SPECIAL FUNCTIONS AND THEIR APPLICATIONS

A DISSERTATION

Submitted in partial fulfilment of the requirements for the Award of the Degree of

MASTER OF SCIENCE IN MATHEMATICS

SUBMITTED BY -

Gunjan Sharma

[2K21/MSCMAT/18]

Saurabh

[2K21/MSCMAT/45]

Under the Supervision of
Mr. Jamkhongam Touthang Sir

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DEPARTMENT OF APPLIED MATHEMATICS

DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College Of Engineering)
Bawana Road , Delhi-110042

DEPARTMENT OF APPLIED MATHEMATICS

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(Formerly Delhi College Of Engineering)
Bawana Road , Delhi-110042

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We Saurabh , 2K21/MSCMAT/45 & Gunjan Sharma , 2K21/MSCMAT/18 students of M.Sc. Mathematics, hereby declare that the project Dissertation titled "Special Functions and Their Applications" , which is submitted by us to the Department Of Applied Mathematics , Delhi Technological University , Delhi in partial fulfilment of the requirement for the award of the degree of Master of Science, is original and not copied from any source without proper citation. This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship or other similar title or recognition.

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(Formerly Delhi College Of Engineering)
Bawana Road , Delhi-110042

CERTIFICATE

We hereby certify that the Project Dissertation titled "SPECIAL FUNCTIONS AND THEIR APPLICATIONS " which is submitted by Gunjan Sharma , Roll No. 2K21/MSCMAT/18 & Saurabh, Roll No. 2K21/MSCMAT/45 [Department of Applied Mathematics], Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Master of Science, is a record of the project work carried out by the students under my supervision. To the best of my knowledge this work has not been submitted in part or full for any Degree or Diploma to this University or elsewhere.

Place - New Delhi
Date - 24/05/2023

Mr. Jamkhongam Touthang
SUPERVISOR

DEPARTMENT OF APPLIED MATHEMATICS

DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College Of Engineering)
Bawana Road , Delhi-110042

ABSTRACT

This report deals with a branch of mathematics of utmost importance to scientists and engineers concerned with actual mathematical calculations. This further deals with the basic theory of some special functions, as well as applications of this theory to specific problems of physics and engineering. In the choice of topics, we have been guided by the goal of giving a sufficiently detailed exposition of those problems which are of greatest practical interest.

Beginning with the basic function and their properties like BETA FUNCTIONS, GAMMA FUNCTIONS and HYPERGEOMETRIC FUNCTIONS, further exploring special functions including CYLINDRICAL and SPHERICAL FUNCTIONS.

Chapters containing BESSELS AND LEGENDERS FUNCTIONS are described according to their different types along with their important properties. Certain applications of these functions in mathematics and physics are used to convey the importance of these functions in various fields. Also solved examples would help in the implication of these formulas derived, in the real world.

The text is also supported with few graphs of the above-mentioned function to provide a good visual representation along with the study of these special functions. Finally, the references have been brought up-to-date.

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Signature

Gunjan Sharma
Saurabh

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CHAPTER : 1

Beta Function

1. Beta Function

The Beta function is represented by β (m,n) (where m,n > 0) and defined by the definite integral as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

1.1. Properties of Beta Function

i.

$$\beta(m,n) = \beta(n,m)$$

ii.

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

iii.

$$\beta(m,n) = 2\int_0^{(\pi)/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta$$

CHAPTER : 2

Gamma Function

2. Gamma Function

The Gamma Function is denoted by Γn ,where n>0 and defined by

$$\Gamma n = \int_0^\infty e^{-x}(x)^{m-1}dx$$

2.1. Properties of Gamma Function

i.

$$\Gamma 1 = 1$$

ii.

$$\Gamma(n+1) = n\Gamma(n) = n!$$

iii.

$$\Gamma(n) = z^n \int_0^\infty e^{-zx}(x)^{n-1}dx \;\; n, z > 0$$

iv.

$$\Gamma(n) = \int_0^1 (\log \frac{1}{y})^{n-1}dy$$

v.

$$\Gamma(n+1) = \int_0^\infty e^{-y^{\frac{1}{n}}}dy$$

vi.

$$\Gamma\frac{1}{2} = \Gamma\pi$$

2.2. Relationship between Beta and Gamma Function

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

2.3. Duplication Formula

$$\Gamma m \Gamma(m+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\sqrt{2m}$$

CHAPTER : 3

Cylinder Function (Bessel's Function)

3. Cylinder Function(Bessel's Function)

By a Cylinder Function we mean a solution of the second order Linear differential equation

$$u" + \frac{1}{z}u' + (1 - \frac{\nu^2}{z^2})u = 0 \qquad (5.1)$$

z = complex number

ν= parameter which can take real or complex values

(5.1) is called the Bessel's equation of order ν

3.1. BESSEL FUNCTION OF NON NEGATIVE INTEGRAL ORDER

One of the solution of Bessel equation with ν being a non negative integer n

$$u" + \frac{1}{z}u' + (1 - \frac{n^2}{z^2})u = 0$$

is $u_1 = J_n(z)$ = BESSEL'S FUNCTION OF FIRST KIND OF ORDER n ,

defined by the series

$$J_n(z) = \sum_{k=0}^{\infty} \frac{-1^k (\frac{z}{2})^{n+2k}}{k!(n+k)!}, \qquad |z| < \infty \qquad (5.1.1)$$

Using the Ratio test , series converges . Therefore it represents entire function of z .

Now denote left hand side of (5.1) by $l(u)$

Consider

$$\alpha_k = \frac{(-1)^k}{2^{n+2k}k!(n+k)!}$$

Therefore,

$$l(u_1) = \sum_{k=0}^{\infty} \left[(n+2k)(n+2k-1) + (n+2k) - n^2\right] \alpha_k z^{n+2k-2} + \sum_{k=0}^{\infty} \alpha_k z^{n+2k}$$

$$= \sum_{k=0}^{\infty} 4\left[k(n+k)\right] \alpha_k z^{n+2k-2} + \sum_{k=0}^{\infty} \alpha_k z^{n+2k}$$

$$= \sum_{k=0}^{\infty} \left[4\alpha_{k+1}(k+1)(n+k+1) + \alpha_k\right] z^{n+2k}$$

$$= 0$$

Therefore $J_n(z)$ satisfies the Bessel's equation i.e. it is a Cylinder Function .

- ORDER 0

$$J_0(z) = \sum_{k=0}^{\infty} \frac{-1^k (\frac{z}{2})^{2k}}{k!(k)!}$$

$$= 1 - \frac{(z/2)^2}{(1!)^2} + \frac{(z/2)^4}{(2!)^2} - \frac{(z/2)^6}{(3!)^2} \dots$$

- ORDER 1

$$J_1(z) = \sum_{k=0}^{\infty} \frac{-1^k (\frac{z}{2})^{1+2k}}{k!(1+k)!}$$

$$= \frac{z}{2} - \frac{(z/2)^3}{(2!)} + \frac{(z/2)^5}{(3!)(2!)} \dots$$

$$= \frac{z}{2} - \frac{(z/2)^3}{(2!)} + \frac{(z/2)^5}{(3!)(2!)} \dots$$

$$= \frac{z}{2} \left[1 - \frac{(z/2)^2}{1!2!} + \frac{(z/2)^4}{3!2!} + \frac{(z/2)^6}{3!4!} + \dots\right]$$

Bessel's function of higher order can be represented in terms of two functions $J_0(z)$ and $J_1(z)$.

By multiplying (5.1.1) by z^n and differentiating with respect to z, we get,

$$\frac{d}{dz}[z^n J_n(z)] = \frac{d}{dz}\left[\sum_{k=0}^{\infty} \frac{(-1)^k z^{2n+2k}}{k!(n+k)!2^{n+2k}}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k) z^{2n+2k-1}}{k!(n+k)!2^{n+2k}}$$

$$= z^n \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k) z^{n+2k-1}}{k!(n+k)!2^{n+2k}}$$

$$= z^n \sum_{k=0}^{\infty} \frac{(-1)^k (n+k) z^{n+2k-1}}{k!(n+k)!2^{n+2k-1}}$$

$$= z^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k-1)!} \left(\frac{z}{2}\right)^{n+2k-1}$$

$$= z^n J_{n-1}(z)$$

Therefore we derived,

$$\frac{d}{dz}[z^n J_n(z)] = z^n J_{n-1}(z)$$

Some other recurrence relation are as follows:

$$\frac{d}{dz}[z^{-n} J_n(z)] = -z^{-n} J_{n+1}(z)$$

PROOF

$$\frac{d}{dz}[z^{-n} J_n(z)] = \frac{d}{dz}\left[\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k!(n+k)!2^{n+2k}}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2k) z^{2k-1}}{k!(n+k)!2^{n+2k}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k-1}}{(k-1)!(n+k)!2^{n+2k-1}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} z^{2m+1}}{(m)!(m+n+1)!2^{n+2m+1}}$$

$$= -z^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)!(m+n+1)!} (\frac{z}{2})^{n+2m+1}$$

$$= -z^{-n} J_{n+1}(z)$$

By performing differentiation in the above derived result, we get

$$[z^n J_n'(z)] + z^{n-1} J_n(z) n = z^n J_{n-1}(z)$$

$$\Rightarrow J_n'(z) + \frac{n}{z} J_n(z) = J_{n-1}(z)$$

$$[z^{-n} J_n'(z)] + z^{-n-1} J_n(z)(-n) = -z^{-n} J_{n+1}(z)$$

$$\Rightarrow J_n'(z) - \frac{n}{z} J_n(z) = -J_{n+1}(z)$$

Therefore ,

$$J_{n-1} + J_{n+1} = \frac{2n}{z} J_n \qquad n = 1, 2, \ldots$$

$$J_{n-1} - J_{n+1} = 2J_n'$$

For $n = 0$ we must be replaced by ,

$$J_0'(z) = J_1(z)$$

3.2. GENERATING FUNCTIONS

The Bessel's function of first kind $J_n(z)$ are related to the coefficient of "Laurent expansion" of ,

$$w(z,t) = e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} c_n(z)t^n$$

$0 < |t| < \infty$

multiplying the power series ,

$$e^{\frac{zt}{2}} = 1 + \frac{(z/2)}{(1!)}t + \frac{(z/2)^2}{2!}t^2 + \dots$$

$$e^{\frac{-z}{2t}} = 1 - \frac{(z/2)}{(1!)}\frac{1}{t} + \frac{(z/2)^2}{2!}\frac{1}{t^2} + \dots$$

We obtain , by combining coefficients of t^n and t^{-n} respectively ,

$$C_n(z) = J_n(z)$$

$n = 0, 1, 2, ...$

$$C_n(z) = (-1)^n J_{-n}(z)$$

$n = -1, -2, ...$

$$w(z,t) = J_0(z) + \sum_{n=1}^{\infty} J_n z[t^n + (-1)^n t^{-n}$$

$0 < |t| < \infty$

called the **generating function** of Bessel's Function.

3.3. BESSEL'S FUNCTION OF ARBITRARY ORDER

Bessel's function of the first kind of order 0 is represented by $J_\nu(z)$ i.e

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$$

$|z| < \infty$, $|argz| < \pi$

This satisfies Bessel's equation with parameter ν .

$$l(u) = u" + \frac{1}{z}u' + (1 - \frac{\nu^2}{z^2})u = 0$$

Let $u_1 = J_\nu(z)$

$$l(u_1) = \sum_{k=0}^{\infty} \left[(\nu+2k)(\nu+2k-1) + (\nu+2k) - \nu^2\right] \alpha_k z^{\nu+2k-2} + \sum_{k=0}^{\infty} \alpha_k z^{\nu+2k}$$

$$= \sum_{k=0}^{\infty} 4\left[k(\nu+k)\right] \alpha_k z^{\nu+2k-2} + \sum_{k=0}^{\infty} \alpha_k z^{\nu+2k}$$

$$= \sum_{k=0}^{\infty} \left[4\alpha_{k+1}(k+1)(\nu+k+1) + \alpha_k\right] z^{\nu+2k} = 0$$

where,

$$\alpha_k = \frac{(-1)^k}{2^{\nu+2k} k!(\nu+k)!}$$

(1) Now for integral $\nu = n$ where $n = 0, 1, 2....$

$\Gamma(k+\nu+1) = (n+k)!$, therefore

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{(k)!(k+n)!}$$

(2) For negative integral $\nu = -n$, where $-n = -1, -2, ...$

the first n terms of $J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$ vanishes and the series becomes,

$$J_{-n}(z) = \sum_{k=n}^{\infty} \frac{(-1)^k (z/2)^{-n+2k}}{(k)!(k-n)!}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^{n+s} (z/2)^{n+2s}}{(s)!(s+n)!}$$

where $k = n + s$

Hence ,

$$J_{-n}(z) = (-1)^n J_n(z) \qquad\qquad n = 1, 2, 3...$$

And so the Bessel's function of negative integral order differs only by the sign from the corresponding function of positive integral order. So the generating function will be,

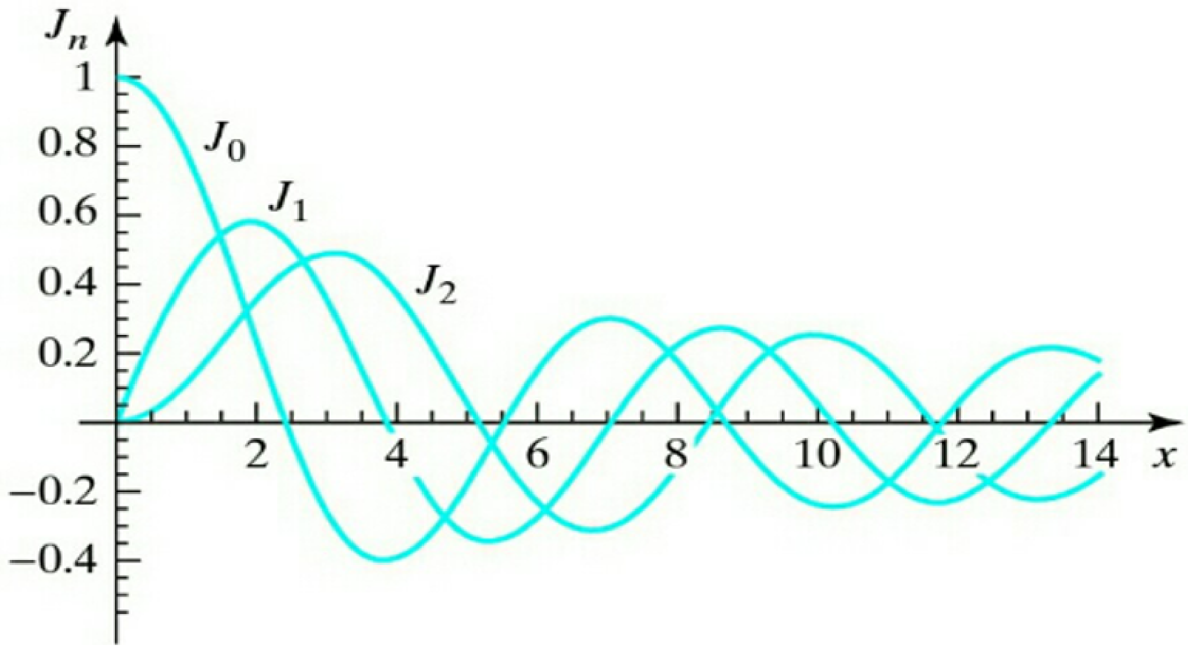

Figure 1: Graph of Bessel's function of the first kind with different orders

$$w(z,t) = e^{\frac{1}{2}z(t-\frac{1}{t})}$$
$$= \sum_{\nu=-\infty}^{\infty} C_\nu(z)t^n$$

where,

$C_\nu(z) = J_\nu(z)$ $\qquad \nu = 0, 1, 2...$

$C_\nu(z) = (-1)^n J_\nu(z)$ $\qquad \nu = -1, -2...$

And finally we get,

$$w(z,t) \sum_{\nu=-\infty}^{\infty} J_\nu(z)t^\nu$$

The recurrence formula for Bessel's function of non-negative integral order remains same for arbitrary order i.e

- $$\frac{d}{dz}[z^\nu J_\nu(z)] = z^\nu J_{\nu-1}(z)$$
- $$\frac{d}{dz}[z^{-\nu} J_\nu(z)] = -z^{-\nu} J_{\nu+1}(z)$$
- $$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{z} J_\nu(z)$$

- $$J_{\nu-1} - J_{\nu+1} = 2J'_{\nu}(z)$$

3.4. BESSEL'S FUNCTION OF SECOND KIND

A second solution of Bessel's equation must be required, which is linearly independent of $J_n(z)$. We name it "$u_2 = Y_n(z)$". Thus general expression for cylinder function of order $\nu = n$ is linear combination of **Bessel's Equation** of first and second kind i.e

$$u = Z_n(Z) = AJ_n(z) + BY_n(z)$$

$n = 0, 1, 2, ...$

Now,

$$l(u) = u" + \frac{1}{2}u' + \left(1 - \frac{\nu^2}{z^2}\right)u = 0$$

hence general form,

$$u = Z_0(z) = C_1u_1(z) + C_2u_2(z)$$

u_1 & u_2 are arbitrary independent solution of Bessel's Equation .

Now choosing $u_1 + J_\nu(z)$ and $u_2 = J_{-\nu}(z)$ is also a solution of Bessel's Equation for non-integral

ν the **ASYMPTOTIC BEHAVIOUR** of these solutions as $z \to 0$ is

$$u_1 \sim \frac{(z/2)^\nu}{\Gamma(1+\nu)}$$

$$u_2 \sim \frac{(z/2)^{-\nu}}{\Gamma(1-\nu)}$$

Therefore , General Cylinder Function -

$$u = Z_\nu = C_1J_\nu(z) + C_2J{-}\nu(z)$$

$\nu \neq 0, \pm1, \pm2.....$

- If ν is a integer , the particular solution $u_1 \& u_2$ are Linearly Dependent and $u = Z_\nu = C_1J_\nu(z) + C_2J{-}\nu(z)$ is no longer solution of Bessel's Equation .

We now introduce **Bessel's Function of second kind** by $Y_\nu(z)$

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}$$

For integral ν , the right hand side of above equation becomes indeterminate therefore we define ,

$$Y_n(z) = \lim_{\nu\to n} Y_\nu(z)$$

Both numerator and denominator are entire function of ν

$$\frac{d}{d\nu}\sin\nu\pi = \pi\cos\pi\nu \neq 0$$

if $\nu = n$

this limit exist and can be calculated by L'Hospital Rule .

Therefore,

$$Y_n(z) = \frac{1}{\pi}\left[\frac{d}{d\nu}J_\nu(z)|_{\nu=n} - (-1)^n\frac{d\nu}{d\nu}J_{-\nu}(z)|_{\nu=n}\right]$$

And so,

$Y_\nu(z)$ is an analytic function of z in the plane cut,which is along $[-\infty, 0]$ and entire function of the parameter ν for fixed z.

Therefore , Solution $u_1 = J_\nu(z)$ & $u_2 = Y_\nu(z)$ are linearly independent for integral ν

Hence,

$$u = Z_\nu(z) = C_1J_\nu(z) + C_2Y_\nu(z)$$

Some of the Recurrence relation are as follows -

1. $$\frac{d}{dz}[z^\nu Y_\nu(z)] = z^\nu Y_{\nu-1}(z)$$

2. $$\frac{d}{dz}[z^{-\nu}Y_\nu(z)] = -z^{-\nu}Y_{\nu+1}(z)$$

3. $$Y_{\nu-1} + Y_{\nu+1} = \frac{2\nu}{z}Y_\nu(z)$$

4. $$Y_{\nu-1} - Y_{\nu+1} = 2Y'_\nu(z)$$

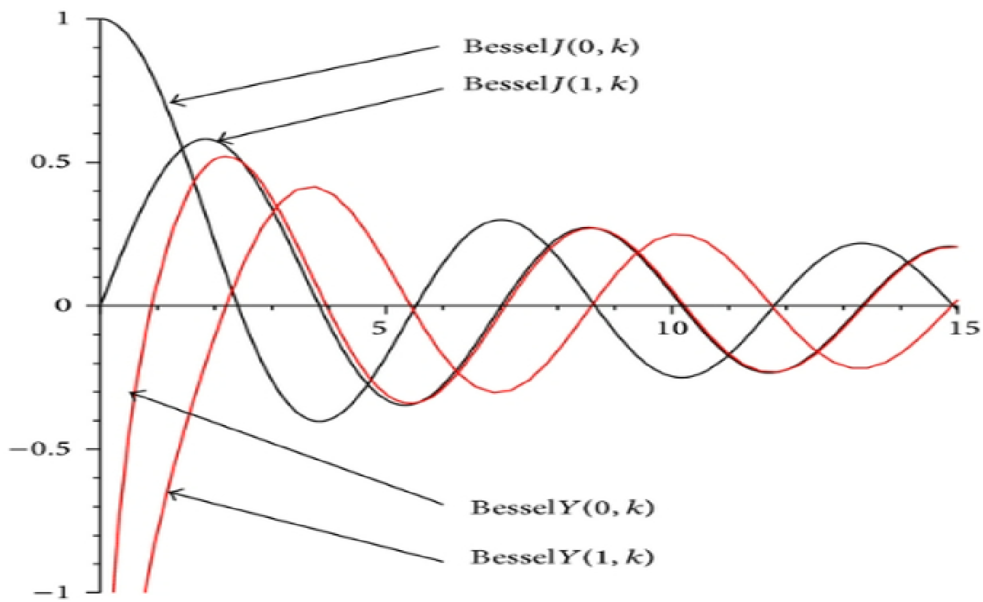

Figure 2: First and second kind of Bessels function comparison

5.

$$Y_{-n}(z) = (-1)^n Y_n(z)$$

$n = 0, 1, ...$

3.5. SERIES EXPANSION OF $Y_n(z)$

Consider,

$$\frac{d}{d\nu} J_\nu(z)|_{\nu=n} = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{k!(n+k)!} \left[\log \frac{z}{2} - \Psi(k+n+1)\right]$$

where, $\Psi(z) = \frac{\Gamma z'}{\Gamma z}$, logarithmic derivative of gamma function.

Similarly,

$$\frac{d}{d\nu} J_{-\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{-\nu+2k}}{k!\Gamma(k-\nu+1)} \left[-\log \frac{z}{2} - \Psi(k-\nu+1)\right]$$

As $\nu \to n$,$\Gamma(k-\nu+1) \to \infty$ and $\Psi(k-\nu+1) \to \infty$.

Therefore first n terms of last series becomes indeterminate. Now,

$$lim_{\nu \to n} \frac{\Psi(k-\nu+1)}{\Gamma(k-\nu+1)} = lim_{\nu \to n} \left[\Gamma\nu - k \sin(\nu - k) \frac{\Psi(\nu-k) + \pi \cot \pi(\nu-k)}{\pi}\right]$$
$$= (-1)^{n-k}(n-k-1)!$$

where $k = 0, 1, 2 \ldots n-1$

Thus introducing $p = k - \nu$ we get,

$$\frac{d}{d\nu}J_{-\nu}|_{\nu=n}(z) = (-1)^n \sum_{k=0}^{n-1} \frac{n-k-1}{k!}(z/2)^{2k-n} + (-1)^n \sum_{p=0}^{\infty} \frac{(-1)^p}{(n+p)!p!}\left[-\log\frac{z}{2} - \Psi(p+1)\right](\frac{z}{2})^{2p+n}$$

Therefore,

$$Y_n(z) = \frac{-1}{\pi}\sum_{k=0}^{n-1} \frac{n-k-1}{k!}(z/2)^{2k-n} + \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{-1^n}{k!(n+k)!}(z/2)^{2k+n}\left[2\log\frac{z}{2} - \Psi(k+1) - \Psi(k+n+1)\right]$$

$n = 0, 1, 2, ...$

Also $\Psi(1) = -\gamma$, $\Psi(m+1) = -\gamma + 1 + \frac{1}{2} + \ldots + \frac{1}{m}$, where $\gamma = 0.577 =$ Euler's constant

Finally we get the series expansion as,

$$Y_n(z) = \frac{2}{\pi}J_n(z)\log\frac{z}{2} - \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{n-k-1}{k!}(z/2)^{2k-n} - \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}(z/2)^{2k+n}\left[\Psi(k+1) + \Psi(k+n+1)\right]$$

For asymptotic representations,

$$Y_0(z) \sim \frac{2}{\pi}\log\frac{z}{2} \qquad z \to 0$$

$$Y_n(z) \sim \frac{-(n-1)!}{\pi}(\frac{z}{2})^{-n} \qquad z \to 0$$

$n = 1, 2, ...$

Which shows ,

$Y_n(z)$becomes infinite as $z \to 0$

3.6. BESSEL'S FUNCTION OF THIRD KIND (Hankel Function)

Hankel Function denoted by $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ where ,

$$H_\nu^{(1)}(z) = J_\nu + \iota Y_\nu \tag{1}$$

$$H_\nu^{(2)}(z) = J_\nu - \iota Y_\nu \tag{2}$$

Linear Combination of J_ν and Y_ν have very simple asymptotic expression for large $|z|$.

Therefore , $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are linearly independent of each other and J_ν ,
Therefore ,

$$u = Z_\nu(z) = A_1 J_\nu(z) + A_2 H_\nu^{(1)}(z)$$

$$= Z_\nu(z) = B_1 J_\nu(z) + B_2 H_\nu^{(2)}(z)$$

$$= Z_\nu(z) = D_1 H_\nu(z) + D_2 H_\nu^{(2)}(z)$$

Also Hankel functions satisfies some recurrence relations -

1.
$$\frac{d}{dz}[z^\nu H_\nu P(z)] = z^\nu H_{\nu-1}^p(z)$$

2.
$$\frac{d}{dz}[z^{-\nu} H_\nu^P(z)] = -z^{-\nu} H_{\nu+1}^p(z)$$

3.
$$H_{\nu-1}^p + H_{\nu+1}^p = \frac{2\nu}{z} H_{\nu+1}^p$$

4.
$$H_{\nu-1}^p - H_{\nu+1}^p = 2\frac{dH_\nu^p}{dz}$$

$p = 1, 2$

3.7. BESSEL FUNCTION OF IMAGINARY ARGUMENT

Consider,

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)\nu + 2k}{\Gamma(k+1)\Gamma(k+\nu+1)} \qquad |z| < \infty, |argz| < \pi$$

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, \qquad \nu \neq 0, \pm 1, \pm 2...$$

Therefore, $I_\nu(z)$ and $K_\nu(z)$ are simply related to Bessel's function of argument $ze^{\pm\frac{\pi i}{2}}$.

(1) If $-\pi < argz < \frac{\pi}{2}$ i.e. $-\pi/2 < argze^{\frac{\pi i}{2}} < \pi$, then

$$J_\nu(ze^{\frac{\pi i}{2}}) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2} e^{\frac{\pi i}{2}})^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$$

$$= e^{\frac{\pi i}{2}} \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$$

$$= e^{\frac{\pi i}{2}} I_\nu(z)$$

therefore,

$$I_\nu(z) = e^{-\frac{\pi i}{2}} J_\nu(ze^{\frac{\pi i}{2}}) \qquad -\pi < argz < \frac{\pi}{2}$$

Similarly,

$$H_\nu(ze^{\frac{\pi i}{2}}) = \frac{J_{-\nu}(ze^{\frac{\pi i}{2}}) - e^{\frac{-\pi i}{2}} J_\nu(ze^{\frac{\pi i}{2}})}{i \sin \nu\pi}$$

$$= \frac{J_{-\nu}(ze^{\frac{\pi i}{2}}) - e^{\frac{-\pi\nu i}{2}} I_\nu(z)}{i \sin \nu\pi}$$

$$= \frac{2}{\pi i} e^{\frac{-\pi\nu i}{2}} K_\nu(z)$$

Hence we get,

$$K_\nu(z) = \frac{\pi i}{2} e^{\frac{\pi\nu i}{2}} H_\nu(ze^{\frac{\pi i}{2}}), \qquad -\pi < argz < \frac{\pi}{2}$$

(2) If $-\pi/2 < argz < \pi$ i.e. $-\pi < argze^{\frac{\pi i}{2}} < \pi$, then

$$I_\nu(z) = e^{\frac{\pi\nu i}{2}} J_\nu(ze^{\frac{-\pi i}{2}})$$

,
which is **MODIFIED BESSEL'S FUNCTION OF FIRST KIND**

$$K_\nu(z) = \frac{\pi i}{2} e^{\frac{\pi\nu i}{2}} H_\nu(ze^{\frac{\pi i}{2}})$$

which is called the **MACDONALD'S FUNCTION**

$I_\nu(z)$ and $K_\nu(z)$ are linearly independent solution of differential equation,$u" + \frac{1}{z}u' + (1 - \frac{\nu^2}{z^2})u = 0$ in which

$$u = C_1 I_\nu(z) + C_2 K_\nu(z)$$

SOME OF THE SIMPLE RECURRENCE RELATIONS ARE-

1.

$$\frac{d}{dz}[z^\nu I_\nu(z)] = z^\nu I_{\nu-1}(z)$$

2.

$$\frac{d}{dz}[z^{-\nu}I_\nu(z)] = z^{-\nu}I_{\nu+1}(z)$$

3.

$$\frac{d}{dz}[z^{\nu}K_\nu(z)] = -z^{\nu}K_{\nu-1}(z)$$

4.

$$\frac{d}{dz}[z^{-\nu}K_\nu(z)] = -z^{-\nu}K_{\nu+1}(z)$$

OTHER TWO USEFUL FORMULAS ARE

(1) $I_{-n}(z) = I_n(z)$ $\quad n = 0, \pm 1, \pm 2...$

(2) $K_{-\nu}(z) = K_\nu(z)$

CHAPTER : 4

(Applications of Cylinder Function)

4. APPLICATIONS OF CYLINDER FUNCTIONS

4.1. Separation of Variables

Consider the partial differential equation ,

$$\Delta^2 u = \frac{1}{a^2}\frac{\delta^2 u}{\delta t^2} + b\frac{\delta u}{\delta t} + cu \qquad (1)$$

Where $\Delta^2 u$ is the Laplacian operator,t is the time, and a,b,c are constants.Many of the mathematical physics differential equation are of the above given form. The boundary conditions on function u often require the use of the **cylindrical coordinates system** given by, r, ϕ, z related to rectangular coordinates x,y,z by

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

Where ,

$0 \leq r < a,$
$-\pi \leq \phi < \pi,$
$-a \leq z < a$

In Cylindrical coordinates ,

$$\frac{1}{r}\frac{d}{dr}(r\frac{du}{dr}) + \frac{1}{r^2}\frac{d^2u}{d\phi^2} + \frac{d^2u}{dz^2} = \frac{1}{a^2}\frac{d^2u}{dt^2} + \frac{du}{dt} + cu \qquad (2)$$

has infinite many solution of form

$$u = R(r)Z(z)\Phi(\phi)T(t) \qquad (3)$$

where each on the functions depends on only one variable
Putting (3) into (2)

$$\frac{1}{r}\frac{d}{dr}\left((r\frac{dR(r)Z(z)\Phi(\phi)T(t)}{dr}\right) + \frac{1}{r^2}\left(\frac{d^2R(r)Z(z)\Phi(\phi)T(t)}{d\phi^2}\right) + \frac{d^2\left(R(r)Z(z)\Phi(\phi)T(t)\right)}{dz^2} = \frac{1}{a^2}\frac{d^{2(}R(r)Z(z)\Phi(\phi)T(t))}{dt^2} + \frac{d(R(r}{}$$

and now dividing by $RZ\Phi T$

$$\frac{1}{Rr}\frac{d}{dr}(r\frac{dR}{dr}) + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} - c = \frac{1}{T}\left(\frac{1}{a^2}\frac{d^2T}{dt^2} + bT\right) \quad (4)$$

Since r, ϕ, z, t are independent of each other , both side of (4) must equal a constant say - χ^2
Therefore , this leads to the following equations

$$\frac{1}{a^2}\frac{d^2T}{dt^2} + bT + T\chi^2 = o \quad (5)$$

$$\frac{1}{Rr}\frac{d}{dr}(r\frac{dR}{dr}) + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2} + \chi^2 = c - \frac{1}{z}\frac{d^2Z}{dz^2} \quad (6)$$

By the same reason , both side of (5) must equal to a constant , say $-\lambda^2$,

$$c - \frac{1}{z}\frac{d^2Z}{dz^2} = -\lambda^2$$

$$\implies (c + \lambda^2)Z = \frac{d^2Z}{dz^2}$$

and

$$r^2\left[\frac{1}{Rr}\frac{d}{dr}(r\frac{dR}{dr}) + (\lambda^2 + \chi^2)\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \quad (7)$$

Again , equating last equation by constant say μ^2,

$$\frac{d^2\Phi}{d\phi^2} + \mu^2\phi = 0 \quad (8)$$

$$\frac{1}{r}\frac{d}{dr}(r\frac{dR}{dr}) + (\lambda^2 + \chi^2 + \frac{\mu^2}{r^2})R = 0 \quad (9)$$

This process is called **separation of variables** , leads to infinite many solution of form (3) depending on the parameters χ, λ, μ

TWO IMPORTANT SPECIAL CASES OF EQUATION ARE

i. LAPLACE'S EQUATION ($\Delta^2 u = 0$)

where we consider (a = b = c = 0)
equation has particular solution which have the following form

$$u = R(r)Z(z)\phi(\phi) \quad (10)$$

where ,

$$\frac{1}{r}\frac{d}{dr}(r\frac{dR}{dr}) + (\lambda^2 - \frac{\mu^2}{r^2})R = 0 \quad (11)$$

(i)

$$\frac{d^2Z}{dz^2} - \lambda^2 Z = 0$$

$$\frac{d^2\Phi}{d\phi^2} + \mu^2\Phi = 0$$

In the special case where the conditions are such that u is not dependent of the angular coordinate ϕ, thus we now have

$$u = R(r)Z(z) \tag{12}$$

where,

$$\frac{1}{r}\frac{d}{dr}(r\frac{dR}{dr}) + \lambda^2 R = 0 \tag{13}$$

$$\frac{d^2Z}{dz^2} - \lambda^2 Z = 0 \tag{14}$$

ii. FINAL SOLUTION OF LAPLACE EQUATION

We have the first differential equation as,

(i)

$$\frac{d^2Z}{dz^2} - \lambda^2 Z = 0$$

Considering , $m^2 - \lambda^2 = 0$

$\Rightarrow m = +\lambda, -\lambda$ (ROOTS ARE DISTINCT)

Hence the solution,

$$Z(z) = C_1 e^{\lambda z} + C_2 e^{-\lambda z}$$

We have the second differential equation as,

(ii)

$$\frac{d^2\Phi}{d\phi^2} + \mu^2\Phi = 0$$

Considering , $m^2 + \lambda^2 = 0$

$\Rightarrow m = +i\lambda, -i\lambda$ (ROOTS ARE COMPLEX)

Hence the solution,

$$\Phi(\phi) = C_3 \cos \lambda\phi + C_4 \sin \lambda\phi$$

We have the third differential equation as,

(iii)

$$\frac{1}{r}\frac{d}{dr}(r\frac{dR}{dr}) + (\lambda^2 - \frac{\mu^2}{r^2})R = 0$$

On simplifying further we get,

$$\frac{1}{r}\left[r\frac{d^2R}{dr^2} + \frac{dR}{dr}\right] + R\lambda^2 - \frac{\mu^2}{r^2}R = 0$$
$$\Rightarrow \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{\mu^2}{r^2}\right)R = 0$$
$$\Rightarrow \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\frac{\lambda^2 r^2 - \mu^2}{r^2}\right)R = 0$$

and as earlier shown, is identified as the Bessel's equation , then its corresponding solution will be,

$$R(r) = C_5 J_{\sqrt{\lambda^2 r^2 - \mu^2}}(r) + C_6 Y_{\sqrt{\lambda^2 r^2 - \mu^2}}(r)$$

where J(r) and Y(r) are solutions of Bessel's equation of the first and second kind , with order $\sqrt{\lambda^2 r^2 - \mu^2}$ respectively.

Also in the special case where the conditions are such that u is independent of the angular coordinate ϕ, we have

$$u = R(r)Z(z)$$

we have the following solution,

$$\frac{1}{r}\frac{d}{dr}(r\frac{dR}{dr}) + \lambda^2 R = 0 \qquad R(r) = C_7 J_0(r\lambda) + C_8 Y_0(r\lambda)$$

$$\frac{d^2Z}{dz^2} - \lambda^2 Z = 0 \qquad Z(z) = C_1 e^{\lambda z} + C_2 e^{-\lambda z}$$

iii. HELMHOLTZ'S EQUATION ($\Delta^2 u + k^2 = 0$)
subjected to the conditions ($a = b = 0, c = -k^2$)

The particular solution from separation of variable obtained is of the form

$$u = R(r)Z(z)\Phi(\phi) \tag{15}$$

where ,

$$\frac{1}{r}\frac{d}{dr}(r\frac{dR}{dr}) + (\lambda^2 - \frac{\mu^2}{r^2})R = 0 \tag{16}$$

solution to which is given by,

$$R(r) = C_1 J_{\sqrt{\lambda^2 r^2 - \mu^2}}(r) + C_2 Y_{\sqrt{\lambda^2 r^2 - \mu^2}}(r)$$

$$\frac{d^2 Z}{dz^2} - (\lambda^2 - k^{2)})Z = 0 \tag{17}$$

solution to which is given by,

$$Z(z) = C_3 e^{\sqrt{\lambda^2 - k^2}z} + C_4 e^{-\sqrt{\lambda^2 - k^2}z}$$

$$\frac{d^2\Phi}{d\phi^2} + \mu^2\Phi = 0 \tag{18}$$

solution to which is given by,

$$\Phi(\phi) = C_5 \cos\lambda\phi + C_6 \sin\lambda\phi$$

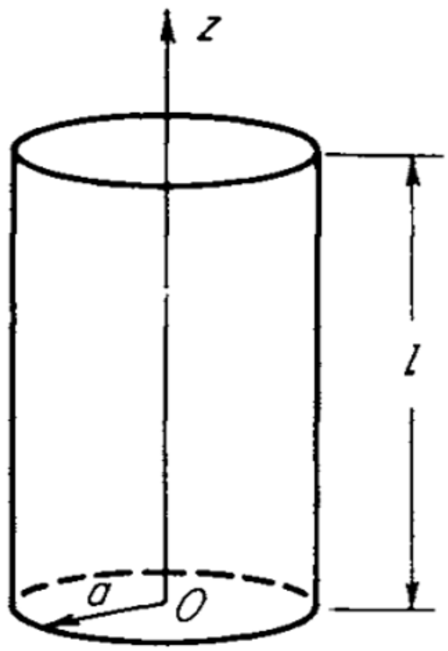

Figure 3: Dirichlet problem for a cylinder

4.2. THE DIRICHLET PROBLEM FOR A CYLINDER

THE BOUNDARY VALUE PROBLEM OF POTENTIAL THEORY

A function $u = u(x, y, z)$is said to be **harmonic** in a domain τ if$u$ and its first and second partial derivatives with respect to$x, yandz$ are continuous and satisfy Laplace's equation $\Delta^2 u = 0$in τ. Consider the problem of deriving a function u which has harmonic nature in τ and satisfies one of the three boundary conditions -

$$u|_\sigma = f$$

$$\frac{\delta u}{\delta n}|_\sigma = f$$

$$\left(\frac{\delta u}{\delta n} + hu\right)|_\sigma = f \qquad h > 0$$

where σ is the boundary of τ, f is a given function of a variable point of σ, and$\frac{\delta u}{\delta n}$ denotes the derivative with respect to the exterior normal to σ. This problem is called the first boundary value problem of potential theory or the **Dirichlet problem** if the boundary condition is $u|_\sigma = f$

We now consider the Dirichlet problem for the case where $\tau$is a cylinder of length l and radius a.

Letr, ϕ, zbe a cylindrical coordinate system, with z-axis along the axis of the cylinder and origin in one face of the cylinder. To satisfy the boundary condition $u|_\sigma = f$, we first solve two simpler problems corresponding to the boundary conditions

$$u|_{r=a} = 0 \ , \qquad u|_{z=0} = f_0 \ , \qquad u|_{z=l} = f_l \qquad u|_{r=a} = F \ , \qquad u|_{z=0} = u|_{z=l} = 0$$

We temporarily assume that the boundary conditions are independent of the angular coordinate ϕ, so that,

$$f_0 = f_0(r) \qquad f_l = f_l(r) \qquad F = F(z)$$

As obtained earlier, the particular solution of Laplace equation independent of ϕ will have the form $u = R(r)Z(z)$ where $R(r), Z(z)$ satisfies the equation obtained by separation of variables method ,

$$\frac{1}{r}\frac{d}{dr}(r\frac{dR}{dr}) + \lambda^2 R = 0$$

$$\frac{d^2 Z}{dz^2} - \lambda^2 Z = 0$$

Solving these equations we get,

$$R(r) = AJ_0(r\lambda) + BY_0(r\lambda)$$

and

$$Z(z) = Ce^{\lambda z} + De^{-\lambda z}$$

Re writing Z(z) in terms of $\sin h$ and $\cos h$ we get,

$$Z(z) = C\cosh\lambda z + D\sinh\lambda z$$

(1) We consider the first three boundary conditions.Since$J_0(\lambda r) \to 1$, $Y_0(\lambda r) \to \infty$ as $r \to 0$, and since the solution R must satisfy the physical requirement of being bounded on the axis of the cylinder, the constant B=0.

Therefore homogeneous boundary condition becomes ,

$$AJ_0(a\lambda) = 0$$

Thus the values of the parameter λ are $\lambda n = x_n/a$ where the x_n are the positive zeros of the Bessel function $J_0(x)$. So we obtain the following set of particular solutions of Laplace's equation

$$u = u_n = \left[M_n \cosh\left(x_n \frac{z}{a}\right) + N_n \sinh\left(x_n \frac{z}{a}\right)\right] J_0(x_n \frac{r}{a})$$

By superposition of these solutions, we can construct a solution of our problem. In fact, suppose each of the functions $f_0(r)$and $f_l(r)$ can be expressed in a Fourier-Bessel series as,

$f_0(r) = \sum_{n=1}^{\infty} f_{0,n} J_0(x_n \frac{r}{a})$ $\qquad$ $f_l(r) = \sum_{n=1}^{\infty} f_{l,n} J_0(x_n \frac{r}{a})$

where ,

$$f_{\rho,n} = \frac{2}{a^2 J_1^2(x_n)} \int_o^a r f_\rho(r) J_0(x_n \frac{r}{a}) dr, \qquad \rho = 0, l$$

therefore we get the series,

$$u = \sum_{n=1}^{\infty} \left[f_{0,n} \frac{\sinh(x_n \frac{l-z}{a})}{\sinh(x_n \frac{l}{a})} + f_{l,n} \frac{\sinh(x_n \frac{z}{a})}{\sinh(x_n \frac{l}{a})} \right] J_0(x_n \frac{r}{a})$$

which satisfies the Laplace's equation and the required boundary conditions.

(2) We consider the remaining boundary conditions. Here we set C= 0 and let,

$$\lambda = \frac{n\pi i}{l}, \qquad n = 1, 2, 3....$$

to satisfy the homogeneous boundary condition . Then the solution obtained would be ,

$$R(r) = AI_0(\frac{n\pi r}{l}) + BK_0(\frac{n\pi r}{l})$$
$$Z(z) = D \sinh(\frac{n\pi z}{l})$$

where I_0 and K_0 are Bessel's function of imaginary argument. Setting B = 0 since $K_0(\frac{n\pi r}{l}) \to \infty$ as $r \to 0$. Therefore solution of Laplace equation is now,

$$u = u_n = M_n I_0(\frac{n\pi r}{l}) \sin(\frac{n\pi z}{l}), \qquad n = 1, 2, ...$$

By applying superposition method, we find the solution as,

$$u = \sum_{n=1}^{\infty} F_n \frac{I_0(\frac{n\pi r}{l})}{I_0(\frac{n\pi a}{l})} \sin(\frac{n\pi z}{l})$$

where F_n are the Fourier coefficients of F(z) such that,

$$F_n = \frac{2}{l} \int_0^l F(z) \sin(\frac{n\pi z}{l}) dz$$

i. SOLVED EXAMPLE ON THE DIRICHLET PROBLEM OF A CYLINDER

Find the stationary distribution of temperature u in a cylinder of length l and radius a, with one end held at temperature u_0, while the rest of the surface is held at temperature 0.

SOLUTION

According to the given condition we set, $f_0 = u_0$ and $f_l = 0$

We derived the solution for the case where f vanishes on the lateral surface of the cylinder, hence we will use the set of solution,

$$u = \sum_{n=1}^{\infty} \left[f_{0,n} \frac{\sinh(x_n \frac{l-z}{a})}{\sinh(x_n \frac{l}{a})} + f_{l,n} \frac{\sinh(x_n \frac{z}{a})}{\sinh(x_n \frac{l}{a})} \right] J_0(x_n \frac{r}{a})$$

where

$$f_{\rho,n} = \frac{2}{a^2 J_1^2(x_n)} \int_0^a r f_\rho(r) J_0(x_n \frac{r}{a}) dr, \qquad \rho = 0, l$$

$f_0(r) = \sum_{n=1}^{\infty} f_{0,n} J_0(x_n \frac{r}{a})$ $\qquad$ $f_l(r) = \sum_{n=1}^{\infty} f_{l,n} J_0(x_n \frac{r}{a})$

Given $f_l(r) = 0$ and $f_0(r) = u_0$ we get the final temperature solution equation as,

$$u = \sum_{n=1}^{\infty} \left[f_{0,n} \frac{\sinh(x_n \frac{l-z}{a})}{\sinh(x_n \frac{l}{a})} \right] J_0(x_n \frac{r}{a})$$

And,

$$f_{0,n} = \frac{2}{a^2 J_1^2(x_n)} \int_0^a r u_0 J_0(x_n \frac{r}{a}) dr$$

We also know that

$$J_\nu(z) = \sum_{k=0}^{a} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$$

$|z| < a$, $|argz| < \pi$

Therefore for order $\nu = 0$ we have

$$J_0(x_n \frac{r}{a}) = \sum_{k=0}^{a} \frac{(-1)^k (x_n \frac{r}{a})^{2k}}{\Gamma(k+1)\Gamma(k+1)}$$

Substituting these values in the final series solution we get,

$$u = \sum_{n=1}^{\infty} \left[\frac{\sinh(x_n \frac{l-z}{a})}{\sinh(x_n \frac{l}{a})} \frac{2u_0}{a^2 J_1^2(x_n)} \int_0^a r J_0(x_n \frac{r}{a}) dr \right] J_0(x_n \frac{r}{a})$$

$$= 2u_0 \sum_{n=1}^{\infty} \left[\frac{\sinh(x_n \frac{l-z}{a})}{\sinh(x_n \frac{l}{a})} \frac{1}{a^2 J_1(x_n)} \int_0^a r \frac{J_0(x_n \frac{r}{a})}{J_1(x_n)} dr \right] J_0(x_n \frac{r}{a})$$

$$= 2u_0 \sum_{n=1}^{\infty} \left[\frac{\sinh(x_n \frac{l-z}{a})}{\sinh(x_n \frac{l}{a})} \frac{1}{a^2 J_1(x_n)} \int_0^a r \left\{ \frac{\sum_{k=0}^{a} \frac{(-1)^k (x_n \frac{r}{a})^{2k}}{\Gamma(k+1)\Gamma(k+1)}}{\sum_{k=0}^{a} \frac{(-1)^k (x_n)^{2k+1}}{\Gamma(k+1)\Gamma(k+2)}} \right\} dr \right] J_0(x_n \frac{r}{a})$$

$$= 2u_0 \sum_{n=1}^{\infty} \left[\frac{\sinh(x_n \frac{l-z}{a})}{\sinh(x_n \frac{l}{a})} \frac{1}{a^2 J_1(x_n)} \int_0^a r \sum_{k=0}^{a} \left\{ \frac{(\frac{r}{a})^{2k}(k+2)}{x_n} \right\} dr \right] J_0(x_n \frac{r}{a})$$

$$= 2u_0 \sum_{n=1}^{\infty} \left[\frac{\sinh(x_n \frac{l-z}{a})}{\sinh(x_n \frac{l}{a})} \frac{1}{x_n J_1(x_n)} \sum_{k=0}^{a} \frac{(k+1)}{a^{2k+2}} \int_0^a r^{2k+1} dr \right] J_0(x_n \frac{r}{a})$$

$$= 2u_0 \sum_{n=1}^{\infty} \left[\frac{\sinh(x_n \frac{l-z}{a})}{\sinh(x_n \frac{l}{a})} \frac{1}{x_n J_1(x_n)} \sum_{k=0}^{a} \frac{(k+1)}{2(k+1)} \right] J_0(x_n \frac{r}{a})$$

$$= u_0 \sum_{n=1}^{\infty} \left[\frac{\sinh(x_n \frac{l-z}{a})}{\sinh(x_n \frac{l}{a})} \frac{J_0(x_n \frac{r}{a})}{x_n J_1(x_n)} \right]$$

which is the required stationary distribution of temperature u in the cylinder.

4.3. COOLING OF A HEATED CYLINDER

Consider the problem of the cooling of an infinitely long cylinder of a radius a ,heated to the temperature $u_0 = f(r)$ and the radiating heat into the surrounding medium at zero temperature.

This problem reduces to solving the equation of heat conduction

$$c\rho \frac{\delta u}{\delta t} = k\Delta^2 u \qquad (1)$$

subject to boundary condition

$$\left(\frac{\delta u}{\delta r} + hu \right) |_{r=a} = 0 \qquad (2)$$

and the initial condition

$$u|_{t=0} = u_0 = f(r) \qquad (3)$$

where k = thermal conductivity of the cylinder
c = heat capacity
ρ = density
λ = emissivity

$$h = \frac{\lambda}{k}$$

Using the separation of variables, and assuming the solution to be

$$u = R(r)T(t)$$

we find the cylindrical equations ,

$$b\frac{dT}{dt} + \chi^2 T = 0$$

,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \chi^2 R = 0$$

Where $-\chi^2$ is the separation constant and $b = C\rho/k$

with the solutions ,

$$R = AJ_0(\chi r) + BY_0(\chi r)$$

,

$$T = Ce^{-\chi^2 t/b}$$

Since $J_0(\chi r) \to 1$,$Y_0(\chi r) \to \infty$ as $r \to 0$ and R must satisfy the physical requirement of being bounded on the axis of the cylinder , the constant B must be zero.

From (2) χ must satisfy the equation,

$$hJ_0(\chi a) - \chi J_1(\chi a) = 0 \qquad (4)$$

If $x = \chi a$, then the above equation becomes,

$$haJ_0(x) - xJ_1(x) = 0 \qquad (5)$$

which has only real roots, with respect to the origin.Let

$$0 < x_1 < x_2 \ldots < x_n < \ldots$$

be the positive roots of the equation (5). Then the value of χ are $\chi_n = \frac{x_n}{a}$ and the set of particular solution of (1) would be

$$u = u_n = M_n J_0\left(x_n \frac{r}{a}\right) e^{\frac{-(x_n)^2 t}{a^2 b}}, \qquad n = 1, 2, \ldots$$

Superposition of these solution gives,

$$u = \sum_{n=1}^{\infty} M_n J_0\left(x_n \frac{r}{a}\right) e^{\frac{-(x_n)^2 t}{a^2 b}} \tag{6}$$

because of the initial condition (3), the coefficients M_n must satisfy the relation,

$$f(r) = \sum_{n=1}^{\infty} M_n J_0\left(x_n \frac{r}{a}\right), \qquad 0 \leq r < a \tag{7}$$

By expanding $f(r)$ in a Dini series we have,

$$M_n = \frac{2}{a^2 \left[J_0^2(x_n + J_1^2(x_n)\right]} \int_0^a r f(r) J_0\left(x_n \frac{r}{a}\right) dr \tag{8}$$

Therefore solution of our heat conduction problem is given by (6).

CHAPTER : 5

(Hyper Geometric Functions)

5. HYPER-GEOMETRIC FUNCTIONS

Hyper-geometric series is the power series

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta_k)}{(\gamma)_k k! z^k}$$

wherezis complex variable ,α, β, γ are parameter which take complex or real values and the symbol $(\lambda)_k$ denotes the quantity,

$$(\lambda)_0 = 1 \qquad (\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \lambda(\lambda+1)\ldots(\lambda+k-1), k = 1, 2, ...$$

Thus, the hyper geometric is defined by,

$$F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta_k)}{(\gamma)_k k! z^k} ------> (18)$$

where,

$$|z| < 1$$

Its integral representation is given by,

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} z^k \int_0^1 t^{\beta-1+k}(1-t)^{\gamma-\beta-1} dt$$

,where reversing the order of summation and integration is justified by an absolute convergence argument.

5.1. SPHERICAL HARMONICS

Solution of the Linear Differential Equation -

$$(1-z^2)u'' - 2zu' + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]u = 0 \tag{1}$$

where z = complex number
ν, μ = parameters belonging to real numbers or complex numbers.

Such equation in mathematical physics is used in orthogonal curvilinear co ordinates to solve the boundary value problems for special domains i.e. sphere , spheroids or torus.

5.2. THE HYPER-GEOMETRIC EQUATION AND ITS SERIES SOLUTION

$$(1-z^2)u'' + [\gamma - (\alpha+\beta+1)z]\,u' - \alpha\beta u = 0$$

Let $u = \sum_{k=0}^{\infty} c_k z^{k+s}, c_0 \neq 0$ be a particular solution.

THEREFORE , $u' = \sum_{k=0}^{\infty} c_k (k+s) z^{k+s-1}$

and $u'' = \sum_{k=0}^{\infty} c_k (k+s)(k+s-1) z^{k+s-2}$

Now substituting the values we get,

$$(z-z^2)\left[\sum_{k=0}^{\infty} c_k(k+s)(k+s-1)z^{k+s-2}\right] + (\gamma-(\alpha+\beta+1)z)\left[\sum_{k=0}^{\infty} c_k(k+s)z^{k+s-1}\right] - \alpha\beta\sum_{k=0}^{\infty} c_k z^{k+s} = 0$$

$$= \sum_{k=0}^{\infty} c_k(k+s)(k+s-1+\gamma)z^{k+s-1} - \sum_{k=0}^{\infty} c_k\left[(k+s)(k+s-1) + (k+s)(\alpha+\beta+1) + \alpha\beta\right]z^{k+s} = 0$$

$$= \sum_{m=-1}^{\infty} c_{m+1}(m+1+s)(m+s+\gamma)z^{m+s} - \sum_{k=0}^{\infty} c_k\left[(k+s)(k+s-1) + (k+s)(\alpha+\beta+1) + \alpha\beta\right]z^{k+s} = 0$$

$$= c_0 z^{s-1}(s)(s-1) + \gamma s + \sum_{k=0}^{\infty} c_{k+1}(k+s+1)(k+s+\gamma)z^{k+s} - \sum_{k=0}^{\infty} c_k\left[(k+s)(k+s-1) + (k+s)(\alpha+\beta+1) + \alpha\beta\right]z^{k+s} = 0$$

$$= C_0 + \sum_{k=0}^{\infty}\left[c_{k+1}\left[(k+s+1)(k+s+\gamma)\right] - \left[(k+s)(k+s-1) + (\alpha+\beta+1)(s+k) + \alpha\beta\right]c_k\right]z^{k+s} = 0$$

Put coefficient of z^{s-1} and z^{k+s} as 0,

z^{s-1} : $c_0\,[s(s-1)+\gamma s]=0$ which implies , $s=0$ or $s=1-\gamma$

z^{s+k} : $c_{k+1}\,[(k+s+1)(k+s+\gamma)]=[(k+s)(k+s-1)+(\alpha+\beta+1)(s+k)+\alpha\beta]\,c_k$

which implies ,

$$c_{k+1}=\frac{[(k+s)(k+s-1)+(\alpha+\beta+1)(s+k)+\alpha\beta]\,c_k}{[(k+s+1)(k+s+\gamma)]}$$

k= 0,1,2....

Now for s = 0,

$$\begin{aligned}c_{k+1}&=\frac{[(k)(k-1)+(\alpha+\beta+1)(k)+\alpha\beta]\,c_k}{[(k+1)(k+\gamma)]}\\&=\frac{(k+\alpha)(k+\beta)c_k}{(k+1)(k+\gamma)}\end{aligned}$$

Therefore $c_k=\frac{(\alpha_k)(\beta_k)c_0}{k!(\gamma_k)}$

hence solution u = $u_1=F(\alpha,\beta,\gamma;z)=\sum_{k=0}^{\infty}\frac{(\alpha_k)(\beta_k)z^k}{k!(\gamma_k)}$

Now for $s=1-\gamma$,

$$c_{k+1}=\frac{[(k+1-\gamma)(k-\gamma)+(\alpha+\beta+1)(1-\gamma+k)+\alpha\beta]}{[(k-\gamma+2)(k+1)]}c_k \qquad k=0,1,2....$$

$$c_k=\frac{[(k-1+\gamma)(k-\gamma)+(\alpha+\beta+1)(k-\gamma)+\alpha\beta]}{[(1+k-\gamma+)(k)}c_{k-1} \qquad k=1,2,3....$$

$$\begin{aligned}&=\frac{[(k-\gamma)(\gamma+k+\alpha\beta)+\alpha\beta]}{[(1+k-\gamma+)(k)}c_{k-1}\\&=\frac{(k+\alpha-\gamma)(k+\beta-\gamma)}{(k)(k+1-\gamma)}c_{k-1}\end{aligned}$$

Therefore,

$$c_k=\frac{(1+\alpha-\gamma)_k(k+\beta-\gamma)_k}{(k!)(2-\gamma)_k}c_0$$

hence solution u = $u_2=z^{1-\gamma}\sum_{k=0}^{\infty}\frac{(1+\alpha-\gamma)_k(1+\beta-\gamma)_k}{(k!)(2-\gamma)_k}z^k$

$$= z^{1-\gamma} F(1 + \alpha - \gamma, 1 + \beta - \gamma; 2 - \gamma; z)$$

Therefore general solution,

$$u = AF(\alpha, \beta; \gamma; z) + Bz^{1-\gamma} F(1 + \alpha - \gamma, 1 + \beta - \gamma; 2 - \gamma; z)$$

CHAPTER : 6

(Legendre's Function)

6. LEGENDRE'S FUNCTION

$(1-z^2)u'' - 2zu' + \nu(\nu+1)u) = 0$ is known as Legendre's equation (1)

which can be reduced to hyper geometric equation by making suitable changes of variables.

$(i) Putting t = (1-z)/2$ in (1) we get ,

$$t(1-t)u'' + [(1-2t)]u' + \nu(\nu+1)u = 0 \qquad (2)$$

which is a special case of hyper-geometric equation where,

$\alpha = -\nu, \beta = \nu+1, \gamma = 1$

$(ii) Putting t = z^{-2}, u = z^{-v-1}\nu$ converts (1) to

$$t(1-t)\frac{d^2v}{dt^2} + \left[\left(\nu+\frac{3}{2}\right) - \left(\nu+\frac{5}{2}\right)t\right]\frac{dv}{dt} - \left(\frac{\nu}{2}+1\right)\left(\frac{\nu}{2}+\frac{1}{2}\right)v = 0$$

which is also a special case of hyper geometric equation where,

$\alpha = \frac{\nu}{2}+1, \beta = \frac{\nu}{2}+\frac{1}{2}, \gamma = \nu+\frac{3}{2}$

Therefore 2 particular solution of (1) are -

$$u = u_1 = F\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right), \qquad |z-1| < 2$$

$$u = u_2 = \frac{\sqrt{\pi}\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})(2z)^{v+1}} F\left(\frac{\nu}{2}+1, \frac{\nu}{2}+\frac{1}{2}; \nu+\frac{3}{2}, \frac{1}{z^2}\right), \qquad |z| > 1, |argz| < \pi, \nu \neq -1, -2, -3....$$

where F$(\alpha, \beta; \gamma, z)$is a hyper geometric series.

THESE SOLUTIONS ARE CALLED THE LEGENDRE'S FUNCTION OF DEGREE ν OF **THE FIRST KIND** AND **THE SECOND KIND** , DENOTED BY P_ν AND Q_ν , RESPECTIVELY.

Therefore,

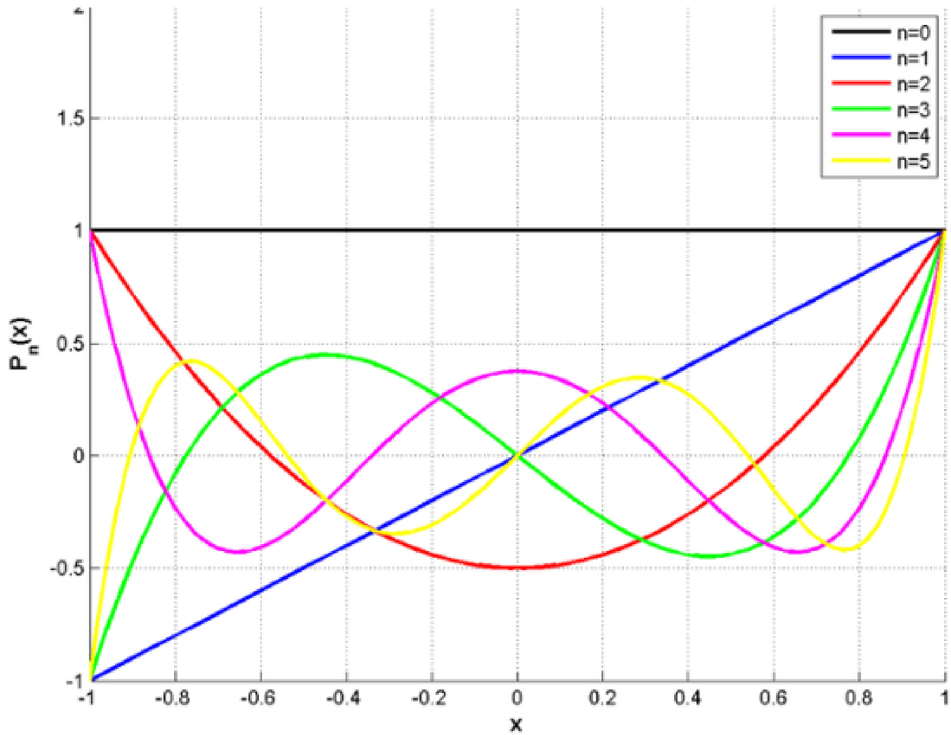

Figure 4: Graph of Legendre's function of first kind of various orders

$$P_\nu(z) = F\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right), \qquad |z-1| < 2$$

$$Q_\nu(z) = \frac{\sqrt{\pi}\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})(2z)^{v+1}} F\left(\frac{\nu}{2}+1, \frac{\nu}{2}+\frac{1}{2}; \nu+\frac{3}{2}, \frac{1}{z^2}\right), \qquad |z| > 1, |argz| < \pi, \nu \neq -1, -2, -3....$$

Thus the general solution u of (1) can be expressed as a linear combination of Legendre's function of the first kind and second kind , i.e.

$$u = AP_\nu(z) + BQ_\nu(z)$$

where

$$|arg(z-1)| < \pi, \qquad \nu \neq -1, -2, ...$$

CHAPTER : 7

(Applications of Legendre's Function)

7. APPLICATIONS OF LEGENDRE'S FUNCTION

7.1. Solutions of LAPLACE'S EQUATION in spherical coordinates

Orthogonal curvilinear coordinates using separation if variables in Laplace's equation in the **spherical coordinates system**r, θ, ϕ to the rectangular coordinates x,y,z by the formula

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta \qquad (1)$$

given that,

$0 \leq r < \infty$, $0 \leq \theta \leq \pi$, $-\pi < \phi \leq \pi$

We consider for triply orthogonal system of surfaces such that sphere $r = const$, circular cone $\theta = constant$ and the plane $\phi = const$ passing through z axis.

Also we know squares of the element of arc length is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \qquad (2)$$

Now in terms of metric-coefficients $h_\alpha, h_\beta, h_\gamma$ the Laplacian operator is defined as

$$\Delta^2 u = \frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\delta}{\delta\alpha}\left(\frac{h_\gamma h_\beta}{h_\alpha}\frac{\delta u}{\delta\alpha}\right) + \frac{\delta}{\delta\beta}\left(\frac{h_\alpha h_\gamma}{h_\beta}\frac{\delta u}{\delta\beta}\right) + \frac{\delta}{\delta\gamma}\left(\frac{h_\alpha h_\beta}{h_\gamma}\frac{\delta u}{\delta\gamma}\right)\right] \qquad (3)$$

Hence according to (1) , the metric-coefficients here would therefore be

$$h_r = 1, h_\theta = r, h_\phi = r\sin\theta$$

And corresponding Laplacian equation is given as

$$\Delta^2 u = \frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta u}{\delta r}\right) + \frac{1}{r^2\sin\theta}\frac{\delta}{\delta\theta}\left(\sin\theta\frac{\delta u}{\delta\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\delta^2 u}{\delta\phi^2} = 0 \qquad (4)$$

By the SEPARATION OF VARIABLES METHOD we get the particular solution of the form

$$u = R(r)\Theta(\theta)\Phi(\phi) \qquad (5)$$

Substituting (5) in (4) we get,

$$\Delta^2\left[R(r)\Theta(\theta)\Phi(\phi)\right] = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d(R(r)\Theta(\theta)\Phi(\phi))}{dr}\right)+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d(R(r)\Theta(\theta)\Phi(\phi))}{d\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{d^2(R(r)\Theta(\theta)\Phi(\phi))}{d\phi^2} = 0$$

Now dividing the above equation by $R\Phi\Theta$ and multiplying by $r^2\sin^2\theta$ we get the following ,

$$\left[\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

,possibly when both sides equal a constant say , μ^2. This leads to 2 equations,

$$\frac{d^2\Phi}{d\phi^2}+\mu^2\Phi = 0$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{\mu^2}{\sin^2\theta} \qquad (7)$$

Now equating the above equation by a constant say, $\nu(\nu+1)$. Therefore we get another 2 set of equations as,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\nu(\nu+1)R = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[\nu(\nu+1)-\frac{\nu^2}{\sin^2\theta}\right]\Theta = 0 \qquad (8)$$

The corresponding Laplace's equation depends on the parameters μ and ν , which can be used to construct solution of boundary values problem of mathematical physics involving spherical domains.

Now the corresponding differential solution for particular equations are as follows ,

(i)

$$\frac{d^2\Phi}{d\phi^2}+\mu^2\Phi = 0$$

Solution to which is,

$$\Phi(\phi) = C_1\cos\mu\phi + C_2\sin\mu\phi$$

(ii)

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[\nu(\nu+1)-\frac{\nu^2}{\sin^2\theta}\right]\Theta = 0$$

which is a Legendre's function , general solution to which is,

$$\Theta(\theta) = AP_\nu(\mu \sin\theta) + BQ_\nu(\mu \sin\theta)$$

(iii)

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \nu(\nu+1)R = 0$$

which is a Bessel's function with arg $= kr$ and order $\nu + \frac{1}{2}$ and substituting $R = r^{1/2}v$ we get ,

$$v'' + \frac{1}{r}v' + \left[k^2 - \frac{(\nu+1/2)^2}{r^2}\right]v = 0$$

Solution to which is,

$$R(r) = r^{1/2}\left[CJ_{\nu+\frac{1}{2}}(kr) + DH_{\nu+\frac{1}{2}}(kr)\right]$$

For **rotationally symmetric** case , we have u independent of Φ , hence we have the general solution as,

$$u = r^{1/2}\left[AJ_{\nu+\frac{1}{2}}(kr) + BH_{\nu+\frac{1}{2}}(kr)\right][CP_\nu(\mu\sin\theta) + DQ_\nu(\mu\sin\theta)]$$

7.2. DIRICHLET PROBLEM OF A SPHERE

Here we consider the interior Dirichlet problem for a spherical domain.We also assume that boundary function f and the solution u are independent of angle ϕ.

Choosing the origin at the center of the sphere of radius a the the z axis along the line of symmetry, we have the following problem

FIND THE FUNCTION $u(r,\theta)$ **SUCH THAT -**

1. u IS HARMONIC IN THE DOMAIN $r < a$ AND CONTINUOUS IN THE CLOSED DOMAIN $r \le a$

2. u SATISFIES THE BOUNDARY CONDITION $u|_{r=a} = f(\theta)$, WHERE $f(\theta)$ IS CONTINUOUS IN THE INTERVAL $0 \le \theta \le \pi$

We obtain the rotational symmetricity by setting $\Phi = 1$ in (5)and $\mu = 0$ in (8)
then the we obtain the differential equation for the Legendre's function of argument $x = \cos\theta$ which for $-1 < x < 1$ has general solution

$$\Theta = AP_\nu(\cos\theta) + BQ_\nu(\cos\theta)$$

where $P_\nu(x)$ and $Q_\nu(x)$ are Legendre functions of the first kind and the second kind whereν is the arbitrary complex number.

Since $x = \cos\theta$ ranges over closed $[-1,1]$ and as $x \to 1$, $Q_\nu(x) \to \infty$ but $P_\nu(x)$ remains bounded.So we must set B= 0 for the problem to be bounded .

Moreover since $P_\nu(x) \to \infty$ as $x \to -1$ unless ν is a non negative integer. Choosing $\nu = n$. Thus the only solution of for $\mu = 0$ which remain bounded in the closed interval $0 \leq \theta \leq \pi$ correspond to non negative integral ν and are of the form,

$$\Theta = AP_\nu(\cos\theta), \qquad n = 0, 1, 2...$$

where the $P_n(x)$ is the Legendre polynomial having degree n. As for the radial equation ,which is an Euler equation , with general solution ,

$$R = Cr^\nu + Dr^{-\nu-1}$$

In the present case $\nu = n$ and the solution must be bounded at the center of the sphere , $D = 0$.

$$R = Cr^\nu \qquad where n = 0, 1, 2.....$$

and hence the appropriate set of particular solution of Laplace's equation inside the sphere is ,

$$u = u_n = M_n r^2 P_n(\cos\theta)$$

where $n = 0, 1, 2.....$

Now we can solve the boundary value problem by superposition of the above solution. The boundary function $f(\theta)$ can be expanded in a series of Legendre's polynomial as

$$f(\theta) = \sum_{n=0} \infty f_n P_n(\cos\theta), \qquad 0 \leq \theta \leq \pi$$

where ,

$$f_n = (n + 1/2) \int_0 \pi f(\theta) P_n(\cos\theta) \sin\theta d\theta$$

Now if the series $f(\theta)$ converges uniformly in the interval $[0, \pi]$, then we let $M_n = f_n a^(- n)$ and thus we get solution as,

$$u = \sum_{n=0} \infty f_n \left(\frac{r}{a}\right) P_n(\cos\theta)$$

which solves the Dirichlet's problem of a sphere when boundary conditions are applied as $u|_{r=a} = f(\theta)$

7.3. DIRICHLET PROBLEM OF A CONE

The ability to separate variables in Laplace's equation written in spherical coordinates also allows us to solve boundary value problems for the domain bounded by the surface of an infinite circular cone.

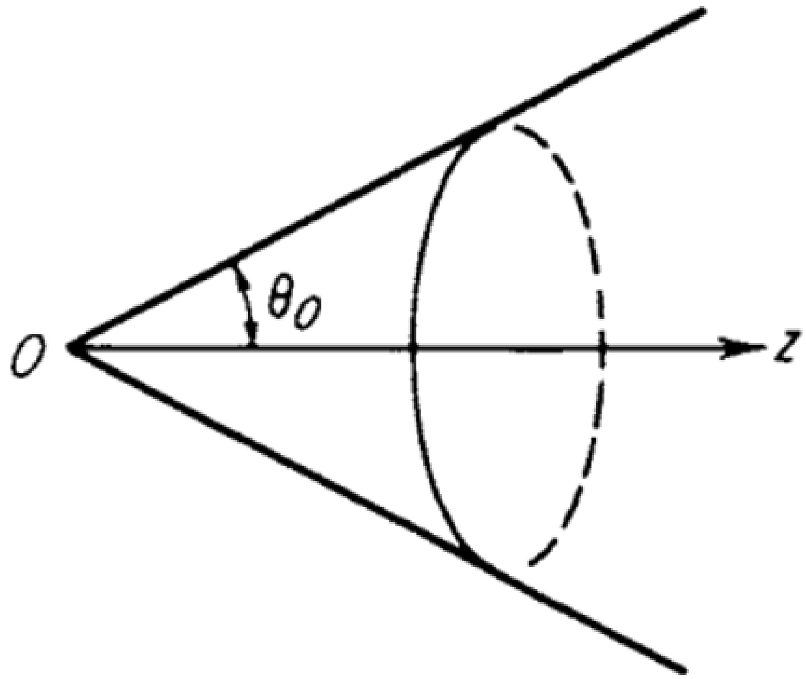

Figure 5: Dirichlet problem for a cone

Choose the origin at the vertex of the cone, and let the z axis lie along the same axis of symmetry of the cone. Then the equation of the cone is $\theta = \theta_0 (\theta_0 < \pi)$, and the Dirichlet problem for the case of axially symmetric boundary conditions can be stated as follows ,

Find the functions $u = u(r, \theta)$ **such that**
1) u is harmonic in the domain $0 < r < \infty, 0 \leq \theta \leq \theta_0$ **and continuous in the closed domain** $0 \leq r < \infty, 0 \leq \theta \leq \theta_0$

2) u satisfies the boundary condition $u|_{\theta=\theta_0} = f(r)$ **and the condition at infinity** $u|_{r\to\infty} \to 0$ **uniformly in** θ**, where** $f(r)$ **is continuous in the interval** $0 \leq r < \infty$ **and** $f(r)|_{r\to\infty} = 0$

Using the solution obtained by Laplace's equations we get the the general solution as,

$$\Theta = AP_\nu(\cos\theta) + BQ_\nu(\cos\theta)$$

Setting B= 0 for the bounded solution on the axis of the cone . Also $P_\nu(\cos\theta)$ is bounded for arbitrary ν if $0 \leq \theta \leq \theta_0$. letting $\nu = -1/2 + i\tau$ where $\tau \geq 0$ Therefore,

$$u = u_\tau = [M_\tau \cos(\tau \log r) + N_\tau \sin(\tau \log r)] r^{-1/2} P_{-1/2+i\tau}(\cos\theta)$$

M_τ and N_τ are continuous functions.

Using the definition of the Legendre's function of first kind we get,

$$P_{-1/2+i\tau}(\cos\theta) = F\left(1/2 - i\tau, 1 + i\tau; 1; \sin^2\frac{\theta}{2}\right)$$

$$= 1 + \frac{1/4 + \tau^2}{(1!)^2} \sin^2 \frac{\theta}{2} + \frac{(1/4 + \tau^2)(\frac{9}{4} + \tau^2)}{(2!^2)} \sin^4 \frac{\theta}{2} + ...$$

which shows that $P_{-1/2+i\tau}(\cos\theta)$ is real and satisfies the following,

$$1 \leq P_{-1/2+i\tau}(\cos\theta), \qquad 0 \leq \theta \leq \pi$$

$$P_{-1/2+i\tau}(\cos\theta) \leq P_{-1/2+i\tau}(\cos\theta_0), \qquad 0 \leq \theta \leq \theta_0$$

Now we suppose that $f(r)$ is such that $\phi(r) = r^{1/2} f(r)$ having Fourier expansion of the form ,

$$g(r) = r^{1/2} f(r) = \int_0^\infty \left[G_c(\tau)\cos(\tau logr) + G_s(\tau)\sin(\tau logr)\right] d\tau$$

where,

$$G_c(\tau) = \frac{1}{\pi}\int_0^\infty f(r) r^{-1/2} \cos(\tau \log r) dr \qquad G_s(\tau) = \frac{1}{\pi}\int_0^\infty f(r) r^{-1/2} \sin(\tau \log r) dr$$

where we have uniform convergent integral in $[r_1, r_2]$ such that $0 < r_1 < r_2 < \infty$. Then letting,

$$M_r = \frac{G_c(\tau)}{P_{-1/2+i\tau}(\cos\theta_0)}$$

$$N_r = \frac{G_s(\tau)}{P_{-1/2+i\tau}(\cos\theta_0)}$$

We then obtain the solution of the problem by integrating with respect to parameter τ from 0 to ∞,

$$u = r^{1/2} \int_0^\infty \left[G_c(\tau)\cos(\tau logr) + G_s(\tau)\sin(\tau logr)\right] \frac{P_{-1/2+i\tau}(\cos\theta)}{P_{-1/2+i\tau}(\cos\theta_0)} d\tau$$

CONCLUSION

This report aimed to present examples of the new application of the special functions to solve problem in the field of theoretical physics and mathematics. Special functions used in the theoretical physics were introduced as a result of the search for the solutions of practical problems. These have immensely important role in aspects related to cylinders, spheres, cone torus, spheroids, ellipses etc.

Diagrammatic representation of these special functions provides good knowledge of the function's behaviour as well as related examples gives clear application of these functions in the real world.

This research work concludes by the goal of giving a sufficiently detailed exposition of some special function and related problems, which are of greatest practical interest.

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