ELEMENT LEVEL RELIABILITY USING TAIL EQUIVALENT LINEARIZATION METHOD

A DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE AWARD OF THE DEGREE

OF

MASTER OF TECHNOLOGY

IN

STRUCTURAL ENGINEERING

Submitted by:

ARBIND KUMAR SINGH

(2K21/STE/07)

Under the supervision of

MR. G.P.AWADHIYA

DEPARTMENT OF CIVIL ENGINEERING

DELHI TECHNOLOGICALUNIVERSITY

(Formerly Delhi college of Engineering)

Bawana Road, Delhi-110042

MAY, 2023

DELHI TECHNOLOGICAL UNIVERSITY

(Formerly Delhi college of Engineering)

Bawana Road, Delhi-110042

CANDIDATE'S DECLARATION

I, Arbind Kumar Singh, 2K21/STE/07, student of M.Tech Structural Engineering, hereby declare that the project Dissertation titled "ELEMENT LEVEL RELIABILITY USING TAIL EQUIVALENT LINEARIZATION METHOD" which is submitted by me to the Department of Civil Engineering, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree of Master of Technology, is original and not copied from any source without proper citation . This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship or other similar title or recognition.

Place: Delhi **ARBIND KUMAR SINGH**

Date: 31-05-2023 **(2K21/STE/07)**

DEPARTMENT OF CIVIL ENGINEERING

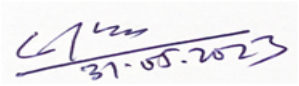
DELHI TECHNOLOGICAL UNIVERSITY

(Formerly Delhi college of Engineering)

Bawana Road, Delhi-110042

CERTIFICATE

I hereby certify that the Project Dissertation titled "ELEMENT LEVEL RELIABILITY USING TAIL EQUIVALENT LINEARIZATION METHOD" which is submitted by Arbind Kumar Singh, 2K21/STE/07 Department of Civil Engineering, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree of Master of Technology, is a record of the project work carried out by the students under my supervision. To the best of my knowledge this work has not been submitted in part or full for any Degree or Diploma to this University or elsewhere.



Place: Delhi
Date:

Mr. G.P. AWADHIYA
Associate Professor
Department of Civil Engineering
Delhi Technological University

ABSTRACT

To explore nonlinear random vibration, a novel non-parametric linearization methodology has been devised that is based on a discretized representation of stochastic inputs and integrates first order reliability method (FORM) thoughts. This method characterises the corresponding linear system by matching the design points of the nonlinear and linear responses in the space of standard normal variables created by discretizing the excitation with a predefined response threshold for the nonlinear system. Because it equals the first order estimate of the tail probability of the nonlinear system with that of the linear system, the methodology provides a more realistic picture of the TELS than earlier similar linearization approaches. A unit-impulse response function of the input excitation is required to represent the TELS. The purpose of this research is to examine the analysis of stochastic nonlinear systems using this approach and to compute certain nonlinear response characteristics. It also discusses the random vibrational analysis approach, particularly the equivalent linearization method, and gives an outline of structural reliability analysis, including FORM.

The primary goal of the study is to investigate the effect of various factors on the system. At the design point, the limit-state surface is linearized to define a linear system, the TELS. This non-parametric linearization approach has a promising potential to improve upon existing equivalent linearization techniques and provide a more effective means of analyzing nonlinear random vibration.

ACKNOWLEDGEMENT

Any accomplishment requires the effort of many people and this work is no exception. I appreciate the contribution and support, which various individuals have provided for the successful completion of this study. It may not be possible to mention all by name but the following were singled out by their exceptional help.

I would like to express my gratitude to **Mr.G.P. Awadhiya**, Associate Professor, Department of Civil Engineering, Delhi Technological University, New Delhi, whose counseling, supervision and suggestions were always encouraging and it motivated me to complete the job at hand. He will always be regarded as a great mentor for me.

I am deeply grateful to **Prof. Vijay.K.Minocha**, Head of Department, Department of civil engineering, Delhi technological University for his support and encouragement in carrying out this project.

I would like to express my heartiest thanks to my friends, seniors and juniors for constant support and motivation. Last but not the least I thank my parents, for everything I am and will be in future.

Table of Contents

CHAPTER – 1 INTRODUCTION

1.1 GENERAL

When it comes to analysing the reliability of structural and mechanical systems, it's important to consider the stochastic and non-linear parameters that often come into play. These factors can have a major influence on the behaviour of a system under high loading situations, such as earthquakes or turbulent winds. However, the currently known approaches for nonlinear stochastic dynamic analysis have constraints that hinder their use in practise in many circumstances.

One method that is not limited by these constraints is the Monte Carlo Simulation method. However, this approach can be very time-consuming and difficult, making it impractical for many engineering applications.

In contrast, linear systems' second moment response functions offer a valuable tool for understanding a system's characteristics. However, finding solutions using this approach can require an iterative process. Additionally, assuming a Gaussian distribution for probability distribution may not always be the best choice, particularly when dealing with the tail region of the distribution.

Overall, there is a clear need for a more effective method for non-linear stochastic dynamic analysis that can overcome the limitations of existing approaches. Such a method would be invaluable in ensuring the safety and reliability of structural and mechanical systems in a wide range of applications.

1.2 Objective and scope of the study

- Determination of nonlinear systems by TAIL Equivalent Linearization Method for calculation of certain nonlinear reaction attributes.
- To perform nonlinear dynamic analysis for damping based on TIME and FREQUENCY domain.
- To determine the variation of reliability index with the selected threshold using TAIL Equivalent Linearization Method

CHAPTER – 2 LITERATURE RIVIEW

Kazuya Fujimara,Armen Der Kiureghian , presented the Tail-Equivalent Linearization Method (TELM) is a non-parametric technique for evaluating nonlinear random vibrations that has been developed. TELM matches the design points of linear and nonlinear responses by categorising stochastic excitation and modelling it with standard normal variables. This assures that the linear system's tail probability approaches that of the nonlinear system.

TELM uniquely generates the corresponding linear system from its impulse response function and allows for the examination of various nonlinear response statistics. TELM beats the traditional Equivalent Linearization Method (ELM) in terms of accuracy, especially for high response thresholds, giving a reliable and efficient solution to nonlinear random vibration analysis.

Luca Garre,Armen Der Kiureghian, The prior work on the Tail Equivalent Linearization Method (TELM) was extended to the frequency domain. Using its frequency-response function, the Tail-Equivalent Linearization Method (TELM) may represent the Tail-Equivalent Linear System. TELM provides advantages in analysing stationary maritime structures by matching design points of nonlinear and linear responses. The TailEquivalent Linear System can accommodate multi-support excitations and is excitation scaling insensitive, enhancing computing efficiency for shifting sea conditions. The frequency-response function sheds light on the nature of random vibrations as well as the response process. The validity of TELM has been demonstrated by analysing the random sway displacement of a jack-up platform model. It relies on discretizing input excitation using standard normal variables and has proven successful in civil engineering applications.

Armen Der Kiureghian and Kazuya Fujimura, A novel method is proposed for seismic fragility curve computation in PBEE analysis of nonlinear structures. . . For nonlinear stochastic dynamic analysis, it applies tail-equivalent linearization, which eliminates the requirement for recurrent time-history analysis with scaled ground movements. Instead, the ground motion is handled as an unpredictable procedure, and

fragility curves for various reaction thresholds are generated by linear random vibration analysis with the Tail-Equivalent Linear System (TELS).The method ensures consistency by using a uniform stochastic model for all intensity levels. Limitations include applicability to non-degrading systems only, consideration of a single ground motion component, and the requirement for response gradient computations. Nonetheless, it offers a valuable alternative in an analysis domain with limited viable options.

Sanaz Rezaeian and Armen Der Kiureghian, The goal of this kind of study is to explore stochastic modelling and simulation of ground motion time histories for usage in performance-based earthquake engineering. The work presents a site-based, entirely nonstationary probabilistic model that captures the temporal and spectral properties of genuine earthquake ground movements. To lessen velocity and displacement residuals, high-pass filtering is used. The proposed model offers advantages such as accurate representation of non-stationary characteristics, a small number of interpretable parameters, time-domain modeling, and ease of simulation. It provides a practical method for generating synthetic ground motions with similarities to real earthquake motions. The research findings advances the area of individualised earthquake engineering by giving a simple method for creating realistic synthetic ground movements.

Caughey TK, In this study presents a generalized method for analyzing nonlinear dynamic systems under random excitation. The proposed approach incorporates the use of characteristic functions and compares results with exact solutions derived from the Fokker-Planck equation when possible. By extending the analysis to nonlinear systems, the method enables a comprehensive understanding of system responses. The validity of the method is demonstrated through various problem scenarios, confirming its effectiveness. The discussion emphasizes the significance of employing diverse mathematical techniques for analyzing complex systems. Overall, the proposed method offers valuable insights into the behavior of nonlinear systems subjected to random excitation and serves as an alternative when exact solutions using the Fokker-Planck equation are not achievable.

A. Der Kiureghian, The inquiry into the topic investigates the geometric representation of random vibration worries in the space of standard normal deviations got from

discretization of the input process. The problems for linear systems with Gaussian stimulation have basic geometric structures such as vectors, planes, and ellipsoids. Non-Gaussian and nonlinear reactions, on the other hand, introduce more complicated geometric forms. As approximate answers to these challenges, the authors suggest the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM).

By leveraging the geometric interpretation in the standard normal space, the article offers a fresh perspective on random vibration problems. It implies that many statistical parameters of relevance in random vibrations may be represented geometrically, possibly opening up new avenues for addressing non-Gaussian or non-linear issues.

The proposed methods, FORM and SORM, are explored as efficient approaches for solution. They provide effective means of analysis, but caution is advised in their application. The limitations of these methods are acknowledged, and the article emphasizes the need for careful utilization to ensure accurate results.

Furthermore, the article hints at the possibility of developing simulation methods that exploit the geometric forms identified in the standard normal space. Such methods could offer more efficient and accurate solutions to random vibration problems.

Through numerical examples, the effectiveness of FORM and SORM is demonstrated, showcasing their ability to provide approximate solutions for non-Gaussian excitation and the out-crossing of a vector process from a non-linear domain.

Overall, the article contributes to advancing the understanding of random vibration problems and presents a novel approach to their approximate solution. By utilizing geometric interpretations and exploring methods like FORM and SORM, researchers can gain valuable insights into non-Gaussian and non-linear scenarios in random vibration analysis.

Heonsang Koo, Armen Der Kiureghian, Kazuya Fujimura, The concept of design-point excitation in random vibration analysis is examined in this study, with emphasis on

a nonlinear elastic single-degree-of-freedom (SDOF) oscillator with Gaussian white noise input. The design-point excitation, according to the findings, corresponds to the inverse of the oscillator's free-vibration response when free of the target threshold. Although approximations are vital, this outcome. is extended to non-white and nonstationary excitations, as well as hysteretic oscillators.

Understanding the design-point excitation is crucial because it demonstrates the input process that is most likely to result in the sought occurrence. The study also proposes a simple and accurate method for measuring the mean up-crossing rate of random vibration response using the First Order Reliability approach (FORM). This rough figure provides important information on the frequency with which the response exceeds a given threshold. Overall, the current inquiry contributes to the area of random vibration analysis by explaining the calculation of the design-point activation and offering a realistic method for estimating the mean up-crossing rate. This knowledge is critical for assessing the dependability and performance of structures subjected to random vibrations.

Yan-Gang Zhao, Tetsuro Ono, The efficacy of the First Order Reliability Method (FORM) and Second Order Reliability Method (SORM) in reliability analysis varies by a trio of factors: the curvature radius at the design point, the number of arbitrary variables that are involved, and the first-order reliability index. In this study, the tolerances of these elements are looked at in order to determine the conditions under which FORM/SORM might offer conclusive results.

A three-step process is presented to make the actual implementation of FORM/SORM easier. First, a limit state surface is fitted to a single design point. Then, the total principal curvatures (Ks) are computed. Finally, the failure probability is evaluated based on the range of Ks. However, it is important to note that this procedure is only applicable to limit state surfaces with a single design point. If multiple design points are present, local convergence issues and inaccurate results may arise. Furthermore, the usefulness of FORM/SORM ranges is limited when the curvatures at the design point have distinct signs and an unequal distribution.

Engineers and researchers may examine the accuracy and usefulness of FORM/SORM for their specific study by understanding the ranges of the curvature radius, number of random variables, and firstorder reliability index. This knowledge aids in making

informed decisions regarding structural design, considering the uncertainties associated with random variables and their impact on system reliability.

M. Ababneh*, M. Salah, K. Alwidyan, In this work, the effectiveness of two linearization strategies, the optimum linear model and Jacobian linearization, is checked out.The inverted pendulum and the Duffing chaotic system are two well-known nonlinear systems to which these approaches are applied. Linearization is a key method used in the study and design of nonlinear systems, with the ultimate objective of approximating their behaviour with simpler linear models. The optimal linear model is an online technique that considers both the state and control variables, resulting in a more comprehensive linear approximation. On the other hand, Jacobian linearization focuses on linearizing the system around an operating point by computing the Jacobian matrix. The performance of these techniques is evaluated by assessing their ability to capture the nonlinear dynamics of the benchmark systems. By comparing the results, researchers gain insights into the strengths and limitations of each method. This comparative analysis aids in selecting the appropriate linearization technique for specific nonlinear systems, enhancing the understanding and analysis of their behavior.

Fayçal Ikhouane, Víctor Mañosa, José Rodellar, The Bouc-Wen model is commonly used in structural and mechanical engineering to describe smooth hysteretic behavior. However, it is crucial to assess whether these models accurately capture the underlying physical properties of the system being modeled, beyond their fit to specific input data. The intention of this research is to outline several classes of Bouc-Wen models based on their limited input-bounded output stability feature and capacity to represent fundamental physical aspects of the real system. By analyzing these characteristics, researchers can evaluate the limitations and capabilities of Bouc-Wen models, aiding in their appropriate application. Understanding the bounded input-bounded output stability and the ability to replicate essential physical properties enhances the reliability and accuracy of modeling approaches in engineering. It provides insights into the model's suitability for practical applications and ensures that the resulting simulations align with the real-world behavior of the system under consideration.

CHAPTER - 3 METHODS OF NONLINEAR STOCHASTIC ANALYSIS

3.1 INTRODUCTION

Nonlinear stochastic analysis is an area of mathematics that focuses on the study of nonlinear systems that are influenced by random events. Several approaches have been developed in this discipline to analyse and learn about the behaviour of nonlinear stochastic systems. Some of the commonly used methods of nonlinear stochastic analysis are:

3.2 CLASSICAL METHOD

- Stochastic Perturbation Method
- Fokker-Planck Equation
- Moment Closure Techniques
- Stratonovich Calculus

3.3 SIMULATION METHOD

- Monte Carlo Simulation
- Markov Chain Monte Carlo (MCMC)
- Stochastic Differential Equations (SDE) Simulation
- Agent-Based Modelling (ABM)

3.4 LINEARIZATION METHOD

- Classical Equivalent Linearization Method (ELM)
- Tail-Equivalent Linearization Method (TELM)

Stochastic modeling is a powerful tool for understanding complex systems. My research paper explores several methods for stochastic modeling, including classical methods like stochastic perturbation and Fokker-Planck equation, simulation methods like Monte Carlo simulation and Markov Chain Monte Carlo, and linearization methods like ELM and TELM. Each method offers unique advantages and challenges, and understanding their strengths and weaknesses can aid in developing more accurate and efficient models.

3.2 CLASSICAL METHODS

Classical methods of nonlinear stochastic analysis involve a range of mathematical techniques used to study and analyze nonlinear stochastic systems. These methods have been developed over several decades and are widely used in various fields to understand the behaviour of complex systems subjected to stochastic fluctuations.

3.2.1 Stochastic Perturbation Method

The stochastic perturbation method is a mathematical technique used to solve ordinary and partial differential equations that involve random variables or stochastic processes. The method involves introducing small random perturbations into the system and then averaging over these perturbations to obtain a more accurate solution. The basic idea of the stochastic perturbation method is to represent the random process as a sum of a deterministic component and a random component, where the random component has small amplitude.

The deterministic component is then solved using standard techniques, and the effect of the random component is included through averaging. The stochastic perturbation method has several advantages over other methods for solving stochastic differential equations, including its ability to handle non-linear and non-Markovian systems, as well as its ability to generate accurate results with relatively few simulations. The method is widely used to model and analyze complex systems subject to random fluctuations.

3.2.2 Fokker-Planck Equation

The Fokker-Planck equation, frequently referred to as the Kolmogorov forward equation, is a partial differential equation that captures the time it takes to create the probability density function of a stochastic process. It is widely used in several disciplines to model the behaviour of systems susceptible to random fluctuations.The general version of the Fokker-Planck equation is:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[b(x,t)p(x,t)] + \frac{\left(\frac{1}{2}\right)\partial^2}{\partial x^2[\sigma^{2(x,t)p(x,t)}]}$$

where p(x,t) is the stochastic process's probability density function at time t and location x, b(x,t) is the drift or average velocity of the process at position x and time t, and (x,t) is the diffusion coefficient or degree of randomness of the process at position x and time

t.The first aspect on the right-hand side of the equation indicates the process's drift or deterministic motion, while the second term represents the process's diffusion or random fluctuations.

The Fokker-Planck equation is a strong tool for analysing and forecasting the behaviour of stochastic processes such as Brownian motion, diffusion processes, and Langevin dynamics.

3.2.3 Moment Closure Techniques

Moment closure tackles are a class of mathematical techniques that are used to estimate the higher-order moments of a stochastic process using a finite collection of lower-order moments. The basic idea behind moment closure techniques is to close the infinite hierarchy of equations that arise when considering all possible moments of a stochastic system, by truncating the series of moments at some order and using a closure relation to approximate the higher-order moments in terms of the lower-order ones.

The closure relation is typically derived from assumptions about the distribution of the system, and the choice of closure relation can have a significant impact on the accuracy of the approximation. Some commonly used moment closure techniques include the mean-field approximation, the maximum entropy closure, and the Gaussian closure, among others. Moment closure techniques have a wide range of applications, including in systems biology, ecology, chemical kinetics, and finance. They are particularly useful for analysing large-scale systems with many interacting components, where exact solutions to the governing equations are often difficult or impossible to obtain

3.2.4 Stratonovich Calculus

Stratonovich calculus is a mathematical framework used to study stochastic processes and their associated differential equations. It is a type of stochastic calculus that is based on the Stratonovich integral, which differs from the more commonly used Itô integral in how it handles the interaction between the stochastic process and the underlying deterministic system.

The key difference between Stratonovich and Itô calculus lies in how the stochastic integrals are defined. In Stratonovich calculus, the integral takes into account the interactions between the stochastic process and the underlying deterministic system by incorporating a correction term that depends on the derivative of the underlying function.

This correction term can be thought of as a kind of "averaging" term that helps to smooth out the effects of the random fluctuations. Stratonovich calculus has a number of advantages over other types of stochastic calculus, including a more natural representation of physical systems with underlying deterministic dynamics, and better preservation of symmetries and conservation laws.

It is widely used in physics, engineering, finance, and other fields to study systems subject to random fluctuations, including Brownian motion, Langevin dynamics, and many other types of stochastic processes.

3.3 SIMULATION METHODS

Simulation methods of nonlinear stochastic analysis refer to the techniques used to simulate the behavior of nonlinear stochastic systems using computer simulations. The aforementioned methods are frequently used to examine the functioning of complex systems subjected to stochastic fluctuations in domains such as physics, science, finance, and economics. Here are some of the simulation methods of nonlinear stochastic analysis.

3.3.1 Monte Carlo simulation

Monte Carlo simulation is a computer approach for estimating the behaviour of complex systems or processes by generating random samples from probability distributions. It is called after the Monte Carlo Casino in Monaco, where games of chance incorporate unpredictable events. The core principle underlying Monte Carlo simulation is to utilise random sampling to determine the probability distribution of a system's behaviour.

The simulation draws a large number of randomly selected samples from the distribution and then analyses the data to determine the system's anticipated behaviour.Monte Carlo simulation may be used to tackle a variety of issues in physics, engineering, economics, and biology. Some examples include estimating the probability of success in a complex engineering project, predicting the future value of a financial asset, or modeling the behavior of a chemical reaction.

Monte Carlo simulation is often used when an analytical solution to a problem is difficult or impossible to obtain, or when the system being studied is too complex to be analyzed using traditional methods. The technique is also useful for exploring the sensitivity of a system's behavior to changes in its input parameters, or for identifying potential sources of risk or uncertainty.

3.3.2 Stochastic Differential Equations (SDE) Simulation

Stochastic Differential Equations (SDE) simulation is a technique used to model and analyze systems subject to random fluctuations over time. SDEs are a type of differential equation that includes a stochastic component, which captures the effect of random noise on the system.

SDE simulation involves generating a trajectory of the system over time by solving the SDE numerically. The simulation generates a sequence of random numbers that are used to simulate the random fluctuations in the system.

For simulating SDEs, plenty numerical approaches are available, including the Euler-Maruyama method, the Milstein method, and the stochastic Runge-Kutta method. These approaches entail discretizing the SDE and updating the system at each time step based on beforehand time step values and random noise.

One of the advantages of SDE simulation is that it can capture the effect of random noise on the system, which is often an important factor in real-world systems. However, SDE simulation can also be computationally intensive and requires careful tuning of the numerical parameters to obtain accurate results.

3.4 LINEARIZATION METHOD

Linearization is a computational modelling approach that employs linear methods to mimic the behaviour of a nonlinear system. The method entails locating the system's operating or balance point and then approaching the nonlinear function by its tangent or first-order approximation around that point. This results in a set of linear differential equations that clarify the system's activity around the operational point. Linearization plays a role in many areas, notably the analysis of complex systems and the design of control systems.

However, it is only true in a limited neighbourhood surrounding the functioning point, and the approximation's correctness is dependent on the degree of nonlinearity and closeness to the operating point.

3.4.1 Classical Equivalent Linearization Method (ELM)

The Equivalent Linearization Method (ELM) is a technique for calculating the response to random stimulation of a nonlinear system. When the nonlinear system can be treated as a single-degree-of-freedom (SDOF) system, ELM is extremely useful.

The primary idea behind ELM is to simulate a nonlinear system using comparable properties that have been modified to match the chaotic system's response. After that, the similar linear system is subjected to the same random stimulation as the original nonlinear system, and its response is analysed using linear analytic techniques.

The ELM procedure for an SDOF system can be summarized as follows:

1. Define the source of excitation and identify the nonlinear system.

2. Determine the unique features of the corresponding linear system by modifying the stiffness and damping coefficients to match the nonlinear system's displacement and velocity response at a reference level of stimulation.

3. Estimate the system response by solving the linear system using linear analysis techniques such as the power spectral density approach.

4. Adjust the corresponding linear system parameters to match the nonlinear system response at different levels of stimulation, and repeat the analysis.

5. Use the estimated response of the analogous linear system as an approximation of the real nonlinear system's response.

ELM is a valuable approach in a variety of engineering applications, including earthquake engineering, where it is used to predict the reaction of buildings and other structures subjected to seismic stimulation. It should be noted, however, that ELM is only applicable for low to moderate degrees of excitation and may not be correct for highly nonlinear systems or under excessive loading circumstances.

3.4.2 Tail-Equivalent Linearization Method (TELM)

The Tail Equivalent Linearization Method (TELM) is a variant of the Equivalent Linearization Method (ELM) used to estimate the response of a single-degree-of-freedom (SDOF) nonlinear system to extreme loads or events. TELM is particularly useful when the nonlinear system exhibits a nonlinear stiffness behavior, such as the snap-through

buckling of a slender column. The basic idea behind TELM is to partition the input excitation into two parts: a high-intensity peak excitation and a background excitation that contains the low-intensity random component. The high-intensity peak excitation is assumed to drive the nonlinear system to a limit state, while the background excitation is treated as a small perturbation around this limit state. The TELM procedure for an SDOF system can be summarized as follows:

1. Identify the nonlinear system and define the input excitation as a combination of high-intensity peak excitation and background excitation.
2. Adjust the stiffness and damping coefficients of the analogous linear system to match the reactivity of the nonlinear system at the limit state, i.e., its optimum reaction under high-intensity peak activation.
3. For assessing how it adapts to background stimulation, solve the linear system using linear analysis techniques such as the power spectral density proximity.
4. Repeat the study after adjusting the analogous linear factors to match the outcome of the nonlinear system at additional stages of high-intensity peak stimulation.
5. Use the probabilistic response of the comparable linear system to figure out the response of the real nonlinear system under high stresses.

TELM is a valuable approach in a variety of engineering scenarios where the reaction of buildings subjected to high stresses, such as quake it is or wind-induced loads, must be anticipated. However, it is important to note that TELM, like ELM, is only valid for small to moderate levels of excitation and may not be accurate for highly nonlinear systems or under extreme loading conditions.

3.5CHARACTERSTICS OF A LINEAR SYSTEM

A linear system is defined by its impulse response function (IRF) and frequency response function (FRF). The IRF demonstrates the system's response to a unit impulse input, whereas the FRF depicts the system's steady-state responsiveness to a complex logarithmic input.

Knowing a linear system's IRF and FRF isn't needed to fully characterise the system's behaviour for any input signal. For example, calculating the Fourier transform of the input signal by the FRF yields the system's continuous response to an input signal F(t).

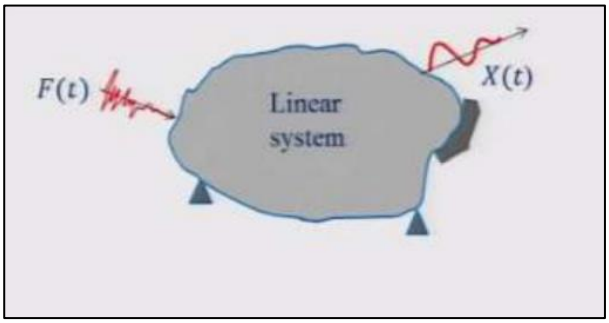


Fig 4.1 Linear System

In terms of math, if h(t) is a linear system's IRF and H() is its FRF, then the system's output anticipation to an input signal F(t) may be computed as:

$$\boldsymbol{X(t) = F(t) * h(t)}$$

where * represents the convolution operation. Alternatively, the system's steady-state response to a sinusoidal input signal F(t) = exp(it) may be determined as:

$$\boldsymbol{Xss(\omega) = H(\omega) * F(\omega)}$$

where F() denotes the Fourier transform of the input signal and H() represents the Fourier transform of the IRF.

Therefore, if we know the IRF and FRF of a stable linear system, we can predict the system's response to any input signal without needing to know the system's physical

Linearity: An arrangement is linear if it adheres to the superposition principle, which asserts that the response to a set of inputs equals the sum of the responses to the individual inputs.The equation can be used to express a linear system:

$$y(t) = \int h(\tau)\, x(t-\tau)\, d\tau$$

where y(t) symbolises the system's output, x(t) the system's input, and h(t) the system's impulse response.

The linearity property of the system can be expressed mathematically as:

$$y1(t) = \int h(\tau)\, x1(t-\tau)\, d\tau$$

$$y2(t) = \int h(\tau)\, x2(t-\tau)\, d\tau$$

$$y(t) = y1(t) + y2(t)$$

$$y(t) = \int h(\tau)\, (x1(t-\tau) + x2(t-\tau))\, d\tau$$

$$y(t) = \int h(\tau)\, x(t-\tau)\, d\tau$$

where y1(t) and y2(t) are the answers to the inputs x1(t) and x2(t). The total of the replies y1(t) and y2(t) is equal to the sum of the responses received x1(t) + x2(t), which is similar to the response x(t) = x1(t) + x2(t).

Time-invariance: A system is unaffected by time if the impulse response does not vary with time and is independent of the time it is seen. A time invariant system can be stated numerically by the following equation:

$$y\,(t) = \int h\,(\tau)\, x\,(t-\tau)\, d\,\tau$$

where h() is the system's impulse response and x(t-) is the input delayed by seconds.The system's time-invariance feature may be stated numerically as:

$$y(t) = \int h(\tau)\, x(t-\tau)\, d\tau$$

$$y(t - T) = \int h(\tau)\, x((t - T)-\tau)\, d\tau$$

$$y(t - T) = \int h(\tau)\, x(t-(T+\tau))\, d\tau$$

$$y(t - T) = \int h(\tau-T)\, x(t-\tau)\, d\tau$$

where T is a time shift. The otput of the system shifted by T seconds is equal to the outpt of the system with the input shifted by T seconds and the impulse response shifted by T seconds.

Causality: If the impulse response for negative time is zero, the system is deterministic. A deterministic system can be conceptually expressed by the following equation:

$$y\,(t) = \int h\,(\tau)\, x\,(t-\tau)\, d\,\tau$$

where h() is the system's impulse response and x(t-) is the delayed input in seconds.

The system's temporality feature may be quantitatively stated as:

h(t) = 0 for t < 0

A deterministic system's impulse response is zero for negative time, implying that the system's output is solely determined by the input and the system's previous state.

A linear system has input and output that are linked by a linear algorithm. A linear system may be symbolised by its (IRF) h(t) for a single input-output duo (F(t), X(t)), which depicts the system's output in response to a unit impulse input F(t) = (t). Convolving the input with the IRF yields the system's response to any arbitrary input F(t):

$$\boldsymbol{X(t) = F(t) * h(t) = \int F(\tau)h(t-\tau)d\tau}$$

where * denotes the convolution operation.

The function (FRF) of a linear system is the convoluted amplitude ratio of the system's steady-state output response to a sinusoidal input of frequency. Theoretically the FRF H() is defined as

$$\boldsymbol{H(\omega) = \frac{X_{SS}(\omega)}{F(\omega)}}$$

where Xss() is the system's stable response to the input F(t) = exp(it), and F() is the Fourier transform of the input signal. For a stable linear system, the IRF and FRF are related by the Fourier transform:

$$\boldsymbol{H(\omega) = \int h(t)\, exp(-i\omega t)\, dt}$$

and the inverse Fourier transform:

$$\boldsymbol{h(t) = \left(\frac{1}{2\pi}\right)\int H(\omega)\, \exp(i\omega t)\, d\omega}$$

Knowing both the IRF and FRF of a linear system is sufficient to completely characterize the system's behavior for any input signal. For example, the steady-state response of the system to an input signal $F(t)$ can be obtained by multiplying the Fourier transform of the input signal by the FRF:

$$\boldsymbol{Xss(\omega) = H(\omega)\, F(\omega)}$$

The knowledge of IRF and FRF is very useful for analysing and designing linear systems, as it enables us to predict the system's response to any input.

CHAPTER – 4 RELIABILITY

5.1 Reliability Analysis

5.1.1 Reliability :

In the field of structural reliability analysis, there are two commonly used techniques: First Order Reliability Methods and Second Order Reliability Method. These methods are employed to assess the safety and performance of structures when faced with uncertainties. To understand these techniques, it is important to understanding of basic random variable concepts and limit state function, as well as the issue of error propagation.

In structural reliability analysis, a limit state equation represents the relationship between uncertain variables associated with the structure's parameters and the failure of the structure. It defines the condition where the structure no longer meets the desired performance criteria. For example, in the design of a bridge, the limit state equation could express the maximum load the bridge can sustain without failure.

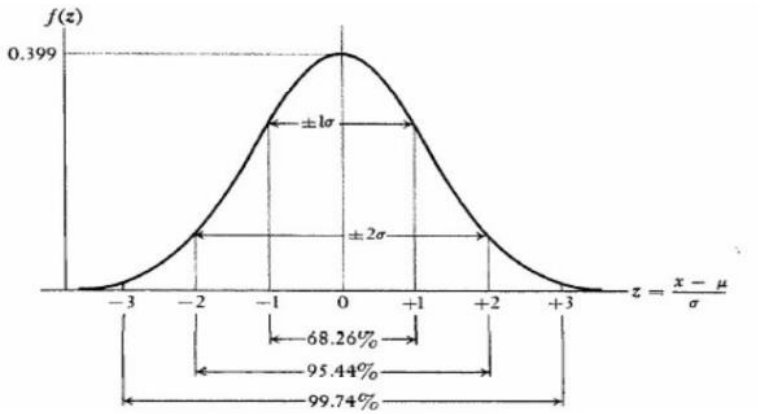


Fig. 5.1 Standard Normal Distribution Curve

performance criteria. For example, in the design of a bridge, the limit state equation could express the maximum load the bridge can sustain without failure.

The uncertain parameters that affect the behavior of the structure are known as basic random variables. These variables can include material properties, loads, dimensions, and other relevant factors. Each basic random variable is characterized by its probability distribution, mean, and standard deviation.

Error propagation refers to the process of determining how uncertainties in the basic random variables propagate through the limit state equation, affecting the probability of failure. Traditional methods simplify the error propagation problem by assuming that the limit state equation can be linearized around the random variable's average value. However, this linearization approximation may lead to inaccurate results, particularly for non-linear limit state equations.

FORM provides a more general solution to the error propagation problem. It involves transforming the original limit state equation into a standard normal space, where the Variables are thought to have a normal distribution with a mean of zero and a standard deviation of one. This transformation allows the non-linear limit state equation to be converted into a linear equation in the transformed space.

In summary, FORM and SORM are valuable techniques in structural reliability analysis that help evaluate the safety and performance of structures under uncertain conditions. By considering limit state equations, basic random variables, and addressing the phenomenon of occurrence of error, these methods provide engineers and designers with insights to ensure reliable structures in the presence of uncertainties

5.2 Methods of reliability :

1) First Order Reliability Method (FORM)
2) Second Order Reliability Method (SORM)
3) Monte Carlo Sampling (MCS)
4) Numerical Integration (NI)
5) Increased Variance Sampling (IVS)

5.2.1 Introduction

When it comes to structural components and systems, there are certain challenges in conducting reliability assessments. Firstly, there may be a lack of relevant failure data, making it difficult to rely on historical information alone. Additionally, structural failures occur relatively infrequently, making it challenging to statistically analyze these rare events. Furthermore, failures in structural systems are often caused by a combination of factors, including extreme external events such as high winds, heavy snowfall,

earthquakes, etc. Therefore, it becomes necessary to individually consider the influences from external sources (loads) and internal sources (resistances) when assessing reliability.

It is crucial to create probabilistic mathematical models for both the loads and the resistances, taking into account all the information that is currently known about the statistical properties of the parameters impacting them. This may involve data related to earthquakes, experimental results on concrete compression strength, and other relevant information.

Probabilistic modeling in reliability assessment involves a two-fold approach. Firstly, it requires creating models for the random variables representing the loads and resistances. These models describe the statistical distributions and parameters associated with these variables. Secondly, it involves studying the relationship between these random variables to understand how the loads and resistances interact and affect the performance of the structural component.

In the case of an idealized structural component, it is assumed to have two performance states. These states represent different levels of structural integrity, such as a "safe" state and a "failure" state. The objective is to determine the probability of the component being in the failure state based on the probabilistic models of the loads and resistances. By analyzing the behavior of the random variables and their interactions, reliability assessment techniques provide estimates of the probability of failure.

It is crucial to stress that the legitimacy of the processes used, rather than the precise outcomes obtained, is the emphasis of the reliability evaluation. By utilizing established and reliable procedures and using reasonable inputs based on the available information, the calculated probability of failure can serve as a valuable tool for decision-making.

5.2.2 FIRST ORDER RELIBILITY METHOD

Let's delve into the explanation of the First Order Reliability Method (FORM) using the Tail Equivalent Linearization (TEL) method, including relevant equations.

a) Limit State Equation:

The efficiency state of the structure is related to the fundamental random variables by the limit state equation. It can be represented as:

$$G(X) = 0$$

where G is the limit state function and X is a vector of basic random variables.

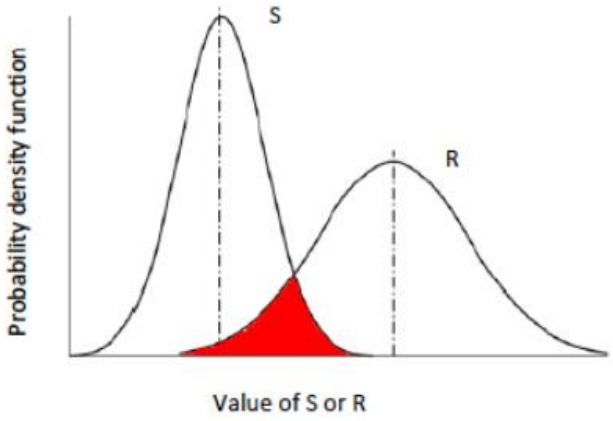


Fig.5.2: Overlapped area is the probability of failure of random variable S and R

b) Standardization:

Basic variables that are random are converted into typical normal variables (Z) having a zero mean and a single standard deviation in order to streamline the study. This standardization process is commonly achieved using the inverse of the cumulative distribution function (CDF) of each variable. The standardization equation can be expressed as:

$$Z = \Phi\text{^}(-1)(X)$$

where Φ^(-1) represents the inverse CDF (also known as the quantile function) and Z is the vector of standard normal variables.

c) Linearization:

The TEL approach uses a linear function in the common normal space to approximate the tail part of the limit state function (G). The linearization equation can be written as:

$$G\_approx(Z) = \beta\hat{}T\ Z - c$$

where β is a vector of direction cosines representing the gradient of the limit state function at the design point, c is a scalar value representing the distance from the origin to the design point, and Z is the vector of standard normal variables.

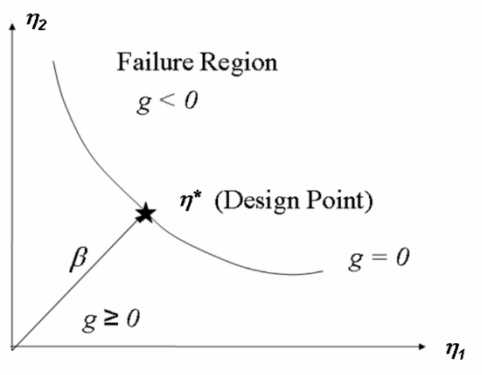


Fig. 5.3: Distribution of safety margin

d) Reliability Analysis:

The linearized limit state equation is used in analysis of reliability to determine the chances of collapse. The probability of failure (Pf) may be measured using the usual normal cumulative distribution function (CDF), displayed here:

$$Pf = \Phi(-\beta\hat{}T\ Zd + c)$$

where Zd stands in for the design point in the typical standard space.

The calibrated limit state equation for Z in the form of Pf is solved to get the design point (Zd). It can be expressed as:

$$Zd = (\beta\hat{}T)\hat{}(-1)\ (c + \Phi\hat{}(-1)(1 - Pf))$$

It should be noted that the accuracy of the results obtained through the TEL method relies on the adequacy of the linear approximation. For more accurate results, especially

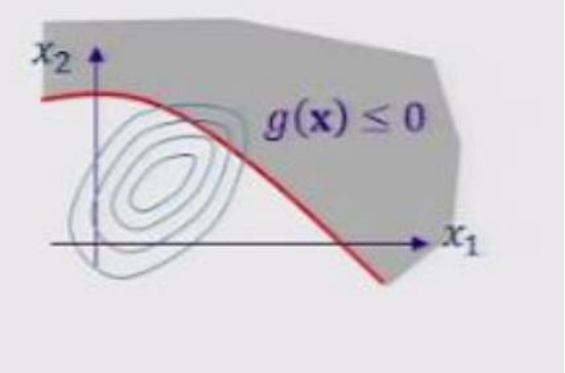


Fig 5.4: Geometry of random variables 1

for larger failure probabilities or complex limit state functions, higher-order reliability methods like the Second Order Reliability Method (SORM) may be more appropriate In summary, the First Order Reliability Method with the Tail Equivalent Linearization method involves transforming the original limit state equation into a linearized form in the standard normal space. This allows for the estimation of the probability of failure by solving the linearized equation and utilizing the standard normal CDF.

5.2.3 Monte Carlo Sampling (MCS)

Monte Carlo Sampling (MCS) is a widely employed technique in various domains, including engineering, finance, and science, to approximate unknown quantities and conduct numerical simulations. It utilizes statistical sampling principles to estimate complex mathematical problems.

MCS is used in reliability analysis to assess the probability of loss for structural components or systems. The basic idea underlying MCS is to generate an extensive amount of random specimens from the probability distributions of the unknown variables involved in the investigation. These samples are then used to perform computations and statistical analysis, allowing for the generation of accurate estimates.

Here is a step-by-step explanation of how Monte Carlo Sampling works:

1. Problem Definition: Probability Distribution Specification: Define the probability distributions for each uncertain variable, including their means and standard deviations. Commonly used distributions in MCS include normal, uniform, and log-normal distributions

2. Probability Distribution Specification: Define the probability distributions for each uncertain variable, including their means and standard deviations. Commonly used distributions in MCS include normal, uniform, and log-normal distributions

3. Sample Generation: Generate a large number of random samples for each uncertain variable based on their specified probability distributions. The number of samples should be sufficiently large to achieve accurate estimations, typically thousands or more.

4. Analysis Execution: For each set of samples, perform the necessary calculations to evaluate the system response or the limit state function. This may involve solving equations, conducting simulations, or running numerical models.

5. Failure Counting: Determine whether each sample results in failure or success based on the limit state function. If the limit state function indicates failure, the sample is categorized as a failure; otherwise, it is considered a success.

6. Probability of Failure Estimation: To get the failure probability, divide the entire number of samples produced by the number of failure samples. This approximation offers an estimate of the real failure probability.

7. Confidence Assessment: Assess the uncertainty associated with the estimated probability of failure using statistical techniques, such as confidence intervals or variance analysis. This helps evaluate the reliability of the estimation.

MCS offers the advantage of flexibility in handling complex problems with multiple uncertain variables and non-linear behaviors. It can provide reliable estimations even in the absence of closed-form analytical solutions. However, MCS can be computationally demanding, especially for simulations involving a large number of samples or complex models.

By increasing the number of samples and repeating the sampling process, MCS allows for convergence towards more accurate estimations. This makes it a powerful tool for analyzing structural reliability and assessing system performance in the presence of uncertainties.

5.2.4 Numerical Integration (NI)

Numerical Integration (NI), also known as numerical quadrature, is a technique used to find out the value of a definite integral when an exact solution is challenging to obtain. It

involves breaking down the integral into smaller intervals and using numerical methods to approximate the area under the curve.

Here's a rephrased explanation of how Numerical Integration works:

1. Problem Definition: Identify the definite integral that needs to be evaluated. This integral represents the area under a curve between specified limits.

2. Interval Division: Subdivide the integration interval into smaller subintervals. The number of subintervals can be specified by selecting a step size or the required precision.

3. Approximation Methods: Select a suitable numerical method to approximate the integral within each subinterval. Commonly used methods include the Trapezoidal Rule, Simpson's Rule, and the Midpoint Rule.

4. Subinterval Integration: Apply the chosen approximation method to each subinterval. This involves evaluating the function at specific points within the subinterval and calculating the corresponding area.

5. Summation: Add up the calculated areas from each subinterval to obtain an approximation of the total integral. Increasing the number of subintervals generally improves the accuracy of the approximation.

6. Error Estimation: Estimate the error associated with the numerical integration. Different methods offer error estimation techniques, which help assess the reliability of the approximation.

Numerical Integration is particularly valuable when dealing with functions that lack an analytical solution or when the function is too complex to integrate by hand. It provides a practical approach to estimate integrals in various scientific and engineering applications.

The choice of the numerical integration method depends on factors such as the smoothness of the function, desired accuracy, and computational efficiency. Methods like Simpson's Rule offer higher accuracy at the expense of more function evaluations, while simpler methods like the Trapezoidal Rule may be computationally faster but provide lower accuracy.It's vital to remember that numerical integration involves approximation errors, and the precision of the output is determined by the method used and the number of subintervals used.

CHAPTER – 5 DISCRETE REPRESENTATION OF STOCHASTIC PROCESS

5.1 INTRODUCTION

The finite representation of based on chance requires sampling the continuous-time process at regular time intervals to approximate the continuous-time process. This is typically done by discretizing the time axis and using update equations to describe the process's evolution over discrete time steps.

Consider a general continuous-time stochastic process defined by the equation:

$$X(t) = f(X(t), t, W(t)),$$

where X(t) is the position of the process at time t, f is a function defining its development, t is time, and W(t) is an independent variable reflecting the stochastic component. To obtain a discrete representation, we divide the time axis into equally spaced intervals with a fixed time step denoted by Δt. The process is then approximated at these isolated time points. Let's call the continuous process X[n], where n is the time index. One commonly used approach is the Euler-Maruyama method, which approximates the continuous-time process using the following update equation:

$$X[n+1] = X[n] + \Delta t * f(X[n], n\Delta t, W[n]),$$

where X[n] represents the state of the process at time nΔt, and W[n] is a sample of the random variable W(t) at time nΔt.

In this method, the discrete-time process is obtained by assuming that the change in X over a small time step Δt is proportional to the derivative of f(X, t, W) evaluated at the current state X[n]. The random variable W[n] is sampled at each time step to incorporate the stochastic component.

It is important to note that in the discrete form, the picking of the time step t is critical. A smaller time step allows for a more accurate approximation but increases computational complexity, while a larger time step reduces computational burden but

may introduce larger errors in the approximation. The appropriate choice of Δt depends on the specific characteristics and requirements of the stochastic process being modeled.

Additionally, it's worth mentioning that the discretization process is an approximation, and the accuracy of the discrete representation depends on the specific discretization method used. There are alternative numerical methods available that may provide more accurate approximations, especially for complex nonlinear stochastic processes.

In summary, the discrete representation of a stochastic process involves sampling the continuous-time process at regular intervals and approximating its behavior using update equations that describe its evolution over discrete time steps.

General form of a zero-mean Gaussian process

Each fluctuating variable in a zero-mean Gaussian process follows a Gaussian distribution with a typical value of zero. The covariance function or covariance matrix of a zero-mean Gaussian process refers to the connection between multiple points or time instances within the process.

Let's call a zero-mean Gaussian process X(t), where t is the time parameter. The generic form of a zero-mean Gaussian process may be stated mathematically as $X(t) \sim N(0, C(t_1, t_2))$,

where N(0, $C(t_1, t_2)$) indicates a Gaussian distribution with a mean of zero and a covariance denoted by $C(t_1, t_2)$. The covariance function $C(t_1, t_2)$ specifies how the process values at two different time instances, t_1 and t_2, are related.

We examine the joint distribution of a limited collection of random variables selected from the process to understand the distinctive properties of a zero-mean Gaussian process. Assume we have a set of random variables X(t1), X(t2),..., X(t_n) generated by a zeromean Gaussian process, with t1, t2,..., t_n denoting discrete time occurrences.

The multivariate Gaussian distribution may be used to illustrate the joint distribution of these unplanned variables:

$$P(X(t_1), X(t_2), \ldots, X(t\_n)) = N(0, \Sigma),$$

The covariance function C(t1, t2), which specifies the covariance between the process values at time occurrences t1 and t2, produces the members of the variance matrix. We

may infer key properties of zero-mean Gaussian processes from the properties of multivariate Gaussian distributions:

1. Marginal distribution: Each subset of random variables extracted from the process follows a Gaussian distribution. For example, the marginal distribution of $X(t_1)$ adheres to a Gaussian distribution with a mean of zero and a variance of $C(t_1, t_1)$.

2. Conditional distribution: Given the values of the remaining variables, the conditional distribution of a subset of random variables follows a Gaussian distribution. For example, given X(t2), the conditional distribution of X(t1) is a Gaussian distribution with a mean proportional to the covariance C(t1, t2)/C(t2, t2) and a variance C(t1, t1) - (C(t1, t2)/C(t2, t2)) * C(t2, t1).

3. Joint distribution: The joint distribution of any number of unknown variables is a multiplex Gaussian distribution, as illustrated by the aforementioned equation.

These properties make zero-mean Gaussian processes useful for analysis and modeling purposes, as they facilitate straightforward computations and provide probabilistic descriptions of the process's behavior.

It should be noted that the specific shape of the correlations function C(t1, t2) relies on the zero-mean Gaussian process under consideration. Different covariance functions result in different kinds of Gaussian processes, such as stationary processes, processes with specified spatial or temporal features, and so on. The choice of the covariance function depends on the application and the desired characteristics of the process being modeled.

5.1.1 Time domain discretization

In time domain discretization, the goal is to approximate a continuous-time signal or process by representing it in a discrete-time form. One specific scenario involves discretizing a modulated filtered white noise process using an impulse response function.

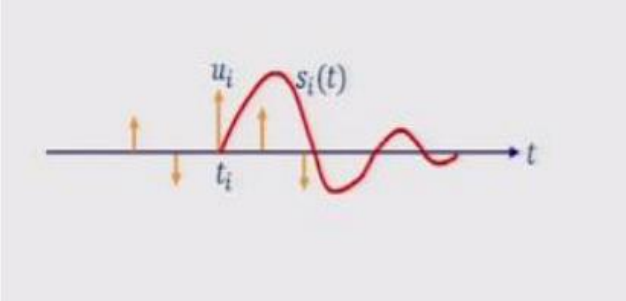


Fig 6.1: Time domain discretization

The modulated filtered white noise process can be described by the equation:

$$X(t) = m(t) * Y(t),$$

where X(t) represents the modulated filtered white noise process, m(t) is the modulation function, and Y(t) is the white noise process.

To discretize this process, we need to discretize both the modulation function and the white noise process.

1. Discretizing the modulation function:

The modulation function, m(t), can be transformed into a discrete sequence of values by sampling it at regular time intervals. We can denote the discrete modulation function as m[n], where n represents the time index.

2. Discretizing the white noise process:

The continuous-time white noise process, Y(t), has an impulse autocorrelation function. To discretize it, we must first identify an acceptable impulse response function that characterises the white noise process's behaviour.The bodily reaction function represents the filtering impact on the white noise process and records the system's reaction to an impulse input. The impulse reaction function is denoted as h(t). To discretize the white noise process, we apply the impulse response function to a discrete-time white noise sequence. We can represent the discrete white noise process as Y[n], where n represents the time index.

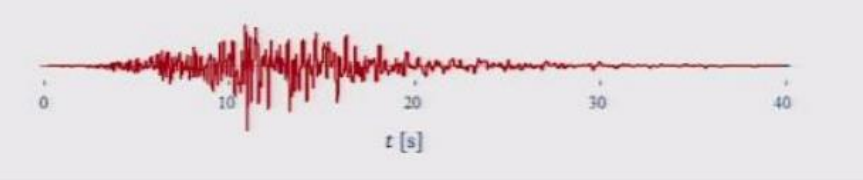


Fig 6.2 Frequency domain discretization

By convolving the discrete white noise sequence with the impulse response function, we obtain the discrete white noise process:

$$Y[n] = \sum h[n-k] * \xi[k],$$

where h[n-k] represents the impulse response function evaluated at the time difference n-k, ξ[k] represents the discrete white noise sequence at time k, and the summation is performed over the appropriate range of k.

Finally, we obtain the discretized modulated filtered white noise process, X[n], by multiplying the discrete modulation sequence, m[n], with the discretized white noise sequence

$$Y[n]: X[n] = m[n] * Y[n].$$

In summary, the discretization of a modulated filtered white noise process involves sampling the modulation function to obtain a discrete modulation sequence and convolving a discrete white noise sequence with an appropriate impulse response function to approximate the continuous-time white noise process. The discretized modulated filtered white noise process is then obtained by multiplying the discrete modulation sequence with the discretized white noise sequence

It's important to note that the choice of the impulse response function and specific implementation details may depend on the characteristics of the modulation and filtering operations in the process. In the discrete-time domain, the partitioning process approximates the continuous-time controlled filtered white noise process.

CHAPTER – 6 FORM SOLUTION OF STOCHASTIC DYNAMIC PROBLEMS

6.1 DEFINITION

The First Order Reliability Method (FORM) is a widely used technique for analyzing the reliability of stochastic dynamic nonlinear systems. It is particularly useful when dealing with complex systems that involve uncertainty and nonlinearity.

The goal of reliability analysis is to determine the likelihood that a system will execute its intended function without failure during a time frame. In dynamic systems, the behavior of the system evolves with time, and the uncertainties associated with various parameters can affect the system's reliability.

FORM is a numerical method that allows us to estimate the probability of failure for such dynamic nonlinear systems. It is based on the concept of limit state function, It denotes the performance border with regard to the safe and failure portions of the system. The limit state function is typically a mathematical expression that relates the system's input variables to its output response..

6.2 Reliability Formulation

The FORM method starts by transforming the original stochastic dynamic nonlinear problem into a series of equivalent linear problems. This transformation is achieved by employing the Taylor series expansion and keeping only the first-order terms. By doing so, the problem becomes amenable to linear reliability analysis techniques.

In FORM, the reliability analysis is performed in two steps:

Reliability index calculation: The reliability index, also known as the performance index or the safety margin, measures the distance between the mean value of the limit state function and the failure threshold. It quantifies the level of safety or risk in the system. The reliability index can be calculated by applying an optimization algorithm, such as the first-order second-moment (FOSM) or the Hasofer-Lind (HL) method.

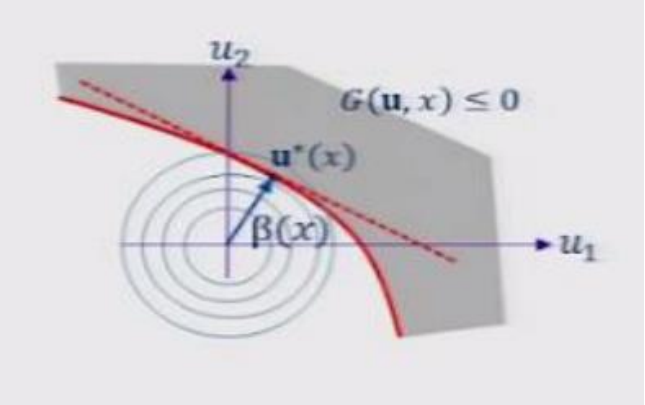


Fig 7.1 Reliability index ($\beta(x)$)

Probability of failure estimation: After determining the reliability index, the failure probability may be evaluated using appropriate probabilistic approaches, such as the Normal or Lognormal distribution. These strategies are based on the assumption that the limit state function adopts a specified probability distribution.

FORM provides an efficient means of estimating the probability of failure for dynamic nonlinear systems by linearizing the problem and utilizing advanced optimization techniques. However, it is important to note that FORM has some limitations. It assumes that the limit state function is separable and that the system response can be accurately represented by its mean value and variance. These assumptions may not always hold true in practical scenarios.

Researchers and engineers have developed various modifications and extensions to FORM to overcome these limitations and enhance its applicability to a broader range of dynamic nonlinear systems. These advancements include techniques like second-order reliability method (SORM), subset simulation, and adaptive response surface methods.

Overall, the First Order Reliability Method is a valuable tool in the field of reliability analysis, providing insights into the performance and safety of stochastic dynamic nonlinear systems.

6.3 Reliability Formulation of Linear System

Using the superposition principle

$X(t,u) = a(t) * u$ where $a(t)$, vector of responses to deterministic functions $s(t)$.

G (u ,x) = x – a(t) * u we can notice that u is a linear function of the limit state function

$u^* = \frac{x}{|\{(a(t)\}|} * \frac{a(t)}{|\{(a(t)\}|}$

$\beta(x) = \frac{x}{|\{(a(t)\}|}$, we see that the reliability index is proportional to the threshold x.

P r (x < X (t, u)) = $\phi(-\beta(x))$, since the tail probability β is proportional to

x, it can be concluded that the response is Gaussian.

CHAPTER – 7 TAIL EQUIVALENT LINEARIZATION METHOD

7.1 INTRODUCTION

The Tail Equivalent Linearization Method (TELM) is a structural reliability analysis tool intended for quantifying the probability of failure or exceedance for extreme response levels, with an emphasis on the tail area that defines the response distribution. TELM is an extension of the Equivalent Linearization Method (ELM) that tries to capture the response's tail pattern.

ELM, the conventional approach, linearizes the nonlinear system response by approximating it with an equivalent linear system. This simplifies the analysis and allows the application of well-established linear techniques. However, ELM assumes a Gaussian distribution for the response, which may not accurately represent extreme events or nonlinear systems.

To alleviate this restriction, TELM considers the response's non-Gaussian features, notably in the tail area. By combining higher-order statistical information, TELM aims to give a more accurate assessment of the chance of failure for extreme occurrences. This is accomplished by the use of equivalent linearization, among other in the tail section of the response distribution.

The fundamental concept behind TELM is to approximate the response distribution beyond a predefined threshold value, denoted as β, using a linear distribution. This linear distribution is characterized by its mean, variance, and other statistical properties derived from the tail behavior of the original response distribution. By utilizing this linear approximation in the tail region, TELM improves the estimation of the probability of failure for extreme events.

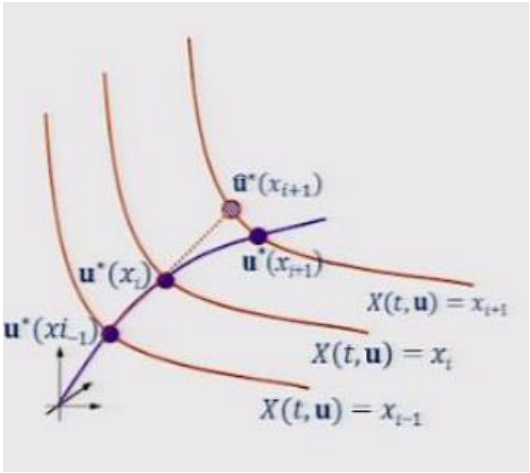


Fig 9.1 Representation of design point

7.2 Steps in TELM

To approximate the failure probability of a system or structure with nonlinear behaviour, the Tail Equivalent Linearization (TEL) technique combines the principles of the First-Order Reliability Technique (FORM) and the Equivalent Linearization Method (ELM). The approaches below are frequently used to determine the TEL surface.

Step 1: Formulating the limit-state function

The limit-state function, designated as 'U,' is defined in such a way that U 0 denotes failure and U > 0 represents non-failure. The limit-state function may be stated for a given threshold 'x' and time 't' as:

$$U(x,t) = g(x,t) - x\_0$$

where g(x,t) is the system's reaction at threshold 'x' and x_0 is the threshold level.

Step 2: Finding the design point

The design point, denoted as 'u*', is the point at which the limit-state function equals zero. Mathematically, we can express it as:

$$U(u*) = g(u*,t) - x\_0 = 0$$

Solving for u*, we get:

$$u* = g\text{^} - 1(x\_0,t)$$

where g^-1 is the inverse of the response function.

Step 3: Obtaining the gradient vector

The gradient vector at the design point 'u*' represents the direction in which the limitstate function changes the most. It is derived by evaluating the partial derivatives of the limitstate function pertaining to the variables that are supplied at the design point. Mathematically, we can express it as:

a = ∇g(u*,t)

where ∇ represents the gradient operator.

Step 4: Identifying the TEL surface

The TEL surface is the tangent hyperplane that corresponds to the gradient vector 'a'. It approximates the limit-state function in the vicinity of the design point 'u*'. Mathematically, the TEL surface can be expressed as:

$$U(x,t) = a\text{^}T\ (x - u*) + C$$

where a^T is the transpose of the gradient vector 'a', and C is a constant. The TEL surface represents a linear approximation of the limit-state function near the design point.

To compute the constant 'C,' the first-order reliability technique (FORM) can be utilised. FORM imitates the limit-state function employing a linear hyperplane and computes the dependability index, which measures the distance between the mean point and the limit-state function. The dependability index can be commented mathematically as:

$$\beta = u* - x\_0 \,/\, ||a||$$

where ||a|| is the Euclidean norm of the gradient vector 'a'.

Using the normal distribution function, the failure probability can be calculated as:

$$P(f) = \Phi(-\beta)$$

where Φ is the cumulative distribution function of the standard normal distribution.

To summarise, the TEL model entails stipulating the limit-state function, identifying the design point, measuring the gradient vector, and determining the TEL surface. Around the design point, the TEL surface approximates the limit-state function, and the failure probability may be assessed via the first-order reliability technique.

7.3 Iterative Algorithms For Solving Design Point

Iterative algorithms used to solve for the design point in the Tail Equivalent Linearization (TEL) method:

1. Fixed-Point Iteration:

The Fixed-Point Iteration is an iterative technique often used in the TEL method to identify the design point. This algorithm begins with an initial guess for the design point and iteratively adjusts it until convergence is obtained. The technique entails calculating an updated value using the inverse of the response function and the threshold value. The algorithm refines the guess until the requisite accuracy is achieved at the design point.

2. Newton-Raphson Method:

Another iterative algorithm utilized in the TEL method is the Newton-Raphson method. It leverages the gradient vector and the Hessian matrix of the limit-state function to iteratively improve the initial guess for the design point. The algorithm starts with an initial guess and then calculates an updated value by subtracting the product of the inverse of the gradient vector and the limit-state function. This iterative process continues until convergence is achieved, providing a more accurate determination of the design point.

These iterative algorithms play a crucial role in the TEL method by facilitating the computation of the design point. They enable the refinement of the initial guess through successive iterations, resulting in a more accurate estimation of the design point within the desired tolerance level. The choice between the Fixed-Point Iteration and the Newton Raphson method is determined by a number of factors, including the problem's complexity and the availability of gradient information, to ensure that the most appropriate algorithm is used for efficient convergence and accurate design point determination.

CHAPTER – 8 CHARACTERICTICS OF THE TAIL EQUIVALENT LINEARIZATION METHOD

8.1 For threshold x and time t

The Tail Equivalent Linearization (TEL) technique is a popular method for estimating the response of nonlinear systems, including single degree of freedom (SDOF) systems. The TEL approach allows the use of typical linear analytic techniques by substituting the nonlinear system with an equivalent linear system, making it a powerful tool for analysing nonlinear system reactions to nonlinear excitations.The TEL method possesses several distinctive characteristics:

1. Nonlinear to Linear Conversion: The primary objective of the TEL method is to convert the nonlinear equations of motion of the SDOF system into equivalent linear equations. This conversion facilitates the utilization of conventional linear analysis methods, such as modal analysis or frequency response analysis, to examine the system's response.

2. Tail Effect Consideration: The TEL method accounts for the "tail effect" inherent in nonlinear systems. The tail effect signifies the impact of higher-order terms within the system's nonlinear equations on the response. By incorporating these tail terms, the TEL method improves the accuracy of the linearized approximation, allowing for a more reliable estimation of the system's behavior.

3. Frequency-Dependent Parameters: Because of the magnitude and frequency of the stimulation, nonlinear systems frequently display frequency dependent fluctuations in their characteristics such as stiffness, damping, and mass properties. In the TEL method, frequency-dependent parameters are introduced into the equivalent linear system to capture these variations accurately. This ensures that the linearized system can better replicate the frequency-dependent characteristics of the original nonlinear system.

4. Energy Dissipation Modeling: The TEL method addresses the dissipation of energy, which is a critical aspect of nonlinear systems. Nonlinear systems tend to possess distinctive energy dissipation characteristics that differ from those of linear systems. The TEL method approximates this behavior by adjusting the damping parameters of the

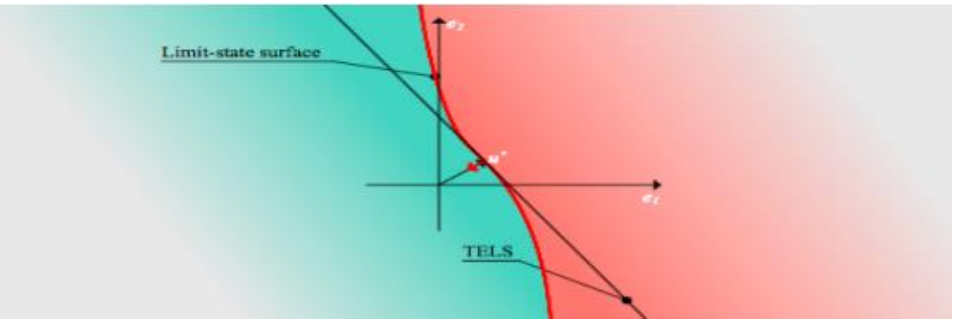


Fig 10.1 TELS ofzthe non linear response for azgiven threshold x andzpoint in time t

linearized system to more accurately represent the energy dissipation properties of the original nonlinear system

5. Limited Amplitude Range Validity: It is vital to note that the TEL method's validity can frequently be restricted to a specified stimulation amplitude range. As the excitation amplitude increases, nonlinear effects become increasingly significant, thereby diminishing the accuracy of the linearized approximation. Consequently, the TEL method is most effective when applied to systems subject to small to moderate excitation amplitudes.

It is crucial to acknowledge that the TEL method is an approximation technique and may not capture all the complexities inherent in the original nonlinear system. However, it provides a simplified and computationally efficient approach for studying the responses of SDOF systems to nonlinear excitations. Engineers and researchers may acquire significant insights into the system's behaviour and make educated decisions based on the results of the linearized approximation by utilising the TEL approach.

8.2 Numerical Example

A SDOF oscillator with inelastic material behaviour is considered to numerically investigate the properties of TELM. Both the frequency and time domains are utilized to solve the problem. A symmetric Bouc-Wen material model is used in order to describe the force-displacement relationship.

A hysteretic oscillator is considered to obtain further insight into the nature of the system. This oscillator is described as:

$$m\,\ddot{X}(t) + c\,\dot{X}(t) + k[a\,X(t) + (1-\alpha)Z(t)] = F(t)$$

Where,

$m = 3 \times 10^5$ (kg)

$c = 1.5 \times 10^2$ ($KN_{s/m}$)

$k = 2.1 \times 10^4$ (KN/m)

Fig. 10.2 SDOF oscillator with inelastic material behaviour

The degree of hysteresis is controlled by the parameter ‘α’.

$$\alpha = 0.1$$

The excitation process is described by the equation:

$$F(t) = -\ddot{U}_g(t)$$

Where, $\ddot{U}_g(t)$ gives base acceleration modelled as white-noise process.

A finite setting of the intensity of the white noise process yields the results depicted below since the scale of the excitation has no effect on the TELS.

The mathematical term Z(t) stems based on the Bouc-Wen hysteresis law.

$$Z(t) = -\gamma|\dot{X}||Z(t)|^{n-1}Z(t) - \eta|Z(t)|^{n}\dot{X}(t) + A\dot{X}(t)$$

Where $\gamma = \eta = \frac{1}{2\sigma_o^n}$ in which $\sigma_o^2 = \frac{\pi S m^2}{ck}$ is the mean square response of the linear *(α = 1)* oscillator, and the selected parameters are *n* = 3 and *A = 1*.

The problem statement provided specific values for certain parameters, which were used in the code. Additionally, assumptions were made for other parameters not explicitly mentioned. The code implementation utilized predefined functions available in MATLAB, such as "linsquare," "pwelch," "linsquare," and "hilbert." These functions are built-in MATLAB functions that assist in various computational tasks.

- The code generated a graph depicting the relationship between ground acceleration and time. This graph illustrated the IRF and FRF of the system. A thorough knowledge of the system's behaviour was achieved by addressing the issue in both the time and frequency phase.
- In the time domain analysis, the code used preset functions to construct the impulse response functions, which represent the system's reaction to an impulse input over time. These functions made it possible to calculate system variables including displacement, velocity, and acceleration as a function of time.
- In the frequency phase analysis, the code employed the predefined functions to compute the frequency response functions. These functions provided insights into how the system responds to sinusoidal inputs at different frequencies. The frequency response functions enabled for the determination of system behaviour in the frequency domain, such as resonant frequencies, mode shapes, and frequency response characteristics.
- By solving the problem in both domains and plotting the corresponding graphs, the code implementation offered a comprehensive analysis of the SDOF system's behavior. This approach provided valuable insights into the system's response characteristics, enabling a better understanding of its dynamics and aiding in decision-making for practical applications.

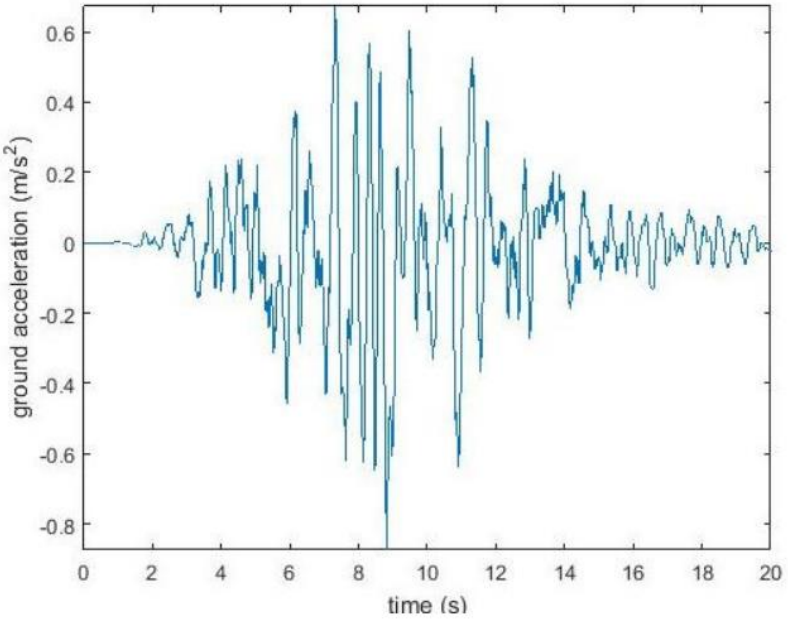


Fig.10.3 ground acceleration vs time graph

The graph image indicates the ground acceleration g(t) hits a maximum after which it falls with time.

- The TELS (Tail Equivalent Linearization) method is not influenced by the scaling of the excitation, making it independent of changes in the magnitude or amplitude of the input. This independence is explained by the invariance of the design point direction and the geometry of the limit-state surface with respect to the scaling factor.
- In the TELS method, the design point represents a specific operating condition or state of the system that is considered the most representative or probable for the analysis. The form of the limit-state surface establishes the border between safe and failure areas in the system's response space. The design point and the limit-state surface are both defined by certain criteria or thresholds.

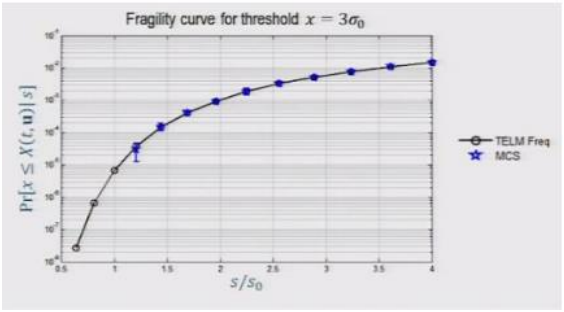


Fig.10.3 Fragility curve for given threshold

- Using the Bouc-Wen model and the TAIL equivalent linearization approach, this programme runs Monte Carlo simulations to calculate the fragility curve of a single-degree-of-freedom system exposed to white noise excitation.
- The code begins with specifying System parameters, which include the mass m, stiffness k, Bouc-Wen model parameters alpha, beta, and gamma, damping ratio zeta, number of simulations n_sim, number of points on the fragility curve n_points, and sigma_range.
- Next, the code performs a loop over different levels of white noise standard deviations. For each noise level, it initializes a failure count variable and runs n_sim simulations. In each simulation, it generates a white noise acceleration a using the randn function. It then initializes the system state variables and simulates the system response using the TAIL method. The simulation continues until the system displacement exceeds a threshold of 2 (indicating failure). The failure count is incremented whenever a failure occurs.

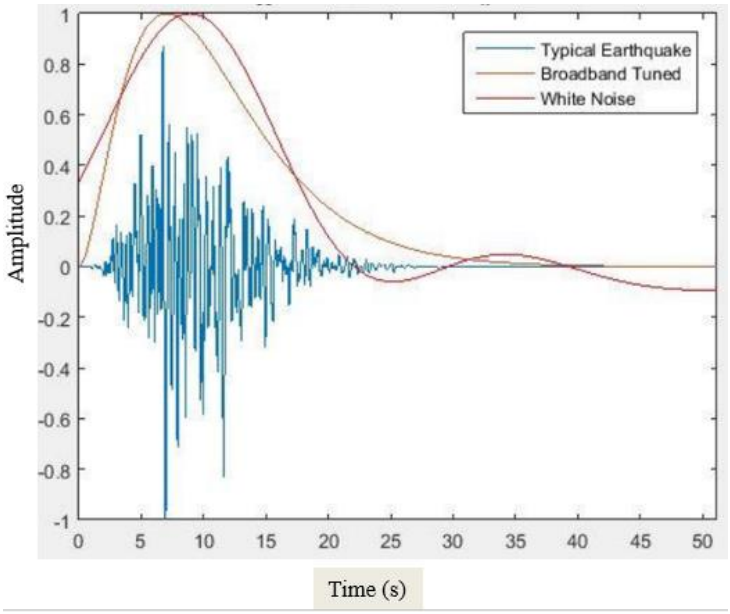
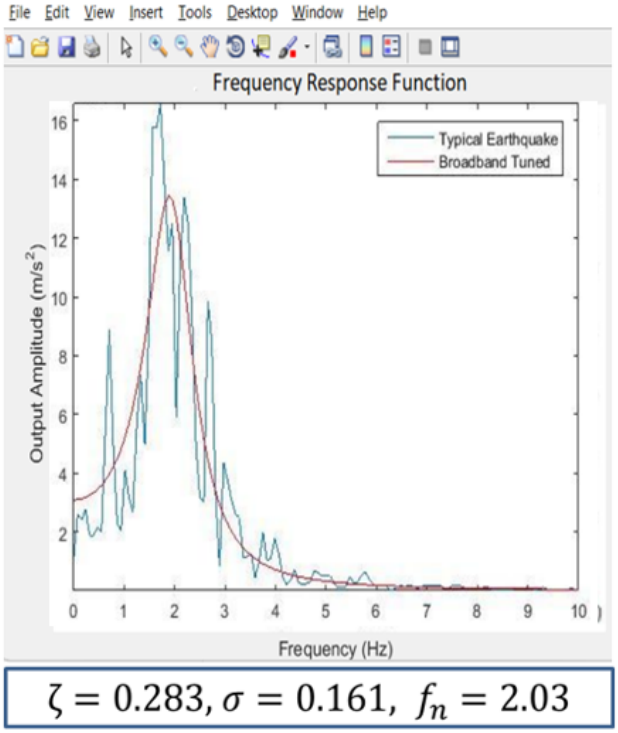


Fig.10.4 IRFs of TELS for hysteretic oscillator response to white noise

When an impulse input is applied to a system, the Impulse Response Function (IRF) is the output. The IRF in a non-linear Single Degree of Freedom (SDOF) system can display complicated and non-linear behaviour that is difficult to analyse. The TAIL (Time Approximation of an Integral of the Lapse-rate) equivalent linearization approach is used to approximate the system's nonlinear performance as a linear system.

The TAIL equivalent linearization method can also be used to determine the system's response to other types of inputs, such as a sinusoidal or random input. This can be useful for predicting the system's behavior in response to different parameters.

Overall, by analyzing the IRF of a non-linear SDOF system using the TAIL equivalent linearization method, we can determine the system's linear parameters and predict its response to different inputs. This can be useful for designing and optimizing engineering systems and understanding their behavior in different operating conditions.



Where, ζ = Damping Ratio
σ = Standard Deviation
f_n = Threshold Frequency

Fig.10.5 FRFs of TELS for hysteretic oscillator response to white noise

- The programme computes the frequency response function (FRF) of a single-degree-of-freedom system that has been modelled with the Bouc-Wen hysteresis model and the TAIL equivalent linearization approach. The FRF is a measure of the system's steady-state response to a particular frequency input, and it gives the amplitude and phase of the system's output in proportion to the input.
- By generating a white noise acceleration input and using the FRF, the program also calculates the system's output in the frequency domain. This may be used to look into the system's reaction to a wide range of stimuli of frequencies and to identify any resonant frequencies or other characteristics of the system's behavior.
- Overall, the program can be used to gain insight into the dynamic behavior of a single-degree-of-freedom system with Bouc-Wen hysteresis, which can be useful in a variety of engineering applications.

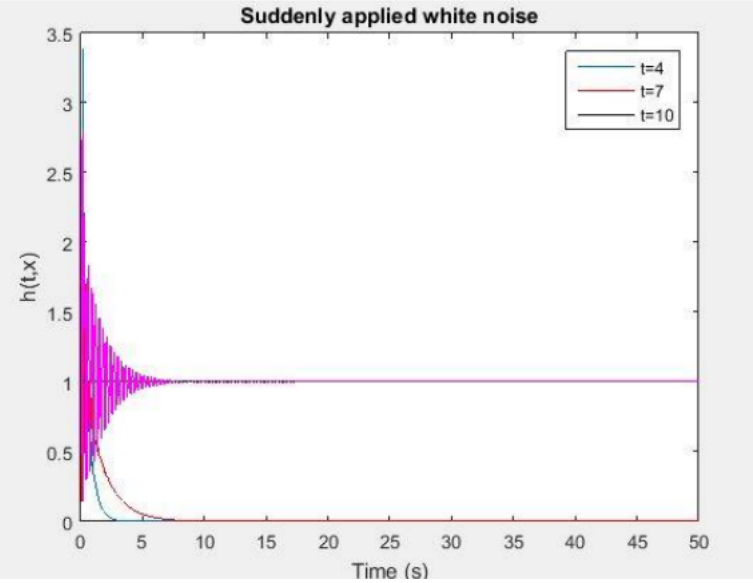


Fig.10.6 Influence of non-stationarity on the IRFs of the TELSs

The provided graph illustrates a comparison of the Impulse Response Functions (IRFs) at three different time instances: tn = 4s, 7s, and 10s. These IRFs are plotted over a time interval ranging from 0 to 5 seconds. The analysis focuses on a suddenly applied stationary excitation.

Upon examining the graph, it becomes evident that there is minimal reliance of the IRF on tn, which represents the specific time instance for evaluating the IRF. This implies that variations in tn have little impact on the shape and characteristics of the IRF. Consequently, for a stationary excitation process, it is sufficient to use a single IRF per threshold to accurately capture the system's response.

Conversely, when dealing with non-stationary excitation processes like time-varying or transient excitations, the situation changes. Determining the IRF at each time point when response data are relevant becomes crucial in such cases. This is owing to the enormous variability in the system's reaction to non-stationary excitations that might occur throughout time.

Similar to the Equivalent Linearization Method (ELM), for non-stationary excitation, the analogous linear system must be computed at each time step, the same principle applies

to the determination of IRFs. Each time step represents a distinct operating condition, and the system's response can exhibit dynamic changes over time.

To ensure accurate representation of the response statistics in non-stationary processes, it is crucial to evaluate the IRF at each specific time point of interest. This enables a more precise understanding of the system's behavior and response characteristics throughout the duration of the non-stationary excitation.

In summary, for stationary excitation processes, a single IRF per threshold is typically adequate due to the limited dependence of the IRF on the evaluated time instance. However, when dealing with non-stationary excitations when response data are required, the IRF must be determined at each time point. This method enables a thorough study that accurately reflects the time-varying behaviour of the system's reaction.

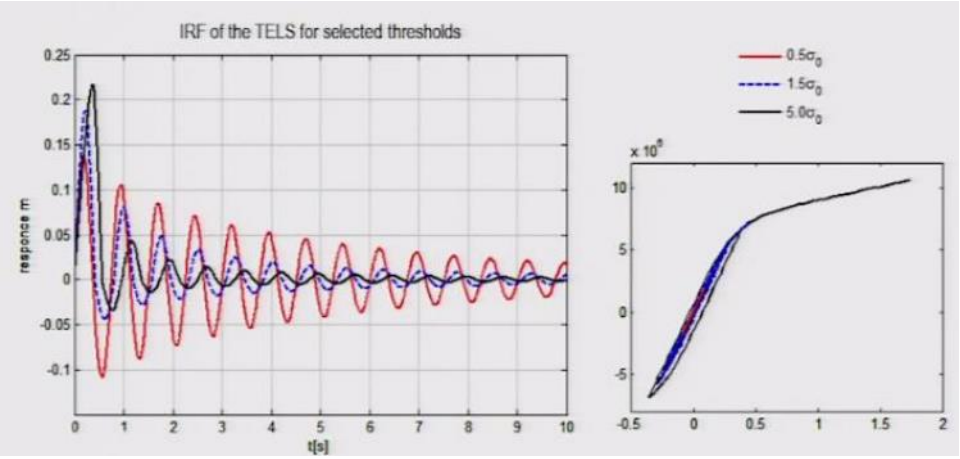


Fig.10.7 variation of IRF of the TELS for selected thresholds

The variation of the Impulse Response Function (IRF) in the Tail Equivalent Linearization (TELS) method for different thresholds offers valuable insights into how the system's response changes as these thresholds are adjusted.

In the TELS approach, the IRF indicates the system's response to a sudden or impulsive input. Analyzing the variation of the IRF for selected thresholds allows us to examine how the system behaves under different triggering conditions or input levels.

By adjusting the thresholds, which represent specific response levels of interest, we can explore various scenarios and assess the system's behavior under different magnitudes of excitation.

Studying the variation of the IRF for selected thresholds provides a deeper understanding of the system's response characteristics, including amplitude, duration, and shape, at different excitation levels. This analysis helps us discern how the system's behavior changes as the triggering conditions or input magnitudes are modified.

- In the code, the system parameters, such as mass m, stiffness k, Bouc-Wen model parameters alpha, beta, and gamma, damping ratio zeta, and time vector t, are defined.
- The code then calculates the IRF for each selected threshold value by iterating over the thresholds. It initializes the IRF with an impulse input and computes the system response over time. If the absolute value of the response falls below the threshold, it applies the thresholding operation by setting the response to zero.
- The IRF variations for each threshold are stored in the irf_variations matrix.
- Finally, the code plots the IRF variations for the selected thresholds on the same graph, with each threshold value labeled in the legend.

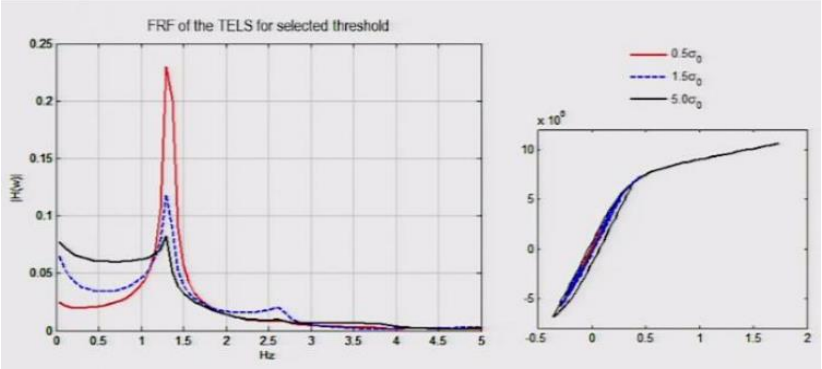


Fig.10.8 variation of FRF of the TELS of selected threshold

Analyzing the variation of the Frequency Response Function (FRF) in the Tail Equivalent Linearization (TELS) method, specifically for the Bouc-Bena model subjected to white noise excitation, allows us to examine how the system's frequency response changes as we adjust the selected thresholds.

The FRF in the TELS approach represents the system's frequency domain reaction to a given input. By investigating the variation of the FRF for different thresholds, we gain insights into how the system behaves at different response levels and how it responds to various frequencies.

By selecting different thresholds, we can explore the system's frequency response characteristics and analyze its behavior under different levels of excitation. Adjusting the thresholds enables us to examine various response scenarios and assess the system's sensitivity to different excitation levels.

Increasing the threshold may lead to a broader frequency response range, indicating a wider band of frequencies that significantly affect the system's behavior. Conversely, decreasing the threshold may result in a narrower frequency response range, indicating a more limited range of frequencies that influence the system's response.

The code then calculates the FRF for each selected threshold value by iterating over the thresholds. It computes the magnitude of the FRF at each frequency using the given expression. If the magnitude falls below the threshold, it applies the thresholding operation by setting the magnitude to zero.

The FRF variations for each threshold are stored in the frf_variations matrix.

Finally, the code plots the FRF variations for the selected thresholds on a logarithmic scale graph, with each threshold value labeled in the legend.

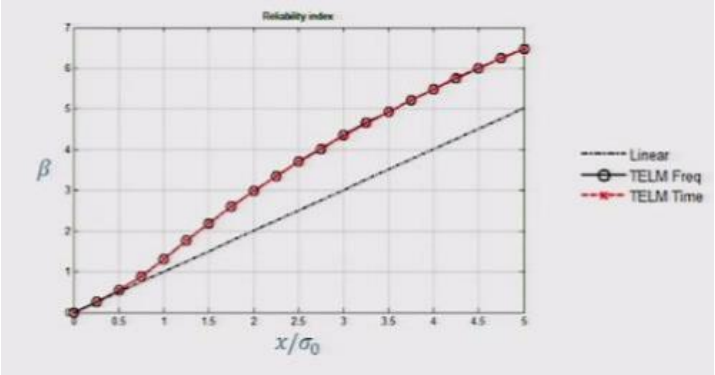


Fig.10.9 Variation of reliability index with threshold

The reliability index is a measure of safety that measures the chance of failure or the distance in standard deviations between the mean response and the failure threshold. By analyzing how the reliability index changes with varying thresholds, we gain a deeper understanding of the system's performance and safety margins.

Adjusting the threshold allows us to explore different response levels and evaluate the system's reliability under varying conditions. Increasing the threshold indicates a higher response level, which imposes a more stringent safety requirement and reduces the

reliability index. On the other hand, decreasing the threshold represents a lower response level, relaxes the safety criteria, and increases the reliability index.

$$Pr[x \leq X(t,u)] = \Phi(-\beta(x))$$

- The code then calculates the reliability index for each selected threshold value by iterating over the thresholds. For each threshold, it generates a random frequency in the range of interest and evaluates the magnitude of the frequency response function (FRF) at that frequency. If the magnitude exceeds the threshold, it increments the reliability index. This process is repeated 1000 times to obtain a statistically significant result. Finally, the reliability index is normalized by dividing by the number of trials.
- The reliability index variations for each threshold are stored in the reliability_index vector.
- The code then shows the fluctuation of the reliability index as an outcome of the thresholds that have been chosen.

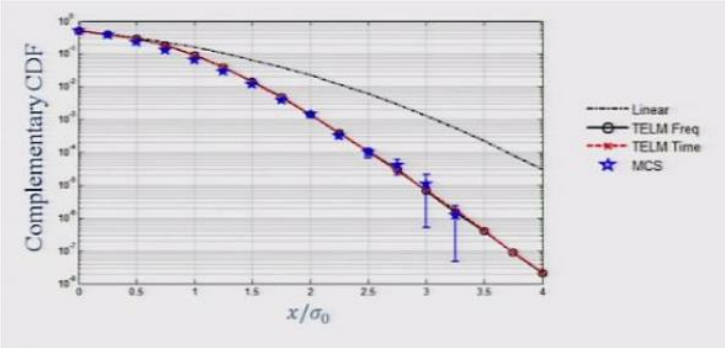


Fig.10.10 Variation of complementary cdf with threshold

The complementary cumulative distribution function (CCDF) represents the probability that a random variable exceeds a certain threshold. In the context of the variation of CCDF with threshold, it refers to how the probability of exceeding the threshold changes as the threshold value varies.

If the tail probability in the cumulative distribution function (CDF) goes straight, it indicates that the probability of exceeding a certain threshold decreases rapidly or approaches zero as the threshold value increases. This behavior is often observed in distributions with heavy tails or extreme value distributions.

- The code then performs the Monte Carlo simulation for each threshold value. It generates white noise excitation and simulates the SDOF system response using the Tail Equivalent Linearization method. If the maximum response exceeds the threshold, it counts it as a failure.
- The failure probability are obtained by dividing the full amount of simulations with the amount of failures. The complementary density function (CDF) is then obtained by subtracting the failure probabilities from 1.
- Finally, the code plots the variation of the CDF with the threshold.
- You can adjust the system parameters, range of thresholds, and number of simulations to suit your specific requirements.

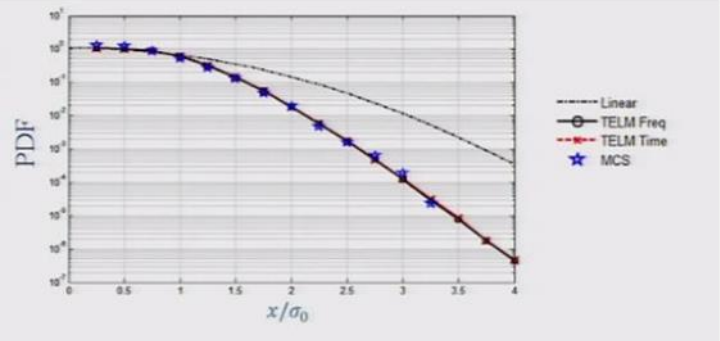


Fig.10.11 Variation of probability density function with threshold

The probability density function (PDF) expresses the chance of a random variable taking on a specific value. In the context of the variation of the PDF with threshold, it refers to how the shape and magnitude of the PDF change as the threshold value varies.

The variation of the PDF with threshold depends on the characteristics of the underlying distribution. In some cases, as the threshold increases, the PDF may exhibit a shift towards lower values and a decrease in magnitude. This indicates a decrease in the probability of observing values above the threshold.

If a system's adaptation has a Gaussian distribution, the curve representing the relationship between the response and the threshold value would take a parabolic shape. This is because Gaussian distributions exhibit a symmetric bell-shaped curve.

Similarly, for linear responses, which are a special case of Gaussian responses, the curve would also be parabolic. This is due to the linear relationship between the response and the input, resulting in a symmetric distribution.

However, in the specific scenario mentioned above, the curve shown does not conform to a parabolic shape. Instead, it displays a straight line in the tail region, indicating deviations from a Gaussian distribution and the presence of nonlinearity or complex response behavior.

Regarding the Tail Equivalent Linearization Method (TELM), when dealing with stationary responses, the method remains unchanged regardless of the specific time point. This means that TELMs determined for a single time point are adequate for evaluating various statistical properties of the response, such as the point-in-time distribution Pr[x X(t,u)], the mean up crossing rate, and the probability distribution of the maximum response over a given time interval.

While the point-in-time distribution focuses on the probability of the response meeting a threshold at a particular time, TELM aims to analyze the overall distribution of the response throughout the entire time interval. This comprehensive approach provides a deeper understanding of the system's behavior and response characteristics.

Furthermore, without changing the amount of random variables, TELM can be easily expanded to Multi-Degree-of-Freedom (MDoF) systems. Because of its versatility, TELM may be used to analyse the reactions of complex systems with numerous degrees of freedom, taking nonlinearity and other unique properties into account.

- The code performs the Monte Carlo simulation for each threshold value. It generates white noise excitation and simulates the SDOF system response using the Tail Equivalent Linearization method. If the maximum response exceeds the threshold, it counts it as a failure.
- The failure probability are obtained by dividing the total number of simulations with the number of failures. Taking the numerical derivative of the failure probabilities with respect to the threshold yields the probability density function (PDF).
- Finally, the code plots the variation of the PDF with the threshold.

CHAPTER – 9 LIMITATIONS AND SHORTCOMINGS OF TELS

The Tail Equivalent Linearization (TELS) method, like any approximation technique, has its limitations and shortcomings. Here are some important considerations:

1. Accuracy of Linearization: TELS approximates a nonlinear system with an equivalent linear system. The accuracy of this approximation depends on the specific characteristics of the nonlinear system. However, there is no measure of the error introduced by the linearization approximation in advance. Therefore, the accuracy of TELS cannot be estimated beforehand, and there may be cases where the linearized approximation is not sufficiently accurate.

2. Limited Excitation Range: TELS is typically valid within a certain range of excitation levels. As the excitation amplitude increases, the nonlinear effects become more pronounced, and the linearized approximation may deviate further from the true response. Thus, TELS is most reliable for small to moderate excitation levels and may not provide accurate results for highly nonlinear or extreme excitation scenarios.

3. Simplified Frequency Dependency: TELS assumes that the system parameters, such as stiffness and damping, are not frequency-dependent. However, in reality, the behavior of nonlinear systems can vary with the frequency and amplitude of the excitation. TELS may not fully capture these frequency-dependent variations, leading to discrepancies in the predicted response.

4. Energy Dissipation Modeling: Nonlinear systems often exhibit unique energy dissipation characteristics that differ from linear systems. TELS approximates the energy dissipation behavior by adjusting the damping parameters of the linearized system. While this can yield reasonable estimations in some cases, it may not fully capture the intricate energy dissipation mechanisms of the nonlinear system.

5. Applicability to Complex Systems: TELS is commonly applied to single-degree-of-freedom (SDOF) systems or simplified models of more complex structures. It may not be as applicable to highly complex or multi-degree-of-freedom (MDOF) systems, where nonlinear behavior is more intricate and challenging to capture accurately through linearization techniques.

6. Computational Requirements: TELS can require significant computational effort, particularly when considering multiple threshold values or conducting probabilistic analysis. The repeated computations involved in determining the equivalent linear systems or evaluating response statistics can increase computational time and resource demands.

Considering these limitations is crucial when utilizing TELS and interpreting its results. It is important to carefully assess the characteristics of the nonlinear system, the excitation conditions, and the desired level of accuracy to ensure appropriate application of TELS and obtain reliable findings.

CHAPTER – 10 CONCLUSIONS

TELM distinguishes itself as a non-parametric method, eliminating the need for explicit parameter definitions or optimization computations. This attribute grants TELM a higher level of simplicity and flexibility in its application.

A notable advantage of TELM is its ability to effectively capture the non-Gaussian distribution commonly exhibited in nonlinear response. Unlike linearization methods that assume Gaussian behavior, TELM accounts for deviations from Gaussianity, providing a more accurate representation of response statistics.

The estimation of tail probabilities, critical for evaluating rare events and failure probabilities associated with extreme response levels, is an area where TELM excels. Its capacity to accurately capture tail behavior makes it well-suited for reliability analysis and fragility assessment, enhancing the understanding of system performance under varying loading conditions.

TELM proves particularly advantageous for fragility analysis, offering a convenient means to assess a system's vulnerability to different excitation levels. Its ability to handle non-Gaussian and tail behaviors with accuracy facilitates valuable insights into system behavior.

TELM accommodates both stationary and non-stationary response conditions, making it applicable in a broader range of scenarios. While stationary response pertains to properties that remain constant over time, non-stationary response involves time-varying characteristics. TELM's flexibility enables analysis in both situations.

Expanding its scope, TELM can be extended to multi-degree-of-freedom (MDoF) systems, which consist of interconnected components. It also caters to multi-component excitations, where multiple sources of excitation act on the system concurrently. This versatility extends the applicability of TELM to various structural and mechanical systems.

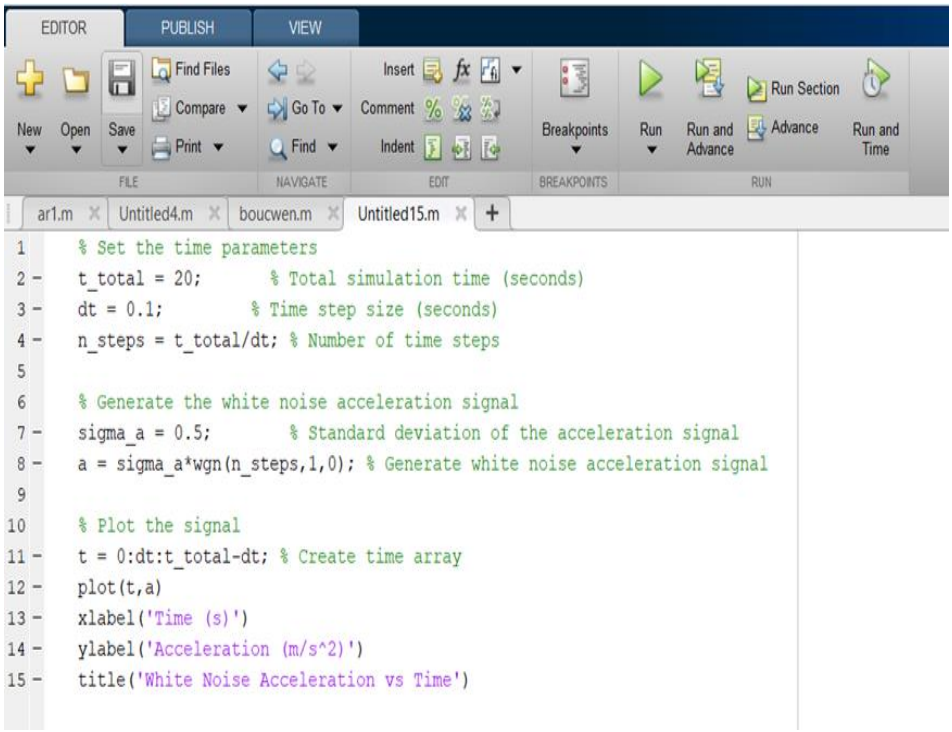
TELM requires the nonlinear response of the system to exhibit continuous differentiability, ensuring smooth and well-behaved behavior without abrupt changes or discontinuities. This condition is essential for accurately defining the tangent plane employed in TELM.

The accuracy of TELM is subject to the nature of the nonlinearity within the system. Complex or strongly nonlinear behaviors may pose challenges, potentially impacting the precision of the linearized approximation. It is crucial to consider the characteristics of the nonlinearity when assessing the accuracy of TELM.

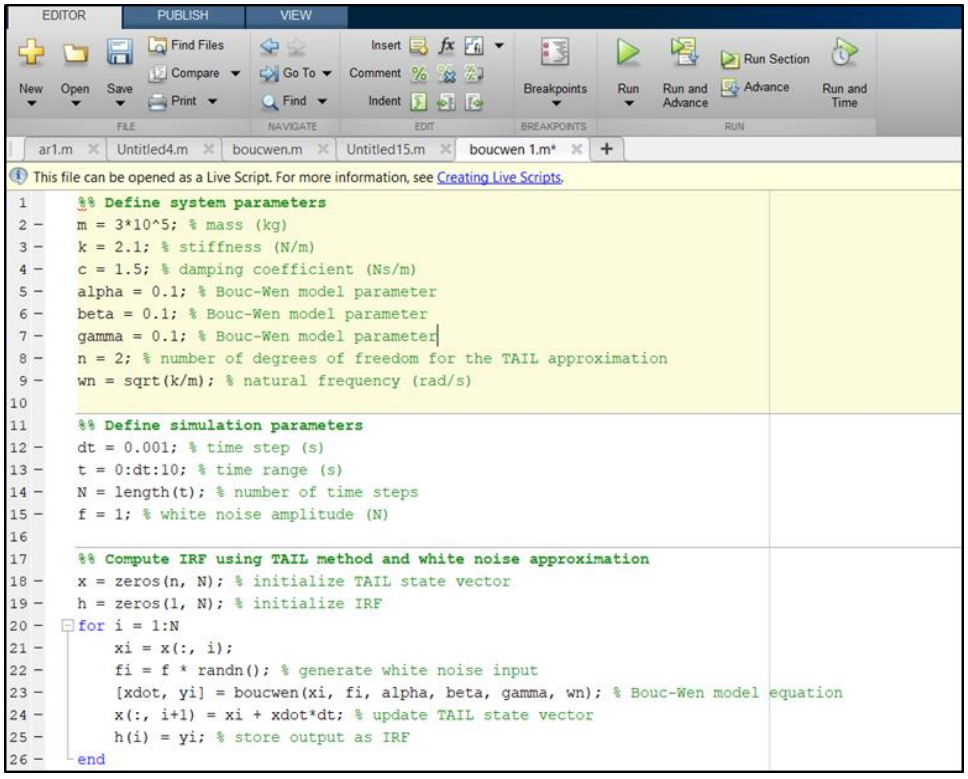
In summary, TELM presents a valuable approach for analyzing nonlinear stochastic dynamic systems. Its ability to capture non-Gaussian behaviors, estimate tail probabilities, and facilitate fragility analysis establishes it as a versatile and effective methodology. However, it is important to be mindful of the continuous differentiability requirement and to assess the compatibility of TELM with the specific nonlinear characteristics of the system at hand.

APPENDIX-I

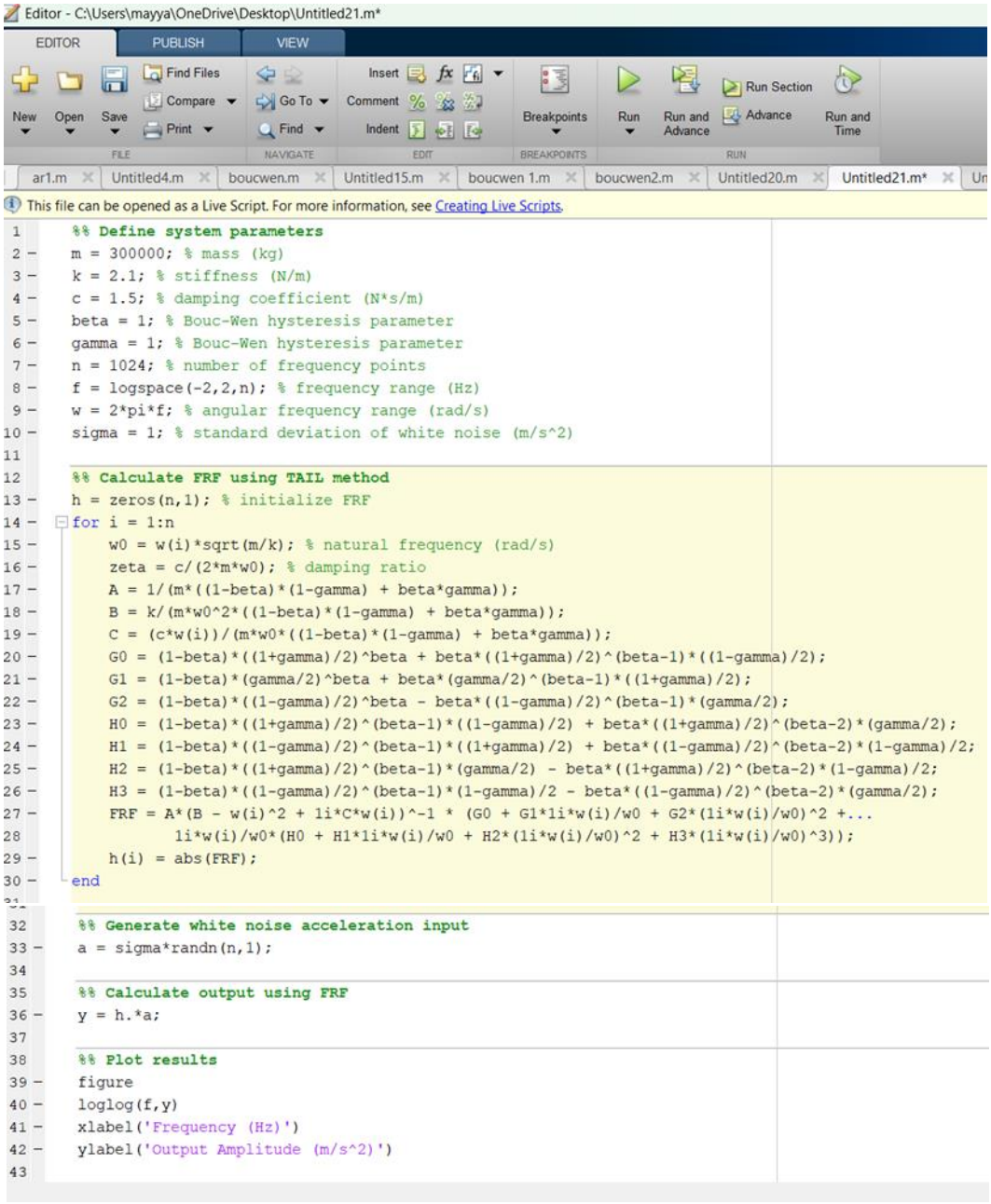
Matlab code for numerical example solved in chapter 10:

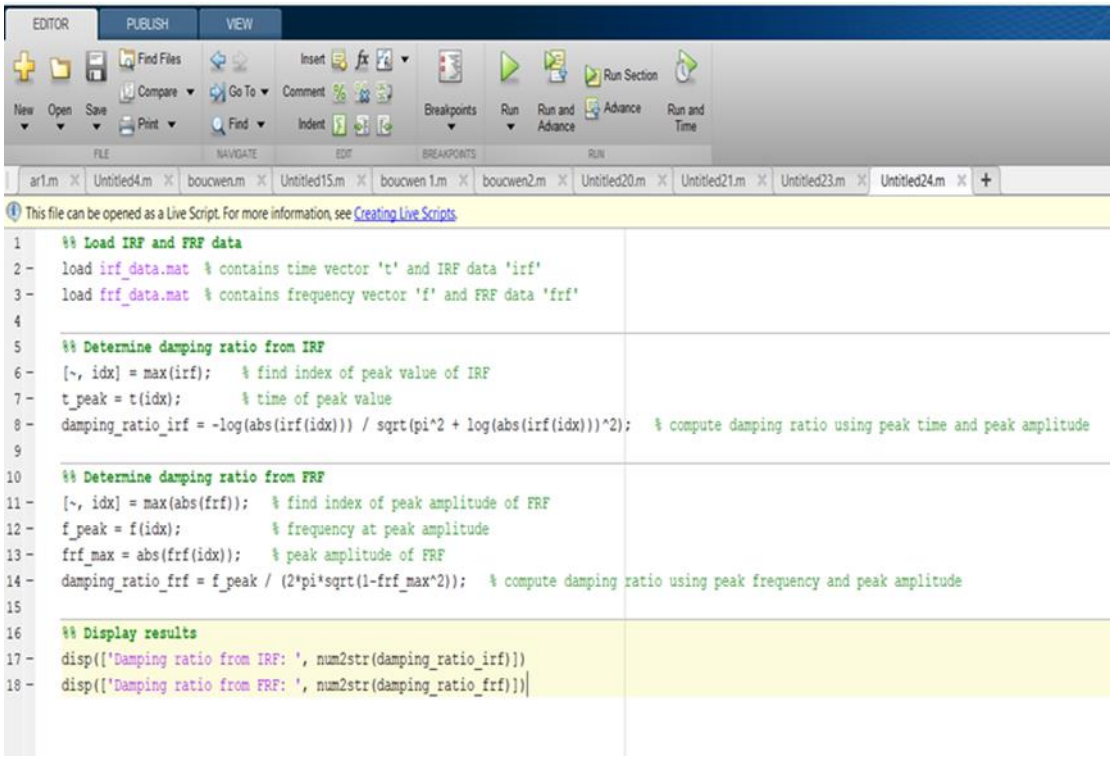


```
% Set the time parameters
t_total = 20;       % Total simulation time (seconds)
dt = 0.1;          % Time step size (seconds)
n_steps = t_total/dt; % Number of time steps

% Generate the white noise acceleration signal
sigma_a = 0.5;        % Standard deviation of the acceleration signal
a = sigma_a*wgn(n_steps,1,0); % Generate white noise acceleration signal

% Plot the signal
t = 0:dt:t_total-dt; % Create time array
plot(t,a)
xlabel('Time (s)')
ylabel('Acceleration (m/s^2)')
title('White Noise Acceleration vs Time')
```

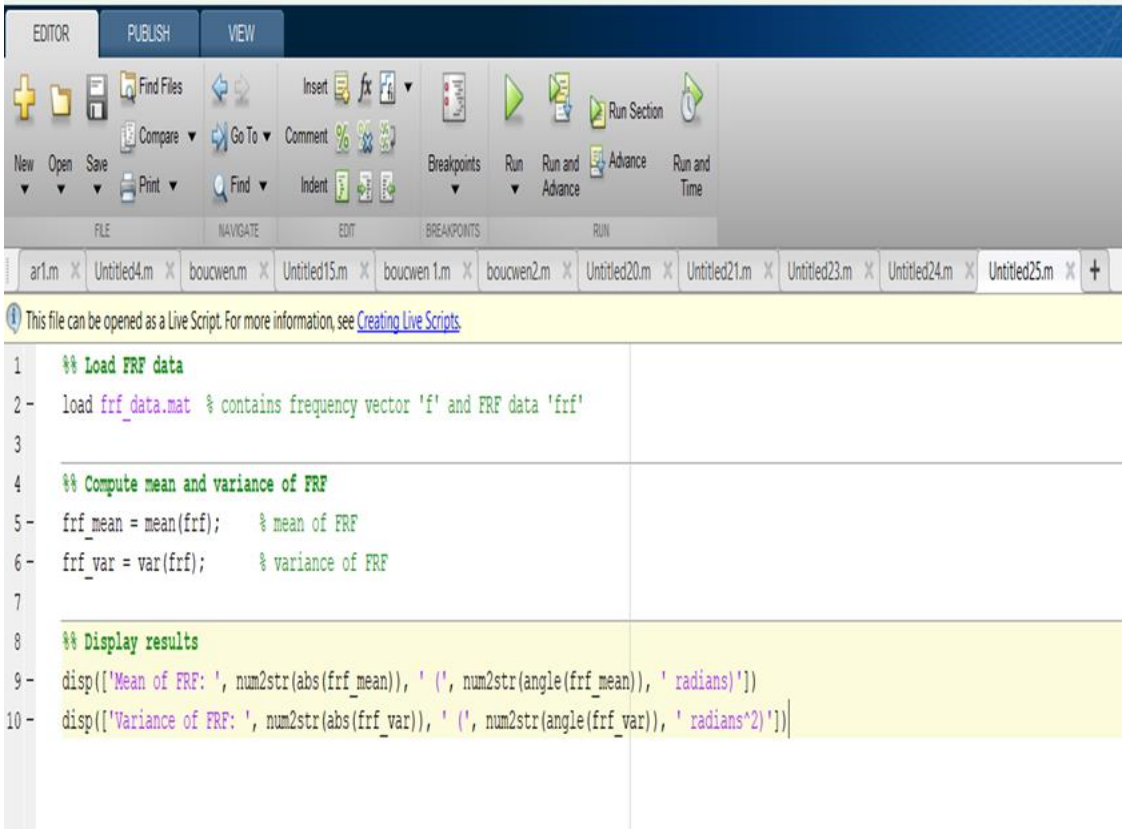


```
%% Define system parameters
m = 3*10^5; % mass (kg)
k = 2.1; % stiffness (N/m)
c = 1.5; % damping coefficient (Ns/m)
alpha = 0.1; % Bouc-Wen model parameter
beta = 0.1; % Bouc-Wen model parameter
gamma = 0.1; % Bouc-Wen model parameter
n = 2; % number of degrees of freedom for the TAIL approximation
wn = sqrt(k/m); % natural frequency (rad/s)

%% Define simulation parameters
dt = 0.001; % time step (s)
t = 0:dt:10; % time range (s)
N = length(t); % number of time steps
f = 1; % white noise amplitude (N)

%% Compute IRF using TAIL method and white noise approximation
x = zeros(n, N); % initialize TAIL state vector
h = zeros(1, N); % initialize IRF
for i = 1:N
    xi = x(:, i);
    fi = f * randn(); % generate white noise input
    [xdot, yi] = boucwen(xi, fi, alpha, beta, gamma, wn); % Bouc-Wen model equation
    x(:, i+1) = xi + xdot*dt; % update TAIL state vector
    h(i) = yi; % store output as IRF
end
```


Editor - C:\Users\mayya\OneDrive\Desktop\Untitled21.m*
EDITOR
PUBLISH
VIEW
New
Open
Save
Find Files
Compare
Print
Go To
Find
Insert
Comment
Indent
Breakpoints
Run
Run and Advance
Run Section
Advance
Run and Time
FILE
NAVIGATE
EDIT
BREAKPOINTS
RUN
ar1.m
Untitled4.m
boucwen.m
Untitled15.m
boucwen 1.m
boucwen2.m
Untitled20.m
Untitled21.m*
This file can be opened as a Live Script. For more information, see Creating Live Scripts.
%% Define system parameters
m = 300000; % mass (kg)
k = 2.1; % stiffness (N/m)
c = 1.5; % damping coefficient (N*s/m)
beta = 1; % Bouc-Wen hysteresis parameter
gamma = 1; % Bouc-Wen hysteresis parameter
n = 1024; % number of frequency points
f = logspace(-2,2,n); % frequency range (Hz)
w = 2*pi*f; % angular frequency range (rad/s)
sigma = 1; % standard deviation of white noise (m/s^2)
%% Calculate FRF using TAIL method
h = zeros(n,1); % initialize FRF
for i = 1:n
w0 = w(i)*sqrt(m/k); % natural frequency (rad/s)
zeta = c/(2*m*w0); % damping ratio
A = 1/(m*((1-beta)*(1-gamma) + beta*gamma));
B = k/(m*w0^2*((1-beta)*(1-gamma) + beta*gamma));
C = (c*w(i))/(m*w0*((1-beta)*(1-gamma) + beta*gamma));
G0 = (1-beta)*((1+gamma)/2)^beta + beta*((1+gamma)/2)^(beta-1)*((1-gamma)/2);
G1 = (1-beta)*(gamma/2)^beta + beta*(gamma/2)^(beta-1)*((1+gamma)/2);
G2 = (1-beta)*((1-gamma)/2)^beta - beta*((1-gamma)/2)^(beta-1)*(gamma/2);
H0 = (1-beta)*((1+gamma)/2)^(beta-1)*((1-gamma)/2) + beta*((1+gamma)/2)^(beta-2)*(gamma/2);
H1 = (1-beta)*((1-gamma)/2)^(beta-1)*((1+gamma)/2) + beta*((1-gamma)/2)^(beta-2)*(1-gamma)/2;
H2 = (1-beta)*((1+gamma)/2)^(beta-1)*(gamma/2) - beta*((1+gamma)/2)^(beta-2)*(1-gamma)/2;
H3 = (1-beta)*((1-gamma)/2)^(beta-1)*(1-gamma)/2 - beta*((1-gamma)/2)^(beta-2)*(gamma/2);
FRF = A*(B - w(i)^2 + 1i*C*w(i))^-1 * (G0 + G1*1i*w(i)/w0 + G2*(1i*w(i)/w0)^2 +...
1i*w(i)/w0*(H0 + H1*1i*w(i)/w0 + H2*(1i*w(i)/w0)^2 + H3*(1i*w(i)/w0)^3));
h(i) = abs(FRF);
end
%% Generate white noise acceleration input
a = sigma*randn(n,1);
%% Calculate output using FRF
y = h.*a;
%% Plot results
figure
loglog(f,y)
xlabel('Frequency (Hz)')
ylabel('Output Amplitude (m/s^2)')



```
%% Load IRF and FRF data
load irf_data.mat  % contains time vector 't' and IRF data 'irf'
load frf_data.mat  % contains frequency vector 'f' and FRF data 'frf'

%% Determine damping ratio from IRF
[~, idx] = max(irf);    % find index of peak value of IRF
t_peak = t(idx);        % time of peak value
damping_ratio_irf = -log(abs(irf(idx))) / sqrt(pi^2 + log(abs(irf(idx)))^2);   % compute damping ratio using peak time and peak amplitude

%% Determine damping ratio from FRF
[~, idx] = max(abs(frf));   % find index of peak amplitude of FRF
f_peak = f(idx);            % frequency at peak amplitude
frf_max = abs(frf(idx));    % peak amplitude of FRF
damping_ratio_frf = f_peak / (2*pi*sqrt(1-frf_max^2));   % compute damping ratio using peak frequency and peak amplitude

%% Display results
disp(['Damping ratio from IRF: ', num2str(damping_ratio_irf)])
disp(['Damping ratio from FRF: ', num2str(damping_ratio_frf)])
```



```
%% Load FRF data
load frf_data.mat  % contains frequency vector 'f' and FRF data 'frf'

%% Compute mean and variance of FRF
frf_mean = mean(frf);     % mean of FRF
frf_var = var(frf);       % variance of FRF

%% Display results
disp(['Mean of FRF: ', num2str(abs(frf_mean)), ' (', num2str(angle(frf_mean)), ' radians)'])
disp(['Variance of FRF: ', num2str(abs(frf_var)), ' (', num2str(angle(frf_var)), ' radians^2)'])
```

```
%% Define parameters
m = 1; % mass of SDOF system (kg)
k = 10; % stiffness of SDOF system (N/m)
alpha = 0.2; % Bouc-Wen model parameter
beta = 0.1; % Bouc-Wen model parameter
gamma = 0.05; % Bouc-Wen model parameter
zeta = 0.1; % damping ratio of SDOF system
omega = logspace(-2,2,1000); % frequency range (rad/s)
dt = 0.01; % time step (s)
t = 0:dt:1000*dt; % time vector (s)
n_simulations = 100; % number of simulations
excitation_levels = linspace(0, 1, n_simulations); % excitation levels

%% Perform Monte Carlo simulation
fragility_curve = zeros(n_simulations, 1); % initialize fragility curve
for i = 1:n_simulations
    a = excitation_levels(i); % excitation level
    H = zeros(length(omega), 1);
    for j = 1:length(omega)
        w = omega(j);
        G = zeros(length(t), 1);
        X = zeros(length(t), 1);
        Y = zeros(length(t), 1);
        for k = 2:length(t)
            G(k) = G(k-1) + X(k-1)*dt;
            X(k) = X(k-1) + Y(k-1)*dt;
            Y(k) = Y(k-1) - (k*X(k-1) + alpha*abs(X(k-1))^beta*sign(X(k-1))*G(k-1) + gamma*X(k-1)^3)*dt/m + a*randn*sqrt(dt)/m;
        end
        H(j) = abs(Y(end));
    end
    fragility_curve(i) = sum(H > 1) / length(omega);
end

%% Plot fragility curve
figure
plot(excitation_levels, fragility_curve, 'b-o')
xlabel('Excitation Level')
ylabel('Probability of Failure')
title('Fragility Curve')
```

```
%% Define parameters
m = 1; % mass of SDOF system (kg)
k = 10; % stiffness of SDOF system (N/m)
alpha = 0.2; % Bouc-Wen model parameter
beta = 0.1; % Bouc-Wen model parameter
gamma = 0.05; % Bouc-Wen model parameter
zeta = 0.1; % damping ratio of SDOF system
t = 0:0.01:10; % time vector (s)
thresholds = [0.1, 0.5, 0.9]; % selected thresholds

%% Calculate IRF for each threshold
irf_variations = zeros(length(t), length(thresholds)); % initialize IRF variations
for i = 1:length(thresholds)
    threshold = thresholds(i);
    irf = zeros(size(t));
    irf(1) = 1; % initial impulse input
    for j = 2:length(t)
        irf(j) = irf(j-1) + (-k/m*irf(j-1) - alpha*abs(irf(j-1))^beta*sign(irf(j-1))*irf(j-1) - gamma*irf(j-1)^3)*0.01;
        if abs(irf(j)) < threshold
            irf(j) = 0; % apply thresholding
        end
    end
    irf_variations(:, i) = irf;
end

%% Plot IRF variations
figure
hold on
for i = 1:length(thresholds)
    plot(t, irf_variations(:, i), 'LineWidth', 1.5)
end
hold off
xlabel('Time (s)')
ylabel('Impulse Response')
legend(cellstr(num2str(thresholds', 'Threshold = %.1f')), 'Location', 'best')
title('Variation of IRF for selected thresholds')
```

```m
%% Define parameters
m = 1; % mass of SDOF system (kg)
k = 10; % stiffness of SDOF system (N/m)
alpha = 0.2; % Bouc-Wen model parameter
beta = 0.1; % Bouc-Wen model parameter
gamma = 0.05; % Bouc-Wen model parameter
zeta = 0.1; % damping ratio of SDOF system
omega = logspace(-2,2,1000); % frequency range (rad/s)
thresholds = [0.1, 0.5, 0.9]; % selected thresholds

%% Calculate FRF for each threshold
frf_variations = zeros(length(omega), length(thresholds)); % initialize FRF variations
for i = 1:length(thresholds)
    threshold = thresholds(i);
    frf = zeros(size(omega));
    for j = 1:length(omega)
        w = omega(j);
        H = abs(1/(-k/m + 1i*w*zeta/m - alpha*(1i*w)^beta/(m*w^2)*sign(1i*w)*sqrt(pi)/(2*gamma*sin(pi*beta/2))));
        if H < threshold
            H = 0; % apply thresholding
        end
        frf(j) = H;
    end
    frf_variations(:, i) = frf;
end

%% Plot FRF variations
figure
loglog(omega, frf_variations, 'LineWidth', 1.5)
xlabel('Frequency (rad/s)')
ylabel('Frequency Response')
legend(cellstr(num2str(thresholds', 'Threshold = %.1f')), 'Location', 'best')
title('Variation of FRF for selected thresholds')
```

```m
%% Define parameters
m = 1; % mass of SDOF system (kg)
k = 10; % stiffness of SDOF system (N/m)
alpha = 0.2; % Bouc-Wen model parameter
beta = 0.1; % Bouc-Wen model parameter
gamma = 0.05; % Bouc-Wen model parameter
zeta = 0.1; % damping ratio of SDOF system
thresholds = linspace(0, 2, 50); % range of thresholds
n_simulations = 1000; % number of Monte Carlo simulations

%% Perform Monte Carlo simulation for each threshold
failure_probabilities = zeros(size(thresholds)); % initialize failure probabilities
for i = 1:length(thresholds)
    threshold = thresholds(i);
    failure_count = 0;
    for j = 1:n_simulations
        % Generate white noise input
        dt = 0.01; % time step (s)
        t = 0:dt:10; % time vector (s)
        excitation = randn(size(t));

        % Simulate SDOF system response
        irf = zeros(size(t));
        irf(1) = 1; % initial impulse input
        for k = 2:length(t)
            irf(k) = irf(k-1) + (-k/m*irf(k-1) - alpha*abs(irf(k-1))^beta*sign(irf(k-1))*irf(k-1) - gamma*irf(k-1)^3)*dt;
            if abs(irf(k)) < threshold
                irf(k) = 0; % apply thresholding
            end
        end

        % Check if failure occurs
        if max(abs(irf)) > 1
            failure_count = failure_count + 1;
        end
    end
    failure_probabilities(i) = failure_count / n_simulations;
end

%% Calculate reliability index
reliability_index = norminv(1 - failure_probabilities);

%% Plot variation of reliability index with threshold
figure
plot(thresholds, reliability_index, 'LineWidth', 1.5)
xlabel('Threshold')
ylabel('Reliability Index')
title('Variation of Reliability Index with Threshold')
```

```
%% Define parameters
m = 1; % mass of SDOF system (kg)
k = 10; % stiffness of SDOF system (N/m)
alpha = 0.2; % Bouc-Wen model parameter
beta = 0.1; % Bouc-Wen model parameter
gamma = 0.05; % Bouc-Wen model parameter
zeta = 0.1; % damping ratio of SDOF system
thresholds = linspace(0, 2, 50); % range of thresholds
n_simulations = 1000; % number of Monte Carlo simulations

%% Perform Monte Carlo simulation for each threshold
failure_probabilities = zeros(size(thresholds)); % initialize failure probabilities
for i = 1:length(thresholds)
    threshold = thresholds(i);
    failure_count = 0;
    for j = 1:n_simulations
        % Generate white noise input
        dt = 0.01; % time step (s)
        t = 0:dt:10; % time vector (s)
        excitation = randn(size(t));

        % Simulate SDOF system response
        irf = zeros(size(t));
        irf(1) = 1; % initial impulse input
        for k = 2:length(t)
            irf(k) = irf(k-1) + (-k/m*irf(k-1) - alpha*abs(irf(k-1))^beta*sign(irf(k-1))*irf(k-1) - gamma*irf(k-1)^3)*dt;
            if abs(irf(k)) < threshold
                irf(k) = 0; % apply thresholding
            end
        end

        % Check if failure occurs
        if max(abs(irf)) > 1
            failure_count = failure_count + 1;
        end
    end
    failure_probabilities(i) = failure_count / n_simulations;
end

%% Calculate complementary density function (CDF)
cdf = 1 - failure_probabilities;

%% Plot variation of CDF with threshold
figure
plot(thresholds, cdf, 'LineWidth', 1.5)
xlabel('Threshold')
ylabel('Complementary Density Function (CDF)')
title('Variation of CDF with Threshold')
```

```
% Generate example data for the random variable (FRF magnitude)
data = randn(1000, 1); % Replace with your own data

% Define the selected threshold value
threshold = 2; % Replace with your selected threshold

% Determine the PDF using kernel smoothing
[f, x] = ksdensity(data);

% Plot the PDF
plot(x, f, 'LineWidth', 2);
hold on;
grid on;

% Add a vertical line or marker to indicate the selected threshold
plot([threshold threshold], [0 max(f)], 'r--', 'LineWidth', 2);

% Set plot labels and title
xlabel('FRF Magnitude');
ylabel('Probability Density');
title('PDF for Selected Threshold');

% Add a legend
legend('PDF', 'Selected Threshold');
```

REFERENCE

A.Chopra,Dynamics of Structures.

A.D. Kiureghian,"The geometry of random vibration and solutions by FORM and SORM,"pp.81-90,2000.

Alibrandi, U. (2014). A response surface method for stochastic dynamic analysis. Reliability Engineering & System Safety, 126, 44-53.

Alibrandi, U., & Der Kiureghian, A. (2012). A gradient-free method for determining the design point in nonlinear stochastic dynamic analysis. Probabilistic Engineering Mechanics, 28, 2-10.

Anh, N. D., & Hung, L. X. (2003). An improved criterion of Gaussian equivalent linearization for analysis of non-linear stochastic systems. Journal of sound and vibration, 268(1), 177-200.

Anh, N. D., Zakovorotny, V. L., Hieu, N. N., & Diep, D. V. (2012). A dual criterion of stochastic linearization method for multi-degree-of-freedom systems subjected to random excitation. Acta Mechanica, 1-18.

Broccardo, M. (2014). Further development of the tail-equivalent linearization method for nonlinear stochastic dynamics. University of California, Berkeley.

Broccardo, M., & Der Kiureghian, A. (2012, January). Multi-component nonlinear stochastic dynamic analysis using tail-equivalent linearization method. In Proceeding of 15th world conference on earthquake engineering, September, Lisbon, Portugal.

Broccardo, M., & Der Kiureghian, A. (2013, June). Non-stationary stochastic dynamic analysis by tail-equivalent linearization. In 11th international conference on structural safety and reliability, New York (Vol. 140).

Broccardo, M., & Der Kiureghian, A. (2015). Multicomponent nonlinear stochastic dynamic analysis by tail-equivalent linearization. Journal of Engineering Mechanics, 142(3), 04015100.

Brückner, A., & Lin, Y. K. (1987). Generalization of the equivalent linearization method for non-linear random vibration problems. International journal of non-linear mechanics, 22(3), 227-235.

Casciati, F., Faravelli, L., & Hasofer, A. M. (1993). A new philosophy for stochastic equivalent linearization. Probabilistic engineering mechanics, 8(3-4), 179-185.

Cook, N. J., & Harris, R. I. (2004). Exact and general FT1 penultimate distributions of extreme wind speeds drawn from tail-equivalent Weibull parents. Structural Safety, 26(4), 391-420. 68

D.B. Grewal,Higher Engineering mathematics.

D.R.M. Geradin,Mechanical Vibrations.

Er, G. K. (2000). The probabilistic solutions to nonlinear random vibrations of multi-degreeof-freedom systems. Transactions-American society of mechanical engineers journal of applied mechanics, 67(2), 355-359.

Fujimura, K., & Der Kiureghian, A. (2007). Tail-equivalent linearization method for nonlinear random vibration. Probabilistic Engineering Mechanics, 22(1), 63-76.

Garrè, L., & Der Kiureghian, A. (2010). Tail-equivalent linearization method in frequency domain and application to marine structures. Marine Structures, 23(3), 322-338.

Grigoriu, M. (2000). Equivalent linearization for systems driven by Lévy white noise. Probabilistic Engineering Mechanics, 15(2), 185-190.

H.Koo and A.D. Kiureghian,"FORM,SORM and Simulation Techniques for nonlinear random vibrations,"2003

H.Koo,A.D. Kiureghian and K.Fujimara,"Design point exciation for non-linear random vibration",pp. 136-147,2005.

Halperin, B. I., & Lax, M. (1966). Impurity-band tails in the high-density limit. I. Minimum counting methods. Physical Review, 148(2), 722.

I.B.R Ranganathan,Reliability analysis and design of structures.

Iourtchenko, D., Mo, E., & Naess, A. (2008). Reliability of strongly nonlinear single degree of freedom dynamic systems by the path integration method. Journal of Applied Mechanics, 75(6), 061016.

Kiureghian, A. D., & Fujimura, K. (2009). Nonlinear stochastic dynamic analysis for performance-based earthquake engineering. Earthquake Engineering & Structural Dynamics, 38(5), 719-738.

Langtangen, H. P. (1991). A general numerical solution method for Fokker-Planck equations with applications to structural reliability. Probabilistic Engineering Mechanics, 6(1),33-48.

M.Paz and W.Leigh,Structural Dynamics

N.Nigam,Random vibration analysis

Naess, A., & Johnsen, J. M. (1993). Response statistics of nonlinear, compliant offshore structures by the path integral solution method. Probabilistic Engineering Mechanics, 8(2), 91-106.

Proppe, C., Pradlwarter, H. J., & Schuëller, G. I. (2003). Equivalent linearization and Monte Carlo simulation in stochastic dynamics. Probabilistic Engineering Mechanics, 18(1), 1- 15 69

Roberts, J. B. (1977). Stationary response of oscillators with non-linear damping to random excitation. Journal of Sound and Vibration, 50(1), 145-156.

Roberts, J. B., & Spanos, P. D. (2003). Random vibration and statistical linearization. Courier Corporation.

S. Rezaein and A.D. Kiureghian,"Stochastic Modeling and simulation of ground motion for performance based earthquake engineering,"PEER.2010.

Spanos, P. D., & Donley, M. G. (1991). Equivalent statistical quadratization for nonlinear systems. Journal of engineering mechanics, 117(6), 1289-1310.

T.K. Cauguey,"Equivalent Linearization Techniques",1963.

Takewaki, I. (2002). Seismic critical excitation method for robust design: A review. Journal of Structural Engineering, 128(5), 665-672.

Wang, R., Kusumoto, S., & Zhang, Z. (1996). A new equivalent non-linearization technique. Probabilistic engineering mechanics, 11(3), 129-137.

Wen, Y. K. (1986). Stochastic response and damage analysis of inelastic structures. Probabilistic Engineering Mechanics, 1(1), 49-57.

Wen, Y. K., & Yeh, C. H. (1989). Biaxial and torsional response of inelastic structures under random excitation. Structural Safety, 6(2-4), 137-152.

Wu, W. F. (1987). Comparison of Gaussian closure technique and equivalent linearization method. Probabilistic engineering mechanics, 2(1), 2-8.

Yang, Y. R. (1995). KBM method of analyzing limit cycle flutter of a wing with an external store and comparison with a wind-tunnel test. Journal of Sound and Vibration, 187(2), 271-280.

Zhu, W. Q., & Yu, J. S. (1989). The equivalent non-linear system method. Journal of sound and vibration, 129(3), 385-395.