A MATHEMATICAL APPROACH FOR THE SOLUTION OF SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS

A Project Dissertation submitted in partial fulfilment of the requirements for the degree of

MASTER OF SCIENCE IN APPLIED MATHEMATICS

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CANDIDATE'S DECLARATION

I, Chaitanya Pathak, who is currently pursuing Master of Science in Applied Mathematics with Roll Number 2K21/MSCMAT/10, hereby declare that the project dissertation submitted by me to the Department of Applied Mathematics at Delhi Technological University to fulfil the requirement for the award of the degree of Master of Science in Applied Mathematics, is original and has not been copied from any source. Furthermore, this work has not been previously used as the basis for conferring a degree, diploma, associate's degree, fellowship, or any other similar title or honour.

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CERTIFICATE

I hereby bear witness that the Project Dissertation submitted by Chaitanya Pathak, Roll Number 2K21/MSCMAT/10, of the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfilment of the requirement for the award of the degree of Master of Applied Mathematics, is a record of the project work completed by the student under my supervision. To the best of my knowledge, neither a portion nor the entirety of this work has ever been submitted to this university or any other institution for a degree or diploma.

Place: Delhi
Date: May 25, 2023

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Abstract

In the natural sciences and engineering, a range of phenomena are modelled using mathematical equations. These equations in mathematics have different parameters. Minor adjustments to these parameters have an impact on the answers of these equations. The perturbation parameter corresponds to this slight modification, which is referred to as a perturbation.
Finding these mathematical equations' exact solutions is challenging. Finding their approximations is therefore the alternate method. The approximation techniques are used to arrive at these solutions. These perturbation methods pave the door for perturbation theory even more.
This paper mainly focuses on the derivation of analytic solutions that accurately capture the physical relevance of the nonlinear phenomena involved, which can be difficult to solve explicitly using numerical schemes, especially when the equations are stiff.
To solve this problem, we provide an iterative analytical strategy based on the Lagrange multiplier method. The Lagrange multiplier can be obtained more accurately and efficiently using variational theory and Liouville-Green transforms in a general setting. This method has been demonstrated to be highly accurate and efficient through illustrative examples. The suggested method provides a clear and succinct answer to the problems with numerical methods and is applicable to various nonlinear evolution equations in mathematical physics.

Acknowledgement

I am incredibly lucky to have had this support and guidance from many people throughout the course of my project because it was essential to its accomplishment and successful completion. So it only makes sense that I thank them all.

I would like to begin by expressing my sincere appreciation to Dr. Aditya Kaushik from the Department of Applied Mathematics at Delhi Technological University, who played a vital role in mentoring me throughout the project. It was his constant support and guidance that helped me to successfully complete the project within the stipulated time frame. Throughout my journey, he has been a great source of inspiration and motivation for me. He has provided valuable guidance and support that has helped me overcome all the obstacles and challenges that I encountered along the way.

I owe a debt of gratitude to my good departmental friends who have supported me in my endeavours and shown me the way forward for my personal and professional development over time. I would want to express my gratitude to my family and friends for always supporting me when I needed it.

I'd like to end by thanking God Almighty for seeing to it that no unforeseen problems occurred while I was working.

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Chapter 1

Introduction

Mathematics has become an indispensable aspect of our daily lives, finding its way into almost every sphere. The progress in applied mathematics can be attributed to the evolution and refinement of various fundamental techniques and approaches. One of the crucial and wide-ranging branches of mathematics today is differential equations. Calculus, has been a subject of both theoretical and functional research for a long time, continues to be an indispensable tool in modern mathematics.

1.1 Differential Equation

Definition 1.1.1 *A differential equation pertains to an equation containing differentials or differential coefficients. In other words, it is an equation that relates a dependent variable and an independent variable with the derivative of the dependent variable with respect to the independent variable.*

In addition, differential equations can be classified into two main categories, each with its own set of subcategories. The two most significant subcategories are ordinary differential equations and partial differential equations.

1.1.1 Ordinary Differential Equation

Definition 1.1.2 *An ordinary differential equation is a type of differential equation where the ordinary derivatives of one or more dependent variables are compared to a single independent variable.*

Example 1.1.1

$$2\frac{d^2u}{dt^2} + tu^2\frac{du}{dt} = 0 \tag{1.1.1}$$

1.1.2 Partial Differential Equation

Definition 1.1.3 *Partial differential equations, which are a specific type of mathematical equation, involve the computation of partial derivatives of one or more dependent variables with respect to several independent variables.*

Example 1.1.2

$$\frac{\delta v}{\delta x} - 2\frac{\delta v}{\delta t} = v^2 \tag{1.1.2}$$

It is important to distinguish between linear and non-linear differential equations because the methods used to solve them can differ significantly. Linear differential equations are generally simpler to solve and have well-known analytical solutions, while non-linear differential equations often require numerical techniques or approximations to obtain solutions. Understanding the type of differential equation at hand is crucial in choosing the appropriate solution method.
After categorizing differential equations in various ways, it is important to understand their origin and applications. By doing this, we may better comprehend differential equations' wide range of applications and the adaptability of its techniques for handling practical issues.
The applications of differential equations span across a wide range of fields in science and engineering, where they are utilized to solve various mathematical models and problems. We will only mention a handful of these issues in this context, which have the potential to be extensively explored in many pages.

1. The difficulty of figuring out how a projectile, rocket, satellite, or planet moves.
2. In an electric car, determining the proper level of charge or current can be very difficult.
3. Examining the population's growth rate or the pace at which radioactive compounds decay.
4. Recognising curves with specific geometric characteristics.

Differential equations arise from mathematical modeling of various real-world problems. Numerous applications use Ordinary Differential Equations (ODEs) with variable coefficients., such as the Euler, Bessel, Legendre, and Laguerre equations. Nonlinear ODEs with variable coefficients, including the Duffing, Thomas-Fermi, and Van der Pol equations, have also received a lot of research in the literature. In practical mathematics, physics and engineering, these linear and nonlinear ODEs with variable coefficients are extremely important.
Researchers sought to develop exact methods for solving a wide range of integral equations and linear and nonlinear equations without any concrete assumptions or

variable discretization. In the 1940s, the idea of singular disturbance first appeared, and has since evolved to approximate solutions to intricate problems.
Mathematical equations used to model natural phenomena and engineering problems often involve parameters, and these issues are frequently expressed in terms of differential equations, usually with at least one minor parameter.

1.2 Perturbation

The concept of small parameter was first introduced by J.H. Poincare. In mathematics, perturbation refers to the study of a system in which some parameters or variables are slightly modified from their known values. This concept is particularly relevant in the context of differential equations, which describe the relationship between a function and its derivatives.

In the case of differential equations, perturbation theory involves analyzing the behavior of a solution to a perturbed differential equation as the perturbation parameter approaches zero. This can be used to investigate how small changes in the system affect the overall behavior of the solution.

Perturbation, a minor alteration in a parameter that affects the solution of these equations, is a common occurrence. To avoid the difficulty of finding exact solutions of mathematical equations, we can instead use approximation techniques, including Perturbation techniques which lead to Perturbation theory. In this theory, the perturbation parameter (ϵ) is a small, dimensionless quantity that is introduced to study the behaviour of solutions and expressing the approximate solution as a formal power series of this parameter. By measuring the effect of small disturbances, Perturbation theory provides an approach to solving problems through local analysis.
Two classes—regular perturbation and singular perturbation—are used to categorise the perturbation.

The classification of perturbations is determined by their impact on the system, with those causing minimal effects being labeled as regular, while those with significant influences are considered singular.

The two problems, i.e., regularly and singularly perturbed are best described as follows-

1.2.1 Regular Perturbation Problem

Definition 1.2.1 *A Regular Perturbation Problem is a type of mathematical problem that involves a small perturbation parameter ϵ in a function $f(y(x), \epsilon) = 0$. The solution of the perturbed problem, $y_\epsilon(x)$, uniformly converges to the solution of the unperturbed problem $y_0(x) = f(y(x), 0)$ over the domain of existence as $\varepsilon \to 0$. This means that the perturbation has a minimal effect on the solution of the problem, and the behavior of the system can be accurately approximated using standard mathematical techniques.*

1.2.2 Singular Perturbation Problem

Definition 1.2.2 *A Singular Perturbation Problem is a type of mathematical problem that involves a perturbation parameter ϵ in a function $f(y(x), \epsilon) = 0$. The solution of the perturbed problem $y_\epsilon(x)$ does not converge uniformly to the solution of the unperturbed problem $y_0(x) = f(y(x), 0)$ as $\varepsilon \to 0$. This means that the perturbation has a significant effect on the behavior of the system, and standard mathematical techniques may not accurately capture its effects. Therefore, specialized methods need to be developed to analyze and solve such problems.*

Initial value problems involving mathematical models are essential in various fields such as science and engineering. Singular perturbation problems with thin boundaries and inner layers are frequently produced by these models, which quantify the relative strength of the highest-order derivative term using a dimensionless parameter. Such problems arise in diverse applications, including chemical reactions, fluid or gas dynamics, heat transfer, the theory of plates and shells, magnetohydrodynamic flow, and neuron variability. The same issues come up when it comes to groundwater transport, turbulence, atmospheric pollution, and vorticity transfer in incompressible Navier-Stokes equations. Nonlinear convection diffusion problems have also gained considerable attention among mathematicians and engineers because of their frequent appearance in various applications in physics, engineering, and biology, such as fluid or gas dynamics, heat transfer, the theory of plates and shells, magneto-hydrodynamic flow, neuron variability, and the study of traveling wave solutions. When dealing with initial value problems, it is not uncommon to encounter dimensionless parameters that measure the relative strength of the highest-order derivative term, which are typically quite small. As a result, the solution often exhibits thin boundary and interior layers, leading to singular perturbation problems. However, using standard higher-order methods such as Galerkin finite elements or central differencing on uniform meshes can result in nonphysical oscillations in the computed solution, which indicates a loss of stability unless the mesh diameter is exceedingly small, making it computationally expensive.

Many researchers have proposed adaptive numeric or asymptotic techniques to address this issue, including non-conforming finite elements, monotone difference methods, local projection stabilization, streamline diffusion methods, fitted schemes, finite volume approximations, and weighted schemes. In recent years, attention has been directed towards problems with a dominant convection term, which pose challenges in accurately resolving the boundary layers. To achieve consistent numerical approximations of layer solutions, locally refined meshes that are standard on the outside and fine in the layer regions are necessary. However, if the locations and widths of the layers are not known in advance, adaptive algorithms are essential to adapt the mesh based on intermediate computed solutions and eventually identify the precise locations and widths of the layers. While nonconforming finite element approximations offer practical benefits such as cheap local communication and efficient parallelization on MIMD machines, stability and convergence problems may arise when using streamline diffusion finite element methods with nonconforming trial spaces. Furthermore, if crucial characteristics like boundary layers are not sufficiently resolved by the underlying mesh, local error estimation may not be accurate. As a result, the use of local error indicators in adaptive refinement algorithms remains an active research area, with issues such as the role of stabilization and the degradation of accuracy with decreasing perturbation parameter values requiring further investigation. In conclusion, traditional numerical techniques are frequently insufficient for solving singularly perturbed situations, as they can exhibit disappointing behavior or be prohibitively expensive in terms of computer memory and processor time.

This paper aims to provide closed-form solutions for nonlinear singularly perturbed initial value problems using the variation iteration method. The method is based on a Lagrange multiplier technique introduced by Inokuti et al., where the Lagrange multiplier is a function rather than a constant. In their work, Inokuti et al. construct ad-joint operators and claim that the Lagrange multiplier can be viewed as a Greens function. This claim is validated, which also shows that Inokuti et al.'s variational technique and He's variation iteration method can be derived using ad-joint operators, Greens function, integration by parts, and the method of weighted residuals. This technique has been successfully used to solve a variety of linear and nonlinear models, such as nonlinear singularly perturbed initial value problems, Burger's equation, and coupled Burger's equation, generalized KdV and coupled Schrodinger-KdV, delay differential equations, autonomous ordinary differential systems, solitary solutions, nonlinear systems of partial differential equations, and nonlinear differential equations of fractional order. Examples with quadratic nonlinear convection terms and quasi-linear terms are provided to demonstrate the accuracy and effectiveness of the approach. The method presented in this paper is highly accurate, brief, and can also be applied to other nonlinear evolution equations of mathematical physics.

Chapter 2

Variation Iteration Method for Ordinary Differential Equation

2.1 Overview of Variation Iteration Method

Ji-huan He, a Chinese mathematician, was the first to introduce the variational iteration approach in 1999. One can employ this method to solve partial and ordinary differential equations, without resorting to unrealistic assumptions that may change the fundamental nature of the solution. The variational iteration method is the most practical and effective for both weak and strong nonlinear equations. Compared to other approaches like the Adomian method, perturbation method, and others, this strategy is also more effective. This method offers successive, rapidly convergent approximations of the exact solution if one exists; in the absence of an exact solution, a few approximations can be employed numerically. Current numerical algorithms have a problem with the treatment of nonlinear components due to limiting assumptions. The VIM lacks a specific criterion for nonlinear operators, such as linearization, small parameters, Adomian polynomials, etc. One of the key advantages of the VIM approach is its ability to reduce calculation size while maintaining excellent numerical accuracy. The approach is also capable of handling a wide variety of theoretical and computational applications in practical issues.

By He's method we introduce the following correction functional corresponding to:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x,t)\{\epsilon u''(t) + r(t)u'(t) + s(t)u(t) - g(t)\}dt, \qquad (2.1.1)$$

where λ is the Lagrange multiplier which can be identified optimally via variational theory. By making the correction functional stationary with restricted variations $\delta u_n(x) = 0$, $\delta u_n'(x) = 0$, we obtain:

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(x,t)\{\epsilon u''(t) + r(t)y'(t) + s(t)u(t) - g(t)\}dt$$

$$\delta u_{n+1}(x) = \delta u_n(x) + \epsilon \int_0^x \lambda(x,t) \frac{d^2}{dt^2} \delta u_n(t) dt + \int_0^x \lambda(x,t) \frac{d}{dt} r(t) \delta u_n(t) + \int_0^x \lambda(x,t) \delta s(t) u_n(t) dt.$$

Integrating by parts, we get:

$$\begin{aligned} \delta u_{n+1}(x) &= \left(1 - \epsilon \frac{\partial \lambda(x,t)}{\partial t} + r(t)\lambda(x,t)\right) \delta u_n(t)_{\downarrow t=x} \epsilon \lambda(x,t) \frac{d}{dt} \delta u_n(t)_{\downarrow t=x} \\ &+ \int_0^x \left(\epsilon \frac{\partial^2 \lambda(x,t)}{\partial t^2} - r(t) \frac{\partial \lambda(x,t)}{\partial t} + s(t)\lambda(x,t)\right) \delta u_n(t) dt. \end{aligned}$$

Therefore, by imposing the above restricted variation terms to the above equation, we obtain the following Euler Lagrange equation:

$$\epsilon \frac{\partial^2 \lambda(x,t)}{\partial t^2} - r(t) \frac{\partial \lambda(x,t)}{\partial t} + s(t)\lambda(x,t) = 0,$$

$$\left(1 - \epsilon \frac{\partial \lambda(x,t)}{\partial t} + r(t)\lambda(x,t)\right)_{\downarrow t=x} = 0,$$

$$\lambda(x,t)_{\downarrow t=x} = 0. \tag{2.1.2}$$

Example 2.1.1 We consider the following simple example:

$$\epsilon u''(x) = 2, u(0) = 1, u(1) = 2.$$

Solving (2.1.2), using the coefficients $r(x) = 0$, $s(x) = 0$, then λ can easily identified as:

$$\lambda(x,t) = \frac{t-x}{\epsilon}$$

Therefore, we have the following iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x,t)\{\epsilon u_n''(t) - 2\} dt.$$

Now we begin with the following initial approximation

$$u_0(x) = A + Bx,$$

where A and B are constants to be determined. By the above iteration formula we have,

$$u_1(x) = A + Bx + \frac{x^2}{\epsilon}.$$

Applying boundary conditions yield $A = 1$ and $B = 1 - \frac{1}{\epsilon}$. Thus

$$u_1(x) = 1 + \left(1 - \frac{1}{\epsilon}\right) x + \frac{x^2}{\epsilon},$$

which is the exact solution.

Example 2.1.2 As a second example we study the singularity equation

$$\epsilon u''(x) - u(x) = 0; u(0) = 1, u(1) = 0.$$

Solving (2.1.2), using the coefficients $r(x) = 0$, $s(x) = -1$, then λ can easily identified as

$$\lambda(x,t) = \frac{e^{\frac{t-x}{\sqrt{\epsilon}}} - e^{\frac{x-t}{\sqrt{\epsilon}}}}{2\sqrt{\epsilon}}.$$

Therefore, we have the following iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(x,t)\{\epsilon u_n''(t) - u_n(t)\}dt.$$

Now we begin with the following initial approximation

$$u_0(x) = Ce^{\frac{x}{\sqrt{\epsilon}}} + De^{\frac{-x}{\sqrt{\epsilon}}},$$

where C and D are constants to be determined. By the above iteration formula we have,

$$u_1(x) = Ce^{\frac{x}{\sqrt{\epsilon}}} + De^{\frac{-x}{\sqrt{\epsilon}}}.$$

Applying boundary conditions yield

$$C = -D = \frac{e^{\frac{1}{\sqrt{\epsilon}}}}{e^{\frac{2}{\sqrt{\epsilon}}}} - 1.$$

Thus,

$$u_1(x) = \frac{e^{\frac{1-x}{\sqrt{\epsilon}}}\left(e^{\frac{1-x}{\sqrt{\epsilon}}} - 1\right)}{e^{\frac{1-x}{\sqrt{\epsilon}}} - 1},$$

which is the exact solution.

2.2 Variation Iteration Method for Ordinary Differential Equation

2.2.1 Brief analysis of the method

In this section, we'll cover the fundamental ideas that guide the variation-iteration process. Look at the nonlinear equation that follows:

$$\mathcal{L}u(x) \equiv L(u(x)) + N(u(x)) = g(x) \tag{2.2.1}$$

where L, in this instance, stands for linear, N is for nonlinear, and $g(s)$ stands for the specified analytic non-homogeneous term. Using variation theory, we can also construct the following correction functional:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\{Lu_n(t) + N\tilde{u}_n(t) - g(t)\}dt, \quad n \geq 0 \tag{2.2.2}$$

where λ denotes Lagrange's multiplier, which may be determined with ease by applying Liouville-Green transforms, integration by parts, and variation theory. Additionally, $u_n(x)$ represents the nth approximation of $u(x)$, and $\tilde{u}_n(x)$ represents the variation which is restricted, signifying $\delta\tilde{u}_n = 0$. Therefore, in the initial stage, we'll determine the Lagrange multiplier's value, λ, and then choose u_0 as a suitable initial function that meets the boundary requirements, then using a correction functional, consecutive approximations, $u_n(x)$ of the function $u(x)$ can be easily obtained. Therefore, exact solution of the problem (2.2.1) :

$$u(x) = \lim_{n\to\infty} u_n(x)$$

2.2.2 Variation iteration method for Ordinary differential equations

Using the variation iteration method, autonomous ordinary differential equations were first solved in 2000. Most differential equations do not have an exact solution that can be expressed in terms of known functions. Therefore, numerical methods are often used to approximate solutions of differential equations. These methods are helpful for addressing nonlinear and linear problems because they provide analytical solutions and have significant advantage over standard numerical techniques. Applying the Variation Iteration Method on the Singularly perturbed problem;

Example 2.2.1 Consider the following problem that has two boundary conditions:

$$\epsilon\frac{d^2u(t)}{dt^2} - 4\frac{du(t)}{dt} = 0, \quad t \in (0,1); \quad u(0) = A_0, \;\; u(1) = A_1, \tag{2.2.3}$$

here ϵ, is a small perturbation.Then the exact solution, with regard to the root $(0, 4/\epsilon)$, of the relevant characteristic polynomial is stated as:

$$\begin{aligned} u(t) &= \frac{-A_1 + A_0\exp(4/\epsilon)}{-1+\exp(4/\epsilon)} - \frac{A_0 - A_1}{-1+\exp(4/\epsilon)}\exp(4t/\epsilon) \\ &= \frac{\exp(4t/\epsilon) - \exp(4/\epsilon)}{1-\exp(4/\epsilon)} \quad \textit{for } A_0 = 1 \textit{ and } A_1 = 0. \end{aligned} \tag{2.2.4}$$

The correction functional with regard to the equation (2.2.3) can be stated as the following:

$$\begin{aligned} u_{n+1}(t) &= u_n(t) + \int_0^t \lambda(s)\left[\epsilon\frac{d^2u_n(s)}{ds^2} - \frac{4du_n(s)}{ds}\right]ds \\ &= u_n(t) - \epsilon\frac{d\lambda(s)}{ds}u_n(s)|_{s=0}^t - 4\lambda(s)u_n(s)|_{s=0}^t + \epsilon\lambda(s)\frac{du_n(s)}{ds}|_{s=0}^t \\ &\quad + \int_0^t \left(\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\frac{d\lambda(s)}{ds}\right)u_n(s)ds. \end{aligned} \tag{2.2.5}$$

Taking a variation with respect to u_n and setting the correction functional, (2.2.5), to zero will make the functional stationary.i.e., $\delta u_{n+1} = 0$:

$$\begin{aligned} \delta u_{n+1}(t) &= \left(1 - \epsilon\frac{d\lambda(s)}{ds} - 4\lambda(s)\right)_{s=t}\delta u_n(t) + \epsilon\lambda(s)|_{s=t}\delta u_n'(t) \\ &\quad + \int_0^t \left(\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\frac{d\lambda(s)}{ds}\right)\delta u_n(s)ds \\ &= 0. \end{aligned}$$

Henceforth,we obtain the Euler-Lagrange equation as

$$\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\frac{d\lambda(s)}{ds} = 0 \tag{2.2.6}$$

and we have the stationary conditions:

$$\left(1 - \epsilon\frac{d\lambda}{ds} - 4\lambda(s)\right)_{s=t} = 0 \quad \text{and} \quad \lambda(s)|_{s=t} = 0. \tag{2.2.7}$$

Thus, Euler equation (2.2.6) with the stationary conditions , (2.2.7), when combined, gives:

$$\lambda(s) = \frac{1}{4}\left(1 - \exp 4\left(\frac{t-s}{\epsilon}\right)\right)$$

As a result, we can now express the variation iteration formula as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{4}\left(1 - \exp 4\left(\frac{t-s}{\epsilon}\right)\right)\left(\epsilon\frac{d^2u(t)}{dt^2} - 4\frac{du(t)}{dt}\right) ds \quad (2.2.8)$$

We can obtain a series of solution for the equation (2.2.3) by considering its linearly independent solutions.We start with the initial approximation $u_0 = B + C\exp(4t)$, where free constants, B and C, can be determined using the boundary conditions. Using (2.2.8), we have the following:

$$\begin{aligned} u_1 &= B + C\exp(t) + C(\epsilon - 4)\frac{1}{4}\int_0^t \left(1 - \exp 4\left(\frac{t-s}{\epsilon}\right)\right)\exp(s)ds \\ &= B + C - C\frac{\epsilon}{4}\left(1 - \exp\left(\frac{4t}{\epsilon}\right)\right) \end{aligned}$$

If we specify the boundary conditions $A_0 = 1$ and $A_1 = 0$, then the solution at the first iteration, which is obtained by using the variation iteration method, gives:

$$u_1(t) = \frac{\exp(4t/\epsilon) - \exp(4/\epsilon)}{1 - \exp(4/\epsilon)}$$

which is indeed the exact solution (2.2.4).

Example 2.2.2 Again, consider the following problem having boundary conditions

$$\epsilon\frac{d^2u(t)}{dt^2} + 4u(t) = 0, \quad t \in (0,1); \quad u(0) = A_0, \;\; u(1) = A_1, \quad (2.2.9)$$

Here ϵ is a small perturbation and the linearly independent solutions, $sin(2/\sqrt{\epsilon})t$ and $cos(2/\sqrt{\epsilon})t$, correspond to the imaginary roots of the relevant characteristic polynomial then the exact solution:

$$\begin{aligned} u(t) &= acos(\frac{2}{\sqrt{\epsilon}})t + \frac{A_1 - A_0 cos(\frac{2}{\sqrt{\epsilon}})}{sin(\frac{2}{\sqrt{\epsilon}})}.sin(\frac{2}{\sqrt{\epsilon}})t \\ &= cos(\frac{2}{\sqrt{\epsilon}})t + \frac{2 - cos(\frac{2}{\sqrt{\epsilon}})}{sin(\frac{2}{\sqrt{\epsilon}})}.sin(\frac{2}{\sqrt{\epsilon}})t \text{ for } A_0 = 1 \text{ and } A_1 = 2 \quad (2.2.10) \end{aligned}$$

Then the correction functional analogous to (2.2.9) is:

$$\begin{aligned} u_{n+1}(t) &= u_n(t) + \int_0^t \lambda(s)\left[\epsilon\frac{d^2u_n(s)}{ds^2} + 4u_n(s)\right] ds \\ &= u_n(t) - \epsilon\frac{d\lambda(s)}{ds}u_n(s)|_{s=0}^t + \epsilon\lambda(s)\frac{du_n(s)}{ds}|_{s=0}^t \\ &\quad + \int_0^t \left(\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\lambda\right) u_n(s)ds. \quad (2.2.11) \end{aligned}$$

by inflicting the variation and taking into consideration the restricted variation i.e $\delta u_{n+1} = 0$, then (2.2.11) becomes:

$$\begin{aligned} \delta u_{n+1}(t) &= \left(1 - \epsilon\frac{d\lambda(s)}{ds}\right)\delta u_n(s) + \epsilon\lambda(s)|_{s=0}^{t}\delta u_n'(s) \\ &\quad + \int_0^t \left(\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\lambda\right)\delta u_n(s)ds. \qquad (2.2.12) \\ &= 0 \end{aligned}$$

Henceforth,we obtain the Euler-Lagrange equation:

$$\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\lambda(s) = 0 \qquad (2.2.13)$$

and we have the stationary conditions as:

$$\left(1 - \epsilon\frac{d\lambda}{ds}\right)_{s=t} = 0 \quad \text{and} \quad \lambda(s)|_{s=t} = 0. \qquad (2.2.14)$$

Thus, equation (2.2.13) with the stationary conditions, when combined, (2.2.14) yields

$$\lambda(s) = \frac{1}{\sqrt{\epsilon}}sin\left(\frac{2s-2t}{\sqrt{\epsilon}}\right).$$

Hence, we may now express the variation iteration formula as:

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\sqrt{\epsilon}}sin\left(\frac{2s-2t}{\sqrt{\epsilon}}\right)\left[\epsilon\frac{d^2u_n(s)}{ds^2} + 4u_n(s)\right]ds \qquad (2.2.15)$$

We can obtain a series of solution for the equation (2.2.9) by considering its linearly independent solutions. We start with the initial approximation $u_0 = C_1\ cos2t + C_2\ sin2t$, where free constants, C_1 and C_2, can be determined using the boundary conditions. Using (2.2.15), we have

$$\begin{aligned} u_1 &= C_1cos2t + C_2sin2t + \frac{4(1-\epsilon)}{\sqrt{\epsilon}}\int_0^t sin\left(\frac{2s-2t}{\sqrt{\epsilon}}\right)(C_1cos2t + C_2sin2t)ds \\ &= cos(2/\sqrt{\epsilon})t + \frac{2 - cos(2/\sqrt{\epsilon})}{sin(2/\sqrt{\epsilon})}.sin(2/\sqrt{\epsilon})t \quad \text{for a=1 and b=2} \qquad (2.2.16) \end{aligned}$$

which is indeed the exact solution (2.2.10)

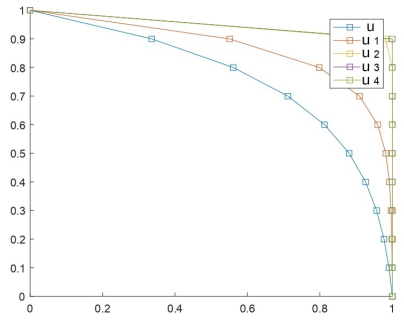

Figure 2.1: The behaviour of the solution to example (2.2.1) is shown as the parameter ϵ is varied.

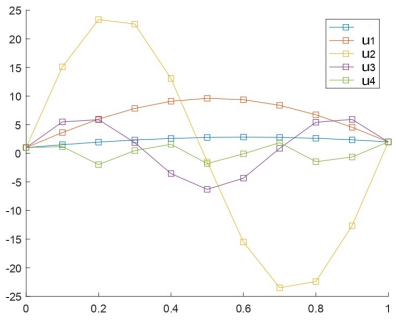

Figure 2.2: The behaviour of the solution to example (2.2.2) is shown as the parameter ϵ is varied.

2.2.3 Problem Description

Take a look at these two-point boundary value problem with nonlinear singular perturbation:

$$\begin{aligned} &\epsilon u'' = f(t, u(t), u'(t)); \quad t \in (0,1) \\ &u(0) = \beta, \quad u(1) = \alpha. \end{aligned} \tag{2.2.17}$$

When dealing with boundary value problems that are singularly perturbed, it is common practice to look for solutions that are stable and provide an indication of where the boundary layers will occur. With the aid of a maximal theory statement, most of these objectives have been achieved. The solution is approximated by values at the boundary of the interval over which it exists. In the subsequent discussion, we will present some of these supporting results.

Existence theorem and a priori bounds

Assume that $p(t)$ and $q(t)$ are functions that are smooth and that they satisfy:

$$p(t) \leq q(t)$$

$$p(0) \leq \xi \leq q(0), p(1) \leq \gamma \leq q(1)$$

$$p''(t) \geq f(t, u(t), u'(t), q''(t) \leq f(t, q(t), q'(t)$$

Furthermore, the following Nagumo condition is true.

$$f(t, u, u') = \mathcal{O}(|u'|^2) as |u'| \to \infty \forall (t, y) \in [a, b] X \mathcal{R}$$

.

Theorem: Suppose there exists bounding functions $p(t)$ and $q(t)$ with the above properties and also suppose that the function f satisfies the Nagumo condition with respect to functions p and q. Then (2.2.17) has a solution $y(t) \in \mathcal{C}^2([0,1])$ satisfying the condition

$$p(t) \leq u(t) \leq q(t), for, t \in [0,1]$$

Moreover, as a direct application of maximum principle the following estimate holds.

Theorem: Suppose that f, function, is continuous with respect to (t, u, u'). The function, f, is of class $\mathcal{C}^1$ with respect to u for (t, u, u') in $[0, 1]$ X $\mathbb{R}^2$ and there exists a positive constant m such that $f_u(t, u, 0) \geq m > 0$ for $(t, u) \in [0, 1]X\mathbb{R}$ then for each $\epsilon > 0$, (2.2.17) has unique solution $u(t, \epsilon) \in [0, 1]$ such that
$|u(t, \epsilon)| \leq M/m$ where

$$M = \max_{[0,1]} |f(t, 0, 0)|, m|\alpha|, m|\beta|$$

Proof:- Define $a(t) = -M/m$ and $b(t) = M/m$. Then,

$$a \leq b, \;\; a(0) \leq \alpha \leq b(0) \; and \; a(1) \leq \beta \leq b(1).$$

For some intermediate point $\xi \in (a, 0)$ an application of Taylor's theorem gives

$$\begin{aligned} f(t, a, 0) &= f(t, 0, 0) + f_u(t, \xi, 0)a \leq |f(t, 0, 0)| + ma \\ &\leq M + m(-M/m) \leq 0 = \epsilon a''. \end{aligned}$$

Similarly, for some intermediate point $\eta \in$ (0,b),

$$\begin{aligned} f(t, b, 0) &= f(t, 0, 0) + f_u(t, \eta, 0) \\ &\geq -M + m(M/m) \geq 0 = \epsilon b''. \end{aligned}$$

Hence, it follows from the previous theorem that for each $\epsilon > 0$ the problem (2.2.17) has a solution $u(t, \epsilon) \equiv u(t)$ on [0,1] satisfying

$$|u(t)| \leq M/m$$

where

$$M = max\{\max_{[0,1]} |f(t, 0, 0)|, m|\alpha|, m|\beta|\}.$$

The solution to the equation is unique because of the maximum principle.

Note: The proof follows from Kaushik, Aditya. "Iterative analytic approximation to nonlinear convection dominated systems." Computer Physics Communications 184.9 (2013): 2061-2069.

2.2.4 Solution Methodology

Now, we'll study using two different possible instances:

$$f(t,u(t),u'(t)) = \begin{cases} h(t,u(t),u'(t)) - m(t)u'(t) - n(t)u(t), & m(t) \neq 0 \text{ and} \\ h(t,u(t),u'(t)), \end{cases} \tag{2.2.18}$$

i.e., In the first instance, the linear component of the function $h(t,y(t),y'(t))$ may be recovered directly, unlike the second instance.

Case-I: $f(t,u(t),u'(t)) = h(t,u(t),u'(t)) - m(t)u'(t) - n(t)u(t);\ m(t) \neq 0.$
As a result, we may develop the correction functional analogous to (2.2.18)

$$\begin{aligned}
u_{n+1}(t) &= u_n(t) + \int_0^t \lambda(\mu)\left(\epsilon u''(\mu) + m(\mu)u'(\mu) + n(\mu)u(\mu) - \tilde{h}(\mu,u,u')\right)d\mu \\
&= u_n(t) + \epsilon\left\{\lambda(\mu)u'(\mu)|_0^t - \int_0^t \frac{d\lambda(\mu)}{d\mu}u'(\mu)d\mu\right\} \\
&\quad + \left\{\lambda(\mu)m(\mu)u(\mu)|_0^t - \int_0^t \left(\frac{d\lambda(\mu)}{d\mu}m(\mu) + \lambda(\mu)\frac{dm(\mu)}{d\mu}\right)u(\mu)d\mu\right\} \\
&\quad + \int_0^t \lambda(\mu)n(\mu)u(\mu)d\mu - \int_0^t \lambda(\mu)\tilde{h}(\mu,u,u')d\mu \\
&= u_n(t) + \epsilon\left\{\lambda(\mu)u'(\mu)|_0^t - \left(\frac{d\lambda(\mu)}{d\mu}u(\mu)|_0^t - \int_0^t \frac{d^2\lambda(\mu)}{d\mu^2}u(\mu)\right)\right\} \\
&\quad + \left\{\lambda(\mu)m(\mu)u(\mu)|_0^t - \int_0^t \left(\frac{d\lambda(\mu)}{d\mu}m(\mu) + \lambda(\mu)\frac{dm(\mu)}{d\mu}\right)m(\mu)d\mu\right\} \\
&\quad + \int_0^t \lambda(\mu)n(\mu)u(\mu)d\mu - \int_0^t \lambda(\mu)\tilde{h}(\mu,u,u')d\mu
\end{aligned} \tag{2.2.19}$$

It should be found that $\lambda(\mu,t) := \lambda(\mu)$ here represents the Lagrange multiplier and the function,$\tilde{h}(\mu,u(\mu),u'(\mu))$, represents the constrained source term variation of the nonlinear source term. (i.e.,$\delta\tilde{h} = 0$).To determine Lagrange's multiplier, we shall apply variational theory and consider variation with respect to an independent variable, u_n (notice that $\delta u_n(0) = 0$) and making the correctional functional stationary in (2.2.19), means., $\delta u_{n+1} = 0$:

$$
\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \epsilon\left\{\lambda(\mu)\delta u'(\mu)|_{\mu=t} - \left(\frac{d\lambda(\mu)}{d\mu}\delta u(\mu)|_{\mu=t} - \int_0^t \frac{d^2\lambda(\mu)}{d\mu^2}\delta u(\mu)\right)\right\} \\
&\quad + \left\{\lambda(\mu)p(\mu)\delta u(\mu)|_{\mu=t} - \int_0^t \left(\frac{d\lambda(\mu)}{d\mu}m(\mu) + \lambda(\mu)\frac{dm(\mu)}{d\mu}\right)\delta u(\mu)d\mu\right\} \\
&\quad + \int_0^t \lambda(\mu)n(\mu)\delta y(\mu)d\mu - \int_0^t \lambda(\mu)\delta\tilde{h}(\mu,u,u')d\mu \\
&= \left(1 - \epsilon\frac{d\lambda(\mu)}{d\mu} + m(\mu)\lambda(\mu)\right)\delta u_n(\mu)|_{\mu=t} + \epsilon\lambda(\mu)\delta\frac{du_n(\mu)}{d\mu}|_{\mu=t} \\
&\quad + \int_0^t \left(\epsilon\frac{d^2\lambda(\mu)}{d\mu^2} - m(\mu)\frac{d\lambda(\mu)}{d\mu} + \left(n(\mu) + \frac{dm(\mu)}{d\mu}\right)\lambda(\mu)\right)\delta u_n(\mu)d\mu \\
&= 0. \qquad (2.2.20)
\end{aligned}
$$

Euler-Lagrange's equation consequently becomes

$$
\left.\begin{aligned}
\epsilon\tfrac{d^2\lambda(\mu)}{d\tau^2} - m(\mu)\tfrac{d\lambda(\mu)}{d\mu} + \left(n(\mu) + \tfrac{dm(\mu)}{d\mu}\right)\lambda(\mu) &= 0 \\
\left(1 - \epsilon\tfrac{d\lambda(\mu)}{d\mu} + m(\mu)\lambda(\mu)\right)_{\mu=t} &= 0 \\
\lambda(\mu)_{\mu=t} &= 0
\end{aligned}\right\} \qquad (2.2.21)
$$

The Lagrange multiplier in this case is λ, which was created by applying Liouville's green transformation. Define Liouville-Green transformation x, $\phi(x)$ and $w(\mu)$ as follows

$$
\left.\begin{aligned}
x = \phi(\mu) &= -\tfrac{1}{\epsilon}\textstyle\int m(\mu)d\mu \\
\psi(\mu) = \phi'(\mu) &= -\tfrac{1}{\epsilon}m(\mu) \\
w(x) &= \psi(\mu)\lambda(\mu).
\end{aligned}\right\} \qquad (2.2.22)
$$

It follows that

$$
\frac{d\lambda(\mu)}{d\mu} = \frac{\phi'(\mu)}{\psi(\mu)}\frac{dw}{dx} - \frac{\psi'(\mu)}{\psi^2(\mu)}w, \quad \text{and} \qquad (2.2.23a)
$$

$$
\frac{d^2\lambda(\mu)}{d\mu^2} = \frac{\phi'^2(\mu)}{\psi(\mu)}\frac{d^2w}{dx^2} + \left(\frac{\phi''(\mu)}{\psi(\mu)} - 2\frac{\phi'(\mu)\psi'(\mu)}{\psi^2(\mu)}\right)\frac{dw}{dx} - \left(\frac{\psi''(\mu)}{\psi^2(\mu)} - 2\frac{\psi'^2(\mu)}{\psi^3(\mu)}\right)w. \qquad (2.2.23b)
$$

Substituting (2.2.23) into (2.2.21), which in turn gives:

$$
\begin{aligned}
&\frac{d^2w}{dx^2} + \left(\frac{\phi''(\mu)}{\psi^2(\mu)} - 2\frac{\phi'(\mu)\psi'(\mu)}{\psi^3(\mu)} - \frac{m(\mu)\phi'(\mu)}{\epsilon\psi^2(\mu)}\right)\frac{dw}{dx} \\
&+ \left(\frac{n(\mu)}{\epsilon\psi^2(\mu)} - \frac{\psi''(\mu)}{\psi^3(\mu)} + 2\frac{\psi'^2(\mu)}{\psi^4(\mu)} + \frac{m(\mu)\psi'(\mu)}{\epsilon\psi^3(\mu)} + \frac{p'(\mu)}{\epsilon\psi^2(\mu)}\right)w = 0.
\end{aligned}
$$

$$
\frac{d^2w}{dx^2} + \frac{dw}{dx} = \epsilon\left(\mathcal{G}(\mu,\epsilon)w(x) - \mathcal{F}(\mu)\frac{dw}{dx}\right) \qquad (2.2.24)
$$

in which

$$\mathcal{F}(\mu)=\frac{m'(\mu)}{m^2(\mu)} \text{ and } \mathcal{G}(\mu,\epsilon)=\left(\epsilon\frac{m''(\mu)}{m^3(\mu)}-2\epsilon\frac{m'^2(\mu)}{m^3(\mu)}-\frac{2m'(\mu)+n(\mu)}{m^2(\mu)}\right).$$

Since $m(\cdot)\in C^2[0,1]$, $n(\cdot)\in C[0,1]$ and $\mathcal{F}(\cdot)$, $\mathcal{G}(\cdot,\epsilon)$ are bounded on $[0,1]$, thus, we have

$$\epsilon\left(\mathcal{G}(\mu,\epsilon)w(x)-\mathcal{F}(\mu)\frac{dw}{dx}\right)\to 0 \quad \text{as} \quad \epsilon\to 0.$$

Therefore, (2.2.24) reduces to

$$\frac{d^2w}{dx^2}+\frac{dw}{dx}\approx 0$$

and hence

$$w(x)=C_1+C_2\exp(-x). \tag{2.2.25}$$

In light of (2.2.22), (2.2.25) results

$$\lambda(\mu,t)=-\frac{\epsilon}{m(\mu)}\left(C_1+C_2\exp\left(\frac{1}{\epsilon}\int_t^\mu m(s)ds\right)\right). \tag{2.2.26}$$

where C_1 and C_2 are made up constants at random. As a result, boundary conditions produce

$$\lambda(\mu,t)=-\frac{1}{m(\mu))}\left(1-\exp\left(\frac{1}{\epsilon}\int_t^\mu m(s)ds\right)\right). \tag{2.2.27}$$

Consequently, the iteration formula for (2.2.18) is provided as

$$\begin{aligned} u_{n+1}(t) &= u_n(t)+\int_0^t -\frac{1}{m(\mu))}\left(1-\exp\left(\frac{1}{\epsilon}\int_t^\mu m(s)ds\right)\right)\times \\ &\left(\epsilon\frac{d^2u(\mu)}{d\mu^2}+m(\mu)\frac{du(\mu)}{d\mu}+n(\mu)y(\mu)-f\left(\mu,u(\mu),\frac{du(\mu)}{d\mu}\right)\right)d\mu \end{aligned} \tag{2.2.28}$$

Case-II: $f(t,u(t),u'(t))=h(t,u(t),u'(t))$; **only contains non-linear or implicit terms.**

The correction functional is presented as follows with regard to (2.2.18)

$$\begin{aligned} u_{n+1}(t) &= u_n(t)+\int_0^t \lambda(\mu)\left(\epsilon u''(\mu)-\tilde{h}(\mu,u,u')\right)d\mu \\ &= u_n(t)+\epsilon\left\{\lambda(\mu)u'(\mu)|_0^t-\int_0^t\frac{d\lambda(\mu)}{d\mu}u'(\mu)d\mu\right\}-\int_0^t\lambda(\mu)\tilde{h}(\mu,u,u')d\mu \\ &= u_n(t)+\epsilon\left\{\lambda(\mu)u'(\mu)|_0^t-\left(\frac{d\lambda(\mu)}{d\mu}u(\mu)|_0^t-\int_0^t\frac{d^2\lambda(\mu)}{d\mu^2}u(\mu)\right)\right\} \\ &\quad -\int_0^t\lambda(\mu)\tilde{h}(\mu,u,u')d\mu \end{aligned} \tag{2.2.29}$$

The Lagrange multiplier is represented in this equation by $\lambda(\mu,t) := \lambda(\mu)$, with $\tilde{h}(\mu,u(\mu),u'(\mu))$ standing for the constrained variation of the nonlinear source term. To determine Lagrange's multiplier, we shall apply variation theory and consider variation with respect to an independent variable, u_n (observe that $\delta u_n(0)=0$) and $\delta u_{n+1}=0$:

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \epsilon\left\{\lambda(\mu)\delta u'(\mu)|_0^t - \left(\frac{d\lambda(\mu)}{d\mu}\delta u(\mu)|_0^t - \int_0^t \frac{d^2\lambda(\mu)}{d\mu^2}\delta u(\mu)\right)\right\} \\
&\quad - \int_0^t \lambda(\mu)\delta\tilde{h}(\mu,u,u')d\mu \\
&= \left(1-\epsilon\frac{d\lambda(\mu)}{d\mu}\right)_{\mu=t}\delta u(t) + \epsilon\lambda(\mu)|_{\mu=t}\delta u'(t) + \epsilon\int_0^t \frac{d^2\lambda(\mu)}{d\mu^2}\delta u(\mu)d\mu \\
&= 0. \qquad (2.2.30)
\end{aligned}$$

Euler-Lagrange's equation consequently becomes

$$\frac{d^2\lambda(\mu)}{d\mu^2}=0; \quad \left(1-\epsilon\frac{d\lambda(\mu)}{d\mu}\right)_{\mu=t}=0, \quad \lambda(\mu)_{\mu=t}=0$$

which in turn yields

$$\lambda = \frac{\mu-t}{\epsilon}.$$

Thus, the iteration formula is as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \left(\frac{\mu-t}{\epsilon}\right)\left(\epsilon\frac{d^2u(\mu)}{d\mu^2} - h\left(\mu,u(\mu),\frac{du(\mu)}{d\mu}\right)\right)d\mu \qquad (2.2.31)$$

Chapter 3

Variation Iteration Method for Partial Differential Equation

3.1 Brief Analysis of the method

In this section, we'll review the core ideas that underlie the variation iteration method. Let us consider the following nonlinear equation as an example:

$$\mathcal{L}z(x) \equiv L(z(x)) + N(z(x)) = h(x,t) \tag{3.1.1}$$

For example $Lz = L_t z + L_x z$, then Equation (3.1.1) can also be written as

$$L_t z + L_x z + Nz - h(x,t) = 0 \tag{3.1.2}$$

with L_x and L_t, both are linear operators with respect to x and t, respectively, N is the Non-Linear operator and $h(x,t)$, a continuous function. Based on the variation iteration method, a correction functional for Equation (3.1.2) in the shape of

$$z_{n+1}(x) = z_n(x,t) + \int_0^t \lambda(\mu)\{L_\mu z_n(x,\mu) + L_x \tilde{z_n}(x,\mu) + N\tilde{z_n}(x,\mu) - h(x,t)d\mu \tag{3.1.3}$$

where λ is the Langrange's multiplier and can be optimally identified using the calculus theory of variation, $\tilde{z_n}$ is a finite variation so that $\delta\tilde{z_n}$ to achieve stationary conditions, and z_0 is chosen based on the initial conditions given.

Let z_n be the approximate solution in the nth iteration $n \geq 0$, then the exact solution of (3.1.1) is given by (3.1.3) :

$$z(x,t) = lim_{n\to\infty} z_n(x,t)$$

$$E_n = |z(x,t) - z_n(x,t)|$$

3.1.1 Problem Description

To construct the form of the correction function, reconsider the following parabolic partial differential equation,

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = \phi(z) + h(x,t) \tag{3.1.4}$$

with boundary conditions $z(0,t) = z(1,t) = 0$,
and initial conditions $z(x,0) = g(x)$.

For example, $L_t = \frac{\partial}{\partial t}$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$, is a differential linear operator, then equation (3.1.4) can be described as

$$L_t z - L_{xx} z = \phi(z) + h(x,t)$$

Or

$$L_t z - L_{xx} z - \phi(z) - h(x,t) = 0 \tag{3.1.5}$$

Component $\phi(z)$, in equation (3.1.5), is a non linear function. Equation (3.1.5) is said to be homogeneous when $h(x,t) = 0$. But on the other hand, if $h(x,t) \neq 0$, then Equation (3.1.5) is said to be non-homogeneous.

3.1.2 Solution Methodology

To solve Equation (3.1.5), then based on the iteration method, the variation of Equation (3.1.5) can be changed to

$$z_{n+1}(x,t) = z_n(x,t) + \int_0^t \lambda(\mu)((L_\mu z_n(x,\mu) - L_{xx}\tilde{z_n}(x,\mu) - \phi(\tilde{z_n}(x,\mu) - h(x,t))d\mu \tag{3.1.6}$$

where $\lambda(\mu)$ is the Lagrange multiplier. The Lagrange multiplier, $\lambda(\mu)$,can be optimally identified through the theory of variation. Therefore, equation (3.1.6) be formed as

$$\delta z_{n+1}(x,t) = \delta z_n(x,t) + \delta \int_0^t \lambda(\mu)(L_\mu z_n(x,\mu) - L_{xx}\tilde{z_n}(x,\mu) - \phi(\tilde{z_n}(x,\mu)) - h(x,t))d\mu \tag{3.1.7}$$

To reach the stationary condition, the condition $\delta\tilde{z_n} = 0$ is needed, so Equation (3.1.7) becomes

$$\delta z_{n+1}(x,t) = \delta z_n(x,t) + \delta \int_0^1 \lambda(\mu)\frac{\partial z_n(x,\mu)}{\partial \mu}d\mu$$

Or

$$\delta z_{n+1} = \delta z_n(x,t)(1 + \lambda|_{\mu=t}) - \int_0^t \lambda'(\mu)\delta\frac{\partial z_n(x,\mu)}{\partial \mu}d\mu \tag{3.1.8}$$

When the condition is stationary, from Equation (3.1.8), we have $\lambda'(\mu)|_{\mu=t} = 0$ and $1 + \lambda(\mu)|_{\mu=t} = 0$ so we get $\lambda = -1$. Then by changing the value of the Lagrange multiplier in equation (3.1.6), now (3.1.6) can be rewritten as

$$z_{n+1}(x,t) = z_n(x,t) - \int_0^t (L_\mu z_n(x,\mu) - L_{xx} z_n(x,\mu) - \phi(z_n(x,\mu)) - h(x,\mu) d\mu \quad (3.1.9)$$

Or

$$z_{n+1}(x,t) = z_n(x,t) - \int_0^t (\frac{\partial z_n(x,\mu}{\partial \mu} - \frac{\partial^2 z_n(x,\mu)}{\partial x^2} - \phi(z_n(x,\mu)) - h(x,\mu)) d\mu \quad (3.1.10)$$

Based on Equation (3.1.10), then by taking the initial value $z_0(x,t)$, then obtained successively $z_1(x,t), z_2(x,t), z_3(x,t), \ldots$ in the form

$$z_1(x,t) = z_0(x,t) - \int_0^t (\frac{\partial z_0(x,\mu}{\partial \mu} - \frac{\partial^2 z_0(x,\mu)}{\partial x^2} - \phi(z_0(x,\mu)) - h(x,\mu)) d\mu$$

$$z_2(x,t) = z_1(x,t) - \int_0^t (\frac{\partial z_1(x,\mu}{\partial \mu} - \frac{\partial^2 z_1(x,\mu)}{\partial x^2} - \phi(z_1(x,\mu)) - h(x,\mu)) d\mu$$

$$z_3(x,t) = z_2(x,t) - \int_0^t (\frac{\partial z_2(x,\mu}{\partial \mu} - \frac{\partial^2 z_2(x,\mu)}{\partial x^2} - \phi(z_2(x,\mu)) - h(x,\mu)) d\mu$$

$$\vdots$$

Example 3.1.1 Parabolic partial differential equation
Let's think about a problem:

$$z_t = z_{xx} - e^{-x}sint + e^{-x}cost \tag{3.1.11}$$

containing the initial conditions $z(x,0) = x$ and boundary conditions:

$$z(0,t) = sint, \quad z(1,t) = \frac{1+sint}{e}$$

Which is obvious to provide the precise answer.

$$z(x,t) = x + e^{-x}(cost - sint) \tag{3.1.12}$$

The correction functional that corresponds to (3.1.11) is given as:

$$z_{n+1}(x,t) = z_n(x,t) + \int_0^t \lambda \left(\frac{\partial z_n(x,\mu)}{\partial \mu} - \frac{\partial^2 z_n(x,\mu)}{\partial x^2} - e^{-x}cost + e^{-x}sint \right) d\mu \tag{3.1.13}$$

where λ, the Lagrange multiplier, can be conveniently determined using variational theory:

$$\delta z_{n+1}(x,t) = \delta z_n(x,t) + \delta \int_0^t \lambda \left(\frac{\partial z_n(x,\mu)}{\partial \mu} - \frac{\partial^2 z_n(x,\mu)}{\partial x^2} - e^{-x}cost + e^{-x}sint \right) d\mu \tag{3.1.14}$$

We obtain $\lambda = -1$ after applying Integration by parts.

$$\delta z_{n+1}(x,t) = \delta z_n(x,t) + \delta \int_0^t (-1) \left(\frac{\partial z_n(x,\mu)}{\partial \mu} - \frac{\partial^2 z_n(x,\mu)}{\partial x^2} - e^{-x}cost + e^{-x}sint \right) d\mu \tag{3.1.15}$$

Now, take an initial approximation that satisfies the initial condition.

$$z_0(x,t) = x$$

substitute initial condition into (3.1.15), then we get

$$z_1(x,t) = x - e^{-x} + e^{-x}sint + e^{-x}cost \tag{3.1.16}$$

similarly,

$$z_2(x,t) = x + 2e^{-x}sint - e^{-x}t \tag{3.1.17}$$

$$z_3(x,t) = x + e^{-x}sint + e^{-x} - e^{-x}cost - \frac{1}{2}e^{-x}t^2 \tag{3.1.18}$$

The remaining iterations can be obtained in a similar manner.

When the outcomes of this analysis were compared to those produced from an exact solution, it became clear that they were nearly identical. This demonstrates that the variational iteration method is a highly accurate and effective mathematical tool for solving differential equations.

t	$z(x,t)_{exact}$	$z(x,t)_{vim}$
0.1	0.190333	0.190333
0.3	0.367396	0.367397
0.5	0.533801	0.531465
0.7	0.68292	0.674006
0.9	0.808783	0.794705

Table 3.1: Numerical results for $x = 0.1$

t	$z(x,t)_{exact}$	$z(x,t)_{vim}$
0.1	0.940589	0.940589
0.3	1.020149	1.020149
0.5	1.094919	1.094911
0.7	1.161919	1.161853
0.9	1.218476	1.218181

Table 3.2: Numerical results for $x = 0.9$

Example 3.1.2 Consider a non linear partial differential equation

$$\frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial t^2} = z^2 - \left(\frac{\partial z}{\partial t}\right)^2 \tag{3.1.19}$$

with initial condition $z(t,0) = e^t$. The variational iteration approach can be used to calculate the nonlinear parabolic equation's value.
The correction functional analogous to (3.1.19)

$$z_{n+1}(x,t) = z_n(x,t) + \int_0^x \lambda \left(\frac{\partial z_n(t,\mu)}{\partial \mu} - \frac{\partial^2 z_n(t,\mu)}{\partial t^2} - z_n^2(t,\mu) + \left(\frac{\partial z(t,\mu)}{\partial t}\right)^2 \right) d\mu \tag{3.1.20}$$

where λ, the lagrange multiplier, can be conveniently determined using variational theory:

$$\delta z_{n+1}(x,t) = \delta z_n(x,t) + \delta \int_0^x \lambda \left(\frac{\partial z_n(t,\mu)}{\partial \mu} - \frac{\partial^2 z_n(t,\mu)}{\partial t^2} - z_n^2(t,\mu) + \left(\frac{\partial z(t,\mu)}{\partial t}\right)^2 \right) d\mu \tag{3.1.21}$$

Applying by parts ,we obtain $\lambda = -1$

$$\delta z_{n+1}(x,t) = \delta z_n(x,t) + \delta \int_0^x (-1) \left(\frac{\partial z_n(t,\mu)}{\partial \mu} - \frac{\partial^2 z_n(t,\mu)}{\partial t^2} - z_n^2(t,\mu) + \left(\frac{\partial z(t,\mu)}{\partial t}\right)^2 \right) d\mu \tag{3.1.22}$$

Now,take arbitrary initial approximation which satisfying the initial condition

$$z(t,0) = e^t$$

substitute initial condition into (3.1.20)

$$\begin{aligned}
z_1(x,t) &= e^t(1+x), \\
z_2(x,t) &= e^t\left(1 + x + \frac{x^2}{2!}\right), \\
z_3(x,t) &= e^t\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right), \\
&\vdots \\
z_n(x,t) &= e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}\right), \\
&\vdots
\end{aligned}$$

We can solve Equation (3.1.19) exactly by taking $n \to \infty$., so we get

$$z_n(x,t) = e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}\cdots\right) \tag{3.1.23}$$

or

$$z(x,t) = e^{x+t} \tag{3.1.24}$$

The accuracy of the solution of (3.1.19) depends on the number of iterations required to solve it. Figures (a) and (b), depict a preliminary data and the comparison between the exact and approximate solutions for the two values of n, i.e., $n = 2$ and $n = 8$, respectively. Figures 2.3 and 2.4 illustrate that the curve produced by $z_8(x,t)$ is

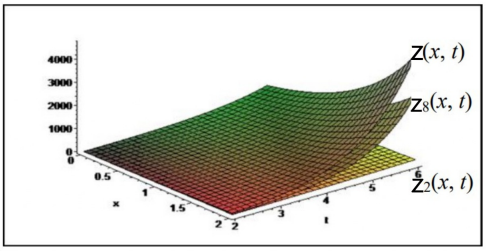

Figure 3.1: (a) Comparison of exact solution for x, $0 \leq x \leq 2$

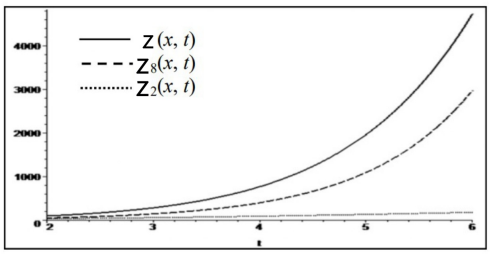

Figure 3.2: (b) Comparison of exact solution for x, $x = 2$

Comparison of the exact solution with the z_2 iteration and z_8 iteration for $2 \leq t \leq 6 : (a)\ \ 0 \leq x \leq 2, (b)\ \ x = 2$

more closely related to $z(x,t)$ than the other curves. This indicates that carrying out additional iterations to refine the approximation will bring you closer to the exact solution.

Chapter 4

Conclusion

Nonlinear problems are frequently solved by linearizing them around a notional solution using the quasi-linearization technique, which is subsequently solved either analytically or numerically. However, this linearization approach may lead to inaccuracies in the solution, particularly for complex problems with severe nonlinearities. To address some of these issues, we proposed an iterative analytic approach that has been shown to be highly accurate and robust with respect to small parameters. Indeed, the suggested approach offers an analytical approximation that is on par with wholly analytical solutions.

Furthermore, this approach is straightforward to implement and can be adapted to tackle a broader array of problems, including those involving discontinuous source terms and the evolution equations of mathematical physics. Unlike numerical methods, the present approach does not entail tiresome algebraic calculations, prior simplification, discretization or linearization, and because it doesn't offer any linear or nonlinear systems of equations, the size of calculations is greatly reduced while still maintaining high accuracy. The approach demonstrates its efficacy by producing promising results in just a few iterations, without any restrictive assumptions. The obtained results are equivalent.

After applying the proposed approach to parabolic partial differential equations, we observed that an infinite number of iterations is needed to obtain the exact solution. The fact that more iterations result in a solution that gets closer to being the exact answer shows how good this method is in solving PDEs. A close approximation to the exact solution can be obtained using this technique. In summary, the proposed iterative analytic method offers a powerful tool for accurately solving nonlinear problems with greater efficiency and ease compared to traditional methods. Moreover, this approach is free from constraining assumptions and justifies its effectiveness by delivering promising results with only a few iterations.

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