequence of linear positive operators to figure out the convergence in Approximation Theory, one useful research is the statistical convergence theory. This chapter is a study of precise approximation properties of Cheney-Sharma Chlodovsky Durrmeyer operators. Using the preliminary results of these operators we verify Bohman-Korovkin's theorem. We study Lipschitz-type space and the second-order modulus of continuity. In the next section, the weighted approximation result is obtained. Moreover, we obtain some approximation properties in terms of A-Statistical convergence of these operators.

---

## 4.1 Introduction

In 1932, Chlodovsky [17] introduced the generalization of Bernstein polynomial known as classical Bernstein-Chlodovsky polynomials on an unbounded interval. These polynomials $C_n : C[0,\infty) \to C[0,\infty), n \in \mathbb{N}$ is defined as

$$C_n(f,x) = \begin{cases} \sum\limits_{\nu=0}^{n} f\left(\frac{\nu}{n}b_n\right)\binom{n}{\nu}\left(\frac{x}{b_n}\right)^{\nu}\left(1-\frac{x}{b_n}\right)^{n-\nu}, & 0 \leq x \leq b_n \\ f(x), & x > b_n \end{cases},$$

where $f \in C[0,\infty), x \in [0,\infty)$ and $0 \leq x \leq b_n$ and $\{b_n\}$ is a positive sequence with $\lim\limits_{n\to\infty} b_n = \infty$, and $\lim\limits_{n\to\infty} \frac{b_n}{n} = 0$.

The Bernstein polynomials and its generalized operators have been extensively studied by several researchers. These operators have many important applications in the field of Mathematics, computer science, and graphic design. For the more information about the Durrmeyer operators and their properties, readers should refer the following articles (cf.[38, 43, 74, 77, 81, 99]).

The eminent Norwegian mathematician Niels Henrik Abel gave famous equality as:

$$(x+y)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} x(x-\nu\beta)^{\nu-1}(y+\nu\beta)^{n-\nu},\ \beta \geq 0.$$

Use of above equality we mention more Abel type formulas

$$\begin{aligned}
(x+y+n\beta)^n &= \sum_{\nu=0}^{n} \binom{n}{\nu} x(x+\nu\beta)^{\nu-1}(y+(n-\nu)\beta)^{n-\nu},\\
(x+y+n\beta)^n &= \sum_{\nu=0}^{n} \binom{n}{\nu} (x+\nu\beta)^{\nu} y(y+(n-\nu)\beta)^{n-\nu-1},\\
(x+y+n\beta)^{n-1} &= \sum_{\nu=0}^{n} \binom{n}{\nu} x(x+\nu\beta)^{\nu-1} y[y+(n-\nu)\beta]^{n-\nu-1},\\
(x+y)(x+y+n\beta)^{n-1} &= \sum_{\nu=0}^{n} \binom{n}{\nu} x(x+\nu\beta)^{\nu-1} y[y+(n-\nu)\beta]^{n-\nu-1},
\end{aligned}$$

where $x, y \in \mathbb{R}$. Using these equalities, Cheney-Sharma [16] introduced a generalization of Bernstein polynomial as follows:

$$\Psi_n(f,x) = (1+n\beta)^{1-n} \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) \binom{n}{\nu} x(x+k\beta)^{\nu-1}(1-x)[1-x+(n-\nu)\beta]^{n-\nu-1}. \tag{4.1}$$

For $\beta = 0$ above operators reduce to the classical Bernstein operators.

In 2020, Söylemez and Taşdelen [82] proposed Cheney-Sharma-Chlodovsky operators depending on some parameters as follows:

$$\begin{aligned}
G_n(f,x) &= (1+n\beta)^{1-n} \sum_{\nu=0}^{n} f\left(\frac{\nu}{n} b_n\right) \binom{n}{\nu} \frac{x}{b_n} \left(\frac{x}{b_n}+\beta\right)^{\nu-1} \left(1-\frac{x}{b_n}\right)\\
&\times \left[1-\frac{x}{b_n}+(n-\nu)\beta\right]^{n-\nu-1}
\end{aligned} \tag{4.2}$$

for $0 \leq x \leq b_n$ and $f(x)$ for $x > b_n$. Here sequence $\{b_n\}$ is the same as defined in classical Bernstein-Chlodovsky polynomials. They proved the order of convergence and discuss some important results of these operators. For $b_n = 1$ the operators (4.2) reduce to (4.1).

Encouraged by the above work, we introduced a new type of Cheney-Sharma-Chlodovsky -Durrmeyer operators, with the help of Lebesgue integrable function on $[0, b_n]$ as

$$\tilde{D}_n(f,x) = \frac{(n+1)}{b_n} \sum_{\nu=0}^{n} \wp_{n,\nu,\beta}\left(\frac{x}{b_n}\right) \int_0^{b_n} \vartheta_{n,\nu}\left(\frac{x}{b_n}\right) f(x)dx, \tag{4.3}$$

where $0 \le x \le b_n$, $\lim_{n\to\infty} b_n \to \infty$, $\lim_{n\to\infty} \frac{b_n}{n} = 0$,

$$\wp_{n,\nu,\beta}\left(\frac{x}{b_n}\right) = (1+n\beta)^{1-n}\binom{n}{\nu}\frac{x}{b_n}\left(\frac{x}{b_n}+\nu\beta\right)^{\nu-1}\left(1-\frac{x}{b_n}\right)\left[1-\frac{x}{b_n}+(n-\nu)\beta\right]^{n-\nu-1},$$

and

$$\vartheta_{n,\nu}\left(\frac{x}{b_n}\right) = \binom{n}{\nu}\left(\frac{t}{b_n}\right)^{\nu}\left(1-\frac{t}{b_n}\right)^{n-\nu}.$$

The aim of this chapter is to study some approximation properties of the operators (4.3). First, we find lemmas and auxiliary results to prove our main results. Then, we verified the Bohman-Korovkin theorem and find direct results in terms of modulus of continuity, Lipschitz-type space, and Ditzian-Totik. We also studied weighted approximation and statistical convergence.

## 4.2 Preliminaries

Here, we provide Moments and Central Moments for the given operators, which help us to show our main results.

**Lemma 4.2.1.** *[82] The moments of the operators $G_n((e_r(t) = t^r), x)$, $r = \overline{0,2}$, we have*

$$\begin{aligned} G_n(e_0,x) &= 1, \\ G_n(e_1,x) &= x, \\ G_n(e_2,x) &\le x^2(1+2n\beta) + x\left(\frac{b_n}{n} + 2n\beta^2 b_n + n^2\beta^3 b_n + 2\beta(1+n\beta)b_n + n\beta^2(1+n\beta)b_n\right). \end{aligned}$$

**Lemma 4.2.2.** *The moments of the operators $\tilde{D}_n(t^r, x)$, $r = \overline{0,2}$, we have*

$$\begin{aligned} \tilde{D}_n(e_0,x) &= 1, \\ \tilde{D}_n(e_1,x) &= \frac{nx+b_n}{n+2}, \\ \tilde{D}_n(e_2,x) &\le \frac{n^2x^2(1+2n\beta) + nx(4+2n\beta+5n^2\beta^2+2n^3\beta^3)b_2 + 2b_n^2}{(2+n)(3+n)}. \end{aligned}$$

*Proof.* After simple computations, we find a relation of the given operators (4.3)

$$\tilde{D}_n(t^r, x) = \frac{(n+1)!}{(n+r+1)!} b_n^r \sum_{\nu=0}^{n} \wp_{n,\nu,\beta}\left(\frac{x}{b_n}\right)\frac{(\nu+r)!}{\nu!}. \tag{4.4}$$

Substitute $r = 0$ in the equation (4.4), we obtain the result $\tilde{D}_n(e_0, x) = 1$.
Again, take $r = 1$ in the equation (4.4), we get

$$\tilde{D}_n(e_1,x) = \frac{(n+1)!}{(n+2)!} b_n \sum_{\nu=0}^{n} \wp_n\left(\frac{x}{b_n}\right)\frac{(\nu+1)!}{\nu!}$$

$$
\begin{aligned}
&= \frac{b_n}{(n+2)}\left\{\sum_{\nu=0}^{n}\left(\frac{\nu}{n}b_n\right)\frac{n}{b_n}\wp_n\left(\frac{x}{b_n}\right)+\sum_{\nu=0}^{n}\wp_n\left(\frac{x}{b_n}\right)\right\}\\
&= \frac{n}{(n+2)}\sum_{\nu=0}^{n}\left(\frac{\nu}{n}b_n\right)\wp_n\left(\frac{x}{b_n}\right)+\frac{b_n}{(n+2)}\sum_{\nu=0}^{n}\wp_n\left(\frac{x}{b_n}\right)\\
&= \frac{n}{(n+2)}G_n(t,x)+\frac{b_n}{(n+2)}G_n(1,x)\\
&= \frac{n}{(n+2)}x+\frac{b_n}{(n+2)},
\end{aligned}
$$

thus

$$\tilde{D}_n(e_1,x)=\frac{nx+b_n}{n+2}.$$

Now, take $r=2$ in the equation (4.4), we find the result

$$
\begin{aligned}
\tilde{D}_n(e_2,x) &= \frac{(n+1)!}{(n+3)!}b_n^2\sum_{\nu=0}^{n}\wp_n\left(\frac{x}{b_n}\right)\frac{(\nu+2)!}{\nu!}\\
&= \frac{b_n^2}{(n+2)(n+3)}\left\{\sum_{\nu=0}^{n}\nu^2\wp_n\left(\frac{x}{b_n}\right)+\sum_{\nu=0}^{n}3\nu\wp_n\left(\frac{x}{b_n}\right)+\sum_{\nu=0}^{n}2\wp_n\left(\frac{x}{b_n}\right)\right\}\\
&= \frac{b_n^2}{(n+2)(n+3)}\left\{\sum_{\nu=0}^{n}\left(\frac{\nu^2}{n^2}b_n^2\right)\left(\frac{n^2}{b_n^2}\right)\wp_n\left(\frac{x}{b_n}\right)+3\sum_{\nu=0}^{n}\left(\frac{\nu}{n}b_n\right)\left(\frac{n}{b_n}\right)\wp_n\left(\frac{x}{b_n}\right)\right.\\
&\quad\left.+2\sum_{\nu=0}^{n}\wp_n\left(\frac{x}{b_n}\right)\right\}\\
&= \frac{n^2}{(n+2)(n+3)}G_n(t^2,x)+\frac{3nb_n}{(n+2)(n+3)}G_n(t,x)+\frac{2b_n^2}{(n+2)(n+3)}G_n(1,x)\\
&\leq \frac{n^2x^2(1+2n\beta)+nx(4+2n\beta+5n^2\beta^2+2n^3\beta^3)b_2+2b_n^2}{(2+n)(3+n)}.
\end{aligned}
$$

Hence

$$\tilde{D}_n(e_2,x)\leq\frac{n^2x^2(1+2n\beta)+nx(4+2n\beta+5n^2\beta^2+2n^3\beta^3)b_2+2b_n^2}{(2+n)(3+n)}.$$

□

The Central moment of the operators $\tilde{D}_n((t-x)^2,x)$, we have

$$\tilde{D}_n((t-x)^2,x) \leq \frac{x^2(6-n+2n^3\beta)+x(-6+2n+2n^2\beta+5n^3\beta^2+2n^4\beta^3)n_n+2b_n^2}{(2+n)(3+n)}.$$

**Lemma 4.2.3.** *Let $f$ be a real-valued continuous bounded function on $[0,1]$, then*

$$|\tilde{D}_n(f)|\leq\|f\|,$$

*where sup-norm $\|.\|$ is defined as $\|f\|=\sup_{x\in[0,b_n]}|f(x)|$.*

## 4.3 Approximation Results

In the present section, we verified the Bohman-Korovkin theorem and estimated local approximation in terms of modulus of continuity, Lipschitz-type space, and Ditzian-totik modulus of smoothness.

**Theorem 4.3.1.** *(Fundamental convergence theorem) Let $\tilde{D}_n$ be the operators given by (4.4) and $h \in C[0,\infty) \cap E$, then the following relations holds*

$$\lim_{n\to\infty} \tilde{D}_n(f,x) = f(x)$$

*uniformly on each compact subset of* $[0,\infty)$*, where*

$$E = \left\{ f : x \in [0,\infty), \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}.$$

*Here $\beta$ as a sequence of positive real numbers such that $\beta = \beta_n$ and $\lim_{n\to\infty} n\beta_n = 0$.*

*Proof.* It is sufficient to prove that the operators $\tilde{D}_n$ verify the conditions, use of Korovkin's theorem

$$\lim_{n\to\infty} \tilde{D}_n(e_r,x) = x^r,\ r = \overline{0,2}$$

uniformly on each compact subset of $[0,\infty)$. From the Lemma 4.2.2,

$$\tilde{D}_n(e_0,x) = 1, \text{ for every } n \in \mathbb{N}$$

tends to 1,

$$\tilde{D}_n(e_1,x) = \frac{nx + b_n}{n+2}$$

tends to $x$, and

$$\tilde{D}_n(e_2,x) \leq \frac{n^2x^2(1+2n\beta) + nx(4+2n\beta+5n^2\beta^2+2n^3\beta^3)b_2 + 2b_n^2}{(2+n)(3+n)}$$

tends to $x^2$ as $n$ large. □

In the following theorem, we obtain the rate of convergence of the operators $\tilde{D}_n$ for functions in $Lip_M^*(\sigma)$.

**Theorem 4.3.2.** *(Lipschitz class) Let $f \in Lip_M^*(\sigma)$ and $0 < \sigma \leq 1$. Then for each $x \in (0,\infty)$, we have*

$$|\tilde{D}_n(f(t);x) - f(x)| \leq M \left( \frac{\tilde{D}_n((t-x)^2;x)}{x} \right)^{\frac{\sigma}{2}}.$$

*Proof.* Let $f \in Lip_M^*(\sigma)$ and $x \in (0,\infty)$, $t \in (0,\infty)$, we have

$$\begin{aligned} |\tilde{D}_n(f(t);x) - f(x)| &\leq \tilde{D}_n(|f(t)-f(x)|;x) \\ &\leq M.\tilde{D}_n\left(\frac{|t-x|^\sigma}{(t+x)^{\frac{\sigma}{2}}};x\right) \\ &\leq \frac{M}{x^{\frac{\sigma}{2}}}\tilde{D}_n(|t-x|^\sigma;x). \end{aligned} \tag{4.5}$$

Taking $p = \frac{2}{\sigma}$ and $q = \frac{2}{2-\sigma}$ and applying Hölder's inequality, we obtain

$$\begin{aligned} \tilde{D}_n(|t-x|^\sigma;x) &\leq \left\{\tilde{D}_n(|t-x|^2;x)\right\}^{\frac{\sigma}{2}}.\left\{\tilde{D}_n(1^{\frac{2}{2-\sigma}};x)\right\}^{\frac{2-\sigma}{2}} \\ &\leq \left\{\tilde{D}_n(|t-x|^2;x)\right\}^{\frac{\sigma}{2}}. \end{aligned} \tag{4.6}$$

Using (4.5) and (4.6), we get the required result

$$|\tilde{D}_n(f(t);x) - f(x)| \leq M\left(\frac{\tilde{D}_n((t-x)^2;x)}{x}\right)^{\frac{\sigma}{2}}.$$

□

**Theorem 4.3.3.** *Let $f \in C_B[0,\infty)$, we have*

$$|\tilde{D}_n(f,x) - f(x)| \leq C\omega_2(f,\sqrt{\delta}) + \omega\left(f,\frac{b_n}{n+2}\right),$$

*where C is a positive constant and $\delta = |\tilde{D}_n((t-x)^2,x)| + \left(\frac{b_n}{n+2}\right)$.*

*Proof.* We define operators $H_n(h,x)$ as follows

$$H_n(h,x) = \tilde{D}_n(h,x) - h\left(\frac{nx+b_n}{n+2}\right) + h(x).$$

The operators $H_n$ are linear and $H_n(t-x,x) = 0$ in view of Lemma 4.2.1. Let $g \in W^2$ from the Taylors expansion of $g$, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Applying the operator $H_n$ on above, we obtain

$$\begin{aligned} H_n(g,x) &= g(x) + g'(x)H_n((t-x),x) + H_n\left(\int_x^t (t-u)g''(u)du,x\right) \\ |H_n(g,x) - g(x)| &= |H_n\left(\int_x^t (t-u)g''(u)du,x\right)| \\ &\leq |\tilde{D}_n\left(\int_x^t (t-u)g''(u)du,x\right)| + \left|\int_x^{\frac{nx+b_n}{n+2}}\left(\frac{nx+b_n}{n+2} - u\right)g''(u)du\right| \end{aligned}$$

$$\begin{aligned}
&\leq \left\{\left|\tilde{D}_n\left(\int_x^t (t-u)du, x\right)\right| + \left|\int_x^{\frac{nx+b_n}{n+2}} \left(\frac{nx+b_n}{n+2} - u\right) du\right|\right\} \|g''\| \\
&\leq \left\{|\tilde{D}_n((t-x)^2, x)| + \left(\frac{b_n}{n+2}\right)^2\right\} \|g''\| \\
&= \delta \|g''\|,
\end{aligned}$$

where $\delta = |\tilde{D}_n((t-x)^2, x)| + \left(\frac{b_n}{n+2}\right)^2$.

Now,

$$\begin{aligned}
|\tilde{D}_n(f,x) - f(x)| &= |H_n(f-g,x) - (f-g)(x)| + |H_n(g,x) - g(x)| \\
&\quad + \left|f\left(\frac{nx+b_n}{n+2}\right) - h(x)\right| \\
&\leq 2\|f-g\| + \delta\|g''\| + \omega\left(f, \frac{b_n}{n+2}\right).
\end{aligned}$$

Taking infimum over all $g \in W^2$, we get

$$|\tilde{D}_n(f,x) - f(x)| \leq K_2(f,\delta) + \omega\left(f, \frac{b_n}{n+2}\right).$$

In view of (1.3), we obtain the result

$$|\tilde{D}_n(f,x) - f(x)| \leq C\omega_2(f, \sqrt{\delta}) + \omega\left(f, \frac{b_n}{n+2}\right).$$

□

## 4.4 Weighted approximation theorem

In this section, we obtained the weighted result of the given operators. Consider the class of function $C_\tau^k(\mathbb{R}^+) = \left\{f \in C_\tau(\mathbb{R}^+); \lim_{x\to\infty} \frac{f(x)}{\tau(x)} = k_f\right\}$ and the norm $\|f\|_\tau = \sup_{x\in\mathbb{R}^+} \frac{|f(x)|}{\tau(x)}$, where $\tau(x) = 1+x^2$ and $k_f$ is a constant depending on $f$.

**Theorem 4.4.1.** *(Weighted approximation) Suppose that $\{b_n\}$ and $\{\beta_n\}$ are sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \frac{b_n}{n} = 0$ and $\lim_{n\to\infty} n\beta_n = 0$, for each $h \in C_\tau^k(\mathbb{R}^+)$ and $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \|\tilde{D}_n(f) - f\|_\tau = 0.$$

*Proof.* Using Lemma 4.2.2, it is sufficient to verify the following conditions

$$\|\tilde{D}_n(t^r, x) - x^r\|_\tau = 0, \ r = \overline{0,2}. \tag{4.7}$$

Since $\lim_{n\to\infty} \|\tilde{D}_n(1,x) - 1||_\tau = 0$, for $r = 0$, (4.7) holds. And $\lim_{n\to\infty} \|\tilde{D}_n(t,x) - x||_\tau = 0$, for $r = 1$, also holds.

Again,

$$\|\tilde{D}_n(t,x)-x^2\|_\tau \leq \sup\left\{\left|\frac{n^2x^2(1+2n\beta)}{(1+x^2)(2+n)(3+n)}\right|+\left|\frac{nx(4+2n\beta+5n^2\beta^2+2n^3\beta^3)b_n}{(1+x^2)(2+n)(3+n)}\right| + \left|\frac{2b_n^2}{(1+x^2)(2+n)(3+n)}\right|\right\}$$

Hence

$$\lim_{n\to\infty}\|\tilde{D}_n(t^2,x)-x^2\|_\tau = 0.$$

the condition (4.7) holds for $r=2$. □

## 4.5 Statistical convergence

In this section, we estimate $A-$statistical convergence of the given operators $\tilde{D}_n$ to identity operators on the weighted spaces.

Firstly, we discuss the weighted Korovkin type approximation theorem for the $A-$statistical convergence given by Duman and Orhan [26] in 2004 and, Dilek and Mehmet [83] in 2017.

**Theorem 4.5.1.** *Let $A=(a_{n\nu})$ be a nonnegative regular summability matrix and $\overline{\rho}_1,\overline{\rho}_2$ weight functions such that*

$$\lim_{|x|\to\infty}\frac{\overline{\rho}_1(x)}{\overline{\rho}_2(x)} = 0. \tag{4.8}$$

*Assume that $(T_n)_{n\geq 1}$ is a sequence of positive linear operators from $C_{\overline{\rho}_1}(\mathbb{R})$ into $B_{\overline{\rho}_2}(\mathbb{R})$. Now $st_A-\lim_n\|T_nf-f\|_{\overline{\rho}_1}$, for all $f\in C_{\overline{\rho}_1}(\mathbb{R})$ iff*

$$st_A-\lim_n\|T_nf_\upsilon-f_\upsilon\|_{\overline{\rho}_1}=0,\ \upsilon=0,1,2,$$

*where*

$$f_\upsilon(x)=\frac{x^\upsilon\overline{\rho}_1(x)}{1+x^2},\ \ \upsilon=0,1,2.$$

**Corollary 4.5.2.** *[26]. Let $A=(a_{n\nu})$ be a nonnegative regular summability matrix and let $(L_n)$ be a sequence of positive linear operators acting from $C_{\rho_o}(R^+)$ into $B_{\rho_\lambda}(R^+)$, $\lambda>0$ one has*

$$st_A-\lim_n\|L_nf-f\|_{\rho_\lambda}=0,\ \ f\in C_{\rho_o}(R^+),$$

*iff*

$$st_A-\lim_n\|L_ne_i-e_i\|_{\rho_o}=0,\ \ i=0,1,2, \tag{4.9}$$

*where* $\rho_o(x) = 1+x^2$ *and* $\rho_\lambda(x) = 1+x^{2+\lambda}$, $\lambda > 0$.

Using the above results we will estimate the korovkin type statical theorem for given operators.

**Theorem 4.5.3.** *Let* $(b_n)$ *and* $(\beta_n)$ *be a sequences of positive numbers such that* $st_A - \lim\limits_{n\to\infty} \frac{b_n}{n} = 0$, $st_A - \lim\limits_{n\to\infty} n\beta_n = 0$ *and* $A = (a_{nk})$ *be a nonnegative regular summability matrix. Then for each* $h \in C_\rho^k(\mathbb{R}^+)$, *we have*

$$st_A - \lim_{n\to\infty} \|\tilde{D}_n f - f\|_{\rho_\lambda} = 0,$$

*where* $\rho_o(x) = 1+x^2$ *and* $\rho_\lambda(x) = 1+x^{2+\lambda}$, $\lambda > 0$.

*Proof.* From the above results, it is sufficient to prove that the operators $\tilde{D}_n$ verify (4.7), we get

$$st_A - \lim_{n\to\infty} \|\tilde{D}_n(e_0;.) - e_0\|_{\rho_o} = 0$$

and

$$st_A - \lim_{n\to\infty} \|\tilde{D}_n(e_1;.) - e_1\|_{\rho_o} = 0.$$

Now,

$$\begin{aligned}
\|\tilde{D}_n(e_2) - x^2\|_{\rho_o} &\leq \sup\left\{ \left| \frac{x^2}{1+x^2} \frac{(2\beta n^3 - 5n - 6)}{(2+n)(3+n)} \right| + \left| \frac{x}{1+x^2} \frac{n(4+2n\beta+5n^2\beta^2+2n^3\beta^3)b_2}{(2+n)(3+n)} \right| \right. \\
&\quad \left. + \left| \frac{1}{1+x^2} \frac{2b_n^2}{(2+n)(3+n)} \right| \right\} \\
&= \frac{2\beta n^3 - 5n - 6}{(2+n)(3+n)} + \frac{n(4+2n\beta+5n^2\beta^2+2n^3\beta^3)b_2}{(2+n)(3+n)} + \frac{2b_n^2}{(2+n)(3+n)} \\
&= J_n.
\end{aligned}$$

For a given $\varepsilon > 0$, we define the following sets

$$\begin{aligned}
N &= \{\nu : \|\tilde{D}_n(e_2;.) - e_2\|_{\rho_o} \geq \varepsilon\}, \\
N_1 &= \left\{\nu : \frac{2\beta n^3 - 5n - 6}{(2+n)(3+n)} \geq \frac{\varepsilon}{3}\right\}, \\
N_2 &= \left\{\nu : \frac{n(4+2n\beta+5n^2\beta^2+2n^3\beta^3)b_2}{(2+n)(3+n)} \geq \frac{\varepsilon}{3}\right\}, \\
N_3 &= \left\{\nu : \frac{2b_n^2}{(2+n)(3+n)} \geq \frac{\varepsilon}{3}\right\}.
\end{aligned}$$

Then, we see that $N \subseteq N_1 \cup N_2 \cup N_3$. Therefore, we get

$$\sum_{n:\|\tilde{D}_n(e_2;.)-e_2\|_{\rho_o} \geq 0} a_{n\nu} \leq \sum_{n\in N_1} a_{n\nu} + \sum_{n\in N_2} a_{n\nu} + \sum_{n\in N_3} a_{n\nu},$$

taking the limit $\nu \to \infty$ in above, we get the result

$$st_A - \lim_n \|\tilde{D}_n(e_2;.) - e_2\|_{\rho_o} = 0.$$

□

## Chapter 5

# Approximation by $\alpha-$Bernstein operators based on certain parameters and the Durrmeyer variant of modified Bernstein polynomials

---

The family of Bernstein polynomials has been extended over a distinct set of operators by Mache and Zhau [66] in the first section of this chapter. We explore a certain approximation depicts for these operators, including Lipschitz space, the rate of convergence via the second-order modulus of continuity, the Voronovskaya and Grüss-Voronovskaya theorems, the Ditzian-Totik moduli of smoothness, and weighted approximation properties. At last, we have used Matlab software to graphically illustrate the convergence of our operators. This chapter continues with a discussion of the Durrmeyer form of modified Bernstein polynomials. We begin with Bohman-Korovkin theorem and give further results. Next, we examine several approximation properties, such as rate of convergence, weighted approximation theorem, Voronovaskaja type, and Ditzian-Totik modulus of continuity via means of these operators. Lastly, a graphic representation of the convergence behavior has been given.

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## 5.1 Approximation results for $\alpha$-Bernstein operators

### 5.1.1 Introduction

In the year 2017, Chen et al. [15] proposed the generalization of Bernstein operators with shape parameter $\beta \in [0,1]$. These operators are defined as follows:

$$B_{n,\beta}(f;x) = \sum_{k=0}^{n} f_k \tilde{b}_{n,k}(\beta;x), \;\; x \in [0,1], \tag{5.1}$$

where $f_k = f\left(\frac{k}{n}\right)$. For $n > 2$, the polynomial $\tilde{b}_{n,k}(\beta;x)$ of degree $n$ is defined by $\tilde{b}_{1,0}(\beta;x) = 1-x$, $\tilde{b}_{1,1}(\beta;x) = x$, and

$$\tilde{b}_{n,k}(\beta;x) = \left[\binom{n-2}{k}(1-\beta)x + \binom{n-2}{k-2}(1-\beta)(1-x) + \binom{n}{k}\beta x(1-x)\right] x^{k-1}(1-x)^{n-k-1}$$

and studied another interesting proof of the Weierstrass theorem. They also studied its fundamental properties, the rate of convergence, and the Voronovskaya type asymptotic estimate formula.
Mache and Zhou [66] also introduced the generalized form of Bernstein operators for some parameters and they are defined as:

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) \frac{\int_0^1 f(t) t^{dk+a}(1-t)^{d(n-k)+b} dt}{\int_0^1 t^{dk+a}(1-t)^{d(n-k)+b} dt}, \;\; a,b > -1$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $d$ be a special sequence in $n$ and $\lambda$ with $n \in \mathbb{N}$, $0 \le \lambda < \infty$ defined as:

$$d := d_n := \left[n^{\lambda}\right],$$

the integral parts. For these operators, they gave some characterization theorem to discuss the importance of their operators concerning the previous several existing operators. They also established some local and global approximation results using second-order modulus of continuity, Lipschitz space, Ditzian-Totik modulus of smoothness, and many more.

Motivated from the above-stated work, we introduce the generalized form of Bernstein operators for $\alpha_n \in [0,1]$, $\rho > 0$ defined as follows:

$$D_n(f;x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\beta;x) \frac{\int_0^1 f\left(\alpha_n t + (1-\alpha_n)\frac{k}{n}\right) t^{\rho k+a}(1-t)^{\rho(n-k)+b} dt}{\int_0^1 t^{\rho k+a}(1-t)^{\rho(n-k)+b} dt}. \tag{5.2}$$

Special Cases:

1. For $\alpha_n = 0$, the operators (5.2) reduce to (5.1) and again take $\beta = 1$, we get well-known Bernstein operators [14].

2. For $\alpha_n = 1$, $\rho = 1$, $a = b = 0$, the operators (5.2) reduce to family of Bernstein Durrmeyer operators proposed by Kajla and Tucer [57] and $\beta = 1$, we get Bernstein-Durrmeyer operators [27].

3. For $\alpha_n = 1$, $a = b = 0$, we get generalized Bernstein Durrmeyer operators considered by Kajla and Goyal [59] and again take $\beta = 1$, we get Păltănea operators [76].

The main aim of this section is to establish the approximation results for our proposed operators using second-order modulus of continuity, Lipschitz space, Ditzian-Totik modulus of smoothness, Voronovskaya type asymptotic and weighted approximation results.

### 5.1.2 Preliminaries

In this section, we discuss some useful Lemmas which will be used in our main results.

**Lemma 5.1.1.** *The moments of the operators $D_n(e_m;x)$, $m = 0, 1$ and 2, we have*

$$\begin{aligned} D_n(e_0;x) &= 1; \\ D_n(e_1;x) &= A_1 x + A_2; \\ D_n(e_2;x) &= B_1 \frac{(-n+n^2-2(1-\alpha))}{n^2} x^2 + \left(B_2 + \frac{B_1(2+n-2\alpha)}{n^2}\right) x + B_3; \end{aligned}$$

where,

$$\begin{aligned} A_1 &= \frac{(2+a+b+\rho n-(2+a+b)\alpha_n)}{(2+a+b+\rho n)}; \\ A_2 &= \frac{(1+a)\alpha_n}{(2+a+b+\rho n)}; \\ B_1 &= \left[1 - \frac{2(2+a+b)\alpha_n}{(2+a+b+\rho n)} + \frac{(6+a^2+5b+b^2+a(5+2b)-\rho n)\alpha_n^2}{(2+a+b+\rho n)(3+a+b+\rho n)}\right]; \\ B_2 &= \left[\frac{(6+2\rho n-2a^2(-1+\alpha_n)-2b(-1+\alpha_n)-6\alpha_n+\rho n\alpha_n+2a(4+b+\rho n-4\alpha_n-b\alpha_n))\alpha_n}{(2+a+b+\rho n)(3+a+b+\rho n)}\right]; \\ B_3 &= \left[\frac{(2+3a+a^2)\alpha_n^2}{(2+a+b+\rho n)(3+a+b+\rho n)}\right]. \end{aligned}$$

*Proof.* We get these results using (5.2) and direct computation with the help of Mathematica software. □

**Lemma 5.1.2.** *The Central moments of the operator $D_n((t-x)^m;x)$, m=1 and 2, we have*

$$D_n((t-x);x) = x(A_1-1)+A_2;$$

$$D_n((t-x)^2;x) = \left(1-2A_1+\frac{B_1(-n+n^2+2(1-\alpha))}{n^2}\right)x^2+\left(-2A_2+B_2+\frac{B_1(2+n-2\alpha)}{n^2}\right)x$$
$$+B_3.$$

*Proof.* The proof follows from (5.2) and Lemma 5.1.1. □

**Lemma 5.1.3.** *The operators $D_n$ verify*

$$\lim_{n\to\infty} nD_n((t-x);x) = \frac{\alpha_n(1+a(1-x)-(2+b)x)}{\rho};$$
$$\lim_{n\to\infty} nD_n((t-x)^2;x) = \frac{(\rho+\alpha_n^2)}{\rho}x(1-x)$$
$$= \frac{(\rho+\alpha_n^2)}{\rho}\phi^2(x);$$
$$\lim_{n\to\infty} n^2D_n((t-x)^4;x) = 3\left(\frac{\rho+\alpha_n^2}{\rho}\right)^2 x^2(1-x)^2$$
$$= 3\left(\frac{\rho+\alpha_n^2}{\rho}\right)^2 \phi^4(x),$$

*where $\phi^2(x)=x(1-x)$.*

**Lemma 5.1.4.** *Let $f\in C[0,1]$, $x\in[0,1]$ and $n\in\mathbb{N}$, then*

$$|D_n(f;x)|\le \|f\|,$$

*where $\|.\|$ is the uniform norm on $[0,1]$.*

*Proof.* We have $D_n(e_0;x)=1$, so

$$|D_n(f;x)|\le D_n(e_0;x)\|f\| = \|f\|.$$

□

### 5.1.3 Direct Results

**Theorem 5.1.5.** *(Fundamental convergence theorem) Let $f\in C[0,1]$ and $n\to\infty$, then the sequence $\{D_n(f;x)\}$ converges uniformly to $f(x)$ in each compact subset of $[0,1]$.*

*Proof.* In view of Lemma 5.1.1, $D_n(1,x)=1$ $D_n(e_1;x)=A_1x+A_2$ tends to $x$ and $D_n(e_2;x)$ tends to $x^2$ as $n\to\infty$, uniformly on every compact subset of $[0,1]$. Hence, by Bohman-Korovkin's theorem [62] the required result holds. □

The following theorem provides the rate of convergence in terms of modulus of continuity.

**Theorem 5.1.6.** *Let $f \in C[0,1]$, we have*

$$|D_n(f;x) - f(x)| \leq C\omega_2(f,\delta) + \omega(f, A_2),$$

*where C is a positive constant,* $\delta = \left[D_n\left((t-x)^2, x\right) + (A_2)^2\right]$.

*Proof.* Define the auxiliary operators $L_n : C[0,1] \to C[0,1]$ as follows:

$$L_n(f;x) = D_n(f;x) - f(A_1x + A_2) + f(x). \tag{5.3}$$

These operators are linear and $L_n(t-x,x) = 0$.
Let $g \in C^2[0,1]$ and $x,t \in [0,1]$. By Taylor's expansion

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Applying the operators $L_n$ on above, we have

$$L_n(g;x) = g(x) + g'(x)L_n((t-x);x) + L_n\left(\int_x^t (t-u)g''(u)du; x\right)$$

implies that

$$\begin{aligned} |L_n(g,x) - g(x)| &\leq L_n\left(\left|\int_x^t (t-u)g''(u)du\right|, x\right) \\ &\leq D_n\left(\left|\int_x^t (t-u)g''(u)du\right|, x\right) \\ &\quad + \left|\int_x^{A_1x+A_2} (A_1x + A_2 - u)\, g''(u)du\right| \\ &\leq \frac{1}{2}D_n\left((t-x)^2, x\right)||g''|| \\ &\quad + \left|\int_x^{A_1x+A_2} (A_1x + A_2 - u)\, du\right| ||g''|| \\ &\leq \frac{1}{2}\left[D_n\left((t-x)^2, x\right) + (A_2)^2\right]||g''|| \\ &\leq \left[D_n\left((t-x)^2, x\right) + (A_2)^2\right]||g''|| \\ &= \delta_n||g''||, \end{aligned} \tag{5.4}$$

where $\delta_n = \left[D_n\left((t-x)^2, x\right) + (A_2)^2\right]$.

$$\begin{aligned} |D_n(f;x) - f(x)| &= |L_n(f-g,x) - (f-g)(x)| + |L_n(g,x) - g(x)| \\ &\quad + |f(A_1x + A_2) - f(x)| \\ &\leq 4||f-g|| + \delta_n||g''|| + \omega(f, A_2). \end{aligned}$$

Taking infimum over all $g \in C^2[0,1]$, we get the desired result.

$$|D_n(f;x) - f(x)| \quad \leq \quad K_2(f,\delta) + \omega(f,A_2).$$

From the equation (1.3), we obtain the required result. □

**Theorem 5.1.7.** *(Lipschitz class) Let $f \in Lip_M^{(\beta,\gamma)}(\sigma)$ and $0 < \sigma \leq 1$ then for all $x \in [0,1]$, the following inequality holds:*

$$|D_n(f;x) - f(x)| \leq M.\left(\frac{\eta_{n,\beta}(x)}{\beta x^2 + \gamma x}\right)^{\frac{\sigma}{2}},$$

*where $\eta_{n,\beta}(x) = D_n((t-x)^2;x)$.*

*Proof.* The result is true for $\sigma = 1$. Then for $f \in Lip_M^{(\beta,\gamma)}(\sigma)$ and $x \in (0,1)$, we have

$$\begin{aligned} |D_n(f;x) - f(x)| &\leq D_n(|f(t) - f(x)|;x) \\ &\leq MD_n\left(\frac{|t-x|}{(t+\beta x^2+\gamma x)^{\frac{1}{2}}};x\right). \end{aligned}$$

Applying Cauchy-Schwarz inequality for sum and $\frac{1}{(t+\beta x^2+\gamma x)^{\frac{1}{2}}} \leq \frac{1}{(\beta x^2+\gamma x)^{\frac{1}{2}}}$, we obtain

$$\begin{aligned} |D_n(f;x) - f(x)| &\leq \frac{M}{(\beta x^2+\gamma x)^{\frac{1}{2}}}\left[D_n((t-x)^2;x)\right]^{\frac{1}{2}} \\ &\leq qM\left\{\frac{\eta_{n,\beta}(x)}{\beta x^2+\gamma x}\right\}^{\frac{1}{2}}. \end{aligned}$$

Therefore result holds for $\sigma = 1$.

Further we prove the result for $0 < \sigma < 1$.

For $f \in Lip_M^{\beta,\gamma}(\sigma)$, using the same argument as in the previous case and applying Holder's inequality with $p = \frac{2}{\sigma}, q = \frac{2}{(2-\sigma)}$, we finally get

$$\begin{aligned} |D_n(f;x) - f(x)| &\leq \frac{M}{(\beta x^2+\gamma x)^{\frac{\sigma}{2}}}\left[D_n((t-x)^2;x)\right]^{\frac{\sigma}{2}} \\ &\leq M\left\{\frac{\eta_{n,\beta}(x)}{\beta x^2+\gamma x}\right\}^{\frac{\sigma}{2}}. \end{aligned}$$

Hence the theorem proved. □

### 5.1.4 Rate of approximation

In this section, we discuss Ditzian-totik modulus of smoothness for the operators $D_n$.

**Theorem 5.1.8.** *(Ditzian-totik modulus of smoothness) For any $f \in C^2[0,1]$ and sufficiently large n the* *following inequality holds*

$$|D_n(f,x) - f(x) - \psi_n(x)| \leq \frac{M\xi}{n\rho}\phi^2(x)\omega_1^{\phi}\left(f'', \sqrt{\frac{\rho+\alpha_n^2}{n\rho}}\right), \tag{5.5}$$

*where $\gamma$ is a positive constant and*

$$\begin{aligned}\psi_n(x) &= (x(A_1-1)+A_2)f'(x) + \left\{\left(1-2A_1+\frac{B_1(-n+n^2+2(1-\alpha))}{n^2}\right)x^2\right.\\ &\quad \left. + \left(-2A_2+B_2+\frac{B_1(2+n-2\alpha)}{n^2}\right)x + B_3\right\}\frac{f''(x)}{2!}.\end{aligned}$$

*Proof.* For $f \in C^2[0,1]$, $x,t \in [0,1]$, by Taylor's expansion

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-y)f''(y)dy$$

or

$$\begin{aligned} f(t) - f(x) - (t-x)f'(x) - (t-x)^2\frac{f''(x)}{2!} &= \int_x^t (t-y)f''(y)dy - \int_x^t (t-y)f''(x)dy \\ &= \int_x^t (t-y)[f''(y)-f''(x)]dy.\end{aligned}$$

Applying operators $D_n(.;x)$ on the above equality, we obtain

$$\begin{aligned} D_n(f;x) - f(x) - D_n((t-x);x)f'(x) - D_n((t-x)^2;x)\frac{f''(x)}{2!} & \\ = D_n\left(\int_x^t (t-y)[f''(y)-f''(x)]dy;x\right) & \\ |D_n(f;x) - f(x) - \psi_n(x)| \leq \left|D_n\left(\int_x^t (t-y)[f''(y)-f''(x)]dy;x\right)\right| & \\ \leq D_n\left(\left|\int_x^t |t-y||f''(y)-f''(x)|dy\right|;x\right). & \end{aligned} \tag{5.6}$$

From ([29], p.337), the inequality $\left|\int_x^t |t-y||f''(y)-f''(x)|dy\right|$

$$\left|\int_x^t |t-y||f''(y)-f''(x)|dy\right| \leq 2\|f''-g\|(t-x)^2 + 2\|\phi g'\|\phi^{-1}(x)(t-x)^3, \tag{5.7}$$

where $x \in (0,1)$, $g \in W_\phi[0,1]$, for $x \in \{0,1\}$ the inequality (5.5) verified.
There exists a constant $\xi > 0$ such that for $n$ sufficiently large and using Lemma 5.1.3, we obtain

$$D_n((t-x)^2;x) \leq \xi\frac{(\rho+\alpha_n^2)}{n\rho}\phi^2(x) \text{ and } D_n((t-x)^4;x) \leq \xi\left(\frac{c+\alpha_n^2}{n\rho}\right)^2\phi^4(x). \tag{5.8}$$

Applying the Cauchy-Schwarz inequality and from relations (5.6), (5.7) and (5.8), we get

$$\begin{aligned}
|D_n(f;x)-f(x)-\psi_n(x)| &\leq 2\|f''-g\|D_n((t-x)^2;x)+2\|\phi g'\|\phi^{-1}(x)D_n((t-x)^3;x)\\
&\leq \xi\frac{(\rho+\alpha_n^2)}{n\rho}\phi^2(x)\|f''-g\|+2\|\phi g'\|\phi^{-1}(x)\times\\
&\quad \{D_n((t-x)^2;x)\}^{\frac{1}{2}}.\{D_n((t-x)^4;x)\}^{\frac{1}{2}}\\
&\leq \xi\frac{(\rho+\alpha_n^2)}{n\rho}\phi^2(x)\|f''-g\|+\phi^2(x)\gamma\frac{(\rho+\alpha_n^2)}{n\rho}\sqrt{\frac{\rho+\alpha_n^2}{n\rho}}\|\phi g'\|\\
&\leq \xi\frac{(\rho+\alpha_n^2)}{n\rho}\phi^2(x)\left\{\|f''-g\|+\sqrt{\frac{\rho+\alpha_n^2}{n\rho}}\|\phi g'\|\right\}.
\end{aligned}$$

Taking infimum over $g\in W_\phi[0,1]$, we get

$$|D_n(f,x)-f(x)-\psi_n(x)| \leq \frac{\xi}{n\rho}.\phi^2(x)K_\phi\left(f'',\sqrt{\frac{\rho+\alpha_n^2}{n\rho}}\right).$$

Using relation (1.4), we obtain the required result. □

**Corollary 5.1.9.** *Let $f\in C^2[0,1]$, then*

$$\lim_{n\to\infty} n\{D_n(f,x)-f(x)-\psi_n(x)\}=0,$$

*where*

$$\begin{aligned}
\psi_n(x) &= (x(A_1-1)+A_2)f'(x)+\left\{\left(1-2A_1+\frac{B_1(-n+n^2+2(1-\alpha))}{n^2}\right)x^2\right.\\
&\quad\left.+\left(-2A_2+B_2+\frac{B_1(2+n-2\alpha)}{n^2}\right)x+B_3\right\}\frac{f''(x)}{2!}.
\end{aligned}$$

### 5.1.5 Grüss-Voronovskaya type theorem

Motivated by the Grüss type inequalities for certain positive linear operators studied in [8, 36], in the following we prove a Grüss-Voronovskaya type theorem of $D_n$ operators.
Define an auxiliary operators:

$$\Omega(fg;x)=D_n(fg;x)-D_n(f;x)D_n(g;x).$$

**Theorem 5.1.10.** *(Grüss-Voronovskaya type theorem) Let $f,g\in C^2[0,1]$ and $\rho>0$. Then for each $x\in[0,1]$,*

$$\lim_{n\to\infty} n\,\Omega(f,g;x)=\frac{(\rho+\alpha_n^2)}{\rho}x(1-x)f'(x)g'(x).$$

*Proof.* Since

$$(fg)(x) = f(x)g(x),\ (fg)'(x) = f'(x)g(x) + f(x)g'(x)$$
$$\text{and } (fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x).$$

Now, we can write

$$\begin{aligned}
\Omega(f,g;x) &= D_n(fg;x) - D_n(f;x)D_n(g;x) \\
&= \left\{D_n(fg;x) - f(x)g(x) - (fg)'(x)D_n((t-x);x) - \frac{(fg)''(x)}{2!}D_n((t-x)^2;x)\right\} \\
&\quad -g(x)\left\{D_n(f;x) - f(x) - f'(x)D_n((t-x);x) - \frac{f''(x)}{2!}D_n((t-x)^2;x)\right\} \\
&\quad -D_n(f;x)\left\{D_n(g;x) - g(x) - g'(x)D_n((t-x);x) - \frac{g''(x)}{2!}D_n((t-x)^2;x)\right\} \\
&\quad +\frac{1}{2!}D_n((t-x)^2;x)\{f(x)g''(x) + 2f'(x)g'(x) - g''(x)D_n(f;x)\} \\
&\quad +D_n((t-x);x)\{f(x)g'(x) - g'(x)D_n(f;x)\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\lim_{n\to\infty} n.\Omega(f,g;x) &= \lim_{n\to\infty} n\{D_n(fg;x) - D_n(f;x)D_n(g;x)\} \\
&= \lim_{n\to\infty} n.g'(x)\{f(x) - D_n(f;x)\}D_n((t-x);x) + \lim_{n\to\infty} n.f'(x)g'(x)D_n((t-x)^2;x) \\
&\quad + \lim_{n\to\infty} n.\frac{g''(x)}{2!}\{f(x) - D_n(f;x)\}D_n((t-x)^2;x).
\end{aligned}$$

From Theorem 5.1.5, it follows that for each $x \in [0,1]$, $D_n(f;x)$ converges to the function $f$, as $n \to \infty$ and in view of Lemma 5.1.3, $\lim_{n\to\infty} nD_n((t-x)^2;x)$ is finite. Hence

$$\lim_{n\to\infty} n\Omega(f,g;x) = \frac{(\rho+\alpha_n^2)}{\rho}x(1-x)f'(x)g'(x).$$

□

### 5.1.6 Weighted Approximation

In this section, we discuss a weighted approximation results for the operators $D_n$.

**Theorem 5.1.11.** *(Weighted approximation) For each $f \in C_\tau^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \|D_n(f;.) - f\|_\tau = 0$$

*Proof.* Using [32], we see that it is sufficient to verify the following conditions

$$\lim_{n\to\infty} \|D_n(t^r;x)-x^r\|_\tau = 0,\ \ r=0,1,2. \tag{5.9}$$

Since $D_n(1;x)=1$,therefore for $r=0$ (5.9) holds.

By Lemma (5.1.1), we have

$$\begin{aligned}\|D_n(t;x)-x\|_\tau &= \sup_{x\in[0,\infty)} \frac{|D_n(t;x)-x|}{1+x^2}\\ &\leq \frac{(1+b)\alpha_n}{(2+a+b+\rho n)},\end{aligned}$$

the condition (5.9) holds for $r=1$ as $n\to\infty$.

Again by Lemma (5.1.1), we have

$$\begin{aligned}\|D_n(t^2;x)-x^2\|_\tau &= \sup_{x\in[0,\infty)} \frac{|D_n(t^2;x)-x^2|}{1+x^2}\\ &\leq \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\Bigg[\Bigg(B_1\frac{(n^2-n-2(1-\alpha))}{n^2}x^2\\ &\quad+\Bigg(B_2+\frac{B_1(2+n-2\alpha)}{n^2}\Bigg)x+B_3\Bigg)-x^2\Bigg]\\ &\leq \left|B_1\frac{(n^2-n-2(1-\alpha))}{n^2}-1\right|+\left|B_2+\frac{B_1(2+n-2\alpha)}{n^2}\right|+|B_3|,\end{aligned}$$

the condition (5.9) holds for $r=2$ as $n\to\infty$.

Hence the theorem proved. □

**Corollary 5.1.12.** *For each $f\in C_\tau^*[0,\infty)$, and $\beta>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)} \frac{|D_n(f;x)-f(x)|}{(1+x^2)^\beta}=0.$$

*Proof.* For any fixed $x_0>0$,

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|D_n(f;x)-f(x)|}{(1+x^2)^\beta} &\leq \sup_{x\leq x_0} \frac{|D_n(f;x)-f(x)|}{(1+x^2)^\beta}+\sup_{x\geq x_0} \frac{|D_n(f;x)-f(x)|}{(1+x^2)^\beta}\\ &\leq \|D_n(f;.)-f\|_{C[0,x_0]}+\|f\|_{x^2}\sup_{x\geq x_0}\frac{|D_n(1+t^2;x)|}{(1+x^2)^\beta}\\ &+ \sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^\beta}.\end{aligned}$$

The first term of the above inequality tends to zero from Theorem 5.1.5. By Lemma 5.1.1 for any fixed $x_0$ it is easily seen that $\sup_{x\geq x_0}\frac{|D_n(1+t^2;x)|}{(1+x^2)^\beta}$ tends to zero as $n\to\infty$. We can choose $x_0$ so large that the last part of above inequality can be made small enough. Thus the proof is completed. □

### 5.1.7 Numerical Results and Discussion

*Example* 1. Let $f(x) = x\cos\left(\frac{7\pi x}{3}\right)$, $\beta = 2/3$, $\rho = 1$, $\alpha_n = 0.5$, $a = 1$, $b = 2$ and $n \in \{10, 20, 30\}$. The convergence of our proposed operators $D_n(f;x)$ towards the function $f(x)$ and absolute error function $E_n(x) = |D_n(f;x) - f(x)|$ are shown in figure: 1(a), and figure: 1(b) respectively.

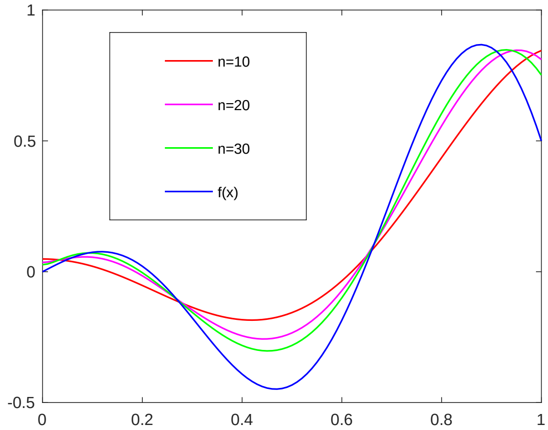


Figure 5.1: Figure:1(a) Uniform Convergence of the given operators

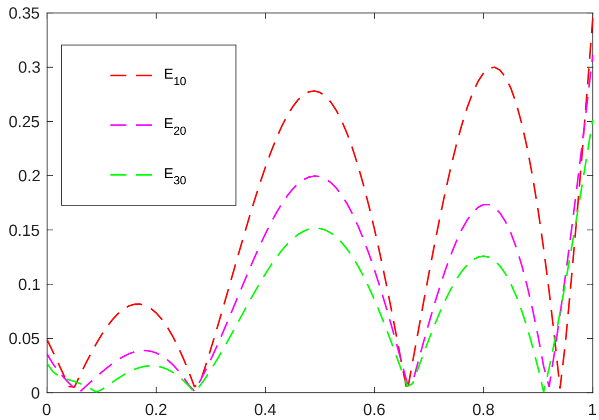


Figure 5.2: Figure:1(b) Absolute error of the proposed operators

*Example* 2. Let $f(x) = x^3 - \sin(1 - \frac{9\pi x}{4})$ and $n = 30$. The convergence of family of Bernstein-Durrmeyer operators [57] $P1(\beta = .65, \rho = 1, \alpha_n = 1, a = 0, b = 0)$, Paltanea type Bernstein operators [59] $P2(\beta = .65, \rho = 2, \alpha_n = 1, a = 0, b = 0)$, Lupas and Meche operators $P3(\beta = 1, \rho = 1, \alpha_n = 1, a = b = -1/2)$ and our proposed operators $P4(\beta = .65, \rho = 2, \alpha_n = 0.25, a = 15, b = 8)$ are shown in figure:2(a). And the absolute error functions $Di = |D_n(f;x) - f(x)|$ for $D_i$, $i = 1, 2, 3$ *and* 4 are shown in figure: 2(b).

The figures: $1(a)$, and figure: $1(b)$ deal with uniform convergence of our proposed operators $(5.2)$ and in figure: $2(a)$, and figure: $2(b)$, it can be easily seen the convergence of our operators are better with comparing of other existing operators. This justifies the study of our proposed operators.

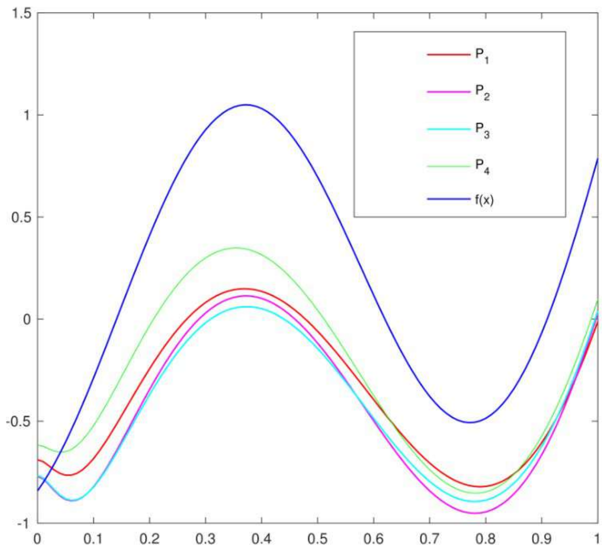


Figure 5.3: Figure: 2(a) Comparison of the existing operators

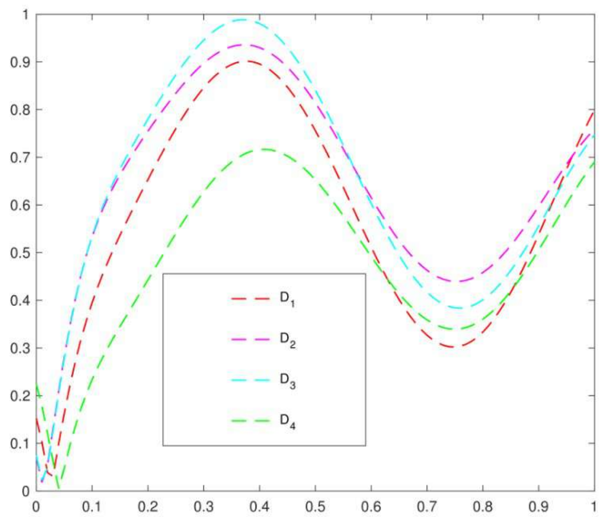


Figure 5.4: Figure: 2(b) Absolute error of the existing operators

## 5.2 Approximation by Durrmeyer variant of modified Bernstein polynomials

In this section, we study on the Durrmeyer variant of modified Bernstein polynomials. First, we provide the auxiliary results and then verify Bohman-Korovkin's theorem. Following that, we find some of the approximation properties for these operators, including the rate of convergence via the Ditzian-Totik modulus of continuity, Voronovaskaja type, and weighted approximation theory. Finally, a graphic depiction of the convergence behavior has been shown.

Continuation recent work in the field of approximation theory, Usta [97] constructed a new modification of Bernstein operators

$$B_n^*(f;x) = \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}f\left(\frac{k}{n}\right), \tag{5.10}$$

where $f \in C(0,1)$, $x \in (0,1)$ and $n \in \mathbb{N}$. Usta found some important results like asymptotic formulas, weighted approximation, rate of convergence and also conclude numerically and graphically of the proposed operators using following lemma:

**Lemma 5.2.1.** *For the operators $B_n^*(l_r;x)$ where $l_r = t^r$, $r = \overline{0,4}$, we have*

$$\begin{aligned}
B_n^*(l_0;x) &= 1;\\
B_n^*(l_1;x) &= \frac{(n-2)}{n}x + \frac{1}{n};\\
B_n^*(l_2;x) &= \frac{(n^2-7n+6)}{n^2}x^2 + \frac{(5n-6)}{n^2}x + \frac{1}{n^2};\\
B_n^*(l_3;x) &= \frac{(n^3-15n^2+38n-24)}{n^3}x^3 + 12\frac{(n^2-4n+3)}{n^3}x^2\\
&\quad + \frac{(13n-14)}{n^3}x + \frac{1}{n^3};\\
B_n^*(l_4;x) &= \frac{(n^4-26n^3+131n^2-226n+120)}{n^4}x^4\\
&\quad + \frac{(22n^3-186n^2+404n-240)}{n^4}x^3\\
&\quad + \frac{(61n^2-211n+150)}{n^4}x^2 + \frac{(29n-30)}{n^4}x + \frac{1}{n^4}.
\end{aligned}$$

To approximate Lebesque integrable functions on the interval $(0,1)$, we have constructed Durrmeyer variant of the operators (5.10), introduced by Usta, as follows:

$$\begin{aligned}
\mathscr{B}_n(f;x) &= \frac{n+1}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\\
&\quad \times \int_0^1 \binom{n}{k}(k-nt)^2t^{k-1}(1-t)^{n-k-1}f(t)dt,
\end{aligned} \tag{5.11}$$

where $f \in C(0,1)$ be a continuous real valued function on the interval $(0,1)$ and $0 < x < 1$, $n \in \mathbb{N}$.

### 5.2.1 Preliminaries

**Lemma 5.2.2.** *The moments of the given operators $\mathscr{B}_n((l_r = t^r);x)$, $r = \overline{0,4}$, we have*

$$\begin{aligned}
\mathscr{B}_n(l_0;x) &= 1;\\
\mathscr{B}_n(l_1;x) &= \frac{(n^2-4n+4)}{n(n+2)}x + \frac{(3n-2)}{n(n+2)};\\
\mathscr{B}_n(l_2;x) &= \frac{(n^3-13n^2+48n-36)}{n(n+2)(n+3)}x^2 + \frac{(12n^2-56n+48)}{n(n+2)(n+3)}x + \frac{14n-12}{n(n+2)(n+3)};
\end{aligned}$$

$$\begin{aligned}
\mathscr{B}_n(l_3;x) &= \frac{(n^4-27n^3+218n^2-480n+288)}{n(n+2)(n+3)(n+4)}x^3+\frac{(27n^3-333n^2+954n-648)}{n(n+2)(n+3)(n+4)}x^2\\
&\quad+\frac{(126n^2-540n+432)}{n(n+2)(n+3)(n+4)}x+\frac{(78n-72)}{n(n+2)(n+3)(n+4)};\\
\mathscr{B}_n(l_4;x) &= \frac{(n^3-6n^2+11n-6)(n-20)^2}{n(n+1)(n+2)(n+3)(n+4)(n+5)}x^4\\
&\quad+\frac{16(3n^4-71n^3+432n^2-844n+480)}{n(n+2)(n+3)(n+4)(n+5)}x^3+\frac{72(7n^3-73n^2+186n-120)}{n(n+2)(n+3)(n+4)(n+5)}x^2\\
&\quad+\frac{96(13n^2-52n+40)}{n(n+2)(n+3)(n+4)(n+5)}x+\frac{24(16n-15)}{n(n+2)(n+3)(n+4)(n+5)}.
\end{aligned}$$

*Proof.* Using the Lemma 5.2.1 and simple calculation in the equation (5.11), we obtain the result $\mathscr{B}_n(l_0;x)=1$.

For $r=1$, we calculate

$$\begin{aligned}
\mathscr{B}_n(l_1;x) &= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\int_0^1\binom{n}{k}(k-nt)^2t^k(1-t)^{n-k-1}dt\\
&= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\left\{\frac{(k(n-2)+2n)\binom{n}{k}\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+3)}\right\}\\
&= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\left\{\frac{k(n-2)}{(n+1)(n+2)}+\frac{2n}{(n+1)(n+2)}\right\}\\
&= \frac{(n-2)}{(n+2)}B_n^*(e_1;x)+\frac{2}{(n+2)}B_n^*(e_0;x)\\
&= \frac{(n^2-4n+4)}{n(n+2)}x+\frac{(3n-2)}{n(n+2)}.
\end{aligned}$$

For $r=2$, we have

$$\begin{aligned}
\mathscr{B}_n(l_2;x) &= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\int_0^1\binom{n}{k}(k-nt)^2t^{k+1}(1-t)^{n-k-1}dt\\
&= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\left\{\frac{(k(n-6)+6n)\binom{n}{k}\Gamma(k+2)\Gamma(n-k+1)}{\Gamma(n+4)}\right\}\\
&= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\left\{\frac{(k+1)(k(-6+n)+6n)}{(n+1)(n+2)(n+3)}\right\}\\
&= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\left\{\frac{k^2(n-6)+k(7n-6)+6n}{(n+1)(n+2)(n+3)}\right\}\\
&= \frac{n(n-6)}{(n+2)(n+3)}B_n^*(e_2;x)+\frac{(7n-6)}{(n+2)(n+3)}B_n^*(e_1;x)+\frac{6}{(n+2)(n+3)}B_n^*(e_0;x)\\
&= \frac{(n^3-13n^2+48n-36)}{n(n+2)(n+3)}x^2+\frac{(12n^2-56n+48)}{n(n+2)(n+3)}x+\frac{14n-12}{n(n+2)(n+3)}.
\end{aligned}$$

For $r=3$, we have

$$\begin{aligned}
\mathscr{B}_n(l_3;x) &= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\\
&\quad\times\int_0^1\binom{n}{k}(k-nt)^2t^{k+2}(1-t)^{n-k-1}dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\\
&\quad\times\left\{\frac{(k(-12+n)+12n)\binom{n}{k}\Gamma(k+3)\Gamma(n-k+1)}{\Gamma(n+5)}\right\}\\
&= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\\
&\quad\times\left\{\frac{k^3(n-12)+k^2(15n-36)+k(38n-24)+24n}{(n+1)(n+2)(n+3)(n+4)}\right\}\\
&= \frac{n^2(n-12)}{(n+2)(n+3)(n+4)}B_n^*(e_3;x)+\frac{n(15n-36)}{(n+2)(n+3)(n+4)}B_n^*(e_2;x)\\
&\quad+\frac{(38n-24)}{(n+2)(n+3)(n+4)}B_n^*(e_1;x)+\frac{24}{(n+2)(n+3)(n+4)}B_n^*(e_0;x)\\
&= \frac{(n^4-27n^3+218n^2-480n+288)}{n(n+2)(n+3)(n+4)}x^3+\frac{(27n^3-333n^2+954n-648)}{n(n+2)(n+3)(n+4)}x^2\\
&\quad+\frac{(126n^2-540n+432)}{n(n+2)(n+3)(n+4)}x+\frac{(78n-72)}{n(n+2)(n+3)(n+4)}.
\end{aligned}
$$

For $r=4$, we have

$$
\begin{aligned}
\mathscr{B}_n(l_4;x) &= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\\
&\quad\times\int_0^1\binom{n}{k}(k-nt)^2t^{k+3}(1-t)^{n-k-1}dt\\
&= \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\\
&\quad\times\left\{\frac{(k(n-20)+20n)\binom{n}{k}\Gamma(4+k)\Gamma(n+1-k)}{\Gamma(6+n)}\right\}\\
&= \frac{(n^3-6n^2+11n-6)(n-20)^2}{n(n+1)(n+2)(n+3)(n+4)(n+5)}x^4\\
&\quad+\frac{16(3n^4-71n^3+432n^2-844n+480)}{n(n+2)(n+3)(n+4)(n+5)}x^3+\frac{72(7n^3-73n^2+186n-120)}{n(n+2)(n+3)(n+4)(n+5)}x^2\\
&\quad+\frac{96(13n^2-52n+40)}{n(n+2)(n+3)(n+4)(n+5)}x+\frac{24(16n-15)}{n(n+2)(n+3)(n+4)(n+5)}.
\end{aligned}
$$

□

**Lemma 5.2.3.** *The Central moments of the operators* $\mathscr{B}_n(f(t)=(t-x)^r,x)$, $r=1,2,4$, *we have*

$$
\begin{aligned}
\mathscr{B}_n(t-x;x) &= \frac{(4-6n)}{n(n+2)}x+\frac{(3n-2)}{n(n+2)};\\
\mathscr{B}_n((t-x)^2;x) &= \frac{-(6n^2-70n+60)}{n(n+2)(n+3)}x^2+\frac{(6n^2-70n+60)}{n(n+2)(n+3)}x+\frac{(14n-12)}{n(n+2)(n+3)};\\
\mathscr{B}_n((t-x)^4;x) &= \frac{12(7n^3-333n^2+1436n-1120)}{n(n+2)(n+3)(n+4)(n+5)}x^4+\frac{-24(7n^3-333n^2+1436n-1120)}{n(n+2)(n+3)(n+4)(n+5)}x^3\\
&\quad+\frac{12(7n^3-411n^2+1958n-1560)}{n(n+2)(n+3)(n+4)(n+5)}x^2+\frac{24(39n^2-261n+220)}{n(n+2)(n+3)(n+4)(n+5)}x\\
&\quad+\frac{24(16n-15)}{n(n+2)(n+3)(n+4)(n+5)}.
\end{aligned}
$$

*Proof.* Using Lemma 5.2.2 and simple calculation, we get the Central moments of the given operators. □

**Lemma 5.2.4.** *We have also find some limiting results as follows:*

$$\begin{aligned}
\lim_{n\to\infty} n\mathscr{B}_n((t-x);x) &= -6x+3;\\
\lim_{n\to\infty} n\mathscr{B}_n((t-x)^2;x) &= 6x(1-x);\\
\lim_{n\to\infty} n^2\mathscr{B}_n((t-x)^4;x) &= 84x^2(x-1)^2.
\end{aligned}$$

**Lemma 5.2.5.** *For $n \in \mathbb{N}$, the bound for second central moment is given by*

$$\mathscr{B}_n((t-x)^2;x) \leq \frac{12x(1-x)}{n}.$$

### 5.2.2 Main Results

**Lemma 5.2.6.** *Let $0 < x < 1$ and $f \in C(0,1)$, we get result*

$$|\mathscr{B}_n(f;x)| \leq \|f\|.$$

*Proof.* Using the given operators (5.11), we have

$$\begin{aligned}
|\mathscr{B}_n(f;x)| &= \left|\frac{n+1}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\right.\\
&\quad \left.\times\int_0^1\binom{n}{k}(k-nt)^2t^{k-1}(1-t)^{n-k-1}f(t)dt,\right|\\
&\leq \frac{n+1}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1}\\
&\quad \times\int_0^1\binom{n}{k}(k-nt)^2t^{k-1}(1-t)^{n-k-1}|f(t)|dt\\
&\leq \|f\|\mathscr{B}_n(1;x)\\
&= \|f\|.
\end{aligned}$$

□

**Theorem 5.2.7.** *(Fundamental convergence theorem) Suppose that $f \in C(0,1)$, then*

$$\|\mathscr{B}_n(f;x) - f(x)\| \to 0,$$

*uniformly as $n \to \infty$.*

*Proof.* From the lemma 5.2.2 and apply Korovkin theorem [62], we get the results for the value $r = \overline{0,4}$

of the given operators

$$\lim_{n\to\infty} \mathscr{B}_n(l_r;x) = x^r.$$

□

The modulus of continuity of the function f is defined as:

$$\omega^*(f,\delta) = \sup_{|t-x|\le\delta}\ \sup_{x\in(0,1)} |f(t)-f(x)|,$$

$$|f(t)-f(x)| \le \left(1+\frac{(t-x)^2}{\delta^2}\right)\omega^*(f,\delta),\ \delta>0 \tag{5.12}$$

and modulus of continuity also satisfy the inequality[9]

$$\omega^*(f,t\delta) \le (1+t)\omega^*(f,\delta),$$

for every $f \in C(0,1)$ and $\delta > 0$.

**Theorem 5.2.8.** *For every* $0<x<1$ *and* $f\in C(0,1)$*, we have*

$$|\mathscr{B}_n(f;x)-f(x)| \quad \le \quad 2\omega^*(f,\delta_n),$$

*where*

$$\delta=\delta_n \quad = \quad \sqrt{\frac{-(6n^2-70n+60)x^2+(6n^2-70n+60)x+(14n-12)}{n(n+2)(n+3)}}. \tag{5.13}$$

*Proof.* Using the linearity property and equation (5.12), we get

$$\begin{aligned}
|\mathscr{B}_n(f;x)-f(x)| &= \Bigg|\frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1} \\
&\quad \times\int_0^1\binom{n}{k}(k-nt)^2t^{k-1}(1-t)^{n-k-1}f(t)dt-f(x)\Bigg| \\
&\le \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1} \\
&\quad \times\int_0^1\binom{n}{k}(k-nt)^2t^{k-1}(1-t)^{n-k-1}|f(t)-f(x)|dt \\
&\le \frac{(n+1)}{n^2}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2x^{k-1}(1-x)^{n-k-1} \\
&\quad \times\int_0^1\binom{n}{k}(k-nt)^2t^{k-1}(1-t)^{n-k-1}\left(1+\frac{(t-x)^2}{\delta^2}\right)\omega^*(f,\delta)dt \\
&= \Bigg(1+\frac{1}{\delta^2}\frac{1}{n(n+2)(n+3)}\{-(6n^2-70n+60)x^2 \\
&\quad +(6n^2-70n+60)x+(14n-12)\}\Bigg)\,\omega^*(f,\delta),
\end{aligned}$$

hence

$$|\mathscr{B}_n(f;x) - f(x)| \leq 2\omega^*(f, \delta_n),$$

where

$$\delta = \delta_n = \sqrt{\frac{-(6n^2-70n+60)x^2+(6n^2-70n+60)x+(14n-12)}{n(n+2)(n+3)}}.$$

□

**Theorem 5.2.9.** *(Lipschitz class) Let $f \in Lip_M^*(\sigma)$ and $0 < \sigma \leq 1$ then we obtain*

$$|\mathscr{B}_n(f;x) - f(x)| \leq M\delta_n^{\sigma}(x),$$

*where $\delta_n(x)$ is defined in equation (5.13).*

*Proof.* Suppose that $f \in Lip_M^*(\sigma)$, $0 < \sigma \leq 1$ and from the equation (6.10) by using the linearity, monotonicity of the given operators $\mathscr{B}_n$, we get

$$\begin{aligned} |\mathscr{B}_n(f;x) - f(x) \quad | &\leq \mathscr{B}_n(|f(t) - f(x)|;x) \\ &\leq M\mathscr{B}_n(|t-x|^{\sigma};x). \end{aligned}$$

Choose $p = \frac{2}{\sigma}, q = \frac{2}{2-\sigma}$ in the Hölder inequality, we get

$$\begin{aligned} |\mathscr{B}_n(f;x) - f(x) \quad | &\leq M\{\mathscr{B}_n((t-x)^2;x)\}^{\frac{\sigma}{2}} \\ &\leq M\sqrt[\sigma]{\frac{-(6n^2-70n+60)x^2+(6n^2-70n+60)x+(14n-12)}{n(n+2)(n+3)}} \\ &\leq M\delta_n^{\sigma}(x). \end{aligned}$$

□

**Theorem 5.2.10.** *(Peetre's K-functional) Let $0 < x < 1$ and $f \in C(0,1)$. Then for each $n \in \mathbb{N}$, there exists a positive constant M such that*

$$|\mathscr{B}_n(f;x) - f(x)| \leq M\omega_2(f, \psi_n^2(x)) + 2\omega(f, \xi_n(x)),$$

*where*

$$\begin{aligned} \psi_n^2(x) = & \frac{1}{2n^2(2+n)^2(3+n)}\left\{12(2x-1)^2 - 12n^4x(x-1) - 16n(23x^2-23x+5)\right. \\ & \left.+n^3(37-152x+152x^2)+n^4(47-220x+220x^2)\right\} \end{aligned}$$

*and*

$$\xi_n(x) = \left| \frac{(4-6n)x+(3n-2)}{n(n+2)} \right|.$$

*Proof.* Define the auxiliary operator $\mathcal{F}_n : C^*(0,1) \to C^*(0,1)$ by

$$\mathcal{F}_n(h;x) = \mathscr{B}_n(h;x) - h\left(\frac{(n-2)^2x+(3n-2)}{n(n+2)}\right) + h(x). \tag{5.14}$$

From Lemma 5.2.2, we have

$$\mathcal{F}_n(1;x) = 1,$$

now

$$\begin{aligned}
\mathcal{F}_n(t-x;x) &= \mathscr{B}_n((t-x);x) - \left(\frac{(n-2)^2x+(3n-2)}{n(n+2)} - x\right) + x - x \\
&= \frac{(4-6n)}{n(n+2)}x + \frac{(3n-2)}{n(n+2)} - \left(\frac{(n-2)^2x+(3n-2)}{n(n+2)} - x\right) + x - x \\
&= 0. \qquad (5.15)
\end{aligned}$$

We write the Taylor's expansion is in this form

$$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-v)h''(v)dv,\ t \in (0,1) \tag{5.16}$$

Applying $\mathcal{F}_n$ operator to both sides of the equation (5.16), we obtain

$$\begin{aligned}
\mathcal{F}_n(h;x) &= \mathcal{F}_n\left(h(x) + (t-x)h'(x) + \int_x^t (t-v)h''(v)dv;x\right) \\
&= h(x) + \mathcal{F}_n((t-x)h'(x);x) + \mathcal{F}_n\left(\int_x^t (t-v)h''(v)dv\right).
\end{aligned}$$

Or

$$\mathcal{F}_n(h;x) - h(x) = h'(x)\mathcal{F}_n((t-x);x) + \mathcal{F}_n\left(\int_x^t (t-v)h''(v)dv\right).$$

From the equation (5.14) and (5.16), we get

$$\begin{aligned}
\mathcal{F}_n(h;x) - h(x) &= \mathcal{F}_n\left(\int_x^t (t-v)h''(v)dv\right) \\
&= \mathscr{B}_n\left(\int_x^t (t-v)h''(v)dv\right) \\
&\quad - \int_x^{\frac{(n-2)^2x+(3n-2)}{n(n+2)}} \left(\frac{(n-2)^2x+(3n-2)}{n(n+2)} - v\right) h''(v)dv \\
&\quad + \int_x^x \left(\frac{(n-2)^2x+(3n-2)}{n(n+2)} - v\right) h''(v)dv. \qquad (5.17)
\end{aligned}$$

Furthermore

$$\left|\int_x^t (t-v)h''(v)dv\right| \leq \int_x^t |t-v||h''(v)|dv \leq \|h''\|\int_x^t |t-u|dv \leq (t-x)^2\|h''\|, \tag{5.18}$$

and

$$\left|\int_x^{\frac{(n-2)^2x+(3n-2)}{n(n+2)}} \left(\frac{(n-2)^2x+(3n-2)}{n(n+2)}-v\right)h''(v)dv\right|$$
$$\leq \|h''\|\int_x^{\frac{(n-2)^2x+(3n-2)}{n(n+2)}} \left(\frac{(n-2)^2x+(3n-2)}{n(n+2)}-v\right)h''(v)dv$$
$$= \frac{\|h''\|}{2}\left(\frac{(n-2)^2x+(3n-2)}{n(n+2)}-x\right)^2$$
$$= \frac{\|h''\|}{2}\left(\frac{(4-6n)x+(3n-2)}{n(n+2)}\right)^2. \tag{5.19}$$

Using the equation (5.18) and (5.19), we rewrite the absolute value of the equation (5.17), we obtain

$$\begin{aligned}|\digamma_n(h;x)-h(x)| &\leq \|h''\|\mathscr{B}_n((t-x)^2;x)+\frac{\|h''\|}{2}\left(\frac{(n-2)^2x+(3n-2)}{n(n+2)}-x\right)^2\\ &= \|h''\|\psi_n^2(x),\end{aligned}$$

where

$$\begin{aligned}\psi_n^2(x) &= \mathscr{B}_n((t-x)^2;x)+\frac{1}{2}\left(\frac{(n-2)^2x+(3n-2)}{n(n+2)}-x\right)^2\\ &= \frac{1}{2n^2(2+n)^2(3+n)}\left\{12(2x-1)^2-12n^4x(x-1)-16n(23x^2-23x+5)\right.\\ &\quad \left.+n^3(37-152x+152x^2)+n^4(47-220x+220x^2)\right\}.\end{aligned}$$

Again we find the bound of the defined operator $\digamma_n(h;x)$ and using Cauchy's Schwarz inequality, we obtain

$$\begin{aligned}|\digamma_n(h;x)| &= \left|\mathscr{B}_n(h;x)-h\left(\frac{(n-2)^2x+(3n-2)}{n(n+2)}\right)+h(x)\right|\\ &\leq |\mathscr{B}_n(h;x)|+\left|h\left(\frac{(n-2)^2x+(3n-2)}{n(n+2)}\right)\right|+|h(x)|\\ &\leq 3\|h\|.\end{aligned}$$

Finally,

$$\begin{aligned}|\mathscr{B}_n(f;x)-f(x)| &= \left|\digamma_n(f;x)-f(x)+f\left(\frac{(n-2)^2x+(3n-2)}{n(n+2)}\right)-f(x)\right.\\ &\quad \left.+h(x)-h(x)+\digamma_n(h;x)-\digamma_n(h;x)\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \ |F_n(f-h;x)-(f-h)(x)|+|F_n(h;x)-h(x)| \\
&\quad +\left|f\left(\frac{(n-2)^2x+(3n-2)}{n(n+2)}\right)-f(x)\right| \\
&\leq \ 4\|f-h\|+\|h''\|\psi_n^2(x)+\omega^*(f,\xi_n(x))\left(\frac{\left|\frac{(n-2)^2x+(3n-2)}{n(n+2)}-x\right|}{\xi_n(x)}+1\right) \\
&= \ 4(\|f-h\|+\|h''\|\psi_n^2(x))+2\omega\left(f,\left|\frac{(4-6n)x+(3n-2)}{n(n+2)}\right|\right), \qquad (5.20)
\end{aligned}
$$

where $\xi_n(x)=\left|\frac{(4-6n)x+(3n-2)}{n(n+2)}\right|$.
Hence, for all $f\in C(0,1)$ by taking the infimum of the equation (5.20), we get

$$
|\mathscr{B}_n(f;x)-f(x)| \ \leq \ 4\mathfrak{k}(h,\psi_n^2(x))+2\omega(f,\xi_n(x)). \qquad (5.21)
$$

The final required results

$$
|\mathscr{B}_n(f;x)-f(x)| \ \leq \ M\omega_2(f,\psi_n^2(x))+2\omega(f,\xi_n(x)).
$$

$\square$

**Theorem 5.2.11.** *(Ditzian-Totik modulus of smoothness) Let $f\in(0,1)$ and $\rho(x)=\sqrt{x(1-x)}$, then*

$$
|\mathscr{B}_n(f;x)-f(x)|\leq C.\omega_\rho\left(f;\sqrt{\frac{1}{n}}\right).
$$

*Proof.* Using the equality $g(s)=g(x)+\int_x^s g'(w)dw$, we rewrite

$$
|\mathscr{B}_n(g;x)-g(x)|=\left|\mathscr{B}_n\left(\int_x^s g'(w)dw;x\right)\right|. \qquad (5.22)
$$

For any $x,s\in(0,1)$, we have

$$
\left|\int_x^s g'(w)dw\right|\leq\|\rho g'\|.\left|\int_x^s\frac{1}{\rho(w)}dw\right|. \qquad (5.23)
$$

Therefore,

$$
\begin{aligned}
\left|\int_x^s\frac{1}{\rho(w)}dw\right| &= \left|\int_x^s\frac{1}{\sqrt{w(1-w)}}dw\right|\leq\left|\int_x^s\left(\frac{1}{\sqrt{w}}+\frac{1}{\sqrt{1-w}}\right)dw\right| \\
&\leq 2\left(|\sqrt{s}-\sqrt{x}|+|\sqrt{1-s}-\sqrt{1-x}|\right) \\
&= 2|s-x|\left(\frac{1}{\sqrt{s}+\sqrt{x}}+\frac{1}{\sqrt{\sqrt{1-s}+\sqrt{1-x}}}\right) \\
&< 2|s-x|\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right)\leq\frac{2\sqrt{2}|s-x|}{\rho(x)}. \qquad (5.24)
\end{aligned}
$$

Combining equations (1.4) and (5.22–5.24) and applying the Cauchy-Schwarz inequality for linear positive operators, we have

$$|\mathscr{B}_n(g;x)-g(x)|<2\sqrt{2}\|\rho g'\|\rho^{-1}(x)\mathscr{B}_n(|l_1-x|;x)\leq 2\sqrt{2}\|\rho g'\|\rho^{-1}(x)\left(\mathscr{B}_n((l_1-x)^2;x)\right)^{\frac{1}{2}}.$$

From the Lemma 5.2.5, we have

$$|\mathscr{B}_n(g;x)-g(x)|<2\sqrt{2}\|\rho g'\|\rho^{-1}(x)\left(\frac{12\rho^2(x)}{n}\right)^{\frac{1}{2}}<C.\sqrt{\frac{1}{n}}\|\rho g'\|. \tag{5.25}$$

It is clear that

$$|\mathscr{B}_n(g)-g| \quad\leq\quad |\mathscr{B}_n(f-g;x)|+|\mathscr{B}_n(g;x)-g(x)|\leq C.\left(\|f-g\|+\sqrt{\frac{1}{n}}\|\rho g'\|\right).$$

Taking infimum on the right-hand side of the above relation over all $g\in W_\rho$, we get

$$|\mathscr{B}_n(f;x)-f(x)|\leq C.\bar{K}_\rho\left(f;\sqrt{\frac{1}{n}}\right)\leq C.\omega_\rho\left(f;\sqrt{\frac{1}{n}}\right).$$

□

### 5.2.3 Voronovskaya Type Theorem

In this section, we verify Voronovskaya-type asymptotic formula for the given operators $\mathscr{B}_n(f;x)$.

**Theorem 5.2.12.** *(Voronvaskaja type theorem) Let the integrable function $f$ on the interval $(0,1)$, and $f',f''$ exist at a fixed point in the interval $(0,1)$. Then we have*

$$\lim_{n\to\infty} n(\mathscr{B}_n(f;x)-f(x))=(-6x+3)f'(x)+3x(1-x)f''(x).$$

*Proof.* The Taylor's formula is defined as

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+R(t,x)(t-x)^2, \tag{5.26}$$

where $R(t,x)=\frac{f''(\varepsilon)-f''(x)}{2}$ is the remainder and $\varepsilon$ is located between $x$ and $t$. Also $\lim_{t\to x}R(t,x)=0$. Apply the given operator $\mathscr{B}_n$ on the equation (5.26), we obtain

$$\begin{aligned}\mathscr{B}_n(f;x)-f(x) \quad&=\quad f'(x)\mathscr{B}_n((t-x);x)+\frac{f''(x)}{2}\mathscr{B}_n((t-x)^2;x)\\ &\quad +\mathscr{B}_n(R(t,x)(t-x)^2;x),\end{aligned} \tag{5.27}$$

multiplying $n$ and take limit $n \to \infty$ of the above equation (5.27), we get

$$\begin{aligned}\lim_{n\to\infty} n(\mathscr{B}_n(f;x)-f(x)) &= \lim_{n\to\infty} nf'(x)\mathscr{B}_n((t-x);x)+\lim_{n\to\infty} n\frac{f''(x)}{2}\mathscr{B}_n((t-x)^2;x)\\ &\quad+\lim_{n\to\infty} n\mathscr{B}_n(R(t,x)(t-x)^2;x).\end{aligned}$$

Using Lemma 5.2.4, we get

$$\begin{aligned}\lim_{n\to\infty} n(\mathscr{B}_n(f;x)-f(x)) &= (-6x+3)f'(x)+3x(1-x)f''(x)\\ &\quad+\lim_{n\to\infty} n\mathscr{B}_n(R(t,x)(t-x)^2;x). \qquad (5.28)\end{aligned}$$

By using the Cauchy-Schwarz inequality in the remainder term, we get

$$n\mathscr{B}_n(R(t,x)(t-x)^2;x) \le \sqrt{n^2\mathscr{B}_n(R^2(t,x);x)}\sqrt{\mathscr{B}_n((t-x)^4;x)}. \qquad (5.29)$$

Since $R^2(.;x)$ is continuous at $t \in (0,1)$ and $\lim\limits_{t\to x} R(t,x)=0$, we analyse that

$$\lim_{n\to\infty} \mathscr{B}_n(R^2(t,x);x) = R^2(x,x) = 0. \qquad (5.30)$$

Thus, from the equation (5.29), (5.30) and Lemma 5.2.3, we obtain

$$\lim_{n\to\infty} n\mathscr{B}_n(R(t,x)(t-x)^2;x) = 0. \qquad (5.31)$$

We obtain the required result using the equations (5.28) and (5.31)

$$\lim_{n\to\infty} n(\mathscr{B}_n(f;x)-f(x)) = (-6x+3)f'(x)+3x(1-x)f''(x).$$

□

The Grüss-Voronovskaja type result for the given operators $\mathscr{B}_n$ as follows:

**Theorem 5.2.13.** *(Grüss-Voronovskaja type theorem) Let $f,g \in (0,1) \to \mathbb{R}$. If $f,g \in C^2(0,1)$, then*

$$\lim_{n\to\infty} n(\mathscr{B}_n((fg);x)-\mathscr{B}_n(f;x).\mathscr{B}_n(g;x)) = f'(x)g'(x)6x(1-x).$$

*Proof.* Suppose that the given relation holds

$$\begin{aligned}\mathscr{B}_n(fg;x)-\mathscr{B}_n(f;x).\mathscr{B}_n(g;x) &= \mathscr{B}_n(fg;x)-f(x)g(x)-(fg)'(x)\mathscr{B}_n((t-x);x)\\ &\quad \frac{1}{2}(fg)''(x)\mathscr{B}_n((t-x)^2;x)-g(x)\left(\mathscr{B}_n(f;x)-f(x)\right.\\ &\quad \left.-f'(x)\mathscr{B}_n((t-x);x)-\frac{1}{2}f''(x)\mathscr{B}_n((t-x)^2;x)\right)\\ &\quad -\mathscr{B}_n(f;x)\left(\mathscr{B}_n(g;x)-g(x)-g'(x)\mathscr{B}_n((t-x);x)\right.\end{aligned}$$

$$-\frac{1}{2}g''(x)\mathscr{B}_n((t-x)^2;x)\Big)+\frac{1}{2}\mathscr{B}_n((t-x)^2;x)$$
$$\times\left(f(x)g''(x)+2f'(x)g'(x)-g''(x)\mathscr{B}_n(f;x)\right)$$
$$+\mathscr{B}_n((t-x);x)(f(x)g'(x)-g'(x)\mathscr{B}_n(f;x)).$$

Taking $\lim_{n\to\infty}$ and multiply $n$ both sides of the above equation, we get
$\lim_{n\to\infty} n\{\mathscr{B}_n(fg;x)-\mathscr{B}_n(f;x).\mathscr{B}_n(g;x)\}$

$$\begin{aligned}
= \quad & \lim_{n\to\infty} n\{\mathscr{B}_n(fg;x)-f(x)g(x)-(fg)'(x)\mathscr{B}_n((t-x);x)\frac{1}{2}(fg)''(x)\mathscr{B}_n((t-x)^2;x)\\
& -g(x)\left(\mathscr{B}_n(f;x)-f(x)-f'(x)\mathscr{B}_n((t-x);x)-\frac{1}{2}f''(x)\mathscr{B}_n((t-x)^2;x)\right)\\
& -\mathscr{B}_n(f;x)\left(\mathscr{B}_n(g;x)-g(x)-g'(x)\mathscr{B}_n((t-x);x)-\frac{1}{2}g''(x)\mathscr{B}_n((t-x)^2;x)\right)\\
& +\frac{1}{2}\mathscr{B}_n((t-x)^2;x)\left(f(x)g''(x)+2f'(x)g'(x)-g''(x)\mathscr{B}_n(f;x)\right)\\
& +\mathscr{B}_n((t-x);x)(f(x)g'(x)-g'(x)\mathscr{B}_n(f;x))\}.
\end{aligned}$$

Using Theorem 5.2.7, 5.2.12 and Lemma 5.2.4, we get the final result

$$\lim_{n\to\infty} n\{\mathscr{B}_n((fg);x)-\mathscr{B}_n(f;x).\mathscr{B}_n(g;x)\}=f'(x)g'(x)6x(1-x).$$

□

### 5.2.4 Weighted Approximation

In this segment, we discuss the korovkin's type theorem for the weighted approximation results of the given modified operators $\mathscr{B}_n$, for each $n\in\mathbb{N}$. Consider $\tau(x)=1+x^2$ as a weighted function which is continuous on $(0,1)$ and $\lim_{|x|\to\infty}\tau(x)=\infty$.

**Lemma 5.2.14.** *(Weighted approximation) Suppose that $f\in C_\tau^*(0,1)$, the given modified operators following the inequality holds:*

$$\|\mathscr{B}_n(f)\|_\tau\leq M\|f\|_\tau.$$

**Theorem 5.2.15.** *(Weighted approximation) Suppose that $f\in C_\tau^*(0,1)$, then the following equalities holds for the operators $\mathscr{B}_n$*

$$\lim_{n\to\infty}\|\mathscr{B}_n(f)-f\|_\tau=0.$$

*Proof.* It is sufficient to prove that

$$\lim_{n\to\infty}\|\mathscr{B}_n(l_r)-l_r\|_\tau=0,\ r=0,1,2,3,4.$$

The result holds trivially for $l_0$, i.e., $\|\mathscr{B}_n(l_0;x)-l_0(x)\|=0$. Next,

$$\begin{aligned}
\|\mathscr{B}_n(l_1)-l_1\|_\tau &= \sup_{x\in(0,1)}\frac{|\mathscr{B}_n(l_1;x)-l_1(x)|}{1+x^2}\\
&= \sup_{x\in(0,1)}\frac{\left|\frac{(n-2)^2x}{n(n+2)}+\frac{(3n-2)}{n(n+2)}-x\right|}{1+x^2}\\
&= \sup_{x\in(0,1)}\frac{1}{1+x^2}\left|\frac{(3n-2)}{n(n+2)}-\frac{(6n-4)}{n(n+2)}x\right|\\
&\leq \left|\frac{(3n-2)}{n(n+2)}\right|\sup_{x\in(0,1)}\frac{1}{1+x^2}+\left|\frac{(6n-4)}{n(n+2)}\right|\sup_{x\in(0,1)}\frac{x}{1+x^2},
\end{aligned}$$

the condition is hold as $n\to\infty$.

Again we have to show that $\|\mathscr{B}_n(l_2)-l_2\|_\tau=0$.

Now,

$$\begin{aligned}
\|\mathscr{B}_n(l_2)-l_2\|_\tau &= \sup_{x\in(0,1)}\frac{|\mathscr{B}_n(l_2;x)-l_2(x)|}{1+x^2}\\
&= \sup_{x\in(0,1)}\frac{1}{1+x^2}\left|\frac{(n^3-13n^2+48n-36)}{n(n+2)(n+3)}x^2+\frac{(12n^2-56n+48)}{n(n+2)(n+3)}x\right.\\
&\quad\left.+\frac{14n-12}{n(n+2)(n+3)}-x^2\right|\\
&= \sup_{x\in(0,1)}\frac{1}{1+x^2}\left|\frac{-2(6-7n)}{n(n+2)(n+3)}-\frac{2(-24+28n-6n^2)}{n(n+2)(n+3)}x\right.\\
&\quad\left.+\frac{-2(18-21n+9n^2)}{n(n+2)(n+3)}x^2\right|\\
&\leq \left|\frac{2(6-7n)}{n(n+2)(n+3)}\right|\sup_{x\in(0,1)}\frac{1}{1+x^2}+\left|\frac{2(-24+28n-6n^2)}{n(n+2)(n+3)}\right|\\
&\quad\times\sup_{x\in(0,1)}\frac{1}{1+x^2}+\left|\frac{2(18-21n+9n^2)}{n(n+2)(n+3)}\right|\sup_{x\in(0,1)}\frac{1}{1+x^2}.
\end{aligned}$$

condition is hold as $n$ large. Similarly the result is hold for others. Hence, the proof is completed □

### 5.2.5 Numerically Analysis

Here, we discuss the convergence behavior of the given operators $\mathscr{B}_n$ numerically and graphically by taking the function $f(x)=x(1-x)^3$, where $x\in(0,1)$ for the different values of $n$. Here, we get the error values and graphs by incorporating the Mathematica tool. Consider $(E_{n,m}f)(x)=|(\mathscr{B}_nf)(x)-f(x)|$, for $n=\{50,100,500,1000\}$ and different values of $x$ be the approximate error functions obtained by the given operators $\mathscr{B}_n$. In 5.5. In the meantime, we compute the error of approximation for $(E_{n,m}f)(x)$, which is shown in the table. Finally, we also shown the error graph 5.6 corresponding to $(E_{n,m}f)(x)$. The conclusion is that when increasing the value of $n$, the rate of convergence of the operators $\mathscr{B}_n$ fast towards the graph of the corresponding function $f(x)$.

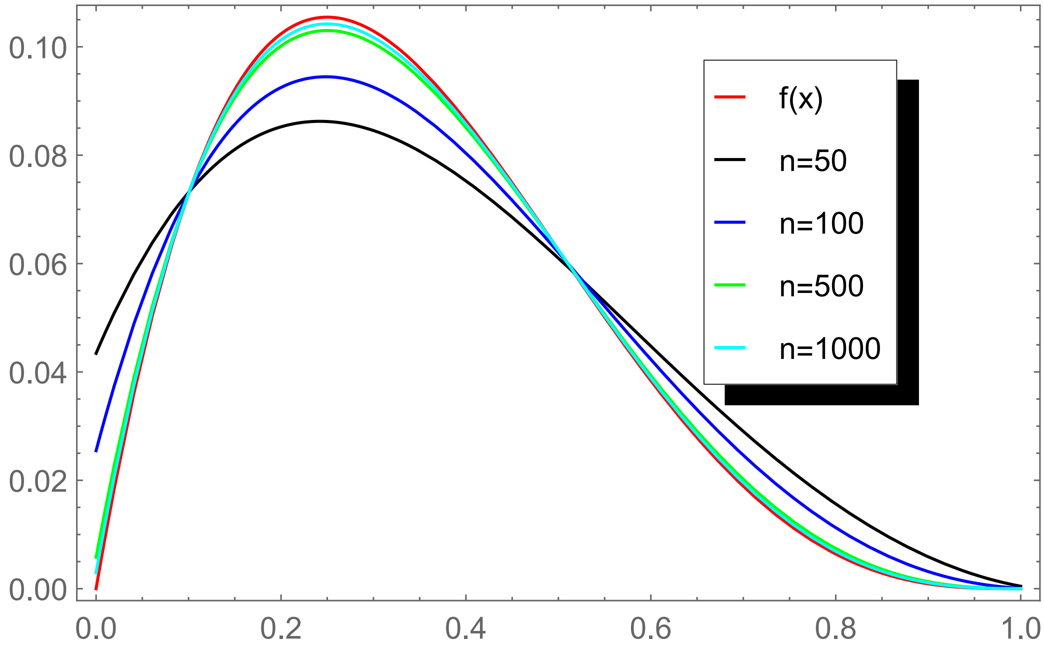


Figure 5.5: Convergence behavior of $\mathscr{B}_n$

Table 2 : Approximation Error $(E_{n,m}f)(x) = |(\mathscr{B}_n f)(x) - f(x)|$, for $n = \{50, 100, 500, 1000\}$

| $x$ | $(E_{50,1}f)(x)$ | $(E_{100,2}f)(x)$ | $(E_{500,3}f)(x)$ | $(E_{1000,3}f)(x)$ |
|---|---|---|---|---|
| 0.0 | 0.0434277 | 0.0254601 | 0.00580391 | 0.00295049 |
| 0.2 | 0.0171665 | 0.00990509 | 0.00223403 | 0.00113433 |
| 0.4 | 0.0111363 | 0.00601322 | 0.00127705 | 0.000643255 |
| 0.6 | 0.00636095 | 0.00390173 | 0.000920816 | 0.000470095 |
| 0.8 | 0.00923331 | 0.00484605 | 0.00100104 | 0.00050229 |
| 1.0 | 0.000454032 | 0.0000663776 | $6.03945\times10^{-7}$ | $7.67344\times10^{-8}$ |

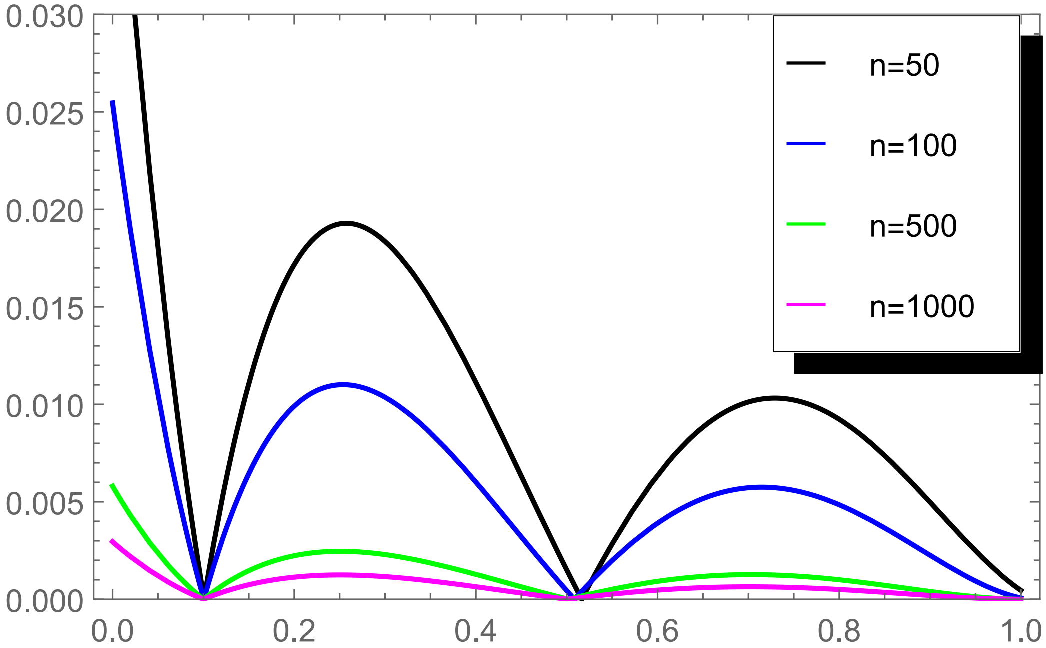


Figure 5.6: Error of Approximation $E_{n,m}$

# Chapter 6

# Approximation by Szász-Păltănea type Operators using the Appell Polynomials of class $A^2$

Numerous polynomial sequences use the theory of Appell polynomials of classes, which is utilized in several mathematical fields, including Combinatorics and number theory. This chapter is a study of a new sequence of operators using Appell polynomials of class $A^2$. We study an estimate of error in approximation in terms of modulus of continuity and rate of convergence in weighted space for these operators. Then, we discuss the convergence for the function having the derivatives of bounded variation.

## 6.1 Introduction

In 1950, Szász summarized the work of the Bernstein polynomials on the infinite interval and obtained the convergence results. These polynomials also play an important role in the field of approximation theory. Szász polynomials are as follows:

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{6.1}$$

where $f \in C[0,\infty)$, for each $x \in [0,\infty)$. Given the work done in the field of Szász operators, Jakimovski et al. [53], gave popularization of Szász operators with the help of the Appell polynomials.

Consider an analytic function in the disc $|z| < \mathbb{R}$, $(\mathbb{R} > 1)$ and $g(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ with $g(1) \neq$

0. The Appell polynomials $p_k(x)$ contains generating function in the form $g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k$. Jakimovski and Leviatan build the following positive linear operators $P_n(f;x)$ defined as:

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \forall\, n \in \mathbb{N}, \tag{6.2}$$

and they deliberate the approximation results on the basis of Szász operators (6.1). In [101], Wood studied the approximation properties of the operators (6.2) on $[0,\infty)$ iff $\frac{a_k}{g(1)} \geq 0$, for $k \in \mathbb{N}$. When $g(u) = 1$, then operators (6.2) reduces to (6.1). Using the above certitude, Ciupa ([18, 19]) gave another variants of the operators and obtained the approximation properties by the goodness of Korovkin's theorem.

In 1974, Ismail [51], studied another generalization of Szász operators in the view of Sheffer polynomials and operators $P_n(f;x)$. Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$, $(a_0 \neq 0)$ and $H(z) = \sum_{k=1}^{\infty} h_k z^k$ $(h_1 \neq 0)$ be analytic functions in the disc $|z| < \mathbb{R}$, $(\mathbb{R} > 1)$, where $a_i$ and $h_i$ in real. In view of the following assumptions

1. $p_k(x) \geq 0$, for $x \in [0,\infty)$.

2. $A(1) \neq 0$ and, $H'(1) = 1$.

The set of polynomials $p_k(x); k \geq 0$ are sheffer polynomials $\Leftrightarrow$ the generating function of the form

$$A(\zeta)e^{xH(\zeta)} = \sum_{k=0}^{\infty} p_k(x)\zeta^k, |\zeta| < \mathbb{R}.$$

Finally, Ismail found the approximation properties of the given function

$$\Im_n(f;x) = \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), n > 0. \tag{6.3}$$

Substitute $H(\zeta) = \zeta$ in the equation (6.3), then the operators reduces to Leviatan operators (6.2). And another way, we supposing $H(\zeta) = \zeta, A(\zeta) = 1$, then the operators (6.3) reduces to the Szász operators (6.1). Continuation on the above work, In 2015, Sezgin Sucu and, Serhan Varma [95] obtained Stancu form of the $\Im_n$ operators (6.3) and estimated their approximation properties. In 2019, M. Mursaleen et. al [69], proposed the Jakimovski-Leviatan-Stancu-Durrmeyer type operators and obtained approximation properties like A-Statistical convergence with the help of Korovkin's theorem and also found $r^{th}$ derivative of a function using proposed operators. Many more researchers have worked on the field of szász operators. These operators are extensively used in computer algebra systems, data science, machine learning, and image processing.

Very recently, Varma and Sucu [98], introduced the generalization of Szász operators using the Appell polynomials of class $A^2$ as follows:

$$T_n(f;x) = \frac{1}{A(1)e^{nx}+B(1)e^{-nx}} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{6.4}$$

with the restrictions $A(1) > 0, B(1) \geq 0$ and $p_k(x) > 0$ for all $k = 0, 1, \ldots$ The Appell polynomials $p_k(x)$ of class $A^2$ are obtained by the following generating function

$$A(\zeta)e^{x\zeta} + B(\zeta)e^{-x\zeta} = \sum_{k=0}^{\infty} p_k(x)\zeta^k, \tag{6.5}$$

where $A(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\zeta^k$ and $B(\zeta) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\zeta^k$ are power series defined over the disc $|z| < \mathbb{R}$ $(\mathbb{R} > 1)$ with $a_0^2 - b_0^2 \neq 0$. They found some axillary results by using newly proposed operators and got approximation results using the modulus of continuity and also studied some kind of sequence of operators containing Gould-Hopper polynomials.

Because of some research articles ([45, 67]), we studied several summation-integral type operators and with the help of Varma and Sucu's work, we introduce a new generalization of Szász-Păltănea operators using the Appell polynomials of class $A^2$ on the interval $[0, \infty)$, define as

$$S_n^{\rho}(f;x) = \frac{1}{A(1)e^{nx}+B(1)e^{-nx}} \left[ \sum_{k=1}^{\infty} p_k(nx) \int_0^{\infty} \phi_{n,k}^{\rho}(\zeta, c) f(\zeta) d\zeta + (a_0 e^{nx} + b_0 e^{-nx}) f(0) \right], \tag{6.6}$$

where

$$\phi_{n,k}^{\rho}(\zeta, c) = \begin{cases} \frac{n\rho}{\Gamma(k\rho)} e^{-n\rho\zeta} (n\rho\zeta)^{k\rho-1}, c = 0 \\ \frac{\Gamma\left(\frac{n\rho}{c} + k\rho\right)}{\Gamma(k\rho)\Gamma\left(\frac{n\rho}{c}\right)} \frac{c^{k\rho}\zeta^{k\rho-1}}{(1+c\zeta)^{\frac{n\rho}{c}+k\rho}}, c = 1, 2, 3, \ldots \end{cases}.$$

It can we easily observed by simple computation that:

$$\int_0^{\infty} \phi_{n,k}^{\rho}(\zeta, c)\, \zeta^r d\zeta = \begin{cases} \frac{\Gamma(k\rho + r)}{\Gamma(k\rho)} \frac{1}{\prod_{j=1}^{r}(n\rho - jc)}, & r \neq 0 \\ 1, & r = 0. \end{cases}$$

On the other hand, given operators (6.6) can be write as

$$S_n^{\rho}(f;x) = \int_0^{\infty} A_n(x, \zeta) f(\zeta) d\zeta, \tag{6.7}$$

where

$$A_n(x,\zeta) = \frac{1}{A(1)e^{nx}+B(1)e^{-nx}}\left[(a_0e^{nx}+b_0e^{-nx})\delta(t)+\sum_{k=1}^{\infty}p_k(nx)\phi^{\rho}_{n,k}(\zeta,c)\right],$$

and $\delta(t)$ is the Dirac-delta function.

For $\beta>0$, let $C_\beta[0,\infty)=\{f\in C[0,\infty): f(\zeta)\leq N(1+\zeta^\beta)\}$, for some $M>0$ endowed with the norm

$$\|f\|_\beta = \sup_{\zeta\in[0,\infty)}\frac{|f(\zeta)|}{(1+\zeta^\beta)}.$$

**Remark 6.1.1.**

- *For $A(\zeta)=1$ and $B(\zeta)=0$ in the equations (6.4) and (6.6), we obtain the famous Szász operators [92] and Szász-Păltănea operators [77].*

- *For $A(\zeta)=1$, $B(\zeta)=0$ and $\rho=1$, the operators* (6.6) *reduces to Philips operators [73].*

## 6.2 Preliminaries

Prior to proceeding to our main results, we prepare some general lemmas which are useful throughout this chapter. Besides, we have used Mathematica software to calculate the moments and central moments of the proposed operators.

In the equation (6.5), substitute $\zeta=1$ and replace $x$ with $nx$, we get

$$\sum_{k=0}^{\infty}p_k(nx) \quad = \quad A(1)e^{nx}+B(1)e^{-nx}.$$

By taking the first four derivatives of the equation (6.5) and substituting $\zeta=1$, and then replacing $x$ with $nx$, we obtained

$$\begin{aligned}
\sum_{k=0}^{\infty}kp_k(nx) &= nx(A(1)e^{nx}-B(1)e^{-nx})+A'(1)e^{nx}+B'(1)e^{-nx},\\
\sum_{k=0}^{\infty}k(k-1)p_k(nx) &= x^2n^2(A(1)e^{nx}+B(1)e^{-nx})+2nx(A'(1)e^{nx}-B'(1)e^{-nx})\\
&\quad +(A''(1)e^{nx}+B''(1)e^{-nx}),\\
\sum_{k=0}^{\infty}k(k-1)(k-2)p_k(nx) &= n^3x^3(A(1)e^{nx}-B(1)e^{-nx})+3x^2n^2(A'(1)e^{nx}+B'(1)e^{-nx})\\
&\quad +3nx(A''(1)e^{nx}-B''(1)e^{-nx})+(A'''(1)e^{-nx}+B'''(1)e^{-nx}),\\
\sum_{k=0}^{\infty}k(k-1)(k-2)(k-3)p_k(nx) &= n^4x^4(A(1)e^{nx}+B(1)e^{-nx})+4n^3x^3(A'(1)e^{nx}+B'(1)e^{-nx})\\
&\quad +6n^2x^2(A''(1)e^{nx}+B''(1)e^{-nx})+4nx(A'''(1)e^{nx}-B'''(1)e^{-nx})\\
&\quad +(A''''(1)e^{nx}+B''''(1)e^{-nx}).
\end{aligned}$$

**Lemma 6.2.1.** *The operators $T_n(\zeta^m;x)$, $m=\overline{0,4}$, we have*

$$
\begin{aligned}
T_n(1;x) &= 1;\\
T_n(\zeta;x) &= \left(\frac{A(1)e^{nx}-B(1)e^{-nx}}{A(1)e^{nx}+B(1)e^{-nx}}\right)x+\frac{A'(1)e^{nx}+B'(1)e^{-nx}}{n[A(1)e^{nx}+B(1)e^{-nx}]};\\
T_n(\zeta^2;x) &= x^2+\left(\frac{[2A'(1)+A(1)]e^{nx}-[2B'(1)+B(1)]e^{-nx}}{n[A(1)e^{nx}+B(1)e^{-nx}]}\right)x\\
&\quad+\frac{[A''(1)+A'(1)]e^{nx}+[B''(1)+B'(1)]e^{-nx}}{n^2[A(1)e^{nx}+B(1)e^{-nx}]};\\
T_n(\zeta^3;x) &= \left(\frac{A(1)e^{nx}-B(1)e^{-nx}}{A(1)e^{nx}+B(1)e^{-nx}}\right)x^3+\frac{1}{n[A(1)e^{nx}+B(1)e^{-nx}]}\big[(3A(1)\\
&\quad+2A'(1))e^{nx}+(3B(1)+2B'(1))e^{-nx}\big]x^2\\
&\quad+\frac{1}{n^2[A(1)e^{nx}+B(1)e^{-nx}]}\big\{(A(1)+(n+6)A'(1)+2nA'')e^{nx}\\
&\quad-(B(1)+(6-n)B'(1)+2nB''(1))e^{-nx}\big\}x\\
&\quad+\frac{(A'(1)+A''')e^{nx}+(B'(1)+B'''(1))e^{-nx}}{n^3[A(1)e^{nx}+B(1)e^{-nx}]};\\
T_n(\zeta^4;x) &= x^4+\frac{\{3(2A(1)+A'(1))e^{nx}-3(2B(1)+B'(1))e^{-nx}\}}{n[A(1)e^{nx}+B(1)e^{-nx}]}x^3\\
&\quad+\frac{1}{n^2[A(1)e^{nx}+B(1)e^{-nx}]}\big\{(7A(1)+13A'(1)+4A''(1))e^{nx}\\
&\quad+(7B(1)+11B'(1)+4B''(1))e^{-nx}\big\}x^2\\
&\quad+\frac{1}{n^4[A(1)e^{nx}+B(1)e^{-nx}]}\big\{((-6-5n)A(1)+n(2(7+3n)A'(1)\\
&\quad+2(1+6n)A''(1)+(2+n)A'''(1)))e^{nx}+((-6+5n)B(1)\\
&\quad+n(2(-7+3n)B'(1)+(2-12n)B''(1)-3B'''(1)))e^{-nx}\big\}x\\
&\quad+\frac{1}{n^5[A(1)e^{nx}+B(1)e^{-nx}]}\big\{((6-5n)A'(1)+n(13A''(1)+7A'''(1)\\
&\quad+A''''(1)))e^{nx}+((6-5n)B'(1)+n(B''(1)+5B'''(1)+B''''(1)))e^{-nx}\big\}.
\end{aligned}
$$

**Lemma 6.2.2.** *For the proposed operators $S_n^{\rho}(\zeta^m;x)$, $m=\overline{0,4}$, we have*

$$
\begin{aligned}
S_n^{\rho}(1;x) &= 1;\\
S_n^{\rho}(\zeta;x) &= \frac{1}{(n\rho-c)[A(1)e^{nx}+B(1)e^{-nx}]}\big\{n\rho x[A(1)e^{nx}-B(1)e^{-nx}]\\
&\quad+\rho[A'(1)e^{nx}+B'(1)e^{-nx}]\big\};\\
S_n^{\rho}(\zeta^2;x) &= \frac{n^2\rho^2}{(n\rho-c)(n\rho-2c)}x^2\\
&\quad+\frac{1}{(n\rho-c)(n\rho-2c)[A(1)e^{nx}+B(1)e^{-nx}]}\big\{(n\rho((A(1)\\
&\quad+2\rho A'(1))e^{nx}-(B(1)+2\rho B'(1))e^{-nx}))x+(\rho((A'(1)\\
&\quad+\rho A''(1))e^{nx}+(B'(1)+\rho B''(1))e^{-nx}))\big\};\\
S_n^{\rho}(\zeta^3;x) &= \frac{1}{(n\rho-c)(n\rho-2c)(n\rho-3c)[A(1)e^{nx}+B(1)e^{-nx}]}\big\{n^3\rho^3(A(1)e^{nx}\\
&\quad-B(1)e^{-nx})x^3+3n^2\rho^2((A(1)+\rho A'(1))e^{nx}+(B(1)
\end{aligned}
$$

$$+\rho B'(1))e^{-nx})x^2+(n\rho((2A(1)+3\rho(2A'(1)+\rho A''(1)))e^{nx}$$
$$-(2B(1)+3\rho(2B'(1)+\rho B'')(1))e^{-nx}))x+\rho((2A'(1)+\rho(3A''(1)$$
$$+\rho A'''(1)))e^{nx}+(2B'(1)+\rho(3B''(1)+\rho B'''(1)))e^{-nx})\};$$

$$S_n^{\rho}(\zeta^4;x) = \frac{n^4\rho^4}{(n\rho-c)(n\rho-2c)(n\rho-3c)(n\rho-4c)}x^4$$
$$+\frac{1}{(n\rho-c)(n\rho-2c)(n\rho-3c)(n\rho-4c)[A(1)e^{nx}+B(1)e^{-nx}]}\times$$
$$\{2n^3\rho^3((3A(1)+2\rho A'(1))e^{nx}-(3B(1)+2\rho B'(1))e^{-nx})x^3$$
$$+n^2\rho^2((11A(1)+6\rho(3A'(1)+\rho A''(1)))e^{nx}+(11B(1))$$
$$+6\rho(3B'(1)+\rho B''(1)))e^{-nx})x^2+(2n\rho((3A(1)+\rho(11A'(1)$$
$$+9\rho A''(1)+2\rho^2A'''(1)))e^{nx})-(3B(1)+\rho(11B'(1)+9\rho B''(1)$$
$$+2\rho^2B'''(1)))e^{-nx})x+\rho((6A'(1)+\rho(11A''(1)+6\rho A'''(1)$$
$$+\rho^2A''''(1)))e^{nx}+(6B'(1)+\rho(11B''(1)$$
$$+6\rho B'''(1)+\rho^2B''''(1)))e^{-nx})\}.$$

*Proof.* This Lemma is an immediate consequence of Lemma 6.2.2 and equation (6.6). Hence the details of its proof are omitted. □

**Lemma 6.2.3.** *The central moments of the operators $S_n^{\rho}((\zeta-x)^m;x)$, for $m=1,2$ and 4, we have*

$$S_n^{\rho}(\zeta-x;x) = \frac{\{-2n\rho B(1)e^{-nx}+c[A(1)e^{nx}+B(1)e^{-nx}]\}}{(n\rho-c)[A(1)e^{nx}+B(1)e^{-nx}]}x$$
$$+\frac{\rho[A'(1)e^{nx}+B'(1)e^{-nx}]}{(n\rho-c)[A(1)e^{nx}+B(1)e^{-nx}]};$$

$$S_n^{\rho}((\zeta-x)^2;x) = \frac{1}{(n\rho-c)(n\rho-2c)[A(1)e^{nx}+B(1)e^{-nx}]}\{cn\rho((A(1)$$
$$+2c^2A(1))e^{nx}+(-7B(1)+(4n^2\rho^2+2c^2)B(1))e^{-nx})x^2$$
$$+\rho((nA(1)+4cA'(1))e^{nx}+(4cB'(1)-n(B(1)$$
$$+4\rho B'(1)))e^{-nx})x+\rho((A'(1)+\rho A''(1))e^{nx}+(B'(1)$$
$$+\rho B''(1))e^{-nx})\};$$

$$S_n^{\rho}((\zeta-x)^4;x) = \frac{1}{(n\rho-c)(n\rho-2c)(n\rho-3c)(n\rho-4c)[A(1)e^{nx}+B(1)e^{-nx}]}$$
$$\{((46c^3n\rho A(1)+24c^4A(1)+3c^2n^2\rho^2A(1))e^{nx}+(-146c^3n\rho B(1)$$
$$-104cn^3\rho^3B(1)+16n^4\rho^4B(1)+24c^4B(1)$$
$$+211c^2n^2\rho^2B(1))e^{-nx})x^4+((96\rho c^3A'(1)+4c^2n(9A(1)$$
$$+5\rho A'(1))+3cn^2\rho A(1))e^{nx}+(96c^3B'(1)-36c^2nB(1)$$
$$-124c^2n\rho B'(1)-4n^3\rho^2(3B(1)+4\rho B'(1))+15cn^2\rho B(1)$$
$$+84cn^2\rho^2B'(1))e^{-nx})x^3+(\rho((72c^2(A'(1)+\rho A''(1))$$
$$+3n^2\rho A(1)-2cn(16A(1)+3\rho(9A'(1)+\rho A''(1)))))e^{nx}$$

$$+(72c^2B'(1)+72c^2\rho B''(1)+n^2\rho(19B(1)+24\rho(2B'(1)$$
$$+\rho B''(1)))+2cn(-16B(1))-3\rho(23B'(1)+15\rho B''(1)))e^{-nx})x^2$$
$$-((2\rho(-8c(2B'(1)+3\rho B''(1))-8cn^3\rho^2B'''(1)+2n^4\rho^3B'''(1)$$
$$+n(3B(1)+\rho(15B'(1)+\rho(15B''(1)+2\rho B'''(1))))))e^{-nx}$$
$$-(3nA(1)+16cA'(1)+n\rho(7A'(1)+3\rho A''(1)))e^{nx})x$$
$$+((\rho(6A'(1)+\rho(11A''(1)+6\rho A'''(1)+\rho^2A''''(1)))e^{nx}$$
$$+(6B'(1)+\rho(11B''(1)$$
$$+6n^3\rho B'''(1)+\rho^2B''''(1)))e^{-nx}))\Big\}.$$

*Proof.* The Proof of the above lemma can be easily found by using the following equalities:

$$\begin{aligned}
S_n^\rho((\zeta-x);x) &= S_n^\rho(\zeta;x)-xS_n^\rho(1;x);\\
S_n^\rho((\zeta-x)^2;x) &= S_n^\rho(\zeta^2;x)-2xS_n^\rho(\zeta;x)+x^2S_n^\rho(1;x);\\
S_n^\rho((\zeta-x)^4;x) &= S_n^\rho(\zeta^4;x)-4xS_n^\rho(\zeta^3;x)+6x^2S_n^\rho(\zeta^2;x)-4x^3S_n^\rho(\zeta;x)+x^4S_n^\rho(1;x).
\end{aligned}$$

□

**Lemma 6.2.4.** *The limiting values of the Lemma 6.2.3, we have*

$$\begin{aligned}
\lim_{n\to\infty} nS_n^\rho((\zeta-x);x) &= \frac{cx}{\rho}+\frac{A'(1)}{A(1)};\\
\lim_{n\to\infty} nS_n^\rho((\zeta-x)^2;x) &= \frac{x(1+cx)}{\rho};\\
\lim_{n\to\infty} n^2S_n^\rho((\zeta-x)^4;x) &= \frac{3x^2(1+cx)^2}{\rho^2}.
\end{aligned}$$

## 6.3 Approximation Results

In the whole chapter, we take $\rho_n(x)=S_n^\rho((\zeta-x)^2;x)$.

Bohman-Korovkin delegated an easy and supporting theorem to provide for continuous functions in a compact space for positive linear operators, which strongly converges to the identity operators. In the following theorem, we obtained an approximation result of the operators $S_n^\rho$ making use of the Bohman-Korovkin's theorem.

**Theorem 6.3.1.** *(Fundamental convergence theorem) Let $f$ be a continuous function on the interval $[0,\infty)$, then*

$$\lim_{n\to\infty} S_n^\rho(f;x)=f(x)$$

*holds uniformly in $x\in[0,\infty)$.*

*Proof.* The proof is based on the conditions by the Korovkin's theorem. By using Lemma 6.2.2, we obtain

$$\lim_{n\to\infty} S_n^{\rho}(\zeta^r;x) = x^r,\ r = \overline{0,4}.$$

□

In the following result, we find the rate of convergence of the given operators $S_n^{\rho}$ for the Lipschitz space. Due to the existence of $x$ in the denominator, we obtain pointwise approximation. From the Szász operators [92], $x$ drop foremost to the uniform convergence.

**Theorem 6.3.2.** *(Lipschitz class) Let $f \in Lip_M^*(\sigma)$ and $\sigma \in (0,1]$. Then the inequality holds:*

$$|S_n^{\rho}(f;x) - f(x)| \leq M\left(\frac{\rho_n(x)}{x}\right)^{\frac{\sigma}{2}},$$

*for all $x \in (0,\infty)$.*

*Proof.* Using the linearity and positivity of the proposed operators (6.7), we write as

$$|S_n^{\rho}(f;x) - f(x)| \leq \int_0^{\infty} A_n(x,\zeta)|f(\zeta) - f(x)|d\zeta.$$

Applying Hölder's inequality with $p = \frac{2}{\sigma}$ and $q = \frac{2}{2-\sigma}$ and Lemma 6.2.2, we obtain

$$\begin{aligned}
|S_n^{\rho}(f;x) - f(x)| &\leq \left(\int_0^{\infty} A_n(x,\zeta)|f(\zeta) - f(x)|^{\frac{2}{\sigma}} d\zeta\right)^{\frac{\sigma}{2}} \left(\int_0^{\infty} A_n(x,\zeta) d\zeta\right)^{\frac{2-\sigma}{2}} \\
&\leq \left(\int_0^{\infty} A_n(x,\zeta)|f(\zeta) - f(x)|^{\frac{2}{\sigma}} d\zeta\right)^{\frac{\sigma}{2}} \\
&\leq M\left(\int_0^{\infty} A_n(x,\zeta)\frac{(\zeta - x)^2}{(\zeta + x)} d\zeta\right)^{\frac{\sigma}{2}} \\
&\leq M\left(\frac{\rho_n(x)}{x}\right)^{\frac{\sigma}{2}}.
\end{aligned}$$

We get the required result. □

In the present theorem, we set up a Voronvaskaja type approximation theorem.

**Theorem 6.3.3.** *(Voronvaskaja type theorem) Let $f \in C^2[0,\infty)$ and $f', f''$ exists at a pont $x \in [0,\infty)$, then the equality holds*

$$\begin{aligned}
\lim_{n\to\infty} n(S_n^{\rho}(f;x) - f(x)) &= \left(\frac{cx}{\rho} + \frac{A'(1)}{A(1)}\right) f'(x) \\
&\quad + \left(\frac{cx^2}{\rho} + \frac{x(A(1) + (2 - A'(1)\rho))}{A(1)\rho}\right)\frac{f''(x)}{2}.
\end{aligned}$$

*If $f''$ is continuous on $[0,\infty)$, then the above result holds uniformly in the interval $[0,b] \subset [0,\infty)$ with $b > 0$.*

*Proof.* We know that the Taylor's series expansion

$$f(\zeta) = f(x) + f'(x)(\zeta - x) + \frac{f''(x)}{2}(\zeta - x)^2 + \sigma(\zeta, x)(\zeta - x)^2, \tag{6.8}$$

where $\sigma(\zeta, x) \in C^2[0,\infty)$ and $\lim_{n\to\infty} \sigma(\zeta, x) = 0$. Applying the given operator $S_n^\rho$ from the equation (6.7) on both sides of the equation (6.8), we have

$$\begin{aligned} \lim_{n\to\infty} n(S_n^\rho(f;x) - f(x)) &= \lim_{n\to\infty} nS_n^\rho((\zeta - x);x)f'(x) + \lim_{n\to\infty} nS_n^\rho((\zeta - x)^2;x)\frac{f''(x)}{2} \\ &\quad + \lim_{n\to\infty} nS_n^\rho(\sigma(\zeta, x)(\zeta - x)^2;x). \end{aligned} \tag{6.9}$$

In the last term of the equation (6.9), using the Cauchy-Schwarz inequality, we obtain

$$nS_n^\rho(\sigma(\zeta, x)(\zeta - x)^2;x) \leq \sqrt{S_n^\rho(\sigma^2(\zeta, x);x)}\sqrt{n^2 S_n^\rho((\zeta - x)^4;x)}.$$

Since $\sigma(\zeta, x) \to 0$, as $t \to x$, applying Theorem 6.3.1, we get $\lim_{n\to\infty} S_n^\rho(\sigma^2(\zeta, x);x) = \sigma^2(x;x) = 0$. And applying Lemma 6.2.4, for large $n$, and $x \in [0,\infty)$, we have

$$n^2 S_n^\rho((\zeta - x)^4;x) = O(1). \tag{6.10}$$

Hence,

$$\lim_{n\to\infty} nS_n^\rho(\sigma(\zeta, x)(\zeta - x)^2;x) = 0. \tag{6.11}$$

The required result are obtained from the equations (6.9), (6.11) and Lemma 6.2.4, we have

$$\begin{aligned} \lim_{n\to\infty} n(S_n^\rho(f;x) - f(x)) &= \left(\frac{cx}{\rho} + \frac{A'(1)}{A(1)}\right) f'(x) \\ &\quad + \left(\frac{cx^2}{\rho} + \frac{x(A(1) + (2 - A'(1)\rho))}{A(1)\rho}\right) \frac{f''(x)}{2}. \end{aligned}$$

□

The uniformity declaration holds from the uniform continuity of $f''$ on $[0,b]$ and in general other results also holds uniformly in $[0,b], b > 0$. In the following theorem, we discussed the approximation results of the proposed operators, using classical modulus of continuity.

**Theorem 6.3.4.** *For $f \in C^2[0,\infty)$, then the inequality holds*

$$|S_n^\rho(f;x) - f(x)| \leq 4M_f(1 + x^2)\rho_n(x) + 2\omega_{b+1}(f;\sqrt{\rho_n(x)}).$$

*where $\omega(f;\rho_n(x))$ is the modulus of continuity of $f$ on $[0,b+1]$.*

*Proof.* From [50], for $\zeta \in (b+1,\infty)$ and $x \in [0,b]$, we have

$$|f(\zeta)-f(x)| \leq 4M_f(t-x)^2(1+x^2)+\left(1+\frac{|\zeta-x|}{\delta}\right)\omega_{b+1}(f,\delta),\ \delta>0.$$

Applying the cauchy-Schwarz inequality, we get

$$\begin{aligned} |S_n^{\rho}(f;x)-f(x)| &\leq 4M_f(1+x^2)S_n^{\rho}((\zeta-x)^2;x) \\ &\quad +\left(1+\frac{(S_n^{\rho}((\zeta-x)^2;x))^{\frac{1}{2}}}{\delta}\right)\omega_{b+1}(f,\delta) \\ &\leq 4M_f(1+x^2)\rho_n(x)+\omega_{b+1}(f,\delta)\left(1+\frac{\sqrt{\rho_n(x)}}{\delta}\right). \end{aligned}$$

Hence suppose $\delta=\sqrt{\rho_n(x)}$, we obtained the result. □

**Theorem 6.3.5.** *(Ditzian-Totik modulus of smoothness) For $f \in C_B[0,\infty)$, then the inequality holds for the large value of n*

$$|S_n^{\rho}(f;x)-f(x)| \leq C\omega^*_{\psi^\tau}\left(f;\frac{\psi^{1-\tau}(x)}{\sqrt{n}}\right),$$

*where C is a constant independent of f and n.*

*Proof.* By the definition of $K-$functional, choose $g \in W_\tau$ such that

$$\|f-g\|+\frac{\psi^{1-\tau}(x)}{\sqrt{n}}\|\psi^\tau g'\| \leq 2K^*_{\psi^\tau}\left(f;\frac{\psi^{1-\tau}(x)}{\sqrt{n}}\right). \tag{6.12}$$

Now,

$$\begin{aligned} |S_n^{\rho}(f;x)-f(x)| &\leq |S_n^{\rho}(f-g;x)|+|S_n^{\rho}(g;x)-g(x)|+|g(x)-f(x)| \\ &\leq 2\|f-g\|+|S_n^{\rho}(g;x)-g(x)|. \end{aligned} \tag{6.13}$$

Consider

$$g(t)=g(x)+\int_x^{\zeta} g'(u)du$$

and

$$|S_n^{\rho}(g;x)-g(x)| \leq S_n^{\rho}\left(\left|\int_x^{\zeta} g'(u)du\right|;x\right). \tag{6.14}$$

Applying Hölder's inequality, we have

$$\left|\int_x^t g'(u)du\right| \leq \|\psi^\tau g'\| \left|\int_x^\zeta \frac{du}{\psi^\tau(u)}\right| \leq \|\psi^\tau g'\| |\zeta - x|^{1-\tau} \left|\int_x^\zeta \frac{du}{\psi(u)}\right|^\tau,$$

we may write

$$\left|\int_x^\zeta \frac{du}{\psi(u)}\right| \leq \left|\int_x^\zeta \frac{du}{\sqrt{u}}\right| \left(\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+\zeta}}\right).$$

Using the inequality $|\alpha+\beta|^r \leq |\alpha|^r + |\beta|^r$, $0 \leq r \leq 1$, we have

$$\begin{aligned}
\left|\int_x^\zeta g'(u)du\right| &\leq \frac{2^\tau \|\psi^\tau g'\| |\zeta - x|}{x^{\frac{\tau}{2}}} \left(\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+\zeta}}\right)^\tau \\
&\leq \frac{2^\tau \|\psi^\tau g'\| |\zeta - x|}{x^{\frac{\tau}{2}}} \left(\frac{1}{(1+x)^{\frac{\tau}{2}}} + \frac{1}{(1+\zeta)^{\frac{\tau}{2}}}\right). \qquad (6.15)
\end{aligned}$$

Thus, from the equation (6.14), (6.15), using Cauchy-Schwartz inequality, Theorem 6.3.1, and sufficiently large $n$, we obtain

$$\begin{aligned}
|S_n^\rho(g;x) - g(x)| &\leq \frac{2^\tau \|\psi^\tau g'\|}{x^{\frac{\tau}{2}}} S_n^\rho \left(|\zeta - x| \left(\frac{1}{(1+x)^{\frac{\tau}{2}}} + \frac{1}{(1+\zeta)^{\frac{\tau}{2}}}\right); x\right) \\
&\leq \frac{2^\tau \|\psi^\tau g'\|}{x^{\frac{\tau}{2}}} \left(\frac{1}{(1+x)^{\frac{\tau}{2}}} \sqrt{S_n^\rho((\zeta - x)^2; x)}\right. \\
&\quad \left. + \sqrt{S_n^\rho((\zeta - x); x)} \sqrt{S_n^\rho((1+\zeta)^{-\tau}; x)}\right) \\
&\leq 2^\tau \|\psi^\tau g'\| \sqrt{S_n^\rho((\zeta - x)^2; x)} \left\{\psi^{-\tau}(x) + x^{\frac{-\tau}{2}} \sqrt{S_n^\rho((1+\zeta)^{-\tau}; x)}\right\} \\
&\leq 2^\tau C \|\phi^\tau g'\| \frac{\phi(x)}{\sqrt{n}} \{\psi^{-\tau}(x) + x^{\frac{-\tau}{2}} (1+x)^{\frac{-\tau}{2}}\} \\
&\leq 2^{\tau+1} \frac{\|\psi^\tau g'\| \psi^{1-\tau}(x)}{\sqrt{n}}. \qquad (6.16)
\end{aligned}$$

Hence, from the equations (6.12 - 6.14), and (6.16), we get the required result

$$\begin{aligned}
|S_n^\rho(f;x) - f(x)| &\leq 2\|f - g\| + 2^{\tau+1} C \|\psi^\tau g'\| \frac{\psi^{1-\tau}}{\sqrt{n}} \\
&\leq C\left\{\|f - g\| + \frac{\psi^{1-\tau}}{\sqrt{n}} \|\psi^\tau g'\|\right\} \leq 2C K_{\psi^\tau}^* \left(f; \frac{\psi^{1-\tau}(x)}{\sqrt{n}}\right) \\
&\leq C\omega_{\psi^\tau}^* \left(f; \frac{\psi^{1-\tau}(x)}{\sqrt{n}}\right).
\end{aligned}$$

□

Here, we will discuss the results are given by Rasás and Steklov function and find approximation results using the second-order modulus of continuity. The modulus of continuity is defined as:

$$\omega_2(\phi,\delta) = \sup_{0<\zeta\leq\delta} \|\phi(.+2\zeta) - 2\phi(.+\zeta) + \phi(.)\|.$$

Now consider $\{L_n\}_{n\geq 0}$ be a sequence of linear positive operators with virtue $L_n(e_i;x) = x^i$. Then according to the Rasás result [34], we have

$$|L_n(g;x) - g(x)| \leq \|g'\|\sqrt{L_n((\zeta-x)^2;x)} + \frac{1}{2}\|g''\|L_n((\zeta-x)^2;x),$$

where $g \in C^2[0,a]$. And for $f \in C[a,b]$, the second-order Steklov function is as follows

$$f_h(x) = \frac{1}{h}\int_{-h}^{h}\left(1-\frac{|\zeta|}{h}\right) f(h;x+\zeta)d\zeta,\ x\in[a,b],$$

where $f(h;.) : [a-h,b+h] \to \mathbb{R}$, $h>0$ by

$$f_h(x) = \begin{cases} P_-(x); & a-h\leq x\leq a \\ f(x); & a\leq x\leq b \\ P_+(x); & b<x\leq b+h \end{cases}$$

and $P_-, P_+$ they are linear best approximation to the function $f$ on the given interval.

**Theorem 6.3.6.** *Let $\phi \in C[0,\infty)$. Then, we obtain*

$$|S_n^\rho(\phi;x) - \phi(x)| \leq \left(\frac{3}{2}+\frac{3a}{4}+\frac{3h^2}{4}\right)\omega_2(\phi;h) + \frac{2h^2}{a}\|\phi\|.$$

*Proof.* From some calculations and using well-known properties, we have

$$\begin{aligned} |S_n^\rho(\phi;x) - \phi(x)| &\leq S_n^\rho(|\phi-\phi_h|;x) + |S_n^\rho(\phi_h;x) - \phi_h(x)| + |\phi_h(x) - \phi(x)| \\ &\leq 2\|\phi-\phi_h\| + |S_n^\rho(\phi_h;x) - \phi_h(x)|, \end{aligned} \tag{6.17}$$

where, $\phi_h \in C^2[0,a)$ be the second-order Steklov function of $\phi$. From the Rasa's result and Landau inequality, we have

$$\begin{aligned} |S_n^\rho(\phi;x) - \phi(x)| &\leq \|\phi_h'\|\sqrt{S_n^\rho((\zeta-x)^2;x)} + \frac{1}{2}\|\phi_h''\|S_n^\rho\left((\zeta-x)^2;x\right) \\ &\leq \left(\frac{2}{a}\|\phi_h\| + \frac{a}{2}\|\phi_h''\|\right)\sqrt{S_n^\rho((\zeta-x)^2;x)} \\ &\quad + \frac{1}{2}\|\phi_h''\|S_n^\rho\left((\zeta-x)^2;x\right) \\ &\leq \left(\frac{2}{a}\|\phi\| + \frac{3a}{4h^2}\omega_2(\phi;h)\right)\sqrt{S_n^\rho((\zeta-x)^2;x)} \\ &\quad + \frac{3}{4h^2}\omega_2(\phi;h)S_n^\rho\left((\zeta-x)^2;x\right). \end{aligned} \tag{6.18}$$

Using the relation given by Zhuk [104], between Steklov function and $\omega_2(\phi;h)$ as: $\|\phi-\phi_h\| \leq \frac{3}{4}\omega_2(\phi;h)$, from the equations (6.17, 6.18), we obtain

$$\begin{aligned} |S_n^{\rho}(\phi;x)-\phi(x)| &\leq \frac{3}{2}\omega_2(\phi;h)+\left(\frac{2}{a}\|\phi\|+\frac{3a}{4h^2}\omega_2(\phi;h)\right)\sqrt{S_n^{\rho}((\zeta-x)^2;x)} \\ &\quad +\frac{3}{4h^2}\omega_2(\phi;h)S_n^{\rho}\left((\zeta-x)^2;x\right). \end{aligned}$$

Choose $h^2=\sqrt{S_n^{\rho}((\zeta-x)^2;x)}$ and, by the simple calculation, we get the required result. □

## 6.4 Weighted Approximation results

In this section, we study the approximation results on the space $C_\tau^*[0,\infty)$ using the weighted modulus of continuity. We know that in general the classical modulus of continuity of first order does not tend to zero on an infinite interval. Now, we develop an approximation theorem using the weighted space of the continuous function $C_\tau^*[0,\infty)$ of the given operators $S_n^{\rho}$.

**Theorem 6.4.1.** *(Weighted approximation) For $f\in C_\tau^*[0,\infty)$ and $\alpha>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|S_n^{\rho}(f;x)-f(x))|}{(\tau(x))^{1+\alpha}}=0.$$

*Proof.* Suppose that $x_0$ be an arbitrary fixed point, then

$$\begin{aligned} \sup_{x\in[0,\infty)}\frac{|S_n^{\rho}(f;x)-f(x))|}{(1+x^2)^{1+\alpha}} &\leq \sup_{x\leq x_0}\frac{|S_n^{\rho}(f;x)-f(x))|}{(1+x^2)^{1+\alpha}}+\sup_{x>x_0}\frac{|S_n^{\rho}(f;x)-f(x))|}{(1+x^2)^{1+\alpha}} \\ &\leq \|S_n^{\rho}(f;.)-f\|_{C[0,x_0]}+\|f\|_2\sup_{x>x_0}\frac{S_n^{\rho}(1+\zeta^2);x}{(1+x^2)^{1+\alpha}}+\sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}. \end{aligned} \tag{6.19}$$

Since $|f(x)|\leq\|f\|_2(1+x^2)$, we have

$$\sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}\leq\frac{\|f\|_2}{(1+x_0^2)^{\alpha}}.$$

Choose a number $\upsilon>0$ and $x_0$ to be large, then

$$\frac{\|f\|_2}{(1+x_0^2)^{\alpha}}<\frac{\upsilon}{6}\Rightarrow\sup_{x>x_0}\frac{\|f\|_2}{(1+x_0^2)^{\alpha}}<\frac{\upsilon}{6}. \tag{6.20}$$

Using Theorem 6.3.1, there exist $n_1\in\mathbb{N}$ such that

$$\begin{aligned} \|f\|_2\frac{S_n^{\rho}(1+\zeta^2;x)}{(1+x^2)^{1_\alpha}} &\leq \frac{\|f\|_2}{(1+x^2)^{\alpha}}\left(1+x^2+\frac{\upsilon}{3\|f\|_2}\right),\ \forall n>n_1 \\ &\leq \frac{\|f\|_2}{(1+x_0^2)^{\alpha}}+\frac{\upsilon}{3},\ \forall n>n_1, x>x_0. \end{aligned}$$

Hence,

$$\|f\|_2 \sup_{x>x_0} \frac{S_n^{\rho}(1+\zeta^2;x)}{(1+x^2)^{1+\alpha}} \quad \leq \quad \frac{\upsilon}{2},\ \forall n > n_1. \tag{6.21}$$

Applying Theorem 6.3.3, there exist $n_2 \in \mathbb{N}$ such that

$$\|S_n^{\rho}(f;.) - f\|_{c[0,x_0]} < \frac{\upsilon}{3},\ n > n_2. \tag{6.22}$$

Consider $n_0 = max(n_1, n_2)$. Thus combining (6.19 - 6.22), we obtained the result

$$\sup_{x\in[0,\infty)} \frac{|S_n^{\rho}(f;x) - f(x)|}{(1+x^2)^{1+\alpha}} < \upsilon,\ \forall n > n_0.$$

In the next theorem, we find the order of approximation for the weighted space corresponding to the proposed operators $S_n^{\rho}$. □

**Theorem 6.4.2.** *(Weighted approximation) Let $f \in C_{\tau}^*[0,\infty)$, and sufficiently large n, we have*

$$|S_n^{\rho}(f;x) - f(x)| \leq C(x)\Omega^*\left(f;\frac{1}{\sqrt{n}}\right). \tag{6.23}$$

*Proof.* For $x \in (0,\infty)$ and $\delta > 0$, using definition of $\Omega^*(f;\delta)$ and Lemma 1.1.2, we have

$$\begin{aligned} |f(\zeta) - f(x)| &\leq (1+(x+|x-t|)^2)\Omega^*(f;|t-x|) \\ &\leq 2(1+x^2)(1+(t-x)^2)\left(1+\frac{|\zeta-x|}{\delta}\right)\Omega^*(f;\delta). \end{aligned}$$

Applying $S_n^{\rho}(.;x)$ on both sides, we get

$$\begin{aligned} |S_n^{\rho}(f;x) - f(x)| &\leq 2(1+x^2)\Omega^*(f;\delta)\left\{1 + S_n^{\rho}((\zeta-x)^2;x)\right. \\ &\quad \left. + S_n^{\rho}\left((1+(\zeta-x)^2\frac{|\zeta-x|}{\delta};x)\right)\right\}. \end{aligned} \tag{6.24}$$

Applying the Cauchy-Schwarz inequality of the above equation, we get

$$\begin{aligned} S_n^{\rho}\left((1+(\zeta-x)^2)\frac{|\zeta-x|}{\delta};x\right) &\leq \frac{(S_n^{\rho}((\zeta-x)^2;x))^{\frac{1}{2}}}{\delta} \\ &\quad + \left\{\frac{1}{\delta}(S_n^{\rho}((\zeta-x)^4;x))^{\frac{1}{2}}\right. \\ &\quad \left. \times (S_n^{\rho}((\zeta-x)^2;x))^{\frac{1}{2}}\right\}, \end{aligned} \tag{6.25}$$

compile the equations (6.23 - 6.25) and, taking $\delta = \frac{1}{\sqrt{n}}$, we get the required result. □

## 6.5 Rate of convergence

In Approximation theory, a fascinating topic of study is the rate of convergence for functions with derivatives of bounded variation.

**Lemma 6.5.1.** *[79] Let $\theta = \theta(n) \to 0$, as $n \to \infty$ and, $\lim_{n\to\infty} n\theta(n) = l \in \mathbb{R}$. For adequately large n, we have*

*(i)* $\xi_n(x,\zeta) = \int_0^{\zeta} A_n(x,\zeta)du \leq \frac{C_1|\varpi(x)|}{(x-\zeta)^2}$

*(ii)* $1-\xi_n(x,\zeta) = \int_{\zeta}^{\infty} A_n(x,\zeta)du \leq \frac{C_1|\varpi(x)|}{(\zeta-x)^2}$,

*where* $x \in (0,\infty)$ *and,* $\varpi(x) = \frac{x(1+cx)}{\rho}$.

*Proof.* From the Lemma 6.2.2, we have

$$\begin{aligned}
\xi_n(x,\zeta) &= \int_0^{\zeta} A_n(x,\zeta)du \\
&\leq \int_0^{\zeta} \left(\frac{x-u}{x-t}\right)^2 A_n(x,\zeta)(x,u)du \\
&\leq \frac{1}{(x-t)^2} S_n^{\rho}((u-x)^2;x) \\
&\leq \frac{C_1|\varpi(x)|}{(x-\zeta)^2},
\end{aligned}$$

when $n$ is large. Similarly proof for $(ii)$. □

**Theorem 6.5.2.** *(Derivative of bounded variation) Let $f \in DBV[0,\infty)$, for every $x \in (0,\infty)$ and, sufficiently large n, we have*

$$\begin{aligned}
|S_n^{\rho}(f;x)-f(x)| \leq\ & \left\{ \frac{\{-2n\rho B(1)e^{-nx} + c[A(1)e^{nx}+B(1)e^{-nx}]\}}{(n\rho-c)[A(1)e^{nx}+B(1)e^{-nx}]} x \right. \\
&\left. + \frac{\rho[A'(1)e^{nx}+B'(1)e^{-nx}]}{(n\rho-c)[A(1)e^{nx}+B(1)e^{-nx}]} \right\} \left| \frac{f'(x+)+f'(x-)}{2} \right| \\
&+ \sqrt{C_1|\varpi(x)|} \left| \frac{f'(x+)-f'(x-)}{2} \right| + \frac{C_1|\varpi(x)|}{x} \sum_{i=1}^{[\sqrt{n}]} \left( \bigvee_{x-\frac{x}{i}}^{x} f'_x \right) \\
&+ \frac{x}{\sqrt{n}} \left( \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f'_x \right) + \left(4M_f + \frac{M_f+|f(x)|}{x^2}\right) C_1|\varpi(x)| \\
&+ |f'(x+)| \sqrt{C_1|\varpi(x)|} \\
&+ \frac{C_1|\varpi(x)|}{x^2} |f(2x)-f(x)-xf'(x+)| + \frac{x}{\sqrt{n}} \left( \bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x \right)
\end{aligned}$$

$$+\frac{C_1|\varpi(x)|}{x}\sum_{i=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}} f'_x\right).$$

*Here $C_1$ be a positive constant and $\bigvee_a^b f$ denotes the total variation of the function $f$ on $[a,b]$ and, $f'_x$ is defined as:*

$$f'_x(\zeta)=\begin{cases} f'(\zeta)-f'(x-),\ 0\leq\zeta<x,\\ 0, \qquad \zeta=x,\\ f'(\zeta)-f'(x+),\ x<\zeta<\infty. \end{cases} \tag{6.26}$$

*Proof.* Suppose for any function $f\in DBV[0,\infty)$, and using equation (6.26), we write as

$$\begin{aligned} f'(u) &= \frac{1}{2}(f'(x)+f'(x-))+f'_x(u)+\frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)\\ &\quad +\delta_x(u)\left(f'(u)-\frac{1}{2}(f'(x+)+f'(x-))\right), \end{aligned} \tag{6.27}$$

where

$$\delta_x(u)=\begin{cases} 1,\ u=x,\\ 0,\ u\neq x \end{cases}.$$

We have $S_n^{\rho}(1;x)=1$ and, using (6.27) for every $x\in(0,\infty)$, we obtain

$$\begin{aligned} S_n^{\rho}(f;x)-f(x) &= \int_0^{\infty}A_n(x,\zeta)(f(\zeta)-f(x))d\zeta\\ &= \int_0^{\infty}A_n(x,\zeta)\left(\int_x^{\zeta}f'(u)\right)d(\zeta)\\ &= -\int_0^{x}\left(\int_{\zeta}^{x}f'(u)du\right)A_n(x,\zeta)d\zeta\\ &\quad +\int_x^{\infty}\left(\int_x^{\zeta}f'(u)du\right)A_n(x,\zeta)d\zeta. \end{aligned} \tag{6.28}$$

Let us suppose that

$$L_1=\int_0^{x}\left(\int_t^{x}f'(u)du\right)A_n(x,\zeta)d\zeta,$$

$$L_2=\int_x^{\infty}\left(\int_x^{\zeta}f'(u)du\right)A_n(x,\zeta)d\zeta.$$

We know that $\int_x^{\zeta}\delta_x(u)du=0$ and, from (6.27), we have

$$L_1 = \int_0^{x}\left\{\int_{\zeta}^{x}\left(\frac{1}{2}(f'(x+)+f'(x-))+f'_x(u)+\frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)\right)du\right\}A_n(x,\zeta)d\zeta$$

$$
\begin{aligned}
&= \frac{1}{2}(f'(x+)+f'(x-))\int_0^x (x-\zeta)A_n(x,\zeta)d\zeta + \int_0^x \left(\int_\zeta^x f'_(x)(u)du\right) A_n(x,\zeta)d\zeta \\
&\quad -\frac{1}{2}(f'(x+)-f'(x-))\int_0^x (x-\zeta)A_n(x,\zeta)d\zeta. \qquad (6.29)
\end{aligned}
$$

Similarly, we find $L_2$

$$
\begin{aligned}
L_2 &= \int_x^\infty \left\{\int_x^\zeta \left(\frac{1}{2}(f'(x+)+f'(x-)) + f'_x(u) + \frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)\right) du\right\} A_n(x,\zeta)d\zeta \\
&= \frac{1}{2}(f'(x+)+f'(x-))\int_x^\infty (\zeta-x)A_n(x,\zeta)d\zeta + \int_x^\infty \left(\int_x^\zeta f'_x(u)du\right) A_n(x,\zeta)d\zeta \\
&\quad +\frac{1}{2}(f'(x+)-f'(x-))\int_x^\infty (\zeta-x)A_n(x,\zeta)d\zeta. \qquad (6.30)
\end{aligned}
$$

Combining the equations (6.28 - 6.29), we get

$$
\begin{aligned}
S_n^\rho(f;x)-f(x) &= \frac{1}{2}(f'(x+)+f'(x-))\int_0^\infty (\zeta-x)A_n(x,\zeta)d\zeta \\
&\quad +\frac{1}{2}(f'(x+)-f'(x-))\int_0^\infty |\zeta-x|A_n(x,t)d\zeta \\
&\quad -\int_0^x \left(\int_\zeta^x f'_x(u)du\right) A_n(x,\zeta)d\zeta + \int_x^\infty \left(\int_x^\zeta f'_x(u)du\right) A_n(x,\zeta)d\zeta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|S_n^\rho(f;x)-f(x)| &\leq \left|\frac{f'(x+)+f'x-}{2}\right| |S_n^\rho((\zeta-x);x)| \\
&\quad +\left|\frac{f'(x+)-f'(x-)}{2}\right| S_n^\rho(|\zeta-x|;x) \\
&\quad +\left|\int_0^x \left(\int_\zeta^x f'_x(u)du\right) A_n(x,\zeta)d\zeta\right| \\
&\quad +\left|\int_x^\infty \left(\int_x^\zeta f'_x(u)du\right) A_n(x,\zeta)d\zeta\right|. \qquad (6.31)
\end{aligned}
$$

Again, take it

$$
C_n(f'_x,x) = \int_0^x \left(\int_\zeta^x f'_x(u)du\right) A_n(x,\zeta)d\zeta
$$

and,

$$
D_n(f'_x,x) = \int_x^\infty \left(\int_x^\zeta f'_x(u)du\right) A_n(x,\zeta)d\zeta.
$$

Our aim is to calculate $C_n(f'_x,x), D_n(f'_x,x)$. From the definition $\xi_n(x,\zeta)$ and applying integration by parts, we get

$$
C_n(f'_x,x) = \int_0^x \left(\int_\zeta^x f'_x(u)du\right) \frac{\partial \xi_n(x,\zeta)}{\partial \zeta} d\zeta = \int_0^x f'_x(\zeta)\xi_n(x,\zeta)d\zeta.
$$

Thus,

$$\begin{aligned}|C_n(f'_x,x)| &= \int_0^x |f'_x(\zeta)|\xi_n(x,\zeta)d\zeta \\ &\leq \int_0^{x-\frac{x}{sqrtn}} |f'_x(\zeta)|\xi_n(x,\zeta)d\zeta + \int_{x-\frac{x}{\sqrt{n}}}^x |f'_x(\zeta)|\xi_n(x,\zeta)d\zeta.\end{aligned}$$

Since $f'_x(x)=0$ and $\xi_n(x,\zeta)\leq 1$, we get

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x |f'_x(\zeta)|\xi_n(x,\zeta)d\zeta &= \int_{x-\frac{x}{\sqrt{n}}}^x |f'_x(\zeta)-f'_x(x)|\xi_n(x,\zeta)d\zeta \leq \int_{x-\frac{x}{\sqrt{n}}}^x \left(\bigvee_t^x f'_x\right) d\zeta \\ &\leq \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right)\int_{x-\frac{x}{\sqrt{n}}}^x d\zeta = \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right).\end{aligned}$$

From the Lemma 6.5.1 and, take $\zeta = x-\frac{x}{u}$, we have

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(\zeta)|\xi_n(x,\zeta)d\zeta &\leq C_1|\varpi(x)|\int_0^{x-\frac{x}{\sqrt{n}}} \frac{|f'_x(\zeta)|}{(x-\zeta)^2}d\zeta \\ &\leq C_1|\varpi(x)|\int_0^{x-\frac{x}{\sqrt{n}}} \left(\bigvee_\zeta^x f'_x\right)\frac{d\zeta}{(x-\zeta)^2} \\ &= \frac{C_1|\varpi(x)|}{x}\int_1^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{u}}^x f'_x\right) \\ &\leq \frac{C_1|\varpi(x)|}{x}\sum_{i=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{i}}^x f'_x\right).\end{aligned}$$

Finally, we get

$$|C_n(f'_x,x)| = \frac{C_1|\varpi(x)|}{x}\sum_{i=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{i}}^x f'_x\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{i}}^x f'_x\right).$$

Similarly integration by parts on $D_n(f'_x,x)$ and using Lemma 6.5.1, we obtain

$$\begin{aligned}|D_n(f'_x,x)| &\leq \left|\int_x^{2x}\left(\int_x^\zeta f'_x(u)du\right)\frac{\partial}{\partial\zeta}(1-\xi_n(x,\zeta))d\zeta\right| \\ &\quad + \left|\int_{2x}^\infty\left(\int_x^\zeta f'_x(u)du\right)A_n(x,\zeta)d\zeta\right| \\ &\leq \left|\int_x^{2x} f'_x(u)du\right||1-\xi_n(x,2x)| + \int_x^{2x}|f'_x(\zeta)|(1-\xi_n(x,\zeta))d\zeta \\ &\quad + \left|\int_{2x}^\infty (f(\zeta)-f(x))A_n(x,\zeta)d\zeta\right| + |f'(x+)|\left|\int_{2x}^\infty(\zeta-x)A_n(x,\zeta)d\zeta\right|.\end{aligned}$$

Also, we have

$$\begin{aligned}\int_x^{2x}|f'_x(\zeta)|(1-\xi_n(x,\zeta))d\zeta &= \int_x^{x+\frac{x}{\sqrt{n}}}|f'_x(\zeta)|(1-\xi_n(x,\zeta))d\zeta\\ &\quad +\int_{x+\frac{x}{\sqrt{n}}}^{2x}|f'_x(\zeta)|(1-\xi_n(x,\zeta))d\zeta\\ &\quad J_1+J_2. \end{aligned} \tag{6.32}$$

Since $f'_x(x)=0$ and, $1-\xi_n(x,\zeta)\leq 1$, we have

$$\begin{aligned} J_1 &= \int_x^{x+\frac{x}{\sqrt{n}}}|f'_x(\zeta)-f'_x(x)|(1-\xi_n(x,\zeta))d\zeta\\ &\leq \int_x^{x+\frac{x}{\sqrt{n}}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x\right)d\zeta\\ &= \frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x\right). \end{aligned}$$

From the Lemma 6.5.1 and, assuming $\zeta=x+\frac{x}{u}$, we get

$$\begin{aligned} J_2 &\leq C_1|\varpi(x)|\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(\zeta-x)^2}|f'_x(\zeta)-f'_x(x)|d\zeta\\ &\leq C_1|\varpi(x)|\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(\zeta-x)^2}\left(\bigvee_x^{x+\frac{x}{u}} f'_x\right)du\\ &= \frac{C_1|\varpi(x)|}{x}\int_1^{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{u}} f'_x\right)du \leq \frac{C_1|\varpi(x)|}{x}\sum_{i=1}^{[\sqrt{n}]}\int_i^{i+1}\left(\bigvee_x^{x+\frac{x}{u}} f'_x\right)du\\ &\leq \frac{C_1|\varpi(x)|}{x}\sum_{i=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{i}} f'_x\right). \end{aligned}$$

Substitute the values of $J_1$ and, $J_2$ in (6.32), we have

$$\int_x^{2x}|f'_x(\zeta)|(1-\xi_n(x,\zeta))d\zeta \leq \frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x\right)+\frac{C_1|\varpi(x)|}{x}\sum_{i=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{u}} f'_x\right).$$

Applying the Cauchy-Schwarz inequality and Lemma 6.5.1, we get

$$\begin{aligned} |D_n(f'_x,x)| &\leq M_f\int_{2x}^{\infty}(1+\zeta^2)A_n(x,\zeta)d\zeta+|f(x)|\int_{2x}^{\infty}A_n(x,\zeta)d\zeta\\ &\quad +|f'(x+)|\sqrt{C_1|\varpi(x)|}+\frac{C_1|\varpi(x)|}{x^2}|f(2x)-f(x)-xf'(x+)|\\ &\quad +\frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x\right)+\frac{C_1|\varpi(x)|}{x}\sum_{i=1}^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{i}} f'_x\right). \end{aligned} \tag{6.33}$$

Since $\zeta \leq 2(\zeta - x)$ and, $x \leq \zeta - x$ when $\zeta \geq 2x$, we have

$$
\begin{aligned}
& M_f \int_{2x}^{\infty} (1+\zeta^2) A_n(x,\zeta) d\zeta + |f(x)| \int_{2x}^{\infty} A_n(x,\zeta) d\zeta \\
& \leq (M_f + |f(x)|) \int_{2x}^{\infty} A_n(x,\zeta) d\zeta + 4M_f \int_{2x}^{\infty} (\zeta - x)^2 A_n(x,\zeta) d\zeta \\
& \leq \frac{M_f + |f(x)|}{x^2} \int_0^{\infty} (\zeta - x)^2 A_n(x,\zeta) d\zeta + 4M_f \int_0^{\infty} (\zeta - x)^2 A_n(x,t) d\zeta \\
& \leq \left( 4M_f + \frac{M_f + |f(x)|}{x^2} \right) C_1 |\varpi(x)|. \qquad (6.34)
\end{aligned}
$$

Using the above inequality, we have

$$
\begin{aligned}
|D_n(f'_x, x)| \quad \leq \quad & \left( 4M_f + \frac{M_f + |f(x)|}{x^2} \right) C_1 |\varpi(x)| + |f'(x+)| \sqrt{C_1 |\varpi(x)|} \\
& + C_1 \frac{1+x^2}{nx^2} |f(2x) - f(x) - xf'(x+)| \\
& + \frac{x}{\sqrt{n}} \left( \bigvee_x^{x+\frac{x}{\sqrt{n}}} f'_x \right) + \frac{C_1 |\varpi(x)|}{x} \sum_{i=1}^{[\sqrt{n}]} \left( \bigvee_x^{x+\frac{x}{i}} f'_x \right). \qquad (6.35)
\end{aligned}
$$

We obtain, the required result using (6.31, 6.33) and, (6.35). □

# Chapter 7

# Conclusions and Future Prospects

In this chapter, findings of studies carried out in this thesis along with significant topics allied to the new aspects of analysis that identify present and future potential research aspects has been presented.

## 7.1 Conclusion

Approximation process is the key to identifying solutions in an analysis that are closest to the exact solutions. This thesis is focused principally on certain approximation operators and analysing the convergence outcomes. Chapter **1** has covered the literature survey, definitions, tools, and historical background of some approximation operators.

Many researchers studied Appell polynomials and established the approximation outcomes. Chapter **2** is based on the Apostol-Genocchi polynomials. We have proposed a new sequence of operators and studied the rate of convergence using multiple tools. Additionally, we have incorporated Kantorovich variant of the given operators to look at the properties of approximation. We show the integral modification and discuss few approximation properties of the proposed operators.

In chapter **3**, we have derived the Bézier variant of Bernstein-Durrmeyer type operators and determined rate of approximation in terms of Ditzian-Totik modulus of continuity and Lipschitz-type space. Also, we have found the derivative of bounded variation and checked the smoothness of functions. In the last, we show the graphical depiction of these operators.

The **fourth chapter** presents Cheney-Sharma Chlodovsky Durrmeyer operators and discusses the convergence via Lipschitz-type space, modulus of continuity, and weighted approximation theorem. We have also studied the A-statistical convergence of the given operators.

In the chapter **5**, first section deals with the inspiration of Mache and Zhou's [66] work on Bernstein polynomials over a different set of operators. We have looked into certain properties of these operators such as the rate of convergence, Lipschitz-type space, Ditzian-Totik modulus of smoothness, Voronovskaya theorem, Grüss-Voronovskaya theorem, and weighted approximation results. We have also established the convergence of above operators graphically. In the second section, we have discussed the Durrmeyer variant of modified Bernstein polynomials. Some approximation properties of the given operators and the convergence behavior has been investigated graphically.

In the chapter **6**, we have considered an Appell polynomial of class $A^2$ and introduced a new generalization of Szász-păltănea operators. Approximation study in terms of modulus of continuity and rate of convergence in the weighted space of given operators also has been done. Moreover, we have obtained the rate of convergence for functions whose derivatives are of bounded variation.

## 7.2 Future plans for academics

Many academics studied the convergence results as they worked on the linear operators. The max-product approach was generalised by Barnabás et al. [13] implementing well-known operators such as Bernstein, Szász, Baskakov, and Picard. Sorin G. Gal [33] worked on non-linear maximum product operators after taking on this concept. In this study, Sorin obtained max-product nonlinear (sublinear) operators by replacing the $\sum$ operators with the max-product operators $\bigvee$. In future, I am inclined to work in this direction on the modified operators to give a higher quality order of approximation with the help of classical and weighted modulus of continuity.

One branch of mathematical analysis known as Summability theory deals with the study of series and sequences, in particular, that might not converse in the routine sense. It provides techniques to connect divergent series with particular limits or average limits to give them meaningful values. This theory in terms of power series was developed by Borwein in 1957. A strong framework for performing uncertainty and inaccuracy in a variety of mathematical fields is provided by fuzzy mathematics. It is used in fields like control systems, optimization, and decision-making.

Recently, Digvijay and Karunesh [80] extended their work in this area. They studied fuzzy operators based on Q-Bernstein-Chlodowsky-Durrmeyer operators in two variables over the space of all fuzzy numbers and also verified the Korovkin theorem. I intend to study this hypothesis and extend it to other operators to find the convergence outcomes.

The exponential-type operators are an important class of operators in approximation theory. A wide range of research has been done, and many of them used the exponential and its generalised approach to study convergence. Recently, Gupta [40], has purely turned out new studies on Laguerre polynomials. He addressed the direct convergence results and presented the composition via the Rathore, Szász-Durrmeyer, Post-Widder, and semi-Post-Widder operators. Since these novel exponential operators are still not extensively explored, I intend to look into them and analyse the approximation results in terms of modulus of continuity.

# References


[1]. Abel, U., and Gupta, V. An estimate of the rate of convergence of a Bézier variant of the Baskakov-Kantorovich operators for bounded variation functions. Demonstratio Math. 36, 1 (2003), 123–136.

[2]. Abel, U., Gupta, V., and Ivan, M. Asymptotic approximation of functions and their derivatives by generalized Baskakov-Százs-Durrmeyer operators. Anal. Theory Appl. 21, 1 (2005), 15–26.

[3]. Abel, U., Gupta, V., and Mohapatra, R. N. Local approximation by a variant of Bernstein-Durrmeyer operators. Nonlinear Anal. 68, 11 (2008), 3372–3381.

[4]. Abel, U., and Heilmann, M. The complete asymptotic expansion for Bernstein-Durrmeyer operators with Jacobi weights. Mediterr. J. Math. 1, 4 (2004), 487–499.

[5]. Abel, U., Ivan, M., and Păltănea, R. The Durrmeyer variant of an operator defined by D. D. Stancu. Appl. Math. Comput. 259 (2015), 116–123.

[6]. Acar, T., Agrawal, P. N., and Neer, T. Bezier variant of the Bernstein-Durrmeyer type operators. Results Math. 72, 3 (2017), 1341–1358.

[7]. Acar, T., and Kajla, A. Blending type approximation by Bézier-summation-integral type operators. Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. 67, 2 (2018), 195–208.

[8]. Acu, A. M., Gonska, H., and Raşa, I. Grüss-type and Ostrowski-type inequalities in approximation theory. Ukrainian Math. J. 63, 6 (2011), 843–864.

[9]. Altomare, F., and Campiti, M. Korovkin-type approximation theory and its applications, vol. 17 of De Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 1994. Appendix A by Michael Pannenberg and Appendix B by Ferdinand Beckhoff.

[10]. Appell, P. Sur une classe de polynômes. Ann. Sci. École Norm. Sup. (2) 9 (1880), 119–144.

[11]. Araci, S. Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus. Appl. Math. Comput. 233 (2014), 599–607.

[12]. Baskakov, V. A. An instance of a sequence of linear positive operators in the space of continuous functions. In Doklady Akademii Nauk (1957), vol. 113, Russian Academy of Sciences, pp. 249–251.

[13]. Bede, B., Coroianu, L., and Gal, S. G. Approximation by max-product type operators. Springer, 2016.

[14]. Bernštein, S. Démonstration du théoreme de weierstrass fondée sur le calcul des probabilities. Comm. Soc. Math. Kharkov 13 (1912), 1–2.

[15]. Chen, X., Tan, J., Liu, Z., and Xie, J. Approximation of functions by a new family of generalized Bernstein operators. J. Math. Anal. Appl. 450, 1 (2017), 244–261.

[16]. Cheney, E. W., and Sharma, A. On a generalization of Bernstein polynomials. Riv. Mat. Univ. Parma (2) 5 (1964), 77–84.

[17]. Chlodovsky, I. Sur le développement des fonctions définies dans un intervalle infini en séries de polynomes de M. S. Bernstein. Compositio Math. 4 (1937), 380–393.

[18]. Ciupa, A. On a generalized Favard-Szasz type operator. In Research Seminar on Numerical and Statistical Calculus, vol. 94 of Preprint. "Babeş-Bolyai" Univ., Cluj-Napoca, 1994, pp. 33–38.

[19]. Ciupa, A. On the approximation by Favard-Szasz type operators. Rev. Anal. Numér. Théor. Approx. 25, 1-2 (1996), 57–61.

[20]. Claesson, A., Kitaev, S., Ragnarsson, K., and Tenner, B. E. Boolean complexes for ferrers graphs. arXiv preprint arXiv:0808.2307 (2008).

[21]. Deo, N., and Dhamija, M. Generalized positive linear operators based on PED and IPED. Iran. J. Sci. Technol. Trans. A Sci. 43, 2 (2019), 507–513.

[22]. Deo, N., Dhamija, M., and Miclăuş, D. Stancu-Kantorovich operators based on inverse Pólya-Eggenberger distribution. Appl. Math. Comput. 273 (2016), 281–289.

[23]. Deo, N., and Pratap, R. $\alpha$-Bernstein-Kantorovich operators. Afr. Mat. 31, 3-4 (2020), 609–618.

[24]. DeVore, R. A., and Lorentz, G. G. Constructive approximation, vol. 303 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1993.

[25]. Ditzian, Z., and Totik, V. Moduli of smoothness, vol. 9 of Springer Series in Computational Mathematics. Springer-Verlag, New York, 1987.

[26]. Duman, O., and Orhan, C. Statistical approximation by positive linear operators. Studia Math. 161, 2 (2004), 187–197.

[27]. Durrmeyer, J. L. Une formule d'inversion de la transformée de Laplace: Applications à la théorie des moments. PhD thesis, 1967.

[28]. Fast, H. Sur la convergence statistique. Colloq. Math. 2 (1951), 241–244 (1952).

[29]. Finta, Z. Remark on Voronovskaja theorem for q-Bernstein operators. Stud. Univ. Babeş-Bolyai Math. 56, 2 (2011), 335–339.

[30]. Fridy, J. A. On statistical convergence. Analysis 5, 4 (1985), 301–313.

[31]. Gadjiev, A. D., and Orhan, C. Some approximation theorems via statistical convergence. Rocky Mountain J. Math. 32, 1 (2002), 129–138.

[32]. Gadžiev, A. D. Theorems of the type of P. P. Korovkin's theorems. Mat. Zametki 20, 5 (1976), 781–786.

[33]. Gal, S. G. Approximation by max-product type nonlinear operators. Stud. Univ. Babeş-Bolyai Math. 56, 2 (2011), 341–352.

[34]. Gavrea, I., and Raşa, I. Remarks on some quantitative Korovkin-type results. Rev. Anal. Numér. Théor. Approx. 22, 2 (1993), 173–176.

[35]. Geng-Zhe, C. Generalized bernstein-bézier polynomials. Journal of Computational Mathematics 1, 4 (1983), 322–327.

[36]. Gonska, H., and Tachev, G. Grüss-type inequalities for positive linear operators with second order moduli. Mat. Vesnik 63, 4 (2011), 247–252.

[37]. Govil, N. K., Gupta, V., and Noor, M. A. Simultaneous approximation for the Phillips operators. Int. J. Math. Math. Sci. (2006), Art. ID 49094, 9.

[38]. Gupta, V. A note on the rate of convergence of Durrmeyer type operators for function of bounded variation. Soochow J. Math. 23, 1 (1997), 115–118.

[39]. Gupta, V. An estimate on the convergence of Baskakov-Bézier operators. J. Math. Anal. Appl. 312, 1 (2005), 280–288.

[40]. Gupta, V. New operators based on laguerre polynomials. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 118, 1 (2024), 19.

[41]. Gupta, V., Acu, A. M., and Srivastava, H. M. Difference of some positive linear approximation operators for higher-order derivatives. Symmetry 12, 6 (2020), 915.

[42]. Gupta, V., and Agarwal, R. P. Convergence estimates in approximation theory. Springer, Cham, 2014.

[43]. Gupta, V., and Heping, W. The rate of convergence of $q$-Durrmeyer operators for $0 < q < 1$. Math. Methods Appl. Sci. 31, 16 (2008), 1946–1955.

[44]. Gupta, V., and López-Moreno, A.-J. Phillips operators preserving arbitrary exponential functions, $e^{at}$, $e^{bt}$. Filomat 32, 14 (2018), 5071–5082.

[45]. Gupta, V., Mohapatra, R. N., and Finta, Z. A certain family of mixed summation-integral type operators. Math. Comput. Modelling 42, 1-2 (2005), 181–191.

[46]. Gupta, V., and Rassias, M. T. Moments of linear positive operators and approximation. SpringerBriefs in Mathematics. Springer, Cham, 2019.

[47]. H. M. Srivastava, Gürhan İçöz, B. c. Approximation properties of an extended family of the szÃąszâĂŞmirakjan beta-type operators. Axioms 8 (2019), 1–13.

[48]. H. M. Srivastava, Faruk özger, S. A. M. Construction of stancu-type bernstein operators based on bÃľzier bases with shape parameter Îż. Symmetry 11 (2019), 1–22.

[49]. He, Y., Araci, S., Srivastava, H. M., and Abdel-Aty, M. Higher-order convolutions for apostol-bernoulli, apostol-euler and apostol-genocchi polynomials. Mathematics 6, 12 (2018), 329.

[50]. İbikli, E., and Gadjieva, E. A. The order of approximation of some unbounded functions by the sequences of positive linear operators. Turkish J. Math. 19, 3 (1995), 331–337.

[51]. Ismail, M. E.-H. On a generalization of Szász operators. Mathematica (Cluj) 16(39), 2 (1974), 259–267 (1977).

[52]. Ispir, N., and Yuksel, I. On the Bezier variant of Srivastava-Gupta operators. Appl. Math. E-Notes 5 (2005), 129–137.

[53]. Jakimovski, A., and Leviatan, D. Generalized Szász operators for the approximation in the infinite interval. Mathematica (Cluj) 11(34) (1969), 97–103.

[54]. Jolany, H., Sharifi, H., and Alikelaye, R. E. Some results for the Apostol-Genocchi polynomials of higher order. Bull. Malays. Math. Sci. Soc. (2) 36, 2 (2013), 465–479.

[55]. Jung, H. S., Deo, N., and Dhamija, M. Pointwise approximation by Bernstein type operators in mobile interval. Appl. Math. Comput. 244 (2014), 683–694.

[56]. Kajla, A. On the Bézier variant of the Srivastava-Gupta operators. Constr. Math. Anal. 1, 2 (2018), 99–107.

[57]. Kajla, A., and Acar, T. Blending type approximation by generalized Bernstein-Durrmeyer type operators. Miskolc Math. Notes 19, 1 (2018), 319–336.

[58]. Kajla, A., and Goyal, M. Blending type approximation by Bernstein-Durrmeyer type operators. Mat. Vesnik 70, 1 (2018), 40–54.

[59]. Kajla, A., and Goyal, M. Generalized Bernstein-Durrmeyer operators of blending type. Afr. Mat. 30, 7-8 (2019), 1103–1118.

[60]. Kantorovich, L. Sur la convergence de la suite des polynoÌĆmes de s. bernstein en dehors de l'intervalle fondamental. Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, 8 (1931), 1103–1115.

[61]. Kolk, E. Matrix summability of statistically convergent sequences. Analysis 13, 1-2 (1993), 77–83.

[62]. Korovkin, P. P. On convergence of linear positive operators in the space of continuous functions. Doklady Akad. Nauk SSSR (N.S.) 90 (1953), 961–964.

[63]. Luo, Q.-M. q-extensions for the apostol-genocchi polynomials. Gen. Math 17, 2 (2009), 113–125.

[64]. Luo, Q.-M. Extensions of the Genocchi polynomials and their Fourier expansions and integral representations. Osaka J. Math. 48, 2 (2011), 291–309.

[65]. Luo, Q.-M., and Srivastava, H. M. Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind. Appl. Math. Comput. 217, 12 (2011), 5702–5728.

[66]. Mache, D. H., and Zhou, D. X. Characterization theorems for the approximation by a family of operators. J. Approx. Theory 84, 2 (1996), 145–161.

[67]. May, C. P. On Phillips operator. J. Approximation Theory 20, 4 (1977), 315–332.

[68]. Mirakjan, G. M. Approximation of continuous functions with the aid of polynomilas. Dokl. Akad. Nauk. 31 (1941), 201–205.

[69]. Mursaleen, M., Rahman, S., and Ansari, K. J. Approximation by Jakimovski-Leviatan-Stancu-Durrmeyer type operators. Filomat 33, 6 (2019), 1517–1530.

[70]. Natanson, I. P. Constructive function theory. Vol. I. Uniform approximation. Frederick Ungar Publishing Co., New York, 1964. Translated from the Russian by Alexis N. Obolensky.

[71]. Özarslan, M. A., and Aktuğlu, H. Local approximation properties for certain King type operators. Filomat 27, 1 (2013), 173–181.

[72]. Ozden, H., and Simsek, Y. Modification and unification of the Apostol-type numbers and polynomials and their applications. Appl. Math. Comput. 235 (2014), 338–351.

[73]. Phillips, R. S. An inversion formula for Laplace transforms and semi-groups of linear operators. Ann. of Math. (2) 59 (1954), 325–356.

[74]. Pratap, R., and Deo, N. Approximation by genuine Gupta-Srivastava operators. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 113, 3 (2019), 2495–2505.

[75]. Pratap, R., and Deo, N. Rate of convergence of Gupta-Srivastava operators based on certain parameters. J. Class. Anal. 14, 2 (2019), 137–153.

[76]. Păltănea, R. Sur un opérateur polynomial défini sur l'ensemble des fonctions intégrables. In Itinerant seminar on functional equations, approximation and convexity (Cluj-Napoca, 1983), vol. 83-2 of Preprint. Univ. "Babeş-Bolyai", Cluj-Napoca, 1983, pp. 101–106.

[77]. Păltănea, R. Durrmeyer type operators on a simplex. Constr. Math. Anal. 4, 2 (2021), 215–228.

[78]. Remmel, J. Ascent sequences, 2+ 2-free posets, upper triangular matrices, and genocchi numbers. In Workshop on Combinatorics, Enumeration, and Invariant Theory, George Mason University, Virginia (2010).

[79]. Sidharth, M., Agrawal, P. N., and Araci, S. Szász-Durrmeyer operators involving Boas-Buck polynomials of blending type. J. Inequal. Appl. (2017), Paper No. 122, 20.

[80]. Singh, D., and Singh, K. K. Fuzzy approximation theorems via power series summability methods in two variables. Soft Computing (2023), 1–9.

[81]. Singh, K. K., and Agrawal, P. N. On szász-durrmeyer type modification using gould hopper polynomials. Mathematical Foundations of Computing 6, 2 (2023), 123–135.

[82]. Söylemez, D., and Taşdelen, F. Approximation by Cheney-Sharma Chlodovsky operators. Hacet. J. Math. Stat. 49, 2 (2020), 510–522.

[83]. Söylemez, D., and Ünver, M. Korovkin type theorems for Cheney-Sharma operators via summability methods. Results Math. 72, 3 (2017), 1601–1612.

[84]. Srivastava, H. M., Finta, Z., and Gupta, V. Direct results for a certain family of summation–integral type operators. Applied mathematics and computation 190, 1 (2007), 449–457.

[85]. Srivastava, H. M., and Gupta, V. Rate of convergence for the Bézier variant of the Bleimann-Butzer-Hahn operators. Appl. Math. Lett. 18, 8 (2005), 849–857.

[86]. Srivastava, H. M., Masjed-Jamei, M., and Beyki, M. R. A parametric type of the apostol-bernoulli, apostol-euler and apostol-genocchi polynomials. Appl. Math. Inf. Sci 12, 5 (2018), 907–916.

[87]. Srivastava, H. M., Masjed-Jamei, M., and Beyki, M. R. Some new generalizations and applications of the apostol-bernoulli, apostol-euler and apostol-genocchi polynomials. Rocky Mountain Journal of Mathematics 49, 2 (2019), 681–697.

[88]. Srivastava, H. M., Özarslan, M. A., and Kaanoğlu, C. Some generalized Lagrange-based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Russ. J. Math. Phys. 20, 1 (2013), 110–120.

[89]. Srivastava, H. M., and Zeng, X.-M. Approximation by means of the Szász-Bézier integral operators. Int. J. Pure Appl. Math. 14, 3 (2004), 283–294.

[90]. Stancu, D. D. Approximation of functions by a new class of linear polynomial operators. Rev. Roumaine Math. Pures Appl. 13 (1968), 1173–1194.

[91]. Stancu, D. D. The remainder in the approximation by a generalized Bernstein operator: a representation by a convex combination of second-order divided differences. Calcolo 35, 1 (1998), 53–62.

[92]. Stark, E. L. Bernstein-Polynome, 1912–1955. In Functional analysis and approximation (Oberwolfach, 1980), vol. 60 of Internat. Ser. Numer. Math. Birkhäuser, Basel-Boston, Mass., 1981, pp. 443–461.

[93]. Stein, J. Table errata: ***Handbook of mathematical functions with formulas, graphs, and mathematical tables*** (Nat. Bur.approximation by integral form of modified bernstein polynomials; International Conference on Analysis and Its Applications (icaa-2023), department of mathematics, shivaji college, university of delhi, delhi. february 27–28, 2023 Standards, Washington, D.C., 1964) edited by Milton Abramowitz and Irene A. Stegun. Math. Comp. 24, 110 (1970), 503.

[94]. Steinhaus, H. Quality control by sampling (a plea for Bayes' rule). Colloq. Math. 2 (1951), 98–108.

[95]. Sucu, S., and Varma, S. Generalization of Jakimovski-Leviatan type Szasz operators. Appl. Math. Comput. 270 (2015), 977–983.

[96]. Szász, O. Generalization of S. Bernstein's polynomials to the infinite interval. J. Research Nat. Bur. Standards (1950), 239–245.

[97]. Usta, F. On new modification of Bernstein operators: theory and applications. Iran. J. Sci. Technol. Trans. A Sci. 44, 4 (2020), 1119–1124.

[98]. Varma, S., and Sucu, S. A generalization of szász operators by using the appell polynomials of class a (2). Symmetry 14, 7 (2022), 1410.

[99]. Verma, D. K., Gupta, V., and Agrawal, P. N. Some approximation properties of Baskakov-Durrmeyer-Stancu operators. Appl. Math. Comput. 218, 11 (2012), 6549–6556.

[100]. Wang, P., and Zhou, Y. A new estimate on the rate of convergence of Durrmeyer-Bézier operators. J. Inequal. Appl. (2009), Art. ID 702680, 7.

[101]. Wood, B. Generalized Szász operators for the approximation in the complex domain. SIAM J. Appl. Math. 17 (1969), 790–801.

[102]. Zeng, X., and Piriou, A. On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions. J. Approx. Theory 95, 3 (1998), 369–387.

[103]. Zeng, X.-M. On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions. II. J. Approx. Theory 104, 2 (2000), 330–344.

[104]. Zhuk, V. V. Functions of the $\mathrm{Lip}\,1$ class and S. N. Bernstein's polynomials. Vestnik Leningrad. Univ. Mat. Mekh. Astronom., vyp. 1 (1989), 25–30, 122–123.