L^p-Spaces in Mathematical analysis

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CONDIDATE'S DECLARATION

We, Anshul Dhariwal(2K21/MSCMAT/08) and Ankush Singh(2K21/MSCMAT/03) of M.Sc.(Mathematics), hereby declare that the project Dissertation titled "L^p-Spaces in Mathematical Analysis" which is submitted by us to the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the Degree Master's of Mathematics, is original and not copied from any source without project citation. This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship or other similar title or recognition.

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CERTIFICATE

I hereby certify that the project dissertation titled "L^p-Spaces in Mathematical Analysis" which is submitted by Anshul Dhariwal(2K21/MSCMAT/08) and Ankush Singh(2K21/MSCMAT/03) to the Department of Applied Mathematics , Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the Degree of Master's of Mathematics, is record of the project word carried out by the students under my supervision. To the best of my knowledge this work has not been submitted in part or full for any other Degree or Diploma of this university or elsewhere.

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ABSTRACT

In functional analysis, a great deal of time is spent with normed linear space, banach space, inner product space, many other spaces like Hilbert space, L^p-space etc.L^p-space is a great field in functional analysis. L^p-space play a central role in many questions in analysis.

Here, we'll focus on the fundamental structural information about the L^p-space. This more abstract viewpoint also has the unanticipated benefit of guiding us to the unexpected finding of a finitely additive measure on all subsets that is consistent with Lebesgue measure.

Here we will be familiar with measurability,measure space, measurable functions and Borel set etc. Lebesgue integral, integration of complex numbers, Lebesgue dominated convergence theorem, dual of L^p-space, two norm convergence in L^p-space etc. also define here.

Contents

CHAPTER 1

Introduction

The L^p-spaces are function spaces that are defined by naturally extending the p-norm to vector spaces of finite dimensions. In honour of Henri Lebesgue, they are sometimes referred to as Lebesgue spaces.

L^p-spaces a crucial class of banach spaces in topological vector spaces and functional analysis. because of their crucial function in the analysis of measure and probability spaces via mathematics. Theoretical discussions of issues in physics, statistics, economics, finance, engineering, and other fields also use Lebesgue spaces.

p-norm in finite dimension is when, $p \geq 1$ for real numbers, the p-norm or L^p-norm of X is defined by,

$$||x||_p = (|x_1|^p + |x_2|^p + ... + |x_n|^P)^{\frac{1}{p}}$$

The norm that corresponds to the rectilinear distance is the 1-norm, while the euclidean norm from above belongs to this class and is the 2-norm.

The maximum norm or l^∞-norm or uniform norm is the limit of L^p-norms for $p \to \infty$.

$$||x||_\infty = max\{|x_1|, |x_2|, ...|x_n|\}$$

.

If the p-norm can be extended to vectors with infinitely many components, the resulting space is called l^p in infinite dimensions is known as the p-norm.

These serve as unique cases:-

a) The group of series with absolute convergence is called l^1.

b) The area of summable square sequences, which are a type of Hilbert space, is l^2.

c) The space of bounded sequences is called l^∞.

General l^p-space is define as the space $l^p(I)$ over a general index set I and $1 \leq p < \infty$ as

$$l^p(I) = \{(x_i)_{i \in I} \in K^{'} : \Sigma_{i \in I} |x_i|^p < +\infty\}$$

with the custom(norm),

$$||x||_p = (\Sigma_{i \in I} |x_i|^p) \frac{1}{p}$$

The region $l^p(I)$ turns into a banach space. This construction results in Rn with the previously mentioned p-norm in the case where I is finite with n elements.

This is the same sequence space l^p described above if I is countably infinite. This non-separable Banach space for uncountable sets I can be thought of as the locally convex direct limit of the l^p-sequence space. Giving the discrete σ-algebra and counting measure to the index set I transforms it into a measure space. Therefore, the space $l^p(I)$ is just a particular instance of the more general L^p-space.

1.1 A normed linear space

A mapping is a norm on a vector space $||.|| : V \to R$ satisfying the following property

- $||x|| \geq 0$
- $||x|| = 0$, if and only if $x = 0$
- $||x + y|| \leq ||x|| + ||y||$
- $||\alpha x|| = |\alpha|.||x||$

1.2 Banach Space

The term "Banach Space" refers to a normed linear space that is complete as a metric space.

1.3 Inner Product Space

Let X be a field of complex numbers in a linear space. If for every pair $(x, y) \in X \times Y$ there corresponds a scalar denoted by $< x, y >$ called inner product of x and y of X such that the following properties hold:-

(IP.1) $< x, y >= \overline{< y, x >}$ where $(x, y) \in X \times X$ and $\overline{< y, x >}$ denotes the conjugate of the complex numbers.

(IP.2) $< \alpha x, y >= \alpha < x, y >$, $\forall\ \alpha \in C$ and $(x, y) \in X \times X$.

(IP.3) $< x + y, z >=< x, z > + < y, z >$, for all $x, y, z \in X$.

(IP.4) $< x, x > \geq 0$ and $< x, x > = 0$ iff $x = \theta$.

Then $(X, < . >)$ is referred to as a pre Hilbert space or an inner product space

1.4 The Hilbert Space

A Hilbert space is a complete inner product space X with respect to a metric $d : X \times X \rightarrow R$ the inner product-induced $<, >$ on $X \times X$, i.e. $d(x, y) =< x - y, x - y >^{1/2}$ for all $x, y \in X$.

1.5 Concept of Measurability

The class of measurable functions is too useful in integration theory. Ut follows some basic definitions which has some basic properties that is defined as below:-

1.5.1 Topology, Topological Space and Open set

If a collection S, which is a subset of X, possesses the following three characteristics, then S is said to be a topology in X :-

a) $\phi \in S$ and $X \in S$

b) If $S_i \in S$ for i = 1,2,3,.....,n then $S_1 \cap S_2 \cap ..... \cap S_n \in S$

c) If S_α is a random grouping of elements from S(finite, countable, or uncountable), then $\cup_\alpha S_\alpha \in S$

Here,

S = topology in X

X = topological space and written as (X, S)

S_i (that is member of S) = open set in X

CONTINUOUS FUNCTION:-

If a mapping is defined as $f : X \to Y$ and X and Y are two topological spaces, then f is said to be a continuous function if $f^{-1}(S)$ is an open set in X for all open sets $S \in Y$.

1.5.2 σ-Algebra, Measurable Space and Measurable set

If a collection r of a set X possesses the qualities listed below, it is said to be a σ-algebra in X.:-

a) $X \in r$

b) If $A \in r$ then $A^c \in r$ where A^c is the complement of A

c) If $A = \cup_{n=1}^{\infty} A_n$ and if $A_n \in r$ $\forall$ n = 1,2,3,....., then $A \in r$

Where,

r = σ-algebra in X

X = measurable space and written as (X, r)

A_n (member of r) = measurable set in X

MEASURABLE FUNCTION:-

If a mapping is defined as $f : X \to Y$ and X is a measurable space, then f is said to be a measurable function if $f^{-1}(A)$ is a measurable set in X for all A in Y.

1.6 Continuity at a point

If every neighbourhood V of a mapping $f : X \to Y$ corresponds to a neighbourhood w of a mapping $f(x_0)$ such that $f(w) \in V$, then the mapping is said to be continuous at $x_0 \in X$.

1.6.1 PROPOSITION:-

Assume that X and Y are topological spaces. A mapping is continuous iff it has the form $f : X \rightarrow Y$. Every point along X has a continuous f.

1.6.2 PROPOSITION:-

Assume that Y and Z are topological spaces. A continuous mapping is $g : Y \rightarrow Z$:-

a) If the topological space X. If both $f : X \rightarrow Y$ and $h = gof$ are continuous, then $h : X \rightarrow Z$ is also continuous.

b) if the space X is measurable. Additionally, if $h = gof$ and $f : X \rightarrow Y$ are both measurable, then $h : X \rightarrow Z$ is also measurable.

1.7 Proposition:-

Let X be a measurable space. Then:-

a) Suppose that f is a complex measurable function on X and that $f = u + iv$, where u and v are real measurable functions on X.

b) Real measurable functions on X are u, v, and $|f|$, if $f = u + iv$ is a complex measurable function on X.

c) f and g must be complex measurable functions on X in order for $f + g$ and fg to be complex measurable functions.

d) In the event that E is a measurable set on X and $\chi_E(x) = 1$ (if $xis \in E$) or 0 (if $xisnot \in E$), then χ_E is a measurable function.

e) If f is a measurable function on X, then $f = \alpha|f|$ exists and there exists a complex measurable function α on X such that $|\alpha| = 1$.

1.8 Lebesgue Integral

Definition:- Let s is a measurable simple function on X as

$$s = \Sigma_{i=1}^{n} \alpha_i \chi_{A_i} ....(1)$$

where s has different values for α_1, α_2,..., α_n. And let's clarify,

$$\int_E s d\mu = \Sigma_{i-1}^{n} \alpha_i \mu(A_i \cap E)...............(2)$$

If $g : X \to [0, \infty]$ is a measurable function and $E \in r$ then we define

$$\int_E g d\mu = Sup \int_E s d\mu.........................(3)$$

The lebesgue integral of g over E, with respect to μ, is denoted by (3) above.

1.9 Integration of Complex Numbers:-

Letting all complex measurable functions g on X in $L^1(\mu)$ be the collection

$$\int_X |g| d\mu < \infty$$

Then, members of $L^1(\mu)$ are referred to be Lebesgue integrals.

Definition:-

In the event that $g = u + iv$, where u and v are real measurable functions on X, and $g \in L^1(\mu)$, we define -

$$\int_E g d\mu = \int_E u^+ d\mu - \int_E u^- d\mu + i \int_E v^+ d\mu - i \int_E v^- d\mu$$

Here

u^+ and u^- = are positive and negative parts of u

v^+ and v^- = positive and negative parts of v

These four functions are measurable.

1.10 Some Basic Results

Result 1.10.1

Let g and $h \in L^1(\mu)$ and u and v are complex numbers. Then $ug + vh \in L^1(\mu)$ and

$$\int_X (ug + vh)d\mu = u \int_X gd\mu + v \int_X hd\mu$$

Result 1.10.2

If $g \in L^1(\mu)$ then,

$$|\int_x gd\mu| \leq \int_X |g|d\mu$$

Result 1.10.3

Lebesgue Dominated convergent Theorem:-

Let $\{g_n\}$ is define as a sequence of complex measurable functions on X such that

$$g(x) = lim_{n\to\infty} g_n(x)$$

for all $x \in X$. If there exist a function $h \in L^1(\mu)$ such that

$|g_n(x)| \leq h(x)$ where n= 1,2,3,....., and $x \in X$ then $g \in L^1(\mu)$
such that

$$lim_{n\to\infty} \int_X |g_n - g|d\mu = 0$$

So,

$$lim_{n\to\infty} \int_X g_n d\mu = \int_X gd\mu$$

CHAPTER 2

Inequalities and Results

2.1 Holder's and Minkowski's Inequality

2.1.1 Minkowski's Inequality

If x, y be two elements fo l_p^n space , i.e., $x = (x_1, x_2, ...x_n), y = (y_1, y_2, ...y_n)$ and the norm of x is defined as

$$||x||p = [\Sigma i = 1^n |x_i|^p]^{1/p}$$

then Minkowski's inequality states that

$$||x + y||_p \leq ||x||_p + ||y||_p$$

.

Proof:-

If $p = 1$, then $||x||p = ||x||_1 = \Sigma i = 1^n |x_i|$

Therefore $||x + y||1 = \Sigma i = 1^n |x_i + y_i| \leq \Sigma_{i=1}^n |x_i| + \Sigma_{i=1}^n |y_i| = ||x||_1 + ||y||_1$

Thus the above inequality holds good for $p = 1$.

Let $p \neq 1$ then

$(||x+y||p)^p = \Sigma i = 1^n |x_i + y_i|^p$ by definition of norm

$=\Sigma_{i=1}^n |x_i + y_i||x_i + y_i|^{p-1} \leq \Sigma_{i=1n}(|x_i|) + (|y_1|)|x_i + y_i|^{p-1}$

$=\Sigma|x_i|(|x_i + y_i|^{p-1}) + \Sigma|y_i|(|x_i + y_i|^{p-1})$

Now by Holder's Inequality

$\Sigma|x_i y_i| \leq (\Sigma|x_i|^p)^{1/p} + (\Sigma|y_i|^q)^{1/q}$ where $\frac{1}{p} + \frac{1}{q} = 1.$

or $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$

$(p-1)q = p$

Applying the above inequality on R.H.S of (1) we get

$(||x+y|_p)^p \leq (\Sigma|x_i|^p)^{1/p}(\Sigma|x_i + y_i|^{(p-1/p)} + (\Sigma|y_i|^p)^{1/p}(\Sigma|x_i + y_i|^{(p-1)q})^{1/q}$
Put $(p-1)q = p$ and $\frac{1}{q} = \frac{p-1}{q}$ in R.H.S we get

$=(\Sigma|x_i|^p)^{1/p}(\Sigma|x_i + y_i|^p)^{p-1/p} + (\Sigma|y_i|^p)^1/p(\Sigma|x_i + y_i|^p)^{p-1/p}$

$=||x||_p(||x+y||_p)^{p-1} + ||y||_p(||x+y||_p)^{p-1}$

$=(||x||_p + ||y||_p)(||x+y||_p)^{p-1}$

Therefore $[||x+y||_p]^p \leq (||x||_p + ||y||_p)(||x+y||_p)^{p-1}$

If $||x+y||p = 0$, both sides vanish, proving that the statement above is true. However, if both sides are divisible by $||x+y||^p p - 1$ and $||x+y||_p \neq 0$, we obtain

$||x+y||_p \leq ||x||_p + ||y||_p$.

Hence Proved

2.1.2 Holder's Inequality

If x, y be two elements fo l_p^n space , i.e., $x = (x_1, x_2, ...x_n), y = (y_1, y_2, ...y_n)$ and the norm of x is defined as

$$||x||p = [\Sigma i = 1^n |x_i|^p]^{1/p}$$

then Holder's inequality states that

$$\Sigma_{i=1}^n |x_i y_i| \leq (\Sigma_{i=1}^n |x_i|^p)^{1/p} (\Sigma_{i=1}^n |x_i|^q)^{1/q}$$

$\Sigma_{i=1}^n |x_i y_i| \leq ||x||_p ||y||_q$, where $\frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty, 1 < q < \infty.$

Proof:-

If $x = 0$ and $y = 0$ then above inequality is obviously true.Thus we suppose that both x and y are not zero.

We know from Lemma proved above that

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q} \qquad where, \frac{1}{p} + \frac{1}{q} = 1$$

Let

$$a_i = [\frac{|x_i|}{||x||_p}]^p, b_i = [\frac{|y_i|}{||y||_q}]^q$$

Using the result (1),for a_i and b_i we have

$$\frac{|x_i|}{||x_p||} \cdot \frac{|y_i|}{||y||_q} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{[||x||_p]^p} + \frac{1}{q} \cdot \frac{|y_i|}{q[||y||_q]^q}$$

Above result is true for each i and hence summing w.r.t i, we get

$$\frac{\sum |x_i||y_i|}{||x||_p||y||_q} \leq \frac{1}{p}.\frac{\sum |x_i|^p}{[||x||_p]^p} + \frac{1}{q}.\frac{\sum |y_i|^q}{[||y||_q]^q} = \frac{1}{p}.\frac{[||x||_p]^p}{[||x||_p]^p} + \frac{1}{q}.\frac{[||y||_q]^q}{[||x||_q]^q}$$

$$= \frac{1}{p}.1 + \frac{1}{q}.1 = 1$$

Therefore $\Sigma_{i=1}^n |x_i y_i| \leq ||x||_p ||y||_q$

or $\Sigma_{i=1}^n |x_i y_i| \leq (\sum |x_i|^p)^(1/p)(\sum |y_i|^q)^1/q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

2.2 Holder's and Minkowski's inequalities for Sequences

Let $x =< x_n >, y =< y_n >$ be the sequence of scalars , such that

$\Sigma_{n=1}^{\infty} |x_n|^p < \infty$ and $\Sigma_{n=1}^{\infty} |y_n|^p < \infty p \geq 1$.

Define $||x||p = (\Sigma i = 1^{\infty} |x_n|^p)^1/p$ then

1. $\Sigma_{n=1}^{\infty} |x_n y_n| \leq (\Sigma_{n=1}^{\infty} |x_n|^p)^1/p(\Sigma_{n=1}^{\infty} |y_n|^q)^1/q = ||x||_p ||y||_q$ where $\frac{1}{p} + \frac{1}{q} = 1$

2. $||x + y||_p = ||x||_p + ||y||_q$

   For any positive integer m we have from Holder's inequality proved before

   $\Sigma_{n=1}^{m} |x_n y_n| \leq (\Sigma_{n=1}^{m} |x_n|^p)^{1/p} (\Sigma_{n=1}^{m} |y_n|^q)^{1/q} \leq (\Sigma_{n=1}^{\infty} |x_n|^p)^{1/p} (\Sigma_{n=1}^{\infty} |y_n|^q)^{1/q} \leq \infty$ by given definition

   Above shows that partial sum $g\Sigma_{n=1}^{m} |x_n y_n|$ of $\Sigma_{n=1} \infty |x_n y_n$ are bounded and hence we conclude that $\Sigma |x_n y_n| < \infty$

Now we know that if $x_n < y_n$ therefore, $\lim_{n\to\infty x_n} \leq lim_{n\to\infty y_n}$, i.e., $x_0 \leq y_0$.

Hence, if in the result (1) we make $m \to \infty$ then by the above result we obtain

$\Sigma_{n=1}^{\infty}|x_n y_n| \leq (\Sigma_{n=1}^{\infty}|x_n|^p)^{1/p}(\Sigma_{n=1}^{\infty}|y_n|^q)^{1/q} = ||x||_p||y||_q$

Proof of Minkowski inequality

In the proof of Minkowski inequality, we had shown that

$\Sigma i = 1^n|x_i+y_i|^p \leq (\Sigma_{i=1}^{n}|x_i|^p)^{1/p}(\Sigma_{i=1}n|x_i+y_i|^p)^{p-1/p}(\Sigma i = 1^n|y_i|^p)^{1/p}(\Sigma_{i=1}^{n}|x_i+y_i|^p)^(p-1/p)$

$\leq (\Sigma_{i=1}^{n}|x_i|^p)^1/p(\Sigma_{i=1}n|x_i + y_i|^p)^{p-1/p}(\Sigma i = 1^n|y_i|^p)^{1/p}(\Sigma_{i=1}^{n}|x_i + y_i|^p)^(p - 1/p)$

$= ||x||p||x + y||p^{p-1} + ||y||p||x + y||p^{p-1}$

Now if we allow n to tend to infinity then

L.H.S $\Sigma_{i=1}^{n}|x_i + y_i|^p$ becomes $\Sigma_{i=1}^{n}|x_i + y_i|^p = ||x + y||_p^p$

Hence from above we get

$= ||x + y||_p^p \leq [||x||_p||y||_p][||x + y||_p]^{p-1}$

In case $||x + y||p = 0$ then both sides vanish and if $||x + y|| \neq 0$ then we can divide , both sides by $||x + y||p^{p-1}$ and we obtain

$$||x + y||_p \leq ||x||_p + ||y||_p.$$

2.3 Conjugate Exponents

If the real values a and b are positive, such that $a + b = ab$ or equivalently,

$$1/a + 1/b = 1$$

where a and $b \in [1, \infty)$ then a and b are called conjugate exponents.

2.4 Inequalities in the form of measurable functions

a and b should be conjugate exponents. Suppose that S is a measure space with the dimension μ, with a range of $[0, \infty]$. Let g and h be measurable functions on S , then-

Holder's Inequality:-

$$\int_S ghd\mu \leq \{\int_S g^a d\mu\}^{1/a} \{\int_S h^b d\mu\}^{1/b}$$

Minkowski's Inequality:-

$$\{\int_S (g+h)^a d\mu\}^{1/a} \leq \{\int_S g^a d\mu\}^{1/a} + \{\int_S h^a d\mu\}^{1/a}$$

Schwart'z Inequality:-

$$\int_S ghd\mu \leq \{\int_S g^2 d\mu\}^{1/a} \{\int_S h^2 d\mu\}^{1/2}$$

2.5 Show that Under the norm, the $L^p(R)$ space is a Banach space.

$$||s||p = [\Sigma_{n=1}^{\infty}|s_n|^p]^{1/p}, \qquad \forall \qquad s =< s_n >\in L^p(R)$$

Proof:

First we will show that this space($L^p(R)$) is a normed linear space

- $||s||p = [\Sigma n = 1^{\infty}|s_n|^p]^{1/p} \geq 0$

  $||s||p = [\Sigma n = 1^{\infty}|s_n|^p]^{1/p} = 0$

  $\Rightarrow \Sigma_{n=1}^{\infty}|s_n|^p = 0$

  $\Rightarrow |s_n|^p = 0$

  $\Rightarrow s_n = 0$

  $\Rightarrow s = 0\forall n$

- $||\lambda \mathrm{s}|| = [\Sigma_{n=1}^{\infty}|\lambda \mathrm{s}_n|^p]^{1/p}$

  $||\lambda \mathrm{s}|| = |\lambda|[\Sigma_{n=1}^{\infty}|s_n|^p]^{1/p}$

  $||\lambda \mathrm{s}|| = |\lambda|||s||$

- $||s + t|| = [\Sigma_{n=1}^{\infty}|s_n + t_n|^p]^{1/p}$

$||s+t|| \leq [\Sigma_{n=1}^{\infty}|s_n|^p]^{1/p}+[\Sigma_{n=1}^{\infty}|t_n|^p]^{1/p} = ||s||+||t||$

$||s+t|| \leq ||s||+||t||$

It is therefore a normed linear space.

Next, we'll demonstrate that it is a banach space: -

Let $< s_n >$ is a cauchy sequence in $L^p(R)$ such that:-

$s^{(1)} =< s_k^{(1)} >, s^{(2)} =< s_k^{(2)}....., s^{(n)} =< s_k^{(n)} >.$

Therefore for all $\epsilon > 0$ a positive number n_0 exists that is suitable for all $m, n \geq n_0$

$||s^m - s^n|| < \epsilon$

$\Rightarrow (\Sigma_{k=1}^{\infty}|s_k^m - s_k^n|^p)^{1/p} < \epsilon$

$\Rightarrow |s_1^m - s_1^n| < \epsilon, |s_2^m - s_2^n| < \epsilon......$

So, we can say that sequence $< s_i^n >$ for all i is a cauchy sequence on the real line each of which will converge to some real numbers as the real line is complete.

Thus,

$s_1^n \rightarrow x_1, s_2^n \rightarrow x_2, ...$ So, we get a sequence $x =< x_n >$.

we wnat to show that $< s_n >$ converges to x.

For a fixed integer i, we know that -

$[\Sigma_{k=1}^{\infty}|s_k^m - s_k^n|^p)^{1/p} < \epsilon$ for all $m, n \geq n_0$

keeping n fixed and let $m \to \infty$, we have,

$[\Sigma_{k=1}^{\infty}|x_k - s_k^n|^p)^{1/p} < \epsilon$, $n \geq n_0$

for $i \to$, we get,

$[\Sigma_{k=1}^{\infty}|x_k - s_k^{n/p}|^p)^{1/p} < \epsilon$, $n \geq n_0$

$\Rightarrow ||x - s^(n)||_p < \epsilon$ for $n \geq n_0 \Rightarrow s^{(n)} \to x$

It remains to prove that $x \in L^p(R)$

$|x_k|^p = |x_k - s_k^{n_0} + s_k^{n_0}|^p \leq 2^p[|x_k - S_k^{n_0}|^p + |s_k^{n_0}|P]$

So, $\Sigma_{k=1}^{\infty}|x_k|^p \leq 2^p(\Sigma_{k=1}^{\infty}|x_k - s_k^{n_0}|^P + \Sigma_{k=1}^{\infty}|s_k^{n_0}|^P)$

$\Sigma_{k=1}^{\infty}|x_k|^p < 2^p.\epsilon^p + 2^p\Sigma_{k=1}^{\infty}|s_k^{n_0}|^P)$

Also,

$s^{n_0} =< s_k^{n_0} >\in L^p(R)$

$\Rightarrow \Sigma_{k=1}^{\infty}|S_k^{N_0}|^p < \infty$

Hence we can say that $L^p(R)$ is a Banach Space.

2.6 Show that Under the following norm $L^\infty(R)$ is a Banach Space

$$||x|| = Sup|x_n|$$

Proof:- :-

In the beginning, we shall show that $L^\infty(R)$ is a normed linear space -

- $||x|| \geq 0$

  $Sup|x_n| \geq 0$

  $|x_n| = 0$ iff $x_n = 0$

  So, $||x|| \geq 0$

- $||\alpha x|| = Sup|\alpha x_n|$

  $||\alpha x|| = |\alpha| Sup|x_n|$

  $||\alpha x|| = |\alpha| ||x||$

- $||x + y|| = Sup|x_n + y_n|$

  $||x + y|| \leq Sup|x_n| + Sup|y_n| = ||x|| + ||y||$

  $\Rightarrow ||x + y|| \leq ||x|| + ||y||$

Becausse all three properties of normed linear space are satisfied. So, We can say that $L^\infty(R)$ is a normed linear space.

We shall now demonstrate that it is a complete space :-

A cauchy sequence in $L^\infty(R)$, let $< x_m >$ be. And assuming that $\epsilon > 0$, there exists an integer N that is positive and

$||x_m - x_n|| < \epsilon \,\forall\, n, m \geq N$

$\Rightarrow |x_{m_k} - x_{n_k}| \leq \epsilon \,\forall\, n, m \geq N$ and $\forall k$ ....(1)

So, $\forall\, k$ the seq $< x_{n_k} >$ is a cauchy sequence in $L^\infty(R)$. Hence there is a sequence $x =< x_k >$ such that $x_{n_k} \to x_k \,\forall\, k$ as $n \to \infty$

Now we have to show that :-

$x \in L^\infty(R)$ [As every cauchy sequence is bounded So, ther exist $k > 0$ such that $||x_n|| < k \,\forall\, k$]

Now,

$||x_{n_k}|| \leq k \,\forall\, n$

$\Rightarrow lim_{n\to\infty} |x_{n_k}|$ = $|x_k| \leq k \,\forall\, n$

$\Rightarrow x \in L^\infty(R)$

Now let $m \to \infty$ in (1), we get,

$|x_k - x_{n_k}| \leq \epsilon \; \forall \; n \geq N$ and $\forall \; k$

So,

$||x_n - x|| = Sup|x_k - x_{m_k}| \leq \epsilon \; \forall \; n \geq N$

So, the arbitrary cauchy sequence $< x_n > \in L^\infty(R)$ convergent to an elemrnt $x \in L^\infty(R)$.

$\Rightarrow L^\infty(R)$ is a whole(complete) space that is a banach space.

Hence Proved.

2.7 Show that $L^2(R)$ is a Banach Space under the norm

$||s||^2 = [\Sigma_{k=1}^{\infty} s_k^2]^{1/2} \; \forall \; s \in L^2(R)$

Proof :-

The given set is closed for addition and saclar multiplication and is a linear space. So, Now we show that it is a normed linear space -

- $||s|| = [\Sigma_{k=1}^{\infty} s_k^2]^{1/2} \geq 0$

  $||s|| = 0$ iff $[\Sigma_{k=1}^{\infty} s_k^2]^{1/2} = 0$

  $\Rightarrow \Sigma_{k=1}^{\infty} s_k^2 = 0 \Rightarrow s_k = 0 \; \forall \; k$

  So, $||s|| \geq 0$

- $||\alpha s|| = [\Sigma_{k=1}^{\infty} \alpha s_k^2]^{1/2}$

  $||\alpha s|| = |\alpha|[\Sigma_{k=1}^{\infty} s_k^2]^{1/2}$

  $||\alpha s|| = |\alpha|||s||$

- $||s + t|| = [\Sigma_{k=1}^{\infty}(s_k + t_k)^2]^{1/2}$

  $||s + t|| \leq [\Sigma_{k=1}^{\infty} s_k^2]^{1/2} + [\Sigma_{k=1}^{\infty} t_k^2]^{1/2} = ||s|| + ||t||$

  $||s + t|| \leq ||s|| + ||t||$

Becausse all three properties of normed linear space are satisfied. So, We can say that $L^2(R)$ is a normed linear space.

Now, We will show that it is a complete space :-

Let $< s^n >$ be a cauchy sequence in $L^2(R)$. And let $\epsilon > 0$ there exist a positive integer such that $||s^m - s^n|| < \epsilon \, \forall \, m, n \geq n_0$

$\Rightarrow [\Sigma_{k=1}^{\infty}(s_k^m - t_k^n)^2]^{1/2} \leq \epsilon$

$\Rightarrow$ Each of the following is less than ϵ that is,

$|s_1^m - s_1^n| \leq \epsilon$, $|s_2^m - s_2^n| \leq \epsilon$, ...

$\Rightarrow$ Each real number in the sequence $< s_i^n > \forall \, i$ will eventually converge to the same real number.

$lim_{n\to\infty}\ s_1^n = x_1$, $lim_{n\to\infty}\ s_2^n = x_2$, ... ,$lim_{n\to\infty}\ s_k^n = x_k$ ......(1)

So, we get a sequence $x =< x_n >$

Now, to show that $< s^n >$ converges to $x =< x_n >$ for a fixed integer p, we have :-

$[\Sigma_{k=1}^{p}(s_k^m - s_k^n)^2]^{1/2} < \epsilon\ \forall\ m, n > n_0$

kepping n fixed and let $m \to \infty$, we have -

$\Sigma_{k=1}^{p}(x_k - s_k^n)^2 < \epsilon\ \forall\ n > n_0$

The Relation holds for each p thus making $p \to \infty$, we get:-

$[\Sigma_{k=1}^{\infty}(x_k - s_k^n)^2]^{1/2} < \epsilon\ \forall\ n \geq n_0$

$\Rightarrow d(x, s^n) \leq \epsilon\ \forall\ n \geq n_0$

$||x - s^n|| \leq \epsilon$

$s^n \to x$

Now we show that $x \in L^2(R)$

$x_k^2 = [x_k - s_k^{n_0} + s_k^{n_0}]^2 = (x_k - s_k^{n_0})^2 + (s_k^{n_0})^2 + 2(x_k - s_k^{n_0})(s_k^{n_0})$

$x_k^2 \leq 2(x_k - s_k^{n_0}) + 2(s_k^{n_0})^2$

So,

$\Sigma_{k=1}^{\infty} x_k^2 \leq 2\Sigma_{k=1}^{\infty}(x_k - s_k^{n_0})^2 + 2\Sigma_{k=1}^{\infty}(s_k^{n_0})^2 < 2\epsilon^2 + 2\Sigma_{k=1}^{\infty}(s_k^{n_0})^2$

$\Rightarrow s_k^{n_0} = (s_k^{n_0}) \in L^2(R)$

$\Rightarrow \Sigma_{k=1}^{\infty}(s_k^{n_0})^2 < \infty$

$\Rightarrow x \in L^2(R)$

Hence every cauchy sequence in $L^2(R)$ convergent to a point in $L^2(R)$.
$\Rightarrow L^2(R)$ is a complete space.
$\Rightarrow L^2(R)$ will be a Banach Space.

Hence Proved.

2.8 Show that space l_2 is a Hilbert Space.

Proof:-

Let $a = x_i$ and $b = y_i$ be elements of l_2.

By, we specify what the inner product of a and b is,

$$(a, b) = \Sigma_{i=1}^{\infty} x_i \bar{y_i}$$

The right-hand series' convergence results from the fact that ,

$x \in l_2$ and $|y_{\bar{i}i}| \leq \frac{|y_i|^2}{2} + \frac{|x_i|^2}{2}$.

Then

$$||a|| = \sqrt{(a,a)} = (\Sigma_{i=1}^{\infty} y_i)$$

It is simple to demonstrate that l_2 satisfies all of the inner product axioms.As stated by, the measure d of l_2 is

$$d(a,b) = ||a-b|| = (a-b, a-b)^{1/2} = (\Sigma_{i=1}^{n} |y_i - x_i|^2)^{1/2}$$

We can immediately observe that l_2 is complete with regard to this metric, making l_2 a Hilbert space.

Hence Proved.

2.9 Show that $L^p(\mathbb{R})$ is not a Hilbert space for $p \neq 2$.

Proof:-

Let $a = (1, 1, 0, 0, ...) \in \mathrm{l}_p$

and $b = (1, -1, 0, 0, ...) \in \mathrm{l}_p$.

Then

$$||a|| = ||b|| = 2^{1/p}$$

and

$$||a+b|| = ||a-b|| = 2$$

Now we see that if $p \neq 2$,the parallelogram law does not hold.

Hence $1 \leq p < \infty l_p(p \neq 2)$ is not a Hilbert space because it is not an inner product space.

2.10 Show that $L_2[p,q]$, is a Hilbert Space.

Proof:-

Define the inner product on $L_2[p,q]$ by

$$< a, b >= \int pq|a(t)b(\bar{t})|dt, \forall a, b \in L_2[p,q]$$

and the norm on $L_2[p,q]$ is given by

$$||a|| = \sqrt{\int_p^q |a(t)|^2 dt}$$

Additionally, it can be demonstrated that $L_2[p,q]$ is complete with regard to this norm and a metric defined by

$$d(a,b) = [\int_p^q |a(t) - b(t)|^2 dt]^{1/2}.$$

So $L_2[p,q]$ is a Hilbert Space.

CHAPTER 3

L^p-Space and Dual of L^p-Space

3.1 L^p-Space

Definition:- Let $p \in (0, \infty)$ and g is a complex measurable function on the variable X. Define-

$$||g||_p = \{\int_X |g|^p d\mu\}^{1/p}............(1)$$

Additionally, let g consist of all $L^p(\mu)$ for which $||g||_p \leq \infty$ where $||g||_p = L^p$ norm of g

$$OR$$

The set of sequence $s =< s_n >$ such that $\Sigma_{n=1}^{\infty} |s_n|^p < \infty$ is called l_p space.

NOTE:-

1) If μ is the Lebesgue measure on R^k, then $L^p(r^k)$ is used in place of $L^p(\mu)$.

2) When a measure is countable on a set X(countable), it is written as $l^p(X)$ or l^p rather than $L^p(X)$.

3) A component of l^p could be thought of as a complex sequence $X = \{x_n\}$ and

$$||X|| = \{\Sigma_{n=1}^{\infty} |x_n|^p\}^{1/p}$$

3.2 COMPLETE L^P-SPACE

If every cauchy sequence in the norm $||.||_{L^p}$ is convergent. Then the every elements where the sequence converges exists in the norm $||.||_{L^p}$ then we say that L^p space is complete L^p-Space.

Most of the spaces are complete spaces but some spaces may exist that is not complete that is very little useful.

L^1, L^2 and L^p spaces are complete spaces.

3.3 Theorem

Show that Under the norm $||.||_{L^p}$, the space $L^p(X, \tau, \mu)$ is complete.

Proof:-

Suppose that $\{a_n\}_{k=1}^{\infty}$ is a cauchy sequence in L^p and let $k \geq 1$ there exist a subseq $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}$ then

$$||a_{n_{k+1}} - a_{n_k}|| \leq 2^{-k}$$

Now assume the series whose convergence is given as below:-

$$a(x) = a_{n_1}(x) + \Sigma_{k=1}^{\infty}(a_{n_{k+1}}(x) - a_{n_k}(x))$$

and

$$b(x) = |a_{n_1}(x)| + \Sigma_{k=1}^{\infty}|(a_{n_{k+1}}(x) - a_{n_k}(x))|$$

the corresponding partial sum is,

$$S_m a(x) = a_{n_1}(x) + \Sigma_{k=1}^{\infty}(a_{n_{k+1}}(x) - a_{n_k}(x))$$

and

$$S_m b(x) = |a_{n_1}(x)| + \Sigma_{k=1}^{\infty}|(a_{n_{k+1}}(x) - a_{n_k}(x))|$$

The triangle inequality for L^p is,

$$||S_m(b)||_{L^p} \leq ||a_{n_1}||_{L^p} + \Sigma_{k=1}^{m}||a_{n_{k+1}} - a_{n_k}||_{L^p}$$

So,

$$||S_m(b)||_{L^p} \leq ||a_{n_1}||_{L^p} + \Sigma_{k=1}^{m} 2^{-k}$$

Now, with $k \to \infty$, prove using the monotonic convergent theorem that $p < \infty$ and the series defining b converge everywhere, and $a \in L^p$.

Now we show that $\{a_n\}$ converges to a. So a is limit of $\{a_n\}$.

Because of the telescopic series, the series' $(m-1)^t h$ sum is accurate a_{n_m}, So $a_{n_m} \to a(x)$.

Now show that $a_{n_m} \to a$ in L^p. Let,

$$|a(x) - S_m a(x)|^p \leq [2max(|a(x)|, |S_m a(x)|)]^p$$

$$|a(x) - S_m a(x)|^p \leq 2^p |a(x)|^p + 2^p |S_m a(x)|^p$$

So,

$$|a(x) - S_m a(x)|^p \leq 2^{p+1} |b(x)|^p$$

The dominated convergence theorem will now be used to obtain $||a_{n_m} - a||_{L^p} \to 0$ as $m \to \infty$. Now because $\{a_n\}$ is cauchy. Let $\epsilon > 0$ and for all $n, m \geq N$ there exist N such that

$$||a_n - a_m||_{L^p} < \epsilon/2$$

Now let choose $n_m > N$ and $||a_{n_m} - a||_{L^p} < \epsilon/2$ then by the triangle equality,

$$||a_n - a||_{L^p} \leq ||a_n - a_{n_k}||_{L^p} + ||a_{n_k} - a||_{L^p} < \epsilon$$

So, the L^p space is complete.

Hence Proved.

3.4 DUAL OF L^P SPACE

Assume that $1/p + 1/q = 1$ and that q is the conjugate exponent of p. Every function $h \in L^q$ provides a bounded linear functional on L^p according to the Holder's inequality. Suppose that,

$$l(g) = \int_X g(x)h(x)d\mu(x) ....(1)$$

And let,

$$||l|| \leq ||h||_{L^q}$$

So if h associates to l then $L^q \subset (L^p)^*$ for $1 \leq p \leq \infty$

Therefore, for some $h \in L^q$, every linear functional on L^p has the form (1).

So, $(L^p)^* = L^q$ for $1 \leq p \leq \infty$, where $(L^p)^*$ is dual of L^p.

Note:-

When $p = \infty$, the dual of L^∞ contains L^1 but it is larger, so the following result is generally false.

3.5 The Riesz Representation Theorem(For the dual of L^p space)

A measurable set S shall be used such that $1 \leq < \infty$ and q be the conjugate of p. Let B define the bounded linear functional on $L^p(S)$ for all $h \in L^q(S)$.

$$B(g) = \int_S h.g \forall h \in L^p(S)$$

Then there exists a single function $h \in$L$^q(S)$ for which $B = F$ and $||F|| = ||h||_q$ for all bounded linear functions F on $L^p(S)$.

Proof:-

Because for all $h \in L^q(E)$, On $L^p(E)$, B is a bounded linear functional such that $||B||_* = ||h||_q$. So by linearity of integration for all $h_1, h_2 \in L^q(S) \Rightarrow B_{h_1} - B_{h_2} = B_{h_1 - h_2}$ so, if $B_{h_1} = B_{h_2}$ then $B_{h_1-h_2} = 0 \Rightarrow ||h_1 - h_2|| = 0$.

$$h_1 = h_2$$

There is only one function $h \in L^q(S)$ for which $B = F$ for any F on $L^p(S)$.

Therefore, we must demonstrate that there is a function $h \in L^q(E)$ for which $F = B$ for all bounded linear functionals F on $L^p(S)$.

Then, for general measurable sets, we confirm this

In the case of any natural number $n \in N$, let F be a bounded linear functional on $L^p(R)$. Explain linear functions

F_n on $L^p[-n, n]$ by ,

$F_n(f) = F(g)$ for all $g \in L^p[-n, n]$

where $g' =$ extension of $f \forall R$

Since $||g||_p = ||g'||_p$

$\Rightarrow$ | F$_n(g)| \leq ||F||_* ||g||_p$ for all $g \in L^p[-n, n]$

$\Rightarrow Thus, ||F_n||_* \leq ||F||_*$

Now let $h_n \in L^q[-n, n]$ for which, $F_n(g) = \int_{-n}^{n} h_n.g$ for all $g \in L^p[-n, n]$ and $||h_n||_q = ||F_n||_* \leq ||F||$
............A

As a result, for every $g \in L^p(R)$ that vanish outside a bounded set, we conclude from the definitions of F_n and h_n coupled with the left hand equally in A.

$$F(g) = \int_R h.g$$

Now by Right hand equality in A $\int_{-n}^{n} |h|^q \leq (||F_*||)^q$ for all n

Since Bounded linear functional B and F agree on dense subspace of $L^p(R)$ so by fatous Lemma $h \in L^q(R)$.

Now let S and F be the general measurable sets.Define the linear functional $\hat{F}$ on $L^p(R)$ by $\hat{f}(g) = F(g|s)$.Then F is bounded linear functional on $L^p(R)$.

Thus we have just shown that there is a function $\hat{f} \in L^q(R)$ for which $\hat{f}$ is represented by integration over R against $\hat{f}$

Hence Proved.

3.6 Theorem

Let $L^p[0,1]$ is the set of all measurable functions $g : [0,1] \to R$ and let dual of $L^p[0,1]$ is $L^p[0,1]^* = \{0\}$ for $0 < p < 1$. Such that the only continuous linear map $L^p[0,1] \to R$ is zero.

Proof:-

Let $a \in L^p[0,1]^*$ with $a \neq 0$. Then a has image in R. As we know that a non zero linear map to a one dimensional space is onto. So, there is some $g \in L^p[0,1]$ such that,

$$|a(g)| \geq 1$$

using let g map $[0,1]$ to R So, $\int_0^s |g(x)|^p dx$ is continuous. There exist some $s \in [0,1]$ such that

$$\int_0^s |g(x)|^p dx = 1/2 \int_0^1 |g(x)|^p > 0$$

Let $h_1 = g\chi_{[0,s]}$ and $h_2 = g\chi_{(s,1]}$ So, $g = h_1 + h_2$ and $|g|^p = |h_1|^p + |h_2|^p$

$$\int_0^1 |h_1(x)|^p dx = \int_0^s |g(x)|^p dx = 1/2 \int_0^1 |g(x)|^p dx$$

Hence,

$$\int_0^1 |h_2(x)|^p dx = 1/2 \int_0^1 |g(x)|^p dx$$

Because, $|a(g)| \geq 1$ and $|a(h_i)| \geq 1/2$ for some i.

Let $g_1 = 2h_i$, So $|a(g_1)| \geq 1$

And,

$$\int_0^1 |g_1(x)|^p dx = 2^p \int_0^1 |h_i(x)|^p dx = 2^{p-1} \int_0^1 |g(x)|^p dx$$

Now we get a sequence $\{g_n\}$ in $L^p[0,1]$ such that $|a(g_n)| \geq 1$

And,

$$d(g_n, 0) = \int_0^1 |g_n(x)|^p dx = (2^{p-1})^n \int_0^1 |g(x)|^p dx \to 0$$

Which contradicts the continuity of a.

Hence Proved.

* <u>TWO NORM CONVERGENCE IN L^p-SPACES</u>

For $1 \leq p \leq \infty$, $L^p[0,1]$ is called a space of all measurable functions g such that $\int_0^1 |g(c)|^p dx < \infty$ and $L^\infty[0,1]$ is called a space of all functions g such that essential $Sup|g| < \infty$ where,

essential $Sup = inf\{N : |g(x)| \leq N \forall [0,1]\}$

If there are two real numbers, p and q, then

$$\frac{1}{p} + \frac{1}{q} = 1$$

the $L^p[0,1]$ is Banach dual of $L^q[0,1]$ and Banach dual of $L^1[0,1]$ is $L^\infty[0,1]$ but Banach dual of $L^\infty[0,1]$ is not $L^1[0,1]$.

3.7 TWO NORM CONVERGENCE IN l^∞

Let $[0,1]$ be the set of all necessary bounded functions in the space $L\infty$. If a function g is bounded almost everywhere then g is called essential bounded functions.

Let $\{g_n\}$ be the sequence of functions is said to be two norm convergence in L^∞, if there exist $N > 0$ such that $||g_n|| < N$ for all n and

$$lim_{n\to\infty} \int_0^1 g_n(x)h(x)dx$$

exists for all absolutely or Lebesgue integrable function h on $[0,1]$.

3.8 SOME IMPORTANT RESULTS

Result:- 3.8.1

Let $\{g_n\}$ is two norm convergence in L^∞, then there exist a function $g \in L^\infty$ such that

$$\int_0^1 g_n h \to \int_0^1 gh$$

as $n \to \infty$ for all Lebesgue integrable function h in $[0,1]$.

Result:- 3.8.2

If h is absolute integrable on $[0,1]$ and

$$M(f) = \int_0^1 g(x)h(x)dx$$

for all $g \in L^\infty$ then M defines a two norm continuous linear function on L^∞.

Result:- 3.8.3

There is an absolute integrable function h such that

$$M(f) = \int_0^1 g(x)h(x)dx$$

for every $g \in L^\infty$ if M is a two norm continuous linear functional on L^∞.

NOTE:-

Norm Convergence in L^p-Space

A seq $\{g_n\}$ of functions in L^p for all $1 \leq p \leq \infty$ is said to be norm convergence to $g \in L^p$ if $||f_n - f||_p \to 0$(convergent to zero), as $n \to \infty$.

3.9 TWO NORM CONVERGENCE IN L^P-SPACE

A seq $\{g_n\}$ of functions in L^p for all $1 \leq p \leq \infty$ is said to be two norm convergence to $g \in L^p$ if there exist $N > 0$ such that,

$$||g_n||_p \leq N$$

for all n and

$$lim_{n\to\infty} \int_0^1 g_n(x)h(x)dx$$

exists for all $h \in L^q$.

LEMMA:-

A function g in L^p for all $1 \leq g < \infty$ if and only if

$$Sup(D)(\Sigma \frac{|M(d) - M(c)|^p}{|d - c|^p - 1}) < \infty$$

where Sup is taken over all of divisions $D = \{[c, d]\}$ of $[0, 1]$ in $[c, d]$ stands for a typical interval in the division.

3.10 SOME IMPORTANT RESULTS

Result:- 3.10.1

If $\{g_n\}$ is two norm convergent in L^p for all $1 \leq p \leq \infty$ then there exist a function $g \in L^p$ such that

$$\int_0^1 g_n h \to \int_0^1 gh$$

as $n \to \infty$.

Result:- 3.10.2

Let $1 \leq p \leq \infty$ if $\{g_n\}$ is norm convergence for all $g \in L^p$ then $\{g_n\}$ is two norm convergence to g in L^p.

Result:- 3.10.3

If $h \in L^q$ for all $1 \leq q \leq \infty$ and

$$F(g) = \int_0^1 g(x)h(x)$$

for all $g \in L^p$ then F defines a two norm continuous linear functional on L^p.

Result:- 3.10.4

If F is two norm continuous functional on L^p, then F is norm continuous functional on L^p for $1 \leq p \leq \infty$

Result:- 3.10.5

There exists a function $h \in L^q$ such that

$$F(g) = \int_0^1 gh$$

for every $g \in L^p$ if $1 \leq p \leq \infty$ and F are two norm continuous linear functional on L^p.

Corollary:-

A linear functional on L^p is two norm continuous if and only if it is norm continuous. Let $1 \leq p \leq \infty$. For $p = \infty$, this corollary does not apply.

CHAPTER 4

Characterisation of L^p-Space

4.1 In L^p Space, Weak Sequential Convergence

Definition:-: If x is a normed linear space, then,Weakly convergent to g ∈x is defined as a sequence g_n ∈x,

$$\lim_{n\to\infty} F(g_n) = F(g) \forall F \in X^*$$

and written as $g_n \to g$ in X.

Thus, g and g_n are members of x, and g_n weakly converges to g.

We continue to write $g_n \to g$ in x to signify that $lim_{n\to\infty}||g_n \to g|| = 0$ and we frequently refer to this style of convergence as strong convergence in x in order to differentiate it from weak convergence. Since

$$|F(g_n) - F(g)| = |F(g_n - g)| \leq ||F||_*||g_n - g|| \forall F \in X^*$$

A sequence converges weakly if it converges strongly, but the opposite is not true.

4.2 Theorem

Let S be a measurable set and $1 \leq p < \infty$.Let $g_n \to g$ in $L^p(S)$.Then g_n is bounded in $L^p(S)$ and $||f||_p \leq \lim inf ||g_n||_p$.

Proof:-

Assume that the conjugate function of g and the conjugate spaces p and q.Ascertaining the right hand inequality first

By Holder's inequality,

$$\int_s g'.g_n \leq ||g'||_q.||g_n||_p = ||g_n||_p \forall n$$

Since g_n converges weakly for all g and g' in $L^q(S)$

$$||g||_p = \int_s g'.g = \lim_{n\to\infty} \int_s g'.g_n \leq \lim inf ||g_n||_p$$

Now, we use contradiction to demonstrate that g_n is constrained in $L^p(S)$.Let $||f_n|| = n.3^n \forall n$ and $||g_n||_p$ be unbounded.

We choose a set of real numbers a_i inductively such that $a_i = \pm\frac{1}{3}^i$ for all k.

$\Rightarrow$a$_1 = \frac{1}{3}$. if i is natural no for which $a_1, a_2, ...a_n$ be defined

$$a_{n+1} = \frac{1}{3^{n+1}} if \int_S [\sum_{i=1}^{n} a_i(g_n)].g_{n+1} \geq 0$$

and $a_{n+1} = -\frac{1}{3^{n+1}}$ [for negative].Therefore by A and the definition of conjugate function

$$|\int_s [\sum_{i=1}^{n} a_i(g_i)'].g_n| \geq \frac{1}{3^i}||g_n||_p = n$$

and $||a_n.g(n)'||_q = \frac{1}{3^n}\forall n$.

Since $||a_i(g_i)'||_q = \frac{1}{3_i}\forall i$

A cauchy sequence in L^q (S) is the partial sum of series sequence $\sum_{i=1}^{\infty} a_i(g_i)^*$. According to the Riesz-Lischer theorem, L^q (E) is complete.

For every n we deduce from the triangle inequality and Holder's inequality, define the function $h \in L^q(E)$ by $h = \sum_{i=1}^{\infty} a_i(g_i)'$.

$$| \int_s h.g_n| = | \int_s [\sum_{i=1}^{\infty} a_i(g_i)'].g_n|$$

$$\geq | \int_s [\sum_{i=1}^{n} a_i(g_i)'].g_n| - | \int_s [\sum_{i=1}^{\infty} a_i(g_i)'].g_n|$$

$$\geq n - | \int_E [\sum_{i=n+1}^{\infty} a_i(g_i)^*].g_n|$$

$$\geq n - [\sum_{i=n+1}^{\infty} \frac{1}{3^i}].||g_n||_p$$

$$= n - \frac{1}{3}^i . \frac{1}{2}.||g_n||_p$$

$$= \frac{n}{2}$$

That is a contradiction since h belongs to $L^q(E)$ and g_n converges weakly in $L^p(S)$.Real number sequence $[\int_s h.g_n]$ converges and is limited as a result.Thus, f_n is constrained by $L^p(S)$.

Corollary:-

Let q be the conjugate of p and S be a measurable set for $1 \leq p\infty$.Let's say that in $L^p(S)$, q_n converges weakly to g while in $L^q(S)$, h_n converges strongly to h.Then

$$\lim_{n \to \infty} \int_s h_n.g_n = \int_s h.g$$

Proof:-

For all n,

$$\int_s h_n.g_n - \int_s h.g = \int_s [h_n - h].g_n + \int_s h.g_n - \int_s h.g$$

Let $K \geq 0$ any constant such that $||g_n||_p \geq k$ for all n.

By Holder's inequality

$$|\int_s h_n.g_n - \int_s h.g| \leq K.||h_n - h||_q + |\int_s h.g_n - \int_s h.g|$$

Due to these disparities and the reality that both

$$\lim_{n\to\infty} ||h_n - h||_q = 0 \lim_{n\to\infty} \int_S h.g_n = \int_s h.g$$

So.

$$\lim_{n\to\infty} \int_s h_n.g_n = \int_s h.g$$

. **Hence Proved.**

4.3 Weak Sequential compactness in L^p space

Definition:-If every sequence g_n in a subset A has a subsequence that weakly converges to $g \in A$, then the subset A is said to be weakly sequentially compact in the normed linear space X.

Theorem:-

Assume that S is a quantifiable set on $1 \leq p < \infty$.Then $g \in L^p(S)$ for all $||g||_p \leq 1$ is weakly sequentially compact in $L^p(S)$.

4.4 Riesz weak compactness theorem

Let $1 < p < \infty$ and S be measurable sets.If g_n is a bounded sequence in $L^p(S)$, then there is a subsequence g_{nk} of g_n and a function g in $L^p(S)$ for which g is a weakly convergent subsequence for every bounded sequence in $L^p(S)$.

$$\lim_{k\to\infty} \int_s g_{nk}h.du = \int_s g.hdu \forall h \in L^q(S)$$

where $\frac{1}{p} + \frac{1}{q} = 1$

4.5 Approximation in L^p-Space

Here, the approximation of L^p spaces with respect to the $L^p(S)$ norm is taken into account. In essence, approximation is derived from the notion of L^p space's density.

Definition:-

A normed linear space with the norm $||.||$ is what we'll call X. Two subsets of X, A and B, with $A \subseteq B$, are given. If there is a function g in A such that $||g - h|| < \epsilon$ exists for each function h in B, then A is said to be dense in B.
It is not difficult to show that the set A is dense in B iff there is a sequence $< g_n >$ such that $lim_{n\to\infty} g_n = h \forall h \in X$

morever it is also useful to observe that for $A \subseteq B \subseteq C \subseteq X$. Given that B is dense in C and A is dense in B, A must also be dense in C.

4.6 Propositions

4.6.1 Proposition:-

Let S be a measurable set, then the subspace of simple functions in $L^p(S)$ is $1 \leq p < \infty$.

Proof:-

Let $h \in L^p(S)$ and $p = \infty$ and $S_0 \subseteq S$ with measure zero for which h is bounded on S.
By lemma There is a set of basic functions on S that converge uniformly on S to h and with respect to the $L^\infty(S)$ norm.
So, Simple function dense in $L^\infty(S)$

Now let $1 \leq p < \infty$ and let h is measurable.
$\Rightarrow$ By simple approximation theorem there exist a sequence $< a_n >$ of simple functions on E such that $< a_n > \to h$ pointwise on S and $|a_n| \leq |h|$ on S for all n.

By integral comparison test $a_n \in L^p(S)$ we claim that $< a_n > \to h \in L^p(E)$. Indeed for all n ,

$$|a_n - h|^p \leq 2^p\{|a_n|^p + |h|^p\} \leq 2^{p+1}|h|^p$$

We conclude from the Lebesgue dominated convergence theorem that $< a_n > \rightarrow h$ in $L^p(S)$ since $|h|^p$ is integrable over S.

4.6.2 Proposition:-

Let $1 \leq p < \infty$ and $[m, n]$ be closed bounded intervals. Therefore, the step function subspace on $[m, n]$ is dense in $L^p[m, n]$.

4.7 Separability in L^p - Space

Definition:-

If a countable subset is dense in a normed linear space X, then X is said to be separable.

Since the rational numbers are a countable dense subset of the real numbers, they can be separated. Because the polynomial with rational coefficents is a countable set that is dense in $C[u, v]$, we may conclude from the Weierstrass approximation theorem that it is separable for $[u, v]$ and closed bounded interval $C[u, v]$ normed by the maximum norm.

4.8 Theorem

Let $1 \leq p < \infty$ and S be a set of measurable elements. When this happens, the normed linear space $L^p(S)$ is separable.

Proof:-

Let $[u, v]$ be a closed and bounded interval and the collection of all step functions on $[u, v]$ is $T[u, v]$.

Let $T^{'}[u, v]$ be the subcollection of $t[u, v]$. comprising step function ϕ on $[u, v]$ that take rational values and for which thre is a partition $P = \{a_0, a_1, ...a_n\}$ on $[u, v]$ with ϕ

constant on (a_{k-1}, a_k) for $1 \leq k \leq n$ and x_k rational for $1 \leq k \leq n-1$.

Now, we can conclude that $T'[u, v]$ is dense in $T[u, v]$ with regard to the $L^p(S)$ norm. This is based on the density of rational numbers in the real numbers. These two inclusions, each of which is dense in relation to the $L^p[u, v]$ norm, are thus present,

$$t'[m, n] \subseteq T[m, n] \subseteq L6p[u, v]$$

So, $\Rightarrow T'[m, n]$ is dense in $L^p[u, v]$ for all n define $A - n$ to be the function on R that vanish outside $[-n, n]$ and those restrictions to $[-n.n]$ belongs to $T^{'}[-n, n]$.

Define $A = \cup_{n \in N} A_n$. Then A is countable collection of functions in $L^p(R)$. By monotonic convergent theorem,

$$lim_{n \to \infty} \int_{[-n,n]} |g|^p = \int_R |g|^p$$

for all $g \in L^p(R)$

$\Rightarrow$ Therefore, A is countable called of functions that are dense in $L^p(R)$ by the selection of each A_n. Let S be a universal measurable set to finish. then the group of limitations on the function's S. Because A is a countably dense subset of $L^p(S)$. $L^p(S)$ is separable.

Hence Proved.

4.9 Characteristic properties of L^p Space

The fundamental component of the theory at L^p spaces, where $p > 1$ is Holder's inequality

$$\int_0^1 g(t).h(t).dt \leq (\int_0^1 |g(t)|^p dt)^{1/p} (\int_0^1 |h(t)|^q dt)^{1/q} ....(A)$$

where , $g(t) \in L^p, h(t) \in L^q$ and $q = \frac{p}{p-1}$

The following particular young's inequality is typically used to demonstrate this inequality

$$\int_0^1 g(t).h(t).dt \leq \frac{1}{p}\int_0^1 |g(t)|^p dt + \frac{1}{q}\int_0^1 |h(t)|^q dt...........(B)$$

For the function , It is well known that ,

$$h(t) = |g(t)|^{p-1} sgng(t) = T.g(t)............(C)$$

If the equality for the pairs of functions holds in B then the equality for that same pair also hold in A this property is characteristic for L^p spaces $(p > 1)$.

Conjugate Similar:-:

The term conjugate similar refers to a transformation T from a universal continuous semi ordered linear space $\mathbb{R}$ into its conjugate space $\mathbb{R}^2$ with the following conditions:

a) $a \geq b \geq 0 \Rightarrow Ta \geq Tb \geq 0$

b) $(Ta)(b) = T(b)(a)$ for any b

c) $T(-a) = -Ta$

4.10 Theorem

Let R be a strictly convex normed universal continuous semi-ordered linear space with at least two linear independent elements.If the following condition is met by a one to one conjugately similar correspondence T,

$$(Ta, a) = ||Ta||.||a||(0 \leq a \in R)..............(1)$$

then we can find a number $p > 1$ such that

$$T\lambda a = \lambda^{p-1} Ta$$

for any $\lambda > 0 a \in R$.

Proof:-

We take advantage of the fact that there is a such T to define on R a modular $m(a)$ that needs the following requirements:

- $0 < m(a) < +\infty$ for all $0 \neq x \in R$
- $m(\lambda a)$ is a convex function of $\lambda > 0$
- $m(a+b) = m(a) + m(b)$ if ab are naturally orthogonal.
- $a \geq b \geq 0 \Rightarrow m(a) \geq m(b)0 \leq a_i \Rightarrow m(a) = Supm_{i \in A}(a_i)$

The modular is defined by $m(a) = \int_0^1 (T\lambda a, a) d\lambda$ when a is non-negative and $m(a) = m(a^+) - m(a^-)$ for any $a \in R$.Conversely , If m is once defined,T is characterized by the following equation

$$(Ta, a) = m(a) + m^-(\bar{a})$$

which is generalization of (C).This is the justification for our claim that our theorem characterizes L^p relating young's and Holders inequality-

Now by (D) $(T\lambda a, a) = ||T\lambda a||.||a||$ for any $\lambda > 0$.Therefore existence of such function $f_a(\lambda)$ is implied by strict convexity of the conjugate norm

$$T\lambda a = f_a(\lambda)Ta...........(E)$$

for all $\lambda > 0$ and $a \in R$ put

$$m(a) = \int_0^1 (T\lambda a, a) d\lambda$$

we get

$$m(\lambda[p]a) = \int_0^\lambda (Tn[p]a, a) dn$$

$$= \int_0^1 (Tnx, [p]a) dn$$

$$\int_0^{\lambda} f_a(n)dn.(T[p]x, x)$$

Hence it follows that -

$$\frac{m(\lambda[p]a)}{m([p]a)} = \frac{\int_0^{\epsilon} f_a(n)dn}{\int_0^{1} f_a(n)dn} = \frac{m(\lambda x)}{m(x)}.............(F)$$

for any $\lambda > 0[p]$ with $[p]x \neq 0$

Now we will prove that if (F) holds for any element a,we can find a no $p > 1$ such that $m(\lambda a) = \lambda^p m(a)$. To show this , we take a positive element a.Since R is at least two dimensional there exists $y > 0$ such that $x \sim y = 0$.Then, putting $C_\lambda = \lambda a + b$ by (F),

$$\frac{m(nC_\lambda)}{m(Z_\lambda)} = \frac{m(n[a]Z_\lambda)}{m([a]Z_\lambda)} = \frac{m(\lambda na)}{m(\lambda a)} \forall(\lambda, n > 0)$$

and

$$\frac{m(nC_\lambda)}{m(Z_\lambda)} = \frac{m(n[b]Z_\lambda)}{m([b]C_\lambda)} = \frac{m(np)}{m(p)}$$

$$\frac{m(\lambda na)}{m(a)} = \frac{m(na)}{m(a)} . \frac{m(\lambda a)}{m(a)}$$

Since $m(\lambda a)$ is continuous with respect to $\lambda > 0$, we can find $p \geq 1$ such that,$m(\lambda a) = \lambda^p m(a)$

Here p must be strictly greater then one,because T is one to one .From the definition of m it follows that $(T\lambda a, a) = \lambda^{p-1}(Ta, a)$

$$\Rightarrow T\lambda a = \lambda^{p-1} Ta$$

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