

DFT (DISCRETE FOURIER TRANSFORM)

A DISERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE AWARD OF THE DEGREE

OF

MASTER OF SCIENCE

IN

APPLIED MATHEMATICS

Submitted by:

SANDEEP NEGI

2K21/MSCMAT/43

Under the supervision of

MR.JAMKHONGAM TOUTHANG



DEPARTMENT OF MATHEMATICS

DELHI TECHNOLOGICAL UNIVERSITY

(Formerly Delhi College of Engineering)

Bawana road, Delhi-110042

May2023

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I Sandeep Negi, Roll No. 2k21/MSCMAT/43 student of M.Sc.(Applied Mathematics), hereby declare that the project Dissertation titled "DFT(Discrete Fourier Transform)" which is submitted by me to the Department of Mathematics, Delhi Technological University, Delhi in partial fulfillment for the degree of Master of Science, is original copied from any source without proper citation. This work has not previously formed the basis for the award of any Degree, Diploma Association, Fellowship or other similar title or recognition.

NAME OF STUDENT

Sandeep Negi

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NAME OF SUPERVISOR

Mr. Jamkhongam Thouthang

SUPERVISOR

ACKNOWLEDGEMENT

It is with great pride, interest, and expectation that I offer you to this Dissertation work. An huge amount of work has gone into the growth of this work and I trust you will see that effort reflected in this. It is significant to keep in mind that it signifies the collective thinking of a set of innovative individuals.

I would like to acknowledge my indebtedness and render my deepest thanks to my supervisor, Professor Mr. Jamkhongam Touthang, who made this work possible. His friendly supervision and skillful advice have been invaluable throughout all phases of the work.

A great work cannot be made great without a dedicated editorial team of publishing supervisor and commentators. I express my deep sense of thankfulness and sincere thanks to all the scholars and employees who have contributed their time and have made this work a great achievement.

Nobody has been more important to me in the search of this project than the members of my family. I would like to thank my parents, whose love and direction are with me in whatever I chase. They are the ultimate role model.

This thesis has been written in my stay at the Mathematics Department of the Delhi Technological University. I would like to acknowledge the Department of Mathematics for providing brilliant working environments.

Regards

Sandeep Negi

ABSTRACT

The outline you have provided covers the main topics related to the Discrete Fourier Transform (DFT) and its applications. Here's a breakdown of the sections you mentioned:

Sinusoids and Exponentials: This section introduces sinusoids, exponentials, complex sinusoids, and their properties. It covers concepts such as t_{60} (decay time), in-phase and quadrature components, the analytic signal, positive and negative frequencies, interference, and the relationship between circular motion and sinusoidal motion. It may also include examples and plots created using Mathematica.

The Discrete Fourier Transform (DFT) Resulting: The DFT is described in this section as a projection of a signal onto a collection of sampled complex sinusoids obtained from the unity roots. It may cover the mathematical formulation of the DFT and its relationship to the time and frequency domains.

Fourier Theorems for the DFT: This section presents various Fourier theorems specific to the DFT. The symmetry relations, the shift theorem, the convolution theorem, the correlation theorem, the power theorem, and theorems pertaining to interpolation and downsampling are some of the theorems in this group. It explores the applications of these theorems in areas such as statistical signal processing, sampling rate conversion, and linear time invariant filtering.

Use cases for the DFT: This section provides actual instances of FFT (Fast Fourier Transform) analysis using MATLAB. It demonstrates the application of the Fourier theorems discussed earlier in understanding and analyzing spectral data. The examples may include topics like spectrum analysis, windowing, zero-padding, and interpreting the results.

Overall, this outline covers the fundamental concepts and applications of the DFT, providing readers with a comprehensive understanding of spectral analysis and the Fourier theorems relevant to the DFT.

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Chapter 1

1 "DFT Primer: An Introduction to the Discrete Fourier Transform"

The Discrete Fourier Transform (DFT) and opinions on the element that will be applied to this bibliophile are presented in this section.

1.1 Defining the Discrete Fourier Transform (DFT)

The computation of the For a signal x, discrete Fourier Transform (DFT) elucidated through

$$X(\omega_k) \triangleq \sum_{n=0}^{N-1} x(t_n) e^{-j\omega_k t_n}, k = 0, 1, 2, \dots, N - 1$$

and its opposite transformation (also known as the IDFT, Inverse Discrete Fourier Transform)

$$x(t_n) = \frac{1}{N} \sum_{k=0}^{N-1} X(\omega_k) e^{j\omega_k t_n}, n = 0, 1, 2, \dots, N - 1$$

here,

$x(t_n) \triangleq$ amplitude of the input signal at phase t_n (sec)

$t_n = nT = nth$ an example second.

$n \triangleq$ trial number (integer)

$T \triangleq$ sample period(sec)

$X(\omega_k) \triangleq$ The frequency spectrum of signal x at a specific radian frequency. ω_k

$\omega_k \triangleq k\omega = kth$ frequency trial

$\omega \triangleq \frac{2\pi}{NT} =$ Sample frequency in radians

$f_s \triangleq 1/T =$ sample rate (samples/sec, or Hz)

N is the total number of time- and frequency-varying trials.

1.2 The DFT's arithmetic

The additional pure form produced by setting the DFT must be used to express the equation. $T = 1$ in the preceding formulation when performing signal processing:

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, k = 0, 1, 2, \dots, N - 1$$
$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N}, n = 0, 1, 2, 3, \dots, N - 1$$

In this context, $x(n)$ represents the input signal at time (sample) n , while $X(k)$ denotes the spectral representation at the k th frequency bin. Although the previous system is more straightforward to comprehend intuitively, this system is mathematically more elegant. Here two remaining signs in the DFT that we have not yet clear:

$$j \triangleq \sqrt{-1}$$

$$e \triangleq \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.7182\dots$$

Complex numbers are based on the first equation, $j = \sqrt{-1}$.

2nd, $e = 2.718\dots$, which is a transcendental number by the aforementioned restriction. We shall derive e from this.

It's crucial to notice that we have them looking in exponents, as in, in addition to complex numbers to contrast with.

$$s_k(n) \triangleq e^{j2\pi nk/N}$$

In a systematic manner, we will explore the concept of imaginary exponents and clarify the meaning behind such mathematical expressions.

We may demonstrate Euler's identity if e , j , and the imaginary exponent are in agreement:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Euler's Identity plays a crucial role in understanding the meaning of terms such as

$$s_k(t_n) \triangleq e^{j\omega kt_n} = \cos(\omega kt_n) + j\sin(\omega kt_n)$$

By examining such an expression, we can comprehend that it represents a complex sinusoid. In the subsequent discussion, we will delve into the details of sinusoids, particularly from an audio perspective.

Finally, it is important to recognize the role of the summation over n in explaining the Discrete Fourier Transform (DFT). We will explore that the summation can be interpreted as

the calculation of the inner product between the signal x and the complex sinusoid s_k . With this knowledge, we can use the following inner-product representation to represent the DFT:

$$X(k) \triangleq \langle x, s_k \rangle$$

where

$$s_k(n) \triangleq e^{j2\pi nk/N}$$

is the complex sinusoid evaluated at the normalized radian frequency $\omega k = 2\pi k/N$, and the inner product operation can be understood as

$$\langle x, y \rangle \triangleq \sum_{n=0}^{N-1} x(n)\overline{y(n)}$$

Since it's complex numbers, We will show that the inner product of x with the k th "basis sinusoid" s_k is a measurement of "how much" and at "what point" of s_k is present in x .

Following the previous, it may be inferred that the inverse DFT is the weighted sum of projections of x onto s_k for $k=0$ to $N-1$

, i.e.,

here

$$X_k \triangleq \frac{\tilde{X}(k)}{N}$$

It should be noted that the projection quantities of the basis sinusoid \tilde{X}_k and s_k are both complex.

Having established a comprehensive understanding of the DFT and its inverse from a mathematical standpoint, we can now proceed to explore several Fourier theorems. These theorems, Theorems, such as the "Theorem of convolution," the "Changing Theorem," and the "Theorem of Parseval," offer a practical foundation for modifying and analysing signals in both the time and frequency domains.

Furthermore, we will delve into practical spectrum analysis applications, utilizing primarily MATLAB as a tool to analyze and visualize signals and their frequency content. By applying these techniques, we will gain practical insights into various cases of spectrum analysis.

1.3 Overview of DFT Math

In conclusion, taking into account the DFT covers the following topics:

1. Intricate Numbers
2. Exponents that are complex
3. Why e?
4. Euler's approach
5. Signal projection onto signal via internal outcome
6. The amount of a signal's x projection into a sinusoid as measured by the DFT
8. A hypothetical accumulation of sinusoidal projections that represents the IDFT
9. A number of Fourier theorems Sets with elementary tie-frequency
10. Useful Matlab spectrum analysis

Additionally, we will go over practical aspects of working with sinusoids, like decibels (dB) and presentation techniques.

Chapter 2

2 Geometric Signal concept

This section provides a general introduction of the geometric signal concept's constituent parts, including vector spaces, norms, inner products, orthogonality, mutual projection, and basic vector space operations.

2.1 The DFT

For a complex sequence of length N , $x(n)$, $n = 0, 1, 2, \dots, N-1$, the discrete Fourier transform (DFT) is obvious.

$$X(\omega_k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, k = 0, 1, 2, \dots, N-1$$

$t_n \triangleq nt$ =nth sample instant (sec)

$\omega_k \triangleq k\Omega$ =trial of kth frequency (rad/sec)

$T \triangleq 1/f_s$ =time sample duration (sec)

$\Omega \triangleq 2\pi f_s/N$ = sample frequency in seconds

We may now fully consider the transform kernel because we are in this position:

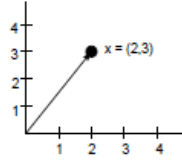
$$e^{-j\omega_k t_n} = \cos(\omega_k t_n) - j\sin(\omega_k t_n)$$

The kernels consist of N discrete frequency ω_k trials of a sinusoid evenly spaced between 0 and the sampling rate $\omega_s \triangleq 2\pi f_s$.

It is sufficient to understand the purpose and operation of the summing over n of the point wise product of $x(n)$ times each complex sinusoid. We'll look at an inner product operation that computes the amount of the signal's projection onto the complex sinusoid, $\cos(\omega_k t_n) + j\sin(\omega_k t_n)$. Since the complex sinusoid at frequency ω_k is present in the input signal x , $X(\omega_k)$, the DFT at that frequency, is a measurement of its amplitude and period. Summations in discrete time, integrals in continuous time, and their kernels all perform this fundamental operation.

2.2 Vectors as Signals

All transmissions and bands for the DFT are N length. A series of length N is characterised by $x(n)$, where $n = 0, 1, 2, \dots, N-1$. Here, $x(n)$ may be real or complex, depending on the number of the length N . We now intend to look at x as a vector in an N -dimensional vector space, with the notation x . To put it another way, every experiment $x(n)$ is regarded as a harmonise in that field. a single N -space point that can be represented by a set of coordinates $(x_0, x_1, x_2, \dots, x_{N-1})$ is referred to as a vector in mathematics and is called a *underlinex*. (The



symbol x_n is equivalent to $x(n)$ in meaning.) Geometrically speaking, it can be represented as an arrow that moves from the source to the point $\underline{x} \triangleq (x_0, x_1, x_2, \dots, x_{N-1})$.

We will now explain the the subsequent ideas and their corresponding counterparts:

$$\underline{x} \triangleq \underline{x} \triangleq x(\cdot) \triangleq (x_0, x_1, x_2, \dots, x_{N-1}) \triangleq [x_0, x_1, x_2, \dots, x_{N-1}] \triangleq [x_0, x_1, x_2, \dots, x_{N-1}]$$

here $x_n \triangleq x(n)$ is the n th trial of the signal (vector) x . From now on, unless precisely stated else, The length of all signals is N .

2.2.1 Vector View Illustration with $N = 2$

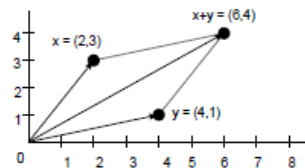
to employ the two-sample signal $x = (2, 3)$

Each trial is a coordinate in the N -dimensional space according to the geometric explanation of a signal of length N . Two trial signals are not very difficult to hear, but they are simple to plot geometrically.

2.3 6.3 Adding vectors

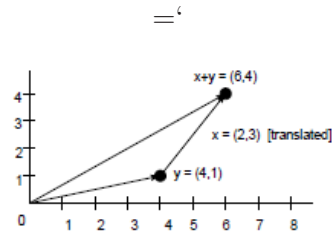
Suppose two vectors are in R^N . $\underline{x} = (x_0, x_1, \dots, x_{N-1})$ and $\underline{y} = (y_0, y_1, \dots, y_{N-1})$, the vector sum is clear by element wise addition. For $n = 0, 1, 2, \dots, N - 1$, $w(n) = x(n) + y(n)$ follows if the sum is denoted by the symbol $w \triangleq x + y$.

The parallelogram rule can be used to create the vector sum for the sum of two vectors,



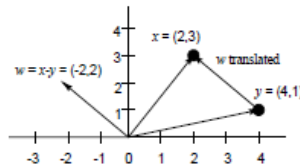
The figure also includes additional dashed lines that complete the parallelogram formed by vectors x and y , representing the vector sum of x and y . These dashed lines correspond to the vectors x and y individually. Since the vector sum forms a parallelogram, these lines complete the geometric shape.

In some cases, where the vector sum is represented as a triangle connecting the origin of one vector to the tip of the other vector. In the figure, vector x is converted to the tip of vector y to form the triangle. However, Vector y can be changed to point at the end of vector x with similar validity and effectiveness, as vector addition is commutative, meaning the order of addition does not affect the result. Therefore, the vector sum $x + y$ is equal to $y + x$.



2.4 Vector subtraction

Figure explains the $\omega = x - y$ vector difference between $x = (2,3)$ and $y = (4,1)$. We calculate $\omega = x - y = (-2,2)$ from the coordinates.



In order to underline the relationship between vectors, the vector ω may be displayed after the tip of y to the tip of x rather than after the origin to the point $(-2,2)$. However, the conversion in the plot has no influence on the mathematical importance or attributes of the vector. However, Commutative subtraction is not possible.

Rewriting the difference vector's definition as $x = y + \omega$ makes it apparent that the vector x should be the sum of the vectors y and ω , which is why the arrowhead is on the right endpoint.

2.5 Signal Metrics

Some helpful uses of the signals are described in this section.

The average value of a signal x is its mean, or more precisely, the "trial mean":

$$\mu_x \triangleq \frac{1}{N} \sum_{n=0}^{N-1} x_n$$

The entire energy of a signal x is defined the sum of squared moduli:

$$\varepsilon_x \triangleq \frac{1}{N} \sum_{n=0}^{N-1} |x_n|^2 (\text{energy of } x)$$

Energy can be defined as the "capacity to do work." Units like "force times distance" or "mass times velocity squared," among others, are used to measure energy and work in the science of physics. In the context of pressure waves, the energy is calculated by calculating the wave impedance divided by the pressure squared integral over time of the medium through which the wave is propagating. For velocity waves, the energy is obtained by integrating the squared velocity multiplied by the wave impedance over time.

A signal x in audio applications often denotes a collection of pressure readings taken from a microphone signal. It might also be used to represent other measures, such as velocity from a magnetic guitar pickup or force from a piezoelectric transducer. In each of these scenarios, the sum of the squared signal samples represents the entire physical energy related to the signal. Music 421² delves deeper into the physical interpretation and ramifications of signal processing.

The energy per trial of a signal with an average power of x is clear:

$$\rho_x \triangleq \frac{\epsilon_x}{N} = \frac{1}{N} \sum_{n=0}^{N-1} |x_n|^2 (\text{average power of } x)$$

Another way to express the average power of a signal x , particularly when x is real, is through the concept of "mean square." When x is a complicated sinusoid with the following form $x(n) = Ae^{j(\omega nT + \phi)}$, where A represents the amplitude, ω is the angular frequency, T is the sampling period, and ϕ is the phase, the average power can be defined as $\rho_x = A^2$. Alternatively, the average power of a complex sinusoid is equal to the instantaneous power, which equates to the amplitude squared.

Physical energy per unit of time is the standard unit of measurement for power. Therefore, it follows logically that the normal signal power is the whole signal energy distributed along its length. Typically, we work with signals, which are time-dependent functions. However, the "power" as defined above still has the clarity of a spatial energy density if the signal instead manifests as a function of distance (for example, displacement tests along a pulsating string). In contrast, power is a density of temporal energy.

Simply put, $\sqrt{(\rho_x)}$ is the root mean square (RMS) level of a signal x . However, take note that after eliminating the mean value, an RMS level can be computed in exercise (especially in audio work). This is referred to as the variance.

The power of the signal with its trial mean subtracted is referred to as the variance (more specifically, the trial variance) of the signal x :

$$\sigma_x^2 \triangleq \frac{1}{N} \sum_{n=0}^{N-1} |x_n - \mu_x|^2$$

The fact that we have real signals is quickly displayed.

$$\sigma_x^2 = \rho_x - \mu_x^2$$

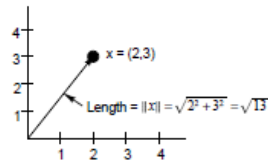
This is described as the "mean square minus the mean squared." When we consider anything other than direct current, we typically think of the variance as the strength of non-constant signal systems. Following the discipline of statistics, particularly the idea of stochastic processes, the words "trial mean" and "trial variance" are used. Time series analysis, random variance, and probability are all statistical concepts that are fundamental to the statistical signal processing area. We shall just touch briefly and in a self-contained manner on a few features of statistical signal processing in this bibliophile. A signal's norm can be determined by taking the square root of its total energy:

$$\|x\| \triangleq \sqrt{\epsilon_x} = \sqrt{\sum_{n=0}^{N-1} |x_n|^2}$$

The length of x in N -space is what we refer to as $\|x\|$. Furthermore, the distance between x and y is seen to be $\|x - y\|$. The "absolute value" or "radius" of a vector can be thought of as the norm.

2.5.1 1. Illustration :

Returning to the straightforward 2D illustration $x = [2, 3]$, we can calculate its norm as $\|x\| = \sqrt{2^2 + 3^2} = \sqrt{13}$. The norm's physical definition as a length measurement is shown in fig.

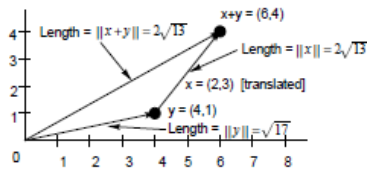


2.5.2 Illustration :

Let's take another look at the vector-sum Illustration that was updated in fig. 6.6.

The vector sum's norm is $\omega = x + y$.

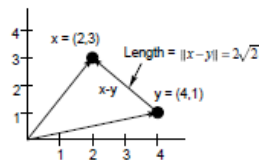
$$\|\omega\| \triangleq \|x + y\| \triangleq \|(2, 3) + (4, 1)\| = \|(6, 4)\| = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$$



X and Y are while, independently, have norms of $\sqrt{13}$ and $\sqrt{17}$. We discover that the triangle inequality is illustrated by the fact that $\|x + y\| \leq \|x\| + \|y\|$. (As may be observed mathematically by examining fig., equality only exists when x and y are collinear.)

2.5.3 Illustration

Think about the vector difference Fig. 1's illustration is diagrammed.



$\omega = x - y$ is the difference vector's norm.

$$\|\omega\| \triangleq \|x - y\| \triangleq \|(2, 3) - (4, 1)\| = \|(-2, 2)\| = \sqrt{(-2)^2 + (2)^2} = 2\sqrt{2}$$

2.6 Additional Norms

We use what is known as a L_2 norm since our core norms are square roots of sums of squares. To emphasise this, we can write $\|x\|_2$.

Similar to how we may have picked a normalised L_2 norms:

$$\|x\|_2 \triangleq \sqrt{\rho_x} = \sqrt{\sum_{n=0}^{N-1} |x_n|^2}$$

(normalized L_2 norm of x)

This only represents the "RMS level" x .

more frequently, the $x \in C^N$ L_2 norm is defined.

$$\|x\|_p \triangleq \left(\sum_{n=0}^{N-1} |x_n|^2 \right)^{\frac{1}{p}}$$

The most intriguing L_p standards include

1. $p = 1$: the L_1 , "absolute value," or "city block" standard.
2. $p = 2$: The "Euclidean," "root energy," or "least squares" norm, all abbreviated as L_2 .
3. The "Chebyshev," "supremum," "minimax," or "uniform" norm, denoted by $p = \infty$.

Keep in mind that the limiting scenario $p = \infty$ results in

$$\|x\|_\infty = \max_{0 \leq n < N} |x_n|$$

There are other additional likely options for the norm. A real-valued signal function $f(x)$ must possess the following three properties in order to be successful as a norm on C^N :

1. $f(\underline{x}) = 0 \iff \underline{x} = 0$
2. $f(\underline{x} + \underline{y}) \leq f(\underline{x}) + f(\underline{y})$
3. $f(c\underline{x}) = |c|f(\underline{x}), c \in C$

The first attribute, "positivity," specifies that only vectors with a value of zero have a norm. The second characteristic is called "subadditivity" and is occasionally referred to as the "triangle inequality" for reasons that may be understood by looking at Fig. According to the third characteristic, As far as scalar multiplication is concerned, the norm is "absolutely homogeneous," which can be complicated and therefore the stage of the scalar is unimportant.

What we are dealing with is known mathematically as a Banach Space, which is a normed linear vector space. To recap, we described a vector as any collection of N real or complex values that we may interpret as being located at certain points in the N -dimensional vector space. We also provided a definition for scalar multiplication, It involves combining a scalar with a vector, which we also regard as a component of the real or complex numbers field. This is also accomplished in a straightforward manner by multiplying each coordinate obtained by adding vectors and scalars. Assuming any two vectors from the vector space, $\underline{x} \in C^N$ and $\underline{y} \in C^N$, and any two scalars from the field of scalars, $c_1 \in C^N$ and $c_2 \in C^N$, it follows that

every linear combination, $c_1x + c_2y$, must also exist in the space. We have defined both our scalars and our vector modules using the field of complex numbers \mathbb{C} (or real numbers \mathbb{R}), and we possess the required properties of closure to guarantee that any linear combination of vectors after C^N lies in C^N . Finally, a vector space becomes a Banach space by virtue of the definition of a norm (any norm).

2.7 The Inner Product

A operation on two vectors called the inner product (or "dot product") results in a scalar. A Hilbert space (also known as a "inner product space") is created by adding an inner product to a Banach space. There are numerous Hilbert space illustrations, but for this bibliophile, we simply need C^N, C (N-dimensional complex vectors and complex scalars).

The inner product between x and y , two (complex) N-vectors, is defined as

$$\langle x, y \rangle \triangleq \sum_{n=0}^{N-1} x(n)\overline{y(n)}$$

In order to tempt a norm in the inner product, the complex conjugation of the second vector is finished.

$$\langle x, x \rangle \triangleq \sum_{n=0}^{N-1} x(n)\overline{x(n)} = \sum_{n=0}^{N-1} |x(n)| \triangleq \epsilon_x = \|x\|^2$$

Because of this, Conjugate symmetrical inner product.

$$\langle y, x \rangle = \overline{\langle x, y \rangle}$$

Keep in mind that the inner product converts \mathbb{C} to $C^N * C^N$. In other words, a complex scalar is mapped to two complex vectors of length N.

2.7.1 Illustration:

In general, we have for $N = 3$

$$\langle x, y \rangle = x_0\overline{y_0} + x_1\overline{y_1} + X_2\overline{y_2}$$

let

$$x = [0, j, 1]$$

$$y = [1, j, j]$$

then

$$\langle x, y \rangle = 0 \cdot 1 + j(-j) + 1(-j) = 0 + 1 + (-j) = 1 - j$$

2.7.2 Internal Production Linearity

It is claimed that a vector $x \in C^N$ is linear if some function $f(x)$ of the vector $x \in C^N$ (also known as an operator on C^N) is linear for all $x_1 \in C^N$ and $x_2 \in C$ as well as for all scalars C_1 and C_2 in C .

$$\implies f(c_1x_1 + c_2x_2) = c_1f(x_1) + c_2f(x_2)$$

Because of this, a linearity operator "commutes with mixing."

Linearity requires two qualities of the elements.

1. additivity: $f(x_1 + x_2) = f(x_1) + f(x_2)$, and
2. homogeneity: $f(c_1x_1) = c_1f(x_1)$.

In its first argument, the inner product $\langle x, y \rangle$ is linear, i.e. $\langle c_1x_1 + c_2x_2, y \rangle = c_1 \langle x_1, y \rangle + c_2 \langle x_2, y \rangle$

This is simple to demonstrate following the explanation:

$$\begin{aligned} \langle c_1x_1 + c_2x_2, y \rangle &\triangleq \sum_{n=0}^{N-1} [c_1x_1(n) + c_2x_2(n)]\overline{y(n)} \\ &= \sum_{n=0}^{N-1} [c_1x_1(n)\overline{y(n)} + \sum_{n=0}^{N-1} c_2x_2(n)]\overline{y(n)} \\ &= c_1 \sum_{n=0}^{N-1} [x_1(n)\overline{y(n)} + c_2 \sum_{n=0}^{N-1} x_2(n)]\overline{y(n)} \\ &\triangleq c_1 \langle x_1, y \rangle + c_2 \langle x_2, y \rangle \end{aligned}$$

The second argument of the internal product, i.e.,

$$\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$$

It is homogeneous in conjugation only in the second argument.

$$\langle x, c_1y_1 \rangle = \overline{c_1} \langle x, y_1 \rangle \neq c_1 \langle x, y_1 \rangle$$

With regard to real scalars, the inner product's second argument is strictly linear:

$$\langle x, r_1y_1 + r_2y_2 \rangle = r_1 \langle x, y_1 \rangle + r_2 \langle x, y_2 \rangle, r_i \in R$$

The term "inner product" is frequently used as a bilinear operator in that case since it is linear in both of its arguments for real scalars.

2.7.3 Norm Induced by the Inner Product

By the inner product, we can characterise a norm on $x \in C^N$:

$$\|\underline{x}\| \triangleq \sqrt{\langle \underline{x}, \underline{x} \rangle}$$

It is easy to demonstrate that a norm's characteristics 1 and 3 are true. The Schwarz Inequality, which is the cause of the following subsection, is simply followed by Property 2. If not, It doesn't take us long to realise that the inner product produces the well-known L_2 norm on C^N .

2.7.4 Cauchy-Schwarz Unfairness

According to the Cauchy-Schwarz Unfairness (sometimes known as the "Schwarz Inequality"), for any $\underline{x} \in C^N$ and $\underline{y} \in C^N$, we have

$$|\langle \underline{x}, \underline{y} \rangle| \leq \|\underline{x}\| \cdot \|\underline{y}\|$$

with equality if and only if $\underline{x} = c\underline{y}$ for some scalar c .

We can quickly display this for real vectors $\underline{x} \in R^N, \underline{y} \in R^N$, as follows: If any \underline{x} or \underline{y} is zero, the inequality holds (as equality). Assuming both are nonzero, let's scale them to unit-length by describing the normalized vectors $\tilde{x} \triangleq \underline{x}/\|\underline{x}\|, \tilde{y} \triangleq \underline{y}/\|\underline{y}\|$, which are unit-length vectors lying on the "unit ball" in R^N (a hypersphere of radius 1). We have

$$\begin{aligned} 0 &\leq \|\tilde{x} - \tilde{y}\|^2 = \langle \tilde{x} - \tilde{y}, \tilde{x} - \tilde{y} \rangle \\ &= \langle \tilde{x}, \tilde{x} \rangle - \langle \tilde{x}, \tilde{y} \rangle - \langle \tilde{y}, \tilde{x} \rangle + \langle \tilde{y}, \tilde{y} \rangle \\ &= \|\tilde{x}\|^2 - [\langle \tilde{x}, \tilde{y} \rangle + \overline{\langle \tilde{x}, \tilde{y} \rangle}] + \|\tilde{y}\|^2 \\ &= 2 - 2\text{re} \langle \tilde{x}, \tilde{y} \rangle \\ &= 2 - 2 \langle \tilde{x}, \tilde{y} \rangle \end{aligned}$$

This suggests

$$\langle \tilde{x}, \tilde{y} \rangle \leq 1$$

alternatively, stopping the normalisation,

$$\text{re} \langle \underline{x}, \underline{y} \rangle \leq \|\underline{x}\| \cdot \|\underline{y}\|$$

similar derivation still applies if \underline{x} is substituted by $-\underline{x}$ yielding

$$-re \langle \underline{x}, \underline{y} \rangle \leq \|x\| \cdot \|y\|$$

The final two equations signify

$$|\langle \underline{x}, \underline{y} \rangle| \leq \|x\| \cdot \|y\|$$

The complex case can be revealed by turning the components of \underline{x} and \underline{y} such that $re\{\langle \tilde{x}, \tilde{y} \rangle\}$ becomes equal to $|\langle \tilde{x}, \tilde{y} \rangle|$.

2.7.5 Triangle Inequality

The triangle inequality asserts that any side of a triangle has a distance that is the total of the distances on the other two sides is less than or equal to that value. Equality only occurs when the triangle scuffs up against a line, according to the triangle inequality. This is equivalent in C^N , to

$$\|\underline{x} + \underline{y}\| \leq \|x\| + \|y\|$$

The Schwarz Inequality makes this point clear-cutly.

$$\begin{aligned} \|\underline{x} + \underline{y}\|^2 &= \langle \underline{x} + \underline{y}, \underline{x} + \underline{y} \rangle \\ &= \|\underline{x}\|^2 + 2re\{\langle \underline{x}, \underline{y} \rangle\} + \|\underline{y}\|^2 \\ &\leq \|\underline{x}\|^2 + 2\{\langle \underline{x}, \underline{y} \rangle\} + \|\underline{y}\|^2 \\ &\leq \|\underline{x}\|^2 + 2\|\underline{x}\| \cdot \|\underline{y}\| + \|\underline{y}\|^2 \\ &= (\|\underline{x}\| + \|\underline{y}\|)^2 \\ &\implies \|\underline{x} + \underline{y}\| \leq \|x\| + \|y\| \end{aligned}$$

2.7.6 Triangle Difference Inequality

The length of any triangle side is bigger than the difference between the other two sides' lengths put together, which is an important variation on the triangle inequality:

$$\|\underline{x} - \underline{y}\| \geq \|x\| - \|y\|$$

Proof:

$$\begin{aligned}
\|\underline{x}\| - \|\underline{y}\|^2 &= \|\underline{x}\|^2 - 2\operatorname{re}\{\langle \underline{x}, \underline{y} \rangle\} + \|\underline{y}\|^2 \\
&\geq \|\underline{x}\|^2 - 2\{\langle \underline{x}, \underline{y} \rangle\} + \|\underline{y}\|^2 \\
&\geq \|\underline{x}\|^2 - 2\|\underline{x}\| \cdot \|\underline{y}\| + \|\underline{y}\|^2 \\
&= [\|\underline{x}\| + \|\underline{y}\|]^2 \\
\implies \|\underline{x} - \underline{y}\| &\geq \|\underline{x}\| - \|\underline{y}\|
\end{aligned}$$

2.7.7 Variable Cosine

There is a formula for the Cauchy-Schwarz inequality.

$$\frac{|\langle \underline{x}, \underline{y} \rangle|}{\|\underline{x}\| \cdot \|\underline{y}\|} \leq 1$$

We can always discover a real number ϕ that satisfies the condition for real vectors $\underline{x}, \underline{y}$

$$\cos(\theta) \triangleq \frac{|\langle \underline{x}, \underline{y} \rangle|}{\|\underline{x}\| \cdot \|\underline{y}\|}$$

As a result, ϕ is understood to be the angle between two vectors in R^N . We may also write $|\cos(\theta)|$ in C^N .

2.7.8 Orthogonality

The x and y vectors (signals) are supposed to be orthogonal if $\langle x, y \rangle = 0$, symbolized $x \perp y$.

$$\boxed{x \perp y \iff \langle x, y \rangle = 0}$$

Keep in mind that the cosine of the angle between actual, orthogonal x and y is zero. The angle formed by two perpendicular lines in plane geometry ($N = 2$) is $\frac{\pi}{2}$, and $\cos\frac{\pi}{2} = 0$, as would be predicted. Likely. More often than not, orthogonality agrees that Physical perpendicularity is established by the straight angle at which two vectors in N -space connect.

2.8 Signal Reform based on projections

We now see how to plan a signal onto other signals. We now want to study how to recreate a signal $x \in C^N$ from its projections onto N different vectors $s_k, k = 0, 1, 2, \dots, N - 1$. This will give us the reverse DFT operation.

Consider the projection of a signal $x \in C^N$ onto the direct manage axes of C^N as a straightforward illustration. Simply $(x_0, 0, \dots, 0)$ are the projection's coordinates on the 0th axis of coordinates. The coordinates for the projection along axis 1 are $(0, x_1, 0, \dots, 0, 0)$, and so forth. The vector sum of the signal's projections onto the coordinate axes is then evidently the original signal x :

$$x = (x_0, \dots, x_{N-1}) = (x_0, 0, \dots, 0) + (0, x_1, 0, \dots, 0) + \dots (0, \dots, 0, x_{N-1})$$

Let's look at the details for the $N = 2$ example to ensure that the previous section has been comprehended. The coordinate axes in 2D must be projected with a random two-trial signal $x = (x_0, x_1)$. Any nonzero vector along a coordinate axis can serve as its symbol. Any vector of the form $(\alpha, 0)$ can denote the horizontal axis, while any vector of the form $(0, \beta)$ can denote the vertical axis. Let's use the positive unit-length representatives for simplicity's sake:

$$e_0 \triangleq (1, 0)$$

$$e_1 \triangleq (0, 1)$$

The projection of \underline{x} onto e_1 is by definition

$$\rho_{e_0}(\underline{x}) \triangleq \frac{\langle \underline{x}, e_0 \rangle}{\|e_0\|^2} e_0 = \langle \underline{x}, e_0 \rangle e_0 = \langle [x_0, x_1], [0, 1] \rangle e_0 = (x_0 \cdot \bar{0} + x_1 \cdot \bar{1}) e_0 = x_1 e_0 = [x_0, 0]$$

Likewise, the forecast of \underline{x} onto e_1 is

$$\rho_{e_1}(\underline{x}) \triangleq \frac{\langle \underline{x}, e_1 \rangle}{\|e_1\|^2} e_1 = \langle \underline{x}, e_1 \rangle e_1 = \langle [x_0, x_1], [0, 1] \rangle e_1 = (x_0 \cdot \bar{0} + x_1 \cdot \bar{1}) e_1 = x_1 e_0 = [x_0, 0]$$

The vector sum of the projections is then used to reconstruct x from its projections onto the coordinate axes:

$$\underline{x} = \rho_{e_0}(\underline{x}) + \rho_{e_1}(\underline{x}) = x_0 e_0 + x_1 e_1 \triangleq x_0(1, 0) + x_1(0, 1) = (x_0, x_1)$$

Given that the coordinates are scalars x_n which may be used with the linear combination of the coordinate vectors e_n to produce an arbitrary vector $x \in C^N$, the projection of a vector onto its coordinate axes is, in some senses, trivial:

$$\underline{x} \triangleq x_0 e_0 + x_1 e_1 + \dots + x_{N-1} e_{N-1}$$

The coordinate vectors' orthogonality should be noted. If they are also unit length, $\|e_n\| = 1$, we claim that the coordinate vectors are orthonormal, $\{e_n\}_{n=0}^{N-1}$. The projection of a signal onto a different collection of vectors from the coordinate set is what is more intriguing. This may be thought of as a change in C^N coordinates. The new vectors will be chosen in the DFT scenario to test complex sinusoids.

2.8.1 General Conditions

2.8.2 Definition

:

Given any two members, a collection of vectors is expected to form a vector space \underline{x} and \underline{y} from the set, The vectors $\underline{x} + \underline{y}$ and \underline{cx} , where c is a scalar, are also included in the collection.

Vectors defined as a list of N complex numbers $\underline{x} = (x_0, \dots, x_{N-1}) \in C^N$, using element wise addition and multiplication by complex scalars $c \in C$, form a vector space, where C^N means the set of all dimension N complex sequences. Similarly, real N -vectors $\underline{x} \in R^N$ and real scalars $\alpha \in R$ form a vector space.

2.8.3 Theorem:

The term "vector space" refers to the collection of all linear combinations of any set of vectors from R^N or C^N .

Proof:

Let's write the first set of vectors as $\underline{s}_0, \dots, \underline{s}_{M-1}$, where M can be any positive integer greater than 0. Any component of the vector space that is x_0 is then by definition a linear combination of them:

$$\underline{x}_0 = \alpha_0 \underline{s}_0 + \dots + \alpha_{M-1} \underline{s}_{M-1}$$

From this we understand that \underline{cx}_0 is also a linear combination of the original vectors, and therefore is in the vector space. Also, given any second vector from the space $\underline{x}_1 = \alpha_1 \underline{s}_0 + \dots + \alpha_{M-1} \underline{s}_{M-1}$, the sum is

$$\begin{aligned} \underline{x}_0 + \underline{x}_1 &= (\alpha_0 \underline{s}_0 + \dots + \alpha_{M-1} \underline{s}_{M-1}) + (\alpha_1 \underline{s}_0 + \dots + \alpha_{M-1} \underline{s}_{M-1}) \\ &= (\alpha_0 + \alpha_1) \underline{s}_0 + \dots + (\alpha_0 + \alpha_1) \underline{s}_{M-1} \end{aligned}$$

It is in vector space because this is essentially an additional linear combination of the original vectors. Here, it is evident how the closure of scalar multiplication and vector addition differs in a "organic" way from the closure of addition and multiplication of real or complex numbers.

2.8.4 Definition:

If a vector space is the set of all linear combinations of a finite collection of vectors, then these vectors should span the space. $\underline{s}_0, \dots, \underline{s}_{M-1}$ in the sentence below.

Illustration:

Each vector in C^N span C^N can be expressed as a linear combination of the coordinate vectors as the coordinate vectors in C^N span C^N .

$$\underline{x} = x_0\underline{e}_0 + x_1\underline{e}_1 + \dots + x_{N-1}\underline{e}_{N-1}$$

here $x_i \in C$, and $\underline{e}_0 = (1, 0, 0, \dots, 0)$, $\underline{e}_1 = (0, 1, 0, \dots, 0)$, $\underline{e}_2 = (0, 0, 1, 0, \dots, 0)$ and so forth till $\underline{e}_{N-1} = (0, \dots, 0, 1)$.

2.8.5 Theorem:

(i) If $\underline{s}_0, \dots, \underline{s}_{M-1}$ span a vector space, and if one of them, say \underline{s}_m , is linearly dependent on the others, then the vector space is spanned by the set got by failing \underline{s}_m from the original set.

(ii) We can constantly choose from them a linearly independent set that spans the same space if $\underline{s}_0, \dots, \underline{s}_M - 1$ span a vector space.

Proof:

A linear combination of the vectors $\underline{s}_0, \dots, \underline{s}_M - 1$ can be used to represent any \underline{x} in the space. The linear combination for \underline{x} produces a other than linearly combining vectors \underline{s}_m by asserting that \underline{s}_m is a additional vectors in the collection are linearly combined. Thus, it is possible to remove \underline{s}_m from the set, establishing (i). We can construct a procedure for initiating the desired subset of the initial vectors to demonstrate (ii). First, allocate (\underline{s}_0) to the set. Assess the linear dependence between \underline{s}_0 and \underline{s}_1 next. If this is the case (i.e., \underline{s}_1 is a scalar times \underline{s}_0), discard \underline{s}_1 ; if not, add it to the new set as well. Check to determine if \underline{s}_2 is linearly dependent on the new collection of vectors by \underline{s}_2 . Remove it if it is (i.e., \underline{s}_1 is some linear combination of \underline{s}_0 and \underline{s}_1); if not, assign it to the new set as well. The new set will only contain linearly independent vectors that span the original space when this operation is finished after processing $\underline{s}_M - 1$.

2.8.6 Definition:

The term "basis" refers to a collection of linearly independent vectors that spans a vector space.

2.8.7 Definition:

The dimension of the space refers to the quantity of vectors contained in a basis for a given space. An N-dimensional space, often known as an N-space, is one in which the dimension is N. We shall only discuss finite-dimensional vector spaces in considerable detail in this bibliophile. Contrarily, as there are an unlimited number of points in both the time and frequency domains, Infinite-dimensional basis sets are necessary for both the discrete-time Fourier transform (DTFT) and Fourier transform.

2.8.8 Orthogonalization through Gram-Schmidt

2.8.9 Theorem:

Considering a collection of N linearly independent vectors $\underline{s}_0, \dots, \underline{s}_{N-1}$ from C^N , we can make an orthonormal set $\tilde{\underline{s}}_0, \dots, \tilde{\underline{s}}_{N-1}$ They cover the same space and are linear combinations of the first set.

Proof:

We prove the theorem by making the preferred orthonormal set $\{\tilde{s}_k\}$ successively from the original set $\{s_k\}$. This process is recognized as Gram-Schmidt orthogonalization.

1. Set $\tilde{s}_0 \triangleq \frac{s_0}{\|s_0\|}$
2. By subtracting the projection of the s_1 onto the s_0 , we can define the y_1 as follows:

$$y_1 \triangleq s_1 - \rho_{\tilde{s}_0}(s_1) \triangleq s_1 - \langle s_1, \tilde{s}_0 \rangle \tilde{s}_0$$

The vector y_1 is orthogonal to \tilde{s}_0 by construction. (We deducted out the part of s_1 that wasn't orthogonal to \tilde{s}_0 .)

3. Set $\tilde{s}_1 \triangleq \frac{y_1}{\|y_1\|}$ (i.e., make the outcome of the preceding step normal).
4. Define y_2 as the s_2 minus the anticipated s_2 onto \tilde{s}_0 and \tilde{s}_1 :

$$y_2 \triangleq s_2 - \rho_{\tilde{s}_0}(s_2) - \rho_{\tilde{s}_1}(s_2) = s_2 - \langle s_2, \tilde{s}_0 \rangle \tilde{s}_0 - \langle s_2, \tilde{s}_1 \rangle \tilde{s}_1$$

5. Normalisation: $\tilde{s}_2 \triangleq \frac{y_2}{\|y_2\|}$

6. This procedure should be followed up till \tilde{s}_{N-1} is given.

Any set of N linearly independent vectors can be converted into an orthonormal basis using the Gram-Schmidt orthogonalization method. It is obvious that we may easily build merely an orthogonal basis without performing normalisation. The crucial step in this process is the deduction out of the projection of the following linearly independent vector onto the vectors assumed up to this point in the set in order to obtain each new orthonormal basis vector. Each new linearly independent vector s_k is, as we can say, In order to make it orthogonal to the entire subspace, Any projection in that subspace that is not zero is deducted from s_k . This projection is made into the subspace spanned by the vectors $\{\tilde{s}_2, \dots, \tilde{s}_{k-1}\}$. To put it another way, we only keep the parts of each new vector s_k that point in the direction of the new dimension. Whatever vector we select initially (s_0 here) determines the first direction, which is arbitrary. The following vector is required to be orthogonal to the preceding one. The third must be opposite to the first two, and so on. This section can be seen as an overview of some of the most important linear algebraic ideas.

Chapter 3

3 Discrete Fourier Transform (DFT) derivation

The Discrete Fourier Transform (DFT), which is a projection of the length N signal $x(n)$ onto the set of N sampled complex sinusoids created by the N roots of unity, is introduced in this section.

3.1 Using The DFT

The Discrete Fourier Transform (DFT) will be discussed after this.

3.1.1 Series in Geometry

Remember that for some complex number $z_1 \in C$, the signal $x(n) \triangleq z_1^n, n = 0, 1, 2, \dots$, explains a sequence of geometrical terms where each phrase is obtained by multiplying the previous term by a (complex) persistent. The sum of a geometric sequence is a straightforward definition of a geometric series:

$$S_N(z_1) \triangleq \sum_{n=0}^{N-1} z_1^n = 1 + z_1 + z_1^2 + z_1^3 + \dots + z_1^{N-1}$$

If $z_1 \neq 1$, the summation can be stated in closed form as

$$S_N(z_1) = \frac{1 - z_1^N}{1 - z_1}$$

Proof: We have

$$S_N(z_1) \triangleq 1 + z_1 + z_1^2 + z_1^3 + \dots + z_1^{N-1}$$

$$z_1 S_N(z_1) = z_1 + z_1^2 + z_1^3 + \dots + z_1^{N-1} + z_1^N$$

$$\implies S_N(z_1) - z_1 S_N(z_1) = 1 - z_1^N$$

$$S_N(z_1) = \frac{1 - z_1^N}{1 - z_1}$$

3.1.2 Orthogonality of Sinusoids

The fact that sinusoids are orthogonal at various frequencies is one of their main characteristics. i .e

$$\omega_1 \neq \omega_2 \implies A_1 \sin(\omega_1 t + \phi_1) \perp A_2 \sin(\omega_2 t + \phi_2)$$

This holds true regardless of how vast and how long they may have been around, whether they are complex or real. The frequencies being different is the only issue. The sinusoidal periods must, however, be infinite.

Sinuoidal signal segments that have been sampled for length N, like those employed by the DFT, exact orthogonality only applies to the harmonics of the sampling rate shared by N, or only across the frequencies $f_k = kf_s/N, k = 0, 1, 2, \dots, N - 1$. The only frequencies with a strict integer number of periods throughout N trials are those mentioned above. matching complex sinusoids to the frequencies f_k

$$s_k(n) \triangleq e^{j\omega_k n T}, \omega_k \triangleq k \frac{2\pi}{N} f_s, k = 0, 1, \dots, N - 1$$

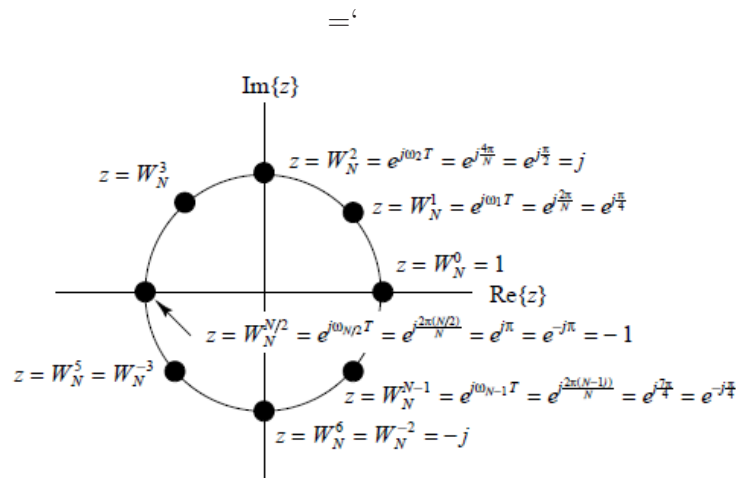
In the complex plane, these sinusoids are generated by the N roots of unity.

$$W_N^k \triangleq e^{j\omega_k T} \triangleq e^{jk2\pi(\frac{f_s}{N})T} = e^{jk(\frac{2\pi}{N})}, k = 0, 1, 2, \dots, N - 1$$

Due to the fact that they all satisfy, these are known as the N roots of oneness.

$$[W_N^k]^N = [e^{j\omega_k T}]^N = [e^{jk(\frac{2\pi}{N})}]^N = e^{jk2\pi} = 1$$

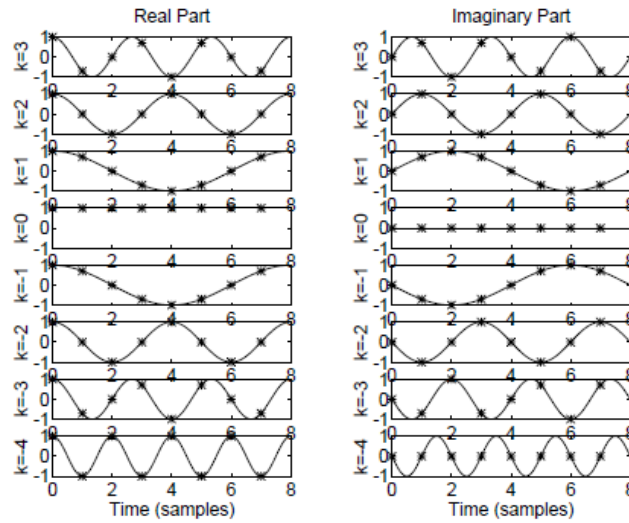
on Fig., the N roots of unity are depicted for $N = 8$ on the complex plane. Overall, a point at $z = 1$ will always exist for some N, and the points similarly split the unit circle. When N is even, there is a point at $z = -1$ (which corresponds to a sinusoid at precisely half the sampling speed), but not if N is odd.



The planned sampling sinusoids for the N roots of unity are shown in Fig. These are the sampled sinusoids that the DFT uses: $(W_N^k)^n = e^{j2\pi k n / N} = e^{j\omega_k n T}$ Note that taking

Consecutively advanced integer powers of the point W_N^k on the unit circle produces trials of the k th DFT sinusoid, giving $[W_N^k]^n, n = 0, 1, 2, \dots, N - 1$. The k th sinusoid originator W_N^k is in turn the k th power of the original N th root of unity $\omega_N \triangleq e^{j2\pi/N}$. The symbolization $W_N, W_N^k,$ and W_N^{nk} are mutual in the digital signal processing literature.

Note that in Fig. the range of k is taken to be $[-N/2, n/2 - 1] = [-4, 3]$ in its place of $[0, N - 1] = [0, 7]$. The best "physical" option is this since it fits with our understanding of "negative frequencies." However, we can increase k by an integer multiple of N without changing the sinusoid that is indexed by k . For all integer m in new parameters, $kpmN$ signifies the same sinusoid.



3.1.3 Sinusoids of the DFT are orthogonal

We now make a point of proving that the DFT sinusoids are only orthogonal. Let

$$s_k(n) \triangleq e^{j\omega_k nT} = e^{j2\pi kn/N}$$

Denote the DFT sinusoid for the k th order. Then

$$\begin{aligned} \langle s_k, s_l \rangle &\triangleq \sum_{n=0}^{N-1} s_k(n) \overline{s_l(n)} = \sum_{n=0}^{N-1} e^{j2\pi kn/N} e^{-j2\pi ln/N} \\ &= \sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = \frac{1 - e^{j2\pi(k-l)}}{1 - e^{j2\pi(k-l)/N}} \end{aligned}$$

where the closed-form equation for the sum of a geometric series was used in the prior step. If $k \neq l$, the numerator is zero while the denominator is nonzero. This shows

$$s_k \perp s_l, k \neq l$$

It is gleefully confirmed that the (nonzero) amplitude and period have no effect on orthogonality despite the fact that we only looked at unit amplitude, zero time complex sinusoids, as utilised by the DFT.

3.1.4 The DFT Sinusoids' norm

When $k = l$, we proceed from the previous step by using the earlier root to obtain

$$\langle s_k, s_k \rangle = \sum_{n=0}^{N-1} e^{j2\pi(k-l)n/N} = N$$

This proves

$$\|s_k\| = \sqrt{N}$$

3.1.5 A set of orthonormal sinusoids

To obtain an orthonormal set, we can standardise the DFT sinusoids:

$$\tilde{s}_k(n) \triangleq \frac{s_k(n)}{\sqrt{N}} = \frac{e^{j2\pi kn/N}}{\sqrt{N}}$$

The signals with an orthonormal sinusoidal basis satisfy

$$\langle \tilde{s}_k, \tilde{s}_l \rangle = \{1, k = l, 0, k \neq l\}$$

The standardised DFT sinusoids are what we refer to as.

3.1.6 DFT, or discrete Fourier Transform

The band is defined by for a signal $x() \in C^N$.

$$X(\omega_k) \triangleq \langle x, s_k \rangle = \sum_{n=0}^{N-1} x(n) \overline{s_k(n)}, k = 0, 1, 2, \dots, N-1$$

or, way it is usually expressed

$$X(\omega_k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}, k = 0, 1, 2, \dots, N-1$$

In other words, the inner product of x and the k th DFT sinusoid s_k defines the k th sample of the band of x , denoted by the symbol $X(\omega_k)$. This formulation is N times the x projection coefficient onto s_k , or

$$\frac{\langle x, s_k \rangle}{\|s_k\|^2} = \frac{X(\omega_k)}{N}$$

The projection of x onto s_k itself is

$$\rho_{s_k}(x) = \frac{X(\omega_k)}{N} s_k$$

The projections' totality is all that the inverse DFT represents:

$$x = \sum_{n=0}^{N-1} \frac{X(\omega_k)}{N} s_k$$

alternatively, as we typically do,

$$x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(\omega_k) e^{j \frac{2\pi kn}{N}}$$

In conclusion, A superposition of the original signal's sinusoidal projections is how the original signal gets reformed in the inverse DFT., and The set of projection coefficients onto the sinusoidal basis set is inversely proportional to the DFT.

3.1.7 Frequencies in the “Cracks”

Only for frequency $\omega_k = 2\pi k f_s / N$ does the DFT appear evident. These really are the only frequencies present in the signal $\omega = \omega_k$. Nevertheless, we investigate random signals from nature using the DFT. What occurs when a signal x contains a frequency ω that is not one of the ω_k DFT-sinusoid frequencies?

Let's plan a length N segment of a sinusoid onto the k th DFT sinusoid at a random frequency ω to find out:

$$x(n) \triangleq e^{j\omega n T}$$

$$s_k(n) \triangleq e^{j\omega_k n T}$$

$$\rho_{s_k}(x) = \frac{\langle x, s_k \rangle}{\langle s_k, s_k \rangle} s_k \triangleq \frac{x(\omega_k)}{N} s_k$$

The coefficient of projection is proportional to

$$X(\omega_k) \triangleq \langle x, s_k \rangle \triangleq \sum_{n=0}^{N-1} x(n) \overline{s_k(n)}$$

$$\sum_{n=0}^{N-1} e^{j\omega n T} e^{-j\omega_k n T} = \sum_{n=0}^{N-1} e^{-j(\omega - \omega_k) n T} = \frac{1 - e^{j(\omega - \omega_k) n T}}{1 - e^{j(\omega - \omega_k) T}}$$

$$e^{j(\omega - \omega_k)(N-1)T/2} \frac{\text{Sin} [(\omega - \omega_k)nT/2]}{\text{Sin} [(\omega - \omega_k)T/2]}$$

by the total of a geometric series in closed-form. The sum is displayed earlier. The total is N for $\omega = \omega_k$ and zero at ω_l , for $l \neq k$, as was previously demonstrated. However, at all other frequencies, the total is not zero.

Any sinusoidal segment may be projected onto the N DFT sinusoids and then rebuilt using merely a linear combination of them because we are only considering N trials. An alternative formulation of this is to say that CN is based on the DFT sinusoids, allowing every length N signal to be described as a linear combination of them. As a result, we need to comprehend what we observe when looking at recorded signal segments. The usual way to approach this in practise is to think about the functioning of a digital filter using DFT.1 What we just added, 2 is the frequency return of this filter, and its magnitude is

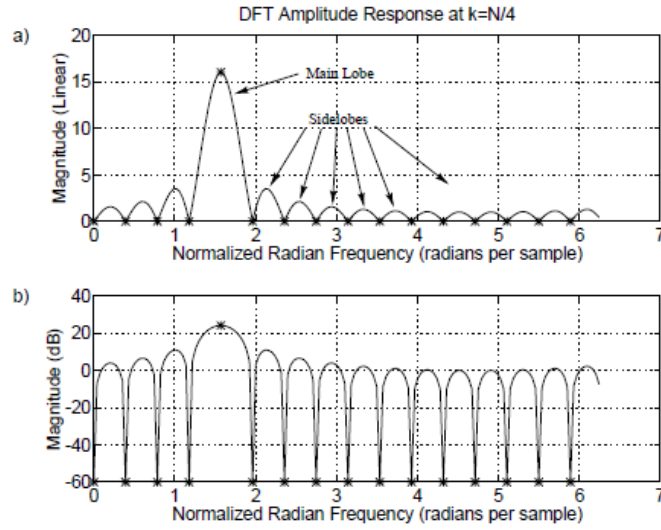
$$|X(\omega_k)| = \left| \frac{\text{Sin} [(\omega - \omega_k)nT/2]}{\text{Sin} [(\omega - \omega_k)T/2]} \right|$$

(shown in Fig. for $k = N/4$). At all other integer values of k , the comeback is the same but moved (circularly) left or right so that the peak is concentrated on ω_k . The secondary peaks away from ω_k are called sidelobes of the DFT response, The comeback is the same for all other integer values of k , but it is relocated (circularly) left or right such that the peak is focused on in Fig. (Clipped at $\text{dB}60$). We see that the sidelobes are really pretty high from an audio perspective. Sinusoids with frequencies near $\omega_k \pm 1.5$, for example, are only attenuated approximately 13dB in the DFT output $X(\omega_k)$.

We can see that all frequencies between dc and the sampling rate are sensitive to $X(\omega_k)$, with the exception of the other DFT-sinusoid frequencies ω for $l \neq k$. This is known as spectral leakage or cross-talk in the band research community. When N is precisely a period or some multiple of a period and the signal being studied is genuinely periodic, there is once more no error. However, in most cases, this is difficult to organise, and spectral leakage can truly cause issues.

Keep in mind that raising N does not concentrate spectral leakage. You may think of it as the result of two sharp sinusoid cuts that were made at the beginning and conclusion of the N -trial time window. Only the DFT sinusoids are left uncut around the window margins. All other frequencies will experience some truncation distortion, and the sidelobes can be attributed to the spectral content of the abrupt cut-off or turn-on transient. Keep in mind that the input signal $x(n)$ is identical to its periodic extension in terms of the DFT. Since the signal is not periodic in N samples, there will be a "glitch" every N samples if we repeat N samples of a sinusoid at frequency $\omega = \omega_k$. This issue can be reflected a new energy source across the entire band.

To lessen spectral leakage (cross-talk from far-off frequencies), we often use a window function, such as a "raised cosine" window, to match the data record nicely to zero at both endpoints of the window. The primary lobe of the DFT response widens and the sidelobes shrink as a result of the smooth tapering. Unless the signal is precisely periodic over the course of N trials, no window can be as effectively observed as a rectangular window of length N . Both Music 420 and the "Illustrations by the DFT" section expand on these topics.



Given that $X(\omega_k)$, the k th spectral sample, is indeed perceived as a gauge of spectral amplitude over a nominal $k - 1/2$ to $k + 1/2$ range of frequencies, this range is frequently referred to as a frequency bin (as in a "storage bin" for spectral energy). The total energy in the k th bin can be observed as $|X(\omega_k)|^2$ (see Parseval's Theorem in the "Fourier Theorems" division). The frequency index k is known as the bin number. Trials of any continuous bandlimited function are mentioned in connection with this; nevertheless, Despite having a mathematically equivalent meaning in the time domain, the term "bin" is exclusively used in the frequency domain.

3.1.8 Standardised DFT

The normalised DFT sinusoids are projected onto for a more "theoretically clean" DFT.

$$\tilde{s}_k(n) \triangleq \frac{e^{j2\pi kn/N}}{\sqrt{N}}$$

In this situation, the normalised DFT of x is

$$\tilde{X}(\omega_k) \triangleq \langle x, \tilde{s}_k \rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

This also precisely corresponds to the coefficient of x projection onto s_k . Then, it is easier to calculate the inverse normalised DFT.

$$x(n) = \sum_{k=0}^{N-1} X(\omega_k) \tilde{s}_k(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(\omega_k) e^{j2\pi kn/N}$$

Although from the perspective of "geometric signal theory" this meaning is considerably clearer, Since it involves more computation than the typical explanation, it is rarely used in exercise. The sign of the exponent in the kernel, however, is the only difference between the forward and inverse transforms in this instance.

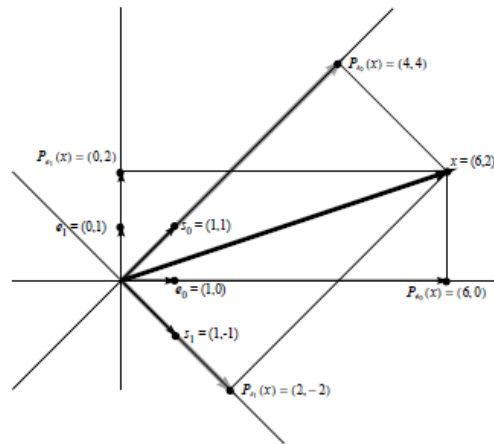
3.2 The Length 2 DFT

Since the basis sinusoids are real, the length 2 DFT is primarily straightforward:

$$\underline{s}_0 = (1, 1)$$

$$\underline{s}_1 = (1, -1)$$

The DFT sinusoid \underline{s}_1 is a sampled sinusoid at half the sample rate, whereas \underline{s}_0 is a sampled constant signal. Figure shows the graphical relationships for the signal's length 2 DFT ($\underline{x} = [6, 2]$).



According to logic, The DFT is identified by us as

$$X(\omega_0) \triangleq \rho_{\underline{s}_0} \triangleq \frac{\langle \underline{x}, \underline{s}_0 \rangle}{\langle \underline{s}_0, \underline{s}_0 \rangle} \underline{s}_0 = \frac{6.1 + 2.1}{1^2 + 1^2} \underline{s}_0 = 4 \underline{s}_0 = (4, 4)$$

$$X(\omega_1) \triangleq \rho_{\underline{s}_1} \triangleq \frac{\langle \underline{x}, \underline{s}_1 \rangle}{\langle \underline{s}_1, \underline{s}_1 \rangle} \underline{s}_1 = \frac{6.1 + 2 \cdot (-1)}{1^2 + (-1)^2} \underline{s}_1 = 2 \underline{s}_1 = (2, -2)$$

Be sure to pay attention to the orthogonal projection lines shown in the illustration. The vectors make up the "time domain" basis. $\{\underline{e}_0, \underline{e}_1\}$, and the projections onto them that are orthogonal to each other are $(6, 0)$ and $(0, 2)$, respectively. The "frequency domain" basis vectors are composed of $\{\underline{s}_0, \underline{s}_1\}$. Additionally, they provide an orthogonal basis set that is rotated by 45 degrees in relation to the time-domain basis vectors.. Giving them an orthogonal projection results in $X(\omega_0) = (4, 4)$ and $X(\omega_1) = (2, -2)$, respectively. The vector sum of the initial signal \underline{x} 's coordinate projections, which represents it in the time domain, or the vector sum of its projections onto the DFT sinusoids, which represents it in the frequency domain, respectively, can be used to represent it. Making \underline{x} the vector sum of its projections onto the DFT sinusoids is known as "taking the inverse DFT," as opposed to computing the coefficients of projection, which is conceptually identical to "taking the DFT."

Chapter 4

4 Fourier Theorems for the DFT

Numerous Fourier theorems are developed in this section for the DFT condition. The shift theorem, convolution theorem, correlation theorem, power theorem, symmetry relations, and theorems relating to interpolation and downsampling are all included. The applications of several theorems, linear time-invariant filtering, sampling rate conversion, and statistical signal processing are discussed.

4.1 DFT and its Inverse

Let $x(n), n = 0, 1, \dots, N - 1$, i.e., $x \in CN$, indicate an n -sample complex sequence. The Discrete Fourier Transform (DFT) is used to define the band of x after that:

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, k = 0, 1, 2, \dots, N - 1$$

The definition of the inverse DFT (IDFT) is

$$x(n) \triangleq \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N}, n = 0, 1, 2, \dots, N - 1$$

Notably, we are not including the sample interval $T = 1/f_s$ in our representation for the first time. Actually, this is the approach used the most frequently in digital signal processing projects. It is common to specify that the sample frequency is $f_s = 1$. But the substitute can set it to any desired value..

$$e^{j2\pi nk/N} = e^{j2\pi k(f_s/N)nT} \triangleq e^{j\omega_k t_n}$$

For the rest of this book over, though, we'll make the more typical (and mathematical) assumption that $f_s = 1$. For the purpose of this chapter solely, we'll use the definition $\omega_k \triangleq 2\pi k/N$. The term "radian frequency" in this context refers to the number of "radians per sample." Throughout this bibliophile, ω_k always refers to "radians per second." (Of course, if the sample rate is indeed 1, there is no change.) In relation to the $f_s = 1$ settlement, we also refer to normalised frequency as follows: The range $[-\pi, \pi)$ encompasses all normalised radian frequencies, whereas the range $[-0.5, 0.5)$ encompasses all normalised Hz frequencies.

4.1.1 Nomenclature and terminology

If X is x 's DFT, then x and X constitute a transform pair, and x and X are written as a pair.

$$\boxed{x \longleftrightarrow X}$$

("x is equivalent to X").

We'll utilise this additional illustration instead:

$$DFT(x) \triangleq X,$$

and

$$DFT_k(x) \triangleq X(k)$$

Time-domain signals are consistently denoted by lowercase letters, such as "x," but frequency-domain signals, such as spectra, are denoted by capital letters ("X").

4.1.2 Modulo Indexing, Periodic Extension

Each of the DFT sinusoids, $s_k(n) \triangleq e^{j\omega_k n}$, has a periodicity with periods that divide N. Consequently, for any integer m, $s_k(n + mN) = s_k(n)$. A linear combination of the time-domain DFT sinusoids can be described as the signal x for length N.

$$x(n) = \frac{1}{N} \sum_k X(k) s_k(n),$$

Given that $x(n + mN) = x(n)$ for any integer m, periodic extension is hence the "automatic" definition of x(n) outside of the range $[0, N - 1]$.

Additionally, the DFT recurs every N trials without a doubt, as

$$X(k + mN) \triangleq \sum_n \langle x, s_{k+mN} \rangle = \sum_{n=0}^{N-1} \langle x, s_k \rangle X(k)$$

because $s_{k+mN}(n) = e^{j2\pi n(k+mN)/N} = e^{j2\pi nk/N} e^{j2\pi nm} = s_k(n)$. (The DFT sinusoids behave in the same way as n and k functions.) For the purpose of DFT investigations, all signals in C^N can be viewed as single periods from an infinitely long periodic signal with period N trials:

4.1.3 Definition

For a particular signal $x \in C^N$, we define

$$x(n + mN) \triangleq x(n)$$

for all integer m.

This agreement allows all indexing of signals and spectra to be understood as N, and we can emphasise this by writing $x(n \bmod N)$. The correct definition of "n mod N" is $n - mN$, where m is selected to provide $n - mN$ in the range $[0, N - 1]$.

When indexing a band X, for instance, we see that $X(N) = X(0)$, proving that the sampling rate for discrete time signals is physically equal to dc. We have what is referred to as the "periodic extension" of x(n) in the time domain.

In other words, the input to the DFT is mathematically regarded as N trials of a periodic signal of period NT seconds. The band is zero between frequency trials ω_k , which is the equivalent assertion in the frequency domain. It is also feasible to adopt the viewpoint that the time-domain signal $x(n)$ is made up of N samples with zeros coming before and after each one. The spectrum is not zero in this situation between the spectral samples ω_k , and band restricted interpolation can be used to restore the band between trials. Except for the topic of "zero padding \longleftrightarrow interpolation below, which is outside the purview of Music, the "time-limited" interpretation of the DFT input is considered in Music 420 in great depth.

4.2 Signal Operators

The following signal operator definitions can be used in the Fourier theorems that follow.

4.2.1 Switch Operators

Definition:

As stated by, the flip operator

$$Flip_n(x) \triangleq x(-n)$$

which is $x(N-n)$ via modulo indexing. As seen in Fig., the Flip() operator flips the order of a sequence's trials 1 through $N-1$, leaving trail 0 alone. Because of modulo indexing, the sequence can also be thought of as being "flipped" about the vertical axis, shown in Fig. When used on a signal or band, the Flip operator is generally referred to as "time reversal" or "frequency reversal," and this interpretation of Fig. is typically what we desire.

4.2.2 Shift Operators

Definition:

The definition of the shift operator is

$$Shift_{\Delta, n}(x) \triangleq x(n - \Delta)$$

and $Shift_{\Delta}(x)$ represents the full shifted signal. Keep in mind that the shift is circular because indexing is done modulo N. However, we typically employ it to illustrate time delay using Δ samples. To prevent the "wrap-around" associated with a circular shift, we usually combine the shift operator with zero padding (adding zeros to the signal x). Figure shows a periodic signal with a first period given by repeated one-trial delays [0, 1, 2, 3, 4].

Illustration:

$$Shift_1([1, 0, 0, 0]) = [0, 1, 0, 0] \text{ (The delay was caused by an impulse).}$$

Illustration:

$$Shift_1([1, 2, 3, 4]) = [4, 1, 2, 3] \text{ (a case of cyclic shifting).}$$

Illustration:

$Shift_{-2}([1, 0, 0, 0]) = [0, 0, 1, 0]$ (Another instance of a circular shift).

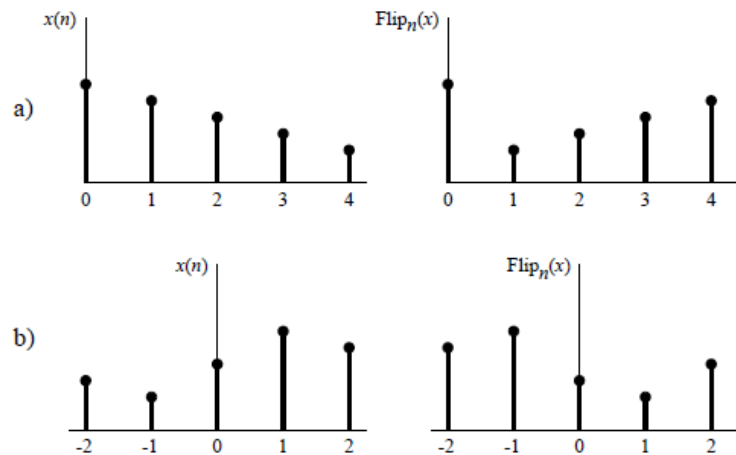
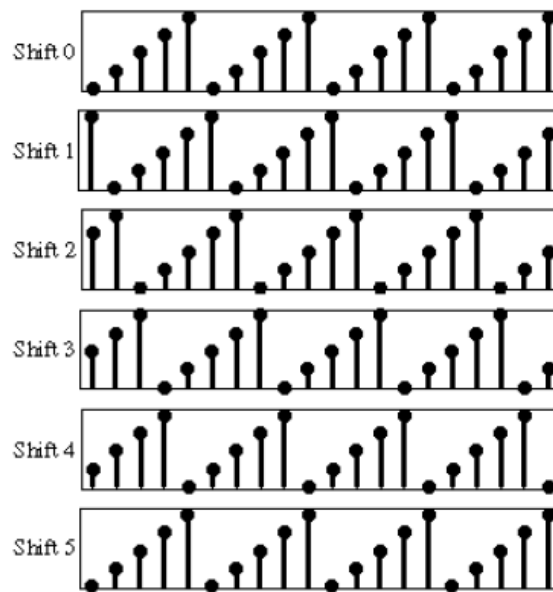


Figure 8.1: Illustration of x and $FLIP(x)$ for $N = 5$ and two different domain interpretations:
a) $n \in [0, N - 1]$.
b) $n \in [-(N - 1)/2, (N - 1)/2]$.



4.2.3 Convolution

Definition:

The definition of the symbol " $x * y$ " for the convolution of x and y signals in C^N are

$$(x * y)_n \triangleq \sum_{m=0}^{N-1} x(m)y(n - m)$$

It should be noted that this convolution is cyclic or "circular". The significance of convolution in the theory of linear systems is covered. Since convolution is commutative,

$$\boxed{x * y = y * x}$$

Proof:

$$(x * y)_n \triangleq \sum_{m=0}^{N-1} x(m)y(n-m) = \sum_{l=n}^{n-N+1} x(n-l)y(l) = \sum_{l=0}^{N-1} y(l)x(n-l) \triangleq (y * x)_n$$

where we changed the summation variable $l \triangleq n-m$ in the first step and used the knowledge that a sum over all N terms is equal to a sum from 0 to $N-1$ in the second step $N = 1$.

Convolution in Graphics

The cyclic convolution operation can be expressed in terms of previously established operators as follows:

$$y(n) \triangleq (x * h)_n \triangleq \sum_{m=0}^{N-1} x(m)h(n-m) = \langle x, SHIFT_n(FLIP(h)) \rangle (hreal)$$

here $x, y \in C^N$ and $h \in R^N$. It helps to visualise how to comprehend the last sentence above.

Figure shows convolution of

$$y = [1, 1, 1, 1, 0, 0, 0, 0]$$

$$h = [1, 0, 0, 0, 0, 1, 1, 1]$$

$$(y * h)(n) = [4, 3, 2, 1, 0, 1, 2, 3]$$

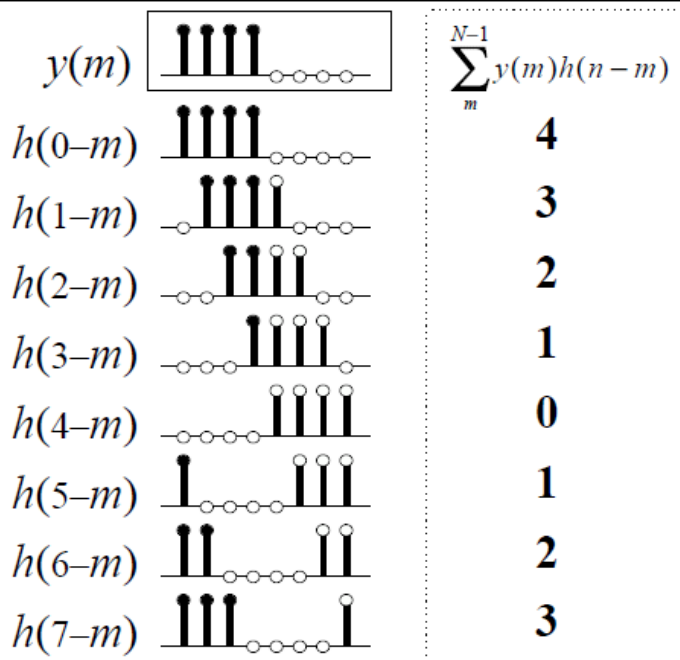


Figure 8.3: Illustration of convolution of $y = [1, 1, 1, 1, 0, 0, 0, 0]$ and its "matched filter" $h=[1,0,0,0,0,1,1,1]$ ($N = 8$).

Take y as an illustration, where y is a "rectangularly windowed signal, zero-padded by a factor of 2," and y is dc (all 1s). The convolution should be as desired.

$$Flip(h) = [1, 1, 1, 1, 0, 0, 0, 0]$$

This is equivalent to y . When h is a matched filter for y^3 , we say that $h = Flip(y)$. In this instance, h is zero-padded by a factor of 2 and matched to check for a "dc component." The purpose of the zero-padding is to mimic cyclic convolution by circular convolution. The calculation of the convolution of y and h is shown in the figure:

$$(y * h) \triangleq \sum_{n=0}^{N-1} y(n).FLIP(h) = [4, 3, 2, 1, 0, 1, 2, 3]$$

Note that the convolution output reaches a big peak at time 0. This huge peak, which is the greatest conceivable if all signals are constrained to a magnitude range of $[-1, 1]$, shows that the matching filter has "found" the dc signal beginning at time 0. Even if additional sinusoids and noise were introduced, this peak would still persist.

4.2.4 Polynomial Multiplication

It should be noted that the coefficients of two polynomials are convolved when they are multiplied together. Let $p(x)$ stand for the m th-order polynomial to illustrate this.

$$p(x) = p_0 + p_1x + p_2x^2 + \dots + p_mx^m$$

$q(x)$ should represent the n th-order polynomial, and the coefficients should be p_i .

$$q(x) = q_0 + q_1x + q_2x^2 + \dots + q_mx^m$$

using q_i coefficients. Then comes [17].

$$p(x)q(x) = p_0q_0 + (p_0q_1 + p_1q_0)x + (p_0q_2 + p_1q_1 + p_2q_0)x^2$$

$$+ (p_0q_3 + p_1q_2 + p_2q_1 + p_3q_0)x^3$$

$$+ (p_0q_4 + p_1q_3 + p_2q_2 + p_3q_1 + p_4q_0)x^4 + \dots$$

$$+ (p_0q_{n+m} + p_1q_{n+m-1} + p_2q_{n+m-2} + p_{n+m-1}q_1 + p_{n+m}q_0)x^{n+m}$$

$p(x)q(x)$ being represented by

$$r(x) \triangleq p(x)q(x) = r_0 + r_1x + r_2x^2 + \dots + r_mx^m + r_{m+n}x^{m+n},$$

We can denote the i th coefficient as follows.

$$r_i = p_0q_i + p_1q_{i-1} + p_2q_{i-2} + \dots + p_{i-1}q_1 + p_iq_0$$

$$= \sum_{j=0}^i p_jq_{i-j} = \sum_{j=-\infty}^{\infty} p_jq_{i-j}$$

$$\triangleq (p * q)(i)$$

Here, The sequences q_i and p_i are doubly infinite that are both evident as zero when i is equal to 0 and when i is equal to m, n .

4.2.5 Adding and subtracting decimal numbers

Since, for instance, decimal values are just polynomials with powers of 10,,

$$7984 = 7.10^3 + 9.10^2 + 8.10^1 + 4.10^0$$

Therefore, adding two numbers together causes their digits to become confused. The only difference is that we have carries instead of the usual polynomial multiplication. In other words, when a convolution result is more than 10, we deduct 10 and raise the digit in the next higher place by 1.

4.2.6 Correlation

Definition:

For two signals in the space, x and $y \in C^N$, The correlation operator is explained as follows:

$$(x * y)_n \triangleq \sum_{m=0}^{N-1} \overline{x(m)} y(n - m)$$

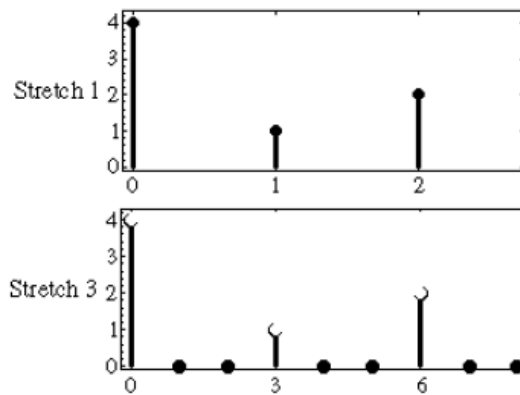
Coefficient of correlation can be interpreted as

$$(x * y)_n = \langle Shift_{-n}(y), x \rangle$$

This is the factor that shifts y unconventionally to the left in a circular motion by n trials onto x . The correlation lag is denoted by the time offset n , and the lagged product is denoted by $\overline{x(m)}y(m + n)$. In 8.8, applications of correlation are examined.

4.2.7 Stretch Operator

The $Stretch_L()$ operator, in contrast to all other operators before it, transforms an input signal of length N to an output signal of length $M \triangleq LN$, where L and N are integers. To emphasise this point, we use "m" as the time index rather than "n".



Definition:

A The definition of stretch by factor L is

$$Stretch_{L,m}(x) \triangleq \{x(m/L), m/L = integer, 0, m/L = integer\}$$

Therefore, add $L-1$ zeros between each pair of trials to expand a signal by the factor L . Figure 1 shows an illustration of a three-fold stretch. It is an illustration.

$$Stretch_3([4, 1, 2]) = [4, 0, 0, 1, 0, 0, 2, 0, 0]$$

Up sampling, or raising the sampling rate by an integer factor, is defined and studied using the stretch operator.

4.2.8 Zero Padding

Definition:

Zero padding is the practise of padding a signal with zeros. It converts a signal of length N into one of length $M \geq N$, albeit An integer multiple of N is not required for M :

$$ZeroPad_{M,m}(x) \triangleq \{x(m), 0 \leq m \leq N - 1, N \leq m \leq M - 1\}$$

For example,

$$ZeroPad_10([1, 2, 3, 4, 5]) = [1, 2, 3, 4, 5, 0, 0, 0, 0, 0]$$

When $x(n)$ denotes a signal beginning at time 0 and encompassing for N trials, the aforementioned concept makes sense. The zero padding is typically introduced between trials if, on the other hand, we are zero padding a band or have a time-domain signal with nonzero trials for negative time indices $(N - 1)/2$ and For N odd, use $(N + 1)/2$ (Keep in mind that $((N + 1)/2 = (N + 1)/2 - N = -(N - 1)/2 \text{ mod } N$), The point $z = -1 (\omega = \pi f_s)$ is where zero padding is added for spectra. This is also true for even N . Figure exemplifies this second interpretation of zero padding. Additionally, it is utilised with zero-phase FFT windows, which are covered in more detail below.

We will be able to demonstrate using Fourier theorems how zero padding in the time domain results in band-limited interpolation in the frequency domain. Additionally, band-limited interpolation in the time domain results from zero padding in the frequency domain. This is an example of an effective optimum sampling rate change.

Band restricted interpolation is the best interpolation for digital signal processing, it is vital to highlight.

4.2.9 Repetition Operator

Like the $Stretch_L()$ operator, the $Repeat_L()$ operator converts a signal of length N to a signal of length $M \triangleq LN$.

Definition:

The repeat L times operator is, as stated by,

$$Repeat_{L,m}(x) \triangleq x(m), m = 0, 1, 2, \dots, M - 1$$

where $M \triangleq LN$. Thus, the $Repeat_L()$ Basically, the operator repeats its input signal L times. In Fig., an illustration of $Repeat_2(x)$ is shown

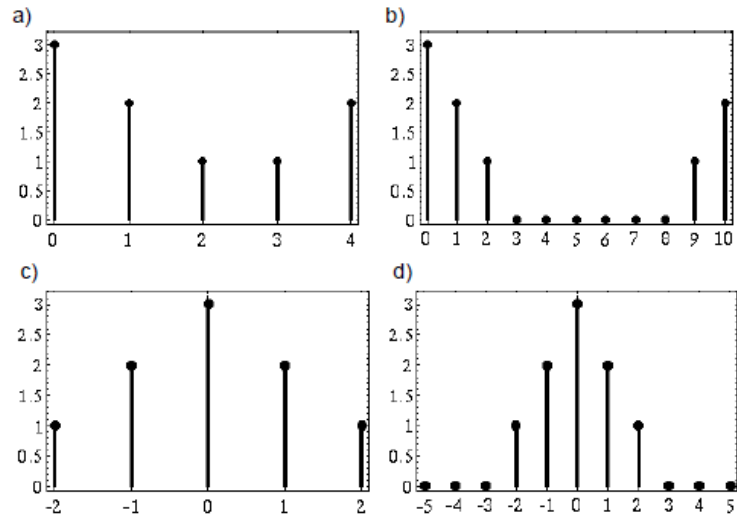


Figure: Here's one of zero padding in the frequency domain

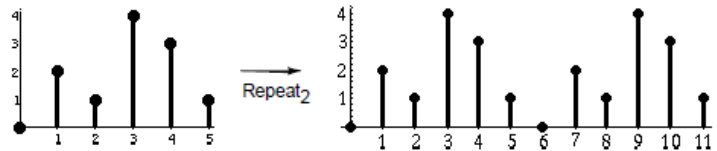
a) The first spectrum $X = [3, 2, 1, 1, 2]$ plotted in the range $[0, N - 1]$ over the domain k where $N = 5$, or as a computer array would normally depict the spectral array.

b) $ZeroPad_{11}(X)$.

c) The identical signal X shown over a more intuitive range for interpreting negative frequencies, $k \in [-(N - 1)/2, (N - 1)/2]$.

d) $ZeroPad_{11}(X)$.

$$REPEAT_2([0, 2, 1, 4, 3, 1]) = [0, 2, 1, 4, 3, 1, 0, 2, 1, 4, 3, 1]$$



The original spectrum X is presented in Fig., and $Repeat_3(X)$ is shown in Fig. as a frequency-domain example. On the right-rear face of the enclosing box is the $z = 1$ point (dc). It should be noted that the spectrum of X appears to be slightly "triangularly shaped" when centred at $k = 0$. The repeating block can be thought of as reflecting the triangularly shaped "baseband" spectrum centred around $z = 1$ or as extending from the point at $z = 1$ to the point far to the left.

To formulate the Fourier theorem, utilise the repeat operator.

$$Stretch_L \longleftrightarrow Repeat_L$$

This means that when a signal is stretched by the factor L , its band repeats L rotations of the unit circle.

4.2.10 Operator for Downsampling

Definition:

Every Lth trial, beginning with trial 0, is taken when downsampling by L.

$$\text{Select}_{L,m}(x) \triangleq x(mL), m = 0, 1, 2, \dots, M - 1 (N = LM, x \in C^N)$$

A length $N = LM$ signal is reduced to a length M signal using the $\text{Select}_L()$ operator. It is the $\text{Stretch}_L()$ operator's opposite (but not the other way around), that is,

$$\text{Select}_L(\text{Stretch}_L(x)) = x$$

$$\text{Stretch}_L(\text{Select}_L(x)) \neq x (\text{in general}).$$

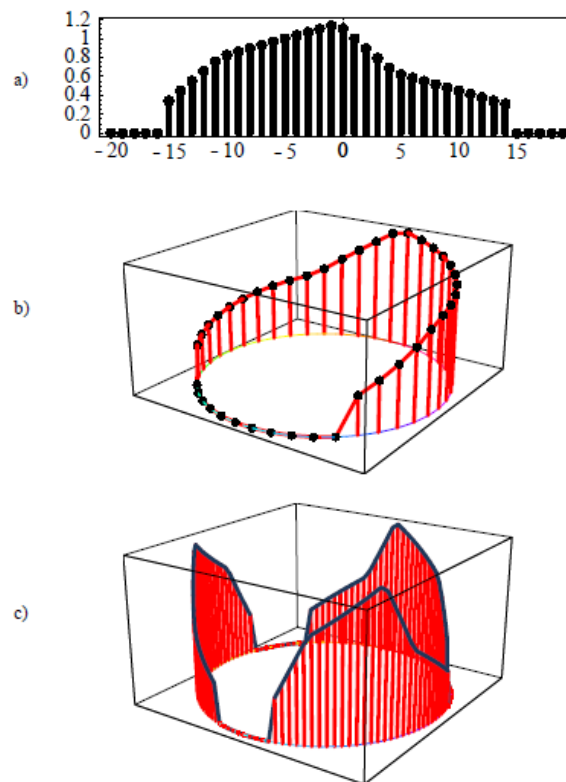


Figure 8.7: Illustration of $\text{REPEAT}_3(X)$.

- a) Conventional plot of X .
- b) Plot of X over the unit circle in the z plane.
- c) $\text{REPEAT}_3(X)$.

Because they are linear time-varying operators, the stretch and downsampling processes do not interchange. Time-varying switches controlled by the trial index n can be used to demonstrate them.

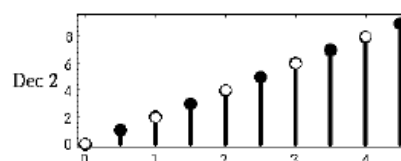


Figure : An example of $Select_2(x)$. The circles with a white fill represent the samples that were kept, while the circles with a black fill represent the samples that were discarded.

Fig. displays a $Select_2(x)$ example. Here's one:

$$Select_2([0, 1, 2, 3, 4, 5, 6, 7, 8, 9]) = [0, 2, 4, 6, 8]$$

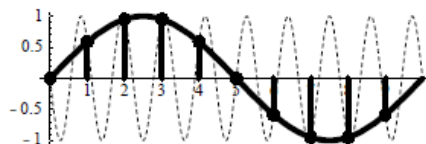
4.2.11 Alias Operator

When a signal is undersampled, aliasing happens. Frequency-domain aliasing results from too low a signal sampling rate (f_s).

When sampling a continuous-time signal, the issue of aliasing typically arises. According to Shannon's Sampling Theorem, aliasing caused by sampling won't occur as long as the sample rate is greater than twice the highest frequency present in the signal being tested. In order to keep the maths as straightforward as possible, we are simply taking discrete-time signals into account in this part. When a discrete-time signal is decimated to lower its sample rate, aliasing can place. Continuous-time sampling can be thought of as the restrictive case when the initial sample rate is infinite.

Figure 1 shows an illustration of aliasing.

Due to aliasing, the high-frequency sinusoid and lower frequency sinusoid in the figure cannot be distinguished from one another. We refer to the lower frequency as aliasing the higher frequency. Time-domain aliasing results from undersampling in the frequency domain. The term "aliasing" typically refers to frequency-domain aliasing when time or frequency is not specified.



It is stated that the aliasing operator is

$$Alias_{L,m}(x) \triangleq \sum_{l=0}^{N-1} x \left(m + l \frac{N}{L} \right), m = 0, 1, 2, \dots, M - 1 (N = LM, x \in C^N)$$

A length $N = LM$ signal is reduced to a length M signal using the $Alias_L()$ operator, just as the $Select_L()$ operator. The best approach to think of it is to divide the initial N samples into L blocks, each of length M , with the first block covering samples 0 through $M - 1$, the second block $2M - 1$, etc. Simply put the blocks after that. This method is known as aliasing. Time domain aliasing is the term used when the original signal is a time signal; frequency-domain aliasing, or simply aliasing, is used when the original signal is a band. Keep in mind that aliasing cannot be reversed. In general, it is impossible to improve the original blocks once they have been joined together.

For Use As An Example

$$Alias_2([0, 1, 2, 3, 4, 5]) = [0, 1, 2] + [3, 4, 5] = [3, 5, 7]$$

$$\text{Alias}_3([0, 1, 2, 3, 4, 5]) = [0, 1] + [2, 3] + [4, 5] = [6, 9]$$

The Fourier theorem is expressed using the alias operator.

$$\text{Select}_L \longleftrightarrow \text{Alias}_L$$

In other words, the signal's spectrum is aliased when it is lowered by the factor L.

Figure displays the outcome of applying Alias_2 to $\text{Repeat}_3(X)$ from Fig. Imagine that the spectrum shown in Figure is drawn on a piece of paper that has been rolled into a cylinder whose edges come together at $z = 1$ (Figure's upper right corner). The paper cylinder can then be rerolled to trim its perimeter in half, simulating the Alias_2 process. Reroll it in such a way that two sheets of paper are always interacting on the new, narrower cylinder. Just incorporate the

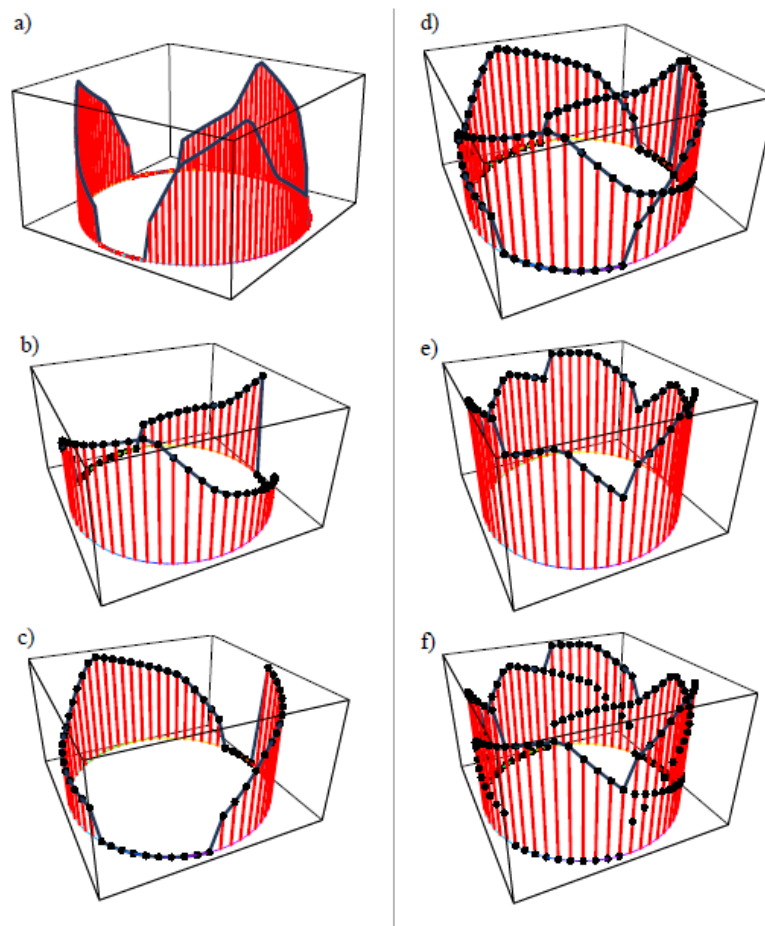


Figure Illustration of aliasing in the frequency domain.

a) $\text{Repeat}_3(X)$ from Fig.

b) The new, smaller unit circle (which is magnified to the original size) was wrapped around the first half of the previous unit circle (0 to π).

c) Second half, also wrapped around the new unit circle (π to 2π).

d) The combination of the summable components.

You may find the Alias₂ of the original band on the unit circle by adding the values on the two intersecting sheets. The cylinder would be further shrunk to alias by three, resulting in three layers of paper in the cylinder, and so on.

The first circular wrap of the paper cylinder is shown in Figure, and the second wrap is shown in Figure. Figure 2 overlays these and Figure 2 adds them all up. Finally, Fig. illustrates the two components' addition and overlay. The first component is reflected to be at its original frequencies, whilst the second component "aliases" to new frequency components. All other unit circles in Figure cover frequencies from 0 to $f_s/2$ if the unit circle in Figure covers frequencies from 0 to f_s .

Overall, aliasing by the factor K equates to a K-fold reduction in sampling rate. Anti-aliasing low pass filters are typically employed to prevent aliasing when the sample rate is decreased. a lowpassAll signal components outside of the frequency range $[-f_s/(2K), f_s/(2K)]$ are reduced by the filter. They will alias if they are not filtered out, and aliasing is typically not reversible.

When the two parts are added, the unit circle is supposedly compressed by $Alias_2$ to a unit circle that is half its original size. A better way to explain "repeating," which is the reverse of aliasing, is as "periodic extension" of the unit circle's perimeter that results in "more band" for the larger unit circle. Downsampling is the stretch operator's counterpart in the time domain. The restriction on factors being integers is lifted in continuous time, and we arrive at the (simpler) similarity theorem (proved in 8.9). Only for integer stretch, downsampling, aliasing, and repetition factors are all of these correlations precise.

4.3 Odd and Even Functions

Even and odd symmetries can be used to express some of the Fourier theorems in a concise manner.

Definition:

If $f(-n) = f(n)$, then a function $f(n)$ is said to be even. Although the term "symmetric" is often used to refer to functions that are symmetric around a point other than 0, an even function is also symmetric.

Definition:

If $f(-n) = -f(n)$, a function $f(n)$ is said to be odd. Anti-symmetric is another name for an odd function.

Take note that $f(0) = 0$ must hold for any odd function $f(n)$. Moreover,

Given that $x(N/2) = -x(-N/2) = -x(-N/2 + N) = -x(N/2)$, i.e., $N/2$ and $-N/2$ index the same point, we also have $x(N/2) = 0$ for every $x \in C^N$ with N even.

Theorem:

Each function $f(n)$ can be broken down into its even and odd parts, denoted by $f_e(n)$ and $f_o(n)$, respective

$$f_e(n) \triangleq \frac{f(n) + f(-n)}{2}$$

$$f_0(n) \triangleq \frac{f(n) + f(-n)}{2}$$

Proof:

According to the definitions above, by construction, $f_e(n)$ is even and $f_o(n)$ is odd. As a whole,

$$f_e(n) + f_o(n) = \frac{f(n) + f(-n)}{2} + \frac{f(n) + f(-n)}{2} = f(n)$$

Theorem:

Odd functions produce an even product, even functions produce an odd product, and odd functions produce an odd product.

Proof is plainly seen.

Even can be thought of as (+) and odd as (-): (+) (+) = (+), (-) (-) = (+), and (+) (-) = (-) (+) = (-). This is because even times even are even, odd times odd are even, and even times odd are odd.

Illustration: Since $\cos(-\theta) = \cos(\theta)$, $\cos(\omega_k n)$ is an even signal.

For instance, since $\sin(-\theta) = -\sin(\theta)$, the signal $\sin(\omega_k n)$ is peculiar.

As an example, the ratio of $\cos(\omega_k n)$ to $\sin(\omega_l n)$ is odd (even when odd).

Example: The difference between $\sin(\omega_k n)$ and $\sin(\omega_l n)$ is even (odd times odd).

Theorem: An odd signal x_o in C^N has zero trials total when added together.

Proof:

By stating the sum as $x_o(0) + [x_o(1) + x_o(-1)] + \dots + x_o(N/2)$, this is easily seen where the final term only appears when N is even. For an odd signal x_o , each term written in this manner is 0.

A good example is $\omega_k = 2\pi k/N$ for all DFT sinusoidal frequencies.

$$\sum_{n=0}^{N-1} \sin(\omega_k n) \cos(\omega_k n) = 0, k = 0, 1, 2, \dots, N-1$$

more broadly,

$$\sum_{n=0}^{N-1} x_e(n) x_o(n) = 0$$

for any x_e even signal and x_o odd signal in C^N .

4.4 Fourier Theorems

The fundamental Fourier theorems are presented and proven in this lesson. It does not matter how straightforward these theorems are when compared to other DFTs, Fourier transforms,

and Fourier series in the DFT situation. The mathematical prerequisites for the existence of the Fourier transform can be rather challenging to explain when infinite summations or integrals are intricate. On these issues, mathematicians have expended a great deal of work. Focusing solely on the DFT situation allows us to understand the fundamental ideas behind the Fourier theorems without getting bogged down in complex mathematical issues.

4.4.1 Linearity

The DFT fulfils the following theorem for all x, y in C^N and α, β in C^N .

$$\boxed{\alpha x + \beta y \longleftrightarrow \alpha X + \beta Y}$$

Because of this, the DFT is a linear operator.

Proof:

$$\begin{aligned} DFT_k(\alpha x + \beta y) &\triangleq \sum_{n=0}^{N-1} [\alpha x(n) + \beta y(n)] e^{-j2\pi nk/N} \\ &= \sum_{n=0}^{N-1} [\alpha x(n)] e^{-j2\pi nk/N} + \sum_{n=0}^{N-1} [\beta y(n)] e^{-j2\pi nk/N} \\ &= \alpha \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} + \beta \sum_{n=0}^{N-1} y(n) e^{-j2\pi nk/N} \\ &\triangleq \alpha X + \beta Y \end{aligned}$$

4.4.2 Judgement and Reversal

Theorem: If x is any number in C^N ,

$$\boxed{\bar{x} \leftrightarrow Flip(\bar{X})}$$

Proof:

$$\begin{aligned} DFT_k(\bar{x}) &\triangleq \sum_{n=0}^{N-1} \overline{x(n)} e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} \overline{x(n) e^{j2\pi nk/N}} \\ &= \overline{\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(-k)/N}} \triangleq FLIP_k(\bar{X}) \end{aligned}$$

Theorem: If x is given in C^N ,

$$\boxed{\bar{x} \leftrightarrow Flip(\bar{X})}$$

Proof: We obtain by altering the summing variable $m \triangleq N - n$

$$\begin{aligned} DFT_k(FLIP(x)) &\triangleq \sum_{n=0}^{N-1} \overline{x(N-n)} e^{-j2\pi nk/N} = \sum_{m=N}^1 \overline{x(m)} e^{-j2\pi(N-m)k/N} \\ &= \sum_{n=0}^{N-1} \overline{x(m)} e^{-j2\pi nk/N} = \overline{\sum_{n=0}^{N-1} x(N-n) e^{-j2\pi nk/N}} \triangleq X(k) \end{aligned}$$

Theorem: For any $x \in C^N$,

$$\boxed{FLIP(x) \leftrightarrow Flip(X)}$$

$$\begin{aligned} DFT_k(FLIP(x)) &\triangleq \sum_{n=0}^{N-1} x(N-n) e^{-j2\pi nk/N} = \sum_{m=0}^{N-1} x(m) e^{-j2\pi(N-m)k/N} \\ &= \sum_{m=0}^{N-1} x(m) e^{j2\pi nk/N} \triangleq X(-k) \triangleq FLIP_k(X) \end{aligned}$$

corollary : For some $x \in R^N$,

$$\boxed{FLIP(x) \leftrightarrow \overline{X}} (x \text{ real})$$

Proof: Prioritising the prior demonstration at the third formula and keeping in mind that x is real

$$\sum_{n=0}^{N-1} x(n) e^{j2\pi nk/N} = \overline{\sum_{n=0}^{N-1} \overline{x(n)} e^{j2\pi nk/N}} = \overline{\sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}} \triangleq \overline{X(k)}$$

when $x(n)$ is an integer.

Therefore, reversal in the time domain corresponds to conjugation in the frequency domain. Negating the spectral phase is a novel approach that causes the signal to move backwards in time.

corollary: If $x \in R$, then

$$\boxed{FLIP(x) \leftrightarrow \overline{X}} (x \text{ real})$$

Proof: The preceding two situations lead to this.

Hermitian symmetry, often known as "conjugate symmetry," is the property $X(-k) = \overline{X(k)}$. It may be referred to as skew-Hermitian if $X(-k) = -\overline{X(k)}$.

An alternative formulation of the previous corollary is

$$\boxed{x \in R^n \leftrightarrow X \text{ is Hermitian}}$$

4.4.3 Symmetry

When x is real, we discovered in the last section that $Flip(X) = \overline{X}$. This fact is highly applicable. Every genuine signal, according to this, has a Hermitian band. This symmetry allows us to discard all of a genuine signal's negative frequency spectral trials and replace them if necessary with fresh positive-frequency trials. Additionally, spectral graphs of actual signals are displayed between 0 Hz and $f_s/2$ Hz. On the other hand, because a complex signal's positive and negative frequency components are independent, A complex signal's spectrum must often be shown between $-f_s/2$ and $f_s/2$ (or from 0 to f_s).

Theorem: If $x \in R^N$, $\text{re}\{X\}$ is even and $\text{im}\{X\}$ is odd.

Proof: The conjugate symmetry of X for real signals x leads to this right away.

Theorem: If $x \in R^N$, $|X|$ is even and $\angle X$ is odd.

Proof: This naturally derives from X 's conjugate symmetry when stated in polar form. $X(k) = |X(k)|e^{j\angle X(k)}$.

One of the most significant symmetry theorems is maybe the conjugate balancing of bands of real signals. However, there are a few more that we can display right away.

A signal that is even has an even transform.

$$\boxed{x_{\text{even}} \leftrightarrow X_{\text{even}}}$$

Proof: By using the formula $x \triangleq x_r + jx_i$, express x in terms of its real and imaginary components. The real and imaginary portions of a complex signal x must be equal for the signal to be even. Then

$$\begin{aligned} X(k) &\triangleq \sum_{n=0}^{N-1} x(n)e^{-j\omega_k n} \\ &= \sum_{n=0}^{N-1} [x_r(n) + jx_i(n)]\cos(\omega_k n) - j[x_r(n) + jx_i(n)]\sin(\omega_k n) \\ &= \sum_{n=0}^{N-1} [x_r(n)\cos(\omega_k n) + x_i(n)\sin(\omega_k n)] + j[x_i(n)\cos(\omega_k n) - x_r(n)\sin(\omega_k n)] \\ &= \sum_{n=0}^{N-1} \text{even.even} + \sum_{n=0}^{N-1} \text{even.odd} + j \sum_{n=0}^{N-1} \text{even.even} - j \sum_{n=0}^{N-1} \text{even.odd} \\ &= \sum_{n=0}^{N-1} \text{even.even} + j \sum_{n=0}^{N-1} \text{even.even} = \text{even} + j.\text{even} = \text{even} \end{aligned}$$

Theorem: The transform of a real even signal is real even:

$$\boxed{x_{\text{realeven}} \leftrightarrow X_{\text{realeven}}}$$

Proof: This follows directly from the previous proof's assumption of $x_i(n) = 0$ and the observation that the DFT of an even and real function reduces to a certain kind of cosine transform.

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos(\omega_k n),$$

or we may present it straight up:

$$\begin{aligned} X(k) &\triangleq \sum_{n=0}^{N-1} x(n) e^{-j\omega_k n} = \sum_{n=0}^{N-1} x(n) \cos(\omega_k n) + j \sum_{n=0}^{N-1} x(n) \sin(\omega_k n) \\ &= \sum_{n=0}^{N-1} x(n) \cos(\omega_k n) (\sum \text{even.odd} = \sum \text{odd} = 0) \\ &\quad \sum_{n=0}^{N-1} \text{even.even} = \sum_{n=0}^{N-1} \text{even} = \text{even} \end{aligned}$$

Zero phase signals are frequently referred to as signals having a real band, such as real, even signals. However, take note that the phase is actually π , not 0, when the band goes negative, which it can do. Even though the phase of such signals fluctuates between 0 and π at the zero-crossings of the band, they are sometimes referred to as "zero phase." In real-world applications, such zero-crossings frequently occur at low amplitudes, like in the "sidelobes" of the DTFT of an FFT window.

4.4.4 Change Theorem

Theorem: Given any integer Δ and any $x \in C^N$,

$$\boxed{DFT_k[SHIFT_{\Delta}(x)] = e^{-j\omega_k \Delta} X(k)}$$

$$\begin{aligned} DFT_k[SHIFT_{\Delta}(x)] &\triangleq \sum_{n=0}^{N-1} x(n - \Delta) e^{-j2\pi n k / N} \\ &= \sum_{m=-\Delta}^{N-1-\Delta} x(m) e^{-j2\pi(m+\Delta)k/N} \\ &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n k / N} e^{-j2\pi k \Delta / N} \\ &= e^{-j2\pi k \Delta / N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi n k / N} \\ &\triangleq e^{-j\omega_k \Delta} X(k) \end{aligned}$$

The shift theorem states that a delay in the time domain agrees with a linear phase component in the frequency domain. In particular, the linear phase term matches a delay of Δ time waveform samples. The spectrum is multiplied by $e^{-j\omega_k\Delta}$, where $\omega_k \triangleq 2\pi k/N$. Simply replace Δ with ΔT to modify the delay from samples to seconds (to consider ω_k as radians per second instead of radians per trial). A linear phase term has no effect on spectral magnitude, it should be emphasised. In other words, $|e^{-j\omega_k\Delta}X(k)| = |X(k)|$.

4.4.5 Phase Linear Terms

Because the phase of the $e^{-j\omega_k\Delta}$ is a linear function of frequency, it is referred to as a linear phase term:

$$\angle e^{-j\omega_k\Delta} = -\Delta \cdot \omega_k$$

The phase against radian frequency map thus exhibits a $-\Delta$ slope. In most circumstances, the slope of the linear phase component is deducted from the samples' time delay. Using the polar version of the original spectrum as an example

$$X(k) = G(k)e^{j\theta(k)},$$

We can see that a linear phase term simply alters the spectral phase in the equation where G and θ are the magnitude and phase of X , respectively (both real) $\theta(k)$:

$$e^{-j\omega_k\Delta}X(k) \triangleq e^{-j\omega_k\Delta}G(k)e^{j\theta(k)} = G(k)e^{j[\theta(k)-\omega_k\Delta]}$$

where $\omega_k \triangleq 2\pi k/N$. The original spectral phase is supplemented with a negatively sloped linear phase by a positive time interruption (waveform shift to the right). The original spectral phase is supplemented with a positively sloped linear phase via a negative time delay (waveform shift to the left). This relationship is one of the most important in exercise, so if it seems like we are overstressing it, that is because we are.

If a signal's phase is of the form kind, it is referred to as linear phase signal.

$$\theta(\omega_k) = \pm\Delta \cdot \omega_k \pm \pi I(\omega_k)$$

where $I(\omega_k)$ is a function that indicates if anything is 0 or 1.

4.4.6 Convolution Theorem

Theorem: For any $x, y \in C^N$,

$$x * y \leftrightarrow X \cdot Y$$

$$DFT_k(x * y) \triangleq \sum_{n=0}^{N-1} (x * y)_n e^{-j2\pi nk/N}$$

$$\begin{aligned}
&\triangleq \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(m)y(n-m)e^{-j2\pi nk/N} \\
&= \sum_{m=0}^{N-1} x(m) \sum_{n=0}^{N-1} y(n-m)e^{-j2\pi nk/N} \\
&\left(\sum_{m=0}^{N-1} x(m)y(n-m)e^{-j2\pi nk/N} \right) Y(k) \\
&\triangleq X(k)Y(k)
\end{aligned}$$

One could argue that this is the most important Fourier theorem ever. It serves as the foundation for many FFT applications. The convolution theorem enables the FFT to do a rapid Fourier transform as well as a fast convolution. It turns out that using the FFT to perform convolutions is only actually more efficient in practise for convolutions that are genuinely long, like $N \geq 100$. When compared to "direct" convolution, the savings increase dramatically for considerably longer convolutions. This occurs because FFT-based convolution only takes on the order of $N \lg(N)$ operations, but direct convolution requires on the order of N^2 operations (multiplications and adds).

4.4.7 Theorem of Convolution Dual

The Convolution's Dual Theorem asserts that convolution in the frequency domain corresponds to multiplication in the time domain:

Theorem:

$$x \cdot y \leftrightarrow \frac{1}{N} X * Y$$

4.4.8 Correlation Theorem

Theorem: For any $x, y \in C^N$,

$$x * y \leftrightarrow \bar{X} \cdot Y$$

Proof:

$$(x * y)_n \triangleq \sum_{m=0}^{N-1} \overline{x(m)} y(n+m)$$

$$= \sum_{m=0}^{N-1} \overline{x(-m)} y(n-m)$$

$$(FLIP(\bar{x} * y))_n$$

$$\leftrightarrow \bar{X}.Y$$

where the Convolution Theorem leads to the final step, and the outcome $Flip(\bar{x}) \leftrightarrow \bar{X}$ from the section on Conjugation and Reversal.

4.4.9 Power Theorem

Theorem: For any $x, y \in C^N$,

$$\langle x, y \rangle = \frac{1}{N} \langle X, Y \rangle$$

Proof:

$$\begin{aligned} \langle x, y \rangle &\triangleq \sum_{n=0}^{N-1} x(n)\overline{y(n)} = (x * y)_0 = DFT_0^{-1}(\bar{Y}.X) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} X(k)\overline{Y(k)} \triangleq \frac{1}{N} \langle X, Y \rangle \end{aligned}$$

The power theorem would be more appealing, you should note ($\langle x, y \rangle = \langle X, Y \rangle$) if the normalised DFT sinusoid were the coefficient of projection onto the DFT, $\tilde{s}_k(n) \triangleq s_k(n)/\sqrt{N}$.

4.4.10 Theorem of Rayleigh Energy (Parseval's Theorem)

Theorem: For any $x \in C^N$,

$$\|x\|^2 = \frac{1}{N} \|X\|^2$$

i.e.

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Proof: A unique instance of the Power Theorem is this.

Remember that if we were using the normalised DFT, the relationship ($\|x\| = \|X\|$) would be clearer.

4.4.11 Stretch Theorem (Repeat Theorem)

Theorem: For any $x \in C^N$,

$$STRETCH_L(x) \leftrightarrow REPEAT_L(X)$$

Proof: Describe the stretcher driver:

$$\text{Stretch}_{L,m}(x) \triangleq \{x(m/L), m/L = \text{integer}, m/L \neq \text{integer}\}$$

Let $y \triangleq \text{Stretch}_L(x)$, where $y \in C^M$, $M = LN$. Define additionally the new denser frequency grid linked to length M by $\omega'_k \Delta = 2\pi k/M$, with $\omega_k = 2\pi k/N$ as usual. Then

$$Y(k) \triangleq \sum_{m=0}^{M-1} y(m) e^{-j\omega'_k m} = \sum_{n=0}^{N-1} x(n) e^{-j\omega'_k nL} (n \triangleq m/L)$$

but

$$\omega'_L \triangleq \frac{2\pi k}{M} L = \frac{2\pi k}{N} = \omega_k$$

Thus, $Y(k) = X(k)$, and when k increases from 0 to 1, L copies of X are created using modulo indexing of X $M - 1 = LN - 1$.

4.4.12 Downsampling Theorem (Aliasing Theorem)

Theorem: For any $x \in C^N$,

$$\boxed{\text{STRETCH}_L(x) \leftrightarrow \frac{1}{L} \text{ALIAS}_L(X)}$$

Proof: Let $k' \in [0, M - 1]$ in the aliased spectrum, denote the frequency index, and let $Y(k') \triangleq \text{Alias}_{L,k'}(X)$. Then Y is length $M = N/L$, where L is the downsampling factor. We have

$$Y(k') \triangleq \text{Alias}_{L,k'}(X) \triangleq \sum_{l=0}^{L-1} Y(k' + lM), k' = 0, 1, 2, \dots, M - 1$$

$$\triangleq \sum_{l=0}^{L-1} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(k'+lM)n/N}$$

$$\sum_{n=0}^{N-1} x(n) e^{-j2\pi k' n/N} \sum_{l=0}^{L-1} e^{-j2\pi l n M/N}$$

Since $M/N = L$, the sum over l becomes

$$\sum_{l=0}^{L-1} e^{-j2\pi l n M/N} = \sum_{l=0}^{L-1} \left[e^{-j2\pi n/L} \right]^l = \frac{1 - e^{-j2\pi n}}{1 - e^{-j2\pi n/L}} = \begin{cases} L, & n = 0 \text{ mod } L \\ 0, & n \neq 0 \text{ mod } L \end{cases}$$

utilising the preceding derived geometric series' closed form statement. We can observe that per L samples, the total over L effectively samples x. The prior formula can be used to represent this by defining $m \triangleq n/L$, which exclusively ranges over the nonzero samples:

$$\begin{aligned}
ALIAS_{L,k'}(X) &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi k'n/N} \sum_{l=0}^{L-1} e^{-j2\pi ln/L} \\
&= L \sum_{m=0}^{N/L-1} x(mL)e^{-j2\pi k'(mL)/N} (m \triangleq n/L) \\
&\triangleq L \sum_{m=0}^{M-1} x(mL)e^{-j2\pi k'm/M} \triangleq L \sum_{m=0}^{M-1} SELECT_{L,m}(x)e^{-j2\pi k'm/M} \\
&\triangleq L.DFT'_k(SELECT_L(x))
\end{aligned}$$

The theorem is proved since the aforementioned deduction is likewise valid in reverse.

4.4.13 Zero Padding Principle

Zero padding is a useful tool for practical band learning. This theorem demonstrates how ideal interpolation in the frequency domain and zero padding in the time domain are equivalent. Let $x \in C^N$ and define $y = ZeroPad_M(x)$. Then $y \in C^M$ with $M \geq N$. Put k in front of the original frequency index to indicate $\omega_k \triangleq 2\pi k/N$ and the new frequency index by k' , where $\omega'_k \triangleq 2\pi k'/M$.

Definition: A spectrum's optimal band-limited interpolation $X(\omega_k) \triangleq DFT_k(x)$, $x \in C^N$, to an arbitrary new frequency $\omega \in [-\pi, \pi)$ is defined as

$$X(\omega) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j\omega n}$$

Keep in mind that this is simply the DFT definition with ω_k and ω altered. Specifically, the band is projected onto the new sinusoid just as if it were a DFT sinusoid in order to interpolate it. When x is taken to be N trials of a time-limited signal, this makes the most sense. This means that the inner product between the signal and any sinusoid will be exactly as in the equation above if the signal truly is zero outside of the time range $[0, N - 1]$. Thus, this type of interpolation is excellent for signals with a limited amount of time.

By applying an integer factor L to a signal, the interpolation operator interpolates it. This is,

$$Interp_{L,k'}(X) \triangleq X(\omega'_k), \omega'_k = 2\pi k'/M, k' = 0, 1, 2, \dots, M-1, M \triangleq LN$$

Since $X(\omega_k) \triangleq DFT_{N,k}(x)$, which at first only defines itself over the N roots of unity, $X(\omega'_k)$ is defined over $M = LN$ roots of unity, we define $X(\omega'_k)$ for $\omega'_k \neq \omega_k$ by ideal bandlimited interpolation.

Theorem: For any $x \in C^N$,

$$\boxed{ZEROPAD_{LN}(x) \leftrightarrow INTERP_L(X)}$$

Proof: Let $M = LN$ with $L \geq 1$. Then

$$DFT_{M,k'}(ZeroPad_M(x)) = \sum_{m=0}^{N-1} x(m)e^{-j2\pi mk'/M} \triangleq X(\omega'_k) = INTERP_L(X)$$

4.4.14 Interpolation in Time with a Band Limit

Zero padding in the frequency domain corresponds to perfect band restricted interpolation in the time domain, according to the dual of the Zero-Padding Theorem. In the time domain, we have not yet established ideal band restricted interpolation in detail. We shall therefore leave the definition of the Zero-Padding Theorem to its dual: Definition: For all $x \in C^N$ and any integer $L \geq 1$,

$$\boxed{Interp_L(x) \triangleq IDFT(ZeroPad_{LN}(X))}$$

In this case, the zero-padding is of the frequency-domain variety, as previously defined and shown in Fig.

It is useful to consider how the Stretch Theorem $Stretch_L(x) \leftrightarrow Repeat_L(X)$ can be used to interpret the Interpolation Theorem. It is practical to define a "zero-centered rectangular window" operator for this purpose:

Theorem: For $x \in C^N$,

$$\boxed{Interp_L(x) = IDFT(RectWin_N(DFT(Stretch_L(x))))}$$

To put it another way, one can achieve perfect band-limited interpolation of x by the factor L by stretching x by the factor L first (by inserting $L - 1$ zeros between successive x -trials), obtaining the DFT, using the ideal low pass filter, and then applying the inverse DFT.

Proof: To begin with, keep in mind that $Stretch_L(x)$ When a signal is stretched by the factor L , or by $Repeat_L(X)$, a new signal $y = Stretch_L(x)$ is created with a spectral grid L times as dense as X and a spectrum Y that has L copies of X repeated around the unit circle. "Baseband copy"

It is possible to define as the breadth N sequence centred on frequency 0 of X . So if we can use a "ideal filter" to "pass" the baseband spectral copy and zero out all others, we can change $Repeat_L(X)$ to $ZeroPad_{LN}(X)$. I.e.,

$$RectWin_N(Repeat_L(X)) = ZeroPad_{LN}(X) \leftrightarrow Interp_L(x)$$

where the final step is time-domain ideal bandlimited interpolation by definition.

It should be noted that the unit's definition of perfect band-limited time-domain interpolation only applies to signals that exhibit periodicity across N trials. Consider the fact that cyclic convolution conforms with rectangular windowing operation in the frequency domain in the time domain⁸, and that cyclic convolution is only identical to acyclic convolution when one of the signals is genuinely periodic in N trials, in order to understand this. There is no issue with the definition of ideal spectral interpolation utilised in linking with the Zero-Padding Theorem

because all spectra $XinC^N$ are actually periodic in N samples. Though we should apply the counterpart of the Zero- Padding Theorem for the DTFT case instead for a more useful formulation of ideal time-domain interpolation. However, the current definition is ideal for signals that are precisely periodic in N trials (a rare circumstance).

Chapter 5

5 Example Applications of the DFT

Certainly! I'd be happy to provide you with some practical illustrations of FFT analysis in MATLAB. The Fast Fourier Transform (FFT) is a powerful tool for analyzing signals in the frequency domain. By applying the FFT algorithm to a time-domain signal, you can obtain its frequency spectrum, which reveals the different frequency components present in the signal.

5.1 Spectrum Analysis of a Sinusoid: Windowing, Zero-Padding, and the FFT

Let's continue with the illustrations that gradually develop the spectrum analysis process. These examples will help you understand the steps involved in analyzing real data using spectrum analysis techniques.

5.1.1 Example 1: FFT of a Simple Sinusoid

The basic sinusoid from our first example is an FFT.

$$x(n) = \cos(\omega_x nT)$$

where we select $T = 1$ (sampling rate set to 1) and $\omega_x = 2\pi(f_s/4)$ (frequency $f_s/4$). The signal length N must be a power of 2 because we're utilising the FFT.

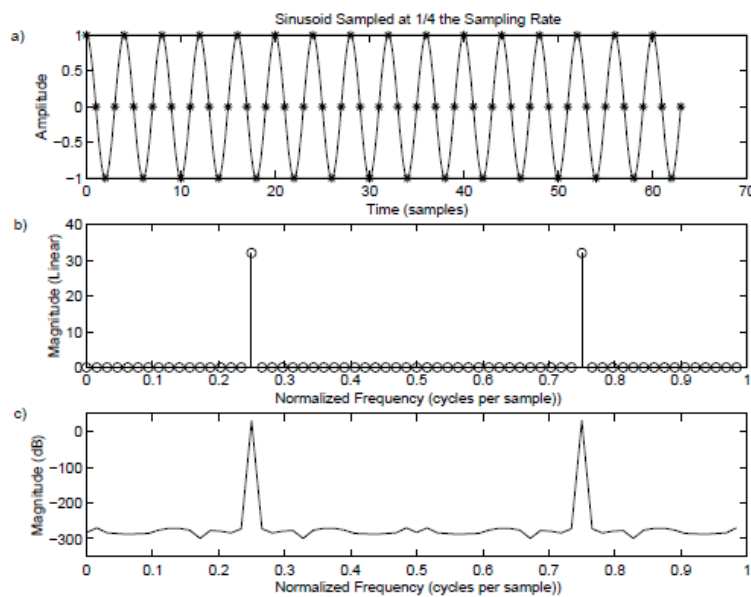


Figure 9.1: Sampled sinusoid at $f = f_s/4$. a) Time waveform. b) Magnitude spectrum. c) DB magnitude spectrum.

The outcomes are shown in Fig. 9.1. Figure 9.1a displays the time-domain signal in sampled and pseudo-continuous forms. Two peaks in the magnitude spectrum can be seen in Fig. 9.1b, located at normalised frequencies $f = 0.25$ and $f = 0.75 = -0.25$ and each measuring magnitude 32 on a linear scale. We anticipate a spectral peak amplitude of $32 = (1/2)64$ because the DFT length is $N = 64$.

$$DFT_k(\text{Cos}(\omega_x n)) \triangleq \sum_{n=0}^{N-1} \frac{e^{j\omega_x n} + e^{-j\omega_x n}}{2} e^{-j\omega_x n} = \sum_{n=0}^{N-1} \frac{e^{j0n}}{2} = \frac{N}{2}$$

as soon as $\omega_k = \pm\omega_x$. For $N = 64$, this occurs at bin numbers $k = (0.25/f_s)N = 16$ and $k = (0.75/f_s)N = 48$. The peaks will really show up at index 17 and 49 in the magX array because, as you may recall, Matlab requires indexing from 1.

You can alter the preceding code to display the magnitude spectrum on a logarithmic scale (dB scale). The resulting spectrum is shown on a logarithmic (dB) scale as well as a linear scale. You can see the near-zero magnitudes in the other bins because the spectral magnitude is better represented by the dB scale.

5.1.2 Illustration 2 Using a Not-So-Simple Sinusoid

To add one-half of a bin to the preceding example's frequency

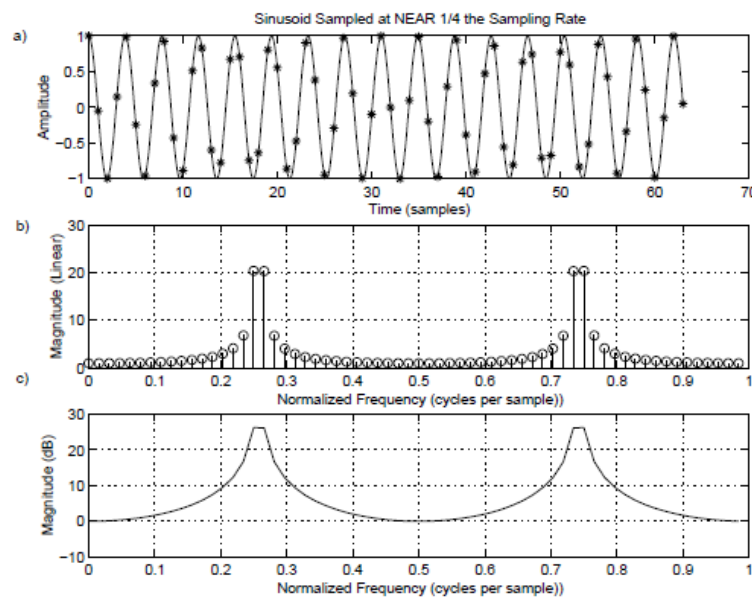


Figure 9.2: Sinusoid at Frequency $f = 0.25 + 0.5/N$. a) Time waveform. b) Magnitude spectrum. c) DB magnitude spectrum.

This enables us to see the repetitive character of the signal and comprehend the phenomena of spectral leakage by analysing the periodic extension of the time waveform and observing the spectral leakage. For additional analysis, the magnitude spectrum is also shown on both linear and dB scales.

Please be aware that Figs. 9.2b and 9.2c are merely placeholders for the figures in the text; the real figures will be determined by how the code is run.

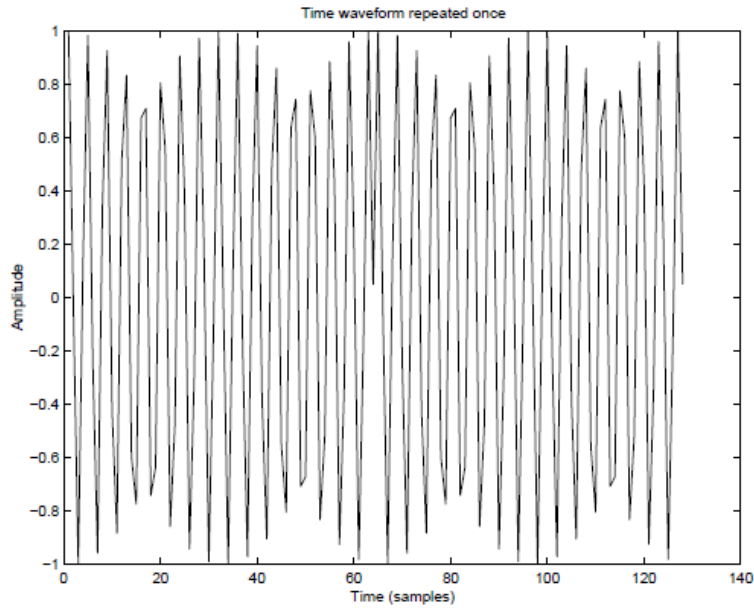


Figure 9.3: Time waveform repeated to show discontinuity introduced by periodic extension (see midpoint).

The outcome is shown in Fig. 9.3. When the signal starts its forced repetition, there is a "glitch" in the middle.

5.1.3 Illustration 3: A Zero-Padded Sinusoid's FFT

Funny enough, if we go back and look at Fig. 9.2c, there are no negative dB values. Let's apply some zero padding in the time domain to provide optimal interpolation in the frequency domain so we can clearly see the band:

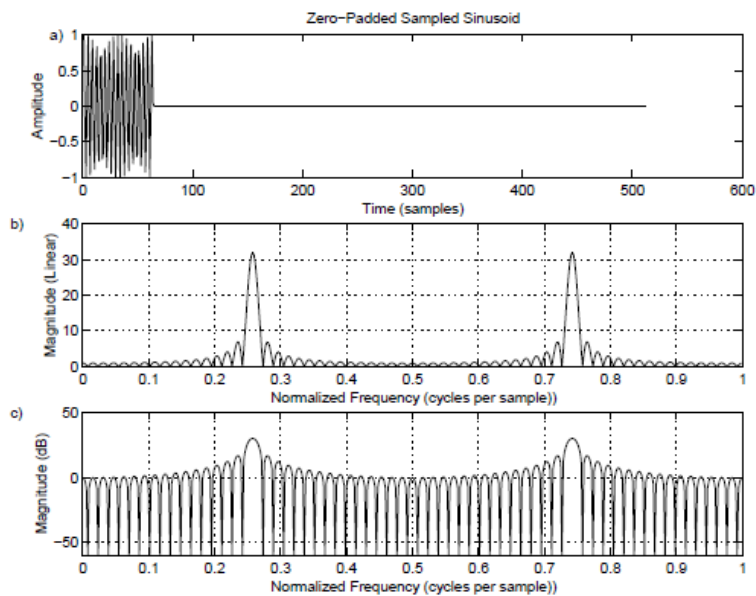


Figure 9.4: Zero-Padded Sinusoid at Frequency $f = 0.25 + 0.5/N$.
a) Time waveform. b) Magnitude spectrum. c) DB magnitude spectrum.

With 0 padding, we can tell that there is a lot going on. The band actually has a typical side lobe configuration. We can now clearly observe that there are negative dB values on the dB scale in Fig. 9.4c. This demonstrates how crucial zero padding is when interpolating spectrum displays so that the eye can properly "fill in" the gaps between the trials.

5.1.4 Illustration 4: Blackman Window

Let's examine the impact of a Blackman window(22), which has respectable but unfavourable features for audio work, to wrap up this sinusoid. The Blackman window is shown in Figure9.5 a, its magnitude band is shown on a dB scale in Figure9.5 b, and a more natural frequency axis is used in Figure9.5 c, which interprets the upper half of the caddy figures as negative frequents

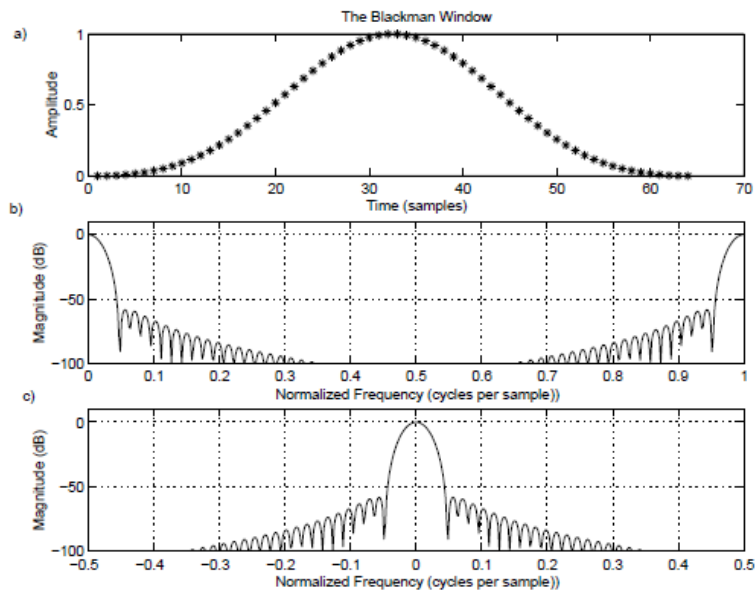


Figure 9.5: The Blackman Window. a) The window itself in the time domain. b) DB Magnitude spectrum of the Blackman window. c) Blackman-window DB magnitude spectrum plotted over frequencies $[-0.5, 0.5)$.

5.1.5 Example 5. Utilising the Blackman Window

Let's connect this window to the sinusoidal data now.

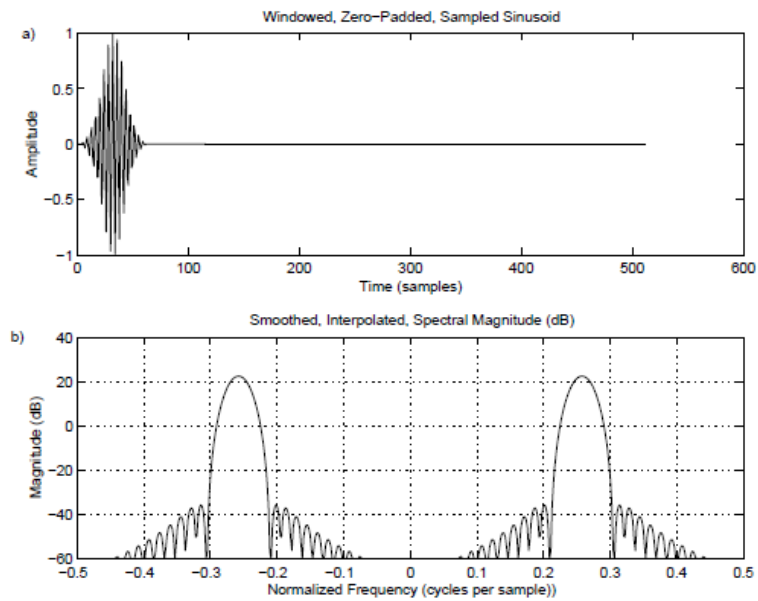


Figure 9.6: Effect of the Blackman window on the sinusoidal data segment.

5.1.6 Illustration 6: Hanning-Windowed Complex Sinusoid

On a complicated sinusoid, the effects of applying the Hanning window against the Blackman window are contrasted. You can see the variations in the spectral properties and the effects of the Hanning window against the Blackman window by contrasting the two magnitude spectra. The Blackman window has a wider main lobe but lower sidelobe levels than the Hanning window, which often has a narrower main lobe and faster sidelobe roll-off.

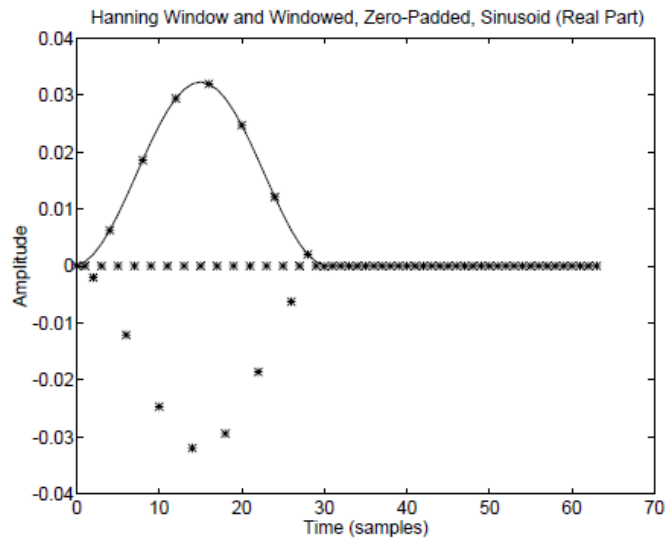


Figure 9.7: A length 31 Hanning Window (“Raised Cosine”) and the windowed sinusoid created using it. Zero-padding is also shown. The sampled sinusoid is plotted with ‘*’ using no connecting lines. You must now imagine the continuous sinusoid threading through the asterisks.

You can change the previous code by adding zero-padding in the time domain to add severely interpolated spectra to visualise the entire band. You can see the whole band and contrast it with the original spectra acquired from the shorter signals by looking at the interpolated spectra. The interpolation aids in giving the signals’ frequency content a clearer representation.

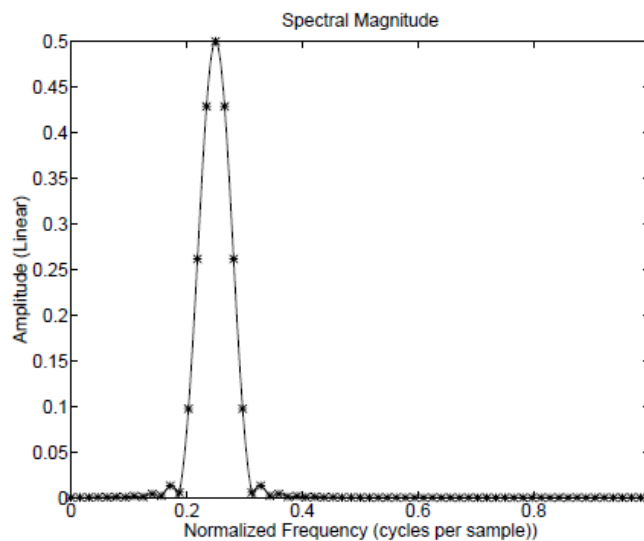


Figure 9.8: Spectral Magnitude, linear scale.

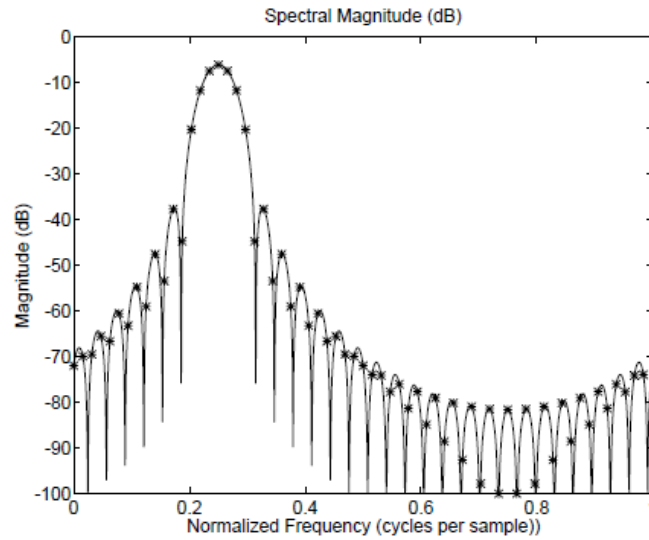


Figure 9.9: Spectral Magnitude, dB scale.

Because we are studying a complex sinusoid $[1, j, -1, -j, 1, j, \dots]$, which is a sampled complex sinusoid frequency $f_s/4$ alone, note that there are no negative frequency components in Fig. 9.8. Consider how difficult it would be to correctly comprehend the shape of the "side lobes" with no padding. The sidelobes in a spectral magnitude plot are easily not visible with symbols that have a zero-padding factor of 2, which was once twice as crucial as necessary to save all spectral information faithfully.

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