CLASSES OF OPERATORS ON BANACH SPACES

A DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE AWARD OF THE DEGREE
OF

MASTERS OF SCIENCE
IN
APPLIED MATHEMATICS

Submitted by

SHAGUN SINGH (2K21/MSCMAT/48)
SAWAN (2K21/MSCMAT/57)

Under the supervision of

JAMKHONGAM TOUTHANG

APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi 110042

MAY, 2023

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042

CANDIDATE's DECLARATION

We, SHAGUN SINGH, SAWAN, Roll No's – 2K21/MSCMAT/48, 2K21/MSCMAT/57 students of M.Sc (Applied Mathematics),hereby declare that the project Dissertation titled "Classes of Operators on Banach Spaces" which is submitted by us to the Applied Mathematics, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of degree of Masters of Science, is original and not copied from any source without proper citation. This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship or other similar title or recognition.

Place: Delhi — Shagun Singh

Date: 24.05.23 — Sawan

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042

CERTIFICATE

I hereby certify that the Project Dissertation titled "Classes of Operators on Banach Spaces " which is submitted by Shagun Singh, Sawan, Roll No's – 2K21/MSCMAT/48, 2K21/MSCMAT/57, Applied Mathematics , Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Masters of Science, is a record of the project work carried out by the students under my supervision. To the best of my knowledge this work has not been submitted in part or full for any Degree or Diploma to this University or elsewhere.

Place: Delhi — Jamkhongam Touthang

Date: 24.05.2023 — **Supervisor**

DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(For merly Delhi College of Engineering)
Bawana Road, Delhi-110042

ACKNOWLEDGEMENT

We wish to express our sincerest gratitude to Dr Jamkhongam Touthang for his continuous guidance and mentorship that he provided us during the project. He showed us the path to achieve our targets by explaining all the tasks to be done and explained to us the importance of this project as well as its industrial relevance. He was always ready to help us and clear our doubts regarding any hurdles in this project. Without his constant support and motivation, this project would not have been successfu l.

Place: Delhi — Shagun Singh

Date: 24.05.2023 — Sawan

Abstract

The study of operators on Banach spaces forms a fundamental branch of functional analysis, with broad applications in various areas of mathematics and physics. This abstract provides an overview of different classes of operators that arise in the context of Banach spaces.

First, we introduce the notion of a bounded linear operator, which is a fundamental class of operators on Banach spaces. Bounded linear operators possess important properties such as continuity and preservation of vector space operations, making them essential in the study of linear transformations .

Next, we delve into more specialized classes of operators, starting with compact operators. Compact operators are characterized by their ability to map bounded sets to relatively compact sets, playing a significant role in the theory of integral equations, spectral analysis, and compactness arguments.

We then explore the realm of self-adjoint operators, which are operators that coincide with their adjoints. Self-adjoint operators possess real spectra and have applications in quantum mechanics, where they correspond to observables with real eigenvalues.

Moving further, we discuss the class of normal operators, which generalize self-adjoint operators and include both self-adjoint and unitary operators as special cases. Normal operators have a rich spectral theory and arise naturally in areas such as quantum mechanics and signal processing.

Additionally, we touch upon the class of positive operators, which are operators that preserve positivity. Positive operators have connections to operator algebras, functional analysis, and the theory of partial differential equations.

Lastly, we examine the concept of bounded invertible operators, known

as isomorphisms, which establish bijective mappings between Banach spaces. Isomorphisms play a central role in the study of isomorphic properties, such as the Banach space isomorphism theorems and isomorphic embeddings.

Throughout this abstract, we highlight the interplay between different classes of operators on Banach spaces, emphasizing their properties, applications, and connections to other areas of mathematics and physics. Understanding these various classes of operators is crucial for developing advanced techniques in functional analysis and for investigating problems across diverse scientific disciplines.

Contents

Chapter 1

INTRODUCTION

We provide a brief context and inspiration of the questions that this thesis asks try to study in this chapter. In addition, we reiterate some definitions and discoveries that will be used in the future and present some notation. We provides a chapter-by-chapter summary of the main findings at the conclusion.

1.1 HISTORY

Mathematicians have long focused on the transformations that serving angles and lengths. An isomorphism is a transformation that maintains the distinction relationship between each pair of elements in a space (metric space, normed linear space). Translation, rotation, and reflection are some examples of similarity. try on Euclidean space. The Fourier transform on L2(R) is another example. The distance conservation criterion makes it easy to define multiple isometric lines.

characteristics of risk switching, such as injection and continuity. Embrace The geometry and structure of Banach spaces require a deep understanding of their isobars. The first to question the composition of a Subjective linear isometri cover a particular Banach space is Stefan Banach Researchers have also begun to study isometric lines of other Banach spaces in the same time period. The projection type is a different type from the transformations needed to understand the architecture of a Banach space. Exponential matrices are simple examples of Euclidean spaces projection. Any diagonalization matrix can be decomposed into a linear matrix sum of power matrices, according to a standard linear algebra result. In the past and more recently, there has been much interest in efforts to characterize projections with the desired quality, such as mononorm projections.

1.2 MOTIVATION

Assume that E is a Banach space. B(E) and G(E) stand for, respectively, the Banach space of all bounded linear operators and the set of all surjective linear isometries on E. Let T B(E) such that For any x E, Tx corresponds to the impact of a subjective linear isometry on x, i.e., there is a Tx G(E) such that T(x) = Tx(x). B(E) and G(E) stand for, respectively, the Banach space of all bounded linear operators and the set of all surjective linear isometries on E.

Definition 1.2.1. Let the algebraic closure of S be defined as

$$S \equiv \{T \in B(E) : Tx \overset{a}{\epsilon} Sx, \forall\ x \in E\}$$

where Sx = {Sx : S ∈ S}.
Also, S ⊂ B(E). If $S = S$, the subset S is called algebraically reflexive. Clearly $S \subseteq S$ Local maps are components of the algebraic closure of S. If we logically must have T S for every map T that is locally associated with S, then algebraically reflexive.

Since every local surjective isometry is surjective. We should keep in mind that an isometry is an isometry. We can see from the description above, if E has finite dimensions, then $G(E)$ is algebraically reflexive; if E is an infinite dimensional Hilbert space, then $G(E)$ is not algebraically reflexive.

Let $G^2_n(E) = TG(E) : Tn = 1$ for $n > N$.
An isometry of order n is an operator $\mathrm{T_n}(E)$. If $\mathrm{G}(\mathrm{C}(X))$ is algebraically reflexive for a compact Hausdorff space, then $\mathrm{G_2}(C(X))$ is also algebraically reflexive. The strong Banach-Stone property of E is a Banach space, and $'OnC_0(X,$ E), where X is a first countable locally compact Hausdorff space and E is a Banach space, this conclusion was expanded to include isometries of order n.

1.3 Research objectives

Definition 1.3.1. A projection P on a Banach space E is referred to as a generalized bi-circular projection if $\alpha \in T \mid \{1\}$ exists such that $P + \alpha(I - P)$ is an isometry on E. In this case, T stands for the complex plane's unit circle.

Any y generalized bi-circular projection, or a projection P such that || P|| = $||I - P|| = 1$, was demonstrated to be bi-contractive. Additionally, if and only if a projection is orthogonal, it qualifies as a generalized bi-circular
Definition 1.3.2. Let $C_{11}(X)$'s subspace A be A. If there is fA such that $|f\,(x_1)| = |f\,(x_2)|$ for pair of different points x_1, x_2 belonging to X, then we say that A is strongly separating.

Definition 1.3.3. Given any subsets of K_E with positive distance $d(A, B)$ = inf $||ab||$: aA, bB, there exists $afA_U\,(K_E)$ such that $|f(x)|1$ for every xA and $|f(y)|1$ for every yB. This closed subalgebra of $^2\ ^2C_U\,(K_E)$ is said to be weakly normal.

In the first section of the thesis, we look at the issue of the sets' algebraic reflexivity:

1. The collection of all surjective linear isometries between $C_0(X)$ subspaces with high degree of separation.

2. The collection of every subjective linear isometry between $C_U\,(K_E)$ subalgebras with weakly normal closed substructures.
3. The collection of each and every surjective linear isometry between $A_U\,(K_E)$ subalgebras whose members vanish at 0 .
4. Each and every subjective linear isometry on the domain of two-time continuously differentiable functions.
5. All collection of finite order surjective linear isometries on the space of 2-times continuously differentiable functions.

Definition 1.3.4. If $e^{i\vartheta T}$ is an isometry for each and every R, then operator $T \in B(X)$ is Hermitian.
Numerous writers have studied hermitian operators on various complex Banach spaces.

1.4 Preliminaries and basic results

Let E and F will be taken as Banach spaces. The set of all surjective linear isometries, from E to F, and the Banach space of all Bounded linear operators, are denoted, respectively, by $B(E,F)$ and $G(E,F).B(E,\ E)$ and $G(E,E)$ are denoted by $B(E)$ and $G(E,E)$, respectively, if $E = F$.

Let K stand for the real/complex number space.The set of all continuous functions with K values on a locally compact Hausdorff space is called X that vanishes at infinity is represented by $C_0(X).f : X \to$ K, which we reviewed earlier, as disappear at infinity if for all > 0, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact. Let K stand for the real/complex number space.The set of all continuous functions with K values on a locally compact Hausdorff space is called X that vanishes at infinity is represented by $C_0(X).f : X \to$ K, which we reviewed earlier, as disappear at infinity if for all > 0, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact. Let K stand for the real/complex number space.The set of all continuous functions with K values on a locally compact Hausdorff space is called X that vanishes at infinity is represented by $C_0(X).f : X \to$ K, which we reviewed earlier, as disappear at infinity if for all > 0, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact . Let K stand for the real/complex number space.The setof all continuous functions with K values on a locally compact Hausdorff space is called X that vanishes at infinity is represented by $C_0(X).f : X \to$ K, which we reviewed earlier, as disappear at infinity if for all > 0, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact. Let K stand for the real/complex number space.The set of all continuous functions with K values on a locally compact Hausdorff space is called X that vanishes at infinity is represented by $C_0(X).f : X \to$ K, which we reviewed earlier, as disappear at infinity if for all > 0, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact.

Definition 1.4.1. Let $C_0(X)$'s subspace A be A. If all of the functions in A reach their maximum on a subset U of X, then U is a boundary for A. The only minimal closed boundary for A is the Shilov boundary, indicated by the symbol A.

The next two theorems describe the design [11] f a strongly separating subspace of $C_0(X)$ into $C_0(Y)$'s into and onto linear isometries.

Theorem 1.4.2. T is the linear isometry of the linear subspace A of $C_0(X)$ into $C_0(Y)$ that separates strongly. Therefore, continuous map $h : Y_0$ onto $\sigma_0 A$, a continuous map $a : Y_0 \to K$, such that $|a(y)| = 1$ for all $y \in Y_0$, and

$$Tf(y) = a(y)f(h(y))forally \in Y_0$$

and for every $f \in A$ are present. These maps are boundaries for $T(A)$. Additionally, Y_0 is closed if σOA is compact.
Theorem 1.4.3. T represents the linear isometry of the linear subspace B of $C_0(Y)$ onto homeomorphism h of $\sigma_0 B$ onto $\sigma_0 A$ exist, ensuring mod of $a(y)$ is 1 for every $y \in \sigma_0 B$ and

$$Tf(y) = a(y)f(h(y))foreveryy \in \sigma_0 B, f \in A.$$

$A_u\left(K_E\right)$ and $A^0\left(K_E\right)$ of $C_u\left(K_E\right)$.

Theorem 1.4.4. Let X and Y are Banach spaces and T is a linear surjective isometry: $A_u\left(K_x\right) \to A_u\left(K_Y\right)$. Then, for every $y \in K_r$ and for every $f \in A_U\left(K_X\right)$, there exists a uniform homeomorphism h from K_Y onto K_X and a function $a \in C_U\left(K_Y\right)$ such that mod of $a(y)$ is 1 for every $y \in K_Y$ and $Tf(y) = a(y)f(h(y))$ for all $y \in K_Y$

Definition 1.4.5
. 1. A projection P on a Banach space E is known as a generalized bi-circular projection if there is $\alpha \in T \mid \{1\}$ such that $P + \alpha(I - P)$ is an isometry on E.
2. The projection P on a Banach space E is referred to as a bi-circular projection if $P + \alpha(I - P)$ is an isometry on E for $all\alpha \in T$
Remark 1.4.6. Let $\alpha \in {}^2T\backslash\{1\}$ and $T \in G(E)$ such that $P + \alpha(I - P) = T^2$ if P is a bi-circular projection on a Banach space E. We'll refer to the isometry T as the isometry connected to P.

Chapter 2

OPERATORS

2.1 Bounded linear operators

A linear map or linear operator T between real (or complex) linear spaces X, Y is a function $T : X \to Y$ such that

$$T(\lambda x + \mu y) = \lambda Tx + \mu Ty \quad for all \lambda, \mu \in \mathrm{R}(or \mathrm{C}) and x, y \in X$$

A linear map $T : X \to X$ is called a linear transformation of X, or a linear operator on X. If $T : X \to Y$ is one-to-one and onto, then we say that T is nonsingular or invertible, and define the inverse map $T^{-1} : Y \to X$ by $T^{-1}y = x$ if and only if $Tx = y$, so that $TT^{-1} = I, T^{-1}T = I$. The linearity of T implies the linearity of T_{-1}. If X, Y are normed spaces, then we can define the notion of a bounded linear map. As we will see, the boundedness of a linear map is equivalent to its continuity.

Definition 2.1.1 Let X and Y be two normed linear spaces. We denote both the X and Y norms by k. A linear map $T : X \to Y$ is bounded if there is a constant $M \geq 0$ such that

kTxk $\leq M$kxk for all $x \in X$.

If no such constant exists, then we say that T is unbounded. If $T : X \to Y$ is a bounded linear map, then we define the operator norm or uniform norm kTk of T by

kTk $= \inf\{M \mid$ kTxk $\leq M$kxk for all $x \in X\}$

We denote the set of all linear maps $T : X \to Y$ by $L(X, Y)$, and the set of all bounded linear maps $T : X \to Y$ by $\mathrm{B}(X, Y)$. When the domain and range spaces are the same, we write $\mathrm{L}(X, X) = \mathrm{L}(X)$ and $\mathrm{B}(X, X) = \mathrm{B}(X)$

Equivalent expressions for kTk are:

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}; \quad \|T\| = \sup_{\|x\| \leq 1} \|Tx\|; \quad \|T\| = \sup_{\|x\|=1} \|Tx\|.$$

We also use the notation $\mathrm{R}^{m \times n}$, or $\mathrm{C}^{m \times n}$, to denote the space of linear maps from R^n to R^m, or C^n to C^m, respectively.

NOTE: The linear $mapA : \mathrm{R} \to \mathrm{R}$ defined by $Ax = ax$, where $a \in \mathrm{R}$, is bounded, and has norm $\|A\| = |a|$

Example 2.1.1 The identity map $I : X \to X$ is bounded on any normed space X, and has norm one. If a map has norm zero, then it is the zero $map0x = 0$.

Linear maps on infinite-dimensional normed spaces need not be bounded.

Example 2.1.2 Let $X = C^\infty([0,1])$ consist of the smooth functions on $[0,1]$ that have continuous derivatives of all orders, equipped with the maximum norm. The space X is a normed space, but it is not a Banach space, since it is incomplete.
The differentiation operator $Du = u^0$ is an unbounded linear $mapD : X \to X$. For example, the function $u(x) = e^{\lambda x}$ is an eigenfunction of D for any $\lambda \in \mathrm{R}$, meaning that $Du = \lambda u$.
Thus $\|Du\|/\|u\| = |\lambda|$ may be arbitrarily large. The unboundedness of differential operators is a fundamental difficulty in their study.

Suppose that $A : X \to Y$ is a linear map between finite-dimensional real linear spaces X, Y with $dimX = n, dimY = m$.
We choose bases $\{e_1, e_2, \ldots, e_n\}$ of X and $\{f_1, f_2, \ldots, f_m\}$ of Y. Then

$$A(e_j) = \sum_{i=1}^{m} a_{ij} f_i$$

for a suitable $m \times n$ matrix (a_{ij}) with real entries. We expand $x \in X$ as

$$x = \sum_{i=1}^{n} x_i e_i$$

where $x_i \in \mathrm{R}$ is the i th component of x. It follows from the linearity of A that

$$A\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{i=1}^{m} y_i f_i$$

where

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

Example 2.1.3 Let $X = {}^\infty(\mathrm{N})$ be the space of bounded sequences $\{(x_1, x_2, \ldots)\}$ with the norm

$$\|(x_1, x_2, \ldots)\|_\infty = \sup_{i \in N} |x_i|$$

A linear $mapA : X \to X$ is represented by an infinite matrix $(a_{ij})^\infty{}_{i,j}$, where

$$y_i = \sum_{j=1}^{n} a_{ij} x_j.$$

Example 3 Let $X = $ (N) be the space of bounded sequences $\{(x_1, x_2 \ldots.)\}$ with the norm

$$\|(x_1, x_2, \ldots)\|_\infty = \sup_{i \in N} |x_i| .$$

A linear map $A : X \to X$ is represented by an infinite matrix $(a_{ij})^\infty{}_{ij=1}$, where

$$(Ax)i = \sum j = 1^\infty a_{ij} x_j.$$

In order for this sum to converge for any $x \in \in^\circ$ (N), we require that

$$\sum_{j=1}^{\infty} |a_{ij}| < \infty$$

for each $i \in$ N, and in order for Ax to belong to œ(N), we require that

$$\sup i \in N \left\{ \sum j = 1^\infty |a_{ij}| \right\} < \infty.$$

Then A is a bounded linear operator on ' (N), and its norm is the maximum row sum,

$$\|A\| \infty = \sup_{i \in N} \left\{ \sum j = 1^\infty |a_{ij}| \right\} .$$

Example 2.1.4 Let $X = C([0, 1])$ with the maximum norm, and

$$k : [0, 1] \times [0, 1] \to R$$

be a continuous function. We define the linear Fredholm integral operator $K : X \to X$ by

$$K f(x) = \int_0^1 k(x, y) f(y) dy$$

Then K is bounded and

$$\|K\| = \max_{0 \le x \le 1} \left\{ \int_0^1 |k(x, y)| dy \right\} .$$

This expression is the "continuous" analog of the maximum row sum for the ∞-norm of a matrix. For linear maps, boundedness is equivalent to continuity.

Theorem2.1.1 A linear map is bounded if and only if it is continuous.

Proof.
First, suppose that $T : X \to Y$ is bounded. Then, for all $x, y \in X$, we have

$$\text{k}Tx - Ty\text{k} = \text{k}T(x - y)\text{k} \le M\text{k}x - y\text{k},$$

where M is a constant for which (5.1) holds. Therefore, we can take $^\delta = \epsilon/M$ in the definition of continuity, and T is continuous.

Second, suppose that T is continuous at 0 . Since T is linear, we have $T(0) = 0$. Choosing $\epsilon = 1$ in the definition of continuity, we conclude that there is a $\delta > 0$ such that k$T x$k ≤ 1 whenever kxk $\le \delta$. For any $x \in X$, with $x6 = 0$, we define $x^\sim$ by

$$\tilde{x} = \delta \frac{x}{\|x\|}$$

Then kx~k $\leq \delta$, so kTx~k ≤ 1. It follows from the linearity of T that

$$\|Tx\| = \frac{\|x\|}{\delta}\|T\tilde{x}\| \leq M\|x\|$$

where $M = 1/\delta$. Thus T is bounded.
The proof shows that if a linear map is continuous at zero, then it is continuous at every point. A nonlinear map may be bounded but discontinuous, or continuous at zero but discontinuous at other points.

The following theorem, sometimes called the BLT theorem for "bounded linear transformation" has many applications in defining and studying linear maps.

Theorem 2.1.2 (Bounded linear transformation) Let X be a normed linear space and Y a Banach space. If M is a dense linear subspace of X and

$$T : M \subset X \to Y$$

is a bounded linear map, then there is a unique bounded linear map $\bar{T}$: $X \to Y$ such that $Tx = Tx$ for all $x \in M$. Moreover, $\|\bar{T}\| = \|T\|$. Proof. For every $x \in X$, there is a sequence (x_n) in M that converges to x. We define

$$\overline{Tx} = \lim Tx_n \cdot n \to \infty$$

This limit exists because (Tx_n) is Cauchy, since T is bounded and (x_n) Cauchy, and Y is complete. We claim that the value of the limit does not depend on the sequence in M that is used to approximate x. Suppose that (x_n) and $({x_n}^0)$ are any two sequences in M that converge to x. Then $\|x_n - x'_n\| \leq \|x_n - x\| + \|x - x'_n\|$ and, taking the limit of this equation as $n \to \infty$, we see that

$$\lim_{n\to\infty} \|x_n - x'_n\| = 0$$

It follows that

$$\|Tx_n - Tx'_n\| \leq \|T\|\,\|x_n - x'_n\| \to 0 \quad asn \to \infty.$$

Hence, (Tx_n) and (Tx'_n) converge to the same limit. The map T is an extension of T, meaning that $Tx = Tx$, for all $x \in M$, because if $x \in M$, we can use the constant sequence with $x_n = x$ for all n to define $\bar{T}x$. The linearity of $\bar{T}$ follows from the linearity of T. The fact that $\bar{T}$ is bounded follows from the inequality

$$\|\bar{T}x\| = \lim_{n\to\infty} \|Tx_n\| \leq \lim_{n\to\infty} \|T\|\,\|x_n\| = \|T\|\|x\|$$

It also follows that $\|\bar{T}\| \leq \|T\|$. $-$ Since $Tx = Tx$ for $x \in M$, we have $\|\bar{T}\| = \|T\|$. Finally, we show that T is the unique bounded linear map from X to Y that point in X, We choose a sequence (x_n) ine M that converges to x. Then, using the coincides with T on M. Suppose that T is another such map, and let x be any continuity of T, the fact that T is an extension of T, and the definition of T, we see that

$$Tx = Tx = \lim_{n\to\infty} = \lim_{n\to\infty} Tx_n = Tx$$

We can use linear maps to define various notions of equivalence between normed linear spaces.

Definition 2.1.3 Two linear spaces X, Y are linearly isomorphic if there is a oneto-one, onto linear map $T : X \to Y$. If X and Y ar normed linear spaces and T, T^{-1} are bounded linear maps, then X and Y are topologically isomorphic. If T also preserves norms, meaning that $\mathrm{k}Tx\mathrm{k} = \mathrm{k}x\mathrm{k}$ for all $x \in X$, then X, Y are isometrically isomorphic.

When we say that two normed linear spaces are "isomorphic" we will usually mean that they are topologically isomorphic. We are often interested in the case when we have two different norms defined on the same space, and we would like to know if the norms define the same topologies.

Theorem 2.1.3 Two norms on a linear space generate the same topology if and only if they are equivalent.

Proof. Let $\mathrm{k} \cdot \mathrm{k}_1$ and $\mathrm{k} \cdot \mathrm{k}_2$ be two norms on a linear space X. We consider the identity map

$$I : (X, \mathrm{k} \cdot \mathrm{k}_1) \to (X, \mathrm{k} \cdot \mathrm{k}_2)$$

From Corollary 4.20, the topologies generated by the two norms are the same if and only if I and I^{-1}
are continuous. Since I is linear, it is continuous if and only if it is bounded. The boundedness of the identity map and its inverse is equivalent to the existence of constants c and C such that (5.10) holds.

Geometrically, two norms are equivalent if the unit ball of either one of the norms is contained in a ball of finite radius of the other norm.

We end this section by stating, without proof, a fundamental fact concerning linear operators on Banach spaces.

Theorem 2.1.4 (Open mapping) Suppose that $T : X \to Y$ is a one-to-one, onto bounded linear map between Banach spaces X, Y. Then $T^{-1} : Y \to X$ is bounded.

This theorem states that the existence of the inverse of a continuous linear map between Banach spaces implies its continuity.

The kernel and range of a linear map

The kernel and range are two important linear subspaces associated with a linear map.

Definition 2.1.5 Let $T : X \to Y$ be a linear map between linear spaces X, Y. The null space or kernel of T, denoted by $kerT$, is the subset of X defined by

$$kerT = \{x \in X \mid Tx = 0\}$$

The range of T, denoted by $ranT$, is the subset of Y defined by $ranT = \{y \in Y \mid$ there

$$existsx \in XsuchthatTx = y\}$$

The word "kernel" is also used in a completely different sense to refer to the kernel of an integral operator. A map $T : X \to Y$ is one-to-one if and only if $kerT = \{0\}$, and it is onto if and only if $ranT = Y$.

Theorem 5 Suppose that $T : X \to Y$ is a linear map between linear spaces X, Y. The kernel of T is a linear subspace of X, and the range of T is a linear subspace of Y. If X and Y are normed linear spaces and T is bounded, then the kernel of T is a closed linear subspace. Proof. If $X_1, X_2 \in kerT$ and $\lambda_1, \lambda_2 \in$ R (or C), then the linearity of T implies that

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T x_1 + \lambda_2 T x_2 = 0$$

so $\lambda_1 X_1 + \lambda_2 x_2 \in kerT$. Therefore, $kerT$ is a linear subspace. If $y_1 y_2 \in ranT$, then there are $x_1, X_2 \in X$ such that $Tx_1 = y_1$ and $Tx_2 = y_2$. Hence

$$T(\lambda_1 x_1 + \lambda_2 X_2) = \lambda_1 T x_1 + \lambda_2 T x_2 = \lambda_1 y_1 + \lambda_2 y_2$$

so $\lambda_1 y_1 + \lambda_2 y_2 \in ranT$. Therefore, $ranT$ is a linear subspace. Now suppose that X and Y are normed spaces and T is bounded. If (x_n) is a sequence of elements in $kerT$ with $x_n \to x$ in X, then the continuity of T implies that

$$Tx = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = 0$$

so $x \in kerT$, and $kerT$ is closed. The nullity of T is the dimension of the kernel of T, and the rank of T is the dimension of the range of T. We now consider some examples. The right shift operator S on $^\infty$(N) is defined by

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$$

and the left shift operator T by

$$T(x_1, X_2, X_{3,\ldots}..) = (x_{2,}, X_{3,}, X_4, \ldots)$$

These maps have norm one. Their matrices are the infinite-dimensional Jordan blocks,

$$[S] = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad [T] = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The kernel of S is $\{0\}$ and the range of S is the subspace

$$ranS = \{(0, x_2, x_3, \ldots)^\circ\infty(\mathrm{N})\}$$

The range of T is the whole space $^\circ$(N), and the kernel of T is the one-dimensional subspace

$$kerT = \{(x_1, 0, 0, \ldots) \mid x_1 \in \mathrm{R}\}$$

The operator S is one-to-one but not onto, and T is onto but not one-to-one. This cannot happen for linear maps $T : X \to X$ on a finite-dimensional space X, such as $X = \mathrm{R}^n$. In that case, $kerT = \{0\}$ if and only if $ranT = X$. An integral operator $K : C([0, 1]) \to C([0, 1])$

$$Kf(x) = \int_0^1 k(x, y)f(y)dy$$

is said to be degenerate if $k(x, y)$ is a finite sum of separated terms of the form

$$k(x, y) = \sum_{i=1}^{n} \varphi_i(x)\psi_i(y)$$

where $\phi_i, \psi_i : [0, 1] \to \mathrm{R}$ are continuous functions. We may assume without loss of generality that $\{\phi_1, \dots, \phi_n\}$ and $\{\psi_1, \dots, \psi_n\}$ are linearly independent. The range of K is the finite-dimensional subspace spanned by $\{\phi_1, \phi_2, \dots, \phi_n\}$, and the kernel of K is the subspace of functions $f \in C([0, 1])$ such that

$$\int_0^1 f(y)\psi_i(y)dy = 0 \quad fori = 1, \dots, n.$$

Both the range and kernel are closed linear subspaces of $C([0, 1])$.
Example 3 Consider the operator $T = I + K$ on $C([0, 1])$, where K is defined in (5.11), which is a perturbation of the identity operator by K. The range of T is the whole space $C([0, 1])$, and is therefore closed. To prove this statement, we observe that $g = Tf$ if and only if

$$f(x) + \int_0^x f(y)dy = g(x)$$

Writing $F(x) = \int_0^x f(y)dy$, we have $F^0 = f$ and

$$F^0 + F = g, \quad F(0) = 0.$$

The solution of this initial value problem is

$$F(x) = \int_0^x e^{-(x-y)}g(y)dy.$$

Differentiating this expression with respect to x, we find that f is given by

$$f(x) = g(x) - \int_0^x e^{-(x-y)}g(y)dy$$

Thus, the operator $T = I + K$ is invertible on $C([0, 1])$ and

$$(I + K)^{-1} = I - L$$

where L is the Volterra integral operator

$$Lg(x) = \int_0^x e^{-(x-y)}g(y)dy.$$

The following result provides a useful way to show that an operator T has closed range. It states that T has closed range if one can estimate the norm of

the solution x of the equation $Tx = y$ in terms of the norm of the right-hand side y .
“In that case, it is often possible to deduce the existence of the following result provides a useful way to show that an operator T has closed range. It states that T has closed range if one can estimate the norm of the solution x of the equation $Tx = y$ in terms of the norm of the right-hand side y

In that case, it is often possible to deduce the existence of solutions

Proposition2.1.1: Let $T : X \to Y$ be a bounded linear map between Banach spaces X, Y. The following statements are equivalent: (a) there is a constant $c > 0$ such that

$$c\|x\| \leq \|Tx\| \quad forall x \in X$$

(b) T has closed range, and the only solution of the equation $Tx = 0$ is $x = 0$.
Proof. First, suppose that T satisfies (a). Then $Tx = 0$ implies that $\|x\| = 0$, so $x = 0$. To show that $ranT$ is closed, suppose that (y_n) is a convergent sequence in ran T, with $y_n \to y \in Y$. Then there is a sequence (x_n) in X such that $Tx_n = y_n$. The sequence (x_n) is Cauchy, since (y_n) is Cauchy and

$$\|x_n - x_m\| \leq \frac{1}{c}\|T(x_n - x_m)\| = \frac{1}{c}\|y_n - y_m\|.$$

Hence, since X is complete, we have $x_n \to x$ for some $x \in X$. Since T is bounded, we have

$$Tx = \lim Tx_n = \lim y_n = y, n \to \infty n \to \infty$$

so $y \in ranT$, and $ranT$ is closed. Conversely, suppose that T satisfies (b). Since ran T is closed, it is a Banach space. Since $T : X \to Y$ is one-to-one, the operator $T : X \to ranT$ is a one-toone, onto map between Banach spaces. The open mapping theorem, Theorem 5.23 , implies that $T^{-1} : ranT \to X$ is bounded, and hence that there is a constant $C > 0$ such that

$$\left\|T^{-1}y\right\| \leq C\|y\| \quad forally \in ranT$$

Setting $y = Tx$, we see that $c\|x\| \leq \|Tx\|$ for all $x \in X$, where $c = 1/C$.
Example 1 Consider the Volterra integral operator $K : C([0,1]) \to C([0,1])$ defined in (5.11). Then

$$K[\cos n\pi x] = \int_0^x \cos n\pi y dy = \frac{\sin n\pi x}{n\pi}$$

We have $\|\cos n\pi x\| = 1$ for every $n \in$ N, but $\|K[\cos n\pi x]\| \to 0$ as $n \to \infty$. Thus, it is not possible to estimate $\|f\|$ in terms of $\|K\|$, consistent with the fact that the range of K is not closed.

Finite-dimensional Banach spaces

In this section, we prove that every finite-dimensional (real or complex) normed linear space is a Banach space, that every linear operator on a finite-dimensional space is continuous, and that all norms on a finite-dimensional space are equivalent. None of these statements is true for infinite-dimensional linear spaces. As a result, topological considerations can often be neglected when dealing with finite-dimensional spaces but are of crucial importance when dealing with infinite dimensional spaces.

We begin by proving that the components of a vector with respect to any basis of a finite dimensional space can be bounded by the norm of the vector.
Lemma: Let X be a finite-dimensional normed linear space with norm $k \cdot k$, and $\{e_1, e_2, \ldots, e_n\}$ any basis of X. There are constants $m > 0$ and $M > 0$ such that if $x = \sum_{i=1}^{n} x_i e_i$, then

$$m \sum_{i=1}^{n} |x_i| \leq \|x\| \leq M \sum_{i=1}^{n} |x_i|$$

Proof. By the homogeneity of the norm, it suffices to prove (5.12) for $x \in X$ such that $\sum_{i=1}^{n} |x_i| = 1$. The "cube"

$$C = \left\{ (x_1, \ldots, x_n) \in R^n \left| \sum_{i=1}^{n} \right| x_i \mid= 1 \right\}$$

is a closed, bounded subset of R^n, and is therefore compact by the Heine-Borel theorem. We define a function $f : C \to X$ by

$$f\left((x_1, \ldots, x_n)\right) = \sum_{i=1}^{n} x_i e_i$$

For $(x_1, \ldots, x_n) \in \mathrm{R}^n$ and $(y_1, \ldots, y_n) \in \mathrm{R}^n$, we have

$$\left\| f\left((x_1, \ldots, x_n)\right) - f\left((y_1, \ldots, y_n)\right) \right\| \leq \sum_{i=1}^{n} |x_i - y_i| \, \|e_i\|$$

so f is continuous. Therefore, since $k \cdot k : X \to \mathrm{R}$ is continuous, the map

$$(x_1, \ldots, x_n)\, 7 \to \mathrm{k} f\left((x_1, \ldots, X_n)\right) \mathrm{k}$$

is continuous. Theorem 1.68 implies that $\mathrm{k} f \mathrm{k}$ is bounded on C and attains its infimum and supremum. Denoting the minimum by $m \geq 0$ and the maximum by $M \geq m$, we obtain (5.12). Let $(\bar{x}_1, \ldots, \bar{x}_n)$ be a point in C where $\mathrm{k} f \mathrm{k}$ attains its minimum, meaning that

$$\mathrm{k}_1 e_1 + \ldots + *_n e_n \mathrm{k} = m$$

The linear independence of the basis vectors $\{e_1, \ldots, e_n\}$ implies that $m6 = 0$, so $m > 0$. This result is not true in an infinite-dimensional space because, if a basis consists of vectors that become "almost" parallel, then the cancellation in linear combinations of basis vectors may lead to a vector having large components but small norm. Theorem 6

Every finite-dimensional normed linear space is a Banach space.
Proof. Suppose that $(x_k)_{k=1}^{\infty}$ is a Cauchy sequence in a finite-dimensional normed linear space X. Let $\{e_1, \ldots, e_n\}$ be a basis of X. We expand x_k as

$$x_k = \sum_{i=1}^{n} x_{i,k} e_i$$

where $x_{i,k} \in$ R. For $1 \leq i \leq n$, we consider the real sequence of i th components, $(x_{i,k})_{k=1}^{\infty}$. Equation (5.12) implies that

$$|x_{i,j} - x_{i,k}| \leq \frac{1}{m} \|x_j - x_k\|$$

so $(x_{i,k})_{k=1}^{\infty}$ is Cauchy. Since R is complete, there is a $y_i \in$ R, such that

$$\lim x_{i,k} = y_i$$

$$k \to \infty We \ \ define y \in X by$$

$$y = \sum_{i=1}^{k} y_i e_i$$

Then, lemma,

$$n \mathrm{k} x_k - y \mathrm{k} \leq M \mathrm{X} |xi, k - y_i| \,\mathrm{k} e_i \mathrm{k}$$

$i = 1$ and hence $x_k \to y$ as $k \to \infty$. Thus, every Cauchy sequence in X converges, and X is complete. Since a complete space is closed, we have the following corollary. Corollary Every finite-dimensional linear subspace of a normed linear space is closed. ”

Theorem 2.1.7 Every linear operator on a finite-dimensional linear space is bounded.

Proof. Suppose that $A : X \to Y$ is a linear map and X is finite dimensional. Let $\{e_{1,\ldots,}, e_n\}$ be a basis of X. If $x = \sum_{i=1}^{n} x_i e_i \in X$, then (5.12) implies that

$$\|Ax\| \leq \sum_{i=1}^{n} |x_i| \, \|Ae_i\| \leq \max 1 \leq i \leq n \{\|Ae_i\|\} \sum i = 1^n |x_i| \leq \frac{1}{m} \max_{1 \leq i \leq n} \{\|Ae_i\|\} \|x\|$$

so A is bounded. Finally, we show that although there are many different norms on a finitedimensional linear space they all lead to the same topology and the same notion of convergence.

Theorem 2.1.8 Any two norms on a finite-dimensional space are equivalent.

Proof. Let $\mathrm{k} \cdot \mathrm{k}1$ and $\mathrm{k} \cdot \mathrm{k}_2$ be two norms on a finite-dimensiona space X. We choose a basis $\{e_1, e_2, \ldots, e_n\}$ of X. Then Lemma implies that there are strictly positive constants m_1, m_2, M_1, M_2 such that if $x = \sum i = 1^n x_i e_i$, then

$$m_1 \sum_{i=1}^{n} |x_i| \leq \|x\|1 \leq M_1 \sum i = 1^n |x_i| \, m_2 \sum_{i=1}^{n} |x_i| \leq \|x\|2 \leq M_2 \sum i = 1^n |x_i| \, .$$

then follows with $c = m_2/M_1$ and $C = M_2/m_1$.

2.2 Compact operators

Compact operators hold significant importance not only due to the well established theory surrounding them but also because they arise in numerous crucial applications .

Consider normed spaces X and Y
A linear operator K from X to Y is classified as compact if its domain $D(K)$ is X, and for any sequence $\{x\_n\} \subset X$ satisfying $X\_n \leq c$, the sequence $\{K\_xn\}$ possesses a subsequence that converges in Y.

The collection of all compact operators from X to Y is denoted as $K(X, Y)$. If X and Y are the same, we can use the shorthand notation $K(X)$ instead of $K(X, X)$.

Definition 2.2.1.
A subset S of a normed space X is deemed compact if every sequence of elements in S possesses a subsequence that converges to an element within S .

Definition 2.2.2.
A subset S of a normed space X is considered relatively compact if every sequence of elements in S has a convergent subsequence that converges to an element of X. It is important to note that the limit of this subsequence may not necessarily belong to S .

Proposition2:
If X and Y are armed spaces, a linear operator $K : X \to Y$, defined everywhere, is compact if and only if $K(B)$, where B is any bounded set contained in X, is relatively compact. It is established ihat K is compact if and only if the image of every bounded subset of X is a relatively compact subset of Y.
Proposition3:
All compact operators belong to the category of bounded operators. In other words, 'he set of compact operators, denoted as $K(X, Y)$, is a subset of the set of bounded operators, represented as $B(X, Y)$.

2.3 Finite Rank Operators

Definition2.3.1
Consider normed spaces X and Y. A linear operator $T : X \to Y$ is categorized as an operator of finite rank if the range of T is finite-dimensional.
The collection of all bounded linear finite rank operators is denoted as BFR (X, Y). It should be noted that not every linear operator of finite rank is necessarily bounded.

Theorem 2.3.1.

If A is an eloment of the set of bounded linear operators from X to Y (denoted as $B(X, Y)$), and K belongs to the set of bounded linear operators of finite rank from Y to Z (denoted as BFR(Y, Z)),
then the product KA belongs to the set of bounded linear operators of finite rank from X to Z (denoted as $BFR(X, Z)$).

Similarly, if L is an element of $B\ FR(X, Y)$ and B belongs to $B(Y, Z)$, then the product BL belongs to $BFR(X, Z)$. In both cases, it can be concluded that the product of a bounded operator and an operator of finite rank, regardless of the order, yields an operator of finite rank.

Theorem 2.3.2.
Every bounded finite rank operator is compact.

Proposition 3 :

If a normed space has finite dimension, then the identity operator associated with that space is compact. Conversely, it the identity operator of a normed space is compact, it implies that [3] he space itself is of finite dimension.

Proposition4:
If X_0 is a subspace of a normed space X, the inclusion operator $I_0 : X_0 \rightarrow X$ is compact if and only if X_0 is finite-dimensional.
Corollary:
In the space of bounded linear operators $B(X)$, [3] compact operator from an infinite-dimensional normed space is non-invertible.

Ideals

Definition 4.
In an arbitrary ring $(R, +,)where"+"representsadditionand"."represents$ multiplication, the additive group $(R, +)$ is denoted. A subset I is considered a two-sided ideal of R if it fulfills the following criteria:

1. $(I, +)$ is a subgroup of $(R, +)$.
2. For any $x \in I$ and $r \in R$, both $x \cdot r$ and $r \cdot x$ belong to I .

The term "two-sided" signifies that we can perform multiplication by any element of R from either the left or right side.

Proposition 5:
K(X) is a two-sided ideal of the normed algebra $B(X)$. When X is a Banach space, $K(X)$ remains a two-sided ideal within the Banach algebra $B(X)$.

Theorem 2.3.1.
Consider X as normed space and Y as a Banach space. If L belongs to

$B(X,Y)$ and there exists a sequence $\{Kn\}$ in $K(X,Y)$ such that $L-Kn$ converges to O^3 as n approaches infinity, then L is an element of $K(X,Y)$.

According to the theorem mentioned above, (X,Y) is a closed subspace within $B(X,Y)$, making it a Banach space. Therefore, the proposition states chat if X is a Banach space, then $K(X)$ is a closed two-sided ideal of the Banach algebra $B(X)$

Approximation Property

Definition2.3.1.
A Banach space for which the finite rank operators are norm-dense in the compact operators is said to have the approximation property.

2.4 Adjoint Operators

Adjoint operators exhibit similar behavior to the transpose of a matrix in real Euclidean space. The transpose AT of a real m × n matrix A satisfies the relationship
⟨Ax, y⟩ = ⟨x, ATy⟩
for all x in Rn and y in Rm, where ⟨·, ·⟩ represents the Euclidean inner product
.

In the context of bounded linear operators, if T is a mapping from a Hilbert space H1 to another Hilbert space H2, denoted as T : H1 → H2, then for a fixed y in H2, the linear functional l is bounded. By the Riesz Representation Theorem, there exists a unique z in H1 such that ⟨Tx, y⟩ = ⟨x, z⟩. This z, determined uniquely by y through T, is denoted as T*y.

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

The adjoint operator T:H260→H1 is a bounded linear operator. It serves as the counterpart of T. For any y^{60} Dounded linear operator T, the norms of T and its adjoint T are equal, denoted as $\|T\| = \|T\|$, and T^* equals T.

To illustrate, consider the linear operator $T : L2[a,b] \to L2[c,d]$, which is generated by the kernel $k(\cdot,\cdot)$ belonging to $C([c,d]\times[a,b])$.

In other words, for $s \in [c,d], (Tf)(s) = \int abk(s,t)f(t)dt$. Then, the inner product $\langle Tf, g\rangle$ can be expressed as ∫ c ∫ abk(s, t)f(t)dt$g$(s)ds, which is equal to $\langle f,$ T* g⟩. Consequently, 1 is represented by the integral operator generated by the kernel $k^(\cdot,\cdot)$, where $k^*(t,s) = k(s,t)$.

In particular cases, if the kernel k is symmetric and $[a,b] = [c,d]$, the operator T is referred to as self-adjoint.

Properties of the adjoint operator

Theorem 4.
Assume $T : H1 \rightarrow H2$ be a bounded linear operator. Then

1. The adjoint of the adjoint operator is equal to the original operator.
2. The norm of the product of T and its adjoint, as well as the norm of the product of the adjoint and T, is equal to the square of the norm of T.
3. The null space of the operator T is the orthogonal complement of the range of its adjoint operator.
4. The orthogonal complement of the null space of T is equal to the range of its adjoint operator.
5. The null space of the adjoint operator T^* is the orthogonal complement of the range of the operator T.
6. The orthogonal complement of the null space of the adjoint operator T^* is equal to the range of the operator T.

2.5 Self-adjoint, Normal and Unitary operators

Definition 2.4.1
In a Hilbert space H, a bounded linear operator T is considered self-adjoint when Tis eaual to its adjoint operator $T*$. If T is both bijective and its adjoint operator T * is equal to its inverse $T - 1$, it is referred to as unitary. A bounded linear operator T is categorized as normal if the product of its adjoint operator $T*$ with T is equal to the product of T with its adjoint operator $T*$, i.e., $T * T = TT*$.

Theorem 5:
Let nH be a Hilbert space and let $P \in B(H)$ is given then P is self-adjoint iff $< Px, x >\in R, x, y \in H$. *Proof:*
Let's consider the assumption that P is self-adjoint, which means P is equal to its adjoint P^*. For an^{4848} and y in the Hilbert space H, we have the following equality:

$$\text{Conjugate of } \left(\left\langle \text{Px}^{41}, x\right\rangle\right) = \langle x, \text{Px}\rangle = \langle \text{P}^*x, x\rangle$$

It is observed that the conjugate of $\langle Px, x >$ is equal to $\langle x, Px >$ and also equal to $\langle \text{P}^*\text{x}, \text{x}\rangle$.

Now, let's assume that $\langle Px, x >$ is a real number for all x in H. Choosing x and y from H, we can deduce the following:
$^4P(x + y), x + y\rangle = \langle Px, x\rangle + \langle Px, y\rangle + \langle Py, x\rangle + \langle Py, y\rangle$

Since $\langle P(x + y), x + y\rangle, \langle Px, x\rangle$, and $\langle Py, y\rangle$ are real numbers, we conclude that:

$\langle Px, y\rangle + \langle Py, x\rangle$ is also real.

Hence, it can be inferred that:

$\langle Px, y\rangle + \langle Py, x\rangle = conjugate(\langle Px, y\rangle + \langle Py, x\rangle) = \langle y, Px\rangle + \langle x, Py\rangle$
..................(1)

Similarly, upon examining the equation:

$\langle P(x + iy), (x + iy)\rangle = \langle Px, x\rangle + \langle Px, iy\rangle + \langle i\text{Py, x}\rangle + \langle \text{iPy, iPy}\rangle$

Expanding the equation, we obtain:
$^4Px, x > -i < Px, y > +i < Py, x > +\langle iPy, iPy >$

From this, we can conclude that:
$\langle Px, y\rangle - \langle Py, x\rangle = -\langle y, Px\rangle + \langle x, Py >$...........(2)

By adding equations (1) and (2), we get:
$2 < Px, y >= 2 < x, Pv >$

$2 < Px, y >= 2 < x, Py >$
This implies :

$$\langle Px, y\rangle = \langle x, Py >$$

Therefore, we have shown that $\langle Px, y\rangle$ is equal to $\langle {}^4P^*x, y\rangle$. Since this holds true for every x and y, we can conclude that $P = P^*$.

Schauder Bases

Definition :
Consider a Banach space X. Let $e = (e_i)\, i \in N$ be a seq4ence in X. If every point x in X can be expressed uniquely as the sum of $x\mathcal{N}_i$, where xi belongs to a field F, then the sequence e is referred to as a Schauder basis of X, also known as a basis.

Weak Compactness in K(E, F):

Consider Banach spaces E and F, and let $L(E, F)$ represent the space of bounded linear operators from E to F. Within $L(E, F), K(E, F)$ is a closed subspace.
Our focus lies on two primary topologies applied to $^7(E, F)$. The first is the weak-operator topology, denoted as w, which is determined by the linear functionals $T \to f^(Te)$, where f belongs to F^*. 7 he second topology, known as the dual weak-operator topology and denoted as $^7w'$, is defined by linear func-

tionals.

$$T- > e^* \left(T' f\right) f^* belongstoF'ebelongstoE^*$$

Theorem 6.
Let A be a subset of $K(E, F)$. A is considered weakly compact if and only if it is w'-compact.

Proof:

Assume A is w'-compact, and let $x(A) = \{X \in r; T$ belongs to $A\}$. Then $x(A)$ is compact according to the pointwise convergence topology in U x V, and thus, $x(A)$ is weakly compact in $C(UxV)$.

Therefore, A is also weakly compact. The converse is evident since the w' topology is weaker than the weak topology of $K(E, F)$.

Corollary 1 : In the case where E is reflexive, a subset A of $K(E, F)$ is considered weakly compact if and only if it is w-compact.

Corollary 2: If both E and F are reflexive and $K(E, F) = L(E, F)$, then $K(E, F)$ is reflexive.

Chapter 3

Geometry of Banach Spaces

Definition 3.1.1 states that a Banach space X is considered strictly convex if, whenever x and y are distinct elements in X with $\|x\| = \|y\| = 1$ and λ is a value between 0 and 1 , the norm of the linear combination $\lambda x+(1-\lambda)y$ is less than 1 .

Proposition 3.1.2 establishes that a Banach space X is strictly convex if and only if, whenever x and y are elements in X such that $\|x+y\| = \|x\|+\|y\|$, then either $y = 0$ or $x = \lambda y$ for some $\lambda \geq 0$.

Proposition 3.1.3 states that if (xk) is a sequence in a uniforno 57 convex Banach space X that weakly converges to $\sigma \in X$, with the norms of xk approaching the norm of x, then the norm of $xk - x$ approaches 0 .

Theorem 3.1.4 states that every uniformly convex Banach space is reflexive.

Proposition 3.1.5 establishes that in a B_{55} banach space X with a strictly convex dual $X*$, a duality map J is m_6 potone, meaning that the real part of the inner product of $x-y$ and $Jx-\ Jy$ is non-negative for all x arif y in X. If X is strictly convex, then J is strictly monotone, meaning that the real part of the inner product is positive for all x and y in X with $x \neq y$.

Definition 3.1.6 defines the approximation property (AP) for a Banach space X, stating that X has the AP if, for any compact set K of X and any $\varepsilon > 0$, there exists a bounded linear operator T with finite rank such that the norm of $Tx - x$ is less than ε for all x in K.

Proposition 3.1.7 presents equivalent statements to X having the AP. Statement (i) states that for any Banach-space Y, any bounded linear operator T from X to Y, any compact subset K of X, and any $\varepsilon > 0$, there exists a bounded linear operator F from X to Y with finite rank such that the norm of $Tx - Fx$ is less than ε for all x in K. Statement (ii) states the same condition as (i), but with the roles of X and Y reversed.

Chapter 4

LOCAL ISOMETERIES

4.1 Local isometries on Strongly Separating Subspaces of $C_0(X)$

In this section, we establish that any local isometry on strongly separating subspaces of C0(X) is, in fact, a surjective isometry. In other words, we demonstrate that the collection of all surjective linear isometries on strongly separating subspaces of $C_0(X)$ possesses algebraic reflexivity.

Remark 3.1.1:
We introduce the sets σA and $\sigma_0 A$, defined as follows: σA comprises elements x_0 in 11 such that, for every neighborhood U of x_0, there exists an f in A where $|f(x)| \leq \|f\|$ holds true for all x in $X - U$.

Meanwhile, σA corresponds to the intersection of σA and the element Y in X for which richere exists an f in A with $f(x) \neq 0$. It is worth noting that existing knowledge affirms that if A is a15 subspace of $\mathrm{C}_0(X)$, then $\partial A = \sigma A$.

Theorem 2.1.2:
Suppose X and Y are locally compact Hausdorff spaces, and A and B are strongly separating linear subspaces of $C_0(X)$ and $C_0(Y)$ respectively. If there exists a nonnegative real-valued injective function g in A, and $\sigma_0 A$ is a compact set, then the operator space G(A, B) possesses algebraic reflexivity.

Proof.
Let $T \in \overline{\mathrm{G(A,B)}}$. exists a subset Y_0 of Y, a continuous surjective map $h : Y_0 \to \sigma_0 A$, and a continuous map $\tau : Y_0 \to K$, where K is a set of complex numbers with $|\tau(y)| = 1$ for all y in Y_0.
Moreover, for every y in Y_0 and f in A, we have $Tf(y) = \tau(y)f(h(y))$.

To demonstrate 'he surjectivity of T, we need to prove that h is a homeomorphism and that $Y_0 = \sigma_0 B$

First, we establish the injectivity of h.
According to the hypothesis, there exists a function g satisfying the 1 ondi-

tions, and we can find T_g in $\mathcal{F}(A, B)$ such that $Tg = T_g g$. Applying Theorem 1.4.3, we find a homeomorphism hg: $\sigma_0 B \rightarrow \sigma_0 A$ and a continuous map $\tau_{\mathrm{g}} : \sigma_0 B \rightarrow K$,
where $|\tau_g(y)| = 1$ for all y in $\sigma_0 B$, and $T_g(y) = \tau_g(y) g(hg(y))$ for y in $\sigma_0 B$.

From the proof of Theorem 1.4.3, we know that Y_0 is a subset of $\sigma 0$ B. By combining $^2 - quations$ (2.1.1) and (2.1.2), we can see that $g(h(y)) = g(hg(y))$ for all y in Y_0. Thus, $h = h_g$ on Y_0, which implies that h is injective. V e conclude that h is a homeomorphism.

To complete the proof, we need to show that $\sigma_0 B$ is a subset of Y_0. or y in $\sigma 0 B$, we have $h_g(y)$ in $\sigma_0 A$.
Since h is onto, there exists y_0 in $Y0$ such that $h(y_0) = h_g(y)$. However, since $h = h_g$ on Y_0, we can deduce that $y = y_0$. This demonstrates that $\sigma_0 B$ is indeed a subset of Y_0.

Therefore, we have established the surjectivity of T.

4.2 Local isometries on various subalgebras of C_U (K_E)

In this section, we prove the algebraic reflexivity of the set of all surjective linear isometries on weakly normal subalgebras of $C_u(K_E)$ and on the subalgebra $A_u^0(K_E)$.

The following remark is crucial in our proofs.

Remark 3.2.1.
1. We can associate the closed subalgebras $A_u(K_E)$ and $A_0(K_E)$ with closed subalgebras $A(E)$ and $A_0(E)$ of $C(\gamma E)$, respectively.

Here, γE represents a compactification of K_E known as the quotient space $\gamma E := \beta K_E / R$, where βK_E is the Stone-Cech compactification of K_E, and R is an equivalence relation defined as $x_1 R x_2$ if $f(x_1) = f(x_2)$ for every $f \in A_u(K_E)$.$A(E)$ and $A_0(E)$ have the property of strongly separating points in γE

Furthermore, it is known that K_E is a subset of the boundary of $A(E)$, and the boundary of $A(E)$ is equal to VE.
Additionally, K_E without the element 0 is contained in the boundary of $A_0(E)$, and the boundary of $A_0(E)$ without the element 0 is equal to γE without the element 0 .

2. It is worth noting that $A(E)$ is a uniform algebra, which means it is a closed subalgebra of $C(\gamma E)$ that separates points and contains the constants. As a result, $A(E)$ does not vanish anywhere, indicating that for every ξ in YE,

there exists an element f in $A(E)$ such that $f(\xi) \neq 0$. Remark 2.1.1 implies that $\sigma_0 A(E)$ is equal to the intersection of $\sigma A(E)$ with the set of ξ in γE where there exists an element f in $A(E)$ such that $f(\xi) \neq 0$.

This is equivalent to the boundary of $A(E)$ intersected with $\gamma(E)$ since $A(E)$ does not vanish anywhere. "Consequently, $\sigma_0 A(E)$ is equal to the boundary of $A(E)$, which is γE.

3. Moreover, $A_0(E)$ strongly separates points in YE, so for ξ in YE where $\xi \neq 0$, there exists an element f in $A_0(E)$ such that $|f(\xi)| = -1 f(0)\ |$. Since $f(0) = 0$, we have $|f(\xi)| \neq 0$, and thus $f(\xi) \neq 0$.
Therefore, the set of ξ in νE where there exists an element f in $A_0(E)$ such that $f(\xi) \neq 0$ is equal to γE without the element 0 . This implies that $\sigma_0 A_0(E)$ is equal to YE without the element 0.

Proposition 3.2.2: If there exists an injective map g in $A_u\left(K_E\right)$ such that $g(x) \geq 1$ for all x in K_{E_1} then the operator space $G\left(A_u\left(K_E\right), A_u\left(K_F\right)\right)$ is algebraically reflexive.
We can establish that T is surjective linear isometry connecting the closed subalgebras $A(E)$ and $A(F)$ of $C(\gamma E)$ and $C(\gamma F)$ respectively.
Theorem guarantees the existence of a homeomorphism h such that:

The function $pF \rightarrow \gamma E$ and the continuous map $\tau: \nu F \rightarrow K$ can be found such that $|\tau(y)| = 1$ for all y in νF, and the equation $Tf(y) = \tau(y) f(h(y))$ holds for all y in νF and f in $A(E)$ (Equation 2.2.1).

In order to establish that $T: A_u\left(K_E\right) \rightarrow A_u\left(K_F\right)$ is a surjective linear isometry, we need to demonstrate that $h: K_F \rightarrow K_E$ is a uniform homeomorphism and that $\mu = \tau \mid K_F$ is a uniformly continuous function.

For the first part we can consider the map g mentioned in the hypothesis. This implies the exis 2^{+}nce of $1 g \in \mathrm{G}\left(\mathrm{Au}\left(K_E\right), \mathrm{Au}\left(K_F\right)\right)$ such that $Tg = T_g g$. Applying Theorem 1.4.4, we can find a uniform homeomorphism $h_g: \mathrm{K}_Y \rightarrow \mathrm{K}_x$ and a function $\tau \mathrm{g} \in \mathrm{Cu}(\mathrm{K}-)$, where $\left|\tau_g(y)\right| = 1$ for all y in KF, and $\mathrm{Tg}(y) = \tau g(y) g\left(h_g(y)\right)$ for all y in KF (Equation 2.2 .2).
By comparing Equations 2.2.1 and 2.2.2 and utilizing the injectivity of g, we can conclude that h = hg on KF, indicating that h is a uniform homeomorphism."

To establish the second part, we assume the contrary that is, suppose μ is not uniformly continuous on K_F.
This implies the existence oi >> and two sequences $\left(x_n\right)$ and $\left(y_n\right)$ in K_F such that $\lim n \rightarrow \infty \left|\left|x_n - y_n\right|\right| = 0$ and $\left|\mu\left(x_n\right) - \mu\left(y_n\right)\right| \geq \varepsilon$ for every n in N. Since T_g is uniformly continuous, we have $\lim n \rightarrow \infty\left(Tg\left(x_n\right) - Tg\left(y_n\right)\right) = 0$ or

$$\lim_{n \rightarrow \infty}\left(\mu\left(x_n\right) g\left(h\left(x_n\right)\right) \quad \mu\left(y_n\right) g\left(h\left(y_n\right)\right)\right) = 0$$

Similarly for the map g^2 we will have

$$\lim_{n\to\infty}\Big(\mu(x_n)\,g^2(h(x_n))-\mu(y_n)\,g^2(h(y_n))\Big)=0$$

Multiplying by $g(h(x_n))$ we get

$$\lim_{n\to\infty}\Big(\mu(x_n)\,g^2(h(x_n))\quad-\mu(y_n)\,g(h(x_n))\,g(h(y_n))\Big)=0$$

Subtracting Equations (2.2.4) and (2.2.5) we will get

$$\lim_{n\to\infty}\Big(\mu(y_n)\,g^2(h(y_n))-\mu(y_n)\,g(h(x_n))\,g(h(y_n))\Big)=0$$

This implies that

$$\lim_{n\to\infty}\left(g(h(x_n))-g(h(y_n))\right)=0$$

Lastly multiplying by $\mu(x_n)$ and subtracting Equation (2.2.3) we get

$$\lim g(h(y_n))\,(\mu(\mathrm{x}_n)-\mu(\mathrm{y}_n))=0$$

. This is a contradiction. Hence, μ is uniformly continuous on K_F.
HENCE PROVED

4.3 Local isometries on $\mathrm{C}^2[0,1]$

In this section, we present and demonstrate the central outcome of this chapter. To ensure a clear understanding, [1] che proof is structured into six distinct steps.
Theorem 3.1.1 states that $\mathrm{G}(\mathrm{C}^2[0,1])$, the set of local isometries on the Banach space $\mathrm{C}^2[0,1]$, exhibits algebraic reflexivity.
Proof .
Let $T\in G\left(C^2[0,1]\right)^a$ and $g\in C^2[0,1]$.
We complete the proof in several steps.

Step I involves defining a linear map $W:C[0,1]\to C[0,1]$ as $W(f)=T\left(Z^2f\right)^{JJ}$.
"We assert that W is a local isometry on $C[0,1]$. To illustrate this, let's consider a function $f\in C[0,1]$. Then $\zeta^2 f$ can be denoted as h, which belongs $to^{12}[0,1]$. As T is a local isometry, there exists $\mathrm{Th}\in G\left(C^2[0,1]\right)$ such that $Th=T_hh$. Consequently, $W(f)=(Th)^{JJ}=(T_hh)^{JJ}=\omega_h\left(h^{JJ}\circ\phi_h\right)=\omega_h(f$ - $\phi_h)$. Thus, [2] V is a local isometry.
As W is also a surjective linear isometry. Hence, there exist continuous functions $\omega:[0,1]\to T$ and a homeomorphism $\phi:[0,1]\to[0,1]$ such that $W(f)=\omega(f\circ\phi)$.

Step II. Let k be defined as $\zeta^2 g^J$. We observe that k belongs to $C^2[0,1]$ and $(Tg)^{JJ}\ (Tk)^J=(Tg-Tk)^{JJ}=r_1(g-k)\Big)^{JJ}$. Since T is a local isometry, there exists T_{g-k} in $G\left(C^2[0,1]\right)$ such that $T(g-k)=T_{g-k}(g-k)$.

This implies that $(T(g-k))^{\jmath J} = (T_{g-k}(g-k))^{\jmath J} = \omega_{g-k}\Big((g-k)^{\jmath J}\ \ \phi_{\mathrm{g-k}}) = 0$.

Hence, $(Tg)^{\jmath J} = (Tk)^{\jmath J} = \Big(T\left(\zeta^2 g^{JJ}\right)\Big)^{\jmath J} = W\left(g^{JJ}\right) = \omega\left(g^{J1} - Q\right)$.

Step III. There exists T_g in $G\left(C^2[0,1]\right)$ such that $Tg = T_g g$. Computing $Tg(x)$ and $(Tg)^J(x)$ at $x = 0$, we have two cases as follows

Case 1: $Tg(0) = \lambda_g\ \ g(0), (Tg)(0) = \mu_{\mathrm{g}} g(0)$
Case 2: $Tg(0) = \lambda_g g(0), (Tg)(0) = \mu_g g(0)$.

Step IV. For the functions $f = 1$ and $f = id$, there exist T_1 and T_{id} in $G\left(C^2[0,1]\right)$ respectively such that $\mathrm{T1} = \mathrm{T11}$ and $\mathrm{Tid} = \mathrm{T}idid.$. Therefore, we have the following four cases:

Case 3 : $\mathrm{T}\ \ 1 = \lambda_1 1, \mathrm{Tid} = \mu_{\mathrm{idid}}$
Case 4 : $\mathrm{T1} = \lambda_1 1, \mathrm{Tid} = \lambda_{id} 1$
Case 5: $\mathrm{T1} = \mu_1\ \ \mathrm{id}, \mathrm{Tid} = \mu_{\mathrm{id}}\mathrm{id}$.
Case 6 : $\mathrm{T}\ \ 1 = \mu_1$ id, $\mathrm{Tid} = \lambda_{id} 1$
Cases 4 and 5 will lead to a contradiction.
We consider Case 4.
Consider $\mathrm{T1} + id$ in $G\left(C^2[0,1]\right)$ such that $\mathrm{T(1+id)} = \mathrm{T1} + id(1+\mathrm{id}\ \ )$. For all x in $[0,1]$, we have:

$$\lambda_1 + \lambda_{id} = T(1)\ \ (x) + T(id)(x) = T(1+id)(x)\ \ = \lambda_{1+id} + \mu_{1+id} x$$

which is a contradiction.
Therefore, Case 5 is not possible." **Step V.** Let $f_1 = g - g(0)1$ and $f_2 = g - g'(0)$ id. There exist $T_{f1},\ T_{f2} \in G\left(C^2[0,1]\right)$ such that $Tf_1 = T_{f1} f\ \ 1$ and $Tf_2 = T_{f2}$ f2.

Using the linearity of T, we can write the following equations:

$$Tf_1(0) = T\ \ g(0) - g(0)T1(0), (Tf_1(0) = (Tg)(0) - g(0)(T1)(0)$$

and

$$T\ \ f_2(0) = Tg(0) - g(0)T1(0), (Tf_2(0) = (Tg)(0) - g(0)(T1)(0)\ \ .$$

Furthermore, using these two Equations and the local structure of T, we obtain the following cases:

Case 7: $Tf_1(0) = 0, (Tf_1)(0) = \mu_{f1} g(0),\ \ Tf_2(0) = \lambda_{f2} g(0), (Tf_2)(0) = 0$.

Case 8: $Tf1(0) = 0, (Tf1)(0) = \mu f1\ \ g(0), Tf2(0) = \lambda \mathrm{f2}\ \ g(0), (Tf_2)'{}'(0) = \mu \mathrm{f2} g(0)$.
Case 9: $Tf_1(0) = \lambda_{f1} g^J(0),\ \ (Tf_1)^{\mathcal{J}}(0) = 0, Tf_2(0) = \lambda_{f2} g(0), (Tf_2)^1(T_1) = 0$.
Case 10: $Tf_1(0) = \lambda_{f1} g^J(0), (Tf_1)(0) = 0, Tf_2(0) = 0,\ \ (Tf_2)^J(0) = \mu_{f2} g(0)$.

Step VI. In this step,consider the cases that arose in steps III, IV, and V one by one, using Equations (3.1.2) and (3.1.3)

Cases 1 and 3: $Tg(0) = \lambda_{\mathrm{g}} g(0), (Tg)(0) = \mu_{\mathrm{g}} g(0), T1 = \lambda_1 1$, and T(id) $= \mu_{idid}$.

(i) If Case 7 holds, then Equations (3.1.2) and (3.1.3) imply that:

$\lambda_g g(0) - g(0)\ \lambda_1 = 0, \mu_g g^J(0) = \mu_{f1} g(0), \lambda_g g(0) = \lambda_{f2} g(0), \mu_g g^J(0) - g(0)\mu_{id} = 0$

Hence, $\lambda_{gg} g(0) = \lambda_1 g(0)$ and $\mu_{\mathrm{g}} g(0) = \mu_{\mathrm{id}} g(0)$.

Substituting the values of $\lambda_{gg} g(0)$ and $\mu_{\mathrm{g}} g^J(0)$ obtained here, as well as the value of (Tg)^J from Equation (3.1.1), into Equation (3.0.3), we get:

$$Tg(x) = Tg(0)+(Tg)^J(0)x+\left(\zeta^2\left((Tg)^{JJ}\right)\right)(x) = \lambda_1 g(0)+\mu_{\mathrm{id}} g^J(0)\ x+\left(\zeta^2\left(\omega\left(g'' \circ \phi\right)\right)\right)(x)$$

ii) when case 8 holds,

$\lambda_{\mathrm{g}} g(0)-g(0)\lambda_1 = 0, \mu_{\mathrm{g}} g(0) = \mu_{\mathrm{f1}} g(0), \lambda_{\mathrm{g}} g(0) = \lambda_{\mathrm{f2}} g(0), \mu_{\mathrm{g}} g(0)-g(0)\mu_{\mathrm{id}} = \mu_{f_2} g(0)$

Thus, $g(0) = 0$ and $\mu_g g'(0) = \mu_{id} g'(0)$.
Equations (3.1.1) and (3.0.3) imply that

$$Tg(\mathrm{x}) = Tg(0)+(Tg)'(0)\mathrm{x}+\left(\zeta^2\left((Tg)''\right)\right)(\mathrm{x}) = \mu_{id} g'(0)\mathrm{x}+\left(\zeta^2\left(\omega\left(g'' \circ \phi\right)\right)\right)(\mathrm{x}\ ).$$

iii) If Case 9 holds, then

$\lambda_g g(0)-g(0)\lambda_1 = \lambda_{f_1} g'(0), \mu_g g'(0) = 0, \lambda_g g(0) = \lambda_{f_2} g(0), \mu_g g'(0)-g\ \mathrm{g}(0)\mu_{id} = 0$
Thus, $g^{\mathrm{J}}(0) = 0$ and $\lambda_g g(0) = \lambda_1 g(0)$ From-2quations (3.1.1) and (3.0.3) we have

$$Tg(x) = Tg(0)+(Tg)'(0)x+\left(\zeta^2\left((Tg)''\right)\right)(x) = \lambda_1 g(0)+\left(\zeta^2(\omega(g\prime \circ \phi))\right)(x)\ .$$

iv) If Case 10 holds, then

$\lambda_g g(0)-\mathrm{g}(0)\lambda_1 = \lambda_{f_1} g'(0), \mu_g g'(0) = 0, \lambda_g g(0) = 0, \mu_g g'(0)-g'(0)\mu_{id} = \mu_{f_2} g(0)$

$\lambda_g g(0)-g(0)\lambda_1 = \lambda_{f_1} g'(0), \mu_g g'(0) = 0, \lambda_g g(0) = 0, \mu_g g\mathrm{l}(0)-g'(0)\mu_{id} = \mu_{f_2} g(0)$

which implies, $g(0) = 0$ and $g(0) = 0$.

Again, we apply Equations (3.1.1) and (3.0.3)

$$Tg(x) = Tg(0) + (Tg)'(0)x + \left(\zeta^2\left((Tg)''\right)\right)(x) = \left(\zeta^2\left(\omega\left(g'' \circ \phi\right)\right)\right)(x)$$

From (i)-(iv) we conclude that $T \in G(C^2[0,1])$. Cases 1 and 6. $Tg(0) = \lambda_g g(0), (\text{Tg})'(0) = \mu_g g'(0), \text{T1} = \mu_1$ id and $\text{T(id)} = \lambda_{id}1$. (i) If Case 7 holds, then

$$\lambda_g g(0) = 0, \mu_g g'(0) - g(0)\mu_1 = \mu_{f_1} g'(0), \lambda_g g(0) - g\ \text{g}(0)\lambda_{id} = \lambda_{f_2} g(0), \mu_g g(0) = 0.$$

Thus, $g(0) = 0$ and $g(0) = 0$. (ii) When Case 8 holds,

$$\lambda_g g(0) = 0, \mu_g g'(0) - g(0)\mu_1 = \mu_{f_1} g'(0), \lambda_g g(0) - g'(0)\lambda_{id} = 0, \mu_g g'(0) = \mu_{f_2} g(0).$$

Whence, $g(0) = 0$ and $g(0) = 0$. (iii) When Case 9 holds, then

$$\lambda_g g(0) = \lambda_{f_1} g'(0), \mu_g g'(0) - \text{g}(0)\mu_1 = 0, \lambda_g g(0) - g'(0)\lambda_{id} = \lambda_{f_2} g(0), \mu_g g'(0) = 0 .$$

Thus, $g(0) = 0$ and $g'(0) = 0$.

(iv) When Case 10 holds, then

$$\lambda_g g(0) = \lambda_{f_1} g(0), \mu_g g^g(0) - g(0)\mu_1 = 0, \lambda_g g(0) - g\ \text{g}(0)\lambda_{id} = 0, \mu_g g\ \text{g}(0) = \mu_{f_2} g(0). Thus \lambda_g g(0$$

Subsituting values of $\lambda_g g(0)$ and $\mu_g g(0)$ from (i)-(iv), the value of (Tg)" fromEquation (5.1.1), in Equation (3.0.3), implies that $T \in G(C^2[0,1])$.

Cases 2 and 3. $Tg(0) = \lambda_g g'(0), (\text{Tg})'(0) = \mu_g g(0), \text{T1} = \lambda_1 1$ and $\text{T(id)} = \mu_{id}$ id .

(i) When Case 7 holds, then

$$\lambda_g g'(0) - g(0)\lambda_1 = 0, \mu_g g(0) = \mu_{f_1} g'(0), \lambda_g g'(0) = \lambda_{f_2} g(0), \mu_g g(0) - g'(0)\mu_{id} = 0. Thus, \lambda_g g'(0)$$

(ii) When Case 8 holds,

$$\lambda_g g(0) - g(0)\lambda_1 = 0, \mu_g g(0) = \mu_{f_1} g(0), \lambda_g g(0) = 0, \mu_g g(0) - g!(0)\mu_{id} = \mu_{f_2} g(0). impliesthatg(0)$$

(iii) When Case 9 holds, then

$$\lambda_g\ \text{g}(0) - \text{g}(0)\lambda_1 = \lambda_{f_1}\text{gl}(0), \mu_g\ \text{g}(0) = 0, \lambda_g\ \text{g}(0) = \lambda_{f_2}\ \text{g}(0), \mu_g\ \text{g}(0) - \text{gl}(0)\ \mu_{id} = 0$$

Thus, $g(0) = 0$ and $g\prime(0) = 0$.

(iv) When Case 10 holds, then

$$\lambda_g g'(0)\ g(0)\lambda_1 = \lambda_{f_1} g'(0), \mu_g g(0) = 0, \lambda_g g'(0) = 0, \mu_g g(0) - g'(0)\mu_{id} = \mu_{f_2} g(0)$$

We conclude that $g(0) = 0$ and $g'(0) = 0$.
Thus, we get $T \in G\ (C^2[0,1])$
HENCE PROVED

4.4 Structure of isometries of finite order on $C^2[0,1]$

Proposition 3.2.3.
The proposition states that T is an element ot G^n $(C^2[0,1])$ if and only if there exist λ and μ in T, a homeomorphism $\phi : [0,1] \to [0,1]$, and a continuous function $\omega : [0,1] \to T$.
These conditions hold true for all f in $C^2[0,1]$ and x in $[0,1]$, and one of the following cases is satisfied.
1. When T is the first type isometry,

$$\lambda^n = \mu^n = 1, \omega(x)\omega(\phi(x))\omega\left(\phi^2(x)\right)\cdots\omega\left(\phi^{n-1}(x)\right) = 1, \quad \phi^n(x) = x$$

2. When T is the second type isometry, h is even and

$$\lambda^{\frac{n}{2}}\mu^{\frac{n}{2}} = 1, \quad \omega(x)\omega(\phi(x))\omega\left(\phi^2(x)\right)\cdots\ \omega\left(\phi^{n-1}(x)\right) = 1\ , \phi^n(x) = x$$

Proof.
Let $T \in \mathrm{G}^n\left(C^2[0,1]\right)$
Since $T \in G\left(C^2[0,1]\right), \exists \lambda, \mu \in T$, a homeomorphism $\phi : [0,1] \to [0,1]$ and a continuous function $\omega : [0,1] \to T$ such that T is of form (Λ) or $(\Lambda\Lambda)$.

Suppose T has the form (Λ), then $T^n = 1$ implies that

$$\lambda^n f(0) + \mu^n f(0)x + \int_x \int_t^t \omega(s)\ \omega(\phi(s))\omega\left(\phi^2(s)\right)\cdots\ \omega\left(\phi^{n-1}(s)\right) f''\left(\phi^n(s)\right) dsdt = f(x).$$

putting $f = 1$, we get $\lambda^n = 1$.
On differentiating and putting $^2 =$ id we get $\mu^n = 1$.
On taking the second derivative,

$$\omega(x)\omega(\phi(x))\omega\left(\phi^2(x)\right)\cdots\ \omega\left(\phi^{n-1}(x)\right) f\prime\left(\phi^n(x)\right) = f''(x).$$

which implies that

$$\omega(x)\omega(\phi(x))\omega\left(\phi^2(x)\right)\cdots\omega\left(\phi^{n-1}(x)\right) = 1, and \phi^n(x) = x$$

Now, T has the form ($\wedge$). We consider the following two cases.

i) If n is odd, then $T^n = \mathrm{I}$ will implies that

$$\int x \int_t$$

$$\omega(s)\ \omega(\phi(s))\omega\left(\phi^2(s)\right)\cdots\ \omega\left(\phi^{n-1}(s)\right)\ f\prime\left(\phi^n(s)\right) dsdt = f(x)$$

if f = id, we get $\lambda^{\frac{n+1}{1}}\mu^{-1} = \ x$, for all $x \in [0,1]$, which is a contradiction.
2 is n is even then

$$\lambda^{\frac{n}{2}}\mu^{\frac{n}{2}}f(0)+\ \lambda^{\frac{n}{2}}\mu^{\frac{n}{2}}f'(0)x$$

$$\int_x\int_0^t\omega(s)\omega(\phi(s))\ \omega\left(\phi^2(s)\right)\cdots\ \omega\left(\phi^{n-1}(s)\right)f''\left(\phi^n(s)\right)dsdt\ = f(x).$$

$$Put$$

f $=1, We\ have \lambda_2\mu^2 = 1$ On double differentiating,

$$\omega(x)\omega(\phi(x))\omega\left(\phi^2(x)\right)\cdots\ \omega\left(\phi^{n-1}(x)\right)f''\left(\phi^n(x)\right) = \ f''(x).$$

It follows that

Put f = 1, We have $\lambda_2\mu^2 = 1$ On double differentiating ,

$$\omega(x)\omega(\phi(x))\omega\left(\phi^2(x)\right)\cdots\ \omega\left(\phi^{n-1}(x)\right)\ f''\left(\phi^n(x)\right) = \ f''(x).$$

It follows that

$$\omega(x)\ \omega(\phi(x))\omega\left(\phi^2(x)\right)\cdots\ \omega\left(\phi^{n-1}(x)\right) = 1\ \ , and \phi^n(x) = x$$

Now, if assertion(1) holds,

$$T^nf(x)\ \ = \lambda^nf(0)+\mu^nf'(0)x+\int_0 t0\omega(s)\omega(\phi(s))\omega\left(\phi^2(s)\right)\cdots\ \omega\left(\phi^{n-1}(s)\right)\ \ f''\left(\phi^n(s)\right)dsdt$$

= f(0)+f'(0) + $f(x) - f(0) - f'(0)$
= $f(x)$.

$$If assertion\ \ (2) holds,$$

= f(0)+f'(0) + $f(x) - f(0) -\ \ f'(0)$
$= f(x)$

4.5 Local isometries of finite order on $C^2[0,1]$

We establish the algebraic reflexivity of the set $\mathrm{G^n}\left(\mathrm{C}^2([0,1])\ \right)^2$ when n is an odd number. The case when n is even is addressed in the concluding remark of this section.

Proposition 3.5.1
When n is odd, G^n $(C^2[0,1])$ is algebraically reflexive.

Proof:

Let $\mathrm{T} \in \mathrm{G}^n$ $(\mathrm{C}^2[0,1])$

By utilizing Theorem 3.1.1, we can deduce that $\mathrm{T} \in \mathrm{G}$ $(\mathrm{C}^2[0,\ 1])$.

Applying Theorem 1.4.7, we obtain the existence of λ, $\mu \in$ T, a homeomorphism $\phi : [0,1] \to [0,1]$, and a continuous function $\omega : [0,1] \to T$ such that T follows the forms () or ().

Additionally, or each $f_{42} \in C^2[0,1]$, there exists $T_f \in$ G^{n} $(\mathrm{C}^2[0,1])$ such that Tf = T_f^{42}. Considering that n is odd, we observe that T will always follow form ().

Moreover, the equation $(Tf)''(x) = (T_f f)'^J(x)$ implies that $\omega(x)'^J(\phi(x)) = \omega f(x) f'^J(\phi f(x))$.
By setting $f = x$^3, we conclude that $\omega(x) = \omega f(x)$ and $\phi(x) = \phi f(x)$ for every $x \in [0,1]$.
Consequently, $\omega(x)\omega(\phi(x))\omega$ $(\phi$^$2(x)) \cdots \omega$ $(\phi$^$(n-1)(x)) = 1$ and ϕ^$n(x) = x$.

Now, let's assume T is in the form (*). By computing Tf and $(Tf)'$ at $x = 0$, we obtain $\lambda f(0) = \lambda_f f(0)$ and $\mu f'(0) = \mu_f f'(0)$.
This implies $_1$ that $\lambda = \lambda_f$ and $\mu = \mu_f$. Therefore, λ^$n = \mu$^$n = 1$, leading to $\in \mathrm{G}^n$ $(C^2[0,1])$.

If T is in the form (**), repeating the same calculations results in $\lambda f'(0) = \lambda_f f(0)$ and $\mu f(0) = \mu_f f'(0)$.
Choosing $f \in C^2[0,1]$ such that $f(0) = 0$ and $f^J(0) \neq 0$ will lead to a contradiction .
HENCE PROVED

Chapter 5

CONCLUSION,CHALLENGES and some FUTURE PLANS

5.1 Summary and Difficulties

In this thesis, we primarily focused on two main aspects. The first aspect involved studying the set of isometries on various Banach spaces with algebraic reflexivity.

We were able to demonstrate that the local maps, which in our case were isometries, exhibit global properties and belong to the specified class of operators in many significant scenarios. It is worth noting that the aforementioned difficulty only arises when dealing with linear algebraic structures, and the local maps are also linear in nature.

However, it is common to explore these issues within a broader context of more generic structures.

The second aspect of our research involved tackling the challenge of characterizing specific classes of norm-one projections on the space C^2 [0, 1].

This presented its own set of complexities and required a distinct approach. We have also looked into how projections in this space relate to isometries. Though comparable findings should hold for the space Cr [0, 1] , the sheer volume of examples becomes overwhelming, especially for r is greater than 4. This still holds true if we consider projections to be a convex combination of at least four isometries.

5.2 Future Plans

5.2.1 Local isometries within subspaces of function spaces with vector-valued functions.

Certain subspaces of vector-valued function spaces could be used to formulate the algebraic reflexivity problems that were explored for $C_0(X)$ subspaces.

Consider the Banach space $C_0(X, E)$, which consists of continuous functions from X to E that vanish at infinity and is equipped with the supremum norm.

Let SE represent the set of elements e belongs to E such that their norm is equal to 1, denoted as e = 1, representing the unit sphere of E.

We define the map $f/otimes$ e : X $\to$ $Easfollows$: $for\ anyx\in$ X, $(f/otimes$ e)(x) is given by the product of f(x) and e for the values of f in C0(X) and e in E.

It is simple to demonstrate that $f\otimes$ e $\in C_0(X, E)$.

Definition 5.2.1.

Let $C_0(X)$'s subspace A be A. Any subspace of $A[A]$ is what we refer to. The set $f\otimes$ e is contained in $C_0(X, E)$ as follows: $\{f\otimes$ e : $f \in A, e \in S_E\}$

5.2.2 Reflexivity in algebra considering non-linear cases

One can consider any mathematical structure A and a^2 class of transformations E acting on A.

We define a map $\phi : A \to A$ to be 2locally associated with E if, for any pair of elements x and y in A, there exists an element $\phi(x, y)$ in E such that $\phi(x) = \phi(x, y)(x)$ and $\phi(y) = \phi(x, y)(y)$.

We use the term "algebraically reflexive" to describe the class E, drawing inspiration from the concept of algebraic reflexivity for linear mappings with a locality of 1 . Specifically, E is algebraically reflexive if every map ϕ that satisfies the 2-local property with respect to E belongs to E as well.

5.2.3 Generalized n-circular projections on Banach spaces

Consider the set $B_s(H)$, which consists ^{36}f all self-adjoint operators on the Hilbert space H.

We define an order relation between elements ^{54}A and B in $B_s(H)$ as $A \leq B$ if the inner product $\langle Ax, x\rangle$ is less than or equal to $\langle Bx, x\rangle$ for every x in H. This establishes the usual ordering on the set.

Now, let's consider bijective map $\Phi : B_s(H) \to B_s(H)$ that preserves the order, meaning $A \leq B$ if and only if $\Phi(A) \leq \Phi(B)$. Such a map is called an order-automorphism.

The question at hand is " whether the group of order-automorphisms of $B_s(H)$ exhibits algebraic reflexivity.

5.2.4 Generalized n-circular projections in Banach spaces

Definition 5.2.2.
If there are $P_1, P_2, \ldots, P^{44}_{n-1}$ of finite order and nontrivial projections on E such that $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in T \backslash \{1\}, \lambda_i, i = 1, 2, \ldots, n-1$ then a projection PO is said to be generalized n-circular projection on a Banach space E refers to a specific type of projection where n is greater than or equal to 2 :

1. $\lambda_i / = \lambda_j$ for i
2. P0 $\bigoplus P_1 \bigoplus \cdots \bigoplus Pn - 1 = 1$,
3. P0 $+ \lambda_1$P1 $\cdots + \lambda n - 1$P$n - 1$ is a surjective isometry.

Given the available resources, fully understanding the structure of generalized n-circular projections on traditional Banach spaces poses a significant challenge.

However, in specific spaces like Lp(Ω, E) where 1 p ¡ , p 2, and (Ω, µ) is a -finite measure space, and E is a separable Banach space with a trivial Lp-structure, we aim to examine this problem specifically for n = 3 or higher

Bibliography:

[1] For proofs of the Hahn-Banach, open mapping, and Banach-Alaoglu theorems, see Folland [12], Reed and Simon [45], or Rudin [48].

[2] The use of linear and Banach spaces in optimization theory is discussed in [34].

[3] Applied functional analysis is discussed in Lusternik and Sobolev [33].

[4] Abu Baker, A.B. and Dutta, S., Projections in the convex hull of three surjective isometries on C(Ω), J. Math. Anal. Appl. 379 (2011), no. 2, 878-888.

[5] Abu Baker, A.B., Botelho F. and Jamison, J. E., Representation of generalized bi circular projections on Banach spaces, Acta Sci. Math. (Szeged) 80 (2014), no. 3-4, 591-601.

[6] Abu Baker, A.B. and Dutta, S., Structure of generalized 3-circular projections for symmetric norms, Proc. Indian Acad. Sci. Math. Sci., 126 (2016), no. 2, 241–252.

[7] Abu Baker, A.B., Generalized 3-circular projections for unitary congruence invariant norms, Banach J. Math. Anal. 10 (2016), no. 3, 451–465.

[8] Abu Baker, A.B., Reflexivity of isometries of order n, Colloq. Math. 159 (2020) 2, 223-230.

[9] Abu Baker, A.B. and Maurya, R., Algebraic Reflexivity of isometries on spaces of continuously differentiable functions, Ricerche di Matematica, https://doi.org/10.1007/s11587-021-00593-1.

[10] Abu Baker, A.B. and Maurya, R., Projections in the convex hull of isometries on C 2 [0, 1], Positivity, https://doi.org/10.1007/s11117-021-00860-3.

[11] Abu Baker, A.B. and Maurya, R., Local isometries on subspaces and subalgebras of function spaces, Operators and Matrices, accepted.

[12] Araujo J. and J. J. Front, Linear isometries between subspace of continuous functions, Trans. Amer. Math. Soc. 349 (1997), no. 1, 413-428.

[13] Araujo J. and J. J. Front, Linear isometries on subalgebras of uniformly continuous functions, Proc. Edinburgh Math. Soc. (2) 43 (2000), no. 1, 139–147.

[14] Banach, S., Theorie des Operations Lineares, Chelsea, Warzaw, 1932.

[15] Behrends, E., M-structure and the Banach-Stone theorem, Lecture Notes in Mathe matics 736, Springer-Verlag, Berlin-Heidelberg-NewYork, 1979.

[16] Berkson, E., Hermitian projections and orthogonality in Banach spaces, Proc. London Math. Soc. (3), 24 (1972), 101-118.

[17] Berkson, E. and Porta, H., Hermitian operators and one-parameter groups of isome tries in Hardy spaces, Trans. Amer. Math. Soc., 185 (1973), 331-344.

[18] Berkson, E. and Sourour, A., The Hermitian operators on some Banach spaces, Studia Math., 52 (1974), 33-41.

[19] Botelho, F., Projections as convex combinations of surjective isometries on C(Ω), J. Math. Anal. Appl. 341 (2008), no. 2, 1163-1169.

[20] Botelho, F., Dey, P. and Iliˇsevi´c, D., Hermitian projections on some Banach spaces and related topics, Linear Algebra Appl. 598 (2020), 92-104.

[21] Botelho, F. and Jamison, J. E., Generalized bi-circular projections on minimal ideals of operators, Proc. Amer. Math. Soc. 136 (2008), no. 4, 1397-1402.

[22] Botelho, F. and Jamison, J. E., Algebraic reflexivity of sets of bounded operators on vector valued Lipschitz functions, Linear Algebra Appl. 432 (2010), no. 12, 3337-3342.

[23] Botelho, F. and Jamison, J. E., Projections as averages of isometries on minimal norm ideals, Linear Algebra Appl. 435 (2011), no. 6, 1344-1355. 52 Bibliography

[24] Botelho, F., Jamison, J. E. and Jim´enez-Vargas, A., Projections and averages of isome tries on Lipschitz spaces, J. Math. Anal. Appl. 386 (2012), no. 2, 910-920.

[25] Botelho, F. and Rao, T. S. S. R. K., On algebraic reflexivity of sets of surjective isometries between spaces of weak∗ continuous functions, J. Math. Anal. Appl. 432 (2015), no. 1, 367-373.

[26] Cabello S´anchez, F. and Moln´ar, L., Reflexivity of the isometry group of some classical spaces, Rev. Mat. Iberoamericana 18 (2002), no. 2, 409-430.

[27] Cuka J. and Iliˇsevi´c, D., ˇ Generalized tricircular projections on minimal norm ideals in B(H), J. Math. Anal. Appl. 434 (2016), no. 2, 1813–1825.

[28] Dutta, S. and Rao, T. S. S. R. K., Algebraic reflexivity of some subsets of the isometry group, Linear Algebra Appl. 429 (2008), no. 7, 1522-1527.

[29] Fleming, R. J. and Jamison, J. E., Hermitian operators and isometries on sums of Banach spaces, Proc. Edinburgh Math. Soc. (2), 32 (1989), no. 2, 169-191.

[30] Fleming, R. J. and Jamison, J. E., Isometries on Banach spaces: function spaces, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 129. Chapman & Hall/CRC, Boca Raton, FL, 2003.

[31] Fleming, R. J. and Jamison, J. E., Isometries on Banach spaces. Vol. 2. Vector-valued function spaces, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 138. Chapman & Hall/CRC, Boca Raton, FL, 2008.

[32] Font, J. J., Linear isometries between certain subspaces of continuous vector-valued functions, Illinois J. Math. 42 (1998), no. 3, 389-397.

[33] Foˇsner, M., Iliˇsevi´c, D. and Li, C.-K., G-invariant norms and bicircular projections, Linear Algebra Appl. 420 (2007), nos. 2-3, 596-608.

[34] Gowers, W. T. and Maurey, B., The unconditional basic sequence problem, J. Amer. Math. Soc. 6 (1993), 851-874.

[35] Moln´ar, L., Zalar, B., Reflexivity of the group of surjective isometries on some Banach spaces, Proc. Edinburgh Math. Soc. (2) 42 (1999), no. 1, 17-36.

[36] Moln´ar, L., Reflexivity of the automorphism and isometry groups of C∗ -algebras in BDF theory, Arch. Math. (Basel) 74 (2000), no. 2, 120-128.

[37] Moln´ar, L., Order-automorphisms of the set of bounded observables, J. Math. Phys. 42 (2001), 5904-5909.

[38] Moln´ar, L., Selected preserver problems on algebraic structures of linear operators and on function spaces, Lecture Notes in Mathematics, 1895. Springer-Verlag, Berlin, 2007.

[39] Munkres J. R., Topology, Second edition, Prentice Hall, Inc., Upper Saddle River, NJ, 2000.

[40] Oi, S., Algebraic reflexivity of isometry groups of algebras of Lipschitz maps, Linear Algebra Appl. 566 (2019), 167-182.

[41] Randrianantoanina, B., Norm-one projections in Banach spaces, Taiwanese J. Math. 5 (2001), no. 1, 35-95.

[42] Rao, T. S. S. R. K., Local isometries of L(X, C(K)), Proc. Amer. Math. Soc. 133 (2005), no. 9, 2729-2732.

[43] Semrl, P., ˇ Local automorphisms and derivations on B(H), Proc. Amer. Math. Soc. 125 (1997), 2677-2680.

PAPER NAME

dissertation report.pdf

WORD COUNT	CHARACTER COUNT
7549 Words	**33269 Characters**
PAGE COUNT	FILE SIZE
45 Pages	**874.0KB**
SUBMISSION DATE	REPORT DATE
May 21, 2023 9:21 PM GMT+5:30	**May 21, 2023 9:22 PM GMT+5:30**

12% Overall Similarity

The combined total of all matches, including overlapping sources, for each database.

- 11% Internet database
- 1% Publications database
- Crossref Posted Content database
- 7% Submitted Works database

Excluded from Similarity Report

- Crossref database
- Bibliographic material
- Quoted material
- Cited material
- Small Matches (Less then 10 words)
- Manually excluded text blocks