Numerical Methods for the Solution of Singularly Perturbed Parabolic Convection-Diffusion Problems with Discontinuous Coefficient and Delay

A Thesis
Submitted for the award of degree of
Doctor of Philosophy
in Mathematics
by

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October, 2023

Declaration

I declare that the research work in this thesis entitled "**Numerical Methods for the Solution of Singularly Perturbed Parabolic Convection-Diffusion Problems with Discontinuous Coefficient and Delay**" for the award of the degree of ***Doctor of Philosophy in Mathematics*** has been carried out by me under the supervision of ***Prof. Aditya Kaushik***, Department of Applied Mathematics, Delhi Technological University, Delhi, India, and has not been submitted by me earlier in part or full to any other university or institute for the award of any degree or diploma.

I declare that this thesis represents my ideas in my own words and where others' ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission.

Date : **(Nitika Sharma)**

Place : Delhi, India.

Certificate

Department of Applied Mathematics
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This is to certify that the research work embodied in the thesis entitled "Numerical Methods for the Solution of Singularly Perturbed Parabolic Convection-Diffusion Problems with Discontinuous Coefficient and Delay" submitted by Nitika Sharma (2K18/PHD/AM/09) is the result of her original research carried out in the Department of Applied Mathematics, Delhi Technological University, Delhi, for the award of **Doctor of Philosophy** under the supervision of **Prof. Aditya Kaushik**.

It is further certified that this work is original and has not been submitted in part or fully to any other university or institute for the award of any degree or diploma.

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

Date:
Place: Delhi , India

Prof. Aditya Kaushik
(Supervisor)

Prof. S. Sivaprasad Kumar
(Head of Department)

Acknowledgements

While presenting my PhD thesis, I must first express my sincere gratitude to the Almighty for His countless blessings and presence in my life, which I experience every day through my family, teachers, and friends.

I would like to thank my supervisor, Prof. Aditya Kaushik, for his patient guidance and unwavering faith in me throughout my PhD. I consider myself incredibly fortunate to have a guide who genuinely cares about his students and pushes them professionally and personally to achieve their goals in life. His determination, positive outlook and confidence have always been a source of inspiration to me.

I would also like to extend my gratitude to the Head of the Department, Prof. S. Sivaprasad Kumar and other faculty members of the department for providing all the facilities necessary for conducting my research. Their constant guidance, kind words of encouragement and concerns offer a healthy working space to carry out research. I am also grateful to the department staff for their kind assistance throughout my PhD.

I am fortunate to have been a part of PhD scholars, Department of Applied Mathematics, DTU. Also, I especially appreciate my research peer group for their everlasting moral support, valuable inputs on the subject and for proofreading my work. A special thanks to all my friends at DTU who made my research journey so enjoyable.

Finally, I owe my deepest gratitude to my parents for their unconditional love and support. My accomplishments and success are because of their constant solace and motivation. I am also highly indebted to my husband, who always supports me and cares deeply about my research. He endured this long process with me, which would not have been possible without his sacrifices.

In closing, I would like to thank almighty God for his abundant blessings and guiding me on the right path to complete this PhD thesis.

Date : **(Nitika Sharma)**

Place : Delhi, India.

To my beloved parents,

who taught me to be strong and fearless,

and my husband,

who encouraged me to always follow my heart.

Abstract

In the present thesis, an attempt has been made to construct, apply, analyse and optimise some simple and efficient parameter-uniform finite difference methods for solving singularly perturbed parabolic convection-diffusion problems with discontinuous coefficients, source term and delay. These problems commonly arise in the different fields of applied mathematics, for example, edge layers in solid mechanics, aerodynamics, oceanography, rafted-gas dynamics, transition points in quantum mechanics, shock and boundary layers in fluid dynamics, magnetohydrodynamics, drift-diffusion equations of semiconductor devices, plasma dynamics, skin layers in electrical applications, Stoke's line in mathematics and rarefied-gas dynamics. These types of problems depend on a small perturbation parameter ε, that multiplies some or all of the highest-order derivative terms. On limiting the value of the perturbation parameter to zero, the solutions to such problems approach a discontinuous limit and exhibit a multiscale character. Often these mathematical problems are extremely difficult (or even impossible) to solve exactly, and in these circumstances, approximate solutions are necessary. Asymptotic analysis and numerical analysis are two principal approaches for solving singular perturbation problems. The classical numerical methods have been known to be effective for solving most problems that arise in applications, but they failed when applied to singular perturbation problems. That is, for the solutions to these problems, classical numerical methods fail to provide good approximations. This motivated us to develop robust numerical methods for solving such types of problems with an emphasis on non-uniform grids.

In this thesis, we have provided numerical schemes for solving three different types of convection-diffusion problems of varying complexity. The thesis consists of six chapters. A brief outline of the chapters is as follows:

Chapter one provides an overview of the fundamentals of singular perturbation theory. Besides, it presents concepts and a historical assessment of the related literature. This chapter also provides a detailed literature survey of various state-of-the-art techniques developed in the recent past. In addition, the chapter illustrates the aim and objectives of the research work.

Chapter two presents an adaptive finite difference method to solve a class of singularly perturbed parabolic delay differential equations with discontinuous convection coefficient and source. The simultaneous presence of discontinuity and the delay makes the problem stiff. The solution to the problem considers the present state of the physical system and its history. The numerical scheme based on the upwind finite difference method is presented on a specially generated mesh to solve the problem. The adaptive mesh is chosen so that most of the mesh points remain in regions with rapid transitions. The proposed numerical method is analysed for consistency, stability and convergence. Extensive theoretical analysis is performed to obtain consistency and error estimates. The proposed method is unconditionally stable, and the convergence obtained is parameter-uniform with first-order convergence in space and first-order convergence in time. The chapter ends with numerical illustrations for the method suggested.

Chapter three extends the idea further and aims to provide a better numerical approximation of the solution to the model problem considered in Chapter two. The chapter presents a higher-order hybrid difference method over an adaptive mesh to solve the problem. The proposed method is a composition of a central difference scheme and a midpoint upwind scheme on a specially generated mesh. Moreover, the time variable is discretised using an implicit finite difference method. The error estimates of the proposed numerical method satisfy parameter-uniform second-order convergence in space and first-order convergence in time. The rigorous numerical analysis of the proposed method on a Shishkin class mesh establishes the supremacy of the proposed scheme.

Chapter four presents a high-order finite difference scheme to solve singularly perturbed parabolic convection-diffusion problems with a large delay and an integral boundary condition. The solution of the problem features a weak interior layer besides a boundary layer. This chapter presents a higher-order accurate numerical method on a specially designed non-uniform mesh. The technique employs the Crank-Nicolson difference scheme in the temporal variable, whereas an upwind difference scheme in space. It is proved that the proposed method is unconditionally stable and converges uniformly, independent of the perturbation parameter. The error analysis indicates that the numerical solution is uniformly stable and shows parameter-uniform second-order convergence in time and first-order convergence in space.

Chapter five presents a robust computational technique to solve a class of two-parameter parabolic convection-diffusion problems with a large delay. The presence of perturbation parameters leads to the twin boundary layers and interior layers in the solution, whose appropriate numerical approximation is the main goal of this chapter. The numerical method is composed of an upwind difference scheme in space, and a Crank-

Nicolson scheme in time is used to find the approximate solution of the problem. It is proved that the method is parameter-uniform with second-order accuracy in time and almost first-order accuracy in space. Numerical examples are provided in support of the theory.

Chapter six concludes the work done and provides insight into the author's thoughts on the future direction of the research.

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Chapter 1

Introduction

1.1 Perturbation Theory

Differential equations generally model physical phenomena in natural sciences and engineering. When considering a mathematical model associated with natural phenomena, we often attempt to deal with the essential quantities while ignoring the negligible ones that involve small parameters. The model obtained by keeping the small parameters is called the perturbed model, whereas the simplified degenerate model is called the unperturbed or reduced model. As a matter of course, we choose unperturbed models because they are relatively simple to deal with. However, a natural question concerning the role of the omitted terms arises. Does the presence or omission of such terms affect the solution or the information obtained from the mathematical model?

Perturbation theory is the study of the effect of small disturbances in a mathematical model of physical phenomena due to these small parameters, and these small parameters are called perturbation parameters. The problems we obtain by retaining the small parameters are called perturbed problems, whereas the simplified degenerate problems are known as unperturbed or reduced problems. The perturbation problems are categorized broadly into two types, namely

1. regular perturbation problems (RPPs), and

2. singular perturbation problems (SPPs).

Let $\mathcal{D}$ be an open bounded set with smooth boundary Γ and $\bar{\mathcal{D}}$ denotes its closure. Consider the boundary value problem

$$\mathcal{P}_\varepsilon : \mathcal{L}_\varepsilon u := \mathcal{L}_0 + \varepsilon \mathcal{L}_1 = f(x, \varepsilon); \quad x \in \mathcal{D} \text{ and } u(\Gamma) \text{ is given.} \tag{1.1}$$

Here ε is a small parameter such that $0 < \varepsilon \ll 1$, $\mathcal{L}_\varepsilon$ is a differential operator, and $f(x, \varepsilon)$ is a given real-valued smooth function. We assume that, for each ε, $\mathcal{P}_\varepsilon$ has a unique smooth solution $u := u_\varepsilon(x)$. Denote by $\mathcal{P}_0$ the corresponding degenerate equation obtained by setting $\varepsilon = 0$ in (1.1) and by u_0 the smooth solution of $\mathcal{P}_0$.

Definition 1.1.1. Problem $\mathcal{P}_\varepsilon$ is called regularly perturbed with respect to some norm $\|\cdot\|$ if there exists a solution u_0 of problem $\mathcal{P}_0$ such that

$$\| u_\varepsilon - u_0 \| \to 0 \text{ as } \varepsilon \to 0.$$

Otherwise, $\mathcal{P}_\varepsilon$ is said to be singularly perturbed with respect to the same norm.
Here $\|\cdot\|$ is the supremum norm (or maximum norm) defined for every continuous function $g : \bar{\Omega} \to \mathbb{R}$ as

$$\| g \|_{\bar{\Omega}} = \sup \left\{ |g(x)| : x \in \bar{\Omega} \right\}.$$

Example 1.1.2. Consider the initial-boundary-value problem $\mathcal{P}_\varepsilon$:

$$u_\varepsilon^{''}(x) + 2\varepsilon u_\varepsilon^{'}(x) - u_\varepsilon(x) = 0, \quad x \in (0,1) \text{ with } u_\varepsilon(0) = 0, \; u_\varepsilon(1) = 1 \text{ and } 0 < \varepsilon \ll 1.$$

The solution of $\mathcal{P}_\varepsilon$ reads

$$u := u_\varepsilon(x) = \frac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}}, \text{ where } m_1 = -\varepsilon + \sqrt{1+\varepsilon^2} \text{ and } m_2 = -\varepsilon - \sqrt{1+\varepsilon^2}.$$

Moreover, it follows that $\lim_{\varepsilon \to 0} u_\varepsilon(x) = \dfrac{\sinh(x)}{\sinh(1)} := u_0$. Clearly, u_0 is the solution of the reduced problem $\mathcal{P}_0$, obtained by setting $\varepsilon = 0$. Therefore, $\mathcal{P}_\varepsilon$ is a regular perturbation problem (RPP).

Example 1.1.3. Consider the initial-boundary-value problem $\mathcal{P}_\varepsilon$:

$$-\varepsilon u_\varepsilon^{''}(x) + u_\varepsilon^{'}(x) = 1, \quad x \in (0,1) \text{ with } u_\varepsilon(0) = u_\varepsilon(1) = 1 \text{ and } 0 < \varepsilon \ll 1.$$

The solution of $\mathcal{P}_\varepsilon$ reads

$$u(x) := u_\varepsilon(x) = x - \frac{e^{\left(-\frac{1-x}{\varepsilon}\right)} - e^{\left(-\frac{1}{\varepsilon}\right)}}{1 - e^{\left(-\frac{1}{\varepsilon}\right)}}.$$

The solution u_ε regarded as function of two variables $u_\varepsilon : [0,1] \times (0,1) \to u(x,\varepsilon)$ satisfies

$$\lim_{x \to c} \lim_{\varepsilon \to 0} u(x,\varepsilon) = c = \lim_{\varepsilon \to 0} \lim_{x \to c} u(x,\varepsilon), \forall c \in [0,1).$$

However,

$$\lim_{x \to 1} \lim_{\varepsilon \to 0} u(x,\varepsilon) = 1 \neq 0 = \lim_{\varepsilon \to 0} \lim_{x \to 1} u(x,\varepsilon).$$

The solution u_ε as a function of two variables possesses a singularity at the point $(1,0)$ in the (x,ε)-plane. Since, $\| u_\varepsilon - u_0 \| \nrightarrow 0$ uniformly over the entire domain as $\varepsilon \to 0$, $\mathcal{P}_\varepsilon$ is a singular perturbation problem.

Example 1.1.4. Consider the initial-boundary-value problem $\mathcal{P}_\varepsilon$:

$$\varepsilon u_\varepsilon^{'}(x) + u_\varepsilon(x) = 0, \quad x \in (0, 1) \text{ with } u_\varepsilon(0) = u_0.$$

Here, $u_0 \in \mathbb{R}$ is a given constant, and the singular perturbation parameter ε is assumed to be non-negative. When $0 < \varepsilon \ll 1$, the problem is singularly perturbed, and a boundary layer may appear depending on the given value of u_0. When $\varepsilon > 0$, the exact solution of the problem is $u_\varepsilon(x) = u_0 e^{-x/\varepsilon}$. Putting $\varepsilon = 0$ in the differential equation gives the reduced equation $u_0(x) = 0$ for all $x \in (0, 1)$. The solution u_ε of the continuous problem has value u_0 at the point $x = 0$, whereas the solution of the degenerate problem vanishes identically. Therefore, these two solutions differ in all cases except for the case when $u_0 = 0$. Excluding this case, there is a small neighbourhood of $x = 0$ in which the solution changes exponentially and has a steep gradient when $0 < \varepsilon \ll 1$. This behaviour of u_ε is called a boundary layer phenomenon, and the problem $\mathcal{P}_\varepsilon$ is called a singular perturbation problem of layer type.

1.2 Singular Perturbation Problems

Singular perturbation problems are widespread in nature and arise in the modelling of various complex phenomena, such as electromagnetic field problems in moving media [110], the theory of plates and shells [124], turbulence model [165], convective heat transfer problems with large Peclet numbers [126], water quality problems in river networks [29], drift-diffusion equation of semiconductor device modelling [231], atmospheric dispersion [235], Black-Scholes model [38], Michaelis-Menton theory for enzyme reactions [198], groundwater transport [31], neuronal variability [275], simulation of oil extraction from underground reservoirs [78], Reissner-Mindlin plate theory [13], Fokker-Planck equation [23], impulses and physiological states of nerve membrane [84], chemical reactor theory [188] to name a few among many others [255, 176, 291, 89, 200, 188, 33].

1.2.1 Historical Overview

More than half a century ago, A.N. Tikhonov [288, 289, 290] began to systematically study singular perturbations, although there had been some previous attempts in this direction [22, 63]. However, the aerodynamic boundary layer was first defined by Prandtl [232]. The term singular perturbation was first used in the work of Friedrichs and Wasow [87]. In 1957, in a fundamental paper [295], M.I. Vishik and L.A. Lyusternik studied linear partial differential equations with singular perturbations, introducing the famous method which is today called the Vishik-Lyusternik method. From that moment on, an entire literature has been devoted to this subject [162, 146, 134, 131, 247].

It is a common experience that when we stand in a breeze or wade in the water, we feel the drag force. We know that the drag is due to fluid friction or viscosity. However, scientists long believed viscosity should not play a role in the picture, as it had such a small value for water and air. Assuming no viscosity, the finest mathematical physicists of the 19th century constructed a large body of elegant results, predicting that the drag on a body in steady flow would be zero. This discrepancy between ideal fluid theory or hydrodynamics and the common experience was known as d' Alembert's paradox. The paradox was only resolved in a revolutionary 1904 paper by L Prandtl, who showed that viscous effects can never be neglected, no matter how small the viscosity [11, 232]. More precisely, the determining factor is the Reynolds number *Re*, a dimensionless measure of the relative importance of inertial to viscous forces in the flow. Prandtl postulated that for certain kinds of high Reynolds numbers or nearly frictionless flows, for example, the flow past a streamlined body like an airfoil, the viscous effects would be confined to thin regions called boundary layers. For certain other kinds of high *Re* flows, such as the flow past a bluff body like a sphere, viscous effects need not be confined to such thin layers; viscosity then has a more dramatic effect than what its low value might suggest. The critical concept of boundary layers has now spread to many other fields; boundary layers often arise in what is known as singular perturbation problems [11, 232].

We reproduced a long-standing paradox from [11] in the paragraph above. For more on historical overview, an introduction to Prandtl's resolution of the Paradox: Flow past a Thin Plate, and for other classical examples, the interested reader is referred to an exciting general article, "*Ludwig Prandtl and boundary layers in fluid flow-How a small viscosity can cause large effects*" by J. H. Arakeri and P. N. Shankar [11].

1.2.2 Classification of Singular Perturbation Problems

The singular perturbation problems are further classified broadly into two categories.

1. **Singular perturbation problem of cumulative type:** A small perturbation parameter ε characterises these problems, and its effect is apparent after a considerable amount of time, typically after an interval of order $O\left(\frac{1}{\varepsilon}\right)$. For instance, let's consider the motion of a satellite around the Earth, where the dominant force is the spherically symmetric gravitational field. If the gravitational field were the only force acting on the satellite, its motion would be periodic. However, small influencing forces from factors such as the thin atmosphere, moon, distant sun, and other

stars significantly alter the satellite's motion after many orbital revolutions due to their cumulative effect.

2. **Singular perturbation problem of layer type:** A small perturbation parameter ε characterises these problems, leading to narrow spatial regions called layer regions in short intervals of time where the solution changes exponentially and exhibits a steep gradient. In contrast, away from these regions, the solution varies smoothly. These problems can be classified into two types based on the position of the layer in the solution.

   (i) **Singular perturbation problems of boundary layer type:** In these types of problems, the layer region is adjacent to the boundary of the domain. Experts commonly refer to this region as the boundary layer, a term Prandtl introduced in the context of fluid mechanics. However, in the context of gas motion, one can also refer to it as shock waves. In electric applications, people know this layer as skin layers, and in mathematics, it sometimes goes by the name of Stoke's surfaces.

   (ii) **Singular perturbation problems of interior layer or free layer type:** In these problems, the layers lie within the domain, away from boundaries. Hence, we refer to them as interior layers or free layers. We can attribute the presence of these interior layers to various factors, such as the existence of turning points, non-smooth coefficients, non-smooth initial/boundary conditions, non-linearities, or lack of compatibility at the domain boundaries. These reasons give rise to interior layers, contributing to the complexity of the problem.

1.2.3 Model Problems

The objective is to determine the solution to these problems by identifying the location, width, and strength of all the layers present in the solution. Consequently, we classify these problems into convection-diffusion problems and reaction-diffusion problems. The characteristics of the layers, such as their strength, width, and location, depending on whether the problem is of convection-diffusion or reaction-diffusion type. The coefficients and initial/boundary conditions specified in the problem also influence these characteristics. Several physical and mathematical models of the convection-diffusion and reaction-diffusion equations have been mentioned in the literature, as referenced by some authors [189, 248, 82, 266, 191].

1. **Convection-diffusion problems:** Convection-diffusion problems model physical phenomena involving convection, reaction, and diffusion processes. In these problems, the order of the degenerate equation reduces by one. Let us consider a two-point boundary value problem on a unit interval $\Omega = (0, 1)$

$$\begin{cases} -\varepsilon u_\varepsilon''(x) + u_\varepsilon'(x) = 0, \ x \in \Omega \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon(1) = u_1, \end{cases}$$

where $u_0, u_1 \in \mathbb{R}$ are some given constants and $0 < \varepsilon \ll 1$. The exact solution of the problem is $u_\varepsilon(x) = \dfrac{u_1 e^{-1/\varepsilon} - u_0}{e^{-1/\varepsilon} - 1} + \dfrac{u_0 - u_1}{e^{-1/\varepsilon} - 1} e^{-(1-x)/\varepsilon}$. The corresponding degenerate equation is of order one, and we can impose only one boundary condition. It needs to be clarified which of the two possible boundary conditions we can impose. Since the characteristic direction aligns with the positive x-axis, we cannot impose a boundary condition at $x = 1$. The corresponding degenerate problem reads

$$\begin{cases} v_0'(x) = 0, \ x \in \Omega \\ v_0(0) = u_0, \end{cases}$$

and its solution is $v_0(x) = u_0$. Therefore, a boundary layer will form near $x = 1$ unless the boundary value of u_ε at $x = 1$ agrees with the value of the reduced solution v_0 at $x = 1$. Thus we may say that the solution exhibit only one boundary layer of width ε in the neighbourhood of $x = 1$.

2. **Reaction-diffusion problems:** Reaction-diffusion problem models physical phenomena involving both reaction and diffusion processes. In these types of problems, the order of the degenerate equation reduces by two. Let us consider a two-point boundary value problem defined on a unit interval $\Omega = (0, 1)$

$$\begin{cases} -\varepsilon u_\varepsilon''(x) + u_\varepsilon(x) = 0, \ x \in \Omega \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon(1) = u_1, \end{cases}$$

where $u_0, u_1 \in \mathbb{R}$ are the given constants and $0 < \varepsilon \ll 1$. The exact solution of the problem is $u_\varepsilon(x) = \dfrac{u_1 - u_0 e^{-1/\sqrt{\varepsilon}}}{1 - e^{-2/\sqrt{\varepsilon}}} e^{-(1-x)/\sqrt{\varepsilon}} + \dfrac{u_0 - u_1 e^{-1/\sqrt{\varepsilon}}}{1 - e^{-2/\sqrt{\varepsilon}}} e^{-x/\sqrt{\varepsilon}}$. Note that the corresponding degenerate equation has order zero. Consequently, we cannot impose any boundary conditions on its solution. The exact solution of the degenerate equation is $v_0(x) = 0$. Therefore, a boundary layer will appear at $x = 0$ unless $u_0 = 0$. Similarly, a boundary layer will occur at $x = 1$ unless $u_1 = 0$. The layer correction function $e^{-x/\sqrt{\varepsilon}}$ in the solution suggests that the solution has a steep gradient in $(0, \sqrt{\varepsilon})$ but not in $(\sqrt{\varepsilon}, 1)$. The behaviour of $e^{-(1-x)/\sqrt{\varepsilon}}$ is analogous. Therefore, we may have two boundary layers of width $\sqrt{\varepsilon}$, each at outflow boundary regions. However, one or no boundary layer may occur for special choices of boundary conditions.

In this thesis, we focus on singular perturbation problems of convection-diffusion type. To gain insights into the strength and location of layers in the solution of the convection-diffusion problems in question, we consider the following simple mathematical representation of a convection-diffusion problem defined on a unit interval $\Omega = (0, 1)$

$$-\varepsilon u''_\varepsilon(x) + a(x)u'_\varepsilon(x) + b(x)u_\varepsilon(x) = f(x), \quad x \in \Omega, \tag{1.2}$$

where $0 < \varepsilon << 1$ is the perturbation parameter and appropriate boundary conditions are specified. The rules tabulated in Table 1.1 are helpful for inferring information about the boundary and interior layers present in the analytical solution of the problem. When the function $a(x)$ in the general convection-diffusion problem specified in (1.2) is equal to zero, the problem transforms to the reaction-diffusion problem. The information about the strength and location of the interior/ boundary layers can be inferred from Table 1.2.

1.3 Delay Differential Equation

A differential-difference equation also called a delay differential equation, is a type of functional differential equation where the system's evolution at a certain time depends not only on the present state of the system but also on the state of the system at an earlier time. The simplest form of a first-order delay differential equation with constant time delays can be expressed as

$$u'(x) = f(x, u(x), u(x - \sigma_1), u(x - \sigma_2), \ldots, u(x - \sigma_k)).$$

An initial value problem for a first-order delay differential equation differs from its counterpart in a notable way. While a differential requires an initial condition specified at a point to determine a unique solution, a delay differential equation necessitates the specification of the solution profile over an interval of length equivalent to delay. For example, consider the following initial value problem for a linear first-order differential equation

$$\frac{du}{dt} = ku, \quad t \in (0, 1], \quad u(0) = 1$$

admits the exponential solution $u(t) = e^{kt}$. In this problem, the value of u at a given point (*i.e.* $u(0)$) is sufficient to obtain a solution at any time t. The past has no role in

Table 1.1: **Strength and location of boundary and interior layers in convection-diffusion SPPs.**

Smoothness of functions			Value of the function	Strength and location of	
$b(x)$	$a(x)$	$f(x)$	$a(x)$	**Boundary Layer**	**Interior Layer**
Smooth			$< 0, \forall x \in \overline{\Omega}$	Strong, at $x = 0$	—
Smooth			$> 0, \forall x \in \overline{\Omega}$	Strong, at $x = 1$	—
Smooth		Discontinuous at $x = d \in \Omega$	$< 0, \forall x \in \overline{\Omega}$	Strong, at $x = 0$	Weak, on right side of $x = d$
Smooth		Discontinuous at $x = d \in \Omega$	$> 0, \forall x \in \overline{\Omega}$	Strong, at $x = 1$	Weak, on left side of $x = d$
Smooth	Discontinuous at $x = d \in \Omega$		$< 0, \forall x \in \overline{\Omega}$	Strong, at $x = 0$	Weak, on right side of $x = d$
Smooth	Discontinuous at $x = d \in \Omega$		$> 0, \forall x \in \overline{\Omega}$	Strong, at $x = 1$	Weak, on left side of $x = d$
Smooth	Discontinuous at $x = d \in \Omega$		$> 0, x \in (0, d)$ and $< 0, x \in (d, 1)$	—	Strong, on both side of $x = d$
Smooth	Discontinuous at $x = d \in \Omega$		$< 0, x \in (0, d)$ and $> 0, x \in (d, 1)$	Solution is unbounded	
—			$= 0$	SPP is of reaction-diffusion type	

Table 1.2: **Strength and location of boundary and interior layers in reaction-diffusion SPPs.**

Smoothness of functions		Strength and location of	
$b(x)$	$f(x)$	**Boundary Layer**	**Interior Layer**
Smooth		Strong, at both endpoints $x = 0$ and $x = 1$	—
Smooth	Discontinuous at $x = d \in \Omega$	Strong, at both endpoints $x = 0$ and $x = 1$	Strong, on both sides of $x = d$

determining the solution. Now, we consider the following initial value problem for a linear first-order delay differential equation

$$\frac{du}{dt} = ku(t-\sigma), \quad t \in (0,T], \quad \sigma > 0, \quad T \geq 2, \quad u(t) = 1, \ -\sigma \leq t \leq 0. \tag{1.3}$$

Here, the derivative of the unknown function u is expressed in terms of u at some previous time $(t-\sigma)$ and σ represents the delay, shift or time lag. We must specify the unknown function u on a finite interval to obtain a unique solution. This specified function is commonly referred to as the memory function, as it provides information about the solution in the past. It is important to note that this requirement, i.e. the specification of the solution profile over a finite interval, makes the delay differential equations infinite dimensional problems, even if there is only a single linear delay differential equation, as an infinite dimensional set of initial conditions between $t = \sigma$ and $t = 0$ have to be defined. Another significant characteristic of delay differential equations is that the solutions of delay differential equations have discontinuities that propagate. In (1.3), assuming $k = -1$ and $\sigma = 1$. Then, note that

$$u'(0^-) = 0 \neq -1 = u'(0^+), \quad u''(1^-) = 0 \neq 1 = u''(1^+)$$

so that the jump in $u'(t)$ at $t = 0$ propagates to a jump in $u''(t)$ at $t = 1$, and so on. More generally, the jump in $u'(t)$ at $t = 0$ propagates to a jump in $u^{n+1}(t)$ at $t = n$, $n \in \mathbb{N}$.

In this thesis, we will focus on parabolic PDEs with spatial delay, or we may also call them shift. Parabolic problems with spatial delay refer to a class of PDEs that incorporate shifts or translations in their formulation. These equations arise in various scientific and engineering fields where spatially distributed systems exhibit dynamic time delays or shifts of mixed type. Understanding the characteristics of parabolic problems with spatial delay is crucial for analyzing and predicting the behaviour of such systems.

One notable characteristic of these equations is the presence of both temporal and spatial dependencies. The delay or shift introduces a memory effect, where the system's current behaviour depends not only on its current state but also on its past states at neighbouring or distant locations. This memory effect can arise from physical phenomena such as diffusion, advection, or transport processes with finite propagation speeds. This spatial interaction can lead to complex dynamics, spatial patterns, and wave propagation phenomena. Such problems frequently appear in epidemics and population dynamics, where these small shifts play an essential role in modelling various real-life phenomena [152]. For example, boundary value problems for delay differential equations arise naturally in studying variational problems in control theory, where the problem is complicated by the effect of delays in signal transmission [75]. In the mathematical model for determining the expected first-exit time in generating action potential in nerve cells by random synaptic inputs in dendrites, the shifts are due to the jumps in the membrane potential, which are very small [275]. Analyzing parabolic problems with spatial delay requires specialized mathematical techniques that extend traditional methods used for parabolic PDEs.

1.4 Methods for Solving Singular Perturbation Problems

In the literature, researchers widely use two approaches to solve singularly perturbed problems: asymptotic and numerical methods.

1.4.1 Asymptotic Methods

The asymptotic methods provide a straightforward way to determine an accurate approximation to the solution of a singular perturbation problem. In these methods, the behaviour of the analytical solution of the problem can be studied through the asymptotic expansion method. The solution is approximated as an asymptotic series in the small parameter ε such as

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots + \varepsilon^n u_n.$$

The values of $u_0, u_1, u_2, \ldots, u_n$ can be obtained by substituting u into the given equation after doing term by term differentiation. After substitution, first few terms are solved to get $u_0, u_1, u_2, \ldots, u_n$ and form an approximate solution to the problem. The asymptotic solution accurately approximates the solution to the problem over a large portion of the domain, i.e., the outer region but is inaccurate over the small region, i.e., the layer region because the effect of the perturbation term in the problem is not negligible in this region. However, the straightforward asymptotic expansion leads to a differential equation of lower order than the original differential equation, and the solution fails to satisfy all the boundary or the initial conditions.

Thus, the method of asymptotic expansion does not properly approximate the exact solution of singular perturbation problem. This limitation of the asymptotic expansion method is removed by using the following methods:

1. **Method of matched asymptotic expansions:** This approach involves obtaining two complementary solutions, the inner solution and the outer solution, in their respective regions by treating a specific portion of the domain as a distinct perturbation problem. Subsequently, the solutions from different regions of the domain are patched or matched to derive an approximate solution for the entire domain.

The inner and outer solutions were first matched by using stretching transformation [86]. During the 1950s, this method was refined and applied to numerous physical problems [167, 212, 137, 163, 138, 294, 295]. For more details on this technique, one can refer to the books [291, 144, 216].

2. **Method of multiple scales:** This method constructs uniformly valid approximations for solving singular perturbation problems by incorporating multiple scales for the independent variable. Additional terms are included by introducing new independent variables to eliminate secular terms and determine a uniformly approximate solution. This method was introduced in the late 1950s and has since been extensively explored in various examples [36]. Later, this method has been employed to solve numerous singular perturbation problems in [143, 64, 268, 58, 123, 34, 159, 164]. The multiple scales method offers a significant advantage in tackling nonlinear problems [159]. However, introducing additional slow scales can lead to potential ambiguities in the perturbation series solution, which must be carefully addressed, as demonstrated in [36].

3. **WKB approximation:** This method is used to obtain a global approximation for solving linear singular perturbation problems. The method assumes an exponential dependence of the solution on the boundary layer, which is a reasonable assumption for linear singular perturbation problems. This assumption significantly reduces the effort of finding an asymptotic approximation for the solution. The method involves initially identifying approximate linearly independent solutions, which are then combined through superposition to form a general solution. Unlike other asymptotic methods like matched asymptotic expansions or multiple scales, the boundary conditions are typically solved exactly at the end of the process rather than being approximated. This method, known as the Wentzel-Kramers-Brillouin (WKB) method, was first utilized in the 1920s to approximate solutions to the Schrödinger equation. The historical development of this method can be found in [116], while the mathematical details about the method can be found in [256]. Applications of the WKB method in quantum mechanics and solid mechanics can be found in [41] and [274], respectively. Further, this method has been applied to solve various SPPs [3, 96, 151].

4. **Other Methods:** In addition to the previously mentioned prominent asymptotic methods, several other asymptotic methods are available for both linear and nonlinear problems. For linear problems, some of these methods include the Poincaré-Lindstedt method [145, 222, 35, 181, 47, 202], the method of strained parame-

ters [203, 267, 273], the method of periodic averaging [57, 136, 6], and the linearized perturbation method. For nonlinear problems, there are methods such as the variational iteration method [112, 115], the modified Poincaré-Lindstedt method [113, 174, 236, 8], the homotopy perturbation method [211, 114, 2], the parameter expansion method [245, 298] and the perturbation-iteration methods [15, 186, 204]. For a comprehensive understanding of the progressive developments in the asymptotic theory of singular perturbations, additional information can be found in books [293, 272, 36, 214].

The asymptotic expansion treats a relatively small class of problems and requires the user to know the boundary layer's location and width. Additionally, the method is not well-suited for tackling two-dimensional problems efficiently. Even for complex one-dimensional nonlinear problems, the asymptotic approximation remains valid only for small perturbation parameter values. Applying these methods requires a thorough understanding of the expected solution behaviour. This motivates one to use numerical methods to solve such problems.

1.4.2 Numerical Methods

Numerical methods are used to obtain an approximate solution for problems where finding a closed-form solution is typically not possible. These methods are intended for a broad range of problems and provide quantitative information about the particular problem. Being quantitative in nature, the solution generated by these methods is quite different from the qualitative solution provided by asymptotic methods.

Researchers have developed several numerical methods to solve SPPs in the past few decades. These methods are generally classified as computational methods and parameter-uniform numerical methods. When the perturbation parameter is set to a critical value, the standard finite difference, finite element, or finite volume methods, collectively known as classical computational methods, are found to be insufficient on uniform meshes and require an extremely large number of mesh points to generate accurate numerical solutions [249]. The reason behind this limitation of the computational methods is the presence of steep gradients in the boundary layer(s) of the analytical solution. These methods fail to reduce the maximum point-wise error until the mesh size and the singular perturbation parameter have the same order of magnitude. However, refining the mesh size to the order of perturbation parameter (say when $\varepsilon = 10^{-6}$) unexpectedly increases the number of mesh points and the associated computational cost. Thus the major limitation of the computational method is the dependence of domain discretization on the perturbation parameter. It is desirable to develop robust computational methods in which the discretization, error, and order of convergence are independent of the perturbation parameter. Such robust methods work uniformly for all values of the perturbation parameter and are known as parameter-uniform numerical methods. These methods are generally classified into two main categories: the fitted finite difference operator and the fitted mesh method. A brief survey enumerating the chronological developments in both classical computational methods and parameter-uniform numerical methods is as follows.

1. **Finite difference methods:** The finite difference method (FDM) is a well-established and straightforward technique for approximating the solution of a singular perturbation problem. Its introduction in numerical applications dates back to the late 1960s, coinciding with the emergence of minicomputers that provided a convenient framework for handling complex problems. In this method, the domain of interest is discretized by dividing it into a mesh or grid. Then, all the derivatives present in the differential equation are replaced with algebraic differential quotients. For instance, the derivative $\frac{du}{dx}$ can be approximated using either the first-order forward difference quotient $\left.\frac{du}{dx}\right|_i \approx \frac{u_{i+1} - u_i}{h}$ or the second-order central difference quotient $\left.\frac{du}{dx}\right|_i \approx \frac{u_{i+1} - u_{i-1}}{2h}$, where u_i is the value of u at the mesh point i and h is the mesh spacing. By replacing the derivatives at all interior mesh points, a system of algebraic equations is generated in which $u_i's$ represent the unknowns. After enforcing the boundary conditions, the number of unknowns in the system will be equal to the number of interior nodes in the mesh. These unknowns can be determined by solving the system of equations either exactly or approximately, utilizing direct methods or iterative methods like the Gauss-Seidel method, Jacobi method, Successive Overrelaxation method, or other advanced techniques.

   The development of a three-point difference scheme on a uniform mesh for a one-dimensional two-point singular perturbation boundary value problem was first introduced in [228]. The approach involved identifying mesh locations where the difference between the computed solution and its neighbouring value exceeded a predetermined threshold value. An iterative procedure was implemented to increase the concentration of mesh points at these identified locations, and a smoothing technique was employed to mitigate accuracy loss caused by abrupt changes in mesh spacing. The Gauss elimination method was applied to solve the linear algebraic equations formed by the difference scheme. The obtained numerical results indicate that the computed solution converged to the exact solution. Later, this method is extended to solve a class of nonlinear problems [229].

These methods require strict constraints on the mesh spacing to maintain stability when the perturbation parameter is very small. For instance, consider the following singular perturbation boundary value problem:

$$\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0, 1), \quad u(0) = p_0, \quad u(1) = p_1,$$

where ε represents the perturbation parameter, $a(x)$, $b(x)$ and $f(x)$ are smooth functions satisfying $a(x), b(x) \geq 0$ on Ω. It was found that the central difference scheme, implemented on a uniform mesh $\{x_i = ih\}$ with mesh spacing ($h = 1/N$), becomes unstable and oscillates when $a(x_i)h/2\varepsilon > 1$[67]. The authors in [125] introduced an upwind scheme to overcome this stability issue. In this scheme, the first derivative is replaced by a one-sided (forward or backward) difference instead of the central difference. The choice of forward or backward difference depends upon the sign of $a(x)$ at a particular mesh point x_i. This scheme is known as the Il'in-Allen-Southwell scheme [7]. The upwind scheme was observed to provide stability and exhibits better convergence when compared to the central difference scheme. Although it is considered as the first fitted operator scheme, it has limitations that it is first-order uniformly convergent in the outer region only.

In [20], the authors solved a singularly perturbed reaction-diffusion problem using the finite difference method on a non-uniform mesh. The non-uniform mesh was constructed by using a continuous mesh generating function $\psi : \bar{\Omega} \to [0, 1]$, which is defined as

$$\psi(t) = \begin{cases} \chi(t) := -\dfrac{\sigma\varepsilon}{\beta} \ln(1 - t/q), & t \in [0, \tau], \\ \phi(t) := \chi(\tau) + \psi'(\tau)(t - \tau), & t \in [\tau, 1/2], \\ 1 - \psi(1 - t), & t \in (1/2, 1], \end{cases}$$

where the transition point τ is the solution of the nonlinear problem $(1 - 2\tau)\phi'(\tau) = 1 - 2\chi(\tau)$. The mesh generated by this method is known as the Bakhvalov mesh. This mesh is considered to have a complicated structure, and extending the mesh to solve the singularly perturbed partial differential equations (PDEs) is difficult.

In [68], a class of SPPs is solved by using an upwind finite difference method. The author compared the asymptotic behaviour of the solution obtained from the difference scheme with the exact solution. Later, the authors extended this method to solve second-order ODEs [69]. They obtained elementary estimates for the solution and its derivatives by using the maximum principle [234]. In [4], the upwind method is further refined and used to solve SPPs with systems of equations. In this method, a parameter was introduced in the difference equation, and it was chosen in such a way that an accurate approximation for the reduced problem is obtained in the interior region as well. Later, this method is extended to solve SPPs with internal turning points [28].

In [142], three-point difference schemes is applied to SPPs without turning points. The authors examined three difference operators(L_h^1, L_h^2 and L_h^3) on a uniform mesh of size h to approximate the solution. The operator L_h^1 provides an approximation of order h. The error bounds for the L_h^2 operator, used in [109], and the L_h^3 operator, used in [28, 125], contain a term of the form $\frac{h^2}{(h+\varepsilon)}$, indicating a reduction in the order of convergence by one as ε approaches 0. Therefore, methods using the L_h^2 and L_h^3 operators exhibit second-order convergence in the outer region and first-order convergence inside the layer region.

In [73], the exponential box scheme is introduced to solve singularly perturbed convection-diffusion problems. This scheme combined the exponential difference operator [7] with Keller's box scheme [141] to achieve a stable and second-order accurate approximation of the solution. Later in [37], the authors proved that the exponential difference [73], when applied on a uniform mesh with a mesh size of h, provides uniformly second-order accuracy for solving convection-diffusion problems. This result demonstrated that the exponential box scheme is reliable and accurate across the entire computational domain, ensuring consistent second-order accuracy of the approximation.

In [16], the authors modified the upwind scheme to enhance its accuracy for convection-diffusion SPPs. This modified scheme achieved second-order accuracy, similar to the central difference scheme, while preserving the stability properties of the upwind scheme. This modification improved the accuracy of the solution and provided better convergence properties.

In [264], the author developed a scheme based on the integro-interpolation method [253] to solve a class of singularly perturbed differential equations of ordinary type and parabolic type. The scheme was developed on a mesh similar to the Bakhvalov mesh and exhibited third-order convergence for ODEs and first-order convergence for PDEs. In [296], the author generalized the Bakhvalov mesh for the finite-difference discretization of one-dimensional nonlinear reaction-diffusion SPPs. This generalization extended the use of the Bakhvalov mesh to handle non-linear problems and achieved a uniform second-order convergence.

In [90], the author developed a family of uniformly accurate finite difference schemes for convection-diffusion SPPs using the high-order differences with identity expansion (HODIE) framework [178, 66]. The discretization error analysis was carried out using the stability results from [209]. The theoretical analysis showed that the uniform convergence of any order could be achieved, depending on the smoothness of the data. Achieving higher-order convergence with this scheme required additional evaluations of the data.

In [80], it is shown that for convection-diffusion SPPs, the fitted finite difference operator is only necessary for the layer region, while the solution in the outer region can be accurately approximated using the standard fitted operator. This observation allowed for more efficient computation by reducing the computational cost in the outer region. In [81], the author investigated a variety of finite difference schemes to derive sufficient conditions for uniform convergence. He showed that these conditions are not only satisfied by uniformly convergent schemes but also by a more general class of upwind schemes.

In [91], an exponentially graded mesh was employed for two-point singularly perturbed boundary value problems. The graded mesh divided the domain into three regions: the inner region with an extremely fine mesh, the transit region where the mesh geometry changes from fine to coarse, and the outer region with a uniform mesh. The number of mesh points in the inner region was significantly higher (approximately k times) than the number of mesh points in the outer region. Various finite difference schemes were applied on the graded mesh and uniform convergence of order $O(h^k)$ was achieved. However, the complexity involved in creating the graded mesh made it challenging to extend the mesh to higher dimensions. Considering this limitation, the author in [265] introduced a relatively simple mesh known as the Shishkin mesh, which could be conveniently extended to higher dimensions. For convection-diffusion problems, he proposed a piecewise uniform mesh with a transition point τ defined as $\tau = \min(1/2, \varepsilon\tau_0 \ln N)$, where $\tau_0 \geq p/\alpha$. The parameter p characterizes the order of convergence of the numerical method. The mesh $\bar{\Omega}^N = \{x_i\}_{i=0}^N$ is constructed by dividing both subintervals $[0, \tau]$ and $[\tau, 1]$ into $N/2$ equal subintervals if a boundary layer is present near the left endpoint of the domain. When $\varepsilon\tau_0 \ln N > 1/2$, (for sufficiently large N compared to $1/\varepsilon$), this mesh transforms into a uniform mesh. Similarly, if there is a boundary layer near the right endpoint, the domain $\bar{\Omega}$ is divided into $[0, 1-\tau]$ and $[1-\tau, 1]$, each with $N/2$ equal subintervals, to obtain a piecewise uniform mesh. For reaction-diffusion problems, the transition parameter τ is defined as $\tau = \min(1/4, \sqrt{\varepsilon}\tau_0 \ln N)$, where $\tau_0 \geq p/\beta$.

A piecewise uniform mesh is constructed by discretizing the domain $\bar{\Omega} = [0, 1]$ into $[0, \tau]$, $[\tau, 1-\tau]$, $[1-\tau, 1]$, where the subintervals contain $N/4$, $N/2$, and $N/4$ equally spaced mesh points, respectively. It is important to note that one limitation of the Shishkin mesh is that it requires prior knowledge about the location and width of the boundary layers. All these meshes, Bakhvalov [20], Vulanović [296], Gartland [91], and Shishkin [265], are constructed based on a priori information about the width and location of the layers in the exact solution and are thus known as apriori meshes.

In [77], the author analysed a defect-correction method for a one-dimensional convection-diffusion problem without turning points. He showed that the kth approximation obtained using the defect-correction method converges uniformly with a rate of $O((\varepsilon_0 - \varepsilon)^k + h^2)$, where ε_0 is of the order $O(h)$ in outer regions. However, the error estimates degrade to $O(1)$ in the inner regions.

In [284], the authors introduced a spline difference scheme that uses quadratic and cubic splines for discretizing reaction-diffusion problems on a non-uniform mesh. In [285], the authors discretized reaction-diffusion problems using quadratic splines on a piecewise uniform Shishkin mesh and achieved an almost second-order accuracy in the discrete maximum norm. In [122], the authors used a cubic spline difference scheme on Bakhvalov mesh to solve reaction-diffusion problems. The result obtained using the Bakhvalov mesh was found to be superior to that achieved with the Shishkin mesh.

In [179], the author introduced a posteriori mesh, which does not require prior information about the width and location of the layers in the exact solution. The method involves computing an approximate solution on an arbitrary mesh and then using the error estimate, based on the difference derivatives of the computed solution to determine a monitor function. This monitor function helps in achieving mesh equidistribution. Authors in [32] further developed this idea by proposing a monitor function that combines a constant term with an appropriate power of the second derivative of the singular component of the solution. This choice of monitor function improved the mesh equidistribution and enhanced the accuracy of the numerical solution. In [149, 169], the authors utilized the arc-length monitor function for mesh equidistribution to solve convection-diffusion problems. In [230], the author introduced numerical methods based on exponential finite difference approximations with h^4 accuracy for solving one and two-dimensional convection-diffusion problems. A nonlinear two-point SPP is considered in [132]. The authors employed quasi-linearization to linearize the original nonlinear equation for each linear case. Then, they used a cubic spline difference scheme on a variable mesh to approximate the linear equations. Continuing their work, the authors in [133] developed an exponentially fitted difference scheme using spline in compression for solving two-point singularly perturbed boundary value problems. In [170], the author presented a survey on layer-adapted meshes for convection-diffusion problems, emphasizing the importance of using appropriate grids to achieve uniform convergence.

In [10], the authors considered a one-dimensional steady-state convection-diffusion problem with Robin boundary conditions. To discretize the problem, they use standard upwind finite difference operators on Shishkin meshes. Furthermore, the authors in [54] developed a finite difference scheme for solving a one-dimensional time-dependent convection-diffusion problem with initial-boundary conditions. They employed the classical Euler implicit method for time discretization and the simple upwind scheme on a Shishkin mesh for spatial discretization.

In [54], the authors developed a finite difference scheme to solve one-dimensional time-dependent convection-diffusion problem. They used the classical Euler implicit scheme and upwind scheme for the time discretization and spatial discretization, respectively. In [254], the authors presented an adaptive finite difference method to solve singularly perturbed convection-diffusion problems. The authors combined a first-order upwind and a second-order central scheme to achieve a higher order of convergence. In [171], the author discretized a singularly perturbed convection-diffusion problem using a simple first-order upwind difference scheme on general meshes. He derived an expansion of the error of the scheme that enables uniform error bounds with respect to the perturbation parameter in the discrete maximum norm for both a defect correction method and the Richardson extrapolation technique.

In [30], the author presented a cubic spline in compression to solve two-point singularly perturbed boundary value problems. In [148], the author analysed that the arc-length monitor function does not yield satisfactory numerical approximations for reaction-diffusion problems. It has been observed that an optimal choice of the monitor function not only depends on the discretization technique and the norm of the error to be minimized but also on the nature of the problem.

In [226], the authors considered a self-adjoint two-point singularly perturbed boundary value problem. They employ a fitted finite difference scheme on a Shishkin mesh for solving the problem by reducing it to normal form. While the authors in [177] proposed a non-standard finite difference scheme for solving self-adjoint two-point singularly perturbed boundary value problems using Micken's finite difference method.

In [258], the authors introduced a finite difference scheme for discretizing singularly perturbed boundary value problems. The presented scheme is a combination of the simple upwind scheme and the central difference scheme on a Shishkin mesh. It is observed that the proposed scheme exhibited higher-order convergence compared to the simple upwind scheme alone.

In [244], the authors used spline in compression to generate second and fourth-order uniformly convergent numerical techniques for singularly perturbed boundary value problems. To deal with Robin-type boundary conditions, authors in [201] applied the central difference method on the regular region of standard Shishkin mesh and cubic splines to discretize the layer region.

In [197], the authors investigated the effect of Richardson extrapolation on two fitted operator finite difference methods (FOFDM), (FOFDM-I) [226] and (FOFDM-II) [177]. They found that FOFDM-I achieved fourth-order accuracy for moderate values of the perturbation parameter, while it attained second-order accuracy for small values of the perturbation parameter. However, they observed that Richardson extrapolation did not improve the order of convergence for FOFDM-I. For FOFDM-II, which is uniformly second-order convergent, the order of convergence can be improved up to fourth order by using Richarson extrapolation. In [172], the author proposed a compact fourth-order finite difference scheme for solving two-point reaction-diffusion SPPs on a Shishkin mesh.

Authors in [242] applied exponential splines to generate an almost second-order uniformly convergent difference scheme on standard Shishkin mesh for semi-linear reaction-diffusion problems. The method exhibits uniform convergence of almost second-order in discrete maximum norm. Later, they devised an exponential spline difference scheme on piecewise uniform Shishkin mesh [243].

In [130], the authors proposed a numerical approach to solve the singularly perturbed time-dependent convection-diffusion problem in one spatial dimension. They employed a semi-discretization technique by applying the backward Euler finite difference method in the temporal direction. To discretize the resulting set of ordinary differential equations, they utilized the midpoint upwind finite difference scheme on a non-uniform mesh of Shishkin type in the spatial direction. In [105], the authors proposed a method in which domain decomposition was combined with higher-order difference discretization for solving two-point singularly perturbed boundary value problems of convection-diffusion type. In [61], the authors used the same scheme combination as in [201] on an equidistributed grid. Their approximation scheme uses cubic splines for the mixed-boundary conditions and the classical central scheme elsewhere.

In [196], the authors considered a time-dependent singularly perturbed reaction-diffusion problem. They employ the classical backward Euler method to discretize the problem in time and a fitted operator finite difference method in space. In [98], the authors proposed a classical upwind finite difference scheme on layer-adapted nonuniform meshes to solve singularly perturbed parabolic convection-diffusion problem. In [195], the authors proposed a uniformly convergent FDM for a coupled system of singularly perturbed ODEs of convection-diffusion type. It was proved that the proposed discrete operator satisfies the stability property in the maximum norm. In [227], the author presented a survey of non-standard FDMs.

In [92], the authors proposed an adaptive finite difference technique using the central difference scheme on a layer-adapted mesh for a linear second-order singularly perturbed boundary value problem. It was shown that the proposed technique has fourth-order convergence. In [180], the authors considered singularly perturbed degenerate parabolic convection-diffusion problems in two-dimension. They used an alternating direction implicit finite difference scheme to discretize the time derivative and an upwind finite difference scheme to discretize the spatial derivative.

In [210], the authors introduced a hybrid difference scheme for solving singularly perturbed convection-diffusion problems. Their scheme combined the upwind scheme on the coarse part of the Shishkin mesh with the central difference scheme on the fine part. In [93], the authors considered a singularly perturbed fourth-order differential equation with a turning point. They used the classical finite difference scheme on an appropriate piecewise uniform Shishkin mesh to solve the problem. In [60], the authors proposed a second-order uniformly convergent numerical method for a singularly perturbed parabolic convection-diffusion initial-boundary-value problem in two-dimension. They used a fractional-step method in the time direction, while a finite difference scheme was used in the spatial direction. In [269], a higher-order Richardson extrapolation scheme is presented for solving a singularly perturbed system of parabolic convection-diffusion problems. Whereas in [175], a septic B-spline method is presented for solving a self-adjoint singularly perturbed two-point boundary value problem.

In [184], the authors proposed a uniformly convergent FDM to solve singularly perturbed time-dependent convection-diffusion problems in the framework of the method of lines. The method uses the fitted operator finite difference method to discretize the spatial derivatives, followed by the Crank–Nicolson method for the time derivative. Moreover, Richardson extrapolation is performed in space to improve the accuracy of the method. In [104], a linear singularly perturbed parabolic reaction-diffusion problem with incompatible boundary-initial data is considered. The method combines the computational solution of a classical finite difference operator on a tensor product of two piecewise-uniform Shishkin meshes with an analytical function that captures the local nature of the incompatibility. In [100], the authors proposed a parameter-uniform numerical method for viscous Burgers' equation. In order to find a numerical approximation, they linearized the equation to obtain a sequence of linear PDEs. The linear PDEs are then solved by a finite difference scheme, which comprises the backward-difference scheme for the time derivative and the upwind finite difference scheme for the spatial derivatives. Whereas in [62], the authors considered a system of singularly perturbed reaction-diffusion problems. In [56], the authors deal with linear parabolic singularly perturbed systems of convection-diffusion type in two-dimension. The numerical method combines the upwind finite difference scheme to discretize in space, and the fractional implicit Euler method, together with a splitting by directions and components of the reaction–convection–diffusion operator, to discretize in time.

In [297], a hybrid higher-order finite-difference scheme is presented for a class of linear singularly perturbed convection-diffusion problems in one dimension. Whereas in [183], the authors presented a hybrid scheme for solving singularly perturbed parabolic problems with Robin-type boundary conditions. The scheme is a combination of FOFDM in space and the backward Euler method in time. They have also proposed a finite difference scheme to solve Volterra integro singularly perturbed differential equation. The proposed scheme used a non-standard finite difference scheme for solving the differential part and Simpson's rule for solving the integral part. The Richardson extrapolation is used to increase the order of convergence to two. In [185], the authors proposed a second-order finite difference scheme to solve a singularly perturbed Volterra integro-differential equation.

In [106], the author proposes a higher-order numerical scheme to solve singularly perturbed reaction-diffusion problems. The proposed scheme is a combination of a fourth-order numerical difference method and a classical central difference method. In [157], the authors presented a parameter-uniform numerical method on equidistributed meshes for solving a class of singularly perturbed parabolic problems with Robin boundary conditions. The discretization consists of a modified Euler scheme in time, a central difference scheme in space, and a special finite difference scheme for the Robin boundary conditions.

In [128], the authors presented a second-order robust method for solving singularly perturbed Burgers' equation. In [88], a specific class of parabolic singularly perturbed convection-diffusion problems is investigated. The problem is discretized using the backward Euler scheme in the temporal direction and the upwind scheme on a Harmonic mesh in the spatial direction. In [158], the authors introduce a high-order convergent numerical method for singularly perturbed time dependent problems using mesh equidistribution. The discretization is based on the backward Euler scheme in time and a high-order non-monotone scheme in space. In [199], numerical approximations are computed for the solution of a system of two reaction-convection-diffusion equations by a fitted mesh finite difference method. In [270], authors conducted a numerical investigation of an initial boundary value problem for a singularly perturbed system of two equations of convection-diffusion type. The authors proposed a numerical method that combined a spline-based scheme with a Shishkin mesh and achieved second-order uniform convergence. While in [271], the authors present a uniformly convergent numerical technique for a time-dependent singularly perturbed system of two equations of reaction-diffusion type. The proposed numerical technique consists of the Crank–Nicolson scheme in the temporal direction over a uniform mesh and the quadratic B-splines collocation technique over an exponentially graded mesh in the spatial direction. In [208], the authors considered singularly perturbed elliptic convection-diffusion differential equations in two dimensions. They discretized the problem by using an upwind difference scheme on a modified exponentially graded Bakhvalov mesh. Further, in[303], the authors analysed a higher order numerical method for a class of two-dimensional parabolic singularly perturbed problem of convection-diffusion type for the case when the convection coefficient is vanishing inside the domain. The Peaceman–Rachford scheme is used on a uniform mesh for time discretization, and a hybrid scheme is applied on the Bakhvalov–Shishkin mesh for spatial discretization. In [55], the authors deal with one-dimensional linear parabolic singularly perturbed systems of convection-diffusion type. The diffusion term in each equation is affected by a small positive parameter of different magnitudes. The numerical algorithm combines the classical upwind finite difference scheme to discretize in space and the fractional implicit Euler method together with an appropriate splitting by components to discretize in time.

In [94], a parameter-uniform numerical method is constructed for singularly perturbed Robin type parabolic convection-diffusion problems having boundary turning points. The problem is discretized by means of the implicit Euler method in time and the non-standard finite difference method in space on a uniform mesh. Moreover, the non-standard finite difference method is used to discretize the Robin boundary conditions. While in [263], the authors deal with a singularly perturbed two-dimensional steady-state convection-diffusion problem with Robin boundary conditions.

In [156], the authors present a domain decomposition method to solve a class of singularly perturbed parabolic problems of reaction-diffusion type having Robin boundary conditions. The method considers three subdomains, of which two are fine mesh and the other is coarse mesh. The partial differential equation associated with the problem is discretized using the finite difference scheme on each subdomain, while the Robin boundary conditions associated with the problem are approximated using a special finite difference scheme to maintain accuracy. In [250], the authors considered a class of time-fractional singularly perturbed convection-diffusion problems. To discretize the problem, a classical $L1$ finite difference scheme is employed on a graded mesh to discretize the time-fractional derivative. Further, a standard upwinding procedure in the spatial direction is used on a piecewise uniform Shishkin mesh.

In [223], the authors considered a second-order singularly perturbed Volterra integro-differential equation. On a layer adapted Shishkin mesh, the problem is solved using finite difference schemes. Whereas in [224] author presents a fitted mesh finite difference method for solving a singularly perturbed Fredholm integro-differential equation.

2. **Finite element methods:** The finite element method (FEM) is a numerical technique commonly used to approximate solutions to ODEs or PDEs. In this method, the dependent variable u in the differential equation is approximated by a function u_h which is a linear combination of basis functions ψ as $u \approx u_h = \sum_i u_i \psi_i$. The basis functions, ψ_i, are chosen such that they form a set of functions that span the solution space. The coefficients of the basis functions, u_i, represent the unknowns that need to be determined. These coefficients correspond to the values of the solution at specific mesh points, i. By substituting the approximation u_h into the original dif-

ferential equation, the problem is transformed into a system of algebraic equations. This system is then solved to determine the values of the coefficients u_i, which in turn define the approximate solution u_h. The finite element method offers flexibility in choosing the shape and size of the basis functions, allowing for adaptability to complex geometries and varying solution behaviour. It is widely used in various fields of engineering and applied sciences for solving a wide range of problems governed by differential equations.

During the late 1970s, researchers began applying Petrov-Galerkin methods, a variant of the FEM, to solve SPPs. In [307], the authors recognized the need for special approaches to address SPPs using finite element analysis. They developed a FEM approach analogous to the upwind scheme by incorporating upwinding into the test function. This modification aimed to obtain an oscillation-free solution for SPPs. In [187], the author proposed a FEM formulation that reduced to a simple upwind scheme in the limiting case, specifically for solving singularly perturbed ODEs.

In [48], the authors introduced a FEM that utilized piecewise linear and quadratic basis functions to solve second-order differential equations. Further, in [118], the authors used an upwind finite element scheme for two-dimensional convective transport equations. Author in [292] proposed a FEM technique employing piecewise polynomials of degree at most k to solve two-point singularly perturbed boundary value problems. The proposed method provided parameter-uniform error estimates of $O(h^{k+1})$ in the maximum norm, indicating convergence rates that depended on the mesh size h. In [119], the authors conducted a survey summarizing various FEMs and upwind schemes employed to solve convection-dominated flow problems.

In [117], the author proposed a FEM that utilized a combination of quadratic trial and cubic test functions to solve the steady-state convection-diffusion equation. A series of papers were published investigating the selection of test spaces to symmetrize the associated bilinear form; see, e.g., [24, 26, 25, 27]. The goal was to obtain an optimal approximate solution similar to the ones achieved by applying Galerkin methods to symmetric problems.

In [70], a convection-dominated diffusion problem is solved by combining the FEM or FDM with the method of characteristics. The authors derived optimal order error estimates in L^2 and $W^{1,2}$ for the FEM and various error estimates for a variety of FDMs. The concept of hinged elements was introduced in [217]. These elements were essentially piecewise linear finite elements that could vary according to problem data and were used to solve one-dimensional linear non self-adjoint two-point singularly perturbed boundary value problems.

In [277], the authors applied the FEM with a Petrov-Galerkin approach using exponential basis elements to solve conservative non self-adjoint singularly perturbed boundary value problems. The method has first-order convergence in L_∞ and second-order convergence at the nodes. The Authors, in [76], introduced an adaptive streamline diffusion finite element method for solving stationary convection-diffusion problems by using shock capturing artificial viscosity technique.

In [282], the authors applied Galerkin FEMs using a piecewise polynomial basis functions on a Shishkin mesh to obtain optimal convergence results for high-order elliptic two-point singularly perturbed boundary value problem of reaction-diffusion type. They also achieved uniform convergence results for a family of Galerkin FEMs on a Shishkin mesh for high-order elliptic two-point singularly perturbed boundary value problems of convection-diffusion type [283].

In [168], the author utilized Galerkin FEM on a Bakhvalov-Shishkin mesh to solve a linear two-dimensional convection-diffusion problem. It was shown that better error estimates were obtained on the Bakhvalov-Shishkin mesh compared to the Shishkin mesh. Further, in [306], the author studied superconvergence approximations of singularly perturbed boundary value problems of reaction-diffusion type and convection-diffusion type. He obtained superconvergence with an error bound of $O((N^{-1}\ln(N+1))^{p+1})$ in a discrete energy norm by applying the standard finite element method of any fixed order p on a modified Shishkin mesh.

In [276], the author presented a FEM approach for solving a non self-adjoint SPP. In [246], the author achieved optimal convergence results for the two-point singularly perturbed boundary value problem of convection-diffusion type in the energy norm. The analysis was conducted on a Bakhvalov mesh.

In [286], the authors considered a two-parameter elliptic SPP. They decomposed the solution into smooth and layer components and derived error bounds for these components and their derivatives. This analysis was then used to analyse the FEM in [287]. In [300], the performance of high-order versions of FEM on various meshes is investigated. The authors proposed the p/h_p FEM, which aimed to approximate the solution to a problem with an exponential rate of convergence independent of the perturbation parameters.

In [85], the authors applied Galerkin FEM to solve elliptic convection-diffusion problems. They analysed the superconvergence property of the method on a

Shishkin mesh and determined that it was almost first-order accurate in the energy norm. In [49], the authors proposed a multiscale FEM to approximate the solution to elliptic SPPs with high contrast coefficients using coarse quasiuniform meshes. The method achieved first-order convergence in the energy norm and second-order convergence in the L_2 norm. Authors in [127] compared the performance of He's homotopy perturbation method with that of FEM for solving the two-dimensional heat conduction equation. They concluded that there was excellent agreement between the analytical results obtained using the homotopy perturbation method and the numerical results obtained using FEM, demonstrating the accuracy and reliability of FEM in solving heat conduction problems.

In [304], the authors solved a singularly perturbed convection-diffusion equation using linear FEM on a Shishkin mesh. They utilised symmetries in the convective term of the bilinear form over adjacent intervals to achieve superconvergence of almost second-order accuracy in general cases. This research highlighted the potential for improving the accuracy and efficiency of FEM through the exploitation of specific problem structures and symmetries. In [299], the authors construct a finite volume element method on the Shishkin mesh for solving a singularly perturbed reaction-diffusion problem. In [239], the authors introduce numerical methods for singularly perturbed convection-diffusion problems with a turning point. As a result of the turning point, the problem typically exhibits exponential-type boundary layers or a cusp-type interior layer. They develop non-symmetric discontinuous Galerkin FEM with interior penalties for both cases. Usual Shishkin mesh is invoked for the problem with boundary layers, whereas generalized Shishkin type mesh is used to tackle the interior layer of cusp-type. In [155], the authors presented a convergence analysis of a weak Galerkin FEM using polygonal meshes for the semilinear singularly perturbed time-dependent convection-diffusion-reaction equations.

Comprehensive discussions on the theoretical foundations and practical implementations of FEM for various problems can be found in several books; see, e.g. [43, 40, 18]. These references extensively cover the theory and applications of FEM.

3. **Finite volume methods:** The finite volume method (FVM) is a numerical technique used to approximate the solution to PDEs. In this method, the domain is discretized into mesh elements known as control volumes. The PDE is integrated over each control volume to obtain a set of balance equations. These balance equa-

tions are then discretized into a set of algebraic equations, resulting in a system of equations with discrete unknowns. The system of equations can be solved either exactly or approximately using direct or iterative methods such as the Gauss-Seidel method and the Jacobi method. Iterative methods iteratively update the values of the unknowns until a desired level of accuracy is achieved. Direct methods, on the other hand, solve the system of equations in one step but may be computationally expensive for large systems. The FVM is an integral scheme, similar to the FEM, whereas FDM is a differential scheme. In FVM, the integral form of the PDEs is used to construct the discrete equations, whereas FDM approximates the derivatives directly using finite difference approximations. Differential schemes are generally faster than integral schemes, but integral schemes, such as FVM, have the advantage of being more accurate than their differential counterparts when dealing with irregular meshes. For a detailed description of FVM, several books are available as references. [191] provides a detailed description of cell-vertex FVMs, [166] focuses on FVMs for hyperbolic problems, and [182] covers FVMs for general PDEs, providing a comprehensive overview of the method and its applications.

1.5 Plan of the Thesis

In this thesis, we study, analyse and develop adaptive numerical schemes for solving various models of singularly perturbed convection-diffusion boundary value problems. An adaptive discretization technique can accommodate problems with different physical and dynamic features by adjusting the resolution, order and type of discretization. It is generally used with an adaptive numerical method that balances the solution accuracy and the associated computational cost. Therefore, an appropriate numerical method and the discretization technique accurately solve the problem and improve convergence. Taking this into account, in this thesis, we propose different methods to solve singularly perturbed parabolic convection-diffusion problems with discontinuous coefficient, source term and delay. We apply the proposed methods to solve three different convection-diffusion problems of varying complexity.

The thesis is organized as follows: Chapter 2 presents an adaptive finite difference method to solve a class of singularly perturbed parabolic delay differential equations with discontinuous convection coefficient and source that reads

$$\left.\begin{aligned}
&\varepsilon u_{xx}(x,t) + a(x)u_x(x,t) - b(x)u(x-1,t) - c(x)u(x,t) - u_t(x,t) = f(x,t),\\
&(x,t) \in S^- \cup S^+ := (0,1)\times(0,T] \cup (1,2)\times(0,T],\\
&u(x,t) = p_0(x,t) \text{ on } \Gamma_1 := \{(x,0),\ x \in [0,2]\},\\
&u(x,t) = p_1(x,t) \text{ in } \Gamma_2 := \{(x,t),\ x \in [-1,0],\ t \in [0,T]\},\\
&u(x,t) = p_2(x,t) \text{ on } \Gamma_3 := \{(2,t),\ t \in [0,T]\},
\end{aligned}\right\} \tag{1.4}$$

where $\varepsilon \ll 1$ is a small positive parameter, b and c are sufficiently smooth functions such that $b(x) < 0$, $c(x) > 0$ and $b(x) + c(x) \geq 0$ for all $x \in [0,2]$. Moreover, we assume that

$$\left.\begin{aligned}
&a(x) = \begin{cases} a_1(x) & \text{if } 0 \leq x \leq 1,\\ a_2(x) & \text{if } 1 < x \leq 2,\end{cases} \qquad f(x,t) = \begin{cases} f_1(x,t) & \text{if } (x,t) \in \overline{S}^-,\\ f_2(x,t) & \text{if } (x,t) \in \overline{S}^+,\end{cases}\\
&a_1(x) < -\gamma_1 < -2\gamma < 0,\quad a_2(x) > \gamma_2 > 2\gamma > 0,\quad |\,[a]\,| \leq C,\quad |\,[f]\,| \leq C,
\end{aligned}\right\} \tag{1.5}$$

where $\gamma = \min\{\gamma_1, \gamma_2\}$. The simultaneous presence of discontinuity and the delay makes the problem stiff. The solution to the problem considers the present state of the physical system and its history. The numerical scheme based on the upwind finite difference method is presented on a specially generated mesh to solve the problem. The adaptive mesh is chosen so that most of the mesh points remain in regions with rapid transitions. The proposed numerical method is analysed for consistency, stability and convergence. Extensive theoretical analysis is performed to obtain consistency and error estimates. The proposed method is unconditionally stable, and the convergence obtained is parameter-uniform with first-order convergence in space and first-order convergence in time. Numerical results are presented for model problems demonstrating the effectiveness of the proposed technique. Convergence obtained in practical satisfies theoretical predictions.

Chapter 3 extends the idea further and aims to provide a better numerical approximation of the solution to the model problem (1.4). The chapter presents a higher-order hybrid difference method over an adaptive mesh to solve the problem (1.4). The proposed method is a composition of a central difference scheme and a midpoint upwind scheme on a specially generated mesh. Moreover, the time variable is discretised using an implicit finite difference method. The error estimates of the proposed numerical method satisfy parameter-uniform second-order convergence in space and first-order convergence in time. The rigorous numerical analysis of the proposed method on a Shishkin class mesh establishes the supremacy of the proposed scheme.

Chapter 4 presents a high-order finite difference scheme to solve singularly perturbed parabolic convection-diffusion problems with a large delay and an integral boundary condition which is given as

$$\left.\begin{aligned}&-\varepsilon u_{xx}(x,t)+p(x)u_x(x,t)+q(x)u(x,t)+r(x)u(x-1,t)+u_t(x,t)=g(x,t),\\&(x,t)\in(0,2)\times(0,T],\\&u(x,t)=\psi_1(x,t)\text{ in }\Gamma_1:=\{(x,t),\ x\in[-1,0],\ t\in[0,T]\},\\&u(x,t)=\psi_2(x,t)\text{ on }\Gamma_2:=\{(x,0),\ x\in[0,2]\},\\&\mathcal{K}u(x,t)=u(2,t)-\varepsilon\int_0^2 f(x)u(x,t)dx=\psi_3(x,t)\text{ on }\Gamma_3=\{(2,t),\ t\in[0,T]\},\end{aligned}\right\}$$

where $\varepsilon \ll 1$ is a small positive parameter, $g(x,t)$, $p(x)$, $q(x)$ and $r(x)$ are sufficiently smooth functions. Also, assume that the initial-boundary data ψ_1, ψ_2 and ψ_3 are smooth and bounded functions such that

$$\left.\begin{aligned}&p(x)\geq p_0>p_0^*>0,\quad q(x)\geq q_0>0,\quad r(x)\leq r_0<0,\\&p_0^*+q_0+r_0>0,\quad q(x)+r(x)\geq 2\eta>0,\\&u(1^-,t)=u(1^+,t),\quad u_x(1^-,t)=u_x(1^+,t).\end{aligned}\right\}$$

Here, $f(x)$ is non-negative, monotonic function such that $\displaystyle\int_0^2 f(x)dx<1$. The solution of the problem features a weak interior layer besides a boundary layer. This chapter presents a higher-order accurate numerical method on a specially designed non-uniform mesh. The technique employs the Crank-Nicolson difference scheme in the temporal variable, whereas an upwind difference scheme in space. It is proved that the proposed method is unconditionally stable and converges uniformly, independent of the perturbation parameter. The error analysis indicates that the numerical solution is uniformly stable and shows parameter-uniform second-order convergence in time and first-order convergence in space. The numerical result for two model problems is presented, which agrees with the theoretical estimates.

Chapter 5 presents a robust computational technique to solve a class of two-parameter parabolic convection-diffusion problems with a large delay which is given as

$$\left.\begin{aligned}&\varepsilon u_{xx}(x,t)+\mu p(x,t)u_x(x,t)-q(x,t)u(x,t)+r(x,t)u(x-1,t)-u_t(x,t)=g(x,t),\\&(x,t)\in(0,2)\times(0,T],\\&u(x,t)=\psi_1(x,t)\text{ in }\Gamma_1:=\{(x,t),\ x\in[-1,0],\ t\in[0,T]\},\\&u(x,t)=\psi_2(x,t)\text{ on }\Gamma_2:=\{(x,0),\ x\in[0,2]\},\\&u(x,t)=\psi_3(x,t)\text{ on }\Gamma_3=\{(2,t),\ t\in[0,T]\},\end{aligned}\right\}$$

where $0<\varepsilon<1$, $0<\mu<1$ are small parameters, the functions $f(x,t)$, $p(x,t)$, $q(x,t)$ and $r(x,t)$ are sufficiently smooth functions and assume that

$$p(x,t)\geq p_0>0,\quad q(x,t)\geq q_0>0,\quad r(x,t)\geq r_0>0,\quad (q-r)(x,t)\geq\kappa>0,$$

$\gamma = \min_{(x,t)\in\bar{D}} \left(\frac{q(x,t) - r(x,t)}{p(x,t)} \right)$ and the initial-boundary data ψ_1, ψ_2 and ψ_3 are smooth functions on their respective domain. The presence of perturbation parameters leads to the twin boundary layers and interior layers in the solution, whose appropriate numerical approximation is the main goal of this chapter. The numerical method is composed of an upwind difference scheme in space and a Crank-Nicolson scheme in time is used to find the approximate solution of the problem. It is proved that the method is parameter-uniform with second-order accuracy in time and almost first-order accuracy in space. Numerical examples are provided in support of the theory.

Finally, Chapter 6 concludes the thesis with a summary of the work highlighting its significant contributions. It opens the discussion about future research directions and points out the challenging steps towards analyzing more complicated problems.

Chapter 2

Parabolic Problems with Discontinuous Coefficient and Delay

2.1 Introduction

Singularly perturbed parabolic PDEs with delay and discontinuous coefficients terms constitute a challenging class of mathematical models that arise in various fields, including physics, engineering, and biology. These equations include time-dependent variables and exhibit sensitivity to small perturbations in the system parameters. The parabolic nature of these PDEs indicates their ability to describe dynamic processes with diffusion-like behaviour, where quantities such as temperature, concentration, or population density evolve over time and space. However, what sets them apart is the presence of delays, discontinuous coefficients and source terms, which introduce additional complexity to the problem.

The incorporation of delays in these equations captures the influence of past states or events on the current evolution of the system. Delays add memory-like behaviour to the PDEs, making their analysis and numerical solution more intricate. Furthermore, the discontinuous coefficients introduce abrupt changes or jumps in the system's material properties or physical characteristics. Moreover, the problem is singularly perturbed, and the solution of these equations exhibits a multiscale character since the corresponding degenerate system fails to satisfy the given boundary data. There are narrow regions across which the solution changes rapidly and displays layer behaviour. Standard numerical methods on uniform meshes fail to consistently approximate solutions in these layer regions. Consequently, traditional solution techniques may not be directly applicable, requiring specialized methods to handle the discontinuities appropriately.

Many researchers have tried to provide consistent numerical approximations to singularly perturbed differential equations with a delay, discontinuous coefficients, and source terms. In [279, 280], a standard numerical method with piecewise linear interpolation on a Shishkin mesh is proposed to solve second and third-order ordinary differential equations. In [79], the authors solve a reaction-diffusion equation with a discontinuous diffusion coefficient. Whereas in [240] author presents an almost first-order uniformly convergent method for a coupled system of two reaction-diffusion equations with discontinuous source terms. Later, researchers extended the technique to n-number of equations in [225].

A discrete approximation of singularly perturbed parabolic partial differential equation with a discontinuous initial condition is proposed in [120]. The problem is bi-singular, where a classical singularity is twisted together with the singular nature of the problem. A numerical scheme is presented based on the fitted operator method defined over a uniform mesh. Indeed, it is an exact finite difference scheme for the error function associated with the discontinuity in the initial condition. However, there appears no further attempt towards developing fitted operator methods for parabolic reaction-diffusion problems until authors in [103] regularize the problem by reinstating the discontinuous initial condition. A linear parabolic problem with a turning point is also brought to attention [220]. The interior layer problem considered by the authors is of the following form

$$\begin{aligned} &-\varepsilon u_{xx} + a u_x + bu + c u_t = f, \quad (x,t) \in (0,1) \times (0,T], \quad b, c \geq 0, \\ &0 < \varepsilon \ll 1, \quad u(0,t), \quad u(1,t), u(x,0) \text{ is specified.} \end{aligned} \tag{2.1}$$

A particular case, when the convection coefficient is discontinuous across the curve $\Gamma_1 := \{(d(t),t) \mid t \in [0,T],\ 0 < d(t) < 1\}$ and having a particular sign pattern $a(x) > 0,\ x < d(t)$ or $a(x) < 0,\ x > d(t)$ is examined in [71]. Whereas, in [221] author presents a parameter uniform method over a specially designed Shishkin mesh. The mesh is obtained by mapping Γ_1 to the vertical line $x = d(0)$. The case when the initial condition $u(x,0)$ contains its interior layer is studied separately in [102]. However, the convection coefficient is assumed to be space independent, smooth, and of one sign. It is to note that the reduced initial condition (set $\varepsilon = 0$) is discontinuous at some point. Moreover, the discontinuity travels with the characteristic curve $\Gamma_2 := \{(d(t),t) \mid t \in [0,T], d'(t) = a(t),\ d(0) = d\}$ associated with the reduced hyperbolic equation $av_x + bv + cv_t = f$. In [193], a hybrid difference scheme is presented to solve the parabolic problem with a discontinuous convection coefficient. On the other hand, in [220], an interior layer appears in the solution of (2.1) since the convective coefficient of the problem contains an interior layer with a hyperbolic tangent profile. This problem appears to be a time-dependent version of the ordinary differential equation examined in [219]. In [72], the authors deal with a parabolic problem with a boundary turning point. In [82], an experimental technique is presented to analyse uniform convergence over piecewise uniform meshes. However, it was presented in context to a singularly perturbed elliptic equation when the convective term degenerates on the boundary of the domain.

In [53, 45], a two-parameter parabolic problem is studied. A hybrid monotone difference scheme is conferred using the averaging method at the discontinuous points [45]. While [53] presents the case when the convective term degenerates inside the spatial domain, and the source term has a discontinuity of the first kind on the degeneration line. A particular case, when $\mu = 0$ was studied in [83]. In [140, 51, 52], researchers studied a similar problem for $\mu = 1$. In [241], authors present a parameter uniformly convergent method for a singularly perturbed parabolic system of equations with a discontinuous source term. In this work, the diffusion term in each equation is affected by a small positive parameter of different magnitudes. Besides, researchers have paid attention to nonlinear problems with an interior layer in [237, 150, 205]. In [150], the authors consider a parabolic periodic boundary value problem and construct the interior layer type formal asymptotics. They establish the existence and asymptotic stability of the solution by using precise lower and upper solutions. The analysis presented is then used to construct an efficient numerical method for a slightly general nonlinear problem in [237]. In [279], researchers used a finite difference method based on linear interpolation to solve an ordinary delay differential equation with a discontinuous coefficient. They later considered a hybrid initial value method for the same problem and demonstrated that it yielded improved results.

The analysis of the special methods for singularly perturbed parabolic partial functional differential equations with discontinuous data and degenerating convective terms has yet to see much development in the literature. This chapter presents a numerical method to solve singularly perturbed parabolic partial differential equations with a delay, discontinuous coefficient and source. Besides, the chapter presents rigorous consistency, stability, and convergence analysis of the proposed scheme and illustrates numerical results.

2.2 Continuous Problem

Let $D = S^- \cup S^+ := (0,1) \times (0,T] \cup (1,2) \times (0,T]$, $\bar{\Omega} = [0,2]$ and consider the non-homogeneous initial-boundary-value problem

$$\left.\begin{aligned}
&\varepsilon u_{xx}(x,t) + a(x)u_x(x,t) - b(x)u(x-1,t) - c(x)u(x,t) - u_t(x,t) = f(x,t) \text{ in } D,\\
&u(x,t) = p_0(x,t) \text{ on } \Gamma_1 := \{(x,0),\ x \in [0,2]\},\\
&u(x,t) = p_1(x,t) \text{ in } \Gamma_2 := \{(x,t),\ x \in [-1,0],\ t \in [0,T]\},\\
&u(x,t) = p_2(x,t) \text{ on } \Gamma_3 := \{(2,t),\ t \in [0,T]\},
\end{aligned}\right\} \quad (2.2)$$

where $\varepsilon \ll 1$ is a small positive parameter, $b(x)$ and $c(x)$ are sufficiently smooth functions such that $b(x) < 0$, $c(x) > 0$ and $b(x) + c(x) \geq 0$ for all $x \in [0,2]$. Moreover, we assume that

$$\left.\begin{aligned}
&a(x) = \begin{cases} a_1(x) & \text{if } 0 \leq x \leq 1,\\ a_2(x) & \text{if } 1 < x \leq 2, \end{cases} \qquad f(x,t) = \begin{cases} f_1(x,t) & \text{if } (x,t) \in \overline{S^-},\\ f_2(x,t) & \text{if } (x,t) \in \overline{S^+}, \end{cases}\\
&a_1(x) < -\gamma_1 < -2\gamma < 0,\quad a_2(x) > \gamma_2 > 2\gamma > 0,\quad |\,[a]\,| \leq C,\quad |\,[f]\,| \leq C,
\end{aligned}\right\} \quad (2.3)$$

where $\gamma = \min\{\gamma_1, \gamma_2\}$. The solution of (2.2) satisfies $[u] = 0$ and $[u_x] = 0$ at $x = 1$. Here, $[u]$ denotes the jump of u defined at the point of discontinuity $x = 1$ as $[u](1,t) = u(1^+,t) - u(1^-,t)$, where $u(1^\pm,t) = \lim_{x\to 1\pm 0} u(x,t)$. The functions p_0, p_1 and p_2 are sufficiently smooth functions and satisfy the compatibility conditions

$$\left.\begin{aligned}
&p_0(0,0) = p_1(0,0),\quad p_0(2,0) = p_2(2,0),\\
&\varepsilon\frac{\partial^2 p_0(0,0)}{\partial x^2} + a(0)\frac{\partial p_0(0,0)}{\partial x} - b(0)p_1(-1,0) - c(0)p_0(0,0) - \frac{\partial p_1(0,0)}{\partial t} = f(0,0),\\
&\varepsilon\frac{\partial^2 p_0(2,0)}{\partial x^2} + a(2)\frac{\partial p_0(2,0)}{\partial x} - b(2)p_0(1,0) - c(2)p_0(2,0) - \frac{\partial p_2(2,0)}{\partial t} = f(2,0).
\end{aligned}\right\}$$

Let us rewrite (2.2) as

$$L_\varepsilon u(x,t) = \mathcal{F}(x,t),$$

where

$$L_\varepsilon u(x,t) = \begin{cases} \varepsilon u_{xx}(x,t) + a_1(x)u_x(x,t) - c(x)u(x,t) - u_t(x,t) & \text{if } (x,t) \in S^-,\\ \varepsilon u_{xx}(x,t) + a_2(x)u_x(x,t) - b(x)u(x-1,t) - c(x)u(x,t) - u_t(x,t) & \text{if } (x,t) \in S^+, \end{cases}$$

and

$$\mathcal{F}(x,t) = \begin{cases} f_1(x,t) + b(x)p_1(x-1,t) & \text{if } (x,t) \in S^-,\\ f_2(x,t) & \text{if } (x,t) \in S^+. \end{cases}$$

Under these assumptions, the solution of (2.2) exists and is unique [9, 139]. The simultaneous presence of discontinuity and delay makes the problem stiff. The solution $u(x,t)$ of (2.2) exhibits a strong interior layer and a weak boundary layer in the neighbourhood of the points $x = 1$ and $x = 2$, respectively. Moreover, it is easy to follow that the differential operator L_ε satisfies the following minimum principle.

Lemma 2.2.1. *Suppose* $\mathcal{P} \in C^0(\bar{D}) \cap C^2(S^- \cup S^+)$ *satisfies* $\mathcal{P}(x,t) \geq 0$ *for all* $(x,t) \in \Gamma := \bar{D} \setminus D$*,* $[\mathcal{P}_x](1,t) \leq 0$*,* $t > 0$ *and* $L_\varepsilon \mathcal{P}(x,t) \leq 0$ *for all* $(x,t) \in S^- \cup S^+$*. Then* $\mathcal{P}(x,t) \geq 0$ *for all* $(x,t) \in \bar{D}$*.*

Proof. Choose $(x^k,t^k) \in \bar{D}$ such that $\mathcal{P}(x^k,t^k) = \min_{(x,t)\in\bar{D}} \mathcal{P}(x,t)$. Consequently,

$$\mathcal{P}_x(x^k,t^k) = 0, \;\; \mathcal{P}_t(x^k,t^k) = 0 \;\text{ and }\; \mathcal{P}_{xx}(x^k,t^k) > 0 \text{ for } (x^k,t^k) \in (S^- \cup S^+). \tag{2.4}$$

Suppose $\mathcal{P}(x^k,t^k) < 0$ and it follows that $(x^k,t^k) \notin \Gamma$.

Case I: If $(x^k,t^k) \in S^-$, then

$$L_\varepsilon \mathcal{P}(x^k,t^k) = \varepsilon \mathcal{P}_{xx}(x^k,t^k) + a_1(x^k)\mathcal{P}_x(x^k,t^k) - \mathcal{P}_t(x^k,t^k) - c(x^k)\mathcal{P}(x^k,t^k).$$

Since $a_1(x) < 0,\; c(x) > 0$. Then, by using (2.4), we have

$$L_\varepsilon \mathcal{P}(x^k,t^k) > 0.$$

Case II: If $(x^k,t^k) \in S^+$, then

$$\begin{aligned} L_\varepsilon \mathcal{P}(x^k,t^k) &= \varepsilon \mathcal{P}_{xx}(x^k,t^k) + a_2(x^k)\mathcal{P}_x(x^k,t^k) - b(x^k)\mathcal{P}(x^k-1,t^k) \\ &\quad - c(x^k)\mathcal{P}(x^k,t^k) - \mathcal{P}_t(x^k,t^k) \\ &= \varepsilon \mathcal{P}_{xx}(x^k,t^k) + a_2(x^k)\mathcal{P}_x(x^k,t^k) - c(x^k)\mathcal{P}(x^k,t^k) \\ &\quad - b(x^k)\left(\mathcal{P}(x^k-1,t^k) - \mathcal{P}(x^k,t^k)\right) - b(x^k)\mathcal{P}(x^k,t^k) - \mathcal{P}_t(x^k,t^k) \\ &= \varepsilon \mathcal{P}_{xx}(x^k,t^k) + a_2(x^k)\mathcal{P}_x(x^k,t^k) - \left(b(x^k) + c(x^k)\right)\mathcal{P}(x^k,t^k) \\ &\quad - b(x^k)\left(\mathcal{P}(x^k-1,t^k) - \mathcal{P}(x^k,t^k)\right) - \mathcal{P}_t(x^k,t^k). \end{aligned}$$

Since $a_2(x) > 0,\;\; b(x) < 0,\;\; c(x) > 0,\;\; b(x) + c(x) > 0$ and by using (2.4), we have

$$L_\varepsilon \mathcal{P}(x^k,t^k) > 0.$$

Case II: If $(x^k,t^k) = (1,t^k)$, then

$$[\mathcal{P}_x](x^k,t^k) = \mathcal{P}_x(x^{k+},t^k) - \mathcal{P}_x(x^{k-},t^k) > 0 \text{ since } \mathcal{P}_x(x^{k+},t^k) < 0.$$

A contradiction to the assumption and consequently the required result follows. □

An important application of the minimum principle is establishing the boundedness of the solution. As an immediate application, we obtain

Lemma 2.2.2. *Let u be the solution of* (2.2). *Then*

$$||u||_{\infty,\bar{D}} \leq ||u||_{\infty,\Gamma} + \frac{1}{\gamma}||f||_{\infty,\bar{D}}, \quad \gamma = \min\{\gamma_1, \gamma_2\}. \tag{2.5}$$

Proof. Consider

$$\psi_{\pm} = \begin{cases} ||u||_{\infty,\Gamma} + \dfrac{x}{\gamma}||f||_{\infty,\bar{D}} \pm u & \text{if } x \leq 1, \\ ||u||_{\infty,\Gamma} + \dfrac{(2-x)||f||_{\infty,\bar{D}}}{\gamma} \pm u & \text{if } x \geq 1. \end{cases}$$

For $(x,t) \in S^-$, it follows that

$$\begin{aligned} L_\varepsilon\psi_{\pm}(x,t) &= \varepsilon\psi_{xx} + a_1(x)\left(\frac{||f||_{\infty,\bar{D}}}{\gamma} \pm u_x\right) - c(x)\left(\frac{x}{\gamma}||f||_{\infty,\bar{D}} + ||u||_{\infty,\Gamma} \pm u\right) \mp u_t \\ &= \pm L_\varepsilon u + a_1(x)\frac{||f||_{\infty,\bar{D}}}{\gamma} - c(x)||u||_{\infty,\Gamma} - c(x)\frac{x||f||_{\infty,\bar{D}}}{\gamma} \leq 0 \end{aligned}$$

since $a_1(x) < 0$ and $c(x) > 0$. Similarly, for $(x,t) \in S^+$, it is easy to verify that $L_\varepsilon\psi_{\pm}(x,t) \leq 0$. Also, $[\psi_{x\pm}](1,t) = \pm[u_x](1,t) = 0$. The required result (2.5) now follows from Lemma 2.2.1. □

In general, one can assume homogeneous boundary conditions $p_0 = p_1 = p_2 = 0$ by subtracting from u some suitable smooth function that satisfies the original boundary conditions [249]. Moreover, as proved in [139], it is an easy exercise to obtain the following estimate.

Lemma 2.2.3. *Let u be the solution of* (2.2). *Then*

$$\left|\frac{\partial^i u}{\partial t^i}(x,t)\right| \leq C \text{ for all } (x,t) \in \bar{D} \text{ and } i = 0,1,2.$$

Proof. The proof follows from the mean value theorem and (2.5). □

2.3 Time Discretization

Let $\mathbb{T}_t^M = \{t_k = kT/M, k = 0,\ldots,M\}$ be a uniform mesh in the time direction. Then, the backward Euler method leads to

$$\begin{aligned} &\varepsilon\Delta t U_{xx}(x,t_{k+1}) + a\Delta t U_x(x,t_{k+1}) - b\Delta t U(x-1,t_{k+1}) - \hat{c}U(x,t_{k+1}) \\ &= -U(x,t_k) + \Delta t f(x,t_{k+1}), \quad x \in (0,1)\cup(1,2) \text{ and } k = 0,1,\ldots,M-1 \end{aligned} \tag{2.6}$$

such that

$$\begin{cases} U(x,0) = p_0(x), \ \ 0 \le x \le 2, \\ U(x,t_{k+1}) = p_1(x,t_{k+1}), \ \ -1 \le x \le 0, \ 0 \le k \le M-1, \\ U(2,t_{k+1}) = p_2(t_{k+1}), \ \ 0 \le k \le M-1, \\ U(1^-,t_{k+1}) = U(1^+,t_{k+1}), \ \ U_x(1^-,t_{k+1}) = U_x(1^+,t_{k+1}), \ \ 0 \le k \le M-1, \end{cases} \tag{2.7}$$

where $\hat{c}(x) = (c(x)\Delta t + 1)$. Let us rewrite (2.6) as

$$\mathfrak{L}^M U(x,t_{k+1}) = \mathcal{F}(x,t_{k+1}), \tag{2.8}$$

where

$$\mathfrak{L}^M U(x,t_{k+1}) = \begin{cases} \varepsilon\Delta t U_{xx}(x,t_{k+1}) + a(x)\Delta t U_x(x,t_{k+1}) - \hat{c}(x)U(x,t_{k+1}) \ \text{ if } \ x \in (0,1), \\ \varepsilon\Delta t U_{xx}(x,t_{k+1}) + a(x)\Delta t U_x(x,t_{k+1}) - b(x)\Delta t U(x-1,t_{k+1}) \\ -\hat{c}(x)U(x,t_{k+1}) \ \text{ if } \ x \in (1,2), \end{cases} \tag{2.9}$$

and

$$\mathcal{F}(x,t_{k+1}) = \begin{cases} \Delta t f(x,t_{k+1}) + \Delta t b(x) p_1(x-1,t_{k+1}) - U(x,t_k) \ \text{ if } \ x \in (0,1), \\ \Delta t f(x,t_{k+1}) - U(x,t_k) \ \text{ if } \ x \in (1,2). \end{cases} \tag{2.10}$$

The operator $\mathfrak{L}^M$ satisfies the following minimum principle.

Lemma 2.3.1. *Let $\phi(x,t_{k+1})$ be a smooth function such that $\phi(x,t_{k+1}) \ge 0$ for $x = 0,2$, $[\phi_x](1,t_{k+1}) \le 0$ and $\mathfrak{L}^M\phi(x,t_{k+1}) \le 0$ for all $x \in (0,1) \cup (1,2)$. Then $\phi(x,t_{k+1}) \ge 0$ for all $x \in [0,2]$.*

Proof. Choose (x^o,t_{k+1}) such that $\phi(x,t_{k+1})$ attains its minimum at $x^o \in [0,2]$. Then

$$\phi_x(x^o,t_{k+1}) = \phi_t(x^o,t_{k+1}) = 0 \text{ and } \phi_{xx}(x^o,t_{k+1}) > 0 \text{ for } x^o \in (0,1) \cup (1,2). \tag{2.11}$$

Suppose $\phi(x^o,t_{k+1}) < 0$ and it follows that $(x^o,t_{k+1}) \notin \Gamma$ since $\phi(x,t_{k+1}) \ge 0$ for $x = 0,2$.

Case I: If $x^o \in (0,1)$

$$\begin{aligned} \mathfrak{L}^M\phi(x^o,t_{k+1}) &= \varepsilon\Delta t\phi_{xx}(x^o,t_{k+1}) + a_1(x^o)\Delta t\phi_x(x^o,t_{k+1}) - \hat{c}(x^o)\phi(x^o,t_{k+1}) \\ &> 0, \ \text{ from (2.3) and (2.11).} \end{aligned}$$

Case II: If $x^o \in (1,2)$

$$\begin{aligned} \mathfrak{L}^M\phi(x^o,t_{k+1}) &= \varepsilon\Delta t\phi_{xx}(x^o,t_{k+1}) + a_2(x^o)\Delta t\phi_x(x^o,t_{k+1}) - b(x^o)\Delta t\phi(x^o-1,t_{k+1}) \\ &\quad -\hat{c}(x^o)\phi(x^o,t_{k+1}) \\ &= \varepsilon\Delta t\phi_{xx}(x^o,t_{k+1}) + a_2(x^o)\Delta t\phi_x(x^o,t_{k+1}) - b(x^o)\Delta t\phi(x^o,t_{k+1}) \\ &\quad -b(x^o)\Delta t(\phi(x^o-1,t_{k+1}) - \phi(x^o,t_{k+1})) - c(x^o)\Delta t\phi(x^o,t_{k+1}) \\ &\quad -\phi(x^o,t_{k+1}) \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon\Delta t\phi_{xx}(x^o, t_{k+1}) - b(x^o)\Delta t(\phi(x^o - 1, t_{k+1}) - \phi(x^o, t_{k+1}))\\
&\quad + a_2(x^o)\Delta t\phi_x(x^o, t_{k+1}) - \Delta t\phi(x^o, t_{k+1})(b(x^o) + c(x^o)) - \phi(x^o, t_{k+1})\\
&> 0, \text{ from } (2.3) \text{ and } (2.11).
\end{aligned}
$$

Case III: If $x^o = 1$

$$[\phi_x](x^o, t_{k+1}) = \phi_x(x^{o+}, t_{k+1}) - \phi_x(x^{o-}, t_{k+1}) > 0 \text{ since } \phi(x^o, t_{k+1}) < 0.$$

It contradicts the assumption and hence, the required result follows. □

The operator $\mathfrak{L}^M$ is inverse monotone and $\|(\mathfrak{L}^M)^{-1}\|_\infty \le C$ [21]. Consequently, the stability of the scheme is immediate.

Lemma 2.3.2. *Let $\hat{e}_{k+1} := \hat{U}(x, t_{k+1}) - u(x, t_{k+1})$ be the local truncation error at $(k+1)$th time step. Then $\|\hat{e}_{k+1}\|_\infty \le C(\Delta t)^2$ for some constant C.*

Moreover, if $E_k := u(x, t_k) - U(x, t_k)$ denotes the global error in the time direction. Then, it follows that

$$\|E_{k+1}\|_\infty = \left\|\sum_{i=1}^{k} \hat{e}_i\right\|_\infty \le \|\hat{e}_1\|_\infty + \|\hat{e}_2\|_\infty + \|\hat{e}_3\|_\infty + \ldots + \|\hat{e}_k\|_\infty \le C\Delta t. \tag{2.12}$$

This in turn ensures the uniform convergence of the time semidiscretization process. Next, we obtain a priori estimate on the solution of the semidiscretized problem (2.6).

Lemma 2.3.3. *Let $U(x, t_{k+1})$ be the solution of* (2.6)*. Then*

$$\|U(x, t_{k+1})\|_{\infty,\bar{\Omega}} \le \max\left\{|U(0, t_{k+1})|, \frac{\|\mathcal{F}\|_{\infty,\bar{\Omega}}}{\gamma}, |U(2, t_{k+1})|\right\} \quad \textit{for all} \quad x \in [0, 2].$$

Proof. Consider $\psi^\pm(x, t_{k+1}) = \max\left\{|U(0, t_{k+1})|, \frac{\|\mathcal{F}\|_{\infty,\bar{\Omega}}}{\gamma}, |U(2, t_{k+1})|\right\} \pm U(x, t_{k+1})$. Then, $\psi^\pm(0, t_{k+1}) \ge 0$ and $\psi^\pm(2, t_{k+1}) \ge 0$. For $x \in (0, 1)$

$$
\begin{aligned}
\mathfrak{L}^M\psi^\pm(x, t_{k+1}) &= -\hat{c}(x)\max\left\{|U(0, t_{k+1})|, \frac{\|\mathcal{F}\|_{\infty,\bar{\Omega}}}{\gamma}, |U(2, t_{k+1})|\right\} \pm \mathfrak{L}^M U(x, t_{k+1})\\
&\le 0
\end{aligned}
$$

since $\hat{c}(x) = c(x)\Delta t + 1 \ge 0$. Similarly, for $x \in (1, 2)$ we compute

$$
\begin{aligned}
\mathfrak{L}^M\psi^\pm(x, t_{k+1}) &= -(b(x)\Delta t + \hat{c}(x))\max\left\{|U(0, t_{k+1})|, \frac{\|\mathcal{F}\|_{\infty,\bar{\Omega}}}{\gamma}, |U(2, t_{k+1})|\right\} \pm \mathfrak{L}^M U(x, t_{k+1})\\
&\le 0
\end{aligned}
$$

since $b(x) + c(x) \ge 0$. Moreover, for $x = 1$, $[\psi_x^\pm](1, t_{k+1}) = \pm[U_x](1, t_{k+1}) = 0$. Consequently, from Lemma 2.3.1 the required result follows. □

The solution $U(x, t_{k+1})$ of the semidiscretized problem (2.6) is known to admit a decomposition into smooth and a singular component [173]. We write

$$U(x, t_{k+1}) := V(x, t_{k+1}) + W(x, t_{k+1}).$$

The smooth component $V(x, t_{k+1})$ is the solution of

$$\begin{cases} \mathfrak{L}^M V(x, t_{k+1}) = \mathcal{F}(x, t_{k+1}), \quad x \in (0,1) \cup (1,2), \quad 0 \le k \le M-1, \\ V(0, t_{k+1}) = V_0(0, t_{k+1}), \\ V(1^-, t_{k+1}) = -(\hat{c}(1))^{-1} \left(\Delta t f(1, t_{k+1}) - U(1, t_k) + \Delta t b(1) p_1(0, t_{k+1})\right), \\ V(1^+, t_{k+1}) = -(\hat{c}(1))^{-1} \left(\Delta t f(1, t_{k+1}) - U(1, t_k) + \Delta t b(1) V_0(0, t_{k+1})\right), \\ V(2, t_{k+1}) = V_0(2, t_{k+1}), \end{cases} \tag{2.13}$$

where $V_0(x, t_{k+1})$ satisfies the corresponding degenerate problem. Also, the singular component $W(x, t_{k+1})$ satisfies a homogeneous problem which reads

$$\begin{cases} \mathfrak{L}^M W(x, t_{k+1}) = 0, \quad x \in (0,1) \cup (1,2), \quad 0 \le k \le M-1, \\ W(0, t_{k+1}) = 0, \\ W(2, t_{k+1}) = U(2, t_{k+1}) - V(2, t_{k+1}), \\ W(1^+, t_{k+1}) - W(1^-, t_{k+1}) = V(1^-, t_{k+1}) - V(1^+, t_{k+1}), \\ W_x(1^+, t_{k+1}) - W_x(1^-, t_{k+1}) = V_x(1^-, t_{k+1}) - V_x(1^+, t_{k+1}). \end{cases} \tag{2.14}$$

The next Lemma provides bounds on the derivative of $V(x, t_{k+1})$ and $W(x, t_{k+1})$ with respect to x.

Lemma 2.3.4. *Let $V(x, t_{k+1})$ and $W(x, t_{k+1})$ be the solutions of* (2.13) *and* (2.14)*, respectively. Then, for $s = 0, 1, 2, 3$*

$$\left|\frac{d^s V(x, t_{k+1})}{dx^s}\right| \le C(1 + \varepsilon^{2-s}) \textit{ for } x \in (0,1) \cup (1,2) \textit{ and}$$

$$\left|\frac{d^s W(x, t_{k+1})}{dx^s}\right| \le \begin{cases} \varepsilon^{-s} \exp\left(\dfrac{-\gamma(1-x)}{\varepsilon}\right) & \textit{for } x \in (0,1), \\ \varepsilon^{-s} \exp\left(\dfrac{-\gamma(x-1)}{\varepsilon}\right) + \varepsilon^{-(s+1)} \exp\left(\dfrac{-\gamma(2-x)}{\varepsilon}\right) & \textit{for } x \in (1,2). \end{cases}$$

Proof. The proof follows from [279]. □

2.4 Spatial Discretization

The solution of the problem exhibits strong interior layer across discontinuity $x = 1$ and a weak boundary layer at $x = 2$. Therefore, we construct a piecewise uniform mesh $\bar{\mathbb{D}}_x^N$ which condenses around $x = 1$ and $x = 2$. We write

$$[0, 2] = [0, 1 - \tau_1] \cup [1 - \tau_1, 1] \cup [1, 1 + \tau_2] \cup [1 + \tau_2, 2 - \tau_2] \cup [2 - \tau_2, 2],$$

where $\tau_1 = \min\left\{0.5, \dfrac{2\varepsilon \ln N}{\gamma}\right\}$ and $\tau_2 = \min\left\{0.25, \dfrac{2\varepsilon \ln N}{\gamma}\right\}$ are the mesh transition parameters. We place $N/4$ mesh points each in intervals $[0, 1-\tau_1]$, $[1-\tau_1, 1]$, $[1+\tau_2, 2-\tau_2]$ and $N/8$ mesh points in intervals $[1, 1+\tau_2]$ and $[2-\tau_2, 2]$. Consequently, we obtain

$$\bar{\mathbb{D}}_x^N = \{x_i\}_0^N = \begin{cases} x_i = 0 & \text{for } i = 0, \\ x_i = x_{i-1} + h_i & \text{for } i = 1, \ldots, N, \end{cases}$$

where

$$h_i = \begin{cases} \frac{4}{N}(1-\tau_1) & \text{for } i = 1, \ldots, N/4, \\ \frac{4}{N}\tau_1 & \text{for } i = N/4+1, \ldots, N/2, \\ \frac{8}{N}\tau_2 & \text{for } i = N/2+1, \ldots, 5N/8, \\ \frac{4}{N}(1-2\tau_2) & \text{for } i = 5N/8+1, \ldots, 7N/8, \\ \frac{8}{N}\tau_2 & \text{for } i = 7N/8+1, \ldots, N. \end{cases}$$

To discretize the differential operator in (2.8), we first define the finite difference operators on the piecewise uniform mesh $\bar{\mathbb{D}}_x^N$ as

$$D_x^+ U_{i,k+1} = \frac{U_{i+1,k+1} - U_{i,k+1}}{h_{i+1}}, \quad D_x^- U_{i,k+1} = \frac{U_{i,k+1} - U_{i-1,k+1}}{h_i},$$
$$D_x^+ D_x^- U_{i,k+1} = \frac{2(D_x^+ U_{i,k+1} - D_x^- U_{i,k+1})}{h_i + h_{i+1}}.$$

The discrete problem on $\bar{D}^{N,M} = \bar{\mathbb{D}}_x^N \times \mathbb{T}_t^M$ thus reads

$$\mathfrak{L}_\varepsilon^{N,M} U_{i,k+1} = \mathcal{F}_{i,k+1}, \tag{2.15}$$

where

$$\mathfrak{L}_\varepsilon^{N,M} U(x_i, t_{k+1}) = \begin{cases} \varepsilon \Delta t D_x^+ D_x^- U_{i,k+1} + a_i \Delta t D_x^- U_{i,k+1} - \hat{c}_i U_{i,k+1} \;\; \text{for } i = 1, \ldots, N/2-1, \\ \varepsilon \Delta t D_x^+ D_x^- U_{i,k+1} + a_i \Delta t D_x^+ U_{i,k+1} - b_i \Delta t U_{i-N/2,k+1} - \hat{c}_i U_{i,k+1} \\ \text{for } i = N/2+1, \ldots, N-1, \end{cases}$$

and

$$\mathcal{F}_{i,k+1} = \begin{cases} \Delta t f_{i,k+1} - U_{i,k} + \Delta t b_i p_1(x_{i-N/2}, t_{k+1}), & \text{for } i = 1, \ldots, N/2-1, \\ \Delta t f_{i,k+1} - U_{i,k}, & \text{for } i = N/2+1, \ldots, N-1. \end{cases}$$

Moreover, for $i = N/2$

$$D_x^+ U_{N/2,k+1} = D_x^- U_{N/2,k+1}$$

with

$$\begin{cases} U_{i,0} = p_0(x_i) & \text{for } i = 0, \ldots, N, \\ U_{N,k+1} = p_2(t_{k+1}) & \text{for } k = 0, \ldots, M-1, \\ U_{i,k+1} = p_1(x_i, t_{k+1}) & \text{for } k = 0, \ldots, M-1 \text{ and } i = -N/2, \ldots 0. \end{cases}$$

Lemma 2.4.1. *Let $\phi_{i,k+1}$ be a mesh function so that $\phi_{i,k+1} \geq 0$ for $i = \{0, N\}$, $\mathfrak{L}_\varepsilon^{N,M}\phi_{i,k+1} \leq 0$ for all $i = 1, \ldots, N/2-1, N/2+1, \ldots, N$ and $D_x^+\phi_{N/2,k+1} - D_x^-\phi_{N/2,k+1} \leq 0$. Then $\phi_{i,k+1} \geq 0$ for all $i = 0, 1, \ldots, N$.*

Proof. Choose $i^* \in \{0, 1, \ldots, N\}$ such that $\phi_{i^*,k+1} = \min_{\bar{\mathbb{D}}_x^N \times \mathbb{T}_t{}^M} \phi_{i,k+1}$. Assume that $\phi_{i^*,k+1} < 0$ and it follows that $i^* \notin \{0, N\}$. For $i^* \in \{1, 2, \ldots, N/2-1\}$

$$
\begin{aligned}
\mathfrak{L}_\varepsilon^{N,M}\phi_{i^*,k+1} &= \varepsilon\Delta t D_x^+D_x^-\phi_{i^*,k+1} + a_i\Delta t D_x^-\phi_{i^*,k+1} - \hat{c}_i\phi_{i^*,k+1} \\
&= \frac{2\varepsilon\Delta t}{\hat{h}_i}\left\{\frac{\phi_{i^*+1,k+1} - \phi_{i^*,k+1}}{h_{i+1}} - \frac{\phi_{i^*,k+1} - \phi_{i^*-1,k+1}}{h_i}\right\} \\
&\quad + a_i\Delta t\left\{\frac{\phi_{i^*,k+1} - \phi_{i^*-1,k+1}}{h_i}\right\} - \hat{c}_i\phi_{i^*,k+1} \\
&> 0.
\end{aligned}
$$

Also, $\mathfrak{L}_\varepsilon^{N,M}\phi_{i^*,k+1} > 0$ for $i^* \in \{N/2+1, \ldots, N-1\}$ and for $i^* = N/2$

$$D_x^+\phi_{i^*,k+1} - D_x^-\phi_{i^*,k+1} > 0 \text{ since } \phi_{i^*,k+1} < 0.$$

A contradiction to the assumption and the required result follows. □

As an immediate consequence of Lemma 2.4.1, we obtain the following estimate.

Lemma 2.4.2. *Let $\phi_{i,k+1}$ be the numerical solution of* (2.15). *Then*

$$\|\phi_{i,k+1}\|_{\infty,\bar{D}^{N,M}} \leq \max\left\{|\phi_{0,k+1}|, |\phi_{N,k+1}|, \|\mathfrak{L}_\varepsilon^{N,M}\phi\|_{\infty,\bar{D}^{N,M}}\right\}, \quad \forall \;\; 0 \leq i \leq N, \quad 0 \leq k \leq M-1.$$

Proof. Let $\psi_{i,k+1}^{\pm} = \max\left\{|\phi_{0,k+1}|, |\phi_{N,k+1}|, \|\mathfrak{L}_\varepsilon^{N,M}\phi\|_{\infty,\bar{D}^{N,M}}\right\} \pm \phi_{i,k+1}$. Clearly, $\psi_{i,k+1}^{\pm} \geq 0$ for $i = 0, N$ and $\mathfrak{L}_\varepsilon^{N,M}\psi_{i,k+1}^{\pm} \leq 0$ for $i \in \{1, \ldots, N-1\}\backslash\{N/2\}$. Moreover, $(D_x^+ - D_x^-)\psi_{i,k+1}^{\pm} = 0$ for $i = N/2$. The required result thus follows from Lemma 2.4.1. □

2.5 Error Estimates

We decompose the solution $U_{i,k+1}$ into smooth and singular components to obtain parameter uniform error bounds. We write $U_{i,k+1} := V_{i,k+1} + W_{i,k+1}$, where V and W satisfy

$$
\begin{cases}
\mathfrak{L}_\varepsilon^{N,M}V_{i,k+1} = \mathcal{F}_{i,k+1} \;\; \text{for} \;\; i \in \{1, \ldots, N-1\} \setminus \{N/2\}, \\
V_{0,k+1} = V(0, t_{k+1}), \\
V_{N/2-1,k+1} = V(1^-, t_{k+1}), \\
V_{N/2+1,k+1} = V(1^+, t_{k+1}), \\
V_{N,k+1} = V(2, t_{k+1}),
\end{cases}
\tag{2.16}
$$

and

$$\begin{cases} \mathfrak{L}_\varepsilon^{N,M} W_{i,k+1} = 0 \ \text{ for } \ i \in \{1,\dots,N-1\} \setminus \{N/2\}, \\ W_{0,k+1} = W(0,t_{k+1}), \\ W_{N,k+1} = W(2,t_{k+1}), \\ V_{N/2+1,k+1} + W_{N/2+1,k+1} = V_{N/2-1,k+1} + W_{N/2-1,k+1}, \\ D_x^- V_{N/2,k+1} + D_x^- W_{N/2,k+1} = D_x^+ V_{N/2,k+1} + D_x^+ W_{N/2,k+1}. \end{cases} \tag{2.17}$$

Moreover, the error $e_{i,k+1}$ is defined as

$$\begin{aligned} e_{i,k+1} &:= U(x_i,t_{k+1}) - U_{i,k+1} \\ &= (V(x_i,t_{k+1}) - V_{i,k+1}) + (W(x_i,t_{k+1}) - W_{i,k+1}). \end{aligned}$$

Theorem 2.5.1. *Let $V(x_i,t_{k+1})$ and $V_{i,k+1}$ be the solution of* (2.13) *and* (2.16)*, respectively. Then*

$$|\mathfrak{L}_\varepsilon^{N,M}(V(x_i,t_{k+1}) - V_{i,k+1})| \le CN^{-1} \ \textit{for} \ \ i \in \{0,\dots,N\} \setminus \{N/2\}.$$

Moreover, if $W(x_i,t_{k+1})$ and $W_{i,k+1}$ be the solution of (2.14) *and* (2.17)*, respectively. Then*

$$|\mathfrak{L}_\varepsilon^{N,M}(W(x_i,t_{k+1}) - W_{i,k+1})| \le CN^{-1}(\ln N)^2 \ \textit{for} \ \ i \in \{0,\dots,N\} \setminus \{N/2\}.$$

Proof. The proof follows from [189, 139]. □

Next, we obtain some priori estimates at $(x_{N/2}, t_{k+1})$. To that end, we consider

$$\begin{aligned} |(D_x^+ - D_x^-)e_{N/2,k+1}| &= \left|(D_x^+ - D_x^-)\left(U(x_{N/2},t_{k+1}) - U_{N/2,k+1}\right)\right| \\ &= \left|(D_x^+ - D_x^-)\left(U(x_{\frac{N}{2}},t_{k+1})\right)\right| \\ &\le \left|\left(D_x^+ - \frac{d}{dx}\right)U(x_{\frac{N}{2}},t_{k+1})\right| + \left|\left(D_x^- - \frac{d}{dx}\right)U(x_{\frac{N}{2}},t_{k+1})\right| \\ &\le \frac{1}{2}h_{\frac{N}{2}}^+ \max_{x_i\in(1,2)}\left|\frac{d^2U(x_i,t_{k+1})}{dx^2}\right| + \frac{1}{2}h_{\frac{N}{2}}^- \max_{x_i\in(0,1)}\left|\frac{d^2U(x_i,t_{k+1})}{dx^2}\right| \\ &\le Ch^* \max_{x_i\in(0,1)\cup(1,2)}\left|\frac{d^2U(x_i,t_{k+1})}{dx^2}\right| \\ &\le \frac{Ch^*}{\varepsilon^2} \end{aligned} \tag{2.18}$$

since $(D_x^+ - D_x^-)U_{N/2,k+1} = 0$ and $h^* = h_{\frac{N}{2}}^- = h_{\frac{N}{2}}^+$.

If we consider

$$\phi_{i,k+1} = \begin{cases} \dfrac{\prod_{k=1}^{i}\left(1+\frac{\gamma}{\varepsilon}h_k\right)}{\prod_{k=1}^{N/2}\left(1+\frac{\gamma}{\varepsilon}h_k\right)} & \text{for } 0 \le i \le N/2, \\ \dfrac{\prod_{k=i}^{N-1}\left(1+\frac{\gamma}{\varepsilon}h_{k+1}\right)}{\prod_{k=N/2}^{N-1}\left(1+\frac{\gamma}{\varepsilon}h_{k+1}\right)} & \text{for } N/2 \le i \le N, \end{cases}$$

then $0 \le \phi_{i,k+1} \le 1$ for $0 \le i \le N$. Moreover, for $i = 0, 1, \dots, N/2$

$$D_x^+\phi_{i,k+1} = \frac{\gamma}{\varepsilon}\phi_{i,k+1},$$

$$D_x^-\phi_{i,k+1} = \frac{1}{h_i}\frac{\prod_{k=1}^{i}\left(1+\frac{\gamma}{\varepsilon}h_k\right)}{\prod_{k=1}^{N/2}\left(1+\frac{\gamma}{\varepsilon}h_k\right)}\left(1-\frac{1}{1+\frac{\gamma}{\varepsilon}h_i}\right) = \frac{\gamma}{\varepsilon}\frac{1}{\left(1+\frac{\gamma}{\varepsilon}h_i\right)}\phi_{i,k+1},$$

and

$$\begin{aligned} \delta^2\phi_{i,k+1} &= \frac{2}{h_i+h_{i+1}}(D_x^+ - D_x^-)\phi_{i,k+1} \\ &= \left(\frac{\gamma}{\varepsilon}\right)^2\frac{2h_i}{h_i+h_{i+1}}\left(\frac{1}{1+\frac{\gamma}{\varepsilon}h_i}\right)\phi_{i,k+1} \\ &\le 2\left(\frac{\gamma}{\varepsilon}\right)^2\phi_{i,k+1}. \end{aligned}$$

Also, for $i = N/2, \dots, N$

$$D_x^+\phi_{i,k+1} = \frac{-\gamma}{\varepsilon}\left(\frac{\phi_{i,k+1}}{1+\frac{\gamma}{\varepsilon}h_{i+1}}\right), \quad D_x^-\phi_{i,k+1} = \frac{-\gamma}{\varepsilon}\phi_{i,k+1}, \quad \delta^2\phi_{i,k+1} \le 2\left(\frac{\gamma}{\varepsilon}\right)^2\phi_{i,k+1}$$

and at $i = \frac{N}{2}$

$$\begin{aligned} (D_x^+ - D_x^-)\phi_{i,k+1} &= -\frac{\gamma}{\varepsilon}\left(\frac{1}{1+\frac{\gamma}{\varepsilon}h^+_{N/2}} + \frac{1}{1+\frac{\gamma}{\varepsilon}h^-_{N/2}}\right)\phi_{N/2,k+1} \\ &= \frac{-\gamma}{\varepsilon}\left(\frac{2}{1+\frac{\gamma}{\varepsilon}h^*_{N/2}}\right)\phi_{N/2,k+1} \\ &\le -\frac{C}{\varepsilon}. \end{aligned}$$

Then, for $0 \le i \le N/2 - 1$

$$\begin{aligned} \mathfrak{L}_\varepsilon^{N,M}\phi_{i,k+1} &= \varepsilon\Delta t\delta^2\phi_{i,k+1} + a_i\Delta t D_x^-\phi_{i,k+1} - \hat{c}_i\phi_{i,k+1} \\ &\le \left(\frac{2\Delta t\gamma^2}{\varepsilon} + \frac{\gamma}{\varepsilon}\frac{a_i\Delta t}{(1+\frac{\gamma}{\varepsilon}h_i)} - \hat{c}_i\right)\phi_{i,k+1}. \end{aligned} \tag{2.19}$$

Similarly, for $N/2+1 \le i \le N$

$$\begin{aligned}\mathfrak{L}_\varepsilon^{N,M}\phi_{i,k+1} &= \varepsilon\Delta t\delta^2\phi_{i,k+1} + a_i\Delta t D_x^+\phi_{i,k+1} - b_i\Delta t\phi_{i-N/2,k+1} - \hat{c}_i\phi_{i,k+1}\\ &\le \left(\frac{2\Delta t\gamma^2}{\varepsilon} - \frac{\gamma a_i\Delta t}{\varepsilon(1+\frac{\gamma}{\varepsilon}h_{i+1})} - \hat{c}_i\right)\phi_{i,k+1} - b_i\Delta t\phi_{i-N/2,k+1}.\end{aligned}$$

Theorem 2.5.2. *Let $U(x_i, t_{k+1})$ and $U_{i,k+1}$ be the solutions of* (2.6) *and* (2.15)*, respectively. Then at* $(k+1)$*th time step*

$$\left|U(x_i,t_{k+1}) - U_{i,k+1}\right| \le CN^{-1}(\ln N)^2 \ \ for\ \ 0 \le i \le N.$$

Proof. Consider the barrier function

$$\psi_{i,k+1}^{\pm} = C_1N^{-1}(\ln N)^2 + C_2\frac{\gamma}{\varepsilon}h^*\phi_{i,k+1} \pm e_{i,k+1} \ \ \text{for} \ \ i = 0,1,\ldots,N,$$

where C_1 and C_2 are constants. Using (2.19), the assumption $a_i < -2\gamma$ and Theorem 2.5.1 to obtain

$$\begin{aligned}\mathfrak{L}_\varepsilon^{N,M}\psi_{i,k+1}^{\pm} &= -C_1\hat{c}_iN^{-1}(\ln N)^2 + C_2\frac{\gamma}{\varepsilon}h^*\left(\frac{2\Delta t\gamma^2}{\varepsilon} + a_i\Delta t\frac{\gamma}{\varepsilon} - \hat{c}_i\right) \pm \mathfrak{L}_\varepsilon^{N,M}e_{i,k+1}\\ &\le -C_1\hat{c}_iN^{-1}(\ln N)^2 + C_2\left(\frac{\gamma}{\varepsilon}\right)^2 h^*\Delta t\left(2\gamma + a_i - \hat{c}_i\frac{\varepsilon}{\gamma}\Delta t\right) \pm CN^{-1}(\ln N)^2\\ &\le 0\end{aligned}$$

for $i = 1,\ldots,N/2-1$. Similarly, for $i = N/2+1,\ldots,N-1$ we obtain

$$\begin{aligned}\mathfrak{L}_\varepsilon^{N,M}\psi_{i,k+1}^{\pm} &= C_1(-\hat{c}_i - b_i\Delta t)N^{-1}(\ln N)^2 + C_2\left(\frac{\gamma}{\varepsilon}\right)h^*\mathfrak{L}_\varepsilon^{N,M}\phi_{i,k+1} \pm \mathfrak{L}_\varepsilon^{N,M}e_{i,k+1}\\ &= C_1(-\hat{c}_i - b_i\Delta t)N^{-1}(\ln N)^2 + C_2\left(\frac{\gamma}{\varepsilon}\right)h^*\frac{\gamma}{\varepsilon}\Delta t\left(2\gamma - a_i - \hat{c}_i\frac{\varepsilon\Delta t}{\gamma}\right)\\ &\quad -C_2\left(\frac{\gamma}{\varepsilon}\right)h^*b_i\Delta t\phi_{i-N/2,k+1} \pm CN^{-1}(\ln N)^2\\ &\le 0.\end{aligned}$$

Also, if $i = N/2$

$$\begin{aligned}(D_x^+ - D_x^-)\psi_{N/2,k+1}^{\pm} &= (D_x^+ - D_x^-)\left(C_1N^{-1}(\ln N)^2 + C_2\left(\frac{\gamma}{\varepsilon}\right)h^*\phi_{N/2,k+1} \pm e_{N/2,k+1}\right)\\ &\le C_2\left(\frac{\gamma}{\varepsilon}\right)h^*\left(\frac{-C}{\varepsilon}\right) \pm \frac{Ch^*}{\varepsilon^2}\\ &\le 0\end{aligned}$$

for some suitable constant C_2. Moreover, $\psi_{0,k+1}^{\pm} \ge 0$, $\psi_{N,k+1}^{\pm} \ge 0$ and $0 \le \phi_{i,k+1} \le 1$. Consequently, the required result follows from Lemma 2.4.1. □

Next, we combine (2.12) and Theorem 2.5.2 to obtain the main result.

Theorem 2.5.3. *Let u and $U_{i,k+1}$ be the solutions of* (2.2) *and* (2.15)*, respectively. Then*

$$\left|u(x_i,t_{k+1}) - U_{i,k+1}\right| \le C(\Delta t + (N^{-1}(\ln N)^2))$$

for $0 \le i \le N$ *and* $0 \le k \le M$.

2.6 Numerical Illustration

In this section, we examine the performance of the proposed method and numerically verify the theoretical estimates. We consider two test problems for numerical computations.

Example 2.6.1. *Consider the following singularly perturbed problem [139]:*

$$\begin{cases} \varepsilon u_{xx}(x,t) + a(x)u_x(x,t) + u(x-1,t) - 3u(x,t) - u_t(x,t) = f(x,t), & (x,t) \in (0,2) \times (0,2], \\ u(x,0) = 0, \quad x \in [0,2], \\ u(x,t) = 0, \quad (x,t) \in [-1,0] \times [0,2], \\ u(2,t) = 0, \quad t \in [0,2], \end{cases}$$

where $a(x) = \begin{cases} -(4+x), & x \in [0,1], \\ (3+x^2), & x \in (1,2], \end{cases}$ *and* $f(x,t) = \begin{cases} -1, & (x,t) \in [0,1] \times [0,2], \\ 1, & (x,t) \in (1,2] \times [0,2]. \end{cases}$

Example 2.6.2. *Consider the following singularly perturbed problem:*

$$\begin{cases} \varepsilon u_{xx}(x,t) + a(x)u_x(x,t) + 2u(x-1,t) - 5u(x,t) - u_t(x,t) = f(x,t), & (x,t) \in (0,2) \times (0,2], \\ u(x,0) = 0, \quad x \in [0,2], \\ u(x,t) = 0, \quad (x,t) \in [-1,0] \times [0,2], \\ u(2,t) = 0, \quad t \in [0,2], \end{cases}$$

where

$a(x) = \begin{cases} -(4+x^2), & x \in [0,1], \\ (8-x^2), & x \in (1,2], \end{cases}$ *and* $f(x,t) = \begin{cases} 4xt^2e^{-t}, & (x,t) \in [0,1] \times [0,2], \\ 4(2-x)t^2e^{-t}, & (x,t) \in (1,2] \times [0,2]. \end{cases}$

The exact solution of the problem is not available for comparison. Therefore, we estimate the error using the double mesh principle [173]. The maximum absolute error ($E_\varepsilon^{N,\triangle t}$) and order of convergence ($R_\varepsilon^{N,\triangle t}$) are calculated using

$$E_\varepsilon^{N,\triangle t} := \max\left|U^{N,\triangle t}(x_i,t_{k+1}) - \tilde{U}^{2N,\triangle t/2}(x_i,t_{k+1})\right| \text{ and } R_\varepsilon^{N,\triangle t} := \log_2\left(\frac{E_\varepsilon^{N,\triangle t}}{E_\varepsilon^{2N,\triangle t/2}}\right),$$

where $U^{N,\triangle t}(x_i,t_{k+1})$ and $\tilde{U}^{2N,\triangle t/2}(x_i,t_{k+1})$ are the numerical solutions obtained on $\bar{\mathbb{D}}_x^N \times \mathbb{T}_t{}^M$ and $\bar{\mathbb{D}}_x^{2N} \times \mathbb{T}_t{}^{2M}$, respectively.

In case, perturbation parameter tends to zero, the solution of the problem exhibit turning point behaviour (Figures 2.1-2.4). Maximum absolute error and order of convergence for Example 2.6.1 and Example 2.6.2 are tabulated for different values of ε, M, and N in Tables 2.1-2.4. Moreover, log-log plots of the maximum absolute error can be had from Figure 2.5 and Figure 2.6. It is evident from it that the errors decreases monotonically as N increases. Numerical solution for Example 2.6.1 and Example 2.6.2 are plotted in Figure 2.1 and Figure 2.3, respectively. Also, the numerical solutions at final time step ($t = 2$) for different values of ε are displayed in Figure 2.2 and Figure 2.4.

Table 2.1: Maximum absolute error and order of convergence for Example 2.6.1 for different values of ε, M, and N when $M = N$.

N	$\varepsilon = 2^{-0}$	2^{-1}	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}
32	0.007388	0.009292	0.013393	0.017789	0.019344	0.019860	0.019962
	0.9895	0.8712	0.8571	0.8718	0.8797	0.8639	0.8490
64	0.003721	0.005080	0.007394	0.009721	0.010513	0.010912	0.011082
	0.9834	0.9274	0.9196	0.9107	0.9253	0.8718	0.8571
128	0.001882	0.002671	0.003909	0.005171	0.005536	0.005963	0.006118
	0.9878	0.9423	0.9418	0.9263	0.9106	0.8876	0.8650
256	0.000949	0.001390	0.002035	0.002721	0.002945	0.003223	0.003359
	0.9924	0.9571	0.9588	0.9332	0.9260	0.9026	0.8809
512	0.000477	0.000716	0.001047	0.001425	0.001550	0.001724	0.001824
	0.9970	0.9721	0.9713	0.9649	0.9488	0.9346	0.9036
1024	0.000239	0.000365	0.000534	0.000730	0.000803	0.000902	0.000975
	0.9984	0.9938	0.9928	0.9931	0.9897	0.9743	0.9339

Table 2.2: Maximum absolute error and order of convergence for Example 2.6.1 for different values of ε, M, and N when $M = 2N$.

N	$\varepsilon = 2^{-0}$	2^{-1}	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}
32	0.004733	0.007924	0.012284	0.016524	0.018423	0.019067	0.019182
	0.9713	0.9124	0.9063	0.8804	0.8825	0.8848	0.8796
64	0.002414	0.004210	0.006554	0.008976	0.009993	0.010326	0.010426
	0.9857	0.9481	0.9329	0.9622	0.8555	0.8931	0.8890
128	0.001219	0.002182	0.003433	0.004607	0.005523	0.005560	0.005630
	0.9917	0.9660	0.9508	0.9246	0.9811	0.9129	0.9115
256	0.000613	0.001117	0.001776	0.002427	0.002798	0.002953	0.002993
	0.9976	0.9731	0.9584	0.9515	0.9444	0.9364	0.9336
512	0.000307	0.000569	0.000914	0.001255	0.001454	0.001543	0.001567
	0.9953	0.9874	0.9781	0.9783	0.9705	0.9567	0.9479
1024	0.000154	0.000287	0.000464	0.000637	0.000742	0.000795	0.0008123
	0.9967	0.9905	0.9893	0.9884	0.9815	0.9781	0.9721

Table 2.3: Maximum absolute error and order of convergence for Example 2.6.2 for different values of ε, M, and N when $M = N$.

N	$\varepsilon = 2^{-0}$	2^{-1}	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}
32	0.003389	0.003688	0.003854	0.003972	0.005890	0.005892	0.005894
	0.8167	0.9260	0.9037	0.9017	0.8897	0.8825	0.8749
64	0.001924	0.001941	0.002060	0.002126	0.003179	0.003196	0.003214
	0.9527	0.9184	0.9038	0.8953	0.8853	0.8805	0.8863
128	0.000994	0.001027	0.001101	0.001143	0.001721	0.001736	0.001746
	0.9487	0.9274	0.9197	0.9130	0.8957	0.8881	0.8841
256	0.000515	0.000540	0.000582	0.000607	0.000925	0.000938	0.000946
	0.9586	0.9475	0.9275	0.9236	0.9049	0.9048	0.9027
512	0.000265	0.000280	0.000306	0.000320	0.000494	0.000501	0.000506
	0.9730	0.9694	0.9445	0.9469	0.9250	0.9297	0.9277
1024	0.000135	0.000143	0.000159	0.000166	0.000260	0.000263	0.000266
	0.9843	0.9795	0.9674	0.9672	0.9512	0.9459	0.9417

Table 2.4: Maximum absolute error and order of convergence for Example 2.6.2 for different values of ε, M, and N when $M = 2N$.

N	$\varepsilon = 2^{-0}$	2^{-1}	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}
32	0.002107	0.002316	0.002723	0.003246	0.003356	0.003378	0.003410
	0.9429	0.9271	0.9191	0.9112	0.9012	0.8859	0.8799
64	0.001096	0.001218	0.001440	0.001726	0.001797	0.001828	0.001853
	0.9713	0.9488	0.9353	0.9330	0.9196	0.9038	0.9029
128	0.000559	0.000631	0.000753	0.000904	0.000950	0.000977	0.000991
	0.9871	0.9661	0.9491	0.9375	0.9289	0.9182	0.9110
256	0.000282	0.000323	0.000390	0.000472	0.000499	0.000517	0.000527
	1.0000	0.9778	0.9635	0.9460	0.9405	0.9319	0.9175
512	0.000141	0.000164	0.000200	0.000245	0.000260	0.000271	0.000279
	1.0102	1.0000	0.9856	0.9708	0.9563	0.9529	0.9442
1024	0.000070	0.000082	0.000101	0.000125	0.000134	0.000140	0.000145
	1.0104	1.0010	0.9921	0.9914	0.9784	0.9749	0.9668

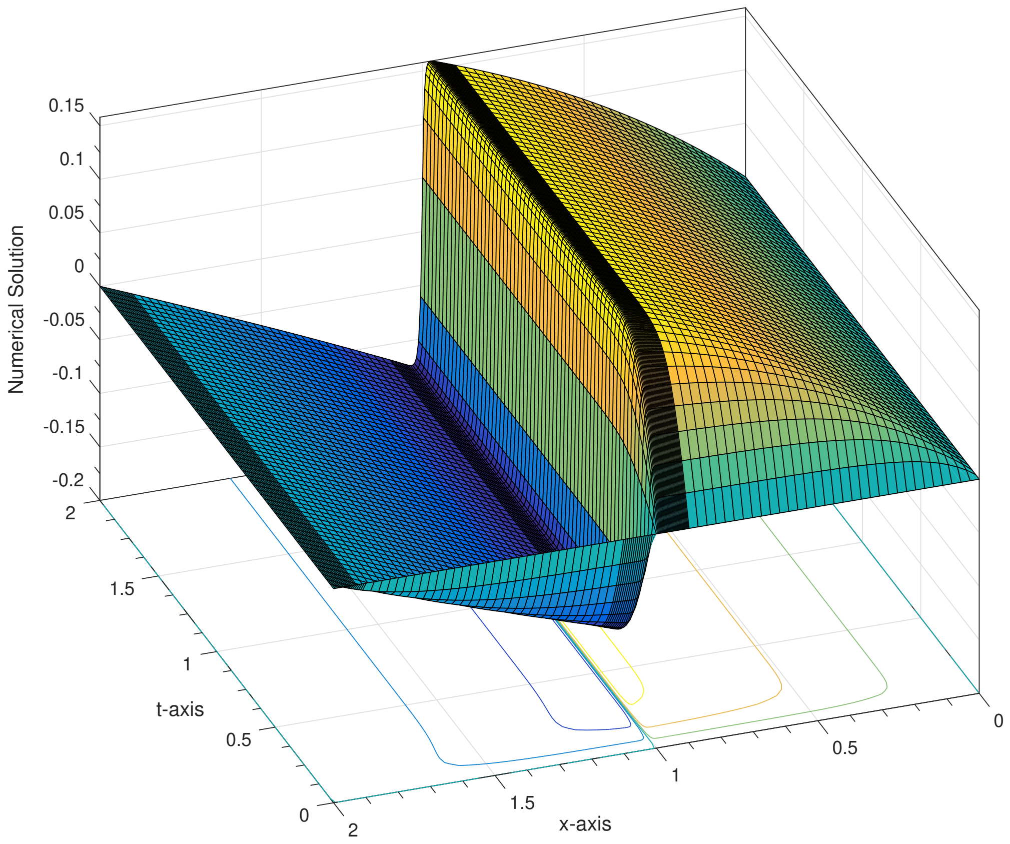

Figure 2.1: Numerical solution of Example 2.6.1 for $\varepsilon = 2^{-6}$ when $M = N = 64$.

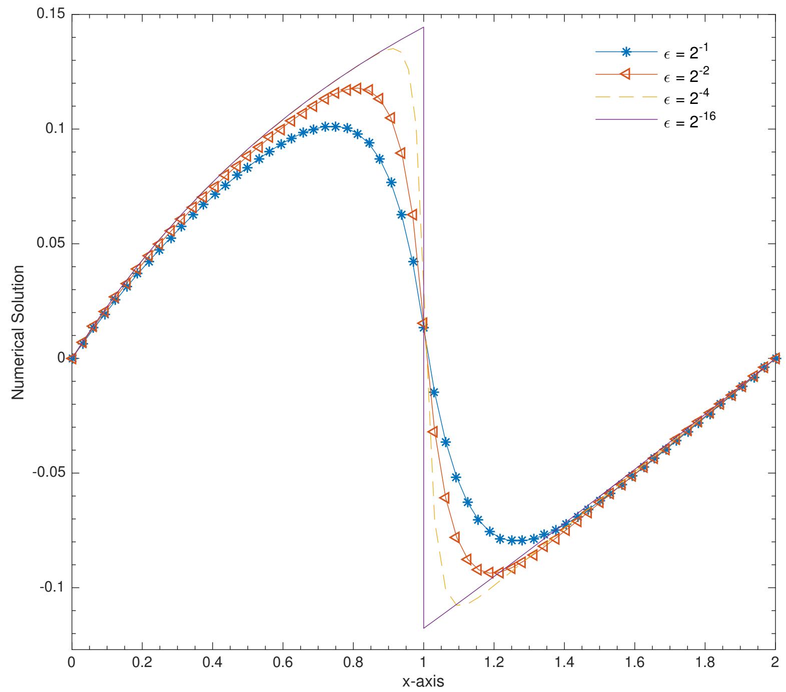

Figure 2.2: Numerical solutions of Example 2.6.1 at $t = 2$ for different values of ε when $N = 64$.

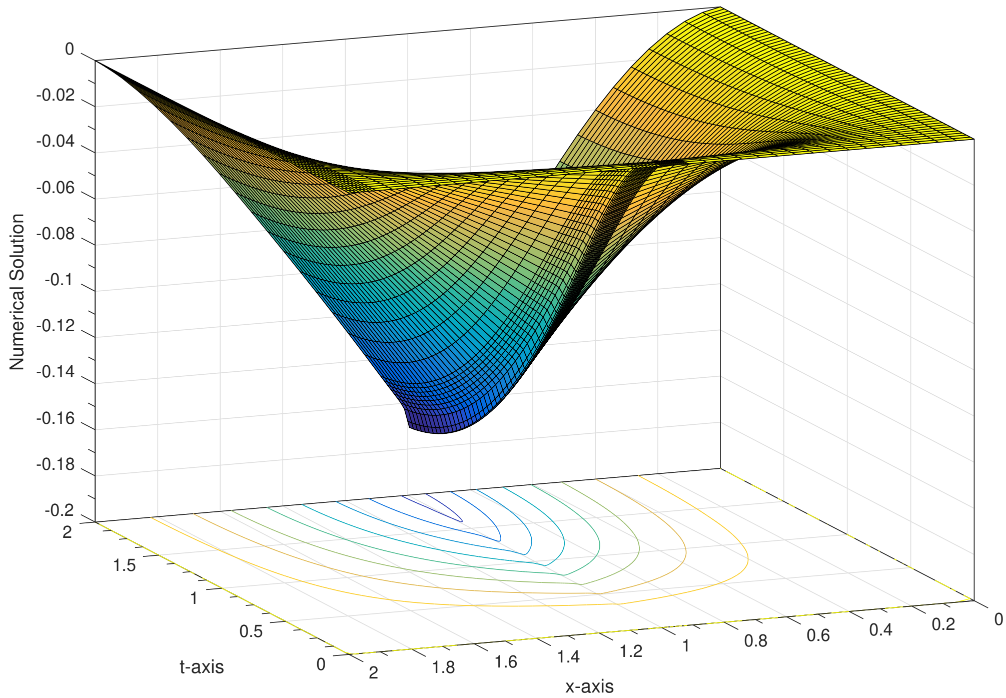

Figure 2.3: Numerical solution of Example 2.6.2 for $\varepsilon = 2^{-6}$ when $M = N = 64$.

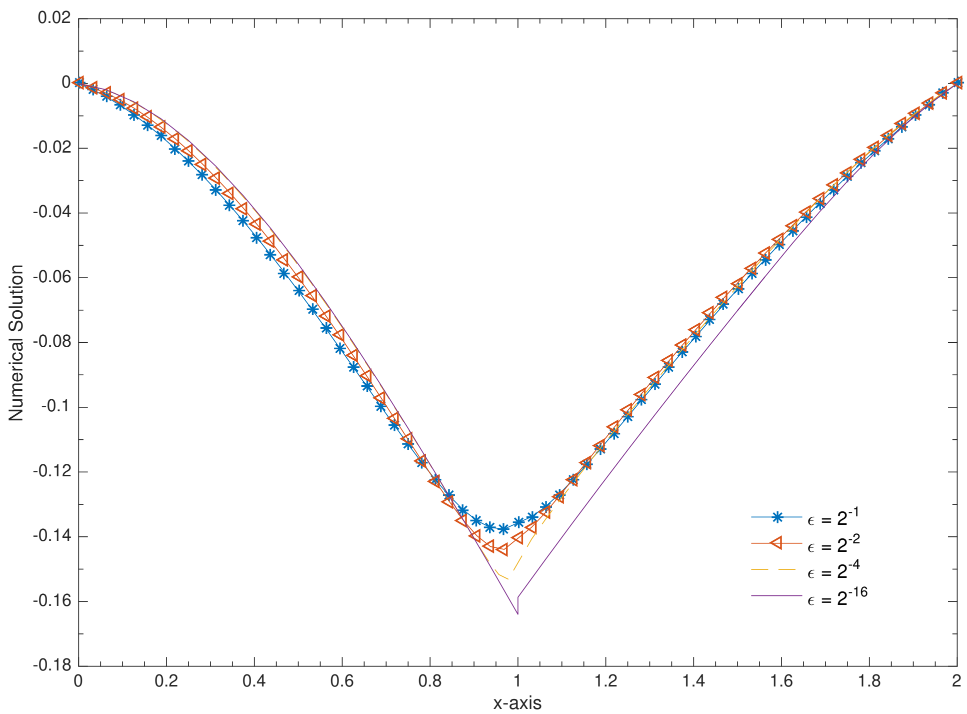

Figure 2.4: Numerical solutions of Example 2.6.2 at $t = 2$ for different values of ε when $N = 64$.

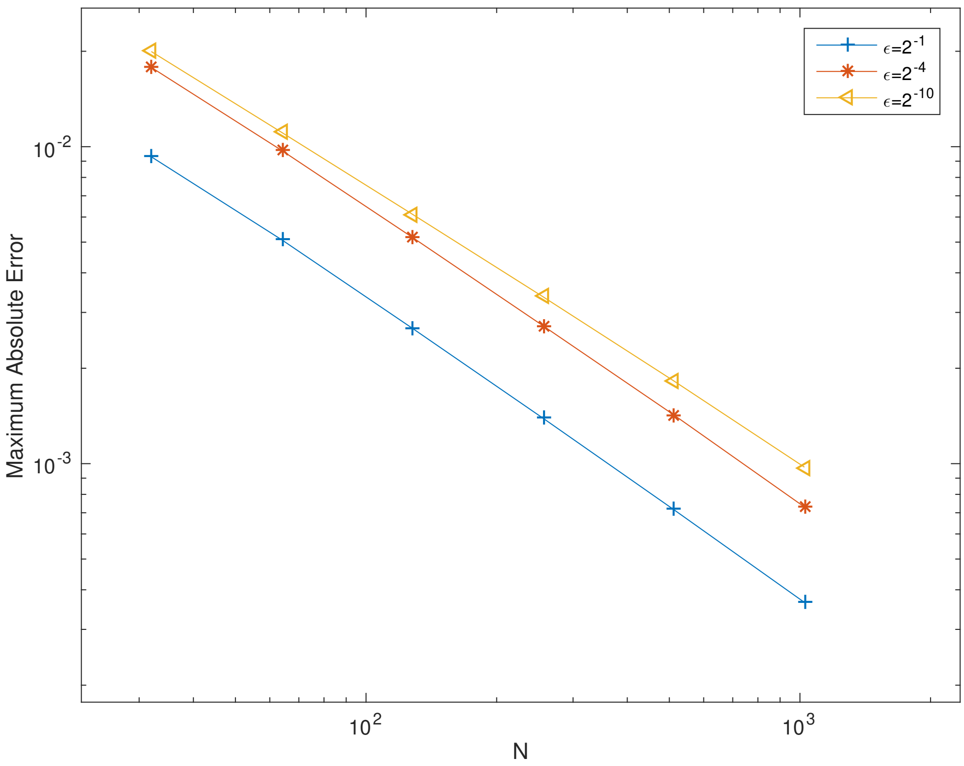

Figure 2.5: Error plot for Example 2.6.1.

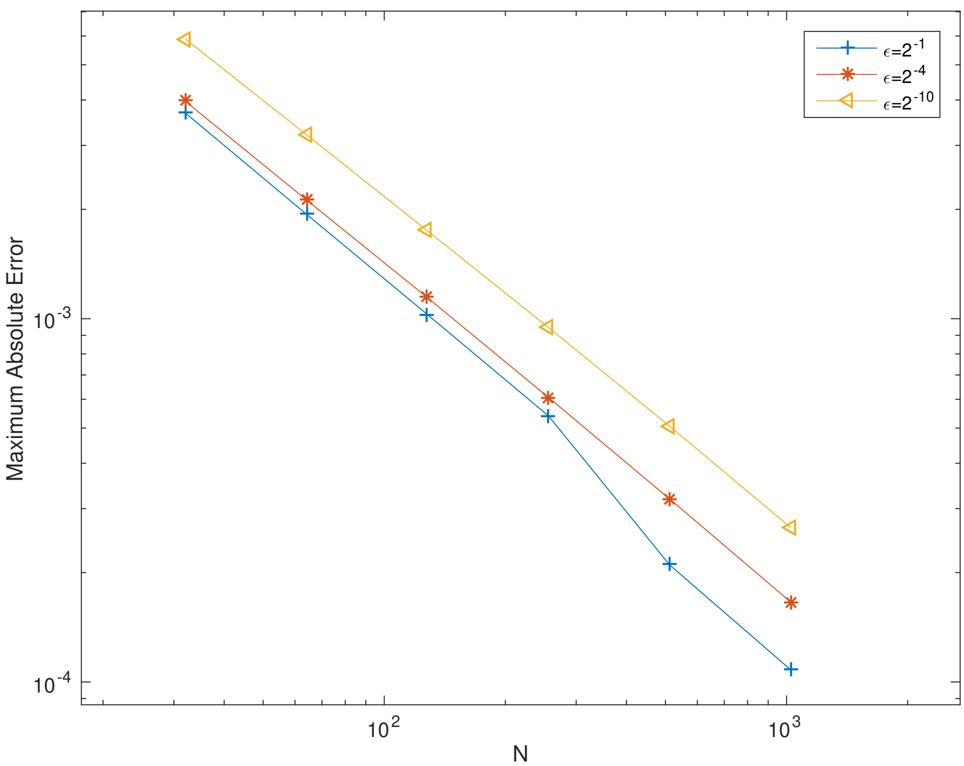

Figure 2.6: Error plot for Example 2.6.2.

2.7 Concluding Remarks

A class of singularly perturbed parabolic partial differential equations with discontinuous coefficients and source terms is solved numerically. The solution of the problem takes into account not just the present state of the physical system but also its history. The simultaneous presence of discontinuous data and delay makes the problem stiff. In the limiting case, the solution of the problem exhibits a multiscale character [214, 249]. There are narrow regions where solution derivatives are exponential and exhibit turning point behaviour across discontinuities besides weak boundary layers.

An implicit numerical scheme based on the upwind finite difference method is presented on a specially generated mesh. The mesh has been chosen such that most of the mesh points remain in the regions with rapid transitions. The proposed numerical method has been analysed for consistency, stability and convergence. Theoretical analysis is performed to obtain consistency and error estimates. It is found that the method proposed is unconditionally stable, and the convergence obtained is parameter uniform. Numerical examples have been presented to demonstrate the effectiveness of the technique. Convergence obtained in practical satisfies theoretical predictions. The method presented is easy to implement and, with a little modification, can easily be extended to even more general situations like problems in higher dimensions and nonlinear evolution equations.

Chapter 3

Parabolic Problems with Discontinuous Coefficient and Delay-A Hybrid Method

3.1 Introduction

In Chapter 2, we proposed a numerical method for solving singularly perturbed parabolic PDEs with a delay, discontinuous coefficient and source subject to the Dirichlet boundary conditions. This Chapter extends our scope of work to study a higher-order hybrid method for such problems.

Hybrid methods leverage the strengths of different numerical techniques to achieve high accuracy and efficiency in approximating the solutions. In recent years, the development of higher-order methods has shown promising results in efficiently and accurately solving singularly perturbed parabolic PDEs with discontinuous coefficients and delay. Combining higher-order methods with other specialized techniques, such as domain decomposition or adaptive mesh refinement, can accurately capture the multiscale character and discontinuities in the solution, allowing for a more faithful representation of the underlying physical phenomena. Additionally, the improved accuracy often leads to significant computational savings compared to first-order methods, as fewer grid points are required to achieve the desired level of accuracy. The significance of hybrid numerical methods lies in their ability to overcome the limitations of individual methods and provide tailored approaches that adapt to the specific characteristics of the problem at hand.

Researchers have developed various hybrid methods for solving singular perturbation problems. In [59], the authors report a hybrid scheme for a time-delayed convection-diffusion problem. The presented method combines an upwind scheme and a central difference scheme, achieving nearly second-order accuracy in space, which is considered optimal compared to the approach in [99]. Furthermore, researchers have investigated a hybrid difference scheme for parabolic partial differential equations with delay and advance terms in [238, 108]. Moreover, authors in [252] use a hybrid scheme for a two-parameter singularly perturbed parabolic problem with a delay. In [233], the authors presented an efficient and higher-order numerical method to solve a singularly perturbed time-delayed parabolic convection-diffusion problem. The hybrid scheme combines the central difference scheme in the layer region and the mid-point upwind scheme in the outer region defined on a Shishkin mesh for discretization in the spatial direction. The Peaceman-Rachford splitting algorithm splits each time step into two partial time steps. This splitting is performed based on the concept of the Crank-Nicholson scheme.

In [251], the researchers consider a singularly perturbed parabolic problem with a simple interior turning point. They employ a fitted mesh finite difference scheme to approximate the numerical solution, which consists of a hybrid finite difference operator on a non-uniform Shishkin mesh in the space direction and a backward implicit Euler scheme on a uniform mesh in the time variable. To improve the accuracy of the resulting scheme in the temporal direction, they utilize the Richardson extrapolation scheme. The study reveals that the extrapolated scheme exhibits almost second-order uniform convergence in space and time variables. In [301], the authors consider a problem involving a singularly perturbed parabolic equation with multiple interior turning points and twin boundary layers. To discretize the problem, they utilize the implicit Euler method on a uniform mesh for the time variable and a hybrid scheme on a generalized Shishkin mesh for the spatial variable. Additionally, in [302], the authors employ a finite difference scheme to solve a one-dimensional singularly perturbed parabolic convection-diffusion problem with an interior turning point. This scheme combines the implicit Euler method on a uniform mesh in the time direction and a hybrid finite difference operator on a generalized Shishkin mesh in the spatial direction.

In [154], a high-order parameter-robust hybrid numerical method is used to solve a Dirichlet problem for a one-dimensional time-dependent singularly perturbed reaction-diffusion equation. To approximate the solution to the problem, the author constructs a numerical method by combining the Crank–Nicolson method on a uniform mesh in the time direction, together with a hybrid scheme which is a suitable combination of a fourth-order compact difference scheme and the standard central difference scheme on a generalized Shishkin mesh in the spatial direction. In contrast, [194] focuses on a two-dimensional singularly perturbed parabolic convection-diffusion initial-boundary value problem. They employ the Peaceman-Rachford alternating direction implicit method to approximate the time derivative on a uniform mesh. For spatial discretization, they propose a hybrid finite difference scheme on a piecewise uniform Shishkin mesh. This scheme incorporates a midpoint upwind scheme on the coarse part of the mesh and the standard central difference scheme on the fine part of the mesh. Furthermore, in [192] and [193], researchers examined the robustness of the similar hybrid scheme for a parabolic convection-diffusion problem with both smooth and non-smooth data.

The literature on the analysis of hybrid methods for singularly perturbed parabolic partial functional differential equations with discontinuous data and a large delay remains relatively limited. This chapter addresses this gap by introducing a hybrid numerical method for solving singular perturbation problems. The method is designed to handle equations with a delay, discontinuous coefficient and source term. Additionally, the chapter provides a thorough analysis of the scheme's consistency, stability, and convergence. The presented numerical results illustrate and validate the effectiveness of the proposed approach.

3.2 Continuous Problem

Let $D := (0,2) \times (0,T]$ and $S^- \cup S^+ := (0,1) \times (0,T] \cup (1,2) \times (0,T]$. Consider the non-homogenous initial-boundary-value problem

$$\left.\begin{aligned} L_\varepsilon u &= \varepsilon u_{xx}(x,t) + a(x)u_x(x,t) - b(x)u(x,t) - u_t(x,t) \\ &= f(x,t) + c(x)u(x-1,t) \text{ in } S^- \cup S^+, \\ u(x,t) &= p_0(x) \text{ on } [0,2] \times \{t=0\}, \\ u(x,t) &= p_1(x,t) \text{ in } [-1,0] \times [0,T], \\ u(x,t) &= p_2(t) \text{ on } \{x=2\} \times [0,T], \end{aligned}\right\} \tag{3.1}$$

where $\varepsilon \ll 1$ is a small positive parameter, $b(x)$ and $c(x)$ are sufficiently smooth functions such that $c(x) > 0$, $b(x) > 0$ for all $x \in [0,2]$. Moreover, we assume that

$$\left.\begin{aligned} &a(x) = \begin{cases} a_1(x) & \text{if } 0 \le x \le 1, \\ a_2(x) & \text{if } 1 < x \le 2, \end{cases} \quad f(x,t) = \begin{cases} f_1(x,t) & \text{if } (x,t) \in \overline{S}^-, \\ f_2(x,t) & \text{if } (x,t) \in \overline{S}^+, \end{cases} \\ &-\gamma_1^* < a_1(x) < -\gamma_1 < 0, \;\; \gamma_2^* > a_2(x) > \gamma_2 > 0, \;\; \|[a]\| \le C, \quad \|[f]\| \le C, \end{aligned}\right\} \tag{3.2}$$

where $\gamma = \min\{\gamma_1, \gamma_2\}$ and $\gamma^* = \max\left\{\gamma_1^*, \gamma_2^*\right\}$. The solution of (3.1) satisfies $[u] = 0$ and $[u_x] = 0$ at $x = 1$. Here, $[u]$ denotes the jump of u defined at the point of discontinuity $x = 1$ as $[u](1,t) = u(1^+,t) - u(1^-,t)$, where $u(1^\pm,t) = \lim\limits_{x \to 1 \pm 0} u(x,t)$. The functions p_0, p_1

and p_2 are Hölder continuous and satisfy the compatibility conditions

$$\left.\begin{aligned} &p_0(0,0) = p_1(0,0), \quad p_0(2,0) = p_2(2,0),\\ &\varepsilon\frac{\partial^2 p_0(0,0)}{\partial x^2} + a(0)\frac{\partial p_0(0,0)}{\partial x} - b(0)p_0(0,0) - c(0)p_1(-1,0) - \frac{\partial p_1(0,0)}{\partial t} = f(0,0),\\ &\varepsilon\frac{\partial^2 p_0(2,0)}{\partial x^2} + a(2)\frac{\partial p_0(2,0)}{\partial x} - b(2)p_0(2,0) - c(2)p_0(1,0) - \frac{\partial p_2(2,0)}{\partial t} = f(2,0). \end{aligned}\right\}$$

On the domain S^-, the delay term $u(x-1,t) = p_1(x-1,t)$. Under these assumptions, the solution of (3.1) exists uniquely and satisfies $u \in C^{1+\lambda}(D) \cap C^{2+\lambda}(S^- \cup S^+)$ [160, 248] . The solution $u(x,t)$ of (3.1) displays a strong interior layer in the neighbourhood of the point $x = 1$ [193]. Moreover, it is easy to follow that the differential operator L_ε satisfies the following maximum principle.

Lemma 3.2.1. *Suppose* $\mathcal{P} \in C^0(\bar{D}) \cap C^2(S^- \cup S^+)$ *satisfies* $\mathcal{P}(x,t) \leq 0$ *for all* $(x,t) \in \Gamma := \bar{D} \setminus D$, $[\mathcal{P}_x](1,t) \geq 0$, $t > 0$ *and* $L_\varepsilon\mathcal{P}(x,t) \geq 0$ *for all* $(x,t) \in S^- \cup S^+$. *Then* $\mathcal{P}(x,t) \leq 0$ *for all* $(x,t) \in \bar{D}$.

Proof. Choose $(x^k,t^k) \in \bar{D}$ such that $\mathcal{P}(x^k,t^k) = \max\limits_{(x,t)\in\bar{D}} \mathcal{P}(x,t)$. Consequently,

$$\mathcal{P}_x(x^k,t^k) = 0, \quad \mathcal{P}_t(x^k,t^k) = 0 \text{ and } \mathcal{P}_{xx}(x^k,t^k) < 0.$$

Suppose $\mathcal{P}(x^k,t^k) > 0$ and it follows that $(x^k,t^k) \notin \Gamma$. If $(x^k,t^k) \in S^- \cup S^+$, note that $L_\varepsilon\mathcal{P}(x^k,t^k) < 0$. Moreover, if $x^k = 1$, then $[\mathcal{P}_x](x^k,t) < 0$. A contradiction to the assumption and the required result follows. □

As an immediate application of the maximum principle, we obtain the following result.

Lemma 3.2.2. *Let u be the solution of* (3.1). *Then*

$$\|u\|_{\infty,\bar{D}} \leq \|u\|_{\infty,\Gamma} + \frac{1}{\gamma}\|f\|_{\infty,\bar{D}}, \quad \gamma = \min\{\gamma_1,\gamma_2\}. \tag{3.3}$$

Proof. Consider

$$\psi_\pm = \begin{cases} -\|u\|_{\infty,\Gamma} - \dfrac{x}{\gamma}\|f\|_{\infty,\bar{D}} \pm u & \text{if } x \leq 1,\\ -\|u\|_{\infty,\Gamma} - \dfrac{(2-x)\|f\|_{\infty,\bar{D}}}{\gamma} \pm u & \text{if } x \geq 1. \end{cases}$$

For $(x,t) \in S^-$, it follows that

$$L_\varepsilon\psi_\pm(x,t) = \pm L_\varepsilon u - a_1(x)\frac{\|f\|_{\infty,\bar{D}}}{\gamma} + b(x)\|u\|_{\infty,\Gamma} + b(x)\frac{x\|f\|_{\infty,\bar{D}}}{\gamma} \geq 0$$

since $a_1(x) < 0$ and $b(x) > 0$. Similarly, for $(x,t) \in S^+$, it is easy to verify that $L_\varepsilon\psi_\pm(x,t) \geq 0$. Also, $[\psi_x\pm](1,t) = \pm[u_x](1,t) = 0$. The required result (3.3) now follows from Lemma 3.2.1. □

To find sharper bounds on the solution and its derivatives, we decompose the solution u into smooth and singular components. We write $u := v + w$. The smooth component v satisfies

$$\left.\begin{aligned} &L_\varepsilon v(x,t) = f(x,t) + c(x)v(x-1,t) \text{ in } S^- \cup S^+,\\ &v(x,t) = u(x,t) \text{ in } [-1,0]\times[0,T],\\ &v(1^-,t) = j_1(t),\quad v(1^+,t) = j_2(t),\quad t\in(0,T],\\ &v(x,t) = u(x,t) \text{ on } \{x=2\}\times(0,T],\\ &v(x,t) = u(x,t) \text{ on } [0,2]\times\{t=0\}, \end{aligned}\right\} \tag{3.4}$$

where the functions $j_1(t)$ and $j_2(t)$ will be computed later using Theorem 3.2.4. The singular component w satisfies

$$\left.\begin{aligned} &L_\varepsilon w(x,t) = c(x)w(x-1,t) \text{ in } S^- \cup S^+,\\ &w(x,t) = 0 \text{ in } [-1,0]\times[0,T],\\ &w(x,t) = 0 \text{ on } \{x=2\}\times(0,T],\\ &w(x,t) = 0 \text{ on } [0,2]\times\{t=0\},\\ &[w](1,t) = -[v](1,t),\quad \left[\tfrac{\partial w}{\partial x}\right](1,t) = -\left[\tfrac{\partial v}{\partial x}\right](1,t),\quad t\in(0,T]. \end{aligned}\right\} \tag{3.5}$$

The following theorem is the direct result from [121] that we will use in Theorem 3.2.4.

Theorem 3.2.3. *Let $a, b, c \in C^2[0,2]$, $f \in C^{2+\lambda}(\bar{D})$ and $a(x) \geq \gamma > 0$ for all $x \in [0,2]$. Also, suppose p_0, p_1 and p_2 are identically zero so that the compatibility conditions hold in $\Gamma_c := (-1,0) \cup (0,0) \cup (2,0) \cup (1,0)$ and*

$$\frac{\partial^{k+m} F}{\partial x^k \partial t^m} = 0, \quad 0 \leq k + 2m \leq 2,$$

where $F(x,t) = f(x,t) + c(x)u(x-1,t)$. Then $u \in C^{4+\lambda}(\bar{D})$ and

$$\left\| \frac{\partial^{k+m} u}{\partial x^k \partial t^m} \right\|_{\infty, D} \leq C\varepsilon^{-k}, \quad 0 \leq k + 2m \leq 4. \tag{3.6}$$

Theorem 3.2.4. *There exist smooth functions $j_1(t)$, $j_2(t)$ such that the smooth and singular component defined in (3.4) and (3.5) satisfies the following bounds for $0 \leq k \leq 3$, $m \geq 0$ and $0 \leq k + 2m \leq 4$*

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial t^m} \right\|_{\infty, S^- \cup S^+} \leq C, \qquad \left\| \frac{\partial^4 v}{\partial x^4} \right\|_{\infty, S^- \cup S^+} \leq C\varepsilon^{-1},$$

and

$$\left| \frac{\partial^{k+m} w}{\partial x^k \partial t^m}(x,t) \right| \leq \begin{cases} C(\varepsilon^{-k} \exp(-(1-x)\gamma_1/\varepsilon)) & \text{for } (x,t) \in S^-,\\ C(\varepsilon^{-k} \exp(-(x-1)\gamma_2/\varepsilon)) & \text{for } (x,t) \in S^+, \end{cases}$$

$$\left| \frac{\partial^4 w}{\partial x^4}(x,t) \right| \leq \begin{cases} C(\varepsilon^{-4} \exp(-(1-x)\gamma_1/\varepsilon)) & \text{for } (x,t) \in S^-,\\ C(\varepsilon^{-4} \exp(-(x-1)\gamma_2/\varepsilon)) & \text{for } (x,t) \in S^+. \end{cases}$$

Proof. We present our analysis separately in the subregions S^- and S^+. Let us begin with the subregion S^+ and define $\bar{D}^*$ such that $\bar{S}^+ \subset \bar{D}^*$. Let $D^* = \Omega^* \times (0, T]$, $\Omega^* = [-2, 2]$ and define function v^* on D^* as

$$v^* = v_0{}^* + \varepsilon v_1{}^* + \varepsilon^2 v_2{}^* + \varepsilon^3 v_3{}^*,$$

where the function v_0^* is the solution of

$$\left.\begin{aligned} & a^* \frac{\partial v_0{}^*}{\partial x} - b^* v_0{}^* - \frac{\partial v_0{}^*}{\partial t} = f^* + c^* v_0^*(x-1,t) \text{ in } D^*, \\ & v_0{}^*(x,t) = p_1{}^*(x,t) \text{ in } [-3,0] \times [0,T], \\ & v_0{}^*(x,t) = p_0{}^*(x) \text{ on } [-2,2] \times \{t=0\}, \end{aligned}\right\} \tag{3.7}$$

the function v_1^* and v_2^* are the solutions of

$$\left.\begin{aligned} & a^* \frac{\partial v_i{}^*}{\partial x} - b^* v_i{}^* - \frac{\partial v_i{}^*}{\partial t} = -\frac{\partial^2 v_{i-1}{}^*}{\partial x^2} + c^* v_i{}^*(x-1,t) \text{ in } D^*, \ i = 1,2, \\ & v_i{}^*(x,t) = 0 \text{ in } [-3,0] \times (0,T], \ \ v_i{}^*(x,t) = 0 \text{ on } [-2,2] \times \{t=0\}, \ i = 1,2, \end{aligned}\right\} \tag{3.8}$$

and $v_3{}^*$ is the solution of

$$\left.\begin{aligned} & \varepsilon \frac{\partial^2 v_3{}^*}{\partial x^2} + a^* \frac{\partial v_3{}^*}{\partial x} - b^* v_3{}^* - \frac{\partial v_3{}^*}{\partial t} = c^* v_3{}^*(x-1,t) - \frac{\partial^2 v_2{}^*}{\partial x^2} \text{ in } D^*, \\ & v_3{}^*(x,t) = 0 \text{ in } [-3,0] \times [0,T], \\ & v_3{}^*(x,t) = 0 \text{ on } [-2,2] \times \{t=0\}, \ \ v_3{}^*(x,t) = 0 \text{ on } \{x=2\} \times (0,T]. \end{aligned}\right\} \tag{3.9}$$

Here, the coefficients a^*, b^* and c^* as well as the function $p_0{}^*$ are smooth extensions of a, b, c and p_0 from the domain $[1, 2]$ to the domain $[-2, 2]$, respectively. The functions f^* and $p_1{}^*$ are the smooth extensions of f and p_1 from the domain $\bar{S}^+$ to the domain $\bar{D}^*$. In a neighbourhood of the point $(-2, 0)$, the functions $p_0{}^*$, $p_1{}^*$ and f^* are built such that $p_0{}^* = p_1{}^* = f^* = 0$. Assuming that a^*, b^*, c^* and f^* are sufficiently smooth on $\bar{D}^*$. We set all the initial-boundary data associated with (3.7) and (3.8) equal to zero. Define $F^* = f^* + c^* v^*(x-1,t)$ and impose the following compatibility condition on the set Γ_c:

$$\frac{\partial^{k+m} F^*}{\partial x^k \partial t^m} = 0 \quad \text{for} \quad 0 \le k+m \le 7. \tag{3.10}$$

Using the results of [39] for the first order differential equations defined in (3.7) and (3.8), we have $v_i{}^* \in C^{9-2i+\lambda}(\bar{D}^*) \cap C^{8-2i}(\bar{D}^*)$, $i = 0, 1, 2$. This implies $\dfrac{\partial^2 v_2{}^*}{\partial x^2} \in C^{2+\lambda}\left(\bar{D}^*\right)$ and therefore $v_3{}^* \in C^{4+\lambda}\left(\bar{D}^*\right)$. Then, it follows that

$$\left\|\frac{\partial^{k+m} v_i{}^*}{\partial x^k \partial t^m}\right\|_{\infty, D^*} \le C, \ i = 0,1,2, \qquad \left\|\frac{\partial^{k+m} v_3{}^*}{\partial x^k \partial t^m}\right\|_{\infty, D^*} \le C\varepsilon^{-k} \ \text{ for } \ 0 \le k + 2m \le 4.$$

The smooth component v is now defined as a restriction of v^* on the domain $\bar{S}^+$. Define $v^*(1,t) = j_2(t) = v(1,t)$. Thus v satisfies

$$L_\varepsilon v = f + c(x)v(x-1,t) \text{ in } S^+,$$
$$v(x,t) = v^*(x,t) \text{ on } \{x=1\}\times(0,T],$$
$$v(x,t) = u(x,t) \text{ on } [1,2]\times\{t=0\}.$$

Since $v^* = v_0^* + \varepsilon v_1^* + \varepsilon^2 v_2^* + \varepsilon^3 v_3^*$, we deduce

$$\begin{aligned}\left\|\frac{\partial^{k+m}v^*}{\partial x^k\partial t^m}\right\|_{\infty,S^+} &\le \left\|\frac{\partial^{k+m}v_0{}^*}{\partial x^k\partial t^m}\right\|_{\infty,S^+} + \varepsilon\left\|\frac{\partial^{k+m}v_1{}^*}{\partial x^k\partial t^m}\right\|_{\infty,S^+} + \varepsilon^2\left\|\frac{\partial^{k+m}v_2{}^*}{\partial x^k\partial t^m}\right\|_{\infty,S^+}\\ &\quad +\varepsilon^3\left\|\frac{\partial^{k+m}v_3{}^*}{\partial x^k\partial t^m}\right\|_{\infty,S^+}\\ &\le C\left(1+\varepsilon^{3-k}\right) \text{ for } 0\le k+2m\le 4.\end{aligned}$$

As v is a restriction of v^* on the domain $\bar{S}^+$, we have

$$\left\|\frac{\partial^{k+m}v}{\partial x^k\partial t^m}\right\|_{\infty,S^+} \le C\left(1+\varepsilon^{3-k}\right) \text{ for } 0\le k+2m\le 4.$$

Thus, for $0\le k\le 3$ and $0\le k+2m\le 4$, the smooth component v satisfies

$$\left\|\frac{\partial^{k+m}v}{\partial x^k\partial t^m}\right\|_{\infty,S^+} \le C \quad\text{and}\quad \left\|\frac{\partial^4 v}{\partial x^4}\right\|_{\infty,S^+} \le C\varepsilon^{-1}.$$

Similarly, the bounds for smooth component v in the subregion S^- can be obtained.

Next define the barrier functions on $S^-\cup S^+$ as

$$\phi^\pm(x,t) = \begin{cases}(\pm v(x,t)-C)\exp\left(\frac{-(1-x)\gamma_1}{\varepsilon}\right) \pm w(x,t) & \text{for } (x,t)\in S^-,\\ (\pm v(x,t)-C)\exp\left(\frac{-(x-1)\gamma_2}{\varepsilon}\right) \pm w(x,t) & \text{for } (x,t)\in S^+,\end{cases} \tag{3.11}$$

where C is a constant. For $(x,t)\in S^-$, using assumption $a_1+\gamma_1<0$ to obtain

$$L_\varepsilon\phi^\pm(x,t) = L_\varepsilon\left((\pm v - C)\exp\left(\frac{-(1-x)\gamma_1}{\varepsilon}\right)\right) \pm L_\varepsilon w(x,t) \ge 0.$$

Similarly, for $(x,t)\in S^+$, we obtain

$$\begin{aligned}L_\varepsilon\phi^\pm(x,t) &= \frac{\gamma_2}{\varepsilon}\exp\left(\frac{-(x-1)\gamma_2}{\varepsilon}\right)\left((\pm v - C)(-a_2+\gamma_2) - 2\varepsilon v_x + \frac{Cb\varepsilon}{\gamma_2}\right)\\ &\quad \pm\frac{\gamma_2}{\varepsilon}\exp\left(\frac{-(x-1)\gamma_2}{\varepsilon}\right)\left(\frac{f+cv(x-1,t)}{\gamma_2}\right)\ge 0.\end{aligned}$$

Moreover, $\phi^\pm\in C^0(\bar{D})$ and $\phi^\pm(x,t)\le 0$ for $(x,t)\in\Gamma$. Consequently, the required bounds on w follows from Lemma 3.2.1. The required estimates for the derivatives of w follow from [190]. □

3.3 Difference Scheme

The solution of the problem exhibits a strong interior layer at $x = 1$. Therefore, we discretize the domain by constructing a rectangular mesh $\bar{D}^{N,M} = \bar{\mathbb{D}}_x^N \times T_t{}^M$ in such a way that it will condense around the point $x = 1$. We write

$$[0, 2] = [0, 1 - \sigma] \cup [1 - \sigma, 1] \cup [1, 1 + \sigma] \cup [1 + \sigma, 2],$$

where $\sigma = \min\left\{\frac{1}{2}, \sigma_0 \varepsilon \ln N\right\}$ and σ_0 is a constant that will be chosen later on. We place $\frac{N}{4}$ mesh points in each of the subintervals. Consequently, we obtain

$$\bar{\mathbb{D}}_x^N = \{x_i : i = 0, \ldots, N\},$$

and

$$h_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, N, \quad \hat{h}_i = h_i + h_{i+1}, \quad i = 1, 2, \ldots, N-1,$$
$$h_i = \begin{cases} H = \dfrac{4(1-\sigma)}{N} & \text{for } i = 1, \ldots, N/4, 3N/4 + 1, \ldots, N, \\ h = \dfrac{4\sigma}{N} & \text{for } i = N/4 + 1, \ldots, N/2, N/2 + 1, \ldots, 3N/4. \end{cases}$$

We define the uniform mesh for the domain $[0, T]$, as follows

$$T_t{}^M = \{t_k = k\Delta t : k = 0, \ldots, M, \ \Delta t = T/M\}.$$

To discretize the differential operator in (3.1), we first define the finite difference operators on the mesh $\bar{D}^{N,M}$ as

$$D_x^+ v_i^k = \frac{v_{i+1}^k - v_i^k}{h_{i+1}}, \quad D_x^- v_i^k = \frac{v_i^k - v_{i-1}^k}{h_i}, \quad D_x^0 v_i^k = \frac{v_{i+1}^k - v_{i-1}^k}{h_{i+1} + h_i}, \quad D_t^- v_i^k = \frac{v_i^k - v_i^{k-1}}{\Delta t},$$

and $\delta_x^2 v_i^k = \dfrac{2(D_x^+ v_i^k - D_x^- v_i^k)}{h_{i+1} + h_i}$. Also, define $v^k_{\frac{i\mp 1}{2}} = \dfrac{\left(v_{i\mp 1}^k + v_i{}^k\right)}{2}$ and $v_{\frac{i\mp 1}{2}} = \dfrac{(v_{i\mp 1} + v_i)}{2}$.

We employ the classical central difference scheme within the intervals $(1 - \sigma, 1)$ and $(1, 1 + \sigma)$ while utilizing the midpoint upwind scheme in the remaining intervals. We utilize second-order one-sided difference approximations to ensure continuity of the spatial derivative at the point of discontinuity. For the time derivative, we employ the backward-Euler method for discretization. Thus, the discrete problem can be expressed as follows

$$\begin{cases} U_{i,0} = p_0(x_i) & \text{for } i = 0,\dots,N, \\ L_{mu}^{N,M,(L)} U_{i,k+1} = f_{\frac{i-1}{2},k+1} + c_{\frac{i-1}{2}} p_1 & \text{for } i = 1,\dots,\frac{N}{4}, \\ L_{cen}^{N,M} U_{i,k+1} = f_{i,k+1} + c_i p_1 & \text{for } i = \frac{N}{4}+1,\dots,\frac{N}{2}-1, \\ L_{cen}^{N,M} U_{i,k+1} = f_{i,k+1} + c_i U_{i-N/2,k+1} & \text{for } i = \frac{N}{2}+1,\dots,\frac{3N}{4}-1, \\ L_{mu}^{N,M,(R)} U_{i,k+1} = f_{\frac{i+1}{2},k+1} + c_{\frac{i+1}{2}} U_{i-N/2,k+1} & \text{for } i = \frac{3N}{4},\dots,N-1, \\ D_x^F U_{i,k+1} - D_x^B U_{i,k+1} = 0 & \text{for } i = \frac{N}{2}, \\ \text{for } k = 0,\dots,M-1, \end{cases} \tag{3.12}$$

where

$$\begin{cases} {L_{mu}}^{N,M,(L)} U_{i,k+1} = \varepsilon\delta_x^2 U_{i,k+1} + a_{\frac{i-1}{2}} D_x^- U_{i,k+1} - b_{\frac{i-1}{2}} U_{\frac{i-1}{2},k+1} - D_t^- U_{\frac{i-1}{2},k+1}, \\ {L_{cen}}^{N,M} U_{i,k+1} = \varepsilon\delta_x^2 U_{i,k+1} + a_i D_x^0 U_{i,k+1} - b_i U_{i,k+1} - D_t^- U_{i,k+1}, \\ {L_{mu}}^{N,M,(R)} U_{i,k+1} = \varepsilon\delta_x^2 U_{i,k+1} + a_{\frac{i+1}{2}} D_x^+ U_{i,k+1} - b_{\frac{i+1}{2}} U_{\frac{i+1}{2},k+1} - D_t^- U_{\frac{i+1}{2},k+1}, \end{cases} \tag{3.13}$$

and

$$\begin{cases} D_x^F U_{\frac{N}{2},k} = \left(-U_{\frac{N}{2}+2,k} + 4U_{\frac{N}{2}+1,k} - 3U_{\frac{N}{2},k}\right)/2h, \\ D_x^B U_{\frac{N}{2},k} = \left(U_{\frac{N}{2}-2,k} - 4U_{\frac{N}{2}-1,k} + 3U_{\frac{N}{2},k}\right)/2h. \end{cases} \tag{3.14}$$

After simplifying the terms in (3.12), we obtain

$$\begin{cases} U_i^0 = p_0(x_i) & \text{for } i = 0,\dots,N, \\ L_\varepsilon^{N,M} U_{i,k+1} = \widetilde{f}_{i,k+1} & \text{for } i = 1,\dots,N-1, \\ U_{i,k+1} = p_1(x_i, t_{k+1}) & \text{for } i = -N/2,\dots,0, \\ U_{N,k+1} = p_2(t_{k+1}) & \text{for } k = 0,\dots,M-1, \end{cases} \tag{3.15}$$

where

$$L_\varepsilon^{N,M} U_{i,k+1} = \begin{cases} \left[r_i^- U_{i-1,k+1} + r_i^0 U_{i,k+1} + r_i^+ U_{i+1,k+1}\right] + \left[p_i^- U_{i-1,k} + p_i^0 U_{i,k} + p_i^+ U_{i+1,k}\right] \\ \text{for } i = 1,\dots,N/2-1, N/2+1,\dots,N-1, \\ q_i^{-,2} U_{i-2,k+1} + q_i^{-,1} U_{i-1,k+1} + q_i^0 U_{i,k+1} + q_i^{+,1} U_{i+1,k+1} + q_i^{+,2} U_{i+2,k+1} \\ \text{for } i = N/2, \end{cases} \tag{3.16}$$

and

$$\widetilde{f}_{i,k+1} = \begin{cases} m_i^- f_{i-1,k+1} + m_i^0 f_{i,k+1} + m_i^+ f_{i+1,k+1} + s_i^+ p_1(x_{i-N/2}, t_{k+1}) + s_i^- U_{i-N/2,k+1} \\ \text{for } i = 1,\dots,N/2-1, \ \ N/2+1,\dots,N-1, \\ 0 \ \text{ for } i = N/2. \end{cases} \tag{3.17}$$

Here, for $i = 1, \dots, N/4$

$$\begin{cases} r_i^- = \left(\dfrac{2\varepsilon}{\hat{h}_i h_i} - \dfrac{a_{i-1/2}}{h_i} - \dfrac{b_{i-1/2}}{2} - \dfrac{1}{2\Delta t}\right), \quad r_i^0 = \left(\dfrac{-2\varepsilon}{h_i h_{i+1}} + \dfrac{a_{i-1/2}}{h_i} - \dfrac{b_{i-1/2}}{2} - \dfrac{1}{2\Delta t}\right), \\ r_i^+ = \dfrac{2\varepsilon}{\hat{h}_i h_{i+1}}, \quad p_i^- = \dfrac{1}{2\Delta t}, \quad p_i^0 = \dfrac{1}{2\Delta t}, \quad p_i^+ = 0, \\ m_i^- = \frac{1}{2}, \quad m_i^0 = \frac{1}{2}, \quad m_i^+ = 0, \quad s_i^+ = c_{i-1/2}, \quad s_i^- = 0, \end{cases} \tag{3.18}$$

for $i = N/4 + 1, \dots, N/2 - 1, N/2 + 1, \dots, 3N/4 - 1$

$$\begin{cases} r_i^- = \left(\dfrac{2\varepsilon}{\hat{h}_i h_i} - \dfrac{a_i}{\hat{h}_i}\right), \quad r_i^0 = \left(\dfrac{-2\varepsilon}{h_i h_{i+1}} - b_i - \dfrac{1}{\Delta t}\right), \quad r_i^+ = \left(\dfrac{2\varepsilon}{\hat{h}_i h_{i+1}} + \dfrac{a_i}{\hat{h}_i}\right), \\ p_i^- = 0, \quad p_i^0 = \frac{1}{\Delta t}, \quad p_i^+ = 0, \quad m_i^- = 0, \quad m_i^0 = 1, \quad m_i^+ = 0, \end{cases} \tag{3.19}$$

for $i = N/4 + 1, \dots, N/2 - 1$

$$s_i^+ = c_i, \quad s_i^- = 0,$$

for $i = N/2 + 1, \dots, 3N/4 - 1$

$$s_i^+ = 0, \quad s_i^- = c_i,$$

for $i = 3N/4, \dots, N - 1$

$$\begin{cases} r_i^- = \dfrac{2\varepsilon}{\hat{h}_i h_i}, \quad r_i^0 = \left(\dfrac{-2\varepsilon}{h_i h_{i+1}} - \dfrac{a_{i+1/2}}{h_{i+1}} - \dfrac{b_{i+1/2}}{2} - \dfrac{1}{2\Delta t}\right), \\ r_i^+ = \left(\dfrac{2\varepsilon}{\hat{h}_i h_{i+1}} + \dfrac{a_{i+1/2}}{\hat{h}_{i+1}} - \dfrac{b_{i+1/2}}{2} - \dfrac{1}{2\Delta t}\right), \\ p_i^- = 0, \quad p_i^0 = \frac{1}{2\Delta t}, \quad p_i^+ = \frac{1}{2\Delta t}, \\ m_i^- = 0, \quad m_i^0 = \frac{1}{2}, \quad m_i^+ = \frac{1}{2}, s^+ = 0, \quad s^- = c_{i+1/2}, \end{cases} \tag{3.20}$$

and lastly,

$$q_{N/2}^{-,2} = \frac{-1}{2h}, \; q_{N/2}^{-,1} = \frac{2}{h}, \; q_{N/2}^{0} = -\frac{3}{h}, \; q_{N/2}^{+,1} = \frac{2}{h}, \quad q_{N/2}^{+,2} = \frac{-1}{2h}. \tag{3.21}$$

3.4 Error Estimates

The difference operator $L_\varepsilon^{N,M}$ in (3.16) fails to fulfil the conditions of discrete maximum principle because, for this, we need $q_i \geq 0$ to prove A to be an M-matrix. Consequently, we must modify (3.15). For $i = N/2$

$$q_{N/2}^{-,2} U_{N/2-2,k+1} + q_{N/2}^{-,1} U_{N/2-1,k+1} + q_{N/2}^{0} U_{N/2,k+1} + q_{N/2}^{+,1} U_{N/2+1,k+1} + q_{N/2}^{+,2} U_{N/2+2,k+1} = 0. \tag{3.22}$$

From (3.15), for $i = N/2 - 1$, we have

$$\begin{aligned} \left(\frac{2\varepsilon - h a_{N/2-1}}{2h^2}\right) U_{N/2-2,k+1} &= f_{N/2-1,k+1} + c_{N/2-1} p_1(x_{-1}, t_{k+1}) - r_{N/2-1}^0 U_{N/2-1,k+1} \\ &\quad + r_{N/2-1}^+ U_{N/2,k+1} + \frac{1}{\Delta t} U_{N/2-1,k} \end{aligned} \tag{3.23}$$

and, for $i = N/2 + 1$

$$\left(\frac{2\varepsilon + ha_{N/2+1}}{2h^2}\right) U_{N/2+2,k+1} = f_{N/2+1,k+1} + c_{N/2+1} U(x_1, t_{k+1}) - r^0_{N/2+1} U_{N/2+1,k+1} + r^+_{N/2+1} U_{N/2,k+1} + \frac{1}{\Delta t} U_{N/2+1,k}. \tag{3.24}$$

Now put the values of $U_{N/2-2,k+1}$ and $U_{N/2+2,k+1}$ from (3.23) and (3.24) into (3.22) to write

$$\begin{aligned} & q^{-,2}_{N/2}\left(\frac{2h^2}{2\varepsilon - ha_{N/2-1}}\right)\Big(f_{N/2-1,k+1} + c_{N/2-1} p_1(x_{-1}, t_{k+1}) - r^0_{N/2-1} U_{N/2-1,k+1} \\ & - r^+_{N/2-1} U_{N/2,k+1} - \frac{1}{\Delta t} U_{N/2-1,k}\Big) + q^{-,1}_{N/2} U_{N/2-1,k+1} + q^0_{N/2} U_{N/2,k+1} + q^{+,1}_{N/2} U_{N/2+1,k+1} \\ & + q^{+,2}_{N/2}\left(\frac{2h^2}{2\varepsilon + ha_{N/2+1}}\right)\Big(f_{N/2+1,k+1} + c_{N/2+1} U(x_1, t_{k+1}) - r^0_{N/2+1} U_{N/2+1,k+1} \\ & - r^+_{N/2+1} U_{N/2,k+1} - \frac{1}{\Delta t} U_{N/2+1,k}\Big) = 0. \end{aligned} \tag{3.25}$$

After rearranging the terms in (3.25), the discrete problem reads

$$\begin{cases} U_{i,0} = p_0(x_i) & \text{for } i = 0, \ldots, N, \\ L^{N,M}_{\tau} U_{i,k+1} = \widetilde{f}_{\tau,i,k+1} & \text{for } i = 1, \ldots, N-1, \\ U_{i,k+1} = p_1(x_i, t_{k+1}) & \text{for } i = -N/2, \ldots, 0, \\ U_{N,k+1} = p_2(t_{k+1}) & \text{for } k = 0, \ldots, M-1, \end{cases} \tag{3.26}$$

where

$$L^{N,M}_{\tau} U_{i,k+1} = \begin{cases} \Big(r^-_i U_{i-1,k+1} + r^0_i U_{i,k+1} + r^+_i U_{i+1,k+1}\Big) & \\ + \Big(p^-_i U^k_{i-1} + p^0_i U^k_i + p^+_i U_{i+1,k}\Big) & \text{if } i = N/2, \\ L^{N,M}_{\varepsilon} U_{i,k+1} & \text{if } i \neq N/2, \end{cases} \tag{3.27}$$

and

$$\widetilde{f}_{\tau,i,k+1} = \begin{cases} \Big[m^-_i f_{i-1,k+1} + m^0_i f_{i,k+1} + m^+_i f_{i+1,k+1}\Big] & \\ + l_1 p_1(x_{-1}, t_{k+1}) + l_2 U(x_1, t_{k+1}) & \text{if } i = N/2, \\ \widetilde{f}_{i,k+1} & \text{if } i \neq N/2. \end{cases}$$

Now, for $i = N/2$

$$\begin{cases} r^-_i = \dfrac{1}{2h}\left(4 - \dfrac{2\left(2\varepsilon + h^2 b_{i-1} + \frac{h^2}{\Delta t}\right)}{2\varepsilon - ha_{i-1}}\right), \quad r^0_i = \dfrac{1}{2h}\left(-6 + \dfrac{2\varepsilon + ha_{i-1}}{2\varepsilon - ha_{i-1}} + \dfrac{2\varepsilon - ha_{i+1}}{2\varepsilon + ha_{i+1}}\right), \\ r^+_i = \dfrac{1}{2h}\left(4 - \dfrac{2\left(2\varepsilon + h^2 b_{i+1} + \frac{h^2}{\Delta t}\right)}{2\varepsilon + ha_{i+1}}\right), \\ p^-_i = \dfrac{h}{(2\varepsilon - ha_{i-1})\Delta t}, \quad p^0_i = 0, \quad p^+_i = \dfrac{h}{(2\varepsilon + ha_{i+1})\Delta t}, \\ m^-_i = \dfrac{h}{(2\varepsilon - ha_{i-1})}, \quad m^0_i = 0, \quad m^+_i = \dfrac{h}{(2\varepsilon + ha_{i+1})}, \\ l_1 = \dfrac{-hc_{i-1}}{2\varepsilon - ha_{i-1}}, \quad l_2 = \dfrac{-hc_{i+1}}{2\varepsilon + ha_{i+1}}, \end{cases}$$

and for $i \neq N/2$, these coefficients are defined in (3.18), (3.19) and (3.20). Let $D^{N,M} = \bar{D}^{N,M} \cap D$ and $\Gamma^{N,M} = \bar{D}^{N,M} \setminus D^{N,M}$.

Lemma 3.4.1. *Let $N > 0$ be such that*

$$\frac{N}{\ln N} \geq 2\sigma_0\gamma^* \tag{3.28}$$

and

$$\left(\| b \|_\infty + \Delta t^{-1}\right) \leq \frac{\gamma N}{2}. \tag{3.29}$$

Let Y be the mesh function such that $Y \leq 0$ on $\Gamma^{N,M}$ and $L_\tau^{N,M} Y \geq 0$ in $D^{N,M}$. Then $Y \leq 0$ in $\bar{D}^{N,M}$.

Proof. Write $L_\tau^{N,M}$ as

$$\begin{aligned} -L_\tau^{N,M} Y_{i,k+1} &= \left[A_{i,i-1}Y_{i-1,k+1} + A_{i,i}Y_{i,k+1} + A_{i,i+1}Y_{i+1,k+1}\right] \\ &\quad - \left[B_{i,i-1}Y_{i-1,k} + B_{i,i}Y_{i,k} + B_{i,i+1}Y_{i+1,k}\right], \end{aligned} \tag{3.30}$$

where $\mathbf{A} := (A_{i,j})$ and $\mathbf{B} := (B_{i,j})$ are written as

$$\begin{aligned} &A_{i,i-1} = -r_i^-, \quad A_{i,i} = -r_i^0, \quad A_{i,i+1} = -r_i^+, \\ &B_{i,i-1} = p_i^-, \quad B_{i,i} = p_i^0, \quad B_{i,i+1} = p_i^+. \end{aligned}$$

Clearly, $B \geq 0$ since $p_i^-, p_i^0, p_i^+ \geq 0$, and it is easy to follow from (3.28) and (3.29) that A is an M-matrix [82]. The remaining part we prove by induction. For that, we assume $Y_{i,k} \leq 0$, $k = 0, \ldots, N-1$. Then, we can rewrite (3.30) as $AY_{i,k+1} = BY_{i,k} - L^{N,M}Y_{i,k+1}$. Note that $B \geq 0$, $Y_{i,k} \leq 0$, $A^{-1} \geq 0$ and $L_\tau^{N,M} Y_{i,k+1} \geq 0$ by hypothesis. Consequently, it follows that $Y_{i,k+1} \leq 0$ in $D^{N,M}$. □

Next, we obtain the following estimate as an immediate consequence of Lemma 3.4.1.

Lemma 3.4.2. *Let U be the solution of* (3.26) *and the conditions* (3.28) *and* (3.29) *hold true. Then*

$$\|U\|_{\infty,\bar{D}^{N,M}} \leq \|U\|_{\infty,\Gamma^{N,M}} + \frac{1}{\gamma}\|\widetilde{f_\tau}\|_{\infty,\bar{D}^{N,M}}.$$

Proof. Let

$$\phi_i^{\pm,k+1} = -\|U\|_{\infty,\Gamma^{N,M}} - \begin{cases} x_i \dfrac{\|\tilde{f}_\tau\|_{\infty,\bar{D}^{N,M}}}{\gamma} \mp U_{i,k+1} & \text{for } i = 0,\ldots,N/2, \\ (2 - x_i)\dfrac{\|\tilde{f}_\tau\|_{\infty,\bar{D}^{N,M}}}{\gamma} \mp U_{i,k+1} & \text{for } i = N/2+1,\ldots,N. \end{cases}$$

Then $\phi_N^{\pm,k+1} \le 0$, $\phi_i^{\pm,k+1} \le 0$ for $i = -N/2, \ldots, 0$ and $\phi_i^{\pm,0} \le 0$ for $i = 0, \ldots, N$. For $i \neq N/2$, $L_\tau^{N,M} \phi_i^{\pm,k+1} \ge 0$. Further, for $i = N/2$

$$L_\tau^{N,M} \phi_{N/2}^{\pm,k+1} = L_\tau^{N,M} \{-\|U\|_{\infty,\Gamma^{N,M}} - \frac{1}{\gamma} \|\tilde{f}_\tau\|_{\infty,\bar{D}^{N,M}} \mp U_{N/2,k+1}\} \ge (D_x^F - D_x^B) \phi_{N/2}^{\pm,k+1},$$

and

$$(D_x^F - D_x^B) \phi_{N/2}^{\pm,k+1} = \frac{1}{2h} \left(-\phi_{N/2+2}^{\pm,k+1} + 4\phi_{N/2+1}^{\pm,k+1} - 6\phi_{N/2}^{\pm,k+1} - \phi_{N/2-2}^{\pm,k+1} + 4\phi_{N/2-1}^{\pm,k+1} \right) \ge 0.$$

Consequently, the required result follows from Lemma 3.4.1. □

Next, we decompose the solution into smooth and singular component. We write $U_{i,k+1} := V_{i,k+1} + W_{i,k+1}$. We define the mesh functions V_L and V_R as the approximation of v to the left and the right of the point of discontinuity $x = 1$. Similarly, we define the mesh functions W_L and W_R as the approximation of w to the left and the right of the point of discontinuity $x = 1$. Here, the functions V_L and V_R satisfy

$$\left.\begin{aligned} &L_\tau^{N,M} V_{L,i,k+1} = \tilde{f}_{\tau,i,k+1} \ \text{ for } \ i = 1, \ldots, N/2-1, \\ &V_{L,i,k+1} = v(x_i, t_{k+1}) \ \text{ for } \ i = -N/2, \ldots, 0, \\ &V_{L,N/2,k+1} = v(1^-, t_{k+1}), \quad k \ge 0, \\ &V_{L,i,0} = v(x_i, 0) \ \text{ for } \ i = 0, \ldots, N/2, \end{aligned}\right\} \tag{3.31}$$

and

$$\left.\begin{aligned} &L_\tau^{N,M} V_{R,i,k+1} = \tilde{f}_{\tau,i,k+1} \ \text{ for } \ i = N/2+1, \ldots, N-1, \\ &V_{R,N/2,k+1} = v(1^+, t_{k+1}), \quad V_{R,N,k+1} = v(2, t_{k+1}), \quad k \ge 0, \\ &V_{R,i,0} = v(x_i, 0) \ \text{ for } \ i = N/2, \ldots, N. \end{aligned}\right\} \tag{3.32}$$

Moreover, the functions W_L and W_R satisfy

$$\left.\begin{aligned} &L_\tau^{N,M} W_{L,i,k+1} = 0 \ \text{ for } \ i = 1, \ldots, N/2-1, \\ &W_{L,i,k+1} = 0 \ \text{ for } \ i = -N/2, \ldots, 0, \\ &W_{L,i,0} = 0 \ \text{ for } \ i = 0, \ldots, N/2, \\ &L_\tau^{N,M} W_{R,i,k+1} = 0 \ \text{ for } \ i = N/2+1, \ldots, N-1, \\ &W_{R,N,k+1} = 0, \quad k \ge 0, \\ &W_{R,i,0} = 0 \ \text{ for } \ i = N/2, \ldots, N, \\ &W_{R,N/2,k+1} + V_{R,N/2,k+1} = W_{L,N/2,k+1} + V_{L,N/2,k+1}, \\ &D_x^F W_{R,N/2,k+1} + D_x^F V_{R,N/2,k+1} = D_x^B W_{L,N/2,k+1} + D_x^B V_{L,N/2,k+1}, \quad k \ge 0. \end{aligned}\right\} \tag{3.33}$$

Now, the numerical solution U can be written as

$$U_{i,k+1} = \begin{cases} V_{L,i,k+1} + W_{L,i,k+1} & \text{for } i = 0, \dots, N/2 - 1, \\ V_{L,i,k+1} + W_{L,i,k+1} = V_{R,i,k+1} + W_{R,i,k+1} & \text{for } i = N/2, \\ V_{R,i,k+1} + W_{R,i,k+1} & \text{for } i = N/2 + 1, \dots, N. \end{cases} \tag{3.34}$$

Lemma 3.4.3. *Let V_L and V_R be the solutions of* (3.31) *and* (3.32), *and v be the solution of* (3.4). *Then under the assumptions (*3.28*) and (*3.29*)*

$$|V_{L,i,k+1} - v(x_i, t_{k+1})| \le C(N^{-2} + \Delta t)x_i \qquad \textit{for } i = 1, \dots, N/2 - 1,$$
$$|V_{R,i,k+1} - v(x_i, t_{k+1})| \le C(N^{-2} + \Delta t)(2 - x_i) \qquad \textit{for } i = N/2 + 1, \dots, N - 1.$$

Proof. Consider $\Psi_{L,i}^{k+1} = -C(N^{-2} + \Delta t)x_i$ for $i = 0, \dots, N/2$. Then

$$L_\tau^{N,M}(V_{L,i,k+1} - v(x_i, t_{k+1})) = (L_\varepsilon - L_\tau^{N,M})v(x_i, t_{k+1}).$$

For $i = 1, \dots, N/2 - 1$, we obtain

$$\begin{aligned} &\left|L_\tau^{N,M}\left(V_{L,i,k+1} - v(x_i, t_{k+1})\right)\right| \\ &\le \begin{cases} C\left[(\varepsilon + h_i)(h_i + h_{i+1})\left\|\frac{\partial^3 v}{\partial x^3}\right\|_\infty + h_i{}^2\left(\left\|\frac{\partial^2 v}{\partial x^2}\right\|_\infty + \left\|\frac{\partial v}{\partial x}\right\|_\infty\right) + \Delta t\left\|\frac{\partial^2 v}{\partial t^2}\right\|_\infty\right] & \text{for } i = 1, \dots, N/4, \\ C\left[h^2\left(\varepsilon\left\|\frac{\partial^4 v}{\partial x^4}\right\|_\infty + \left\|\frac{\partial^3 v}{\partial x^3}\right\|_\infty\right) + \Delta t\left\|\frac{\partial^2 v}{\partial t^2}\right\|_\infty\right] & \text{for } i = N/4 + 1, \dots, N/2 - 1. \end{cases} \end{aligned}$$

Using Theorem 3.2.4, conditions $h_i \le CN^{-1}$ and $\varepsilon \le N^{-1}$ to find

$$\left|L_\tau^{N,M}(V_{L,i,k+1} - v(x_i, t_{k+1}))\right| \le C(N^{-2} + \Delta t) \le L_\tau^{N,M}\psi_{L,i}^{k+1}.$$

Furthermore, from Lemma 3.4.1, we have

$$\left|V_{L,i,k+1} - v(x_i, t_{k+1})\right| \le C(N^{-2} + \Delta t)x_i, \quad 1 \le i \le N/2 - 1.$$

Now, consider $\psi_{R,i}^{k+1} = -C\left(N^{-2} + \Delta t\right)(2 - x_i)$ for $i = N/2, \dots, N$. Similarly, it follows that for $i = N/2 + 1, \dots, N - 1$

$$\begin{aligned} &\left|L_\varepsilon^{N,M}\left(V_{R,i,k+1} - v(x_i, t_{k+1})\right)\right| \\ &\le \begin{cases} C\left[h^2\left(\varepsilon\left\|\frac{\partial^4 v}{\partial x^4}\right\|_\infty + \left\|\frac{\partial^3 v}{\partial x^3}\right\|_\infty\right) + \Delta t\left\|\frac{\partial^2 v}{\partial t^2}\right\|_\infty\right] \text{ for } i = N/2 + 1, \dots, 3N/4 - 1, \\ C\left[(\varepsilon + h_{i+1})(h_i + h_{i+1})\left\|\frac{\partial^3 v}{\partial x^3}\right\|_\infty + h_{i+1}{}^2\left(\left\|\frac{\partial^2 v}{\partial x^2}\right\|_\infty + \left\|\frac{\partial v}{\partial x}\right\|_\infty\right) + \Delta t\left\|\frac{\partial^2 v}{\partial t^2}\right\|_\infty\right] \\ \text{for } i = 3N/4, \dots, N - 1. \end{cases} \end{aligned}$$

For $i = N/2 + 1, \dots, N - 1$, a similar argument for $(V_R - v)$ yields

$$\left|V_{R,i,k+1} - v(x_i, t_{k+1})\right| \le C(N^{-2} + \Delta t)(2 - x_i).$$

□

Next, we define two mesh functions on $\bar{\mathbb{D}}_x^N = \{x_i\}_0^N$ given by

$$S_i = \prod_{j=1}^{i}\left(1+\frac{\alpha h_j}{\varepsilon}\right) \text{ for } i = 1,\ldots,N/2 \text{ and } Q_i = \prod_{j=1}^{N-i}\left(1+\frac{\alpha h_j}{\varepsilon}\right) \text{ for } i = N/2,\ldots,N-1$$

so that $S_0 = 1$ and $Q_N = 1$, where α is a positive constant.

Lemma 3.4.4. *Let $\alpha \le \gamma/2$. Then the functions S_i and Q_i satisfy*

$$-L_\tau^{N,M} S_i \ge \begin{cases} \dfrac{C}{\varepsilon+\alpha H} S_i & \text{if } i = 1,\ldots,N/4, \\ \dfrac{C}{\varepsilon+\alpha h} S_i & \text{if } i = N/4+1,\ldots,N/2-1, \end{cases}$$

and

$$-L_\tau^{N,M} Q_i \ge \begin{cases} \dfrac{C}{\varepsilon+\alpha h} Q_i & \text{if } i = N/2+1,\ldots,3N/4-1, \\ \dfrac{C}{\varepsilon+\alpha H} Q_i & \text{if } i = 3N/4,\ldots,N-1. \end{cases}$$

Proof. For $i = 1,\ldots,N/4$

$$-L_\tau^{N,M} S_i \;=\; -\frac{2\varepsilon}{\hat{h}_i}\left[\left(\frac{S_{i+1}-S_i}{h_{i+1}}\right)-\left(\frac{S_i-S_{i-1}}{h_i}\right)\right] - a_{\frac{i-1}{2}}\left(\frac{S_i-S_{i-1}}{h_i}\right) + b_{\frac{i-1}{2}} S_{\frac{i-1}{2}}.$$

Since $S_i = \left(1+\dfrac{\alpha h_i}{\varepsilon}\right) S_{i-1}$, we obtain

$$S_i - S_{i-1} = \frac{\alpha h_i}{\varepsilon} S_{i-1} \text{ and } a_i \le -\gamma_1 \le -2\alpha.$$

It follows that

$$-L_\tau^{N,M} S_i = -\frac{2\alpha}{\hat{h}_i}(S_i - S_{i-1}) - a_{\frac{i-1}{2}} S_{i-1} + b_{\frac{i-1}{2}} S_{\frac{i-1}{2}} \ge \frac{C}{\varepsilon+\alpha h_i} S_i.$$

Moreover, for $i = N/4+1,\ldots,N/2-1$

$$-L_\tau^{N,M} S_i = -\varepsilon\delta_x^2 S_i - a_i D_x^0 S_i + b_i S_i + D_t^- S_i \ge \frac{C}{\varepsilon+\alpha h} S_i.$$

In case $i = N/2+1,\ldots,3N/4-1$

$$\begin{aligned} -L_\tau^{N,M} Q_i &= -\varepsilon\delta_x^2 Q_i - a_i D_x^0 Q_i + b_i Q_i + D_t^- Q_i \\ &= -\frac{\varepsilon}{h}\left[\left(\frac{Q_{i+1}-Q_i}{h_{i+1}}\right)-\left(\frac{Q_i-Q_{i-1}}{h_i}\right)\right] - a_i\left(\frac{Q_{i+1}-Q_{i-1}}{\hat{h}_i}\right) + b_i Q_i. \end{aligned}$$

Since $Q_{i+1} - Q_i = \dfrac{-\alpha h_{N-i}}{\varepsilon} Q_{i+1}$ and $a_i \ge \gamma_2 \ge 2\alpha$, we have

$$-L_\tau^{N,M} Q_i \ge \frac{\alpha}{h}(Q_{i+1}-Q_i) + \frac{a_i}{2}\frac{\alpha}{\varepsilon}(Q_{i+1}-Q_{i-1}) \ge \frac{C}{(\varepsilon+\alpha h)} Q_i.$$

For $i = 3N/4, \ldots, N-1$

$$-L_\tau^{N,M} Q_i \geq -\frac{2\varepsilon}{\hat{h}_i}\left[\left(\frac{Q_{i+1}-Q_i}{h_{i+1}}\right)-\left(\frac{Q_i-Q_{i-1}}{h_i}\right)\right]-a_{\frac{i+1}{2}}\left(\frac{Q_{i+1}-Q_i}{h_{i+1}}\right).$$

Since $Q_{i+1} - Q_i = \dfrac{-\alpha h_{N-i}}{\varepsilon} Q_{i+1}$ and $a_i \geq \gamma_2 \geq 2\alpha$, we can write

$$-L_\tau^{N,M} Q_i \geq \frac{2\alpha}{\hat{h}_i}(Q_{i+1}-Q_i) + a_{\frac{i+1}{2}}\frac{\alpha}{\varepsilon}Q_{i+1} \geq \frac{C}{(\varepsilon+\alpha h_{N-i})}Q_i.$$

□

Lemma 3.4.5. *Let $\sigma_0 \geq 2/\alpha$. Then*

$$\left(\frac{S_i}{S_{N/2}}\right) \leq CN^{-4\left(\frac{1-2i}{N}\right)}, \quad i = N/4, \ldots, N/2-1, \tag{3.35}$$

and

$$\left(\frac{Q_i}{Q_{N/2}}\right) \leq CN^{-4\left(\frac{2i}{N-1}\right)}, \quad i = N/2+1, \ldots, 3N/4. \tag{3.36}$$

Proof. For $i = N/4, \ldots, N/2-1$, $h_i = h$ and therefore

$$\left(\frac{S_i}{S_{N/2}}\right) = \left(1 - \frac{\alpha h}{\varepsilon + \alpha h}\right)^{(N/2-i)}.$$

Taking log on both sides

$$\left(\frac{S_i}{S_{N/2}}\right) \leq \exp\left((N/2-i)\left(\frac{-\alpha h}{\varepsilon+\alpha h}\right)\right) \leq CN^{-4(1-2i/N)}$$

because $N\dfrac{8(\alpha\sigma_0)^2(1-2i/N)(N^{-1}\ln N)}{(1+4\alpha\sigma_0 N^{-1}\ln N)}$ is bounded. Further , for $i = N/2+1, \ldots, 3N/4$

$$\left(\frac{Q_i}{Q_{N/2}}\right) = \frac{\prod_{j=1}^{N-i}(1+\frac{\alpha h}{\varepsilon})}{\prod_{j=1}^{N/2}(1+\frac{\alpha h}{\varepsilon})} = \left(1+\frac{\alpha h}{\varepsilon}\right)^{-(i-N/2)}.$$

Similarly, it is easy to follow that $\left(\dfrac{Q_i}{Q_{N/2}}\right) \leq CN^{-4(2i/N-1)}$. □

Now, we will calculate the errors for the layers components W_L and W_R in $((0, 1-\sigma] \cup [1+\sigma, 2)) \times (0, T]$.

Lemma 3.4.6. *Let $\alpha \leq \gamma/2$ and $\sigma_0 \geq 2/\alpha$. Then under the assumptions* (3.28) *and* (3.29)*, the errors associated to the layer components satisfy*

$$|W_{L,i,k+1} - w(x_i, t_{k+1})| \leq CN^{-2} \text{ for } i = 1, \ldots, N/4,$$
$$|W_{R,i,k+1} - w(x_i, t_{k+1})| \leq CN^{-2} \text{ for } i = 3N/4, \ldots, N-1.$$

Proof. From (3.34), we have

$$|U_{N/2,k+1}| = |V_{R,N/2,k+1} + W_{R,N/2,k+1}| = |V_{L,N/2,k+1} + W_{L,N/2,k+1}| \le |V_{L,N/2,k+1}| + |W_{L,N/2,k+1}|.$$

Using Theorem 3.2.4 and Lemma 3.4.2 to obtain $|W_{N/2,k+1}| \le C$. Let us next consider

$$\phi_{L,i}^{k+1} = -C\left(\frac{S_i}{S_{N/2}}\right) \quad \text{for} \quad i = 0, \ldots, N/2,$$

where C is chosen sufficiently large. For $i = 0, \ldots, N/2$ an application of Lemma 3.4.4 yields

$$L_\tau^{N,M}\left(\phi_{L,i}^{k+1} \pm W_{L,i,k+1}\right) = L_\tau^{N,M}\phi_{L,i}^{k+1} \pm L_\tau^{N,M} W_{L,i,k+1} \ge C\frac{S_i}{S_{N/2}} \ge 0,$$

and $W_{L,0,k+1} = 0 = W_{L,i,0}$. Thus

$$\phi_{L,0}^{k+1} \pm W_{L,0,k+1} = \phi_{L,0}^{k+1} = \frac{-CS_0}{S_{N/2}} \le 0.$$

Similarly, we can show that $\phi_{L,i}^{0} \pm W_{L,i,0} \le 0$ and $\phi_{L,N/2}^{k+1} \pm W_{L,N/2,k+1} \le 0$. Next we use Lemma 3.4.1 and observe that

$$|W_{L,i,k+1}| \le C\left(\frac{S_i}{S_{N/2}}\right), \quad i = 1, \ldots, N/2 - 1. \tag{3.37}$$

Furthermore, $|W_{L,i,k+1}| \le C\left(\frac{S_i}{S_{N/2}}\right) \le C\left(\frac{S_{N/4}}{S_{N/2}}\right)$ for $i = 1, \ldots, N/4$ and from Lemma 3.4.5, we have

$$|W_{L,i,k+1}| \le CN^{-2}, \quad i = 1, \ldots, N/4. \tag{3.38}$$

Since $\alpha \le \gamma/2 < \gamma$, $\sigma = \sigma_0 \varepsilon \ln N$, $\sigma_0 \ge 2/\alpha$, we see using Theorem 3.2.4 that

$$|w(x_i, t_{k+1})| \le C \exp\left(\frac{-(1 - x_i)\gamma_1}{\varepsilon}\right) \le CN^{-2}, \quad i = 1, \ldots, N/4. \tag{3.39}$$

Combine (3.38) and (3.39) to obtain

$$|W_{L,i,k+1} - w(x_i, t_{k+1})| \le CN^{-2}, \quad i = 1, \ldots, N/4.$$

Let us next consider $\phi_{R,i}^{k+1} = -C\left(\frac{Q_i}{Q_{N/2}}\right)$ for $i = N/2, \ldots, N$. Then using Lemma 3.4.4 for $i = N/2 + 1, \ldots, N - 1$ we calculate

$$L_\tau^{N,M}\left(\phi_{R,i}^{k+1} \pm W_{R,i,k+1}\right) = L_\tau^{N,M}\phi_{R,i}^{k+1} \pm L_\tau^{N,M} W_{R,i,k+1} \ge C\frac{Q_i}{Q_{N/2}} \ge 0$$

and $W_{R,N,k+1} = 0 = W_{R,i,0}$. Thus $\phi^{k+1}_{R,N} \pm W_{R,N,k+1} = \phi^{k+1}_{R,N} \leq 0$.
Similarly, we can show that $\phi^{0}_{R,i} \pm W_{R,i,0} \leq 0$, $\phi^{k+1}_{R,N/2} \pm W_{R,N/2,k+1} \leq 0$ and Lemma 3.4.1 leads to $\phi^{k+1}_{R,i} \pm W_{R,i,k+1} \leq 0$ for $i = N/2, \ldots, N$. Thus

$$|W_{R,i,k+1}| \leq |\phi^{k+1}_{R,i}| \leq C\left(\frac{Q_i}{Q_{N/2}}\right) \text{ for } i = N/2+1, \ldots, N-1$$
$$\leq C\left(\frac{Q_{3N/4}}{Q_{N/2}}\right) \text{ for } i = 3N/4, \ldots, N-1$$

because Q is decreasing and from Lemma 3.4.5, we have $\left(\frac{Q_{3N/4}}{Q_{N/2}}\right) \leq CN^{-2}$. Thus

$$|W_{R,i,k+1}| \leq CN^{-2}, \quad i = 3N/4, \ldots, N-1. \tag{3.40}$$

Now, using Theorem 3.2.4 and doing the same calculation as we did for W_L, we get

$$|w(x_i, t_{k+1})| \leq C\exp\left(\frac{-(x_i-1)\gamma_2}{\varepsilon}\right) \leq CN^{-2}, \quad i = 3N/4, \ldots, N-1. \tag{3.41}$$

Using (3.40) and (3.41), we find

$$|W_{R,i,k+1} - w(x_i, t_{k+1})| \leq CN^{-2}, \quad i = 3N/4, \ldots, N-1.$$

□

Next, we will state and prove some results required to obtain parameter uniform error bounds.

Lemma 3.4.7. *The following inequalities hold true:*

$$\exp\left(-\alpha(1-x_i)/\varepsilon\right) \leq \left(\frac{S_i}{S_{N/2}}\right), \quad i = 1, \ldots, N/2-1 \tag{3.42}$$

and

$$\exp\left(-\alpha(x_i-1)/\varepsilon\right) \leq \left(\frac{Q_i}{Q_{N/2}}\right), \quad i = N/2+1, \ldots, N-1. \tag{3.43}$$

Proof. For each j, from Lemma 2.5 of [278], it follows that

$$\exp\left(\frac{-\alpha h_j}{\varepsilon}\right) = \left(\exp\left(\frac{\alpha h_j}{\varepsilon}\right)\right)^{-1} \leq \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}. \tag{3.44}$$

Consequently, for $j = i+1, \ldots, N/2$

$$\prod_{i+1}^{N/2} \exp\left(\frac{-\alpha h_j}{\varepsilon}\right) \leq \prod_{i+1}^{N/2}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}$$

$$\exp\left(\frac{-\alpha}{\varepsilon}(h_{i+1} + h_{i+2} + \ldots + h_{N/2})\right) = \frac{\prod_{1}^{i}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)}{\prod_{1}^{i}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)\prod_{i+1}^{N/2}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)}$$

$$\exp\left(\frac{-\alpha}{\varepsilon}(1-x_i)\right) \leq \frac{S_i}{S_{N/2}}$$

and for $j = N-(i+1), \dots, N/2$

$$\prod_{N-(i+1)}^{N/2} \exp\left(\frac{-\alpha h_j}{\varepsilon}\right) \leq \prod_{N-(i+1)}^{N/2} \left(1 + \frac{\alpha h_j}{\varepsilon}\right)^{-1}$$

$$\exp\left(\frac{-\alpha}{\varepsilon}(h_{N-(i+1)} + h_{N-(i+2)} + \dots + h_{N/2})\right) = \frac{\prod_{1}^{N-i}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)}{\prod_{N-(i+1)}^{N/2}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)\prod_{1}^{N-i}\left(1 + \frac{\alpha h_j}{\varepsilon}\right)}$$

$$\exp\left(\frac{-\alpha}{\varepsilon}(x_i - 1)\right) \leq \frac{Q_i}{Q_{N/2}}.$$

□

Lemma 3.4.8. *The difference operators D_x^B and D_x^F satisfy*

$$D_x^B S_{N/2} \geq \frac{C}{\varepsilon + \alpha h} S_{N/2} \text{ and } -D_x^F Q_{N/2} \geq \frac{C}{\varepsilon + \alpha h} Q_{N/2}.$$

Proof. Since $(S_i - S_{i-1}) = \frac{\alpha h_i}{\varepsilon} S_{i-1}$, we have

$$\begin{aligned}
D_x^B S_{N/2} &= \frac{1}{2h}\left(S_{N/2-2} - 4S_{N/2-1} + 3S_{N/2}\right) \\
&= \frac{1}{2}\frac{\alpha}{\varepsilon}\left(3S_{N/2-1} - S_{N/2-2}\right) \\
&= \frac{\alpha}{2\varepsilon}\left(2S_{N/2-1} + S_{N/2-1} - S_{N/2-2}\right) \\
&= \frac{\alpha}{2\varepsilon}\left(2S_{N/2-1} + \frac{\alpha h}{\varepsilon} S_{N/2-2}\right) \\
&= \frac{\alpha}{2\varepsilon}\left(\frac{2S_{N/2}}{\left(1 + \frac{\alpha h}{\varepsilon}\right)} + \frac{\alpha h}{\varepsilon}\frac{S_{N/2-1}}{\left(1 + \frac{\alpha h}{\varepsilon}\right)}\right) \\
&= \frac{\alpha}{2\varepsilon}\frac{S_{N/2}}{\left(1 + \frac{\alpha h}{\varepsilon}\right)^2}\left(2 + \frac{3\alpha h}{\varepsilon}\right) \\
&= \frac{\alpha}{2\varepsilon}\frac{S_{N/2}}{\left(1 + \frac{\alpha h}{\varepsilon}\right)^2}\left(2\left(1 + \frac{\alpha h}{\varepsilon}\right) + \frac{\alpha h}{\varepsilon}\right) \\
&\geq \frac{\alpha}{\varepsilon}\frac{S_{N/2}}{\left(1 + \frac{\alpha h}{\varepsilon}\right)^2}\left(1 + \frac{\alpha h}{\varepsilon}\right) \\
&\geq \frac{C}{\varepsilon + \alpha h} S_{N/2}.
\end{aligned}$$

Also, since $Q_{i+1} - Q_i = \frac{-\alpha h_{N-i}}{\varepsilon} Q_{i+1}$, we have

$$-D_x^F Q_{N/2} = \frac{1}{2h}\left(Q_{N/2+2} - 4Q_{N/2+1} + 3Q_{N/2}\right)$$

$$
\begin{aligned}
&= \frac{1}{2h}\left(Q_{N/2+2} - Q_{N/2+1} + 3(Q_{N/2} - Q_{N/2+1})\right)\\
&= \frac{1}{2h}\left((Q_{N/2+2} - Q_{N/2+1}) - 3(Q_{N/2+1} - Q_{N/2})\right)\\
&= \frac{1}{2}\frac{\alpha}{\varepsilon}\left(3Q_{N/2+1} - Q_{N/2+2}\right)\\
&= \frac{1}{2h}\frac{\alpha h}{\varepsilon}\left(2Q_{N/2+1} + Q_{N/2+1} - Q_{N/2+2}\right)\\
&= \frac{\alpha}{2\varepsilon}\left(2Q_{N/2+1} + \frac{\alpha h}{\varepsilon}Q_{N/2+2}\right)\\
&= \frac{\alpha}{2\varepsilon}\left(\frac{2Q_{N/2}}{\left(1+\frac{\alpha h}{\varepsilon}\right)} + \frac{\alpha h}{\varepsilon}\frac{Q_{N/2+1}}{\left(1+\frac{\alpha h}{\varepsilon}\right)}\right)\\
&= \frac{\alpha}{2\varepsilon}\left(\frac{2Q_{N/2}}{\left(1+\frac{\alpha h}{\varepsilon}\right)} + \frac{\alpha h}{\varepsilon}\frac{Q_{N/2}}{\left(1+\frac{\alpha h}{\varepsilon}\right)^2}\right)\\
&= \frac{\alpha}{2\varepsilon}Q_{N/2}\frac{1}{\left(1+\frac{\alpha h}{\varepsilon}\right)^2}\left(2\left(1+\frac{\alpha h}{\varepsilon}\right) + \frac{\alpha h}{\varepsilon}\right)\\
&\geq \frac{\alpha}{\varepsilon}\frac{\left(1+\frac{\alpha h}{\varepsilon}\right)}{\left(1+\frac{\alpha h}{\varepsilon}\right)^2}Q_{N/2}\\
&\geq \frac{C}{\varepsilon+\alpha h}Q_{N/2}.
\end{aligned}
$$

□

Theorem 3.4.9. *Let u and $U_{i,k+1}$ be the solutions of* (3.1) *and* (3.26)*, respectively. Then under the assumptions* (3.28)*,* (3.29) *and $\alpha \leq \gamma/2$ with $\sigma_0 \geq 2/\alpha$*

$$
|U_{i,k+1} - u(x_i, t_{k+1})| \leq \begin{cases} C\left(N^{-2} + \Delta t\right) & \text{for } i = 1, \ldots, N/4, 3N/4, \ldots, N-1, \\ C\left(N^{-2}\ln^2 N + \Delta t\right) & \text{for } i = N/4+1, \ldots, 3N/4-1. \end{cases} \tag{3.45}
$$

Proof. We compute the error separately in the layer region and outside the layer region.

Case I: For $i = 1, \ldots, N/4, 3N/4, \ldots, N-1$. The triangle inequality, Lemma 3.4.3 and Lemma 3.4.6 yields

$$
\begin{aligned}
|U_{i,k+1} - u(x_i, t_{k+1})| &\leq |V_{L,i,k+1} - v(x_i, t_{k+1})| + |W_{L,i,k+1} - w(x_i, t_{k+1})|\\
&\leq C(N^{-2} + \Delta t)x_i + CN^{-2}\\
&\leq C(N^{-2} + \Delta t).
\end{aligned} \tag{3.46}
$$

Case II: For $i = N/4 + 1, \ldots, 3N/4 - 1$. Consider

$$\begin{aligned}
L_\tau^{N,M}(U_{i,k+1} - u(x_i, t_{k+1})) &= L_\tau^{N,M} U_{i,k+1} - L_\tau^{N,M} u(x_i, t_{k+1}) \\
&= \tilde{f}_\tau - L_\tau^{N,M} u(x_i, t_{k+1}) \\
&= L_\varepsilon u(x_i, t_{k+1}) - L_\tau^{N,M} u(x_i, t_{k+1}) \\
&= (L_\varepsilon - L_\tau^{N,M}) u(x_i, t_{k+1}) \\
&= \left(\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta x^2\right) + a\left(\frac{\partial}{\partial x} - D_x^0\right) - \left(\frac{\partial}{\partial t} - D_t^-\right)\right) u(x_i, t_{k+1}).
\end{aligned}$$

Using Theorem 3.2.4, we have

$$\begin{aligned}
|L_\tau^{N,M}(U_{i,k+1} - u(x_i, t_{k+1}))| &\le h \int_{x_{i-1}}^{x_{i+1}} \left(\varepsilon \left\|\frac{\partial^4 u}{\partial x^4}\right\|_\infty + \left\|\frac{\partial^3 u}{\partial x^3}\right\|_\infty\right) dx + C\Delta t \left\|\frac{\partial^2 u}{\partial t^2}\right\|_\infty \\
&\le h \int_{x_{i-1}}^{x_{i+1}} \left(\varepsilon \left\|\frac{\partial^4 v}{\partial x^4}\right\|_\infty + \varepsilon \left\|\frac{\partial^4 w}{\partial x^4}\right\|_\infty\right) dx \\
&\quad + h \int_{x_{i-1}}^{x_{i+1}} \left(\left\|\frac{\partial^3 v}{\partial x^3}\right\|_\infty + \left\|\frac{\partial^3 w}{\partial x^3}\right\|_\infty\right) dx \\
&\quad + C\Delta t \left(\left\|\frac{\partial^2 v}{\partial t^2}\right\|_\infty + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_\infty\right) \\
&\le \left(C\frac{h}{\varepsilon^2}\left[\exp(-(1 - x_{i+1})\gamma_1/\varepsilon) - \exp(-(1 - x_{i-1})\gamma_1/\varepsilon)\right]\right) \\
&\quad + Ch^2 + C\Delta t \\
&= \left(C\frac{h}{\varepsilon^2}\left[\exp(-(1 - x_i - h)\gamma_1/\varepsilon) - \exp(-(1 - x_i + h)\gamma_1/\varepsilon)\right]\right) \\
&\quad + Ch^2 + C\Delta t \\
&= \left(C\frac{h}{\varepsilon^2}\exp(-(1 - x_i)\gamma_1/\varepsilon)\left[\exp(h\gamma_1/\varepsilon) - \exp(-h\gamma_1/\varepsilon)\right]\right) \\
&\quad + Ch^2 + C\Delta t \\
&\le C\left[h^2 + \frac{h}{\varepsilon^2}\exp\left(\frac{-(1 - x_i)\gamma_1}{\varepsilon}\right)\sinh\left(\frac{h\gamma_1}{\varepsilon}\right)\right] + \Delta t. \qquad (3.47)
\end{aligned}$$

From (3.16), we have

$$2\sigma_0\gamma^* \le \frac{N}{\ln N} 4\sigma_0\gamma_1 \le \frac{2N}{\ln N}, \quad \frac{4\sigma\gamma_1}{\varepsilon \ln N} \le \frac{2N}{\ln N} \text{ and } \frac{h\gamma_1}{\varepsilon} \le 2.$$

Since $\sinh x \le cx$, $x \in [0, 2]$. This implies $\sinh\left(\frac{h\gamma_1}{\varepsilon}\right) \le C\left(\frac{h\gamma_1}{\varepsilon}\right)$. Therefore, (3.47) becomes

$$|L_\tau^{N,M}(U_{i,k+1} - u(x_i, t_{k+1}))| \le C\left[\left(h^2 + \frac{h^2}{\varepsilon^3}\exp\left(\frac{-(1 - x_i)\gamma_1}{\varepsilon}\right)\right) + \Delta t\right]. \qquad (3.48)$$

Similarly, for $i = N/2 + 1, \ldots, 3N/4 - 1$, we have

$$|L_\tau^{N,M}(U_{i,k+1} - u(x_i, t_{k+1}))| \le C\left[\left(h^2 + \frac{h^2}{\varepsilon^3}\exp\left(\frac{-(x_i - 1)\gamma_2}{\varepsilon}\right)\right) + \Delta t\right]. \qquad (3.49)$$

Furthermore, at $x_{N/2} = 1$

$$\begin{aligned}
|L_\tau^{N,M}(U_{N/2,k+1} - u(x_{N/2}, t_{k+1}))| &= \left|L_\tau^{N,M} U_{N/2,k+1} - L_\tau^{N,M} u(x_{N/2}, t_{k+1})\right| \\
&= \left|\tilde{f}_{\tau,N/2,k+1} - L_\tau^{N,M} u_{N/2,k+1}\right| \\
&\leq \left|m_{N/2}^- f_{N/2-1,k+1} + m_{N/2}^+ f_{N/2+1,k+1} - L_\tau^{N,M} u_{N/2,k+1}\right| \\
&\leq m_{N/2}^- \left|f_{N/2-1,k+1} - L_\tau^{N,M} u_{N/2-1,k+1}\right| \\
&\quad + m_{N/2}^+ \left|f_{N/2+1,k+1} - L_\tau^{N,M} u_{N/2+1,k+1}\right| \\
&\quad + \left|(D_x^F - D_x^B) U_{N/2,k+1}\right| \\
&\leq C \left|L_\tau^{N,M}(U_{N/2-1,k+1} - u(x_{N/2-1}, t_{k+1}))\right| \\
&\quad + C \left|L_\tau^{N,M}(U_{N/2+1,k+1} - u(x_{N/2+1}, t_{k+1}))\right| \\
&\quad + \left|(D_x^F - D_x^B) U_{N/2,k+1} - \left[\frac{\partial u}{\partial x}\right](x_{N/2}, t_{k+1})\right| \\
&\leq C\left(\frac{h^2}{\varepsilon^3} + \Delta t\right).
\end{aligned}$$

Let us next define the function

$$\Theta_i^k = \begin{cases} -C(N^{-2} + \Delta t)\,(1 + (x_i - (1-\sigma))) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{S_i}{S_{N/2}}\right) & \text{for } i = N/4, \ldots, N/2, \\ -C(N^{-2} + \Delta t)\,(1 + ((1+\sigma) - x_i)) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{Q_i}{Q_{N/2}}\right) & \text{for } i = N/2+1, \ldots, 3N/4. \end{cases}$$

Then

$$\begin{aligned}
L_\tau^{N,M}\Theta_i^{k+1} &= \begin{cases} -C(N^{-2} + \Delta t) L_\tau^{N,M}(x_i) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{L_\tau^{N,M} S_i}{S_{N/2}}\right) & \text{for } i = N/4, \ldots, N/2, \\ C(N^{-2} + \Delta t) L_\tau^{N,M}(x_i) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{L_\tau^{N,M} Q_i}{Q_{N/2}}\right) & \text{for } i = N/2+1, \ldots, 3N/4, \end{cases} \\
&= \begin{cases} -C(N^{-2} + \Delta t)(a_i - b_i x_i) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{L_\tau^{N,M} S_i}{S_{N/2}}\right) & \text{for } i = N/4, \ldots, N/2, \\ C(N^{-2} + \Delta t)(a_i - b_i x_i) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{L_\tau^{N,M} Q_i}{Q_{N/2}}\right) & \text{for } i = N/2+1, \ldots, 3N/4. \end{cases}
\end{aligned}$$

Using assumption $\alpha \leq \gamma/2$, Lemma 3.4.6 and Lemma 3.4.7 to obtain

$$L_\tau^{N,M}\Theta_i^{k+1} \geq \begin{cases} C\gamma_1(N^{-2} + \Delta t) + C\frac{h^2}{\varepsilon^2}\exp\left(-(1 - x_i)\gamma_1/\varepsilon\right) & \text{for } i = N/4, \ldots, N/2, \\ C\gamma_2(N^{-2} + \Delta t) + C\frac{h^2}{\varepsilon^2}\exp\left(-(x_i - 1)\gamma_2/\varepsilon\right) & \text{for } i = N/2+1, \ldots, 3N/4. \end{cases}$$

From assumption (3.28) and $\sigma = \sigma_0 \varepsilon \ln N$ implies that $\dfrac{N}{\ln N} \geq \dfrac{2\sigma\gamma^*}{\varepsilon \ln N}$. Since $h = \dfrac{4\sigma}{N}$, we have $\dfrac{h}{\varepsilon} \leq \dfrac{2}{\gamma^*}$. Now, using Lemma 3.4.8 we calculate that

$$\begin{aligned}
L_\tau^{N,M}\Theta_i^{k+1} &\geq (D_x^F - D_x^B)\Theta_i^{k+1} \\
&= -C(N^{-2} + \Delta t)(D_x^F(x_i) - D_x^B(x_i)) + \frac{ch^2}{\varepsilon^2}\left[\left(\frac{D_x^B S_{N/2}}{S_{N/2}}\right) - \left(\frac{D_x^B Q_{N/2}}{Q_{N/2}}\right)\right] \\
&= 2C(N^{-2} + \Delta t)\frac{h^2}{\varepsilon^2}\frac{1}{\varepsilon + \alpha h}\left[\frac{S_{N/2}}{S_{N/2}} + \frac{Q_{N/2}}{Q_{N/2}}\right] \\
&\geq 2C(N^{-2} + \Delta t) + 2C\frac{h^2}{\varepsilon^3}.
\end{aligned} \tag{3.50}$$

Therefore, it follows from (3.46) – (3.50) that

$$\begin{cases}
L_\tau^{N,M}\Theta_i^{k+1} \geq |L_\tau^{N,M}(U_{i,k+1} - u(x_i, t_{k+1}))| & \text{for } i = N/4 + 1, \dots, 3N/4 - 1, \\
-\Theta_i^{k+1} \geq |U_{i,k+1} - u(x_i, t_{k+1})| & \text{for } i = N/4, 3N/4 \text{ and} \\
-\Theta_i^0 \geq |U_{i,0} - u(x_i, t_0)| & \text{for } i = N/4, \dots, 3N/4.
\end{cases}$$

Then applying the discrete maximum principle to $\Theta_i^{k+1} \pm (U_{i,k+1} - u(x_i, t_{k+1}))$ over the domain $\bar{D}^{N,M} \cap ([1-\sigma, 1+\sigma] \times [0, T])$, we obtain

$$\left|U_{i,k+1} - u(x_i, t_{k+1})\right| \leq C\left(\frac{h^2}{\varepsilon^2} + \Delta t\right) \leq C(N^{-2}\ln^2 N + \Delta t) \text{ for } i = N/4 + 1, \dots, 3N/4 - 1.$$

□

3.5 Numerical Illustrations

The performance of the proposed method is examined in this section and the theoretical estimates are numerically verified. We consider two test problems for numerical computations.

Example 3.5.1. *Consider the following singularly perturbed problem:*

$$\begin{cases}
\varepsilon u_{xx}(x,t) + a(x)u_x(x,t) - x(2-x)u(x,t) - u_t(x,t) = f(x,t) + u(x-1,t), \\
(x,t) \in (0,2) \times (0,2], \\
u(x,0) = 0, \quad x \in [0,2], \\
u(x,t) = t^2, \quad (x,t) \in [-1,0] \times [0,2], \\
u(2,t) = 0, \quad t \in (0,2],
\end{cases}$$

where

$$a(x) = \begin{cases}
-(2 + x(2-x)), & x \in [0,1], \\
(2 + x(2-x)), & x \in (1,2],
\end{cases}$$

and

$$f(x,t) = \begin{cases} 2(1+x^2)t^2, & (x,t) \in [0,1]\times[0,2], \\ 3(1+x^2)t^2, & (x,t) \in (1,2]\times[0,2]. \end{cases}$$

Example 3.5.2. *Consider the following singularly perturbed problem:*

$$\begin{cases} \varepsilon u_{xx}(x,t) + a(x)u_x(x,t) - 5u(x,t) - u_t(x,t) = f(x,t) + 2u(x-1,t), & (x,t) \in (0,2)\times(0,2], \\ u(x,0) = 0, \quad x \in [0,2], \\ u(x,t) = 0, \quad (x,t) \in [-1,0]\times[0,2], \\ u(2,t) = 0, \quad t \in (0,2], \end{cases}$$

where

$$a(x) = \begin{cases} -(4+x^2), & x \in [0,1], \\ (6-x^2), & x \in (1,2], \end{cases}$$

and

$$f(x,t) = \begin{cases} 4xt^2\exp(-t), & (x,t) \in [0,1]\times[0,2], \\ 4(2-x)t^2\exp(-t), & (x,t) \in (1,2]\times[0,2]. \end{cases}$$

The exact solutions for the problems are unknown for comparison. Therefore, we use the double mesh principle to estimate the error. The maximum absolute error ($E_\varepsilon^{N,\triangle t}$) and order of convergence ($R_\varepsilon^{N,\triangle t}$) are calculated using

$$E_\varepsilon^{N,\triangle t} := \max\left|U^{N,\triangle t}(x_i,t_{k+1}) - \tilde{U}^{2N,\triangle t/2}(x_i,t_{k+1})\right| \text{ and } R_\varepsilon^{N,\triangle t} := \log_2\left(\frac{E_\varepsilon^{N,\triangle t}}{E_\varepsilon^{2N,\triangle t/2}}\right),$$

where $U^{N,\triangle t}(x_i,t_{k+1})$ and $\tilde{U}^{2N,\triangle t/2}(x_i,t_k)$ are the approximate solutions obtained on the mesh $\bar{D}^{N,M}$ and $\bar{D}^{2N,2M}$, respectively. When, the perturbation parameter approaches zero, the problem's solution exhibits turning point behaviour (Figures 3.1-3.4). Maximum absolute error and order of convergence for Examples 3.5.1 and 3.5.2 are tabulated in Tables 3.1-3.2. Moreover, the maximum absolute errors for Examples 3.5.1 and 3.5.2 are plotted in Figures 3.5 and 3.6, respectively. The surface plot of the numerical solution for Examples 3.5.1 and 3.5.2 are plotted in Figures 3.1 and 3.3, respectively. Also, the numerical solutions at final time step ($t = 2$) for different values of ε are displayed in Figures 3.2 and 3.4.

The numerical results tabulated in Tables 3.1-3.2 do not clearly depict the theoretical order of convergence for spatial discretization. It is to be noted that the error in numerical solution is due to spatial and temporal discretization. Consequently, the errors given in

Tables 3.1 and 3.2 are a combination of temporal and spatial errors, with the layer regions playing a significant role.

The hybrid difference scheme improves accuracy in space only. To verify this, we performed numerical experiments for $M = N^2$ and numerical results are tabulated in Table 3.3. In Table 3.4, we have fixed $\varepsilon = 2^{-6}$ and $N = 512$ and reduce Δt by half and the associated errors are presented at different values of x. It can be observed that the errors reduce by almost half which confirms the first-order convergence in time. Figures 3.7 and 3.8 have also been illustrated to show the errors in layer region and outside layer region. This demonstrates that the numerical method is second-order spatially accurate outside of the interior layer and the errors are reduced in the layer region as claimed in Theorem 3.4.9.

Table 3.1: Maximum absolute error and order of convergence for Example 3.5.1 for different values of ε, M and N when $M = N$.

N	$\varepsilon = 2^{-2}$	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
32	1.099e-01	8.697e-01	8.723e-01	8.712e-01	8.709e-01	8.708e-01
	1.6274	1.5675	1.2466	1.2572	1.2601	1.2609
64	3.558e-02	2.934e-01	3.676e-01	3.644e-01	3.635e-01	3.633e-01
	1.3320	2.0478	1.5866	1.6018	1.6060	1.6071
128	1.413e-02	7.096e-02	1.223e-01	1.200e-01	1.194e-01	1.192e-01
	1.1428	1.9332	1.7309	1.7302	1.7308	1.7311
256	6.400e-03	1.858e-02	3.687e-02	3.619e-02	3.598e-02	3.592e-02
	1.0336	1.7107	1.6093	1.6294	1.6359	1.6376
512	3.126e-03	5.676e-03	1.208e-02	1.169e-02	1.157e-02	1.154e-02
	1.0066	1.5189	1.5926	1.6132	1.6207	1.6241

Table 3.2: Maximum absolute error and order of convergence for Example 3.5.2 for different values of ε, M and N when $M = N$.

N	$\varepsilon = 2^{-2}$	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
32	1.886e-03	4.150e-03	2.485e-03	2.146e-03	2.288e-03	2.323e-03
	1.4071	1.6239	1.1244	1.0741	1.0986	1.1044
64	7.113e-04	1.346e-03	1.139e-03	1.019e-03	1.068e-03	1.080e-03
	1.5199	1.6981	1.5245	1.3442	1.3740	1.3811
128	2.480e-04	4.149e-04	3.961e-04	4.015e-04	4.122e-04	4.149e-04
	1.5620	1.4068	1.5615	1.6111	1.6576	1.6714
256	8.400e-05	1.565e-04	1.342e-04	1.313e-04	1.304e-04	1.302e-04
	1.5102	1.5403	1.5137	1.6794	1.7007	1.7062
512	2.949e-05	5.380e-05	4.700e-05	4.101e-05	4.014e-05	3.991e-05
	1.4117	1.5972	1.4798	1.4726	1.4368	1.4272

Table 3.3: Maximum absolute error and order of convergence for Example 3.5.1 and 3.5.2 for different values of M and N when $M = N^2$ and $\varepsilon = 2^{-10}$.

	For Example 3.5.1			For Example 3.5.2		
N	left region $[0, 1-\sigma]$	interior layer region $(1-\sigma, 1+\sigma)$	right region $[1+\sigma, 2]$	left region $[0, 1-\sigma]$	interior layer region $(1-\sigma, 1+\sigma)$	right region $[1+\sigma, 2]$
32	7.602e-03	9.073e-01	2.237e-02	1.863e-04	2.215e-03	6.044e-05
	1.9665	1.3111	1.9344	1.9878	1.1088	1.8829
64	1.973e-03	3.656e-01	5.687e-03	4.652e-05	1.026e-03	1.632e-05
	1.9760	1.6430	1.9033	1.9624	1.3972	1.7645
128	5.083e-04	1.170e-01	1.478e-03	1.182e-04	3.895e-04	4.790e-06
	1.9592	1.7816	1.8513	1.9592	1.7332	1.5841
256	1.323e-04	3.404e-02	3.992e-04	3.064e-06	1.186e-04	1.594e-06
	1.9438	1.9236	1.8205	1.9382	1.8560	1.5632

Table 3.4: Maximum absolute error and order of convergence for Example 3.5.2 for different values of M and x when $N = 512$ and $\varepsilon = 2^{-6}$.

x	$M = 32$	64	128	256	512
$x_{N/2+1}$	3.837e-04	2.058e-04	1.071e-04	5.486e-05	2.773e-05
	0.8988	0.9423	0.9651	0.9843	0.9981
$x_{N/2+4}$	3.999e-04	2.122e-4	1.093e-04	6.530e-05	3.225e-05
	0.9144	0.9569	0.7432	1.0177	1.0184

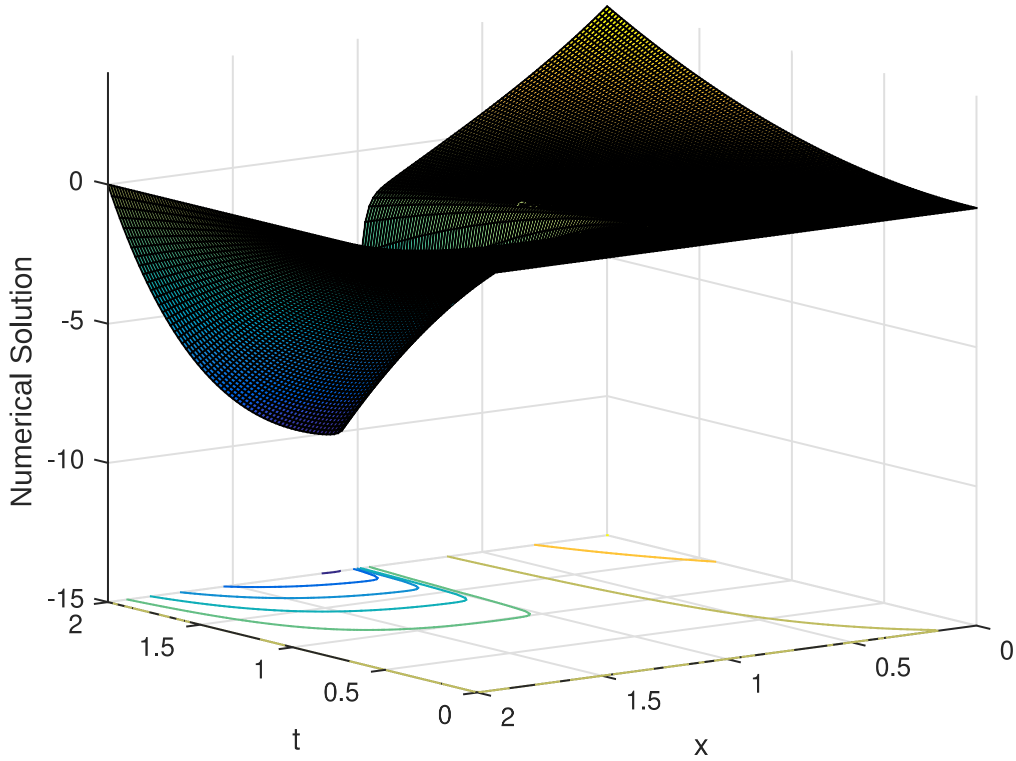

Figure 3.1: Numerical solution of Example3.5.1 for $\varepsilon = 2^{-4}$ when $M = N = 128$.

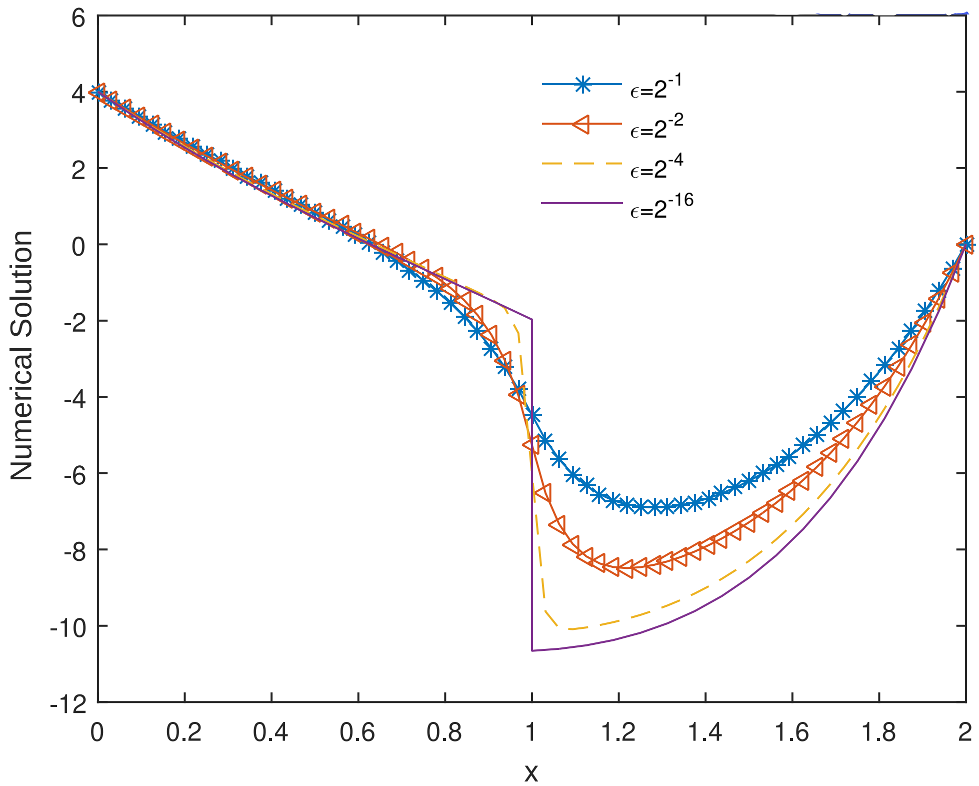

Figure 3.2: Numerical solutions of Example 3.5.1 at $t = 2$ for different values of ε when $N = 128$.

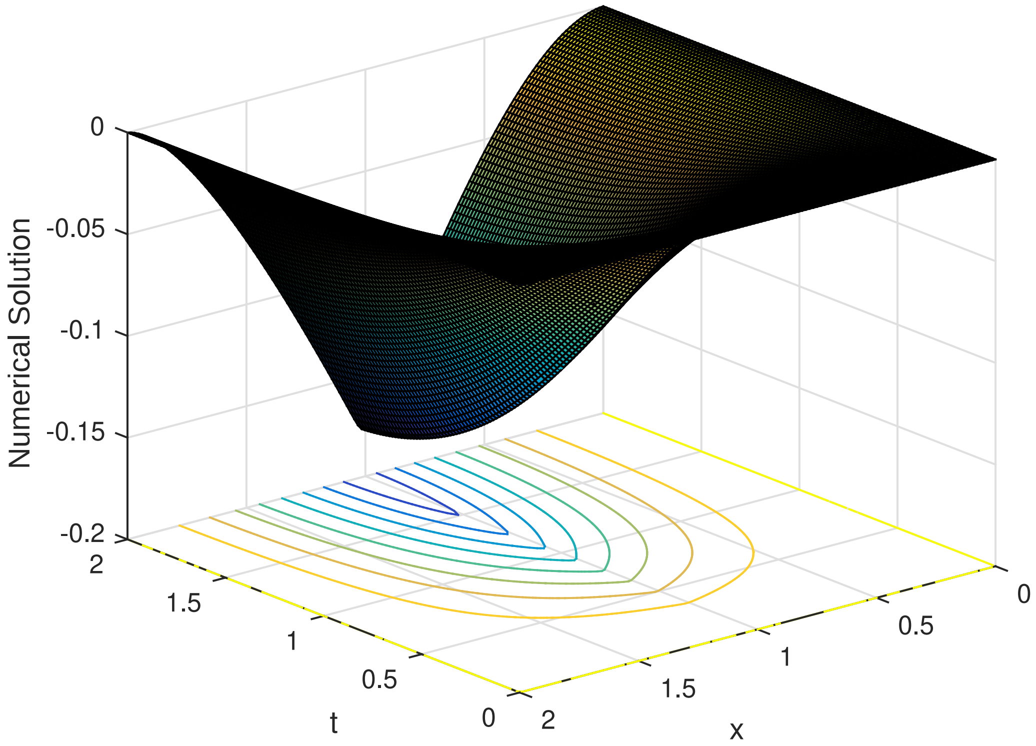

Figure 3.3: Numerical solution of Example 3.5.2 for $\varepsilon = 2^{-4}$ when $M = N = 128$.

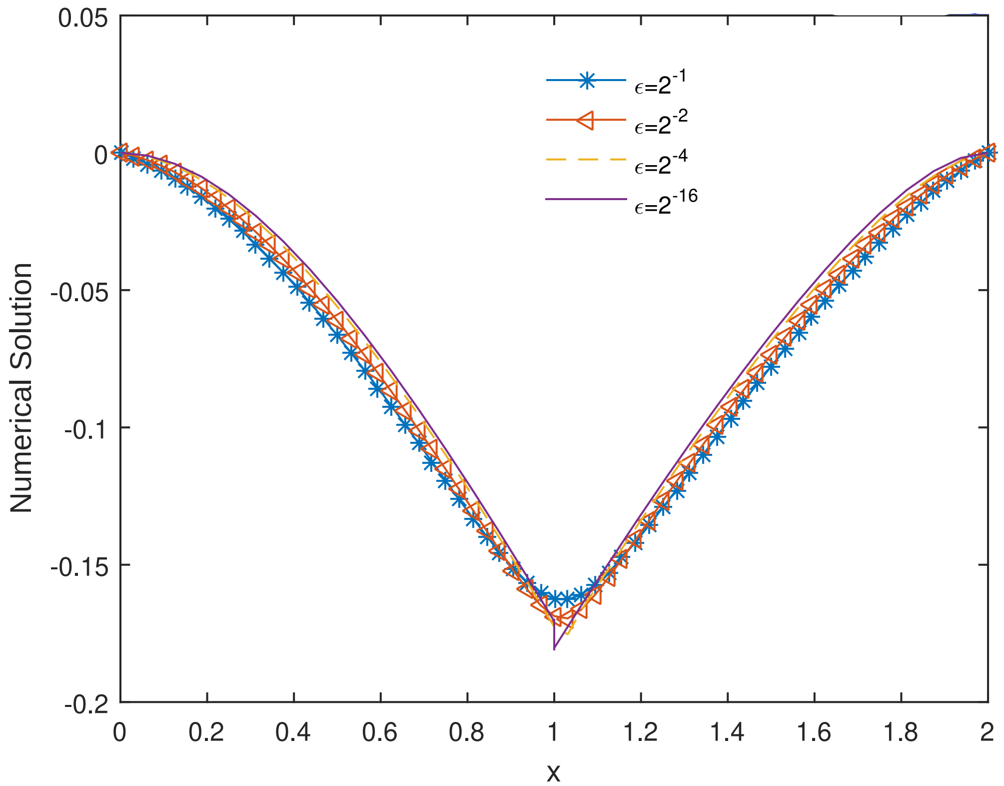

Figure 3.4: Numerical solutions of Example 3.5.2 at $t = 2$ for different values of ε when $N = 64$.

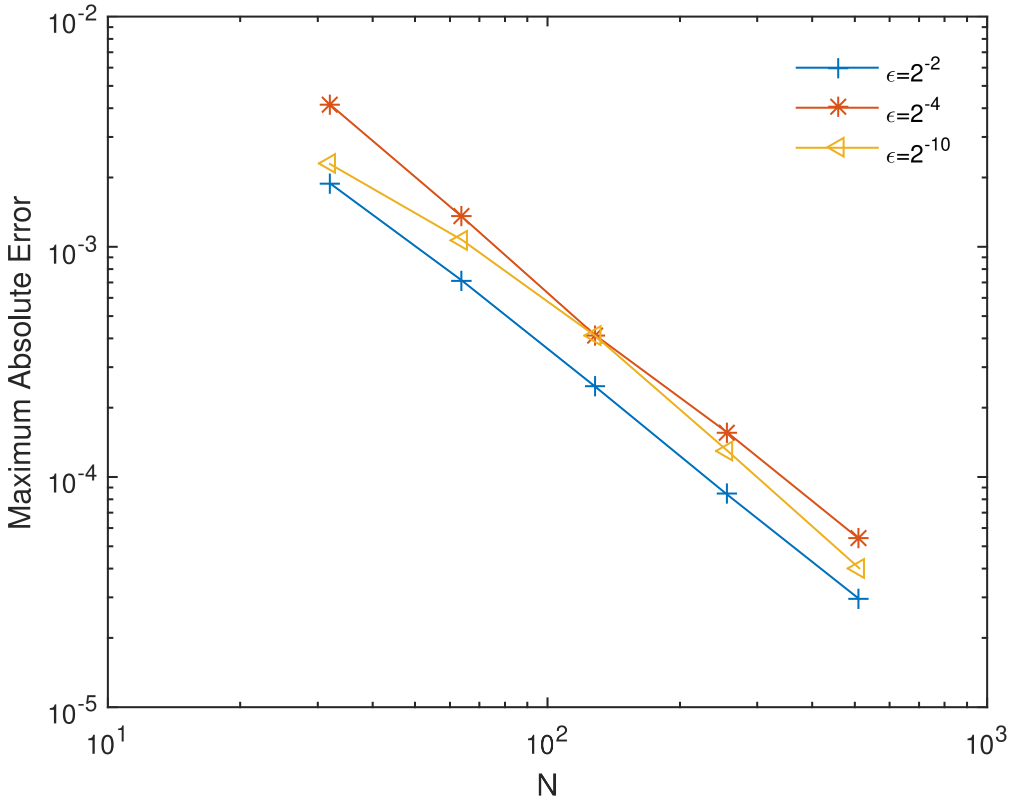

Figure 3.5: Error plot for Example 3.5.1.

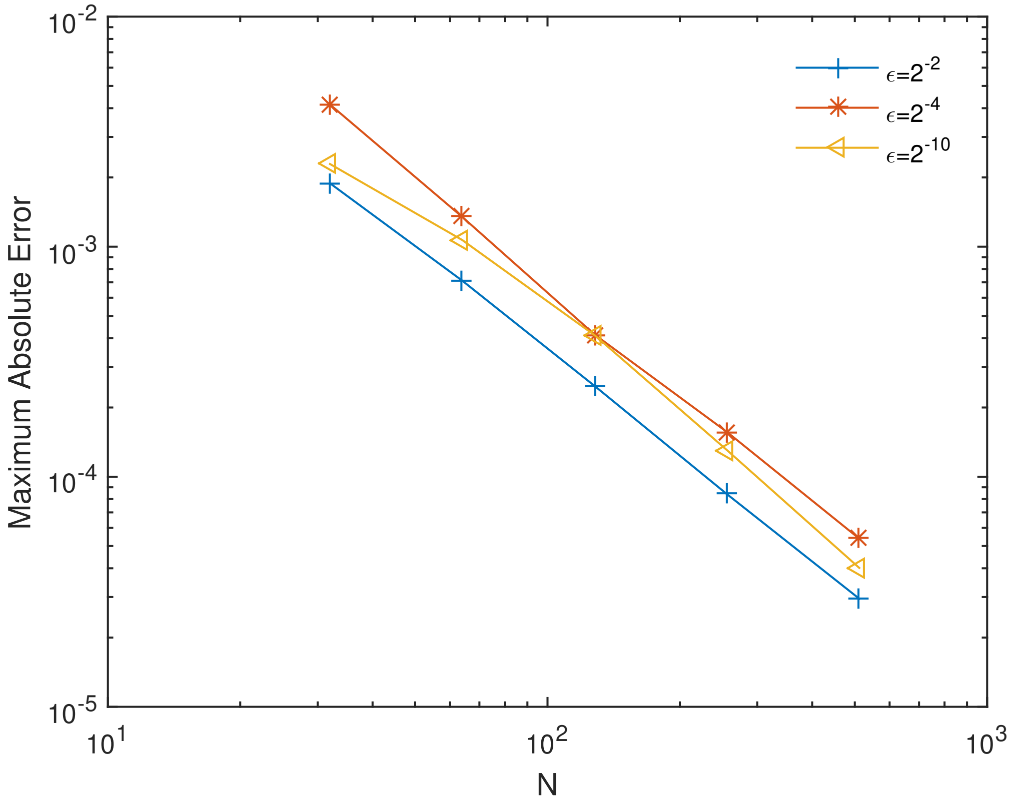

Figure 3.6: Error plot for Example 3.5.2.

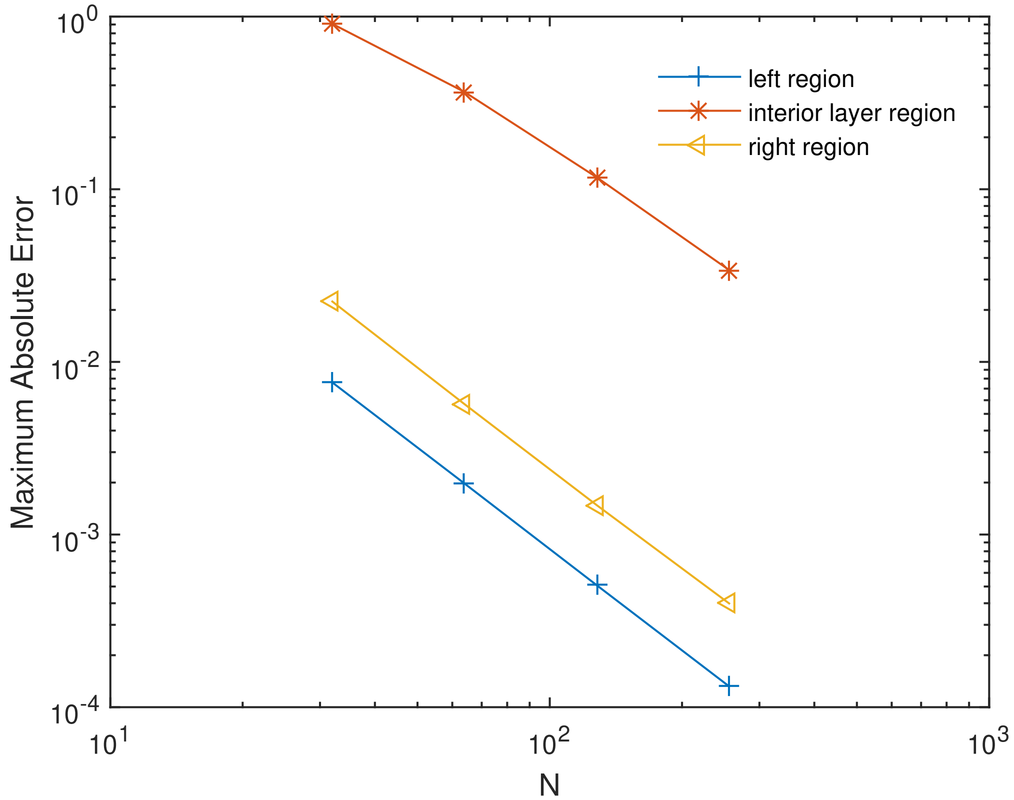

Figure 3.7: Error plot of the spatial order of convergence for Example 3.5.1.

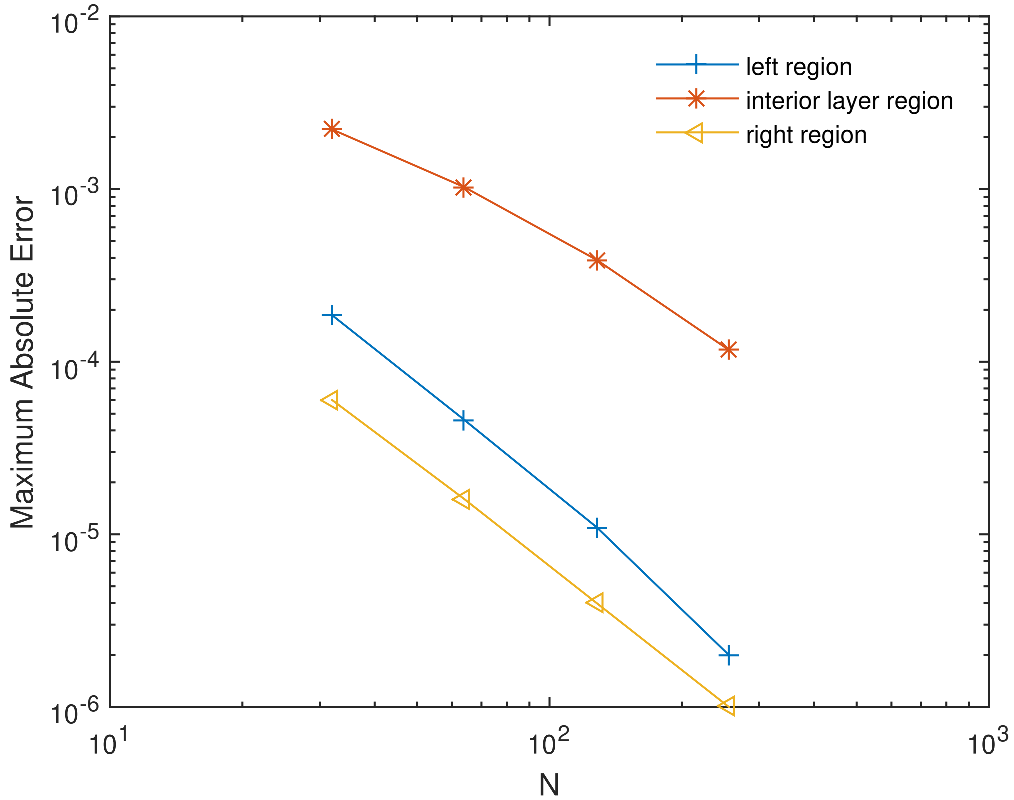

Figure 3.8: Error plot of the spatial order of convergence for Example 3.5.2.

3.6 Concluding Remarks

Singularly perturbed parabolic functional differential equation with discontinuous coefficient and source term is numerically solved. The problem's solution takes into account not just the present state of the physical system but also its history. The simultaneous presence of discontinuous data and delay makes the problem stiff. In the limiting case, the solution of the problem exhibits multiscale character. There are narrow regions where solution derivatives grow exponentially and exhibit turning point behaviour, leading to sharp interior layers across discontinuities.

A hybrid numerical scheme composed of a central difference scheme in the layer region and a midpoint upwind scheme outside the layer region is used to discretize space variable over a specially generated mesh. Whereas an implicit finite difference scheme is used to discretize the time variable. The mesh has been chosen so that most of the mesh points remain in the regions with rapid transitions. The proposed numerical method has been analysed for consistency, stability and convergence. Theoretical analysis is performed to obtain consistency and error estimates. It is found that the method proposed is unconditionally stable, and the convergence obtained is parameter uniform. Numerical illustrations are presented for two test examples that demonstrate the effectiveness of the technique. Convergence obtained in practical satisfies theoretical predictions.

Chapter 4

Parabolic Problems with Delay and Integral Boundary Conditions

4.1 Introduction

In Chapters 2 and 3, we proposed numerical schemes for solving singularly perturbed parabolic PDEs with delay and discontinuous coefficients with Dirichlet boundary conditions. In this chapter, we extend the scope of our work to solve singularly perturbed parabolic PDEs with a large delay and an integral boundary condition.

Singularly perturbed parabolic PDEs with delay and integral boundary conditions are mathematical problems that arise in various applications, including fluid dynamics, heat transfer, and chemical engineering. These problems are characterized by the presence of a small parameter in the highest-order derivative term, leading to multiple timescales within the system. The interplay between fast and slow dynamics, non-local effects induced by delays, and the influence of integral boundary conditions give rise to complex phenomena such as boundary and interior layers. These layers are regions of rapid variation in the solution, and they can pose significant challenges for numerical solutions.

Many authors have developed numerical methods to solve singularly perturbed parabolic partial differential equations with delay and integral boundary conditions. Often these methods involve the use of non-uniform meshes and special time-stepping schemes. In some cases, it may also be necessary to use adaptive mesh refinement techniques to resolve the boundary and interior layers. For the existence, uniqueness and well-posedness of such problems, the readers can see [12, 42, 19] and references therein. In [257], the authors studied singularly perturbed delay differential equations with integral boundary conditions using an upwind finite difference method on piecewise uniform Shishkin mesh. In [259], the authors utilized a hybrid difference scheme to obtain the approximate solution for singularly perturbed differential equations with a large delay and integral boundary condition. They demonstrated that the method exhibits nearly second-order convergence, which is optimal compared to the results reported in [153]. In [111], the author investigates the study of singularly perturbed parabolic convection-diffusion equations with integral boundary conditions and a large negative shift. They employ the implicit Euler method for the temporal direction and apply the exponentially fitted finite difference scheme for the spatial direction to formulate a parameter-uniform numerical method. In [95], the author addresses the numerical solution of singularly perturbed parabolic partial differential equations with a large negative shift in the spatial variable and integral boundary condition on the right side of the domain. The author formulates a numerical method that combines the Crank-Nicolson difference scheme for the temporal direction on a uniform mesh and utilizes the exponentially fitted operator finite difference method for spatial discretization. Numerical integration techniques are employed to handle the integral boundary condition. The convergence rate is optimized using the Richardson extrapolation technique. The paper [74] focuses on singularly perturbed delay differential equations with integral boundary conditions. To address this problem, the author suggests a finite difference scheme utilizing an appropriate piecewise Shishkin-type mesh. It is proven that the proposed approach exhibits almost first-order convergence. Additionally, an error estimate is calculated using the discrete norm. In [5], the author develops and improves a higher-order Haar wavelet approach for solving nonlinear singularly perturbed differential equations with various pairs of boundary conditions, such as initial, boundary, two points, integral, and multi-point integral boundary conditions. The theoretical convergence and computational stability of the method are also presented. A comparison between the proposed higher-order Haar wavelet method and recently published works, including the well-known Haar wavelet method, is performed in terms of convergence and accuracy. In the case of nonlinear equations, the author adopts the quasilinearization technique.

The theory and the area of numerical approximation for time-dependent singular perturbation problems with a large delay and integral boundary conditions still need to be developed. This chapter presents a parameter uniform numerical method to solve such a class of singularly perturbed parabolic partial differential equations. Moreover, the chapter presents rigorous consistency, stability and convergence analysis of the proposed method and illustrates numerical results to support theoretical estimates.

4.2 Continuous Problem

Let $D = \Omega \times \Delta := (0, 2) \times (0, T]$ and consider the parabolic delay differential equation with integral boundary condition

$$\left.\begin{aligned}
&\mathcal{L}u(x,t) = -\varepsilon u_{xx}(x,t) + p(x)u_x(x,t) + q(x)u(x,t) + r(x)u(x-1,t) + u_t(x,t) = g(x,t) \text{ in } D,\\
&u(x,t) = \psi_1(x,t) \text{ in } \Gamma_1 := \{(x,t),\ x \in [-1,0],\ t \in [0,T]\},\\
&u(x,t) = \psi_2(x,t) \text{ on } \Gamma_2 := \{(x,0),\ x \in [0,2]\},\\
&\mathcal{K}u(x,t) = u(2,t) - \varepsilon \int_0^2 f(x)u(x,t)dx = \psi_3(x,t) \text{ on } \Gamma_3 := \{(2,t),\ t \in [0,T]\},
\end{aligned}\right\} \tag{4.1}$$

where $\varepsilon \ll 1$ is a small positive parameter, $g(x,t)$, $p(x)$, $q(x)$ and $r(x)$ are sufficiently smooth functions. Also, assume that the initial-boundary data ψ_1, ψ_2 and ψ_3 are smooth and bounded functions such that

$$\left.\begin{aligned}
&p(x) \geq p_0 > p_0^* > 0,\quad q(x) \geq q_0 > 0,\quad r(x) \leq r_0 < 0,\\
&p_0^* + q_0 + r_0 > 0,\quad q(x) + r(x) \geq 2\eta > 0,
\end{aligned}\right\} \tag{4.2}$$

Here, $f(x)$ is a non-negative, monotonic function such that $\int_0^2 f(x)dx < 1$. Moreover, the given data satisfies the compatibility conditions

$$\begin{cases}
\psi_2(0,0) = \psi_1(0,0),\quad \psi_2(2,0) = \psi_3(2,0),\\
-\varepsilon\dfrac{\partial^2\psi_2(0,0)}{\partial x^2} + p(0)\dfrac{\partial\psi_2(0,0)}{\partial x} + q(0)\psi_2(0,0) + r(0)\psi_1(-1,0) + \dfrac{\partial\psi_1(0,0)}{\partial t} = g(0,0),\\
-\varepsilon\dfrac{\partial^2\psi_2(2,0)}{\partial x^2} + p(2)\dfrac{\partial\psi_2(2,0)}{\partial x} + q(2)\psi_2(2,0) + r(2)\psi_2(1,0) + \dfrac{\partial\psi_3(2,0)}{\partial t} = g(2,0).
\end{cases}$$

Rewriting (4.1) as

$$\mathcal{L}u(x,t) = \mathcal{G}(x,t),$$

where

$$\mathcal{L}u(x,t) = \begin{cases}
\mathcal{L}_1u(x,t) = -\varepsilon u_{xx}(x,t) + p(x)u_x(x,t) + q(x)u(x,t) + u_t(x,t)\\
\quad \text{if } (x,t) \in D_1 := (0,1) \times [0,T],\\
\mathcal{L}_2u(x,t) = -\varepsilon u_{xx}(x,t) + p(x)u_x(x,t) + q(x)u(x,t) + r(x)u(x-1,t) + u_t(x,t)\\
\text{if } (x,t) \in D_2 := (1,2) \times [0,T],
\end{cases}$$

and

$$\mathcal{G}(x,t) = \begin{cases}
g(x,t) - r(x)\psi_1(x-1,t) & \text{if } (x,t) \in D_1,\\
g(x,t) & \text{if } (x,t) \in D_2
\end{cases}$$

with

$$\begin{cases} u(x,t) = \psi_1(x,t) \text{ in } \Gamma_1 := \{(x,t),\ x \in [-1,0],\ t \in [0,T]\}, \\ u(x,t) = \psi_2(x,t) \text{ on } \Gamma_2 := \{(x,0),\ x \in [0,2]\}, \\ \mathcal{K}u(x,t) = u(2,t) - \varepsilon \displaystyle\int_0^2 f(x)u(x,t)dx = \psi_3(x,t) \text{ on } \Gamma_3 := \{(2,t),\ t \in [0,T]\}, \\ u(1^-,t) = u(1^+,t), \quad u_x(1^-,t) = u_x(1^+,t), \end{cases}$$

These assumptions confirm the existence and uniqueness of the solution [9, 160]. The solution $u(x,t)$ of (4.1) exhibits a weak interior layer at $x = 1$ and a strong boundary layer at $x = 2$.

Lemma 4.2.1. *Suppose* $\mathcal{P}(x,t) \in C^{2,1}(\bar{D})$ *satisfies* $\mathcal{P}(0,t) \geq 0$, $\mathcal{P}(x,0) \geq 0$, $\mathcal{K}\mathcal{P}(2,t) \geq 0$ *with* $\mathcal{L}\mathcal{P}(x,t) \geq 0$ *for all* $(x,t) \in D_1 \cup D_2$ *and* $[\mathcal{P}_x](1,t) = \mathcal{P}_x(1^+,t) - \mathcal{P}_x(1^-,t) \leq 0$. *Then* $\mathcal{P}(x,t) \geq 0$ *for all* $(x,t) \in \bar{D}$.

Proof. Let $(x^k,t^k) \in D$ and $\mathcal{P}(x^k,t^k) = \min_{(x,t)\in\bar{D}} \mathcal{P}(x,t)$. Consequently,

$$\mathcal{P}_x(x^k,t^k) = 0, \quad \mathcal{P}_t(x^k,t^k) = 0 \text{ and } \mathcal{P}_{xx}(x^k,t^k) > 0. \tag{4.3}$$

Suppose $\mathcal{P}(x^k,t^k) < 0$, it follows that $(x^k,t^k) \notin \Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

Case I: If $x^k \in (0,1)$, then

$$\begin{aligned} \mathcal{L}_1\mathcal{P}(x^k,t^k) &= -\varepsilon\mathcal{P}_{xx}(x^k,t^k) + p(x^k)\mathcal{P}_x(x^k,t^k) + q(x^k)\mathcal{P}(x^k,t^k) + \mathcal{P}_t(x^k,t^k) \\ &< 0, \text{ from (4.2) and (4.3).} \end{aligned}$$

Case II: If $x^k \in (1,2)$, then

$$\begin{aligned} \mathcal{L}_2\mathcal{P}(x^k,t^k) &= -\varepsilon\mathcal{P}_{xx}(x^k,t^k) + p(x^k)\mathcal{P}_x(x^k,t^k) + q(x^k)\mathcal{P}(x^k,t^k) \\ &\quad + r(x^k)\mathcal{P}(x^k-1,t^k) + \mathcal{P}_t(x^k,t^k) \\ &\leq -\varepsilon\mathcal{P}_{xx}(x^k,t^k) + q(x^k)\mathcal{P}(x^k,t^k) + r(x^k)\mathcal{P}(x^k,t^k) \\ &< 0, \text{ from (4.2) and (4.3).} \end{aligned}$$

Case III: If $x^k = 1$, then

$$[\mathcal{P}_x](x^k,t^k) = \mathcal{P}_x(x^{k+},t^k) - \mathcal{P}_x(x^{k-},t^k) > 0 \text{ since } \mathcal{P}(x^k,t^k) < 0.$$

A contradiction to the assumption and consequently the required result follows. □

As a consequence of Lemma 4.2.1, obtaining the following stability estimate is straightforward.

Lemma 4.2.2. *Let u be the solution of* (4.1). *Then*

$$\|u\|_{\infty,\bar{D}} \le \|u\|_{\infty,\Gamma} + \frac{1}{\eta}\|\mathcal{G}\|_{\infty,\bar{D}}. \tag{4.4}$$

Proof. Define $\theta_{\pm}(x,t) = \|u\|_{\infty,\Gamma} + \frac{1}{\eta}\|\mathcal{G}\|_{\infty,\bar{D}} \pm u(x,t)$, $(x,t) \in \bar{D}$. For $(x,t) \in \Gamma$, $\theta_{\pm}(x,t) \ge 0$, and if $(x,t) \in D_1 \cup D_2$, it follows that

$$\mathcal{L}\theta_{\pm}(x,t) = (q+r)(x)\left(\|u\|_{\infty,\Gamma} + \frac{\|\mathcal{G}\|_{\infty,\bar{D}}}{\eta}\right) \pm \mathcal{L}u(x,t) \ge 2\eta\left(\|u\|_{\infty,\Gamma} + \frac{\|\mathcal{G}\|_{\infty,\bar{D}}}{\eta}\right) \pm \mathcal{G} \ge 0.$$

Moreover, if $x = 1$, then $[\theta_{x\pm}](1,t) = \pm[u_x](1,t) = 0$. The required result (4.4) now follows as a consequence of Lemma 4.2.1. □

Generally, we may take homogeneous boundary data $\psi_1 = \psi_2 = \psi_3 = 0$ by subtracting some appropriate smooth function from u that satisfies the original boundary data [249].

Lemma 4.2.3. *Let u be the solution of* (4.1). *Then*

$$\left|\frac{\partial^i u(x,t)}{\partial t^i}\right| \le C \text{ for all } (x,t) \in \bar{D} \text{ and } i = 0,1,2.$$

Proof. For $i = 0$, the result follows from Lemma 4.2.2. The assumption $\psi_1(x,t) = \psi_3(x,t) = 0$ gives $u = 0$ along the left and right hand sides of D, which implies $u_t = 0$ along these sides. Also, $\psi_2(x,t) = 0$ gives $u = 0$ along the line $t = 0$. Thus $u_x = 0 = u_{xx}$ along the line $t = 0$. Now, put $t = 0$ in (4.1) to obtain

$$-\varepsilon u_{xx}(x,0) + p(x)u_x(x,0) + q(x)u(x,0) + r(x)u(x-1,0) + u_t(x,0) = g(x,0)$$

implying $u_t(x,0) = g(x,0)$ since $u_x(x,0) = u_{xx}(x,0) = u(x,0) = u(x-1,0) = 0$. Thus $|u_t| \le C$ on Γ as $g(x,t)$ is continuous on $\bar{D}$. On applying differential operator $\mathcal{L}$ on $u_t(x,t)$, we get

$$\mathcal{L}u_t(x,t) = g_t(x,t) \Rightarrow |\mathcal{L}u_t| = |g_t| \le C.$$

An application of the Lemma 4.2.1 yields $|u_t| \le C$ on $\bar{D}$.

Now $u_{tt} = 0$ on $\Gamma_1 \cup \Gamma_3$ as $u_t = 0$ on Γ_1 and Γ_3. Differentiating (4.1) with respect to t and put $t = 0$ to obtain

$$-\varepsilon u_{xxt}(x,0) + p(x)u_{xt}(x,0) + q(x)u_t(x,0) + r(x)u_t(x-1,0) + u_{tt}(x,0) = g_t(x,0). \tag{4.5}$$

Since $u_t(x,0) = g(x,0)$, thus

$$u_{xt}(x,0) = g_x(x,0), \quad u_{xxt}(x,0) = g_{xx}(x,0)$$

and by definition, we have

$$u_t(x-1,0) = \lim_{\Delta t \to 0} \frac{u(x-1,\Delta t) - u(x-1,0)}{\Delta t} = 0.$$

From (4.5), it follows that

$$u_{tt}(x,0) = \varepsilon g_{xx}(x,0) - p(x)g_x(x,0) - q(x)g_x(x,0) + g_t(x,0).$$

Thus on Γ_2, $|u_{tt}| \le C$ for significantly large value of C as g is continuous on $\bar{D}$. Therefore, $|u_{tt}| \le C$ on Γ. Now, applying the differential operator $\mathcal{L}$ on u_{tt} gives

$$|\mathcal{L}u_{tt}(x,t)| = |g_{tt}(x,t)| \le C \text{ on } D.$$

Therefore, from Lemma 4.2.1, we have $|u_{tt}(x,t)| \le C$ on $\bar{D}$. □

4.3 Time Discretization

Let $\mathbb{T}_t{}^M = \{t_k = kT/M, k = 0,\ldots,M\}$ be an equidistant mesh that partitions the domain $[0,T]$ into M number of subintervals. We semidiscretize (4.1) using the Crank-Nicholson scheme in the time variable. The resulting semidiscrete problem on $\mathbb{T}_t{}^M$ thus reads

$$\begin{aligned} &\frac{-\varepsilon}{2}U_{xx}(x,t_{k+1}) + \frac{p(x)}{2}U_x(x,t_{k+1}) + l(x)U(x,t_{k+1}) + \frac{r(x)}{2}U(x-1,t_{k+1}) \\ &\quad = \frac{\varepsilon}{2}U_{xx}(x,t_k) - \frac{p(x)}{2}U_x(x,t_k) + m(x)U(x,t_k) - \frac{r(x)}{2}U(x-1,t_k) \\ &\quad + \frac{g(x,t_{k+1}) + g(x,t_k)}{2}, \quad x \in (0,1)\cup(1,2) \text{ and } k = 0,1,\ldots,M-1 \end{aligned} \tag{4.6}$$

such that

$$\begin{cases} U(x,0) = \psi_2(x,0), \quad 0 \le x \le 2, \\ U(x,t_{k+1}) = \psi_1(x,t_{k+1}), \quad -1 \le x \le 0, \quad 0 \le k \le M-1, \\ U(2,t_{k+1}) = \psi_3(2,t_{k+1}), \quad 0 \le k \le M-1, \\ U(1^-,t_{k+1}) = U(1^+,t_{k+1}), \quad U_x(1^-,t_{k+1}) = U_x(1^+,t_{k+1}), \quad 0 \le k \le M-1, \end{cases} \tag{4.7}$$

where, $l(x) = \dfrac{\Delta t q(x) + 2}{2\Delta t}$ and $m(x) = \dfrac{2 - \Delta t q(x)}{2\Delta t}$. Let us rewrite (4.6) as

$$\mathfrak{L}_{CN}U(x,t_{k+1}) = \hat{\mathcal{G}}(x,t_{k+1}), \tag{4.8}$$

where

$$\mathfrak{L}_{CN}U(x,t_{k+1}) = \begin{cases} \mathfrak{L}_{CN1}U = \frac{-\varepsilon}{2}U_{xx}(x,t_{k+1}) + \frac{p(x)}{2}U_x(x,t_{k+1}) + l(x)U(x,t_{k+1}) & \text{if } x \in (0,1), \\ \mathfrak{L}_{CN2}U = \frac{-\varepsilon}{2}U_{xx}(x,t_{k+1}) + \frac{p(x)}{2}U_x(x,t_{k+1}) + l(x)U(x,t_{k+1}) & \\ \quad + \frac{r(x)}{2}U(x-1,t_{k+1}) & \text{if } x \in (1,2), \end{cases}$$

and

$$\hat{\mathcal{G}}(x,t_{k+1}) = \begin{cases} \hat{\mathcal{G}}_1 = \frac{\varepsilon}{2}U_{xx}(x,t_k) - \frac{p(x)}{2}U_x(x,t_k) + m(x)U(x,t_k) - \frac{r(x)}{2}U(x-1,t_k) \\ \quad +\frac{g(x,t_{k+1})+g(x,t_k)}{2} - \frac{r(x)}{2}\psi_1(x-1,t_{k+1}) \text{ if } x \in (0,1), \\ \hat{\mathcal{G}}_2 = \frac{\varepsilon}{2}U_{xx}(x,t_k) - \frac{p(x)}{2}U_x(x,t_k) + m(x)U(x,t_k) - \frac{r(x)}{2}U(x-1,t_k) \\ \quad +\frac{g(x,t_{k+1})+g(x,t_k)}{2} \text{ if } x \in (1,2). \end{cases} \tag{4.9}$$

The operator $\mathfrak{L}_{CN}$ satisfies the following maximum principle.

Lemma 4.3.1. *Let $\chi(x,t_{k+1})$ be a smooth function such that $\chi(x,t_{k+1}) \geq 0$ for $x = 0,2$, $\mathfrak{L}_{CN}\chi(x,t_{k+1}) \geq 0$ for all $x \in (0,1)\cup(1,2)$ and $[\chi_x](1,t_{k+1}) \leq 0$. Then $\chi(x,t_{k+1}) \geq 0$ for all $x \in \bar{\Omega}$.*

Proof. Let $\chi(\xi,t_{k+1}) = \min_{x\in\bar{\Omega}} \chi(x,t_{k+1})$ for some $\xi \in \bar{\Omega}$. Then

$$\chi_x(\xi,t_{k+1}) = 0 \quad \text{and} \quad \chi_{xx}(\xi,t_{k+1}) > 0. \tag{4.10}$$

Suppose $\chi(\xi,t_{k+1}) < 0$ and it follows that $(\xi,t_{k+1}) \notin \Gamma$ since $\chi(\xi,t_{k+1}) \geq 0$ for $x = 0,2$.

Case I: If $\xi \in (0,1)$

$$\begin{aligned} \mathfrak{L}_{CN1}\chi(\xi,t_{k+1}) &= \frac{-\varepsilon}{2}\chi_{xx}(\xi,t_{k+1}) + \frac{p(\xi)}{2}\chi_x(\xi,t_{k+1}) + l(\xi)\chi(\xi,t_{k+1}) \\ &< 0, \text{ from (4.2) and (4.10).} \end{aligned}$$

Case II: If $\xi \in (1,2)$

$$\begin{aligned} \mathfrak{L}_{CN2}\chi(\xi,t_{k+1}) &= \frac{-\varepsilon}{2}\chi_{xx}(\xi,t_{k+1}) + \frac{p(\xi)}{2}\chi_x(\xi,t_{k+1}) + l(\xi)\chi(\xi,t_{k+1}) + \frac{r(\xi)}{2}\chi(\xi-1,t_{k+1}) \\ &\leq \frac{-\varepsilon}{2}\chi_{xx}(\xi,t_{k+1}) + \frac{p(\xi)}{2}\chi_x(\xi,t_{k+1}) + l(\xi)\chi(\xi,t_{k+1}) + \frac{r(\xi)}{2}\chi(\xi,t_{k+1}) \\ &= \frac{-\varepsilon}{2}\chi_{xx}(\xi,t_{k+1}) + \left(\frac{\Delta t q(\xi)+2}{2\Delta t}\right)\chi(\xi,t_{k+1}) + \frac{r(\xi)}{2}\chi(\xi,t_{k+1}) \\ &= \frac{-\varepsilon}{2}\chi_{xx}(\xi,t_{k+1}) + \left(\frac{q(\xi)+r(\xi)}{2}\right)\chi(\xi,t_{k+1}) + \frac{1}{\Delta t}\chi(\xi,t_{k+1}) \\ &\leq \frac{-\varepsilon}{2}\chi_{xx}(\xi,t_{k+1}) + \eta\chi(\xi,t_{k+1}) + \frac{1}{\Delta t}\chi(\xi,t_{k+1}) \\ &< 0, \text{ from (4.2) and (4.10).} \end{aligned}$$

Case III: If $\xi = 1$

$$[\chi_x](\xi,t_{k+1}) = \chi_x(\xi^+,t_{k+1}) - \chi_x(\xi^-,t_{k+1}) > 0 \text{ since } \chi(\xi,t_{k+1}) < 0.$$

The required result thus follows from contradiction. □

Lemma 4.3.2. *Let* $U(x, t_{k+1})$ *be the solution of* (4.6). *Then*

$$\|U(x, t_{k+1})\|_{\infty,\bar{\Omega}} \leq \max\left\{|U(0, t_{k+1})|, |U(2, t_{k+1})|, \frac{\Delta t}{\eta\Delta t + 1}\|\hat{\mathcal{G}}\|_{\infty,\bar{\Omega}}\right\}.$$

Proof. Consider $\zeta_{\pm}(x, t_{k+1}) = \max\left\{|U(0, t_{k+1})|, \frac{\Delta t}{\eta\Delta t + 1}\|\hat{\mathcal{G}}\|_{\infty,\bar{\Omega}}\right\} \pm U(x, t_{k+1})$ for $x \in [0, 1)$. Then, $\zeta_{\pm}(0, t_{k+1}) \geq 0$. Moreover

$$\begin{aligned}
\mathcal{L}_{CN}\zeta_{\pm}(x, t_{k+1}) &= l(x)\max\left\{|U(0, t_{k+1})|, \frac{\Delta t}{\eta\Delta t + 1}\|\hat{\mathcal{G}}_1\|\right\} \pm \mathcal{L}_{CN1}U(x, t_{k+1}) \\
&\geq \frac{\Delta t q(x) + 2}{2\Delta t}\frac{\Delta t}{\eta\Delta t + 1}\|\hat{\mathcal{G}}_1\| \pm \hat{\mathcal{G}}_1 \\
&\geq \frac{(q(x) + r(x))\,\Delta t + 2}{2(\eta\Delta t + 1)}\|\hat{\mathcal{G}}_1\| \pm \hat{\mathcal{G}}_1 \\
&\geq 0, \text{ from (4.2).}
\end{aligned}$$

Similarly, consider $\zeta_{\pm}(x, t_{k+1}) = \max\left\{|U(2, t_{k+1})|, \frac{\Delta t}{\eta\Delta t + 1}\|\hat{\mathcal{G}}\|_{\infty,\bar{\Omega}}\right\} \pm U(x, t_{k+1})$ for $x \in (1, 2]$. Then, $\zeta_{\pm}(2, t_{k+1}) \geq 0$. Also

$$\begin{aligned}
\mathcal{L}_{CN}\zeta_{\pm}(x, t_{k+1}) &= \mathcal{L}_{CN2}\zeta_{\pm}(x, t_{k+1}) \\
&= \left(l(x) + \frac{r(x)}{2}\right)\max\left\{|U(2, t_{k+1})|, \frac{\Delta t}{\eta\Delta t + 1}\|\hat{\mathcal{G}}_2\|\right\} \pm \mathcal{L}_{CN2}U(x, t_{k+1}) \\
&\geq \left(\frac{\Delta t q(x) + 2}{2\Delta t} + \frac{r(x)}{2}\right)\frac{\Delta t}{\eta\Delta t + 1}\|\hat{\mathcal{G}}_2\| \pm \hat{\mathcal{G}}_2 \\
&\geq \left(\frac{q(x) + r(x)}{2} + \frac{1}{\Delta t}\right)\frac{\Delta t}{\eta\Delta t + 1}\|\hat{\mathcal{G}}_2\| \pm \hat{\mathcal{G}}_2 \\
&\geq 0, \text{ from (4.2).}
\end{aligned}$$

Moreover, $[\zeta_{x\pm}](1, t_{k+1}) = \pm[U_x](1, t_{k+1}) = 0$. Consequently, the required result follows from Lemma 4.3.1. □

Next, we compute global error using local error bound. From (4.8)

$$\mathcal{L}_{CN}\tilde{U}(x, t_{k+1}) = \tilde{\mathcal{G}}(x, t_{k+1}), \tag{4.11}$$

where $\mathcal{L}_{CN}$ is as defined in (4.8), and

$$\tilde{\mathcal{G}}(x, t_{k+1}) = \begin{cases} \frac{\varepsilon}{2}u_{xx}(x, t_k) - \frac{p(x)}{2}u_x(x, t_k) + m(x)u(x, t_k) - \frac{r(x)}{2}u(x-1, t_k) \\ +\frac{g(x, t_{k+1}) + g(x, t_k)}{2} - \frac{r(x)}{2}\psi_1(x-1, t_{k+1}) & \text{if } x \in (0, 1], \\ \frac{\varepsilon}{2}u_{xx}(x, t_k) - \frac{p(x)}{2}u_x(x, t_k) + m(x)u(x, t_k) - \frac{r(x)}{2}u(x-1, t_k) \\ +\frac{g(x, t_{k+1}) + g(x, t_k)}{2} & \text{if } x \in (1, 2) \end{cases}$$

with

$$\begin{cases} \tilde{U}(x,0) = \psi_2(x), \quad x \in [0,2], \\ \tilde{U}(x,t_{k+1}) = \psi_1(x,t_{k+1}), \quad -1 \le x \le 0, \ -1 \le k \le M-1, \\ \tilde{U}(2,t_{k+1}) = \psi_3(t_{k+1}), \quad -1 \le k \le M-1. \end{cases}$$

Lemma 4.3.3. *Let* $\hat{e}_{k+1} := \tilde{U}(x,t_{k+1}) - u(x,t_{k+1})$ *be the local truncation error at* $(k+1)$*th time step. Then* $\|\hat{e}_{k+1}\|_\infty \le C(\Delta t)^3$ *for some constant* C.

Proof. For proof, see [50]. □

Moreover, local truncation error at each time step contributes to the estimate for global error $E_{k+1} := u(x,t_{k+1}) - U(x,t_{k+1})$. Then, it follows that

$$\|E_{k+1}\|_\infty = \left\| \sum_{i=1}^{k} \hat{e}_i \right\|_\infty \le \|\hat{e}_1\|_\infty + \|\hat{e}_2\|_\infty + \ldots + \|\hat{e}_k\|_\infty \le C\Delta t^2. \tag{4.12}$$

As a result, the time semidiscretization procedure achieves uniform convergence.

The solution $U(x,t_{k+1})$ of the semidiscretized problem (4.6) is known to admit a decomposition into smooth and singular components [173]. We write

$$U(x,t_{k+1}) := X(x,t_{k+1}) + Z(x,t_{k+1}).$$

Here, the smooth component $X(x,t_{k+1})$ satisfies

$$\begin{cases} \mathcal{L}_{CN1} X(x,t_{k+1}) = \hat{\mathcal{G}}_1(x,t_{k+1}), \\ X(0,t_{k+1}) = X_0(0,t_{k+1}), \\ X(1,t_{k+1}) = (l(x))^{-1} \hat{\mathcal{G}}_1(1,t_{k+1}) \end{cases} \tag{4.13}$$

in $(0,1)$, and in $(1,2)$ satisfies

$$\begin{cases} \mathcal{L}_{CN2} X(x,t_{k+1}) = \hat{\mathcal{G}}_2(x,t_{k+1}), \\ X(1,t_{k+1}) = (l(x))^{-1} \left(\hat{\mathcal{G}}_2(1,t_{k+1}) - \dfrac{r(1)}{2} X(0,t_{k+1}) \right), \\ X(2,t_{k+1}) = X_0(2,t_{k+1}), \end{cases} \tag{4.14}$$

where $X_0(x,t_{k+1})$ satisfies the associated reduced problem. Also, the singular component $Z(x,t_{k+1})$ satisfies

$$\begin{cases} \mathcal{L}_{CN} Z(x,t_{k+1}) = 0, \quad x \in (0,1) \cup (1,2), \\ Z(0,t_{k+1}) = 0, \\ Z(2,t_{k+1}) = U(2,t_{k+1}) - X(2,t_{k+1}), \\ Z(1^+,t_{k+1}) - Z(1^-,t_{k+1}) = X(1^-,t_{k+1}) - X(1^+,t_{k+1}), \\ Z_x(1^+,t_{k+1}) - Z_x(1^-,t_{k+1}) = X_x(1^-,t_{k+1}) - X_x(1^+,t_{k+1}). \end{cases} \tag{4.15}$$

Further, decompose $Z(x,t_{k+1})$ as $Z(x,t_{k+1}) := Z_I(x,t_{k+1}) + Z_B(x,t_{k+1})$, where $Z_B(x,t_{k+1})$ satisfies

$$\begin{cases} \mathcal{L}_{CN} Z_B(x,t_{k+1}) = 0, \quad x \in (0,1)\cup(1,2), \\ Z_B(0,t_{k+1}) = 0, \\ Z(2,t_{k+1}) = U(2,t_{k+1}) - X(2,t_{k+1}), \end{cases} \tag{4.16}$$

and $Z_I(x,t_{k+1})$ satisfies

$$\begin{cases} \mathcal{L}_{CN} Z_I(x,t_{k+1}) = 0, \quad x \in (0,1)\cup(1,2), \\ Z_I(0,t_{k+1}) = 0, \quad Z_I(2,t_{k+1}) = 0, \\ \dfrac{dZ_I}{dx}(1^+,t_{k+1}) - \dfrac{dZ_I}{dx}(1^-,t_{k+1}) = \dfrac{dX}{dx}(1^-,t_{k+1}) - \dfrac{dX}{dx}(1^+,t_{k+1}). \end{cases} \tag{4.17}$$

The following lemma provides bounds on the derivatives of the smooth component $X(x,t_{k+1})$ and singular component $Z(x,t_{k+1})$.

Lemma 4.3.4. *Let $X(x,t_{k+1})$ be the solution of* (4.13)-(4.14) *and $Z(x,t_{k+1})$ be the solution of* (4.15)-(4.17). *Then, for $k = 0,1,2,3$*

$$\left|\frac{d^k X(x,t_{k+1})}{dx^k}\right| \le C(1+\varepsilon^{2-k}) \text{ for } x \in (0,1)\cup(1,2),$$

$$\left|\frac{d^k Z_B(x,t_{k+1})}{dx^k}\right| \le C\varepsilon^{-k}\exp\left(\frac{-p_0^*(2-x)}{\varepsilon}\right) \text{ for } x \in (0,1)\cup(1,2), \text{ and}$$

$$\left|\frac{d^k Z_I(x,t_{k+1})}{dx^k}\right| \le \begin{cases} C\varepsilon^{1-k}\exp\left(\dfrac{-p_0^*(1-x)}{\varepsilon}\right) & \text{for } x \in (0,1), \\ C\varepsilon^{1-k} & \text{for } x \in (1,2). \end{cases}$$

Proof. For proof, see [257]. □

4.4 Spatial Discretization

The solution of the problem exhibits a strong boundary layer at $x = 2$ and a weak interior layer at $x = 1$. Therefore, to generate a piecewise-uniform mesh $\bar{\mathbb{D}}_x^N$, we partition the given interval $[0,2]$ into four subintervals as

$$[0,2] = [0,1-\beta]\cup[1-\beta,1]\cup[1,2-\beta]\cup[2-\beta,2],$$

where $\beta = \min\left\{0.5, \dfrac{2\varepsilon\ln N}{p_0^*}\right\}$ is the mesh transition parameter. Each subinterval contains $N/4$ mesh points. Consequently, we obtain

$$\bar{\mathbb{D}}_x^N = \{x_i\}_0^N = \begin{cases} x_i = 0 & \text{for } i = 0, \\ x_i = x_{i-1} + h_i & \text{for } i = 1,\ldots,N, \end{cases}$$

where

$$h_i = \begin{cases} \frac{4}{N}(1-\beta) & \text{for } i = 1,\dots,N/4, N/2+1,\dots,3N/4, \\ \frac{4}{N}\beta & \text{for } i = N/4+1,\dots,N/2, 3N/4+1,\dots,N. \end{cases}$$

The discrete problem on $\bar{D}^{N,M} = \bar{\mathbb{D}}_x^N \times \mathbb{T}_t{}^M$ thus reads

$$\mathcal{L}_{CN}^N U_{i,k+1} = \hat{\mathcal{G}}_{i,k+1}, \; i = 1,\dots,N-1, \tag{4.18}$$

where

$$\mathcal{L}_{CN}^N U_{i,k+1} = \begin{cases} \mathcal{L}_{CN1}^N U_{i,k+1} = \dfrac{-\varepsilon}{2}\delta_x^2 U_{i,k+1} + \dfrac{p_i}{2} D_x^- U_{i,k+1} + l_i U_{i,k+1} \\ \text{for } i = 1,\dots,N/2-1, \\ \mathcal{L}_{CN2}^N U_{i,k+1} = \dfrac{-\varepsilon}{2}\delta_x^2 U_{i,k+1} + \dfrac{p_i}{2} D_x^- U_{i,k+1} + l_i U_{i,k+1} + \dfrac{r_i}{2} U_{i-N/2,k+1} \\ \text{for } i = N/2+1,\dots,N-1, \end{cases}$$

and

$$\hat{\mathcal{G}}_{i,k+1} = \begin{cases} \hat{\mathcal{G}}_1(x_i, t_{k+1}) = \dfrac{\varepsilon}{2}\delta_x^2 U_{i,k} - \dfrac{p_i}{2} D_x^- U_{i,k} + m_i U_{i,k} - \dfrac{r_i}{2} U_{i-N/2,k} + \dfrac{1}{2}\left(g_{i,k+1} + g_{i,k}\right) \\ \qquad -\dfrac{r_i}{2}\psi_1(x_{i-N/2}, t_{k+1}) \text{ for } i = 1,\dots,N/2-1, \\ \hat{\mathcal{G}}_2(x_i, t_{k+1}) = \frac{\varepsilon}{2}\delta_x^2 U_{i,k} - \frac{p_i}{2} D_x^- U_{i,k} + m_i U_{i,k} - \frac{r_i}{2} U_{i-N/2,k} + \frac{1}{2}\left(g_{i,k+1} + g_{i,k}\right) \\ \qquad \text{for } i = N/2+1,\dots,N-1. \end{cases}$$

Moreover, for $i = N/2$

$$D_x^+ U_{N/2,k+1} = D_x^- U_{N/2,k+1}$$

with

$$\begin{cases} U_{i,0} = \psi_{2,i,0} \text{ for } i = 0,\dots,N, \\ U_{i,k+1} = \psi_{1,i,k+1} \text{ for } i = -N/2, -N/2+1, \dots 0, \; k = 0,1,\dots,M-1, \\ \mathcal{K}^N U_{N,k+1} = U_{N,k+1} - \varepsilon \displaystyle\sum_{i=1}^{N} \frac{f_{i-1} U_{i-1,k+1} + f_i U_{i,k+1}}{2} h_i = \psi_{3,N,k+1} \text{ for } k = 0,1,\dots,M-1. \end{cases}$$

The operator $\mathcal{L}_{CN}^N$ satisfies the following discrete maximum principle.

Lemma 4.4.1. *Let $Z_{i,k+1}$ be the mesh function such that $Z_{i,k+1} \geq 0$ for $i = \{0, N\}$, $\mathcal{L}_{CN}^N Z_{i,k+1} \geq 0$ for all $i = 1,\dots,N/2-1, N/2+1,\dots,N$ and $D_x^+ Z_{N/2,k+1} - D_x^- Z_{N/2,k+1} \leq 0$. Then $Z_{i,k+1} \geq 0$ for all $i = 0, 1 \dots, N$.*

Proof. Choose $j^* \in \{0,1,\dots,N\} \setminus \{N/2\}$ such that $Z_{j^*,k+1} = \min_{\bar{\mathbb{D}}_x^N \times \mathbb{T}_t{}^M} Z_{i,k+1}$. Assume that $Z_{j^*,k+1} < 0$ and it follows that $j^* \notin \{0, N\}$.

Case I: For $j^* \in \{1, 2, \ldots, N/2 - 1\}$

$$
\begin{aligned}
\mathcal{L}_{CN1}^N Z_{j^*,k+1} &= \frac{-\varepsilon}{2}\delta_x^2 Z_{j^*,k+1} + \frac{p_{j^*}}{2} D_x^- Z_{j^*,k+1} + l_{j^*} Z_{j^*,k+1} \\
&= \frac{-\varepsilon}{\hat{h}_{j^*}} \left\{ \frac{Z_{j^*+1,k+1} - Z_{j^*,k+1}}{h_{j^*+1}} - \frac{Z_{j^*,k+1} - Z_{j^*-1,k+1}}{h_{j^*}} \right\} \\
&\quad + \frac{p_{j^*}}{2} \left\{ \frac{Z_{j^*,k+1} - Z_{j^*-1,k+1}}{h_{j^*}} \right\} + l_{j^*} Z_{j^*,k+1} \\
&< 0.
\end{aligned}
$$

Case II: For $j^* \in \{N/2 + 1, \ldots, N - 1\}$

$$
\begin{aligned}
\mathcal{L}_{CN2}^N Z_{j^*,k+1} &= \frac{-\varepsilon}{2}\delta_x^2 Z_{j^*,k+1} + \frac{p_{j^*}}{2} D_x^- Z_{j^*,k+1} + l_{j^*} Z_{j^*,k+1} + \frac{r_{j^*}}{2} Z_{j^*-N/2,k+1} \\
&\leq \frac{-\varepsilon}{2}\delta_x^2 Z_{j^*,k+1} + \frac{p_{j^*}}{2} D_x^- Z_{j^*,k+1} + l_{j^*} Z_{j^*,k+1} + \frac{r_{j^*}}{2} Z_{j^*,k+1} \\
&= \frac{-\varepsilon}{2}\delta_x^2 Z_{j^*,k+1} + \frac{p_{j^*}}{2} D_x^- Z_{j^*,k+1} + (l_{j^*} + r_{j^*}) Z_{j^*,k+1} \\
&= \frac{-\varepsilon}{2}\delta_x^2 Z_{j^*,k+1} + \frac{p_{j^*}}{2} D_x^- Z_{j^*,k+1} + \left(\frac{q_{j^*} + r_{j^*}}{2} + \frac{1}{\Delta t} \right) Z_{j^*,k+1} \\
&\leq \frac{-\varepsilon}{2}\delta_x^2 Z_{j^*,k+1} + \frac{p_{j^*}}{2} D_x^- Z_{j^*,k+1} + \left(\eta + \frac{1}{\Delta t} \right) Z_{j^*,k+1} \\
&= \frac{-\varepsilon}{\hat{h}_{j^*}} \left\{ \frac{Z_{j^*+1,k+1} - Z_{j^*,k+1}}{h_{j^*+1}} - \frac{Z_{j^*,k+1} - Z_{j^*-1,k+1}}{h_{j^*}} \right\} \\
&\quad + \frac{p_{j^*}}{2} \left\{ \frac{Z_{j^*,k+1} - Z_{j^*-1,k+1}}{h_{j^*}} \right\} + \left(\eta + \frac{1}{\Delta t} \right) Z_{j^*,k+1} \\
&< 0.
\end{aligned}
$$

Case III: For $j^* = N/2$, $D_x^+ Z_{N/2,k+1} - D_x^- \phi_{N/2,k+1} > 0$.

The required result follows from contradiction. □

Consequently, we obtain the following stability estimate of the discrete operator $\mathcal{L}_{CN}^N$.

Lemma 4.4.2. *Let $Z_{i,k+1}$ be the solution of* (4.18). *Then*

$$
\|Z_{i,k+1}\|_{\infty,\bar{D}^{N,M}} \leq \max \left\{ |Z_{0,k+1}|, |Z_{N,k+1}|, \frac{\Delta t}{\eta \Delta t + 1} \|\mathcal{L}_{CN}^N Z_{i,k+1}\|_{\infty,\bar{D}^{N,M}} \right\}, \quad \forall \ 0 \leq i \leq N, \ \ 0 \leq k \leq M-1.
$$

Proof. Let $\chi^{\pm}_{i,k+1} = \max\left\{|Z_{0,k+1}|, \dfrac{\Delta t}{\eta\Delta t+1}\|\mathcal{L}^N_{CN1}Z_{i,k+1}\|\right\} \pm Z_{i,k+1}$ for $i = 0,\dots,N/2-1$. Then, $\chi^{\pm}_{0,k+1} \geq 0$ and

$$\begin{aligned}
\mathcal{L}^N_{CN1}\chi^{\pm}_{i,k+1} &= l_i \max\left\{|Z_{0,k+1}|, \frac{\Delta t}{\eta\Delta t+1}\|\mathcal{L}^N_{CN1}Z_{i,k+1}\|\right\} \pm \mathcal{L}^N_{CN1}Z_{i,k+1} \\
&= l_i \max\left\{|Z_{0,k+1}|, \frac{\Delta t}{\eta\Delta t+1}\|\mathcal{L}^N_{CN1}Z_{i,k+1}\|\right\} \pm \hat{\mathcal{G}}_1(x_i, t_{k+1}) \\
&\geq \frac{l_i\Delta t}{\eta\Delta t+1}\|\hat{\mathcal{G}}_1\| \pm \hat{\mathcal{G}}_1 \\
&\geq \frac{(q_i+r_i)\Delta t+2}{2\Delta t}\frac{\Delta t}{\eta\Delta t+1}\|\hat{\mathcal{G}}_1\| \pm \hat{\mathcal{G}}_1 \\
&\geq \frac{2\eta\Delta t+2}{2\Delta t}\frac{\Delta t}{\eta\Delta t+1}\|\hat{\mathcal{G}}_1\| \pm \hat{\mathcal{G}}_1 \\
&\geq 0.
\end{aligned}$$

Next, define $\chi^{\pm}_{i,k+1} = \max\left\{|Z_{N,k+1}|, \dfrac{\Delta t}{\eta\Delta t+1}\|\mathcal{L}^N_{CN2}Z_{i,k+1}\|\right\} \pm Z_{i,k+1}$ for $i = N/2+1,\dots,N$. Then, $\chi^{\pm}_{N,k+1} \geq 0$ and

$$\begin{aligned}
\mathcal{L}^N_{CN2}\chi^{\pm}_{i,k+1} &= \left(l_i + \frac{r_i}{2}\right)\max\left\{|Z_{N,k+1}|, \frac{\Delta t}{\eta\Delta t+1}\|\mathcal{L}^N_{CN2}Z_{i,k+1}\|\right\} \pm \mathcal{L}^N_{CN2}Z_{i,k+1} \\
&= \left(l_i + \frac{r_i}{2}\right)\max\left\{|Z_{N,k+1}|, \frac{\Delta t}{\eta\Delta t+1}\|\mathcal{L}^N_{CN2}Z_{i,k+1}\|\right\} \pm \hat{\mathcal{G}}_2(x_i, t_{k+1}) \\
&\geq \left(\frac{q_i+r_i}{2} + \frac{1}{\Delta t}\right)\left(\frac{\Delta t}{\eta\Delta t+1}\right)\|\hat{\mathcal{G}}_2\| \pm \hat{\mathcal{G}}_2 \\
&\geq \left(\eta + \frac{1}{\Delta t}\right)\left(\frac{\Delta t}{\eta\Delta t+1}\right)\|\hat{\mathcal{G}}_2\| \pm \hat{\mathcal{G}}_2 \\
&\geq 0.
\end{aligned}$$

Moreover, if $i = N/2$, $(D^+_x - D^-_x)\chi^{\pm}_{i,k+1} = 0$. Thus, the required result follows from Lemma 4.4.1. □

4.5 Error Estimates

Let us decompose $U_{i,k+1}$ into smooth and singular components to obtain a parameter uniform error estimate. We write $U_{i,k+1} := X_{i,k+1} + Z_{i,k+1}$, where the smooth component X and the singular component Z satisfy

$$\begin{cases}
\mathcal{L}^N_{CN1}X_{i,k+1} = \hat{\mathcal{G}}_1(x_i, t_{k+1}) \text{ for } i \in \{1,2,\dots,N/2-1\}, \\
X_{0,k+1} = X(0, t_{k+1}),\ X_{N/2-1,k+1} = X(1^-, t_{k+1}), \\
\mathcal{L}^N_{CN2}X_{i,k+1} = \hat{\mathcal{G}}_2(x_i, t_{k+1}) \text{ for } i \in \{N/2+1,\dots,N-1\}, \\
X_{N/2+1,k+1} = X(1^+, t_{k+1}),\ \ X_{N,k+1} = X(2, t_{k+1}),
\end{cases} \tag{4.19}$$

and

$$\begin{cases} \mathcal{L}_{CN}^{N} Z_{i,k+1} = 0, \text{ for } i \in \{1, \ldots, N-1\} \setminus \{N/2\}, \\ Z_{0,k+1} = Z(0, t_{k+1}), \; Z_{N,k+1} = Z(2, t_{k+1}), \\ X_{N/2+1,k+1} + Z_{N/2+1,k+1} = X_{N/2-1,k+1} + Z_{N/2-1,k+1}, \\ D_x^- X_{N/2,k+1} + D_x^- Z_{N/2,k+1} = D_x^+ X_{N/2,k+1} + D_x^+ Z_{N/2,k+1}. \end{cases} \tag{4.20}$$

The error $e_{i,k+1}$ is defined as

$$e_{i,k+1} := U(x_i, t_{k+1}) - U_{i,k+1} = (X(x_i, t_{k+1}) - X_{i,k+1}) + (Z(x_i, t_{k+1}) - Z_{i,k+1}).$$

Theorem 4.5.1. *Let $U(x_i, t_{k+1})$ and $U_{i,k+1}$ be the solutions of* (4.6) *and* (4.18)*, respectively. Then*

$$\left| U(x_i, t_{k+1}) - U_{i,k+1} \right| \le CN^{-1} \ln^2 N, \text{ for } 0 \le i \le N.$$

Proof. The proof follows on the lines similar to the one presented in [257] for ordinary differential equations. □

Finally, we combine (4.12) and Theorem 4.5.1 to obtain the principle convergence result below.

Theorem 4.5.2. *Let u and $U_{i,k+1}$ be the solutions of the continuous problem* (4.1) *and the discrete problem* (4.18)*, respectively. Then*

$$\left| u(x_i, t_{k+1}) - U_{i,k+1} \right| \le C(\Delta t^2 + (N^{-1} \ln^2 N))$$

for $0 \le i \le N$ and $0 \le k \le M$.

4.6 Numerical Illustrations

In this section, we consider two model problems, present numerical results using the proposed method, and verify the theoretical estimates numerically.

Example 4.6.1. *Consider the following singularly perturbed problem with integral boundary condition:*

$$\begin{cases} [-\varepsilon u_{xx} + (2 + x(2-x))u_x + 3u + u_t](x,t) - u(x-1,t) = 4xe^{-t}t^2, \quad (x,t) \in (0,2) \times (0,2], \\ u(x,t) = 0, \quad (x,t) \in \Gamma_1, \\ u(x,t) = 0, \quad (x,t) \in \Gamma_2, \\ u(2,t) = \dfrac{\varepsilon}{6} \displaystyle\int_0^2 u(x,t)dx, \quad (x,t) \in \Gamma_3. \end{cases}$$

Example 4.6.2. *Consider the following singularly perturbed problem with integral boundary condition:*

$$\begin{cases} [-\varepsilon u_{xx} + 3u_x + (x+10)u + u_t](x,t) - u(x-1,t) = 2(1+x^2)t^2, \quad (x,t) \in (0,2) \times (0,2], \\ u(x,t) = t^2, \quad (x,t) \in \Gamma_1, \\ u(x,t) = 0, \quad (x,t) \in \Gamma_2, \\ u(2,t) = \dfrac{\varepsilon}{6} \displaystyle\int_0^2 x \sin(x) u(x,t) dx, \quad (x,t) \in \Gamma_3. \end{cases}$$

The exact solutions of the above examples are not known for comparison. Therefore, the double mesh principle [173] is used to estimate the proposed method's error and rate of convergence. The maximum absolute error ($E_\varepsilon^{N,\triangle t}$) and order of convergence ($R_\varepsilon^{N,\triangle t}$) are defined as

$$E_\varepsilon^{N,\triangle t} := \max \left| U_{N,\triangle t}(x_i, t_{k+1}) - \tilde{U}_{2N,\triangle t/2}(x_i, t_{k+1}) \right| \text{ and } R_\varepsilon^{N,\triangle t} := \log_2 \left(\frac{E_\varepsilon^{N,\triangle t}}{E_\varepsilon^{2N,\triangle t/2}} \right),$$

where, $U_{N,\triangle t}(x_i, t_{k+1})$ and $\tilde{U}_{2N,\triangle t/2}(x_i, t_{k+1})$ denote the numerical solutions on $\bar{\mathbb{D}}_x^N \times \mathbb{T}_t{}^M$ and $\bar{\mathbb{D}}_x^{2N} \times \mathbb{T}_t{}^{2M}$, respectively.

The maximum point-wise error ($E_\varepsilon^{N,\triangle t}$) and the corresponding order of convergence ($R_\varepsilon^{N,\triangle t}$) for Example 4.6.1 and 4.6.2 are tabulated for different values of ε, M, and N in Tables 4.1 and 4.3, respectively. In addition to this, Tables 4.2 and 4.4 depict the order of convergence in time variable for Examples 4.6.1 and 4.6.2 when $N = 512$ and $\varepsilon = 2^{-6}$.

The presence of interior and boundary layers is apparent from the surface plots of the numerical solution for Examples 4.6.1 and 4.6.2 displayed in Figures 4.1 and 4.3, respectively. Figures 4.2 and 4.4 further illustrate the presence of the layers when $t = 2$ for Examples 4.6.1 and 4.6.2. In contrast, Figures 4.5-4.8 present the solution for different time t and for different values of ε for given examples. It is to observed that as ε approaches the limiting value, it attributes the stiffness to the system and leads to exponential changes across the interior and boundary layers. The log-log plots of errors are given in Figures 4.9-4.10 for Examples 4.6.1 and 4.6.2, respectively. It agrees with the expected convergence rate for the proposed method on the specially generated mesh.

Table 4.1: Maximum absolute error and order of convergence for Example 4.6.1 for different values of ε, M and N when $M = N$.

ε	$M = N = 32$	64	128	256	512	1024
2^{-1}	1.081e-02	5.961e-03	3.117e-03	1.595e-03	8.078e-04	2.857e-04
	0.8587	0.9354	0.9666	0.9814	1.4995	1.5612
2^{-3}	3.050e-02	1.726e-02	7.182e-03	2.791e-03	9.452e-04	3.352e-04
	0.8213	1.2650	1.3636	1.5621	1.4956	1.5516
2^{-5}	1.720e-02	9.412e-03	4.872e-03	2.232e-03	8.921e-04	3.172e-04
	0.8698	0.9499	1.1262	1.3232	1.4918	1.5245
2^{-7}	1.753e-02	9.578e-03	4.235e-03	2.094e-03	8.232e-04	3.099e-04
	0.8720	1.1774	1.0161	1.3473	1.4094	1.4126
2^{-9}	1.785e-02	9.914e-03	4.246e-03	1.875e-03	7.582e-04	2.859e-04
	0.8483	1.2234	1.1792	1.3062	1.4071	1.4221
2^{-11}	1.794e-02	1.043e-02	5.021e-03	2.098e-03	9.620e-04	3.900e-04
	0.7824	1.0547	1.2590	1.1249	1.3026	1.3861

Table 4.2: Maximum absolute error and order of convergence for Example 4.6.1 for different values of M and x when $N = 512$ and $\varepsilon = 2^{-6}$.

x	$M = 32$	64	128	256	512	1024
$x_{N/2+1}$	8.416e-03	2.258e-03	5.966e-04	1.547e-04	3.926e-05	9.972e-06
	1.9881	1.9202	1.9473	1.9783	1.9771	1.9916
$x_{N/2+4}$	8.699e-03	2.366e-03	6.274e-04	1.652e-04	4.172e-05	1.053e-05
	1.8784	1.9150	1.9252	1.9854	1.9862	1.9885
$x_{N/2+6}$	8.920e-03	2.428e-03	6.447e-04	1.633e-04	3.926e-05	9.720e-06
	1.8773	1.9131	1.9811	2.0557	2.0148	2.0557

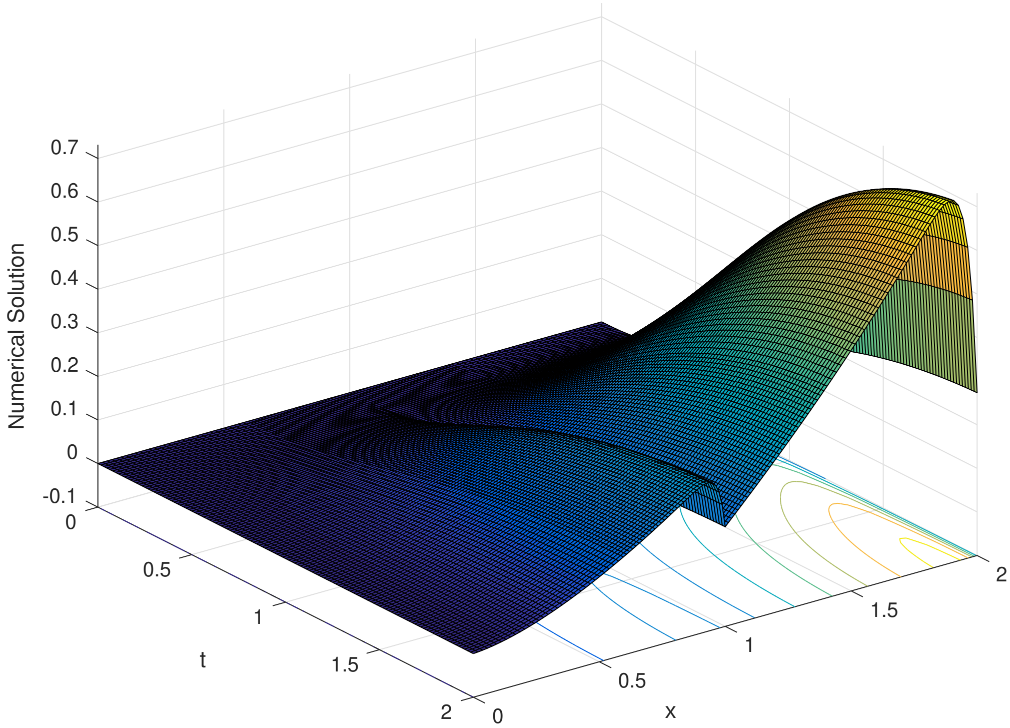

Figure 4.1: Numerical solution of Example 4.6.1 for $\varepsilon = 2^{-4}$ when $M = N = 128$.

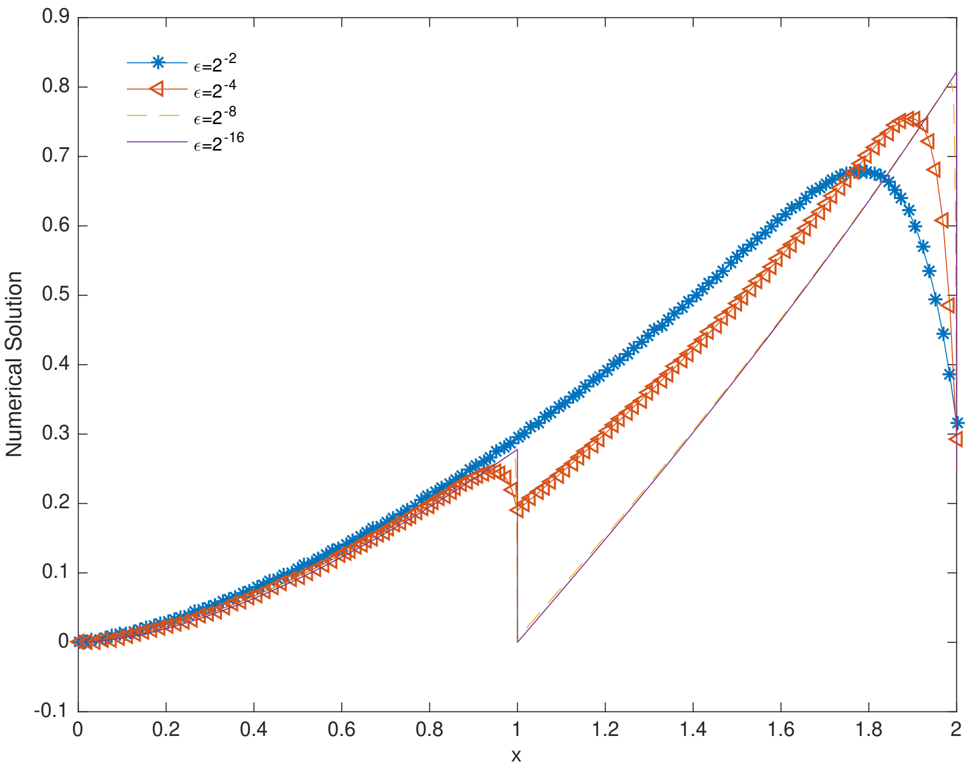

Figure 4.2: Numerical solutions of Example 4.6.1 at $t = 2$ for different values of ε when $N = 128$.

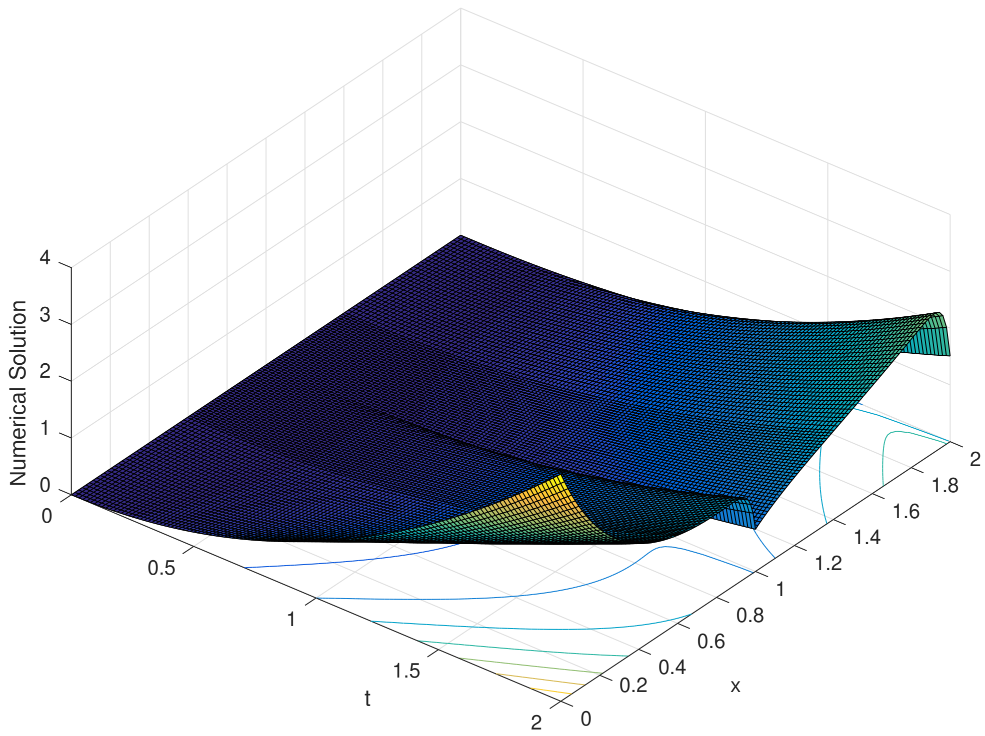

Figure 4.3: Numerical solution of Example 4.6.2 for $\varepsilon = 2^{-4}$ when $M = N = 128$.

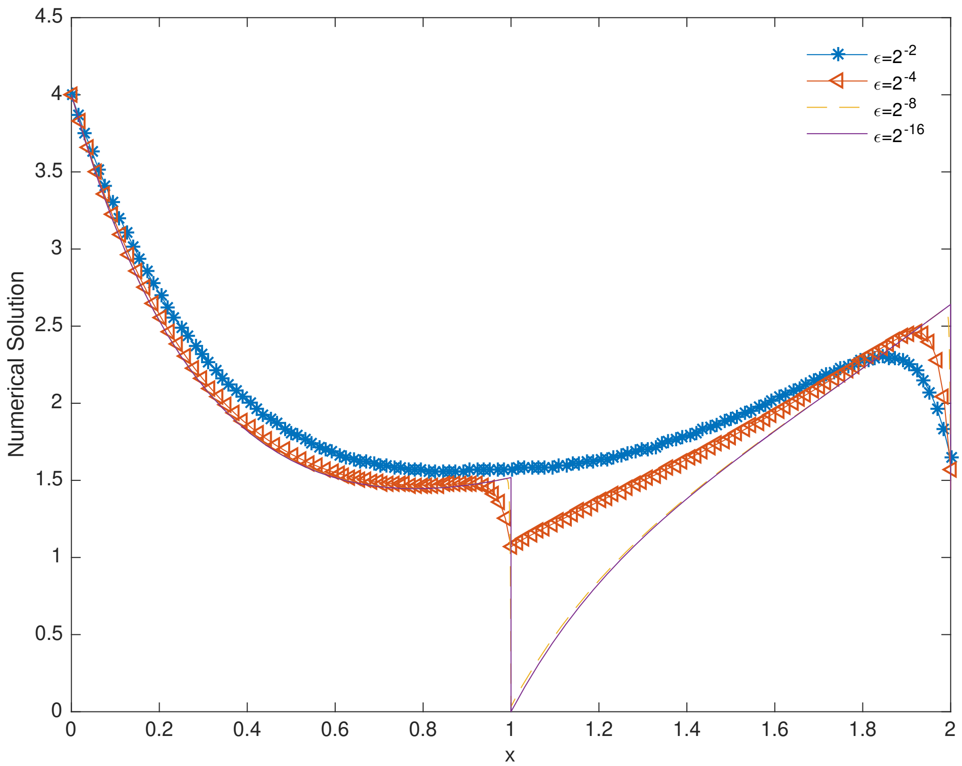

Figure 4.4: Numerical solutions of Example 4.6.2 at $t = 2$ for different values of ε when $N = 128$.

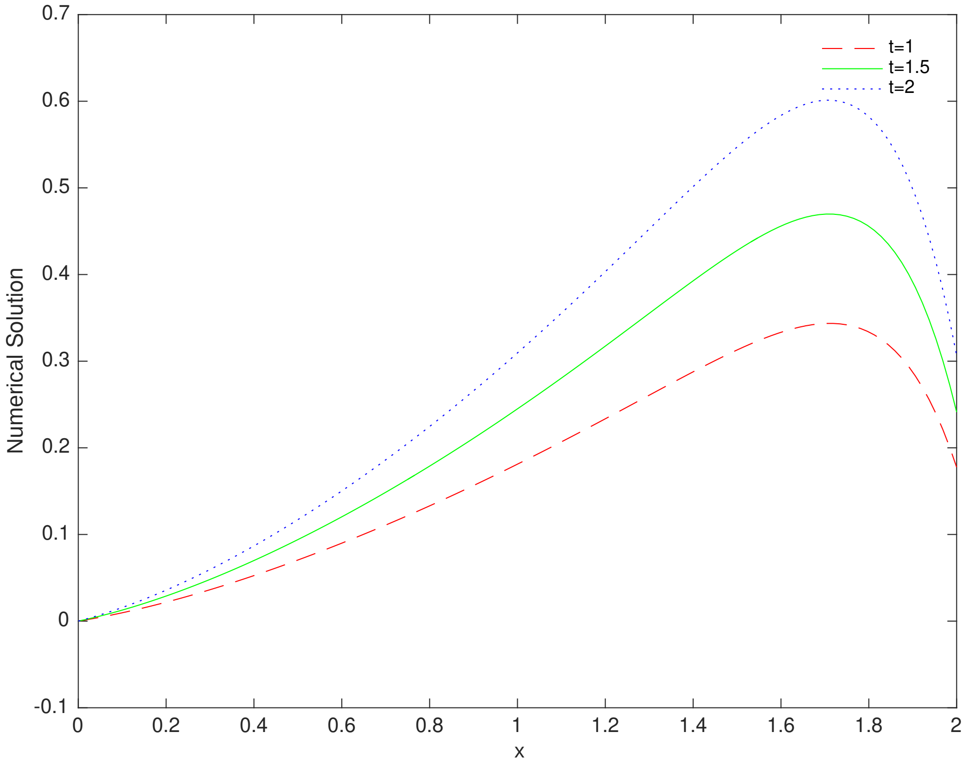

Figure 4.5: Numerical solutions of Example 4.6.1 for different values of t when $\varepsilon = 2^{-1}$.

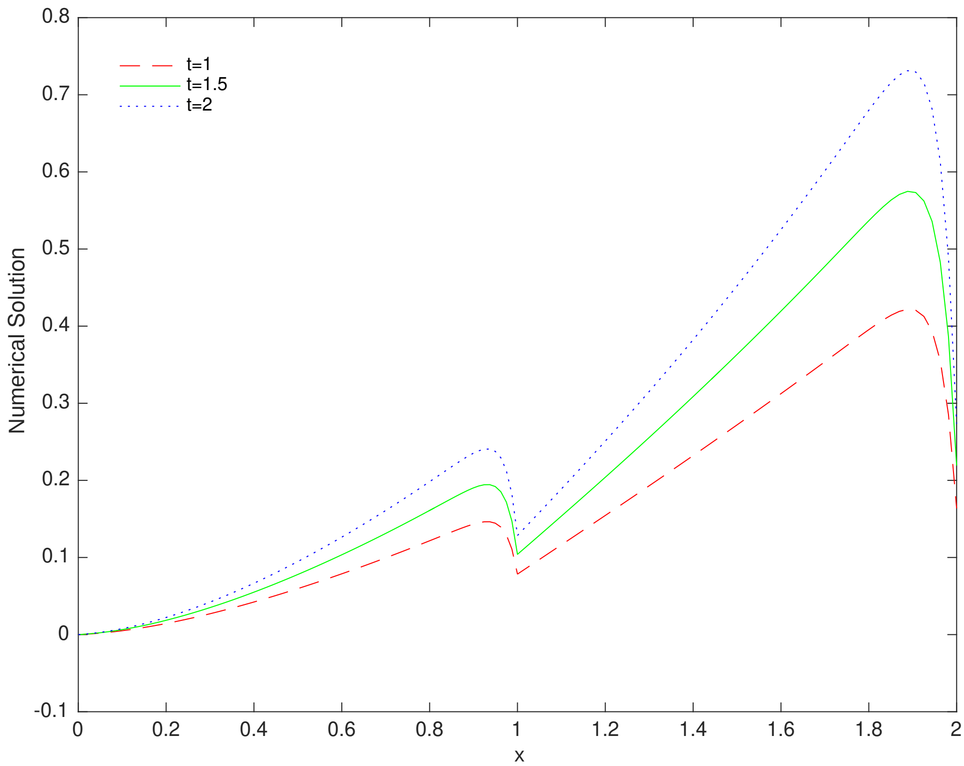

Figure 4.6: Numerical solutions of Example 4.6.1 for different values of t when $\varepsilon = 2^{-4}$.

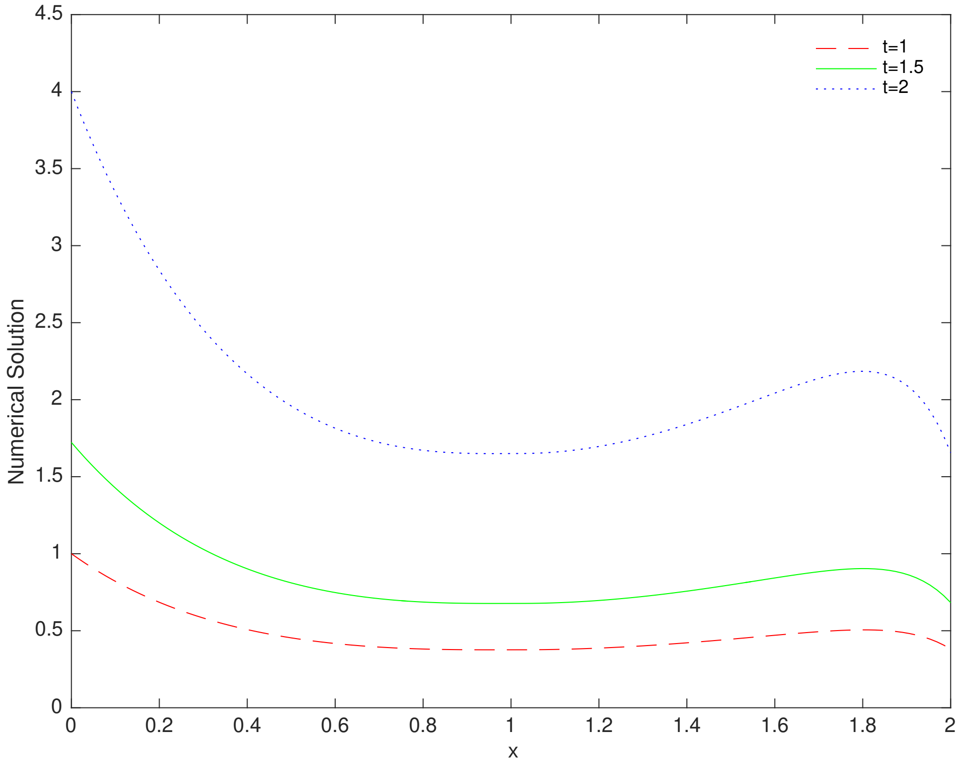

Figure 4.7: Numerical solution of Example 4.6.2 for different different values of t when $\varepsilon = 2^{-1}$.

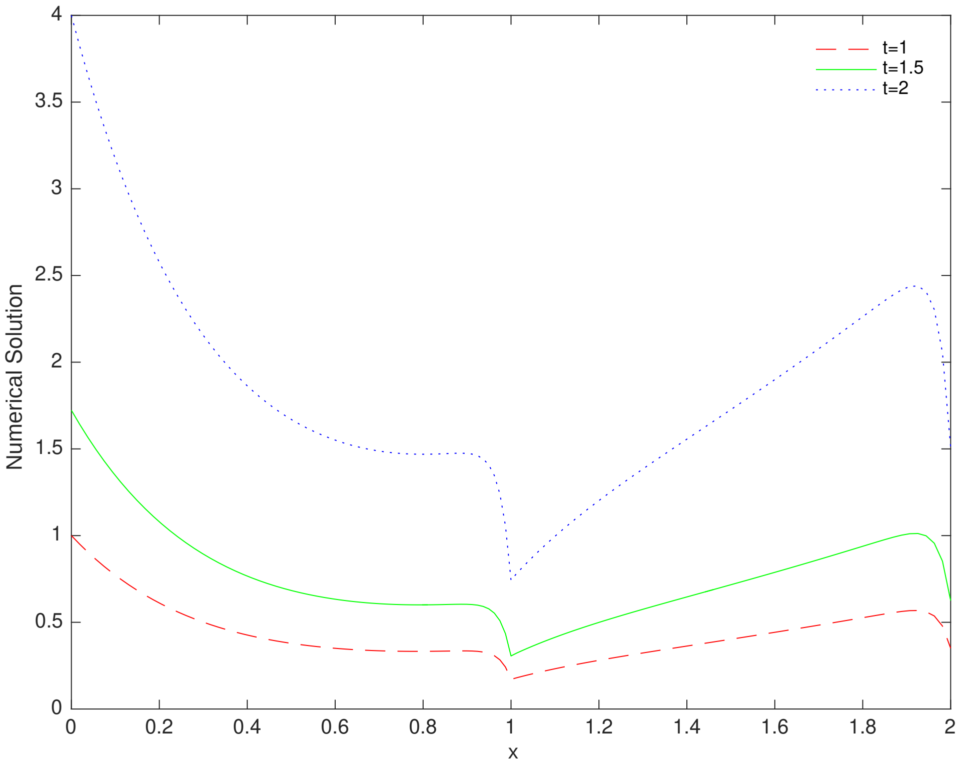

Figure 4.8: Numerical solutions of Example 4.6.2 for different different values of t when $\varepsilon = 2^{-4}$.

Table 4.3: Maximum absolute error and order of convergence for Example 4.6.2 for different values of ε, M and N when $M = N$.

ε	$M = N = 32$	64	128	256	512	1024
2^{-1}	2.822e-02	9.562e-03	4.223e-03	1.832e-03	6.140e-04	1.902e-04
	1.5613	1.1790	1.2048	1.5771	1.6907	1.6973
2^{-3}	1.831e-02	7.134e-03	2.917e-03	1.025e-03	3.872e-04	1.325e-04
	1.3598	1.2902	1.5089	1.4045	1.5471	1.6603
2^{-5}	2.090e-02	8.523e-03	4.245e-03	1.834e-03	7.179e-04	2.625e-04
	1.2941	1.0056	1.2108	1.3531	1.4515	1.5658
2^{-7}	2.387e-02	1.145e-02	4.675e-03	1.893e-03	8.633e-04	3.236e-04
	1.0595	1.2923	1.3043	1.1327	1.4157	1.5139
2^{-9}	2.461e-02	1.156e-02	4.692e-03	1.852e-03	7.278e-04	2.728e-04
	1.0901	1.3009	1.3411	1.3475	1.4157	1.4652
2^{-11}	2.480e-02	1.256e-02	5.281e-03	2.356e-03	9.320e-04	3.615e-04
	0.9815	1.2502	1.1645	1.3379	1.3663	1.4913

Table 4.4: Maximum absolute error and order of convergence for Example 4.6.2 for different values of M and x when $N = 512$ and $\varepsilon = 2^{-6}$.

x	$M = 32$	64	128	256	512	1024
$x_{N/2+1}$	7.810e-03	2.034e-03	5.184e-04	1.313e-04	3.295e-05	8.048e-06
	1.9410	1.9722	1.9812	1.9945	2.0336	2.0341
$x_{N/2+4}$	7.825e-03	2.014e-03	5.220e-04	1.341e-04	3.425e-05	8.712e-06
	1.9580	1.9479	1.9607	1.9691	1.9750	1.9917
$x_{N/2+6}$	8.164e-03	2.098e-03	5.495e-04	1.412e-04	3.602e-05	9.102e-06
	1.9603	1.9328	1.9604	1.9709	1.9845	1.9906

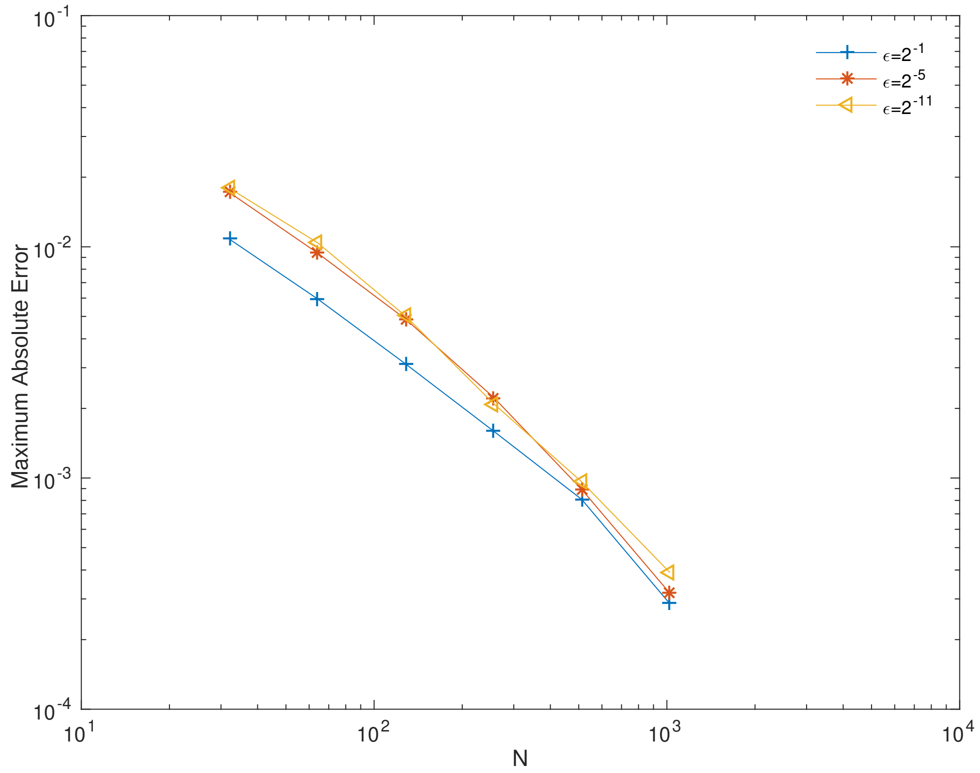

Figure 4.9: Error plot for Example 4.6.1.

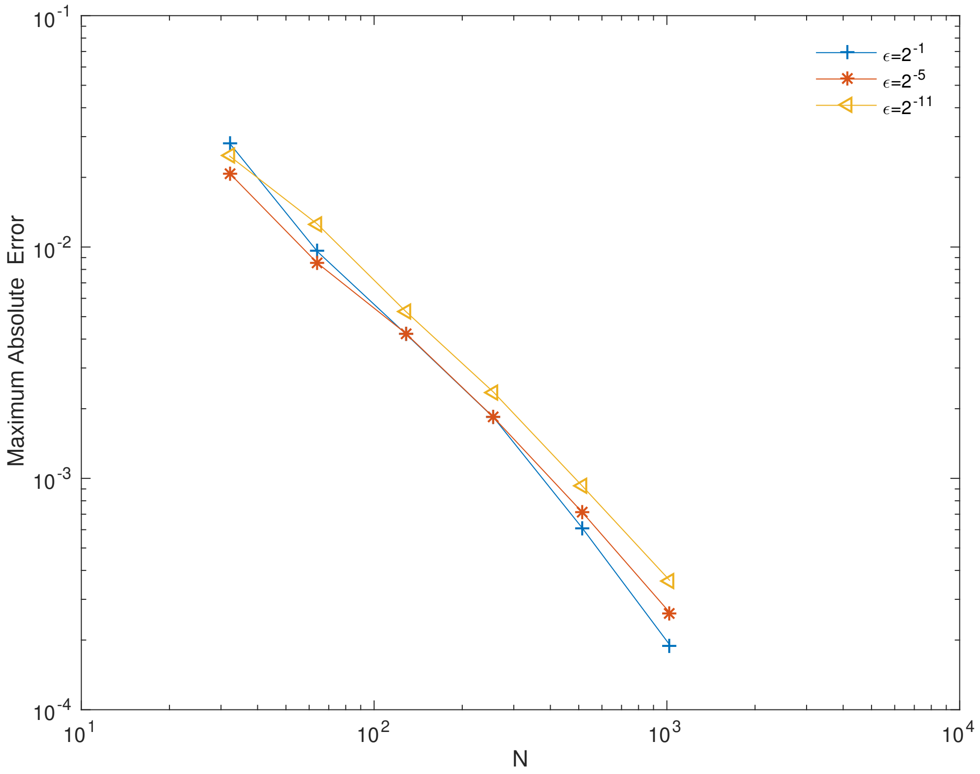

Figure 4.10: Error plot for Example 4.6.2.

4.7 Concluding Remarks

A class of singularly perturbed parabolic partial differential equations with a large delay and an integral boundary condition is solved numerically. The proposed method consists of an upwind finite difference scheme on a non-uniform mesh in space and a Crank-Nicolson scheme on a uniform mesh in the time variable. The non-uniform mesh in the spatial direction is chosen so that most of the mesh points remain in the regions with rapid transitions. The method is investigated for consistency, stability, and convergence. The error analysis of the proposed method reveals the parameter uniform convergence of first-order in space and second-order in time. Numerical experiments corroborate the theoretical findings.

Chapter 5

Parabolic Two-Parameter Problems with Delay

5.1 Introduction

In the previous chapters, we proposed numerical methods for different classes of singularly perturbed parabolic initial boundary value problems that involve only one small perturbation parameter. This chapter extends our study to a class of two-parameter parabolic initial boundary value problems.

The problem we consider involves a parabolic equation with two small parameters which control the system's behaviour and a large delay. This type of problem forms an essential basis for several physical, biological and chemical processes, including chemical flow [46], lubrication theory [65], and reactor theory [214], to name a few among numerous others. The presence of two small parameters leads to interesting phenomena and challenges in the analysis and numerical solution of these problems. The solution exhibits twin boundary layers due to the presence of two parameters and an interior layer due to the large delay. O'Malley has started the study of these problems [214, 215, 213]. He has shown that the nature of the problem and the occurrence of boundary layers in the solution is affected by the magnitude of the two parameters involved. The nature of these layers depends on the ratio of μ^2 and ε, as noted in reference [218].

Many researchers have worked to develop uniformly convergent numerical methods for the solution of two-parameter singularly perturbed parabolic problems. In [206], the researchers consider a two-parameter singularly perturbed time-delay parabolic equation. They employ a fitted operator finite difference scheme to approximate the numerical solution. The first step involves discretising the time variables using the Crank-Nicolson method. The semidiscrete problem is further discretised in space using the exponentially fitted tension-spline finite difference method. The study reveals that the numerical scheme exhibits almost second-order uniform convergence in space and time variables. In [207], they consider a two-parameter singularly perturbed parabolic problem with time delay. They discretised the problem using an exponentially fitted scheme in the spatial direction and the Crank–Nicolson method in the time direction on a uniform mesh. In [17], the authors employ a fitted cubic spline scheme to solve a two-parameter singularly perturbed time-delayed problem. This scheme combines the θ-method on a uniform mesh in the time direction and a cubic spline scheme on a uniform mesh in the spatial direction. In [1], the authors focus on a two-parameter singularly perturbed system of partial differential equations with discontinuous coefficients. This discontinuity and small values of the perturbation parameters cause interior and boundary layers to appear in the solution. They employ a central finite-difference approach on a piecewise uniform Shishkin mesh in the spatial direction and an implicit Euler scheme on a uniform mesh in the time direction. In [261], the authors used a Hermite wavelet-based numerical method to solve two parameters singularly perturbed non-linear Benjamina-Bona-Mohany equation. The method involves time discretisation of Hermite wavelet series approximations with collocation technique. In [305], the authors used a finite element method on a Bakhvalov-type mesh to solve a two-parameter singularly perturbed two-point boundary value problem. Utilising individual interpolation, they obtain the errors for Lagrange interpolation and then prove the optimal order of convergence. In [14], the authors consider a singularly perturbed initial-boundary value problem with two parameters. They propose a fully-discrete numerical method combining the Crank-Nicolson scheme for time variables and the streamline-diffusion finite element method for the spatial variable. In [262], the authors consider a two-dimensional singularly perturbed convection-reaction-diffusion elliptic type problem with two parameters. Furthermore, the authors assume that jump discontinuities exist in the source term along the x-axis and y-axis. They employ an upwind finite-difference technique with an appropriate layer-adapted piecewise uniform Shishkin mesh to approximate the numerical solution.

In [218], the author proposed a numerical method based on an upwind finite difference operator to solve a two-parameter problem. They proved that the method is first-order convergent in space and time. The paper [129] presents a higher-order uniformly convergent method based on finite elements for a two-parameter parabolic singular perturbation problem. In [107], researchers have developed a hybrid scheme for one-dimensional singularly perturbed parabolic problems with two small parameters. The method presented is a composition of an upwind, midpoint upwind and central difference operator on a piecewise uniform Shishkin mesh. In contrast, the implicit Euler method is used for time stepping on a uniform mesh. A numerical approximation of a two-parameter parabolic problem with discontinuous data is proposed in [44]. In [97], the authors present an almost first-order convergent scheme for a two-parameter singularly perturbed problem with a time delay. The method combines an upwind scheme for spatial discretisation and the implicit Euler scheme for time discretisation. A hybrid difference approach for a similar problem is studied in [281]. The author established almost second-order convergence in space direction and first-order convergence in time.

Due to their large applications, it is essential to develop robust numerical methods for solving two-parameter singular perturbation problems. This field of research is still developing, and this chapter is a step forward in developing a parameter-uniform numerical method to solve a two-parameter singular perturbation problem with a large spatial delay. Moreover, the chapter presents rigorous consistency, stability and convergence analysis of the proposed method and illustrates numerical results to support theoretical estimates.

5.2 Continuous Problem

Let $D = \Omega \times \Delta := (0,2) \times (0,T]$ and consider the following two-parameter, non-homogeneous intial boundary value problem

$$\left.\begin{aligned} \mathcal{L}u(x,t) &= \varepsilon u_{xx}(x,t) + \mu p(x,t)u_x(x,t) - q(x,t)u(x,t) + r(x,t)u(x-1,t) - u_t(x,t) \\ &= g(x,t) \text{ in } D, \\ u(x,t) &= \psi_1(x,t) \text{ in } \Gamma_1 := \{(x,t),\ x \in [-1,0],\ t \in [0,T]\}, \\ u(x,t) &= \psi_2(x,t) \text{ on } \Gamma_2 := \{(x,0),\ x \in [0,2]\}, \\ u(x,t) &= \psi_3(x,t) \text{ on } \Gamma_3 := \{(2,t),\ t \in [0,T]\}, \end{aligned}\right\} \tag{5.1}$$

where $0 < \varepsilon < 1$ and $0 < \mu < 1$ are small parameters, $g(x,t)$, $p(x,t)$, $q(x,t)$ and $r(x,t)$ are sufficiently smooth functions such that

$$p(x,t) \geq p_0 > 0, \quad q(x,t) \geq q_0 > 0, \quad r(x,t) \geq r_0 > 0, \quad (q-r)(x,t) \geq \kappa > 0. \tag{5.2}$$

Moreover, we assume that the initial-boundary data ψ_1, ψ_2 and ψ_3 are smooth functions on their respective domain and satisfy the compatibility conditions given below

$$\begin{cases} \psi_2(0,0) = \psi_1(0,0), \quad \psi_2(2,0) = \psi_3(2,0), \\ \varepsilon \dfrac{\partial^2 \psi_2(0,0)}{\partial x^2} + \mu p(0,0)\dfrac{\partial \psi_2(0,0)}{\partial x} - q(0,0)\psi_2(0,0) + r(0,0)\psi_1(-1,0) - \dfrac{\partial \psi_1(0,0)}{\partial t} = g(0,0), \\ \varepsilon \dfrac{\partial^2 \psi_2(2,0)}{\partial x^2} + \mu p(2,0)\dfrac{\partial \psi_2(2,0)}{\partial x} - q(2,0)\psi_2(2,0) + r(2,0)\psi_2(1,0) - \dfrac{\partial \psi_3(2,0)}{\partial t} = g(2,0). \end{cases} \tag{5.3}$$

Let $D_1 = \Omega_1 \times \Delta := (0, 1) \times (0, T]$ and $D_2 = \Omega_2 \times \Delta := [1, 2) \times (0, T]$. Rewriting (5.1) as

$$\mathcal{L}u(x,t) = \mathcal{G}(x,t),$$

where

$$\mathcal{L}u(x,t) = \begin{cases} \mathcal{L}_1 u = \varepsilon u_{xx}(x,t) + \mu p(x,t)u_x(x,t) - q(x,t)u(x,t) - u_t(x,t) \ \text{ if } \ (x,t) \in D_1, \\ \mathcal{L}_2 u = \varepsilon u_{xx}(x,t) + \mu p(x,t)u_x(x,t) - q(x,t)u(x,t) + r(x,t)u(x-1,t) \\ \qquad -u_t(x,t) \ \text{ if } \ (x,t) \in D_2, \end{cases} \tag{5.4}$$

and

$$\mathcal{G}(x,t) = \begin{cases} g(x,t) - r(x,t)\psi_1(x-1,t) & \text{if } \ (x,t) \in D_1, \\ g(x,t) & \text{if } \ (x,t) \in D_2 \end{cases} \tag{5.5}$$

with

$$\begin{cases} u(0,t) = \psi_1(0,t), \ \ u(1^-,t) = u(1^+,t), \ \ u_x(1^-,t) = u_x(1^+,t), \\ u(2,t) = \psi_3(2,t), \ \ u(x,0) = \psi_2(x,0). \end{cases} \tag{5.6}$$

Under these assumptions, the solution of (5.1) exists and is unique [160]. The solution exhibits twin boundary layers due to the presence of perturbation parameters and an interior layer due to the presence of delay [218]. Let $\gamma = \min_{(x,t)\in\bar{D}} \left(\dfrac{q(x,t) - r(x,t)}{p(x,t)} \right)$. If $\dfrac{\mu^2}{\varepsilon} \leq \dfrac{\gamma}{p_0}$, the boundary layers of width $O(\sqrt{\varepsilon})$ appear near the boundaries $x = 0$ and $x = 2$, as well as on the left and right neighbourhoods of $x = 1$ due to the large delay. If $\dfrac{\mu^2}{\varepsilon} \geq \dfrac{\gamma}{p_0}$, the boundary layers of width $O(\mu)$ appear in a left neighbourhood of $x = 1$ and $x = 2$, while a boundary layer of width $O\left(\dfrac{\varepsilon}{\mu}\right)$ appears in a right neighbourhood of $x = 0$ and $x = 1$.

Lemma 5.2.1. *Suppose $\mathcal{P}(x,t) \in C^0(\bar{D}) \cap C^2(D)$ satisfies $\mathcal{P}(0,t) \geq 0$, $\mathcal{P}(2,t) \geq 0$, $\mathcal{P}(x,0) \geq 0$ with $\mathcal{L}_1\mathcal{P}(x,t) \leq 0$ for all $(x,t) \in (0,1) \times [0,T]$ and $\mathcal{L}_2\mathcal{P}(x,t) \leq 0$ for all $(x,t) \in [1,2) \times [0,T]$. Then $\mathcal{P}(x,t) \geq 0$ for all $(x,t) \in \bar{D}$.*

Proof. Choose $(x^k, t^k) \in \bar{D}$ such that $\mathcal{P}(x^k, t^k) = \min_{(x,t)\in\bar{D}} \mathcal{P}(x,t)$. Consequently,

$$\mathcal{P}_x(x^k,t^k) = 0, \ \ \mathcal{P}_t(x^k,t^k) = 0 \ \text{ and } \ \mathcal{P}_{xx}(x^k,t^k) > 0. \tag{5.7}$$

Suppose $\mathcal{P}(x^k,t^k) < 0$ and it follows that $(x^k,t^k) \notin \Gamma^* := \Gamma_1^* \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_1^* := \{(0,t), \ t \in [0,T]\}$.

Case I: If $(x^k,t^k) \in (0,1) \times [0,T]$, then

$$\begin{aligned} \mathcal{L}_1\mathcal{P}(x^k,t^k) &= \varepsilon\mathcal{P}_{xx}(x^k,t^k) + \mu p(x^k,t^k)\mathcal{P}_x(x^k,t^k) - q(x^k,t^k)\mathcal{P}(x^k,t^k) - \mathcal{P}_t(x^k,t^k) \\ &> 0, \text{ from (5.2) and (5.7).} \end{aligned}$$

Case II: If $(x^k, t^k) \in [1,2) \times [0,T]$, then

$$\begin{aligned}\mathcal{L}_2\mathcal{P}(x^k,t^k) &= \varepsilon\mathcal{P}_{xx}(x^k,t^k) + \mu p(x^k,t^k)\mathcal{P}_x(x^k,t^k) - q(x^k,t^k)\mathcal{P}(x^k,t^k)\\ &\quad + r(x^k,t^k)\mathcal{P}(x^k-1,t^k) - \mathcal{P}_t(x^k,t^k)\\ &\geq \varepsilon\mathcal{P}_{xx}(x^k,t^k) - (q-r)(x^k,t^k)\mathcal{P}(x^k,t^k)\\ &> 0, \text{ from (5.2) and (5.7).}\end{aligned}$$

A contradiction to the assumption that $\mathcal{L}_1\mathcal{P} \leq 0$ for all $(x,t) \in (0,1)\times[0,T]$ and $\mathcal{L}_2\mathcal{P} \leq 0$ for all $(x,t) \in [1,2)\times[0,T]$. Consequently, the required result follows from a contradiction. □

As a consequence of Lemma 5.2.1, obtaining the following stability estimate is straightforward.

Lemma 5.2.2. *Let u be the solution of* (5.1). *Then* $\|u\|_{\bar{D}} \leq \|u\|_{\Gamma^*} + \dfrac{1}{\kappa}\|\mathcal{G}\|_{\bar{D}}$.

Proof. Define $\theta_{\pm}(x,t) = \|u\|_{\Gamma^*} + \dfrac{1}{\kappa}\|\mathcal{G}\|_{\bar{D}} \pm u(x,t),\ (x,t) \in \bar{D}$. Then $\theta_{\pm}(x,t) \geq 0$ for all $(x,t) \in \Gamma^*$ and if $(x,t) \in (0,1)\times[0,T]$, it follows that

$$\begin{aligned}\mathcal{L}_1\theta_{\pm}(x,t) &= -q(x,t)\left(\|u\|_{\Gamma^*} + \frac{\|\mathcal{G}\|_{\bar{D}}}{\kappa}\right) \pm \mathcal{L}_1 u(x,t)\\ &= -q(x,t)\left(\|u\|_{\Gamma^*} + \frac{\|\mathcal{G}\|_{\bar{D}}}{\kappa}\right) \pm \mathcal{G}(x,t) \leq 0\end{aligned}$$

Similarly, for $(x,t) \in [1,2)\times[0,T]$

$$\mathcal{L}_2\theta_{\pm}(x,t) = -(q-r)(x,t)\left(\|u\|_{\Gamma^*} + \frac{\|\mathcal{G}\|_{\bar{D}}}{\kappa}\right) \pm \mathcal{L}_2 u(x,t) \leq 0.$$

The required result thus follows from Lemma 5.2.1. □

Lemma 5.2.3. *Let u be the solution of* (5.1). *Then, for* $1 \leq k + 2m \leq 3$

$$\left\|\frac{\partial^{k+m}u}{\partial x^k \partial t^m}\right\|_{\bar{D}} \leq \begin{cases} \dfrac{C}{(\sqrt{\varepsilon})^k} & \text{if } p_0\mu^2 \leq \gamma\varepsilon,\\ C\left(\dfrac{\mu}{\varepsilon}\right)^k\left(\dfrac{\mu^2}{\varepsilon}\right)^m & \text{if } p_0\mu^2 \geq \gamma\varepsilon,\end{cases}$$

where C is a constant independent of ε and μ.

Proof. We obtain the bounds of the solution and its derivatives by splitting the argument into two cases $p_0\mu^2 \leq \gamma\varepsilon$ and $p_0\mu^2 \geq \gamma\varepsilon$.

Case I: Let $p_0\mu^2 \le \gamma\varepsilon$. Consider the stretching variables $\tilde{x} = \dfrac{x}{\sqrt{\varepsilon}}$, $\tilde{t} = t$ to obtain the transformed problem on the domain $\tilde{D} = (0, 2/\sqrt{\varepsilon}) \times (0, T]$:

$$\begin{cases} \left(\tilde{u}_{\tilde{x}\tilde{x}} + \frac{\mu}{\sqrt{\varepsilon}}\tilde{p}\tilde{u}_{\tilde{x}} - \tilde{q}\tilde{u} - \tilde{u}_{\tilde{t}}\right)(\tilde{x},\tilde{t}) + \tilde{r}(\tilde{x},\tilde{t})\tilde{u}(\tilde{x}-1,\tilde{t}) = \tilde{g}(\tilde{x},\tilde{t}) \text{ in } \tilde{D}, \\ \tilde{u}(\tilde{x},\tilde{t}) = \tilde{\psi}_1(\tilde{x},\tilde{t}) \text{ on } \tilde{\Gamma}_1, \\ \tilde{u}(\tilde{x},\tilde{t}) = \tilde{\psi}_2(\tilde{x},\tilde{t}) \text{ on } \tilde{\Gamma}_2, \\ \tilde{u}(\tilde{x},\tilde{t}) = \tilde{\psi}_3(\tilde{x},\tilde{t}) \text{ on } \tilde{\Gamma}_3. \end{cases}$$

Using the condition $p_0\mu^2 \le \gamma\varepsilon$ and result from [160], we have

$$\left\| \frac{\partial^{k+m}\tilde{u}}{\partial\tilde{x}^k \partial\tilde{t}^m} \right\|_{N_{\lambda,\xi}} \le C(1 + \|\tilde{u}\|_{\bar{\tilde{D}}}), \quad 1 \le k + 2m \le 3,$$

where $N_{\lambda,\xi}$ is the rectangle $(\lambda - \xi, \lambda + \xi) \times (0, T] \cap \tilde{D}$ for any $\lambda \in (0, 2/\sqrt{\varepsilon})$ and $\xi > 0$. Now, we return back to the original variable to get

$$\begin{aligned} \left\| \frac{\partial^{k+m}u}{\partial x^k \partial t^m} \right\|_{\bar{D}} &\le C\varepsilon^{\frac{-k}{2}}(1 + \|u\|_{\bar{D}}), \quad 1 \le k + 2m \le 3 \\ &\le \frac{C}{(\sqrt{\varepsilon})^k}, \text{ from Lemma 5.2.2.} \end{aligned}$$

Case II: Let $p_0\mu^2 \ge \gamma\varepsilon$. Consider the stretching variables $\tilde{x} = \dfrac{\mu x}{\varepsilon}$, $\tilde{t} = \dfrac{\mu^2 t}{\varepsilon}$ to obtain the following transformed problem on the domain $\tilde{D} = (0, 2\mu/\varepsilon) \times (0, \mu^2 T/\varepsilon]$:

$$\begin{cases} \left(\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{p}\tilde{u}_{\tilde{x}} - \frac{\tilde{q}\varepsilon}{\mu^2}\tilde{u} - \tilde{u}_{\tilde{t}}\right)(\tilde{x},\tilde{t}) + \frac{\tilde{r}\varepsilon}{\mu^2}(\tilde{x},\tilde{t})\tilde{u}(\tilde{x}-1,\tilde{t}) = \frac{\varepsilon}{\mu^2}\tilde{g}(\tilde{x},\tilde{t}) \text{ in } \tilde{D}, \\ \tilde{u}(\tilde{x},\tilde{t}) = \tilde{\psi}_1(\tilde{x},\tilde{t}) \text{ on } \tilde{\Gamma}_1, \\ \tilde{u}(\tilde{x},\tilde{t}) = \tilde{\psi}_2(\tilde{x},\tilde{t}) \text{ on } \tilde{\Gamma}_2, \\ \tilde{u}(\tilde{x},\tilde{t}) = \tilde{\psi}_3(\tilde{x},\tilde{t}) \text{ on } \tilde{\Gamma}_3. \end{cases}$$

Repeating the argument given in Case I, we get

$$\left\| \frac{\partial^{k+m}u}{\partial x^k \partial t^m} \right\|_{\bar{D}} \le C\left(\frac{\mu}{\varepsilon}\right)^k \left(\frac{\mu^2}{\varepsilon}\right)^m, \quad 1 \le k + 2m \le 3.$$

□

Corollary 5.2.4. *Let $u(x,t)$ be the solution of* (5.1) *and Lemmas 5.2.1 and 5.2.2 hold. Then*

$$\|u_{tt}\|_{\bar{D}} \le \begin{cases} C & \text{if } p_0\mu^2 \le \gamma\varepsilon, \\ C\mu^4\varepsilon^{-2} & \text{if } p_0\mu^2 \ge \gamma\varepsilon. \end{cases}$$

5.3 Time Discretization

Let $\mathbb{T}_t^M = \{t_k = kT/M, k = 0, \ldots, M\}$ be an equidistant mesh in the time direction. We use the Crank-Nicolson method to discretize the problem (5.1) in the time variable. The semidiscrete problem on $\mathbb{T}_t^M$ thus reads

$$\left.\begin{aligned}&(\varepsilon U_{xx} + \mu p U_x - lU)(x, t_{k+1}) + r(x, t_{k+1})U(x-1, t_{k+1})\\ &\quad = (-\varepsilon U_{xx} - \mu p U_x + mU)(x, t_k) - r(x, t_k)U(x-1, t_k) + g(x, t_{k+1}) + g(x, t_k),\\ &\qquad x \in \Omega, \quad k = 0, 1, \ldots, M-1\end{aligned}\right\} \tag{5.8}$$

such that

$$\left.\begin{aligned}&U(x, t_{k+1}) = \psi_1(x, t_{k+1}), \quad -1 \le x \le 0, \quad -1 \le k \le M-1,\\ &U(x, 0) = \psi_2(x, 0), \quad 0 \le x \le 2,\\ &U(2, t_{k+1}) = \psi_3(2, t_{k+1}), \quad -1 \le k \le M-1,\end{aligned}\right\} \tag{5.9}$$

where $l(x, t_{k+1}) = q(x, t_{k+1}) + \dfrac{2}{\Delta t}$, $m(x, t_k) = q(x, t_k) - \dfrac{2}{\Delta t}$ and $U(x, t_{k+1})$ is the numerical approximation of the continuous solution $u(x, t)$ at $(k+1)$th time step.

Let us rewrite (5.8) as

$$\tilde{\mathcal{L}}_n U(x, t_{k+1}) = \tilde{\mathcal{G}}_n(x, t_{k+1}), \; n = 1, 2, \tag{5.10}$$

where

$$\begin{aligned}&\tilde{\mathcal{L}}_1 U(x, t_{k+1}) = (\varepsilon U_{xx} + \mu p U_x - lU)(x, t_{k+1}) && \text{if } x \in (0, 1),\\ &\tilde{\mathcal{L}}_2 U(x, t_{k+1}) = (\varepsilon U_{xx} + \mu p U_x - lU)(x, t_{k+1}) + r(x, t_{k+1})U(x-1, t_{k+1}) && \text{if } x \in [1, 2),\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{G}}_1(x, t_{k+1}) &= (-\varepsilon U_{xx} - \mu p U_x + mU)(x, t_k) - r(x, t_{k+1})\psi_1(x-1, t_{k+1})\\ &\quad - r(x, t_k))\psi_1(x-1, t_k) + g(x, t_{k+1}) + g(x, t_k) \qquad \text{if } x \in (0, 1),\\ \tilde{\mathcal{G}}_2(x, t_{k+1}) &= (-\varepsilon U_{xx} - \mu p U_x + mU)(x, t_k) - r(x, t_k))\psi_1(x-1, t_k)\\ &\quad + g(x, t_{k+1}) + g(x, t_k) \qquad \text{if } x \in [1, 2).\end{aligned}$$

The operator $\tilde{\mathcal{L}}_n$ satisfies the following minimum principle.

Lemma 5.3.1. *Let $\chi(x, t_{k+1})$ be a smooth function such that $\chi(x, t_{k+1}) \ge 0$ for $x = 0, 2$ with $\tilde{\mathcal{L}}_1\chi(x, t_{k+1}) \le 0$ for all $x \in (0, 1)$ and $\tilde{\mathcal{L}}_2\chi(x, t_{k+1}) \le 0$ for all $x \in [1, 2)$. Then $\chi(x, t_{k+1}) \ge 0$ for all $x \in \bar{\Omega}$.*

Proof. Let $\chi(\alpha, t_{k+1}) = \min_{x \in \bar{\Omega}} \chi(x, t_{k+1})$ for some $\alpha \in \bar{\Omega}$. Then

$$\chi_x(\alpha, t_{k+1}) = 0 \quad \text{and} \quad \chi_{xx}(\alpha, t_{k+1}) > 0. \tag{5.11}$$

Suppose $\chi(\alpha, t_{k+1}) < 0$, therefore $\alpha \notin \{0, 2\}$ because $\chi(x, t_{k+1}) \ge 0$ for $x = 0, 2$.

Case I: If $\alpha \in (0,1)$

$$\begin{aligned}\tilde{\mathcal{L}}_1\chi(\alpha, t_{k+1}) &= (\varepsilon\chi_{xx} + \mu p\chi_x - l\chi)(\alpha, t_{k+1}) \\ &> 0, \text{ from (5.2) and (5.11).}\end{aligned}$$

Case II: If $\alpha \in [1,2)$

$$\begin{aligned}\tilde{\mathcal{L}}_2\chi(\alpha, t_{k+1}) &= (\varepsilon\chi_{xx} + \mu p\chi_x - l\chi)(\alpha, t_{k+1}) + r(\alpha, t_{k+1})\chi(\alpha-1, t_{k+1}) \\ &\geq \varepsilon\chi_{xx}(\alpha, t_{k+1}) - l(\alpha, t_{k+1})\chi(\alpha, t_{k+1}) + r(\alpha, t_{k+1})\chi(\alpha, t_{k+1}) \\ &\geq \varepsilon\chi_{xx}(\alpha, t_{k+1}) - \left(q(\alpha, t_{k+1}) + \frac{2}{\Delta t} - r(\alpha, t_{k+1})\right)\chi(\alpha, t_{k+1}) \\ &= \varepsilon\chi_{xx}(\alpha, t_{k+1}) - (q-r)(\alpha, t_{k+1})\chi(\alpha, t_{k+1}) - \frac{2}{\Delta t}\chi(\alpha, t_{k+1}) \\ &> 0, \text{ from (5.2) and (5.11).}\end{aligned}$$

It contradicts the assumption, and hence the result follows from a contradiction. □

Lemma 5.3.2. *Let $U(x, t_{k+1})$ be the solution of* (5.10). *Then*

$$\|U(x, t_{k+1})\|_{\bar{\Omega}} \leq \max\left\{|U(0, t_{k+1})|, |U(2, t_{k+1})|, \frac{1}{\kappa}\|\tilde{\mathcal{G}}\|_{\bar{\Omega}}\right\} \quad \textit{for all} \quad x \in [0,2].$$

Proof. Consider $M_{\pm}(x, t_{k+1}) = \max\left\{|U(0, t_{k+1})|, \frac{1}{\kappa}\|\tilde{\mathcal{G}}_1\|_{\bar{\Omega}_1}\right\} \pm U(x, t_{k+1})$ for all $x \in (0,1)$. Then, $M_{\pm}(0, t_{k+1}) \geq 0$. Moreover, for $x \in (0,1)$

$$\begin{aligned}\tilde{\mathcal{L}}_1 M_{\pm}(x, t_{k+1}) &= -l(x, t_{k+1})\max\left\{|U(0, t_{k+1})|, \frac{1}{\kappa}\|\tilde{\mathcal{G}}_1\|_{\bar{\Omega}_1}\right\} \pm \tilde{\mathcal{L}}_1 U(x, t_{k+1}) \\ &= -l(x, t_{k+1})\max\left\{|U(0, t_{k+1})|, \frac{1}{\kappa}\|\tilde{\mathcal{G}}_1\|_{\bar{\Omega}_1}\right\} \pm \tilde{\mathcal{G}}_1(x, t_{k+1}) \\ &\leq 0, \text{ from (5.2).}\end{aligned}$$

Similarly, consider $M_{\pm}(x, t_{k+1}) = \max\left\{|U(2, t_{k+1})|, \frac{1}{\kappa}\|\tilde{\mathcal{G}}_2\|_{\bar{\Omega}_2}\right\} \pm U(x, t_{k+1})$ for all $x \in [1,2)$. Then, $M_{\pm}(2, t_{k+1}) \geq 0$ and for $x \in [1,2)$ we compute

$$\begin{aligned}\tilde{\mathcal{L}}_2 M_{\pm}(x, t_{k+1}) &= -(l-r)(x, t_{k+1})\max\left\{|U(2, t_{k+1})|, \frac{1}{\kappa}\|\tilde{\mathcal{G}}_2\|_{\bar{\Omega}_2}\right\} \pm \tilde{\mathcal{L}}_2 U(x, t_{k+1}) \\ &= -\left(q(x, t_{k+1}) + \frac{2}{\Delta t} - r(x, t_{k+1})\right)\max\left\{|U(2, t_{k+1})|, \frac{1}{\kappa}\|\tilde{\mathcal{G}}_2\|_{\bar{\Omega}_2}\right\} \\ &\quad \pm\tilde{\mathcal{G}}_2(x, t_{k+1}) \\ &\leq 0, \text{ from (5.2).}\end{aligned}$$

Consequently, from Lemma 5.3.1 the required result follows. □

The local truncation error of the semidiscretized problem (5.10) is given by $e_{k+1} := \hat{U}(x, t_{k+1}) - u(x, t_{k+1})$, where $\hat{U}$ is the solution of

$$\left.\begin{aligned}
&\tilde{\mathcal{L}}_n \hat{U}(x, t_{k+1}) = \hat{\mathcal{G}}(x, t_{k+1}),\ x \in \Omega,\ 0 \le k \le M-1,\\
&\hat{U}(x, t_{k+1}) = \psi_1(x, t_{k+1}),\ \ -1 \le x \le 0,\ -1 \le k \le M-1,\\
&\hat{U}(x, 0) = \psi_2(x, 0),\ \ x \in \bar{\Omega},\\
&\hat{U}(2, t_{k+1}) = \psi_3(2, t_{k+1}),\ \ -1 \le k \le M-1,
\end{aligned}\right\} \tag{5.12}$$

where $\tilde{\mathcal{L}}_n$ is defined in (5.10), and

$$\hat{\mathcal{G}} = \begin{cases}
(-\varepsilon u_{xx} - \mu p u_x + mu)(x, t_k) - r(x, t_{k+1})\psi_1(x-1, t_{k+1}) & \\
-r(x, t_k)\psi_1(x-1, t_k) + g(x, t_{k+1}) + g(x, t_k) & \text{if } x \in (0,1),\\
(-\varepsilon u_{xx} - \mu p u_x + mu)(x, t_k) - r(x, t_k))\psi_1(x-1, t_k) & \\
+g(x, t_{k+1}) + g(x, t_k) & \text{if } x \in [1,2).
\end{cases}$$

Lemma 5.3.3. *For some constant C, the local truncation error at $(k+1)$th time step satisfies the following bound*

$$\|e_{k+1}\|_\infty \le C(\Delta t)^3,\ \ -1 \le k \le M-1.$$

Proof. For proof, see [50]. □

The global error $E_k := u(x, t_k) - U(x, t_k)$ of the semidiscretized problem is the contribution of the local truncation error at each time step. Then, it follows that

$$\|E_{k+1}\|_\infty = \left\| \sum_{i=1}^{k} \hat{e}_i \right\|_\infty \le \|\hat{e}_1\|_\infty + \|\hat{e}_2\|_\infty + \ldots + \|\hat{e}_k\|_\infty \le C\Delta t^2. \tag{5.13}$$

This in turn ensures the uniform convergence of the time semidiscretization process.

Lemma 5.3.4. *Let $U(x, t_{k+1})$ be the solution of* (5.10). *Then*

$$\left\|\frac{d^k U}{dx^k}\right\| \le \frac{C}{(\sqrt{\varepsilon})^k}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^k\right)\max\left\{\|U\|, \|\tilde{\mathcal{G}}\|\right\},\ \ k = 1, 2,$$
$$\left\|\frac{d^3 U}{dx^3}\right\| \le \frac{C}{(\sqrt{\varepsilon})^3}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right)\max\left\{\|U\|, \|\tilde{\mathcal{G}}\|\right\} + \frac{1}{\varepsilon}\left\|\frac{d\tilde{\mathcal{G}}}{dx}\right\|,$$

where C is the constant independent of μ and ε.

Proof. For proof, see [135]. □

The solution $U(x, t_{k+1})$ of the semidiscretized problem (5.10) admit a decomposition into smooth and singular components [173]. We write

$$U(x, t_{k+1}) := X(x, t_{k+1}) + Z(x, t_{k+1}),$$

where

$$X(x, t_{k+1}) = \begin{cases} b(x, t_{k+1}), & x \in [0, 1), \\ c(x, t_{k+1}), & x \in [1, 2], \end{cases} \tag{5.14}$$

and

$$Z(x, t_{k+1}) = \begin{cases} z_1(x, t_{k+1}), & x \in [0, 1), \\ z_2(x, t_{k+1}), & x \in [1, 2]. \end{cases} \tag{5.15}$$

Here, the smooth component $b(x, t_{k+1})$ is the solution of

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_1 b(x, t_{k+1}) = \tilde{\mathcal{G}}_1(x, t_{k+1}), \quad x \in (0, 1), \\ &b(0, t_{k+1}),\ b(1, t_{k+1}) \text{ are to be chosen lately,} \end{aligned}\right\} \tag{5.16}$$

and $c(x, t_{k+1})$ is the solution of

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_2 c(x, t_{k+1}) = \tilde{\mathcal{G}}_2(x, t_{k+1}), \quad x \in (1, 2), \\ &c(x, t_{k+1}) = b(x, t_{k+1}) \text{ on } [0, 1), \\ &c(1, t_{k+1}),\ c(2, t_{k+1}) \text{ are to be chosen lately.} \end{aligned}\right\} \tag{5.17}$$

Moreover, the singular components $z_1(x, t_{k+1})$ and $z_2(x, t_{k+1})$ satisfy the following

$$\tilde{\mathcal{L}}_1 z_1(x, t_{k+1}) = 0, \quad x \in (0, 1), \quad \tilde{\mathcal{L}}_2 z_2(x, t_{k+1}) = 0, \quad x \in (1, 2), \tag{5.18}$$

$$z_1(0, t_{k+1}) = U(0, t_{k+1}) - b(0, t_{k+1}),$$

$$z_2(1, t_{k+1}) - z_1(1, t_{k+1}) = b(1, t_{k+1}) - c(1, t_{k+1}), \tag{5.19}$$

$$\frac{dz_2}{dx}(1, t_{k+1}) - \frac{dz_1}{dx}(1, t_{k+1}) = \frac{db}{dx}(1, t_{k+1}) - \frac{dc}{dx}(1, t_{k+1}), \tag{5.20}$$

$$z_2(2, t_{k+1}) = U(2, t_{k+1}) - c(2, t_{k+1}).$$

The next lemmas provide bounds on the derivatives of $X(x, t_{k+1})$ and $Z(x, t_{k+1})$ with respect to x. We derive bounds for these components separately for $p_0\mu^2 \le \gamma\varepsilon$ and $p_0\mu^2 \ge \gamma\varepsilon$.

5.3.1 Estimates for the Smooth Components When $p_0\mu^2 \le \gamma\varepsilon$

Lemma 5.3.5. *Let $b(x, t_{k+1})$ and $c(x, t_{k+1})$ be the solutions of* (5.16) *and* (5.17)*, respectively. Then for $p_0\mu^2 \le \gamma\varepsilon$ and $k = 0, 1, 2, 3$*

$$\|b^{(k)}\|_{\bar{\Omega}_1} \le C, \quad \|c^{(k)}\|_{\bar{\Omega}_2} \le C.$$

Proof. Decompose the smooth components $b(x, t_{k+1})$ and $c(x, t_{k+1})$ as

$$b(x, t_{k+1}) = b_0(x, t_{k+1}) + \sqrt{\varepsilon} b_1(x, t_{k+1}) + (\sqrt{\varepsilon})^2 b_2(x, t_{k+1}) + (\sqrt{\varepsilon})^3 b_3(x, t_{k+1}), \quad x \in \bar{\Omega}_1,$$

$$c(x, t_{k+1}) = c_0(x, t_{k+1}) + \sqrt{\varepsilon} c_1(x, t_{k+1}) + (\sqrt{\varepsilon})^2 c_2(x, t_{k+1}) + (\sqrt{\varepsilon})^3 c_3(x, t_{k+1}), \quad x \in \bar{\Omega}_2,$$

where the functions b_0 and c_0 are the solutions of

$$\left.\begin{aligned} -(lb_0)(x,t_{k+1}) &= (mb_0)(x,t_k) - r(x,t_{k+1})\psi_1(x-1,t_{k+1}) - r(x,t_k)\psi_1(x-1,t_k)\\ &\quad +g(x,t_k)+g(x,t_{k+1}),\quad x\in\bar{\Omega}_1,\\ -(lc_0)(x,t_{k+1}) + r(x,t_{k+1})b_0(x-1,t_{k+1}) &= (mc_0)(x,t_k) - r(x,t_k)b_0(x-1,t_k)\\ &\quad +g(x,t_k)+g(x,t_{k+1}),\quad x\in\bar{\Omega}_2, \end{aligned}\right\}$$

the functions b_1 and c_1 are the solutions of

$$\left.\begin{aligned} (lb_1)(x,t_{k+1}) &= (\sqrt{\varepsilon}b_0'' + \tfrac{\mu}{\sqrt{\varepsilon}}pb_0')(x,t_{k+1}) + (\sqrt{\varepsilon}b_0'' + \tfrac{\mu}{\sqrt{\varepsilon}}pb_0' - mb_1)(x,t_k),\quad x\in\bar{\Omega}_1,\\ (lc_1)(x,t_{k+1}) - r(x,t_{k+1})b_1(x-1,t_{k+1}) &= (\sqrt{\varepsilon}c_0'' + \tfrac{\mu}{\sqrt{\varepsilon}}pc_0')(x,t_{k+1}) + r(x,t_k)b_1(x-1,t_k)\\ &\quad +(\sqrt{\varepsilon}c_0'' + \tfrac{\mu}{\sqrt{\varepsilon}}pc_0' - mc_1)(x,t_k),\quad x\in\bar{\Omega}_2, \end{aligned}\right\}$$

the functions b_2 and c_2 are the solutions of

$$\left.\begin{aligned} (lb_2)(x,t_{k+1}) &= (\sqrt{\varepsilon}b_1'' + \tfrac{\mu}{\sqrt{\varepsilon}}pb_1')(x,t_{k+1}) + (\sqrt{\varepsilon}b_1'' + \tfrac{\mu}{\sqrt{\varepsilon}}pb_1' - mb_2)(x,t_k),\quad x\in\bar{\Omega}_1,\\ (lc_2)(x,t_{k+1}) - r(x,t_{k+1})b_2(x-1,t_{k+1}) &= (\sqrt{\varepsilon}c_1'' + \tfrac{\mu}{\sqrt{\varepsilon}}pc_1')(x,t_{k+1}) + r(x,t_k)b_2(x-1,t_k)\\ &\quad +(\sqrt{\varepsilon}c_1'' + \tfrac{\mu}{\sqrt{\varepsilon}}pc_1' - mc_2)(x,t_k),\quad x\in\bar{\Omega}_2, \end{aligned}\right\}$$

and lastly, the functions b_3 and c_3 are the solutions of

$$\left.\begin{aligned} \tilde{\mathcal{L}}_1 b_3(x,t_{k+1}) &= (-\sqrt{\varepsilon}b_2'' - \tfrac{\mu}{\sqrt{\varepsilon}}pb_2')(x,t_{k+1}) + (-\sqrt{\varepsilon}b_2'' - \tfrac{\mu}{\sqrt{\varepsilon}}pb_2')(x,t_k)\\ &\quad -(\varepsilon b_3'' + \mu p b_3' - mb_3)(x,t_k),\quad x\in\Omega_1,\\ b_3(0,t_{k+1}) &= 0,\quad b_3(1,t_{k+1}) = 0, \end{aligned}\right\} \tag{5.21}$$

and

$$\left.\begin{aligned} \tilde{\mathcal{L}}_2 c_3(x,t_{k+1}) &= (-\sqrt{\varepsilon}c_2'' - \tfrac{\mu}{\sqrt{\varepsilon}}pc_2')(x,t_{k+1}) - (\sqrt{\varepsilon}c_2'' + \tfrac{\mu}{\sqrt{\varepsilon}}pc_2')(x,t_k)\\ &\quad -(\varepsilon c_3'' + \mu p c_3' - mc_3)(x,t_k) - r(x,t_k)b_3(x-1,t_k),\quad x\in\Omega_2,\\ c_3(1,t_{k+1}) &= 0,\quad c_3(2,t_{k+1}) = 0,\quad c_3(x,t_{k+1}) = b_3(x,t_{k+1})\ \text{ on } [0,1). \end{aligned}\right\} \tag{5.22}$$

Since, the functions $p(x,t_{k+1})$, $r(x,t_{k+1})$, $m(x,t_{k+1})$, $l(x,t_{k+1})$, $\psi_1(x,t_{k+1})$ and $g(x,t_{k+1})$ are sufficiently smooth and $p_0\mu^2 \le \gamma\varepsilon$, we have

$$\begin{aligned} &\|b_0^{(k)}\|_{\bar{\Omega}_1} \le C,\quad \|c_0^{(k)}\|_{\bar{\Omega}_2} \le C,\ \text{for } 0\le k\le 7,\\ &\|b_1^{(k)}\|_{\bar{\Omega}_1} \le C,\quad \|c_1^{(k)}\|_{\bar{\Omega}_2} \le C,\ \text{for } 0\le k\le 5,\\ &\|b_2^{(k)}\|_{\bar{\Omega}_1} \le C,\quad \|c_2^{(k)}\|_{\bar{\Omega}_2} \le C,\ \text{for } 0\le k\le 3. \end{aligned}$$

Now, using Lemma 2 from [101] to obtain $\|b_3\|_{\bar{\Omega}_1} \le C$ and $\|c_3\|_{\bar{\Omega}_2} \le C$. Since $p_0\mu^2 \le \gamma\varepsilon$, Lemma 5.3.4 asserts that $\|b_3'\|_{\bar{\Omega}_1} \le \dfrac{C}{\sqrt{\varepsilon}}$ and $\|c_3'\|_{\bar{\Omega}_2} \le \dfrac{C}{\sqrt{\varepsilon}}$. Moreover, from (5.21) and (5.22), we obtain

$$\|b_3^{(k)}\|_{\bar{\Omega}_1} \le \frac{C}{(\sqrt{\varepsilon})^k},\ \|c_3^{(k)}\|_{\bar{\Omega}_2} \le \frac{C}{(\sqrt{\varepsilon})^k},\ \text{for } k=2,3.$$

Next, we choose $b(0,t_{k+1})$, $b(1,t_{k+1})$, $c(1,t_{k+1})$ and $c(2,t_{k+1})$ to be

$$b(0,t_{k+1}) = (b_0 + \sqrt{\varepsilon}b_1 + \varepsilon b_2)(0,t_{k+1}), \quad b(1,t_{k+1}) = (b_0 + \sqrt{\varepsilon}b_1 + \varepsilon b_2)(1,t_{k+1}),$$
$$c(1,t_{k+1}) = (c_0 + \sqrt{\varepsilon}c_1 + \varepsilon c_2)(1,t_{k+1}), \quad c(2,t_{k+1}) = (c_0 + \sqrt{\varepsilon}c_1 + \varepsilon c_2)(2,t_{k+1}).$$

Since $b = b_0 + \sqrt{\varepsilon}b_1 + \varepsilon b_2 + \varepsilon^{3/2}b_3$, for $k = 0,1,2,3$, we have

$$\|b^{(k)}\|_{\bar{\Omega}_1} \le \|b_0^{(k)}\|_{\bar{\Omega}_1} + \sqrt{\varepsilon}\|b_1^{(k)}\|_{\bar{\Omega}_1} + \varepsilon\|b_2^{(k)}\|_{\bar{\Omega}_1} + \varepsilon^{3/2}\|b_3^{(k)}\|_{\bar{\Omega}_1} \le C.$$

Similarly, we get $\|c^{(k)}\|_{\bar{\Omega}_2} \le C$, for $k = 0,1,2,3$. □

5.3.2 Estimates for the Smooth Components When $p_0\mu^2 \ge \gamma\varepsilon$

Similar to the previous case, here also, the decomposition of $b(x,t_{k+1})$ and $c(x,t_{k+1})$ is considered, but at two levels. We define operators $\tilde{\mathcal{L}}_1^*$, $\tilde{\mathcal{L}}_2^*$, $\tilde{\mathcal{L}}_3$, $\tilde{\mathcal{L}}_3^*$, $\tilde{\mathcal{L}}_4$, and $\tilde{\mathcal{L}}_4^*$ for this purpose. These operators are defined as:

$$\tilde{\mathcal{L}}_1^* = \varepsilon D^2 + \mu p(x,t_k)D - m(x,t_k)I, \quad \tilde{\mathcal{L}}_2^* = \varepsilon D^2 + \mu p(x,t_k)D - m(x,t_k)I + r(x,t_k)E,$$
$$\tilde{\mathcal{L}}_3 = \mu p(x,t_k)D - l(x,t_k)I, \quad \tilde{\mathcal{L}}_3^* = \mu p(x,t_k)D - m(x,t_k)I,$$
$$\tilde{\mathcal{L}}_4 = \mu p(x,t_k)D - l(x,t_k)I + r(x,t_k)E \text{ and } \tilde{\mathcal{L}}_4^* = \mu p(x,t_k)D - m(x,t_k)I + r(x,t_k)E.$$

Here, $DU(x,t_k) = \dfrac{dU}{dx}(x,t_k)$ and $D^2U(x,t_k) = \dfrac{d^2U}{dx^2}(x,t_k)$ are the differential operators. Additionally, $IU(x,t_k) = U(x,t_k)$ is the identity operator and $EU(x,t_k) = EU(x-1,t_k)$ is the shift operator. Finally, we consider the decomposition of $b(x,t_{k+1})$ and $c(x,t_{k+1})$ as

$$b(x,t_{k+1};\varepsilon,\mu) = b_0(x,t_{k+1}) + \varepsilon b_1(x,t_{k+1};\mu) + \varepsilon^2 b_2(x,t_{k+1};\mu) + \varepsilon^3 b_3(x,t_{k+1};\mu), \quad x \in \bar{\Omega}_1,$$
$$c(x,t_{k+1};\varepsilon,\mu) = c_0(x,t_{k+1}) + \varepsilon c_1(x,t_{k+1};\mu) + \varepsilon^2 c_2(x,t_{k+1};\mu) + \varepsilon^3 c_3(x,t_{k+1};\mu), \quad x \in \bar{\Omega}_2,$$

where the functions b_0 and c_0 are the solutions of

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_3 b_0(x,t_{k+1}) = -\tilde{\mathcal{L}}_3^* b_0(x,t_k) - r(x,t_{k+1})\psi_1(x-1,t_{k+1}) - r(x,t_k)\psi_1(x-1,t_k)\\ &\qquad\qquad + g(x,t_k) + g(x,t_{k+1}), \quad x \in [0,1),\\ &b_0(1,t_{k+1}) \text{ is to be chosen lately,}\end{aligned}\right\} \tag{5.23}$$

and

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_4 c_0(x,t_{k+1}) = -\tilde{\mathcal{L}}_4^* c_0(x,t_k) + g(x,t_k) + g(x,t_{k+1}), \quad x \in [1,2),\\ &c_0(2,t_{k+1}) \text{ is to be chosen lately,}\\ &c_0(x,t_{k+1}) = b_0(x,t_{k+1}) \text{ on } [0,1),\end{aligned}\right\} \tag{5.24}$$

the functions b_1 and c_1 are the solutions of

$$\left.\begin{aligned}&\tilde{\mathcal{L}}_3 b_1(x,t_{k+1}) = -b_0''(x,t_{k+1}) - b_0''(x,t_k) - \tilde{\mathcal{L}}_3^* b_1(x,t_k), \quad x\in[0,1),\\ &b_1(1,t_{k+1}) \text{ is to be chosen lately,}\end{aligned}\right\} \tag{5.25}$$

and

$$\left.\begin{aligned}&\tilde{\mathcal{L}}_4 c_1(x,t_{k+1}) = -c_0''(x,t_{k+1}) - c_0''(x,t_k) - \tilde{\mathcal{L}}_4^* c_1(x,t_k), \quad x\in[1,2),\\ &c_1(2,t_{k+1}) \text{ is to be chosen lately,}\\ &c_1(x,t_{k+1}) = b_1(x,t_{k+1}) \text{ on } [0,1),\end{aligned}\right\} \tag{5.26}$$

the functions b_2 and c_2 are the solutions of

$$\left.\begin{aligned}&\tilde{\mathcal{L}}_3 b_2(x,t_{k+1}) = -b_1''(x,t_{k+1}) - b_1''(x,t_k) - \tilde{\mathcal{L}}_3^* b_2(x,t_k), \quad x\in[0,1),\\ &b_2(1,t_{k+1}) = 0,\end{aligned}\right\} \tag{5.27}$$

and

$$\left.\begin{aligned}&\tilde{\mathcal{L}}_4 c_2(x,t_{k+1}) = -c_1''(x,t_{k+1}) - c_1''(x,t_k) - \tilde{\mathcal{L}}_4^* c_2(x,t_k), \quad x\in[1,2),\\ &c_2(2,t_{k+1}) = 0,\\ &c_2(x,t_{k+1}) = b_2(x,t_{k+1}) \text{ on } [0,1),\end{aligned}\right\} \tag{5.28}$$

and lastly, the functions b_3 and c_3 are the solutions of

$$\left.\begin{aligned}&\tilde{\mathcal{L}}_1 b_3(x,t_{k+1}) = -b_2''(x,t_{k+1}) - b_2''(x,t_k) - \tilde{\mathcal{L}}_1^* b_3(x,t_k), \quad x\in(0,1),\\ &b_3(0,t_{k+1}) = 0, \quad b_3(1,t_{k+1}) = 0,\end{aligned}\right\} \tag{5.29}$$

and

$$\left.\begin{aligned}&\tilde{\mathcal{L}}_2 c_3(x,t_{k+1}) = -c_2''(x,t_{k+1}) - c_2''(x,t_k) - \tilde{\mathcal{L}}_2^* c_3(x,t_k), \quad x\in(1,2),\\ &c_3(1,t_{k+1}) = 0, \quad c_3(2,t_{k+1}) = 0,\\ &c_3(x,t_{k+1}) = b_3(x,t_{k+1}) \text{ on } [0,1).\end{aligned}\right\} \tag{5.30}$$

Lemma 5.3.6. *Let $\psi(x,t_{k+1})$ satisfies $\psi(1,t_{k+1}) \geq 0$ and $\tilde{\mathcal{L}}_3\psi(x,t_{k+1}) \leq 0$ for all $x \in [0,1)$. Then $\psi(x,t_{k+1}) \geq 0$ for all $x \in [0,1]$.*

Proof. Define $\psi(\alpha,t_{k+1}) = \min_{\alpha\in\bar{\Omega}_1} \psi(x,t_{k+1})$ for some $\alpha \in \bar{\Omega}_1$. Let $\psi(\alpha,t_{k+1}) < 0$, then $\alpha \neq 1$ and $\psi'(\alpha,t_{k+1}) = 0$. For $\alpha \in [0,1)$, we have $\tilde{\mathcal{L}}_3\psi(\alpha,t_{k+1}) = (\mu p\psi'(\alpha,t_{k+1}) - l\psi(\alpha,t_{k+1})) > 0$, which contradicts the assumption. Therefore, we can conclude that $\psi(\alpha,t_{k+1}) \geq 0$ which implies $\psi(x,t_{k+1}) \geq 0$ for all $x \in [0,1]$. □

Lemma 5.3.7. *Let $b(x,t_{k+1})$ and $c(x,t_{k+1})$ be the solutions of* (5.23)-(5.30). *Then for $p_0\mu^2 \geq \gamma\varepsilon$ and $k = 0,1,2,3$*

$$\|b^{(k)}\|_{\bar{\Omega}_1} \leq C\left(1 + \left(\frac{\varepsilon}{\mu}\right)^{3-k}\right), \quad \|c^{(k)}\|_{\bar{\Omega}_2} \leq C\left(1 + \left(\frac{\varepsilon}{\mu}\right)^{3-k}\right).$$

Proof. To establish the bounds for the derivatives of the smooth components b and c, we use the bounds for the derivatives of b_0, b_1, b_2, b_3, c_0, c_1, c_2 and c_3. Let's first establish the bounds for b_0 and c_0. To do this, we consider the decomposition of b_0 and c_0 as

$$b_0(x,t_{k+1};\mu) = b_{0,0}(x,t_{k+1}) + \mu b_{0,1}(x,t_{k+1}) + \mu^2 b_{0,2}(x,t_{k+1}) + \mu^3 b_{0,3}(x,t_{k+1};\mu) \text{ for } x \in \bar{\Omega}_1,$$
$$c_0(x,t_{k+1};\mu) = c_{0,0}(x,t_{k+1}) + \mu c_{0,1}(x,t_{k+1}) + \mu^2 c_{0,2}(x,t_{k+1}) + \mu^3 c_{0,3}(x,t_{k+1};\mu) \text{ for } x \in \bar{\Omega}_2,$$

where the functions $b_{0,0}$ and $c_{0,0}$ are the solutions of

$$\left.\begin{aligned} -(lb_{0,0})(x,t_{k+1}) &= (mb_{0,0})(x,t_k) - r(x,t_{k+1})\psi_1(x-1,t_{k+1}) - r(x,t_k)\psi_1(x-1,t_k)\\ &\quad +g(x,t_k) + g(x,t_{k+1}), \quad x \in \bar{\Omega}_1,\\ -(lc_{0,0})(x,t_{k+1}) + r(x,t_{k+1})b_{0,0}(x-1,t_{k+1}) &= (mc_{0,0})(x,t_k) - r(x,t_k)b_{0,0}(x-1,t_k)\\ &\quad +g(x,t_k) + g(x,t_{k+1}), \quad x \in \bar{\Omega}_2, \end{aligned}\right\}$$

the functions $b_{0,1}$ and $c_{0,1}$ are the solutions of

$$\left.\begin{aligned} (lb_{0,1})(x,t_{k+1}) &= (pb'_{0,0})(x,t_{k+1}) + (pb'_{0,0} - mb_{0,1})(x,t_k), \quad x \in \bar{\Omega}_1,\\ (lc_{0,1})(x,t_{k+1}) - r(x,t_{k+1})b_{0,1}(x-1,t_{k+1}) &= (pc'_{0,0})(x,t_{k+1}) + (pc'_{0,0} - mc_{0,1})(x,t_k)\\ &\quad +r(x,t_k)b_{0,1}(x-1,t_k), \quad x \in \bar{\Omega}_2, \end{aligned}\right\}$$

the functions $b_{0,2}$ and $c_{0,2}$ are the solutions of

$$\left.\begin{aligned} (lb_{0,2})(x,t_{k+1}) &= (pb'_{0,1})(x,t_{k+1}) + (pb'_{0,1} - mb_{0,2})(x,t_k), \quad x \in \bar{\Omega}_1,\\ (lc_{0,2})(x,t_{k+1}) - r(x,t_{k+1})b_{0,2}(x-1,t_{k+1}) &= (pc'_{0,1})(x,t_{k+1}) + (pc'_{0,1} - mc_{0,2})(x,t_k)\\ &\quad +r(x,t_k)b_{0,2}(x-1,t_k), \quad x \in \bar{\Omega}_2, \end{aligned}\right\}$$

and lastly, the functions $b_{0,3}$ and $c_{0,3}$ are the solutions of

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_3 b_{0,3}(x,t_{k+1}) = -(pb'_{0,2})(x,t_{k+1}) - (pb'_{0,2})(x,t_k) - \tilde{\mathcal{L}}_3^* b_{0,3}(x,t_k), \quad x \in [0,1),\\ &b_{0,3}(1,t_k) = 0, \end{aligned}\right\} \tag{5.31}$$

and

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_4 c_{0,3}(x,t_{k+1}) = -(pc'_{0,2})(x,t_{k+1}) - (pc'_{0,2})(x,t_k) - \tilde{\mathcal{L}}_4^* c_{0,3}(x,t_k), \quad x \in [1,2),\\ &c_{0,3}(2,t_k) = 0,\\ &c_{0,3}(x,t_{k+1}) = b_{0,3}(x,t_{k+1}), \quad x \in [0,1). \end{aligned}\right\} \tag{5.32}$$

Since, the functions $l(x,t_{k+1})$, $m(x,t_{k+1})$, $r(x,t_{k+1})$, $\psi_1(x,t_{k+1})$, $\psi_2(x,t_{k+1})$, $\psi_3(x,t_{k+1})$, and $g(x,t_{k+1})$ are sufficiently smooth on their respective domain. Thus, we have

$$\begin{aligned} &\|b_{0,0}^{(k)}\|_{\bar{\Omega}_1} \le C, \;\; \|c_{0,0}^{(k)}\|_{\bar{\Omega}_2} \le C, \;\text{ for } 0 \le k \le 10,\\ &\|b_{0,1}^{(k)}\|_{\bar{\Omega}_1} \le C, \;\; \|c_{0,1}^{(k)}\|_{\bar{\Omega}_2} \le C, \;\text{ for } 0 \le k \le 9,\\ &\|b_{0,2}^{(k)}\|_{\bar{\Omega}_1} \le C, \;\; \|c_{0,2}^{(k)}\|_{\bar{\Omega}_2} \le C, \;\text{ for } 0 \le k \le 8. \end{aligned} \tag{5.33}$$

Define $\psi^{\pm}(x,t_{k+1}) = C \pm b_{0,3}(x,t_{k+1}),\ x \in \bar{\Omega}_1$. Now, apply Lemma 5.3.6 to $\psi^{\pm}(x,t_{k+1})$ to obtain $||b_{0,3}||_{\bar{\Omega}1} \le C$. Then, we use (5.31) to get

$$||b_{0,3}^{(k)}||_{\bar{\Omega}_1} \le \frac{C}{\mu^k}, \text{ for } 0 \le k \le 8. \tag{5.34}$$

Similarly, consider the function $\psi^{\pm}(x,t_{k+1}) = C \pm c_{0,3}(x,t_{k+1}),\ x \in \bar{\Omega}_2$. Applying Lemma 5.3.6 to $\psi^{\pm}(x,t_{k+1})$ yields $||c_{0,3}||_{\bar{\Omega}_2} \le C$. Using (5.32), we have

$$||c_{0,3}^{(k)}||_{\bar{\Omega}_2} \le \frac{C}{\mu^k}, \text{ for } 0 \le k \le 8. \tag{5.35}$$

As defined in (5.23) and (5.24), choose $b_0(1,t_{k+1})$ and $c_0(2,t_{k+1})$ to be

$$b_0(1,t_{k+1}) = (b_{0,0} + \mu b_{0,1} + \mu^2 b_{0,2})(1,t_{k+1}),\ c_0(2,t_{k+1}) = (c_{0,0} + \mu c_{0,1} + \mu^2 c_{0,2})(2,t_{k+1}).$$

Therefore, using (5.33)-(5.35) to obtain

$$||b_0^k||_{\bar{\Omega}_1} \le C\left(1 + \frac{1}{\mu^{k-3}}\right),\ \ ||c_0^k||_{\bar{\Omega}_2} \le C\left(1 + \frac{1}{\mu^{k-3}}\right),\ \ 0 \le k \le 8. \tag{5.36}$$

Now, decompose the functions b_1 and c_1 as

$$\begin{aligned} b_1(x,t_{k+1};\mu) &= b_{1,0}(x,t_{k+1}) + \mu b_{1,1}(x,t_{k+1}) + \mu^2 b_{1,2}(x,t_{k+1};\mu), \quad x \in \bar{\Omega}_1,\\ c_1(x,t_{k+1};\mu) &= c_{1,0}(x,t_{k+1}) + \mu c_{1,1}(x,t_{k+1}) + \mu^2 c_{1,2}(x,t_{k+1};\mu), \quad x \in \bar{\Omega}_2, \end{aligned}$$

where the functions $b_{1,0}$ and $c_{1,0}$ are the solution of

$$\left.\begin{aligned} &(lb_{1,0})(x,t_{k+1}) = b_0''(x,t_{k+1}) + b_0''(x,t_k) - (mb_{1,0})(x,t_k), \quad x \in \bar{\Omega}_1,\\ &(lc_{1,0})(x,t_{k+1}) - r(x,t_{k+1})c_{1,0}(x-1,t_{k+1}) = c_0''(x,t_{k+1}) + c_0''(x,t_k) - (mc_{1,0})(x,t_k)\\ &\qquad\qquad\qquad\qquad\qquad + r(x,t_k)c_{1,0}(x-1,t_k), \quad x \in \bar{\Omega}_2, \end{aligned}\right\}$$

the functions $b_{1,1}$ and $c_{1,1}$ are the solutions of

$$\left.\begin{aligned} &(lb_{1,1})(x,t_{k+1}) = (pb_{1,0}')(x,t_{k+1}) + (pb_{1,0}' - mb_{1,1})(x,t_k), \quad x \in \bar{\Omega}_1,\\ &(lc_{1,1})(x,t_{k+1}) - r(x,t_{k+1})b_{1,1}(x-1,t_{k+1}) = (pc_{1,0}')(x,t_{k+1}) + (pc_{1,0}' - mc_{1,1})(x,t_k)\\ &\qquad\qquad\qquad\qquad\qquad + r(x,t_k)b_{1,1}(x-1,t_k), \quad x \in \bar{\Omega}_2, \end{aligned}\right\}$$

and the functions $b_{1,2}$ and $c_{1,2}$ are the solution of

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_3 b_{1,2}(x,t_{k+1}) = -(pb_{1,1}')(x,t_{k+1}) - (pb_{1,1}')(x,t_k) - \tilde{\mathcal{L}}_3^* b_{1,2}(x,t_k), \quad x \in [0,1),\\ &b_{1,2}(1,t_k) = 0,\\ &\tilde{\mathcal{L}}_4 c_{1,2}(x,t_{k+1}) = -(pc_{1,1}')(x,t_{k+1}) - (pc_{1,1}')(x,t_k) - \tilde{\mathcal{L}}_4^* c_{1,2}(x,t_k), \quad x \in [1,2),\\ &c_{1,2}(2,t_k) = 0,\\ &c_{1,2}(x,t_{k+1}) = b_{1,2}(x,t_{k+1}), \quad x \in [0,1). \end{aligned}\right\}$$

To establish the bounds for the derivatives of b_1 and c_1, we use a similar argument as we did for b_0 and c_0. We apply Lemma 5.3.6 to suitable barrier functions and obtain the bounds for the derivatives of the components of b_1 and c_1 as follows

$$||b_{1,0}^{(k)}||_{\bar{\Omega}_1} \leq C\left(1 + \frac{1}{\mu^{k-1}}\right),\ ||c_{1,0}^{(k)}||_{\bar{\Omega}_2} \leq C\left(1 + \frac{1}{\mu^{k-1}}\right),\ 0 \leq k \leq 6,$$

$$||b_{1,1}^{(k)}||_{\bar{\Omega}_1} \leq C\left(1 + \frac{1}{\mu^{k}}\right),\ ||c_{1,1}^{(k)}||_{\bar{\Omega}_2} \leq C\left(1 + \frac{1}{\mu^{k}}\right),\ 0 \leq k \leq 5,$$

$$||b_{1,2}^{(k)}||_{\bar{\Omega}_1} \leq \left(\frac{C}{\mu^{k+1}}\right),\ ||c_{1,2}^{(k)}||_{\bar{\Omega}_2} \leq \left(\frac{C}{\mu^{k+1}}\right),\ 0 \leq k \leq 5.$$

Therefore, we get the following bounds for the derivatives of b_1 and c_1

$$||b_{1}^{(k)}||_{\bar{\Omega}_1} \leq C\left(1 + \frac{1}{\mu^{k-1}}\right),\ ||c_{1}^{(k)}||_{\bar{\Omega}_2} \leq C\left(1 + \frac{1}{\mu^{k-1}}\right),\ 0 \leq k \leq 5.$$

Similarly, we can establish the bounds for the derivatives of b_2 and c_2 as given below

$$||b_{2}^{(k)}||_{\bar{\Omega}_1} \leq \left(\frac{C}{\mu^{k+1}}\right),\ ||c_{2}^{(k)}||_{\bar{\Omega}_2} \leq \left(\frac{C}{\mu^{k+1}}\right),\ 0 \leq k \leq 3. \tag{5.37}$$

From (5.29) and (5.30) and Lemma 2 of [101], we get the following bounds for b_3 and c_3

$$||b_3||_{\bar{\Omega}_1} \leq \left(\frac{C}{\mu^3}\right),\ ||c_3||_{\bar{\Omega}_2} \leq \left(\frac{C}{\mu^3}\right).$$

From Lemma 5.2.3, it follows that

$$||b_3^{'}||_{\bar{\Omega}_1} \leq \left(\frac{C}{\varepsilon\mu^2}\right),\ \ ||c_3^{'}||_{\bar{\Omega}_2} \leq \left(\frac{C}{\varepsilon\mu^2}\right), \tag{5.38}$$

and from (5.29) and (5.30), we have

$$||b_3^{''}||_{\bar{\Omega}_1} \leq \left(\frac{C}{\varepsilon^2\mu}\right),\ \ ||c_3^{''}||_{\bar{\Omega}_2} \leq \left(\frac{C}{\varepsilon^2\mu}\right). \tag{5.39}$$

Differentiating equations (5.29) and (5.30) and using $p_0\mu^2 \geq \gamma\varepsilon$, we compute

$$||b_3^{(3)}||_{\bar{\Omega}_1} \leq \left(\frac{C}{\varepsilon^3}\right),\ \ ||c_3^{(k)}||_{\bar{\Omega}_2} \leq \left(\frac{C}{\varepsilon^3}\right). \tag{5.40}$$

Therefore, we can write

$$||b_3^{(k)}||_{\bar{\Omega}_1} \leq \left(\frac{C}{\varepsilon^k\mu^{3-k}}\right),\ \ ||c_3^{(k)}||_{\bar{\Omega}_2} \leq \left(\frac{C}{\varepsilon^k\mu^3 - k}\right),\ \ k = 0, 1, 2, 3.$$

As defined in (5.16), choose $b(0, t_{k+1})$ and $b(1, t_{k+1})$ to be

$$b(0, t_{k+1}) = b_0(0, t_{k+1}) + \varepsilon b_1(0, t_{k+1}) + \varepsilon^2 b_2(0, t_{k+1}),\ b(1, t_{k+1}) = b_0(1, t_{k+1}) + \varepsilon b_1(1, t_{k+1}),$$

and from (5.17), choose $c(1, t_{k+1})$ and $c(2, t_{k+1})$ to be

$$c(1, t_{k+1}) = c_0(1, t_{k+1}) + \varepsilon c_1(1, t_{k+1}) + \varepsilon^2 c_2(1, t_{k+1}),\ c(2, t_{k+1}) = c_0(2, t_{k+1}) + \varepsilon c_1(2, t_{k+1}).$$

We now obtain the bounds for the functions b and c by using the estimates for their components $b_0, b_1, b_2, b_3, c_0, c_1, c_2$ and c_3 and get

$$||b^{(k)}||_{\bar{\Omega}_1} \leq C\left(1 + \left(\frac{\varepsilon}{\mu}\right)^{3-k}\right),\ \ ||c^{(k)}||_{\bar{\Omega}_2} \leq C\left(1 + \left(\frac{\varepsilon}{\mu}\right)^{3-k}\right).$$

□

5.3.3 Estimates for the Singular Components

Let us decompose the singular component $Z(x, t_{k+1})$ defined in (5.15) as

$$\begin{aligned} Z(x, t_{k+1}) &:= Z_L(x, t_{k+1}) + Z_R(x, t_{k+1}) \\ &:= (z_1^L + z_2^L)(x, t_{k+1}) + (z_1^R + z_2^R)(x, t_{k+1}), \end{aligned}$$

where the functions z_1^L and z_2^L satisfy

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_1 z_1^L(x, t_{k+1}) = 0, \quad x \in (0,1), \\ &z_1^L(0, t_{k+1}) = U(0, t_{k+1}) - X(0, t_{k+1}) - j_1(\varepsilon, \mu), \quad z_1^L(1, t_{k+1}) = 0, \end{aligned}\right\} \tag{5.41}$$

and

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_2 z_2^L(x, t_{k+1}) = 0, \quad x \in (1,2), \\ &z_2^L(1, t_{k+1}) = k_1(\varepsilon, \mu) - j_2(\varepsilon, \mu), \quad z_2^L(2, t_{k+1}) = 0, \\ &z_2^L(x, t_{k+1}) = z_1^L(x, t_{k+1}), \quad x \in [0,1), \end{aligned}\right\} \tag{5.42}$$

and the functions z_1^R and z_2^R satisfy

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_1 z_1^R(x, t_{k+1}) = 0, \quad x \in (0,1), \\ &z_1^R(0, t_{k+1}) = j_1(\varepsilon, \mu), \quad z_1^R(1, t_{k+1}) = k_2(\varepsilon, \mu), \end{aligned}\right\} \tag{5.43}$$

and

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_2 z_2^R(x, t_{k+1}) = 0, \quad x \in (1,2), \\ &z_2^R(1, t_{k+1}) = j_2(\varepsilon, \mu), \quad z_2^R(2, t_{k+1}) = U(2, t_{k+1}) - X(2, t_{k+1}), \\ &z_2^R(x, t_{k+1}) = z_1^R(x, t_{k+1}), \quad x \in [0,1). \end{aligned}\right\} \tag{5.44}$$

Here, $k_1(\varepsilon, \mu)$ and $k_2(\varepsilon, \mu)$ are constants to be chosen to satisfy the jump conditions at $x = 1$ given in (5.19) and (5.20). Moreover, constants $j_1(\varepsilon, \mu)$ and $j_2(\varepsilon, \mu)$ are to be chosen separately for the cases $p_0\mu^2 \leq \gamma\varepsilon$ and $p_0\mu^2 \geq \gamma\varepsilon$ to satisfy the requirements for the bounds of the singular component.

Lemma 5.3.8. *Let* $Z_L(x, t_{k+1})$ *and* $Z_R(x, t_{k+1})$ *be the solutions of* (5.41)-(5.44). *Then for* $p_0\mu^2 \geq \gamma\varepsilon$ *and* $k = 0, 1, 2, 3$

$$\begin{aligned} &\|z_1^{L,(k)}\|_{\bar{\Omega}_1} \leq C\left(\frac{\mu}{\varepsilon}\right)^k, \quad \|z_2^{L,(k)}\|_{\bar{\Omega}_2} \leq C\left(\frac{\mu}{\varepsilon}\right)^k, \\ &\|z_1^{R,(k)}\|_{\bar{\Omega}_1} \leq \frac{C}{\mu^k}, \quad \|z_2^{R,(k)}\|_{\bar{\Omega}_2} \leq \frac{C}{\mu^k}. \end{aligned}$$

Proof. Since $U(0, t_{k+1})$ and $U(1, t_{k+1})$ are bounded by constants independent of ε and μ, $|j_1|$, $|j_2|$, $|k_1|$ and $|k_2|$ are also bounded by constants independent of ε and μ. Next, we use Lemma 5.3.4 to compute

$$\|z_1^{L,(k)}(x, t_{k+1})\|_{\bar{\Omega}_1} \leq C\left(\frac{\mu}{\varepsilon}\right)^k, \quad \|z_2^{L,(k)}(x, t_{k+1})\|_{\bar{\Omega}_2} \leq C\left(\frac{\mu}{\varepsilon}\right)^k.$$

To find bounds for z_1^R and z_2^R, consider the following decomposition of z_1^R and z_2^R as

$$\begin{aligned} z_1^R(x,t_{k+1};\varepsilon,\mu) &= z_{1,0}^R(x,t_{k+1};\mu) + \varepsilon z_{1,1}^R(x,t_{k+1};\mu) + \varepsilon^2 z_{1,2}^R(x,t_{k+1};\mu) \\ &\quad + \varepsilon^3 z_{1,3}^R(x,t_{k+1};\varepsilon,\mu), \\ z_2^R(x,t_{k+1};\varepsilon,\mu) &= z_{2,0}^R(x,t_{k+1};\mu) + \varepsilon z_{2,1}^R(x,t_{k+1};\mu) + \varepsilon^2 z_{2,2}^R(x,t_{k+1};\mu) \\ &\quad + \varepsilon^3 z_{2,3}^R(x,t_{k+1};\varepsilon,\mu), \end{aligned} \tag{5.45}$$

where the functions $z_{1,0}^R$ and $z_{2,0}^R$ are the solutions of

$$\tilde{\mathcal{L}}_3 z_{1,0}^R(x,t_{k+1}) = 0, \quad x \in [0,1), \quad z_{1,0}^R(1,t_{k+1}) = k_2, \tag{5.46}$$

and

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_4 z_{2,0}^R(x,t_{k+1}) = 0, \quad x \in [1,2), \quad z_{2,0}^R(2,t_{k+1}) = U(2,t_{k+1}) - X(2,t_{k+1}), \\ &z_{2,0}^R(x,t_{k+1}) = z_{1,0}^R(x,t_{k+1}), \quad x \in [0,1), \end{aligned}\right\} \tag{5.47}$$

the functions $z_{1,1}^R$ and $z_{2,1}^R$ are the solutions of

$$\tilde{\mathcal{L}}_3 z_{1,1}^R(x,t_{k+1}) = -z_{1,0}^{R,(2)}(x,t_{k+1}), \quad x \in [0,1), \quad z_{1,1}^R(1,t_{k+1}) = 0, \tag{5.48}$$

and

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_4 z_{2,1}^R(x,t_{k+1}) = -z_{2,0}^{R,(2)}(x,t_{k+1}), \quad x \in [1,2), \quad z_{2,1}^R(2,t_{k+1}) = 0, \\ &z_{2,1}^R(x,t_{k+1}) = z_{1,1}^R(x,t_{k+1}), \quad x \in [0,1), \end{aligned}\right\} \tag{5.49}$$

the functions $z_{1,2}^R$ and $z_{2,2}^R$ are the solutions of

$$\tilde{\mathcal{L}}_3 z_{1,2}^R(x,t_{k+1}) = -z_{1,1}^{R,(2)}(x,t_{k+1}), \quad x \in [0,1), \quad z_{1,2}^R(1,t_{k+1}) = 0, \tag{5.50}$$

and

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_4 z_{2,2}^R(x,t_{k+1}) = -z_{2,1}^{R,(2)}(x,t_{k+1}), \quad x \in [1,2), \quad z_{2,2}^R(2,t_{k+1}) = 0, \\ &z_{2,2}^R(x,t_{k+1}) = z_{1,2}^R(x,t_{k+1}), \quad x \in [0,1), \end{aligned}\right\} \tag{5.51}$$

and lastly, the functions $z_{1,3}^R$ and $z_{2,3}^R$ are the solutions of

$$\left.\begin{aligned} &\tilde{\mathcal{L}}_1 z_{1,3}^R(x,t_{k+1}) = -z_{1,2}^{R,(2)}(x,t_{k+1}), \quad x \in (0,1), \quad z_{1,3}^R(0,t_{k+1}) = 0, \quad z_{1,3}^R(1,t_{k+1}) = 0, \\ &\tilde{\mathcal{L}}_2 z_{2,3}^R(x,t_{k+1}) = -z_{2,2}^{R,(2)}(x,t_{k+1}), \quad x \in (1,2), \quad z_{2,3}^R(1,t_{k+1}) = 0, \quad z_{2,3}^R(2,t_{k+1}) = 0, \\ &z_{2,3}^R(x,t_{k+1}) = z_{1,3}^R(x,t_{k+1}), \quad x \in [0,1). \end{aligned}\right\}$$

Define $\psi^{\pm}(x,t_{k+1}) = C \pm z_{1,0}^R(x,t_{k+1})$, $x \in [0,1]$ and apply Lemma 5.3.6 to the function $\psi^{\pm}(x,t_{k+1})$ to obtain $\|z_{1,0}^R(x,t_{k+1})\|_{\bar{\Omega}_1} \leq C$. Now we use (5.46) to find

$$\|z_{1,0}^{R,(k)}\|_{\bar{\Omega}_1} \leq \frac{C}{\mu^k}, \quad 0 \leq k \leq 5. \tag{5.52}$$

Similarly, it is easy to follow that

$$\|z_{2,0}^{R,(k)}\|_{\bar{\Omega}_2} \leq \frac{C}{\mu^k}, \quad 0 \leq k \leq 5. \tag{5.53}$$

Now, define $\psi^{\pm}(x, t_{k+1}) = C\|z_{1,0}^{R,(2)}\|_{\Omega_1} \pm z_{1,1}^{R}(x, t_{k+1}),\ x \in [0,1]$ and apply Lemma 5.3.6 to the function $\psi^{\pm}$ we find $\|z_{1,1}^{R}\|_{\bar{\Omega}_1} \leq \dfrac{C}{\mu^2}$. From (5.48), we have

$$\|z_{1,1}^{R,(k)}\|_{\bar{\Omega}_1} \leq \frac{C}{\mu^{k+2}}, \quad 0 \leq k \leq 4. \tag{5.54}$$

Similarly, we can obtain the following bounds for $z_{1,2}^R$, $z_{2,1}^R$ and $z_{2,2}^R$

$$\|z_{1,2}^{R,(k)}\|_{\bar{\Omega}_1} \leq \frac{C}{\mu^{k+4}}, \quad 0 \leq k \leq 3, \tag{5.55}$$

$$\|z_{2,1}^{R,(k)}\|_{\bar{\Omega}_2} \leq \frac{C}{\mu^{k+2}}, \quad 0 \leq k \leq 4, \quad \|z_{2,2}^{R,(k)}\|_{\bar{\Omega}_2} \leq \frac{C}{\mu^{k+4}}, \quad 0 \leq k \leq 3. \tag{5.56}$$

Now, we apply Lemma 1 from [101] to the function $\psi^{\pm}(x, t_{k+1}) = C\|z_{1,2}^{R,(2)}\|_{\Omega_1} \pm z_{1,3}^{R}(x, t_{k+1})$, $x \in [0,1]$, we get $\|z_{1,3}^{R}\|_{\bar{\Omega}_1} \leq \dfrac{C}{\mu^6}$. Using Lemma 5.3.4 to write

$$\|z_{1,3}^{R,(k)}\|_{\bar{\Omega}_1} \leq \frac{C}{(\sqrt{\varepsilon})^k \mu^6}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^k\right), \quad k = 1, 2, \tag{5.57}$$

$$\|z_{1,3}^{R,(3)}\|_{\bar{\Omega}_1} \leq \frac{C}{(\sqrt{\varepsilon})^3 \mu^6}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right) + \frac{C}{\varepsilon\mu^7}. \tag{5.58}$$

Similarly, we can estimate the following bounds for $z_{2,3}^R$

$$\|z_{2,3}^{R,(k)}\|_{\bar{\Omega}_2} \leq \frac{C}{(\sqrt{\varepsilon})^k \mu^6}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^k\right), \quad k = 0, 1, 2, \tag{5.59}$$

$$\|z_{2,3}^{R,(3)}\|_{\bar{\Omega}_2} \leq \frac{C}{(\sqrt{\varepsilon})^3 \mu^6}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right) + \frac{C}{\varepsilon\mu^7}, \quad k = 1, 2. \tag{5.60}$$

Since $p_0\mu^2 \geq \gamma\varepsilon$ and using (5.52)-(5.60) in (5.45), we get

$$\|z_1^{R,(k)}\|_{\bar{\Omega}_1} \leq \frac{C}{\mu^k} \text{ and } \|z_2^{R,(k)}\|_{\bar{\Omega}_2} \leq \frac{C}{\mu^k}, 0 \leq k \leq 3.$$

□

Lemma 5.3.9. *Let $Z_L(x, t_{k+1})$ and $Z_R(x, t_{k+1})$ be the solutions of* (5.41)-(5.44)*. Then for $p_0\mu^2 \leq \gamma\varepsilon$ and $k = 0, 1, 2, 3$*

$$\|z_1^{L,(k)}(x, t_{k+1})\|_{\bar{\Omega}_1} \leq C\varepsilon^{\frac{-k}{2}}, \quad \|z_2^{L,(k)}(x, t_{k+1})\|_{\bar{\Omega}_2} \leq C\varepsilon^{\frac{-k}{2}},$$
$$\|z_1^{R,(k)}(x, t_{k+1})\|_{\bar{\Omega}_1} \leq C\varepsilon^{\frac{-k}{2}}, \quad \|z_2^{R,(k)}(x, t_{k+1})\|_{\bar{\Omega}_2} \leq C\varepsilon^{\frac{-k}{2}}.$$

Proof. Using Lemma 5.3.4, we note that z_1^L and z_2^L satisfy

$$\|z_1^{L,(k)}(x,t_{k+1})\|_{\bar{\Omega}_1} \le C\varepsilon^{\frac{-k}{2}}, \ \ \|z_2^{L,(k)}(x,t_{k+1})\|_{\bar{\Omega}_2} \le C\varepsilon^{\frac{-k}{2}}, \ 0 \le k \le 3.$$

For the bounds of z_1^R and z_2^R, choose $j_1(\varepsilon,\mu)$ and $j_2(\varepsilon,\mu)$ to be zero and using Lemma 5.3.4, we get

$$\|z_1^{R,(k)}\|_{\bar{\Omega}_1} \le C\varepsilon^{\frac{-k}{2}} \text{ and } \|z_2^{R,(k)}\|_{\bar{\Omega}_2} \le C\varepsilon^{\frac{-k}{2}}, \ \ 0 \le k \le 3.$$

□

The following Lemma establishes sharper estimates for the bounds of the singular components.

Lemma 5.3.10. *Let the layer components z_1^L, z_2^L, z_1^R and z_2^R be the solutions of* (5.41)-(5.44). *Then*

$$|z_1^L(x,t_{k+1})| \le Ce^{-\theta_1 x}, \ x \in [0,1], \quad |z_2^L(x,t_{k+1})| \le Ce^{-\theta_1(x-1)}, \ x \in [1,2],$$
$$|z_1^R(x,t_{k+1})| \le Ce^{-\theta_2(1-x)}, \ x \in [0,1], \quad |z_2^R(x,t_{k+1})| \le Ce^{-\theta_2(2-x)}, \ x \in [1,2],$$

where

$$\theta_1 = \begin{cases} \dfrac{\sqrt{\gamma p_0}}{\sqrt{\varepsilon}} & \text{if } \mu^2 \le \dfrac{\gamma\varepsilon}{p_0}, \\ \dfrac{p_0\mu}{\varepsilon} & \text{if } \mu^2 \ge \dfrac{\gamma\varepsilon}{p_0}, \end{cases} \quad \textit{and} \quad \theta_2 = \begin{cases} \dfrac{\sqrt{\gamma p_0}}{2\sqrt{\varepsilon}} & \text{if } \mu^2 \le \dfrac{\gamma\varepsilon}{p_0}, \\ \dfrac{\gamma}{2\mu} & \text{if } \mu^2 \ge \dfrac{\gamma\varepsilon}{p_0}. \end{cases}$$

Proof. For proof, see [135]. □

5.4 Spatial Discretization

The solution of the problem exhibits boundary layers at $x = 0$, $x = 2$ and an interior layer at $x = 1$ [218]. To resolve the layers, we design the mesh to condense in the inner layer regions and to remain coarse in the outer regions, away from the layers. Consequently, to generate a piecewise-uniform meh $\bar{\mathbb{D}}_x^N$, we partition the interval $[0,2]$ into six subintervals as

$$[0,2] = [0,\beta_1] \cup [\beta_1, 1-\beta_2] \cup [1-\beta_2, 1] \cup [1, 1+\beta_1] \cup [1+\beta_1, 2-\beta_2] \cup [2-\beta_2, 2],$$

where β_1 and β_2 are the mesh transition parameters defined as

$$\beta_1 = \begin{cases} \min\left\{\dfrac{1}{4}, \dfrac{2\sqrt{\varepsilon}}{\sqrt{\gamma p_0}}\ln N\right\} & \text{if } \mu^2 \le \dfrac{\gamma\varepsilon}{p_0}, \\ \min\left\{\dfrac{1}{4}, \dfrac{2\varepsilon}{p_0\mu}\ln N\right\} & \text{if } \mu^2 \ge \dfrac{\gamma\varepsilon}{p_0}, \end{cases} \text{ and } \beta_2 = \begin{cases} \min\left\{\dfrac{1}{4}, \dfrac{2\sqrt{\varepsilon}}{\sqrt{\gamma p_0}}\ln N\right\} & \text{if } \mu^2 \le \dfrac{\gamma\varepsilon}{p_0}, \\ \min\left\{\dfrac{1}{4}, \dfrac{2\mu}{\varepsilon}\ln N\right\} & \text{if } \mu^2 \ge \dfrac{\gamma\varepsilon}{p_0}. \end{cases}$$

We place $N/8$ mesh points each in intervals $[0,\beta_1]$, $[1-\beta_2,1]$, $[1,1+\beta_1]$, $[2-\beta_2,2]$ and $N/4$ mesh points in intervals $[\beta_1,1-\beta_2]$ and $[1+\beta_1,2-\beta_2]$. Consequently, we obtain

$$\bar{\mathbb{D}}_x^N = \{x_i\}_0^N = \begin{cases} ih_1 & \text{for } i=0,\ldots,N/8, \\ \beta_1 + \left(i-\frac{N}{8}\right)h_2 & \text{for } i=N/8+1,\ldots,3N/8, \\ (1-\beta_2)+\left(i-\frac{3N}{8}\right)h_3 & \text{for } i=3N/8+1,\ldots,N/2, \\ 1+\left(i-\frac{N}{2}\right)h_1 & \text{for } i=N/2+1,\ldots,5N/8, \\ (1+\beta_1)+\left(i-\frac{5N}{8}\right)h_2 & \text{for } i=5N/8+1,\ldots,7N/8, \\ (2-\beta_2)+\left(i-\frac{7N}{8}\right)h_3 & \text{for } i=7N/8+1,\ldots,N, \end{cases}$$

where

$$h_i = \begin{cases} h_1 = \dfrac{4\beta_1}{N} & \text{for } i=1,\ldots,N/8, N/2+1,\ldots,5N/8, \\ h_2 = \dfrac{4}{N}(1-\beta_1-\beta_2) & \text{for } i=N/8+1,\ldots,3N/8, 5N/8+1,\ldots,7N/8, \\ h_3 = \dfrac{8\beta_2}{N} & \text{for } i=3N/8+1,\ldots,N/2, 7N/8+1,\ldots,N. \end{cases}$$

The fully discrete problem on $\bar{\mathbb{D}}_x^N \times \mathbb{T}_t{}^M$ thus reads

$$\hat{\mathcal{L}}_n\tilde{U}(x_i,t_{k+1}) = \hat{\mathcal{G}}_n(x_i,t_{k+1}), \quad n=1,2, \tag{5.61}$$

where

$$\hat{\mathcal{L}}_1\tilde{U}_{i,k+1} = \varepsilon\delta_x^2\tilde{U}_{i,k+1} + \mu p_{i,k+1}D_x^+\tilde{U}_{i,k+1} - l_{i,k+1}\tilde{U}_{i,k+1} \text{ for } i=1,\ldots,N/2-1,$$
$$\hat{\mathcal{L}}_2\tilde{U}_{i,k+1} = \varepsilon\delta_x^2\tilde{U}_{i,k+1} + \mu p_{i,k+1}D_x^+\tilde{U}_{i,k+1} - l_{i,k+1}\tilde{U}_{i,k+1} + r_{i,k+1}\tilde{U}_{i-N/2,k+1} \text{ for } i=N/2,\ldots,N-1$$

and

$$\begin{aligned} \hat{\mathcal{G}}_1(x_i,t_{k+1}) &= -\varepsilon\delta_x^2\tilde{U}_{i,k} - \mu p_{i,k}D_x^+\tilde{U}_{i,k} + m_{i,k}\tilde{U}_{i,k} - r_{i,k}\psi_1(x_i-1,t_k) - r_{i,k+1}\psi_1(x_i-1,t_{k+1}) \\ &\quad + g_{i,k+1} + g_{i,k} \text{ for } i=1,\ldots,N/2-1, \\ \hat{\mathcal{G}}_2(x_i,t_{k+1}) &= -\varepsilon\delta_x^2\tilde{U}_{i,k} - \mu p_{i,k}D_x^+\tilde{U}_{i,k} + m_{i,k}\tilde{U}_{i,k} - r_{i,k}\tilde{U}(x_i-1,t_k) + g_{i,k+1} + g_{i,k} \\ &\quad \text{for } i=N/2,\ldots,N-1 \end{aligned}$$

with

$$\begin{cases} \tilde{U}(x_i,t_{k+1}) = \psi_1(x_i-1,t_{k+1}) & \text{for } i=0,1,\ldots N/2, \quad k=0,\ldots,M-1, \\ \tilde{U}(x_i,t_0) = \psi_2(x_i,0) & \text{for } i=0,1,\ldots,N, \\ \tilde{U}(x_N,t_{k+1}) = \psi_3(x_N,t_{k+1}) & \text{for } k=0,\ldots,M-1. \end{cases}$$

The operator $\hat{\mathcal{L}}_n$ satisfies the following discrete minimum principle.

Lemma 5.4.1. *Let $Z_{i,k+1}$ be a mesh function such that $Z_{i,k+1} \geq 0$ for $i = \{0, N\}$, $\hat{\mathcal{L}}_1 Z_{i,k+1} \leq 0$ for $i = 1, \ldots, N/2-1$ and $\hat{\mathcal{L}}_2 Z_{i,k+1} \leq 0$ for $i = N/2, \ldots, N-1$. Then $Z_{i,k+1} \geq 0$ for $i = 0, 1, \ldots, N$.*

Proof. Choose $q^* \in \{0, 1, \ldots, N\}$ such that $Z_{q^*,k+1} = \min_{\bar{\mathbb{D}}_x^N \times \mathbb{T}_t^M} Z_{i,k+1}$. Suppose $Z_{q^*,k+1} < 0$ and it follows that $q^* \notin \{0, N\}$, $Z_{q^*+1,k+1} - Z_{q^*,k+1} \geq 0$ and $Z_{q^*,k+1} - Z_{q^*-1,k+1} \leq 0$.

Case I: For $q^* \in \{1, 2, \ldots, N/2-1\}$

$$\begin{aligned}
\hat{\mathcal{L}}_1 Z_{q^*,k+1} &= \varepsilon \delta_x^2 Z_{q^*,k+1} + \mu p_{q^*,k+1} D_x^+ Z_{q^*,k+1} - l_{q^*,k+1} Z_{q^*,k+1} \\
&= \frac{2\varepsilon}{h_{q^*} + h_{q^*+1}} \left\{ \frac{Z_{q^*+1,k+1} - Z_{q^*,k+1}}{h_{q^*+1}} - \frac{Z_{q^*,k+1} - Z_{q^*-1,k+1}}{h_{q^*}} \right\} \\
&\quad + \mu p_{q^*,k+1} \left\{ \frac{Z_{q^*+1,k+1} - Z_{q^*,k+1}}{h_{q^*+1}} \right\} - l_{q^*,k+1} Z_{q^*,k+1} \\
&> 0.
\end{aligned}$$

Case II: For $q^* \in \{N/2, \ldots, N-1\}$

$$\begin{aligned}
\hat{\mathcal{L}}_2 Z_{q^*,k+1} &= \varepsilon \delta_x^2 Z_{q^*,k+1} + \mu p_{q^*,k+1} D_x^+ Z_{q^*,k+1} - l_{q^*,k+1} Z_{q^*,k+1} + r_{q^*,k+1} Z_{q^*-N/2,k+1} \\
&\geq (r_{q^*,k+1} - l_{q^*,k+1}) Z_{q^*,k+1} \\
&> 0.
\end{aligned}$$

The required result follows from a contradiction. □

Consequently, we can prove the following stability estimate for discrete operator $\hat{\mathcal{L}}_n$.

Lemma 5.4.2. *Let $Z_{i,k+1}$ be the solution of* (5.61). *Then*

$$|Z_{i,k+1}| \leq \max \left\{ |Z_{0,k+1}|, |Z_{N,k+1}|, \|\hat{\mathcal{L}}_1 Z_{i,k+1}\|, \|\hat{\mathcal{L}}_2 Z_{i,k+1}\| \right\}.$$

Proof. For proof, see [260]. □

5.5 Error Estimates

Let us decompose $\tilde{U}_{i,k+1}$ into smooth and singular components to obtain parameter uniform error estimates. We write $\tilde{U}_{i,k+1} := \tilde{X}_{i,k+1} + \tilde{Z}_{i,k+1}$, where

$$\tilde{X}(x_i, t_{k+1}) = \begin{cases} \tilde{X}_1(x_i, t_{k+1}) \text{ for } 0 \leq i \leq N/2-1, \\ \tilde{X}_2(x_i, t_{k+1}) \text{ for } N/2 \leq i \leq N, \end{cases}$$

and

$$\tilde{Z}(x_i, t_{k+1}) = \begin{cases} \tilde{Z}_1(x_i, t_{k+1}) \text{ for } 0 \leq i \leq N/2-1, \\ \tilde{Z}_2(x_i, t_{k+1}) \text{ for } N/2 \leq i \leq N. \end{cases}$$

The smooth component $\tilde{X}$ satisfies

$$\left.\begin{aligned}
&\hat{\mathcal{L}}_1\tilde{X}_1(x_i,t_{k+1}) = \hat{\mathcal{G}}_1(x_i,t_{k+1}) \text{ for } i\in\{1,2,\dots,N/2-1\},\\
&\tilde{X}_1(x_0,t_{k+1}) = b(0,t_{k+1}),\ \tilde{X}_1(x_{N/2},t_{k+1}) = b(1,t_{k+1}),\\
&\hat{\mathcal{L}}_2\tilde{X}_2(x_i,t_{k+1}) = \hat{\mathcal{G}}_2(x_i,t_{k+1}) \text{ for } i\in\{N/2+1,\dots,N-1\},\\
&\tilde{X}_2(x_{N/2},t_{k+1}) = c(1,t_{k+1}),\ \ \tilde{X}_2(x_N,t_{k+1}) = c(2,t_{k+1}),\\
&\tilde{X}_2(x_i-1,t_{k+1}) = \tilde{X}_1(x_{i-N/2},t_{k+1}),\ \ \text{for } i\in\{N/2,\dots,N-1\},
\end{aligned}\right\} \tag{5.62}$$

and the singular component $\tilde{Z}$ satisfies

$$\left.\begin{aligned}
&\hat{\mathcal{L}}_1\tilde{Z}_1(x_i,t_{k+1}) = 0 \text{ for } i\in\{1,2,\dots,N/2-1\},\\
&\tilde{Z}_1(x_0,t_{k+1}) = z_1(0,t_{k+1}),\ \tilde{Z}_1(x_{N/2},t_{k+1}) = Z_1(1,t_{k+1}),\\
&\hat{\mathcal{L}}_2\tilde{Z}_2(x_i,t_{k+1}) = 0 \text{ for } i\in\{N/2+1,\dots,N-1\},\\
&\tilde{Z}_2(x_{N/2},t_{k+1}) = z_2(1,t_{k+1}),\ \ \tilde{Z}_2(x_N,t_{k+1}) = z_2(2,t_{k+1}).
\end{aligned}\right\} \tag{5.63}$$

Moreover, the error $e_{i,k+1}$ is given by

$$e_{i,k+1} = \tilde{U}_{i,k+1} - U_{i,k+1} = (\tilde{X}_{i,k+1} - X_{i,k+1}) + (\tilde{Z}_{i,k+1} - Z_{i,k+1}).$$

Lemma 5.5.1. *Let $X(x_i,t_{k+1})$ and $\tilde{X}_{i,k+1}$ be the solutions of* (5.14)*,* (5.16)*,* (5.17) *and* (5.62)*, respectively. Then*

$$|\tilde{X}_{i,k+1} - X(x_i,t_{k+1})| \le CN^{-1},\ \ 0\le i\le N.$$

Proof. Consider

$$\begin{aligned}
|\tilde{X}(x_i,t_{k+1}) - X(x_i,t_{k+1})| &= |(\tilde{X}_1+\tilde{X}_2)(x_i,t_{k+1}) - (b+c)(x_i,t_{k+1})|\\
&= |(\tilde{X}_1-b)(x_i,t_{k+1}) + (\tilde{X}_2-c)(x_i,t_{k+1})|\\
&\le |(\tilde{X}_1-b)(x_i,t_{k+1})| + |(\tilde{X}_2-c)(x_i,t_{k+1})|.
\end{aligned} \tag{5.64}$$

For $i = 1,\dots,N/2-1$

$$\begin{aligned}
|\hat{\mathcal{L}}_1(\tilde{X}_1-b)(x_i,t_{k+1})| &= |\hat{\mathcal{L}}_1\tilde{X}_1(x_i,t_{k+1}) - \hat{\mathcal{L}}_1 b(x_i,t_{k+1})|\\
&\le \varepsilon\left|\left(\delta_x^2 - \frac{d^2}{dx^2}\right)b(x_i,t_{k+1})\right| + \mu|p(x_i,t_{k+1})|\left|\left(D_x^+ - \frac{d}{dx}\right)b(x_i,t_{k+1})\right|\\
&\le C\max_{1\le i\le \frac{N}{2}-1} h_i(\varepsilon\|b\|_3 + \mu\|b\|_2).
\end{aligned}$$

Case I: For $p_0\mu^2 \le \gamma\varepsilon$. Lemma 5.3.5 leads to

$$|\hat{\mathcal{L}}_1(\tilde{X}_1-b)(x_i,t_{k+1})| \le CN^{-1}.$$

Case II: For $p_0\mu^2 \geq \gamma\varepsilon$. Lemma 5.3.7 leads to

$$|\hat{\mathcal{L}}_1(\tilde{X}_1 - b)(x_i, t_{k+1})| \leq CN^{-1}.$$

Define $\psi^{\pm}(x_i, t_{k+1}) = CN^{-1} \pm (\tilde{X}_1 - b)(x_i, t_{k+1})$, $0 \leq i \leq N/2 - 1$ for sufficiently large C. Then $\psi^{\pm}(x_0, t_{k+1}) \geq 0$, $\psi^{\pm}(x_{N/2}, t_{k+1}) \geq 0$ and $\hat{\mathcal{L}}_1\psi^{\pm}(x_i, t_{k+1}) \leq 0$. Now, using Lemma 5.4.1, we obtain

$$|(\tilde{X}_1 - b)(x_i, t_{k+1})| \leq CN^{-1}, \; 0 \leq i \leq N/2.$$

A similar argument for $(\tilde{X}_2 - c)$ leads us to the following estimates

$$|(\tilde{X}_2 - c)(x_i, t_{k+1})| \leq CN^{-1}, \quad N/2 \leq i \leq N.$$

Therefore, from (5.64), it follows that $|\tilde{X}(x_i, t_{k+1}) - X(x_i, t_{k+1})| \leq CN^{-1}$, $0 \leq i \leq N$. □

Let us decompose $\tilde{Z}_1(x, t_{k+1})$ and $\tilde{Z}_2(x, t_{k+1})$ further as $\tilde{Z}_1(x_i, t_{k+1}) := (\tilde{Z}_1^L + \tilde{Z}_1^R)(x_i, t_{k+1})$ and $\tilde{Z}_2(x_i, t_{k+1}) := (\tilde{Z}_2^L + \tilde{Z}_2^R)(x_i, t_{k+1})$, where $\tilde{Z}_1^L$ and $\tilde{Z}_2^L$ satisfy

$$\left.\begin{aligned}
&\hat{\mathcal{L}}_1\tilde{Z}_1^L(x_i, t_{k+1}) = 0, \quad 0 < i < N/2,\\
&\tilde{Z}_1^L(x_0, t_{k+1}) = z_1^L(0, t_{k+1}), \quad \tilde{Z}_1^L(x_{N/2}, t_{k+1}) = 0,\\
&\hat{\mathcal{L}}_2\tilde{Z}_2^L(x_i, t_{k+1}) = 0, \quad N/2 < i < N,\\
&\tilde{Z}_2^L(x_{N/2}, t_{k+1}) = z_2^L(1, t_{k+1}), \quad \tilde{Z}_2^L(x_N, t_{k+1}) = 0,\\
&\tilde{Z}_2^L(x_{N/2} - 1, t_{k+1}) = \tilde{Z}_1^L(x_{i-N/2}, t_{k+1}), \quad N/2 \leq i < N,
\end{aligned}\right\} \tag{5.65}$$

and $\tilde{Z}_1^R$ and $\tilde{Z}_2^R$ satisfy

$$\left.\begin{aligned}
&\hat{\mathcal{L}}_1\tilde{Z}_1^R(x_i, t_{k+1}) = 0, \quad 0 < i < N/2,\\
&\tilde{Z}_1^R(x_0, t_{k+1}) = z_1^R(0, t_{k+1}), \quad \tilde{Z}_1^R(x_{N/2}, t_{k+1}) = z_1^R(1, t_{k+1}),\\
&\hat{\mathcal{L}}_2\tilde{Z}_2^R(x_i, t_{k+1}) = 0, \quad N/2 < i < N,\\
&\tilde{Z}_2^R(x_{N/2}, t_{k+1}) = z_2^R(1, t_{k+1}), \quad \tilde{Z}_2^R(x_N, t_{k+1}) = z_2^R(2, t_{k+1}),\\
&\tilde{Z}_2^R(x_i - 1, t_{k+1}) = \tilde{Z}_1^R(x_{i-N/2}, t_{k+1}), \quad 0 \leq i < N/2.
\end{aligned}\right\} \tag{5.66}$$

Define the mesh functions

$$I_{1,i} = \begin{cases} \prod_{j=1}^{i}(1 + \theta_1 h_j)^{-1}, & 1 < i \leq N/2,\\ 1, & i = 0, \end{cases} \quad \text{and} \quad I_{2,i} = \begin{cases} \prod_{j=i+1}^{N/2}(1 + \theta_2 h_j)^{-1}, & 0 \leq i < N/2,\\ 1, & i = N/2, \end{cases}$$

where θ_1 and θ_2 are defined in Lemma 5.3.10. For $i = N/8, \ldots, N/2$

$$\begin{aligned} I_{1,i} &\leq I_{1,N/8} \\ &= \prod_{j=1}^{N/8} (1+\theta_1 h_{N/8})^{-1} \\ &= (1+\theta_1 h_{N/8})^{-N/8} \\ &= \left(1+\theta_1 \frac{8\beta_1}{N}\right)^{-N/8} \\ &\leq (1+8N^{-1}\ln N)^{-N/8} \leq CN^{-1}, \end{aligned} \tag{5.67}$$

and for $i = 0, \ldots, 3N/8$

$$\begin{aligned} I_{2,i} &\leq I_{2,3N/8} \\ &= \left(1+\theta_2 \frac{8\beta_2}{N}\right)^{-N/8} \\ &\leq (1+8N^{-1}\ln N)^{-N/8} \\ &\leq CN^{-1}. \end{aligned} \tag{5.68}$$

Lemma 5.5.2. *Let $\tilde{Z}_1^L$, $\tilde{Z}_2^L$, $\tilde{Z}_1^R$ and $\tilde{Z}_2^R$ be the solutions of* (5.65) *and* (5.66). *Then*

$$|\tilde{Z}_1^L(x_i, t_{k+1})| \leq CI_{1,i}, \quad |\tilde{Z}_1^R(x_i, t_{k+1})| \leq CI_{2,i}, \quad 0 \leq i \leq N/2 \ \text{ and}$$

$$|\tilde{Z}_2^L(x_i, t_{k+1})| \leq CI_{1,i-\frac{N}{2}}, \quad |\tilde{Z}_2^R(x_i, t_{k+1})| \leq CI_{2,i-\frac{N}{2}}, \quad N/2 \leq i \leq N.$$

Proof. For $i = 0, \ldots N/2$, define $Q^{\pm}(x_i, t_{k+1}) = |\tilde{Z}_1^L(x_0, t_{k+1})| I_{1,i} \pm \tilde{Z}_1^L(x_i, t_{k+1})$. Then, $Q^{\pm}(x_0, t_{k+1}) = |\tilde{Z}_1^L(x_0, t_{k+1})| \pm \tilde{Z}_1^L(x_0, t_{k+1}) \geq 0$ and $Q^{\pm}(x_{N/2}, t_{k+1}) \geq 0$. Also, note that $\hat{\mathcal{L}}_1 I_{1,i} = (\varepsilon\delta_s^2 + \mu p D_s^+ - l) I_{1,i} \leq 0$ which implies $\hat{\mathcal{L}}_1 Q^{\pm}(x_i, t_{k+1}) \leq 0$. Using Lemma 5.4.1, we have

$$|\tilde{Z}_1^L(x_i, t_{k+1})| \leq CI_{1,i}, \quad 0 \leq i \leq N/2.$$

A similar argument for $\tilde{Z}_1^R$, $\tilde{Z}_2^L$ and $\tilde{Z}_2^R$ leads to the following estimates

$$\begin{aligned} &|\tilde{Z}_1^R(x_i, t_{k+1})| \leq CI_{2,i}, \quad 0 \leq i \leq N/2, \\ &|\tilde{Z}_2^L(x_i, t_{k+1})| \leq CI_{1,i-N/2}, \quad N/2 \leq i \leq N, \\ &|\tilde{Z}_2^R(x_i, t_{k+1})| \leq CI_{2,i-N/2}, \quad N/2 \leq i \leq N. \end{aligned}$$

□

To estimate the error bound for $(\tilde{Z} - Z) = (\tilde{Z}_L - Z_L) + (\tilde{Z}_R - Z_R)$, we compute the error bound for $(\tilde{Z}_L - Z_L)$ and $(\tilde{Z}_R - Z_R)$, separately. For this, define $\tilde{Z}_L$ and $\tilde{Z}_R$ as

$$\tilde{Z}_L(x_i, t_{k+1}) = \begin{cases} \tilde{Z}_1^L, & 0 \leq i \leq N/2 - 1, \\ \tilde{Z}_2^L, & N/2 \leq i \leq N, \end{cases} \quad \text{and} \quad \tilde{Z}_R(x_i, t_{k+1}) = \begin{cases} \tilde{Z}_1^R, & 0 \leq i \leq N/2 - 1, \\ \tilde{Z}_2^R, & N/2 \leq i \leq N. \end{cases}$$

Therefore

$$(\tilde{Z}_L - Z_L)(x_i, t_{k+1}) = \begin{cases} (\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1}), & 0 \le i \le N/2 - 1, \\ (\tilde{Z}_2^L - z_2^L)(x_i, t_{k+1}), & N/2 \le i \le N, \end{cases}$$

and

$$(\tilde{Z}_R - Z_R)(x_i, t_{k+1}) = \begin{cases} (\tilde{Z}_1^R - z_1^R)(x_i, t_{k+1}), & 0 \le i \le N/2 - 1, \\ (\tilde{Z}_2^R - z_2^R)(x_i, t_{k+1}), & N/2 \le i \le N. \end{cases}$$

Lemma 5.5.3. *Let Z_L, Z_R be the solutions of* (5.41)-(5.44) *and $\tilde{Z}_L$, $\tilde{Z}_R$ be the solutions of* (5.65)-(5.66)*, respectively. Then*

$$|(\tilde{Z}_L - Z_L)(x_i, t_{k+1})| \le \begin{cases} CN^{-1}\ln N & \text{if } p_0\mu^2 \le \gamma\varepsilon, \\ CN^{-1}(\ln N)^2 & \text{if } p_0\mu^2 \ge \gamma\varepsilon, \end{cases}$$

$$|(\tilde{Z}_R - Z_R)(x_i, t_{k+1})| \le \begin{cases} CN^{-1}\ln N & \text{if } p_0\mu^2 \le \gamma\varepsilon, \\ CN^{-1}\ln N & \text{if } p_0\mu^2 \ge \gamma\varepsilon. \end{cases}$$

Proof. Using (5.67), (5.68) and Lemma 5.5.2, we have

$$|\tilde{Z}_1^L(x_i, t_{k+1})| \le CN^{-1}, \text{ for } N/8 \le i \le N/2, \tag{5.69}$$

$$|\tilde{Z}_1^R(x_i, t_{k+1})| \le CN^{-1}, \text{ for } 0 \le i \le 3N/8, \tag{5.70}$$

$$|\tilde{Z}_2^L(x_i, t_{k+1})| \le CN^{-1}, \text{ for } 5N/8 \le i \le N, \tag{5.71}$$

$$|\tilde{Z}_2^R(x_i, t_{k+1})| \le CN^{-1}, \text{ for } N/2 \le i \le 7N/8. \tag{5.72}$$

Case I: For $p_0\mu^2 \le \gamma\varepsilon$ and $1 \le i \le N/8 - 1$

$$\begin{aligned} |\hat{\mathcal{L}}_1(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1})| &= |\hat{\mathcal{L}}_1\tilde{Z}_1^L(x_i, t_{k+1}) - \hat{\mathcal{L}}z_1^L(x_i, t_{k+1})| \\ &= |\hat{\mathcal{L}}_1\tilde{Z}_1^L(x_i, t_{k+1}) - \tilde{\mathcal{L}}_1 z_1^L(x_i, t_{k+1})| \\ &\le \varepsilon\left(\delta_x^2 - \frac{d^2}{dx^2}\right)|z_1^L(x_i, t_{k+1})| + \mu\left(D_x^+ - \frac{d}{dx}\right)|z_1^L(x_i, t_{k+1})| \\ &\le Ch_i\left(\varepsilon|z_1^L|_3 + \mu|z_1^L|_2\right) \\ &\le Ch_i\left(\frac{1}{\sqrt{\varepsilon}} + \frac{\mu}{\sqrt{\varepsilon}}\right) \\ &\le \frac{Ch_i}{\sqrt{\varepsilon}}. \end{aligned}$$

In case $\beta_1 = \dfrac{1}{4}$, we have $\dfrac{1}{4} \le \dfrac{2\sqrt{\varepsilon}}{\sqrt{\gamma p_0}}\ln N$, which implies $\dfrac{1}{\sqrt{\varepsilon}} \le C\ln N$. Thus

$$|\hat{\mathcal{L}}_1(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1})| \le CN^{-1}\ln N.$$

In case $\beta_1 = \dfrac{2\sqrt{\varepsilon}}{\sqrt{\gamma p_0}} \ln N$, we compute

$$|\hat{\mathcal{L}}_1(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1})| \leq \frac{Ch_i}{\sqrt{\varepsilon}} \leq \frac{C}{\sqrt{\varepsilon}}\frac{8\beta_1}{N} \leq CN^{-1}\ln N.$$

Let us next consider $\phi^{\pm}(x_i, t_{k+1}) = CN^{-1}\ln N \pm (\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1})$ for $0 \leq i \leq N/8$. Then $\phi^{\pm}(x_i, t_{k+1}) \geq 0$ for $i = \{0, N/8\}$ and $\hat{\mathcal{L}}_1\phi^{\pm}(x_i, t_{k+1}) \leq 0$. Using Lemma 5.4.1, we get

$$|(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1})| \leq CN^{-1}\ln N, \quad 0 \leq i \leq N/8. \tag{5.73}$$

For $N/8 \leq i \leq N/2 - 1$, using (5.69) and Lemma 5.3.10, we calculate that

$$|(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1}) \leq |\tilde{Z}_1^L(x_i, t_{k+1})| + |z_1^L(x_i, t_{k+1})| \leq CN^{-1}. \tag{5.74}$$

Thus from (5.73) and (5.74), we get

$$|(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1}) \leq CN^{-1}\ln N, \ 0 \leq i \leq N/2 - 1.$$

Similarly, we can compute that

$$|(\tilde{Z}_2^L - z_2^L)(x_i, t_{k+1})| \leq CN^{-1}\ln N, \ N/2 \leq i \leq N.$$

Case II: For $p_0\mu^2 \geq \gamma\varepsilon$ and $1 \leq i \leq N/8 - 1$

$$\begin{aligned}
|\hat{\mathcal{L}}_1(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1})| &= |\hat{\mathcal{L}}_1\tilde{Z}_1^L(x_i, t_{k+1}) - \tilde{\mathcal{L}}_1 z_1^L(x_i, t_{k+1})| \\
&\leq Ch_i\left(\varepsilon|z_1^L|_3 + \mu|z_1^L|_2\right) \\
&\leq Ch_i\left(\frac{\mu^3}{\varepsilon^2} + \frac{\mu^3}{\varepsilon^2}\right) \\
&\leq Ch_i\mu\left(\frac{\mu}{\varepsilon}\right)^2.
\end{aligned}$$

In case $\beta_1 = \dfrac{1}{4}$, we have $\dfrac{1}{4} \leq \dfrac{2\varepsilon}{p_0\mu}\ln N$, which implies $\dfrac{\mu}{\varepsilon} \leq C\ln N$. Thus

$$|\hat{\mathcal{L}}_1(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1})| \leq CN^{-1}(\ln N)^2.$$

If $\beta_1 = \dfrac{2\varepsilon}{p_0\mu}\ln N$, then we compute

$$|\hat{\mathcal{L}}_1(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1})| \leq Ch_i\mu\left(\frac{\mu}{\varepsilon}\right)^2 \leq C\frac{\mu^2}{\varepsilon}N^{-1}\ln N.$$

Consider,

$$\phi^{\pm}(x_i, t_{k+1}) = CN^{-1} + CN^{-1}\ln N(\beta_1 - x_i)\frac{\mu}{\varepsilon} \pm (\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1}), \ 0 \leq i \leq N/8.$$

Then, $\hat{\mathcal{L}}_1\phi^{\pm}(x_i, t_{k+1}) \le 0$ for $1 \le i \le N/8 - 1$ and $\phi^{\pm}(x_i, t_{k+1}) \ge 0$ for $i = \{0, N/8\}$. Using Lemma 5.4.1 to get

$$\begin{aligned} |(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1})| &\le CN^{-1} + CN^{-1}\ln N\beta_1\frac{\mu}{\varepsilon} \\ &\le CN^{-1}(\ln N)^2, \quad 0 \le i \le N/8. \end{aligned} \tag{5.75}$$

For $N/8 \le i \le N/2 - 1$, it follows from (5.69) and Lemma 5.3.10 that

$$|(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1}) \le CN^{-1}. \tag{5.76}$$

Thus, from (5.75) and (5.76)

$$|(\tilde{Z}_1^L - z_1^L)(x_i, t_{k+1}) \le CN^{-1}(\ln N)^2, \quad 0 \le i \le N/2 - 1. \tag{5.77}$$

Similarly, for $N/2 \le i \le N$, we obtain

$$|(\tilde{Z}_2^L - z_2^L)(x_i, t_{k+1})| \le CN^{-1}(\ln N)^2.$$

Thus,

$$|(\tilde{Z}_L - Z_L)(x_i, t_{k+1})| \le \begin{cases} CN^{-1}\ln N & \text{if } p_0\mu^2 \le \gamma\varepsilon, \\ CN^{-1}(\ln N)^2 & \text{if } p_0\mu^2 \ge \gamma\varepsilon. \end{cases}$$

Following the similar steps and argument for $(\tilde{Z}_R - Z_R)$, it is straightforward to establish that for $p_0\mu^2 \le \gamma\varepsilon$ and $p_0\mu^2 \ge \gamma\varepsilon$,

$$|(\tilde{Z}_R - Z_R)(x_i, t_{k+1})| \le CN^{-1}\ln N, \; 0 \le i \le N.$$

□

Theorem 5.5.4. *Let $\tilde{U}(x_i, t_{k+1})$ and $U(x_i, t_{k+1})$ be the solutions of* (5.61) *and* (5.10)*, respectively. Then for* $0 \le i \le N$

$$\left|(\tilde{U} - U)(x_i, t_{k+1})\right| \le C \begin{cases} N^{-1}\ln N & \text{if } p_0\mu^2 \le \gamma\varepsilon \\ N^{-1}(\ln N)^2 & \text{if } p_0\mu^2 \ge \gamma\varepsilon. \end{cases}$$

Proof. The proof follows from Lemma 5.5.1 and Lemma 5.5.3. □

Finally, we combine (5.13) and Theorem 5.5.4 to obtain the principle convergence estimate that reads.

Theorem 5.5.5. *Let u and $\tilde{U}$ be the solutions of the continuous problem* (5.1) *and the discrete problem* (5.61)*, respectively. Then*

$$\left|\tilde{U}(x_i, t_{k+1}) - u(x_i, t_{k+1})\right| \le C \begin{cases} (\Delta t^2 + N^{-1}\ln N) & \text{if } p_0\mu^2 \le \gamma\varepsilon \\ (\Delta t^2 + N^{-1}(\ln N)^2) & \text{if } p_0\mu^2 \ge \gamma\varepsilon \end{cases}$$

for $0 \le i \le N$ *and* $0 \le k \le M$.

5.6 Numerical Illustrations

In this section, we consider two model problems, present numerical results using the proposed method, and verify the theoretical estimates numerically.

Example 5.6.1. *Consider the following two-parameter singularly perturbed problem:*

$$\begin{cases} [\varepsilon u_{xx} + \mu(1+x)u_x - (4+\sin x)u - u_t](x,t) + u(x-1,t) = -e^x, \quad (x,t) \in (0,2)\times(0,2], \\ u(x,t) = 5 + t^2, \quad (x,t) \in \Gamma_1, \\ u(x,t) = 5, \quad (x,t) \in \Gamma_2, \\ u(x,t) = 5, \quad (x,t) \in \Gamma_3. \end{cases}$$

Example 5.6.2. *Consider the following two-parameter singularly perturbed problem:*

$$\begin{cases} [\varepsilon u_{xx} + \mu(2+x+t)u_x - (2+xt)u - u_t](x,t) + u(x-1,t) = (e^{t^2}-1)(1+xt), \\ (x,t) \in (0,2)\times(0,2], \\ u(x,t) = 0, \quad (x,t) \in \Gamma_1, \\ u(x,t) = 0, \quad (x,t) \in \Gamma_2, \\ u(x,t) = 0, \quad (x,t) \in \Gamma_3. \end{cases}$$

The exact solutions of the above examples are not known for comparison. Therefore, the double mesh principle [173] is used to estimate the maximum absolute error and rate of convergence. The maximum absolute error ($E_\varepsilon^{N,\triangle t}$) and order of convergence ($R_\varepsilon^{N,\triangle t}$) are defined as

$$E_\varepsilon^{N,\triangle t} := \max \left| U_{N,\triangle t}(x_i, t_{k+1}) - \tilde{U}_{2N,\triangle t/2}(x_i, t_{k+1}) \right| \text{ and } R_\varepsilon^{N,\triangle t} := \log_2 \left(\frac{E_\varepsilon^{N,\triangle t}}{E_\varepsilon^{2N,\triangle t/2}} \right).$$

Here, $U_{N,\triangle t}(x_i, t_{k+1})$ and $\tilde{U}_{2N,\triangle t/2}(x_i, t_{k+1})$ denotes the numerical solutions on $\bar{\mathbb{D}}_x^N \times \mathbb{T}_t{}^M$ and $\bar{\mathbb{D}}_x^{2N} \times \mathbb{T}_t{}^{2M}$, respectively.

The maximum absolute error ($E_\varepsilon^{N,\triangle t}$) and corresponding order of convergence ($R_\varepsilon^{N,\triangle t}$) for Example 5.6.1 and Example 5.6.2 are tabulated for different values of ε, μ, M, and N in Tables 5.1-5.4. Moreover, the log-log plot of the maximum absolute error can be had from Figures 5.5-5.8. It is evident that the errors decrease monotonically as N increases.

The presence of both interior and boundary layers is apparent from the surface plots of the numerical solution of Examples 5.6.1 and 5.6.2 are displayed in Figures 5.1-5.4, for different values of ε and μ.

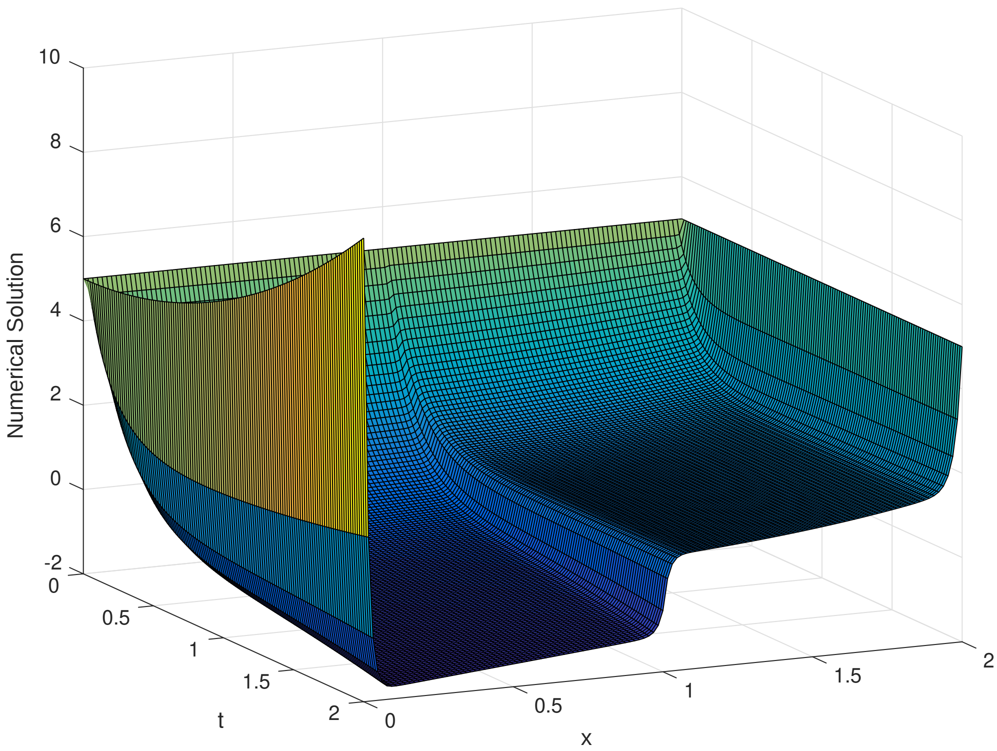

Figure 5.1: Numerical solution of Example 5.6.1 with $M = N = 128$, $\mu = 2^{-22}$ and $\varepsilon = 2^{-17}$.

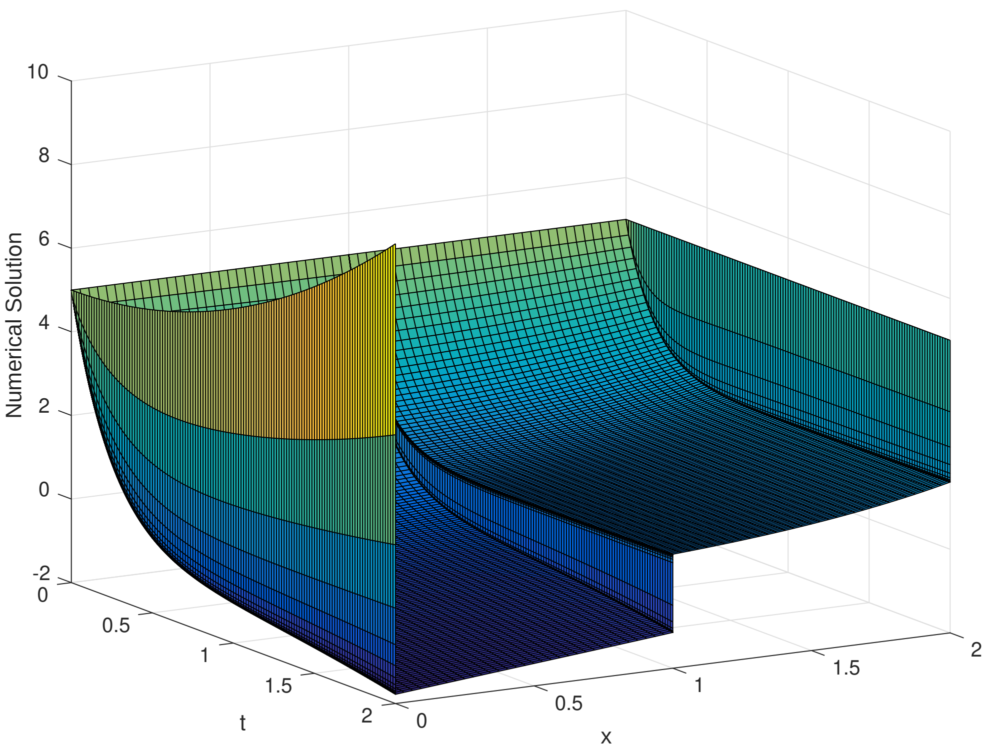

Figure 5.2: Numerical solution of Example 5.6.1 with $M = N = 128$, $\mu = 2^{-6}$ and $\varepsilon = 2^{-14}$.

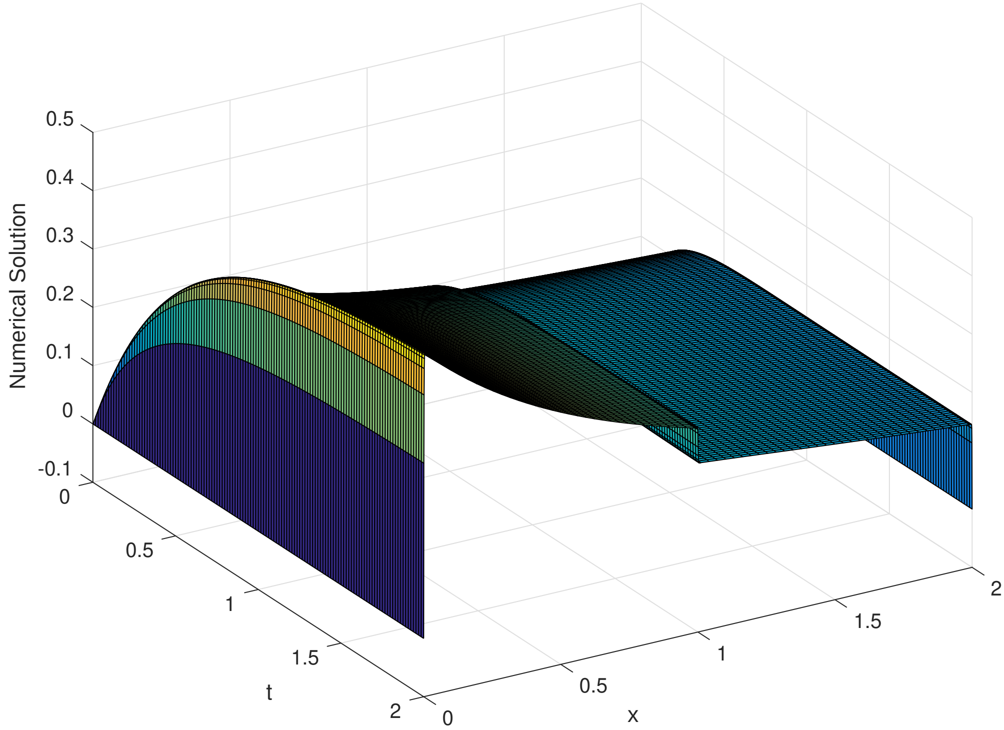

Figure 5.3: Numerical solution of Example 5.6.2 with $M = N = 128$, $\mu = 2^{-18}$ and $\varepsilon = 2^{-20}$.

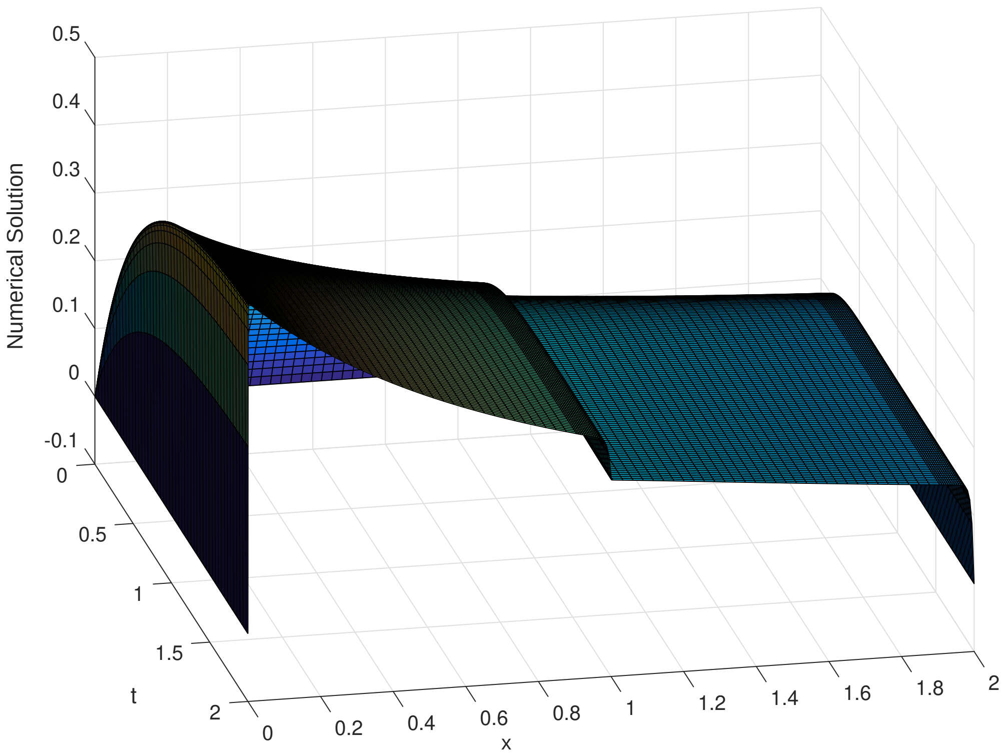

Figure 5.4: Numerical solution of Example 5.6.2 with $M = N = 128$, $\mu = 2^{-8}$ and $\varepsilon = 2^{-18}$.

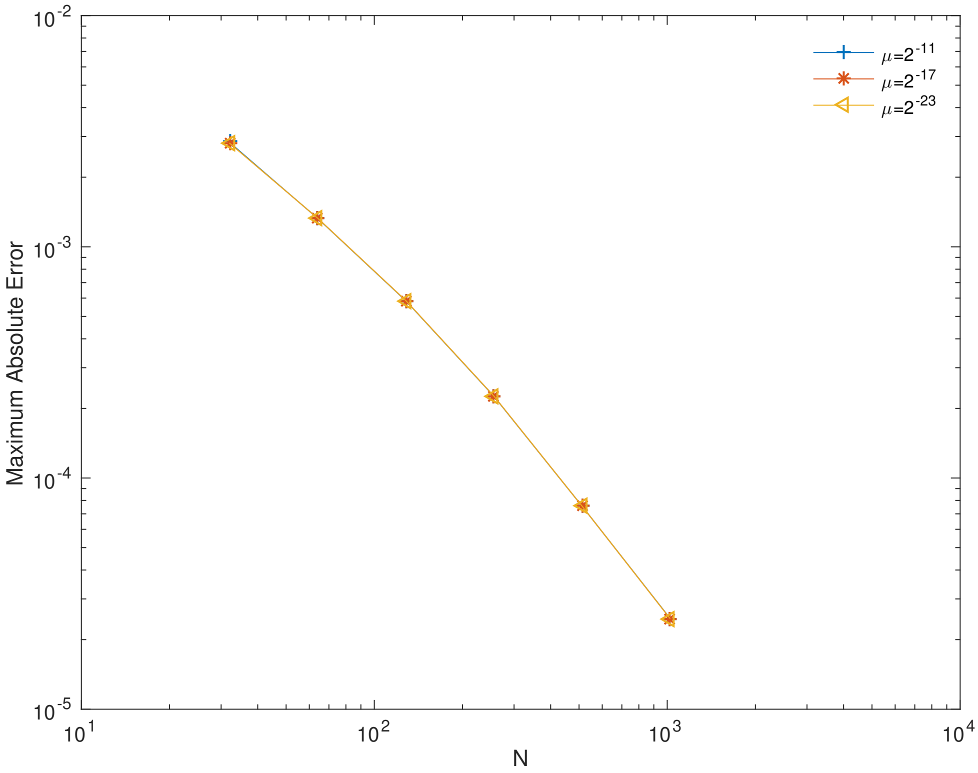

Figure 5.5: Error plot for Example 5.6.1 for Case 1 when $\varepsilon = 2^{-10}$.

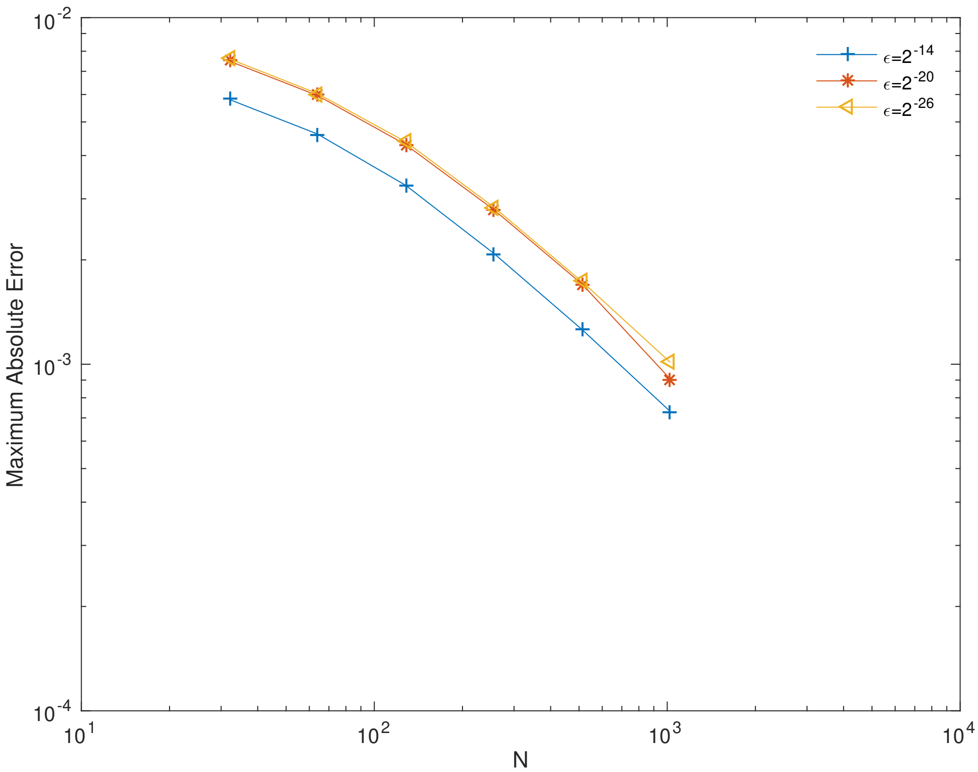

Figure 5.6: Error plot for Example 5.6.1 for Case 2 when $\mu = 2^{-6}$.

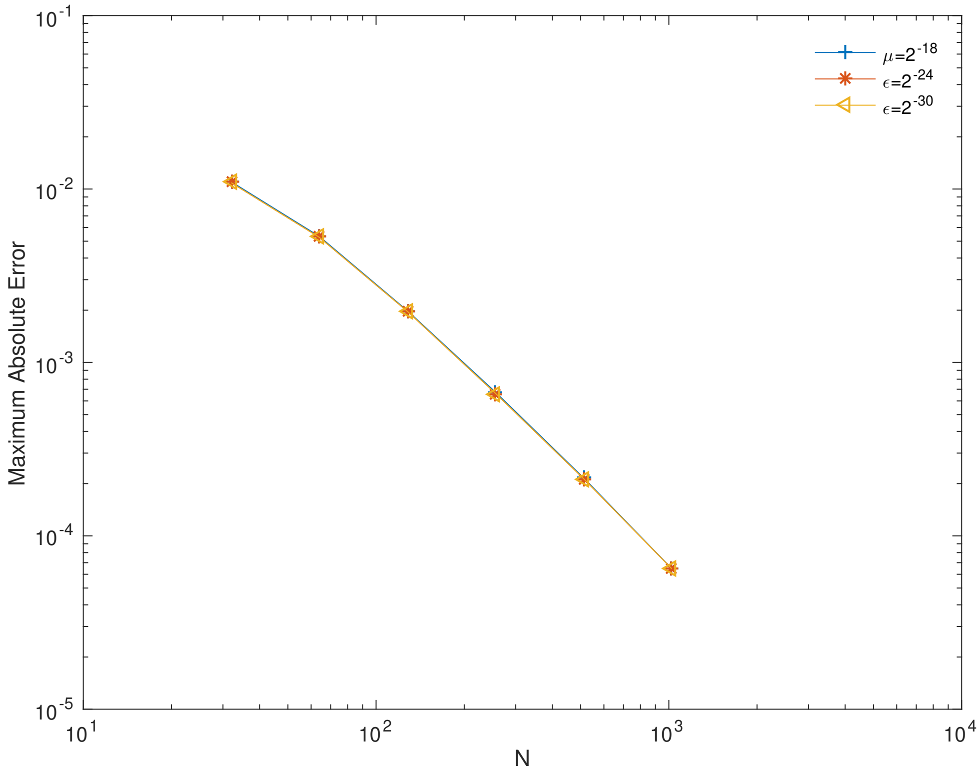

Figure 5.7: Error plot for Example 5.6.2 for Case 1 when $\varepsilon = 2^{-20}$.

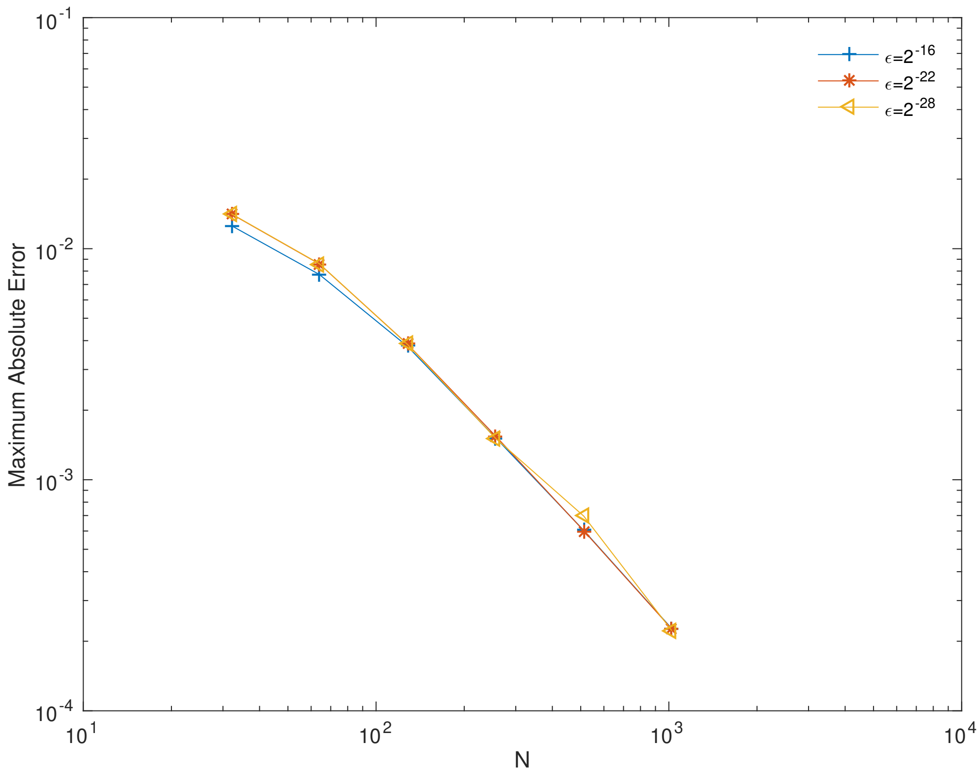

Figure 5.8: Error plot for Example 5.6.2 for Case 2 when $\mu = 2^{-8}$.

Table 5.1: Maximum absolute error and order of convergence for Example 5.6.1 for different values of μ, M and N when $\varepsilon = 2^{-10}$.

μ	$M = N = 32$	64	128	256	512	1024
2^{-11}	2.8356e-03	1.3289e-03	5.8495e-04	2.2566e-04	7.5573e-05	2.4680e-05
	1.0934	1.1838	1.3742	1.5782	1.6145	1.6390
2^{-13}	2.8186e-03	1.3309e-03	5.8587e-04	2.2603e-04	7.5589e-05	2.4635e-05
	1.0826	1.1837	1.3741	1.5803	1.6175	1.6422
2^{-15}	2.8143e-03	1.3314e-03	5.8611e-04	2.2612e-04	7.5593e-05	2.4624e-05
	1.0798	1.1837	1.3741	1.5808	1.6182	1.6487
2^{-17}	2.8133e-03	1.3315e-03	5.8617e-04	2.2615e-04	7.5594e-05	2.4621e-05
	1.0792	1.1837	1.3740	1.5809	1.6184	1.6525
2^{-19}	2.8130e-03	1.3315e-03	5.8618e-04	2.2615e-04	7.5595e-05	2.4620e-05
	1.0791	1.1836	1.3741	1.5809	1.6185	1.6588
2^{-21}	2.8129e-03	1.3315e-03	5.8618e-04	2.2616e-04	7.5595e-05	2.4620e-05
	1.0789	1.1836	1.3740	1.5810	1.6185	1.6588
2^{-23}	2.8129e-03	1.3315e-03	5.8619e-04	2.2616e-04	7.5595e-05	2.4620e-05
	1.0789	1.1836	1.3740	1.5810	1.6185	1.6588
2^{-25}	2.8129e-03	1.3315e-03	5.8619e-04	2.2616e-04	7.5595e-05	2.4620e-05
	1.0789	1.1836	1.3740	1.5810	1.6185	1.6588

Table 5.2: Maximum absolute error and order of convergence for Example 5.6.1 for different values of ε, M and N when $\mu = 2^{-6}$.

ε	$M = N = 32$	64	128	256	512	1024
2^{-14}	5.8061e-03	4.6143e-03	3.2823e-03	2.0873e-03	1.2587e-03	7.3043e-04
	0.3315	0.4914	0.6530	0.7297	0.7851	0.8412
2^{-16}	6.9481e-03	5.2511e-03	3.5537e-03	2.3566e-03	1.4543e-03	8.5388e-04
	0.4040	0.5632	0.5926	0.6963	0.7682	0.8403
2^{-18}	7.1592e-03	5.7505e-03	4.0863e-03	2.6675e-03	1.6513e-03	9.5415e-04
	0.3161	0.4928	0.6153	0.7094	0.7913	0.8577
2^{-20}	7.4927e-03	5.9608e-03	4.2997e-03	2.7905e-03	1.7028e-03	9.0641e-04
	0.3299	0.4712	0.6236	0.7126	0.7730	0.8524
2^{-22}	7.5787e-03	6.0219e-03	4.3625e-03	2.8267e-03	1.7237e-03	1.0087e-03
	0.3317	0.4650	0.6260	0.7136	0.7730	0.8571
2^{-24}	7.5956e-03	6.0352e-03	4.3788e-03	2.8362e-03	1.7291e-03	1.0120e-03
	0.3317	0.4628	0.6265	0.7139	0.7728	0.8592
2^{-26}	7.6080e-03	6.0390e-03	4.3822e-03	2.8385e-03	1.7305e-03	1.0129e-03
	0.3332	0.4626	0.6265	0.7139	0.7726	0.8611
2^{-28}	7.6156e-03	6.0421e-03	4.3830e-03	2.8389e-03	1.7308e-03	1.0131e-03
	0.3339	0.4631	0.6266	0.7138	0.7726	0.8583

Table 5.3: Maximum absolute error and order of convergence for Example 5.6.2 for different values of μ, M and N when $\varepsilon = 2^{-20}$.

μ	$M = N = 32$	64	128	256	512	1024
2^{-18}	1.024e-02	5.3320e-03	1.9856e-03	6.7329e-04	2.1566e-04	6.4616e-05
	1.0478	1.4250	1.5602	1.6424	1.7388	1.8226
2^{-20}	1.0914e-02	5.2812e-03	1.9729e-03	6.6492e-04	2.1370e-04	6.4556e-05
	1.0471	1.4205	1.5690	1.6375	1.7270	1.8192
2^{-22}	1.0887e-02	5.2688e-03	1.9697e-03	6.6289e-04	2.1332e-04	6.4510e-05
	1.0470	1.4195	1.5711	1.6357	1.7254	1.8093
2^{-24}	1.0880e-02	5.2657e-03	1.9689e-03	6.6238e-04	2.1323e-04	6.4506e-05
	1.0470	1.4192	1.5716	1.6352	1.7249	1.8046
2^{-26}	1.0878e-02	5.2649e-03	1.9687e-03	6.6225e-04	2.1320e-04	6.4489e-05
	1.0469	1.4191	1.5717	1.6351	1.7251	1.7963
2^{-28}	1.0878e-02	5.2647e-03	1.9686e-03	6.6222e-04	2.1320e-04	6.4489e-05
	1.0469	1.4191	1.5718	1.6351	1.7251	1.7963
2^{-30}	1.0878e-02	5.2647e-03	1.9686e-03	6.6221e-04	2.1320e-04	6.4489e-05
	1.0469	1.4191	1.5718	1.6350	1.7251	1.7963
2^{-32}	1.0878e-02	5.2646e-03	1.9686e-03	6.6221e-04	2.1320e-04	6.4489e-05
	1.0469	1.4191	1.5718	1.6350	1.7251	1.7963

Table 5.4: Maximum absolute error and order of convergence for Example 5.6.2 for different values of ε, M and N when $\mu = 2^{-8}$.

ε	$M = N = 32$	64	128	256	512	1024
2^{-16}	1.2594e-02	7.7325e-03	3.7780e-03	1.5195e-03	6.0382e-04	2.2666e-04
	0.7037	1.0333	1.3140	1.3314	1.4135	1.5628
2^{-18}	1.3692e-02	8.3363e-03	3.8621e-03	1.5195e-03	6.0025e-04	2.2632e-04
	0.7158	1.1096	1.3462	1.3321	1.4072	1.5646
2^{-20}	1.4047e-02	8.5373e-03	3.8621e-03	1.5182e-03	5.0182e-04	2.2602e-04
	0.7184	1.1444	1.3470	1.3350	1.4129	1.5673
2^{-22}	1.4143e-02	8.5913e-03	3.8622e-03	1.5467e-03	6.0154e-04	2.2583e-04
	0.7191	1.1535	1.3467	1.3351	1.4134	1.5771
2^{-24}	1.4167e-02	8.6051e-03	3.8633e-03	1.5172e-03	6.0131e-04	2.2566e-04
	0.7127	1.1554	1.3484	1.3352	1.4140	1.5842
2^{-26}	1.4174e-02	8.6086e-03	3.8531e-03	1.5119e-03	5.9672e-04	2.2322e-04
	0.7193	1.1598	1.3497	1.3412	1.4186	1.5886
2^{-28}	1.4175e-02	8.6096e-03	3.8526e-03	1.5112e-03	5.9626e-04	2.2209e-04
	0.7193	1.1601	1.3501	1.3417	1.4248	1.5904
2^{-30}	1.4175e-02	8.6096e-03	3.8526e-03	1.5112e-03	5.9626e-04	2.2209e-04
	0.7193	1.1601	1.3501	1.3417	1.4248	1.5904

5.7 Concluding Remark

A convergent numerical method is presented for a two-parameter singularly perturbed parabolic problem with a large spatial delay. In general, the solution of this kind of problem has both boundary and interior layers, which makes the numerical analysis different. Here, we provide a convergent solution by discretizing the continuous problem with the Crank-Nicolson method for the time variable on a uniform mesh and an upwind scheme on a piecewise uniform mesh for the spatial variable. The theoretical analysis shows that the numerical method is almost first-order accurate in space and second-order in time, which is also validated by several numerical experiments. Note that boundary layers appear because of the presence of perturbation parameters. However, it is observed that the interior layers can appear because of the presence of a large delay in reaction term.

Chapter 6

Conclusion and Future Work

6.1 Summary and Conclusion

Many problems in applied mathematics lead to questions of the type: Given is a differential equation with a small parameter ε. This parameter occurs so that the corresponding degenerate differential equation is of lower order than the original one. What happens then to the solution of a boundary value problem of the original differential equation if ε tends to zero in that solution? It is by no means obvious as it sounds and not even always true, as we have seen that the solution of such a boundary value problem tends to a solution of the limiting problem. But even when this is the case, the question arises what are the boundary conditions satisfied by the limiting function. As a solution of a differential equation of lower order than the original one, it cannot, in general, satisfies all the boundary conditions prescribed. In those cases, the solution of the problem shows a peculiar behaviour for very small values of ε. Some of the solution derivatives assume very large values in a narrow region near the boundary. In the most important applications of phenomena of this type the first derivative of the solution and, of course, all the higher derivatives diverge at parts of the boundary, as ε tends to zero. In the physical interpretations, this means the occurrence of “Boundary layers" in which the quantity to be investigated increases or decreases very rapidly.

The systems/problems in which the suppression of a small parameter is responsible for the degeneration (or reduction) of dimension (or order) of the problem are labelled as singularly perturbed systems/problems, which are a special representation of the general class of time scale systems. In many situations, these systems take into account not just the present state of the physical system but also its past history. These models are described by a certain class of functional differential equations. In most applications, a delay is introduced when there are some hidden variables and processes which are not well understood but are known to cause a time-lag. Time delays are natural components of dynamic processes in physics, mechanics, biology, ecology, economics, physiology, population dynamics, and aeronautics, to name a few. Even if the process does not include delay phenomena, the actuators or sensors that are involved in its automatic control usually introduce time-lags. In fact, the majority of the applications lead to complicated equations for which a complete mathematical treatment has not yet been attempted. However, a variety of asymptotic and numerical methods have been developed. But, there are known difficulties linked with these methods. For example, finding the correct asymptotic expansion is not a routine exercise. Moreover, a high degree of compatibility and sufficiently smooth boundary data are required. This is too restraining because engineers often encounter problems that, in general, do not satisfy these smoothness assumptions. Also, numerical discretization of such systems for small ε is burdened with difficulties; for example, FDM and FEM necessitate a mesh to sustain approximations, which is not a routine exercise in higher dimensions or in regions with crooked boundaries. In fact, the discrete solutions obtained using standard Galerkin or centered finite differences methods demonstrate oscillatory behaviour for small discretization parameters. Moreover, when streamline diffusion finite element methods are applied to singular perturbation problems using nonconforming trial spaces, it is observed that stability and convergence problems may occur. However, nonconforming finite element approximations are appropriate, for they have the outstanding practical benefit that each degree of freedom belongs to at most two elements. This results in reasonable local communication, and the method can be parallelized in a highly efficient manner on MIMD-machines. But, singular perturbation problems have local features which make them ripe for adaptive refinement algorithms based on local error indicators, although there remain some dazzling issues: first, the role of stabilization, e.g., using streamline upwind discretizations, and second, the fact that local error estimation may be very inaccurate if critical features like boundary layers are not adequately resolved by the underlying mesh. The layer adapted meshes seem promising in the discretization of such systems. But, they do have associated limitations; the chief among them is the extension of these methods to multi-dimensions or over complex domains. One way is to use a fairly simple discretization in concurrence with a rightfully chosen non-uniform grid. It is correct that the appearance and dimensions of boundary layers are often a-priori unknown. However, in the first step, a mesh can be chosen according to experiences with similar problems which should be adjusted in a recursive manner.

This thesis presents numerical methods for solving singularly perturbed parabolic partial differential equations with discontinuous coefficients and delays. Chapter 1 recalls the definitions of regular and singular perturbation problems and provides a brief literature survey.

Chapter 2 presents and analyses an adaptive finite difference method to solve a class of singularly perturbed parabolic delay differential equations with discontinuous convection coefficient and source. The numerical method involves an upwind finite difference scheme on a layer-adapted Shishkin mesh in space and a backward Euler method in time on a uniform mesh. The proposed numerical method is analysed for consistency, stability, and convergence. Extensive theoretical analysis is performed to obtain consistency and error estimates. The proposed method is unconditionally stable, and the obtained convergence is parameter-uniform with first-order convergence in space and time. Numerical results are presented for model problems, demonstrating the effectiveness of the proposed technique. Chapter 3 extends the idea and presents a higher-order hybrid finite difference method to solve the model problem. The proposed hybrid difference method is a composition of a central difference scheme and a midpoint upwind scheme on a specially generated mesh. Moreover, the time variable is discretised using an implicit finite difference method. The error estimates of the proposed numerical method satisfy parameter-uniform second-order convergence in space and first-order convergence in time. The rigorous numerical analysis of the proposed method on a Shishkin class mesh establishes the supremacy of the proposed scheme.

Chapter 4 proposes a robust higher-order accurate numerical method on a specially designed non-uniform mesh for solving singularly perturbed convection-diffusion problems with delay and integral boundary conditions. An upwind finite difference scheme is used on a non-uniform mesh in space, while a Crank-Nicolson scheme is used on a uniform mesh in the time variable. The proposed method is unconditionally stable and converges uniformly, independent of the perturbation parameter. The error analysis indicates that the numerical solution is parameter-uniform second-order convergence in time and first-order convergence in space.

Chapter 5 presents a robust computational technique to solve a class of two-parameter parabolic convection-diffusion problems with delay. The method involves a Crank-Nicolson scheme for the time variable on a uniform mesh and an upwind scheme on a piecewise uniform mesh for the spatial variable. The proposed scheme gives parameter uniform second-order convergence in time and almost first-order convergence in space.

6.2 Future Work

In this section, we outline some of the interesting problems to which the approach/idea presented in the thesis can be extended. It would be interesting to consider the following problems for future work.

1. Consider a parabolic partial functional differential equation of type

$$\frac{\partial u(x,t)}{\partial t} - \varepsilon\frac{\partial^2 u(x,t)}{\partial x^2} + b(x,t)u(x,t) + a(x,t)u(x,t-\tau) = f(x,t), \quad (x,t) \in D$$

$$u(x,s) = \varphi(x,s); \quad x \in \Omega \equiv]0,1[, \quad s \in [-\tau, 0[,$$
$$u(x,0) = \varphi(x,0) = \varphi(x); \quad x \in \Omega \equiv]0,1[,$$
$$u(0,t) = \Phi, \quad u(1,t) = \Psi; \quad t \in]0,T],$$

where $0 < \varepsilon \ll 1$, $D = (0,1) \times (0,T]$, $\Phi = \Phi(t)$ and $\Psi = \Psi(t)$ are smooth functions defined on the left and right side of the domain corresponding to $x = 0$ and $x = 1$. For simplicity one may consider $T = n\tau$ for some integer $n > 1$. The coefficients $b(x,t)$ and $a(x,t)$ of reaction terms are sufficiently smooth functions satisfying

$$b(x,t) \geq \beta > 0 \quad \text{and} \quad a(x,t) \geq \alpha > 0.$$

For sufficiently small ε, it is clear that the problem exhibits boundary layers on the left and right end of the outflow boundary region. Moreover, the characteristics of the reduced problems being vertical lines $x = \eta$ and the boundary layers are of parabolic type. The existence of a unique solution for the said problem is shown in [161] with the assumption that given data is sufficiently smooth and satisfies appropriate compatibility conditions at the corners. However, for hybrid weighted schemes, we may need more regularity.

2. Consider the following class of two parameter singularly perturbed parabolic problems posed on the domain $G = \Omega \times (0,T]$, $\Omega = (0,1)$, $\Gamma = \tilde{G} \backslash G$

$$\varepsilon\frac{\partial^2 u(x,t)}{\partial x^2} + \mu a\frac{\partial u(x,t)}{\partial x} + bu(x,t-\tau) + cu(x,t) - \frac{\partial u(x,t)}{\partial t} = f(x,t),$$
$$u(x,s) = S(x,s); \quad x \in \Omega \equiv (0,1) = \Gamma_b, \quad s \in [-\tau, 0),$$
$$u = q(t); \quad on \ \Gamma_l \bigcup \Gamma_r,$$

where $\Gamma_b = (x,0) : 0 \leq x \leq 1$, $\Gamma_l = (0,t) : 0 \leq t \leq T$, and $\Gamma_l = (1,t) : 0 \leq t \leq T$. Note that $0 < \varepsilon \leq 1$ and $0 < \mu \leq 1$ are perturbation parameters. We assume sufficient regularity and compatibility at the corners so that the solution and its regular component are sufficiently smooth for our analysis. Our interest lies in constructing parameter uniform numerical methods [144] for this class of singularly perturbed problems. By this we mean numerical methods whose solutions converge uniformly with respect to the singular perturbation parameters. When the parameter $\mu = 1$, the problem is the well-studied parabolic convection diffusion problem [147] and in this case a boundary layer of width $O(\varepsilon)$ appears in the neighbourhood of the edge $x = 0$. When $\mu = 0$ we have a parabolic reaction-diffusion problem [190] and boundary layers of width $O(\sqrt{\varepsilon})$ appear in the neighbourhood of both $x = 0$ and $x = 1$.

Among others problems of interest is a characteristic example from numerical control given by the equation

$$\frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} = v(g(u(x,t-\tau)))\frac{\partial u(x,t)}{\partial x} + c[f(u(x,t-\tau)) - u(x,t)]$$

which models a furnace used to process metal sheets. Here, $u(x,t)$ is the temperature distribution in a metal sheet, moving at a velocity v and heated by a source specified by the function f. Here, both v and f are dynamically adapted by a controlling device monitoring the current temperature distribution. The finite speed of the controller, however, introduces a fixed delay of length τ. Another problem of particular interest is from population dynamics, the so-called Britton-model,

$$\frac{\partial u(x,t)}{\partial t} = \varepsilon \Delta u + u(1 - g * u) \quad \text{with} \quad g * u = \int_{t-\tau}^{t} \int_{\Omega} g(x-y, t-s) u(y,s) dy\, ds.$$

Here, $u(x,t)$ denotes a population density, which evolves through random migration (modelled by the diffusion term) and reproduction (modelled by the nonlinear reaction term). The latter involves a convolution operator with a kernel $g(x,t)$, which models the distributed age-structure dependence of the evolution and its dependence on the population levels in the neighbourhood.

References

[1] Aarthika, K., Shanthi, V., and Ramos, H., 2022, "A computational approach for a two-parameter singularly perturbed system of partial differential equations with discontinuous coefficients," *Appl. Math. Comput.* **434**, 127409.

[2] Abbasbandy, S., 2006, "Iterated He's homotopy perturbation method for quadratic riccati differential equation," *Appl. Math. Comput.* **175**, 581–589.

[3] Abdallah, N., and Pinaud, O., 2006, "Multiscale simulation of transport in an open quantum system: Resonances and WKB interpolation," *J. Comp. Phys.* **213**, 288–310.

[4] Abrahamsson, L., Keller, H., and Kreiss, H., 1974, "Difference approximations for singular perturbations of systems of ordinary differential equations," *Numer. Math.* **22**, 367–391.

[5] Ahsan, M., Bohner, M., Ullah, A., Khan, A. A., and Ahmad, S., 2023, "A haar wavelet multi-resolution collocation method for singularly perturbed differential equations with integral boundary conditions," *Math. Comput. Simulation* **204**, 166–180.

[6] Akulenko, L., 2002, "Higher-order averaging schemes in systems with fast and slow phases," *J. Appl. Math. Mech.* **66**, 153–163.

[7] Allen, D. N. G., and Southwell, R. V., 1955, "Relaxation methods applied to determine the motion, in two dimensions, of a viscous fluid past a fixed cylinder," *Quart. J. Mech. Appl. Math.* **8**, 129–145.

[8] Amore, P., and Aranda, A., 2005, "Improved Lindstedt–Poincaré method for the solution of nonlinear problems," *J. Sound Vib.* **283**, 1115–1136.

[9] Ansari, A., Bakr, S., and Shishkin, G., 2007, "A parameter robust finite difference method for singularly perturbed delay parabolic partial differential equations," *J. Comput. Appl. Math.* **205**, 552–566.

[10] Ansari, A., and Hegarty, A., 2003, "Numerical solution of a convection diffusion problem with robin boundary conditions," *J. Comput. Appl. Math.* **156**, 221–238.

[11] Arakeri, J., and Shankar, P., 2000, "Ludwig Prandtl and boundary layers in fluid flow - How a small viscosity can cause large effects," *Resonance –Journal of Science Education* **5**, 48–63.

[12] Ashyralyev, A., and Sharifov, Y. A., 2013, "Existence and uniqueness of solutions for nonlinear impulsive differential equations with two-point and integral boundary conditions," *Adv. Difference Equ.* **173**, 1–11.

[13] Assiff, T., and Yen, D., 1987, "On the solutions of clamped reissner-mindlinplates under transverse loads," *Quart. Appl. Math.* **45**, 679–690.

[14] Avijit, D., and Natesan, S., 2022, "Convergence analysis of a fully-discrete fem for singularly perturbed two-parameter parabolic pde," *Math. Comput. Simulation* **197**, 185–206.

[15] Awrejcewicz, J., Andrianov, I., and Manevitch, L., 2012, *Asymptotic approaches in nonlinear dynamics: New trends and applications* (Springer Science & Business Media).

[16] Axelsson, O., and Gustafsson, I., 1979, "A modified upwind scheme for convective transport equations and the use of a conjugate gradient method for the solution non-symmetric systems of equations," *IMA J. Appl. Math.* **23**, 321–337.

[17] Ayele, M., Tiruneh, A., and Derese, G., 2023, "Fitted cubic spline scheme for two-parameter singularly perturbed time-delay parabolic problems," *Results Appl. Math.* **18**, 100361.

[18] Aziz, A., 2014, *The mathematical foundations of the finite element method with applications to partial differential equations* (Academic Press).

[19] Bahuguna, D., and Dabas, J., 2008, "Existence and uniqueness of a solution to a semilinear partial delay differential equation with an integral condition," *Nonlinear Dyn. Syst. Theory* **8**, 7–19.

[20] Bakhvalov, N., 841-859, "On the optimization of methods for solving boundary value problems with boundary layers," *Zh. Vychisl. Mat. Mat. Fis.* **9**, 1969.

[21] Bansal, K., and Sharma, K., 2018, "Parameter-robust numerical scheme for time-dependent singularly perturbed reaction-diffusion problem with large delay," *Numer. Funct. Anal. Optim.* **39**, 127–154.

[22] Barbu, L., and Moroşanu, G., 2007, *Singularly Perturbed Boundary Value Problems* (Birkhäusar, Basel).

[23] Barcilon, V., 1996, "Singular perturbation analysis of the fokker-planck equation: Kramer's underdamped problem," *SIAM J. Appl. Math.* **56**, 446–479.

[24] Barrett, J., and Morton, K., 1980, "Optimal finite element solutions to diffusion-convection problems in one dimension," *Int. J. Numer. Methods Eng.* **15**, 1457–1474.

[25] Barrett, J., and Morton, K., 1981, "Optimal finite element approximation for diffusion-convection problems," in *Conference on Mathematics of Finite Elements and Their Applications.*

[26] Barrett, J., and Morton, K., 1981, "Optimal Petrov-Galerkin methods through approximate symmetrization," *IMA J. Numer. Anal.* **1**, 439–468.

[27] Barrett, J., and Morton, K., 1984, "Approximate symmetrization and petrov-galerkin methods for diffusion-convection problems," *Comput. Methods Appl. Mech. Eng.* **45**, 97–122.

[28] Barrett, K., 1974, "The numerical solution of singular-perturbation boundary-value problems," *Quart. J. Mech. Appl. Math.* **27**, 57–68.

[29] Baumert, H., Braun, P., Glos, E., Müller, W.-D., and Stoyan, G., 2005, "Modelling and computation of water quality problems in river networks," in *Optimization Techniques: Proceedings of the 9th IFIP Conference on Optimization Techniques Warsaw, September 4-8, 1979* (Springer). pp. 482–491.

[30] Bawa, R., 2005, "Spline based computational technique for linear singularly perturbed boundary value problems," *Appl. Math. Comput.* **167**, 225–236.

[31] Bear, J., and Verruijt, A., 1987, *Modelling Groundwater Flow and Pollution* (Springer Science & Business Media).

[32] Beckett, G., and Mackenzie, J., 2000, "Convergence analysis of finite difference approximations on equidistributed grids to a singularly perturbed boundary value problem," *Appl. Numer. Math.* **35**, 87–109.

[33] Bejan, A., 1984, *Convection Heat Transfer* (John Wiley, New York).

[34] Belhaq, M., and Lakrad, F., 2000, "Elliptic multiple scales method for a class of autonomous strongly non-linear oscillators," *J. Sound Vib.* **234**, 547–553.

[35] Benbachir, S., 2000, "Lindstedt–Poincaré method and periodic families of the Barbanis-Contopoulos Hamiltonian system," *Math. Comput. Simul.* **51**, 579–596.

[36] Bender, C., and Orszag, S., 1978, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York).

[37] Berger, A., Solomon, J., and Ciment, M., 1981, "An analysis of a uniformly accurate difference method for a singular perturbation problem," *Math. Comp.* **37**, 79–94.

[38] Black, J., and Scholes, M., 1973, "The pricing of options and corporate liabilities," *J. Polit. Econ.* **81**, 637–654.

[39] Bobisud, L., 1967, "Second-order linear parabolic equations with a small parameter," *Arch. Rational Mech. Anal.* **27**, 385–397.

[40] Boffi, D., Brezzi, F., and Fortin, M., 2013, *Mixed finite element methods and applications* (Springer).

[41] Borowitz, S., 1967, *Fundamentals of quantum mechanics, particles, waves, and wave mechanics* (Benjamin).

[42] Boucherif, A., 2009, "Second-order boundary value problems with integral boundary conditions," *Nonlinear Anal.* **70**, 364–371.

[43] Brezzi, F., and Fortin, M., 2012, *Mixed and hybrid finite element methods* (Springer Science & Business Media).

[44] Chandru, M., Das, P., and Ramos, H., 2018, "Numerical treatment of two parameter singularly perturbed parabolic convection diffusion problems with nonsmooth data," *Math. Methods Appl. Sci.* **41**, 5359–5387.

[45] Chandru, M., Prabha, T., and Shanthi, V., 2017, "A parameter robust higher order numerical method for singularly perturbed two parameter problems with nonsmooth data," *J. Comput. Appl. Math.* **309**, 11–27.

[46] Chen, J., and O'Malley, R., 1974, "On the asymptotic solution of a two-parameter boundary value problem ofchemical reactor theory," *SIAM J. Appl. Math.* **26**, 717–729.

[47] Chen, S., Huang, J., and Sze, K., 2007, "Multidimensional Lindstedt-Poincaré method for nonlinear vibration of axially moving beams," *J. Sound Vib.* **306**, 1–11.

[48] Christie, I., Griffiths, D., Mitchell, A., and Zienkiewicz, O., 1976, "Finite element methods for second order differential equations with significant first derivatives," *Int. J. Numer. Meth. Engng.* **10**, 1389–1396.

[49] Chu, C.-C., Graham, I., and Hou, T.-Y., 2010, "A new multiscale finite element method for high-contrast elliptic interface problems," *Math. Comput.* **79**, 1915–1955.

[50] Clavero, C., Gracia, J. L., and Jorge, J. C., 2004, "High-order numerical methods for one-dimensional parabolic singularly perturbed problems with regular layers," *Numer. Methods Partial Differ. Equ.* **21**, 148–169.

[51] Clavero, C., Gracia, J., Shishkin, G., and Shishkina, L., 2013, "Grid approximation of a singularly perturbed parabolic equation with degenerating convective term and discontinuous right hand side," *Int. J. Numer. Anal. Model.* **10**, 795–814.

[52] Clavero, C., Gracia, J., Shishkin, G., and Shishkina, L., 2015, "Schemes convergent ϵ-uniformly for parabolic singularly perturbed problems with a degenerating convective term and a discontinuous source," *Math. Model. Anal.* **20**, 641–657.

[53] Clavero, C., Gracia, J., Shishkin, G., and Shishkina, L., 2017, "An efficient numerical scheme for 1d parabolic singularly perturbed problems with an interior and boundary layers," *J. Comput. Appl. Math.* **318**, 634–645.

[54] Clavero, C., Jorge, J., and Lisbona, F., 2003, "A uniformly convergent scheme on a nonuniform mesh for convection–diffusion parabolic problems," *J. Comput. Appl.Math.* **154**, 415–429.

[55] Clavero, C., and Jorge, J., 2023, "A splitting uniformly convergent method for one-dimensional parabolic singularly perturbed convection-diffusion systems," *Appl. Numer. Math.* **183**, 317–332.

[56] Clavero, C., and Jorge, J. C., 2019, “An efficient numerical method for singularly perturbed time dependent parabolic 2d convection-diffusion systems,” *J. Comput. Appl. Math.* **354**, 431–444.

[57] Couchouron, J., Kamenski, M., and Precup, R., 2003, “A nonlinear periodic averaging principle,” *Nonlinear Anal.* **54**, 1439–1467.

[58] Coulaud, O., 1997, “Multiple time scales and perturbation methods for high frequency electromagnetic-hydrodynamic coupling in the treatment of liquid metals,” *Nonlinear Anal.* **30**, 3637–3643.

[59] Das, A., and Natesan, S., 2015, “Uniformly convergent hybrid numerical scheme for singularly perturbed delay parabolic convection diffusion problems on Shishkin mesh,” *Appl. Math. Comput.* **271**, 168–186.

[60] Das, A., and Natesan, S., 2018, “Higher-order convergence with fractional-step method for singularly perturbed 2d parabolic convection-diffusion problems on shishkin mesh,” *Comput. Math. Appl.* **75**, 2387–2403.

[61] Das, P., and Natesan, S., 2012, “Higher order parameter uniform convergent schemes for Robin type reaction diffusion problems using adaptively generated grid,” *Int. J. Comput. Methods* **9**, 125–152.

[62] Das, P., and Vigo-Aguiar, J., 2019, “Parameter uniform optimal order numerical approximation of a class of singularly perturbed system of reaction diffusion problems involving a small perturbation parameter,” *J. Comput. Appl. Math.* **354**, 533–544.

[63] De-Jager, E., and Furu, J., 1996, *The Theory of Singular Perturbations* (Elsevier, Amsterdam).

[64] Dessi, D., Mastroddi, F., and Morino, L., 2004, “A fifth-order multiple-scale solution for Hopf bifurcations,” *Comput. Struct.* **82**, 2723–2731.

[65] DiPrima, R., 1968, “Asumptotic methods for an infinitely long slider squeeze-film bearing,” *Trans. ASME Ser.F. J. Lub. Tech.* **90**, 173–183.

[66] Doedel, E., 1978, “The construction of finite difference approximations to ordinary differential equations,” *SIAM J. Numer. Anal.* **15**, 450–465.

[67] Doolan, E., Miller, J., and Schilders, W., 1980, *Uniform numerical methods for problems with initial and boundary layers* (Boole Press).

[68] Dorr, F., 1970, "The numerical solution of singular perturbations of boundary value problems," *SIAM J. Numer. Anal.* **7**, 281–313.

[69] Dorr, F., Parter, S., and Shampine, L., 1973, "Applications of the maximum principle to singular perturbation problems," *SIAM Review* **15**, 43–88.

[70] Douglas, J., and Russell, T., 1982, "Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures," *SIAM J. Numer. Anal.* **19**, 871–885.

[71] Dunne, R., and O'Riordan, E., 2007, "Interior layers arising in linear singularly perturbed differential equations with discontinuous coefficients," *in: I. Farago, P. Vabishchevich, and L. Vulkov (Eds.), Proc. of the Fourth International Conference on Finite Difference Methods: Theory & Applications*, 29–38.

[72] Dunne, R., O'Riordan, E., and Shishkin, G., 2003, "A fitted mesh method for a class of singularly perturbed parabolic problems with a boundary turning point," *Comput. Methods Appl. Math.* **3**, 361–372.

[73] El-Mistikawy, T., and Werle, M., 1978, "Numerical method for boundary layers with blowing-the exponential box scheme," *AIAA Journal* **16**, 749–751.

[74] Elango, S., 2023, "Second order singularly perturbed delay differential equations with non-local boundary condition," *J. Comput. Appl. Math.* **417**, 114498.

[75] El'sgol'ts, L., 1964, *Qualitative methods in mathematical analysis. Translations of Mathematical Monographs* (American Mathematical Society, Providence, R.I.).

[76] Eriksson, K., and Johnson, C., 1993, "Adaptive streamline diffusion finite element methods for stationary convection-diffusion problems," *Math. Comput.* **60**, 167–188.

[77] Ervin, V., and Layton, W., 1989, "An analysis of a defect-correction method for a model convection-diffusion equation," *SIAM J. Numer. Anal.* **26**, 169–179.

[78] Ewing, R., 1983, *The Mathematics of Reservoir Simulation* (SIAM, Philadelphia).

[79] Falco, C., and E.O'Riordan, 2010, "Interior layers in a reaction diffusion equation with a discontinuous diffusion coefficient," *Int. J. Numer. Anal. Model.* **7**, 444–461.

[80] Farrell, P., 1987, "Sufficient conditions for uniform convergence of a class of difference schemes for a singularly perturbed problem," *IMA J. Numer. Anal.* **7**, 459–472.

[81] Farrell, P., 1988, "Sufficient conditions for the uniform convergence of a difference scheme for a singularly perturbed turning point problem," *SIAM J. Numer. Anal.* **25**, 619–643.

[82] Farrell, P., Hegarty, A., Miller, J., O'Riordan, E., and Shishkin, G., 2000, *Robust Computational Techniques for Boundary Layers* (Chapman and Hall/CRC, Boca Raton).

[83] Farrell, P., Miller, J., O'Riordan, E., and Shishkin, G., 2000, "Singularly perturbed differential equations with discontinuous source terms," *in: J.J.H. Miller, G.I. Shishkin, L. Vulkov (Eds.), Proc. of Analytical and Numerical Methods for Convection Dominated and Singularly Perturbed Problems*, 23–32.

[84] FitzHugh, R., 1961, "Impulses and physiological states in theoretical models of nerve membrane," *Biophysical journal* **1**, 445–466.

[85] Franz, S., and Linß, T., 2008, "Superconvergence analysis of the Galerkin FEM for a singularly perturbed convection-diffusion problem with characteristic layers," *Numer. Methods Partial Differ. Equ.* **24**, 144–164.

[86] Friedrichs, K., 1941, "The mathematical structure of the boundary layer problem," *Fluid Mechanics*, 171–174.

[87] Friedrichs, K., and Wasow, W., 1946, "Singular perturbations of nonlinear oscillations," *Duke Math.* **13**, 367–381.

[88] G. Babu, G., Prithvi, M., Sharma, K., and Ramesh, V., 2022, "A uniformly convergent numerical algorithm on harmonic (h(l)) mesh for parabolic singularly perturbed convection-diffusion problems with boundary layer," *Differ. Equ. Dyn. Syst.*, 1–14.

[89] Gajic, Z., and Lim, M., 2001, *Optimal Control of Singularly Perturbed Linear Systems and Applications* (Marcel-Dekker, New York).

[90] Gartland, E., 1987, "Uniform high-order difference schemes for a singularly perturbed two-point boundary value problem," *Math. Comput.* **48**, 551–564.

[91] Gartland, E., 1988, "Graded mesh difference schemes for singularly perturbed two point boundary value problems," *Math. Comput.* **51**, 631–657.

[92] Gayaz, K., and Denys, D., 2017, "On supraconvergence phenomenon for second order centered finite differences on non-uniform grids," *J. Comput. Appl. Math.* **326**, 1–14.

[93] Geetha, N., and Tamilselvan, A., 2018, "Parameter uniform numerical method for fourth order singularly perturbed turning point problems exhibiting boundary layers," *Ain Shams Eng. J.* **9**, 845–853.

[94] Gelu, F. W., and Duressa, G. F., 2023, "A parameter-uniform numerical method for singularly perturbed robin type parabolic convection-diffusion turning point problems," *Appl. Numer. Math.* **190**, 50–64.

[95] Gobena, W. T., and Duressa, G. F., 2022, "An optimal fitted numerical scheme for solving singularly perturbed parabolic problems with large negative shift and integral boundary condition," *Results Control Optim.* **9**, 100172.

[96] Gosse, L., and Mauser, L., 2006, "Multiphase semiclassical approximation of an electron in a one-dimensional crystalline lattice–III. from ab initio models to WKB for Schrödinger-Poisson," *J. Comp. Phys.* **211**, 326–346.

[97] Govindarao, L., Mohapatra, J., and Sahu, S. R., 2019, "Uniformly convergent numerical method for singularly perturbed two parameter time delay parabolic problem," *Int. J. Appl. Comput. Math.* **5**, Paper No. 91, 9.

[98] Gowrisankar, S., and Natesan, S., 2014, "Robust numerical scheme for singularly perturbed convection diffusion parabolic initial boundary value problems on equidistributed grids," *Comput. Phys. Commun.* **185**, 2008–2019.

[99] Gowrisankar, S., and Natesan, S., 2017, "ϵ-uniformly convergent numerical scheme for singularly perturbed delay parabolic partial differential equations," *Int. J. Comput. Math.* **94**, 902–921.

[100] Gowrisankar, S., and Natesan, S., 2019, "An efficient robust numerical method for singularly perturbed burgers' equation," *Appl. Math. Comput.* **346**, 385–394.

[101] Gracia, J. L., O'Riordan, E., and Pickett, M., 2006, "A parameter robust second order numerical method for a singularly perturbed two-parameter problem," *Appl. Numer. Math.* **56**, 962–980.

[102] Gracia, J., and O'Riordan, E., 2012, "A singularly perturbed convection diffusion problem with a moving interior layer," *Int. J. Numer. Anal. Model.* **9**, 823–843.

[103] Gracia, J., and O'Riordan, E., 2012, "A singularly perturbed parabolic problem with a layer in the initial condition," *Appl. Math. Comput.* **219**, 498–510.

[104] Gracia, J. L., and O'Riordan, E., 2019, "Parameter-uniform numerical methods for singularly perturbed parabolic problems with incompatible boundary-initial data," *Appl. Numer. Math.* **146**, 436–451.

[105] Grossmann, C., Mohanty, R., and Roos, H.-G., 2011, "A direct higher order discretization in singular perturbations via domain split-A computational approach," *Appl. Math. Comput.* **217**, 9302–9312.

[106] Gupta, A., and Kaushik, A., 2021, "A higher-order accurate difference approximation of singularly perturbed reaction-diffusion problem using grid equidistribution," *Ain Shams Eng. J.* **12**, 4211–4221.

[107] Gupta, V., Kadalbajoo, M., and Dubey, R., 2019, "A parameter-uniform higher order finite difference scheme for singularly perturbed time-dependent parabolic problem with two small parameters," *Int. J. Comput. Math.* **96**, 474–499.

[108] Gupta, V., Kumar, M., and Kumar, S., 2018, "Higher order numerical approximation for time dependent singularly perturbed differential-difference convection-diffusion equations," *Numer. Methods Partial Differ. Equ.* **34**, 357–380.

[109] Gushchin, V., and Shchennikov, V., 1974, "A monotonic difference scheme of second-order accuracy," *USSR Comput. Math. Math. Phys.* **14**, 252–256.

[110] Hahn, S., Bigeon, J., and Sabonnadiere, J., 1987, "An upwind finite element method for electromagnetic field problems in moving media," *lnt. J. Numer. Methods Eng.* **24**, 2071–2086.

[111] Hailu, W. S., and Duressa, G. F., 2023, "Accelerated parameter-uniform numerical method for singularly perturbed parabolic convection-diffusion problems with a large negative shift and integral boundary condition," *Results Appl. Math.* **18**, 100364.

[112] He, J., 1999, "Variational iteration method - a kind of non-linear analytical technique: some examples," *Internat. J. Non-Linear Mech.* **34**, 699–708.

[113] He, J., 2002, "Modified Lindstedt-Poincare methods for some strongly non-linear oscillations: Part I: expansion of a constant," *Internat. J. Non-Linear Mech.* **37**, 309–314.

[114] He, J., 2006, "Homotopy perturbation method for solving boundary value problems," *Phys. Lett. A* **350**, 87–88.

[115] He, J., Wan, Y., and Guo, Q., 2004, "An iteration formulation for normalized diode characteristics," *Internat. J. Circuit. Theory. Appl.* **32**, 629–632.

[116] Heading, J., 1962, *An introduction to phase-integral methods* (Methuen, London).

[117] Heinrich, J., 1980, "On quadratic elements in finite element solutions of steady-state convection-diffusion equation," *Int. J. Numer. Methods Eng.* **15**, 1041–1052.

[118] Heinrich, J., Huyakorn, P., Mitchell, A., and Zienkiewicz, O., 1977, "An upwind finite element scheme for two-dimensional convective transport equations," *Int. J. Numer. Methods Eng.* **11**, 131–143.

[119] Heinrich, J., and O.C.Zienkiewicz, 1979, "The finite element method and 'upwinding' techniques in the numerical solution of convection dominated flow problems," in *Finite Element Methods for Convection Dominated Flows*, pp. 105–136.

[120] Hemker, P., and Shishkin, G., 1994, "Discrete approximation of singularly perturbed parabolic PDEs with a discontinuous initial condition," *Comp. Fluid Dynam. J.* **2**, 375–392.

[121] Hemker, P., Shishkin, G., and Shishkina, L., 2002, "High-order time-accurate schemes for singularly perturbed parabolic convection-diffusion problems with robin boundary conditions," *Comput. Methods Appl. Math.* **2**, 3–25.

[122] Herceg, D., Surla, K., and Rapajic, S., 1998, "Cubic spline difference scheme on a mesh of Bakhvalov type," *Novi Sad J. Math* **28**, 41–49.

[123] Horssen, W. V., 2001, "On integrating vectors and multiple scales for singularly perturbed ordinary differential equations," *Nonlinear Anal.* **46**, 19–43.

[124] Huan-wen, C., 1984, "Some application of the singular perturbation methodto the bending problems of thin plates and shells," *Appl. Math. Mech.* **5**, 1449–1457.

[125] Il'in, A., 1969, "Differencing scheme for a differential equation with a small parameter affecting the highest derivative," *Mathematical Notes of the Academy of Sciences of the USSR* **6**, 596–602.

[126] Jacob, M., 1949, "Heat transfer," *John Wiley & Sons* **1**, 655.

[127] Jalaal, M., Ganji, D., and Mohammadi, F., 2010, "He's homotopy perturbation method for two-dimensional heat conduction equation: Comparison with finite element method," *Heat Transfer-Asian Res.* **39**, 232–245.

[128] Kabeto, M. J., and Duressa, G. F., 2022, "Second-order robust finite difference method for singularly perturbed burgers' equation," *Heliyon* **8**, e09579.

[129] Kadalbajoo, M., and Yadaw, A., 2012, "Parameter-uniform finite element method for two-parameter singularly perturbed parabolic reaction-diffusion problems," *Int. J. Comput. Methods* **9**, 1250047.

[130] Kadalbajoo, M., and Awasthi, A., 2011, "The midpoint upwind finite difference scheme for time-dependent singularly perturbed convection-diffusion equations on non-uniform mesh," *Int. J. Comput. Methods Eng. Sci. Mech.* **12**, 150–159.

[131] Kadalbajoo, M., and Gupta, V., 2010, "A brief survey on numerical methods for solving singularly perturbed problems," *Appl. Math. Comput.* **217**, 3641–3716.

[132] Kadalbajoo, M., and Patidar, K., 2002, "Spline techniques for solving singularly-perturbed nonlinear problems on nonuniform grids," *J. Optim. Theory Appl.* **114**, 573–591.

[133] Kadalbajoo, M., and Patidar, K., 2003, "Exponentially fitted spline in compression for the numerical solution of singular perturbation problems," *Comput. Math. Appl.* **46**, 751–767.

[134] Kadalbajoo, M., and Patidar, K., 2003, "Singularly perturbed problems in partial differential equations: A survey," *Appl. Math. Comput.* **134**, 371–429.

[135] Kalaiselvan, S. S., Miller, J. J. H., and Sigamani, V., 2019, "A parameter uniform numerical method for a singularly perturbed two-parameter delay differential equation," *Appl. Numer. Math.* **145**, 90–110.

[136] Kamenskii, M., and Nistri, P., 2003, "An averaging method for singularly perturbed systems of semilinear differential inclusions with analytic semigroups," *Nonlinear Anal.* **53**, 467–480.

[137] Kaplun, S., 1954, "The role of coordinate systems in boundary-layer theory," *Zeitschrift für angewandte Mathematik und Physik ZAMP* **5**, 111–135.

[138] Kaplun, S., and Lagerstrom, P., 1957, "Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers," *J. Math. Mech.*, 585–593.

[139] Kaushik, A., 2014, "Error estimates for a class of partial functional differential equation with small dissipation," *Appl. Math. Comput.* **226**, 250–256.

[140] Kaushik, A., Sharma, K., and Sharma, M., 2010, "A parameter uniform difference scheme for parabolic partial differential equation with a retarded argument," *Appl. Math. Model.* **34**, 4232–4242.

[141] Keller, H., 1968, *Numerical Methods for Two-Point Boundary-Value Problems* (Blaisdell Publication Co., Waltham, Massachusetts).

[142] Kellogg, R., and Tsan, A., 1978, "Analysis of some difference approximations for a singular perturbation problem without turning points," *Math. Comp.* **32**, 1025–1039.

[143] Kevorkian, J., 1962, *The two variable expansion procedure for the approximate solution of certain non-linear differential equations*, Tech. Rep. (Santa Monica, California).

[144] Kevorkian, J., and Cole, J., 2013, *Perturbation Methods in Applied Mathematics* (Springer, Berlin).

[145] Khrustalev, O., and Vernov, S., 2001, "Construction of doubly periodic solutions via the Poincare-Lindstedt method in the case of massless $\phi 4$ theory," *Math. Comput. Simul.* **57**, 239–252.

[146] Kokotovic, P., 1984, "Applications of singular perturbation techniques to control problems," *SIAM Review* **26**, 501–550.

[147] Kopteva, N., 2001, "Uniform pointwise convergence of difference schemes for convection diffusion problems on layer adapted meshes," *Computing* **66**, 179–197.

[148] Kopteva, N., Madden, N., and Stynes, M., 2005, "Grid equidistribution for reaction diffusion problems in one dimension," *Numer. Algor.* **40**, 305–322.

[149] Kopteva, N., and Stynes, M., 2001, "A robust adaptive method for a quasi-linear one-dimensional convection-diffusion problem," *SIAM J. Numer. Anal.* **39**, 1446–1467.

[150] Kopteva, N., and Stynes, M., 2011, "Stabilised approximation of interior layer solutions of a singularly perturbed semilinear reaction diffusion problem," *Numer. Math.* **119**, 787–810.

[151] Krivec, R., and Mandelzweig, V., 2006, “Quasilinearization method and WKB,” *Comput. Phys. Commun.* **174**, 119–126.

[152] Kuang, Y., 1993, *Delay Differential Equations with Applications in Population Dynamics* (Academic Press, New York).

[153] Kumar, D., and Kumari, P., 2020, “A parameter-uniform collocation scheme for singularly perturbed delay problems with integral boundary conditions,” *J. Appl. Math. Comput.* **63**, 813–828.

[154] Kumar, M., and Rao, S., 2010, “High order parameter robust numerical method for time dependent singularly perturbed reaction diffusion problems,” *Computing* **90**, 15–38.

[155] Kumar, N., Singh, J., and Jiwari, R., 2023, “Convergence analysis of weak galerkin finite element method for semilinear parabolic convection dominated diffusion equations on polygonal meshes,” *Comput. Math. Appl.* **145**, 141–158.

[156] Kumar, S., Aakansha, Singh, J., and Ramos, H., 2023, “Parameter-uniform convergence analysis of a domain decomposition method for singularly perturbed parabolic problems with robin boundary conditions,” *J. Appl. Math. Comput.* **69**, 2239–2261.

[157] Kumar, S., Ramos, H., *et al.*, 2021, “Parameter-uniform approximation on equidistributed meshes for singularly perturbed parabolic reaction-diffusion problems with robin boundary conditions,” *Appl. Math. Comput.* **392**, 125677.

[158] Kumar, S., Vigo-Aguiar, J., *et al.*, 2022, “A high order convergent numerical method for singularly perturbed time dependent problems using mesh equidistribution,” *Math. Comput. Simulation* **199**, 287–306.

[159] Kuzmak, G., 1959, “Asymptotic solutions of nonlinear second order differential equations with variable coefficients,” *J. Appl. Math. Mech.* **23**, 730–744.

[160] Ladyzhenskaya, O., Solonnikov, V., and Ural’ceva, N., 1968, *Linear and quasilinear equations of parabolic type. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23* (American Mathematical Society, Providence, R.I.).

[161] Ladyzhenskaya, O., and Ural’tseva, N., 1968, *Linear and Quasilinear Elliptic Equations* (Academic Press, New York).

[162] Lagerstrom, P., and Casten, R., 1972, "Basic concepts underlying singular perturbation techniques," *SIAM Review* **14**, 63–120.

[163] Lagerstrom, P., and Cole, J., 1955, "Examples illustrating expansion procedures for the Navier-Stokes equations," *J. Ration. Mech. Anal.* **4**, 817–882.

[164] Lakrad, F., and Belhaq, M., 2002, "Periodic solutions of strongly non-linear oscillators by the multiple scales method," *J. Sound Vib.* **258**, 677–700.

[165] Launder, B., and Spalding, D., 1972, *Mathematical models of turbulence* (Academic press, New York).

[166] LeVeque, R., 2002, *Finite volume methods for hyperbolic problems* (Cambridge University Press).

[167] Levinson, N., 1950, "The first boundary value problem for ε-δ $U + A(X,Y)UX + B(X,Y)UY + C(X,Y)U = D(X,Y)$ for small ε," *Annals of Mathematics* **51**, 428–445.

[168] Linß, T., 2000, "Analysis of a Galerkin finite element method on a Bakhvalov-Shishkin mesh for a linear convection diffusion problem," *IMA J. Numer. Anal.* **20**, 621–632.

[169] Linß, T., 2001, "Uniform pointwise convergence of finite difference schemes using grid equidistribution," *Computing* **66**, 27–39.

[170] Linß, T., 2003, "Layer-adapted meshes for convection-diffusion problems," *Comput. Methods Appl. Mech. Eng.* **192**, 1061–1105.

[171] Linß, T., 2004, "Error expansion for a first-order upwind difference scheme applied to a model convection-diffusion problem," *IMA J. Numer. Anal.* **24**, 239–253.

[172] Linß, T., 2008, "Robust convergence of a compact fourth order finite difference scheme for reaction diffusion problems," *Numer. Math.* **111**, 239–249.

[173] Linß, T., 2010, *Layer Adapted Meshes for Reaction Convection Diffusion Problems* (Springer, Berlin).

[174] Liu, H., 2005, "Approximate period of nonlinear oscillators with discontinuities by modified Lindstedt–Poincare method," *Chaos Solitons Fractals* **23**, 577–579.

[175] Lodhi, R. K., and Mishra, H. K., 2018, "Septic b-spline method for second order self-adjoint singularly perturbed boundary-value problems," *Ain Shams Eng. J.* **9**, 2153–2161.

[176] Lu, P., 1973, *Introduction to the Mechanics of Viscous Fluids* (Holt-Rinehart & Winston, New York).

[177] Lubuma, J., and Patidar, K., 2006, "Uniformly convergent non-standard finite difference methods for self-adjoint singular perturbation problems," *J. Comput. Appl. Math.* **191**, 228–238.

[178] Lynch, R., and Rice, J., 1980, "A high-order difference method for differential equations," *Math. Comput.* **34**, 333–372.

[179] Mackenzie, J., 1999, "Uniform convergence analysis of an upwind finite-difference approximation of a convection-diffusion boundary value problem on an adaptive grid," *IMA J. Numer. Anal.* **19**, 233–249.

[180] Majumdar, A., and Natesan, S., 2017, "Alternating direction numerical scheme for singularly perturbed 2d degenerate parabolic convection-diffusion problems," *Appl. Math. Comput.* **313**, 453–473.

[181] Marasco, A., 2000, "Lindstedt-Poincarè method and Mathematica applied to the motion of a solid with a fixed point," *Comput. Math. Appl.* **40**, 333–343.

[182] Mazumder, S., 2015, *Numerical methods for partial differential equations: finite difference and finite volume methods* (Academic Press).

[183] Mbroh, N., Noutchie, S., and Massoukou, R., 2020, "A uniformly convergent finite difference scheme for robin type singularly perturbed parabolic convection diffusion problem," *Math. Comput. Simulation* **174**, 218–232.

[184] Mbroh, N. A., and Munyakazi, J. B., 2019, "A fitted operator finite difference method of lines for singularly perturbed parabolic convection–diffusion problems," *Math. Comput. Simulation* **165**, 156–171.

[185] Mbroh, N. A., Noutchie, S. C. O., and Massoukou, R. Y. M., 2020, "A second order finite difference scheme for singularly perturbed volterra integro-differential equation," *Alex. Eng. J.* **59**, 2441–2447.

[186] Mickens, R., 1981, *An introduction to nonlinear oscillations* (CUP Archive).

[187] Miller, J., 1975, "A finite element method for a two point boundary value problem with a small parameter affecting the highest derivative," *Presented to the Seminar, Mathematical Models and Numerical Methods* **3**, 143–146.

[188] Miller, J., 1997, *Singular Perturbation Problems in Chemical Physics* (John Wiley & Sons, New York).

[189] Miller, J., O'Riordan, E., and Shishkin, G., 2012, *Fitted Numerical Methods for Singular Perturbation Problems* (World Scientific, Singapore).

[190] Miller, J., 'Riordan, E. O., Shishkin, G., and Shishkina, L., 1998, "Fitted mesh methods for problems with parabolic boundary layers," *Math. Proc. Royal Irish Acad.* **98**, 173–190.

[191] Morton, K., 2010, *Numerical Solution of Convection Diffusion Problems* (Chapman & Hall, London).

[192] Mukherjee, K., and Natesan, S., 2009, "Parameter-uniform hybrid numerical scheme for time-dependent convection-dominated initial-boundary-value problems," *Computing* **84**, 209–230.

[193] Mukherjee, K., and Natesan, S., 2011, "ϵ-uniform error estimate of hybrid numerical scheme for singularly perturbed parabolic problems with interior layers," *Numer. Algorithms* **58**, 103–141.

[194] Mukherjee, K., and Natesan, S., 2019, "Parameter-uniform fractional step hybrid numerical scheme for 2d singularly perturbed parabolic convection-diffusion problems," *J. Appl. Math. Comput.* **60**, 51–86.

[195] Munyakazi, J., 2015, "A uniformly convergent nonstandard finite difference scheme for a system of convection-diffusion equations," *Comp. Appl. Math.* **34**, 1153–1165.

[196] Munyakazi, J., and K.C.Patidar, 2013, "A fitted numerical method for singularly perturbed parabolic reaction-diffusion problems," *Comput. Appl. Math.* **32**, 509–519.

[197] Munyakazi, J., and Patidar, K., 2008, "On Richardson extrapolation for fitted operator finite difference methods," *Appl. Math. Comput.* **201**, 465–480.

[198] Murray, J., 1977, *Lectures on nonlinear-differential-equation models in biology* (Clarendon Press, Oxford).

[199] Nagarajan, S., 2022, "A parameter robust fitted mesh finite difference method for a system of two reaction-convection-diffusion equations," *Appl. Numer. Math.* **179**, 87–104.

[200] Naidu, D., and Rao, A., 1985, *Singular Perturbation Analysis of Discrete Control Systems* (Springer-Verlag, Berlin).

[201] Natesan, S., and Bawa, R., 2007, "Second-order numerical scheme for singularly perturbed reaction-diffusion robin problems," *J. Numer. Anal. Ind. Appl. Math.* **2**, 177–192.

[202] Navarro, J., 2008, "On the implementation of the Poincaré-Lindstedt technique," *Appl. Math. Comput.* **195**, 183–189.

[203] Nayfeh, A., 1979, *Perturbation Methods* (Wiley, New York).

[204] Nayfeh, A., 2011, *Introduction to perturbation techniques* (John Wiley & Sons).

[205] Nefedov, N., Recke, L., and Schneider, K., 2013, "Existence and asymptotic stability of periodic solutions with an interior layer of reaction advection diffusion equations," *J. Math. Anal. Appl.* **405**, 90–103.

[206] Negero, N., 2023, "Fitted cubic spline in tension difference scheme for two-parameter singularly perturbed delay parabolic partial differential equations," *Partial Differ. Equ. Appl. Math.* **8**, 100530.

[207] Negero, N., 2023, "A parameter-uniform efficient numerical scheme for singularly perturbed time-delay parabolic problems with two small parameters," *Partial Differ. Equ. Appl. Math.* **7**, 100518.

[208] Nhan, T. A., Mai, V. Q., Mohapatra, J., and Hammouch, Z., 2023, "A new upwind difference analysis of an exponentially graded bakhvalov-type mesh for singularly perturbed elliptic convection-diffusion problems," *J. Comput. Appl. Math.* **418**, 114622.

[209] Niederdrenk, K., and Yserentant, H., 1983, "Die gleichmäßige stabilität singulär gestörter diskreter und kontinuierlicher randwertprobleme," *Numer. Math.* **41**, 223–253.

[210] Ning, L., Haiyan, S., Dongwei, G., and Xinlong, F., 2018, "Multiquadric RBF-FD method for the convection-dominated diffusion problems based on Shishkin nodes," *Int. J. Heat Mass Transfer* **118**, 734–745.

[211] Nofel, T., 2014, "Application of the homotopy perturbation method to nonlinear heat conduction and fractional Van der Pol damped nonlinear oscillator," *Appl. Math.* **5**, 852–861.

[212] Oleǐnik, O., 1952, "On boundary problems from equations with a small parameter in the highest derivatives," *Dokl. Akad. Nauk, SSR* **85**, 493–495.

[213] O'Malley, R., 1967, "Two-parameter singular perturbation problems for second order equations," *J. Math. Mech.* **16**, 1143–1164.

[214] O'Malley, R., 1974, *Introduction to Singular Perturbations* (Academic Press, New York).

[215] O'Malley, R., 1991, *Singular Perturbation Methods for Ordinary Differential Equations* (Springer, New York).

[216] O'Malley, R., 2014, *Historical developments in singular perturbations* (Springer).

[217] O'Riordan, E., 1984, "Singularly perturbed finite element methods," *Numer. Math.* **44**, 425–434.

[218] O'Riordan, E., Pickett, M., and Shishkin, G., 2006, "Parameter-uniform finite difference schemes for singularly perturbed parabolic diffusion-convection-reaction problems," *Math. Comp.* **75**, 1135–1154.

[219] O'Riordan, E., and Quinn, J., 2012, "A singularly perturbed convection diffusion turning point problem with an interior layer," *Comput. Methods Appl. Math.* **12**, 206–220.

[220] O'Riordan, E., and Quinn, J., 2015, "A linearised singularly perturbed convection diffusion problem with an interior layer," *Appl. Numer. Math.* **98**, 1–17.

[221] O'Riordan, E., and Shishkin, G., 2004, "Singularly perturbed parabolic problems with non-smooth data," *J. Comput. Appl. Math.* **166**, 233–245.

[222] Öziş, T., and Yıldırım, A., 2007, "Determination of periodic solution for a $u^{\frac{1}{3}}$ force by He's modified Lindstedt–Poincaré method," *J. Sound Vib.* **301**, 415–419.

[223] Panda, A., and Mohapatra, J., 2023, "On the convergence analysis of efficient numerical schemes for singularly perturbed second order volterra integro-differential equations," *J. Appl. Math. Comput.*, 1–24.

[224] Panda, A., and Mohapatra, J., 2023, "A robust finite difference method for the solutions of singularly perturbed fredholm integro-differential equations," *Mediterr. J. Math.* **20**, 198.

[225] Paramasivam, M., Miller, J., and Valarmathi, S., 2014, "Parameter uniform convergence for a finite difference method for a singularly perturbed linear reaction diffusion system with discontinuous source terms," *Int. J. Numer. Anal. Model.* **11**, 385–399.

[226] Patidar, K., 2005, "High order fitted operator numerical method for self-adjoint singular perturbation problems," *Appl. Math. Comput.* **171**, 547–566.

[227] Patidar, K., 2016, "Nonstandard finite difference methods: recent trends and further developments," *J. Difference Equ. Appl.* **22**, 817–849.

[228] Pearson, C. E., 1968, "On a differential equation of boundary layer type," *J. Math. and Phy.* **47**, 134–154.

[229] Pearson, C. E., 1968, "On a nonlinear differential equation of boundary layer type," *J. Math. and Phy.* **47**, 351–358.

[230] Pillai, A. R., 2001, "Fourth-order exponential finite difference methods for boundary value problems of convective diffusion type," *Int. J. Numer.Methods Fluids* **37**, 87–106.

[231] Polak, S., Heijer, C. D., Schilders, W., and Markowich, P., 1987, "Semiconductor device modelling from the numerical point of view," *Int. J. Numer. Methods Eng.* **24**, 763–838.

[232] Prandtl, L., 1904, *Über Flüssigkeitsbewegung bei sehr kleiner Reibung: in Verhandlungen des III Internationalen Mathematiker-Kongresses* (Teubner, Leipzig).

[233] Priyadarshana, S., Mohapatra, J., and Pattanaik, S., 2023, "A second order fractional step hybrid numerical algorithm for time delayed singularly perturbed 2d convection-diffusion problems," *Appl. Numer. Math.* **189**, 107–129.

[234] Protter, M., and Weinberger, H., 1984, *Maximum Principles in Differential Equations* (Springer-Verlag, New York).

[235] Pudykiewicz, J., 1989, "Simulation of the Chernobyl dispersion with a 3-D hemispheric tracer model," *Tellus B: Chemical and Physical Meterology* **41**, 391–412.

[236] Pušenjak, R., 2008, "Extended Lindstedt-Poincare method for non-stationary resonances of dynamical systems with cubic nonlinearities," *J. Sound Vib.* **314**, 194–216.

[237] Quinn, J., 2015, "A numerical method for a nonlinear singularly perturbed interior layer problem using an approximate layer location," *J. Comput. Appl. Math.* **290**, 500–515.

[238] Ramesh, V., and Priyanga, B., 2021, "Higher order uniformly convergent numerical algorithm for time-dependent singularly perturbed differential-difference equations," *Differ. Equ. Dyn. Syst.* **29**, 239–263.

[239] Ranjan, K. R., and Gowrisankar, S., 2022, "Uniformly convergent nipg method for singularly perturbed convection diffusion problem on shishkin type meshes," *Appl. Numer. Math.* **179**, 125–148.

[240] Rao, S., and Chawla, S., 2013, "Interior layers in coupled system of two singularly perturbed reaction-diffusion equations with discontinuous source term," *Lect. Notes Comput. Sci.* **8236**, 445–453.

[241] Rao, S., and Chawla, S., 2018, "Numerical solution of singularly perturbed linear parabolic system with discontinuous source term," *Appl. Numer. Math.* **127**, 249–265.

[242] Rao, S., and Kumar, M., 2009, "Parameter-uniformly convergent exponential spline difference scheme for singularly perturbed semilinear reaction-diffusion problems," *Nonlinear Anal.* **71**, e1579–e1588.

[243] Rao, S., and Kumar, M., 2010, "A uniformly convergent exponential spline difference scheme for singularly perturbed reaction-diffusion problems," *Neural Parallel Sci. Comput.* **18**, 121–136.

[244] Rashidinia, J., Ghasemi, M., and Mahmoodi, Z., 2007, "Spline approach to the solution of a singularly perturbed boundary value problems," *Appl. Math. Comput.* **189**, 72–78.

[245] Ren, Y., and Zhang, H., 2006, "A generalized F-expansion method to find abundant families of Jacobi elliptic function solutions of the (2 + 1)-dimensional Nizhnik-Novikov-Veselov equation," *Chaos Solitons Fractals* **27**, 959–979.

[246] Roos, H.-G., 2006, "Error estimates for linear finite elements on Bakhvalov type meshes," *Appl. Math.* **51**, 63–72.

[247] Roos, H.-G., 2012, "Robust numerical methods for singularly perturbed differential equations: A survey covering 2008-2012," *ISRN Appl. Math.* **2012**, 1–30.

[248] Roos, H.-G., Stynes, M., and Tobiska, L., 1996, *Numerical methods for singularly perturbed differential equations: Convection-diffusion and flow problems* (Springer-Verlag, New york).

[249] Roos, H.-G., Stynes, M., and Tobiska, L., 2008, *Robust Numerical Methods for Singularly Perturbed Differential Equations* (Springer, Berlin).

[250] Sahoo, S. K., and Gupta, V., 2023, "A robust uniformly convergent finite difference scheme for the time-fractional singularly perturbed convection-diffusion problem," *Comput. Math. Appl.* **137**, 126–146.

[251] Sahoo, S., and Gupta, V., 2023, "An almost second-order robust computational technique for singularly perturbed parabolic turning point problem with an interior layer," *Math. Comput. Simulation* **211**, 192–213.

[252] Sahu, S., and Mohapatra, J., 2021, "Numerical investigation of time delay parabolic differential equation involving two small parameters," *Eng. Comput.* **38**, 2882–2899.

[253] Samarskii, A., 1977, *Theory of difference schemes* (Nauka, Moscow).

[254] Schamberg, T., and Heinrichs, W., 2004, "An adaptive refinement method for convection-diffusion problems," pp. 718–719.

[255] Schlichting, H., and Gersten, K., 2003, *Boundary Layer Theory* (Springer-Verlag, Berlin).

[256] Schlissel, A., 1977, "The initial development of the WKB solutions of linear second order ordinary differential equations and their use in the connection problem," *Hist. Math.* **4**, 183–204.

[257] Sekar, E., and Tamilselvan, A., 2019, "Singularly perturbed delay differential equations of convection-diffusion type with integral boundary condition," *J. Appl. Math. Comput.* **59**, 701–722.

[258] Shahraki, M., and Hosseini, S., 2006, "Comparison of a higher order method and the simple upwind and non-monotone methods for singularly perturbed boundary value problems," *Appl. Math. Comput.* **182**, 460–473.

[259] Sharma, A., and Rai, P., 2022, "A hybrid numerical scheme for singular perturbation delay problems with integral boundary condition," *J. Appl. Math. Comput.* **68**, 3445–3472.

[260] Sharma, N., and Kaushik, A., 2023, "A uniformly convergent difference method for singularly perturbed parabolic partial differential equations with large delay and integral boundary condition," *J. Appl. Math. Comput.* **69**, 1071–1093.

[261] Shiralashetti, S., and Hanaji, S., 2021, "Hermite wavelet based numerical method for the solution of two parameters singularly perturbed non-linear benjamina-bona-mohany equation," *Sci. African* **12**, e00770.

[262] Shiromani, R., Shanthi, V., and Ramos, H., 2023, "A computational method for a two-parameter singularly perturbed elliptic problem with boundary and interior layers," *Math. Comput. Simulation* **206**, 40–64.

[263] Shiromani, R., Shanthi, V., and Ramos, H., 2023, "Numerical treatment of a singularly perturbed 2-d convection-diffusion elliptic problem with robin-type boundary conditions," *Appl. Numer. Math.* **187**, 176–191.

[264] Shishkin, G., 1983, "A difference scheme on a non-uniform mesh for a differential equation with a small parameter in the highest derivative," *USSR Comput. Math. Math. Phys.* **23**, 59–66.

[265] Shishkin, G., 1988, "A difference scheme for a singularly perturbed equation of parabolic type with discontinuous boundary conditions," *USSR Comput. Math. Math. Phys.* **28**, 32–41.

[266] Shishkin, G., and Shishkina, L., 2009, *Difference Methods for Singular Perturbation Problems* (Chapman and Hall/CRC, Boca Raton).

[267] Shivamoggi, B., 2003, "The method of strained coordinates/parameters," in *Perturbation Methods for Differential Equations*, pp. 41–111.

[268] Shooshtari, A., and Zanoosi, A. P., 2010, "A multiple times scale solution for nonlinear vibration of mass grounded system," *Appl. Math. Model.* **34**, 1918–1929.

[269] Singh, M. K., and Natesan, S., 2018, "Richardson extrapolation technique for singularly perturbed system of parabolic partial differential equations with exponential boundary layers," *Appl. Math. Comput.* **333**, 254–275.

[270] Singh, S., and Kumar, D., 2023, "Parameter uniform numerical method for a system of singularly perturbed parabolic convection-diffusion equations," *Math. Comput. Simulation* **212**, 360–381.

[271] Singh, S., Kumar, D., and Vigo-Aguiar, J., 2023, "A robust numerical technique for weakly coupled system of parabolic singularly perturbed reaction–diffusion equations," *J. Math. Chem.* **61**, 1313–1350.

[272] Skinner, L., 2011, *Singular perturbation theory* (Springer Science & Business Media).

[273] Stahara, S., Crisalli, A., and Spreiter, J., 1980, "Evaluation of a strained-coordinate perturbation procedure-nonlinear subsonic and transonic flows," in *18th Aerospace Sciences Meeting*, p. 339.

[274] Steele, C., 1976, "Application of the WKB method in solid mechanics," *Mech. Today* **3**, 243–295.

[275] Stein, R., 1965, "A theoretical analysis of neuronal variability," *Biophys. J.* **5**, 173–194.

[276] Stojanović, M., 2003, "A uniformly accurate finite elements method for singular perturbation problems," *Glasnik Matematički* **38**, 185–197.

[277] Stynes, M., and O'Riordan, E., 1991, "An analysis of a singularly perturbed two-point boundary value problem using only finite element techniques," *Math. Comput.* **56**, 663–675.

[278] Stynes, M., and Roos, H.-G., 1997, "The midpoint upwind scheme," *Appl. Numer. Math.* **23**, 361–374..

[279] Subburayan, V., 2016, "A parameter uniform numerical method for singularly perturbed delay problems with discontinuous convection coefficient," *Arab J. Math. Sci.* **22**, 191–206.

[280] Subburayan, V., and Mahendran, R., 2018, "An ε-uniform numerical method for third order singularly perturbed delay differential equations with discontinuous convection coefficient and source term," *Appl. Math. Comput.* **331**, 404–415.

[281] Sumit, Kumar, S., Kuldeep, and Kumar, M., 2020, "A robust numerical method for a two-parameter singularly perturbed time delay parabolic problem," *Comput. Appl. Math.* **39**, Paper No. 209, 25.

[282] Sun, G., and Stynes, M., 1995, "Finite-element methods for singularly perturbed high-order elliptic two-point boundary value problems. I: Reaction-diffusion-type problems," *IMA J. Numer. Anal.* **15**, 117–139.

[283] Sun, G., and Stynes, M., 1995, "Finite element methods for singularly perturbed higher order elliptic two point boundary value problems ii: convection diffusion type," *IMA J. Numer. Anal.* **15**, 197–219.

[284] Surla, K., and Jerković, V., 1990, *An exponentially fitted quadratic spline difference scheme on a non-uniform mesh* (Univ. u Novom Sadu Zb. Rad. Prirod. Mat. Fak. Ser. Mat.).

[285] Surla, K., and Uzelac, Z., 1997, "A spline difference scheme on a piecewise equidistant grid," *ZAMM* **77**, 901–909.

[286] Teofanov, L., and Roos, H.-G., 2007, "An elliptic singularly perturbed problem with two parameters I: Solution decomposition," *J. Comput. Appl. Math.* **206**, 1082–1097.

[287] Teofanov, L., and Roos, H.-G., 2008, "An elliptic singularly perturbed problem with two parameters II: Robust finite element solution," *J. Comput. Appl. Math.* **212**, 374–389.

[288] Tikhonov, A., 1948, "The dependence of the solutions of differential equations on a small parameter," *Math. Sb.* **22**, 193–204.

[289] Tikhonov, A., 1950, "On systems of differential equations containing parameters," *Math. Sb.* **27**, 147–156.

[290] Tikhonov, A., 1952, "Systems of differential equations containing small parameters multiplying some of derivatives," *Math. Sb.* **31**, 575–586.

[291] Van-Dyke, M., 1964, *Perturbation Methods in Fluid Dynamics* (Academic Press, New York).

[292] Veldhuizen, M., 1978, "Higher order methods for a singularly perturbed problem," *Numer. Math.* **30**, 267–279.

[293] Verhulst, F., 2005, *Methods and applications of singular perturbations: boundary layers and multiple timescale dynamics* (Springer Science & Business Media).

[294] Visik, M., and Lyusternik, L., 1957, "On elliptic equations containing small parameters in the terms with higher derivatives," *Dokl. Akad. Nauk, SSR* **113**, 734–737.

[295] Visik, M., and Lyusternik, L., 1957, "Regular degeneration and boundary layer for linear differential equations with small parameter," *Usp. Mat. Nauk* **12**, 3–122.

[296] Vulanović, R., 1983, "On a numerical solution of a type of singularly perturbed boundary value problem by using a special discretisation mesh," *Rev. Res. Math. Ser. Univ. Novi Sad* **13**, 187–201.

[297] Vulanović, R., and Nhan, T. A., 2020, "Robust hybrid schemes of higher order for singularly perturbed convection-diffusion problems," *Appl. Math. Comput.* **386**, 125495.

[298] Wang, D., and Zhang, H., 2005, "Further improved F-expansion method and new exact solutions of konopelchenko-dubrovsky equation," *Chaos Solitons Fractals* **25**, 601–610.

[299] Wang, Y., Meng, X., and Li, Y., 2021, "The finite volume element method on the shishkin mesh for a singularly perturbed reaction-diffusion problem," *Comput. Math. Appl.* **84**, 112–127.

[300] Xenophontos, C., and Oberbroeckling, L., 2007, "A numerical study on the finite element solution of singularly perturbed systems of reaction diffusion problems," *Appl. Math. Comput.* **187**, 1351–1367.

[301] Yadav, S., and Rai, P., 2020, "A higher order numerical scheme for singularly perturbed parabolic turning point problems exhibiting twin boundary layers," *Appl. Math. Comput.* **376**, 125095.

[302] Yadav, S., and Rai, P., 2021, "An almost second order hybrid scheme for the numerical solution of singularly perturbed parabolic turning point problem with interior layer," *Math. Comput. Simulation* **185**, 733–753..

[303] Yadav, S., and Rai, P., 2023, "A parameter uniform higher order scheme for 2d singularly perturbed parabolic convection-diffusion problem with turning point," *Math. Comput. Simulation* **205**, 507–531.

[304] Zhang, J., and Liu, X., 2019, "Superconvergence of finite element method for singularly perturbed convection-diffusion equations in 1-D," *Appl. Math. Lett.* **98**, 278–283.

[305] Zhang, J., and Yanhui, L., 2021, "High-order finite element method on a bakhvalov-type mesh for a singularly perturbed convection–diffusion problem with two parameters," *Appl. Math. Comput.* **397**, 125953.

[306] Zhang, Z., 2002, "Finite element superconvergence approximation for one-dimensional singularly perturbed problems," *Numer. Methods Partial Differ. Equ.* **18**, 374–395.

[307] Zienkiewicz, O., Gallagher, R., and Hood, P., 1975, "Newtonian and non-Newtonian viscous impompressible flow. temperature induced flows and finite elements solutions. the mathematics of finite elements and applications," *Academic Press, London*, 235–267.

List of Publications

1. Aditya Kaushik, Nitika Sharma, An adaptive difference scheme for parabolic delay differential equation with discontinuous coefficients and interior layers, *Journal of Difference Equations and Applications*, 26:11-12 (2020), 1450-1470. https://doi.org/10.1080/10236198.2020.1843645. **(SCIE, Impact Factor 1.476)**

2. Aditya Kaushik, Vijayant Kumar, Manju Sharma, Nitika Sharma, A modified graded mesh and higher order finite element method for singularly perturbed reaction–diffusion problems, *Mathematics and Computers in Simulation*, 185 (2021), 486-496. https://doi.org/10.1016/j.matcom.2021.01.006. **(SCIE, Impact Factor 2.463)**

3. Nitika Sharma, Aditya Kaushik, A hybrid finite difference method for singularly perturbed delay partial differential equations with discontinuous coefficient and source, *Journal of Marine Science and Technology-Taiwan*, 30:3 (2022), 217-236. https://jmstt.ntou.edu.tw/journal/vol30/iss3/4. **(SCIE, Impact Factor 0.667)**

4. Nitika Sharma, Aditya Kaushik, A uniformly convergent difference method for singularly perturbed parabolic partial differential equations with large delay and integral boundary condition, *Journal of Applied Mathematics and Computing*, 69:1 (2023), 1071-1093. https://doi.org/10.1007/s12190-022-01783-2. **(SCIE, Impact Factor 2.196)**

5. Nitika Sharma, Aditya Kaushik, An efficient numerical method for two-parameter singularly perturbed parabolic problems with large delay. **Communicated**

Papers presented in International Conferences

1. A parameter-uniform difference method for parabolic partial difference differential equation with discontinuous coefficients; *International Conference on Innovations in Physical Sciences*, Chaudhary Charan Singh University, Meerut. August 09-11, 2019.

2. An efficient difference scheme for parabolic delay differential equation with interior layers; *International Conference on Integrated Interdisciplinary Innovations in Engineering*, Panjab University, Chandigarh. August 28-30, 2020.

3. Robust numerical scheme for singularly perturbed convection-diffusion type partial differential equation with large delay and integral boundary condition; *International Conference on Computational Methods in Sciences and Engineering*, BITS-Pilani, Hyderabad. April, 22-24, 2022.