A Major Project Report on

Bernstein Operator and its Modifications

Submitted in partial fulfillment of requirements for the award of the Degree of

Master's in Mathematics

Submitted By:

Kanita (2K19/MSCMAT/05)

Under the guidance of:

Dr. Naokant Deo

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY

Shahbad Daulatpur, Main Bawana Road, Delhi-110042, India

May, 2021

DEPARTMENT OF APPLIED MATHEMATICS

DELHI TECHNOLOGICAL UNIVERSITY

(Formerly Delhi College of Engineering)
Bawana Road, New Delhi-110042

DECLARATION

I hereby declare that the work presented in this report entitled “Bernstein Operator and its Modifications”, in fulfillment of the requirement for the award of the degree Master’s in Mathematics, submitted in the Applied Mathematics Department of Delhi Technological University, New Delhi, is an authentic record of my own work carried out during my degree under the guidance of Dr. Naokant Deo.

The work reported in this has not been submitted by me for award of any other degree or diploma.

Kanita

Date: 20th May 2021

Name:
Kanita (2K19/MSCMAT/05)

DEPARTMENT OF APPLIED MATHEMATICS

DELHI TECHNOLOGICAL UNIVERSITY

(Formerly Delhi College of Engineering)
Bawana Road, New Delhi-110042

CERTIFICATE

This is to certify that the Project work entitled "Bernstein Operator and its Modifications" submitted by Kanita in fulfillment for the requirements of the award of Master's Degree in Mathematics is an authentic work carried out by her under my supervision and guidance. To the best of my knowledge, the matter embodied in the project has not been submitted to any other University / Institute for the award of any Degree.

Date: 20th May, 2020

Dr. Naokant Deo
Professor
Department of Applied Mathematics

DEPARTMENT OF APPLIED MATHEMATICS

DELHI TECHNOLOGICAL UNIVERSITY

(Formerly Delhi College of Engineering)
Bawana Road, New Delhi-110042

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ABSTRACT

S.N. Bernstein was the mathematician who proved the Weierstrass Theorem by defining Bernstein Polynomials and Operators. This paper illustrates the various forms of the Bernstein Operator and the different modifications done by other mathematicians in order to study and prove more theorems in Approximation Theory.

In this paper, we will be dealing with the classical Bernstein Operator, Bernstein-Kantorovich Operator, q-Bernstein Operator and Bernstein-Durrmeyer Operator.

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CHAPTER 1

BERNSTEIN POLYNOMIAL

1.1 Introduction

Polynomials are very helpful mathematical tools because of their simplified definitions. They can be calculated manually, and complicated ones can be simplified using programming languages. They represent a large variety of functions. Their derivatives and integrals can also be calculated easily, and as we know they can be used to approximate any function to any amount of accuracy as desired with the help of Taylor's and McLaurin's Expansion.

Polynomials are written in the following form

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

The above expression defines a polynomial as a linear combination of some defined elementary polynomials $\{1, t, t^2, \ldots, t^n\}$.

In general, we can write any polynomial function with degree less than or equal to n in the above notation as

- The set of all polynomials with degree less than or equal to n generates a vector space: sum of two polynomials will again be a polynomial, product of a scalar and a polynomial will again be in the set of polynomials and all the other properties of a vector space will also be true.
- The set $\{1, t, t^2, \ldots, t^n\}$ forms a basis of the vector space of polynomials with degree less than or equal to n – that is, such a polynomial can be uniquely expressed as a linear combinations of the functions of this set.

The above basis is known as the power basis. Though this is not the only basis of the vector space defined above. There are infinite basis of the vector space of all polynomials with degree less than or equal to n.

With this background of polynomials let us move forward and define the Bernstein Polynomial and then Bernstein Operator.

1.2 Bernstein Polynomial

The Bernstein Polynomials of degree n are defined by

$$B_{(k,n)}(t) = {}^{n}C_{k}\, t^{k}(1-t)^{n-k} \qquad \textit{for } k = 1,2,\ \dots\ n$$

$$B_{(k,n)}(t) = 0 \qquad \textit{for } k < 0 \textit{ or } k > n$$

There are (n+1) n^{th} degree Bernstein polynomials.

The first few polynomials are as follows.

- The Bernstein polynomials of degree 1 are

  $B_{1,1}(t) = t$

  $B_{0,1}(t) = 1-t$

  For $0 \le t \le 1$, they will represent the following curves

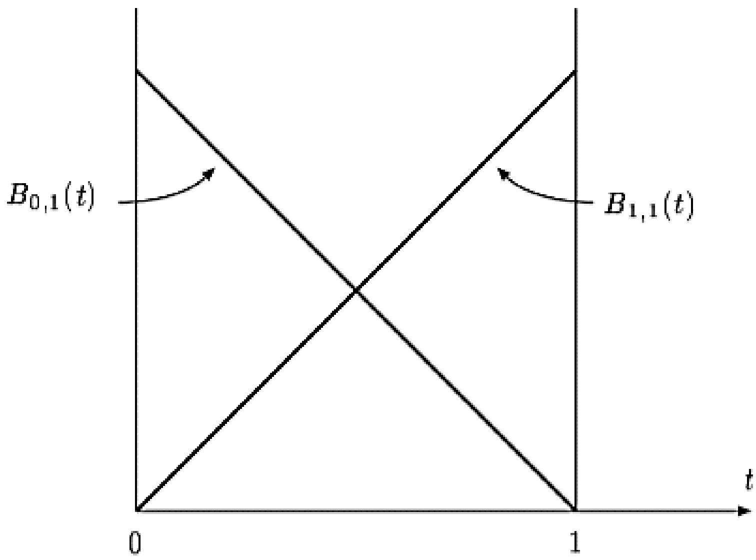

Figure 1.1

- The Bernstein polynomials of degree 2 are

$B_{0,2}(t) = (1-t)^2$

$B_{1,2}(t) = 2t(1-t)$

$B_{2,2}(t) = t^2$

For $0 \leq t \leq 1$, they will represent the following curves

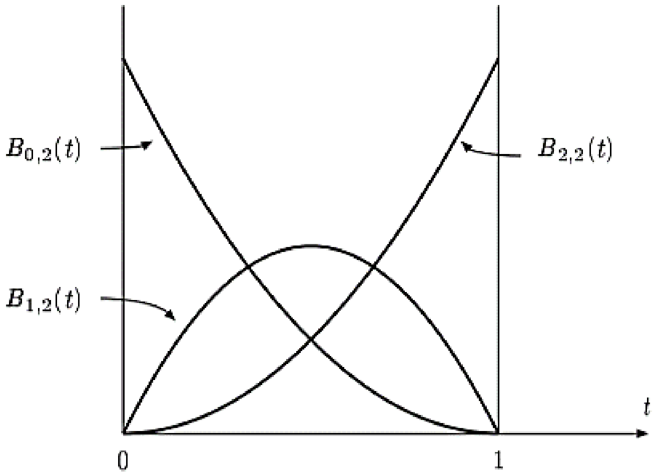

Figure 1.2

- The Bernstein polynomials of degree 3 are

$B_{0,3}(t) = (1-t)^3$

$B_{1,3}(t) = 3t(1-t)^2$

$B_{2,3}(t) = 3t^2(1-t)$

$B_{3,3}(t) = t^3$

For $0 \leq t \leq 1$, they will represent the following curves

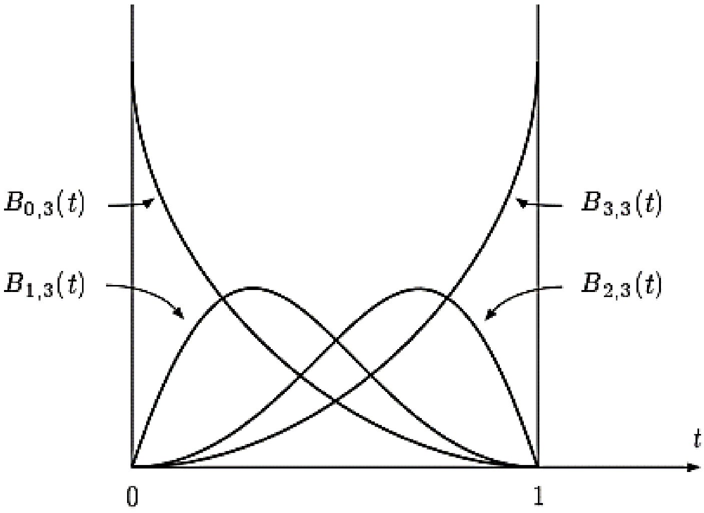

Figure 1.3

1.3 Bernstein Polynomials in Recursive Form

The Bernstein polynomials of degree n can be defined by a combination of two Bernstein polynomials of degree n − 1. That is, the kth, nth-degree Bernstein polynomial can be written as

$$B_{k,n}(t) = (1\text{-}t)B_{k,n-1}(t) + tB_{k-1,n-1}(t)$$

<u>Proof,</u>

$$(1\text{-}t)B_{k,n-1}(t) + tB_{k-1,n-1}(t) = (1\text{-}t)^{n-1}C_k\, t^k(1\text{-}t)^{n-1-k} + t\,^{n-1}C_{k-1}t^{k-1}(1\text{-}t)^{n-1-(k-1)}$$

$$= {}^{n-1}C_k\, t^k(1\text{-}t)^{n-k} + {}^{n-1}C_{k-1}\, t^k\,(1\text{-}t)^{n-k}$$

$$= [\,{}^{n-1}C_k + {}^{n-1}C_{k-1}\,]\, t^k\,(1\text{-}t)^{n-k}$$

$$= {}^{n-1}C_k\, t^k\,(1\text{-}t)^{n-k}$$

$$= B_{k,n}(t)$$

1.4 Properties of Bernstein Polynomial

- Symmetry:

$$B_{k,n}(t) = B_{n-k,n}(1-t)$$

- Non-negativity:

$$B_{k,n}(t) \geq 0 \qquad\qquad for\ 0 \leq t \leq 1$$

- Normalization:

$$\sum_{k=0}^{n} B_{k,n}(t) = 1$$

CHAPTER 2

BERNSTEIN OPERATOR

2.1 Bernstein Polynomial on *f*

Let *f* be a bounded real-valued function defined on the interval [0,1]

Let $B_n(f)$ be the polynomial on [0,1] defined as follows

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

This $B_n(f)$ is the nth Bernstein Polynomial for *f* also known as the Bernstein Operator.

- When $f(x) = 1$

$$B_n(1) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

$$= [x + (1-x)]^n$$

$$= 1$$

- When $f(x) = x$

Differentiating the binomial theorem,

$$\frac{d}{dp}\left(\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}\right) = \frac{d}{dp}\left((p+q)^n\right)$$

$$= n(p+q)^{n-1}$$

Thus, we get

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} p^k q^{n-k} = (p+q)^{n-1} p$$

Replacing q by 1-*x* and p by *x* we will arrive at the following

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^k (1-x)^{n-k}$$

$$= x$$

Similarly, we can derive the following identities:

$$B_n(x^2) = \sum_{k=0}^{n} \binom{n}{k} \frac{k^2}{n^2} x^k (1-x)^{n-k}$$

$$= \frac{(n-1)x^2}{n} + \frac{x}{n}$$

$$B_n(x^3) = \sum_{k=0}^{n} \binom{n}{k} \frac{k^3}{n^3} x^k (1-x)^{n-k}$$

$$= \frac{(n-1)(n-2)x^3}{n^2} + \frac{3(n-1)x^2}{n^2} + \frac{x}{n^2}$$

$$B_n(x^4) = \sum_{k=0}^{n} \binom{n}{k} \frac{k^4}{n^4} x^k (1-x)^{n-k}$$

$$= \frac{(n-1)(n-2)(n-3)x^4}{n^3} + \frac{6(n-1)(n-2)x^3}{n^3} + \frac{7(n-1)x^2}{n^3} + \frac{x}{n^3}$$

Combining the above equations, we get the following equations

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} = x(1-x)\frac{1}{n}$$

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^4 x^k(1-x)^{n-k} = x(1-x)\frac{(3n-6)x(1-x)+1}{n^3}$$

2.2 Theorem:

Let f be a real-valued function defined, and bounded by M on the interval [0, 1]. For each point x of continuity of f, $B_n(f)(x) \to f(x)$ as $n \to \infty$.

If f is continuous on [0, 1], then the Bernstein polynomial $B_n(f)$ tends uniformly to f as $n \to \infty$.

With x a point of differentiability of f, $B'_n(f)(x) \to f'(x)$ as $n \to \infty$.

If f is continuously differentiable on [0, 1], then $B'_n(f)$ tends to f' uniformly as $n \to \infty$.

Proof:

Consider $$B_n(f)(x) - f(x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right) - f(x)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}$$
$$= \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left[f\left(\frac{k}{n}\right) - f(x)\right]$$

Thus, for each *x* in [0,1]

$$|B_n(f)(x) - f(x)| \leq \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\left|f\left(\frac{k}{n}\right) - f(x)\right|$$

To calculate the RHS we break the summation into two parts- first part where $\left|\frac{k}{n} - x\right| < \delta$ and the second part contains those terms for which $\left|\frac{k}{n} - x\right| \geq \delta$ denoted by $\sum'$ and $\sum''$ respectively.

Suppose that *f* is continuous at *x*

Then for any $\varepsilon > 0$, there exists a $\delta \geq 0$ such that

$$|f(x') - f(x)| < \frac{\varepsilon}{2} \qquad \text{when } |x' - x| < \delta$$

And hence, for the first summation $\sum'$

$$\sum' \binom{n}{k} x^k(1-x)^{n-k}\left|f\left(\frac{k}{n}\right)-f(x)\right| < \sum' \binom{n}{k} x^k(1-x)^{n-k}\frac{\varepsilon}{2}$$

$$\leq \frac{\varepsilon}{2}\sum_{k=0}^{n}\binom{n}{k} x^k(1-x)^{n-k}$$

$$= \frac{\varepsilon}{2}$$

Now, for the second summation $\sum''$ we have

$$\delta \leq \left|\frac{k}{n} - x\right| \Rightarrow \delta^{2} \leq \left|\frac{k}{n} - x\right|^2$$

$$\delta^2 \sum'' \binom{n}{k} x^k(1-x)^{n-k}\left|f\left(\frac{k}{n}\right)-f(x)\right|$$

$$\leq \sum'' \binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^k(1-x)^{n-k}\left|f\left(\frac{k}{n}\right)-f(x)\right|$$

$$\leq \sum'' \binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^k(1-x)^{n-k}2M$$

$$\leq 2M\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^k(1-x)^{n-k}$$

$$= 2M\frac{x(1-x)}{n}$$

$$\leq \frac{2M}{n}$$

Thus,

$$\sum'' \binom{n}{k} x^k(1-x)^{n-k}\left|f\left(\frac{k}{n}\right)-f(x)\right| \leq \frac{2M}{\delta^2 n}$$

For this δ, by Archimedean Principle we can choose n_0 so that, when $n \geq n_0$ $\frac{2M}{\delta^2 n}$ then $\leq \frac{\varepsilon}{2}$

For this n_0, we will have

$$
\begin{aligned}
|B_n(f)(x) - f(x)| &\le \sum{}' + \sum{}'' \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

And hence, $B_n(f)(x) \to f(x)$ as $n \to \infty$ for each point x where the function f is continuous.

Moreover, if f is continuous at each point of [0,1] then f is uniformly continuous on [0,1].

For this given ε, we choose δ such that:

$$|f(x') - f(x)| < \frac{\varepsilon}{2}\ \forall\, x', x \in [0,1] \qquad \text{whenever} \qquad |x' - x| < \delta$$

Thus, for this chosen n_0 and for this δ when $n \ge n_0 |B_n(f)(x) - f(x)| \le \varepsilon \qquad \forall\, x \in [0,1]$

Thus, $\|B_n(f) - f\| \le \varepsilon$

And $B_n(f)$ tends uniformly to f as $n \to \infty$.

Now, for $x \in [0,1]$

$$
\begin{aligned}
B_n'(f) &= \frac{d}{dx}\left(\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right)\right) \\
&= \sum_{k=1}^{n}\binom{n}{k}kx^{k-1}(1-x)^{n-k}f\left(\frac{k}{n}\right) - \sum_{k=0}^{n-1}\binom{n}{k}(n-k)x^k(1-x)^{n-k-1}f\left(\frac{k}{n}\right) + \left[nx^{n-1}f(1) - n(1-x)^{n-1}f(0)\right] \\
&= \sum_{k=0}^{n}\binom{n}{k}\left[k(1-x) - (n-k)x\right]x^{k-1}(1-x)^{n-k-1}f\left(\frac{k}{n}\right) \\
&= n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n} - x\right)x^{k-1}(1-x)^{n-k-1}f\left(\frac{k}{n}\right)
\end{aligned}
$$

Note: *When k = 0 then* $\left(\frac{k}{n} - x\right)x^{k-1} = -1$

When k = n then $\left(\frac{k}{n} - x\right)(1-x)^{n-k-1} = 1$

Also,

$$0 = f(x)\frac{d}{dx}(1) = f(x)\frac{d}{dx}\left(\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\right)$$
$$= f(x)n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)x^{k-1}(1-x)^{n-k-1}$$

Thus, $\forall\, x \in [0,1]$

$$B_n'(f)(x) = n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)x^{k-1}(1-x)^{n-k-1}\left[f\left(\frac{k}{n}\right)-f(x)\right]$$

Now, let us suppose that *f* is differentiable at *x*.

Let $\varepsilon > 0$ be given

Then, we write

$$\frac{f\left(\frac{k}{n}\right)-f(x)}{\frac{k}{n}-x} = f'(x)+\xi_k$$

Now, since *f* is differentiable at *x*

Hence, there exists δ such that whenever $0 < |x' - x| < \delta$ then $\left|\frac{f(x')-f(x)}{x'-x} - f'(x)\right| < \frac{\varepsilon}{2}$

Thus, when $0 < \left|\frac{k}{n} - x\right| < \delta$ then $|\xi_k| = \left|\frac{f\left(\frac{k}{n}\right)-f(x)}{\frac{k}{n}-x} - f'(x)\right| < \frac{\varepsilon}{2}$

It follows that

$$B_n'(f)(x) = n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)x^{k-1}(1-x)^{n-k-1}\left[f\left(\frac{k}{n}\right)-f(x)\right]$$

$$= n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)x^{k-1}(1-x)^{n-k-1}\left[\left(\frac{k}{n}-x\right)f'(x)+\left(\frac{k}{n}-x\right)\xi_k\right]$$

$$= f'(x)n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^2x^{k-1}(1-x)^{n-k-1} + n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^2x^{k-1}(1-x)^{n-k-1}\xi_k$$

$$= f'(x) + n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^2x^{k-1}(1-x)^{n-k-1}\xi_k$$

Again, we calculate this last sum by breaking it into the two summations $\sum'$ and $\sum''$- first part where $\left|\frac{k}{n} - x\right| < \delta$ and the second part contains those terms for which $\left|\frac{k}{n} - x\right| \geq \delta$

For the first summation $\sum'$, we have

$$\begin{aligned}
|n\sum\nolimits'| &\leq n\sum\nolimits' \binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^{k-1}(1-x)^{n-k-1}|\xi_k| \\
&< n\sum\nolimits' \binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^{k-1}(1-x)^{n-k-1}\frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2}n\sum_{k=0}^{n} \binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^{k-1}(1-x)^{n-k-1} \\
&= \frac{\varepsilon}{2}
\end{aligned}$$

For the second summation, $\left|\frac{k}{n} - x\right| \geq \delta$ so that

$$|\xi_k| \leq \left|\frac{f\left(\frac{k}{n}\right)-f(x)}{\frac{k}{n}-x}\right| + |f'(x)| \leq \frac{2M}{\delta} + |f'(x)|$$

And $\left|\frac{k}{n} - x\right|^2 \geq \delta^{\,2}$ will imply that

$$\begin{aligned}
\delta^2|n\sum\nolimits''| &\leq \delta^2 n\sum\nolimits'' \binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^{k-1}(1-x)^{n-k-1}|\xi_k| \\
&\leq n\sum\nolimits'' \binom{n}{k}\left(\frac{k}{n}-x\right)^4 x^{k-1}(1-x)^{n-k-1}|\xi_k| \\
&\leq n\sum\nolimits'' \binom{n}{k}\left(\frac{k}{n}-x\right)^4 x^{k-1}(1-x)^{n-k-1}\left(\frac{2M}{\delta} + |f'(x)|\right) \\
&\leq n\sum_{k=0}^{n} \binom{n}{k}\left(\frac{k}{n}-x\right)^4 x^{k-1}(1-x)^{n-k-1}\left(\frac{2M}{\delta} + |f'(x)|\right) \\
&= n\frac{(3n-6)x(1-x)+1}{n^3}\left(\frac{2M}{\delta} + |f'(x)|\right) \\
&\leq n\frac{3}{n^2}\left(\frac{2M}{\delta} + |f'(x)|\right) \\
&= \frac{6M+3\delta|f'(x)|}{n\delta}
\end{aligned}$$

Thus

$$\left|n\sum''\right| \leq \frac{6M+3\delta|f'(x)|}{n\delta^3}$$

For this δ, by Archimedean Principle we choose n_0 so that, when $n \geq n_0$ $\frac{6M+3\delta|f'(x)|}{n\delta^3} < \frac{\varepsilon}{2}$

And hence,

$$\begin{aligned}\left|B_n'(f)(x) - f'(x)\right| &\leq \left|n\sum'\right| + \left|n\sum''\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Hence, for each point x where f is differentiable, $B'_n(f)(x) \to f'(x)$ as $n \to \infty$.

We now show that

The sequence $\{B'_n(f)\}$ tends uniformly to f' where f is continuously differentiable on [0,1].

For all points x where f is differentiable, we have:

$$B_n'(f)(x) = f'(x) + n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^{k-1}(1-x)^{n-k-1}\xi_k$$

Assuming that f is differentiable for all x in [0, 1] and the derivative of f is continuous on [0,1]

Let M' be the supremum of $|f'(x)|$ where $x \in [0,1]$

Choose $\delta > 0$ such that $|f'(x') - f'(x)| < \frac{\varepsilon}{2}$ whenever $|x'-x| < \delta$

Now, $\forall\, x \in [0,1]$ we had defined

$$\xi_k = \frac{f\left(\frac{k}{n}\right) - f(x)}{\frac{k}{n} - x} - f'(x) \quad \text{when } \frac{k}{n} \neq x, \quad \text{and}$$

$$\xi_k = 0 \qquad \text{when } \frac{k}{n} = x$$

Since f is differentiable on [0,1]

Hence, we can apply the Mean Value Theorem on $x_k \in \left(\frac{k}{n}, x\right)$

$$f\left(\frac{k}{n}\right) - f(x) = f'(x_k)\left(\frac{k}{n} - x\right)$$

For the case when $\frac{k}{n} = x$, choose $x = x_k$

With these choices $\xi_k = f'(x_k) - f'(x)$

Thus we get

$$B_n'(f)(x) - f'(x) = n\sum_{k=0}^{n} \binom{n}{k}\left(\frac{k}{n} - x\right)^2 x^{k-1}(1-x)^{n-k-1}\left(f'(x_k) - f'(x)\right)$$

In this case again we estimate the summation in the RHS of this inequality by breaking it into two parts $\sum'$ and $\sum''$ like done before for the approximation of the derivative at x.

Here however, $\xi_k = f'(x_k) - f'(x)$

And we chose this δ by the definition of uniform continuity of f' on [0,1]

such that $|f'(x_k) - f'(x)| < \frac{\varepsilon}{2}$ whenever $|x_k - x| < \delta$

Now for the first summation $\sum'$ over those k such that $\left|\frac{k}{n} - x\right| < \delta$

$$n\sum{}' \binom{n}{k}\left(\frac{k}{n} - x\right)^2 x^{k-1}(1-x)^{n-k-1}|f'(x_k) - f'(x)| < \frac{\varepsilon}{2} n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n} - x\right)^2 x^{k-1}(1-x)^{n-k-1}$$
$$= \frac{\varepsilon}{2}$$

For the second summation $\sum''$ over those k such that $\delta \le \left|\frac{k}{n} - x\right|$

We have $\delta^2 \le \left|\frac{k}{n} - x\right|^2$

For this case $|f'(x_k) - f'(x)| < 2M'$

And thus,

$$\begin{aligned}
&\delta^2 n \sum'' \binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^{k-1}(1-x)^{n-k-1}|f'(x_k)-f'(x)| \\
\leq & n \sum'' \binom{n}{k}\left(\frac{k}{n}-x\right)^4 x^{k-1}(1-x)^{n-k-1} 2M' \\
\leq & 2M' n \sum_{k=0}^{n} \binom{n}{k}\left(\frac{k}{n}-x\right)^4 x^{k-1}(1-x)^{n-k-1} \\
= & 2M' n \frac{(3n-6)x(1-x)+1}{n^3} \\
\leq & \frac{6M'}{n}
\end{aligned}$$

Again, by Archimedean Principle, when n > n_0

$$n \sum'' \binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^{k-1}(1-x)^{n-k-1}|f'(x_k)-f'(x)| \leq \frac{6M'}{\delta^2 n} < \frac{\varepsilon}{2}$$

And finally, ∀ x ∈ [0,1],

$$\begin{aligned}
\left|B_n'(f)(x)-f'(x)\right| \leq & n \sum' \binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^{k-1}(1-x)^{n-k-1}|f'(x_k)-f'(x)| \\
& + n \sum'' \binom{n}{k}\left(\frac{k}{n}-x\right)^2 x^{k-1}(1-x)^{n-k-1}|f'(x_k)-f'(x)| \\
< & \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
= & \varepsilon
\end{aligned}$$

Thus, $\|B_n(f') - f'\| \leq \varepsilon$

And $\{B'_n(f)\}$ tends to f' uniformly.

CHAPTER 3

BERNSTEIN - KANTOROVICH OPERATOR

Let f be a bounded real-valued function defined on the interval [0,1]

Then the classical Bernstein Polynomial of f, $B_n(f)$ on [0,1], is defined as follows

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

or

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right) \qquad \text{where} \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{(n-k)}$$

Now, let f: [0,1] $\longrightarrow$ $\mathbb{R}$ be an integrable function

The classical Bernstein - Kantorovich operators are defined by

$$K_n(f;x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt, \;\; x \in [0,1], \;\; n \in \mathbb{N}$$

It can also be written as

$$K_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 f\left(\frac{k+t}{n+1}\right) dt \qquad \text{where} \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{(n-k)}$$

Over the years, there have been several modifications on Bernstein and Kantorovich Operator. One such modification is when we replace t be t^α for $\alpha > 0$ in the classical Bernstein- Kantorovich Operator.

This modification does not affect the linearity or positivity of K_n, rather it generates a new sequence as follows:

$$K_{n,\alpha}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 f\left(\frac{k+t^\alpha}{n+1}\right) dt \quad \text{where } p_{n,k} \text{ is the same as defined above}$$

A similar modification is the q-type generalization of Bernstein - Kantorovich polynomial operators as follows

$$B^*_{n,q}(f,x) := \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t \qquad \text{where } f \in C[0,1],\ 0 < q < 1$$

3.1 *q* – Bernstein - Kantorovich operator

Definitions: Let q > 0

For any n ϵ ℕ ∪ {0}, the q-integer [n] = $[n]_q$ is defined by

$[n] = 1 + q + ... + q^{n-1}$

[0] = 0

The q-factorial [n]! = $[n]_q$! is defined by

[n]! = [1][2]…[n]

[0]! = 1

For integers $0 \le k \le n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\ [n-k]!}$$

The q-analogue of integration in the interval [0,A] is defined by

$$\int_0^A f(t)\, d_q t := A(1-q) \sum_{n=0}^{\infty} f(Aq^n)\, q^n \qquad \text{for } 0 < q < 1$$

Now, based on the above definitions, we define the q-type Kantorovich modification as follows

$$B^*_{n,q}(f,x) = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f\left(\frac{[k]+q^k t}{[n+1]}\right) d_q t \qquad 0 \le x \le 1,\ n \in \mathbb{N}$$

where,

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)_q^{n-k} \quad \text{and} \quad (1-x)_q^n := \prod_{s=0}^{n-1}(1-q^s x)$$

Remark:

We notice that, as $q \to 1^-$ the q-Bernstein-Kantorovich operator becomes the classical Bernstein-Kantorovich operator.

Lemma: We have the following recurrence formula for $B^*_{n,q}$

For $n \in \mathbb{N},\ x \in [0,1]\ and\ 0 < q \le 1$, we have

$$B^*_{n,q}(t^m,x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n-1)^i B_{n,q}\left(t^{j+i},x\right)$$

Proof:

$$\begin{aligned}
B^*_{n,q}(t^m,x) &= \sum_{k=0}^{n} p_{n,k}(q;x) \sum_{j=0}^{m} \int_0^1 \binom{m}{j} \frac{[k]^j q^{k(m-j)} t^{m-j}}{[n+1]^m} d_q t \\
&= \sum_{k=0}^{n} p_{n,k}(q;x) \sum_{j=0}^{m} \binom{m}{j} \frac{q^{k(m-j)} [k]^j}{[n+1]^m [m-j+1]} \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{k=0}^{n} \left(q^k - 1 + 1\right)^{m-j} \frac{[k]^j}{[n]^j} p_{n,k}(q;x) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{k=0}^{n} \sum_{i=0}^{m-j} \binom{m-j}{i} \left(q^k - 1\right)^i \frac{[k]^j}{[n]^j} p_{n,k}(q;x) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i \sum_{k=0}^{n} \frac{[k]^{j+i}}{[n]^{j+i}} p_{n,k}(q;x) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i B_{n,q}\left(t^{j+i},x\right).
\end{aligned}$$

Hence Proved.

Using the above lemma, we obtain $B^*_{n,q}(1,x)$, $B^*_{n,q}(t,x)$ and $B^*_{n,q}(t^2,x)$ explicitly as follows:

For all $n \in \mathbb{N}, x \in [0,1]$ *and* $0 < q \leq 1$, we have

$$B^*_{n,q}(1,x) = 1$$

$$B^*_{n,q}(t,x) = \frac{2q}{[2]}\frac{[n]}{[n+1]}x + \frac{1}{[2]}\frac{1}{[n+1]}$$

$$B^*_{n,q}\left(t^2,x\right) = \frac{q(q+2)}{[3]}\frac{q[n][n-1]}{[n+1]^2}x^2 + \frac{4q+7q^2+q^3}{[2][3]}\frac{[n]}{[n+1]^2}x + \frac{1}{[3]}\frac{1}{[n+1]^2}$$

Verification:

$$B^*_{n,q}(t,x) = \frac{1}{[n+1][2]}\left(B_{n,q}(1,x) + (q^n-1)B_{n,q}(t,x)\right) + \frac{[n]}{[n+1]}B_{n,q}(t,x)$$

$$= \left(\frac{q^n-1}{[2][n+1]} + \frac{[n]}{[n+1]}\right)x + \frac{1}{[2][n+1]}$$

$$= \frac{2q}{[2]}\frac{[n]}{[n+1]}x + \frac{1}{[2][n+1]}$$

$$B^*_{n,q}\left(t^2,x\right) = \frac{1}{[3][n+1]^2}\left(B_{n,q}(1,x) + 2(q^n-1)B_{n,q}(t,x) + (q^n-1)^2 B_{n,q}\left(t^2,x\right)\right)$$

$$+ \frac{2[n]}{[2][n+1]^2}\left(B_{n,q}(t,x) + (q^n-1)B_{n,q}\left(t^2,x\right)\right) + \frac{[n]^2}{[n+1]^2}B_{n,q}\left(t^2,x\right)$$

$$= \frac{1}{[3][n+1]^2} + \left(\frac{[n]^2}{[n+1]^2} + \frac{2[n](q^n-1)}{[2][n+1]^2} + \frac{(q^n-1)^2}{[3][n+1]^2}\right)\left(1 - \frac{1}{[n]}\right)x^2$$

$$+ \left(\frac{[n]^2}{[n][n+1]^2} + \frac{2[n](q^n-1)}{[2][n][n+1]^2} + \frac{(q^n-1)^2}{[3][n][n+1]^2} + \frac{2[n]}{[2][n+1]^2} + \frac{2(q^n-1)}{[3][n+1]^2}\right)x$$

$$= \frac{2q+3q^2+q^3}{[2][3]}\frac{q[n][n-1]}{[n+1]^2}x^2 + \frac{4q+7q^2+q^3}{[2][3]}\frac{[n]}{[n+1]^2}x + \frac{1}{[3][n+1]^2}$$

Hence Proved.

Note: *Again, for q = 1 we see that we get the moments as that of the Bernstein-Kantorovich operator.*

Theorem:

Let $\{q_n\}$ be a sequence of real numbers such that $0 < q_n < 1$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$.

Then we have

$$\lim_{n\to\infty} [n]_{q_n} B^*_{n,q_n}(t-x;x) = -\frac{1+a}{2}x + \frac{1}{2}$$

$$\lim_{n\to\infty} [n]_{q_n} B^*_{n,q_n}\left((t-x)^2;x\right) = -\frac{1}{3}x^2 - \frac{2}{3}ax^2 + x$$

Proof:

Using the formulae of $B^*_{n,q_n}(t;x)$ and $B^*_{n,q_n}(t^2;x)$ obtained above we get the following

$$\lim_{n\to\infty} [n]_{q_n} B^*_{n,q_n}(t-x;x) = \lim_{n\to\infty}\left\{[n]_{q_n}\left(\frac{2q_n}{[2]_{q_n}}\frac{[n]_{q_n}}{[n+1]_{q_n}} - 1\right)x + \frac{1}{[2]_{q_n}}\frac{[n]_{q_n}}{[n+1]_{q_n}}\right\}$$

$$= \lim_{n\to\infty}\left\{-\frac{[n]_{q_n}}{[n+1]_{q_n}}\frac{1+q_n^{n+1}}{[2]_{q_n}}x + \frac{1}{[2]_{q_n}}\frac{[n]_{q_n}}{[n+1]_{q_n}}\right\}$$

$$= -\frac{1+a}{2}x + \frac{1}{2}$$

$$\lim_{n\to\infty} [n]_{q_n} B^*_{n,q}\left((t-x)^2, x\right) = \lim_{n\to\infty} [n]_{q_n}\left(B^*_{n,q}\left(t^2, x\right) - x^2 - 2xB^*_{n,q}(t-x,x)\right)$$

$$= \lim_{n\to\infty} [n]_{q_n}\left(\frac{q_n(q_n+2)}{[3]_{q_n}}\frac{[n]^2_{q_n} - [n]_{q_n}}{[n+1]^2_{q_n}} - 1\right)x^2$$

$$+ \lim_{n\to\infty} [n]_{q_n}\frac{4q_n + 7q_n^2 + q_n^3}{[2]_{q_n}[3]_{q_n}}\frac{[n]_{q_n}}{[n+1]^2_{q_n}}x - \lim_{n\to\infty} [n]_{q_n} 2xB^*_{n,q_n}(t-x,x)$$

$$= \lim_{n\to\infty} q_n(1-q_n^n)\left(2q_n + q_n^2 + 2\right)x^2 - \lim_{n\to\infty}\left(4q_n + 3q_n^2 + 2q_n^3\right)x^2$$

$$+ \lim_{n\to\infty} \frac{4q_n + 7q_n^2 + q_n^3}{[2]_{q_n}[3]_{q_n}} x - \lim_{n\to\infty} [n]_{q_n} 2xB^*_{n,q_n}(t-x,x)$$

$$= \frac{5}{3}(1-a)x^2 - 3x^2 + 2x + (1+a)x^2 - x$$
$$= -\frac{1}{3}x^2 - \frac{2}{3}ax^2 + x.$$

Hence Proved.

Now we will see another modification of the Bernstein Operator, known as the

Bernstein - Durrmeyer Operator

CHAPTER 4

BERNSTEIN-DURRMEYER OPERATOR

The operator introduced by Durrmeyer are defined as

$$D_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt, \quad x \in [0,1] \qquad \text{where,} \quad p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$$

V. Gupta introduced a different type of Durrmeyer modification of the Bernstein Polynomials stated as follows

$$B_n(f,x) = n\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 b_{n,k}(t)f(t)dt, \quad x \in [0,1]$$

where, $\quad p_{n,k}(x) = (-1)^k \frac{x^k}{k!}\phi_n^{(k)}(x),$

$$b_{n,k}(t) = (-1)^{k+1}\frac{t^k}{k!}\phi_n^{(k+1)}(t) \qquad \text{and}$$

$$\phi_n(x) = (1-x)^n$$

The value of $p_{n,k}(x)$ used in both the above definitions is the same, only the representation has changed.

Also, we can easily see that $\quad \sum_{k=0}^{n} p_{n,k}(x) = 1,$

$$\int_0^1 b_{n,k}(t)dt = 1 \quad \text{and}$$

$$b_{n,n}(t) = 0$$

We can further study on another integral modification of the Bernstein Polynomial defined as follows

$$B_{n,\alpha}(f,x) = \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x)\int_0^1 b_{n,k}(t)f(t)dt, \quad x \in [0,1]$$

where,

$$Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$$

and, $$J_{n,k}(x) = \sum_{j=k}^{n} p_{n,j}(x) \qquad \text{when } k \le n$$

$$J_{n,k}(x) = 0 \qquad \text{otherwise.}$$

Properties of $J_{n,k}(x)$

(i) $J_{n,k}(x) - J_{n,k+1}(x) = p_{n,k}(x), k = 0, 1, 2, 3, \ldots;$

(ii) $J'_{n,k}(x) = np_{n-1,k-1}(x), k = 1, 2, 3, \ldots;$

(iii) $J_{n,k}(x) = n\int_0^x p_{n-1,k-1}(u)du, k = 1, 2, 3, \ldots;$

(iv) $J_{n,0}(x) > J_{n,1}(x) > J_{n,2}(x) > \cdots > J_{n,n}(x) > 0, 0 < x < 1$

That is, for every natural number k,

$J_{n,k}(x)$ is strictly increasing from 0 to 1.

Alternatively, we may rewrite the operator as

$$B_{n,\alpha}(f, x) = \int_0^1 K_{n,\alpha}(x,t)f(t)dt, \quad 0 \le x \le 1 \qquad \text{where} \quad K_{n,\alpha}(x,t) = \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x)b_{n,k}(t)$$

Note: *For* $\alpha = 1$, *this modified version of the operator reduces to the classical version of the Bernstein-Durrmeyer Operator as defined by V. Gupta.*

CONCLUSION

In this paper, we studied the various forms of the Bernstein Operator and their convergence properties. The idea of approximating functions using Bernstein Polynomials was proposed by Sergei Natanovich Bernstein in 1912, after whom the operator is named. Since then, many mathematicians have worked their theories on and used them as a guide to prove multiple theorems in Approximation Theory. Later in the years, mathematician L.V. Kantorovich devised a modification of these operators, known as the Bernstein-Kantorovich polynomials. The advantage of this modification was that they were defined over a larger class of functions and not just polynomials (unlike the classical Bernstein Operator). Another interesting modification of the Bernstein polynomials was introduced by J.L. Durrmeyer for approximating functions. The recurrence formula of these operators help us understand more in the field of convergence of these operators.

There are various other operators, such as Szász, Lupas, and Baskakov operators. The Kantorovich and Durremeyer modifications are applied to these operators as well. However, in the paper, we focused mainly on the Bernstein Operator and its modifications. In this paper, we also defined q-integers and used them to studied various properties of the modified operators.

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