FOURIER TRANSFORM AND IT'S APPLICATION

A Project Dissertation submitted in partial fulfilment of the requirements for the degree of

MASTER OF SCIENCE IN APPLIED MATHEMATICS

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CANDIDATE'S DECLARATION

We, Hitesh Yadav and Pushpender Kumar, who are currently pursuing a Master of Science in Applied Mathematics with Roll Number 2K21/MSCMAT/22 and 2K21/MSCMAT/37 respectively, hereby declare that the project dissertation submitted by us to the Department of Applied Mathematics at Delhi Technological University to fulfil the requirement for the award of the degree of Master of Science in Applied Mathematics, is original and has not been copied from any source. Furthermore, this work has not been previously used as the basis for conferring a degree, diploma, associate's degree, fellowship, or any other similar title or honour.

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CERTIFICATE

We hereby bear witness that the Project Dissertation submitted by Hitesh Yadav, Roll Number 2K21/MSCMAT/22 and Pushpender Kumar,Roll number 2K21/MSCMAT/37 of the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Master of Applied Mathematics, is a record of the project work completed by the student under my supervision. To the best of my knowledge, neither a portion nor the entirety of this work has ever been submitted to this university or any other institution for a degree or diploma.

Place: Delhi
Date: 24 May 2023

Mr. Jamkhongam Touthang
Supervisor

Abstract

The Fourier Transform is a powerful mathematical tool that decomposes a function or a signal into its constituent frequencies, revealing the underlying frequency content and providing insights into its behavior. This thesis presents a comprehensive exploration of the Fourier Transform, covering its theoretical foundations, various techniques for implementation, and a wide range of applications across different fields.

The thesis begins by providing a comprehensive overview of Fourier series, introducing the key concepts and mathematical foundations. The fundamental properties of periodic functions and the Fourier series representation are elucidated, including the convergence, linearity, and symmetry properties.

Furthermore, this thesis explores the broad spectrum of applications of the Fourier Transform. It examines its role in signal processing, such as filtering, spectral analysis, and noise reduction. Additionally, it discusses the use of Fourier Transform in image processing, including image enhancement, compression, and pattern recognition.

In conclusion, this thesis provides a comprehensive overview of the Fourier Transform, elucidating its theoretical foundations, implementation techniques, and versatile applications. It highlights the transformative impact of the Fourier Transform in diverse fields and emphasizes its significance as a fundamental tool for understanding and manipulating signals and data in the frequency domain.

Acknowledgement

We are incredibly lucky to have had this support and guidance from many people throughout the course of my project because it was essential to its accomplishment and successful completion.

We would like to begin by expressing our sincere appreciation to **Mr. Jamkhongam Touthang** from the **Department of Applied Mathematics** at Delhi Technological University, who played a vital role in mentoring me throughout the project. It was his constant support and guidance that helped me to successfully complete the project within the stipulated time frame. Throughout our journey, he has been a great source of inspiration and motivation for us. He has provided valuable guidance and support that has helped us overcome all the obstacles and challenges that we encountered along the way.

We owe a debt of gratitude to my good departmental friends who have supported us and shown us the way forward for our personal and professional development over time. We want to express our gratitude to our family and friends for always supporting us when we needed it.

We would like to end by thanking God Almighty for seeing to it that no unforeseen problems occurred while we were working.

Contents

Chapter 1

INTRODUCTION

Infinite series are used frequently in modern analysis today.With the introduction of arithmetic operations in the field of numbers, the idea of ”infinite series” originated.There is solid proof that Greek mathematicians like Archimedes, Eudoxous, and others employed infinite series for good in geometry, including calculating the area enclosed by simple curves, the volume of simple bodies, and other subjects.With the exception of a few, mathematicians employed infinite series instead of having a clear understanding of the amount that the series represented because they saw infinite series as a prolongation of the process of computing finite sums.They thought that the typical approach that works for finite sums would ipso facto work for infinite series as well.Due to this idea of infinite series, mathematicians ran into a lot of inconsistencies that caused them to question the validity of mathematics as a whole because of its unfailing ability to arrive at the right result.Mathematicians with taste sought to employ infinite series as little as possible.However, infinite series became a necessary tool after calculus was discovered in the seventeenth century, and the issue became more serious because infinite series could no longer be avoided.The only corrective measures taken at the time by mathematicians were to choose series on which arithmetic operations were unquestionably applicable and to utilise those series in the demonstrations.However, an infinite series’ sum was no longer an arithmetic sum of numbers.When Cauchy’s theory of the convergence of infinite series was developed, it vanished.In 1821, A.L. Cauchy provided a method to calculate the sum of an infinite series based on the idea of limit in his book ”Analyse Algebrique.” According to this concept, let $\sum_{n=0}^{\infty} u_n$ be a given infinite series and s_n denotes the n^{th} partial sum of the series $\sum_{n=0}^{\infty} u_n$. That is

$$s_n = \sum_{n=0}^{n} u_k \quad ; n = 0, 1, 2, 3, ...$$

Suppose there is a number s, such that for any $\epsilon > 0$ there is a natural number m (depends upon ϵ) such that

$$|s_n - s| < \epsilon, for\, all\, n \geq m$$

, then the sum of the series $\sum_{n=0}^{\infty} u_n$ is s,that is the limit of the sequence s_n. Since the limit is unique, no infinite series can have two different sums. Cauchy called a series for which the sum s of an infinite series $\sum_{n=0}^{\infty} u_n$ exists, a convergent series and such of the other series for which the sum based on the above concept did not exist were termed as non-convergent or divergent series. The infinite series that were non-convergent were outside the understandable domain of mathematics and can be treated as useless phantoms of mathematics. As a result, Cauchy's concept of convergence could give a sum to only a few infinite series, which were excluded from the valid domain of mathematics. Because it was so simple and effective, Cauchy's method for calculating the sum of an infinite series was adopted almost without exception by the mathematic community of the time.For a while, it seemed as though the sum and applications of infinite series problem had been entirely and definitively solved. Soon later, it became apparent that some divergent series caused issues with the use of mathematics in physics. Despite the approximation of some series used in dynamical astronomy agreeing with the evaluated data, the actual series was divergent. Even though Fourier series had many uses in wave mechanics, they weren't always convergent in the Cauchy sense. The concept of series convergence put forth by Cauchy had a far-reaching impact and helped put many oddities to rest, but mathematicians soon realised that this was not the end of the road. Researchers resumed their investigation into divergent series.Fejer developed a theorem in 1904 that states the arithmetic mean of the partial sums of a continuous function's Fourier series always converges to the function itself. It didn't take long for curious mathematicians to realise that Cauchy's idea of a sum to the infinite series is only a procedure, and that comparable methods could be created to assign sums to a larger class of infinite series. This mathematical insight led to the discovery of summability methodologies' foundation. Great mathematicians including Abel, Cesàro, Holder, Nörlund, Reisz, Borel, and Hausdorf developed some of the most well-known procedures, which are named after them.The use of functional analysis techniques in recent years has had an impact on a number of contemporary fields, including probability theory, number theory, and functional analysis. It has also aided in the development of a rich and fruitful summability theory, which has helped in the elimination of numerous puzzles from the field of mathematical analysis.

In the preceding sentences, we provide a detailed breakdown of the many concepts and terminologies in the summability theory that are pertinent to the chapter's argument.

P is a summability technique. A series is said to be summable using the procedure P if it assigns a sum to the infinite series $\sum_{n=0}^{\infty} u_n$. The phrase $\sum_{n=0}^{\infty} u_n$ in P is also used. Similar to the last illustration, $\sum_{n=0}^{\infty} u_n$ means that $\sum_{n=0}^{\infty} u_n$ cannot be summed up with P. The idea of convergence has simply been enlarged to include summability. The introduction of absolutely summability methods is analogous to the generalisation of absolutely convergence.

Summability methods are devices to associate a sum in a reasonable way to some non-convergent series.

A summability method will not be worthwhile, if it fails to assign a sum to a series which is not convergent in Cauchy's sense. Further more if the new sum of the series coincides with Cauchy's sum, the method will be more useful. Accordingly we define the followings

(1)A summability technique P is deemed conservative if a series' convergence implies the method's P summability.

(2) If absolute convergence of a series means that it is absolutely summable by the method P, then the method P is said to be absolutely conservative.

(3) If a summability method P is conservative and maintains the sum of the convergent series (which coincides with Cauchy's sum), it is considered to be regular.

(4) If a summability method "P" is both absolutely conservative and regular, it is said to be absolutely regular.

(5) If two summability methods do not sum a series to two different sums, they are considered to be consistent. As usual, "N, Z, Q, R" stands for the set of natural, integer, rational, and real numbers, respectively. "C" or "K" stand for the set of complex numbers, respectively.

(6) Let $\{x_n\}$ be a sequence. Then the difference operator Δ on $\{x_n\}$ is recursively defined by

$$\Delta x_n = x_n - x_{n+1}, n = 0, 1, 2, \ldots.$$
$$\Delta^k x_n = \Delta\left(\Delta^{k-1} x_n\right) \text{ for } k \in N.$$

Basic Technique

The process of summability approaches involves transforming a given infinite series or sequence of partial sums into another series or sequence, which can then be analyzed and evaluated using Cauchy's method as the fundamental technique. Summability approaches fall into one of two groups, depending on the type of change. The T-Process and the φ Process are them.

T-Processes

Let's have a look at the collection of all real or complex number sequences. The sequences over a complex or real field form an infinite-dimensional vector space, which we'll denote as S. "Within this space, we define a linear transformation T that maps the sequence space onto itself. Which is

$$T\left(x\left\{s_n\right\}+y\left\{t_n\right\}\right)=xT\left(\left\{s_n\right\}\right)+yT\left(\left\{t_n\right\}\right)$$

for any two sequences $\left\{s_n\right\},\left\{t_n\right\}$ of the sequence space S and x, y are elements of the field.

A summability method P belongs to the T-process if it is a linear transformation from a sequence space into itself. If P can be represented by an infinite matrix, denoted as $\left(a_{mn}\right)_{\infty\times\infty}$, then we refer to P as a matrix transformation." Thus a matrix transformation is a matrix $P=\left(a_{mn}\right)_{\infty\times\infty}$ that transformations a sequence $\left\{s_n\right\}$ into another sequence $\left\{t_n\right\}$ as follows:

$$\left\{t_n\right\}=P\left\{s_n\right\}$$

or

$$\left(\left\{t_m\right\}\right)=P\left(\left(s_n\right)\right)$$

$$=\left(\left(a_{mn}\right)\right)\left(\left(s_n\right)\right)$$

$$t_m=\sum_{n=0}^{\infty}a_{mn}s_n, m=0,1,2,\ldots$$

The necessary and sufficient condition for the matrix method $P = (a_{mn})_{\infty\times\infty}$ to be regular are :

$$\sup_m \sum_{n=0}^{\infty} |a_{mn}| \leq H$$

where H is an absolute constant,

$$\lim_{m\to\infty} a_{mn} = 0, \text{ for every fixed } n,$$
$$\lim_{m\to\infty} \sum_{n=0}^{\infty} a_{mn} = 1$$

Toeplitz initially established these conditions for triangular matrices, while Steinhaus later demonstrated that they also hold true for general matrices. If the condition holds for every value of $m = 0, 1, 2, \ldots$, the matrix T is referred to as completely regular.

Sequence of partial sums $\{s_n\}$ of a series $\sum_{n=0}^{\infty} u_n$ with its is said to be absolutely P-summable if the P-transform of $\{s_n\}$ is a function of bounded variation. Therefore $\sum_{n=0}^{\infty} u_n$ is absolutely P-summable or simply $\sum_{n=0}^{\infty} u_n \in |P|$ if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| = O(1)$$

where t_n is as definition. The necessary and sufficient conditions for absolutely regularity of P-methods are:

$$\sum_{n=0}^{\infty} a_{mn} \text{ is convergent for all } m$$
$$\sum_{m=1}^{\infty} \left| \sum_{n=1}^{p} (a_{mn} - a_{m-1,n}) \right| \leq K$$

K is an absolute constant.

Chapter 2

FOURIER SERIES

The synthetic theory has had a significant impact on Fourier analysis. It became evident that Cauchy's method of assigning sums to infinite series was inadequate when applied to Fourier series. This led to the development of summation theory, which aimed to determine the value of the generating function whenever a Fourier series of a continuous function converged. Fejer and Lebesgue's findings confirmed that the Fourier series of any summable function $(C, 1)$ could be summed almost everywhere. This result resolved the problem of convergence that Cauchy's method was unable to address.

The recognition of this fact sparked the rapid advancement of synthesis theory. Fourier analysis greatly benefited from the utilization of summability methods, as it allowed for the resolution of various anomalous situations encountered in both ordinary and absolute summability methods. Simultaneously, the application of summability methods to infinite series contributed significantly to their progress and refinement.

Given an integrable function $f(x)$ of period 2π, the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called the Fourier series of $f(x)$; where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kxdx, \text{ for } k = 1, 2, 3, \ldots \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kxdx, \text{ for } k = 1, 2, 3, \ldots
\end{aligned}$$

$\{a_k\}$ and $\{b_k\}$ are called Fourier constant of $f(x)$. This indicates that the series is the Fourier series of $f(x)$ and then we write

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

As per Hurwitz, it is crucial to emphasize that there is no assumption made regarding the series being convergent or converging to $f(x)$ when expressing it. However, it is known that if $f(x)$ is a periodic function with period 2π, integrable over the interval $[-\pi, \pi]$, and exhibits continuity or finite discontinuity at a particular point, there exists a positive value δ such that for all h in the interval $[0, \delta]$, the two ratios

$$\frac{f(x+h) - f(x+)}{h} \quad \text{and} \quad \frac{f(x-h) - f(x-)}{h}$$

are integrable over the interval $[0, \delta]$.

Let

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$
$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(t)$$

be the Fourier series of $f(t)$. series

$$\sum_{m=1}^{\infty}(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t)$$

is called conjugate series of the Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

Let $\{\lambda_n\}$ be a sequence of real constants. Then the string $\sum \lambda_n B_n(x)$ is called the associative conjugate.

We use the following notation

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$$

and

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

We can easily have

$$A_n(t) = \frac{2}{\pi}\int_0^\pi \phi(t)\cos ntdt$$

and

$$A_n(t) = \frac{2}{\pi}\int_0^\pi \varphi(t)\cos ntdt$$

We also have

$$\Psi_0(t) = \psi(t)$$
$$\Psi_\beta(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-u)^{\beta-1}\psi(u)du, \beta > 0$$
$$\psi_\beta(t) = \Gamma(\beta+1)t^{-\beta}\bar{\Psi}_\beta(t), \beta \geq 0$$
$$[x] = \text{ greatest integer not exceeding } x,$$
$$U = \left[\frac{1}{u}\right], \tau = \left[\frac{1}{t}\right].$$

2.1 METHODS OF SUMMABILITY

CESÀRO METHOD OF SUMMABILITY

There is a familiar particularization of T-process where we defined in , we take

where $A_n^\alpha = \begin{pmatrix} n+\alpha \\ \alpha \end{pmatrix}$

We obtain a specific summability technique known as the Cesàro method of By applying a specific sequence-to-sequence transformation, we obtain a summability technique known as the Cesàro method of summation. In this method, the sequence of partial sums of the infinite series is denoted as s_n, while the transformed sequence is represented by $\sum u_n$.

$$V_n = \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}s_k, n \in N$$

Cesàro is credited with developing the summing method for positive integral orders.

Knopp later expanded it to include all positive fractional orders, and Chapman and Hardy and Chapman expanded it to include negative orders > -1.

It is obvious that the Cesàro technique of order $0.$, which is equivalent to convergence and absolute convergence, respectively, is (C, O) and $|C, 0|$. The $(C, 1)$ approach and the method of summing by arithmetic methods are same as well. It is possible to confirm the regularity and consistency of all Cesàro procedures for summing of positive order $(\alpha > 0)$.

Fekete established the concept of the absolute Cesàro technique of summing for positive integral orders. Then Kogbetliantz widened it to incorporate all orders. He also defined the inclusion relations for both ordinary and absolute Cesàro summabilities for $\beta \geq \alpha > -1$.

$$(C, \alpha) \subseteq (C, \beta)$$

and

$$|C, \alpha| \subseteq |C, \beta|$$

NÖRLUND METHOD OF SUMMABILITY

In the T-method, if we take,

$$a_{nk} = \begin{cases} \frac{p_{n-k}}{P_n}, & \text{if } P_n \neq 0 \text{ for } k \leq n \\ 0, & \text{if } k > n \end{cases}$$

where $\{p_n\}$ is a sequence of complex or real constants and $P_n = \sum_{k=0}^{n} p_k \neq 0$, we obtain a synthesis method known as Nörlund synthesis. Thus, the sequence transformation

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k, \quad P_n \neq 0, \quad n \in \mathbb{N}$$

determines the Nörlund mean of the series $\sum u_n$ or the sequence $\{s_n\}$ generated by the coefficient series $\{p_n\}$. If $\lim_{n\to\infty} t_n = s$ is a finite number, then the series $\sum u_n$ is said to be Nörlund summable, or more specifically, (N, p_n) can be summed to s. Furthermore if

$$\sum |t_n - t_{n-1}| < \infty$$

the string $\sum u_n$ is said to be completely composable in a Nörlund fashion, or simply $|N, p_n|$ is composable in s. The regularity conditions of this method are

$$\text{(i)} \quad \lim_{n\to\infty} \frac{p_n}{P_n} = 0$$

and

$$\text{(ii)} \sum_{k=0}^{n} |p_k| \leq C\,|P_n|$$

where C is a constant.

In 1919, Nörlund introduced the method a independent way. Since then, the method has remained widely identified by its name. If special, we take

$$P_n = A_n^{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma\alpha}, \alpha > 0$$

then the method (N, p_n) reduces to the method (C, α).

RIESZ SUMMABILITY

If in the defined T method, we take

$$a_{nk} = \left\{ \begin{array}{ll} \frac{p_k}{P_n}, P_n \neq 0, & \text{for } k \leq n \\ 0, & \text{if for } k > n \end{array}, \right\}.$$

where $\{p_n\}$ is a sequence of real or complex constants and $P_n = \sum_{k=0}^{n} p_k \neq 0$, we get a summation method called Riesz summability or $\left(\bar{N}, p_n\right)$ summability. If $\{t_n\}$ converges to s, a finite numbers, then the sequence $\sum u_n$ is Riesz composable or $(\ barN, p_n)$ - can add up to s. Also, if $\{t_n\} \in BV$, ie

$$\sum |t_n - t_{n-1}| < \infty$$

The sequence-to-sequence transformation given by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k, P_n \neq 0, n \in N$$

determines the average $(\bar{N}, p_n)$ of the series $\sum u_n$ or the sequence $\{s_n\}$ with the coefficient sequence $\{p_n\}$. the string $\sum u_n$ is said to be fully composable according to Riesz, or simply $\left|\bar{N}, p_n\right|$ is composable to s

Specifically, if we take $p_n = 1$, for every n then $(\bar{N}, p_n)$ means reduce to $(C, 1)$ means . Also, for $p_n = e^n$ for every n, $(\bar{N}, p_n)$ is equivalent to convergence.

The regularity conditions of the method $(\bar{N}, p_n)$ are

$$\text{(I)} \qquad \lim_{n \to \alpha} |P_n| = \infty$$

And

$$\text{(ii)} \qquad \sum_{k=0}^{n} |p_k| \leq C\,|P_n|$$

where C is an absolute constant. This summing method was first introduced by Riesz in 1909.

ABEL SUMMABILITY

If $\sum u_n$ is an infinite series with a sequence of partial sums $\{s_n\}$, we can define a sequence transformation into a function by

$$t(x) = (1 - x) \sum_{n=0}^{\infty} x^n s_n$$

where $t(x)$ is assumed to exist for all x in the domain $0 \leq x < 1$, we define the functional transformation $t(x)$ of Abel summation for the series $\sum u_n$. If $\lim_{x \to 1} t(x) = b$, a finite number, then the sequence $\sum u_n$ is said to be Abel composable, or simply possible sums up (A), to s. Furthermore, if $t(x) \in BV$ in $[0, 1)$ then the sequence $\sum u_n$ called Abel is completely composable, or simply $|A|$-is composable.

Abel's method is widely recognized as a robust and effective summation technique, regarded as both conventional and absolute. It encompasses the methods of Cesàro and Nörlund, making it particularly powerful. However, it is important to note that Whittakar has demonstrated through examples that the convergence of a series does not always guarantee the existence of a sum under the $|A|$ (Abel) method. Even before Abel, mathematicians extensively employed this method to assign sums to infinite series. Euler, in particular, frequently utilized this technique to assign finite values to non-convergent series. With the introduction of the concept of a limit by Cauchy, Abel precisely described this method in terms of limits, and as a result, it became associated with his name.

(N, p, q) SUMIMABILITY METHOD

$$\text{Let} \quad a_{mn} = \begin{cases} \frac{p_{m-n}q_n}{r_m} \text{ for } n \le m, \\ 0 \text{ for } n > m, \end{cases} ,$$

where $\{p_n\}$ and $\{q_n\}$ are two sequences of real or complex constants such that

$$r_m = \sum_{n=0}^{m} p_{m-n}q_n \neq 0, \text{ for all } m$$

then convert the sequence to the sequence given by

$$t_m = \frac{1}{r_m} \sum_{n=0}^{m} p_{m-n}q_n s_n$$

is m-th (N, p, q) mean of $\{s_n\}$. This method was first introduced by Borwein. In this method if we put $\{p_n\} = \{1\}$ then reduces to $(\bar{N}, q)$ mean of $\{s_n\}$ and if we put $\{q_n\} = \{1\}$, then it reduces to (N, P) mean of $\{s_n\}$

Chapter 3

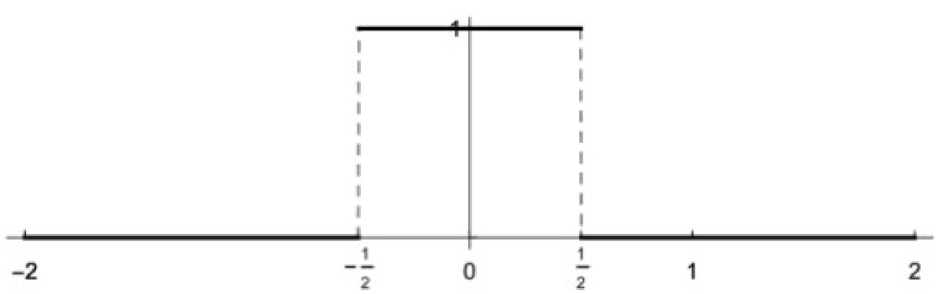
FOURIER TRANSFORM

When transitioning from Fourier series to Fourier transform, the term ”transition” is appropriate because it signifies the shift from analyzing periodic functions to aperiodic functions. As we extend the period of a function, we consider aperiodic functions as a limiting case. However, this process does not directly yield the desired result. Obtaining the Fourier transform from the Fourier coefficients requires some additional adjustments, but it leads to a smooth transition and an intriguing exploration.

Example: Square function and it’s Fourier transform. Let’s look at a concrete, simple example. Consider the rectangle function defined by , or “Rect” for short.

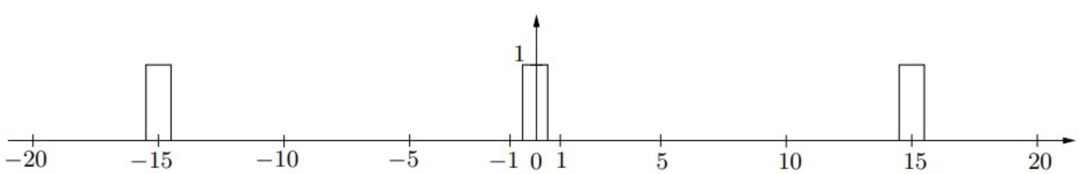
$$\Pi(s) = \begin{cases} 1, & |s| < 1/2 \\ 0, & |s| \geq 1/2 \end{cases}$$

This is a not-so-complicated graphic.

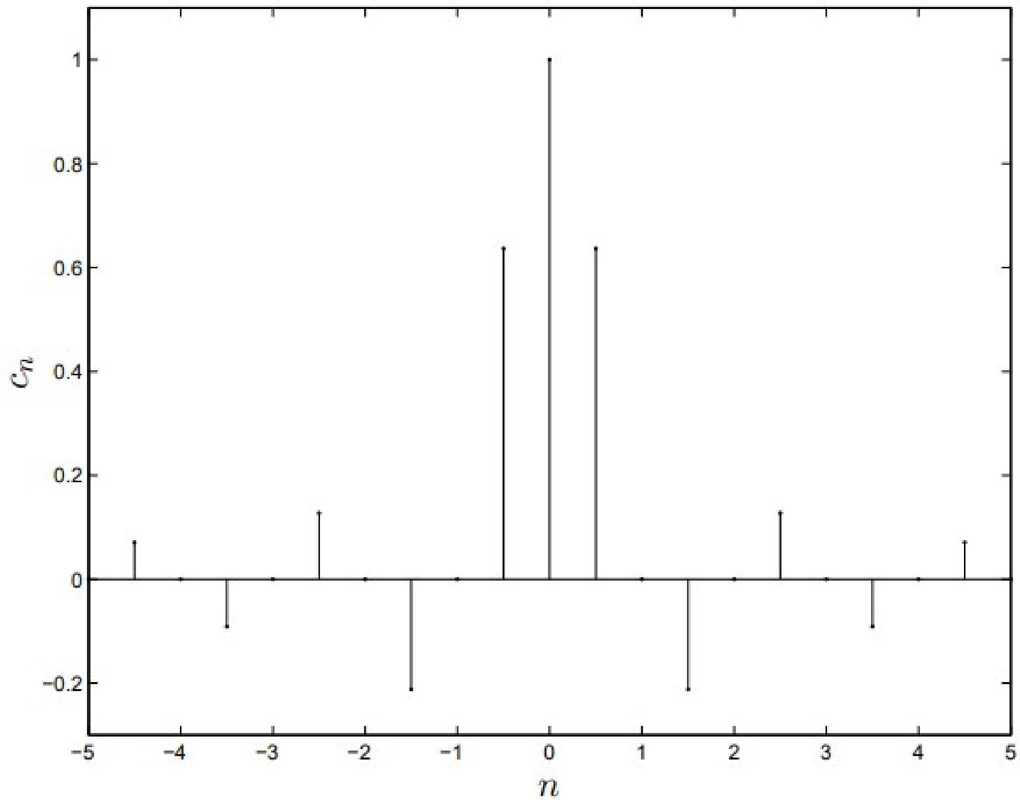


$\Pi(t)$ is even - centerod at the origin - and has width 1 . One function that we will consider is $\Pi(t)$, which can be envisioned as representing a switch that is turned on for one second and off for the remaining time. Π is commonly referred to as the top hat function due to its graphical shape.

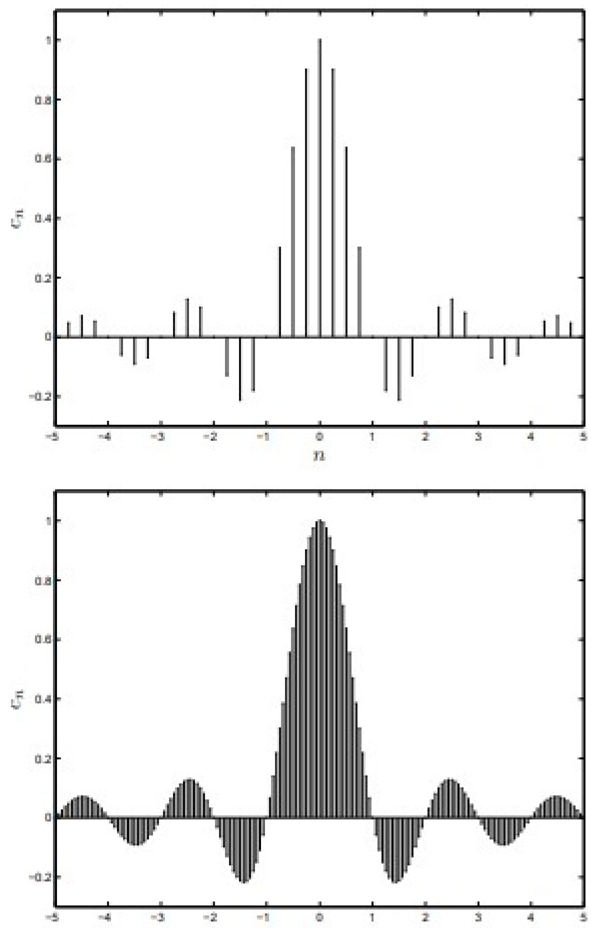
$\Pi(t)$ is not periodic. There is no Fourier series. In case you're having trouble, I've experimented with periodization a bit and want to do it with Π for a specific purpose.$\Pi(t)$, when viewed as a periodic version, repeats the non-zero segment of the function at regular intervals, with longer intervals where the function is zero. A visualization of this can be imagined by turning the switch on for 1 second and then repeating this pattern, leaving it off for an extended period while intermittently turning the switch on. This concept is often associated with the term "duty cycle." Below is a plot of $\Pi(t)$ periodized with a period of 15.



The provided graphs depict the Fourier coefficients of periodic rectangular functions with periods 2, 4, and 16. Since the function is real and even, the Fourier coefficients are real in all cases. Therefore, these graphs display the actual coefficients, and the magnitude squared is not shown.



1
0.8
0.6
0.4
0.2
0
-0.2
c_n
−5
−4
−3
−2
−1
0
1
2
3
4
5
n



It can be seen that the frequencies get closer and closer as the period increases. It will be as follows. Indeed, the coefficients of the Fourier series exhibit a specific pattern or curve for the given example. Analyzing this particular example and combining it with further examples can lead to general statements and insights. By studying the behavior of the coefficients, we can uncover broader principles and understanding in the context of Fourier analysis.

The Fourier series has the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T}$$

In the spectrum of a signal, the frequencies are spaced apart by $1/T$, where T represents the period of the signal. As the period T increases, the points in the spectrum become more closely packed, leading to a higher density of frequencies. This observation is evident in the provided pictures, where the spacing between the points in the spectrum becomes smaller as the period T increases. The n^{th} Fourier coefficient is given by

$$c_n = \frac{1}{T}\int_0^T e^{-2\pi int/T} f(t)dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)dt$$

We can find the Fourier coefficient for $\Pi(t)$:

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T}\Pi(t)dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi int/T}\cdot 1dt \\ &= \frac{1}{T}\left[\frac{1}{-2\pi in/T}e^{-2\pi int/T}\right]_{t=-1/2}^{t=1/2} = \frac{1}{2\pi in}\left(e^{\pi in/T} - e^{-\pi in/T}\right) = \frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right). \end{aligned}$$

The spectrum is represented by a discrete set of points indexed by n, where each point in the spectrum corresponds to n/T for $n = 0, \pm 1, \ldots$.

We're approaching the desired point, but we haven't quite reached it yet. If we intend to consider a limit such as $T \to \infty$ for every n, then n/T will become very small as T becomes very large,

$$\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right) \text{ is about size } \frac{1}{T} \quad (\text{remember } \sin\theta \approx \theta \text{ if } \theta \text{ small})$$

That is, for each n of this so-called transformation,

$$\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right)$$

It tends to 0, like $1/T$. To compensate for this, we scale up by T. i.e. instead of

$$(\text{ periodicized } \Pi)\left(\frac{n}{T}\right) = T\frac{1}{\pi n}\sin\left(\frac{\pi n}{T}\right) = \frac{\sin(\pi n/T)}{\pi n/T}.$$

In fact, the scaled transformation plot is shown above. And if T is large, you can consider replacing the dense discrete points n/T with a continuous variable like s.

$$(\text{ periodicized } \Pi)(s) = \frac{\sin \pi s}{\pi s}.$$

$$
\begin{aligned}
\left(\frac{n}{T}\right) &= T \cdot c_n \\
&= \int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t)dt.
\end{aligned}
$$

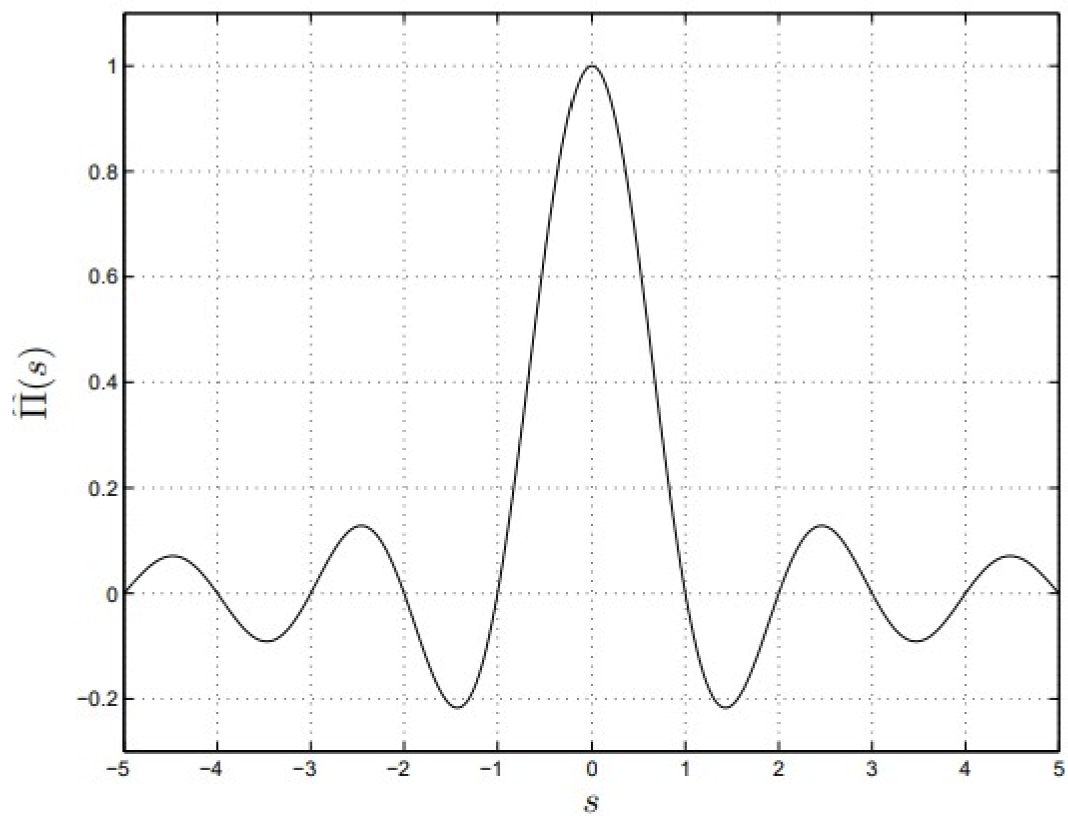
Now, let's envision the scenario where we take the limit as T approaches infinity and replace the discrete variable n/T with a continuous variable s. We can also shift the limits of integration to $-\infty$ and $+\infty$. In this context, the function Π would refer to the rectangular function defined previously. The (limited) conversion formula for

$$
\widehat{\Pi}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} \Pi(t)dt
$$

Let's solve the integral. (I've seen the discrete form before, so I know what the answer is.)

$$
\widehat{\Pi}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} \Pi(t)dt = \int_{-1/2}^{1/2} e^{-2\pi s t} \cdot 1dt = \frac{\sin \pi s}{\pi s}
$$

Here are the graphics. Now, if we observe the continuous curve, it will closely follow and shadow the plot of the discrete scaled Fourier coefficients.



A function $\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ is encountered frequently in this subject and is commonly referred to as the sinc function.

$$\operatorname{sinc} t = \frac{\sin \pi t}{\pi t}$$

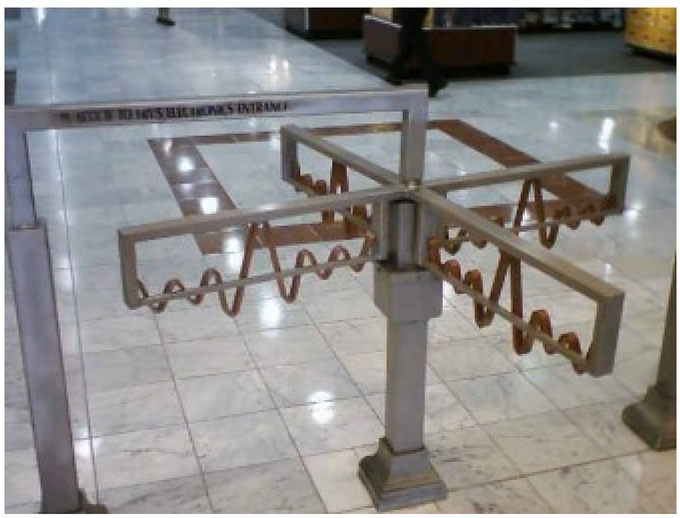
As

$$\operatorname{sinc} 0 = 1$$

by well-known limit

$$\lim_{t \to 0} \frac{\sin t}{t} = 1$$

It is quite common for electrical engineers to be familiar with and encounter the sinc function frequently in their field of study and work.



How common is that? If you start periodicizing almost any function with the intention of $T \to \infty$ you'll run into the same idea, namely scaling Fourier coefficients with T. Suppose $f(t)$ is outside $|t|.Zero \leq 1/2$. (It can be any interval; we just want to assume that the function outside the interval is zero.)

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi int/T} f(t)dt = \frac{1}{T}\int_{-1/2}^{1/2} e^{-2\pi int/T} f(t)dt.$$

$$\begin{aligned}
|c_n| &= \frac{1}{T}\left|\int_{-1/2}^{1/2} e^{-2\pi int/T} f(t)dt\right| \\
&\leq \frac{1}{T}\int_{-1/2}^{1/2} \left|e^{-2\pi int/T}\right| |f(t)|dt = \frac{1}{T}\int_{-1/2}^{1/2} |f(t)|dt = \frac{A}{T},
\end{aligned}$$

where

$$A = \int_{-1/2}^{1/2} |f(t)| dt$$

which remains constant regardless of the values of n and T. It is worth noting that c_n approaches zero as $1/T$, so we can rescale it by multiplying with T and examine

$$(\text{ Scaled transform of } f) \left(\frac{n}{T}\right) = Tc_n = \int_{-T/2}^{T/2} e^{-2\pi int/T} f(t) dt$$

In the limit as $T \to \infty$ we replace n/T by s and consider As we take the limit of T approaching infinity, we substitute n/T with s and consider the following expression:

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t) dt$$

We have now arrived at the integral formula for the Fourier transform.

The Fourier transform of a function $f(t)$ is defined as follows:

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t) dt$$

Let's consider this as an initial definition. We will later explore the conditions under which such an integral exists. Suppose we have a function $f(t)$ defined for all real numbers t. For each s in the set of real numbers $\mathbb{R}$, we can evaluate the integral of the product of $f(t)$ and $e^{-2\pi ist}$ with respect to t. This integral results in a complex-valued function of s, denoted as $\hat{f}(s)$, which represents the Fourier transform of $f(t)$. It is worth noting that if t has the dimension of time, then s must have the dimension of $1/\text{time}$ to ensure that the argument st of the exponential function $e^{-2\pi ist}$ is dimensionless.

Fourier transforms can escape the desire to find spectral information about aperiodic functions, but the added complexity and richness of the results will quickly feel like you're in a whole other world. . The definitions given here are good in that they are rich in content, albeit complex.

The spectrum of a periodic function consists of a discrete set of frequencies, which may be infinite if there are sharp transitions in the function. On the other hand, the Fourier transform of an aperiodic signal results in a continuous spectrum, representing a continuum of frequencies. In some cases, the Fourier transform $\hat{f}(s)$ can be identically zero for sufficiently large values of $|s|$, which is a characteristic of a significant class of signals known as bandlimited signals. Alternatively, a non-zero value of $\hat{f} expands(s)$ to $\pm\infty$ or $\hat{f}(s)$ expands to some value of s can be null only.

The Fourier transform allows us to break down a signal into its constituent frequency components. However, the process of reconstructing the original signal in the time domain from its frequency domain representation might not be immediately clear. How can we recover the time domain signal $f(t)$ using the information provided in the frequency domain $\hat{f}(s)$?

Recovering $f(t)$ from $\hat{f}(s)$

To retrieve the original function $f(t)$ from its Fourier transform $\hat{f}(s)$, we can utilize the idea of representing non-periodic functions as limits of periodic functions. Suppose we have a situation where $f(t)$ is zero outside a specific interval." By introducing a large period T and making $f(t)$ periodic, we can express it as a Fourier series expansion,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}$$

The Fourier coefficients can be expressed in terms of the Fourier transform of f calculated at the points $s_n = n/T$.

$$\begin{aligned} c_n =& \frac{1}{T}\int_{-T/2}^{T/2} e^{-2\pi i n t/T} f(t) dt = \frac{1}{T}\int_{-\infty}^{\infty} e^{-2\pi i n t/T} f(t) dt \\ & \text{(we can extend the limits to } \pm\infty \text{ since } f(t) \text{ is zero outside of } [-T/2, T/2]) \\ =& \frac{1}{T}\hat{f}\left(\frac{n}{T}\right) = \frac{1}{T}\hat{f}(s_n). \end{aligned}$$

Putting into the expression for $f(t)$:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}\hat{f}(s_n) e^{2\pi i s_n t}$$

Now, if we consider the points $s_n = \frac{n}{T}$, we observe that these points are spaced $\frac{1}{T}$ apart. To simplify notation, we can denote $\frac{1}{T}$ as Δs." Then, the sum above can be interpreted as a Riemann sum, which approximates the integral of the function.

$$\sum_{n=-\infty}^{\infty} \frac{1}{T}\hat{f}(s_n) e^{2\pi i s_n t} = \sum_{n=-\infty}^{\infty} \hat{f}(s_n) e^{2\pi i s_n t}\Delta s \approx \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i s t} ds.$$

As the period T approaches infinity, the spacing between the points s_n becomes infinitesimally small. Consequently, the Riemann sum approaches an integral, and the limits on the integral extend from negative infinity to positive infinity. This allows us to capture the entire frequency spectrum of the function $f(t)$, including both positive and negative frequencies. we expect

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(s)e^{2\pi ist}ds$$

By applying the inverse Fourier transform, we have successfully retrieved the original function $f(t)$ from its Fourier transform $\hat{f}(s)$.

The inverse Fourier transform, denoted as $g(s)$, is defined as the function obtained by applying the inverse Fourier transform operation to a given function. The integral expression we derived earlier can be considered as a standalone "transform that computes the inverse Fourier transform.That is

$$\check{g}(t) = \int_{-\infty}^{\infty} e^{2\pi ist}g(s)ds \quad \text{(upside down hat - cute)}$$

Again, for the moment we will treat this formally and not discuss the conditions under which the integral makes sense. With this in mind, I also created the Fourier Inversion Theorem. That is

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi ist}\hat{f}(s)ds$$

$$(\hat{f})^2 = f$$

The inverse Fourier transform bears a resemblance to the Fourier transform, with the main difference being the presence of a minus sign." In subsequent discussions, we will delve deeper into the remarkable symmetry that exists between the Fourier transform and its inverse.

By the way, you can also start with $\hat{f}$ as a primitive instead of f and consider the whole argument above. Doing this gives the complementary result of the inverse Fourier transform.

$$(\check{g}) = g.$$

Here's a brief summary of our progress in the guide so far. We have covered essential concepts that are crucial to understanding the topic. It is challenging to condense all the information into a concise summary, and we appreciate your patience as we work on completing the guide.

- The Fourier transform of $f(t)$ is

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ist}dt$$

This function $\hat{f}(s)$ is a function with complex values for different values of s. However, it's worth noting that one particular value is easy to compute and worth highlighting: when $s = 0$, we have

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(t)dt$$

The integral of a function represents the area under its graph. In calculus, the concept of integration involves calculating the area beneath the graph of a function. "When applying the Fourier transform to a real-valued function, such as $f(t)$, the Fourier transform value at $\hat{f}(0)$ is also a real number. This is because $\hat{f}(0)$ represents the average value or the DC component of the function, which is a real quantity. However, it is worth noting that other values in the Fourier transform may be complex numbers, indicating the existence of frequency components with both magnitude and phase information.

- To add to that, the spectrum of a signal represents the distribution of energy of the signal across all frequencies. In other words, the spectrum describes how much of the signal's energy is concentrated in different frequency bands. The Fourier transform provides a way to analyze a signal and determine its spectrum, which is often useful in a variety of applications, including signal processing, communications, and control systems.

- Not all frequencies necessarily appear in the Fourier transform $\hat{f}(s)$ of a function. The value of $\hat{f}(s)$ can be zero for certain values of s, indicating the absence of those frequencies in the signal.

$$\hat{f}(s) = 0 \quad \text{for } |s| \text{ large}$$

- The inverse Fourier transform is defined by

$$\check{g}(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} g(s) ds$$

Combining the Fourier transform and its inverse, the Fourier inversion theorem provides a way to switch between two (equivalent) representations of a signal.

One important consequence of Fourier inversion is that

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(s) ds$$

It is difficult to computationally comprehend this outcome. The integral of a complex-valued function is represented on the right-hand side of the equation, but the outcome is real (presuming that $f(0)$ is real).

The Fourier transform of a periodic function is the accumulation of δ functions. The same is required of us, but it requires work. where $\hat{f}(s)$ is a complex-valued function that has been converted and may be equivalent" to $f(t)$, but has entirely distinct characteristics. Do keep that in mind. Is it accurate to say that we can just put $\hat{f}(s)$ into the inverse Fourier transform equation if it exists? With the exception of the negative sign forward transformation, this is likewise a faulty integral. - And does $f(t)$ actually yield anything? Really? It merits consideration.

- The squared quantity $|\hat{f}(s)|^2$ is called the power spectrum (especially when used in communications) or the power spectral density (especially when used in optics). spectrum (especially in other contexts).

The Parseval identity in the Fourier transform establishes an important relationship between the energy of a signal in the time domain and its energy distribution in the frequency domain.

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(s)|^2 ds$$

Notational warning: Please note that the notational choices in the context of the Fourier transform are not perfect and vary depending on the specific operation and circumstances. This can lead to frustration and confusion due to the need to switch between transforms and their inverses, variable naming conventions, presence or absence of variable representation, alteration of signs, and incorporation of complex conjugates. These routine operations, if not carefully managed, can result in significant confusion. To illustrate this point, I can provide several examples that highlight the common challenges and complaints associated with Fourier transform notation.

In the context of the Fourier transform, it is common to use uppercase letters, such as F, to represent the transformed function when the original function is denoted by a lowercase letter, like f. This convention applies to various variables, such as a and A, z and Z", and so on. However, it's important to note that the variables in the original and transformed functions usually have different names, for example, $f(x)$ (or $f(t)$) and $F(s)$. This use of "capitalization" is prevalent in engineering, but it often leads to confusion regarding the concept of "duality" explained below.

Moreover, when we view the Fourier transform as an operation that takes a function and produces a new function, it can be beneficial to represent this operation using specific notation.For instance, it is common to express $\hat{f}(s)$ as $\mathcal{F}f(s)$, which fully indicates the transformation being applied. The complete definition can be defined as:

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi ist} f(t)dt$$

This notation provides clarity and helps minimize ambiguity. Similarly, the process of calculating the inverse Fourier transform is symbolized as $\mathcal{F}^{-1}$, emphasizing the inverse relationship of the transformation.So,

$$\mathcal{F}^{-1}g(t) = \int_{-\infty}^{\infty} e^{2\pi ist} g(s)ds$$

Use the $\mathcal{F}f$ notation more often. Again, this is far from ideal as keeping variables straight is a problem - as you can see. After all, a function and its Fourier transform must form a "Fourier pair". To represent this sibling relationship, various notations have been devised. one is

$$f(t) \rightleftharpoons F(s)$$

It is important to note that while the canonical definition of the Fourier transform exists, it is not the only one. There are different conventions regarding where to include the factor of 2π. Some approaches incorporate it as an exponential factor, while others treat it as a separate element or omit it altogether. Additionally, there is the question of which operation represents the Fourier transform and which represents its inverse, including the consideration of the minus sign in the exponential conversion. These various conventions and rules are employed in the field on a daily basis. I mention this to ensure that when discussing Fourier transforms with a friend, you are both aware of the rules and avoid misunderstandings.

Chapter 4

APPLICATION OF FOURIER TRANSFORM

The exploration of the Fourier transform can be compared to the study of calculus in many respects. When delving into calculus, individuals begin by comprehending the distinct formulas for differentiation and integration applicable to various functions and function types, such as powers, exponentials, and trigonometric functions. Moreover, they acquire knowledge of the fundamental principles and rules of differentiation and integration, including concepts like the product rule, chain rule, and inverse functions, enabling them to handle more intricate combinations of functions. Similarly, in the examination of the Fourier transform, it is necessary to establish a collection of specific functions and their transforms that one can rely upon, while concurrently developing general principles and outcomes pertaining to the operations and characteristics of the Fourier transform.

Examples

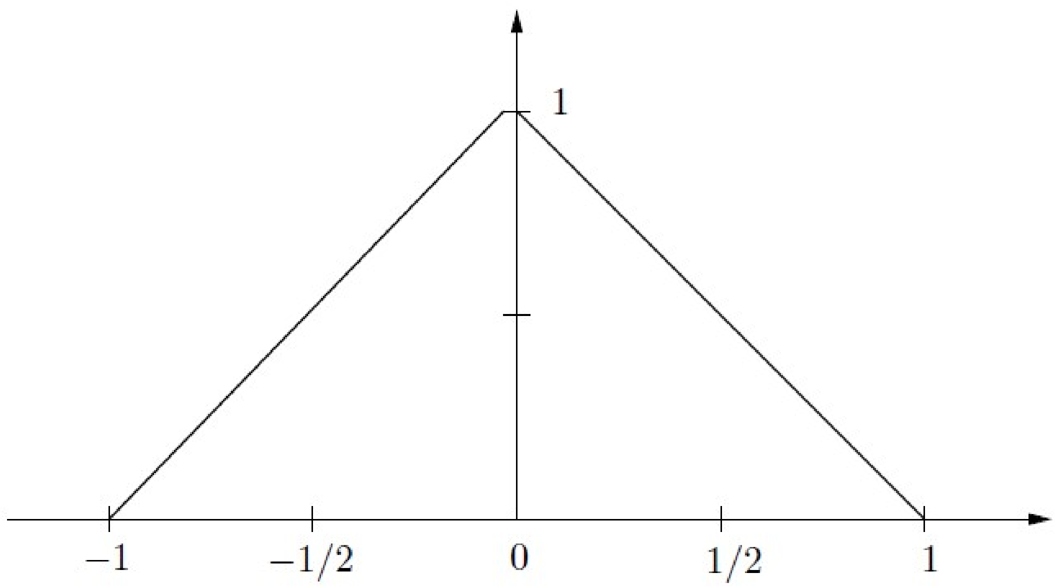
We have previously examined the example of Fourier transform of the rectangular function:

$$\widehat{\Pi} = \text{sinc} \quad \text{or} \quad \mathcal{F}\Pi(t) = \text{sinc}\, t$$

utilizing the $\mathcal{F}$ notation. Now, let's explore few additional examples.

The triangle function , defined by

$$\Lambda(y) = \begin{cases} 1 - |y| & |y| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

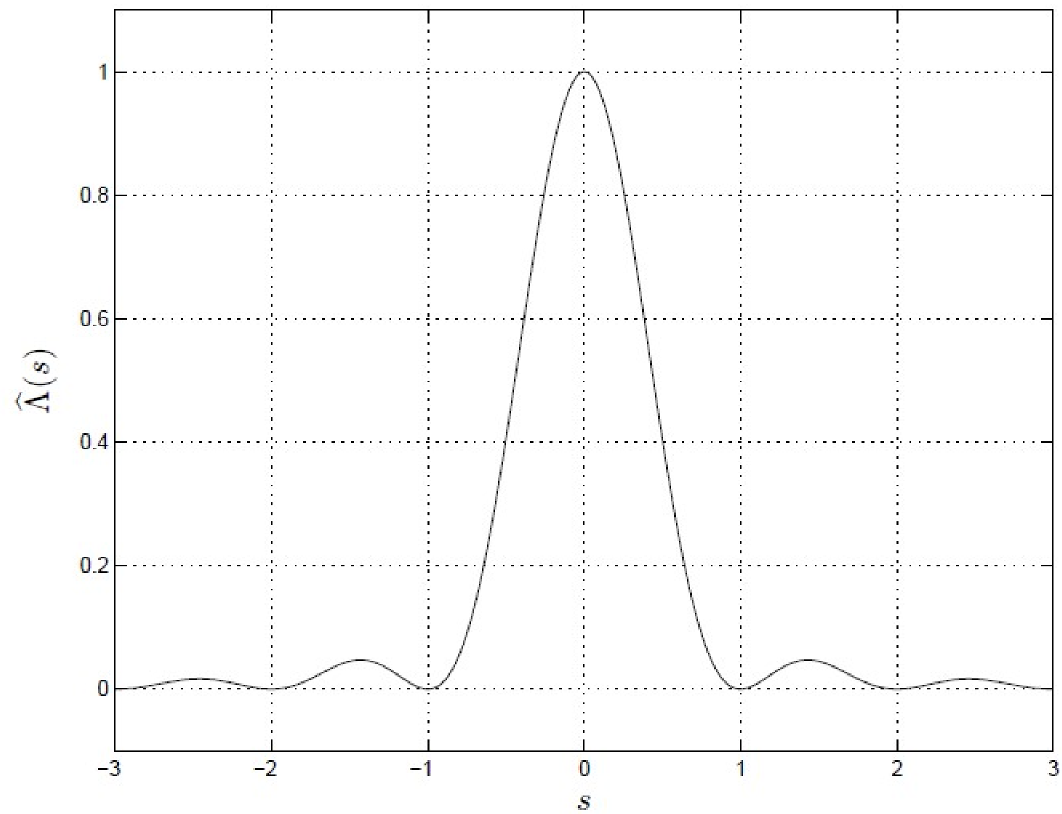


Now, For the Fourier transform we compute :

$$\begin{aligned}
\mathcal{F}\Lambda(t) &= \int_{-\infty}^{\infty} \Lambda(y) e^{-2\pi i t y} dy = \int_{-1}^{0} (1+y) e^{-2\pi i t y} dy + \int_{0}^{1} (1-y) e^{-2\pi i t y} dy \\
&= \left(\frac{1 + 2i\pi t}{4\pi^2 t^2} - \frac{e^{2\pi i t}}{4\pi^2 t^2} \right) - \left(\frac{2i\pi t - 1}{4\pi^2 t^2} + \frac{e^{-2\pi i t}}{4\pi^2 t^2} \right) \\
&= -\frac{e^{-2\pi i t} \left(e^{2\pi i t} - 1 \right)^2}{4\pi^2 t^2} = -\frac{e^{-2\pi i t} \left(e^{\pi i t} \left(e^{\pi i t} - e^{-\pi i t} \right) \right)^2}{4\pi^2 t^2} \\
&= -\frac{e^{-2\pi i t} e^{2\pi i t} (2i)^2 \sin^2 \pi t}{4\pi^2 t^2} = \left(\frac{\sin \pi t}{\pi t} \right)^2 = \operatorname{sinc}^2 t.
\end{aligned}$$

It is no coincidence that the Fourier transform of trigonometric functions is the square of the Fourier transform of rectangular functions. This has to do with convolution.

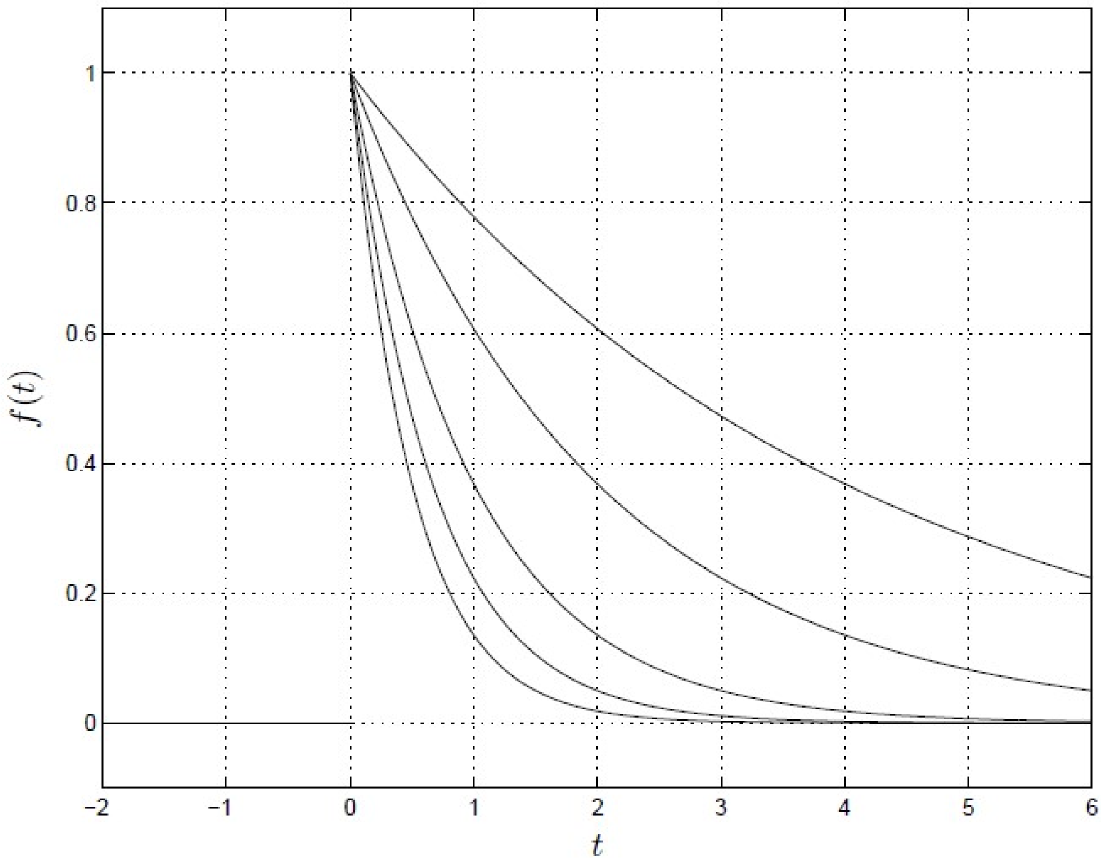
The graph of $\operatorname{sinc}^2 s$ is:



The exponential decay, which is frequently encountered, can be defined as follows:

$$f(x) = \begin{cases} 0 & y \leq 0 \\ e^{-ay} & y > 0 \end{cases}$$

where 'a' is a constant that is positive. The function represents a signal that initiates from zero, gradually increases, and then exponentially decays. The graphs for $a = 2, 1.5, 1.0, 0.5, 0.25$ are shown below.



To return to the exponential decline, we may easily determine its Fourier transform.

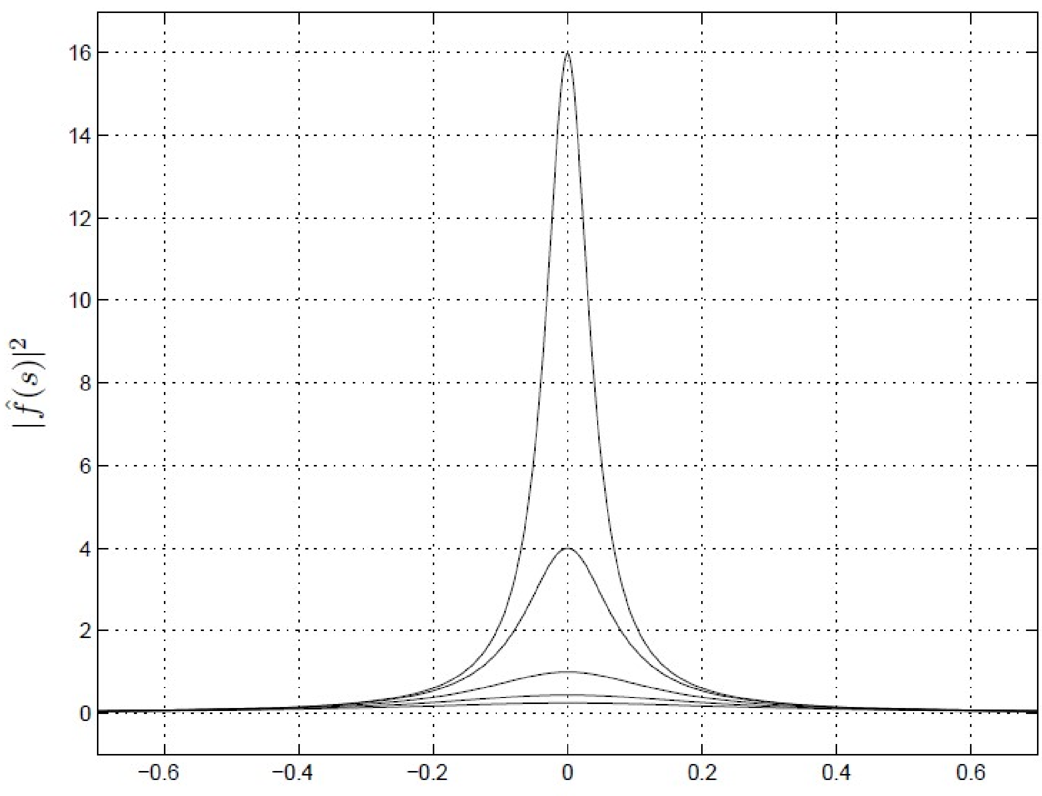
$$
\begin{aligned}
\mathcal{F}f(s) &= \int_0^\infty e^{-2\pi i s t} e^{-bt} dt = \int_0^\infty e^{-2\pi i s t - bt} dt \\
&= \int_0^\infty e^{(-2\pi i s - b)t} dt = \left[\frac{e^{(-2\pi i s - b)t}}{-2\pi i s - b} \right]_{t=0}^{t=\infty} \\
&= \frac{e^{(-2\pi i s)t}}{-2\pi i s - b} e^{-bt} \bigg|_{t=\infty} - \frac{e^{(-2\pi i s - b)t}}{-2\pi i s - b} \bigg|_{t=0} = \frac{1}{2\pi i s + b}
\end{aligned}
$$

Unlike the Fourier transforms of the rectangular function and the triangular function, the Fourier transform of the exponential decay function is complex. This is due to the absence of even symmetry in the exponential decay function. We will explore the concept of symmetry in more detail later. However, since the exponential decay function lacks this symmetry, its Fourier transform becomes complex.

The power spectrum of exponential decay function is given by the expression:

$$|\mathcal{F}f(s)|^2 = \frac{1}{|2\pi is + b|^2} = \frac{1}{b^2 + 4\pi^2 s^2}$$

Below are graphs of the power spectrum function for the same values of 'b' as shown in the graphs of the exponential decay function:



Which comes first? It's a significant issue that you'll quickly learn to recognise in relation to the time-domain images. Also take notice that, despite $\mathcal{F}f(s)$ not being an even function of s, $|\mathcal{F}f(s)|^2$. Indeed, although the power spectrum $|\mathcal{F}f(s)|^2$ exhibits a shape reminiscent of a "bell curve," it is not a Gaussian function. The Gaussian function will be discussed later. It is worth noting that this power spectrum is frequently encountered in the analysis of transition probabilities and lifetimes of excited states in atoms. The curve is known as a Lorenz profile.

It is universally accepted that, as a fundamental requirement, one must be acquainted with the remarkable equation:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

The direct application of the Fundamental Theorem of Calculus is not applicable in obtaining the integral because the function $f(x) = e^{-x^2}$ does not possess an elementary antiderivative. One of the most well-known mathematical tricks is the fact that it may be assessed precisely. You shouldn't go through life without understanding it since it is due to Euler. It's also worthwhile to watch again even if you've already watched it; see the discussion that follows this paragraph.

Regardless of the specific problem at hand, normalizing the Gaussian function such that the total area under the curve is equal to 1 consistently proves to be advantageous. There are various methods to accomplish this, but as we shall see, the greatest option for Fourier analysis

$$f(x) = e^{-\pi x^2}$$

The integral of e^{-x^2} is

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$$

Now calculating the Fourier transform,

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi isx} dx$$

Differentiate w.r.t. s :

$$\frac{d}{ds}\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-\pi x^2}(-2\pi ix)e^{-2\pi isx} dx$$

The integration by parts method is well-suited for this situation. Let's designate $dv = -2\pi ixe^{-\pi x^2}dx$ and $u = e^{-2\pi isx}$. By integrating dv, we obtain $v = ie^{-\pi x^2}$.

When we evaluate the product uv at the limits of $\pm\infty$, it becomes clear that it equals zero.Thus

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}f(t) &= -\int_{-\infty}^{\infty} ie^{-\pi x^2}(-2\pi it)e^{-2\pi itx}dx \\
&= -2\pi s\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi itx}dx \\
&= -2\pi s\mathcal{F}f(t)
\end{aligned}
$$

So $\mathcal{F}f(t)$ satisfies the differential equation

$$\frac{d}{dt}\mathcal{F}f(t) = -2\pi s\mathcal{F}f(t)$$

The solution that satisfies the given initial condition is unique and can be expressed as follows:

$$\mathcal{F}f(t) = \mathcal{F}f(0)e^{-\pi t^2}$$

But

$$\mathcal{F}f(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}dx = 1$$

Hence

$$\mathcal{F}f(t) = e^{-\pi t^2}$$

We have made an extraordinary discovery that the Gaussian function, represented as $f(x) = e^{-\pi y^2}$, is equal to its own Fourier transform.

We aim to calculate:

$$I = \int_{-\infty}^{\infty} e^{-y^2}dy$$

The variable of integration can be assigned any arbitrary name, allowing us to express the integral as:

$$I = \int_{-\infty}^{\infty} e^{-x^2}dx$$

$\therefore$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)$$

Since the variables are not "coupled" in this case, we can combine this into a double integral.

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)e^{-y^2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)}dxdy$$

Now, we will introduce a variable change using polar coordinates, represented as (r, θ). Initially, let's examine the integration limits.Allowing both x and y to range from $-\infty$ to ∞ covers the entire plane. In polar coordinates, encompassing the entire plane corresponds to r ranging from 0 to ∞, and θ ranging from 0 to 2π.

Next, we replace the term $e^{-\left(x^2+y^2\right)}$ with e^{-r^2}, as it is expressed more conveniently in polar coordinates. Furthermore, the area element $dxdy$ is transformed into $rdrd\theta$. It is crucial to observe the additional factor of r in the area element, as it plays a significant role in the subsequent computations. With this change to polar coordinates, we can continue our analysis. we have

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)}dxdy = \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}rdrd\theta$$

$$\int_{0}^{\infty} e^{-r^2}rdr = -\frac{1}{2}e^{-r^2}\Big]_{0}^{\infty} = \frac{1}{2}$$

$$I^2 = \int_{0}^{2\pi}\frac{1}{2}d\theta = \pi$$

$$\int_{-\infty}^{\infty} e^{-x^2}dx = I = \sqrt{\pi}$$

General Properties and Formulae

We have begun to compile a database of certain transformations. Let's now go the other route for a moment and create some generic characteristics. We will set aside any concerns you may have about transformations existing, integrals converging, or anything else for the purposes of this discussion, as well as for the majority of our work in the next lectures. Unwind and relish the journey.

4.1 Fourier transform pairs and duality

The Fourier transform and inverse Fourier transform exhibit a significant characteristic of symmetry, which is not present in Fourier series. In Fourier series, the ”inverse transform” corresponds to the series itself, where the coefficients are determined by an integral (transforming $f(t)$ into $\hat{f}(n)$). The key distinction between the Fourier transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ lies in the sign of the exponential term. Specifically, replacing s with $-s$ in the Fourier transform formula gives the inverse Fourier transform. This symmetry enhances our understanding of the relationship between the transforms and allows for a more unified perspective.

Likewise, the Fourier transform can be obtained by replacing t :

$$\mathcal{F}f(-s) = \int_{-\infty}^{\infty} e^{-2\pi i(-s)t} f(t)dt = \int_{-\infty}^{\infty} e^{2\pi ist} f(t)dt = \mathcal{F}^{-1}f(s)$$

and with $-t$ in the equation for the inverse Fourier transform.We get,

$$\mathcal{F}^{-1}f(-t) = \int_{-\infty}^{\infty} e^{2\pi is(-t)} f(s)ds = \int_{-\infty}^{\infty} e^{-2\pi ist} f(s)ds = \mathcal{F}f(t)$$

Typically, the two variables, s and t, are associated with distinct domains in the context of the forward and inverse transforms. One represents the frequency domain while the other corresponds to the time domain. This differentiation can sometimes be confusing and requires careful consideration.

However, it is important to recognize that both domains are intertwined and play significant roles in the expressions and equations involved. This apparent complexity may arise intermittently, but it is a challenge that can be surmounted. Let us approach it with mathematical reasoning: Transformations are manipulations of functions that create new functions. When writing a formula that involves evaluating a transform of a variable, it is important to note that the variable itself is just a symbol, and its name is arbitrary. What truly matters is that it is clear and understandable what the variable represents and how it functions within the formula. Also note what the expression notation indicates and, equally important, what it does not. For example, the first expression demonstrates the outcome of performing the Fourier transform on f and subsequently evaluating it with $-s$. It’s not a $\mathcal{F}(f(-s))$ expression like ”change s in f expression to $-s$ and then convert”. Put the first equation, denoted as $(\mathcal{F}f)(-s) = \mathcal{F}^{-1}f(s)$, by enclosing $\mathcal{F}$ f in parentheses. can also be written in I put a to emphasize it, but I found it too clumsy. Please be careful.

The equations

$$\mathcal{F}f(-s) = \mathcal{F}^{-1}f(s)$$
$$\mathcal{F}^{-1}f(-t) = \mathcal{F}f(t)$$

The symmetry observed in the formulas of the Fourier transform and its inverse is rooted in a fundamental mathematical principle. According to this principle, if the original function is defined in certain groups (which are not explicitly specified here), the transformation (also well-defined) is defined in the corresponding dual group.'" In the case of Fourier series, where the function is periodic, its natural domain corresponds to a circle (visualize a circle representing the interval $[0, 1]$ with the endpoints identified). Remarkably, it is found that the dual of the group of circles is the set of integers, leading to the Fourier transform $\hat{f}$ yielding integer values for n. Similarly, when considering the group of real numbers **R**, its dual group is once again **R**. Consequently, the Fourier transform of a function defined on **R** is also defined on **R**. Examining the general definitions of the Fourier transform and its inverse within this context, we arrive at the symmetrical outcome we observe. This property is commonly referred to as the duality property of the transform. It signifies that the pair of Fourier transforms, f and $\mathcal{F}f$, are connected through duality, indicating an interchanging relationship between the two. Although they may initially appear as distinct statements, they can be transformed from one to the other.

Here's an example that demonstrates the use of duality. As we know

$$\mathcal{F}\Pi = \text{sinc}$$

and

$$\mathcal{F}^{-1}\text{sinc} = \Pi$$

Using duality, we can find $\mathcal{F}$ sinc:

$$\mathcal{F}\text{sinc}(t) = \mathcal{F}^{-1}\text{sinc}(-t) = \Pi(-t)$$

Now, with the additional knowledge that Π is an even function ($-\Pi(-t) = \Pi(t)$), we can conclude that

$$\mathcal{F}\text{sinc} = \Pi$$

Let's employ the same reasoning to determine $\mathcal{F}\,\text{sinc}^2$. We know that

$$\mathcal{F}\Lambda = \text{sinc}^2$$

$$\mathcal{F}^{-1}\,\text{sinc}^2 = \Lambda$$

But

$$\mathcal{F}\,\text{sinc}^2(t) = \left(\mathcal{F}^{-1}\,\text{sinc}^2\right)(-t) = \Lambda(-t)$$

since Λ is even,

$$\mathcal{F}\,\text{sinc}^2 = \Lambda$$

Duality and Reverse Signals:

Duality can be understood in slightly different ways, but I prefer the interpretation that keeps variables in check and is easy to remember. Let's start with a signal $f(s)$ and define the inverse signal f^- as

$$f^-(s) = f(-s).$$

Note that reversing the signal twice gives back the original signal.

It's worth noting that the conditions for a function to be even or odd can be conveniently expressed using the reversed signals:

$$f \text{ is odd if } f^- = -f,$$
$$f \text{ is even if } f^- = f,$$

In other words, a signal is even if its reversal does not change the signal, and it is odd if its reversal changes the sign." We will delve into this concept further in the next section.

Reversing the signal simply means reversing time, but this operation is applicable in a general sense, regardless of the signal type or the variable being considered. By utilizing this notation, we can rephrase the initial duality equation $\mathcal{F}f(-s) = \mathcal{F}^{-1}f(s)$ as:

$$(\mathcal{F}f)^- = \mathcal{F}^{-1}f.$$

Similarly, we can rewrite the second duality equation $\mathcal{F}^{-1}f(-t) = \mathcal{F}f(t)$ as:

$$\left(\mathcal{F}^{-1} f\right)^{-}=\mathcal{F} f .$$

Indeed, these equations essentially convey the same idea, with one being the reverse or mirror image of the other.

Moreover, employing this notation facilitates a quicker derivation of results such as $\mathcal{F}\,\mathrm{sinc} = \Pi$.

$$\mathcal{F}\,\mathrm{sinc}=\left(\mathcal{F}^{-1}\,\mathrm{sinc}\right)^{-}=\Pi^{-}=\Pi .$$

Let's explore the behavior of $\mathcal{F} f^{-}$, which represents the Fourier transform of the reversed signal. By definition, we have:

$$\mathcal{F} f^{-}(s)=\int_{-\infty}^{\infty} e^{-2 \pi i s t} f^{-}(t) d t=\int_{-\infty}^{\infty} e^{-2 \pi i s t} f(-t) d t$$

To simplify the integral, we perform a change of variable. Let's consider the substitution $v=-t$, which implies $dv=-dt$ or $dt=-dv$. As t ranges from $-\infty$ to ∞, the variable $v=-t$ ranges from ∞ to $-\infty$. With this change of variable, we obtain the following expression:

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-2 \pi i s t} f(-t) d t & =\int_{\infty}^{-\infty} e^{-2 \pi i s(-v)} f(v)(-d v) \\
& =\int_{-\infty}^{\infty} e^{2 \pi i s u} f(v) d v \\
& =\mathcal{F}^{-1} f(s)
\end{aligned}$$

Hence, we find that $\mathcal{F} f^{-}=\mathcal{F}^{-1} f$.

Moreover, substituting $\mathcal{F}^{-1} f=(\mathcal{F} f)^{-}$, we have:

$$\mathcal{F} f^{-}=(\mathcal{F} f)^{-}$$

It's important to note the placement of parentheses in this equation.

To explore the behavior of $\mathcal{F}^{-1} f^{-}$, we can utilize the previous duality results:

$$\mathcal{F}^{-1} f^{-}=\left(\mathcal{F} f^{-}\right)^{-}=\left(\mathcal{F}^{-1} f\right)^{-}$$

In simpler terms, the duality relation states that if we reverse a signal and then apply the inverse Fourier transform, it is equivalent to applying the inverse Fourier transform first and then reversing the resulting signal. Therefore, we have the relationship: $\mathcal{F}^{-1}(f^-) = \mathcal{F}f$.

In essence, the entire list of duality relations can be reduced to just two :

$$\begin{aligned} \mathcal{F}f &= \left(\mathcal{F}^{-1}f\right)^- \\ \mathcal{F}f^- &= \mathcal{F}^{-1}f \end{aligned}$$

4.2 Even and odd symmetries and the Fourier transform

The concept of even and odd functions has been useful in several instances, taking advantage of their symmetry. When dealing with real-valued functions, the notion of even and odd functions has a clear interpretation in terms of graph symmetry. However, when considering complex-valued functions, graphing becomes challenging as we cannot visually represent complex values on a graph, leading to a loss of geometric intuition. "Even though the algebraic definitions of even and odd functions apply to both complex-valued and real-valued functions, the graphical representation is limited to real-valued functions.

How is the symmetry of a function reflected in the properties of the Fourier transform? I won't go into detail, but I'll point out some important cases.

- if a function $f(x)$ is either even or odd, then its Fourier transform will also exhibit the same evenness or oddness, respectively.

Using the concept of reversed signals, we need to demonstrate that if a function f is even, then its reversed Fourier transform, denoted as $(\mathcal{F}f)^-$, is equal to the regular Fourier transform of f, denoted as $\mathcal{F}f$. Similarly, if f is odd, then $(\mathcal{F}f)^-$ is equal to the negative of the regular Fourier transform of f, represented as $-\mathcal{F}f$. This result can be derived quickly using the equations we previously obtained.

$$(\mathcal{F}f)^- = \mathcal{F}f^- = \begin{cases} \mathcal{F}(-f) = -\mathcal{F}f & \text{if } f \text{ is odd} \\ \mathcal{F}f, & \text{if } f \text{ is even} \end{cases}$$

Due to the complex-valued nature of the Fourier transform of a function, we can also explore additional symmetries concerning the behavior under complex conjugation for $\mathcal{F}f(s)$.

The derivation process is essentially the same, it can be beneficial to review it as an exercise to observe the similarities.

$$\begin{aligned}(\mathcal{F}f)^{-}(s) &= \mathcal{F}^{-1}f(s)\\ &= \int_{-\infty}^{\infty} e^{2\pi ist} f(t)dt\\ &= \overline{\int_{-\infty}^{\infty} e^{-2\pi ist} f(t)dt}\\ &= \overline{\mathcal{F}f(s)}\end{aligned}$$

Furthermore, it is interesting to note that one can verify $\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}f)))(s) = f(s)$, which means that applying the Fourier transform four times yields the original function, implying that $\mathcal{F}^4$ is the identity transformation.

If the function $f(t)$ exhibits symmetry, we can further explore its properties by combining the previously discussed results and considering the nature of complex numbers. Specifically, a complex number is considered real if it is equal to its conjugate, and it is purely imaginary if it is equal to its conjugate with a negative sign. Taking this into account, we can deduce the following expression:

- If the function f is even and real valued, it follows that its Fourier transform is also even. This symmetry property is a characteristic of even functions and their corresponding Fourier transforms.

- If the function f is real-valued and an odd function, its Fourier transform possesses the characteristics of being an odd function and purely imaginary.

This principle becomes evident when we examine the Fourier transform of specific functions such as the rectangular function $\Pi(t)$ and the trigonometric function $\Lambda(t)$." In both cases, these functions possess even symmetry, meaning they are symmetric about the y-axis. As a result, their respective Fourier transforms, sinc and sinc^2, also exhibit the property of even symmetry and are real-valued.

4.3 Linearity

The Fourier transform possesses an important property called linearity, which allows it to operate on functions in a straightforward manner.

- The Fourier transform of the sum of two functions, f and g, is equal to the sum of their individual Fourier transforms: $\mathcal{F}(f+g)(t) = \mathcal{F}f(t) + \mathcal{F}g(t)$.
- The Fourier transform of a scalar multiple of a function, αf, is equal to the scalar multiple of its Fourier transform: $\mathcal{F}(\alpha f)(t) = \alpha \mathcal{F}f(t)$, where α can be any real or complex number.

These linear properties can be easily verified by comparing them to the corresponding properties of integrals. For example, to show the linearity of the sum, we can evaluate the integral:

$$\begin{aligned}
\mathcal{F}(f+g)(t) &= \int_{-\infty}^{\infty} (f(x) + g(x)) e^{-2\pi i t x} dx \\
&= \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x} dx + \int_{-\infty}^{\infty} g(x) e^{-2\pi i t x} dx \\
&= \mathcal{F}f(t) + \mathcal{F}g(t).
\end{aligned}$$

In previous discussions, I applied the property of multiplication without explicitly mentioning it when stating $\mathcal{F}(-f) = -\mathcal{F}f$ in relation to odd functions and their transformations. Although we hadn't formally listed the property, it is indeed a valid application of linearity.

4.4 The shift theorem

A time shift or delay in the variable t has a straightforward impact on the Fourier transform. The magnitude of the Fourier transform“ $|\mathcal{F}f(s)|$ is expected to remain unchanged because a time shift in the original signal should not alter the energy at any point in the spectrum. The only difference is a phase shift in $\mathcal{F}f(s)$, and that is indeed what occurs.

To find the value of Fourier transform $f(t+a)$, where a is a constant, we can employ the following expression:

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(t+a)e^{-2\pi ist}dt = &\int_{-\infty}^{\infty} f(v)e^{-2\pi is(v-a)}dv \\
&(\text{ substituting } v = t+a\text{; the limits still go from } -\infty \text{ to } \infty) \\
= &\int_{-\infty}^{\infty} f(u)e^{-2\pi isv}e^{2\pi isa}dv \\
= &e^{2\pi isa}\int_{-\infty}^{\infty} f(v)e^{-2\pi isv}dv = e^{2\pi isa}\hat{f}(s).
\end{aligned}
$$

The most appropriate notation to convey this property is commonly the pair notation, $f \rightleftharpoons F$.[7] Thus:

- If $f(t) \rightleftharpoons F(s)$ then $f(t+a) \rightleftharpoons e^{2\pi isa}F(s)$.

A little more generally, $f(t \pm a) \rightleftharpoons e^{\pm 2\pi isa}F(s)$.

It is worth noting that, as promised, the magnitude of the Fourier transform remains unchanged when a time shift is applied. This is due to the fact that the factor in front of the transform has a magnitude of 1 :

$$
\left|e^{\pm 2\pi isa}F(s)\right| = \left|e^{\pm 2\pi isa}\right| |F(s)| = |F(s)|
$$

4.5 The stretch (similarity) theorem

How does stretching or shrinking the variable in the time domain impact the Fourier transform? Specifically, we want to understand the changes in the Fourier transform of $f(at)$ when we scale the variable t by a factor of a. Let's first consider the case where a is positive. Then

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(at)e^{-2\pi ist}dt = &\int_{-\infty}^{\infty} f(u)e^{-2\pi is(u/a)}\frac{1}{a}du \\
= &\frac{1}{a}\int_{-\infty}^{\infty} f(u)e^{-2\pi i(s/a)u}du = \frac{1}{a}\mathcal{F}f\left(\frac{s}{a}\right)
\end{aligned}
$$

When $a < 0$, if we perform the variable permutation $u = ax$, the limit of the integral is inverted, resulting in a transformed function given by $(-1/a)\mathcal{F}f(s/a)$ with an increase. we can combine both cases to provide a comprehensive expression of the stretching theorem.

- If $f(x) \rightleftharpoons F(s)$ then $f(ax) \rightleftharpoons \frac{1}{|a|} F\left(\frac{s}{a}\right)$

The aforementioned transformation, where the variable is changed from x to ax, is commonly referred to as the similarity theorem. This term is used because altering the variable by a factor of a represents a change in scale, which is often associated with the concept of similarity.

The Stretch Law reveals an important observation. Assuming a is positive for clarity, when a is large , the graph of $f(at)$ in the time domain is horizontally compressed compared to $f(t)$. However, in the frequency domain, two distinct changes take place. Firstly, the Fourier transform becomes $(1/a)F(s/a)$, indicating that if a is large, $F(s/a)$ is stretched instead of being compressed in comparison to $F(s)$. Secondly, multiplying by $1/a$ compresses the values of the transform.

On the other hand, when a is small, the graph of $f(at)$ in the time domain is stretched horizontally compared to $f(t)$. However, in the frequency domain, a different effect occurs. The Fourier transform is compressed horizontally while being stretched vertically. This phenomenon is commonly referred to as the inability to precisely locate the signal and is a topic of frequent discussion.

Now, let's address some typographical concerns. The notation $\mathcal{F}f(t+b)$ may cause confusion regarding what is being transformed and the connection to the variable s. The hat notation, such as $f(\widehat{t+b})$, poses similar challenges due to limited space for the variable s. It is important to converge in both the time and frequency domains to clarify this principle more precisely.

To summarize, we can conclude that when a function is expanded in the time domain, it becomes compressed in the frequency domain, and conversely, when a function is compressed in the time domain, it becomes expanded in the frequency domain. This resembles what happens in the spectrum of long-term or short-term periodic functions." Assuming the period is T, where points are spaced by $1/T$ in the spectrum, if T is large, the function is distributed over a longer time period, but the spectrum becomes narrower since $1/T$ is smaller. On the contrary, if the value of T is small, the function experiences compression in the time domain, indicating that it repeats more rapidly. However, in the frequency domain, the spectrum widens as the value of $1/T$ increases.

It is important to note that in the above discussion, I have attempted to avoid focusing solely on the graphical properties of transformations, although there may have been some inadvertent implications. Indeed, the presence of complex values in transforms adds to the complexity of the phenomenon. Nonetheless, the compression and diffusion effects can be visualized geometrically by analyzing the plot of the function. These graphical representations provide insights into the observed compression and broadening phenomena.

example: The term ”Stretched Rectangle” is not precisely accurate, but it is frequently used in various applications. It can be defined as $p > 0$, where p represents a positive value.

$$\Pi_p(t) = \begin{cases} 1 & |t| < p/2 \\ 0 & |t| \geq p/2 \end{cases}$$

In other words, $\Pi_p(t)$ can be seen as a rectangular function with a width of p. We can determine its Fourier transform through direct integration, or alternatively, we can apply the stretch theorem if we observe that,

$$\Pi_p(t) = \Pi(t/p).$$

the definition of Π is,:

$$\Pi(t/p) = \left\{ \begin{array}{ll} 1 & |t/p| < 1/2 \\ 0 & |t/p| \geq 1/2 \end{array} \right. = \left\{ \begin{array}{ll} 1 & |t| < p/2 \\ 0 & |t| \geq p/2 \end{array} \right. = \Pi_p(t).$$

Now since $\Pi(t) \rightleftharpoons \operatorname{sinc} s$, by the stretch theorem

$$\Pi(t/p) \rightleftharpoons p \operatorname{sinc} ps$$

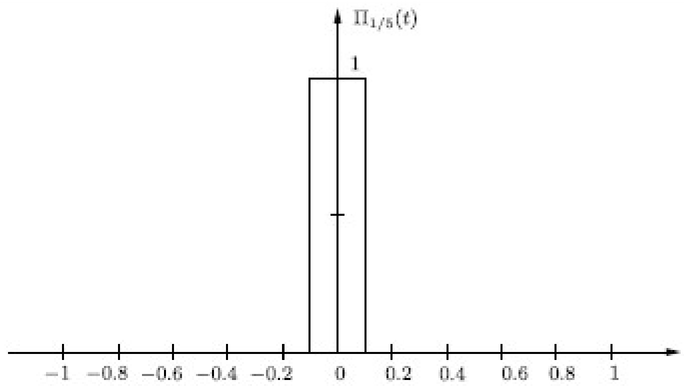
and so

$$\mathcal{F}\Pi_p(s) = p \operatorname{sinc} ps$$

The graphs of the Fourier transform pairs for $p = 1/5$ and $p = 5$ are provided, and it’s crucial to note the scales on the axes.

This observation is intriguing as it aligns with the well-known Heisenberg Uncertainty Principle in quantum mechanics, which exemplifies similar behavior.

It would be valuable to revisit the example of the one-sided exponential decay and its Fourier transform. By doing so, you can compare the graphs of $|\mathcal{F}f|$ for different parameter values and establish relevant connections.


$\Pi_{1/5}(t)$
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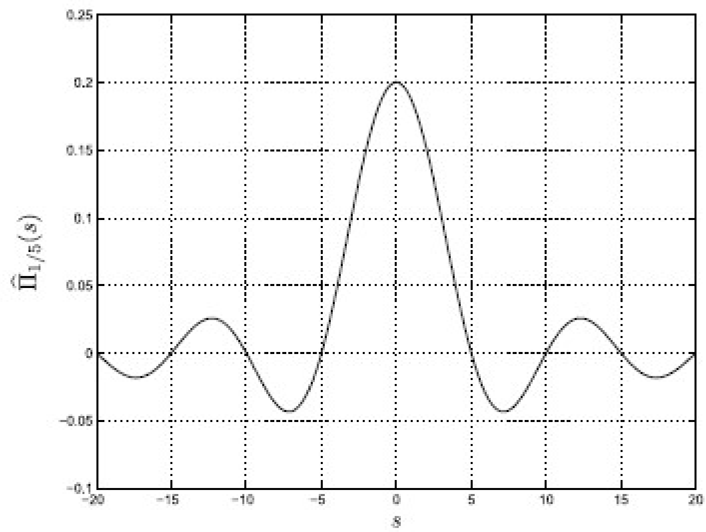
−1 −0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8 1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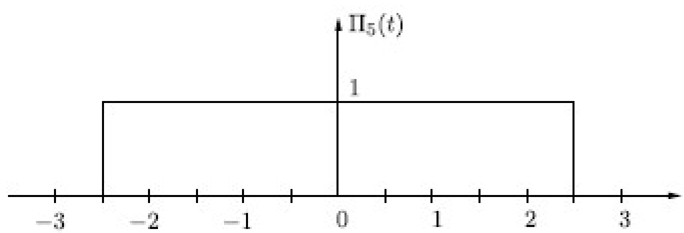
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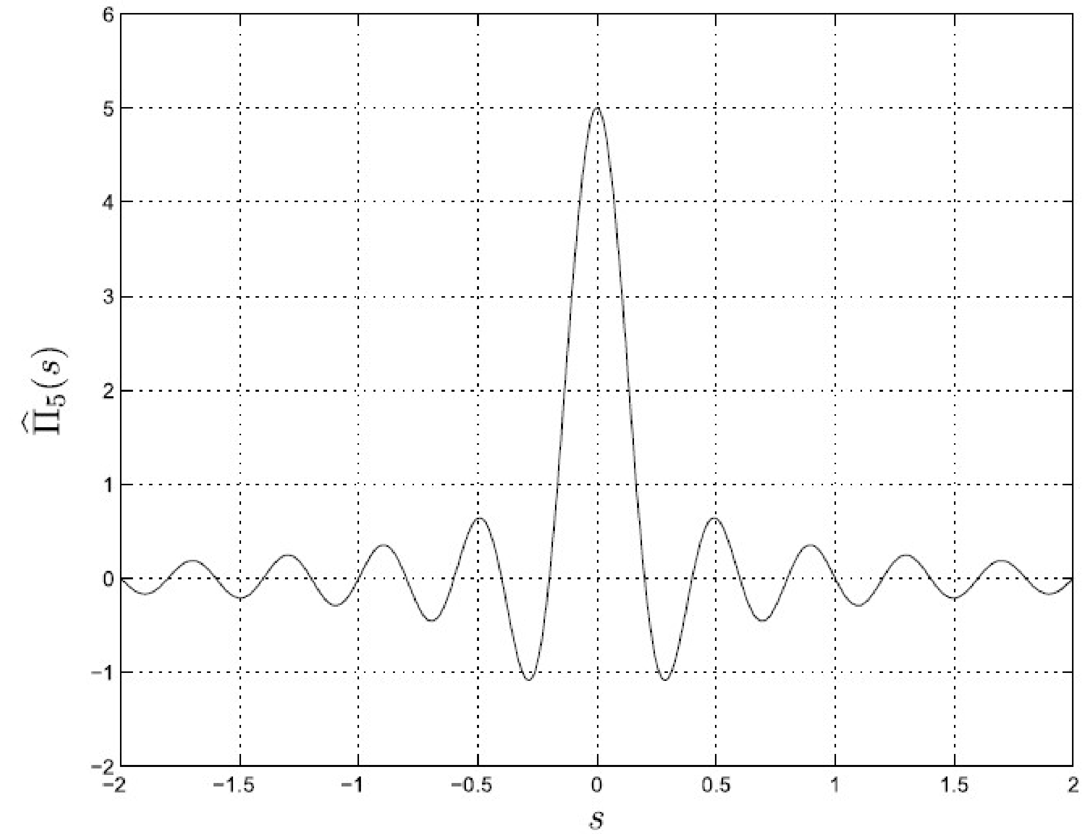
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Chapter 5

Conclusion

In conclusion, this thesis has delved into the topic of Fourier Transform and its significance in various domains. The Fourier Transform is a fundamental mathematical tool that has revolutionized signal processing, data analysis, and scientific research.

Through this research, we have gained a deep understanding of the mathematical principles behind the Fourier Transform. It provides a powerful technique to decompose complex signals or functions into simpler sinusoidal components. By representing signals in the frequency domain, we can extract valuable information about their spectral content, enabling us to analyze, manipulate, and interpret them more effectively.

The applications of the Fourier Transform are extensive and far-reaching. In the field of signal processing, it has been instrumental in areas such as telecommunications, audio and video compression, image analysis, and filtering. The ability to analyze signals in the frequency domain has led to significant advancements in communication systems, allowing for efficient data transmission and improved audiovisual experiences.

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