SOLUTIONS OF LORENZ MODEL

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I, **Deepak Yadav (2K21/MSCMAT/13)** and **Jyoti Yadav (2K21/MSCMAT/26)** student of Master in Science (Mathematics), declaring that the project's dissertation titled SOLUTIONS OF LORENZ MODEL is original and not copied from any source without proper citation and is presented by us to the Department of Applied Mathematics, Delhi Technological University, Delhi, in partial fulfilment of the requirement for the award of the degree of Master of Science in Mathematics. The awarding of a degree, diploma, associateship, fellowship, or any other equivalent title or honour has not historically been based on this work.

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CERTIFICATE

I hereby certify that the project dissertation titled SOLUTIONS OF LORENZ MODEL which is submitted by Deepak Yadav ($2K21/MSCMAT/13$) and Jyoti Yadav ($2K21/MSCMAT/26$) to Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Masters of Science in Mathematics, is a record of the project work completed by the students under my supervision. To the best of my knowledge, this work has not been submitted in full or in part to this university for any degree or diploma.

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ABSTRACT

Edward Lorenz, a mathematician and meteorologist, was the one who originally explored the Lorenz system, a system of ordinary differential equations. For specific parameter values and beginning conditions, it is noteworthy for having chaotic solutions. The Lorenz attractor, in particular, is a collection of chaotic Lorenz system solutions. In popular culture, the term ”butterfly effect” refers to the Lorenz attractor’s real-world implications, which state that in a chaotic physical system, without perfect knowledge of the initial conditions (even the minute disturbance of the air caused by a butterfly flapping its wings), we will never be able to predict its future course. This demonstrates how physically deterministic systems can yet be unpredictable due to their inherent nature.

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LIST OF ABBREVIATIONS

ODE:	Ordinary Differential Equation
DE:	Differential Equation
PDE:	Partial Differential Equation
IF:	Integrating Factor method
DDE’s:	Delay Differential Equations
SDE’s:	Stochastic Differential Equations
LPDE’S:	Linear Elliptic Partial Differential Equations
IVP:	Initial Value Problem
BVP:	Boundary Value Problem
LHS:	Left Hand Side (of an expression)
RHS:	Right Hand Side (of an expression)

Chapter 1

Preliminaries

1.1 Introduction

Mathematics is the language of nature. If we look into the nature and try to understand its basic fundamentals and the processes that keeps on occurring, right from very simple to very complex structures, math's is the tool through which we can see and understand it. For many processes that are linear in nature, we have found the direct equations that governs it but for most of them which are nonlinear in nature, we still have not got the exact expression to deal with them. But we have found out, other ways to deal with them.

This is where" Numerical Analysis" comes into picture. It gives us the tool to find the required solution with minimum error, in the situation where obtaining the exact solution is not possible. Thus, one of the most crucial areas of mathematics is numerical analysis. when it comes to the questions of dealing with real life problems.

There are various techniques available to solve various kind of problems, for example: Newton Rapson Method, Euler's Method, Finite Difference Method etc. The main point that should be kept in mind while using any numerical technique is the accuracy, stability, and consistency, without which the numerical solution technique is meaningless.

The acronym "Differential Equations" pertains to any kind of equation with variables and its derivatives regarding one or more independent variables. The linguistic make up of differential equation is quite an important and significant one. Majority of the laws in nature is expressed in this language. Subjects like. Physics, Chemistry, Biology, Medical Sciences, Astronomy find their most useful and natural expression in this languages. In all the natural process, the variables that are involved are connected with their rates of change through some basic scientific principle that governs them. This often results in a Differential Equation.

Analytical solution process of a differential equations gives us an exact solution, but when we are dealing with real life situation analytical method fails. Linear problems can be easily dealt through analytic approach, but nonlinearity is one of the main characteristics of real life problems. So mathematicians were forced to think of new ways to deal such situation which gave to the birth of Numerical Approach of solving problems. Numerical Partial Differential Equation is very vast area of study. It consists of three aspects as its major components namely-applications, mathematics and computation. There are various numerical techniques that handle Non-Linear Partial Differential Equations. Each has its own strength and weaknesses in its own domain. Some of the major methods are: Finite Difference Methods, Finite Volume Method, Finite Element Method etc.

Numerical programming tools like MATLAB, MATHEMATICA, MAPLE have become a very helpful source for computing the results of such methods, where calculating by hand have almost become an impossible scenario.

1.2 Definitions:

1. **Differential Equation:** -

Any equation containing variables and their rates of change is called as differential equation.

2. **Different Kinds of Differential Equations:**

(i) **Ordinary Differential Equation:**

An ordinary differential equation is a differential equation accompanied by only one dependent variable.

For example:

$$\frac{dy}{dx} = 2y$$

(ii) **Partial Differential Equation:**

A partial differential equation is that which comprises multiple dependent variables; as a consequence, the related derivatives are partial derivatives.

3. **Order of a Differential Equation:** The greatest derivative in the equation determines what is known as the order of the differential equation.

4. **Degree of a Differential Equation:**

A differential equation's degree is laid out by the highest derivative that appears in it, given there aren't any radicals or fractions in the equation when examining the derivative.

For example:

$$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = sinx$$

Here the degree of the equation is 3.

5. Differential Equation is said to be:

(i) **Well Posed:**

When there is a unique solution satisfying given auxiliary conditions, and the solution is completely dependent on given data, then the problem is said to be well posed.

(ii) **Well Conditioned:**

We say that a problem is well posed if a slight change in the data of a well-posed problem causes a comparatively modest change in the solution. If there is a substantial change in the solution, the problem has been termed ill conditioned.

6. Linear and Non-linear Differential Equation:

Whenever the partial derivatives of the dependent variable u and all of its corresponding ones present in the equation, the partial differential equation is referred to as linear.

Therefore, the form of a linear partial differential equation.

$$Lu = g(x, y)$$

where: Lu is a sum of terms each of which is a product of a function of x and y with u or once of its partial derivatives.

A partial differential equation shall be referred to as nonlinear if it is not linear.

7. Homogeneous and Non-homogeneous Differential Equation:

A linear equation is called homogeneous when

$$g(x, y) = 0$$

in

$$Lu = g(x, y)$$

Otherwise, we call it non homogeneous.

1.3 Partial Differential Equation:

Partial differential equations, or PDEs, are certain kinds of differential equations that involve the partial derivatives of one or more dependent variables besides one or more independent variables.

A more general formulation of a PDE for the function $u(x_1, x_2, \ldots\ldots\ldots x_n)$ can be retrieved as -

$$F(x_1, x_2, \ldots, x_n, u, u_{x1}, \ldots, u_{xn}, u_{x1x1}, \ldots, u_{xnxn}, \ldots) = 0 \tag{1.1}$$

where, u_{xj} , ..., u_{xixj} are partial derivatives of u.

The difference between linear and non-linear equations is a first crucial distinction.

If and only if F is linear with regard to u and all of its derivatives, equation (1.1) is linear; otherwise, it is non linear.

The sorts of nonlinearity are the subject of the second distinction.

1. **Semi linear**:

   When F is solely linear with respect to all of its components but nonlinear with respect to u.

2. **Quasi linear:**

   When the function F's highest order derivatives are linear with respect to u.

3. **Fully Non linear:**

   Whenever there are nonlinear interactions between F and the highest order derivatives of u.

1.3.1 Significance of Second order PDE:

Many physical problems, such as rigid body dynamics, elasticity, heat transfer, and fluid mechanics, are described by second order PDEs.. While fourth order partial differential equations (PDEs) do occasionally appear in problems, A fourth order PDE is frequently separated into two second order PDEs, along with the required boundary and initial conditions, and solved concurrently. This is similar to how we divide higher order ODEs into systems of first order equations.

Understanding how to solve second order PDEs is therefore essential for solving PDEs.

solving difficulties in the actual world is crucial

PDE of second order classified:

Take into account the second order equation in two independent variables are depicted in turn:

$$G\frac{\partial^2 u}{\partial x^2} + H\ \frac{\partial^2 u}{\partial x\ \partial y} + I\ \frac{\partial^2 u}{\partial y^2} + J\frac{\partial u}{\partial x} + K\ \frac{\partial u}{\partial y} + Lu = M \qquad (1.2)$$

where G, H, I, J, K& L are all functions of x, y, u,

We categorise the aforementioned equation (1.2) as a discriminant sign.
$D = H^2 - 4GI$

If $H^2 - 4GI\ > 0$ then Hyperbolic PDE is the term for equation (1.2).

If $H^2 - 4GI\ = 0$ then Parabolic PDE is the way of referring to equation (1.2)

If $H^2 - 4GI\ < 0$ subsequently, equation (1.2) is characterized as an elliptic PDE.

All Parabolic and Hyperbolic equations are classified as initial value problem and Elliptic equations always occur as boundary value problems.

Now we are giving some examples based on classification of PDE :

1. Wave Equation :

$$u_{tt} = c^2 \, u_{xx}$$

It is a example Hyperbolic PDE.

2. Heat Equation :

$$u_t = k \, u_{xx}$$

It is a example Parabolic PDE.

3. Laplace Equation :

$$u_{xx} + u_{yy} = 0$$

It is a example Elliptic PDE.

CHAPTER 2

LITERATURE REVIEW

2.0 Introduction

It demonstrates the many numerical and analytical strategies put forth by various authors and researchers who have either overseen this work or have carried out related research in order to solve ordinary differential equations. An account of the development of ordinary differential equations and the hunt for more efficient ways to solve them opens this section.

2.1 Ordinary Differential Equations:-

Differential equations can be used to explain almost any system that is undergoing change. In the domains of science, engineering, business, medical services, social science, and the field of economics among others as well. They are prevalent everywhere. Many investigations and mathematical studies have been conducted on differential equations and an assortment of other complex systems that can be elucidated by mathematical expressions.

When we encounter or create a differential equation, several inquiries come to mind right away:

1. Does there exist a solution to the equation?

2. If so, is the solution exclusive?

3. What is the conclusion?

4.Is there a methodical approach to solving this equation?

Ordinary differential equations (ODEs) may generally be solved using a variety of techniques. All other methods are subsets of the analytical and numerical approaches, which are the primary techniques for solving ordinary differential equations.

2.2 Ordinary Differential Equations in Antiquity

2.2.1 Overview

Physical problem-solving efforts gradually gave rise to mathematical models employing an equation that heavily relies on a function and its derivatives. Nonetheless, a select few mathematical puzzles served as the inspiration for the conceptual growth of this modern area of mathematics, known as ordinary differential equations. These problems and their solutions led to the creation of a distinct field where resolving these equations became its own objective.

Newton figured out his first differential equation in 1676. The phrase "differential equations" (aequatio differentialis, Latin) was coined by Leibniz in the same year to explain the connection between the dx and dy differentials of the two variables x and y.

In 1693, Leibniz discovered how to solve a differential equation., and Newton made the findings public. of earlier techniques of solving differential equations in the same year. This year is regarded as the beginning of differential equations as a separate branch of mathematics.

One of the first to grasp Leibniz's formulation of differential calculus were the Bernoulli brothers, Swiss mathematicians who lived in Basel (1654–1705). The brothers both decried Newton's theories and proclaimed that the contention of fluxions was a misappropriation of Leibniz' original theories because they refuted the hypothesis that

Newton had proven—that the earth and the planets travel around the sun in elliptical patterns.

They went to considerable pains to use differential calculus to refute Newton's Principia. The first book on the subject of differential equations is generally regarded as being Italian mathematician Gabriele Manfredi's 1707 work On the Invention of First-degree Differential Equations, which was written between 1701 and 1704 and published in Latin. The main source of inspiration or focus of the work was the concepts of The Bernoulli siblings and Leibniz in the vast bulk of partial differential equations and differential equations
in the 18th century, including those by Leonhard Euler, Daniel Bernoulli, Joseph Lagrange, and Pierre Laplace, looked to be based on the version developed by Leibniz.

In order to create differential equations which, in finite form it takes, happened to be integrable., Swiss mathematician Leonhard Euler first used the integrating factor in 1739.

In terms of James Maxwell's later 1871 restylized "curl" notation (test of integrability), George Green's work from around 1828 appears to relate to in some way creating the evaluation of a "integrable" or cautious field of force (or via William Thomson has some reciprocity to thermodynamics) (or perhaps Peter Tait's earlier work). In or around 1839, Green said:

“The total sum for any given part of the mass will always be the precise differential of some function if all internal forces are multiplied by the elements of their respective directions”

According to some, the Green's function of strain-energy shares a similar theme with Hermann Helmholtz free energy and Willard Gibbs thermodynamic potentials. As least as early as 1841, the terms "precise differential" and "full differential" were both in widespread usage.

2.2.2 Changes in ordinary differential equation solutions as depicted in writing: -

When Isaac Newton (1642–1727) did three class divisions of first order differential equations , the search for universal approaches to integrating differential equations got under way. Newton would use the powers of the dependent variables to represent the right side of the equation and would use an infinite series as a solution. The coefficients of the infinite series were thereafter concluded.

Despite the fact Newton pointed out ,the constant coefficient might be chosen at will and came to the conclusion the fact that a first order equation's general solution relies on a random constant, the assertion that the equation has an infinite number of specific solutions wasn't fully known until the middle of the 18th century.

James Bernoulli published a method for integrating a first-order homogeneous differential equation in 1692, and shortly after that integrating a first-order linear equation was broken down into quadratures.

During the Bernoulli dynasty, almost every elementary method for resolving differential equations A Swiss family of scientists known as the Bernoullis made significant contributions to differential equations in the late seventeenth and early eighteenth centuries. The founder of this renowned mathematical family was Nikolaus Bernoulli I (1623–1708). The most prominent Bernoulli members of the family who made substantial contributions to this novel area of differential equations are James I, John I, and Daniel I.
and of the first order was initially identified.

Leibniz made an implicit discovery of the method of variable separation in 1691 as a result of the inverse issue of tangents. On May 9, 1694, Leibniz received a message.
,John Bernoulli was the one to coin the explicit method and the phrase seperatio indeterminatarum, or separation of variables.

Leonhard Euler (1707-1783) provided the next significant development when he posed and solved the problem of *reducing a particular class of second order differential equations to that of first order.* In order to derive a second solution from a known one, He blends the finding of an integrating factor with the reduction of a second order problem to a first order equation. Additionally, Euler demonstrated that the answer to a first-order differential equation.

In a message to John Bernoulli dated September 15, 1739, Euler began his study of the homogeneous linear differential equation with constant coefficients.

which was later printed in Miscellanea Berolinensia in 1743. Euler successfully dealt with repeating quadratic components for a year before moving on to the non-homogeneous linear equation.

Equations that are integrable in finite form were the first to be produced using a strategy which calls for progressively reducing an equation's order with the assistance of integrating factors.
These equations were step-by-step reduced and then integrated by Euler. Euler utilised the integrating-by-series technique for equations that could not be integrated in a finite form.

The famous method of Euler was published in his three-volume work *Institutiones Calculi Integralis* in the years 1768 to 1770, republished in his collected works (Euler, 1913).

Joseph Louis Lagrange (1736–1813) created the adjoint equation while attempting to identify an integrating factor on account of the general linear equation. In addition to figuring out an integrating factor for the general linear equation, Lagrange established the universal replies to to an n-order homogeneous linear equation. Lagrange also invented the parameter-variation method.

Building on the work of Lagrange ,Jean Le Rond d'Alembert (1717–1783) determined the circumstances in which a linear differential equation's order could be lowered. D'Alembert was able to handle the issue of linear equations with constant coefficients, which served as the basis for his investigation of linear differential systems, by coming up with a strategy for handling the exceptional circumstances.. Partial differential equations were first introduced to D'Alembert in a 1747 dissertation on vibrating strings. He spent the most of his time working on this area.

One hundred years after Leibniz introduced the integral sign, the period of early discovery of general methods for integrating ordinary differential equations came to an end in 1775. For many problems, the formal methods were inadequate. When solutions with unique qualities were needed, it became more crucial to have criteria that ensured the existence of such solutions. Boundary value problems resulted in exploration of the Laguerre, Legendre, and Hermite polynomials owing to and they were regarded as the source of common differential equations like Bessel's equation. Modern numerical methods evolved. as a result of the study of these and other functions that are answers to hyper geometric type equations.

Hence, by 1775, the search for generic techniques for integrating ordinary differential equations came to an abrupt end as more and more focus was placed on analytical techniques and existential concerns.

The idea of generalizing the Euler method, by allowing for a number of evaluations of the derivative to take place in a step, is generally attributed to Runge (1895). Further contributions were made by Heun (1900) and Kutta (1901). The latter fully described the set of order four Runge-Kutta procedures and suggested the first order five approaches. Special methods for second order differential equations were proposed by Nyström

(1925), who also contributed to the development of methods for first order equations. Sixth order approaches weren't introduced until Huta's (1956, 1957) study.

The journal of initial value problems (IVPs) in order to ordinary differential equations: Accuracy Analysis of Numerical Solutions was published by Islam, Md. A. (2015). (ODE) The error estimators for Runge-Kutta Techniques were compared by Shampine, L.F., and Watts, H.A. in 1971.

The division of variables that is The Fourier technique is another name for the Separation of Variables method of solving partial differential equations [Renze, John and Weisstein, Eric W. This strategy works because each function must independently be a constant if the sum of functions of independent variables is a constant. [Renze, John, and Weisstein, Eric W., "Separation of Variables," 1750] L'Hospital was the first to use the method.

The Laplace transformation is a special type of integral transform created by French mathematician Pierre-Simon Laplace.. A British physicist named Oliver Heaviside methodically created the Laplace transformation. It is the most often used integral transform. This transformation technique is motivated by its simplicity in application and ease of comprehension. Laplace transformation is used in many instances to find the general answer.

This review shows that there have been significant advancements in the quest for more effective methods of solving ordinary differential equations. These techniques still need to be researched, evaluated for efficacy, and classified in accordance with their intended uses (This is particularly useful for individuals who simply want to find a more efficient method for solving their ordinary differential equations, without necessarily delving deeply into the subject matter)

The literature research also shows that the Runge-Kutta method and the Euler method have been applied extensively to numerically solve initial value problems. The Runge-Kutta approach, the midpoint method, and the Euler method are just a few of the methods that authors have used to try to quickly and reliably solve initial value problems (IVP). The essay focused on accurate fourth-order Runge-Kutta solutions to initial value problems for ordinary differential equations (ODE), as well as initial value problems (IVP) for ODE's numerical answers are accurate analysed. In various circumstances, it is investigated how to resolve starting value problems in ordinary differential equations using numerical methods. Numerical solutions were used to study initial value problems for ordinary differential equations.

In the aforementioned study, initial value problems involving ordinary differential equations are tackling via the Euler and Runge-Kutta methods without the use of any discretization, transformation, or constrictive assumptions.

Numerous studies on analytical strategies and numerical techniques for solving ODEs have been conducted, to the researcher's knowledge and based on the study., despite the fact that these topics are frequently discussed separately in the literature. The contribution of this paper is a compilation of many numerical and analytical methods for ODES solution.

2.3 Techniques for solving ordinary differential equations numerically

Ordinary differential equations (ODEs). have approximate numerical solutions that can be found using numerical techniques. They are additionally employed in "numerical integration," however this term is occasionally understood to mean computing integrals.

Many differential equations are analytically solvable, but many others are not. These differential equations result from simulating actual issues.. In general, the chances of discovering an exact mathematical solution are favourable when the modelling results in a linear differential equation. Nevertheless, non-linear differential equations are far more challenging and rarely have accurate solutions. In order to solve problems that are otherwise intractable, there is a need for numerical techniques that can approximately approximate solutions. This field has developed quickly as a result of the introduction of potent computers that can carry out computations at extremely high speeds, and there are currently numerous numerical approaches available.

Although numerical methods are advantageous because a numerical approximation to the solution is frequently sufficient for practical applications, such as in engineering, In practice, a large number of differential equations are not amenable to symbolic calculation. ("analysis"). The algorithms used in this study can be used to generate such an approximation.

CHAPTER 3

STABILITY ANALYSIS OF THE CHAOTIC LORENZ SYSTEM

3.0 Introduction

The Lorenz model exhibits extreme sensitivity to beginning circumstances and the peculiar attractor phenomena, making it a benchmark system in chaotic dynamics. Using different control techniques, it is feasible to change a peculiar attractor into a non-chaotic one even though the system has a tendency to amplify perturbations. A geometric approach is utilised to determine the controlled Lorenz system's overall stability, and to assess the stability of the controlled Lorenz system at its equilibrium locations, Routh-Hurwitz testing is used. The controlled Lorenz system is shown to have a single universally stable equilibrium point for the set of parameter values taken into account.

It is frequently required to take into account the transport of heat between bodies in thermal contact in research and engineering. Temperature differences cause heat transfers, which are carried out through a combination of electromagnetic radiation, convection, and direct molecule collisions (conduction). In this chapter, we'll talk about the thermosyphon, a heat-transfer technique that uses a fluid-filled loop and gravity-driven convection. After going over some fundamental ideas we construct a mathematical model of a thermosyphon device in fluid dynamics and find that it reduces to the classical Lorenz system upon nondimensionalization of the variables.

We start with an introduction to fluid mechanics that has been partially adapted from. Fluid mechanics is the name given to the theory of fluid (hydrodynamics) and gas (aerodynamics) flow and the continuum approach is helpful in describing. We may safely ignore the fluid's discrete particle make-up and use average attributes over these smaller volumes because even the smallest volume considered will contain a significant number of molecules, according to the continuum approximation. In this manner, the mean particle velocity v of a small volume element can be related to any point Q in the fluid. The velocity fleld v = v(r; t) represents the resulting velocity dispersion in space and time, where r stands for location.

Three key conservation laws from classical mechanics and thermodynamics, namely the laws of conservation of energy, conservation of momentum, and conservation of space, can be used to create the equations that determine the temporal and spatial evolution of these fields.

1. First one is mass conservation

2. Second is linear momentum conservation, and

3. Third is energy conservation

The Reynolds transport theorem can be used to create a continuity equation that can be used to express all three of these conservation rules. The following treatment of fluid mechanics was based on a more extensive treatment contained in. To inspire our model of the thermosyphon, we just give a brief summary of the findings here.

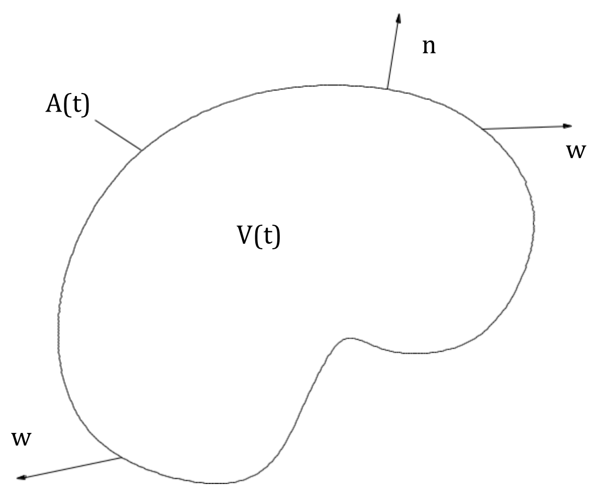


Figure 3.1: Control volume

THE LORENZ SYSTEM

The Lorenz equations are one of the most ancient instances of a system that may go through chaotic evolution specifically in parameter regimes. Its basic form belies the sophisticated behaviour it exhibits, which is one of the reasons the model has attracted so much attention since it was developed.

It can be seen in a wide range of situations, such as circuit oscillations, single-mode laser system behaviour, disc dynamo activity, and a disorderly waterwheel's motions. Using a linear approximation to the system, we examine the behaviour of the Lorenz model in this chapter, identify its equilibrium points, and talk about the stability of solutions near those equilibrium points.

1 Analytical Solutions of Non-Linear System of Differential Equations

In Lorenz model:

x: in proportion to how strongly convection is moving
y: equivalent to the disparity in temperature between the descending and ascending currents
z: in proportion to the vertical distortion
curve of the temperature from linearity
t: dimensionless time

σ, b, r are positive parameters

where,
σ is called Prandtl number.
(It involves the viscosity and thermal conductivity of the fluid)
r is the control parameter
b measures the width-to-height ratio of the connection layer

$$\frac{dx}{dt} = \sigma(-x+y) \qquad \rightarrow k(x,y,z)$$
$$\frac{dy}{dt} = rx - y - xz \qquad \rightarrow l(x,y,z)$$
$$\frac{dz}{dt} = -bz + xy \qquad \rightarrow m(x,y,z)$$

Equilibrium or Fixed Points:

$$\frac{dx}{dt} = 0 \qquad \frac{dy}{dt} = 0 \qquad \frac{dz}{dt} = 0$$

$$\sigma(-x+y)=0 \tag{1}$$
$$rx-y-xz=0 \tag{2}$$
$$-bz+xy=0 \tag{3}$$

From eq(i):

$$x=y$$

Put this in eq(iii)

$$bz=x^2$$

We get

$$z=\frac{x^2}{b} \tag{4}$$

Let $x=0, y=0$

then

$$z=0$$

Hence,

$$P_1=(0,0,0)$$

Let us assume

$$z=(r-1)$$

Put it in eq(iv)

$$z=\frac{x^2}{b}$$

⇒

$$(r-1)=\frac{x^2}{b}$$

⇒

$$x^2=b(r-1)$$

⇒

$$x^2=b(r-1)$$

$\Rightarrow$

$$x = \pm\sqrt{b(r-1)}$$

And as

$$y = x$$

Hence

$$y = \pm\sqrt{b(r-1)}$$

So, the fixed points are

$$P_1 = (0,0,0)$$

$$P_2 = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$$

$$P_3 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$$

Now Jacobian Matrix

$$J = \begin{pmatrix} k_x & k_y & k_z \\ l_x & l_y & l_z \\ m_x & m_y & m_z \end{pmatrix}$$

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

Let us say, $J = A(matrix)$

and $tr(A) = -(\sigma + 1 + b)$

Now,

$$\left|A - \lambda I\right| = \begin{vmatrix} -\sigma - \lambda & \sigma & 0 \\ r - z & -1 - \sigma & -x \\ y & x & -b - \lambda \end{vmatrix}$$

$$= -\sigma - \lambda((1 + \lambda)(b + \lambda) + x^2) - \sigma((r - z)(-b - \lambda) + xy)$$
$$= -\sigma - \lambda(b + \lambda + b\lambda + \lambda^2 + x^2) + \sigma((r - z)(b + \lambda) - xy) \quad (5)$$

Now for the fixed point $P_1(0, 0, 0)$ the characteristic equation is:

$C_A(\lambda) = -\sigma - \lambda(b + (b + 1)\lambda + \lambda^2 + 0) + \lambda((r - 0)(b + \lambda) - 0.0)$

$C_A(\lambda) = -\sigma - \lambda(\lambda^2 + (b + 1)\lambda + b) + \sigma(r(b + \lambda))$

$$= \sigma\lambda^2 - \sigma\lambda(b + 1) - \sigma b - \lambda^3 - \lambda^2(b + 1) - b\lambda + \sigma r(b + \lambda)$$

$$= -\lambda^3 - \lambda^2(\sigma + b + 1) - \lambda(\sigma b + \sigma + b - \sigma r) + \sigma b(r - 1)$$

$$= -[\lambda^3 + \lambda^2(\sigma + b + 1) + \lambda(\sigma + b + \sigma b - \sigma r) - \sigma b(r - 1)]$$

By Hit and Trial method let,

$$\lambda_1 = -b$$

then

$$C_A(\lambda_1) = -[-b^3 + b^2(\sigma + b + 1) + (-b)(\sigma + b + \sigma b - \sigma r) - \sigma b(r - 1)]$$

Hence, $\lambda_1 = -b$ is one root of characteristic equation $C_A(\lambda)$.
And other roots of characteristic equation $C_A(\lambda)$ are:

$$\lambda_2 = \frac{1}{2}(-1 - \lambda + \sqrt{D})$$

$$\lambda_3 = \frac{1}{2}(-1 - \lambda - \sqrt{D})$$

where $D = 4\sigma r + \sigma^2 - 2\sigma + 1$

Since D is quadratic equation in terms of σ
Hence

$$\sigma = \frac{2 - 4r \pm \sqrt{(4r-2)^2 - 4 \times 1 \times 1}}{2}$$

$$\sigma = \frac{2 - 4r \pm \sqrt{16r^2 + 4 - 16r - 4}}{2}$$

$$\sigma = 1 - 2r \pm 2\sqrt{r^2 - r}$$

Case (i): $0 < r < 1$
then

$$-1 - \sigma \pm \sqrt{D} < 0$$

Then all our eigen values are negative real values.

Hence, it is stable.

Case (ii): $r = 1$
then

$$\lambda_1 = -b$$

$$\lambda_2 = 0$$

$$\lambda_3 = -(1 + \sigma)$$

Hence, it is unstable.

Case (iii): $r > 1$
then

$$-1 - \sigma + \sqrt{D} > 0$$

$$-1 - \sigma - \sqrt{D} < 0$$

So, λ_2 will be positive eigen value.

Hence, it is unstable.

2 Global Stability of origin for $0 < r < 1$:

Since every trajectory moves towards the origin as $t \to \infty$, the origin is considered to be universally stable.

Let us consider the Lapnov's function:

$$H(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$$

(i) Clearly,

$$H(0, 0, 0) = 0$$

(ii) It is obvious that

$$H(x, y, z) > 0 \qquad\qquad \not{H}(x, y, z) \neq (0, 0, 0)$$

(iii)

$$\frac{dH}{dt} = \frac{\delta H}{\delta x} \cdot \frac{dx}{dt} + \frac{\delta H}{\delta y} \cdot \frac{dy}{dt} + \frac{\delta H}{\delta z} \cdot \frac{dz}{dt}$$

$$= \frac{1}{\sigma} \cdot \sigma(-x + y) + 2y(rx - y - xz) + 2z(-bz + xy)$$

$$= -2x^2 + 2xy + 2rxy - 2y^2 - 2xyz - 2bz^2 + 2xyz$$

$$= 2[(r + 1)xy - x^2 - y^2 - bz^2]$$

By completing the square method we get,

$$\frac{dH}{dt} = -2(x - \frac{(r+1)}{2}y)^2 - 2(1 + (\frac{(r+1)}{2})^2)y^2 - 2bz^2$$

As $0 < r < 1$, so the potential function is strictly decreasing.

Hence, as $t \to \infty$, $H(t) \to 0$
Thus, origin is global attracter for $0 < r < 1$.

Since our Lorenz system is symmetric in (x, y) i.e., if $(x(t), y(t), z(t))$ is a solution then $(-x(t), -y(t), -z(t))$ is also a solution.

Our other fixed points are:

$$P_2 = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, (r-1))$$

$$P_3 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, (r-1))$$

Because of the symmetry, we consider P_2 only.
From equation(v), now our characteristic equation becomes

$$C_A(\lambda) = -\sigma - \lambda(\lambda^2 + (b+1)\lambda + b + b(r-1)) + \sigma((r-r+1)(b+\lambda) - b(r-1))$$

$$= -\sigma - \lambda(\lambda^2 + (b+1)\lambda + br) + \sigma(b + \lambda - br + b))$$

$$= -\lambda^3 - \lambda^2(b+1) - \lambda br - \sigma\lambda^2 - \sigma\lambda(b+1) - \sigma br + \sigma\lambda - \sigma br + 2\sigma b$$

$$= -[\lambda^3 + (\sigma + b + 1)\lambda^2 + b(r+\sigma)\lambda + 2\sigma b(r-1)] \qquad \text{eq(vi)}$$

Case (i):

If $1 < r < 1.3$

then all our eigen values are negative real values.

Hence, it is stable.

3 Hopf - Bifurcation $r = r^*$

A Hopf bifurcation occurs when the real parts of the eigen values vanishes at $r = r^*$
From eq (vi),

$$C_A = -[\lambda^3 + (\sigma + b + 1)\lambda^2 + b(r + \sigma)\lambda + 2\sigma b(r - 1)]$$

Plugging in for $r*$ and solving this then we get eigen values are:

$$\lambda_1 = -(\sigma + b + 1)$$

$$\lambda_{2,3} = \pm i\sqrt{\frac{2\sigma(\sigma + 1)}{\sigma - b - 1}}$$

This is a Hopf Bifurcation.

The value of r^* can be easily determined considering purely imaginary roots.

Let $\lambda = ir$ for $\mu \in IR$ plugging this back into the characteristic equation.

$$C_A(ir) = -[-ir^3 - (\sigma + b + 1)\mu^2 + ib(r + \sigma)\mu + 2\sigma b(r - 1)] = 0$$

Taking real and imaginary parts we have:

$$\mu^2 = \frac{2\sigma b(r - 1)}{\sigma + b + 1}$$

and

$$r^3 = rb(\sigma + r)$$

As $r \neq 0$, then solving these two equations for r and r^* we have

$$r^* = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

Lorenz's results are based on the following values of the physical parameters taken from Saltzman's paper (1962).

$$\sigma = 10 \text{ and } b = \frac{8}{3}$$

Therefore we get

$$r^* = \frac{470}{19} \approx 24.74$$

Case (ii): If

$$1.3 < r < 24.74$$

then

$$\lambda_1 < 0$$

and λ_2, λ_3 are complex roots with negative real parts.

Hence, it is stable.

Case (iii): If

$$r > 24.74$$

then

$$\lambda_1 < 0$$

and λ_2, λ_3 are complex roots with positive real part.

Hence, it is unstable.

Summary:

Fixed Points $\rightarrow$	P_1	P_2	P_3
$0 < r < 1$	Asymptotic Stable	Does not exist	Does not exist
$1 < r < 1.3$	Unstable	Asymptotic Stable	Asymptotic Stable
$1.3 < r < 24.74$	Unstable	Asymptotic Stable	Asymptotic Stable
$r > 24.74$	Unstable	Unstable	Unstable

The standard fourth order Runge-Kutta algorithm with a standard step size of h = 0:00625 is employed in this study. An endless collection of R3 trajectories with variable properties depends on R's value is the solution. One trajectory is chosen from an infinite array by choosing an initial condition.

Diagrams depict the behaviour of the Lorenz system for various values of R as time series and phase space representations, respectively. As soon as the flow of fluid stops for R = 0, the system instantly achieves mechanical equilibrium, leaving conduction to provide thermal equilibrium. R = 10 causes the system to rapidly evolve to one of the two stable flxed sites because at that moment convective motion is encouraged due to the significant vertical temperature difference. The two flxed sites are unstable for R = 28 and the trajectory is caught on an odd attractor.

The system repeatedly jumps from one nearly periodic orbit to the next, exhibiting the typical, seemingly random behaviour of a chaotic system. Keep in mind that the growth of the horizontal temperature difference y and the fluid flow x are intimately connected for both R = 10 and R = 28 in the progression of its development.

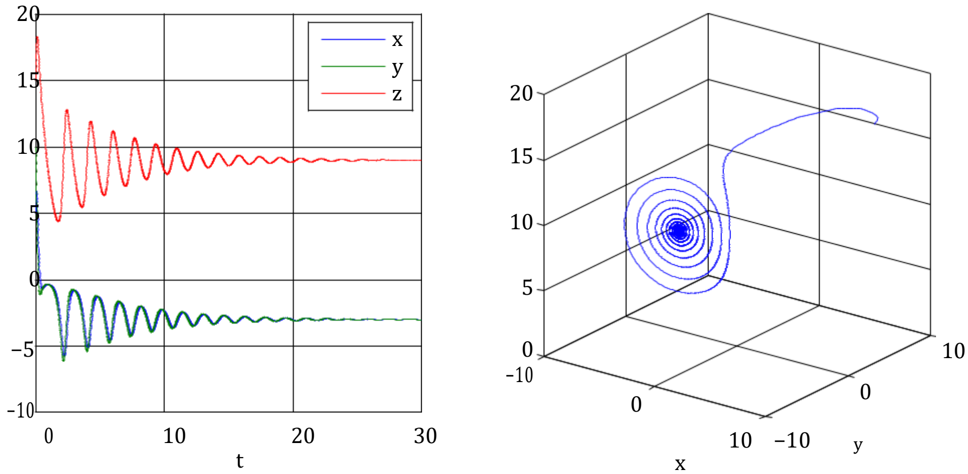


Figure 3.2: Lorenz System with R=10

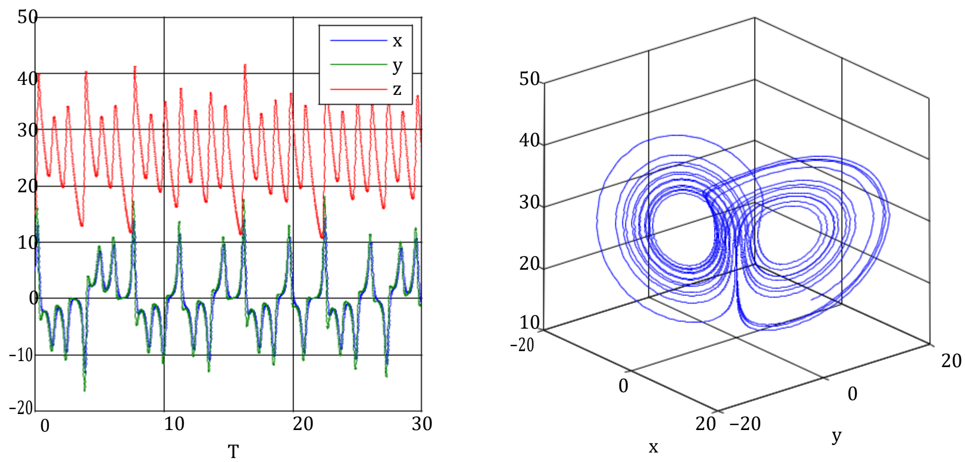
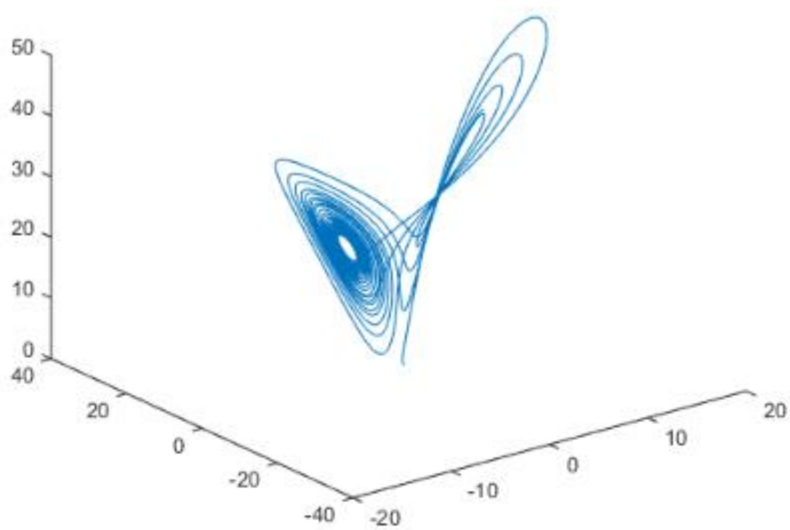


Figure 3.3: Lorenz System with R=28

We create a histogram for each value of R by cutting an x, y, or z interval and counting the number of times the system goes into each bin.

As it follows its trajectory, the system regularly returns to that value. Modified Feigenbaum diagramming is another name for this style of plot development.
A closer look at the region between 0 and R and 15 when chaotic behaviour starts.

When R rises, we observe that the equilibrium point shifts farther from the starting point and in phase space, the system trajectories stretch out considerably.



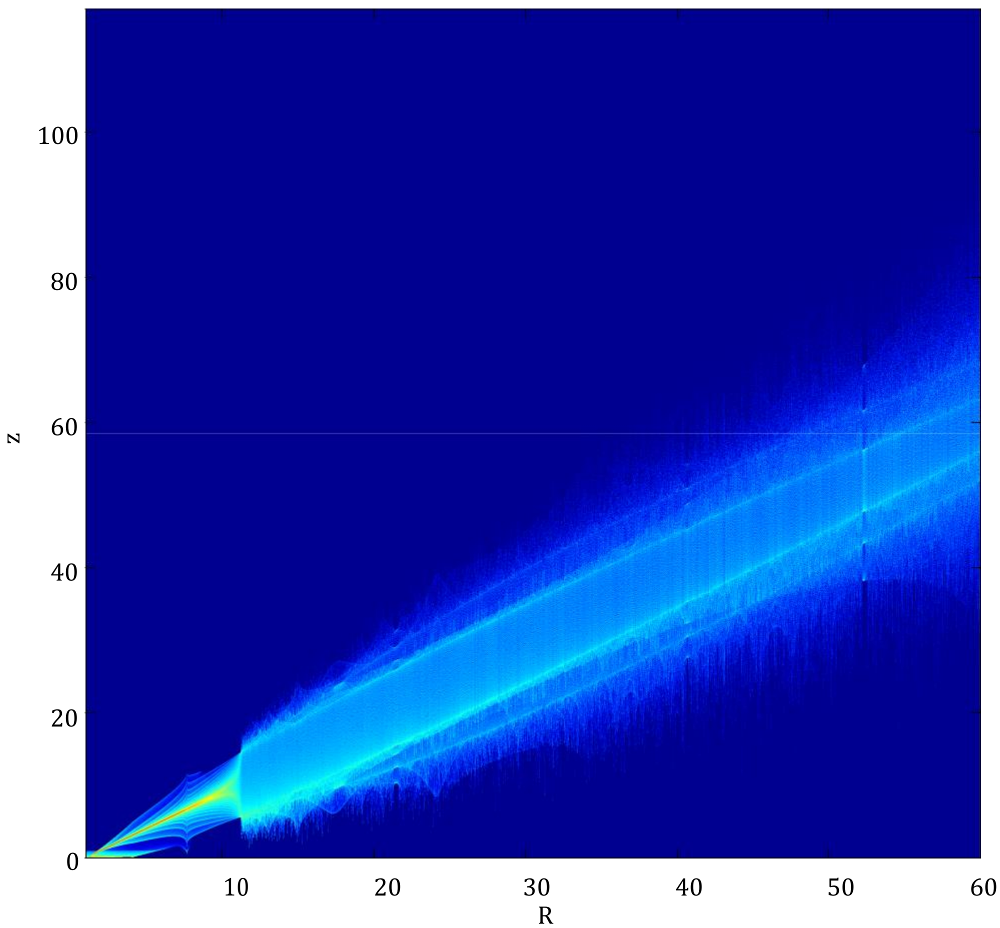


Figure 3.4: Behavior of the z-coordinate for $0 < R < 60$

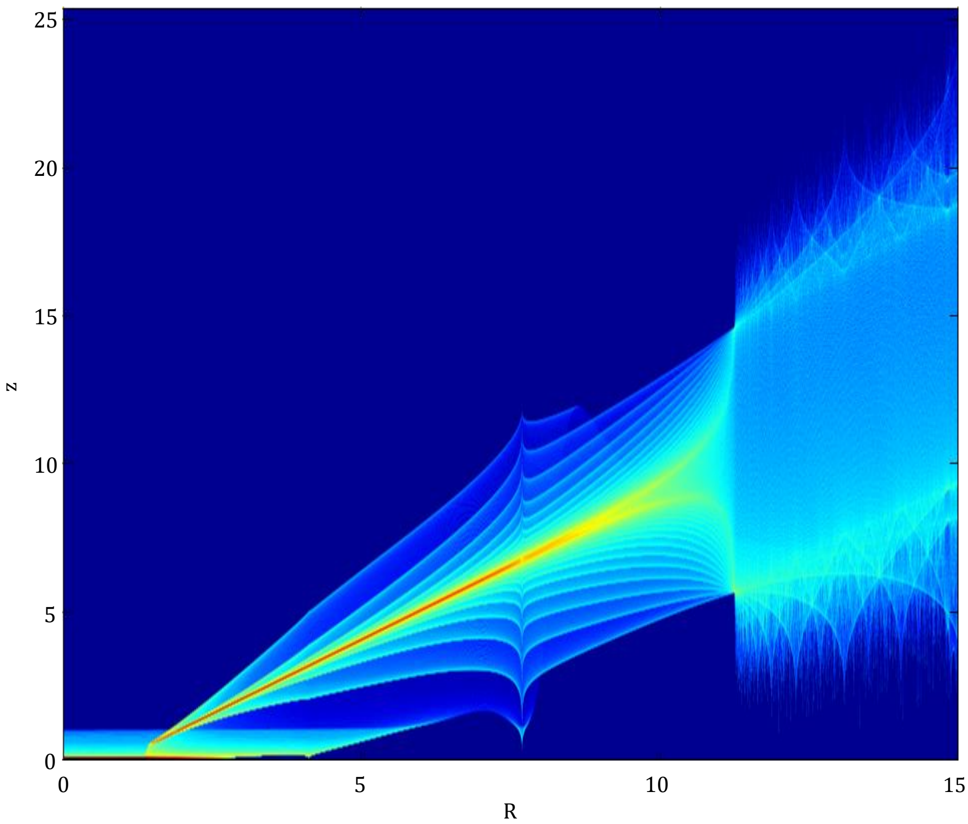
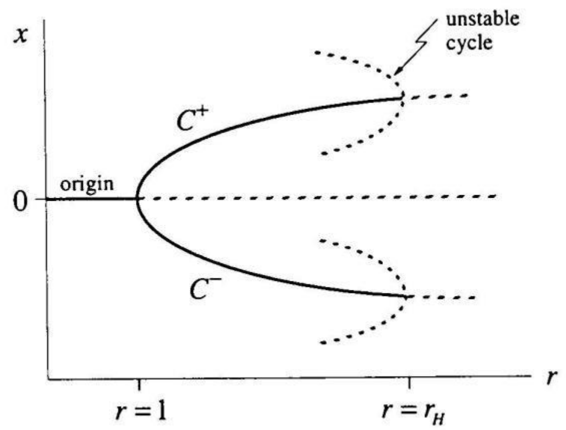


Figure 3.5: Behavior of the z-coordinate for $0 < R < 15$



(Diagram depicting Partial bifurcation)

Instability characterises all limit cycles for $r > r_H$

therefore this second attractor must have some peculiar characteristics. As a result, unstable objects are one after another are repelling the trajectories for $r > r_H$ They cannot move in this set for all time without intersecting even if they are constrained to a restricted set with zero volume at the same time—what an odd attractor!

A Strange Attractor in Chaos
a Strange Attractor in Chaos

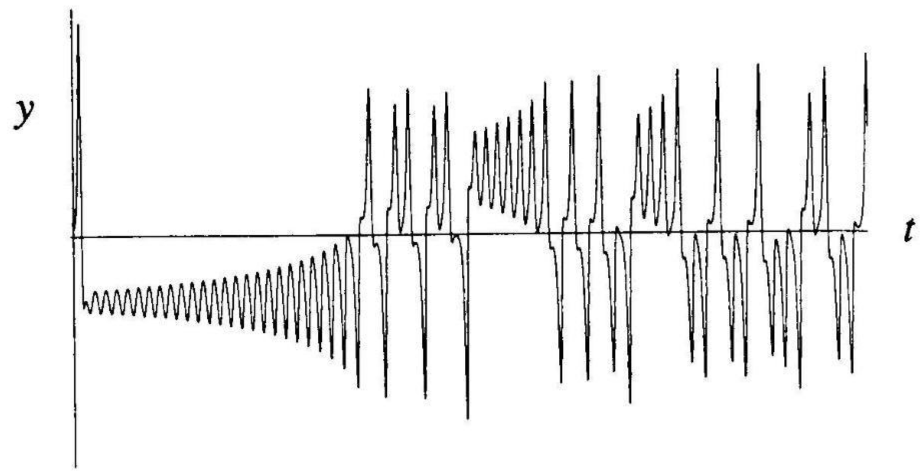
When

$$(x_0, y_0, z_0) = (0,1,0)$$

and σ = 10, b = 8/3, and r = 28, Lorenz took this into account.

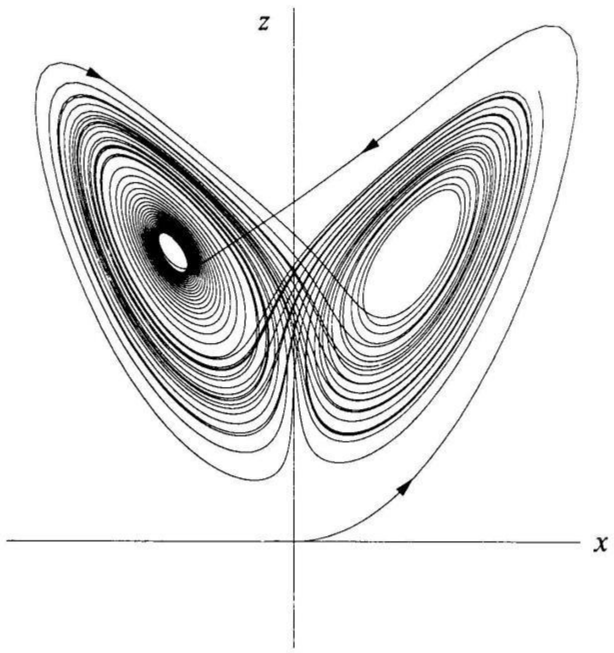
$r_H = \sigma(\sigma + b + 3)/(\sigma - b - 1) = 24.74$, therefore $r > r_H$.

The final answer y(t) resembles...



Following a short phase, the solution settles into an erratic oscillation that lasts as long as t but never precisely repeats. A periodic oscillation is present.

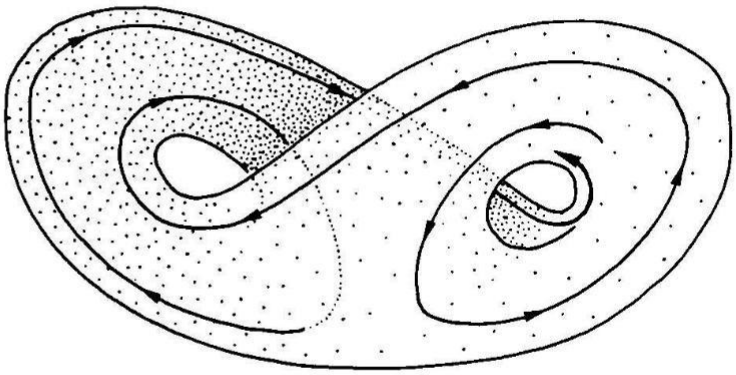
Lorenz found when the answer is viewed as a trajectory in phase space, a lovely structure manifests itself. For example , the well-known but-terfly wing pattern can be seen when plotting x(t) versus z(t).



- The 3-dimensional route was projected onto a 2-dimensional plane, which caused the trajectory to appear to cross itself repeatedly. There are no crossings in 3D

- From one cycle then to next, the counting of circuits made on either aspect varies erratically. The order for the number of circuits in each lobe resembles a random sequence in many ways!

- A thin set that mimics a pair of butterfly wings appears to be where the trajectory settles when viewed in all three dimensions. This attractor, which we refer to as a peculiar attractor, can be represented schematically as...



Conclusion

One could utilize the Lorenz System as a straightforward introduction to chaos theory after studying it, before moving on to far more complex systems. In fact, there have been a lot more, more intricate, higher dimensional expansions to Lorenz's theory.

We learn that chaos is a physical system's default state and that much more research is necessary to comprehend what's happening and how to compute these systems. For the time being, it appears that our options are limited to numerical system analysis, which still explains a lot about the system and the butterfly effect, which describes the sensitivity to initial conditions.

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