RADIUS, COEFFICIENT CONSTANTS AND DIFFERENTIAL SUBORDINATIONS OF CERTAIN UNIVALENT FUNCTIONS

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By

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DECLARATION

I declare that the work in this thesis titled "**Radius, Coefficient Constants and Differential Subordinations of Certain Univalent Functions**" for the award of the degree of *Doctor of Philosophy in Mathematics* is original and has been carried out by me under the supervision of *Prof. S. Sivaprasad Kumar*, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

This thesis has not been submitted by me earlier in part or full to any other university or institute for the award of any degree or diploma. Wherever applicable, proper acknowledgment is given to the other's work.

Date- 30-11-2022 **KAMALJEET**

CERTIFICATE

On the basis of declaration submitted by **Mr. KAMALJEET**, a research scholar under my supervision, I hereby certify that the thesis titled **"Radius, Coefficient Constants and Differential Subordinations of Certain Univalent Functions"** submitted to the Department of Applied Mathematics, Delhi Technological University, Delhi, India for the award of the degree of *Doctor of Philosophy in Mathematics*, is a record of bonafide research work carried out by him.

To the best of my knowledge the work reported in this thesis is original and has not been submitted to any other Institution or University in any form for the award of any Degree or Diploma.

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Date: 30-11-2022 **KAMALJEET**

Place: Delhi, India.

Dedicated

to

my parents and wife

Contents

Preface

This thesis contributes several new results to the basic theory of the univalent function theory. Although there are several elegant and beautiful articles available in the literature, but it was Ma and Minda's paper "*A unified treatment of some special classes of univalent functions. Proceedings of the Conference on Complex Analysis, Tianjin, Conf Proc Lecture Notes Anal., I Int Press, Cambridge, MA. 157–169 (1992)*", which paved the way for new research in univalent function theory. It is worth here to mention that the Ma-Minda class covers the classical classes of the univalent starlike and convex functions. After this, various problems were studied for a specific subclass of univalent functions, particular focus has been on the subclasses of starlike functions. In this thesis, we study a variety of problems for the Ma-Minda classes. Hence, either we generalize the known results or establish some new results for this class. In brief, we generalize certain results, which trace their origin to the following defining articles: "*T.H. MacGregor, Majorization by univalent functions. Duke Math. J.* **34**, *95–102 (1967).*" Interest in special functions in view of radius problem can be seen from the papers "*R.K. Brown, Univalence of Bessel functions. Proc. Amer. Math. Soc. 11, 278–283 (1960)*", "*R.K. Brown, Univalent solutions of* $W'' + pW = 0$. *Canadian J. Math. 14, 69–78 (1962)*", "*H.S. Wilf, The radius of univalence of certain entire functions. Illinois J. Math. 6, 242–244 (1962)*" and "*E. Kreyszig and J. Todd, The radius of univalence of Bessel functions. I, Illinois J. Math. 4, 143–149 (1960).*" About absolute power series sum and its connection to the univalent function theory follows from the papers "*H. Bohr, A Theorem Concerning Power Series, Proc. London Math. Soc. (2)* **13** *(1914), 1–5*" and "*Y.A. Muhanna, Bohr's phenomenon in subordination and bounded harmonic classes, Complex Var. Elliptic Equ.* **55** *(2010), no. 11, 1071–1078.*" The work on convolution and relevant radius problem comes from the papers "*G. Szegö, Zur Theorie der schlichten Abbildungen, Math. Ann.* **100** *(1928), no. 1, 188–211*" and "*H. Silverman, Radii problems for sections of convex functions, Proc. Amer. Math. Soc.* **104** *(1988), no. 4, 1191–1196.*"

In the context of the above, we prove the classical results and establish topics of current interest for the general Ma-Minda classes. In certain investigations, particularly chapter 3 and chapter 5 pave the way for future research.

List of Symbols

$\mathbb{C}$	Set of complex numbers
$\mathbb{D}$	Open unit disk $\{z \in \mathbb{C} : \|z\| < 1\}$
$\mathscr{A}_n$	The class of analytic functions of the form $f(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$, where $\mathscr{A}_1 := \mathscr{A}$
$\mathscr{S}$	The class of univalent functions normalized by $f(0) = 0 = f'(0) - 1$
$\mathscr{H}_0$	The class of harmonic functions f with $f_{\bar{z}}(0) = 0$ and $f = h + \bar{g}$, where $h(z) = z + \sum_{m=2}^{\infty} a_m z^m$ and $g(z) = \sum_{m=2}^{\infty} b_m z^m$ are analytic functions in $\mathbb{D}$
$\mathscr{S}^*$	The class of starlike functions in $\mathscr{S}$
$\mathscr{C}$	The class of convex functions in $\mathscr{S}$
$\mathscr{P}$	The class of Carathéodory functions defined by $p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n$ and $\Re p(z) > 0$
$\mathscr{S}^*(\psi)$	The class of Ma-Minda starlike functions, where $\psi \in \mathscr{P}$
$\mathscr{C}(\psi)$	The class of Ma-Minda convex functions, where $\psi \in \mathscr{P}$
$\mathscr{K}$	The class of close-to-convex functions
$H_q(n)$	Hankel determinant
$g \prec f$	g is subordinate to f
$\mathscr{F}(\psi)$	The class of analytic functions $\left\{f \in \mathscr{A} : \frac{zf'(z)}{f(z)} - 1 \prec \psi(z),\ \psi(0) = 0\right\}$, where ψ is univalent
$f * g$	Convolution $f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ of two analytic functions f and g in $\mathscr{A}$
$M_r^N(f)$	N^{th}-tail sum $\sum_{n=N}^{\infty} \|a_n\|\|z^n\|$ of an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers

Chapter 1

Introduction

This chapter gives a glimpse into the Geometric function theory. The purpose of this chapter is to define various classes of analytic functions and also to introduce basic definitions and concepts which will be needed in the subsequent chapters. It also covers some basic notations and includes the synopsis of the thesis with some significant findings of the present study.

The study of geometric function theory aims at obtaining the qualitative characteristics of complex valued functions mathematically. One of the main branch of geometric function theory is theory of univalent functions. The subject is classified under geometric function theory due to the fact that from simple geometrical consideration a large number of remarkable properties of univalent functions have been found.

Definition 1.0.1. A function $f(z)$ is said to be *univalent* in a domain $\mathscr{D} \subset \mathbb{C}$ if it is one-to-one, i.e, for $z_1, z_2 \in \mathscr{D}$, $f(z_1) = f(z_2)$ implies that $z_1 = z_2$.

An analytic function f in the domain $\mathscr{D}$ is said to be locally univalent at $z_0 \in \mathscr{D}$, if $f'(z_0) \neq 0$. analytic univalent functions in the domain $\mathscr{D}$ are also called *conformal mappings* in $\mathscr{D}$ because they preserve angles (both in magnitude and direction). In 1851, Riemann stated that a remarkable result which is known as Riemann Mapping Theorem given below.

Theorem A (Riemann Mapping Theorem)**.** Let $\mathscr{D} \subset \mathbb{D}$ be a simply connected domain with $a \in \mathscr{D}$. Then there exists a unique analytic function $g : \mathscr{D} \to \mathbb{C}$ such that

$(A.)$ $g(a) = 0$ and $g'(a) > 0$;

$(B.)$ g is univalent;

$(C.)$ $g(\mathscr{D}) = \Omega$, where Ω is also a simply connected domain.

Thus, the properties of the univalent function defined on the open unit disk $\mathbb{D}$ can be easily translated into the properties of the original function defined in the simply connected domain $\mathscr{D}$. Therefore, it is sufficient to study analytic functions on the unit disk $\mathbb{D}$. Since the quantity

$$f_1(z) = \frac{f(z) - f(0)}{f'(0)}, \quad f'(0) \neq 0$$

represents the shifting and contraction (or expansion) with rotation of the image domain $f(\mathbb{D})$ and any property of the function $f_1(z)$ is immediately translated into a corresponding property of $f(z)$, so we consider the normalization, namely $f(0) = 0$ and $f'(0) = 1$. Let $\mathscr{A}$ be the class of all such normalized analytic functions. Now we can proceed to discuss some important subclasses of $\mathscr{A}$. Moreover, the importance of normalization can be seen in the existence of solution to the coefficient related problem, and its connection to the compactness of a given function space. For this, we refer to see [58, Chapter 4, Section 5].

1.1 Classes of univalent and starlike functions

Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of univalent functions. If $f \in \mathscr{S}$, then the Taylor Series expansion of f is given by:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1.1}$$

In the year 1907, Koebe [80] proved that for the class $\mathscr{S}$, there exists an absolute constant $k > 0$ such that boundary of the image $f(\mathbb{D})$ can not be distorted so far as to come within a distance less than k of the origin. In 1916, Bieberbach [32] established the beautiful result that $|a_2| \leq 2$ for every function $f \in \mathscr{S}$ and using this, determined the value of k as 1/4. This shows the geometrical connection of coefficient bounds on the geometry of functions. Shortly, we shall see the importance of coefficients bounds in the concept of Bohr phenomenon. Bieberbach also conjectured that $|a_n| \leq n$. Meanwhile, the validity of this conjecture was found true for many subclasses of $\mathscr{S}$. In 1925, J. E. Littlewood [98] proved that $|a_n| \leq en$ for all n, showing that Bieberbach conjecture is true up to a factor of $e = 2.718\cdots$. Finally in 1985, Louis De Branges [36] proved this conjecture, by using special functions. Before the proof of Bieberbach conjecture, several subclasses and other fascinating results appeared to solve it. A systematic study in this direction can be seen in some known standard books. Books by Nehari [119], Pommerenke [131], Goodman [58], Duren [49], Graham and Kohr [63], Jenkins [69], Thomas et al. [166] and survey articles of Hayman [66] and Duren [48] are excellent sources of information on univalent function theory.

Coming back, we first describe a geometrical property, which further leads to an important subclass of univalent functions.

Definition 1.1.1. A domain D is said to be starlike w.r.t a point $w_0 \in D$ if each ray with initial point w_0 intersects the interior of D in a set which is either a line segment or a ray. If a function $f(z)$ maps $\mathbb{D}$ onto a domain which is starlike w.r.t $w_0 = 0$, we say that $f(z)$ is a starlike function.

Analytically, a function $f(z) \in \mathscr{S}$ is starlike with respect to origin if and only if $\Re(zf'(z)/f(z)) > 0$.

The class of starlike functions is defined as

$$\mathscr{S}^* := \left\{ f \in \mathscr{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\}.$$

Definition 1.1.2. A set D in the plane is said to be convex if for every pair of points w_1 and w_2, the line joining w_1 and w_2 is contained in D. If a function $f(z)$ maps $\mathbb{D}$ onto a convex domain, we say that $f(z)$ is a convex function.

Analytically, a function $f(z) \in \mathscr{S}$ is said to be convex if and only if $\Re(1 + zf''(z)/f'(z)) > 0$. The class of convex functions is defined as

$$\mathscr{C} := \left\{ f \in \mathscr{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}.$$

It is easy to note that every convex function is starlike with respect to every point in the region $f(\mathbb{D})$, hence every convex function is a starlike function but the converse need not be true. One of the example is $f(z) = z + z^2/2$. Later in 1936, Robertson [140] generalized the classes $\mathscr{S}^*$ and $\mathscr{C}$ by introducing:

Definition 1.1.3. A function $f \in \mathscr{S}$ is said to be Convex function of order α if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (0 \leq \alpha < 1, z \in \mathbb{D}).$$

The class of such functions is denoted by $\mathscr{C}(\alpha)$. Note that $\mathscr{C}(0) = \mathscr{C}$, the class of convex functions.

Definition 1.1.4. A function $f \in \mathscr{S}$ is said to be Starlike of order α if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (0 \leq \alpha < 1, z \in \mathbb{D}).$$

The class of such functions is denoted by $\mathscr{S}^*(\alpha)$. Note that $\mathscr{S}^*(0) = \mathscr{S}^*$, the class of starlike functions.

The well-known Alexander's transformation given below establishes a two-way bridge between the classes $\mathscr{C}(\alpha)$ and $\mathscr{S}^*(\alpha)$ defined as

$$f \in \mathscr{C}(\alpha) \Longleftrightarrow zf'(z) \in \mathscr{S}^*(\alpha).$$

In view of the analytic characterization of the above-discussed classes, let us recall the following important class:

Definition 1.1.5. A function $p(z)$ which is analytic in $\mathbb{D}$ is said to be in the Carathéodory class $\mathscr{P}$, if it satisfy

$$p(0) = 1 \text{ and } \Re p(z) > 0$$

and have the form: $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$.

Thus, the classes $\mathscr{P}$ and $\mathscr{S}^*$ can now be related to each other as follows:

$$f \in \mathscr{S}^* \iff zf'(z)/f(z) \in \mathscr{P}.$$

Hence, the relation between the coefficients p_n of the function p in the class $\mathscr{P}$ becomes prominent and serves as a tool to solve the many coefficient problems related to the subclasses of $\mathscr{S}$. A recent survey article entitled *"A survey on coefficient estimates for Carathéodory functions"* by N. E. Cho et al. [40] deals with carathéodory coefficients p_n. To proceed further, we need to recall

Definition 1.1.6. (Subordination) An analytic function f is subordinate to another analytic function g, denoted by $f \prec g$, if there is an analytic function w with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. Further, If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

The principle of subordination owes its origin to Lindelöf (1908), but the basic theory was developed, later on, by Littlewood [98] and Rogosinski [142]. This principle enables us to derive information about an analytic function f, if certain geometric details of the conformal map associated with this function are known. The class $\mathscr{P}$ is also known as Carathéodory class of functions and plays an important role in the characterization of the many subclasses of $\mathscr{S}$. In 1992, using subordination, Ma-Minda [102] introduced the unified class of starlike and convex functions defined as follows:

$$\mathscr{S}^*(\psi) := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \psi(z) \right\} \quad \text{and} \quad \mathscr{C}(\psi) := \left\{ f \in \mathscr{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) \right\}, \tag{1.1.2}$$

where

1. $\psi(z) \in \mathscr{P}$ is an analytic univalent function in $|z| < 1$.

2. $\Re\psi(z) > 0$, $\psi'(0) > 0$, $\psi(0) = 1$ and $\psi(\mathbb{D})$ is symmetric about real axis.

In literature, there are still several problems that are not solved in general. This fact motivated several authors, see Table 1.1, to study specific subclasses in view of (3.1.3).

Class	$\psi(z)$	Reference
$\mathscr{S}^*_C$	$1 + 4z/3 + 2z^2/3$	Sharma, Jain and Ravichandran [150]
$\mathscr{S}^*_{SG}$	$2/(1+e^{-z})$	Goel and Kumar [60]
$\mathscr{S}^*_s$	$1 + \sin z$	Cho, Kumar and Kumar [82]
$\mathscr{S}^*_L$	$\sqrt{1+z}$	Sokół and Stankiewicz [161]
$\mathscr{S}^*_{Ne}$	$1 + z - z^3/3$	Wani and Swaminathan [170]
$\mathscr{S}^*_{q_b}$	$\sqrt{1+bz}, \quad b \in (0,1]$	Sokół [158]
Δ^*	$z + \sqrt{1+z^2}$	Raina and Sokół [134]
$\mathscr{S}^*_e$	e^z	Mendiratta, Nagpal and Ravichandran [109]
$\mathscr{S}^*_{\varphi_o}$	$1 + \frac{z}{k}\left(\frac{k+z}{k-z}\right), \ k = 1 + \sqrt{2}$	Kumar and Ravichandran [89]
$\mathscr{S}^*_{RL}$	$\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$	Mendiratta, Nagpal and Ravichandran [108]

Table 1.1: Some subclasses of starlike functions obtained for different Choices of $\psi(z)$

1.2 Concept of differential subordination

In very simple terms, a differential subordination in the complex plane is the generalization of a differential inequality on the real line. Obtaining information about the properties of a function from properties of its derivatives plays an important role in the functions of a real variable. Let us directly compare the following two results:

If $f'(x) > 0$, then f is an increasing function.

A simple analog of this is the Noshiro-Warschawski theorem: if f is analytic in the unit disk $\mathbb{D}$, then

$\Re f'(z) > 0$ implies f is univalent in $\mathbb{D}$.

For further interest in differential subordination implication, we strongly refer to read the book *"Miller, S.S., Mocanu, P.T.: Differential subordinations, Monographs and Textbooks in Pure and Applied Mathematics. New York: Marcel Dekker, Inc, (2000)."* Now we only require to state the following concept, which shall use as tool in our results:

Definition 1.2.1. (Differential Subordination) [110] Let $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and let h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfy the (second order) differential subordination

$$\phi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.2.1}$$

then p is called a solution of the differential subordination. An analytic function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.2.1). A univalent dominant $\tilde{q}$ that satisfy $\tilde{q} \prec q$ for all dominant q of (1.2.1) is called the best dominant. Note that the best dominant is unique up to the rotation of $\mathbb{D}$. For more information on differential subordination see [110].

Definition 1.2.2. [110] Let Q be the set of functions q that are analytic and injective on $\overline{\mathbb{D}} \backslash \mathbf{E}(q)$, where

$$\mathbf{E}(q) = \{\zeta \in \partial\mathbb{D} : \lim_{z \to \zeta} = \infty\} \tag{1.2.2}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \backslash \mathbf{E}(q)$.

Definition 1.2.3. [110] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$, consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\psi(r, s, t; z) \notin \Omega$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\Re \dfrac{t}{s} + 1 \geq m \Re \left[\dfrac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right]$, $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D} \backslash \mathbf{E}(q)$ and $m \geq n$.

To study differential subordination implications using the technique of admissible function so that the implication

$$\{\Psi(p(z), zp'(z), z^2p''(z); z) | z \in \mathbb{D}\} \prec h(z) \Rightarrow p(z) \prec q(z) \tag{1.2.3}$$

holds, one needs to follow Definition 1.2.2 and Definition 1.2.3. Here we see our interest in the class

$$\mathscr{S}^*_{\wp} := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec 1 + ze^z =: \wp(z) \right\},$$

where $\wp$ maps the unit disk onto a cardioid domain, since the function $\wp$ does not follow Definition 1.2.2. This makes this class important for our study not only in view of differential subordination but in other aspects also. In this thesis, we use the concept of differential subordination implications as a technique for establishing our results, for instance in majorization problems.

1.3 Synopsis of the Thesis

Univalent function theory is the classical part of complex analysis, it falls under Geometric function theory as it deals with the geometrical aspects of the image domain $f(\mathbb{D})$ of analytic functions f defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In fact, it begins with the sharp estimation of the second coefficient bound of the univalent functions f, given by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. The brief history of $|a_2|$ estimation began with Bieberbach's work and its application in Koebe's famous $1/4$-th Quarter theorem [80]. In an attempt to get the bounds for other coefficients during the second decade of 20^{th} century, many coefficients functional like $a_2^2 - a_3$ were considered, which yields $|a_3| \leq 5$, a crude estimate of the coefficient for univalent functions. Now the sharp estimates of the coefficients like $|a_n| \leq n$ are well-known for the univalent functions. In fact, the sharp bounds of coefficients are yet to be known for several subclasses of starlike functions. However, the sharp bounds for the initial coefficients a_2, a_3, a_4 for functions in $\mathscr{S}^*(\psi)$, the Ma-Minda class of starlike functions, defined below, are now known and derived in the recent years and the estimate for a_5, subject to certain conditions, is obtained recently in [56]. Further, the importance of such coefficient bounds has been noticed recently in the concept of Bohr's radius problems, see [34, 111, 115]. Moreover, the need of estimations of such coefficients and other coefficients functional like $a_2^2 - \beta a_3$ can be found in higher-order Hankel determinants, see the recent survey article titled *"A survey on coefficient estimates for Carathéodory functions"* by N. E. Cho et al. [40]. Let $\mathscr{P}$ be the class of analytic Carathéodory functions defined by $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, such that $\Re p(z) > 0$ in $\mathbb{D}$. Many subclasses of univalent starlike functions were studied in the past, associated with the interesting Carathéodory regions represented by the Carathéodory functions $\sqrt{1+z}$, $1 + \sin z$, $z + \sqrt{1+z^2}$, e^z and $2/(1+e^{-z})$, introduced and studied by Sokół and Stankiewicz [161], Kumar et al. [38], Raina et al. [134], Mendiratta et al. [109], and Goel and Kumar [60] respectively. For $-1 \leq E < D \leq 1$, let $\mathscr{S}^*[D,E] := \mathscr{S}^*((1+Dz)/(1+Ez))$ be the class of Janowski starlike functions. Recently, coefficient problems are considered for the general class $\mathscr{S}^*(\psi)$, which includes the estimation of well-known Zalcman functional, Fekete-Szegö and Hankel determinant bounds, involving the successive coefficients of functions in the desired class. Further, the coefficient bound estimation eventually helps us to find the radius estimation, for instance, one may observe it in the case of Bohr radius problems. In 1992, Ma and Minda [102] unified various subclasses of starlike and convex univalent functions, which are respectively given by:

$$\mathscr{S}^*(\psi) := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \psi(z) \right\} \text{ and } \mathscr{C}(\psi) := \left\{ f \in \mathscr{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) \right\},$$

where $\psi \in \mathscr{P}$ is univalent and $\psi(\mathbb{D})$ symmetric about real axis and starlike with respect to 1. The symbol $\prec$ denotes the usual subordination. Note that, for different choices of ψ, we obtain different subclasses of $\mathscr{S}^*$. For instance if $\psi(z) := (1+z)/(1-z), (1+(1+(1-2\alpha))z)/(1-z)$ and $((1+z)/(1-z))^\gamma$, then the class $\mathscr{S}^*(\psi)$ reduces respectively to the class of starlike functions $\mathscr{S}^*$, Robertson's [140] class $\mathscr{S}^*(\alpha)$ of starlike functions of order α and Stankiewicz's [162] class $\mathscr{S}\mathscr{S}^*(\gamma)$ of strongly starlike functions of order γ, where $0 \leq \alpha < 1$ and $0 < \gamma \leq 1$. Note that $\mathscr{S}^*(0) = \mathscr{S}\mathscr{S}^*(1) = \mathscr{S}^*$.

Now we brief what is Radius problem: "*To find the maximal radius $r < 1$ so that a function f has the property P in $\mathbb{D}_r$ or $f(\mathbb{D}_r)$ has the property P*" is known as the radius problem. For instance, let f be a starlike function, which is not convex in $\mathbb{D}$. Then the maximal radius r for which $f(\mathbb{D}_r)$ is convex, is known as the radius of convexity for such starlike functions.

Let f and g be the analytic functions in a domain $D \subset \mathbb{C}$. The function f is said to be *majorized* by g, provided $|f(z)| \leq |g(z)|$ for each z in D. In 1967, Macgregor [104] discussed the concept of majorization for the univalent functions and derived the following worthy result:

Theorem B. (Macgregor, [104]) Suppose f and g be analytic in $|z| < 1$. Let $g(0) = 0$ and f be majorized by g in $|z| < 1$. Then

$(A.)$ If $0 \leq r \leq \sqrt{2} - 1$, then $\max_{|z|=r} |f'(z)| \leq \max_{|z|=r} |g'(z)|$.

$(B.)$ If g is univalent for $|z| < 1$, then f' is majorized by g' in $|z| \leq 2 - \sqrt{3}$.

$(C.)$ If g maps one-to-one onto a convex domain, then f' is majorized by g' in $|z| \leq 1/3$.

Further, he [104] deduced coefficients bounds for analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ majorized by the analytic function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $|z| < 1$, where g is either univalent or spiral-like or convex function. In the present work, we deal with the generalization of the above Theorem B.

Other interesting problem for the analytic functions is due to Bohr [34], who proved in 1914 that analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $|f(z)| < 1$, satisfies the inequality

$$\sum_{n=0}^{\infty} |a_n||z|^n < 1, \tag{1.3.1}$$

which holds for $|z| < 1/3$, where $1/3$ is best possible constant. The inequality (1.3.1) is known as Bohr's sum or sometimes classical Bohr's inequality. This inequality has attracted a lot of attention in recent times. For such functions, Rogosinski gave the partial sum's inequality, that is

$$\sum_{n=0}^{N-1} |a_n||z|^n < 1,$$

which holds for $|z| < 1/2$. The constant $1/2$ is the best possible. Let us now write an important Bohr-

Rogosinski inequality for analytic functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$, see Kayumov et al. [76] and Muhanna [111], which is described as follows:

$$|f(z)|+\sum_{n=N}^{\infty}|a_n||z|^n\leq |f(0)|+d(f(0),f(\mathbb{D})),$$

where d here refers to the distance. Let $g(z)=\sum_{n=1}^{\infty}b_nz^n$ and $S(f)=\{g:g\prec f\}$, the families of analytic subordinates of f, where f is univalent, it is proved that:

Theorem C. (Kayumov et al., [76]) Assume that f and g are analytic in $\mathbb{D}$ such that f is univalent in $\mathbb{D}$ and $g\in S(f)$. Then for each $m,N\in\mathbb{N}$, the inequality

$$|g(z^m)|+\sum_{n=N}^{\infty}|b_n||z|^n\leq |f(0)|+d(f(0),f(\mathbb{D})) \tag{1.3.2}$$

holds for $|z|\leq r^f_{m,N}$, where $r^f_{m,N}$ is the positive root of the following equation

$$4r^m-(1-r^m)^2+4r^N(N(1-r)+r)\left(\frac{1-r^m}{1-r}\right)^2=0.$$

The radius $r^f_{m,N}$ is sharp due to Koebe function $z/(1-z)^2$.

In the present thesis, we investigate several radius results such as Bohr, Rogosinski and Bohr-Rogosinki radius involving the Ma-Minda classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$, as well as some other subclasses of univalent functions. Radius problems for special functions are also an active area of research. For the normalized form of certain special functions, the radius of starlikeness and convexity are known, for special function's related properties, see [21–23,25,27–29]. However, $\mathcal{S}^*(\psi)$-radius and $\mathcal{C}(\psi)$-radius for special functions are still open, we deal them in Chapter 6. Further, several types of radius problems were also discussed in the past, see [58, Chapter 13] and [49, Chapter 6, Section 6.4 Majorization, pg. 202]. However, their generalization for the Ma-Minda classes of univalent starlike and convex functions, yet not established. Motivated by these observations, we further study extensively the concepts related to the radius problems especially for the general classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$, which finally made this thesis end up with nice and significant collections of radius problems for $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$. The findings of this thesis are either new or provide new insight into the literature. We now brief in the following, the chapter-wise description, skipping the first introductory chapter, which essentially deals with basic definitions, notations, and notions needed for the study.

Chapter 2

In this chapter, titled "*A Cardioid Domain and Starlike functions*", we introduce and study a new class of starlike functions given by

$$\mathcal{S}^*_{\wp}:=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec 1+ze^z=:\wp(z)\right\},$$

where $\wp$ maps the unit disk onto a cardioid domain. We find the radius of convexity of $\wp(z)$ and establish the inclusion relations between the class $\mathscr{S}^*_{\wp}$ and some well-known classes. Further, we derive sharp radius constants and coefficient-related results for the class $\mathscr{S}^*_{\wp}$. We enlist below a few results pertaining to this chapter:

1. The $\mathscr{S}^*_{\wp,n}$-radius of the class $\mathscr{S}^*_n[D,E]$ is given by

   (i) $R_{\mathscr{S}^*_{\wp,n}}(\mathscr{S}^*_n[D,E]) = \min\left\{1, \left(\frac{1/e}{D-(1-1/e)E}\right)^{\frac{1}{n}}\right\}$, when $0 \leq D < E \leq 1$.

   (ii) $R_{\mathscr{S}^*_{\wp,n}}(\mathscr{S}^*_n[D,E]) = \begin{cases} R_1, & \text{if } R_1 \leq r_1 \\ R_2, & \text{if } R_1 > r_1 \end{cases}$ when $-1 \leq E < 0 \leq \leq 1$,

   where

   $$R_1 := R_{\mathscr{S}^*_{\wp,n}}(\mathscr{S}^*_n[D,E]) \text{ as defined in part (i)},$$

   $$R_2 = \min\{1, (e/(D-(e+1)E))^{1/n}\}$$

   and

   $$r_1 = \left(\frac{(e^2-1)/2e}{((e^2+2e-1)/2e)E^2 - DE}\right)^{1/2n}.$$

   In particular, for the class $\mathscr{S}^*$, we have $R_{\mathscr{S}^*_{\wp}}(\mathscr{S}^*) = 1/(2e-1)$.

2. Let $f \in \mathscr{S}^*_{\wp}$. Then the following necessary condition for the 3-fold and 2-fold symmetric functions, respectively hold:

   (i) $\hat{f} \in \mathscr{S}^{*(3)}_{\wp}$ implies that $|H_3(1)| \leq 1/9$.

   (ii) $\hat{f} \in \mathscr{S}^{*(2)}_{\wp}$ implies that $|H_3(1)| \leq 1/16$.

   The above bounds are sharp.

Several inclusion results are nicely depicted in a graph. All the results of this chapter are sharp and some are kept as open problems.

Chapter 3

In this chapter, titled "*Classes of Analytic functions associated with univalent functions*", motivated by the class of analytic functions associated with the functions, namely Booth Lemniscate and Cissoid of Diocles, we introduce and study a new class of analytic functions given by:

$$\mathscr{F}(\psi) := \left\{ f \in \mathscr{A} : \left(\frac{zf'(z)}{f(z)} - 1\right) \prec \psi(z),\ \psi(0) = 0 \right\},$$

where ψ is univalent. We establish the growth and covering theorems with some geometric conditions on ψ, which are as follows:

1. (Growth Theorem) Suppose that $\max_{|z|=r} \Re\psi(z) = \psi(r)$ and $\min_{|z|=r} \Re\psi(z) = \psi(-r)$. Then the

function $f \in \mathscr{F}(\psi)$ satisfies the sharp inequalities:

$$r\exp\left(\int_0^r \frac{\psi(-t)}{t}dt\right) \leq |f(z)| \leq r\exp\left(\int_0^r \frac{\psi(t)}{t}dt\right), \quad (|z| = r).$$

2. (Covering Theorem) If $f \in \mathscr{F}(\psi)$, then either f is a rotation of f_0 or

$$\{w \in \mathbb{C} : |w| \leq -f_0(-1)\} \subset f(\mathbb{D}),$$

where $-f_0(-1) = \lim_{r\to 1}(-f_0(-r))$ and f_0 is given by

$$f_0(z) = z\exp\int_0^z \frac{\psi(t)}{t}dt.$$

Note that the functions in $\mathscr{F}(\psi)$ need not be univalent. As an application, we obtain the growth theorem for the complete range of α and β for the functions in the classes $\mathscr{BS}(\alpha) := \{f \in \mathscr{A} : (zf'(z)/f(z)) - 1 \prec z/(1-\alpha z^2),\ \alpha \in [0,1)\}$ and $\mathscr{S}_{cs}(\beta) := \{f \in \mathscr{A} : (zf'(z)/f(z)) - 1 \prec z/((1-z)(1+\beta z)),\ \beta \in [0,1)\}$, respectively which improves the earlier known bounds. The sharp Bohr-radii for the classes given by $S(\mathscr{BS}(\alpha)) = \{g : g \prec f, g(z) = \sum_{k=1}^{\infty} b_k z^k$ and $f \in \mathscr{BS}(\alpha)\}$ and $\mathscr{BS}(\alpha)$ are also obtained.

All the results obtained here are more general as well as sharp.

Chapter 4

In this chapter, titled "*Some general results for the Ma-Minda classes*", motivated by the well-known work of MacGregor, Campbell and Szegö, we consider functions analytic in the unit disk that are subordinate to functions of the same type that are defined by certain differential subordinations. We prove several sharp majorization theorems, a product theorem, convolution conditions for necessary and sufficient conditions, and some radius problems related to the Ma-Minda classes. Some of the interesting results we obtained on majorization are given below:

1. Let $\Re\phi(z) > 0$ and ϕ be convex in $\mathbb{D}$ with $\phi(0) = 1$. Suppose ψ be the function such that $m_r := \min_{|z|=r}|\psi(z)|$ and also satisfies the differential equation

$$\psi(z) + \frac{z\psi'(z)}{\psi(z)} = \phi(z).$$

Let $g \in \mathscr{A}$ and $f \in \mathscr{C}(\phi)$. If g is majorized by f in $\mathbb{D}$, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq r_\psi$, where r_ψ is the least positive root of the equation

$$(1-r^2)m_r - 2r = 0.$$

The result is sharp for the case $m_r = \psi(-r)$.

2. Let ϕ be convex in $\mathbb{D}$, with $\Re\phi(z) > 0$, $\phi(0) = 1$ and suppose $f \in \mathscr{A}$ satisfies the differential

subordination

$$\frac{zf'(z)}{f(z)} + z\left(\frac{zf'(z)}{f(z)}\right)' \prec \phi(z).$$

If g is majorized by f, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq r_0$, where r_0 is the least positive root of the equation

$$(1-r^2)\min_{|z|=r} \Re\psi(z) - 2r = 0,$$

where

$$\psi(z) := \frac{1}{z}\int_0^z \phi(t)dt.$$

The result is sharp for the case $\min_{|z|=r} \Re\psi(z) = \psi(\pm r)$.

The contributions of this chapter are mostly the generalization of well-known results or new establishments in the above-mentioned direction.

Chapter 5

This chapter titled "*Bohr and Rogosinski phenomenon for certain classes of univalent functions* ", deals with a popular radius problem generally known as Bohr's phenomenon, which has been studied in various settings, however little is known about Rogosinski radius. For a fixed $f \in \mathscr{S}^*(\psi)$ or $\mathscr{C}(\psi)$, the class of analytic subordinants $S_f(\psi) := \{g : g \prec f\}$ is studied for the Bohr-Rogosinski phenomenon. It's applications to the classes $\mathscr{S}^*(\psi)$ and $\mathscr{C}(\psi)$ are also shown. The phenomenon is also studied for a class of harmonic functions to answer the problem for certain classes of univalent functions with negative coefficients. We conclude the chapter with a natural generalization of the Bohr-Rogosinski sum for general Ma-Minda classes. Some of the important results are as follows:

1. Let $f_0(z)$ be given by the relation $zf'(z)/f(z) = \psi(z)$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}^*(\psi)$. Assume $f_0(z) = z + \sum_{n=2}^{\infty} t_n z^n$ and $\hat{f}_0(r) = r + \sum_{n=2}^{\infty} |t_n| r^n$. If $g \in S_f(\psi)$. Then

   $$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \partial\Omega)$$

   holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the unique positive root of the equation:

   $$\hat{f}_0(r^m) + \hat{f}_0(r) - p_{\hat{f}_0}(r) = -f_0(-1),$$

   where

   $$p_{\hat{f}_0}(r) = \begin{cases} 0, & N = 1; \\ r, & N = 2 \\ r + \sum_{n=2}^{N-1} |t_n| r^n, & N \geq 3 \end{cases}$$

   The result is sharp when $r_b = r_0$ and $t_n > 0$.

2. Let $\{\phi_n(r)\}_{n=1}^{\infty}$ be a non-negative sequence of continuous functions in $[0,1]$ such that the series

$$\phi_1(r)+\sum_{n=2}^{\infty}\left|\frac{f_0^{(n)}(0)}{n!}\right|\phi_n(r)$$

converges locally uniformly with respect to each $r\in[0,1)$. If

$$\beta|f'(z^m)|+(1-\beta)|f(z^m)|+\phi_1(r)+\sum_{n=2}^{\infty}\left|\frac{f_0^{(n)}(0)}{n!}\right|\phi_n(r)<-f_0(-1)$$

and $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{S}^*(\psi)$. Then

$$\beta|f'(z^m)|+(1-\beta)|f(z^m)|+\sum_{n=1}^{\infty}|a_n|\phi_n(r)\leq d(0,\partial\Omega) \tag{1.3.3}$$

holds for $|z|=r\leq r_b=\min\{1/3,r_0\}$, where $m\in\mathbb{N}$, $\Omega=f(\mathbb{D})$ and r_0 is the smallest positive root of the equation:

$$\beta f_0'(r^m)+(1-\beta)f_0(r^m)+\sum_{n=2}^{\infty}\left|\frac{f_0^{(n)}(0)}{n!}\right|\phi_n(r)=-f_0(-1)-\phi_1(r),$$

where

$$f_0(z)=z\exp\int_0^z\frac{\psi(t)-1}{t}dt.$$

Moreover, the inequality (1.3.3) also holds for the class $S_f(\psi)$ in $|z|=r\leq r_b$. When $r_b=r_0$, then the radius is best possible.

The concept of the Bohr-phenomenon has been studied explicitly in this chapter for the Ma-Minda classes. Also, certain fundamental inequalities are established involving subordination.

Chapter 6

This chapter titled "*$\mathscr{S}^*(\psi)$ and $\mathscr{C}(\psi)$- radii of Special functions* ", deals with the radius problem explaining the geometric properties of the normalized forms of some special functions, which has been of special interest among the Geometric function theorist. Further, we show that the classes $\mathscr{S}^*(1+\alpha z)$ and $\mathscr{C}(1+\alpha z)$, $0<\alpha\leq1$ solve the problem of finding the sharp $\mathscr{S}^*(\psi)$-radii and $\mathscr{C}(\psi)$-radii for some normalized special functions, whenever $\psi(-1)=1-\alpha$. Radius of strongly starlikeness is also considered. One of our radius results is as follows: Let

$$\Phi(\kappa,\delta,z)=\sum_{n\geq0}\frac{z^n}{n!\Gamma(n\kappa+\delta)},$$

called as Wright function, where $\kappa > -1$ and $z, \delta \in \mathbb{C}$. The function Φ is entire for $\kappa > -1$. Consider the following normalized Wright functions:

$$\begin{aligned} f_{\kappa,\delta}(z) &= \left[z^{\delta}\Gamma(\delta)\Phi(\kappa,\delta,-z^2)\right]^{1/\delta} \\ g_{\kappa,\delta}(z) &= z\Gamma(\delta)\Phi(\kappa,\delta,-z^2) \\ h_{\kappa,\delta}(z) &= z\Gamma(\delta)\Phi(\kappa,\delta,-z). \end{aligned}$$

For simplicity, we write $W_{\kappa,\delta}(z) := \Phi(\kappa,\delta,-z^2)$. We denote $\mathscr{S}^*(\psi)$-radius by $R[\mathscr{S}^*(\psi)]$.

1. Let $\kappa, \delta > 0$ and $\alpha \in (0,1]$ such that the largest disk $\{w : |w-1| < \alpha\} \subseteq \psi(\mathbb{D})$. Then

$$R[\mathscr{S}^*(\psi)] = R[\mathscr{S}^*(1+\alpha z)],$$

   where $\psi(-1) = 1-\alpha$ and $\mathscr{S}^*(1+\alpha z)$-radii for the functions $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ are the smallest positive roots of the following equations respectively:

   (i) $rW'_{\kappa,\delta}(r) + \delta\alpha W_{\kappa,\delta}(r) = 0;$

   (ii) $rW'_{\kappa,\delta}(r) + \alpha W_{\kappa,\delta}(r) = 0;$

   (iii) $\sqrt{r}W'_{\kappa,\delta}(\sqrt{r}) + 2\alpha W_{\kappa,\delta}(\sqrt{r}) = 0.$

2. Let $\kappa, \delta > 0$ and $\alpha \in (0,1]$ such that the largest disk $\{w : |w-1| < \alpha\} \subseteq \psi(\mathbb{D})$. Then

$$R[\mathscr{C}(\psi)] = R[\mathscr{C}(1+\alpha z)],$$

   where $\psi(-1) = 1-\alpha$ and $R[\mathscr{C}(1+\alpha z)]$ for the functions $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ are the smallest positive roots of the following equations respectively:

   (i) $\frac{r\Psi''_{\kappa,\delta}(r)}{\Psi'_{\kappa,\delta}(r)} + \left(\frac{1}{\delta}-1\right)\frac{r\Psi'_{\kappa,\delta}(r)}{\Psi_{\kappa,\delta}(r)} + \alpha = 0;$

   (ii) $rg''_{\kappa,\delta}(r) + \alpha g'_{\kappa,\delta}(r) = 0;$

   (iii) $rh''_{\kappa,\delta}(z) + \alpha h'_{\kappa,\delta}(r) = 0,$

   where $\Psi_{\kappa,\delta}(z) = z^{\delta}\Phi(\kappa,\delta,-z^2)$.

The sharp $\mathscr{S}^*(\psi)$ and $\mathscr{C}(\psi)$-radii of the normalized form of Wright functions, Lommel functions, Struve functions and Legendre polynomials of odd degree, etc. are established in the chapter. The results not only generalize the known work, but also provide a simple proof, which in turn reduces a lot of calculations.

Further, we have provided a concluding remark as well as the future scope of the study in the thesis.

1.4 List of Publications

The above chapters discuss the work from the following articles:

1. S. Sivaprasad Kumar and Kamaljeet Gangania, A cardioid domain and starlike functions, Anal. Math. Phys., 11, no. 2, 54 (2021). (**SCIE, IF-1.1705**)

2. Kamaljeet Gangania and S. Sivaprasad Kumar, On Geometrical Properties of Certain Analytic functions, Iran. J. Sci. Technol. Trans. A Sci., 45, 1437-1445 (2021). (**SCIE, IF-1.1553**)

3. Kamaljeet Gangania and S. Sivaprasad Kumar, On Certain Generalizations of $\mathscr{S}^*(\psi)$, Comput. Methods Funct. Theory, 22, 215–227 (2022). (**SCIE, IF-1.155**)

4. Kamaljeet Gangania and S. Sivaprasad Kumar, Bohr-Rogosinski phenomenon for $\mathscr{S}^*(\psi)$ and $\mathscr{C}(\psi)$, Mediterr. J. Math., 19, 161 (2022). (**SCIE, IF-1.4**)

5. Kamaljeet Gangania and S. Sivaprasad Kumar, Bohr radius for some classes of Harmonic mappings, Iran. J. Sci. Technol. Trans. A Sci., 46, 883-890 (2022). (**SCIE, IF-1.1553**)

6. Kamaljeet Gangania and S. Sivaprasad Kumar, $\mathscr{S}^*(\psi)$ and $\mathscr{C}(\psi)$-radii for some special functions, Iran. J. Sci. Technol. Trans. A Sci., 46, 955-966 (2022). (**SCIE, IF-1.1553**)

7. Kamaljeet Gangania and S. Sivaprasad Kumar, A generalized Bohr-Rogosinski phenomenon, Iran. J. Sci. Technol. Trans. A Sci. (2022) (**SCIE, IF-1.1553**) arxiv.org/abs/2202.02560 (**accepted**)

8. Kamaljeet Gangania and S. Sivaprasad Kumar, Certain Radii problems for $\mathscr{S}^*(\psi)$ and Special functions, Mathematica Slovaca (To appear) (2023). arxiv.org/abs/2007.07816v2, (**SCIE, IF-0.996**)

9. Kamaljeet Gangania and S. Sivaprasad Kumar, On Certain Generalizations of $\mathscr{S}^*(\psi) : II$, arxiv.org/abs/2106.04962 (communicated)

Chapter 2

A Cardioid Domain and Starlike Functions

We introduce a new class of starlike functions defined by

$$\mathcal{S}^*_{\wp} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + ze^z =: \wp(z) \right\},$$

where $\wp$ maps the unit disk onto a cardioid domain. We find the radius of convexity of $\wp(z)$ and establish the inclusion relations between the class $\mathcal{S}^*_{\wp}$ and some well-known classes. Further, we derive sharp radius constants and coefficient-related results for the class $\mathcal{S}^*_{\wp}$.

2.1 Introduction

The investigation of properties of starlike and convex functions emerged soon after the Bieberbach conjecture on univalent functions. Goodman [59] and Ronning [143] started investigating the properties like uniform starlikeness and uniformly convexity. Then the paper by Sokół [158] and, Kanas and Wiśniowska [73] gave insights into subclasses of starlike and convex functions associated with the Lemniscate of Bernoulli and Conic domains, respectively. But in 1992, a unified treatment of subclasses of starlike and convex functions was given by Ma and Minda [102]. Since then several fascinating articles in view of the Ma-Minda class appeared concerning several types of radius and coefficient problems.

Let $\mathcal{A}_n$ be the class of analytic functions of the form $f(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$. Let $\mathcal{A} := \mathcal{A}_1$ and $\mathcal{S}$ be the subclass of $\mathscr{A}$ consisting of univalent functions f having the following power series expansion:

$$f(z) = z + b_2 z^2 + b_3 z^3 + \cdots . \tag{2.1.1}$$

In view of Ma and Minda classes defined in (1.3), we also consider the class of starlike functions related to the contracted cardioid regions represented by the function $\wp_\alpha(z) = 1 + \alpha z e^z$, $0 < \alpha \leq 1$. More precisely,

$$\mathscr{S}^*(\wp, \alpha) := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec 1 + \alpha z e^z \right\} \quad (z \in \mathbb{D}).$$

Observe that for $0 < \alpha < \beta \leq 1$, $\wp_\alpha(\mathbb{D}) \subset \wp_\beta(\mathbb{D})$. Let $\wp(z) := \wp_1(z)$. Note that the function $\wp$ does not satisfy the Definition 1.2.2 and thus it is an independent interesting case study for the differential subordination implication related to this function itself using admissible conditions in view of Definition 1.2.3. In contrast to the literature, the image domain $\wp(\mathbb{D})$ is not convex having a cusp at $\wp(-1)$, and $\wp$ also has no explicit inverse representation. Motivated by these facts, we study in particular the following class:

$$\mathscr{S}^*_\wp := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \wp(z) \right\} \quad (z \in \mathbb{D}). \tag{2.1.2}$$

A function $f \in \mathscr{S}^*_\wp$ if and only if there exists a function $p \in \mathscr{P}$ and $p \prec \wp$ such that

$$f(z) = z \exp\left(\int_0^z \frac{p(t) - 1}{t} dt \right). \tag{2.1.3}$$

If we take $p(z) = \wp(z)$, then we obtain from (2.1.3) the function

$$f_1(z) := z \exp(e^z - 1) = \sum_{n=0}^{\infty} B_n \frac{z^{n+1}}{n!} = z + z^2 + z^3 + \frac{5}{6} z^4 + \frac{5}{8} z^5 + \frac{13}{30} z^6 + \cdots , \tag{2.1.4}$$

where B_n are the Bell numbers satisfying the recurrence relation given by

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{2.1.5}$$

We now state below the following common results meant for $\mathscr{S}^*_\wp$ using results in [102], by omitting the proof.

Theorem 2.1.1. Let $f \in \mathscr{S}^*_\wp$ and f_1 as defined in (2.1.4). Then

(i) Growth theorem: $-f_1(-|z|) \leq |f(z)| \leq f_1(|z|)$.

(ii) Covering theorem: $\{w : |w| \leq -f_1(-1) \approx 0.5314\} \subset f(\mathbb{D})$.

(iii) Rotation theorem: $|\arg f(z)/z| \leq \max_{|z|=r} \arg(f_1(z)/z)$.

(iv) $f(z)/z \prec f_1(z)/z$ and $|f'(z)| \leq f_1'(|z|)$.

As a consequence of growth theorem, for $|z| = r$, we obtain

$$\log\left|\frac{f(z)}{z}\right| \leq \int_0^r e^t dt \leq \int_0^1 e^t dt = e - 1,$$

which implies $|f(z)| \leq e^{e-1}$ and the bound can not be further improved as $z\exp(e^z-1)$ acts as an extremal function.

The geometric properties of the cardioid domain $\wp(\mathbb{D})$ and the inclusion relationship between $\mathcal{S}^*_{\wp}$ and the classes $\mathcal{S}\mathcal{S}^*(\gamma)$, $\mathcal{S}^*(\alpha)$ and many more are examined in the present chapter. Various sharp radius results associated with $\mathcal{S}^*_{\wp}$ are also obtained. Further, we find the coefficient estimates for $f \in \mathcal{S}^*_{\wp}$ and the sharp bound for the first five coefficients. Conjecture related to the sharp bound of n^{th} coefficient is also posed. Using the expression of the carathéodory coefficient p_4 in terms of p_1, where the technique has not been used much so far, we also obtain the estimate for the third Hankel determinant for the class $\mathcal{S}^*_{\wp}$. For the classes of two-fold and three-fold symmetric functions associated with $\mathcal{S}^*_{\wp}$, sharp estimates are also obtained for the third Hankel determinant. In addition, problems related to coefficients are also discussed.

2.2 Properties of cardioid domain

Since $1+ze^z$ maps $\mathbb{D}$ onto a starlike domain, our first result aims in finding the radius of convexity of the same:

Theorem 2.2.1. The radius of convexity of the function $\wp(z) = 1+ze^z$ is the smallest positive root of the equation $r^3-4r^2+4r-1=0$, which is given by

$$r_c = (3-\sqrt{5})/2 \approx 0.381966.$$

Proof. Now it is to find the constant $r_c \in (0,1]$ so that

$$\Re\left(1+\frac{z\wp''(z)}{\wp'(z)}\right) > 0 \quad (|z| < r_c). \tag{2.2.1}$$

Since

$$\Re\left(1+\frac{z\wp''(z)}{\wp'(z)}\right) = \frac{r^3\cos\theta + r^2(3+\cos 2\theta)+4r\cos\theta+1}{1+2r\cos\theta+r^2} =: g(r,\theta)$$

and g is symmetric about the real axis as $g(r,\theta) = g(r,-\theta)$. Thus we only need to consider $\theta \in [0,\pi]$. Further, we have $1+2\cos\theta r + r^2 > 0$ for $r \in (0,1)$ and $\theta \in [0,\pi])$. So we may consider the numerator of $g(r,\theta)$ as

$$g_N(r,\theta) := r^3\cos\theta + r^2(3+\cos 2\theta)+4r\cos\theta+1.$$

Now to arrive at (2.2.1), we only need to show

$$g_N(r,\theta) > 0 \quad (r \in (0,r_c)). \tag{2.2.2}$$

It is evident that for any fixed $r = r_0$, $g_N(r_0,\theta)$ attains its minimum at $\theta = \pi$, which is given by $g_N(r_0,\pi) = -r_0^3+4r_0^2-4r_0+1$. Since $g_N(0,\pi) > 0$ and if r_c is the least positive root of $r^3-4r^2+4r-1=0$ then (2.2.2) follows. □

Using elementary calculus, one can easily find the following sharp bounds that are associated with the function $\wp(z) = 1 + ze^z$, which are used extensively in obtaining our subsequent results.

Lemma 2.2.1. (***Function Bounds***) Let $\wp_R(\theta)$ and $\wp_I(\theta)$ denote the real and imaginary parts of $\wp(e^{i\theta})$, respectively.

(i) Let $\wp_R(\theta) = 1 + e^{\cos\theta}\cos(\theta + \sin\theta)$, then $\wp_R(\theta_0) \leq \Re\wp(z) \leq 1 + e$ where $\theta_0 \approx 1.43396$ is the solution of $3\theta/2 + \sin\theta = \pi$ and $\wp_R(\theta_0) \approx 0.136038$.

(ii) Let $\wp_I(\theta) = e^{\cos\theta}\sin(\theta + \sin\theta)$, then $|\operatorname{Im}\wp(z)| \leq \wp_I(\theta_0)$, where $\theta_0 \approx 0.645913$ is the solution of $3\theta/2 + \sin\theta = \pi/2$ and $\wp_I(\theta_0) \approx 2.10743$.

(iii) $|\arg \wp(z)| = |\arctan(\wp_I(\theta)/\wp_R(\theta))| \leq (0.89782)\pi/2$.

(iv) $|\wp(z)| \leq 1 + re^r$, whenever $|z| = r < 1$.

The bounds are the best possible.

The following lemma aims at finding the largest (or smallest) disk centered at the sliding point $(a, 0)$ inside (or containing) the cardioid domain $\wp(\mathbb{D})$.

Lemma 2.2.2. Let $\wp(z) = 1 + ze^z$. Then we have

1. $\{w : |w - a| < r_a\} \subset \wp(\mathbb{D})$, where

$$r_a = \begin{cases} (a-1) + 1/e, & 1 - 1/e < a \leq 1 + (e - e^{-1})/2; \\ e - (a-1), & 1 + (e - e^{-1})/2 \leq a < 1 + e. \end{cases}$$

2. $\wp(\mathbb{D}) \subset \{w : |w - a| < R_a\}$, where

$$R_a = \begin{cases} 1 + e - a, & 1 - 1/e < a \leq (e + e^{-1})/2 \\ \sqrt{d(\theta_a)}, & (e + e^{-1})/2 < a < 1 + e. \end{cases}$$

where $\theta_a \in (0, \pi)$, is the root of the following equation:

$$\sin(\theta/2) + (1 - a)\sin(3\theta/2 + \sin\theta) = 0. \tag{2.2.3}$$

Proof. We begin with the first part. The curve $\wp(e^{i\theta}) = \wp_R(\theta) + i\wp_I(\theta) = 1 + e^{\cos\theta}(\cos(\theta + \sin\theta) + i\sin(\theta + \sin\theta))$ represents the boundary of $\wp(\mathbb{D})$ and is symmetric about the real axis. So it is enough to consider θ in $[0, \pi]$. Now, square of the distance of $(a, 0)$ from the points on the curve $\wp(e^{i\theta})$ is given by

$$\begin{aligned} d(\theta) &:= (a - 1 - e^{\cos\theta}\cos(\theta + \sin\theta))^2 + e^{2\cos\theta}\sin^2(\theta + \sin\theta) \\ &= e^{2\cos\theta} - 2(a-1)e^{\cos\theta}\cos(\theta + \sin\theta) + (a-1)^2. \end{aligned}$$

Case(i): If $1 - e^{-1} < a \leq (e + e^{-1})/2$, then $d(\theta)$ decreases in $[0, \pi]$. Therefore, we get

$$r_a = \min_{\theta \in [0,\pi]} \sqrt{d(\theta)} = \sqrt{d(\pi)} = (a-1) + 1/e.$$

Now for the range $(e+e^{-1})/2 \leq a < 1+(e-e^{-1})/2$, it is easy to see that the equation

$$d'(\theta) = -4e^{\cos\theta}\cos(\theta/2)(\sin(\theta/2)-(a-1)\sin(3\theta/2+\sin\theta)) = 0$$

has three real roots $0, \theta_a$ and π, where θ_a is the root of the equation given in (2.2.3) and we have $\theta_{a_1} < \theta_{a_2}$ whenever $a_1 < a_2$. Further, we see that $d(\theta)$ increases in $[0, \theta_a]$ and decreases in $[\theta_a, \pi]$. Also,

$$d(\pi) - d(0) = 2(e+e^{-1})(a-(1+(e-e^{-1})/2)) < 0.$$

Therefore, $\min\{d(0), d(\theta_a), d(\pi)\} = d(\pi)$ and we have

$$r_a = \sqrt{d(\pi)} = (a-1) + 1/e.$$

Case(ii): If $1+(e-e^{-1})/2 \leq a < 1+e$, we see that $d(\theta)$ is an increasing function for $\theta \in [0, \theta_a]$ and decreasing for $\theta \in [\theta_a, \pi]$, where θ_a is the root of the equation defined in (2.2.3). Also,

$$d(\pi) - d(0) = 2(e+e^{-1})(a-(1+(e-e^{-1})/2)) > 0.$$

Therefore, $\min\{d(0), d(\theta_a), d(\pi)\} = d(0)$ and we have

$$r_a = \sqrt{d(0)} = e - (a-1).$$

This completes the proof of the first part. The proof of the second part is much akin to the first part so is skipped here. □

Remark 2.2.1. We obtain the largest disk $D_L := |w-a| < r_a$ contained in $\wp(\mathbb{D})$ when $a = 1+(e-e^{-1})/2$ and $r_a = (e+e^{-1})/2$ and the smallest disk $D_S := |w-a| < R_a$, which contains $\wp(\mathbb{D})$ when $a = (e+e^{-1})/2$ and $R_a = 1+(e-e^{-1})/2$. Thus $D_L \subset \wp(\mathbb{D}) \subset D_S$.

Our next result deals with the inclusion relations of the class $\mathcal{S}^*_{\wp}$ involving various classes including the following, see [12, 73, 168]:

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \Re\frac{zf'(z)}{f(z)} < \beta, \quad \beta > 1 \right\},$$

$$k-\mathcal{ST} := \left\{ f \in \mathcal{A} : \Re\frac{zf'(z)}{f(z)} > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad k \geq 0 \right\}$$

and

$$\mathcal{ST}_p(a) := \left\{ f \in \mathcal{A} : \Re\frac{zf'(z)}{f(z)} + a > \left|\frac{zf'(z)}{f(z)} - a\right|, \quad a > 0 \right\}.$$

Theorem 2.2.2. ***Inclusion Relations:***

(i) $\mathcal{S}^*_{\wp} \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^*$ for $0 \leq \alpha \leq \omega_0 \approx 0.136038$.

(ii) $\mathcal{S}^*_{\wp} \subset \mathcal{M}(\beta)$ for $\beta \geq 1+e$ and $\mathcal{S}^*_{\wp} \subset \mathcal{S}^*(\omega_0) \cap M(1+e)$.

(iii) $\mathcal{S}^*_{\wp} \subset \mathcal{SS}^*(\gamma) \subset \mathcal{S}^*$ for $0.897828 \approx \gamma_0 \leq \gamma \leq 1$.

(iv) $\mathcal{S}^*_{\wp} \subset \mathcal{S}\mathcal{T}_p(a)$ for $a \geq b \approx 1.58405$.

(v) $k-\mathcal{S}\mathcal{T} \subset \mathcal{S}^*_{\wp}$ for $k \geq e-1$,

where $\omega_0 = \min \Re\wp(z)$. These best possible inclusion relations are clearly depicted in Figure 6.4.3.

Proof. Proof of (i), (ii) and (iii) directly follows from Lemma 2.2.1. We begin with the proof of part (iv).

(iv) Note that boundary $\partial\Omega_a$ of the domain $\Omega_a = \{w \in \mathbb{C} : \Re w + a > |w-a|\}$ is a parabola. Now $\mathcal{S}^*_{\wp} \subset \mathcal{S}\mathcal{T}_p(a)$, provided $\Re w + a > |w-a|$, where $w = 1 + ze^z$. Upon taking $z = e^{i\theta}$, we have

$$T(\theta) := \frac{e^{2\cos\theta}\sin^2(\theta+\sin\theta)}{4(1+e^{\cos\theta}\cos(\theta+\sin\theta))} < a.$$

Further, $T'(\theta) = 0$ if and only if $\theta \in \{0, \theta_0, \pi\}$, where $\theta_0 \approx 1.23442$ is the unique root of the equation

$$\cos(3\theta/2+\sin\theta)(2+e^{\cos\theta}\cos(\theta+\sin\theta)) + e^{\cos\theta}\cos(\theta/2) = 0 (0 < \theta < \pi).$$

Therefore, $\max_{0\leq\theta\leq\pi} T(\theta) = \max\{T(0), T(\theta_0), T(\pi)\} = T(\theta_0) \approx 1.58405$. Since $\mathcal{S}\mathcal{T}_p(a_1) \subset \mathcal{S}\mathcal{T}_p(a_2)$ for $a_1 < a_2$, it follows that $\mathcal{S}^*_{\wp} \subset \mathcal{S}\mathcal{T}_p(a)$ for $a \geq b \approx 1.58405$.

(v) Let $f \in k-\mathcal{S}\mathcal{T}$ and $\Gamma_k = \{w \in \mathbb{C} : \Re w > k|w-1|\}$. For $k > 1$, the boundary curve $\partial\Gamma_k$ is an ellipse $\gamma_k : x^2 = k^2(x-1)^2 + k^2y^2$ which can be rewritten as

$$\frac{(x-x_0)^2}{u^2} + \frac{(y-y_0)^2}{v^2} = 1,$$

where $x_0 = k^2/(k^2-1)$, $y_0 = 0$, $u = k/(k^2-1)$ and $v = 1/\sqrt{k^2-1}$. Observe that $u > v$. Therefore, for the ellipse γ_k to lie inside $\wp(\mathbb{D})$, we must ensure that $x_0 \in (1-1/e, 1+e)$, which holds for $k \geq \sqrt{(1+e)/e}$ and by Lemma 2.2.2, we have

$$1-1/e \leq x_0 - u \quad \text{and} \quad x_0 + u \leq 1+e,$$

whenever

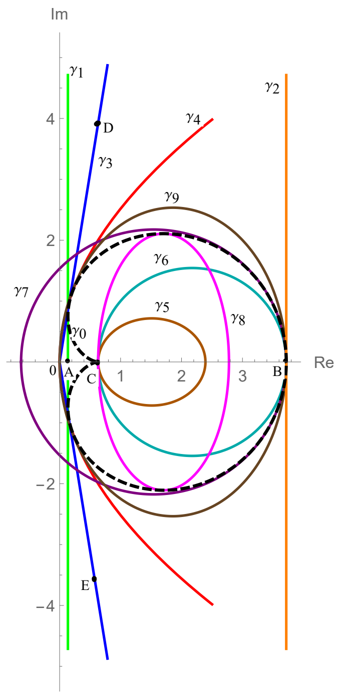
$$k \geq \max\left\{\sqrt{(1+e)/e},\ e-1,\ (1+e)/e\right\} = e-1.$$

Since $\Gamma_{k_1} \subseteq \Gamma_{k_2}$ for $k_1 \geq k_2$, it follows that $k-\mathcal{S}\mathcal{T} \subset \mathcal{S}^*_{\wp}$ for $k \geq e-1$. □

Remark 2.2.2. **Inclusion relation of cardioid with vertical Ellipse:**
Consider the equation $\frac{(x-h)^2}{v^2} + \frac{y^2}{u^2} = 1$, where $x = h + v\cos\theta$, $y = u\sin\theta$ and $\theta \in (0,\pi)$. Case(i) Let $u = \max \operatorname{Im} \wp(z)$ and $v = h - (1-1/e)$, where $h = \Re\wp(z)$, which corresponds to $\max \operatorname{Im} \wp(z) \approx 1.70529$ for the largest vertical ellipse, V_L inside $\wp(\mathbb{D})$. Case(ii) Let $h = v = (1+e)/2$ and $u = e - (1/2e)$ for the smallest vertical ellipse, V_S containing $\wp(\mathbb{D})$.

In view of the Remark 2.2.2 and the class of starlike functions linked to the conic domains considered by Kanas and Wiśniowska [73], we pose as an independent interest the following problem:

Open Problem 1. Find the explicit form of the functions $\Psi \in \mathcal{P}$, which maps $\mathbb{D}$ onto a vertical elliptical domain.



Legend -

$\gamma_0 : \wp(z) = 1 + ze^z$
$\gamma_1 : \Re w = A$
$\gamma_2 : \Re w = 1 + e$
$\gamma_3 : |\arg w| = (0.897828)\pi/2$
$\gamma_4 : |w - 1.58405| - \mathrm{Re}\, w = 1.58405$
$\gamma_5 : \Re w = (e-1)|w-1|$
$\gamma_6 : |w - (1 + \frac{e^2-1}{2e})| = \frac{(e^2+1)}{2e}$
$\gamma_7 : |w - \frac{e^2+1}{2e}| = 1 + \frac{e^2-1}{2e}$
$\gamma_8 : \frac{(\Re w - 1.7052)^2}{(2.1074)^2} + \frac{(\mathrm{Im}\, w)^2}{(1.0731)^2} = 1$
$\gamma_9 : \frac{(2\Re w - B)^2}{B^2} + \frac{4(\mathrm{Im}\, w)^2}{(e+BC)^2} = 1$
$A = \min \Re \wp(z) \approx 0.136038$
$B = 1 + e$
$C = 1 - e^{-1}$
$\arg(D) = -\arg(E) = (0.897828)\pi/2$

Figure 2.1: Boundary curves of best dominants and subordinants of $\wp(z) = 1 + ze^z$.

For our next result, we need the following class and some related results:

$$\mathscr{P}_n[D,E] := \left\{ p(z) = 1 + \sum_{k=n}^{\infty} c_n z^n : p(z) \prec \frac{1+Dz}{1+Ez}, \quad |E| \leq 1, D \neq E \right\},$$

where $n \in \mathbb{N}$, $\mathscr{P}_n(\alpha) := \mathscr{P}_n[1-2\alpha, -1]$ and $\mathscr{P}_n := \mathscr{P}_n(0) \quad (0 \leq \alpha < 1)$.

Lemma 2.2.3. *[149]* If $p \in \mathscr{P}_n(\alpha)$, then for $|z| = r$,

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2(1-\alpha)nr^n}{(1-r^n)(1+(1-2\alpha)r^n)}.$$

Lemma 2.2.4. *[137]* If $p(z) \in \mathscr{P}_n[D,E]$, then for $|z| = r$,

$$\left| p(z) - \frac{1 - DEr^{2n}}{1 - E^2 r^{2n}} \right| \leq \frac{|D-E| r^n}{1 - E^2 r^{2n}}.$$

Particularly, if $p \in \mathscr{P}_n(\alpha)$, then

$$\left| p(z) - \frac{1 + (1-2\alpha) r^{2n}}{1 - r^{2n}} \right| \leq \frac{2(1-\alpha) r^n}{1 - r^{2n}}.$$

Theorem 2.2.3. Let $-1 < E < D \leq 1$. If one of the following two conditions holds.

(i) $2(e-1)(1-E^2) < 2e(1-DE) \leq (e^2+2e-1)(1-E^2)$ and $E - 1 \leq e(1-D)$;

(ii) $(e^2+2e-1)(1-E^2) \leq 2e(1-DE) < 2e(1+e)(1-E^2)$ and $D - E \leq e(1+E)$.

Then $\mathscr{S}^*[D,E] \subset \mathscr{S}^*_{\wp}$.

Proof. If $f \in \mathscr{S}^*[D,E]$, then $zf'(z)/f(z) \in \mathscr{P}[D,E]$. Therefore, by Lemma 2.2.4, we have

$$\left|\frac{zf'(z)}{f(z)} - a\right| \leq \frac{(D-E)}{1-E^2}, \tag{2.2.4}$$

where $a := (1-DE)/(1-E^2)$. Suppose that the conditions in (i) hold. Now multiplying by $(E+1)$ on both sides of the inequality $(E-1) \leq e(1-D)$ and then dividing by $1-E^2$, gives $(D-E)/(1-E^2) \leq a-(1-1/e)$. Similarly, the inequality $2(e-1)(1-E^2) < 2e(1-DE) \leq (e^2+2e-1)(1-E^2)$ is equivalent to $1-1/e < (1-DE)/(1-E^2) \leq 1+(e-e^{-1})/2$. Therefore from (2.2.4), we see that $zf'(z)/f(z) \in \{w \in \mathbb{C} : |w-a| < r_a\}$, where $r_a = a-(1-1/e)$ and $1-1/e < a \leq 1+(e-e^{-1})/2$. Hence, $f \in \mathscr{S}^*_{\wp}$ by Lemma 2.2.2. Similarly, we can show that $f \in \mathscr{S}^*_{\wp}$, if the conditions in (ii) hold. □

2.3 Radius Problems

In this section, we consider several radius problems for $\mathscr{S}^*_{\wp}$. In the following theorem, we find the largest radius $r_\varepsilon < 1$ for which the functions in $\mathscr{S}^*_{\wp}$ are in the desired class when ε is given. We denote the A-radius of the class B by $R_A(B)$.

Theorem 2.3.1. Let $f \in \mathscr{S}^*_{\wp}$. Then

(i) $f \in \mathscr{S}^*(\alpha)$ in $|z| < r_\alpha$, $\alpha \in (\alpha_0, 1)$, where $\alpha_0 = 1 + \frac{\sqrt{5}-3}{2} e^{\frac{\sqrt{5}-3}{2}}$ and $r_\alpha \in (0,1)$ is the smallest root of the equation

$$1 - re^{-r} - \alpha = 0.$$

(ii) $f \in \mathscr{M}(\beta)$ in $|z| < r_\beta$, where

$$r_\beta = \begin{cases} r_0(\beta) & \text{for} \quad 1 < \beta < 1+e \\ 1 & \text{for} \quad \beta \geq 1+e \end{cases}$$

and $r_0(\beta) \in (0,1)$ is the smallest root of $1 + re^r = \beta$.

(iii) $f \in \mathscr{S}\mathscr{S}^*(\gamma)$ in $|z| < r_\gamma$, $\gamma \in (0,1]$, where

$$r_\gamma = \min\{1, r_0(\gamma)\}$$

and $r_0(\gamma) \in (0,1)$ is the smallest root of the following equation:

$$\arcsin\left(\frac{1}{r}\ln\left(\frac{r}{\sin(\gamma\pi/2)}\right)\right) + \sqrt{r^2 + \ln^2\left(\frac{r}{\sin(\gamma\pi/2)}\right)} = \frac{\gamma\pi}{2}. \tag{2.3.1}$$

Proof. Since $zf'(z)/f(z) \prec \wp$, it suffices to consider the cardioid domain $\wp(\mathbb{D})$ so that certain geometry can be performed.

(i) Since $f \in \mathscr{S}^*_{\wp}$, there exists a function $\omega(z) \in \Omega$ such that

$$\frac{zf'(z)}{f(z)} = 1 + \omega(z)e^{\omega(z)}.$$

Since $|\omega(z)| \leq |z|$, we can assume $\omega(z) = Re^{i\theta}$, where $R \leq |z| = r$ and $-\pi \leq \theta \leq \pi$. A calculation shows that

$$|Re^{i\theta}e^{Re^{i\theta}}| = Re^{R\cos\theta} =: T(\theta).$$

Since $T(\theta) = T(-\theta)$, it is sufficient to consider $\theta \in [0, \pi]$. Further, $T'(\theta) \leq 0$ implies

$$|\omega(z)e^{\omega(z)}| = T(\theta) \leq T(0) \leq Re^R \leq re^r.$$

Therefore, we obtain

$$\Re\frac{zf'(z)}{f(z)} \geq 1 - |\omega(z)e^{\omega(z)}| \geq 1 - re^r \geq \alpha,$$

whenever $1 - re^r - \alpha \geq 0$.

(ii) Since $f \in \mathscr{S}^*_{\wp}$, therefore using subordination principle and Lemma 2.2.1, we have

$$\Re\frac{zf'(z)}{f(z)} \leq \Re\wp(\omega(z)) \leq |\wp(\omega(z))| \leq 1 + re^r \quad (|z| = r), \tag{2.3.2}$$

where $\omega \in \Omega$. Thus $f \in \mathscr{M}(\beta)$ in $|z| < r$, whenever $1 + re^r < \beta$.

(iii) Let $f \in \mathscr{S}^*_{\wp}$, then $f \in \mathscr{S}\mathscr{S}^*(\gamma)$ in $|z| < r$ provided

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leq |\arg(\wp(z))| \leq \gamma\pi/2 \quad (|z| = r).$$

Assuming $z = re^{i(\theta+\pi/2)}$,

$$\theta + r\cos\theta = \gamma\pi/2 \text{ and } re^{-r\sin\theta} = \sin(\gamma\pi/2), \tag{2.3.3}$$

we have

$$\begin{aligned}1 + ze^z &= 1 + re^{-r\sin\theta}(-\sin(\theta + r\cos\theta) + i\cos(\theta + r\cos\theta))\\ &= 1 + \sin(\gamma\pi/2)(-\sin(\gamma\pi/2) + i\cos(\gamma\pi/2))\\ &= \cos^2(\gamma\pi/2) + i\sin(\gamma\pi/2)\cos(\gamma\pi/2),\end{aligned}$$

which implies $|\arg(\wp(z))| \leq \gamma\pi/2$. Now we obtain equation (2.3.1) by eliminating θ from the equations given in (2.3.3), a geometrical observation ensures the existence of the unique root for the equation (2.3.1). Thus the result now follows by considering the smallest root of (2.3.1). The result is further sharp as we can find $z_0 = r_0 e^{i(\theta_0+\pi/2)}$ for any fixed γ at which the function f_1, given by (2.1.4) satisfies

$$\begin{aligned}\left|\arg\frac{z_0 f_1'(z_0)}{f_1(z_0)}\right| &= |\arg(1 + ze^z)|_{z=z_0}\\ &= |\arg(\cos^2(\gamma\pi/2) + i\sin(\gamma\pi/2)\cos(\gamma\pi/2))|\\ &= |\arctan(\tan(\gamma\pi/2))|\\ &= \gamma\pi/2.\end{aligned}$$

□

Theorem 2.3.2. Let $f \in \mathscr{S}^*_{\wp}$. Then $f \in \mathscr{C}(\alpha)$ in $|z| < r_\alpha$, where $r_\alpha \in (0,1)$ is the smallest root of the equation

$$(1-r)(1-re^r)(1-re^r-\alpha)-re^r=0. \tag{2.3.4}$$

Proof. If $f \in \mathscr{S}^*_{\wp}$, then there exists a Schwarz function ω such that

$$\frac{zf'(z)}{f(z)} = 1+\omega(z)e^{\omega(z)}.$$

Now a computation yields

$$1+\frac{zf''(z)}{f'(z)} = 1+\omega(z)e^{\omega(z)} + \frac{z\omega'(z)e^{\omega(z)}(1+\omega(z))}{1+\omega(z)e^{\omega(z)}}. \tag{2.3.5}$$

From (2.3.5), we obtain

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) \geq 1+\Re(\omega(z)e^{\omega(z)}) - \frac{|z\omega'(z)||(\omega(z)+1)e^{\omega(z)}|}{1-|\omega(z)e^{\omega(z)}|}. \tag{2.3.6}$$

Since $|\omega(z)| \leq |z|$, we can assume that $\omega(z) = Re^{i\theta}$, where $R \leq |z| = r$ and $-\pi \leq \theta \leq \pi$. Now using triangle inequality together with the Schwarz-Pick inequality:

$$\frac{|\omega'(z)|}{1-|\omega(z)|^2} \leq \frac{1}{1-|z|^2},$$

we have $|z\omega'(z)e^{\omega(z)}(1+\omega(z))| \leq re^r/(1-r)$. Also $|\omega(z)e^{\omega(z)}| \leq Re^R \leq re^r$. Upon using these inequalities in (2.3.6), we get

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) \geq 1-re^r-\frac{re^r}{(1-r)(1-re^r)} \geq \alpha, \tag{2.3.7}$$

and with the least root of (2.3.4), the above inequality (2.3.7) hold and hence the result. □

By taking $\alpha = 0$ in Theorem 2.3.2, we obtain the following result:

Corollary 2.3.3. Let $f \in \mathscr{S}^*_{\wp}$. Then $f \in \mathscr{C}$ whenever $|z| < r_0 \approx 0.256707$.

Remark 2.3.1. Let $\omega(z) = z = re^{i\theta}$ and $\alpha = 0$ in Theorem 2.3.2. Then for the function given by (2.1.4), we have

$$\Re\left(1+\frac{zf_1''(z)}{f_1'(z)}\right) = \Re\left(1+ze^z+\frac{z(1+z)e^z}{1+ze^z}\right) =: F(r,\theta),$$

where

$$F(r,\theta) = -\frac{1+r\cos\theta+R\cos\theta_1+rR\cos(\theta_1-\theta)}{1+2R\cos\theta_1+R^2}+2+r\cos\theta+R\cos\theta_1$$

and

$$z=re^{i\theta}, R=re^{r\cos\theta}, \theta_1=\theta+r\sin\theta.$$

Numerically, we note that for all $\theta \in [0,\pi]$, $F(r,\theta) \geq 0$ whenever $r \leq r_0 \approx 0.599547$, but when θ approaches to π, then $F(r,\theta) < 0$ for $r = r_0 + \varepsilon$, $\varepsilon > 0$. Thus we previse that the sharp radius of convexity for the class $\mathcal{S}^*_{\wp}$ is r_0.

For the next theorems 2.3.4-2.3.7, we need to recall some classes: Let $f \in \mathcal{A}_n$. If we set $p(z) = zf'(z)/f(z)$, then the class $\mathcal{P}_n[D,E]$ reduces to $\mathcal{S}^*_n[D,E]$, the class of Janowski starlike functions and $\mathcal{S}^*_n(\alpha) := \mathcal{S}^*_n[1-2\alpha,-1]$. Further, let

$$\mathcal{S}^*_{\wp,n} := \mathcal{A}_n \cap \mathcal{S}^*_{\wp} \quad \text{and} \quad \mathcal{S}^*_n(\alpha) := \mathcal{A}_n \cap \mathcal{S}^*(\alpha).$$

Ali et al. [9] studied the classes, $\mathcal{F}_n := \{f \in \mathcal{A}_n : f(z)/z \in \mathcal{P}_n\}$, $\mathcal{S}^*_n[D,E]$ and the subclass consisting of close-to-starlike functions of type α given by

$$\mathcal{CS}_n(\alpha) := \left\{ f \in \mathcal{A}_n : \frac{f(z)}{g(z)} \in \mathcal{P}_n,\ g \in \mathcal{S}^*_n(\alpha) \right\}.$$

We find the $\mathcal{S}^*_{\wp,n}$-radii for the classes defined above.

Theorem 2.3.4. The sharp $\mathcal{S}^*_{\wp,n}$-radius of the class $\mathcal{F}_n$ is given by:

$$R_{\mathcal{S}^*_{\wp,n}}(\mathcal{F}_n) = (\sqrt{1+n^2e^2} - ne)^{1/n}.$$

Proof. If $f \in \mathcal{F}_n$, then the function $h(z) := f(z)/z \in \mathcal{P}_n$ and

$$\frac{zf'(z)}{f(z)} - 1 = \frac{zh'(z)}{h(z)}.$$

Using Lemmas 2.2.2 and 2.2.3, we get

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2nr^n}{1-r^{2n}} \leq \frac{1}{e}.$$

Upon simplifying the last inequality, we get $r^{2n} + 2ner^n - 1 \leq 0$. Thus, the $\mathcal{S}^*_{\wp,n}$-radius of $\mathcal{F}_n$ is the least positive root of $r^{2n} + 2ner^n - 1 = 0$ in $(0,1)$. Since for the function $f_0(z) = z(1+z^n)/(1-z^n)$, $\operatorname{Re}(f_0(z)/z) > 0$ in $\mathbb{D}$. We have $f_0 \in \mathcal{F}_n$ and $zf_0'(z)/f_0(z) = 1 + 2nz^n/(1-z^{2n})$. Moreover, the result is sharp as we have at $z = R_{\mathcal{S}^*_{\wp,n}}(\mathcal{F}_n)$:

$$\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{2nz^n}{1-z^{2n}} = \frac{1}{e}.$$

This completes the proof. □

Let $\mathcal{F} := \mathcal{F}_1$, which is $\mathcal{F} := \{f \in \mathcal{A} : f(z)/z \in \mathcal{P}\}$. MacGregor [103] showed that $r_0 = \sqrt{2} - 1$ for the class $\mathcal{F}$ is the radius of univalence and starlikeness. The $\mathcal{S}^*_{\wp}$-radius is shown below:

Corollary 2.3.5. The $\mathcal{S}^*_{\wp}$-radius of the class $\mathcal{F}$ is given by

$$R_{\mathcal{S}^*_{\wp}}(\mathcal{F}) = \sqrt{1+e^2} - e \approx 0.178105.$$

Theorem 2.3.6. The sharp $\mathcal{S}^*_{\wp,n}$-radius of the class $\mathcal{CS}_n(\alpha)$ is given by

$$R_{\mathcal{S}^*_{\wp,n}}(\mathcal{CS}_n(\alpha)) = \left(\frac{1/e}{\sqrt{(1+n-\alpha)^2-(1/e)(2(1-\alpha)-1/e)}+1+n-\alpha}\right)^{1/n}.$$

Proof. Let $f\in\mathcal{CS}_n(\alpha)$ and $g\in\mathcal{S}^*_n(\alpha)$. Then, we have $h(z):=f(z)/g(z)\in\mathcal{P}_n$, which implies:

$$\frac{zf'(z)}{f(z)}=\frac{zg'(z)}{g(z)}+\frac{zh'(z)}{h(z)}.$$

Using Lemma 2.2.3 with $\alpha=0$ and Lemma 2.2.4, we have

$$\left|\frac{zf'(z)}{f(z)}-a\right|\leq\frac{2(1+n-\alpha)r^n}{1-r^{2n}}, \tag{2.3.8}$$

where $a:=(1+(1-2\alpha)r^{2n})/(1-r^{2n})\geq 1$. Note that $a\leq 1+(e-e^{-1})/2$ if and only if $r^{2n}\leq(e^2-1)/(e^2-1+4e(1-\alpha))$. Let $r\leq R_{\mathcal{S}^*_{\wp,n}}(\mathcal{CS}_n(\alpha))$. Then

$$\begin{aligned} r^{2n} &\leq \left(\frac{1}{e(2-\alpha)+\sqrt{e^2(2-\alpha)^2-e(2-2\alpha-\frac{1}{e})}}\right)^2\\ &\leq \frac{1}{2e^2\alpha^2-(8e^2-2e)\alpha+(8e^2-2e+1)}. \end{aligned}$$

Further, the expression on the right of the above inequality is less than or equal to $\frac{e^2-1}{(e^2-1)+4e(1-\alpha)}$, provided

$$T(\alpha):=2e^2(e^2-1)\alpha^2-(8e^4-2e^3-8e^2-2e)\alpha+(8e^4-2e^3-8e^2-2e)\geq 0.$$

Since $T'(\alpha)<0$ and $\min_{0<\alpha<1}T(\alpha)=\lim_{\alpha\to 1}T(\alpha)=2e^2(e^2-1)>0$. Therefore, $a\leq 1+(e-e^{-1})/2$. Using Lemma 2.2.2, it follows that the disk, given by (2.3.8) is contained in the cardioid $\wp(\mathbb{D})$, if

$$\frac{1-2(1+n-\alpha)r^n+(1-2\alpha)r^{2n}}{1-r^{2n}}\geq 1-\frac{1}{e},$$

which is equivalent to $(2-2\alpha-1/e)r^{2n}-2(1+n-\alpha)r^n+1/e\geq 0$, and holds when $r\leq R_{\mathcal{S}^*_{\wp,n}}(\mathcal{CS}_n(\alpha))$. For sharpness, we consider the following functions

$$f_0(z):=\frac{z(1+z^n)}{(1-z^n)^{(n+2-2\alpha)/n}} \quad\text{and}\quad g_0(z):=\frac{z}{(1-z^n)^{2(1-\alpha)/n}}, \tag{2.3.9}$$

such that $f_0(z)/g_0(z)=(1+z^n)/(1-z^n)$ and $zg_0'(z)/g_0(z)=(1+(1-2\alpha)z^n)/(1-z^n)$. Moreover, we have $\Re(f_0(z)/g_0(z))>0$ and $\Re(zg_0'(z)/g_0(z))>\alpha$ in $\mathbb{D}$. Hence $f_0\in\mathcal{CS}_n(\alpha)$ and

$$\frac{zf_0'(z)}{f_0(z)}=\frac{1+2(1+n-\alpha)z^n+(1-2\alpha)z^{2n}}{1-z^{2n}}.$$

For $z=R_1e^{i\pi/n}$, we have $zf_0'(z)/f_0(z)=1-1/e$. □

Theorem 2.3.7. The $\mathcal{S}^*_{\wp,n}$-radius of the class $\mathcal{S}^*_n[D,E]$ is given by

$$(i)\ R_{\mathcal{S}^*_{\wp,n}}(\mathcal{S}^*_n[D,E]) = \min\left\{1,\ \left(\frac{1/e}{D-(1-1/e)E}\right)^{\frac{1}{n}}\right\},\ \text{when } 0 \le E < D \le 1.$$

$$(ii)\ R_{\mathcal{S}^*_{\wp,n}}(\mathcal{S}^*_n[D,E]) = \begin{cases} R_1, & \text{if } R_1 \le r_1 \\ R_2, & \text{if } R_1 > r_1 \end{cases} \quad \text{when } -1 \le E < 0 \le D \le 1.$$

where

$$R_1 := R_{\mathcal{S}^*_{\wp,n}}(\mathcal{S}^*_n[D,E]) \text{ as defined in part (i)},$$

$$R_2 = \min\{1, (e/(D-(e+1)E))^{1/n}\}$$

and

$$r_1 = \left(\frac{(e^2-1)/2e}{((e^2+2e-1)/2e)E^2 - DE}\right)^{1/2n}.$$

In particular, for the class $\mathcal{S}^*$, we have $R_{\mathcal{S}^*_{\wp}}(\mathcal{S}^*) = 1/(2e-1)$.

Proof. Let $f \in \mathcal{S}^*_n[D,E]$. Using Lemma 2.2.4, we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-DEr^{2n}}{1-E^2r^{2n}}\right| \le \frac{(D-E)r^n}{1-E^2r^{2n}}. \tag{2.3.10}$$

(i) If $0 \le E < D \le 1$, then

$$a := \frac{1-DEr^{2n}}{1-E^2r^{2n}} \le 1.$$

Further, by Lemma 2.2.2 and equation (2.3.10), we see that $f \in \mathcal{S}^*_{\wp,n}$ if

$$\frac{DEr^{2n} + (D-E)r^n - 1}{1-E^2r^{2n}} \le \frac{1}{e} - 1,$$

which upon simplification, yields

$$r \le \left(\frac{1/e}{D-(1-1/e)E}\right)^{1/n}.$$

The result is sharp due to the function $f_0(z)$, given by

$$f_0(z) = \begin{cases} z(1+Ez^n)^{\frac{D-E}{nE}}; & E \ne 0, \\ z\exp\left(\frac{Dz^n}{n}\right); & E = 0. \end{cases} \tag{2.3.11}$$

(ii) If $-1 \le E < 0 < D \le 1$, then

$$a := \frac{1-DEr^{2n}}{1-E^2r^{2n}} \ge 1.$$

Let us first assume that $R_1 \le r_1$. Note that $r \le r_1$ if and only if $a \le 1 + (e-e^{-1})/2$. In particular, for

$0 \leq r \leq R_1$, we have $a \leq 1+(e-e^{-1})/2$. Further, from Lemma 2.2.2, we have $f \in \mathscr{S}^*_{\wp,n}$ in $|z| \leq r$, if

$$\frac{(D-E)r^n}{1-E^2r^{2n}} \leq (a-1)+\frac{1}{e},$$

which holds whenever $r \leq R_1$. Let us now assume that $R_1 > r_1$. Thus $r \geq r_1$ if and only if $a \geq 1+(e-e^{-1})/2$. In particular, for $r \geq R_1$, we have $a \geq 1+(e-e^{-1})/2$. Further, from Lemma 2.2.2, we have $f \in \mathscr{S}^*_{\wp,n}$ in $|z| \leq r$ whenever

$$\frac{(D-E)r^n}{1-E^2r^{2n}} \leq e-(a-1),$$

which holds when $r \leq R_2$. Hence, the result follows with sharpness due to $f_0(z)$ given in (2.3.11). □

Theorem 2.3.8. The $\mathscr{S}^*_{\wp,n}$-radius of the class $\mathscr{M}^*_n(\beta)$, $(\beta > 1)$ is given by

$$R_{\mathscr{S}^*_{\wp,n}}(\mathscr{M}^*_n(\beta)) = (2e(\beta-1)+1)^{-1/n}.$$

Proof. If $f \in \mathscr{M}_n(\beta)$, then $zf'(z)/f(z) \prec (1+(2\beta-1)z)/(1+z)$. Now using Lemma 2.2.4, we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+(1-2\beta)r^{2n}}{1-r^{2n}}\right| \leq \frac{(\beta-1)2r^n}{1-r^{2n}}.$$

Note that for $\beta > 1$, we have $(1+(1-2\beta)r^{2n})/(1-r^{2n}) < 1$. Therefore by Lemma 2.2.2, we get $f \in \mathscr{S}^*_{\wp,n}$ in $|z| < r$, provided

$$\frac{(\beta-1)2r^n}{1-r^{2n}} - \frac{1+(1-2\beta)r^{2n}}{1-r^{2n}} \leq \frac{1}{e}-1,$$

which holds whenever $r \leq R_{\mathscr{S}^*_{\wp,n}}(\mathscr{M}^*_n(\beta))$. The result is sharp due to

$$f_0(z) := \frac{z}{(1-z^n)^{2(1-\beta)/n}},$$

as we see that $zf_0'(z)/f_0(z) = (1+(1-2\beta)z^n)/(1-z^n) = 1-1/e$ when $z = R_{\mathscr{S}^*_{\wp,n}}(\mathscr{M}^*_n(\beta))$. □

In the following theorem, we attempt to find the sharp $\mathscr{S}^*_{\wp}$-radii of the class $\mathscr{S}^*(\psi)$, for different choices of ψ such as $1+\sin(z)$, $\sqrt{2}-c\sqrt{(1-z)/(1+2cz)}$, $1+4z/3+2z^2/3$, $z+\sqrt{1+z^2}$, e^z and $\sqrt{1+z}$, where $c := \sqrt{2}-1$. Authors in [38, 108, 109, 134, 150, 159] introduced and studied these subclasses of starlike functions which we denote by $\mathscr{S}^*_s$, $\mathscr{S}^*_{RL}$, $\mathscr{S}^*_e$, Δ^*, $\mathscr{S}^*_C$ and $\mathscr{S}^*_L$, respectively.

Theorem 2.3.9. The sharp $\mathscr{S}^*_{\wp}$-radii of $\mathscr{S}^*_L, \mathscr{S}^*_{RL}, \mathscr{S}^*_e, \mathscr{S}^*_C, \mathscr{S}^*_s$ *and* Δ^* are:

(i) $R_{\mathscr{S}^*_{\wp}}(\mathscr{S}^*_L) = (2e-1)/e^2 \approx 0.600423.$

(ii) $R_{\mathscr{S}^*_{\wp}}(\mathscr{S}^*_{RL}) = \frac{1+2(\sqrt{2}-1)e}{e^2(\sqrt{2}-1)(\sqrt{2}-1+2(\sqrt{2}-1+e^{-1})^2)} \approx 0.648826.$

(iii) $R_{\mathscr{S}^*_{\wp}}(\mathscr{S}^*_e) = 1-\ln(e-1) \approx 0.458675.$

(iv) $R_{\mathscr{S}^*_{\wp}}(\mathscr{S}^*_C) = 1-\sqrt{1-3/2e} \approx 0.330536.$

(v) $R_{\mathscr{S}^*_{\wp}}(\mathscr{S}^*_s) = \arcsin(1/e) \approx 0.376727.$

(vi) $R_{\mathscr{S}^*_{\wp}}(\Delta^*) = (2e-1)/(2e(e-1)) \approx 0.474928.$

Proof. (i) If $f \in \mathscr{S}^*_L$, then for $|z| = r$

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \sqrt{1-r} \leq \frac{1}{e},$$

provided $r \leq (2e-1)/e^2 =: R_{\mathscr{S}^*_{\wp}}(\mathscr{S}^*_L)$. Now consider the function

$$f_0(z) := \frac{4z\exp(2(\sqrt{1+z}-1))}{(1+\sqrt{1+z})^2}.$$

Since $zf_0'(z)/f_0(z) = \sqrt{1+z}$, it follows that $f_0 \in \mathscr{S}^*_L$ and for $z = -R_{\mathscr{S}^*_{\wp}}(\mathscr{S}^*_L)$, we get $zf_0'(z)/f_0(z) = 1 - 1/e$. Hence the result is sharp.

(ii) Let $f \in \mathscr{S}^*_{RL}$. Then for $|z| < r$,

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \sqrt{2} + (\sqrt{2}-1)\sqrt{\frac{1+r}{(1-2(\sqrt{2}-1)r)}} \leq \frac{1}{e},$$

provided

$$r \leq \frac{1+2(\sqrt{2}-1)e}{e^2(\sqrt{2}-1)(\sqrt{2}-1+2(\sqrt{2}-1+e^{-1})^2)} =: R_{\mathscr{S}^*_{\wp}}(\mathscr{S}^*_{RL}).$$

For sharpness, consider

$$f_0(z) := z\exp\left(\int_0^z \frac{q_0(t)-1}{t}dt\right),$$

where

$$q_0(z) = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{(1+2(\sqrt{2}-1)z)}}.$$

Now for $z = -R_{\mathscr{S}^*_{\wp}}(\mathscr{S}^*_{RL})$, we have

$$\frac{zf_0'(z)}{f_0(z)} = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{(1+2(\sqrt{2}-1)z)}} = 1 - \frac{1}{e}.$$

(iii) Let $\rho = 1 - \ln(e-1)$, $q(z) = e^z$ and $f \in \mathscr{S}^*_e$. To prove that $f \in \mathscr{S}^*_{\wp}$ in $|z| < \rho$, it only suffices to show that $q(\rho z) \prec \wp(z)$ for $z \in \mathbb{D}$ and thus for $|z| = r$, we must have $e^{-r} \geq \wp(-1)$, which gives $r \leq \rho$. Now the difference of the square of the radial distances from the point $(1,0)$ to the corresponding points on the boundary curves $\partial\wp(e^{i\theta})$ and $\partial q(\rho e^{i\theta})$ is given by

$$T(\theta) := e^{2\cos\theta} - e^{2\rho\cos\theta} - 1 + 2e^{\rho\cos\theta}\cos(\rho\sin\theta) \quad (0 \leq \theta \leq \pi).$$

Since $T'(\theta) \leq 0$ and $T(0) > 0$, it follows that the condition $r \leq \rho$ is also sufficient for $q(\rho z) =$

$e^{\rho z} \prec 1+ze^z = \wp(z)$. For sharpness, consider the function

$$f_0(z) := z\exp\left(\int_0^z \frac{e^t-1}{t}dt\right).$$

Since $zf_0'(z)/f_0(z) = e^z$, which implies $f_0 \in \mathscr{S}_e^*$ and for $z = -R_{\mathscr{S}_\wp^*}(\mathscr{S}_e^*)$, we have $e^z = 1-1/e$.

(iv) Let $\rho = 1-\sqrt{1-3/2e}$, $q(z) = 1+4z/3+2z^2/3$ and $f \in \mathscr{S}_C^*$. To prove that $f \in \mathscr{S}_\wp^*$ in $|z|<\rho$, we make use of the fact that for $|z| = r < 1$, the minimum distance of $w = q(z)$ from 1 must be less than $1/e$. Therefore, for $f \in \mathscr{S}_\wp^*$ in $|z|<r$, it is necessary that

$$\frac{4}{3}r - \frac{2}{3}r^2 \leq \frac{1}{e},$$

which gives $r \leq \rho$. Since $r_0 = 1/2(>\rho)$ is the radius of convexity of $q(z)$ and it is symmetric about the real axis, we have $q(-r) \leq \Re q(z) \leq q(r)$ for $|z| = r < 1/2$. Thus $1-1/e \leq \Re q(\rho z) \leq 1+4\rho/3+2\rho^2/3 < 1+e$. Therefore, to prove that $q(\rho z) \prec \wp(z)$, it suffices to show that $\max_{|z|=1}|\arg(q(\rho z)| \leq \max_{|z|=1}|\arg(\wp(z)|$. Since

$$\begin{aligned}
\max_{0\leq\theta\leq\pi} \arg(q(\rho e^{i\theta})) &= \max_{0\leq\theta\leq\pi} \arctan\left(\frac{\frac{4}{3}\rho\sin\theta + \frac{2}{3}\rho^2\sin 2\theta}{1+\frac{4}{3}\rho\cos\theta + \frac{2}{3}\rho^2\cos 2\theta}\right)\\
&\leq \arctan\left(\frac{\frac{4}{3}\rho\sin\theta_0 + \frac{2}{3}\rho^2\sin 2\theta_0}{1-\frac{4}{3}\rho + \frac{2}{3}\rho^2}\right)\\
&\approx (0.401955)\pi/2 < \max_{0\leq\theta\leq\pi} \arg\wp(e^{i\theta}) \approx (0.89782)\pi/2,
\end{aligned}$$

where $\theta_0 \in (0,\pi)$ is the only root of $\cos\theta + \rho\cos 2\theta = 0$. Hence, the condition $r \leq \rho$ is also sufficient for $q(\rho z) \prec \wp(z)$. Let us consider the function

$$f_0(z) := z\exp\left(\frac{4z+z^2}{3}\right).$$

Since $zf_0'(z)/f_0(z) = q(z)$, it follows that $f_0 \in \mathscr{S}_C^*$ and for $z = -R_{\mathscr{S}_\wp^*}(\mathscr{S}_C^*)$, we get $q(z) = 1-1/e$. Hence the result is sharp.

(v) Let $\rho = \sin^{-1}(1/e)$, $q(z) = 1+\sin(z)$ and $f \in \mathscr{S}_s^*$. Then using the similar argument as in part (iv) together with a result ([38], Theorem 3.3) and Lemma 2.2.2, we have $f \in \mathscr{S}_\wp^*$ in $|z|<r$, provided $\sin r \leq 1/e$ which in turn gives $r \leq \rho$. For the sharpness, we consider the function

$$f_0(z) := z\exp\left(\int_0^z \frac{\sin t}{t}dt\right).$$

Since $zf_0'(z)/f_0(z) = q(z)$, we have $f_0 \in \mathscr{S}_s^*$. For $z = -R_{\mathscr{S}_\wp^*}(\mathscr{S}_s^*)$, we arrive at $q(z) = 1-1/e$.

(vi) Let $\rho = (2e-1)/(2e(e-1))$, $q(z) = z+\sqrt{1+z^2}$ and $f \in \Delta^*$. Clearly $\min|z+\sqrt{1+z^2}-1| = 1+r-\sqrt{1+r^2}$ whenever $|z| = r < 1$. Therefore using Lemma 2.2.2, for $f \in \mathscr{S}_\wp^*$ we must have $\sqrt{1+r^2}-r \leq 1-1/e$, which gives $r \leq \rho$. Following the similar argument as in part (iv), we see that $r \leq \rho$ is also a sufficient condition for $q(\rho z) \prec \wp(z)$ to hold. Now for the function

$$f_0(z) := z\exp\left(\int_0^z \frac{t+\sqrt{1+t^2}-1}{t}dt\right),$$

we have $zf_0'(z)/f_0(z) = q(z)$, which implies $f_0 \in \Delta^*$. For $z = -R_{\mathscr{S}^*_{\wp}}(\Delta^*)$, we get $q(z) = 1-1/e$, which shows that the result is sharp.

□

Now for our next radius problem, we need to consider some classes: Here below we presume the value of α to be either 0 or $1/2$.

$$\mathscr{F}_1(\alpha) := \left\{ f \in \mathscr{A}_n : \Re\frac{f(z)}{g(z)} > 0 \text{ and } \Re\frac{g(z)}{z} > \alpha,\ g \in \mathscr{A}_n \right\},$$

$$\mathscr{F}_2 := \left\{ f \in \mathscr{A}_n : \left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ and } \Re\frac{g(z)}{z} > 0,\ g \in \mathscr{A}_n \right\}$$

and

$$\mathscr{F}_3 := \left\{ f \in \mathscr{A}_n : \left|\frac{f(z)}{g(z)} - 1\right| < 1 \text{ and } g \in \mathscr{C},\ g \in \mathscr{A}_n \right\}.$$

Theorem 2.3.10. The sharp $\mathscr{S}^*_{\wp,n}$-radii of functions in the classes $\mathscr{F}_1(\alpha)$, $\mathscr{F}_2$ and $\mathscr{F}_3$, respectively, are:

(i) $R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_1(0)) = \left(\sqrt{4n^2e^2+1} - 2ne\right)^{1/n}.$

(ii) $R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_1(1/2)) = \left(2/(\sqrt{(3ne+2)^2-8ne} + 3ne)\right)^{1/n}.$

(iii) $R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_2) = \left(2/(\sqrt{(3ne+2)^2-8ne} + 3ne)\right)^{1/n}.$

(iv) $R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_3) = \left(\frac{\sqrt{(n+1)^2+4(n-1+1/e)/e}-(1+n)}{2(n-1+1/e)}\right)^{1/n}.$

Proof. Let us consider the functions $p, h : \mathbb{D} \to \mathbb{C}$, defined by $p(z) = g(z)/z$ and $h(z) = f(z)/g(z)$. We write $p_0(z) = g_0(z)/z$ and $h_0(z) = f_0(z)/g_0(z)$.

(i) If $f \in \mathscr{F}_1(0)$, then $p, h \in \mathscr{P}_n$ such that $f(z) = zp(z)h(z)$. Thus it follows from Lemma 2.2.3 that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{4nr^n}{1-r^{2n}} \leq \frac{1}{e},$$

provided $r \leq \left(\sqrt{4n^2e^2+1} - 2ne\right)^{1/n} =: R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_1(0))$. Thus $f \in \mathscr{S}^*_{\wp,n}$ whenever $r \leq R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_1(0))$. Now for the functions

$$f_0(z) = z\left(\frac{1+z^n}{1-z^n}\right)^2 \quad \text{and} \quad g_0(z) = z\left(\frac{1+z^n}{1-z^n}\right),$$

we have $\Re h_0(z) > 0$ and $\operatorname{Re} p_0(z) > 0$. Hence $f_0 \in \mathscr{F}_1(0)$. For $z = R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_1(0))e^{i\pi/n}$, we see that

$$\frac{zf_0'(z)}{f_0(z)} = 1 + \frac{4nz^n}{1-z^{2n}} = 1 - \frac{1}{e}.$$

Thus the result is sharp.

(ii) Let $f \in \mathscr{F}_1(1/2)$. Then $h \in \mathscr{P}_n$ and $p \in \mathscr{P}_n(1/2)$. Since $f(z) = zp(z)h(z)$, it follows from Lemma 2.2.3 that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{2nr^n}{1-r^{2n}} + \frac{nr^n}{1-r^n} = \frac{3nr^n + nr^{2n}}{1-r^{2n}} \leq \frac{1}{e},$$

provided

$$r \leq \left(\frac{\sqrt{9n^2e^2 + 4(ne+1)} - 3ne}{2(ne+1)}\right)^{1/n} =: R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_1(1/2)).$$

Thus $f \in \mathscr{S}^*_{\wp,n}$ whenever $r \leq R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_1(1/2))$. For the functions

$$f_0(z) = \frac{z(1+z^n)}{(1-z^n)^2} \quad \text{and} \quad g_0(z) = \frac{z}{1-z^n},$$

we have $\Re h_0(z) > 0$ and $\operatorname{Re} p_0(z) > 1/2$. Hence $f \in \mathscr{F}_1(1/2)$. The result is sharp, since for $z = R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_1(1/2))$ we have

$$\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{3nz^n + nz^{2n}}{1-z^{2n}} = \frac{1}{e}.$$

(iii) Let $f \in \mathscr{F}_2$. Then $p \in \mathscr{P}_n$. Since $|h(z)-1| < 1$ if and only if $\Re(1/h(z)) > 1/2$. Therefore $1/h \in \mathscr{P}_n(1/2)$. Since $f(z)/h(z) = zp(z)$, using Lemma 2.2.3 we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3nr^n + nr^{2n}}{1-r^{2n}} \leq \frac{1}{e},$$

provided $r^n \leq 2/(\sqrt{(3ne+2)^2 - 8ne} + 3ne)$. For sharpness, consider

$$f_0(z) := \frac{z(1+z^n)^2}{1-z^n} \quad \text{and} \quad g_0(z) := \frac{z(1+z^n)}{1-z^n}.$$

Since

$$|h_0(z) - 1| = |z^n| < 1 \text{ and } \Re p_0(z) = \Re\frac{1+z^n}{1-z^n} > 0.$$

Therefore $f_0 \in \mathscr{F}_2$ and for $z = R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_2)e^{i\pi/n}$, we have

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \left|\frac{3nz^n - nz^{2n}}{1-z^{2n}}\right| = \frac{1}{e}.$$

(iv) Let $f \in \mathscr{F}_3$. Then $1/h(z) = g(z)/f(z) \in \mathscr{P}_n(1/2)$ and

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{zh'(z)}{h(z)}. \tag{2.3.12}$$

Using a result due to Marx-Strohhäcker that every convex function is starlike of order $1/2$, it follows from Lemma 2.2.4 that

$$\left|\frac{zg'(z)}{g(z)} - \frac{1}{1-r^{2n}}\right| \leq \frac{r^n}{1-r^{2n}}. \tag{2.3.13}$$

Now using Lemma 2.2.3 and equation (2.3.13), we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-r^{2n}}\right| \leq \frac{r^n}{1-r^{2n}} + \frac{nr^n}{1-r^n} = \frac{(n+1)r^n + nr^{2n}}{1-r^{2n}}.$$

Thus using Lemma 2.2.2 we have $f \in \mathscr{S}^*_{\wp,n}$, provided

$$\frac{(n+1)r^n + nr^{2n}}{1-r^{2n}} \leq \left(\frac{1}{1-r^{2n}} - 1\right) + \frac{1}{e},$$

which implies $r \leq R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_3)$. Now consider the functions

$$f_0(z) = \frac{z(1+z^n)}{(1-z^n)^{1/n}} \quad \text{and} \quad g_0(z) = \frac{z}{(1-z^n)^{1/n}}.$$

Since $g_0 \in \mathscr{C}$ and $|h_0(z)-1| = |z^n| < 1$. Therefore, the function $f_0 \in \mathscr{F}_3$ and for $z = R_{\mathscr{S}^*_{\wp,n}}(\mathscr{F}_3)e^{i\pi/n}$, we have $zf_0'(z)/f_0(z) = 1 - 1/e$, which confirms the sharpness of the result.

□

2.4 Coefficient Problems

Using the coefficients of $f(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$, Pommerenke [130], Noonan and Thomas [120] considered the Hankel determinant $H_q(n)$, defined by

$$H_q(n) := \begin{vmatrix} b_n & b_{n+1} & \dots & b_{n+q-1} \\ b_{n+1} & b_{n+2} & \dots & b_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n+q-1} & b_{n+q} & \dots & b_{n+2(q-1)} \end{vmatrix}, \tag{2.4.1}$$

where $b_1 = 1$. Finding the upper bound of $|H_3(1)|$, $|H_2(2)|$ and $|H_2(1)|$ for the functions belonging to various subclasses of $\mathscr{A}$ in $\mathscr{S}$ is a usual phenomenon in GFT. Note that the Fekete-Szegö functional $b_3 - b_2^2$, coincide with $H_2(1)$, which was analyzed by Bieberbach in 1916. In fact, Fekete-Szegö considered the generalized functional $b_3 - \mu b_2^2$, where μ is real and $f \in \mathscr{S}$. For the class $\mathscr{S}^*$, it is well-known that $|H_2(2)| \leq 1$. Recently, bound for the second Hankel determinant, $H_2(2) = b_2 b_4 - b_3^2$

is obtained by Alarif et al. [7] for the class $\mathcal{S}^*(\psi)$. The estimation of the third Hankel determinant is more difficult in comparison with the second Hankel determinant, especially when sharp bounds are needed, where the third Hankel determinant is given by

$$H_3(1) = b_3(b_2b_4 - b_3^2) - b_4(b_4 - b_2b_3) + b_5(b_3 - b_2^2). \tag{2.4.2}$$

The upper bound for $|H_3(1)|$ can be obtained by estimating each term of (2.4.2), see [139]. For more work in this direction see [18, 81, 82, 92, 172]. The following lemmas are needed to prove our coefficient results.

Lemma 2.4.1. *[102]* Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ be the Carathéodory function. Then for a complex number τ, we have

$$|p_2 - \tau {p_1}^2| \leq 2\max(1, |2\tau - 1|).$$

Lemma 2.4.2. *[138]* Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ be the Carathéodory function. Then for $n, m \in \mathbb{N}$,

$$|p_{n+m} - \gamma p_n p_m| \leq \begin{cases} 2, & 0 \leq \gamma \leq 1; \\ 2|2\gamma - 1|, & \text{elsewhere.} \end{cases}$$

Here below, we partially disclose the lemma given in [11], which is required in the sequel. Here, we need the following two relevant sets.

$$D_8 := \left\{ (\mu, \nu) : \frac{1}{2} \leq |\mu| \leq 2, -\frac{2}{3}(|\mu| + 1) \leq \nu \leq \frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \right\}$$

and

$$D_9 := \left\{ (\mu, \nu) : |\mu| \geq 2, -\frac{2}{3}(|\mu| + 1) \leq \nu \leq \frac{2|\mu|(|\mu| + 1)}{\mu^2 + 2|\mu| + 4} \right\}.$$

In view of these sets, the following lemma provides a sharp bound for the certain combination of the coefficient of the power series of a Schwarz function.

Lemma 2.4.3. *[11]* If $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ be the Schwarz function, then

$$|c_3 + \mu c_1 c_2 + \nu c_1^3| \leq \Psi(\mu, \nu),$$

where

$$\Psi(\mu, \nu) = \frac{2}{3}(|\mu| + 1)\left(\frac{|\mu| + 1}{3(1 + \nu + |\mu|)} \right)^{\frac{1}{2}} \quad \text{for} \quad (\mu, \nu) \in D_8 \cup D_9.$$

The following lemma carries the expression for p_2 and p_3 in terms of p_1, derived in [96, 97] and p_4 in terms of p_1 obtained in [91].

Lemma 2.4.4. Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ be the Carathéodory function. Then for some complex numbers ζ, η and ξ with $|\zeta| \leq 1$, $|\eta| \leq 1$ and $|\xi| \leq 1$, we have

$$2p_2 = p_1^2 + \zeta(4 - p_1^2), \quad 4p_3 = p_1^3 + 2p_1\zeta(4 - p_1^2) - p_1\zeta^2(4 - p_1^2) + 2(4 - p_1^2)(1 - |\zeta|^2)\eta$$

and

$$8p_4 = p_1^4 + (4-p_1^2)\zeta(p_1^2(\zeta^2-3\zeta+3)+4\zeta) - 4(4-p_1^2)(1-|\zeta|^2)(p_1(\zeta-1)\eta + \bar{\zeta}\eta^2 - (1-|\eta|^2)\xi).$$

We now define the function f_n such that $f_n(0) = f_n'(0) - 1 = 0$ and

$$\frac{zf_n'(z)}{f_n(z)} = \wp(z^n) \qquad (n = 1,2,3,\cdots),$$

which acts as an extremal function for many subsequent results and we have

$$f_n(z) = z\exp((e^{z^n}-1)/n) \tag{2.4.3}$$

Theorem 2.4.1. Let $f(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathscr{S}^*_{\wp}$ and $\alpha = (1+e)^2$, then

$$\sum_{k=2}^{\infty}(k^2-\alpha)|b_k|^2 \leq \alpha - 1. \tag{2.4.4}$$

Proof. If $f \in \mathscr{S}^*_{\wp}$, then $zf'(z)/f(z) = \wp(\omega(z))$, where ω is a Schwarz function. Now for $0 \leq r < 1$, we have

$$\begin{aligned}
2\pi\sum_{k=1}^{\infty} k^2|b_k|^2 r^{2k} &= \int_0^{2\pi} |re^{i\theta}f'(re^{i\theta})|^2 d\theta = \int_0^{2\pi} |f(re^{i\theta}) + f(re^{i\theta})\omega(re^{i\theta})e^{\omega(re^{i\theta})}|^2 d\theta \\
&\leq \int_0^{2\pi}\left(|f(re^{i\theta})| + |f(re^{i\theta})\omega(re^{i\theta})e^{\omega(re^{i\theta})}|\right)^2 d\theta \\
&= \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta + \int_0^{2\pi} |f(re^{i\theta})\omega(re^{i\theta})e^{\omega(re^{i\theta})}|^2 d\theta + 2\int_0^{2\pi} |f(re^{i\theta})|^2 |\omega(re^{i\theta})e^{\omega(re^{i\theta})}| d\theta \\
&\leq \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta + \int_0^{2\pi} |f(re^{i\theta})e^{\omega(re^{i\theta})}|^2 d\theta + 2\int_0^{2\pi} |f(re^{i\theta})|^2 |e^{\omega(e^{i\theta})}| d\theta \\
&\leq \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta + e^{2r}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta + 2e^r\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \\
&\leq 2\pi(1+e^r)^2 \sum_{k=1}^{\infty} |b_k|^2 r^{2k},
\end{aligned}$$

which finally yields

$$\sum_{k=1}^{\infty}(k^2-(1+e^r)^2)|b_k|^2 r^{2k} \leq 0, \quad 0 \leq r < 1.$$

Letting $r \to 1^-$, we get the desired result. □

Corollary 2.4.2. Let $f(z) = z + \sum_{k=4}^{\infty} b_k z^k \in \mathscr{S}^*_{\wp}$ and $\alpha = (1+e)^2$, then

$$|b_k| \leq \sqrt{\frac{\alpha-1}{k^2-\alpha}}, \text{ for all } k \geq 4.$$

Example 2.4.3. (i) $z/(1-Az)^2 \in \mathscr{S}^*_{\wp}$ if and only if $|A| \leq 1/(2e-1)$.

(ii) $f(z) = z + b_k z^k \in \mathscr{S}^*_{\wp}$ if and only if $|b_k| \leq 1/(e(k-1)+1)$, where $k \in \mathbb{N} - \{1\}$.

(*iii*) $f(z) = z\exp(Az) \in \mathcal{S}^*_{\wp}$ if and only if $|A| \leq 1/e$.

Proof. (i) If $A=1$, then $z/(1-z)^2 \notin \mathcal{S}^*_{\wp}$, since $|f(z)| \leq e^{e-1}$ for $f \in \mathcal{S}^*_{\wp}$. Now let $K(z) = z/(1-Az)^2$. If $A \neq 1$, then the disk

$$\left| w - \frac{1+|A|^2}{1-|A|^2} \right| < \frac{2|A|}{1-|A|^2}, \tag{2.4.5}$$

is the image of $\mathbb{D}$ under the bilinear transformation $w = zK'(z)/K(z) = (1+Az)/(1-Az)$ with diameter's endpoints $x_L := (1-|A|)/(1+|A|)$ and $x_R := (1+|A|)/(1-|A|)$. Now for the disk (2.4.5) to be inside the cardioid $\wp(\mathbb{D})$, it is necessary that $x_L \geq 1-1/e$ which gives $|A| \leq 1/(2e-1)$. Conversely, let $|A| \leq 1/(2e-1)$. Then we have

$$a := \frac{1+|A|^2}{1-|A|^2} \leq \frac{2e-e^{-1}+2}{2(e-1)} \quad \text{and} \quad r := \frac{2|A|}{1-|A|^2} \leq \frac{2e-1}{2e(e-1)}.$$

Since $r_a > r$, thus Lemma 2.2.2 ensures that disk $\{w : |w-a| < r\} \subset \wp(\mathbb{D})$. Hence, $K \in \mathcal{S}^*_{\wp}$.
(ii) Since $zf'(z)/f(z) = (1+kb_kz^{k-1})/(1+b_kz^{k-1})$ maps $\mathbb{D}$ onto the disk $\{w \in \mathbb{C} : |w-a| < r\}$, where

$$a := \frac{1-k|b_k|^2}{1-|b_k|^2} \quad \text{and} \quad r := \frac{(k-1)|b_k|}{1-|b_k|^2}.$$

Further $f(z) = z + b_kz^k \in \mathcal{S}^*$ if and only if $|b_k| \leq 1/k$, which ensures $(1-k|b_k|^2)/(1-|b_k|^2) \leq 1$. Therefore in view of Lemma 2.2.2, $\{w \in \mathbb{C} : |w-a| < r\} \subset \wp(\mathbb{D})$ if and only if

$$\frac{(k-1)|b_k|}{1-|b_k|^2} \leq \frac{1-k|b_k|^2}{1-|b_k|^2} - 1 + \frac{1}{e},$$

which is equivalent to $(ke-e+1)|b_k|^2 + (ke-e)|b_k| - 1 \leq 0$. Hence, $|b_k| \leq 1/(e(k-1)+1)$.
(iii) Since $zf'(z)/f(z) = 1+Az$ maps $\mathbb{D}$ onto the disk $\{w \in \mathbb{C} : |w-1| < |A|\}$. Therefore in view of Lemma 2.2.2, the inequality

$$|w-1| < |A| \leq 1/e,$$

yields the necessary and sufficient condition $|A| \leq 1/e$ for $1+Az \prec \wp(z)$. □

Remark 2.4.1. Note that when $k = \sqrt{2}+1$, we have $q_0(z) = 1 + (z/k)((k+z)/(k-z)) \prec \wp(z)$. Therefore, the class of starlike functions $\mathcal{S}^*(q_0)$ introduced in [89] is contained in $\mathcal{S}^*_{\wp}$. Further the sharp $\mathcal{S}^*(\psi)$-radius for the class $\mathcal{S}^*$ is also given by the relation

$$R_{\mathcal{S}^*(\psi)}(\mathcal{S}^*) = \max |A|, \tag{2.4.6}$$

where A is defined in such a way that $z/(1-Az)^2 \in \mathcal{S}^*(\psi)$. Thus if $z/(1-Az)^2 \in \mathcal{S}^*(q_0) \subset \mathcal{S}^*_{\wp}$, then by Example 2.4.3 we see that $|A| \leq 1/(2e-1)$ and, therefore in view of (2.4.6), we now state a result ([89], theorem 2.3, pg 203) in its correct form using the result ([89], theorem 3.2, pg 206):

$$z/(1-Az)^2 \in \mathcal{S}^*(q_0) \text{ if and only if } |A| \leq \frac{3-2\sqrt{2}}{2\sqrt{2}-1} < \frac{1}{2e-1}.$$

The authors [89] proved that $|A| \leq 1/3$.

Theorem 2.4.4. Let $f(z)=z+\sum_{k=2}^{\infty}b_kz^k\in\mathscr{S}^*_{\wp}$, then

$$|b_2|\leq 1,\quad |b_3|\leq 1,\quad |b_4|\leq 5/6\quad\text{and}\quad |b_5|\leq 5/8. \tag{2.4.7}$$

The bounds are sharp.

Proof. Let $p(z)\in\mathscr{P}$. Since there exists one-one correspondence between the classes Ω and $\mathscr{P}$ via the following functions:

$$\omega(z)=\frac{p(z)-1}{p(z)+1}\quad\text{and}\quad p(z)=\frac{1+\omega(z)}{1-\omega(z)}.$$

Therefore, for $f\in\mathscr{S}^*_{\wp}$ we have

$$\frac{zf'(z)}{f(z)}=\wp(\omega(z))=\wp\left(\frac{p(z)-1}{p(z)+1}\right),$$

where

$$\wp\left(\frac{p(z)-1}{p(z)+1}\right)=1+\frac{p_1}{2}z+\frac{p_2}{2}z^2+\left(-\frac{p_1^3}{16}+\frac{p_3}{2}\right)z^3+\left(\frac{p_1^4}{24}-\frac{3p_1^2p_2}{16}+\frac{p_4}{2}\right)z^4+\cdots \tag{2.4.8}$$

and

$$\frac{zf'(z)}{f(z)}=1+b_2z+(2b_3-b_2^2)z^2+(3b_4-3b_2b_3+b_2^3)z^3+(-b_2^4+2b_3(2b_2^2-b_3)-4b_2b_4+4b_5)z^4+\cdots. \tag{2.4.9}$$

On comparing the coefficients of z^k $(k=1,2,3,4)$ in (2.4.8) and (2.4.9), we get

$$b_2=\frac{p_1}{2},\ b_3=\frac{1}{4}\left(p_2+\frac{p_1^2}{2}\right),\ b_4=\frac{1}{6}\left(p_3+\frac{3}{4}p_1p_2\right)$$

and

$$b_5=\frac{1}{8}\left(\frac{1}{48}p_1^4+\frac{1}{4}p_2^2+\frac{2}{3}p_1p_3-\frac{1}{8}p_1^2p_2+p_4\right). \tag{2.4.10}$$

Now using the fact $|p_k|\leq 2$, Lemma 2.4.1 with $\tau=-1/2$ and Lemma 2.4.2 with $\gamma=-3/4$, we obtain $|b_2|\leq 1$, $|b_3|\leq 1$ and $|b_4|\leq 5/6$ respectively.

For b_5, using proper rearrangement of terms and then applying triangle inequality, we see that

$$\begin{aligned}
|b_5|&=\frac{1}{8}\left|\frac{1}{48}p_1^4+\frac{1}{4}p_2^2+\frac{2}{3}p_1p_3-\frac{1}{8}p_1^2p_2+p_4\right|\\
&=\frac{1}{8}\left|\frac{1}{48}p_1^4+(p_4+\frac{2}{3}p_1p_3)+\frac{1}{4}p_2(p_2-\frac{1}{2}p_1^2)\right|\\
&\leq\frac{1}{8}\left(\frac{1}{48}|p_1|^4+|p_4+\frac{2}{3}p_1p_3|+\frac{1}{4}|p_2||p_2-\frac{1}{2}p_1^2|\right)\\
&\leq\frac{1}{8}\left(\frac{1}{48}|p_1|^4-\frac{1}{4}|p_1|^2+\frac{4}{3}|p_1|+3\right)=:G(p_1).
\end{aligned}$$

Now to maximize the above expression, without loss of generality, we write

$$G(p)=\frac{1}{48}p^4-\frac{1}{4}p^2+\frac{4}{3}p+3 \quad (p\in[0,2]),$$

then $G'(p)\geq 0$. Thus $G(p)\leq 5$, which implies that $|b_5|\leq 5/8$. The bounds for b_k (k=1,2,3,4) are sharp with the extremal function f_1 defined in (2.4.3). □

Now in view of Theorem 2.4.4, we conjecture the following:

Conjecture 2.4.1. (Open) Let $f(z)\in\mathcal{S}^*_{\wp}$. Then the following sharp estimates hold:

$$|b_k|\leq\frac{B_{k-1}}{(k-1)!} \quad \text{for all} \quad k\geq 1,$$

where B_k are Bell numbers satisfying the recurrence relation defined in (2.1.5) and the extremal function f_1 is given by (2.4.3).

Remark 2.4.2. The logarithmic coefficients d_k for $f\in\mathcal{S}$ are defined by the following series expansion:

$$\log\frac{f(z)}{z}=2\sum_{k=1}^{\infty}d_kz^k, \quad z\in\mathbb{D}. \tag{2.4.11}$$

Recently, Cho [1] obtained the sharp logarithmic coefficient bounds for the Ma-Minda class $\mathcal{S}^*(\psi)$. Consequently, we have the following sharp result:

Let $f\in\mathcal{S}^*_{\wp}$. Then the logarithmic coefficients of f given by (2.4.11) satisfies

$$|d_k|\leq 1/2.$$

Remark 2.4.3. Now if $f(z)\in\mathcal{S}^*_{\wp}$, then from (2.4.10), using triangle inequality together with Lemma 2.4.1, we obtain the following estimates for the Fekete-Szegö functional:

$$|b_3-\mu b_2^2|=\frac{1}{4}\left|p_2-\left(\mu-\frac{1}{2}\right)p_1^2\right|\leq\frac{1}{2}\max\left(1,2|\mu-1|\right). \tag{2.4.12}$$

Equality cases holds for the function $f_1(z)=z\exp(e^z-1)$, when $\mu\in[1/2,3/2]$ and the function $f_2(z)=z\exp((e^{z^2}-1)/2)$, when $\mu\leq 1/2$ or $\mu\geq 3/2$ given by (2.4.3). In particular for $\mu=1$, we have $|H_2(1)|=|b_3-b_2^2|\leq 1/2$.

Now the Covering theorem stated in Theorem 2.1.1 ensures that for every f in $\mathcal{S}^*_{\wp}$, $f(\mathbb{D})$ contains a disk of radius $e^{1/e-1}$ centered at the origin. Hence, every function $f\in\mathcal{S}^*_{\wp}$ has an inverse f^{-1} which given by

$$\begin{aligned} f^{-1}(w) &= w+\sum_{k=2}^{\infty}A_kw^k \\ &= w-b_2w^2+(2b_2^2-b_3)w^3-(5b_2^3-5b_2b_3+b_4)w^4+\cdots, \end{aligned}$$

then we have $f^{-1}(f(z))=z$ and $f(f^{-1}(w))=w$ for $|w|<r_0(f)$ and $r_0>e^{1/e-1}$. Thus using Theorem 2.4.4 and equation 2.4.12, we easily obtain

$$|A_2| \leq 1 \quad \text{and} \quad |A_3| \leq 1.$$

The bounds are sharp with extremal function f_1^{-1}, where f_1 is defined in (2.4.3).

Theorem 2.4.5. Let $f \in \mathcal{S}^*_{\wp}$. Then for $f^{-1}(\omega) = \omega + \sum_{k=2}^{\infty} A_k \omega^k$, we have

$$|A_4| \leq \frac{5}{6} \quad \text{and} \quad |A_3 - \mu A_2^2| \leq \begin{cases} 3-\mu, & \mu \leq 5/2; \\ 1/2, & 5/2 \leq \mu \leq 7/2; \\ \mu - 3, & \mu \geq 7/2. \end{cases}$$

The bounds are sharp.

Proof. Consider the inverse function $f^{-1}(\omega) = \omega + \sum_{k=2}^{\infty} A_k \omega^k$, where we have $A_4 = -5b_2^3 + 5b_2b_3 - b_4$, which can be now rewritten in terms of Carathéodory coefficients using (2.4.10) as

$$A_4 = -\frac{1}{6}\left(p_3 - 3p_1p_2 + \frac{15}{8}p_1^3\right).$$

Now using Lemma 2.4.1 with $\tau = 5/8$ and $|p_k| \leq 2$,

$$|b_4| = \frac{1}{6}\left|p_3 - 3p_1\left(p_2 - \frac{5}{8}p_1^2\right)\right| \leq \frac{1}{6}(|p_3| + 3|p_1||p_2 - \frac{5}{8}p_1^2|) \leq \frac{5}{6}.$$

The bound is sharp with extremal function f_1^{-1}, where f_1 is defined in (2.4.3). Now for the Fekete-Szegö type inequality for the inverse function f^{-1} we have

$$|A_3 - \mu A_2^2| = |b_3 - tb_2^2|, \quad t = \mu - 2.$$

Thus using (2.4.12) the desired sharp result follows. □

Theorem 2.4.6. Let $f(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{S}^*_{\wp}$, then

$$|b_2b_3 - b_4| \leq \frac{2}{3}\sqrt{\frac{2}{5}}.$$

The bound is sharp.

Proof. Let $f \in \mathcal{S}^*_{\wp}$. Then

$$\frac{zf'(z)}{f(z)} = \wp(\omega(z)), \tag{2.4.13}$$

where $\omega \in \Omega$. Then proceeding as in Theorem 2.4.4, from (2.4.13) we have

$$b_2 = c_1, \quad b_2 = \frac{1}{2}(c_2 + 2c_1^2) \quad \text{and} \quad b_4 = \frac{1}{6}(2c_3 + 7c_1c_2 + 5c_1^3). \tag{2.4.14}$$

Therefore, with $\mu = 2, \nu = -1/2$ and $\psi(\mu,\nu) = |c_3 + \mu c_1 c_2 + \nu c_1^3|$, we have

$$|b_2 b_3 - b_4| = \frac{1}{3}|c_3 + 2c_1c_2 - c_1^3/2| = \frac{1}{3}\psi(\mu,\nu).$$

Now using Lemma 2.4.3, we obtain

$$|b_2 b_3 - b_4| \leq \frac{2}{3}\sqrt{\frac{2}{5}}.$$

The bound is sharp as there is an extremal function

$$f(z) = z\exp\int_0^z \frac{\wp(\omega(t))-1}{t}dt,$$

where $w(z) = z(\sqrt{2/5}-z)/(1-\sqrt{2/5}z)$. □

We now enlist below in the remark certain special cases of earlier known results pertaining to our class $\mathscr{S}^*_{\wp}$:

Remark 2.4.4. We obtain the following result by using a result ([7], theorem 2.2, pg 230): *Let* $f(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathscr{S}^*_{\wp}$, *then*

$$|H_2(2)| = |b_2 b_4 - {b_3}^2| \leq 1/4,$$

where equality is attained for the function f_2 given by (2.4.3).

Remark 2.4.5. Now using Theorems 2.4.4, 2.4.6 and Remark 2.4.4 together with the estimate given in (2.4.12) and triangle inequality, we obtain the following result:
Let the function $f(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathscr{S}^*_{\wp}$, *then*

$$|H_3(1)| \leq 0.913864\cdots.$$

Remark 2.4.6. Until now the bound on the third Hankel determinant is obtained using the triangle inequality approach. But note that using the method applied in Theorem 2.4.8, we can substantially improve the known bounds for many subclasses of starlike functions such as $\mathscr{S}^*_s$, $\mathscr{S}^*_C$ and $\mathscr{S}^*_e$.

In 2010, Babalola [17] showed that $|H_3(1)| \leq 16$ for the class $\mathscr{S}^*$. In 2018, Lecko [94] obtained the sharp inequality $|H_3(1)| \leq 1/9$ for the class $\mathscr{S}^*(1/2)$. For the class $\mathscr{S}^*$, it is proved in [92] that $|H_3(1)| \leq 8/9$ (not sharp), which improves the earlier known bound $|H_3(1)| \leq 1$ established by Zaprawa [172]. Since $\mathscr{S}^*_{\wp} \subset \mathscr{S}^*$, it seems reasonable that the bound on $|H_3(1)|$ for $\mathscr{S}^*_{\wp}$ can be further improved. A function f in $\mathscr{A}$ is called n-fold symmetric if $f(e^{2\pi i/n}z) = e^{2\pi i/n}f(z)$ holds for all $z \in \mathbb{D}$, where n is a natural number. We denote the set of n-fold symmetric functions by $\mathscr{A}^{(n)}$. Let $f \in \mathscr{A}^{(n)}$, then f has power series expansion $f(z) = z + b_{n+1}z^{n+1} + b_{2n+1}z^{2n+1} + \cdots$. Therefore, for $f \in \mathscr{A}^{(3)}$ and $f \in \mathscr{A}^{(2)}$ respectively, we have

$$H_3(1) = -{b_4}^2 \quad \text{and} \quad H_3(1) = b_3(b_5 - {b_3}^2). \tag{2.4.15}$$

Thus we can now find estimates on the third Hankel determinant $|H_3(1)|$ in the classes $\mathcal{S}^{*(2)}_{\wp}$ and $\mathcal{S}^{*(3)}_{\wp}$.

Theorem 2.4.7. Let $f \in \mathcal{S}^*_{\wp}$. Then

(i) $\hat{f} \in \mathcal{S}^{*(3)}_{\wp}$ implies that $|H_3(1)| \leq 1/9$.

(ii) $\hat{f} \in \mathcal{S}^{*(2)}_{\wp}$ implies that $|H_3(1)| \leq 1/16$.

The result is sharp.

Proof. (i) Since $f(z) = z + b_2z^2 + \cdots \in \mathcal{S}^*_{\wp}$ if and only if $\hat{f}(z) = (f(z^3))^{1/3} = z + \beta_4 z^4 + \cdots \in \mathcal{S}^{*(3)}_{\wp}$. We have $\beta_4 = b_2/3$. Hence for $\hat{f} \in \mathcal{S}^{*(3)}_{\wp}$, from (2.4.7) and (2.4.15), we obtain

$$|H_3(1)| = |\beta_4|^2 = \frac{1}{9}|b_2|^2 \leq \frac{1}{9}.$$

The above estimate is sharp for $\hat{f}_1$, where f_1 is given by (2.4.3).
(ii) Since $f(z) = z + b_2z^2 + \cdots \in \mathcal{S}^*_{\wp}$ if and only if $\hat{f}(z) = (f(z^2))^{1/2} = z + \alpha_3 z^3 + \alpha_5 z^5 + \cdots \in \mathcal{S}^{*(2)}_{\wp}$. Upon comparing the coefficients in the following:

$$z^2 + b_2z^4 + b_3z^6 + \cdots = (z + \alpha_3 z^3 + \alpha_5 z^5 + \cdots)^2,$$

we obtain

$$\alpha_3 = \frac{1}{2}b_2 \quad \text{and} \quad \alpha_5 = \frac{1}{2}b_3 - \frac{1}{8}{b_2}^2. \tag{2.4.16}$$

If $\hat{f} \in \mathcal{S}^{*(2)}_{\wp}$, then from (2.4.15) we have

$$H_3(1) = \alpha_3(\alpha_5 - {\alpha_3}^2).$$

Now using (2.4.10), (2.4.16) and Lemma 2.4.4, we obtain

$$|H_3(1)| = \frac{1}{4}\left|b_2\left(b_3 - \frac{3}{4}{b_2}^2\right)\right| = \frac{1}{64}|p_1||({p_1}^2 - p_1) + \xi(4 - {p_1}^2)|,$$

where $|\xi| \leq 1$. Since $H_3(1) = \alpha_3(\alpha_5 - {\alpha_3}^2)$ is rotationally invariant, so we may assume $p_1 := p \in [0,2]$. Thus using triangle inequality, we easily get $|H_3(1)| \leq (3p^3 - 4p^2 + 4p)/256 =: g(p)$. Since $g'(p) > 0$ for all $p \in [0,2]$. Therefore, $\max_{0 \leq p \leq 2} g(p) = g(2)$. Hence

$$|H_3(1)| \leq \frac{1}{16}.$$

The above estimate is sharp for $\hat{f}_1$, where f_1 is given by (2.4.3). □

In the following result, the bound obtained in the Remark 2.4.5 is improved.

Theorem 2.4.8. Let $f \in \mathcal{S}^*_{\wp}$. Then $|H_3(1)| \leq 0.150627$.

Proof. From (2.4.2) and (2.4.10), we have

$$H_3(1) = \frac{1}{9216}(-21p_1^6 + 60{p_1}^4 p_2 + 96{p_1}^3 p_3 + 192 p_1 p_2 p_3 - 144{p_1}^2{p_2}^2 - 144{p_1}^2 p_4 - 72{p_2}^3 - 256{p_3}^2 + 288 p_2 p_4)$$

and using Lemma 2.4.4 and writing p_1 as p and $t = 4 - p_1^2$, we have

$$H_3(1) = \frac{1}{9216}\left(\Upsilon_1(p,\zeta) + \Upsilon_2(p,\zeta)\eta + \Upsilon_3(p,\zeta)\eta^2 + \Upsilon_4(p,\zeta,\eta)\xi\right), \tag{2.4.17}$$

where $\zeta, \eta, \xi \in \overline{\mathbb{D}}$ and

$$\begin{aligned}
\Upsilon_1(p,\zeta) &= -4p^6 + t(t(-25p^2\zeta^2 + 19p^2\zeta^3 + 2p^2\zeta^4 + 36\zeta^3) + 5p^4\zeta - 16p^4\zeta^2 - 24p^2\zeta^3),\\
\Upsilon_2(p,\zeta) &= t(1-|\zeta|^2)(t(64p\zeta^2 - 80p\zeta) + 32p^3),\\
\Upsilon_3(p,\zeta) &= -t^2(1-|\zeta|^2)(64 + 8|\zeta|^2),\\
\Upsilon_4(p,\zeta,\eta) &= 72t^2(1-|\zeta|^2)^2\zeta.
\end{aligned}$$

Let $x = |\zeta| \in [0,1]$ and $y = |\eta| \in [0,1]$. Now using $|\xi| \leq 1$ and triangle inequality, from (2.4.17) we obtain

$$|H_3(1)| \leq \frac{1}{9216}\left(f_1(p,x) + f_2(p,x)y + f_3(p,x)y^2 + f_4(p,x)\right) \tag{2.4.18}$$

$$=: \frac{F(p,x,y)}{9216}, \tag{2.4.19}$$

where

$$\begin{aligned}
f_1(p,x) &= 4p^6 + t(t(25p^2x^2 + 19p^2x^3 + 2p^2x^4 + 36x^3) + 5p^4x + 16p^4x^2 + 24p^2x^3),\\
f_2(p,x) &= t(1-x^2)(t(80px + 64px^2) + 32p^3),\\
f_3(p,x) &= t^2(1-x^2)(64 + 8x^2)
\end{aligned}$$

and $f_4(p,x) = 72t^2x(1-x^2)^2$. Since $f_2(p,x)$ and $f_3(p,x)$ are non-negative functions over $[0,2]\times[0,1]$. Therefore, from (2.4.18) together with $y = |\eta| \in [0,1]$ we obtain

$$F(p,x,y) \leq F(p,x,1).$$

Thus, $F(p,x,1) = f_1(p,x) + f_2(p,x) + f_3(p,x) + f_4(p,x) =: G(p,x)$.
Now we shall maximize $G(p,x)$ over $[0,2]\times[0,1]$. For this we consider the following possible cases:

1. when $x = 0$, we have

$$G(p,0) = 1024 - 512p^2 + 128p^3 + 64p^4 - 32p^5 + 4p^6 =: g_1(p).$$

   Since $g_1'(p) < 0$ on $[0,2]$. Therefore, $g_1(p)$ is an decreasing function over $[0,2]$. Thus, the function $g_1(p)$ attains its maximum value at $p = 0$ which is equal to 1024.

2. when $x = 1$, we have

$$G(p,1) = 576 + 544p^2 - 272p^4 + 29p^6 =: g_2(p).$$

Since $g_2'(p) = 0$ has a critical point at $p_0 = 2\sqrt{(68 - 7\sqrt{34})/87} \approx 1.11795$. Therefore, it is easy to see that $g_2(p)$ is an increasing function for $p \leq p_0$ and decreasing for $p_0 \leq p$. Thus the function $g_2(p)$ attains its maximum at $p := p_0$, which is approximately equal to 887.674.

3. when $p = 0$, we have

$$G(0,x) = 1024 - 896x^2 + 576x^3 - 128x^4 =: g_3(x).$$

Since $g_3'(x) < 0$ on $[0,1]$. Therefore, the function $g_3(x)$ attains its maximum at $x = 0$, which is equal to 1024 and for the case, when $p = 2$, we easily obtain $G(p,x) \leq 256$.

4. when $(p,x) \in (0,2) \times (0,1)$, a numerical computation shows that there exists a unique real solution for the system of equations

$$\partial G(p,x)/\partial x = 0 \quad \text{and} \quad \partial G(p,x)/\partial p = 0$$

inside the rectangular region: $[0,2] \times [0,1]$, at $(p,x) \approx (0.531621, 0.482768)$. Consequently, we obtain $G(p,x) \leq 1388.18$.

Hence, from the above cases we conclude that

$$F(p,x,y) \leq 1388.18 \quad \text{on} \quad [0,2] \times [0,1] \times [0,1],$$

which implies that

$$H_3(1) \leq \frac{1}{9216} F(p,x,y) \leq 0.150627.$$

□

Conjecture 2.4.2. If $f \in \mathscr{S}^*_{\wp}$, then the sharp bound for the third Hankel determinant is given by

$$|H_3(1)| \leq \frac{1}{9} \approx 0.1111\cdots,$$

with the extremal function $f(z) = z\exp\left(\frac{1}{3}(e^{z^3} - 1)\right) = z + \frac{1}{3}z^4 + \frac{2}{9}z^7 + \cdots$.

Highlights of the chapter

In this chapter, we investigated the geometrical properties of the cardioid function and used them to establish several inclusion and radius results for the class of cardioid starlike functions. We derived the sharp fifth coefficient bound for functions in our class and also established initial sharp bounds for the associated inverse functions. Further, we obtained the sharp third hankel determinant for the two and three-fold symmetric functions for our class and also obtained a better estimate for the third hankel determinant. However, based on the observations made in this chapter, we proposed some conjectures pertaining to the coefficient bounds and the third hankel determinant.

Chapter 3

A class of Analytic functions associated with univalent function

In 2019, Kargar et al. [74] introduced a new class using the starlikeness expression, the first of its kind:

$$\mathscr{BS}(\alpha) := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} - 1 \prec \frac{z}{1-\alpha z^2},\ \alpha \in [0,1) \right\},$$

where $z/(1-\alpha z^2) =: \psi(z)$ (Booth lemniscate function [126] and [127]) is an analytic univalent function and symmetric with respect to the real and imaginary axes. Here, we note that the function

$$1 + \frac{z}{1-\alpha z^2}$$

is univalent but not Carathéodory for $\alpha \in (0,1)$. In fact, functions in this class $\mathscr{BS}(\alpha)$ may not be univalent. To further examine the such types of classes in general, we introduce a new class as follows:

$$\mathscr{F}(\psi) := \left\{ f \in \mathscr{A} : \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \psi(z),\ \psi(0) = 0 \right\},$$

where ψ is univalent and establish the growth theorem with some geometric conditions on ψ and obtain the Koebe domain with some related sharp inequalities. Its applications are studied for certain classes defined on the basis of geometry.

3.1 Introduction

Let $f \in \mathcal{A}$ in the open unit disk $\mathbb{D} := \{z : |z| < 1\}$. Let $f(z) = w$ and Γ_w be the image of Γ_z (the circle $C_r : z = re^{i\theta}$) under the function f in $\mathcal{A}$. The curve Γ_w is said to be starlike with respect to $w_0 = 0$ if $\arg(w - w_0)$ is a non-decreasing function of θ, that is,

$$\frac{d}{d\theta}\arg(w - w_0) \geq 0, \quad \theta \in [0, 2\pi],$$

which is equivalent to

$$\frac{d}{d\theta}\arg(w - w_0) = \Re\left(\frac{zf'(z)}{f(z)}\right) \geq 0. \tag{3.1.1}$$

If the inequality (3.1.1) holds for each circle $|z| = r < 1$, then it characterizes a special class $\mathcal{S}^*$, the class of starlike functions in $\mathbb{D}$. It is obvious from (3.1) that for each $0 < r < 1$, the curve Γ_w is not allowed to have a loop which ensures the univalency of the function. But if for some $0 \neq z \in \mathbb{D}$, $\Re(zf'(z)/f(z)) < 0$, then f is not starlike with respect to 0, or equivalently we can say that the image curve $\Gamma_w : f(|z| = r)$ is definitely not starlike, but still it may or may not be univalent. From (3.1.1), we also see the importance of the Caratheódory functions by writing (3.1.1) in terms of subordination as:

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \quad (z \in \mathbb{D}), \tag{3.1.2}$$

where the symbol $\prec$ stands for the usual subordination. In 1992, Ma and Minda [102] generalized (3.1.2) by unifying all the subclasses of starlike functions as follows:

$$\mathcal{S}^*(\Psi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \Psi(z) \right\}, \tag{3.1.3}$$

where Ψ has positive real part, $\Psi(\mathbb{D})$ symmetric about the real axis with $\Psi'(0) > 0$ and $\Psi(0) = 1$. For some special classes, refer [60, 73, 151] and the references therein. In view of the above, we consider the following definition:

Definition 3.1.1. Let ψ be the analytic univalent function in $\mathbb{D}$ such that $\psi(0) = 0$, $\psi(\mathbb{D})$ is starlike with respect to 0. Then

$$\mathcal{F}(\psi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} - 1 \prec \psi(z),\ \psi(0) = 0 \right\}. \tag{3.1.4}$$

Note that when $1 + \psi(z) \not\prec (1+z)/(1-z)$, then the functions in the class $\mathcal{F}(\psi)$ may not be univalent in $\mathbb{D}$ which also implies $\mathcal{F}(\psi) \not\subseteq \mathcal{S}^*$. Thus in case, when the function $1 + \psi =: \Psi$ has positive real part, $\Psi(\mathbb{D})$ symmetric about the real axis with $\Psi'(0) > 0$, then $\mathcal{F}(\psi)$ reduces to the class $\mathcal{S}^*(\Psi)$. The functions in the class defined in (3.1.3) are univalent which helps a lot in studying the geometrical properties of the functions, for example, Growth and Distortion theorems, etc. But this may not be quite easy to study the analogous results in the class $\mathcal{F}(\psi)$. In this direction, recently, Kargar et al. [74]

considered the following class, the first of its kind:

$$\mathcal{BS}(\alpha) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} - 1 \prec \frac{z}{1-\alpha z^2},\ \alpha \in [0,1) \right\}, \tag{3.1.5}$$

where $z/(1-\alpha z^2) =: \psi(z)$ (Booth lemniscate function [126] and [127]) is an analytic univalent function and symmetric with respect to the real and imaginary axes. Note that the function $(1+z/(1-\alpha z^2))$ assumes negative values for $\alpha \neq 0$, therefore functions in this class may not be univalent. For f belonging to $\mathcal{BS}(\alpha)$, using the vertical strip domain $\{w \in \mathbb{C} : \mu < \Re w < \nu$, where $\mu < 1 < \nu\}$, Kargar et al. [74] proved that $|f(z)/z|$ is bounded and obtained the coefficient estimates when $0 \leq \alpha \leq 3-2\sqrt{2}$ along with Fekete-Szegö inequality for the associated $k-th$ root transformation. In 2018, Najmadi et al. [118] obtained the bounds for the quantities $\Re f(z)$, $|f(z)|$ and $|f'(z)|$ when $0 \leq \alpha \leq 3-2\sqrt{2}$. Recently, Kargar et al. [75] obtained the best dominant of the subordination $f(z)/z \prec F(z)$ for the range $0 < \alpha \leq 3-2\sqrt{2}$ using the convolution technique, where $F(z) = \left(1+z\sqrt{\alpha}\right)/\left(1-z\sqrt{\alpha}\right)^{\frac{1}{2\sqrt{\alpha}}}$. Cho et al. [39] dealt with the first-order differential subordination implications and also solved the various sharp radius problems pertaining to the class $\mathcal{BS}(\alpha)$.

In 2019, Masih et al. [107] considered the following class with $\beta \in [0,1/2]$:

$$\mathcal{S}_{cs}(\beta) := \left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{z}{(1-z)(1+\beta z)},\ \beta \in [0,1) \right\}. \tag{3.1.6}$$

They proved the growth theorem and also obtained the sharp estimates for the logarithmic coefficients but only for the range $\beta \in [0,1/2]$. Note that for $\beta \in [0,1/2]$, $\mathcal{S}_{cs}(\beta)$ is a Ma-Minda subclass, but for the other range, functions in this class may not be univalent. We also introduce and study the following class in section 3.3.3:

$$\mathcal{S}_{\gamma}(\eta) := \left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{\gamma z}{(1+\eta z)^2},\ \eta \in [0,1),\ \gamma > 0 \right\}.$$

In this chapter, we establish the sharp growth theorem for the class $\mathcal{F}(\psi)$ with certain geometric conditions on ψ and obtain a covering theorem. Further, we provide some examples to illustrate our result, and some newly defined classes are also discussed. As an application, we obtain the growth theorem for the complete range of α and β for the functions in the classes $\mathcal{BS}(\alpha)$ and $\mathcal{S}_{cs}(\beta)$, respectively that improves the earlier known bounds. Finally, the sharp Bohr-radii for the classes $S(\mathcal{BS}(\alpha))$ and $\mathcal{BS}(\alpha)$ are obtained. For some classes, we study the geometrical behavior of an analytic function of the form $f(z)/z$ which arises frequently while working with the class $\mathcal{S}^*(\Psi)$ and play an important role, for example, in obtaining the bounds for $\Re(f(z)/z)$ and $\arg(f(z)/z)$. Further, geometrical properties and coefficients estimation for the class $\mathcal{F}(\psi)$ are still open.

3.2 Distortions Theorems

Let $\mathcal{F}(\psi)$ be the class as defined in (3.1.4). Now we begin with the following:

Theorem 3.2.1 (Growth Theorem)**.** Suppose that $\max_{|z|=r} \Re\psi(z) = \psi(r)$ and $\min_{|z|=r} \Re\psi(z) = \psi(-r)$.

Then $f \in \mathscr{F}(\psi)$ satisfies the sharp inequalities:

$$r \exp\left(\int_0^r \frac{\psi(-t)}{t} dt\right) \leq |f(z)| \leq r \exp\left(\int_0^r \frac{\psi(t)}{t} dt\right), \quad (|z| = r). \tag{3.2.1}$$

Proof. Let $f \in \mathscr{F}(\psi)$. For $z = re^{i\theta}$, we have

$$\phi(-r) \leq \Re\psi(re^{i\theta}) \leq \phi(r). \tag{3.2.2}$$

Let $\Phi(z) = \psi(\omega(z))$, where ω is a Schwarz function. Then from (3.1.4), we have

$$\log \frac{f(z)}{z} = \int_0^z \frac{\Phi(\zeta)}{\zeta} d\zeta.$$

Now by taking $\zeta = te^{i\beta}$ so that $d\zeta = e^{i\beta} dt$, where β is fixed but arbitrary and $z = re^{i\beta}$, we have

$$\log \frac{f(z)}{z} = \int_0^r \frac{\Phi(te^{i\beta})}{t} dt. \tag{3.2.3}$$

From the Maximum-minimum modulus principle, we find that Φ also satisfies the inequality (3.2.2). Therefore, without loss of generality, we may replace Φ by ψ and β by θ in (3.2.3). Then by equating real parts on either side of (3.2.3), we have

$$\log \left|\frac{f(z)}{z}\right| = \int_0^r \frac{\Re\Phi(te^{i\theta})}{t} dt \tag{3.2.4}$$

and now using the inequalities (3.2.2) in (3.2.4), we obtain

$$\int_0^r \frac{\psi(-t)}{t} dt \leq \log \left|\frac{f(z)}{z}\right| \leq \int_0^r \frac{\psi(t)}{t} dt,$$

and (3.2.1) follows. The result is sharp for the function

$$f_0(z) = z \exp \int_0^z \frac{\psi(t)}{t} dt. \tag{3.2.5}$$

This completes the proof. □

Remark 3.2.1. In the above theorem, we choose $\max_{|z|=r} \Re\psi(z) = \psi(r)$ and $\min_{|z|=r} \Re\psi(z) = \psi(-r)$ for computational convenience. However, these conditions may change according to the choice of ψ in that case, appropriately these may be replaced.

Remark 3.2.2. If $1 + \psi$ is a Carathéodory univalent function, then Theorem 3.2.1 reduces to the result [102, Corollary 1, p. 161] and moreover, letting r tends to 1 in Theorem 3.2.1, we obtain the covering theorem (Koebe-radius) for the class $\mathscr{F}(\psi)$.

Corollary 3.2.2 (Covering Theorem)**.** If $f \in \mathscr{F}(\psi)$ and f_0 as defined in (3.2.5), then either f is a rotation of f_0 or

$$\{w \in \mathbb{C} : |w| \leq -f_0(-1)\} \subset f(\mathbb{D}),$$

where $-f_0(-1) = \lim_{r \to 1}(-f_0(-r))$.

Let $L(f,r)$ denote the length of the boundary curve $f(|z|=r)$. Note that for $z=re^{i\theta}$, we have $L(f,r):=\int_0^{2\pi}|zf'(z)|d\theta$. Now we obtain the following result:

Corollary 3.2.3. Assume that $\max_{|z|=r}|\psi(z)|=\psi(r)$ and further let $M(r)=\exp\left(\int_0^r\frac{\psi(t)}{t}dt\right)$. If $f\in\mathscr{F}(\psi)$, then for $|z|=r$, we have

$$\Re\frac{f(z)}{z}\leq M(r),\ \ |f'(z)|\leq(1+\psi(r))M(r)$$

and

$$L(f,r)\leq 2\pi r(1+\psi(r))M(r).$$

Let

$$\psi(z)=\begin{cases}\beta z/(1+\alpha z), & \beta>0,\,0<\alpha<1\,;\\ \eta z, & \eta>0.\end{cases}$$

Then the above two choices of ψ are clearly convex univalent and $\psi(\mathbb{D})$ are symmetric about real axis as $\overline{\psi(z)}=\psi(\bar{z})$. It is further evident that $1+\psi(z)\not\prec(1+z)/(1-z)$ except for the second choice of ψ when $0<\eta\leq 1$. We now obtain the following sharp result from Theorem 3.2.1:

Example 3.2.4. Let $f\in\mathscr{F}(\beta z/(1+\alpha z))$, where $\beta>0$ and $0<\alpha<1$ and $|z|=r$. Then

$$r(1-\alpha r)^{\frac{\beta}{\alpha}}\leq|f(z)|\leq r(1+\alpha r)^{\frac{\beta}{\alpha}},$$

which implies:

$$\left\{w:|w|\leq(1-\alpha)^{\frac{\beta}{\alpha}}\right\}\subset f(\mathbb{D}),\ \ |f'(z)|\leq\left(1+\frac{\beta r}{1+\alpha r}\right)(1+\alpha r)^{\frac{\beta}{\alpha}}$$

and

$$\Re\frac{f(z)}{z}\leq(1+\alpha r)^{\frac{\beta}{\alpha}}.$$

Example 3.2.5. Let $f\in\mathscr{F}(\eta z)$, where $\eta>0$ and $|z|=r$. Then we have

$$r\exp(-\eta r)\leq|f(z)|\leq r\exp(\eta r),$$

which implies:

$$\{w:|w|\leq\exp(-\eta)\}\subset f(\mathbb{D}),\ \ |f'(z)|\leq(1+\eta r)\exp(\eta r)$$

and

$$\Re\frac{f(z)}{z}\leq\exp(\eta r).$$

From the above examples, it is clear that $f\in\mathscr{F}(\psi)$ if and only if

$$\frac{zf'(z)}{f(z)}\in\begin{cases}\Omega_1, & \psi(z)=\beta z/(1+\alpha z);\\ \Omega_2, & \psi(z)=\eta z,\end{cases}$$

where $\Omega_1=\{w\in\mathbb{C}:|w-1|<|\beta-\alpha(w-1)|\}$ and $\Omega_2=\{w\in\mathbb{C}:|w-1|<\eta\}$, respectively for $z\in\mathbb{D}$.

3.3 Some Applications and Further results

3.3.1 On Booth-Lemniscate

Let $\mathcal{BS}(\alpha)$ be the class as defined in (3.1.5).

Theorem 3.3.1. Let $0<\alpha<1$ and $f\in\mathcal{BS}(\alpha)$, then for $|z|=r$ we have

$$-\hat{f}(-r)\leq |f(z)|\leq \hat{f}(r), \tag{3.3.1}$$

where

$$\hat{f}(z)=z\left(\frac{1+z\sqrt{\alpha}}{1-z\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}}. \tag{3.3.2}$$

The result is sharp.

Proof. Let $\psi(z):=z/(1-\alpha z^2)$ and $f\in\mathcal{BS}(\alpha):=\mathcal{F}(\psi)$. For $z=re^{i\theta}$, we have

$$-\frac{r}{1-\alpha r^2}\leq \Re\psi(re^{i\theta})\leq \frac{r}{1-\alpha r^2}$$

and

$$-\int_0^r \frac{1}{1-\alpha t^2}dt\leq \log\left|\frac{f(z)}{z}\right|\leq \int_0^r \frac{1}{1-\alpha t^2}dt,$$

where

$$\int_0^r \frac{1}{1-\alpha t^2}dt=\frac{1}{2\sqrt{\alpha}}\log\frac{1+\sqrt{\alpha}r}{1-\sqrt{\alpha}r}.$$

Hence, the result follows from Theorem 3.2.1. □

Remark 3.3.1. Theorem 4.5.1 improves the upper bound of $\Re f(z)$ and bounds of $|f(z)|$, obtained in [118, Theorem 2, p. 116] and [118, Theorem 3, p. 116] respectively.

We now extend [75, Theorem 2.6, p. 1238] for the complete range of α using Theorem 4.5.1:

Corollary 3.3.2. Let $f\in\mathcal{BS}(\alpha)$, $\alpha\in(0,1)$ and $|z|=r$, then

$$\Re\frac{f(z)}{z}\leq\left(\frac{1+r\sqrt{\alpha}}{1-r\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}}\quad\text{and}\quad |f'(z)|\leq\left(1+\frac{r}{1-\alpha r^2}\right)\left(\frac{1+r\sqrt{\alpha}}{1-r\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}}.$$

The result is sharp for the function $\hat{f}$ given in (3.3.2).

Corollary 3.3.3. Let $\alpha\in(0,1)$ be fixed. Then $f\in\mathcal{BS}(\alpha)$ satisfies the inequality

$$L(f,r)\leq 2\pi r\left(1+\frac{r}{1-\alpha r^2}\right)\left(\frac{1+r\sqrt{\alpha}}{1-r\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}},\quad (|z|=r).$$

Corollary 3.3.4 (Koebe-radius)**.** Let $0<\alpha<1$ and $\hat{f}$ as given in (3.3.2). If $f\in\mathcal{BS}(\alpha)$, then either $f$is a rotation of $\hat{f}$ or

$$\{w\in\mathbb{C}:|w|\leq -\hat{f}(-1)\}\subset f(\mathbb{D}).$$

Proof. The proof follows by letting r tends to 1 in the inequality $-\hat{f}(-r) \le |f(z)|$, given in (3.3.1). $\square$

Theorem 3.3.5. Let $\alpha \in (0, 3-2\sqrt{2}]$ be fixed. Then $f \in \mathscr{BS}(\alpha)$ satisfies the sharp inequality

$$\left|\arg \frac{f(z)}{z}\right| \le \max_{|z|=r} \arg \left(\frac{1+z\sqrt{\alpha}}{1-z\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}}.$$

Proof. From [75, Theorem 2.5, p. 1238], we have $f(z)/z \prec \hat{f}(z)/z$ for $0 < \alpha \le 3-2\sqrt{2}$, where $\hat{f}$ is defined in (3.3.2). Since the function $\hat{f}(z)/z$ is convex and symmetric about the real axis in $\mathbb{D}$, therefore we easily see that

$$\left(\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}} > 0.$$

Thus $\hat{f}(z)/z$ is a Carathéodory function and the result follows. $\square$

For our next result, we need the following definition and a related class:

Definition 3.3.6. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ are analytic in $\mathbb{D}$ and $f(\mathbb{D}) = \Omega$. Consider a class of analytic functions $S(f) := \{g : g \prec f\}$ or equivalently $S(\Omega) := \{g : g(z) \in \Omega\}$. Then the class $S(f)$ is said to satisfy Bohr-phenomenon, if there exists a constant $r_0 \in (0,1]$ such that the inequality $\sum_{k=1}^{\infty} |b_k| r^k \le d(f(0), \partial\Omega)$ holds for all $|z| = r \le r_0$, where $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of $\Omega = f(\mathbb{D})$. The largest such r_0 for which the inequality holds is called the Bohr-radius.

See the articles [10, 31] and the references therein for more. Let us now introduce the following class:

$$S(\mathscr{BS}(\alpha)) := \left\{ g : g \prec f,\ g(z) = \sum_{k=1}^{\infty} b_k z^k \text{ and } f \in \mathscr{BS}(\alpha) \right\}.$$

Theorem 3.3.7 (Booth-Bohr-radius)**.** The class $S(\mathscr{BS}(\alpha))$ satisfies the Bohr-phenomenon in $|z| \le r(\alpha)$, where $r(\alpha)$ is the unique positive root of the equation

$$r\left(\frac{1+r\sqrt{\alpha}}{1-r\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}} - \left(\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}} = 0, \tag{3.3.3}$$

whenever $0 < \alpha \le 3-2\sqrt{2}$. The result is sharp for the function $\hat{f}$ given in (3.3.2).

Proof. Since $g \in S(\mathscr{BS}(\alpha))$, we have $g \prec f$ for a fixed $f \in \mathscr{BS}(\alpha)$. From Corollary 3.3.4, we obtain the Koebe-radius $r_* = -\hat{f}(-1)$ such that $r_* \le d(0, \partial\Omega) = |f(z)|$ for $|z| = 1$. Also using [75, Theorem 2.5, p. 1238], we have

$$\frac{f(z)}{z} \prec \frac{\hat{f}(z)}{z}. \tag{3.3.4}$$

Recall the result [31, Lemma 1, p.1090], which reads as: let f and g be analytic in $\mathbb{D}$ with $g \prec f$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$. Then $\sum_{k=0}^{\infty} |b_k| r^k \le \sum_{n=0}^{\infty} |a_n| r^n$ for $|z| = r \le 1/3$. Now using the result for $g \prec f$ and (4.7.4), we have

$$\sum_{k=1}^{\infty} |b_k| r^k \le r + \sum_{n=2}^{\infty} |a_n| r^n \le \hat{f}(r) \quad \text{for } |z| = r \le 1/3.$$

Finally, to establish the inequality $\sum_{k=1}^{\infty}|b_k|r^k \le d(f(0),\partial\Omega)$, it is enough to show $\hat{f}(r) \le r_*$. But this holds whenever $r \le r(\alpha)$, where $r(\alpha)$ is the least positive root of the equation $\hat{f}(r) = r_*$. Now let $T(r) := \hat{f}(r) - r_*$, then

$$T'(r) = \left(\frac{1+r\sqrt{\alpha}}{1-r\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}} + r\left(\frac{1+r\sqrt{\alpha}}{1-r\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}-1}\frac{1}{(1-r\sqrt{\alpha})^2}.$$

Since $(1+r\sqrt{\alpha})/(1-r\sqrt{\alpha}) > 0$, therefore $T'(r) > 0$ and so T is an increasing function of r. Also $T(0) < 0$ and $T(1) > 0$. Thus the existence of the root $r(\alpha)$ is ensured by the Intermediate Value theorem for the continuous functions. By a computation, it can easily be seen that $r(\alpha) < 1/3$ and hence the result. □

Corollary 3.3.8. Let $0 < \alpha \le 3-2\sqrt{2}$. The Bohr-radius for the class $\mathcal{BS}(\alpha)$ is $r(\alpha)$, where $r(\alpha)$ is the unique positive root of the Eq. (3.3.3).

3.3.2 On Cissoid of Diocles

Let us consider

$$S_\beta(z) = \frac{z}{(1-z)(1+\beta z)} = \frac{1}{1+\beta}\left(\frac{1}{1-z} - \frac{1}{1+\beta z}\right) = \sum_{n=1}^{\infty}\frac{1-(-\beta)^n}{1+\beta}z^n,$$

where $\beta \in [0,1)$. Clearly, it is analytic, symmetric about the real-axis and maps the unit disk $\mathbb{D}$ onto the domain bounded by *Cissoid of Diocles*:

$$CS(\beta) := \left\{u+iv \in \mathbb{C} : \left(u - \frac{1}{2(\beta-1)}\right)(u^2+v^2) + \frac{2\beta}{(1+\beta)^2(\beta-1)}v^2 = 0\right\}.$$

Let us now consider the class $\mathcal{S}_{cs}(\beta)$ as defined in (3.1.6). Masih et al. [107] considered this class with $\beta \in [0,1/2]$ since $\Re(1+z/((1-z)(1+\beta z)) \ge (2\beta-1)/(2(\beta-1)) \ge 0$. Clearly, $\mathcal{S}_{cs}(\beta) = \mathcal{F}(S_\beta(z))$ for $\beta \in [0,1)$ and we have the following result:

Theorem 3.3.9. Let $f \in \mathcal{S}_{cs}(\beta)$ and $\beta \in [0,1)$. Then for $|z| = r$ we have the following sharp inequality:

$$-\tilde{f}(-r) \le |f(z)| \le \tilde{f}(r),$$

where

$$\tilde{f}(z) = z\left(\frac{1+\beta z}{1-z}\right)^{\frac{1}{1+\beta}}. \tag{3.3.5}$$

Proof. Let $\psi(z) := z/((1-z)(1+\beta z))$ and $f \in \mathcal{S}^*_{cs}(\beta) := \mathcal{F}(\psi)$. Following the proof of [107, Theorem 3.1, p. 5], it is easy to see that for $z = re^{i\theta}$, where $\theta \in [0,2\pi]$, we have

$$\min_{|z|=r}\Re\psi(z) = \frac{-r+(\beta-1)r^2+\beta r^3}{(1+r)^2(1-\beta r)^2} = \psi(-r)$$

and

$$\max_{|z|=r} \Re\psi(z) = \lim_{\theta\to 0} \frac{-r^2+\beta r^2-\beta r^3\cos\theta + r\cos\theta}{(1+r^2-2r\cos\theta)(1+\beta^2r^2+2\beta r\cos\theta)}$$
$$\leq \frac{\beta-1}{2(1+\beta)^2}$$
$$= \max_{|z|=1} \Re\psi(z).$$

Thus, we have $\psi(-r) \leq \Re\psi(z) \leq \psi(r)$ for $r \neq 1$, and for $r = 1$

$$1/(2(\beta-1)) = \psi(-1) \leq \Re\psi(z) \leq (\beta-1)/(2(\beta+1)^2).$$

Also, note that

$$\tilde{f}(z) = z\exp\int_0^z \frac{\psi(t)}{t}dt = z\left(\frac{1+\beta z}{1-z}\right)^{\frac{1}{1+\beta}}.$$

Now the result follows from Theorem 3.2.1. □

Remark 3.3.2. Let $\tilde{F}(z) = \tilde{f}(z)/z$ and $|z| = 1$, where $\tilde{f}$ is as defined in Theorem 3.3.9. A calculation show that

$$1+\frac{\tilde{F}''(z)}{\tilde{F}'(z)} = 1+\frac{-\beta z}{(1+\beta z)(1-z)}+\frac{2z}{1-z},$$

which implies that

$$\Re\left(1+\frac{\tilde{F}''(z)}{\tilde{F}'(z)}\right) \geq \beta\Re\left(\frac{-z}{(1+\beta z)(1-z)}\right).$$

Since

$$\Re\left(\frac{-z}{(1+\beta z)(1-z)}\right) = \frac{1+\beta}{2(1+\beta^2-2\beta\cos\theta)} =: g(\theta),$$

and a simple calculation shows that g attains its minimum at $\theta = 0$. Therefore, we have

$$\Re\left(1+\frac{\tilde{F}''(z)}{\tilde{F}'(z)}\right) \geq \frac{\beta(1+\beta)}{2(1-\beta)^2} \geq 0.$$

Hence $\tilde{F}$ is convex univalent in $\mathbb{D}$.

Remark 3.3.3. Observe that the function $S_\beta(z)$ is not convex when $\beta \neq 0$ and the result, $f(z)/z \prec \tilde{F}(z)$ similar to theorem 3.3.14 is still open for $f \in \mathscr{S}_{cs}(\beta)$.

By letting r tends to 1 in the above Theorem 3.3.9, we obtain:

Corollary 3.3.10 (Koebe-radius)**.** Let $\tilde{f}$ as given in (3.3.5). If $f \in \mathscr{S}_{cs}(\beta)$, then either $f$is a rotation of $\tilde{f}$ or

$$\left\{w \in \mathbb{C} : |w| \leq -\tilde{f}(-1) = \left(\frac{1-\beta}{2}\right)^{1/(1+\beta)}\right\} \subset f(\mathbb{D}).$$

Remark 3.3.4. We improved the result [107, Corollary 4.3.1, p. 8] in Theorem 3.3.9 and Corollary 3.3.10 by extending the range of β.

3.3.3 Modified Koebe function

The Koebe function $k(z) = z/(1-z)^2$ has a pole at $z = 1$ and maps unit disk onto the domain $\mathbb{C}\backslash(-\infty, 1/4]$, which is a slit domain. We now introduced the modified Koebe function:

$$K(z) := \frac{z}{(1+\eta z)^2}, \quad 0 \le \eta < 1, \tag{3.3.6}$$

which is bounded in $\mathbb{D}$ and symmetric about the real-axis. It is interesting to observe the geometry of the domain $K(\mathbb{D})$, which assumes different shapes for different choices of η such as a convex or a Bean or a Cardioid-shaped domain. Especially when η tends to 1, we see that one of the rotations of the image domain $K(\mathbb{D})$ will converge to $k(\mathbb{D})$ and thereby justifying the name of $K(z)$. Since $k(z) = (u^2(z)-1)/4$, where $u(z) = (1+z)/(1-z)$, in a similar fashion, we can write

$$K(z) = \frac{1}{4\eta}(1-v^2(z)),$$

where $v(z) = (1-\eta z)/(1+\eta z)$ and $\eta \neq 0$.

Lemma 3.3.1. The function $K(z)$ as defined in (3.3.6) is convex for $0 \le \eta \le 2-\sqrt{3}$.

Proof. Let $K(z) = z/(1+\eta z)^2$. When $\eta = 0$, $K(z)$ is the identity function and hence is convex. So let us consider $0 < \eta < 1$. By a computation, we obtain that

$$1 + \frac{zK''(z)}{K'(z)} = \frac{1-4\eta z+\eta^2 z^2}{(1-\eta z)(1+\eta z)}.$$

Putting $z = e^{i\theta}$, we have

$$\Re\left(1 + \frac{zK''(z)}{K'(z)}\right) = \frac{1-4\eta(1-\eta^2)\cos\theta - \eta^4}{((1+\eta^2)^2-(2\eta\cos\theta)^2)}. \tag{3.3.7}$$

Since $((1+\eta^2)^2-(2\eta\cos\theta)^2) > 0$ for all θ and for each fixed η. Therefore, we now only need to consider the numerator in (3.3.7). A computation reveals that

$$N(\theta) := 1-4\eta(1-\eta^2)\cos\theta - \eta^4$$

is increasing in $0 \le \theta \le \pi$ (note that $N(\theta) = N(-\theta)$) with $N(\theta) \ge 0$ when $0 < \eta \le 2-\sqrt{3}$, while $N(\theta)$ takes negative values when $\eta > 2-\sqrt{3}$. Hence by the definition of convexity, result follows. $\square$

Now let us consider the function

$$\psi(z) := \frac{\gamma z}{(1+\eta z)^2} = \gamma K(z), \quad \text{where} \quad \gamma > 0.$$

Then following the class $\mathcal{F}(\psi)$ defined in (3.1.4), we introduce a related class defined as follows:

$$\mathcal{S}_\gamma(\eta) := \left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{\gamma z}{(1+\eta z)^2},\ \eta \in [0,1),\ \gamma > 0 \right\}. \tag{3.3.8}$$

Note that if γ and η satisfies the condition $(1-\eta)^2 \geq \gamma$, then the class $\mathscr{S}_\gamma(\eta)$ reduces to a Ma-Minda subclass of univalent starlike functions. Also letting $\eta = 1/4$ and $\gamma = 25(\sqrt{2}-1)/16$, we see that the class $\mathscr{S}^*(\sqrt{1+z}) \subset \mathscr{S}_\gamma(\eta)$.

Theorem 3.3.11. Let $f \in \mathscr{S}_\gamma(\eta)$ and $\eta \in [0, 2-\sqrt{3}]$. Then for $|z| = r$ we have the following sharp inequality:

$$-\kappa(-r) \leq |f(z)| \leq \kappa(r),$$

where

$$\kappa(z) := z\exp\left(\frac{\gamma z}{(1+\eta z)^2}\right).$$

Proof. Since $\psi(z) = \gamma K(z)$, using Lemma 3.3.1, we see that for $|z| = r$,

$$\psi(-r) \leq \Re\psi(z) \leq \psi(r).$$

Also, we have $\kappa(z) = z\exp\int_0^z(\psi(t)/t)dt$. Hence, the result follows from Theorem 3.2.1. □

We also note that $\Re\psi(z) \geq \psi(-r)$ for all $\eta \in [0,1)$ which implies $-\kappa(-r) \leq |f(z)|$ holds for $\eta \in [0,1)$ in Theorem 3.3.11. So we have the following result:

Corollary 3.3.12 (Koebe-radius)**.** Let $f \in \mathscr{S}_\gamma(\eta)$ and $\eta \in [0,1)$. Then either f is a rotation of κ or

$$\left\{w \in \mathbb{C} : |w| \leq -\kappa(-1) = \exp\left(\frac{-\gamma}{(1-\eta)^2}\right)\right\} \subset f(\mathbb{D}).$$

Recall that a function $f \in \mathscr{A}$ is starlike of order $\alpha \in [0,1)$, if $\Re(zf'(z)/f(z)) > \alpha$. Thus using $\Re(zf'(z)/f(z)) \geq \Re\psi(z) \geq \psi(-r)$ for all $\eta \in [0,1)$, we have the following result:

Theorem 3.3.13 (Radius of starlikeness)**.** Let $f \in \mathscr{S}_\gamma(\eta)$, $\gamma > 0$ and $\eta \in [0,1)$. Then f is starlike (univalent) of order $\alpha \in [0,1)$ inside the disk $|z| < r_0$, where r_0 is the smallest positive root of the equation

$$(1-\alpha)\eta^2 r^2 - (2(1-\alpha)\eta + \gamma)r + (1-\alpha) = 0.$$

Remark 3.3.5. Let $F_\kappa(z) := \kappa(z)/z = \exp(\gamma z/(1+\eta z)^2)$. We see that for $\eta = 0$ and $\gamma \leq 1$, F_κ is clearly convex. So consider $0 < \eta < 1$. After some calculations, we obtain that

$$G(z) := 1 + \frac{zF''_\kappa(z)}{F'_\kappa(z)} = \frac{\eta^4 z^4 + (2\eta^3 + \gamma\eta^2)z^3 - (6\eta^2 + 2\eta\gamma)z^2 + (\gamma - 2\eta)z + 1}{(1+\eta z)^3(1-\eta z)}.$$

Now for $z = e^{i\theta}$, the denominator of the real part of G is $(1+\eta^2 - 2\eta\cos\theta)(1+\eta^2+2\eta\cos\theta)^3 > 0$, since $(1-\eta)^2 > 0$ and therefore, it suffices to consider the numerator. After a rigorous computation, we find that numerator of the real part of G is non-negative if and only if $0 < \gamma < 1$ and $0 < \eta \leq \eta_0$, where η_0 (depends on γ) is the smallest positive root of the equation

$$(1-\gamma) + (3\gamma - 10)\eta^2 + 12\eta^3 + (8-3\gamma)\eta^4 - 16\eta^5 + (2+\gamma)\eta^6 + 4\eta^7 - \eta^8 = 0. \tag{3.3.9}$$

Hence, F_κ convex for $0 < \gamma < 1$ and $0 < \eta \leq \eta_0$.

For our next result, we need to recall the following result of Ruscheweyh and Stankiewicz [145]:

Lemma 3.3.2 ([145]). Let the analytic functions F and G be convex univalent in $\mathbb{D}$. If $f \prec F$ and $g \prec G$, then

$$f * g \prec F * G \quad (z \in \mathbb{D}).$$

Theorem 3.3.14. Let $\eta \in [0, 2-\sqrt{3}]$. If f belongs to the class $\mathscr{S}_\gamma(\eta)$, then

$$\frac{f(z)}{z} \prec F_\kappa(z), \quad (z \in \mathbb{D})$$

where $F_\kappa(z) = \kappa(z)/z$ is the best dominant and κ as defined in Theorem 3.3.11.

Proof. Let $f \in \mathscr{S}_\gamma(\eta)$, then by definition we have

$$\phi(z) := \frac{zf'(z)}{f(z)} - 1 \prec \psi(z). \tag{3.3.10}$$

It is well-known that the function

$$g(z) = \log\left(\frac{1}{1-z}\right) = \sum_{n=1}^{\infty} \frac{z^n}{n} \in \mathscr{C},$$

where $\mathscr{C}$ is the usual class of normalized convex(univalent) function and thus for $f \in \mathscr{A}$, we get

$$\phi(z) * g(z) = \int_0^z \frac{\phi(t)}{t} dt \quad \text{and} \quad \psi(z) * g(z) = \int_0^z \frac{\psi(t)}{t} dt. \tag{3.3.11}$$

From Lemma 3.3.1, we see that ψ is convex for $\eta \in [0, 2-\sqrt{3}]$. Thus applying Lemma 3.3.2 in (3.3.10), we get

$$\phi(z) * g(z) \prec \psi(z) * g(z). \tag{3.3.12}$$

Now from (3.3.11) and (3.3.12), we obtain

$$\int_0^z \frac{\phi(t)}{t} dt \prec \int_0^z \frac{\psi(t)}{t} dt,$$

which implies that

$$\frac{f(z)}{z} := \exp \int_0^z \frac{\phi(t)}{t} dt \prec \exp \int_0^z \frac{\psi(t)}{t} dt =: \frac{\kappa(z)}{z}.$$

This completes the proof. □

Corollary 3.3.15. Let $0 < \gamma < 1$ and $0 < \eta \leq \min\{2-\sqrt{3}, \eta_0\}$, where η_0 is the least positive root of the equation (3.3.9) and also let $0 < \gamma \leq \pi/2$ when $\eta = 0$. If $f \in \mathscr{S}_\gamma(\eta)$, then f satisfies the sharp inequality

$$\left|\arg \frac{f(z)}{z}\right| \leq \max_{|z|=r} \arg \exp\left(\frac{\gamma z}{(1+\eta z)^2}\right).$$

Proof. Let $F_\kappa(z) := \kappa(z)/z = \exp(\gamma z/(1+\eta z)^2)$ which is symmetric about the real axis. From Theorem 3.3.14, have $f(z)/z \prec F_\kappa(z)$ for $0 \leq \eta \leq 2-\sqrt{3}$. Since for $\eta = 0$, $\Re F_\kappa(z) > 0$ if and only $\gamma \leq \pi/2$. The result is obvious. Now from Remark 3.3.5, we see that if $0 < \gamma < 1$ and $0 < \eta \leq \min\{2-\sqrt{3}, \eta_0\}$, where η_0 is the least positive root of the equation (3.3.9) then F_κ is convex which implies

$$\Re F_\kappa(z) \geq \exp\left(\frac{-\gamma}{(1-\eta)^2}\right) > 0,$$

and F_κ is also a Carathéodory function in this case. Hence the result follows. □

Now using Theorem 3.3.11, Remark 3.3.5 and Theorem 3.3.14, we obtain the following result:

Theorem 3.3.16. Let $f \in \mathcal{S}_\gamma(\eta)$, then

$$\Re\left(\frac{f(z)}{z}\right) \leq \exp\left(\frac{\gamma r}{(1+\eta r)^2}\right) \quad \text{for} \quad \eta \in [0,1)$$

and

$$\min_{|z|=r} \exp\left(\frac{\gamma z}{(1+\eta z)^2}\right) \leq \Re\left(\frac{f(z)}{z}\right) \quad \text{for} \quad \eta \in [0, 2-\sqrt{3}].$$

In particular, if $0 < \gamma < 1$ and $0 < \eta \leq \min\{2-\sqrt{3}, \eta_0\}$, where η_0 is the least positive root of the equation (3.3.9), then

$$\exp\left(\frac{-\gamma r}{(1-\eta r)^2}\right) \leq \Re\left(\frac{f(z)}{z}\right).$$

The result is sharp.

Remark 3.3.6. It is interesting to observe that even in the class $\mathcal{F}(\psi)$, functions may not be univalent. But with the conditions on the bounds for the real part of ψ, a similar result holds as obtained by Ma-Minda [102] which is quite important to obtain the Koebe domain. From Remark 3.3.2 and Remark 3.3.5, we also note that the function $f_0(z)/z$, where f_0 as defined in (3.2.5) behaves quite differently in the particular classes. With this perspective, we conclude this chapter by introducing the following three new subclasses of $\mathcal{F}(\psi)$:

$$\mathcal{T} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} - 1 \prec \log(1-z) \right\},$$

which means $zf'(z)/f(z) \in \{w \in \mathbb{C} : |\exp(w-1) - 1| < 1\}$,

$$\mathcal{S}_p := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 - \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \right\},$$

or equivalently $zf'(z)/f(z) \in \{w \in \mathbb{C} : |1-w| < \Re((1-w)+\pi^2)\}$, a parabola with opening in left half-plane and

$$\mathcal{L}(\beta) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} - 1 \prec \frac{z}{\cos(\beta z)}, \ \beta \in [0,1] \right\}.$$

The above new classes are still open to study. Also see figure 6.4.3. Note that for the classes $\mathcal{T}$ and $\mathcal{L}(\beta)$, the function f_0 defined in (3.2.5) takes the respective particular form

$$f_{\mathcal{T}}(z) := z\exp(-Li_2(z)),$$

where

$$-\int_0^z \frac{\log(1-t)}{t} dt = \sum_{n=1}^{\infty} \frac{z^n}{n^2} =: Li_2(z)$$

known as dilogarithm function and

$$f_{\mathscr{L}}(z) := z \exp \int_0^z \frac{1}{\cos \beta t} dt = z(\sec \beta z + \tan \beta z)^{1/\beta}), \quad \beta \neq 0.$$

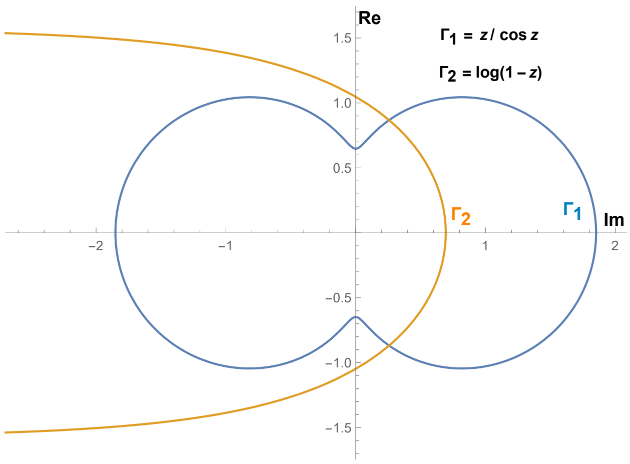


Figure 3.1: Boundary curves of the functions $z/\cos z$ and $\log(1-z)$

Highlights of the chapter

In this chapter, we initiated a systematic way to study non-univalent functions by introducing the class $\mathscr{F}(\psi)$. For this, we established growth theorems and also derived radii of starlikeness. For the particular subclass $\mathscr{BS}(\alpha)$ and a related class of subordinants $S(\mathscr{BS}(\alpha))$, we established Bohr's phenomenon. Likewise, there are several types of radius and coefficient problems, which can be covered in future attempts.

Chapter 4

Some general results for the Ma-Minda classes

In this chapter, we consider functions analytic in the unit disk that are subordinate to functions of the same type that are defined by certain differential subordinations. We prove several sharp majorization theorems and a product theorem. Further, necessary and sufficient conditions along with other aspects of radius problems are studied by employing the technique of subordination and convolution.

4.1 Introduction

In 1967, MacGregor [104] started the radius problems, which says that if $|f(z)|-|g(z)|>0$ for each $|z|<1$, then $|f'(z)|-|g'(z)|>0$ in $|z|<2-\sqrt{3}$, when f is univalent. But the problem when f belongs to others classes is still open. To proceed next, first let us recall an equivalent definition of majorization.

Definition 4.1.1 ([104]). Let f and g be analytic in $\mathbb{D}$. A function $g(z)$ is said to be majorized by $f(z)$, denoted by $g << f$, if there exists an analytic function $\Phi(z)$ in $\mathbb{D}$ satisfying $|\Phi(z)|\leq 1$ and $g(z)=\Phi(z)f(z)$ for all $z\in\mathbb{D}$.

Theorem D ([104]). Let g be majorized by f in $\mathbb{D}$ and $g(0)=0$. If $f(z)$ is univalent in $\mathbb{D}$, then $|g'(z)|\leq|f'(z)|$ in $|z|\leq 2-\sqrt{3}$. The constant $2-\sqrt{3}$ is sharp.

Recently Tang and Deng [164] obtained the majorization results for $\mathcal{S}^*(\psi)$ for some specific choices of ψ, motivated by this in section 6.2.1, we devise a general approach to handle the same for $\mathcal{C}(\psi)$, which is precisely stated as: if $g\in\mathcal{A}$, $f\in\mathcal{C}(\psi)$ and g is majorized by f in $\mathbb{D}$, then we find the largest radius $r_\psi\leq 1$ such that $|g'(z)|\leq|f'(z)|$ in $|z|\leq r_\psi$. Several other results in this direction are also obtained. In section 4.3, we consider the radius problem posed by Obradović and Ponnusamy [121] namely: Let $g \in \mathcal{S}^*(\psi_1)$ and $h \in \mathcal{S}^*(\psi_2)$, then find the largest radius $r_0 \leq 1$ such that the function $F(z) = (g(z)h(z))/z$ belongs to certain well-known class of starlike functions in $|z| < r_0$. As a special case, we also obtain a result of Obradović and Ponnusamy [121]. Throughout this chapter, we shall assume that the function ϕ in $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ has real coefficients in its power series expansion.

4.2 Majorization for starlike and convex functions

Let us consider the analytic function $\psi(z) := 1 + B_1 z + B_2 z^2 + \cdots$. Here $B_1 = \psi'(0)$, the coefficient of z, plays a major role in deciding the orientation of the function ψ. Thus ψ is positively or negatively oriented depends on whether B_1 is positive or negative. Ma-Minda only considered the case $\psi'(0) > 0$, as it may be possible that for the case when $\psi'(0) < 0$, many postulates for the class $\mathcal{S}^*(\psi)$ need not remain same. With this perspective, we begin with the following:

Theorem 4.2.1. Let $\Re\phi(z) > 0$ and ϕ be convex in $\mathbb{D}$ with $\phi(0) = 1$. Suppose ψ be the function such that $m_r := \min_{|z|=r} |\psi(z)|$ and also satisfies the differential equation

$$\psi(z) + \frac{z\psi'(z)}{\psi(z)} = \phi(z). \tag{4.2.1}$$

Let $g \in \mathcal{A}$ and $f \in \mathcal{C}(\phi)$. If g is majorized by f in $\mathbb{D}$, then

$$|g'(z)| \leq |f'(z)| \quad \text{in } |z| \leq r_\psi, \tag{4.2.2}$$

where r_ψ is the least positive root of the equation

$$(1 - r^2)m_r - 2r = 0. \tag{4.2.3}$$

The result is sharp for the case $m_r = \psi(-r)$.

Proof. Let us define $p(z) := zf'(z)/f(z)$. Since $f \in \mathcal{C}(\phi)$, therefore we have $1 + zf''(z)/f'(z) \prec \phi(z)$, which can be equivalently written as

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z). \tag{4.2.4}$$

Since $\Re\phi(z) > 0$ and ϕ is convex in $\mathbb{D}$, therefore using [110, Theorem 3.2d, p. 86] the solution ψ of the differential equation (4.2.1) is analytic in $\mathbb{D}$ with $\Re\psi(z) > 0$ and has the following integral form given by

$$\psi(z) := h(z)\left(\int_0^z \frac{h(t)}{t}dt\right)^{-1},$$

where

$$h(z) = z\exp\int_0^z \frac{\phi(t) - 1}{t}dt.$$

Since $\Re\psi(z) > 0$ and p satisfies the subordination (4.2.4), therefore using [110, Lemma 3.2e, p. 89] we conclude that ψ is univalent and $p \prec \psi$, where ψ is the best dominant. Thus we have obtained that

$f \in \mathscr{C}(\phi)$ implies $zf'(z)/f(z) \prec \psi(z)$ and ψ is the best dominant (which is important for the sharpness of result), which is a univalent Carathéodory function. Now as $g \in \mathscr{A}$ and $f \in \mathscr{C}(\phi)$, therefore we obtain the following well defined equality

$$\frac{f(z)}{f'(z)} = \frac{z}{\psi(\omega(z))}, \quad (z \in \mathbb{D})$$

where ω is a Schwarz function. Hence, using $\min_{|z|=r} |\psi(\omega(z))| \geq \min_{|z|=r} |\psi(z)|$ and the hypothesis $\min\limits_{|z|=r} |\psi(z)| = m_r$, we obtain that

$$\left|\frac{f(z)}{f'(z)}\right| \leq \frac{r}{m_r}, \quad (0 < r < 1). \tag{4.2.5}$$

Now if g is majorized by f, then by definition, we have $g(z) = \Psi(z)f(z)$, where Ψ is analytic and satisfies $|\Psi(z)| \leq 1$ in $\mathbb{D}$ such that $g'(z) = \Psi(z)f'(z) + \Psi'(z)f(z)$. Thus using (4.2.5) together with the following Schwarz-Pick inequality

$$|\Psi'(z)| \leq \frac{1-|\Psi(z)|^2}{1-|z|^2},$$

we obtain

$$|g'(z)| \leq |f'(z)| \left(|\Psi(z)| + \frac{1-|\Psi(z)|^2}{1-r^2} \frac{r}{m_r} \right) = |f'(z)| h(\beta, r), \tag{4.2.6}$$

where $|\Psi(z)| := \beta$ and

$$h(\beta, r) = \beta + \frac{1-\beta^2}{1-r^2} \frac{r}{m_r}.$$

Thus to arrive at (4.2.2), it suffices to show that $h(\beta, r) \leq 1$, which is equivalent to show that

$$k(\beta, r) := (1-r^2)m_r - (\beta+1)r \geq 0. \tag{4.2.7}$$

Since $\frac{\partial}{\partial \beta} k(\beta, r) = -r < 0$, Therefore, (4.2.7) holds whenever

$$k(r) := \min_{\beta} k(\beta, r) = k(1, r) \geq 0.$$

Note that $k(r)$ is a continuous function of r and further $k(0) = m_0 = \psi(0) = 1 > 0$ and $k(1) < 0$. Thus there exists a point $r_\psi \in (0,1)$ such that $k(r) \geq 0$ for all $r \in [0, r_\psi]$, where r_ψ is the least positive root of (4.2.3).

Proof of sharpness: Now let $m_r = \psi(-r)$. Choose $f(z) \in \mathscr{C}(\phi)$ such that $zf'(z)/f(z) = \psi(-z)$ and $\Psi(z) = (z+\alpha)/(1+\alpha z)$, where $-1 \leq \alpha \leq 1$. Let r_0 be the second consecutive positive root (if exists) of the equation (4.2.3), otherwise choose $r_0 = 1$. We show that for each $r_\psi < r < r_0$ we can choose α so that $g'(r) > f'(r) > 0$, which implies that g' is not majorized by f' outside $|z| \leq r_\psi$. First, note that

$$\frac{f(r)}{f'(r)} = \frac{r}{\psi(-r)}. \tag{4.2.8}$$

Since

$$g'(r) = f'(r) \left(\frac{r+\alpha}{1+\alpha r} + \frac{1-\alpha^2}{(1+\alpha r)^2} \frac{f(r)}{f'(r)} \right) =: f'(r) h(r, \alpha)$$

and $h(r,1) = 1$, it suffices to show that $\partial h(r,\alpha)/\partial \alpha < 0$ at $\alpha = 1$ in order to establish that $h(r, 1-\varepsilon) > 1$, and hence $g'(r) > f'(r) > 0$. But at $\alpha = 1$, we have:

$$\begin{aligned}\frac{\partial h(r,\alpha)}{\partial \alpha} &= \frac{2}{(1+r)^2}\left(\frac{1-r^2}{2} - \frac{f(r)}{f'(r)}\right)\\ &= \frac{2}{(1+r)^2}\left(\frac{1-r^2}{2} - \frac{r}{\psi(-r)}\right)\\ &< 0,\end{aligned}$$

using the equations (4.2.3), (4.2.7), (4.2.8) and the fact that $k(r) < 0$ for all $r \in (r_\psi, r_0)$. □

Remark 4.2.1. The following result was proved by MacGregor [104]: Let $g \in \mathscr{A}$ and $f \in \mathscr{C}$. If g is majorized by f in $\mathbb{D}$, then $|g'(z)| \le |f'(z)|$ in $|z| \le 1/3$. The result is sharp.

In our next result, we show the application to the Janowski class [72], which covers many well-known classes. Here $\mathscr{C}[D,E] := \mathscr{C}((1+Dz)/(1+Ez))$.

Corollary 4.2.2. Let f belongs to $\mathscr{C}[D,E]$, where $-1 \le E < D \le 1$ along with $1 + D/E \ge 0$ and $-1 \le E < 0$. If g is majorized by f, then

$$|g'(z)| \le |f'(z)| \quad \text{in} \quad |z| \le r_0,$$

where r_0 is the smallest positive root of the equation

$$(1-r^2)\left({}_2F_1\left(1-\frac{D}{E}, 1, 2; \frac{-Er}{1-Er}\right)\right)^{-1} - 2r = 0.$$

The result is sharp.

Proof. In Theorem 4.2.1, put $\phi(z) = (1+Dz)/(1+Ez)$. Then we have $\psi(z) := 1/q(z)$, where

$$q(z) = \begin{cases} \int_0^1 \left(\frac{1+Etz}{1+Ez}\right)^{\frac{D-E}{E}} dt, & \text{if } E \ne 0;\\ \int_0^1 e^{D(t-1)z} dt, & \text{if } E = 0,\end{cases}$$

which further can be represented in terms of confluent and Gaussian hypergeometric functions, respectively as follows:

$$q(z) = \begin{cases} {}_2F_1\left(1-\frac{D}{E}, 1, 2; \frac{Ez}{1+Ez}\right), & \text{if } E \ne 0;\\ {}_1F_1\left(1, 2; -Dz\right), & \text{if } E = 0.\end{cases}$$

Since $1 + D/E \ge 0$ and $-1 \le E < 0$, therefore we have

$$\min_{|z|=r} \Re\psi(z) = \psi(-r) = \frac{1}{q(-r)} = \left({}_2F_1\left(1-\frac{D}{E}, 1, 2; \frac{-Er}{1-Er}\right)\right)^{-1}.$$

Since $\Re\psi(z) > 0$ and $\min_{|z|=r} \Re\psi(z) = \psi(-r)$, therefore we conclude that $\min_{|z|=r} |\psi(z)| = \psi(-r)$ and hence, the result follows from Theorem 4.2.1. □

Now we have the result for the class of convex functions of order α using Corollary 4.2.2:

Corollary 4.2.3. Let f belongs to $\mathscr{C}[1-2\alpha,-1]$, where $0\leq\alpha<1$. If g is majorized by f, then $|g'(z)|\leq|f'(z)|$ in $|z|\leq r_0$, where r_0 is the smallest positive root of the equation

$$(1-r^2)\left({}_2F_1\left(2(1-\alpha),1,2;\frac{r}{1+r}\right)\right)^{-1}-2r=0.$$

The result is sharp.

Corollary 4.2.4. Let f belongs to $\mathscr{C}[D,0]$. If g is majorized by f, then $|g'(z)|\leq|f'(z)|$ in $|z|\leq r_0$, where r_0 is the smallest positive root of the equation

$$(1-r^2)(Dre^{-Dr}/(e^{-Dr}-1))+2r=0.$$

The result is sharp.

Proof. From the proof of Corollary 4.2.2, we obtain that $\psi(z)=Dze^{Dz}/(e^{Dz}-1)$, when $\phi(z)=1+Dz$. Now with a little computation, we find that the function $l(z)=ze^z/(e^z-1)$ is convex univalent in $\mathbb{D}$. Therefore, the function $\psi(z)=l(Dz)$ is also convex in $\mathbb{D}$ for each fixed $0<D\leq1$. Since ψ is also symmetric about the real axis, we conclude that $\min_{|z|=r}|\psi(z)|=\psi(-r)$. □

Theorem 4.2.5. Let ϕ be convex in $\mathbb{D}$, with $\Re\phi(z)>0$, $\phi(0)=1$ and suppose $f\in\mathscr{A}$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}+z\left(\frac{zf'(z)}{f(z)}\right)'\prec\phi(z). \tag{4.2.9}$$

If g is majorized by f, then $|g'(z)|\leq|f'(z)|$ in $|z|\leq r_0$, where r_0 is the least positive root of the equation

$$(1-r^2)\min_{|z|=r}\Re\psi(z)-2r=0,$$

where

$$\psi(z):=\frac{1}{z}\int_0^z\phi(t)dt.$$

The result is sharp for the case $\min_{|z|=r}\Re\psi(z)=\psi(\pm r)$.

Proof. Let $p(z)=zf'(z)/f(z)$. Then the subordination (4.2.9) can equivalently be written as:

$$p(z)+zp'(z)\prec\phi(z).$$

A simple calculation show that the analytic function $\psi(z):=(1/z)\int_0^z\phi(t)dt$ satisfies

$$\psi(z)+z\psi'(z)=\phi(z).$$

Now from the Hallenbeck and Ruscheweyh result [110, Theorem 3.1b, p. 71], we have $p\prec\psi$, where ψ is the best dominant and also convex. Further, since $\Re\phi(z)>0$, using the integral operator [110, Theorem 4.2a, p. 202] preserving functions with positive real part, we see that ψ is a Carathéodory function. Thus we have

$$\frac{f(z)}{zf'(z)}\prec\frac{1}{\psi(z)}\quad\text{which implies}\quad\left|\frac{f(z)}{f'(z)}\right|\leq\frac{r}{\min_{|z|=r}|\psi(z)|}=\frac{r}{\min_{|z|=r}\Re\psi(z)}.$$

Now proceeding same as in the Theorem 4.2.1 result follows. □

Corollary 4.2.6. Suppose $f \in \mathcal{A}$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)} + z\left(\frac{zf'(z)}{f(z)}\right)' \prec \frac{1+z}{1-z}.$$

If g is majorized by f, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq r_0$, where r_0 is the least positive root of the equation

$$(1-r^2)(2\log(1+r)-r)-2r^2 = 0.$$

The result is sharp.

Corollary 4.2.7. Suppose $f \in \mathcal{A}$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)} + z\left(\frac{zf'(z)}{f(z)}\right)' \prec e^z.$$

If g is majorized by f, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq r_0$, where r_0 is the least positive root of the equation

$$(1-r^2)(1-e^{-r})-2r^2 = 0.$$

The result is sharp.

Theorem 4.2.8. Let ϕ be convex in $\mathbb{D}$, with $\Re\phi(z) > 0$, $\phi(0) = 1$ and suppose $f \in \mathcal{A}$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + 2z\left(\frac{zf'(z)}{f(z)}\right)'\right) \prec \phi(z), \quad \alpha \in [0,1). \tag{4.2.10}$$

If g is majorized by f, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq r_0$, where r_0 is the least positive root of the equation

$$(1-r^2)\min_{|z|=r}|\sqrt{\psi(z)}| - 2r = 0,$$

where

$$\psi(z) := \frac{1}{z}\int_0^z \phi(t)dt.$$

The result is sharp when $\min_{|z|=r}|\sqrt{\psi(z)}| = \sqrt{\psi(\pm r)}$.

Proof. Let $p(z) = zf'(z)/f(z)$. Then the subordination (4.2.10) can be equivalently written as:

$$p^2(z) + 2zp(z)p'(z) \prec \phi(z),$$

which using the change of variable $P(z) = p^2(z)$ becomes

$$P(z) + zP'(z) \prec \phi(z).$$

Now proceeding as in Theorem 4.2.5, we see that $p(z) \prec \sqrt{\psi(z)}$ and $\sqrt{\psi(z)}$ is the best dominant. Further, since $\Re\phi(z) > 0$, using [110, Theorem 4.2a, p. 202], we see that ψ is a Carathéodory function.

Therefore,

$$|\arg\sqrt{\psi(z)}| = \frac{1}{2}|\arg\psi(z)| \leq \frac{\pi}{4},$$

which implies $\Re\sqrt{\psi(z)} > 0$. Thus we have

$$\frac{f(z)}{zf'(z)} \prec \frac{1}{\sqrt{\psi(z)}} \quad \text{which implies} \quad \left|\frac{f(z)}{f'(z)}\right| \leq \frac{r}{\min_{|z|=r}|\sqrt{\psi(z)}|}.$$

Now proceeding same as in the Theorem 4.2.1 result follows. □

Corollary 4.2.9. Suppose $f \in \mathscr{A}$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + 2z\left(\frac{zf'(z)}{f(z)}\right)'\right) \prec \frac{1+(2\alpha-1)z}{1+z}.$$

If g is majorized by f, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq r_0$, where r_0 is the least positive root of the equation

$$(1-r^2)\min_{|z|=r}\Re\sqrt{\psi(z)} - 2r = 0,$$

where

$$\psi(z) := \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt.$$

Corollary 4.2.10. Suppose $f \in \mathscr{A}$ satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + 2z\left(\frac{zf'(z)}{f(z)}\right)'\right) \prec 1+\alpha z, \quad (\alpha \in (0,1]).$$

If g is majorized by f, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq r_0$, where r_0 is the least positive root of the equation

$$(1-r^2)\sqrt{1-\beta r} - 2r = 0, \quad \text{where} \quad \beta = \alpha/2.$$

The result is sharp.

Now we state the following result without proof as it follows from Theorem 4.2.1:

Theorem 4.2.11. Let $\psi \in \mathscr{P}$ be a univalent function such that

$$m_r := \min_{|z|=r}|\psi(z)| = \begin{cases} \psi(-r), & \text{if } \psi'(0) > 0; \\ \psi(r), & \text{if } \psi'(0) < 0. \end{cases}$$

Let $g \in \mathscr{A}$ and $f \in \mathscr{S}^*(\psi)$. If g is majorized by f in $\mathbb{D}$, then

$$|g'(z)| \leq |f'(z)| \quad \text{in } |z| \leq r_\psi,$$

where r_ψ is the least positive root of the equation

$$(1-r^2)m_r - 2r = 0.$$

The result is sharp.

Example 4.2.12. Let us consider the analytic functions $\psi_1(z) = \sqrt{1-z}$ and $\psi_2(z) = \sqrt{1+z}$. Note that $\psi_1'(0) < 0$, $\psi_2'(0) > 0$ and for $|z| = r$,

$$m_{r_1} = \min_{|z|=r} |\psi_1(z)| = \psi_1(r) = \sqrt{1-r} = \psi_2(-r) = \min_{|z|=r} |\psi_2(z)| = m_{r_2}.$$

Now from Theorem 4.2.11, we obtain the following result:
If $g \in \mathscr{A}$, $f \in \mathscr{S}^*(\psi_i)$, where $i = 1,2$ and g is majorized by f, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq r_0$, where r_0 is the least positive root of the equation

$$(1-r^2)\sqrt{1-r} - 2r = 0.$$

Interestingly, the desired radius in both cases remains the same as $\psi_1(\mathbb{D}) = \psi_2(\mathbb{D})$, though ψ_1 and ψ_2 are oppositely oriented.

Remark 4.2.2. Taking $\alpha = 0$ or $\eta = 1$ in Corollary 4.2.13, case (ii) and (iii), respectively, we obtain the result proved by T. H. MacGregor [104], namely: *Let $g \in \mathscr{A}$ and $f \in \mathscr{S}^*$. If $g << f$ in $\mathbb{D}$, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq 2-\sqrt{3}$.* The result is sharp.

Now we obtain the following majorization results for some known classes as well those introduced and studied in [38, 60, 109, 134].

Corollary 4.2.13. Let $g \in \mathscr{A}$ and $f \in \mathscr{S}^*(\psi)$. If $g << f$ in $\mathbb{D}$, then $|g'(z)| \leq |f'(z)|$ in $|z| \leq r_\psi$, where r_ψ is the least positive root of the equation $P(r) = 0$ and the result follow for each one of the following cases:

(i) $P(r) = (1-r^2)((1-Dr)/(1-Er)) - 2r$ when $\psi(z) = \frac{1+Dz}{1+Ez}$, where $-1 \leq E < D \leq 1$.

(ii) $P(r) = (1-r)(1-(1-2\alpha)r) - 2r$ when $\psi(z) = \frac{1+(1-2\alpha)z}{1-z}$, where $0 \leq \alpha < 1$.

(iii) $P(r) = (1-r^2)((1-r)/(1+r))^\eta - 2r$ when $\psi(z) = \left(\frac{1+z}{1-z}\right)^\eta$, where $0 < \eta \leq 1$.

(iv) $P(r) = (1-r^2)\left(\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}\right) - 2r$ when $\psi(z) = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$.

(v) $P(r) = (1-r^2)(b(1-r))^{1/a} - 2r$ when $\psi(z) = (b(1+z))^{1/a}$, where $a \geq 1$ and $b \geq 1/2$.

(vi) $P(r) = (1-r^2) - 2re^r$ when $\psi(z) = e^z$.

(vii) $P(r) = (1-r^2)(\sqrt{1+r^2} - r) - 2r$ when $\psi(z) = z + \sqrt{1+z^2}$.

$(viii)$ $P(r) = (1-r^2) - r(1+e^r)$ when $\psi(z) = \frac{2}{1+e^{-z}}$.

(ix) $P(r) = (1-r^2)(1-\sin r) - 2r$ when $\psi(z) = 1+\sin z$.

The results are sharp.

Remark 4.2.3. In Corollary 4.2.13, case (ix), we obtained the radius $r_\psi \approx 0.312478$ which improves the majorization radius $r_s \approx 0.309757$ obtained in [165].

Let $\psi(z) = 1 + z/(1-\alpha z^2)$, $0 \leq \alpha < 1$, introduced and studied by Kargar et al. [74]. Clearly $\psi \in \mathscr{P}$ only when $\alpha = 0$ and hence Theorem 4.2.11 holds when $\psi(z) = 1+z$. Moreover, for some $r > 0$, the quantity $z/\psi(z)$ does not exist for all $|z| = r$. In view of the same, the result proved by Tang and Deng [164], needs correction and the corrected version is stated in the following corollaries:

Corollary 4.2.14. Let $g \in \mathscr{A}$ and $f \in \mathscr{S}^*(1+\beta z)$, $0 < \beta \leq 1$. If $g << f$ in $\mathbb{D}$, then

$$|g'(z)| \leq |f'(z)| \quad \text{in} \quad |z| \leq r_\beta,$$

where r_β is the least positive root of the equation

$$(1-r^2)(1-\beta r) - 2r = 0.$$

The result is sharp.

Now we obtain the result related to $\mathscr{BS}(\alpha)$, the class of Booth Lemniscate starlike functions when $\alpha \neq 0$.

Corollary 4.2.15. Let $0 < \alpha < 1$ and r_α be the unique root of the equation

$$\alpha r^2 + r - 1 = 0. \tag{4.2.11}$$

Let $g \in \mathscr{A}$ and $g << f$ in $\mathbb{D}$, where $f \in \mathscr{BS}(\alpha)$. Then

$$|g'(z)| \leq |f'(z)| \quad \text{in} \quad |z| \leq r_{B(\alpha)} := \min\{r_\alpha, r_0\},$$

where r_0 is the least positive root of the equation

$$(1-r^2)\left(1 - \frac{r}{1-\alpha r^2}\right) - 2r = 0.$$

The result is sharp.

Proof. Observe that $\Re\left(1+\frac{z}{1-\alpha z^2}\right) > 0$ for $|z| < r_\alpha$, where r_α is the unique root of (4.2.11). Thus the inequality in (4.2.5) holds for $|z| = r < r_\alpha$ and the result follows at once. □

4.3 Product of starlike functions

Assume that ψ_1 and ψ_2 belong to $\mathscr{P}$ and satisfy the following conditions for $|z| = r$ and $i = 1,2$

$$\max_{|z|=r} \Re\psi_i(z) = \psi_i(r) \quad \text{and} \quad \min_{|z|=r} \Re\psi_i(z) = \psi_i(-r). \tag{4.3.1}$$

Motivated by Obradović and Ponnusamy [121], in this section, we consider the radius problem to generalize their result and also establish a similar result for the Uralegaddi class $\mathscr{M}(\beta) := \{f \in \mathscr{A} : \Re(zf'(z)/f(z)) < \beta,\ \beta > 1\}$.

Theorem 4.3.1. Let $g \in \mathscr{S}^*(\psi_1)$ and $h \in \mathscr{S}^*(\psi_2)$, where ψ_i satisfy the first condition in (4.4.2). Then the function F defined by

$$F(z) = \frac{g(z)h(z)}{z} \tag{4.3.2}$$

belongs to $\mathscr{M}(\beta)$ in the disk $|z| < r_\beta = \min\{1, r_0(\beta)\}$, where $r_0(\beta)$ is the least positive root of the equation

$$\psi_1(r) + \psi_2(r) - 1 - \beta = 0. \tag{4.3.3}$$

The radius r_β is sharp.

Proof. Let $g \in \mathcal{S}^*(\psi_1)$ and $h \in \mathcal{S}^*(\psi_2)$. Then in view of (4.4.2) and subordination principle, it follows that

$$\Re \frac{zg'(z)}{g(z)} \leq \psi_1(r) \quad \text{and} \quad \Re \frac{zh'(z)}{h(z)} \leq \psi_2(r)$$

in $|z| \leq r < 1$. Since

$$\frac{zF'(z)}{F(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} - 1,$$

we have for $|z| = r$,

$$\Re \frac{zF'(z)}{F(z)} \leq \psi_1(r) + \psi_2(r) - 1 \leq \beta,$$

whenever $r \leq \min\{1, r_0(\beta)\}$, where $r_0(\beta)$ is the least positive root of the equation (4.3.3). The sharpness follows by considering the functions

$$g(z) = z \exp \int_0^z \frac{\psi_1(t) - 1}{t} dt \quad \text{and} \quad h(z) = z \exp \int_0^z \frac{\psi_2(t) - 1}{t} dt.$$

□

Corollary 4.3.2. Let $g \in \mathcal{S}^*(\gamma)$ and $h \in \mathcal{S}^*(\tau)$. Then the function F defined in (4.3.2) belongs to $\mathcal{M}(\beta)$ in the disk $|z| < \min\{1, r_0(\beta)\}$, where

$$r_0(\beta) = \frac{\beta - 1}{3 + \beta - 2(\gamma + \tau)}.$$

The proof of the following result is much akin to Theorem 4.3.1, so is omitted here.

Theorem 4.3.3. Let $g \in \mathcal{S}^*(\psi_1)$ and $h \in \mathcal{S}^*(\psi_2)$, where ψ_i satisfy the second condition in (4.4.2). Then the function F defined in (4.3.2) is starlike of order γ in the disk $|z| < r_\gamma$, where r_γ is the least positive root of the equation

$$\psi_1(-r) + \psi_2(-r) - 1 - \gamma = 0.$$

The radius r_γ is sharp.

Now using Theorem 4.3.3, we obtain the following result proved by Obradović and Ponnusamy [121]:

Remark 4.3.1. Let $g \in \mathcal{S}^*(\gamma)$ and $h \in \mathcal{S}^*(\tau)$. Then the function F defined in (4.3.2) is starlike of order γ_0 in the disk

$$|z| < \frac{1 - \gamma_0}{\gamma_0 + 3 - 2(\gamma + \tau)}.$$

Remark 4.3.2. Note that the identity function $z \in \mathcal{S}^*(\psi)$. Thus if we choose $g(z) = z$ (or $h(z) = z$) in (4.3.2), then the problem reduces to obtaining the $\mathcal{M}(\beta)$-radius (or $\mathcal{S}^*(\gamma)$-radius) of the class $\mathcal{S}^*(\psi_2)$ (or $\mathcal{S}^*(\psi_1)$). It is also evident that the conditions given in (4.4.2) establish the inclusion relations $\mathcal{S}^*(\psi) \subseteq \mathcal{M}(\psi(1))$ and $\mathcal{S}^*(\psi) \subseteq \mathcal{S}^*(\psi(-1))$, respectively.

4.4 Convolution properties and sufficient conditions

The convolution of two power series $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n.$$

Ruscheweyh and Sheil-Small [144] proved that if $f\in\mathscr{C}$ and $g\in\mathscr{C}$ ($g\in\mathscr{S}^*$), then $f*g\in\mathscr{C}$ ($f*g\in\mathscr{S}^*$). Later on, Ma-Minda [102] proved that if $f\in\mathscr{C}$ and $g\in\mathscr{S}^*(\psi)$, then $f*g\in\mathscr{S}^*(\psi)$, when $\psi(\mathbb{D})$ is convex. Szegö [163] in 1928 discussed the radii of convexity for the sections $f_k(z)=z+\sum_{n=2}^{k}a_nz^n$ of the functions $f\in\mathscr{C}$, while in 1988 Silverman [155] considered the radii of starlikeness of f_k. Also see Silverman et al. [153, 154] work on convolution for the Janowski classes. We shall consider these problems for the Ma-Minda classes $\mathscr{C}(\psi)$ and $\mathscr{S}^*(\psi)$ following the idea of Goodman and Schoenberg [57]. In the following, we extend the results of the Bulboacă and Tuneski [35] for the class $\mathscr{S}^*(\psi)$:

Theorem 4.4.1. Let h be analytic with $h(0)=0$, $h'(0)\neq 0$. Suppose that h satisfies

$$\Re\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2} \tag{4.4.1}$$

and

$$\frac{1}{z}\int_0^z h(t)dt\prec\frac{\psi(z)-1}{\psi(z)}. \tag{4.4.2}$$

If $f\in\mathscr{A}$, then

$$\frac{f(z)f''(z)}{(f'(z))^2}\prec h(z)\quad\text{implies}\quad f\in\mathscr{S}^*(\psi).$$

Proof. Using the result [35, Theorem 3.1, p. 3], we see that $f(z)f''(z)/(f'(z))^2\prec h(z)$ implies

$$\frac{1}{z}\int_0^z\left(1-\left(\frac{f(t)}{f'(t)}\right)'\right)dt=1-\frac{f(z)}{zf'(z)}\prec\frac{1}{z}\int_0^z h(t)dt.$$

From the above subordination, we have

$$\frac{f(z)}{zf'(z)}\prec 1-\frac{1}{z}\int_0^z h(t)dt.$$

Now to prove that $f\in\mathscr{S}^*(\psi)$, it suffices to consider

$$1-\frac{1}{z}\int_0^z h(t)dt\prec\frac{1}{\psi(z)},$$

which is equivalent to (4.4.2). This completes the proof. ∎

Corollary 4.4.2. Let $-1\le E<D\le 1$ and $0<\beta\le 1$ such that

$$h(z)=1-\left(\frac{1+Ez}{1+Dz}\right)^{\beta}\left(\frac{1+(D+E-\beta(D-E))z+DEz^2}{(1+Dz)(1+Ez)}\right)$$

and satisfies (4.4.1). If $f \in \mathscr{A}$ and

$$\frac{f(z)f''(z)}{(f'(z))^2} \prec h(z).$$

Then $f \in \mathscr{S}^*\left(\left(\frac{1+Dz}{1+Ez}\right)^{\beta}\right)$.

Remark 4.4.1. 1. Choosing $E=-1$ and $D=1$ in Corollary 4.4.2 gives [35, Corollary 3.5].

2. Choosing $E=0$ and $\beta=1$, then Corollary 4.4.2 reduces to [35, Example 4.5].

In the following, choosing $D=1-2\alpha$, $E=-1$ and $\beta=1$ in Corollary 4.4.2 yields [35, Corollary 3.4] (Note that $\alpha \neq 1$). Also, in [35, Example 4.4] correct range of α is $[0,1)$.

Corollary 4.4.3. If $f \in \mathscr{A}$, $0 \leq \alpha < 1$ and

$$\frac{f(z)f''(z)}{(f'(z))^2} \prec \frac{2(1-\alpha)((1-2\alpha)z^2+2z)}{(1+(1-2\alpha)z)^2}.$$

Then f is starlike of order α.

Considering $\psi(z)=\sqrt{1+cz}$, where $0<c\leq 1$ in Theorem 4.4.1, we get the following:

Corollary 4.4.4. Let $0<c<c_0<1$, where $c_0 \approx 0.845276$ is the unique positive root of the equation

$$30-\frac{75}{2}c^2-\frac{201}{32}c^4=0.$$

If $f \in \mathscr{A}$ and

$$\frac{f(z)f''(z)}{(f'(z))^2} \prec 1-\frac{2+cz}{2(1+cz)^{3/2}}.$$

Then $f \in \mathscr{S}^*(\sqrt{1+cz})$.

Let us consider the following function in Theorem 4.4.1:

$$h(z)=1-\frac{e^{\lambda z}(1-\lambda z)}{e^{2\lambda z}},$$

where $0<\lambda \leq 1$. Then

$$1+\frac{zh''(z)}{h'(z)}=\frac{2-4\lambda z+\lambda^2z^2}{2-\lambda z} \quad \text{and} \quad \Re\left(1+\frac{zh''(z)}{h'(z)}\right)=\frac{2-4\lambda+\lambda^2}{2-\lambda}.$$

Thus, we get

Corollary 4.4.5. Let $0<\lambda \leq \frac{1}{4}(9-\sqrt{33})<1$. If $f \in \mathscr{A}$ and

$$\frac{f(z)f''(z)}{(f'(z))^2} \prec 1-\frac{e^{\lambda z}(1-\lambda z)}{e^{2\lambda z}}. \quad \text{Then} \quad f \in \mathscr{S}^*(e^{\lambda z}).$$

In 1985, Silverman and Silvia [154] obtained some necessary and sufficient conditions in terms of convolution operators for the functions to be in the Janowski classes, and generalized the results of Silverman et. al. [153]. We now extend their results for the class $\mathscr{F}(\phi)$ defined in (3.1.4), which consequently yield results for the classes $\mathscr{S}^*(\psi)$ and $\mathscr{C}(\psi)$ when $\Re(1+\phi(z))>0$.

Theorem 4.4.6. Let $f \in \mathscr{A}$ such that $f(z)/z \neq 0$. Assume that $\psi = 1+\phi$ Then $f \in \mathscr{F}(\phi)$ if and only if

$$\frac{1}{z}\left(f(z) * \frac{z-\lambda z^2}{(1-z)^2}\right) \neq 0, \tag{4.4.3}$$

where $\lambda = \psi(e^{it})/(1-\psi(e^{it}))$ and $t \in [0, 2\pi)$.

Proof. Since $f \in \mathscr{F}(\phi)$ if and only if $zf'(z)/f(z) \prec \psi(z)$, which is further equivalent to

$$\frac{zf'(z)}{f(z)} \neq \psi(e^{it}), \quad (z \in \mathbb{D}, t \in [0, 2\pi)),$$

(means that the values $zf'(z)/f(z)$ does not lie on the boundary Ω of the domain $\psi(\mathbb{D})$) which, using the hypothesis that $f(z)/z \neq 0$, can be equivalently written as

$$\frac{1}{z}\left(zf'(z) - \psi(e^{it})f(z)\right) \neq 0. \tag{4.4.4}$$

Since

$$zf'(z) = f(z) * \frac{z}{(1-z)^2} \quad \text{and} \quad f(z) = f(z) * \frac{z}{1-z}.$$

Therefore, (4.4.4) becomes

$$\frac{1}{z}\left(f(z) * \frac{z-\psi(e^{it})(z-z^2)}{(1-z)^2}\right) \neq 0,$$

and further, with a little computation, it reduces to (4.4.3). □

Note that we can also write

$$\frac{1}{z}\left(f(z) * \frac{z-\lambda z^2}{(1-z)^2}\right) = \frac{1}{z}(zf'(z) - \lambda(zf'(z) - f(z))).$$

Thus using the power series expansion of $f(z)$, (4.4.3) becomes

$$\sum_{k=2}^{\infty} (\lambda(k-1)-k)a_k z^{k-1} \neq 1,$$

where λ is as defined in Theorem 4.4.6, and yields the following sufficient condition in terms of coefficients for the functions from $\mathscr{A}$ to be in $\mathscr{S}^*(\psi)$:

Corollary 4.4.7. If $f \in \mathscr{A}$ such that $f(z)/z \neq 0$ and satisfies $\sum_{k=2}^{\infty} |\lambda(k-1)-k||a_k| < 1$. Then the function $f \in \mathscr{F}(\phi)$.

In particular, we have the following sufficient condition for the class of Janowski starlike functions:

Corollary 4.4.8. The function $f \in \mathscr{S}^*((1+Dz)/(1+Ez))$, if

$$\sum_{k=2}^{\infty} \left(\frac{1+|D|}{D-E} + \left(\frac{1+|D|-E+D}{D-E}\right)k\right)|a_k| < 1.$$

The following result with some rearrangement and specific values of D and E reduces to [154, Theorem 6, Theorem 7].

Corollary 4.4.9. Let $\psi(z) = (1+Dz)/(1+Ez)$, where $-1 \leq E < D \leq 1$. Then $f \in \mathcal{S}^*(\psi)$ if and only if

$$\frac{1}{z}\left(f(z) * \frac{z + \frac{\zeta+D}{D-E}z^2}{(1-z)^2}\right) \neq 0, \quad |\zeta| = 1.$$

Corollary 4.4.10. Let $\psi(z) = 1 + ze^z$. Then $f \in \mathcal{S}^*(\psi)$ if and only if

$$\frac{1}{z}\left(f(z) * \frac{z + (1+\zeta e^{\zeta})z^2}{(1-z)^2}\right) \neq 0, \quad |\zeta| = 1.$$

The class $\mathcal{S}^*(1+\sin z)$ was introduced in [38].

Corollary 4.4.11. Let $\psi(z) = 1 + \sin z$. Then $f \in \mathcal{S}^*(\psi)$ if and only if

$$\frac{1}{z}\left(f(z) * \frac{z - \arcsin\zeta(1+\sin\zeta)z^2}{(1-z)^2}\right) \neq 0, \quad |\zeta| = 1.$$

Note that $f \in \mathcal{F}(\phi)$ if and only if $g(z) = \int_0^z (f(t)/t)dt \in \mathcal{CF}(\phi)$, where

$$\mathcal{CF}(\phi) := \left\{ f \in \mathcal{A} : \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\}.$$

Therefore, the condition given in (4.4.3) is equivalent to the following:

$$\frac{1}{z}\left(zg'(z) * \frac{z - \lambda z^2}{(1-z)^2}\right). \tag{4.4.5}$$

Now using the convolution fact that $zg'(z) * f(z) = g(z) * zf'(z)$ in (4.4.5), we obtain the following result, which is the convex analogue of Theorem 4.4.6:

Theorem 4.4.12. Let $f \in \mathcal{A}$ such that $f(z)/z \neq 0$. Assume that $\psi = 1 + \phi$ Then $f \in \mathcal{CF}(\phi)$ if and only if

$$\frac{1}{z}\left(f(z) * \frac{z + (1-2\lambda)z^2}{(1-z)^3}\right) \neq 0,$$

where λ is as defined in Theorem 4.4.6.

Note that we can also write

$$\frac{1}{z}\left(f(z) * \frac{z + (1-2\lambda)z^2}{(1-z)^3}\right) = f'(z) + (1-\lambda)zf''(z).$$

Thus, using the power series expansion of $f(z)$, we equivalently get

$$\sum_{k=2}^{\infty} (\lambda(k-1) - k)ka_k z^{k-1} \neq 1,$$

which gives the following sufficient condition in terms of coefficients:

Corollary 4.4.13. If $f \in \mathcal{A}$ such that $f(z)/z \neq 0$ and satisfies $\sum_{k=2}^{\infty} k|\lambda(k-1) - k||a_k| < 1$. Then $f \in \mathcal{CF}(\phi)$.

In particular, we have the following sufficient condition for the class of Janowski convex functions:

Corollary 4.4.14. The function $f \in \mathscr{C}((1+Dz)/(1+Ez))$, if

$$\sum_{k=2}^{\infty}\left(\frac{1+|D|}{D-E}+\left(\frac{1+|D|-E+D}{D-E}\right)k\right)k|a_k|<1.$$

Corollary 4.4.15. Let $\psi(z)=(1+Dz)/(1+Ez)$, where $-1\leq E<D\leq 1$. Then $f\in\mathscr{C}(\psi)$ if and only if

$$\frac{1}{z}\left(f(z)*\frac{z+\frac{(3D-E)+2\zeta}{D-E}z^2}{(1-z)^3}\right)\neq 0,\quad |\zeta|=1.$$

Corollary 4.4.16. Let $\psi(z)=1+ze^z$. Then $f\in\mathscr{C}(\psi)$ if and only if

$$\frac{1}{z}\left(f(z)*\frac{z+(3+2\zeta e^{\zeta})z^2}{(1-z)^3}\right)\neq 0,\quad |\zeta|=1.$$

Corollary 4.4.17. Let $\psi(z)=1+\sin z$. Then $f\in\mathscr{C}(\psi)$ if and only if

$$\frac{1}{z}\left(f(z)*\frac{z-\arcsin\zeta(2+\sin\zeta)z^2}{(1-z)^3}\right)\neq 0,\quad |\zeta|=1.$$

It is well known that the class $\mathscr{C}(\psi)$ is closed under convolution. Also the class $\mathscr{S}^*(\psi)$ is closed under convolution when convoluted with convex functions when $\psi(\mathbb{D})$ is convex. With this motivation, the following result provides the largest radius r_0 such that in $|z|<r_0$ the class $\mathscr{S}^*(\psi)$ is closed under convolution.

Theorem 4.4.18. Let r_ψ be the largest radius such that $F(z)=z+\sum_{n=2}^{\infty}n^2z^n$ belongs to $\mathscr{S}^*(\psi)$ for $|z|<r_\psi$. If $f,g\in\mathscr{S}^*(\psi)$, where ψ is convex. Then $f*g$ belongs to $\mathscr{S}^*(\psi)$ for $|z|<r_\psi$. The radius is the best possible.

Proof. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$. Then

$$\begin{aligned}G(z):=f(z)*g(z)&=\left(z+\sum_{n=2}^{\infty}n^2z^n\right)*\left(z+\sum_{n=2}^{\infty}\frac{a_n}{n}z^n\right)*\left(z+\sum_{n=2}^{\infty}\frac{b_n}{n}z^n\right)\\&=F(z)*\left(z+\sum_{n=2}^{\infty}\frac{a_n}{n}z^n\right)*\left(z+\sum_{n=2}^{\infty}\frac{b_n}{n}z^n\right).\end{aligned}$$

Recall that

$$f\in\mathscr{S}^*(\psi)\quad\text{if and only if}\quad\int_0^z\frac{f(t)}{t}dt\in\mathscr{C}(\psi).$$

Similarly, for the function g. Therefore, $z+\sum_{n=2}^{\infty}\frac{a_n}{n}z^n$ and $z+\sum_{n=2}^{\infty}\frac{b_n}{n}z^n$ belong to $\mathscr{C}(\psi)$. Now let

$$\begin{aligned}H(z):=&\left(z+\sum_{n=2}^{\infty}\frac{a_n}{n}z^n\right)*\left(z+\sum_{n=2}^{\infty}\frac{b_n}{n}z^n\right)\\=&\,z+\sum_{n=2}^{\infty}\frac{a_nb_n}{n^2}z^n\in\mathscr{C}(\psi)\subseteq\mathscr{C}\end{aligned}$$

so that

$$G(z) = F(z) * H(z).$$

Now from the hypothesis, we have

$$\frac{F(r_\psi z)}{r_\psi} \in \mathscr{S}^*(\psi), \tag{4.4.6}$$

that is, F belongs to $\mathscr{S}^*(\psi)$ for $|z| < r_\psi$. Since $H \in \mathscr{C}$. Therefore, using (4.4.6), we get

$$B(z) := \left(\frac{F(r_\psi z)}{r_\psi}\right) * H(z) \in \mathscr{S}^*(\psi), \tag{4.4.7}$$

whenever ψ is convex. Thus, we conclude that

$$G(z) = r_\psi B\left(\frac{z}{r_\psi}\right) \in \mathscr{S}^*(\psi), \quad (|z| < r_\psi).$$

Note that the radius r_ψ is independent of the choices of the functions f and g, and thus the sharpness of the result follows from (4.4.6). □

Remark 4.4.2. Note that

$$F(z) = \frac{z(1+z)}{(1-z)^3} = z + \sum_{n=2}^{\infty} n^2 z^n.$$

Now the logarithmic differentiation of the above yields:

$$\frac{zF'(z)}{F(z)} = \frac{1+4z+z^2}{1-z^2} = \frac{1+z}{1-z} - \frac{1}{1+z} + \frac{1}{1-z}. \tag{4.4.8}$$

It follows from (4.4.8) that

$$\Re\left(\frac{zF'(z)}{F(z)}\right) \geq \frac{1-4r+r^2}{1-r^2}. \tag{4.4.9}$$

Moreover, the following sharp inequality also holds:

$$\left|\frac{zF'(z)}{F(z)} - 1\right| \leq \frac{2r(2+r)}{1-r^2}. \tag{4.4.10}$$

Furthermore, the following inequality also holds:

$$\left|\frac{zF'(z)}{F(z)} - \frac{1+r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}. \tag{4.4.11}$$

Now applying Theorem 4.4.18, we get the following result:

Corollary 4.4.19. Let $f, g \in \mathscr{S}^*(\psi)$, where ψ is convex. Then $f * g$ belongs to $\mathscr{S}^*(\psi)$ for $|z| < r_\psi$, where

(i) $r_\psi = (2 - \sqrt{3+\alpha^2})/(1+\alpha)$, when $\psi(z) = (1+(1-2\alpha)z)/(1-z)$.

(ii) $r_\psi = (-2+\sqrt{5})/(1+\sqrt{2})$, when $\psi(z) = \sqrt{1+z}$.

(iii) $r_\psi = (-2+\sqrt{4+b(2+b)})/(2+b)$, where $b = (e-1)/(e+1)$, when $\psi(z) = 2/(1+e^{-z})$.

(iv) $r_\psi = (2-\sqrt{4-b^2})/b$, where $b = \sin\pi\gamma/2$, when $\psi(z) = ((1+z)/(1-z))^\gamma$.

The radii are sharp.

Proof. The part (i) follows using the inequality (4.4.9) such that $(1-4r+r^2)(1-r^2) \geq \alpha$, which holds for $r \leq (2-\sqrt{3+\alpha^2})/(1+\alpha)$. Further

$$\frac{z_0 F'(z_0)}{F(z_0)} = \alpha \quad \text{for} \quad z_0 = \frac{\sqrt{3+\alpha^2}-2}{1+\alpha},$$

implies the sharpness of the radius. Note that the disks $\{w : |w-1| < \sqrt{2}-1\}$ and $\{w : |w-1| < (e-1)/(e+1)\}$ are contained in $\psi(\mathbb{D})$ where $\psi(z) = \sqrt{1+z}$ and $2/(1+e^{-z})$, respectively. Therefore, the parts (ii) and (iii) follow using the inequality (4.4.10) such that

$$\frac{2r(2+r)}{1-r^2} \leq \sqrt{2}-1 \quad \text{and} \quad \frac{2r(2+r)}{1-r^2} \leq \frac{e-1}{e+1},$$

which holds for $r \leq (-2+\sqrt{5})/(1+\sqrt{2})$ and $r \leq (-2+\sqrt{4+b(2+b)})/(2+b)$, where $b = (e-1)/(e+1)$ respectively. Since

$$\frac{z_0 F'(z_0)}{F(z_0)} = 1+\sqrt{2} \quad \text{for} \quad z_0 = \frac{-2+\sqrt{5}}{1+\sqrt{2}}$$

and

$$\frac{z_0 F'(z_0)}{F(z_0)} = \frac{2}{1+e} \quad \text{for} \quad z_0 = (\sqrt{4+b(2+b)}-2)/(2+b),$$

therefore the radii obtained are sharp. Part (iv) follows by using the inequality (4.4.11) and the fact that the disk $\{w : |w-a| < r_a\}$ is contained in the sector $|\arg w| \leq \pi\gamma/2$, whenever $r_a \leq a\sin(\pi\gamma/2)$. □

The following lemma was introduced by Kumar and Gangania [83] to obtain certain radius constants (see [83, p.12-14]) related to the operators like Livingston and Bernardi etc. to cover the case when ψ is starlike but not convex.

Lemma 4.4.1. Let r_c be the radius of convexity of ψ. If $g \in \mathcal{C}$ and $f \in \mathcal{S}^*(\psi)$. Then $f * g \in \mathcal{S}^*(\psi)$ for $|z| < r_\psi = \min\{r_c, 1\}$.

Now using the above lemma, we may write (4.4.7) as follows:

$$B(z) := \left(\frac{F(r_0 z)}{r_0}\right) * H(z) \in \mathcal{S}^*(\psi),$$

where $r_0 = \min\{r_\psi, r_c\}$. Note that $r_c = (3-\sqrt{5})/2$ when $\psi(z) = 1+ze^z$. Thus, we have the following result:

Corollary 4.4.20. Let $f, g \in \mathcal{S}^*(1+ze^z)$. Then the function $f * g$ belongs to $\mathcal{S}^*(1+ze^z)$ for $|z| < (2e-\sqrt{4e^2-2e+1})/(2e-1) \approx 0.0957$. The radius is sharp.

Corollary 4.4.21. Let $f, g \in \mathcal{S}^*(1+\sin z)$. Then the function $f * g$ belongs to $\mathcal{S}^*(1+\sin z)$ for $|z| < (\sqrt{4+\sin 1(2+\sin 1)} - 2)/(2+\sin 1) \approx 0.1858$. The radius is sharp.

In the next theorem, we continue to extend the ideas of Szegö [163] and Silverman [155] on the starlikeness and convexity of sections $f_k(z) = z + \sum_{n=2}^{k} a_n z^n$ of f in $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$, respectively.

Theorem 4.4.22. Let $g_k(z) = (z - z^{k+1})/(1-z)$. If $f \in \mathcal{C}(\psi)$, where ψ be convex. Then

1. $f_k \in \mathcal{C}(\psi)$ in $|z| < r_0$, where r_0 is the radius of convexity of g_k.
2. $f_k \in \mathcal{S}^*(\psi)$ in $|z| < r_\psi$, whenever g_k belongs to $\mathcal{S}^*(\psi)$ in $|z| < r_\psi$.

The radii are the best possible.

Proof. Let $t = r_0$ in the first part and r_ψ in the second part, respectively. Then the proof follows by observing that $f_k(z) = t h_k(z/t)$, where $h_k(z) = f(z) * \dfrac{g_k(tz)}{t}$. □

Remark 4.4.3. If we choose $\psi(z) = (1+z)/(1-z)$. Then Theorem 4.4.22-(ii) reduces to the Silverman's result [155, Theorem 1, p. 1192].

Jackson [71] introduced and studied the q-derivative defined as

$$d_q f(z) := \frac{f(qz) - f(z)}{(q-1)z} = \frac{1}{z}\left(z + \sum_{n=2}^{\infty} [n]_q a_n z^n\right), \quad z \neq 0$$

and $d_q f(0) = f'(0)$, where $[n]_q = \frac{1-q^n}{1-q}$.

Theorem 4.4.23. Let r_ψ be the largest radius in $(0,1]$ and $q \in (0,1)$ such that

$$\frac{z}{(1-qz)(1-z)} \in \mathcal{S}^*(\psi) \quad \text{for} \quad |z| < r_\psi,$$

where ψ is convex. If $f \in \mathcal{C}$, then we have

$$z d_q f(z) = z + \sum_{n=2}^{\infty} [n]_q a_n z^n \in \mathcal{S}^*(\psi) \quad \text{for} \quad |z| < r_\psi.$$

The radius is the best possible.

Proof. Observe that for each $q \in \mathbb{C}$ where $|q| \leq 1$, $q \neq 1$, we have

$$h_q(z) = \frac{1}{1-q} \log\left(\frac{1-qz}{1-z}\right) = \sum_{n=1}^{\infty} \left(\frac{1-q^n}{1-q}\right) \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{[n]_q}{n} z^n \in \mathcal{C},$$

which implies that

$$z h_q'(z) = \frac{z}{(1-qz)(1-z)} = \sum_{n=1}^{\infty} [n]_q z^n \in \mathcal{S}^*.$$

We note that

$$z d_q f(z) = \left(z + \sum_{n=2}^{\infty} a_n z^n\right) * \left(\sum_{n=1}^{\infty} [n]_q z^n\right) = f(z) * \frac{z}{(1-qz)(1-z)} = f(z) * z h_q'(z).$$

Now for simplicity, let us write $H(z) = zh'_q(z)$. Then from hypothesis, we see that

$$\frac{H(r_\psi z)}{r_\psi} \in \mathcal{S}^*(\psi). \tag{4.4.12}$$

Since $f \in \mathcal{C}$, and $\mathcal{C} * \mathcal{S}^*(\psi) = \mathcal{S}^*(\psi)$ whenever ψ is convex. Therefore, we have

$$f(z) * \frac{H(r_\psi z)}{r_\psi} \in \mathcal{S}^*(\psi),$$

which is equivalent to saying that

$$zd_q f(z) \in \mathcal{S}^*(\psi) \quad \text{for} \quad |z| < r_\psi.$$

From the proof, we note that the sharpness of the radius r_ψ follows from (4.4.12). □

Remark 4.4.4. From the function $H(z) = z/((1-qz)(1-z))$, we have:

$$\frac{zH'(z)}{H(z)} = 1 + \frac{z}{1-z} + \frac{qz}{1-qz}. \tag{4.4.13}$$

It follows from (4.4.13) that

$$\Re\left(\frac{zH'(z)}{H(z)}\right) \geq \frac{1-qr^2}{(1+r)(1+qr)}. \tag{4.4.14}$$

Moreover, the following sharp inequality also holds:

$$\left|\frac{zH'(z)}{H(z)} - 1\right| \leq \frac{r(1+q-2qr)}{(1-r)(1-qr)}. \tag{4.4.15}$$

Now proceeding in a similar way as in Corollary 4.4.19 using (4.4.14) when the function $\psi(z) = (1+(1-2\alpha)z)/(1-z)$, and (4.4.15) when $\psi(z) = \sqrt{1+z}$ and $2/(1+e^{-z})$, we have

Corollary 4.4.24. If $f \in \mathcal{C}$, then for all $0 < q < 1$, we have

$$zd_q f(z) = z + \sum_{n=2}^{\infty} [n]_q a_n z^n \in \mathcal{S}^*(\psi) \quad \text{for} \quad |z| < r_\psi,$$

where

1. $r_\psi = (\sqrt{\alpha^2(1-q^2)+4q} - \alpha(q+1))/(2q(1+\alpha))$, when $\psi(z) = (1+(1-2\alpha)z)/(1-z)$ for $\alpha \geq (1-q)/2(1+q)$.
2. $r_\psi = ((1+q) - \sqrt{1+q^2})/(q\sqrt{2}(\sqrt{2}+1))$, when $\psi(z) = \sqrt{1+z}$.
3. $r_\psi = ((1+q)(1+b) - \sqrt{((1+q)(1+b))^2 - 4bq(2+b)})/(2q(2+b))$, where $b = (e-1)/(e+1)$, when $\psi(z) = 2/(1+e^{-z})$.

The radii are sharp.

Remark 4.4.5. Note that part (i) of the above corollary includes the result [[128], Theorem 2.1]. Also $r_\psi = 1$ for $\alpha \in [0, (1-q)/2(1+q)]$.

4.5 Modified Distortion theorem for Ma-Minda starlike functions

The class $\mathcal{S}^*(\psi)$ unifies various subclasses of starlike functions, which are obtained for an appropriate choice of ψ. See the special cases in [38, 60, 84, 109, 134, 159]. Ma-Minda discussed many properties of the class $\mathcal{S}^*(\psi)$, in particular, they proved the distortion theorem [102, Theorem 2, p.162] with some restriction on ψ, namely

$$\min_{|z|=r}|\psi(z)| = \psi(-r) \quad \text{and} \quad \max_{|z|=r}|\psi(z)| = \psi(r). \tag{4.5.1}$$

The importance of the above conditions can be seen in achieving the sharpness for majorization results in [51]. Here, we modify the distortion theorem by relaxing this restriction on ψ to obtain a more general result, which leads to a special case study. In particular, Ma-Minda [102] used the assumption that $|\psi(z)|$ attains its maximum and minimum value respectively at $z=r$ and $z=-r$, see eq. (4.5.1). Now, what if ψ does not satisfy the condition (4.5.1) and why the condition (4.5.1) is so important? To answer this, we first need to recall the following result:

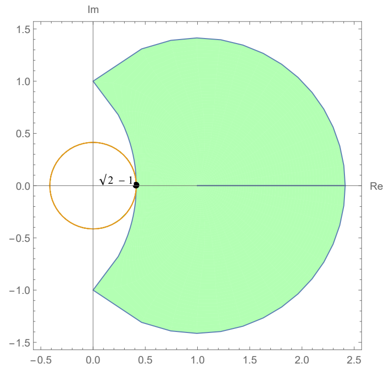
Lemma 4.5.1. *([102]) Let $f \in \mathcal{S}^*(\psi)$ and $|z_0| = r < 1$. Then $-f_0(-r) \leq |f(z_0)| \leq f_0(r)$. Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of f_0, where $zf_0(z)/f_0(z) = \psi(z)$ such that*

$$f_0(z) = z\exp\int_0^z \frac{\psi(t)-1}{t}dt. \tag{4.5.2}$$

We see that a Ma-Minda starlike function, in general, need not satisfy the condition (4.5.1). To examine the same, let us consider two different Ma-Minda starlike functions, namely $\psi_1(z) := z+\sqrt{1+z^2}$ and $\psi_2(z) := 1+ze^z$. The unit disk images of ψ_1 and ψ_2 are displayed in figure 4.1 and figure 4.2.

We know that the radius of a circle, centered at the origin and touching only the boundary points of an image domain of a complex function, yields the optimal values of the modulus of the function. For example, see figure 4.1 to locate the lower bound of the modulus for a crescent function. Therefore it is evident from figure 4.1 that both the bounds $\psi_1(-r)$ and $\psi_1(r)$ of $|\psi_1|$ are attained on the real line and we have $\psi_1(-r) \leq |\psi_1(z)| \leq \psi_1(r)$ for each $|z| = r$. Whereas, from figure 4.2, we see that only the upper bound $\psi_2(r)$ of $|\psi_2|$ is attained on the real line and $|\psi_2(z)| \leq \psi_2(r)$ for each $|z| = r$. Although both ψ_1 and ψ_2 are Ma-Minda functions but the distortion theorem of Ma-Minda [102, Theorem 2, p. 162] does not accommodate the function ψ_2, as the lower bound for $|\psi_2(z)|$ is not attained on the real line for all $|z| = r > (3-\sqrt{5})/2$, see figure 4.3. To overcome this limitation, we modify the distortion theorem, wherein we theoretically assume the modulus bounds of the function and obtain a more general result. Thus the Ma-Minda functions, for which modulus bounds are not attained on the real line but could be computed can now be entertained for distortion theorem using the following result:

Theorem 4.5.1 (Modified Distortion Theorem)**.** Let ψ be a Ma-Minda function. Assume that $\min_{|z|=r}|\psi(z)| = |\psi(z_1)|$ and $\max_{|z|=r}|\psi(z)| = |\psi(z_2)|$, where $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$ for some $\theta_1, \theta_2 \in [0,\pi]$. Let the func-



Let $\min\limits_{|z|=r}|\psi_2(z)| =: \gamma_i(r)$, where $z = r_i e^{i\theta}$, $0 \leq \theta \leq \pi$, then from table 4.1, we have $\gamma_1(1) = 0.372412$, $\gamma_2(4/5) = 0.527912$, $\gamma_3(2/3) = 0.611553$, $\gamma_4(1/2) = 0.693287$, $\gamma_5(r) = 1 - re^{-r}$, where $r \leq (3-\sqrt{5})/2$.

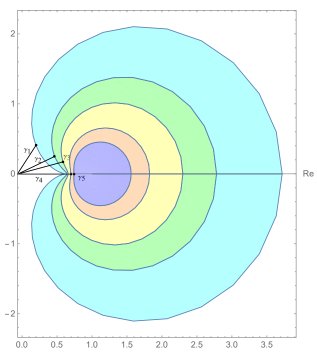


Figure 4.2: Image of unit disk under $\psi_2(z) := 1 + ze^z$

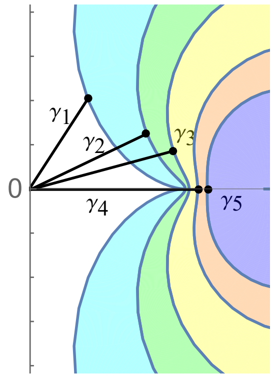


Figure 4.3: A zoom of figure 4.2

tion $f \in \mathscr{S}^*(\psi)$ and $|z_0| = r < 1$. Then

$$|\psi(z_1)|\left(\frac{-f_0(-r)}{r}\right) \leq |f'(z_0)| \leq \left(\frac{f_0(r)}{r}\right)|\psi(z_2)|. \tag{4.5.3}$$

Proof. Let $p(z) = zf'(z)/f(z)$. Then $f \in \mathscr{S}^*(\psi)$ if and only if $p(z) \prec \psi(z)$. Using a result [102, Theorem 1, p.161], we have

$$\frac{f(z)}{z} \prec \frac{f_0(z)}{z}, \tag{4.5.4}$$

where f_0 is given by (4.5.2). Now using Maximum-Minimum principle of modulus, (4.5.4) and by Lemma 4.5.1, $-f_0(-r)/r \leq |f(z_0)/z| \leq f_0(r)/r$, we easily obtain for $|z_0| = r$

$$\begin{aligned}
|\psi(z_1)|\left(\frac{-f_0(-r)}{r}\right) &= \min_{|z|=r}|\psi(z)|\min_{|z|=r}\left|\frac{f_0(z)}{z}\right| \\
&\leq \left|p(z_0)\frac{f(z_0)}{z_0}\right| = |f'(z_0)| \\
&\leq \max_{|z|=r}|\psi(z)|\max_{|z|=r}\left|\frac{f_0(z)}{z}\right| \\
&= \left(\frac{f_0(r)}{r}\right)|\psi(z_2)|,
\end{aligned}$$

that is,

$$|\psi(z_1)|\left(\frac{-f_0(-r)}{r}\right) \leq |f'(z_0)| \leq \left(\frac{f_0(r)}{r}\right)|\psi(z_2)|,$$

where z_1 and z_2 are as defined in the hypothesis. □

To illustrate Theorem 4.5.1, we consider the function $\psi(z) = 1 + ze^z$. Then we have the following expression for its modulus:

$$|\psi(z)| = \sqrt{1 + re^{r\cos\theta}(re^{r\cos\theta} + 2\cos(\theta + r\sin\theta))}. \tag{4.5.5}$$

Using equation (4.5.5) and Theorem 4.5.1, we obtain the following table, providing the minimum for various choices of r:

r	$0 \leq \theta_1 \leq \pi$	$\lvert\psi(re^{i\theta_1})\rvert$	$m(r,\theta_1) = \lvert\psi(re^{i\theta_1})\rvert(-f_0(-r)/r)$
1	1.88438	0.372412	0.197923
4/5	2.01859	0.527912	0.304374
2/3	2.17677	0.611553	0.375966
1/2	2.58169	0.693287	0.467769
$r \leq (3-\sqrt{5})/2$	π	$\psi_2(-r)$	$f_0'(-r)$

Table 4.1: The lower bounds for $|1 + ze^z|$ for different choices of $r = |z|$.

Now using Theorem 4.5.1, we obtain the following distortion theorem for the class $\mathcal{S}^*(1 + ze^z)$:

Example 4.5.2. Let $\psi(z) = 1 + ze^z$ and $\min_{|z|=r}|\psi(z)| = |\psi(z_1)|$, where $z_1 = re^{i\theta_1}$ for some $\theta_1 \in [0, \pi]$. Let $f \in \mathcal{S}^*(\psi)$ and $|z_0| = r < 1$. Then

$$m(r,\theta_1) \leq |f'(z_0)| \leq f_0'(r), \quad \left(r > \tfrac{3-\sqrt{5}}{2}\right)$$

and

$$f_0'(-r) \leq |f'(z_0)| \leq f_0'(r), \quad \left(r \leq \tfrac{3-\sqrt{5}}{2}\right),$$

where $f_0(z) = z\exp(e^z - 1)$ and $m(r,\theta_1)$ is provided in table 4.1 for some specific values of r.

Remark 4.5.1. In Theorem 4.5.1, if we assume that $\theta_1 = \pi$ and $\theta_2 = 0$, then extremes in equation (4.5.3) simplifies to $f_0'(-r)$ and $f_0'(r)$, respectively since $zf_0'(z)/f_0(z) = \psi(z)$. Thus, the extremes in the equation (4.5.3) are in terms of r alone and also lead to the sharp bounds. Consequently, we obtain the following distortion theorem of Ma-Minda [102] as a special case of Theorem 4.5.1 :

Let $\min_{|z|=r}|\psi(z)| = \psi(-r)$ and $\max_{|z|=r}|\psi(z)| = \psi(r)$. If $f \in \mathcal{S}^*(\psi)$ and $|z_0| = r < 1$. Then

$$f_0'(-r) = \psi(-r)\frac{f_0(-r)}{-r} \leq |f'(z_0)| \leq \frac{f_0(r)}{r}\psi(r) = f_0'(r).$$

Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of f_0.

4.6 An extremal problem for the class $\mathscr{S}^*(\psi)$

In 1961, Goluzin [61] obtained the set of extremal functions $f(z)=z/(1-xz)^2$, $|x|=1$ for the problem of maximization of the quantity $\Re\Phi(\log(f(z)/z))$ or $|\Phi(\log(f(z)/z))|$ over the class $\mathscr{S}^*$, where Φ is a non-constant entire function. In 1973, MacGregor [105] proved the result for the class $\mathscr{S}^*(\alpha) := \{f\in\mathscr{A} : \Re(zf'(z)/f(z))>\alpha, \alpha\in[0,1)\}$. Later on Barnard [30] discussed this for Bounded starlike functions. Now, we present the result for the Ma-Minda class:

Theorem 4.6.1. Suppose Φ is a non-constant entire function and $0<|z_0|<1$ and assume that the class $\mathscr{S}^*(\psi)$ is closed. Then maximum of either

$$\Re\Phi\left(\log\frac{f(z_0)}{z_0}\right) \quad \text{or} \quad \left|\Phi\left(\log\frac{f(z_0)}{z_0}\right)\right| \tag{4.6.1}$$

for functions in the class $\mathscr{S}^*(\psi)$ is attained only when the function is of the form

$$f(z)=z\exp\int_0^{\zeta z}\frac{\psi(t)-1}{t}dt, \quad \text{where} \quad |\zeta|=1. \tag{4.6.2}$$

Proof. Since the class $\mathscr{S}^*(\psi)$ is compact, therefore the problem under consideration has a solution. Moreover, in view of a result of Goluzin [61], in (4.6.1) it suffices to consider the continuous functional

$$\Re\Phi\left(\log\frac{f(z_0)}{z_0}\right).$$

Let $f\in\mathscr{S}^*(\psi)$. Then using a result from [102], $f(z)/z\prec f_0(z)/z =: F(z)$, where $f_0(z)=z\exp\int_0^z\frac{\psi(t)-1}{t}dt$ or equivalently $\log(f(z)/z)\prec\log F(z)$. Thus,

$$g(z)=\Phi\left(\log\frac{f(z)}{z}\right)\prec\Phi(\log F(z))=G(z).$$

Note that G is also non-constant as is Φ. Thus for each $r\in(0,1)$ by subordination principle, we obtain $g(\overline{\mathbb{D}}_r)\subset G(\overline{\mathbb{D}}_r)=\Omega$. Since $G(xz)\prec G(z)$ for $|x|\leq 1$ is obvious, therefore for $|z_0|=r$, we have $\{g(z_0) : g\prec G \text{ in } \mathbb{D}\}=\Omega$. Now by considering a support line to the compact set Ω, we conclude that

$$\max_{f\in\mathscr{S}^*(\psi)}\Re\Phi\left(\log\frac{f(z_0)}{z_0}\right)=\Re w_1, \quad w_1\in\partial\Omega.$$

Since G is also an open map, therefore there exists a point z_1 where $|z_1|=r$ and $G(z_1)=w_1$ such that among finitely many w_1, for one suitable w_1, we have

$$\Phi\left(\log\frac{f(z_0)}{z_0}\right)=w_1,$$

where f is the solution for the extremal problem. Now by the well known Lindelöf principle, we have

$$\Phi\left(\log\frac{f(z)}{z}\right)=\Phi(\log F(xz)), \tag{4.6.3}$$

that is, if f is the desired solution, then (4.6.3) holds for some x, $|x| = 1$. Since Φ is a non-constant analytic function, so we may write

$$\Phi(w) = c_0 + c_n w^n + c_{n+1} w^{n+1} + \cdots;\ c_n \neq 0.$$

If we set $\log(f(z)/z) = \alpha_1 z + \alpha_2 z^2 + \cdots$ and $\log(F(z)) = \beta_1 z + \beta_2 z^2 + \cdots$, then from (4.6.3), comparing the coefficients, we get $c_n \alpha_1^n = c_n \beta_1^n$. Or equivalently, $\alpha_1^n = \beta_1^n$, which in particular implies that $|\alpha_1| = |\beta_1|$. Since $\log(f(z)/z) \prec \log F(xz)$, $|\alpha_1| = |\beta_1|$ is possible only if $\log(f(z)/z) = \log F(xyz)$ for some $|y| = 1$. Therefore, we conclude that

$$f(z) = z \exp \int_0^{uz} \frac{\psi(t) - 1}{t} dt,$$

where $|u| = 1$ if f is a solution to the extremal problem. □

Remark 4.6.1. Note that the analogous result for the class $\mathscr{C}(\psi)$ also holds.

Now as an application of the Theorem 4.6.1, we obtain the result due to MacGregor [105]:

Corollary 4.6.2. [105] Suppose Φ is a non-constant entire function and $0 < |z_0| < 1$. Then the maximum of the expression (4.6.1) for functions in the class $\mathscr{S}^*(\alpha)$ is attained only when the function is of the form

$$f(z) = \frac{z}{(1 - \zeta z)^{2-2\alpha}},\ |\zeta| = 1.$$

Proof. If $f \in \mathscr{S}^*(\alpha)$, then $f(z)/z \prec 1/(1-z)^{2-2\alpha}$ and the result follows. □

Corollary 4.6.3. Suppose Φ is a non-constant entire function and $0 < |z_0| < 1$. Then the maximum of the expression (4.6.1) for functions in the class $\mathscr{S}^*_{\wp}$ is attained only when the function is of the form

$$f(z) = z \exp(e^{\zeta z} - 1),\ |\zeta| = 1.$$

Proof. If $f \in \mathscr{S}^*_{\wp}$, then $f(z)/z \prec \exp(e^z - 1)$ and the result follows. □

4.7 The class of subordinants of a starlike function

In 1914, Harald Bohr [34] proved the following remarkable result related to the power series:

Theorem E. (Bohr's Theorem) [34] Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function in $\mathbb{D}$ and $|g(z)| < 1$ for all $z \in \mathbb{D}$, then $\sum_{k=0}^{\infty} |a_k| r^k \leq 1$ for all $z \in \mathbb{D}$ with $|z| = r \leq 1/3$.

Bohr actually proved the above result for $r \leq 1/6$. Further Wiener, Riesz and Shur independently sharpened the result for $r \leq 1/3$. Presently the Bohr inequality for functions mapping unit disk onto different domains, other than unit disk is an active area of research. For the recent development on Bohr-phenomenon, see the articles [10, 31, 33, 113, 125] and references therein. The concept of Bohr-phenomenon in terms of subordination can be described as: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ are analytic in $\mathbb{D}$ and $f(\mathbb{D}) = \Omega$. For a fixed f, consider a class of analytic functions $S(f) := \{g : g \prec f\}$ or equivalently $S(\Omega) := \{g : g(z) \in \Omega\}$. Then the class $S(f)$ is said to satisfy Bohr-phenomenon, if there exists a constant $r_0 \in (0,1]$ satisfying the inequality $\sum_{k=1}^{\infty} |b_k| r^k \leq d(f(0), \partial\Omega)$ for all $|z| = r \leq r_0$ and $g(z) \in S(f)$, where $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of $\Omega = f(\mathbb{D})$. The largest such r_0 for which the inequality holds, is called the Bohr-radius. In 2014, Muhanna et al. [113] proved the Bohr-phenomenon for $S(W_\alpha)$, where $W_\alpha := \{w \in \mathbb{C} : |\arg w| < \alpha\pi/2, 1 \leq \alpha \leq 2\}$, which is a Concave-wedge domain (or exterior of a compact convex set) and the class $R(\alpha, \beta, h)$ defined by $R(\alpha, \beta, h) := \{g \in \mathscr{A} : g(z) + \alpha z g'(z) + \beta z^2 g''(z) \prec h(z)\}$, where h is a convex function (or starlike) and $R(\alpha, \beta, h) \subset S(h)$. In 2018, Bhowmik and Das [31] proved the Bohr-phenomenon for the classes given by $S(f) = \{g \in \mathscr{A} : g \prec f$ and $f \in \mu(\lambda)\}$, where $\mu(\lambda) = \{f \in \mathscr{A} : |(z/f(z))^2 f'(z) - 1| < \lambda, 0 < \lambda \leq 1\}$ and $S(f) = \{g \in \mathscr{A} : g \prec f$ and $f \in \mathscr{S}^*(\alpha), 0 \leq \alpha \leq 1/2\}$, where $\mathscr{S}^*(\alpha)$ is the well-known class of starlike functions of order α.

Here, for any fixed $f \in \mathscr{S}^*(\psi)$, we introduce and study the Bohr-phenomenon inside the disk $|z| \leq 1/3$ for the following class of analytic subordinants:

Definition 4.7.6. Let $f \in \mathscr{S}^*(\psi)$. Then the class of analytic subordinants functions is defined as

$$S_f(\psi) := \left\{ g(z) = \sum_{k=1}^{\infty} b_k z^k : g \prec f \right\}. \tag{4.7.1}$$

Note that $\mathscr{S}^*(\psi) \subset \bigcup_{f \in \mathscr{S}^*(\psi)} S_f(\psi)$. As an application, we obtain the Bohr-radius for the class $S(f)$, where $f \in \mathscr{S}^*((1+Dz)/(1+Ez))$, the class of Janowski starlike functions, with some additional restriction on D and E apart from $-1 \leq E < D \leq 1$.

Note that "the Bohr radius of the class $\mathscr{X}$ is at least r_x", this holds for every result in this section. In general, Bohr radius is estimated for a specific class provided the sharp coefficients bounds of the functions in that class are known. For instance, consider the class of starlike univalent functions, where we have the sharp coefficient bounds: $|a_n| \leq n$. However, for most of the Ma-Minda subclasses, the better coefficients bounds are yet not known. Hence, we encounter the following problem, especially in context of Ma-minda classes, which we deal with here to a certain extent:

Problem 4.7.1. If coefficients bounds are not known, how one can find a good lower estimate for the Bohr radius of a given class?

To readily understand the above problem, consider the class $\mathscr{S}^*(1+ze^z)$, where the sharp coefficients bounds for functions in this class are unknown. In this situation, how one can find the Bohr radius for this class or is there any way out with the lower bounds all alone? Here below we state Theorem 4.7.7, where we find a solution for this problem. Note that the Bohr radius $3-2\sqrt{2} \approx 0.1713$ for the class $\mathscr{S}^*$ serves as a lower bound for the class $S_f(\psi)$ and is also a special case of Theorem 4.7.7.

Let $\mathbb{B}(0,r) := \{z \in \mathbb{C} : |z| < r\}$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$. For any $g \in S_f(\psi)$, we find the radius r_b so that $S_f(\psi)$ obey the following Bohr-phenomenon:

$$\sum_{k=1}^{\infty} |b_k| r^k \leq d(f(0), \partial\Omega) \quad \text{for} \quad |z| = r \leq r_b, \tag{4.7.2}$$

where $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of $\Omega = f(\mathbb{D})$. Now we prove our main result:

Theorem 4.7.7. Let r_* be the Koebe-radius for the class $\mathscr{S}^*(\psi)$, $f_0(z)$ be given by the equation (4.5.2) and $g(z)=\sum_{k=1}^{\infty} b_k z^k \in S_f(\psi)$. Assume $f_0(z)=z+\sum_{n=2}^{\infty} t_n z^n$ and $\hat{f}_0(r)=r+\sum_{n=2}^{\infty}|t_n|r^n$. Then $S_f(\psi)$ satisfies the Bohr-phenomenon

$$\sum_{k=1}^{\infty}|b_k|r^k \leq d(f(0),\partial\Omega), \quad \text{for } |z|=r\leq r_b, \tag{4.7.3}$$

where $r_b=\min\{r_0,1/3\}$, $\Omega=f(\mathbb{D})$ and r_0 is the least positive root of the equation

$$\hat{f}_0(r)=r_*.$$

The result is sharp when $r_b=r_0$ and $t_n>0$.

Proof. Since $g\in S_f(\psi)$, we have $g\prec f$ for a fixed $f\in\mathscr{S}^*(\psi)$. By letting r tends to 1 in Lemma 4.5.1, we obtain the Koebe-radius $r_*=-f_0(-1)$. Therefore $\mathbb{B}(0,r_*)\subset f(\mathbb{D})$, which implies that

$$r_*\leq d(0,\partial\Omega)=|f(z)| \quad \text{for} \quad |z|=1.$$

Also using the result [102, Theorem 1, p.161], we have

$$\frac{f(z)}{z}\prec\frac{f_0(z)}{z}. \tag{4.7.4}$$

Recall the result [31, Lemma 1, p.1090], which reads as: let f and g be analytic in $\mathbb{D}$ with $g\prec f$, where $f(z)=\sum_{n=0}^{\infty}a_n z^n$ and $g(z)=\sum_{k=0}^{\infty}b_k z^k$. Then $\sum_{k=0}^{\infty}|b_k|r^k\leq\sum_{n=0}^{\infty}|a_n|r^n$ for $|z|=r\leq 1/3$. Now using the result for $g\prec f$ and (4.7.4), we have

$$\sum_{k=1}^{\infty}|b_k|r^k\leq r+\sum_{n=2}^{\infty}|a_n|r^n\leq\hat{f}_0(r) \quad \text{for } |z|=r\leq 1/3.$$

Finally, to establish the inequality (4.7.3), it is enough to show $\hat{f}_0(r)\leq r_*$. But this holds whenever $r\leq r_0$, where r_0 is the least positive root of the equation $\hat{f}_0(r)=r_*$. The existence of the root r_0 is ensured by the relations

$$\hat{f}_0(1)\geq|f_0(1)|\geq r_* \quad \text{and} \quad \hat{f}_0(0)<r_*.$$

Thus, if $r_b=\min\{r_0,1/3\}$ then $\sum_{k=1}^{\infty}|b_k|r^k\leq d(0,\partial\Omega)$ holds. The case of sharpness follows for the function f_0. □

Remark 4.7.1. Let us further assume that the coefficients B_n of ψ are positive. Then the function $f_0(z)=z+\sum_{n=2}^{\infty}t_n z^n$ defined by integral representation (4.5.2) can be written as

$$f_0(z)=z\exp\left(\sum_{n=1}^{\infty}\frac{B_n}{n}z^n\right),$$

which implies $f_0(r)=\hat{f}_0(r)$ for $|z|=r$.

From the proof of Theorem 4.7.7, we have the following:

Theorem 4.7.8. Let r_* be the Koebe-radius for the class $\mathscr{S}^*(\psi)$, $f_0(z)$ be given by the equation (4.5.2) and $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}^*(\psi)$. Assume $f_0(z)=z+\sum_{n=2}^{\infty}t_nz^n$ and $\hat{f}_0(r)=r+\sum_{n=2}^{\infty}|t_n|r^n$. Then $\mathcal{S}^*(\psi)$ satisfies the Bohr-phenomenon

$$r+\sum_{n=2}^{\infty}|a_n|r^n\leq d(f(0),\partial\Omega),\quad \text{for } |z|=r\leq r_b,$$

where $r_b=\min\{r_0,1/3\}$, $\Omega=f(\mathbb{D})$ and r_0 is the least positive root of the equation

$$\hat{f}_0(r)=r_*.$$

The result is sharp when $r_b=r_0$ and $t_n>0$.

If we choose $\psi(z)=(1+Dz)/(1+Ez)$, $-1\leq E<D\leq 1$, then $\mathcal{S}^*(\psi)$ denotes the class of Janowski starlike functions and we have

$$r_*=\begin{cases}(1-E)^{\frac{D-E}{E}}, & E\neq 0;\\ e^{-D}, & E=0.\end{cases}\tag{4.7.5}$$

and

$$f_0(z)=\begin{cases}z(1+Ez)^{\frac{D-E}{E}}, & E\neq 0;\\ z\exp(Dz), & E=0.\end{cases}\tag{4.7.6}$$

Recall that the n^{th} $(n\geq 2)$ coefficients of $f_0(z)$ satisfy

$$|t_n|=\prod_{k=0}^{n-2}\frac{|E-D+Ek|}{k+1}=M(n),\tag{4.7.7}$$

Thus from Theorem 4.7.7, we have the following result:

Corollary 4.7.9. Let $\psi(z)=(1+Dz)/(1+Ez)$, $-1\leq E<D\leq 1$. Then $S_f(\psi)$ (and $\mathcal{S}^*(\psi)$) satisfies the Bohr-phenomenon (4.7.2) for $|z|=r\leq r_b$, where $r_b=\min\{r_0,1/3\}$ and r_0 is the least positive root of the equation

$$r+\sum_{n=2}^{\infty}|t_n|r^n-(1-E)^{\frac{D-E}{E}}=0,$$

where t_n is as defined in (4.7.7).

Now as an application of Corollary 4.7.9, we obtain the following result.

Corollary 4.7.10. (Bohr-radius with Janowski class) Let $\psi(z)=(1+Dz)/(1+Ez)$, $-1\leq E<D\leq 1$.

(i) If $E=0$ and $D\geq\frac{3}{4}\log 3$. Then $S_f(\psi)$ (and $\mathcal{S}^*(\psi)$) satisfies the Bohr-phenomenon (4.7.2) for $|z|=r\leq r_0$, where r_0 is the only real root of the equation

$$1-re^{D(1+r)}=0.\tag{4.7.8}$$

(ii) If $E\neq 0$ and further satisfies

$$3(1-E)^{\frac{D-E}{E}}\leq(1+E/3)^{\frac{D-E}{E}}.\tag{4.7.9}$$

Then $S_f(\psi)$ (and $\mathcal{S}^*(\psi)$) satisfies the Bohr-phenomenon (4.7.2) for $|z|=r\leq r_0$, where r_0 is the only real root of the equation

$$(1-E)^{\frac{D-E}{E}} - r(1+Er)^{\frac{D-E}{E}} = 0. \tag{4.7.10}$$

The result is sharp for the function f_0 defined in (4.7.6).

Proof. (i): Since $E=0$, we have $r_* = e^{-D}$. Moreover $\hat{f}_0(r) = f_0(r) = r\exp(Dr)$. Now we need to show

$$r\exp(Dr) \leq e^{-D} \tag{4.7.11}$$

or equivalently $T(r) := 1 - re^{D(1+r)} \geq 0$ holds for $r \leq r_0$. Which obviously holds for $\frac{3}{4}\log 3 \leq D \leq 1$. Since $d(f_0(0), \partial f_0(\mathbb{D})) = r_*$, therefore we see from inequality (4.7.11) that Bohr-radius is sharp for the function f_0 given by (4.7.6).

(ii): Proceeding as in case (i), it is sufficient to show the inequality

$$r(1+Er)^{\frac{D-E}{E}} \leq (1-E)^{\frac{D-E}{E}} \tag{4.7.12}$$

or equivalently $g(r) := (1-E)^{\frac{D-E}{E}} - r(1+Er)^{\frac{D-E}{E}} \geq 0$ holds for $r \leq r_0$. This obviously follows whenever D and E satisfy (4.7.9). In view of the inequality (4.7.12), the sharp Bohr-radius is achieved for the function f_0 given by (4.7.6). □

Remark 4.7.2. (Bohr-radius with starlike functions of order α) Let $\psi(z) := (1+(1-2\alpha)z)/(1-z)$, where $0 \leq \alpha < 1$. We see $\mathcal{S}^*(\psi) := \mathcal{S}^*(\alpha)$ and for this class, we have

$$r_* = \frac{1}{2^{2(1-\alpha)}} \quad \text{and} \quad f_0(z) = \frac{z}{(1-z)^{2(1-\alpha)}}.$$

Observe that here $\hat{f}_0(r) = f_0(r)$. Now as an application of Corollary 4.7.10, we obtain the result due to Bhowmik et al. [31], namely:

If $0 \leq \alpha \leq 1/2$. Then $S_f(\psi)$ satisfies the Bohr-phenomenon $\sum_{k=1}^{\infty} |b_k| r^k \leq d(f(0), \partial\Omega)$, for $|z| = r \leq r_b$, where $r_b = \min\{r_0, 1/3\} = r_0$ and r_0 is the only real root of the equation $(1-r)^{2(1-\alpha)}/r = 2^{2(1-\alpha)}$. The result is sharp.

Now from the above remark, in particular, we have:

Corollary 4.7.11. If $0 \leq \alpha \leq 1/2$. Then the class $\mathcal{S}^*((1+(1-2\alpha)z)/(1-z))$ satisfies the Bohr-phenomenon (4.7.2) for $|z| = r \leq r_0$, where r_0 is the only real root of the equation

$$(1-r)^{2(1-\alpha)}/r = 2^{2(1-\alpha)}.$$

The result is sharp. In particular, the Bohr radius for the class $\mathcal{S}^*$ is $3-2\sqrt{2} \approx 0.1713$.

If we choose $\psi(z) = \sqrt{1+z}$, then $\mathcal{S}^*(\psi) := \mathcal{SL}^*$, the class of lemniscate starlike functions and for this class we have:

$$f_0(z) = \frac{4z\exp(2\sqrt{1+z}-2)}{(1+\sqrt{1+z})^2} \quad \text{and} \quad r_* = -f_0(-1) \approx 0.541341. \tag{4.7.13}$$

Also in this case $\hat{f}_0(r) = f_0(r)$ and therefore, we obtain the following corollary:

Corollary 4.7.12. The class $S_f(\psi)$ (and $\mathcal{SL}^*$), where $\psi(z) = \sqrt{1+z}$ satisfies the Bohr-phenomenon (4.7.2) for $|z| = r \leq 1/3$.

If we consider $\psi(z) = 1 + ze^z$, then $\mathcal{S}^*(\psi) := \mathcal{S}^*_{\wp}$, the class of cardioid starlike functions introduced in [84] and for this class, we have:

$$f_0(z) = z\exp(e^z - 1) \quad \text{and} \quad r_* = -f_0(-1) \approx 0.531464. \tag{4.7.14}$$

Here we can also see that $\hat{f}_0(r) = f_0(r)$ and we obtain the following corollary:

Corollary 4.7.13. The class $S_f(\psi)$ (and $\mathcal{S}^*_{\wp}$), where $\psi(z) = 1 + ze^z$ satisfies the Bohr-phenomenon (4.7.2) for $|z| = r \leq 1/3$.

Ali et al. [10] also showed that the coefficient bound of a function in a class has a role in the estimation of the Bohr radius. Observed that for each $f \in \mathcal{S}^*(\psi)$, the class $S_f(\psi)$ satisfies the Bohr-phenomenon for $r \leq \min(1/3, r_0)$, where r_0 is the least positive root of $\hat{f}_0(r) - r_* = 0$. Since $\mathcal{S}^*(\psi) \subset \bigcup_{f \in \mathcal{S}^*(\psi)} S_f(\psi)$, therefore the Bohr-radius for the class $\mathcal{S}^*(\psi)$ is $r \geq \min(1/3, r_0)$. In Corollary 4.7.13, we find $r_0 \approx 0.349681$ (an upper bound for Bohr radius), which is almost close to $1/3 \approx 0.33333$ and is the unique root of $f_0(r) - r_* = 0$. Moreover, the bound for the coefficients of the functions belonging to $\mathcal{S}^*_{\wp}$ and $\mathcal{SL}^*$ have been conjectured [84, 159] with the extremals given in (4.7.14) and (4.7.13) respectively. Thus by using Theorem 4.7.7 and the approach dealt in [10] (assuming that conjectures are true), we propose the following conjectures:

Conjecture 4.7.1. The Bohr-radius for the class $\mathcal{S}^*_{\wp}$ is $r_0 \approx 0.349681$ which is the unique root in $(0,1)$ of the equation $re^{e^r} = e^{1/e}$.

Conjecture 4.7.2. The Bohr-radius for the class $\mathcal{SL}^*$ is $r_0 \approx 0.439229$, which the unique root in $(0,1)$ of the equation $e^2 r\exp(2\sqrt{1+r}-2) = (1+\sqrt{1+r})^2$.

Highlights of the chapter

In this chapter, we have attempted some generalizations for the Ma-Minda classes, which are new. We established the majorization results for the Ma-Minda classes. We extended and generalized some classical results pertaining to convolution for the class $\mathcal{F}(\phi)$ and studied their applications to classes of Ma-Minda starlike and convex functions along with an extremal problem studied by Goluzin. We also established the concept of Bohr-phenomenon to the Ma-Minda classses.

Chapter 5

Bohr and Rogosinski phenomenon for certain classes of univalent functions

In Geometric function theory, occasionally attempts have been made to solve a particular problem for the Ma-Minda classes, $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$ of univalent starlike and convex functions, respectively. Recently, a popular radius problem generally known as Bohr's phenomenon has been studied in various settings. However, little is known about the Rogosinski radius. In this chapter, for a fixed $f \in \mathcal{S}^*(\psi)$ or $\mathcal{C}(\psi)$, the class of analytic subordinants $S_f(\psi) := \{g : g \prec f\}$ is studied for the Bohr-Rogosinski phenomenon. It's applications to the classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$ are also shown. The phenomenon will be further explored in several relevant directions considering certain natural generalzations connecting the known results and leading to new ones.

5.1 Introduction

The idea of the Bohr phenomenon is being considered and developed in several directions in recent times from classes of analytic functions in one variable to the Banach spaces. Now following the discussion of Section 4.7, for systematic study over the ma-Minda classes, we need to recall the concept of the Bohr phenomenon in terms of subordination:

Definition 5.1.1 (Muhanna, [111]). Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ are analytic in $\mathbb{D}$ and $f(\mathbb{D}) = \Omega$. For a fixed f, consider a class of analytic functions $S(f) := \{g : g \prec f\}$ or equivalently $S(\Omega) := \{g : g(z) \in \Omega\}$. Then the class $S(f)$ is said to satisfy the Bohr phenomenon, if there exists a constant $r_0 \in (0, 1]$ satisfying the inequality $\sum_{k=1}^{\infty} |b_k| r^k \leq d(f(0), \partial\Omega)$ for all $|z| = r \leq r_0$ and $g \in S(f)$, where $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of $\Omega = f(\mathbb{D})$. The largest such r_0 is called the Bohr-radius.

In the work of Muhanna et al. [113] and Bhowmik and Das [31], the role of the sharp coefficient's bound of f was prominent in achieving the respective Bohr radius for the class $S(f)$, see [10, 85, 86]. But in general, the sharp coefficient's bounds for functions in a given class are not available, one can see [38, 84, 85, 89, 150], thus certain power series inequalities are needed. In this direction, Bhowmik and Das obtained the following vital inequality to achieve the Bohr radius for the class $S(f)$, where $f \in \mu(\lambda)$ and $\mathcal{S}^*(\alpha), 0 \leq \alpha \leq 1/2\}$ respectively:

Lemma 5.1.1. [31] let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be analytic in $\mathbb{D}$ and $g \prec f$. Then

$$\sum_{k=0}^{\infty} |b_k| r^k \leq \sum_{n=0}^{\infty} |a_n| r^n, \quad \text{for} \quad |z| \leq \frac{1}{3}.$$

Motivated by the class $S(f)$, Kumar and Gangania in [86, Sec. 5] (also see [64]) further used the above Lemma 5.1.1 in the absence of the sharp coefficient's bounds of f to study the Bohr phenomenon for the class $S_f(\psi)$, which eventually holds for the class $\mathcal{S}^*(\psi)$:

Definition 5.1.2. Let $f \in \mathcal{S}^*(\psi)$ or $\mathcal{C}(\psi)$ be fixed. Then the class of subordinants functions g is defined as:

$$S_f(\psi) := \left\{ g(z) = \sum_{k=1}^{\infty} b_k z^k : g \prec f \right\}.$$

Definition 5.1.3 (Koebe-radius)**.** Let $f \in \mathcal{S}^*(\psi)$. Then either f is a rotation of f_0 or

$$\{w \in \mathbb{C} : |w| \leq r_* = -f_0(-1) \subset f(\mathbb{D})\},$$

where $r_* = \lim_{r \to 1}(-f_0(-r))$. Equality holds for the function

$$f_0(z) = z \exp \int_0^z \frac{\psi(t) - 1}{t} dt.$$

Theorem 5.1.4. [86, Theorem 5.1](also see [64]) Let r_* be the Koebe-radius for the class $\mathcal{S}^*(\psi)$, $f_0(z)$ be given by the equation (4.5.2) and $g(z) = \sum_{k=1}^{\infty} b_k z^k \in S_f(\psi)$. Assume $f_0(z) = z + \sum_{n=2}^{\infty} t_n z^n$ and $\hat{f}_0(r) = r + \sum_{n=2}^{\infty} |t_n| r^n$. Then $S_f(\psi)$ satisfies the Bohr-phenomenon

$$\sum_{k=1}^{\infty} |b_k| r^k \leq d(f(0), \partial\Omega), \quad \text{for } |z| = r \leq r_b,$$

where $r_b = \min\{r_0, 1/3\}$, $\Omega = f(\mathbb{D})$ and r_0 is the least positive root of the equation

$$\hat{f}_0(r) = r_*.$$

The result is sharp when $r_b = r_0$ and $t_n > 0$.

Again if we look at the classical Bohr inequality and try to replace the initial coefficients $a_k, (k = 0, 1)$ by $|f(z)|$ and $|f'(z)|$, and further z by some suitable choice of functions $\omega(z)$ such that $|\omega(z)| < 1$. Or replace the Taylor coefficients a_k completely by the higher order derivatives of f. Then the combinations obtained lead us to the new inequalities, which are usually called Bohr-type inequalities. We now mention a few such combinations:

Suppose that $f(z)=\sum_{k=0}^{\infty}a_kz^k$ be analytic in $\mathbb{D}$ and $a=|a_0|$ and $||f_0||_r^2=\sum_{k=1}^{\infty}|a_{2k}|r^{2k}$, where the function f_0 is given by $f_0(z)=f(z)-a_0$.

1. $|f(z)|^n+\sum_{k=1}^{\infty}|a_k|r^k$, $n=0$ or 1

2. $|f(z)|+|f'(z)||z|+\sum_{k=2}^{\infty}|a_k|r^k$

3. $|f(z)|+\sum_{k=N}^{\infty}\left|\frac{f^{(k)}(z)}{k!}\right|r^k$

4. $|f(\omega(z))|+\sum_{k=1}^{\infty}|a_k|r^k+\frac{1+ar}{(1+a)(1-r)}||f_0||_r^2$

For some important work in this direction, we refer to see [65, 101].

Note that Muhanna et al. [114] recently discussed the Bohr type inequalities for the k-th section of the analytic functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ using the Bohr Operator

$$M_r(f)=\sum_{n=0}^{\infty}|a_n||z^n|=\sum_{n=0}^{\infty}|a_n|r^n.$$

Paulsen and Singh [124] using this operator provided a simple elementary proof of Bohr's Theorem E and extended it to the Banach algebras (for the basic important discussion, see [114, 124]). Now for the simplicity and further discussion, we define the following basic operator for f, where $S^N(f(z))=\sum_{n=N}^{\infty}a_nz^n$:

$$M_r^N(f)=\sum_{n=N}^{\infty}|a_n||z^n|=\sum_{n=N}^{\infty}|a_n|r^n, \tag{5.1.1}$$

and thus the following observations hold for $|z|=r$ for each $z\in\mathbb{D}$

1. $M_r^N(f)\geq 0$, and $M_r^N(f)=0$ if and only if $f\equiv 0$

2. $M_r^N(f+g)\leq M_r^N(f)+M_r^N(g)$

3. $M_r^N(\alpha f)=|\alpha|M_r^N(f)$ for $\alpha\in\mathbb{C}$

4. $M_r^N(f.g)\leq M_r^N(f).M_r^N(g)$

5. $M_r^N(1)=1$.

Using this operator, we now can get similar results as obtained by Muhanna et al. [114] for the interim k-th sections $S_k^N(f(z))=\sum_{n=N}^{k}a_nz^n$ and the function $S^N(f(z))$.

In analogy with Bohr's Theorem, there is also the notion of Rogosinski radius, however, a little is known about Rogosinski radius as compared to Bohr radius, which is defined as follows, also see [93, 141, 148]:

Theorem F (Rogosinski Theorem)**.** If $g(z)=\sum_{k=0}^{\infty} b_k z^k$ with $|g(z)|<1$, then for every $N\geq 1$ we have

$$\left|\sum_{k=0}^{N-1} b_k z^k\right| \leq 1, \quad \text{for} \quad |z|\leq \frac{1}{2}.$$

The radius $1/2$ is called the Rogosinski radius.

Kayumov et al. [76] considered a new quantity, called Bohr-Rogosinski sum, which is described as follows:

$$|g(z)|+\sum_{k=N}^{\infty}|b_k||z|^k, \quad |z|=r.$$

For the case, $N=1$, note that this sum is similar to Bohr's sum, where $g(0)$ is replaced by $|g(z)|$. We also refer the readers to see [3, 13]. Now we say the family $S(f)$ has Bohr-Rogosinski phenomenon, if there exists $r_N^f \in (0,1]$ such that the inequality:

$$|g(z)|+\sum_{k=N}^{\infty}|b_k||z|^k \leq |f(0)|+d(f(0),\partial\Omega)$$

holds for $|z|=r\leq r_N^f$. The largest such r_N^f is called the Bohr-Rogosinski radius. Authors [76] also proved the following interesting results:

Theorem G. [76, Theorems 5-6] Let $g\in S(f)$, where f is univalent in $\mathbb{D}$. Then for each $m,N\in\mathbb{N}$, the inequality

$$|g(z^m)|+\sum_{k=N}^{\infty}|b_k||z|^k \leq |f(0)|+d(f(0),\partial\Omega)$$

holds for $|z|=r\leq r_{m,N}^f$, where $r_{m,N}^f$ is the smallest positive root of:

$$4r^m-(1-r^m)^2+4r^N(N(1-r)+r)\left(\frac{1-r^m}{1-r}\right)^2=0.$$

The radius is sharp for the Koebe function $z/(1-z)^2$. Moreover, if f is convex (univalent) in $\mathbb{D}$, then $r_{m,N}^f$ is the smallest positive root of:

$$3r^m-1+2r^N\left(\frac{1-r^m}{1-r}\right)=0.$$

The radius is sharp for the convex function $z/(1-z)$.

Motivated by the above work, let us now introduce the Bohr-Rogosinski phenomenon for the class of analytic subordinants $S_f(\psi)$:

Definition 5.1.8. The class $S_f(\psi)$ has a Bohr-Rogosinski phenomenon, if there exists an $0<r_0\leq 1$ such that for each $g\in S_f(\psi)$,

$$|g(z)|+\sum_{k=N}^{\infty}|b_k||z|^k \leq d(f(0),\partial\Omega)$$

for $|z| = r \leq r_0$, where $N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of Ω.

Note that $\mathscr{S}^*(\psi) \subset \bigcup_{f\in\mathscr{S}^*(\psi)} S_f(\psi)$. Further, the connection between the Bohr-Rogosinski and Bohr phenomenon can be seen through Definition 5.1.8, if we replace $|g(z)|$ by $|g(z^m)|$, where $m \in \mathbb{N}$, and then consider the special case by taking $m \to \infty$ with $N = 1$.

5.2 Bohr-Rogosinski phenomenon for $\mathscr{S}^*(\psi)$ and $\mathscr{C}(\psi)$

In this section, for a fixed $f \in \mathscr{S}^*(\psi)$ or $\mathscr{C}(\psi)$, the class of subordinants $S_f(\psi) := \{g : g \prec f\}$ is studied for the Bohr-Rogosinski phenomenon in general settings along with its applications to the standard classes of univalent starlike and convex functions. First, we need the following fundamental result, which is an extension of the Lemma 5.1.1:

Lemma 5.2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be analytic in $\mathbb{D}$ and $g \prec f$, then

$$\sum_{k=N}^{\infty} |b_k| r^k \leq \sum_{n=N}^{\infty} |a_n| r^n \tag{5.2.1}$$

for $|z| = r \leq \frac{1}{3}$ and $N \in \mathbb{N}$.

Proof. Since $g \prec f$, we have $g(z) = f(\omega(z))$, where ω is a Schwarz function. For the case $\omega(z) = cz$, $|c| = 1$, the function g is a rotation of f or $g = f$, and the inequality (5.2.1) easily holds. So consider the case: $\omega(z) \neq cz$, $|c| = 1$. Now the coefficient b_k of the function g is given by: for any $k \geq N \in \mathbb{N}$

$$b_k = \sum_{n=N}^{k} a_n {\beta_k}^{(n)},$$

where the t-th power of the analytic function ω is represented as $\omega^t(z) = \sum_{l\geq t} {\beta_l}^{(t)} z^l$, $t \in \mathbb{N}$. Now we see that

$$\begin{aligned}
\sum_{k=N}^{m} |b_k| r^k &= \sum_{k=N}^{m} \left| \sum_{n=N}^{n} a_n {\beta_k}^{(n)} \right| r^k \\
&\leq \sum_{k=N}^{m} \sum_{n=N}^{n} |a_n| |{\beta_k}^{(n)}| r^k \\
&= \sum_{n=N}^{m} |a_n| {M_m}^{(n)}(r),
\end{aligned}$$

where ${M_m}^{(n)}(r) = \sum_{k=n}^{m} |{\beta_k}^{(n)}| r^k$ and $m \in \mathbb{N}$. Since $|\omega^n(z)/z^n| < 1$ for any $n \geq 1$, using Bohr's Theorem E we have

$$\sum_{k=n}^{m} |{\beta_k}^{(n)}| r^{k-n} \leq \sum_{k=n}^{\infty} |{\beta_k}^{(n)}| r^{k-n} \leq 1, \quad r \leq \frac{1}{3},$$

that is, $M_m{}^{(n)}(r) \leq r^n$ holds for $r \leq 1/3$. Hence, for any $m \geq N \geq 1$ and $r \leq 1/3$

$$\sum_{k=N}^{m} |b_k| r^k \leq \sum_{n=N}^{m} |a_n| r^n.$$

The result now follows by taking $m \to \infty$. □

Proof. (Alternate proof of the Lemma 5.2.1) Since $g(z) = f(\omega(z))$, where ω is the Schawrz function, we have

$$M_r^N(g) = M_r^N\left(\sum_{k=N}^{\infty} a_k(\omega(z))^k\right) \leq \sum_{k=N}^{\infty} |a_k| \left(M_r(\omega(z))\right)^k \leq \sum_{k=N}^{\infty} |a_k||z|^k$$

for $|z| = r \leq 1/3$. □

Remark 5.2.1. In Lemma 5.2.1, taking $N \to 1$ and the fact the $g(0) = f(0)$ we obtain Lemma 5.1.1.

Moreover, the following results are obtained using the properties of the operator $M_r^N(f)$ and Lemma 5.2.1:

Corollary 5.2.1. Let the analytic functions f, g and h satisfies $g(z) = h(z) f(\omega(z))$ in $\mathbb{D}$, where ω is the Schwarz function. Assume $|h(z)| \leq \tau$ for $|z| < \tau \leq 1$. Then

$$M_r^N(g) \leq \tau M_r^N(f), \quad 0 \leq |z| = r \leq \frac{\tau}{3}.$$

Corollary 5.2.2. Let $\tau = 1$ in Corollary 5.2.1. Then

$$M_r^N(g) \leq M_r^N(f), \quad 0 \leq |z| = r \leq \frac{1}{3}.$$

We need the following lemma for our next result.

Lemma 5.2.2. *([102]) Let* $f \in \mathcal{S}^*(\psi)$ and $|z_0| = r < 1$. Then $f(z)/z \prec f_0(z)/z$ and

$$-f_0(-r) \leq |f(z_0)| \leq f_0(r).$$

Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of f_0, where $z f_0(z)/f_0(z) = \psi(z)$ such that

$$f_0(z) = z \exp \int_0^z \frac{\psi(t) - 1}{t} dt. \tag{5.2.2}$$

Our next results discuss Bohr-Rogosinski phenomenon for the classes $S_f(\psi)$ and $\mathcal{S}^*(\psi)$, respectively.

Theorem 5.2.3. Let $f_0(z)$ be given by the equation (5.2.2) and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\psi)$. Assume that $f_0(z) = z + \sum_{n=2}^{\infty} t_n z^n$ and $\hat{f}_0(r) = r + \sum_{n=2}^{\infty} |t_n| r^n$. If $g \in S_f(\psi)$. Then

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \partial\Omega) \tag{5.2.3}$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the unique positive root of the equation:

$$\hat{f}_0(r^m)+\hat{f}_0(r)-p_{\hat{f}_0}(r)=-f_0(-1), \tag{5.2.4}$$

where

$$p_{\hat{f}_0}(r)=\begin{cases} 0, & N=1; \\ r, & N=2 \\ r+\sum_{n=2}^{N-1}|t_n|r^n, & N\geq 3 \end{cases}$$

The result is sharp when $r_b=r_0$ and $t_n>0$.

Proof. Let $g(z)=\sum_{k=1}^{\infty}b_kz^k\prec f(z)$, where $f\in\mathscr{S}^*(\psi)$. Now by Lemma 5.2.1, for $r\leq 1/3$, we have

$$\sum_{k=N}^{\infty}|b_k|r^k\leq\sum_{n=N}^{\infty}|a_n|r^n.$$

Again applying Lemma 5.2.1 on $f(z)/z\prec f_0(z)/z$ (Lemma 5.2.2), we get that for $r\leq\frac{1}{3}$

$$\sum_{k=N}^{\infty}|b_k|r^k\leq\sum_{n=N}^{\infty}|a_n|r^n\leq\sum_{n=N}^{\infty}|t_n|r^n. \tag{5.2.5}$$

Now $g\prec f$ implies that $g(z)=f(\omega(z))$, which using the Lemma 5.2.2 yields

$$|g(|z|\leq r)|=|f(\omega(|z|\leq r))|\leq\max_{|z|=r}|f(|z|\leq r)|\leq f_0(r),$$

that is,

$$|g(z)|=|f(\omega(z))|\leq f_0(r)$$

for $|z|=r$, where ω is a Schwarz function. Moreover, replacing z by z^m in the above inequality along with the definition of $\hat{f}_0$ gives

$$|g(z^m)|\leq\hat{f}_0(r^m). \tag{5.2.6}$$

Also, by letting r tends to 1 in Lemma 5.2.2, we obtain the Koebe-radius $r_*=-f_0(-1)$. Therefore, the open ball $\mathbb{B}(0,r_*)\subset f(\mathbb{D})$, which implies that for $|z|=1$

$$r_*\leq d(0,\partial\Omega). \tag{5.2.7}$$

Now using the equations (5.2.5), (5.2.6) and (5.2.7), we have

$$\begin{aligned}|g(z^m)|+\sum_{k=N}^{\infty}|b_k||z|^k &\leq\hat{f}_0(r^m)+\sum_{n=N}^{\infty}|t_n|r^n\\ &=\hat{f}_0(r^m)+\hat{f}_0(r)-p_{\hat{f}_0}(r)\\ &\leq r_*\\ &\leq d(0,\partial\Omega)\end{aligned}$$

holds whenever $|z|=r\leq\min\{\frac{1}{3},r_0\}$, where r_0 is the smallest positive root of the equation:

$$G(r):=\hat{f}_0(r^m)+\hat{f}_0(r)-p_{\hat{f}_0}(r)-r_*=0.$$

Note that $G(0) < 0$, and since $\hat{f}_0(1) \geq |f_0(1)| \geq r_*$ (see Lemma 5.2.2), we see that

$$2\hat{f}_0(1) - \sum_{n=1}^{N-1} |t_n| - r_* = (\hat{f}_0(1) - \sum_{n=1}^{N-1} |t_n|) + (\hat{f}_0(1) - r_*) > 0$$

where $t_1 = 1$, which implies $G(1) > 0$. Clearly, for $0 \leq r \leq 1$

$$G'(r) = \hat{f'}_0(r^m) + (\hat{f'}_0(r) - p'_{\hat{f}_0}(r)) > 0,$$

which implies G is a continuous increasing function in $[0,1]$. Thus $G(r) = 0$ has a root in the interval $(0,1)$. Let us take $g = f = f_0$, then the sharpness follows for the function f_0 with the equality in (5.2.3) as

$$f_0({r_b}^m) + \sum_{n=N}^{\infty} t_n {r_b}^n = r^* = d(0, \partial\Omega)$$

when $r_b = r_0$ and $t_n > 0$. □

Remark 5.2.2. Let $\psi(z) = (1+z)/(1-z)$, then Theorem 5.2.3 reduces to [76, Theorem 5].

Remark 5.2.3. Observe that if we take $m \to \infty$ and $N = 1$, then Theorem 5.2.3 reduces to [86, Theorem 5.1].

Corollary 5.2.4. Let $f_0(z)$ be given by the equation (5.2.2). Assume $f_0(z) = z + \sum_{n=2}^{\infty} t_n z^n$ and $\hat{f}_0(r) = r + \sum_{n=2}^{\infty} |t_n| r^n$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\psi)$. Then

$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(0, \partial\Omega) \tag{5.2.8}$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the unique positive root of the equation:

$$\hat{f}_0(r^m) + \hat{f}_0(r) - p_{\hat{f}_0}(r) = -f_0(-1),$$

where $p_{\hat{f}_0}$ is as defined in Theorem 5.2.3. The radius is sharp for the function f_0 when $r_b = r_0$ and $t_n > 0$.

Corollary 5.2.5. Let $\psi(z) = 1 + \dfrac{4}{3}z + \dfrac{2}{3}z^2$, $f_0(r) = r\exp\left(\dfrac{4}{3}r + \dfrac{r^2}{3}\right)$ and $m = 1$. If $g \in S_f(\psi)$. Then the inequality (5.2.3) holds for $|z| = r \leq r_N$, where $N \in \mathbb{N}$ and $r_N (< 1/3)$ is the unique positive root of the equation:

$$2r\exp\left(\frac{4}{3}r + \frac{r^2}{3}\right) - p_{f_0}(r) - \exp(-1) = 0,$$

where $p_{f_0} = p_{\hat{f}_0}$ is as defined in Theorem 5.2.3 with $|t_n| = t_n = {f_0}^n(0)/n!$. Moreover, if $f \in \mathcal{S}^*(\psi)$. Then the inequality (5.2.8) also holds for $r \leq r_N$. The radius r_N is sharp.

Remark 5.2.4. In Corollary 5.2.5, we observe that the radius r_N approaches $r_0 = 0.25588\cdots$ for large value of N, where r_0 is the unique positive root of

$$r\exp\left(\frac{4}{3}r + \frac{r^2}{3}\right) - \exp(-1) = 0.$$

Moreover, if $m \geq 2$ then the inequalities (5.2.3) and (5.2.8) hold for $r \leq 1/3$.

Corollary 5.2.6. Let $\psi(z) = 1 + ze^z$ and $m = 1$. If $g \in S_f(\psi)$. Then the inequality (5.2.3) holds for $|z| = r \leq r_N = \min\{r_0, 1/3\}$, where $N \in \mathbb{N}$ and r_0 is the unique positive root of the equation:

$$2r\exp(e^r - 1) - T(r) - \exp(e^{-1} - 1) = 0,$$

where

$$T(r) = \begin{cases} 0, & N = 1; \\ r, & N = 2; \\ \sum_{n=1}^{N-1} \frac{B_{n-1}}{(n-1)!} r^n, & N \geq 3 \end{cases}$$

and B_n are the bell numbers such that $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. Moreover, if $f \in \mathcal{S}^*(\psi)$. Then the inequality (5.2.8) also holds for $r \leq r_N$. The radius $r_N < 1/3$ is sharp for $N \leq 3$.

Corollary 5.2.7. Let $\psi(z) = 1 + \frac{z}{k}\left(\frac{k+z}{k-z}\right)$ with $k = \sqrt{2} + 1$. If $g \in S_f(\psi)$. Then the inequality (5.2.3) holds for $|z| = r \leq r_b = \min\{1/3, r_0\}$, where $N \in \mathbb{N}$ and r_0 is the unique positive root of the equation:

$$\frac{r^m}{e^{r^m}}\left(\frac{k}{k - r^m}\right)^{2k} + \frac{r}{e^r}\left(\frac{k}{k - r}\right)^{2k} - p_{f_0}(r) - e\left(\frac{k}{k+1}\right)^{2k} = 0,$$

where $p_{f_0} = p_{\hat{f}_0}$ is as defined in Theorem 5.2.3 and $t_n = |t_n|$ are the Taylor coefficients of the function $f_0(r) = \frac{r}{e^r}\left(\frac{k}{k-r}\right)^{2k}$. Moreover, if $f \in \mathcal{S}^*(\psi)$. Then the inequality (5.2.8) also holds for $r \leq r_b$. The radius r_b is sharp when $m = 1$ and $N \leq 4$.

Since all the Taylor coefficients of the function $1 + \sin z$ are not positive, $\hat{f}_0 \neq f_0$. So we consider the radius r_N up to three decimal places only, which also reveals the connection of positive coefficients of ψ to the sharp Bohr-Rogosinski radius.

Corollary 5.2.8. Let $\psi(z) = 1 + \sin z$ and $m = 1$. If $g \in S_f(\psi)$. Then the inequality (5.2.3) holds for $|z| = r \leq r_N$, where $N \in \mathbb{N}$ and $r_N(< 1/3)$ is the unique positive root of the equation:

$$2r\exp(Si(r)) - \exp(Si(-1)) - p_{f_0}(r) = 0,$$

where $f_0(r) = r\exp(Si(r))$, where $Si(x)$ is the Sin Integral defined as:

$$Si(x) := \int_0^x \frac{\sin(x)}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

Moreover, if $f \in \mathcal{S}^*(\psi)$. Then the inequality (5.2.8) also holds for $r \leq r_N$.

Remark 5.2.5. In Corollary 5.2.8, the numerical computations reveal that the Bohr-Rogosinski radius $r_N \approx 0.290 * \cdots < 1/3$ for any $N > 4$, where $* = 6$ or 7. Also $r_N < 1/3$ for $N \leq 4$. Moreover, as $N \to \infty$, the required radius $r_0 \approx 0.290 * \cdots$ is the unique positive root of

$$r\exp(Si(r)) - \exp(Si(-1)) = 0.$$

Next, we discuss the Bohr-Rogosinski phenomenon for the celebrated Janowski class, $\mathcal{S}^*((1 + Dz)/(1 + Ez)) \equiv \mathcal{S}[D, E]$, where $-1 \leq E < D \leq 1$ of univalent starlike functions. We need to recall:

Lemma 5.2.3. [16, Theorem 3] If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{S}[D,E]$. Then for $n\geq 2$, the following sharp bounds occur:

$$|a_n|\leq\prod_{k=0}^{n-2}\frac{|E-D+Ek|}{k+1}.$$

Theorem 5.2.9. Let $\psi(z)=(1+Dz)/(1+Ez)$, $-1\leq E<D\leq 1$. If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{S}^*(\psi)$. Then

$$|f(z^m)|+\sum_{n=N}^{\infty}|a_n||z|^n\leq d(0,\partial\Omega) \tag{5.2.9}$$

holds for $|z|=r\leq r_0$, where $m,N\in\mathbb{N}$, $\Omega=f(\mathbb{D})$ and r_0 is the unique positive root of the equations:

$$r^m(1+Er^m)^{\frac{D-E}{E}}+A(r)+\sum_{n=N}^{\infty}\prod_{k=0}^{n-2}\frac{|E-D+Ek|}{k+1}r^n-(1-E)^{\frac{D-E}{E}}=0,\quad \text{if } E\neq 0,$$

where $A(r)=r$ for $N=1$ and 0 otherwise, and

$$r^me^{Dr^m}+re^{Dr}-J(r)-e^{-D}=0,\quad \text{if } E=0,$$

where

$$J(r)=\begin{cases}0, & N=1;\\ r, & N=2;\\ \sum_{n=2}^{N-1}\prod_{k=0}^{n-2}\frac{D}{k+1}r^n, & N\geq 3.\end{cases} \tag{5.2.10}$$

The radius r_0 is sharp.

Proof. Let us consider the function f_0 such that $zf_0'(z)/f_0(z)=(1+Dz)/(1+Ez)$, which is given by

$$f_0(z)=\begin{cases}z(1+Ez)^{\frac{D-E}{E}}, & E\neq 0;\\ ze^{Dz}, & E=0.\end{cases} \tag{5.2.11}$$

Now using the Lemma 5.2.2 and Lemma 5.2.3, we have $|f(z^m)|\leq f_0(r^m)$, the Koebe radius $r_*=-f_0(-1)$ and

$$\sum_{n=N}^{\infty}|a_n||z|^n\leq\sum_{n=N}^{\infty}\prod_{k=0}^{n-2}\frac{|E-D+Ek|}{k+1}r^n,\quad N\geq 2.$$

Now proceeding as in Theorem 5.2.3, for r_0 as defined in the statement, the result follows. To prove the sharpness of the radius r_0, we see that at $|z|=r=r_0$ and $f=f_0$ given in (5.2.11):

$$\begin{aligned}&|f(z^m)|+\sum_{n=N}^{\infty}|a_n||z|^n\\ &=\begin{cases}(r_0)^m(1+E(r_0)^m)^{\frac{D-E}{E}}+A(r_0)+\sum_{n=N}^{\infty}\prod_{k=0}^{n-2}\frac{|E-D+Ek|}{k+1}(r_0)^n, & E\neq 0;\\ (r_0)^me^{D(r_0)^m}+(r_0)e^{Dr_0}-J(r_0), & E=0.\end{cases}\\ &=\begin{cases}(1-E)^{\frac{D-E}{E}}, & E\neq 0;\\ e^{-D}, & E=0.\end{cases}\\ &=-f_0(-1)=d(0,\partial\Omega),\end{aligned}$$

where $J(r)$ is as defined in (5.2.10), and $A(r)=r$ for $N=1$ and 0 otherwise for the case $E\neq 0$. $\square$

Remark 5.2.6. Taking $m \to \infty$ and $N = 1$ in Corollary 5.2.9, we obtain the Bohr radius for the class $\mathcal{S}[D,E]$, which covers many classical cases.

In Theorem 5.2.9, putting $D = 1-2\alpha$ and $E = -1$, where $0 \leq \alpha < 1$, we get the result for the class of univalent starlike functions of order α, that is, $\mathcal{S}^*(\alpha)$:

Corollary 5.2.10. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\alpha)$. Then the inequality (5.2.9) holds for $|z| = r \leq r_0$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the smallest positive root of the equations:

$$\frac{r^m}{(1-r^m)^{2(1-\alpha)}} + A(r) + \sum_{n=N}^{\infty} \prod_{k=0}^{n-2} \frac{k+2(1-\alpha)}{k+1} r^n - \frac{1}{4^{1-\alpha}} = 0,$$

where $A(r) = r$ for $N = 1$ and 0 otherwise. The radius r_0 is sharp.

Putting $\alpha = 0$ in Corollary 5.2.10, we get the following:

Corollary 5.2.11. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*$. Then the inequality (5.2.9) holds for $|z| = r \leq r_0$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the smallest positive root of the equations:

$$4r^m - (1-r^m)^2 + 4r^N(N(1-r)+r)\left(\frac{1-r^m}{1-r}\right)^2 = 0.$$

The radius r_0 is sharp.

To proceed further, we need to recall the following fundamental result:

Lemma 5.2.4. [102] Let $f \in \mathcal{C}(\psi)$. Then $zf''(z)/f'(z) \prec zl_0''(z)/l_0'(z)$ and $f'(z) \prec l_0'(z)$. Also, for $|z| = r$ we have

$$-l_0(-r) \leq |f(z)| \leq l_0(r),$$

where

$$1 + zl_0''(z)/l_0'(z) = \psi(z). \tag{5.2.12}$$

Further, $r_* = -l_0(-1) = \lim_{r\to 1}(-l_0(-r))$ is the Koebe-radius for the class $\mathcal{C}(\psi)$.

Now we discuss the results for the convex analogue $\mathcal{C}(\psi)$ of $\mathcal{S}^*(\psi)$.

Theorem 5.2.12. Let r_* be the Koebe-radius for the class $\mathcal{C}(\psi)$, $l_0(z)$ be given by the equation (5.2.12) and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}(\psi)$. Assume $l_0(z) = z + \sum_{n=2}^{\infty} l_n z^n$ and $\hat{l}_0(r) = r + \sum_{n=2}^{\infty} |l_n| r^n$. If $g \in S_f(\psi)$. Then

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \partial\Omega) \tag{5.2.13}$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the unique positive root of the equation:

$$\hat{l}_0(r^m) + \hat{l}_0(r) - p_{\hat{l}_0}(r) = -l_0(-1),$$

where

$$p_{\hat{l}_0}(r) = \begin{cases} 0, & N = 1; \\ r, & N = 2; \\ r + \sum_{n=2}^{N-1} |l_n| r^n, & N \geq 3. \end{cases}$$

The result is sharp when $r_b = r_0$ and $l_n > 0$.

Proof. Let $g(z) = \sum_{k=1}^{\infty} b_k z^k \prec f(z)$, where $f \in \mathscr{C}(\psi)$. From the Alexander's relation, it is known that $f \in \mathscr{C}(\psi)$ if and only if

$$zf'(z) = \tilde{g}(z), \quad \text{or equivalently} \quad f(z) = \int_0^z \frac{\tilde{g}(t)}{t} dt$$

for some $\tilde{g} \in \mathscr{S}^*(\psi)$. Now by Lemma 5.2.1, for $r \leq 1/3$, we have

$$\sum_{k=N}^{\infty} |b_k| r^k \leq \sum_{n=N}^{\infty} |a_n| r^n = \sum_{n=N}^{\infty} \frac{|\tilde{b}_n|}{n} r^n, \tag{5.2.14}$$

where $\tilde{b}_n$ are the Taylor coefficients of $\tilde{g}$. Again applying Lemma 5.2.1 on $f'(z) \prec l_0'(z)$ (Lemma 5.2.4), we get that

$$M_{\tilde{g}}(r) - p_{\tilde{g}}(r) \leq M_h(r) - p_h(r), \quad r \leq \frac{1}{3}, \tag{5.2.15}$$

where $M_g(x) := \sum_{k=1}^{\infty} |b_k| x^k$, and h is given by the relation $zl_0'(z) = h(z)$. Now using the equations (5.2.14) and (5.2.15), we have for $r \leq 1/3$

$$\begin{aligned} \sum_{k=N}^{\infty} |b_k||z|^k &\leq \sum_{n=N}^{\infty} \frac{|\tilde{b}_n|}{n} r^n \\ &= \int_0^r \frac{M_{\tilde{g}(t)} - p_{\tilde{g}(t)}}{t} dt \\ &\leq \int_0^r \frac{M_h(t) - p_h(t)}{t} dt \\ &= \sum_{n=N}^{\infty} |l_n| r^n \\ &= \hat{l}_0(r) - p_{\hat{l}_0}(r). \end{aligned} \tag{5.2.16}$$

Now $g \prec f$ implies that $g(z) = f(\omega(z))$, which using the Lemma 5.2.4 yields

$$|g(|z| \leq r)| = |f(\omega(|z| \leq r))| \leq \max_{|z|=r} |f(|z| \leq r)| \leq l_0(r),$$

that is,

$$|g(z)| = |f(\omega(z))| \leq l_0(r)$$

for $|z| = r$, where ω is a Schwarz function. Moreover, replacing z by z^m gives

$$|g(z^m)| \leq \hat{l}_0(r^m). \tag{5.2.17}$$

Also, by letting r tends to 1 in Lemma 5.2.4, we obtain the Koebe-radius $r_* = -l_0(-1)$. Therefore, the open ball $\mathbb{B}(0, r_*) \subset f(\mathbb{D})$, which implies that for $|z| = 1$

$$r_* \leq d(0, \partial\Omega). \tag{5.2.18}$$

Hence, using the inequalities (5.2.16), (5.2.17) and (5.2.18), we have

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \le \hat{l}_0(r^m) + \hat{l}_0(r) - p_{\hat{l}_0}(r)$$
$$\le r_*$$
$$\le d(0, \partial\Omega)$$

holds whenever $|z| = r \le \min\{\frac{1}{3}, r_0\}$, where r_0 is the smallest positive root of the equation:

$$H(r) := \hat{l}_0(r^m) + \hat{l}_0(r) - p_{\hat{l}_0}(r) - r_* = 0.$$

Clearly, H is continuous and $H'(r) > 0$ for $0 \le r < 1$. Note that $H(0) < 0$, and since $\hat{l}_0(1) \ge |l_0(1)| \ge r_*$ as r tends to 1 (see Lemma 5.2.4), we see that

$$2\hat{l}_0(1) - \sum_{n=1}^{N-1} |l_n| - r_* = (\hat{l}_0(1) - \sum_{n=1}^{N-1} |l_n|) + (\hat{l}_0(1) - r_*) > 0,$$

which implies $H(1) > 0$. Thus $H(r) = 0$ has a root in the interval $(0,1)$. The sharpness follows for the function l_0 by taking $g = f = l_0$ with the equality in (5.2.13) as

$$l_0(r_b^m) + \sum_{n=N}^{\infty} l_n r_b^n = r_* = d(0, \partial\Omega)$$

when $r_b = r_0$ and $l_n > 0$. □

Remark 5.2.7. Let $\psi(z) = (1+z)/(1-z)$, then Theorem 5.2.12 reduces to [76, Theorem 6].

Remark 5.2.8. Let $g \in S_f(\psi)$, where $f \in \mathscr{C}(\psi)$. In (5.2.14) if we use the Rogosinski's well known result: "If $g \prec f$ and f is a normalized convex function then $|b_n| \le 1$ for $n \ge 1$", then we can remove the role of Lemma 5.2.1 in the proof of Theorem 5.2.12 and get

$$\sum_{k=N}^{\infty} |b_k| r^k \le \sum_{n=N}^{\infty} r^k = \frac{r^N}{1-r}, \quad r \in (0,1). \tag{5.2.19}$$

Further combining (5.2.19) and the inequality (5.2.17)

$$|g(z^m)| \le \hat{l}_0(r^m).$$

we get

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k| r^k \le \hat{l}_0(r^m) + \frac{r^N}{1-r}, \quad r \in (0,1).$$

Thus the inequality (5.2.13) follows for $|z| = r \le r_0$, where r_0 is the smallest positive root of

$$\hat{l}_0(r^m) + \frac{r^N}{1-r} + l_0(-1) = 0. \tag{5.2.20}$$

Also, the radius r_0 can not be improved follows if we choose

$$g(z) = f(z) = \frac{z}{1-z} = l_0(z).$$

Note that if the coefficients of ψ are positive then $\hat{l}_0 = l_0$, and in this case (5.2.20) reduces to

$$l_0(r^m) + \frac{r^N}{1-r} + l_0(-1) = 0.$$

Moreover, taking $N = 1$ and $m \to \infty$, we get the result obtained by Hamada [64, Theorem 4.1].

The following result is explicitly for the class $\mathscr{C}(\psi)$.

Corollary 5.2.13. Let r_* be the Koebe-radius for the class $\mathscr{C}(\psi)$, $l_0(z)$ be given by the equation (5.2.12). Assume $l_0(z) = z + \sum_{n=2}^{\infty} l_n z^n$ and $\hat{l}_0(r) = r + \sum_{n=2}^{\infty} |l_n| r^n$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{C}(\psi)$. Then

$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(0, \partial\Omega) \tag{5.2.21}$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the unique positive root of the equation:

$$\hat{l}_0(r^m) + \hat{l}_0(r) - p_{\hat{l}_0}(r) = r_*,$$

where $p_{\hat{l}_0}$ is as defined in Theorem 5.2.12. The radius is sharp for the function l_0 when $r_b = r_0$ and $l_n > 0$.

Remark 5.2.9. The special case of taking $m \to \infty$ and $N = 1$ in Theorem 5.2.12 and Corollary 5.2.13 establish the Bohr phenonmenon for the classes $S_f(\psi)$ and $\mathscr{C}(\psi)$, respectively.

After some little computations when $\psi(z) = (1+z)/(1-z)$, the Corollary 5.2.13 yields:

Corollary 5.2.14. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{C}$. Then the inequality (5.2.21) holds for $|z| = r \leq r_0$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the unique positive root of the equations:

$$3r^m - 1 + 2r^N \left(\frac{1-r^m}{1-r} \right) = 0.$$

The radius r_0 is sharp.

Corollary 5.2.15. Let $\psi(z) = 1 + ze^z$ and $m = 1$. If $g \in S_f(\psi)$. Then the inequality (5.2.13) holds for $|z| = r \leq r_N$, where $N \in \mathbb{N}$ and $r_N (< 1/3)$ is the unique positive root of the equation:

$$2r(1+re^r)\exp(e^r - 1) - H(r) - (1-e^{-1})e^{e^{-1}-1} = 0,$$

where

$$H(r) = \begin{cases} 0, & N = 1; \\ r, & N = 2; \\ \sum_{n=0}^{N-1} \left(\frac{(n+1)B_n}{n!} \right) r^{n+1}, & N \geq 3. \end{cases}$$

and B_n are the bell numbers such that $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. Moreover, if $f \in C(\psi)$. Then the inequality (5.2.21) also holds for $r \leq r_N$. The radius r_N is sharp.

5.3 Bohr radius for some classes of Harmonic mappings

There are certain classes of univalent functions whose natural extensions to the harmonic functions have been studied, see [53, 116]. In this section, we aim to study Bohr radius problems for such classes. As an application, we derive the Bohr radius for their analytic analog.

Let $\mathcal{H}$ denotes the class of complex-valued harmonic functions f (which satisfy the Laplacian equation $\Delta f = 4 f_{z\bar{z}} = 0$) defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, then we can write $f = h + \bar{g}$, where h and g are analytic and satisfy $f(0) = g(0)$. If the Jacobian $J_f := |h'|^2 - |g'|^2 > 0$, we say f is sense-preserving in $\mathbb{D}$. Let $\mathcal{H}_0$ be the class of functions f with $f_{\bar{z}}(0) = 0$ and $f = h + \bar{g}$, where

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m \quad \text{and} \quad g(z) = \sum_{m=2}^{\infty} b_m z^m$$

are analytic functions in $\mathbb{D}$. For $g \equiv 0$, $\mathcal{H}_0$ reduces to the class $\mathcal{A}$ of analytic functions f with normalization $f(0) = 0 = f_z(0) - 1$. Let $\mathcal{S}^0_{\mathcal{H}}$ denotes the class of harmonic and univalent functions, which clearly includes the well-known class of normalized univalent functions $\mathcal{S}$.

Motivated by the Sakaguchi class of starlike functions with respect to the symmetric points using subordination [146], Cho and Dziok [37] considered a subclass of $\mathcal{S}^0_{\mathcal{H}}$, which is given by

$$\mathcal{S}^{**}_{\mathcal{H}}(C,D) := \left\{ f \in \mathcal{H}_0 : \frac{2\mathcal{D}_{\mathcal{H}} f(z)}{f(z) - f(-z)} \prec \frac{1+Cz}{1+Dz}, -D \leq C < D \leq 1 \right\},$$

where $\mathcal{D}_{\mathcal{H}} f(z) := zh'(z) - \overline{zg'(z)}$. Further, in light of Silverman's work [152], they defined the class $\mathcal{S}^{**}_{\tau}(C,D) := \tau^0 \cap \mathcal{S}^{**}_{\mathcal{H}}(C,D)$, where $\tau^a (a \in \{0,1\})$ is the class of functions in $\mathcal{H}_0$ with the coefficients a_m and b_m are replaced respectively by $-a_m$ and $(-1)^a b_m$ for all m. By involving Janowski functions, Dziok [43] studied the following classes:

$$\mathcal{S}^{*}_{\mathcal{H}}(C,D) := \left\{ f \in \mathcal{H}_0 : \frac{2\mathcal{D}_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1+Cz}{1+Dz}, -D \leq C < D \leq 1 \right\},$$

and $\mathcal{S}^{c}_{\mathcal{H}}(C,D) := \{f \in \mathcal{S}^0_{\mathcal{H}} : D_{\mathcal{H}} f(z) \in \mathcal{S}^*_{\mathcal{J}}(C,D)\}$ in addition to the classes $\mathcal{S}^*_{\tau}(C,D) := \tau^0 \cap \mathcal{S}^*_{\mathcal{H}}(C,D)$ and $\mathcal{S}^c_{\tau}(C,D) := \tau^1 \cap \mathcal{S}^c_{\mathcal{H}}(C,D)$. Singh [157] studied the subclass given by $\mathcal{F}(\lambda) := \{f \in \mathcal{A} : |f'(z) - 1| < \lambda, \lambda \in (0,1]\}$ of close-to-convex functions. Later, Nagpal and Ravichandran [116] examined its harmonic extension defined as:

$$\mathcal{F}^0_{\mathcal{H}}(\lambda) := \{f = h + \bar{g} : |f_z(z) - 1| < \lambda - |f_{\bar{z}}(z)|, \lambda \in (0,1]\}.$$

Gao and Zohu [53] investigated a subclass of close-to-convex functions given by $W(\mu,\rho) := \{f \in \mathcal{A} : \Re(h'(z) + \mu z h''(z)) > \rho,\ \mu \geq 0, 0 \leq \rho < 1\}$. Rajbala and Prajapat [133] also explored the subclass of close-to-convex harmonic mappings defined as:

$$W^0_{\mathcal{H}}(\mu,\rho) := \{f = h + \bar{g} \in \mathcal{H}_0 : \Re(h'(z) + \mu z h''(z) - \rho) > |g'(z) + \mu z g''(z)|\},$$

where $\mu \geq 0$ and $0 \leq \rho < 1$, which is the harmonic extension of $W(\mu,\rho)$. This generalizes the classes studied in [55, 117]. In a similar way, Dixit and Porwal [42] considered the class $R_{\mathscr{H}}(\beta) = \{f = h+\bar{g} : \Re\{h'(z)+g'(z)\} \leq \beta, 2 \geq \beta > 1\}$, where $h(z) = z+\sum_{m=2}^{\infty}|a_m|z^m$, $g(z) = \sum_{m=1}^{\infty}|b_m|\bar{z}^m$ with $|b_1| < 1$. Now if we take $b_1 = 0$, then we get the class

$$R^0_{\mathscr{H}}(\beta) := \{f = h+\bar{g} \in \mathscr{H}_0 : \Re\{h'(z)+g'(z)\} \leq \beta, \beta > 1\},$$

comprising of functions with positive coefficients and reduces to the class $R(\beta)$ explored by Uralegaddi for the case $g \equiv 0$. Altinkaya et al. [14] studied the class $k-\widetilde{ST}_q^{-}(\alpha)$ of functions in $\mathscr{A}$ with negative coefficients associated with the conic domains defined by $\Re(p(z)) > k|p(z)-1|+\alpha$, where $0 \leq k < \infty$, $0 \leq \alpha < 1$, $0 < q < 1$, $p(z) = z(\widetilde{D}_q f)(z)/f(z)$ and

$$(\widetilde{D}_q f)(z) = \begin{cases} \dfrac{f(qz)-f(q^{-1}z)}{(q-q^{-1})z}, & z \neq 0 \\[2ex] f'(0), & z = 0 \end{cases} \quad \text{and} \quad [\widetilde{m}]_q = \frac{q^m-q^{-m}}{q-q^{-1}}.$$

In the literature, sufficient conditions for many classes of harmonic and analytic mappings are obtained in terms of coefficients. See [14, 37, 42, 43, 116, 133]. In a similar way, we introduce a new subclass of $\mathscr{H}_0$ as follows:

Definition 5.3.1. Let us consider the class

$$\mathscr{B}^0_{\mathscr{H}}(M) := \{f = h+\bar{g} \in \mathscr{H}_0 : \sum_{m=2}^{\infty}(\gamma_m|a_m|+\delta_m|b_m|) \leq M,\ M > 0\},$$

where

$$\gamma_m,\ \delta_m \geq \alpha_2 := \min\{\gamma_2, \delta_2\} > 0, \tag{5.3.1}$$

for all $m \geq 2$.

Note that the classes $S^{**}_{\tau}(C,D)$, $S^{*}_{\tau}(C,D)$, $S^{c}_{\tau}(C,D)$ and $k-\widetilde{ST}_q^{-}(\alpha)$ are all become subclasses of $\mathscr{B}^0_{\mathscr{H}}(M)$ for some suitable choices of γ_m, δ_m and M, which is evident from equations (5.3.5), (5.3.6), (5.3.7) and Lemma 5.3.1 respectively.

Now let us recall the Bohr phenomenon for the harmonic mappings:

Definition 5.3.2. Let $f(z) = h(z)+\overline{g(z)} = z+\sum_{m=2}^{\infty}a_m z^m+\overline{\sum_{m=2}^{\infty}b_m z^m} \in \mathscr{H}_0$. Then the Bohr-phenomenon is to find the constant $r^* \in (0,1]$ such that the inequality $r+\sum_{m=2}^{\infty}(|a_m|+|b_m|)r^m \leq d(f(0),\partial\Omega)$ holds for all $|z| = r \leq r^*$, where $d(f(0),\partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of $\Omega := f(\mathbb{D})$. The largest such r^* is called the Bohr radius.

Here, we find the sharp growth theorem, covering theorem and finally derive the Bohr radius for the class $\mathscr{B}^0_{\mathscr{H}}(M)$. As an application of our results, we obtain the Bohr radius for the classes $\mathscr{S}^{**}_{\tau}(C,D)$, $S^{*}_{\tau}(C,D)$, $S^{c}_{\tau}(C,D)$, $k-\widetilde{ST}_q^{-}(\alpha)$ and many more. Further, Bohr radii for the classes $W^0_{\mathscr{H}}(\mu,\rho)$ and $\mathscr{F}^0_{\mathscr{H}}(\lambda)$ are also derived.

5.3.1 The class $\mathscr{B}^0_{\mathscr{H}}(M)$ and its applications

Theorem 5.3.3 (Growth Theorem)**.** Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m} \in \mathscr{B}^0_{\mathscr{H}}(M)$. Then

$$r - \frac{M}{\alpha_2} r^2 \leq |f(z)| \leq r + \frac{M}{\alpha_2} r^2 \quad (|z| = r).$$

The inequalities are sharp for the functions $f(z) = z \pm \frac{M}{\alpha_2} z^2$ and $f(z) = z \pm \frac{M}{\alpha_2} \bar{z}^2$ with the suitable choice of α_2.

Proof. From the condition (5.3.1), we see that

$$\sum_{m=2}^{\infty} (\gamma_m |a_m| + \delta_m |b_m|) \leq M \tag{5.3.2}$$

implies

$$\sum_{m=2}^{\infty} (|a_m| + |b_m|) \leq \frac{M}{\alpha_2}, \tag{5.3.3}$$

where γ_m, δ_m and α_2 are as defined in (5.3.1) and the equality in (5.3.3) holds for the function $f(z) = z + (M/\alpha_2)z^2$. Now using the inequality $|a| - |b| \leq |a \pm b| \leq |a| + |b|$ and then (5.3.3), we have for $|z| = r$

$$|z| - \left| \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m} \right| \leq |f(z)| \leq |z| + \left| \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m} \right|,$$

which immediately yields the required inequality. □

Here, we observe that functions in the class $\mathscr{B}^0_{\mathscr{H}}(M)$ are bounded. Taking r tending to 1^- yields:

Corollary 5.3.4 (Covering Theorem)**.** Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m} \in \mathscr{B}^0_{\mathscr{H}}(M)$. Then

$$\left\{ w \in \mathbb{C} : |w| \leq 1 - \frac{M}{\alpha_2} \right\} \subset f(\mathbb{D}).$$

Now we are ready to obtain the Bohr-radius for the class $\mathscr{B}^0_{\mathscr{H}}(M)$.

Theorem 5.3.5. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m} \in \mathscr{B}^0_{\mathscr{H}}(M)$. Then

$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|)|z|^m \leq d(f(0), \partial f(\mathbb{D})) \quad \text{for} \quad |z| \leq r^*,$$

where

$$r^* = \frac{-1 + \sqrt{1 + 4\left(\frac{M}{\alpha_2}\right)\left(1 - \frac{M}{\alpha_2}\right)}}{2\left(\frac{M}{\alpha_2}\right)}.$$

Bohr radius r^* is achieved by the function $f(z) = z - \frac{M}{\alpha_2} z^2$.

Proof. From the Growth Theorem 5.3.3 and Corollary 5.3.4(Covering Theorem), we see that the distance between origin and the boundary of $f(\mathbb{D})$ satisfies

$$d(f(0),\partial f(\mathbb{D})) \geq 1-\frac{M}{\alpha_2}. \tag{5.3.4}$$

Let us consider the continuous function defined in $(0,1)$ as

$$H(r) := r+\frac{M}{\alpha_2}r^2-\left(1-\frac{M}{\alpha_2}\right),$$

such that $H'(r)>0$ for $r\in(0,1)$ with $H(0)<0$ and $H(1)>0$. Therefore, by the Intermediate Value Theorem for continuous functions, we let r^* be the unique positive root in $(0,1)$ as mentioned in the Theorem statement. Thus for $r=r^*$ we have

$$r^*+\frac{M}{\alpha_2}(r^*)^2 = 1-\frac{M}{\alpha_2}.$$

Now from (5.3.3), (5.3.4) and the above equality, it follows that

$$|z|+\sum_{m=2}^{\infty}(|a_m|+|b_m|)|z|^m \leq r+\frac{M}{\alpha_2}r^2 \leq r^*+\frac{M}{\alpha_2}(r^*)^2 \leq d(f(0),\partial f(\mathbb{D}))$$

for $|z|=r\leq r^*$. Let us consider the analytic function $f:\mathbb{D}\to\mathbb{C}$

$$f(z) = z-\frac{M}{\alpha_2}z^2,$$

which by suitably choosing α_2 and using (5.3.2) belongs to $\mathscr{B}^0_{\mathscr{H}}(M)$. Further, for $|z|=r^*$ we have

$$|z|+\sum_{m=2}^{\infty}(|a_m|+|b_m|)|z|^m = d(f(0),\partial f(\mathbb{D})).$$

Hence the result is sharp. □

Remark 5.3.1. Note that we can extend our results by considering the analytic functions of the following form:

$$h(z) = z+\sum_{m=k}^{\infty}a_m z^m \quad \text{and} \quad g(z)=\sum_{m=k}^{\infty}b_m z^m, \quad (k\geq 2)$$

and the class

$$\mathscr{B}^0_{\mathscr{H}}(k,M) := \{f=h+\bar{g} : \sum_{m=k}^{\infty}(\gamma_m|a_m|+\delta_m|b_m|)\leq M,\ M>0\},$$

where

$$\gamma_m,\ \delta_m \geq \alpha_k := \min\{\gamma_k,\delta_k\}>0,$$

for all $m\geq k$. Thus we see that $\mathscr{B}^0_{\mathscr{H}}(2,M)\equiv\mathscr{B}^0_{\mathscr{H}}(M)$. Precisely, for the class $\mathscr{S}^0_{h+\bar{g}}(k,M)$, we have

$$r-\frac{M}{\alpha_k}r^k \leq |f(z)| \leq r+\frac{M}{\alpha_k}r^k, \quad (|z|=r)$$

and the Bohr radius r^* is the unique positive root in $(0,1)$ of the equation

$$r+\frac{M}{\alpha_k}(r^k)-\left(1-\frac{M}{\alpha_k}\right)=0.$$

Now we can also obtain the Bohr radius for the classes (see [44, Theorem 8, 9]) studied by Dziok [44].

5.3.2 Applications to certain classes of univalent functions

We begin with the following corollary.

Corollary 5.3.6. Let $f(z)=z+\sum_{m=2}^{\infty}a_mz^m+\overline{\sum_{m=2}^{\infty}b_mz^m}\in\mathscr{S}_{\tau}^{**}(C,D)$. Then

$$r-\frac{D-C}{2(1+D)}r^2\le|f(z)|\le r+\frac{D-C}{2(1+D)}r^2 \quad (|z|=r)$$

and

$$|z|+\sum_{m=2}^{\infty}(|a_m|+|b_m|)|z|^m\le d(f(0),\partial f(\mathbb{D}))$$

for $|z|\le r^*$, where

$$r^*=\frac{-1+\sqrt{1+4\left(\frac{D-C}{\alpha_2}\right)\left(1-\frac{D-C}{\alpha_2}\right)}}{2\left(\frac{D-C}{\alpha_2}\right)},$$

where α_2 is as defined in (5.3.5). Bohr radius r^* is achieved by the function $f(z)=z-\frac{D-C}{2(1+D)}z^2$.

Proof. Cho and Dziok [37] showed that $f\in\mathscr{S}_{\tau}^{**}(C,D)$ if and only if

$$\sum_{m=2}^{\infty}(|\alpha_m||a_m|+|\beta_m||b_m|)\le D-C, \tag{5.3.5}$$

where

$$\alpha_m=m(1+D)-\frac{(1+C)(1-(-1)^m)}{2}$$

and

$$\beta_m=m(1+D)+\frac{(1+C)(1-(-1)^m)}{2}.$$

Note that for all $m\ge 2$, we have $\alpha_m<\beta_m$ which shows that $0<\alpha_2\le\alpha_m<\beta_m$. Therefore from (5.3.5) we obtain that

$$\sum_{m=2}^{\infty}(|a_m|+|b_m|)\le\frac{D-C}{\alpha_2}$$

and also the condition in (5.3.1) holds by choosing $\gamma_m=\alpha_m$, $\delta_m=\beta_m$ and $M=D-C$. Thus using Theorem 5.3.3 and Theorem 5.3.5, we get the result. □

Corollary 5.3.7. Let $f(z)=z+\sum_{m=2}^{\infty}a_mz^m+\overline{\sum_{m=2}^{\infty}b_mz^m}\in S_{\tau}^{*}(C,D)$. Then

$$r-\frac{D-C}{1+2D-C}r^2\le|f(z)|\le r+\frac{D-C}{1+2D-C}r^2, \quad (|z|=r)$$

and

$$|z|+\sum_{m=2}^{\infty}(|a_m|+|b_m|)|z|^m \leq d(f(0),\partial f(\mathbb{D}))$$

for $|z|\leq r^*$, where

$$r^* = \frac{-1+\sqrt{1+4\left(\frac{D-C}{1+2D-C}\right)\left(1-\frac{D-C}{1+2D-C}\right)}}{2\left(\frac{D-C}{1+2D-C}\right)}.$$

Bohr radius r^* is achieved by the function $f(z)=z-\frac{D-C}{1+2D-C}z^2$.

Proof. Dziok [43] proved that $f\in S^*_\tau(C,D)$ if and only if

$$\sum_{m=2}^{\infty}(\alpha_m|a_m|+\beta_m|b_m|)\leq D-C \tag{5.3.6}$$

and $f\in S^c_\tau(C,D)$ if and only if

$$\sum_{m=2}^{\infty}(m\alpha_m|a_m|+m\beta_m|b_m|)\leq D-C, \tag{5.3.7}$$

where $\alpha_m=m(1+D)-(1+C)$ and $\beta_m=m(1+D)+(1+C)$. We note that $\beta_m>\alpha_m\geq\alpha_2>0$ for all $m\geq 2$. Therefore from (5.3.6), we obtain that

$$\sum_{m=2}^{\infty}(|a_m|+|b_m|)\leq\frac{D-C}{\alpha_2}.$$

Now choosing $\gamma_m=\alpha_m$, $\delta_m=\beta_m$ and $M=D-C$, the condition (5.3.1) holds and thus using Theorem 5.3.3 and Theorem 5.3.5, we get the desired radius. □

Again using $\beta_m>\alpha_m\geq\alpha_2$ for all $m\geq 2$ in (5.3.7), we obtain that if $f\in S^c_\tau(C,D)$ then the following inequality holds:

$$\sum_{m=2}^{\infty}(|a_m|+|b_m|)\leq\frac{D-C}{2\alpha_2}.$$

Now choosing $\gamma_m=m\alpha_m$, $\delta_m=m\beta_m$ and $M=D-C$, the condition (5.3.1) holds and thus using Theorem 5.3.3 and Theorem 5.3.5, we obtain the following result:

Corollary 5.3.8. Let $f(z)=z+\sum_{m=2}^{\infty}a_mz^m+\overline{\sum_{m=2}^{\infty}b_mz^m}\in S^c_\tau(C,D)$. Then

$$r-\frac{D-C}{2(1+2D-C)}r^2\leq|f(z)|\leq r+\frac{D-C}{2(1+2D-C)}r^2, \quad (|z|=r)$$

and

$$|z|+\sum_{m=2}^{\infty}(|a_m|+|b_m|)|z|^m \leq d(f(0),\partial f(\mathbb{D}))$$

for $|z|\leq r^*$, where α_2 is defined in (5.3.7) and

$$r^* = \frac{-1+\sqrt{1+2\left(\frac{D-C}{\alpha_2}\right)\left(1-\frac{D-C}{\alpha_2}\right)}}{\left(\frac{D-C}{\alpha_2}\right)}.$$

Bohr radius r^* is achieved by the function $f(z) = z - \frac{D-C}{2(1+2D-C)}z^2$.

Now we obtain the result for the class $\mathscr{F}^0_{\mathscr{H}}(\lambda)$. Using the condition

$$\sum_{m=2}^{\infty} m(|a_m| + |b_m|) \leq \lambda, \tag{5.3.8}$$

convolution properties, the radius of starlikeness, and certain inclusion relationships were studied in [116] for the class $\mathscr{F}^0_{\mathscr{H}}(\lambda)$. Now with the same condition, we arrive at the following result using Theorem 5.3.3 and Theorem 5.3.5:

Corollary 5.3.9. If $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m} \in \mathscr{F}^0_{\mathscr{H}}(\lambda)$ and also satisfy the condition that $\sum_{m=2}^{\infty} m(|a_m| + |b_m|) \leq \lambda$. Then

$$r - \frac{\lambda}{2}r^2 \leq |f(z)| \leq r + \frac{\lambda}{2}r^2, \quad (|z| = r)$$

and

$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|)|z|^m \leq d(f(0), \partial f(\mathbb{D}))$$

for $|z| \leq r^*$, where

$$r^* = \frac{-1 + \sqrt{1 + 2\lambda\left(1 - \frac{\lambda}{2}\right)}}{\lambda}$$

and the Bohr radius r^* for $\mathscr{F}^0_{\mathscr{H}}(\lambda)$ is obtained when $f(z) = z - \frac{\lambda}{2}z^2$.

If $g \equiv 0$ then $\mathscr{F}^0_{\mathscr{H}}(\lambda)$ reduces to the Singh [157] class $\mathscr{F}(\lambda)$, which is contained in the MacGregor subclass $\mathscr{F} := \{f \in \mathscr{A} : |f'(z) - 1| < 1\}$ of close-to-convex functions.

Corollary 5.3.10. Bohr radius for the classes $\mathscr{F}(\lambda)$ and $\mathscr{F}^0_{\mathscr{H}}(\lambda)$ is same, whenever the condition (5.3.8) holds.

The following two corollaries are for the classes $k - \widetilde{ST}_q^-(\alpha)$ and $R^0_{\mathscr{H}}(\beta)$ respectively:

Lemma 5.3.1. [14] Let $0 \leq k < \infty$, $0 < q < 1$ and $0 \leq \alpha < 1$. Then $f \in k - \widetilde{ST}_q^-(\alpha)$ if and only if

$$\sum_{m=2}^{\infty} ([\widetilde{m}]_q(k+1) - (k+\alpha))a_m \leq 1 - \alpha.$$

From Lemma 5.3.1, we see that choosing $\gamma_m = [\widetilde{m}]_q(k+1) - (k+\alpha)$, $\delta_m = 0$ and $M = 1 - \alpha$, condition in (5.3.1) holds. Therefore, applying Theorem 5.3.3 and Theorem 5.3.5, we get the following result:

Corollary 5.3.11. Let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m \in k - \widetilde{ST}_q^-(\alpha)$. Then

$$r - \frac{q(1-\alpha)}{(q^2+1)(k+1) - q(k+\alpha)}r^2 \leq |f(z)| \leq r + \frac{q(1-\alpha)}{(q^2+1)(k+1) - q(k+\alpha)}r^2,$$

where $|z| = r$ and

$$|z| + \sum_{m=2}^{\infty} a_m |z|^m \leq d(f(0), \partial f(\mathbb{D}))$$

for $|z| \leq r^*$, where

$$r^* = \frac{-1 + \sqrt{1 + 4\left(\frac{q(1-\alpha)}{(q^2+1)(k+1)-q(k+\alpha)}\right)\left(1 - \frac{q(1-\alpha)}{(q^2+1)(k+1)-q(k+\alpha)}\right)}}{2\left(\frac{q(1-\alpha)}{(q^2+1)(k+1)-q(k+\alpha)}\right)}$$

and the Bohr radius r^* is obtained when $f(z) = z - \frac{q(1-\alpha)}{(q^2+1)(k+1)-q(k+\alpha)} z^2$.

Lemma 5.3.2. [42] Let $f \in R^0_{\mathscr{H}}(\beta)$. Then the following inequality

$$\sum_{m=2}^{\infty} (m(|a_m| + |b_m|)) \leq \beta - 1$$

is necessary and sufficient for the functions to be in $R^0_{\mathscr{H}}(\beta)$.

From Lemma 5.3.2, we see that choosing $\gamma_m = \delta_m = m$ and $M = \beta - 1$, condition in (5.3.1) holds. Therefore applying Theorem 5.3.3 and Theorem 5.3.5, we get the following result:

Corollary 5.3.12. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=2}^{\infty} b_m z^m} \in R^0_{\mathscr{H}}(\beta)$. Then

$$r - \frac{\beta - 1}{2} r^2 \leq |f(z)| \leq r + \frac{\beta - 1}{2} r^2, \quad (|z| = r)$$

and

$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|)|z|^m \leq d(f(0), \partial f(\mathbb{D}))$$

for $|z| \leq r^*$, where

$$r^* = \frac{-1 + \sqrt{1 + 2(\beta - 1)\left(1 - \frac{\beta-1}{2}\right)}}{\beta - 1}$$

and the Bohr radius r^* for the class $R^0_{\mathscr{H}}(\beta)$ is obtained when $f(z) = z + \frac{\beta-1}{2} z^2$.

Corollary 5.3.13. Bohr radius for the classes $R(\beta)$ and $R^0_{\mathscr{H}}(\beta)$ is same.

Silverman considered the class with negative coefficients as follows:

$$\mathscr{T} := \left\{ f \in \mathscr{S} : f(z) = z - \sum_{m=2}^{\infty} a_m z^m, a_m \geq 0 \right\}.$$

Using this recently, Ali et al. [10] considered the following general class defined as:

$$\mathscr{T}(\alpha) := \left\{ f \in \mathscr{T} : \sum_{m=2}^{\infty} g_m a_m \leq 1 - \alpha \right\},$$

where $g_m \geq g_2 > 0$ and $\alpha < 1$. Note that if we choose in (5.3.1), $\gamma_m = g_m$, $\delta_m = 0$ and $M = 1 - \alpha$, then the class $\mathscr{B}^0_{\mathscr{H}}(M)$ contains $\mathscr{T}(\alpha)$, which satisfies the required conditions. Thus using Theorem 5.3.3 and Theorem 5.3.5, we have the following result obtained in [10]:

Theorem 5.3.14. Let $f(z)=z-\sum_{m=2}^{\infty}a_m z^m \in \mathscr{T}(\alpha)$. Then

$$r-\frac{1-\alpha}{\gamma_2}r^2 \leq |f(z)| \leq r+\frac{1-\alpha}{\gamma_2}r^2, \quad (|z|=r)$$

and

$$|z|+\sum_{m=2}^{\infty}|a_m||z|^m \leq d(f(0),\partial f(\mathbb{D}))$$

for $|z|\leq r^*$, where

$$r^*=\frac{2(\gamma_2+\alpha-1)}{\gamma_2+\sqrt{\gamma_2^2+4\gamma_2(1-\alpha)-4(1-\alpha)^2}}$$

and the Bohr radius r^* is obtained by the function $f(z)=z-\frac{1-\alpha}{\gamma_2}z^2$.

Choosing $\gamma_m=m-\alpha$ and $\gamma_m=m(m-\alpha)$ in (5.3.1), the class $\mathscr{B}^0_{\mathscr{H}}(M)$ contains the Silverman classes $\mathscr{TST}(\alpha):=\mathscr{T}\cap\mathscr{ST}(\alpha)$ and $\mathscr{TCV}(\alpha):=\mathscr{T}\cap\mathscr{CV}(\alpha)$ of starlike and convex functions with negative coefficients respectively and Theorem 5.3.5 provides the Bohr radius as obtained in [10, Theorem 2.3] and [10, Theorem 2.4].

Ali et al. [10] considered the class $\mathscr{TF}_\alpha:=\mathscr{T}\cap\mathscr{F}_\alpha$, $0\leq\alpha\leq 1$, where $\mathscr{F}_\alpha$ is the class of close to convex functions and showed that if $f\in\mathscr{TF}_\alpha$, then $\sum_{m=2}^{\infty}m(2n+\alpha)a_m\leq 2+\alpha$. Thus choosing $\gamma_m=m(2n+\alpha)$ and $\delta_m=0$ and $M=2+\alpha$ in (5.3.1), the class $\mathscr{TF}_\alpha$ satisfies all conditions of $\mathscr{B}^0_{\mathscr{H}}(M)$, and Theorem 5.3.5 (or Theorem 5.3.14) reduces to [10, Corollary 2.6].

For $\alpha>1$, Owa and Nishiwaki [122] considered the classes of analytic functions $\mathscr{M}(\alpha):=\{f\in\mathscr{A}:\Re(zf'(z)/f(z))<\alpha\}$ and $\mathscr{N}(\alpha):=\{f\in\mathscr{A}:1+\Re(zf''(z)/f'(z))<\alpha\}$. They showed that the conditions:

$$\sum_{n=2}^{\infty}((m-\mu)+|m+\mu-2\alpha|)|a_m|\leq 2(\alpha-1)$$

and

$$\sum_{n=2}^{\infty}m(m-\mu+1+|m+\mu-2\alpha|)|a_m|\leq 2(\alpha-1),$$

where $0\leq\mu\leq 1$ are sufficient for the function $f(z)=z+\sum_{m=2}^{\infty}a_m z^m$ to be in $\mathscr{M}(\alpha)$ and $\mathscr{N}(\alpha)$, respectively. It is easy to see that the above two conditions also become necessary for the classes $\mathscr{TM}(\alpha):=\mathscr{T}\cap\mathscr{M}(\alpha)$ and $\mathscr{TN}(\alpha):=\mathscr{T}\cap\mathscr{N}(\alpha)$ respectively. Thus choosing $\gamma_m=(m-\mu)+|m+\mu-2\alpha|$, $\gamma_m=m(m-\mu+1+|m+\mu-2\alpha|)$ with $\delta_m=0$ and $M=2(\alpha-1)$, from Theorem 5.3.3 and Theorem 5.3.5 we obtain the following result respectively:

Corollary 5.3.15. Let $f(z)=z-\sum_{m=2}^{\infty}a_m z^m\in\mathscr{TM}(\alpha)$ (or $\mathscr{TN}(\alpha)$). Then

$$r-\frac{2(\alpha-1)}{\gamma_2}r^2\leq|f(z)|\leq r+\frac{2(\alpha-1)}{\gamma_2}r^2, \quad (|z|=r),$$

where $\gamma_2 := (2-\mu)+|\mu-2(1-\alpha)|$ (or $2(3-\mu+|\mu-2(1-\alpha)|)$) and

$$|z|+\sum_{m=2}^{\infty}|a_m||z|^m \leq d(f(0),\partial f(\mathbb{D}))$$

for $|z| \leq r^*$, where

$$r^* = \frac{-1+\sqrt{1+4\left(\frac{2(\alpha-1)}{\gamma_2}\right)\left(1-\frac{2(\alpha-1)}{\gamma_2}\right)}}{2\left(\frac{2(\alpha-1)}{\gamma_2}\right)}.$$

The radius r^*, achieved for the function $f(z) = z - \frac{2(\alpha-1)}{\gamma_2}z^2$ is the Bohr radius for the class $\mathscr{TM}(\alpha)$ (or $\mathscr{TN}(\alpha)$).

5.3.3 Bohr-radius for the class $W^0_{\mathscr{H}}(\mu,\rho)$

Rajbala and Prajapat [133] introduced and studied the following subclass of close-to-convex harmonic mappings:

$$W^0_{\mathscr{H}}(\mu,\rho) := \{f = h+\bar{g} \in \mathscr{H}_0 : \Re(h'(z)+\mu z h''(z)-\rho) > |g'(z)+\mu z g''(z)|\},$$

where $\mu \geq 0$ and $0 \leq \rho < 1$. They obtained the sharp estimates for the coefficients and for the growth theorems as follows.

Lemma 5.3.3. [133] let $f = h+\bar{g} \in W^0_{\mathscr{H}}(\mu,\rho)$. Then for $m \geq 2$ the following sharp inequality holds:

$$|a_m|+|b_m| \leq \frac{2(1-\rho)}{m(1+\mu(m-1))}.$$

Lemma 5.3.4. [133] let $f = h+\bar{g} \in W^0_{\mathscr{H}}(\mu,\rho)$. Then for $|z| = r$, we have the sharp inequality

$$|f(z)| \geq |z| - 2\sum_{m=2}^{\infty}\frac{(-1)^{m-1}(1-\rho)}{m(1+\mu(m-1))}|z|^m.$$

Now for this class, we establish the Bohr phenomenon.

Theorem 5.3.16. Let $f(z) = z+\sum_{m=2}^{\infty}a_m z^m + \overline{\sum_{m=2}^{\infty}b_m z^m} \in W^0_{\mathscr{H}}(\mu,\rho)$. Then

$$|z|+\sum_{m=2}^{\infty}(|a_m|+|b_m|)|z|^m \leq d(f(0),\partial f(\mathbb{D}))$$

for $|z| \leq r^*$, where r^* is the unique positive root in $(0,1)$ of

$$r+\sum_{m=2}^{\infty}\frac{2(1-\rho)}{m(1+\mu(m-1))}r^m = 1-2\sum_{m=2}^{\infty}\frac{(-1)^{m-1}(1-\rho)}{m(1+\mu(m-1))}.$$

The radius r^* is the Bohr radius for the class $W^0_{\mathscr{H}}(\mu,\rho)$.

Proof. From Lemma 5.3.4, it follows that the distance between the origin and the boundary of $f(\mathbb{D})$

satisfies

$$d(f(0),\partial f(\mathbb{D})) \geq \left(1-2\sum_{m=2}^{\infty}\frac{(-1)^{m-1}(1-\rho)}{m(1+\mu(m-1))}\right). \tag{5.3.9}$$

Let us consider the continuous function

$$H(r) := r+\sum_{m=2}^{\infty}\frac{2(1-\rho)}{m(1+\mu(m-1))}r^m - \left(1-2\sum_{m=2}^{\infty}\frac{(-1)^{m-1}(1-\rho)}{m(1+\mu(m-1))}\right).$$

Now

$$H'(r) = 1+\sum_{m=2}^{\infty}\frac{2m(1-\rho)}{m(1+\mu(m-1))}r^{m-1} > 0$$

for all $r\in(0,1)$, which implies that H is a strictly increasing continuous function. Note that $H(0)<0$ and

$$H(1) = \sum_{m=2}^{\infty}\frac{2(1-\rho)}{m(1+\mu(m-1))}+\sum_{m=2}^{\infty}\frac{2(-1)^{m-1}(1-\rho)}{m(1+\mu(m-1))} > 0.$$

Thus by the Intermediate Value Theorem for continuous function, we let r^* be the unique root of $H(r)=0$ in $(0,1)$. Now using Lemma 5.3.3 and the inequality (5.3.9), we have

$$\begin{aligned}|z|+\sum_{m=2}^{\infty}(|a_m|+|b_m|)|z|^m &\leq r+\sum_{m=2}^{\infty}\frac{2(1-\rho)}{m(1+\mu(m-1))}r^m \leq r^*+\sum_{m=2}^{\infty}\frac{2(1-\rho)}{m(1+\mu(m-1))}(r^*)^m\\ &= \left(1-2\sum_{m=2}^{\infty}\frac{(-1)^{m-1}(1-\rho)}{m(1+\mu(m-1))}\right) \leq d(f(0),\partial f(\mathbb{D})),\end{aligned}$$

which hold for $r\leq r^*$. Now consider the analytic function

$$f(z) = z+\sum_{m=2}^{\infty}\frac{2(-1)^{m-1}(1-\rho)}{m(1+\mu(m-1))}z^m.$$

Then clearly $f\in W^0_{\mathscr{H}}(\mu,\rho)$ and at $|z|=r^*$, we get $|z|+\sum_{m=2}^{\infty}(|a_m|+|b_m|)|z|^m = d(f(0),\partial f(\mathbb{D}))$. Hence the radius r^* is the Bohr radius for the class $W^0_{\mathscr{H}}(\mu,\rho)$. □

Now Theorem 5.3.16 yields the Bohr radius for the classes $W^0_{\mathscr{H}}(\mu,0) = W^0_{\mathscr{H}}(\mu)$, $W^0_{\mathscr{H}}(0,\rho) = P^0_{\mathscr{H}}(\rho)$, $W^0_{\mathscr{H}}(1,0) = W^0_{\mathscr{H}}$ and $W^0_{\mathscr{H}}(0,0) = P^0_{\mathscr{H}}$. Here we mention the following:

Corollary 5.3.17. Let $f(z) = z+\sum_{m=2}^{\infty}a_m z^m+\overline{\sum_{m=2}^{\infty}b_m z^m}\in P^0_{\mathscr{H}}$. Then

$$|z|+\sum_{m=2}^{\infty}(|a_m|+|b_m|)|z|^m \leq d(f(0),\partial f(\mathbb{D}))$$

for $|z|\leq r^*$, where the Bohr radius r^* is the unique positive root in $(0,1)$ of

$$r+\sum_{m=2}^{\infty}\frac{2}{m}r^m = 1-2\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}. \tag{5.3.10}$$

For the case $g\equiv 0$, the class $W^0_{\mathscr{H}}(\mu,\rho)$ reduces to $W(\mu,\rho)$, introduced by Gao and Zohu [53].

Corollary 5.3.18. The Bohr radius of the classes $W(\mu,\rho)$ and $W^0_{\mathscr{H}}(\mu,\rho)$ is same.

Remark 5.3.2. We studied the Bohr phenomenon for the class $\mathscr{B}^0_{\mathscr{H}}(M)$ and pointed out its several applications in the context of various known classes. Further, the Bohr radius for the classes of q-starlike and q-convex functions studied in [15] can be obtained by merely an application of our result. Similarly, many other classes can also be dealt with for the Bohr radius. Since $\mathscr{S}^{**}_{\tau}(C,D) \subset \mathscr{S}^{**}_{\mathscr{H}}(C,D)$, $S^{*}_{\tau}(C,D) \subset \mathscr{S}^{*}_{\mathscr{H}}(C,D)$ and $S^{c}_{\tau}(C,D) \subset \mathscr{S}^{c}_{\mathscr{H}}(C,D)$. If we let r_* be the Bohr radius for a well defined class $\mathscr{F}$, then we conclude that $r_* \leq r^*$ whenever $\mathscr{F} = \mathscr{S}^{**}_{\mathscr{H}}(C,D)$ or $\mathscr{S}^{*}_{\mathscr{H}}(C,D)$ or $\mathscr{S}^{c}_{\mathscr{H}}(C,D)$. However, finding r_* is still open.

5.4 A generalized Bohr-Rogosinski phenomenon

In this section, we generalize the Bohr-Rogosinski sum for the Ma-Minda classes of starlike and convex functions. Also the phenomenon is studied for the classes of starlike functions with respect to symmetric points and conjugate points along with their convex cases. Further, the connections between the derived results and the known ones are established with suitable examples. For reader's convenience, we again recall the following prominent results :

Theorem H (Bohr's Theorem, [34])**.** Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function in $\mathbb{D}$ and $|g(z)| < 1$ for all $z \in \mathbb{D}$, then

$$\sum_{k=0}^{\infty} |a_k||z|^k \leq 1, \quad \text{for} \quad |z| \leq \frac{1}{3}.$$

Theorem I (Rogosinski Theorem)**.** If $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be an analytic function in $\mathbb{D}$ with $|g(z)| < 1$, then for every $N \geq 1$, we have

$$\left| \sum_{k=0}^{N-1} b_k z^k \right| \leq 1, \quad \text{for} \quad |z| \leq \frac{1}{2}.$$

The radius $1/2$ is called the Rogosinski radius.

Recently, Kumar and Sahoo [90] obtained the generalized classical Bohr's Theorem for functions satisfying $\Re(f)(z) < 1$. Also see, Kayumov et al. [77].

Theorem J. [90, Theorem 2.2] Let $\{v_k(r)\}_{k=0}^{\infty}$ be a sequence of non-negative continuous functions in $[0,1)$ such that the series

$$v_0(r) + \sum_{k=1}^{\infty} v_k(r)$$

converges locally uniformly with respect to $r \in [0,1)$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\Re f(z) < 1$ and $p \in (0,1]$. If

$$v_0(r) > \frac{2(1+r^m)}{p(1-r^m)} \sum_{k=1}^{\infty} v_k(r).$$

Then the following sharp inequality holds:

$$|f(z^m)| v_0(r) + \sum_{k=1}^{\infty} |a_k| v_k(r) \leq v_0(r) \quad \text{for all} \quad |z| = r \leq R_1,$$

where R_1 is the minimal positive root of the equation:

$$\nu_0(r) = \frac{2(1+r^m)}{p(1-r^m)} \sum_{k=1}^{\infty} \nu_k(r).$$

In case when

$$\nu_0(r) < \frac{2(1+r^m)}{p(1-r^m)} \sum_{k=1}^{\infty} \nu_k(r)$$

in some interval $(R_1, R_1+\varepsilon)$, then the number R_1 can not be improved.

If we choose $\nu_k(r) = r^k$ in Theorem J, we get Bohr's Theorem H. At this juncture, it is natural to pose the following problem:

Problem 5.4.1. Can we establish the analogue of Theorem J for the Ma-Minda classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$?

In the context of the above problem and Muhanna [111], we now describe the notion of generalized Bohr-Rogosinski phenomenon here below, in terms of subordination, following the recent development as seen in [77, 90, 100].

Definition 5.4.11. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ are analytic in $\mathbb{D}$. Let $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of $\Omega = f(\mathbb{D})$. For a fixed f, consider a class of analytic functions

$$S(f) := \{g : g \prec f\}$$

or equivalently,

$$S(\Omega) := \{g : g(z) \in \Omega\}.$$

Then we say $S(f)$ satisfies the *Generalized Bohr-Rogosinski phenomenon*, if there exists a constant $r_0 \in (0,1]$ such that

$$\mathcal{P}(r,g,f) + \sum_{k=1}^{\infty} |b_k| \phi_k(r) \leq d(f(0), \partial\Omega),$$

holds for all $|z| = r \leq r_0$, where

(*i*) $\mathcal{P}(r,g,f)$ represent some function of r or certain proper combination of moduli of g, f and their derivatives.

(*ii*) $\{\phi_k(r)\}$ be a sequence of non-negative continuous functions in $[0,1)$ such that the series of the form

$$p_0\phi_0(r) + \sum_{k=1}^{\infty} p_k \phi_k(r)$$

converges locally uniformly with respect to $r \in [0,1)$, where p_k depends on the function f and provide bounds for b_k.

For $\mathcal{P}(r,g,f) = |g(z)|$, $\phi_k(r) = r^k$ $(k \geq N)$ and 0 otherwise in the Definition 5.4.11 gives the quantity considered by Kayumov et al. [76], which is known as the Bohr-Rogosinski sum, given by

$$|g(z)| + \sum_{k=N}^{\infty} |b_k||z|^k, \quad |z| = r.$$

The link between the Bohr-Rogosinski and Bohr phenomenon can be noticed, if we replace $|g(z)|$ by $g(0)$ with $N=1$. We also refer the readers to see [3, 13]. Now we see that the family $S(f)$ has Bohr-Rogosinski phenomenon provided there exists $r_N^f \in (0,1]$ such that the inequality:

$$|g(z)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq |f(0)| + d(f(0), \partial\Omega)$$

holds for $|z| = r \leq r_N^f$. The largest such r_N^f is called the Bohr-Rogosinski radius.

In case when the function f is normalized, then Kumar and Gangania [52] studied Bohr-Rogosinski phenomenon for the class $S_f(\psi)$, which is given below.

Theorem 5.4.12. [52, Theorem 2.3, Page no. 7] Let $f_0(z)$ be given by the equation (4.5.2) and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\psi)$. Assume $f_0(z) = z + \sum_{n=2}^{\infty} t_n z^n$ and $\hat{f}_0(r) = r + \sum_{n=2}^{\infty} |t_n| r^n$. If $g \in S_f(\psi)$. Then

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \partial\Omega)$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the unique positive root of the equation:

$$\hat{f}_0(r^m) + \hat{f}_0(r) - p_{\hat{f}_0}(r) = -f_0(-1),$$

where

$$p_{\hat{f}_0}(r) = \begin{cases} 0, & N=1; \\ r, & N=2; \\ r + \sum_{n=2}^{N-1} |t_n| r^n, & N \geq 3. \end{cases}$$

The result is sharp when $r_b = r_0$ and $t_n > 0$.

For the class $\mathcal{C}(\psi)$, see [52, Theorem 2.13, page no. 12]. In the context of the above, also see [3, 13, 31, 76]. Recall the Bohr Operator

$$M_r(f) = \sum_{n=0}^{\infty} |a_n||z^n| = \sum_{n=0}^{\infty} |a_n| r^n$$

for the analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We considered its extension defined in (5.1.1), which is given by

$$M_r^N(f) = \sum_{n=N}^{\infty} |a_n||z^n| = \sum_{n=N}^{\infty} |a_n| r^n.$$

For the properties of $M_r^N(f)$ in terms of the sequence $\{\nu_n(r)\}_{n=0}^{\infty}$, see [129, Lemma 1]. Using this operator, a simple proof of [31, Lemma 1] was achieved by Gangania and Kumar [52] to settle the Bohr-Rogosinski Phenomenon for the classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$, which in terms of interim k-th section $f_k(z) := \sum_{n=N}^{k} a_n z^n$ is as follows:

Lemma 5.4.1. [52] let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be analytic in $\mathbb{D}$ and $g \prec f$, then

$$M_r^N(g_k) \leq M_r^N(f_k) \tag{5.4.1}$$

for all $|z| = r \leq 1/3$ and $k, N \in \mathbb{N}$.

When $N = 1$ and $k \to \infty$, the lemma was obtained by Bhowmik and Das [31]. Various interesting applications of this lemma can be seen in [31, 51, 52, 64, 85].

While we establish the Bohr-type inequalities for the general class $\mathcal{S}^*(\psi)$ or $\mathcal{C}(\psi)$, the main difficulty that we come across is the unavailability of the sharp coefficients bounds. Here, we require the use of the Lemma 5.2.1 or its proper modifications. Interestingly, the Lemma 5.2.1 (or Lemma 5.4.1) also implies that if $f \prec g$, then within the disk $|z| \leq 1/3$, we have $|a_n| \leq |b_n|$ for all $n \in \mathbb{N} \cup \{0\}$, where b_n are the coefficients of the function g in Lemma 5.2.1. Further, this readily gives the following:

Lemma 5.4.2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in $\mathbb{D}$. Let $\{\nu_k(r)\}_{k=0}^{\infty}$ be a sequence of non-negative functions, continuous in $[0,1)$ such that the series

$$\sum_{n=0}^{\infty} |b_n| \nu_k(r)$$

converges locally uniformly with respect to $r \in [0,1)$. If $g \prec f$, then

$$\sum_{n=0}^{\infty} |a_n| \nu_n(r) \leq \sum_{n=0}^{\infty} |b_n| \nu_n(r) \quad \text{for all} \quad |z| = r \leq \frac{1}{3}.$$

For a different version of the Lemma 5.4.2 with the conditions $\nu_{m+n}(r) \leq \nu_m(r)\nu_n(r)$ and $\nu_0(r) = 1$, see [129, Theorem 3].

To discuss further the Bohr phenomenon, let us recall some well-known classes. A function $f \in \mathcal{A}$ is in the class $\mathcal{K}$ of close-to-convex if $\Re(zf'(z)/g(z)) > 0$, where $g \in \mathcal{S}^*$. In 2004, Ravichandran studied the amalgamated treatment for the classes of starlike and convex functions with respect to the symmetric points for the growth and distortion theorems:

Definition 5.4.13. [136] A function $f \in \mathcal{A}$ is in the class $\mathcal{S}_s^*(\psi)$ and $\mathcal{C}_s(\psi)$ if it satisfy $zf'(z)/h(z) \prec \psi(z)$ and $(zf'(z))'/h'(z) \prec \psi(z)$ respectively, where $2h(z) = f(z) - f(-z)$.

Now if we take $2h(z) = f(z) + \overline{f(\bar{z})}$ in Definition 5.4.13, we obtain the classes $\mathcal{S}_c^*(\psi)$ and $\mathcal{C}_c(\psi)$ of starlike and convex functions with respect to conjugate points, respectively. For the choice $2h(z) = f(z) - \overline{f(-\bar{z})}$, we have the classes $\mathcal{S}_{sc}^*(\psi)$ and $\mathcal{C}_{sc}(\psi)$ of starlike and convex functions with respect to the symmetric conjugate points, respectively. See [136].

Motivated, by the class $\mathcal{S}_s^*((1+z)/(1-z))$ [147], Gao and Zhou [54] studied the class $\mathcal{K}_s$ of close-to-convex functions f, which is characterized as:

$$\Re\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) > 0,$$

where g is some starlike function of order $1/2$. In view of the Definition 5.4.13, the generalized class $\mathcal{K}_s(\psi)$ was studied by Cho et al. [41] and Wang et al. [169].

In this section, we positively answer the Problem 5.4.1 for the classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$. Further, we study the Bohr-Rogosinski phenomenon for the classes $\mathcal{K}_s(\psi)$, $\mathcal{S}_c^*(\psi)$, $\mathcal{C}_c(\psi)$ and $\mathcal{C}_s(\psi)$. For convenience, we denote $\hat{f}(z) = \sum_{n=0}^{\infty} |a_n| z^n$, whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

5.4.1 Generalized Bohr's sum for Ma-Minda starlike functions

We here solve the Problem 5.4.1. But as we do not have sharp coefficient's bound for each a_n for the given class in general. Thus for certain valid assumptions to solve it, we need the following:

Lemma 5.4.3. The families $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$ are normal and compact.

Proof. From Montel's Theorem [58], we see that the class $\mathcal{S}^*(\psi)$ is a normal family. Now let us prove that $\mathcal{S}^*(\psi)$ is compact. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions from $\mathcal{S}^*(\psi)$. Suppose that $\{f_n\}$ be convergent. Then it is well-known that

$$\lim_{n\to\infty} f_n := f \in \mathcal{S}.$$

We show that $f \in \mathcal{S}^*(\psi)$. If possible, suppose that there exists a nonzero point $z_0 \in \mathbb{D}$ such that $g(z_0) \notin \psi(\mathbb{D})$, where $g(z) = zf'(z)/f(z)$. Note that the corresponding sequence $\{g_n\}$ converges to g, where $g_n(z) = zf_n'(z)/f_n(z)$. Now let

$$\varepsilon = dist(g(z_0), \partial\psi(\mathbb{D})).$$

Then the open ball $B(g(z_0), \varepsilon) \not\subset \psi(\mathbb{D})$. Since $g_n \to g$, in particular, $g_n(z_0) \to g(z_0)$. There exists $n(\varepsilon) \in \mathbb{N}$ such that

$$g_n(z_0) \in B(g(z_0), \varepsilon), \quad \forall n \geq n(\varepsilon),$$

which implies $g_n(z_0) \notin \psi(\mathbb{D}), \quad \forall n \geq n(\varepsilon)$. But as $f_n \in \mathcal{S}^*(\psi)$,

$$g_n(z_0) = \frac{z_0 f_n'(z_0)}{f_n(z_0)} \in \psi(\mathbb{D}), \quad \forall n.$$

Hence, we must have $f \in \mathcal{S}^*(\psi)$, that means the family $\mathcal{S}^*(\psi)$ is compact. With similar arguments, it is easy to see that the family $\mathcal{C}(\psi)$ is also compact. $\square$

Remark 5.4.1. (*Existence of sharp coefficients Bounds)* Let us consider the real-valued functional $\mathcal{J}$ defined on $\mathcal{S}^*(\psi)$ as

$$\mathcal{J}(f) = \max\{|a_n|\} \quad \textit{for every} \quad f \in \mathcal{S}^*(\psi),$$

where n is fixed and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. From Lemma 5.4.3, $\mathcal{S}^*(\psi)$ is normal and compact. Further, since $\mathcal{S}^*(\psi) \subseteq \mathcal{S}^*$, we have $|a_n| \leq n$, that means $\mathcal{J}$ is a bounded functional. Hence, following the discussion in the Goodman's Book [58, page no. 44-45], we conclude that

$$\mathcal{J}(f) = \max\{|a_n|\} \quad \textit{exists in the family} \quad \mathcal{S}^*(\psi).$$

Thus, let us say that

$$\max_{f\in\mathcal{S}^*(\psi)} \{|a_n|\} := M(n)$$

for each $n \in \mathbb{N}$. For instance, $M(n) = n$ for the class of univalent starlike functions. For the Janowski starlike functions, it is given by [16, Theorem 3].

We can now state our result in a general setting whose complement is Theorem 5.4.15:

Theorem 5.4.14. Let $\{\phi_n(r)\}_{n=1}^{\infty}$ be a sequence of non-negative continuous functions in $(0,1)$ such that the series

$$\phi_1(r) + \sum_{n=2}^{\infty} M(n)\phi_n(r)$$

converges locally uniformly with respect to each $r \in [0,1)$. If for $\beta \in [0,1]$

$$\beta f_0'(r^m) + (1-\beta) f_0(r^m) + \sum_{n=1}^{\infty} M(n)\phi_n(r) < -f_0(-1). \tag{5.4.2}$$

and the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\psi)$. Then the following inequality

$$\beta |f'(z^m)| + (1-\beta)|f(z^m)| + \sum_{n=1}^{\infty} |a_n|\phi_n(r) \leq d(0, \partial\Omega)$$

holds for $|z| = r \leq r_0$, where $m \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the smallest positive root of the equation:

$$\beta f_0'(r^m) + (1-\beta) f_0(r^m) + \sum_{n=1}^{\infty} M(n)\phi_n(r) = -f_0(-1), \tag{5.4.3}$$

where $M(1) = 1$ with $M(n)$ as described in remark 5.4.1 and

$$f_0(z) = z \exp \int_0^z \frac{\psi(t) - 1}{t} dt.$$

In case when f_0 or it's rotation serves as an extremal for the coefficient's bounds $M(n)$, then the radius r_0 is sharp.

Proof. From Lemma 5.4.3 and Remark 5.4.1, we see that: (a) $\mathcal{J}$ is a bounded real valued continuous functional, (b) $\mathcal{S}^*(\psi)$ is a normal family, and (c) $\mathcal{S}^*(\psi)$ is a compact family in $\mathbb{D}$. Thus, the sharp bounds for each a_n exists. In view of Remark 5.4.1, we have

$$|a_n| \leq M(n),$$

which yields that

$$\sum_{n=1}^{\infty} |a_n|\phi_n(r) \leq \sum_{n=1}^{\infty} M(n)\phi_n(r). \tag{5.4.4}$$

The Koebe-radius for the functions in $\mathcal{S}^*(\psi)$ satisfies

$$d(0, \partial f(\mathbb{D})) \geq -f_0(-1). \tag{5.4.5}$$

Now combining it with the growth and distortion theorems [102], and using the condition 5.4.2, the inequalities (5.4.4) and (5.4.5) gives

$$\begin{aligned}\beta|f'(z^m)|+(1-\beta)|f(z^m)|+\sum_{n=1}^{\infty}|a_n|\phi_n(r)\\ &\leq \beta f_0'(r^m)+(1-\beta)f_0(r^m)+\sum_{n=1}^{\infty}M(n)\phi_n(r)\\ &\leq d(0,\partial\Omega),\end{aligned}$$

which holds in $|z|=r\leq r_0$, where r_0 is the minimal positive root of the equation (5.4.3). The existence of the root r_0 follows from the Intermediate value theorem for continuous function in $(0,1)$. To see the sharpness case, let us consider the function

$$f_0(z)=z\exp\int_0^z \frac{\psi(t)-1}{t}dt.$$

such that it's Taylor series coefficients $a_n(f_0)$ satisfies $|a_n(f_0)|=M(n)$. For this function we have

$$d(0,\partial f(\mathbb{D}))=-f_0(-1),$$

and the following equality holds for $|z|=r_0$:

$$\beta f_0'(r^m)+(1-\beta)f_0(r^m)+\sum_{n=1}^{\infty}M(n)\phi_n(r)=d(0,\partial f(\mathbb{D})),$$

and therefore, if f_0 is extremal for each coefficient's bound, then the radius r_0 can not be improved. □

Question 5.4.1. What if we do not have $M(n)$?

For such cases, the following result complements Theorem 5.4.14:

Theorem 5.4.15. Let $\{\phi_n(r)\}_{n=1}^{\infty}$ be a non-negative sequence of continuous functions in $[0,1]$ such that the series

$$\phi_1(r)+\sum_{n=2}^{\infty}\left|\frac{f_0^{(n)}(0)}{n!}\right|\phi_n(r)$$

converges locally uniformly with respect to each $r\in[0,1)$. If

$$\beta|f'(z^m)|+(1-\beta)|f(z^m)|+\phi_1(r)+\sum_{n=2}^{\infty}\left|\frac{f_0^{(n)}(0)}{n!}\right|\phi_n(r)<-f_0(-1),\quad \beta\in[0,1]$$

and $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}^*(\psi)$. Then

$$\beta|f'(z^m)|+(1-\beta)|f(z^m)|+\sum_{n=1}^{\infty}|a_n|\phi_n(r)\leq d(0,\partial\Omega) \tag{5.4.6}$$

holds for $|z|=r\leq r_b=\min\{1/3,r_0\}$, where $m\in\mathbb{N}$, $\Omega=f(\mathbb{D})$ and r_0 is the smallest positive root of the equation:

$$\beta f_0'(r^m)+(1-\beta)f_0(r^m)+\sum_{n=2}^{\infty}\left|\frac{f_0^{(n)}(0)}{n!}\right|\phi_n(r)=-f_0(-1)-\phi_1(r),$$

where

$$f_0(z)=z\exp\int_0^z\frac{\psi(t)-1}{t}dt.$$

Moreover, the inequality (5.4.6) also holds for the class $S_f(\psi)$ in $|z|=r\leq r_b$. When $r_b=r_0$, then the radius is best possible.

Proof. Since $f\in\mathscr{S}^*(\psi)$, it is known that $f(z)/z\prec f_0(z)/z$. Applying Lemma 5.4.2, we see that

$$\sum_{n=N}^{m}|a_n||z|^n\leq\sum_{k=N}^{m}\left|\frac{f_0^k(0)}{k!}\right||z|^k\quad\text{for}\quad|z|=r\leq\frac{1}{3}.$$

Now choosing $N=m$, we conclude that

$$|a_n|\leq\left|\frac{f_0^n(0)}{n!}\right|$$

holds for each n within the disk $|z|=r\leq 1/3$. Hence, it suffices to see

$$\begin{aligned}
&\beta|f'(z^m)|+(1-\beta)|f(z^m)|+\sum_{n=1}^{\infty}|a_n|\phi_n(r)\\
\leq\;&\beta|f_0'(r^m)|+(1-\beta)|f_0(r^m)|+\phi_1(r)+\sum_{n=2}^{\infty}\left|\frac{f_0^{(n)}(0)}{n!}\right|\phi_n(r)\\
\leq\;&-f_0(-1)\\
\leq\;&d(0,\partial\Omega),
\end{aligned}$$

holds in $|z|=r\leq\min\{r_0,1/3\}$. If $r_0\leq 1/3$, then the equality case can be seen for the function $f=f_0$, whenever Taylor coefficients of ψ are positive. □

The result for the class $\mathscr{C}(\psi)$ also follows on similar lines, so we omit the details of the proof.

Theorem 5.4.16. Let $\{\phi_n(r)\}_{n=1}^{\infty}$ be a non-negative sequence of continuous functions in $[0,1]$ such that the series

$$\phi_1(r)+\sum_{n=2}^{\infty}\left|\frac{f_0^{(n)}(0)}{n!}\right|\phi_n(r)$$

converges locally uniformly with respect to each $r\in[0,1)$. If

$$\beta|f'(z^m)|+(1-\beta)|f(z^m)|+\phi_1(r)+\sum_{n=2}^{\infty}\left|\frac{f_0^{(n)}(0)}{n!}\right|\phi_n(r)<-f_0(-1),,\quad\beta\in[0,1]$$

and $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{C}(\psi)$. Then

$$\beta|f'(z^m)|+(1-\beta)|f(z^m)|+\sum_{n=1}^{\infty}|a_n|\phi_n(r)\le d(0,\partial\Omega) \tag{5.4.7}$$

holds for $|z|=r\le r_b=\min\{1/3,r_0\}$, where $m\in\mathbb{N}$, $\Omega=f(\mathbb{D})$ and r_0 is the smallest positive root of the equation:

$$\beta f_0'(r^m)+(1-\beta)f_0(r^m)+\sum_{n=2}^{\infty}\left|\frac{f_0^{(n)}(0)}{n!}\right|\phi_n(r)=-f_0(-1)-\phi_1(r),$$

where f_0 satisfy

$$1+\frac{zf_0''(z)}{f_0'(z)}=\psi(z).$$

Moreover, the inequality (5.4.7) also holds for the class $S_f(\psi)$ in $|z|=r\le r_b$. When $r_b=r_0$, then the radius is best possible.

Remark 5.4.2. 1. By taking $\phi_n(r)=r^n$ in Theorem 5.4.15 give [86, Theorem 5.1], and [64, Theorem 3.1] for the choice $g=f$ with Taylor coefficients of ψ being positive.

2. Taking $\phi_n=r^n$ for $n\ge N$ and 0 elsewhere in Theorem 5.4.15 yields [52, Theorem 5, Corollary 3].

Let us discuss the generalized Bohr-Rogosinski phenomenon for the celebrated Janowski class of univalent starlike functions. For simplicity, write $\mathscr{S}^*((1+Dz)/(1+Ez))\equiv\mathscr{S}[D,E]$, where $-1\le E<D\le 1$.

Corollary 5.4.17. Let $\{\phi_n(r)\}_{n=1}^{\infty}$ be a sequence of non-negative continuous functions in $(0,1)$ such that the series

$$\phi_1(r)+\sum_{n=2}^{\infty}\prod_{k=0}^{n-2}\frac{|E-D+Ek|}{k+1}\phi_n(r)$$

converges locally uniformly with respect to each $r\in[0,1)$. If for $\beta\in[0,1]$

$$\beta f_0'(r^m)+(1-\beta)f_0(r^m)+\sum_{n=1}^{\infty}|a_n(f_0)|\phi_n(r)<-f_0(-1).$$

and the function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{S}[D,E]$. Then the following sharp inequality

$$\beta|f'(z^m)|+(1-\beta)|f(z^m)|+\sum_{n=1}^{\infty}|a_n|\phi_n(r)\le d(0,\partial\Omega) \tag{5.4.8}$$

holds for $|z|=r\le r_0$, where $m\in\mathbb{N}$, $\Omega=f(\mathbb{D})$ and r_0 is the minimal positive root of the equations:
If $E\ne 0$

$$\beta(1+Dr^m)(1+Er^m)^{\frac{D-2E}{E}}+(1-\beta)r^m(1+Er^m)^{\frac{D-E}{E}}$$
$$+\sum_{n=2}^{\infty}\prod_{k=0}^{n-2}\frac{|E-D+Ek|}{k+1}\phi_n(r)=(1-E)^{\frac{D-E}{E}}-\phi_1(r),$$

and if $E = 0$

$$e^{Dr^m}(\beta + (1-\beta(1-D))r^m) + \sum_{n=2}^{\infty}\prod_{k=0}^{n-2}\frac{D}{k+1}\phi_n(r) = e^{-D} - \phi_1(r),$$

where

$$f_0(z) = \begin{cases} z(1+Ez)^{\frac{D-E}{E}}, & E \neq 0; \\ ze^{Dz}, & E = 0. \end{cases} \tag{5.4.9}$$

The radius r_0 can not be improved.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}[D,E]$. Then for $n \geq 2$, [16, Theorem 3] states that:

$$|a_n| \leq \prod_{k=0}^{n-2}\frac{|E-D+Ek|}{k+1} = M(n),$$

where the function f_0 given by (5.4.9) gives equality. The result now follows by Theorem 5.4.14. □

Remark 5.4.3. Taking $m \to \infty$ and $\beta = 0$ in Corollary 5.4.17 yields: If

$$\sum_{n=1}^{\infty}|a_n(f_0)|\phi_n(r) < -f_0(-1).$$

Then the sharp inequality $\sum_{n=1}^{\infty}|a_n|\phi_n(r) \leq d(0,\partial\Omega)$ holds for $|z| = r \leq r_0$, where $m \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the minimal positive root of the equations:
If $E \neq 0$

$$\sum_{n=2}^{\infty}\prod_{k=0}^{n-2}\frac{|E-D+Ek|}{k+1}\phi_n(r) = (1-E)^{\frac{D-E}{E}} - \phi_1(r),$$

and if $E = 0$

$$\sum_{n=2}^{\infty}\prod_{k=0}^{n-2}\frac{D}{k+1}\phi_n(r) = e^{-D} - \phi_1(r),$$

where f_0 is given in (5.4.9).

In Corollary 5.4.17, putting $D = 1-2\alpha$ and $E = -1$, where $0 \leq \alpha < 1$, we get the result for the class of univalent starlike functions of order α, that is, $\mathcal{S}^*(\alpha)$:

Corollary 5.4.18. Let $\{\phi_n(r)\}_{n=1}^{\infty}$ be a sequence of non-negative continuous functions in $(0,1)$ such that the series

$$\phi_1(r) + \sum_{n=2}^{\infty}\prod_{k=0}^{n-2}\frac{k+2(1-\alpha)}{k+1}\phi_n(r)$$

converges locally uniformly with respect to each $r \in [0,1)$. If for $\beta \in [0,1]$

$$\frac{\beta(1+(1-2\alpha)r^m)}{(1-r^m)^{2(1-\alpha)+1}} + \frac{(1-\beta)r^m}{(1-r^m)^{2(1-\alpha)}} + \sum_{n=2}^{\infty}\prod_{k=0}^{n-2}\frac{k+2(1-\alpha)}{k+1}\phi_n(r) < \frac{1}{4^{1-\alpha}} - \phi_1(r).$$

and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*(\alpha)$. Then the sharp inequality (5.4.8) holds for $|z| = r \le r_0$, where $m \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the minimal positive root of the equations:

$$\frac{\beta(1+(1-2\alpha)r^m)}{(1-r^m)^{2(1-\alpha)+1}} + \frac{(1-\beta)r^m}{(1-r^m)^{2(1-\alpha)}} + \sum_{n=2}^{\infty}\prod_{k=0}^{n-2}\frac{k+2(1-\alpha)}{k+1}\phi_n(r) = \frac{1}{4^{1-\alpha}} - \phi_1(r).$$

The radius r_0 is sharp.

Putting $\alpha = 0$ in Corollary 5.4.18, we get the following:

Corollary 5.4.19. Let the sequence $\{\phi_n(r)\}_{n=1}^{\infty}$ satisfy the hypothesis of Corollary 5.4.18 with $\alpha = 0$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}^*$. Then the inequality (5.4.8) holds for $|z| = r \le r_0$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the smallest positive root of the equations:

$$\frac{\beta(1+r^m)}{(1-r^m)^3} + \frac{(1-\beta)r^m}{(1-r^m)^2} + \sum_{n=1}^{\infty} n\phi_n(r) = \frac{1}{4}.$$

The radius r_0 is sharp.

The following series of examples explore the choices of sequence $\phi_n(r)$:

Example 5.4.20. Let us consider $\phi_n(r) = 0$ for $1 \le n \le N$, and $\phi_n(r) = r^n$ for $n \ge N$ in Corollary 5.4.17. Then the following sharp inequality

$$\beta|f'(z^m)| + (1-\beta)|f(z^m)| + \sum_{n=N}^{\infty} |a_n| r^n \le d(0, \partial\Omega)$$

holds for $|z| = r \le R_{m,\beta,N}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and $R_{m,\beta,N}$ is the unique positive root of the equations:
If $E \neq 0$

$$\beta(1+Dr^m)(1+Er^m)^{\frac{D-2E}{E}} + (1-\beta)r^m(1+Er^m)^{\frac{D-E}{E}}$$
$$+ \sum_{n=N}^{\infty}\prod_{k=0}^{N-2}\frac{|E-D+Ek|}{k+1} r^n = (1-E)^{\frac{D-E}{E}},$$

and if $E = 0$

$$e^{Dr^m}(\beta + (1-\beta(1-D))r^m) + \sum_{n=2}^{\infty}\prod_{k=0}^{n-2}\frac{D}{k+1}\phi_n(r) = e^{-D}.$$

Example 5.4.21. Taking $\phi_{2n-1}(r) = 0$ and $\phi_{2n}(r) = r^{2n}$ in Corollary 5.4.18 yields

$$\beta|f'(z^m)| + (1-\beta)|f(z^m)| + \sum_{n=1}^{\infty} |a_{2n}| r^{2n} \le d(0, \partial\Omega)$$

which holds for $|z| = r \le R_{m,\beta,\alpha}$, where $m \in \mathbb{N}$, $\beta \in [0,1]$, $\Omega = f(\mathbb{D})$ and $R_{m,\beta,\alpha}$ is the unique positive root of the equations:

$$\frac{\beta(1+(1-2\alpha)r^m)}{(1-r^m)^{2(1-\alpha)+1}}+\frac{(1-\beta)r^m}{(1-r^m)^{2(1-\alpha)}}+\sum_{n=1}^{\infty}\prod_{k=0}^{2(n-1)}\frac{k+2(1-\alpha)}{k+1}r^{2n}=\frac{1}{4^{1-\alpha}}.$$

The radius is sharp.

Example 5.4.22. Letting $\phi_{2n}(r)=0$ and $\phi_{2n-1}(r)=r^{2n-1}$ in Corollary 5.4.19 gives the following sharp inequality

$$\beta|f'(z^m)|+(1-\beta)|f(z^m)|+\sum_{n=1}^{\infty}|a_{2n-1}|r^{2n-1}\leq d(0,\partial\Omega)$$

in $|z|=r\leq R_{m,\beta}$ where $m\in\mathbb{N}$, $\beta\in[0,1]$, $\Omega=f(\mathbb{D})$ and $R_{m,\beta}$ is the unique positive root of the equations:

$$\frac{\beta(1+r^m)}{(1-r^m)^{2+1}}+\frac{(1-\beta)r^m}{(1-r^m)^2}+\frac{r(1+r^2)}{(1-r^2)^2}=\frac{1}{4}.$$

The radius is sharp.

5.4.2 Bohr-Rogosinski sum for starlike and convex functions with respect to conjugate and symmetric points

To discuss the Bohr-Rogosinski phenomenon, we first need to recall some related basic definitions, where

1. $\psi(z)\in\mathscr{P}$ is an analytic univalent function in $|z|<1$.
2. $\Re\psi(z)>0$, $\psi'(0)>0$, $\psi(0)=1$ and $\psi(\mathbb{D})$ is symmetric about real axis.

Definition 5.4.23. [41] Let us consider the subclass of close-to-convex functions given by

$$\mathscr{K}_s(\psi)=\left\{f\in\mathscr{A}:-\frac{z^2f'(z)}{h(z)h(-z)}\prec\psi(z)\right\}\quad\text{for some}\quad h\in\mathscr{S}^*(1/2).$$

Definition 5.4.24. [136] The class of starlike functions with respect to conjugate points is given by

$$\mathscr{S}_c^*(\psi)=\left\{f\in\mathscr{A}:\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\prec\psi(z)\right\}.$$

Definition 5.4.25. [136] The class of convex functions with respect to conjugate points given by

$$\mathscr{C}_c(\psi)=\left\{f\in\mathscr{A}:\frac{2(zf'(z))'}{(f(z)+\overline{f(\bar{z})})'}\prec\psi(z)\right\}.$$

Definition 5.4.26. [136] The class of convex function with respect to symmetric points is given by

$$\mathscr{C}_s(\psi)=\left\{f\in\mathscr{A}:\frac{2(zf'(z))'}{f'(z)+f'(-z)}\prec\psi(z)\right\}.$$

We also need to recall the following fundamental results for the classes of Ma-Minda starlike and convex functions:

Lemma 5.4.4. *([102])* Let $f \in \mathcal{S}^*(\psi)$ and $|z_0| = r < 1$. Then $f(z)/z \prec f_0(z)/z$ and $-f_0(-r) \leq |f(z_0)| \leq f_0(r)$. Equality holds for some $z_0 \neq 0$ if and only if f is a rotation of f_0, where

$$f_0(z) = z \exp \int_0^z \frac{\psi(t)-1}{t} dt. \tag{5.4.10}$$

Lemma 5.4.5. *([102])* Let $f \in \mathcal{C}(\psi)$. Then $zf''(z)/f'(z) \prec zl_0''(z)/l_0'(z)$ and $f'(z) \prec l_0'(z)$. Also, for $|z| = r$ we have $-l_0(-r) \leq |f(z)| \leq l_0(r)$, where

$$1 + zl_0''(z)/l_0'(z) = \psi(z). \tag{5.4.11}$$

Remark 5.4.4. Throughout this section, we shall assume that $\min_{|z|=r} |\psi(z)| = \psi(-r)$ and $\max_{|z|=r} |\psi(z)| = \psi(r)$, as under these conditions growth theorems for the above-defined classes in Definitions 5.4.23, 5.4.24, 5.4.25 and 5.4.26 are known.

The following result yields the Bohr-Rogosonski radius for the analytic functions subordinated by close-to-convex functions.

Theorem 5.4.27. Let $f \in \mathcal{K}_s(\psi)$ and $\Omega = f(\mathbb{D})$. If $g \in S_f(\psi)$. Then $|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \partial\Omega)$ holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$ and r_0 is the minimal positive root of the equation:

$$\int_0^{r^m} \frac{\psi(t)}{1-t^2} dt + R^N(r) = \int_0^1 \frac{\psi(-t)}{1+t^2} dt, \tag{5.4.12}$$

where

$$R^N(r) = \int_0^r \frac{M_t^N(\psi) t^{2N}}{t^2(1-t^2)} dt \quad \text{and} \quad f_0(z) = \int_0^z \frac{\psi(t)}{1-t^2} dt.$$

Proof. Let $g(z) = \sum_{k=1}^{\infty} b_k z^k \prec f(z)$. Then by Lemma 5.2.1, for $r \leq 1/3$

$$\sum_{k=N}^{\infty} |b_k| r^k \leq \sum_{n=N}^{\infty} |a_n| r^n = \sum_{n=N}^{\infty} \frac{|\tilde{b}_n|}{n} r^n, \tag{5.4.13}$$

where $\tilde{b}_n$ are the power series coefficient of $\tilde{G}(z)$ defined below. From Definition 5.4.23, we have

$$zf'(z) = G(z)\psi(\omega(z)) =: \tilde{G}(z),$$

where

$$G(z) = \frac{-h(z)h(-z)}{z} =: z + \sum_{n=2}^{\infty} h_{2n-1} z^{2n-1},$$

which is an odd starlike function. Now a simple integration gives that

$$f(z) = \int_0^z \frac{G(t)\psi(\omega(t))}{t} dt.$$

For $f \in \mathscr{A}$, let us consider the operator

$$M_r^N(f) = \sum_{n=N}^{\infty} |a_n||z^n| = \sum_{n=N}^{\infty} |a_n| r^n.$$

Then

$$M_r^N(\tilde{G}) \le M_r^N(G) M_r^N(\psi \circ \omega). \tag{5.4.14}$$

Since $\psi \circ \omega \prec \psi$, we get

$$M_r^N(\psi \circ \omega) \le M_r^N(\psi), \quad \text{for} \quad r \le 1/3. \tag{5.4.15}$$

It is known that odd starlike functions satisfies $|h_{2n-1}| \le 1$. Thus

$$M_r^N(G) \le \sum_{n=N}^{\infty} r^{2n-1} = \frac{1}{r}\left(\frac{r^{2N}}{1-r^2}\right). \tag{5.4.16}$$

Now combining the inequalities (5.4.13), (5.4.14), (5.4.15) and (5.4.16), the following sequence of inequalities holds for $r \le 1/3$:

$$\begin{aligned} \sum_{k=N}^{\infty} |b_k| r^k \le \sum_{n=N}^{\infty} \frac{|\tilde{b}_n|}{n} r^n = \int_0^r \frac{M_t^N(\tilde{G})}{t} dt \le \int_0^r \frac{M_t^N(G) M_t^N(\psi \circ \omega)}{t} dt \\ \le \int_0^r \frac{M_t^N(\psi) t^{2N}}{t^2(1-t^2)} dt := R^N(r). \end{aligned} \tag{5.4.17}$$

Since, also see growth theorem in [41, Theorem 2, page no. 4],

$$|g(z)| = |f(\omega(z))| \le \max_{|z|=r} |f(|z| \le r)| \le \int_0^r \frac{\psi(t)}{1-t^2} dt = f_0(r),$$

it follows that

$$|g(z^m)| \le f(r^m) \le \hat{f}_0(r^m), \quad \text{where} \quad f_0(z) = \int_0^z \frac{\psi(t)}{1-t^2} dt. \tag{5.4.18}$$

Finally, note that

$$d(0, \Omega) \ge \int_0^1 \frac{\psi(-t)}{1+t^2} dt.$$

Hence, from (5.4.17) and (5.4.18), we get

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \le d(0, \partial\Omega)$$

for $|z| = r \le r_b = \min\{1/3, r_0\}$, where $r_0 \in (0,1)$ is the minimal root of the equation (5.4.12). The existence of the root follows by the Intermediate value theorem in the interval $[0,1]$. □

Remark 5.4.5. Taking $m \to \infty$ and $N = 1$, then Theorem 5.4.27 reduces to [5, Theorem 2.2].

Corollary 5.4.28. If $f \in \mathscr{K}_s(\psi)$ and $\Omega = f(\mathbb{D})$. Then

$$|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \le d(0, \partial\Omega)$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$ and r_0 is the minimal positive root of the equation:

$$\int_0^{r^m} \frac{\psi(t)}{1-t^2} dt + R^N(r) = \int_0^1 \frac{\psi(-t)}{1+t^2} dt,$$

where

$$R^N(r) = \int_0^r \frac{M_t^N(\psi) t^{2N}}{t^2(1-t^2)} dt$$

and

$$f_0(z) = \int_0^z \frac{\psi(t)}{1-t^2} dt.$$

Our next result provides the Bohr-Rogosonski radius for the analytic functions subordinated by starlike function with respect to conjugate points.

Theorem 5.4.29. Let h_ψ be given by (4.5.2) and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_c^*(\psi)$. If $g \in S_f(\psi)$. Then

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \partial\Omega) \tag{5.4.19}$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the unique positive root of the equation:

$$h_\psi(r^m) + R^N(r) + h_\psi(-1) = 0, \tag{5.4.20}$$

where

$$R^N(r) = \int_0^r \frac{M_t^N(h_\psi) M_t^N(\psi)}{t} dt$$

The result is sharp when $r_b = r_0$ and $t_n > 0$.

Proof. Since the function $G(z) = (f(z) + \overline{f(\bar{z})})/2$ belongs to $\mathscr{S}^*(\psi)$. Therefore, by Lemma [102] we have

$$\frac{G(z)}{z} \prec \frac{h_\psi(z)}{z},$$

which using Lemma 5.2.1 yields

$$M_r^N(G) \leq M_r^N(h_\psi) \quad \text{for} \quad r \leq \frac{1}{3}. \tag{5.4.21}$$

From Definition 5.4.24, we get $zf'(z) = G(z)\psi(\omega(z))$ which after integration gives

$$f(z) = \int_0^z \frac{G(t)\psi(\omega(t))}{t} dt. \tag{5.4.22}$$

Since $\psi \circ \omega \prec \psi$,

$$M_r^N(\psi \circ \omega) \leq M_r^N(\psi) \quad \text{for} \quad r \leq \frac{1}{3}. \tag{5.4.23}$$

Thus, combining (5.4.21), (5.4.22) and (5.4.23), we see that

$$\sum_{k=N}^{\infty}|b_k||z|^k = M_r^N(g) \leq M_r^N(f)$$
$$= \int_0^r \frac{M_t^N(G)M_t^N(\psi\circ\omega)}{t}dt$$
$$\leq \int_0^r \frac{M_t^N(h_\psi)M_t^N(\psi)}{t}dt =: R^N(r),$$

holds for $r \leq 1/3$. Also, using Maximum-principle of modulus and growth theorem [136], $g \prec f$ implies that

$$|g(|z|\leq r)| = |f(\omega(|z|\leq r))| \leq \max_{|z|=r}|f(|z|\leq r)| = h_\psi(r),$$

which yields $|g(z^m)| \leq h_\psi(r^m)$. Finally, note that

$$d(0,\Omega) \geq -h_\psi(-1).$$

Hence, the Bohr-Rogosinski inequality (5.2.13) holds for $|z| \leq \min\{1/3, r_0\}$, where r_0 is the root of the equation (5.4.20). The existence of the root follows by the Intermediate value theorem for continuous function in $[0,1]$. For the sharpness, note that for the function h_ψ, $d(0,\Omega) = -h_\psi(-1)$ such that if $r_b = r_0$, then for the choice $g = f = h_\psi$:

$$|h_\psi(z^m)| + \sum_{n=N}^{\infty}|t_n||z|^n = d(0,\partial\Omega)$$

holds for $|z| = r_b$ with $t_n > 0$, where $h_\psi(z) = z + \sum_{n=2}^{\infty} t_n z^n$ as given in (4.5.2). □

Remark 5.4.6. Let $\psi(z) = (1+z)/(1-z)$, then Theorem 5.4.29 reduces to [76, Theorem 6].

The following result is explicitly for the class $\mathcal{S}_c^*(\psi)$.

Corollary 5.4.30. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_c^*(\psi)$. Then

$$|f(z^m)| + \sum_{n=N}^{\infty}|a_n||z|^n \leq d(0,\partial\Omega)$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the unique positive root of the equation:

$$h_\psi(r^m) + R^N(r) + h_\psi(-1) = 0,$$

where

$$R^N(r) = \int_0^r \frac{M_t^N(h_\psi)M_t^N(\psi)}{t}dt$$

and h_ψ be given by (4.5.2). The result is sharp when $r_b = r_0$ and $t_n > 0$.

Remark 5.4.7. Taking $m \to \infty$ and $N = 1$ in Theorem 5.4.29 and Corollary 5.4.30 establish the Bohr phenonmenon for the classes $S_f(\psi)$ and $\mathcal{S}_c^*(\psi)$, respectively given in [5, Lemma 2.12] and [5, Theorem 2.9].

In the following, we obtain Bohr-Rogosonski radius for the analytic functions subordinated by the convex function with respect to conjugate points.

Theorem 5.4.31. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{C}_c(\psi)$. If $g\in S_f(\psi)$. Then

$$|g(z^m)|+\sum_{k=N}^{\infty}|b_k||z|^k\leq d(0,\partial\Omega)$$

holds for $|z|=r_b\leq\min\{\frac{1}{3},r_0\}$, where $m,N\in\mathbb{N}$, $\Omega=f(\mathbb{D})$ and r_0 is the minimal positive root of the equation:

$$k_{\psi}(r^m)+R^N(r)+k_{\psi}(-1)=0, \tag{5.4.24}$$

where

$$R^N(r)=\int_0^r\frac{1}{s}\int_0^s M_t^N(k'_{\psi})M_t^N(\psi)dtds,$$

and $k_{\psi}(z)=z+\sum_{n=2}^{\infty}l_nz^n$ is given by (5.4.11). The result is sharp when $r_b=r_0$ and $l_n>0$.

Proof. Consider the function

$$G(z)=\frac{f(z)+\overline{f(\bar{z})}}{2}.$$

Then $G\in\mathscr{C}(\psi)$. Now from Definition 5.4.25, we see that

$$(zf'(z))'=G'(z)\psi(\omega(z)). \tag{5.4.25}$$

This gives

$$f(z)=\int_0^z\frac{1}{y}\int_0^y G'(t)\psi(\omega(t))dtdy. \tag{5.4.26}$$

As $G'\prec k'_{\psi}$, see Lemma 5.4.5, it follows using Lemma 5.2.1 that

$$M_r^N(G')\leq M_r^N(k'_{\psi})\quad\text{for}\quad r\leq\frac{1}{3}. \tag{5.4.27}$$

Hence, using (5.4.25), (5.4.26) and (5.4.27)

$$\begin{aligned}|g(z^m)|+M_r^N(f)&\leq k_{\psi}(r^m)+\int_0^r\frac{1}{y}\int_0^y M_t^N(G')M_t^N(\psi\circ\omega)dtdy\\&\leq k_{\psi}(r^m)+\int_0^r\frac{1}{y}\int_0^y M_t^N(k'_{\psi})M_t^N(\psi)dtdy\\&\leq -k_{\psi}(-1)\leq d(0,\partial\Omega),\end{aligned}$$

holds for $|z|=r\leq r_b=\min\{1/3,r_0\}$, where r_0 is minimal root of the equation (5.4.24). The existence of $0<r_0<1$ can be seen by the Intermediate value theorem for continuous function in $[0,1]$. The case of equality

$$|g(z^m)|+\sum_{k=N}^{\infty}|b_k||z|^k=d(0,\partial\Omega)$$

follows with the choice $g=f=k_{\psi}$, whenever $r_b=r_0$ and $l_n>0$. □

Corollary 5.4.32. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{C}_c(\psi)$. Then

$$|f(z^m)| + \sum_{k=N}^{\infty} |a_n||z|^n \leq d(0, \partial\Omega)$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the minimal positive root of the equation:

$$k_\psi(r^m) + R^N(r) + k_\psi(-1) = 0,$$

where

$$R^N(r) = \int_0^r \frac{1}{s} \int_0^s M_t^N(k'_\psi) M_t^N(\psi) dt ds.$$

The result is sharp when $r_b = r_0$ and $l_n > 0$.

Remark 5.4.8. Taking $m \to \infty$ and $N = 1$ in Corollary 5.4.32 gives [5, Theorem 2.23].

Now, we omit the details of the proof of our next result as it works on similar lines discussed in the above theorems.

Theorem 5.4.33. Let $k_\psi(z) = z + \sum_{n=2}^{\infty} l_n z^n$ be given by (5.4.11) and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{C}_s(\psi)$. If $g \in S_f(\psi)$. Then

$$|g(z^m)| + \sum_{k=N}^{\infty} |b_k||z|^k \leq d(0, \partial\Omega)$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is the unique positive root of the equation:

$$\int_0^r \frac{1}{s} \int_0^s \psi(t)(k'_\psi(t^2))^{1/2} dt ds + R^N(r) = \int_0^1 \frac{1}{s} \int_0^s \psi(-t)(k'_\psi(-t^2))^{1/2} dt ds,$$

where $K'(z) = (k'_\psi(z^2))^{1/2}$ and

$$R^N(r) = \int_0^r \frac{1}{s} \int_0^s M_t^N(K') M_t^N(\psi) dt ds.$$

The result is sharp when $r_b = r_0$ and $l_n > 0$.

Corollary 5.4.34. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{C}_s(\psi)$. Then

$$|f(z^m)| + \sum_{k=N}^{\infty} |a_n||z|^n \leq d(0, \partial\Omega)$$

holds for $|z| = r_b \leq \min\{\frac{1}{3}, r_0\}$, where $m, N \in \mathbb{N}$, $\Omega = f(\mathbb{D})$ and r_0 is as given in Theorem 5.4.33. The result is sharp when $r_b = r_0$ and $l_n > 0$.

Remark 5.4.9. Taking $m \to \infty$ and $N = 1$ in Corollary 5.4.34 gives [5, Theorem 2.25].

Highlights of the chapter

In this chapter, we studied extensively the radius problem in view of Bohr and Rogosinski phenomenon and their natural generalizations for the various classes of analytic functions, especially for the Ma-Minda classes of starlike and convex functions, which is a new attempt and would give a way to further investigations in this direction. We also established the Bohr phenomenon for the classes of certain harmonic functions and studied their application to certain classes of analytic univalent functions.

Chapter 6

$\mathscr{S}^*(\phi)$ and $\mathscr{C}(\phi)$-radii of Some Special functions

In this chapter, we consider the Ma-Minda classes of analytic functions $\mathscr{S}^*(\phi) := \{f \in \mathscr{A} : (zf'(z)/f(z)) \prec \phi(z)\}$ and $\mathscr{C}(\phi) := \{f \in \mathscr{A} : (1+zf''(z)/f'(z)) \prec \phi(z)\}$ defined on the unit disk $\mathbb{D}$ and show that the classes $\mathscr{S}^*(1+\alpha z)$ and $\mathscr{C}(1+\alpha z)$, $0<\alpha\leq 1$ solve the problem of finding the sharp $\mathscr{S}^*(\phi)$-radii and $\mathscr{C}(\phi)$-radii for some normalized special functions, whenever $\phi(-1)=1-\alpha$. The radius of strongly starlikeness is also considered.

6.1 Introduction

The involvement of the special functions and their geometrical properties has been observed vastly in the theory of univalent functions. We also used certain properties of hypergeometric functions in our majorization problems. This motivates us to study another radius problem, which concerns the radii of starlikeness and convexity of the special functions for the classes $\mathscr{S}^*(\phi)$ and $\mathscr{C}(\phi)$.

A real entire function L maps real line into itself is said to be in the Laguerre-Pólya class $\mathscr{L}\mathscr{P}$, if it can be expressed as follows:

$$L(x) = cx^m e^{-ax^2+\beta x}\prod_{k\geq 1}\left(1+\frac{x}{x_k}\right)e^{-\frac{x}{x_k}},$$

where $c,\beta,x_k\in\mathbb{R}$, $a\geq 0$, $m\in\mathbb{N}\cup\{0\}$ and $\sum {x_k}^{-2}<\infty$, see [21], [47, p. 703] and the references therein. The class $\mathscr{L}\mathscr{P}$ consists of entire functions which can be approximated by polynomials with only real zeros, uniformly on the compact sets of the complex plane and it is closed under differentiation.

Recall that $\mathcal{S}^*(\phi)$-radius for a given normalized function f in $\mathcal{A}$ is defined as the largest radius r_0 such that $f \in \mathcal{S}^*(\phi)$ in $|z| \leq r_0$. Similarly $\mathcal{C}(\phi)$-radius can be defined. For more radius problems we refer to [29, 38, 50, 109].

In view of Ma-Minda classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$, the specific choices of D and E in the Janowski classes yields: $\mathcal{S}^*[\alpha,0]$ and $\mathcal{C}[\alpha,0]$, which are the extensions of the Ram Singh [156] classes $\mathcal{S}^*[1,0]$ and $\mathcal{C}[1,0]$ respectively, where $0<\alpha\leq 1$, and $\mathcal{S}^*(\gamma) := \mathcal{S}^*[1-2\gamma,-1]$ and $\mathcal{C}(\gamma) := \mathcal{C}[1-2\gamma,-1]$, where $0 \leq \gamma < 1$ are the classes of starlike and convex functions of order γ, that is,

$$\mathcal{S}^*(\gamma) = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \gamma \right\}$$

and

$$\mathcal{C}(\gamma) = \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \gamma \right\}.$$

Recently, $\mathcal{S}^*(\gamma)$-radius and $\mathcal{C}(\gamma)$-radius for the normalized special functions, which can be represented as Hadamard factorization [95] under certain conditions were studied in [6, 21, 22, 25–29, 45]. Some of the special functions, which are studied recently in terms of radius problems are Bessel functions [6, 22, 26] (see Watson's treatise [171] for more on Bessel function), Struve functions [6, 21], Wright functions [27], Lommel functions [6,21], Legendre polynomials of odd degree [29] and Ramanujan type entire functions [45]. Evidently, the zeros of these special functions and the $\mathcal{L}\mathcal{P}$ class played an important role in the derivation of the above radius results. Now to proceed further, we consider the following assumption as it covers many classical classes and the recently introduced classes, see Corollary 6.7.1:

Assumption 6.1.1. Let ϕ be analytic and univalent with $\Re\phi(z) > 0$, $\phi'(0) > 0$ and $\phi(\mathbb{D})$ is starlike with respect to $\phi(0) = 1$. Assume that the maximal disk $\{w : |w-1| < \alpha\} \subseteq \phi(\mathbb{D})$ and $\phi(-1) = 1-\alpha$. Here, α is the largest such number depending upon $\phi(\mathbb{D})$.

In view of the above literature, we now concisely state our problem:

Problem 6.1.1. Let g be the normalized form of a given special function. Then find the $\mathcal{S}^*(\phi)$-radius and $\mathcal{C}(\phi)$-radius of g, i.e find the largest radius r_0 such that $g \in \mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ respectively in $|z| \leq r_0$.

That is, $\mathcal{S}^*(\phi)$-radius and $\mathcal{C}(\phi)$-radius of g is defined as follows:

$$r_0(g) = \sup \left\{ r \in \mathbb{R}^+ : \frac{zg'(z)}{g(z)} \in \phi(\mathbb{D}), z \in \mathbb{D}_r \right\}$$

and

$$r_0(g) = \sup \left\{ r \in \mathbb{R}^+ : 1 + \frac{zg''(z)}{g'(z)} \in \phi(\mathbb{D}), z \in \mathbb{D}_r \right\}.$$

Note that here the fixed domain $\phi(\mathbb{D})$ will be given.

In this chapter, we obtain the $\mathcal{S}^*(\phi)$-radius and $\mathcal{C}(\phi)$-radius of certain normalized special functions using the extension of the Ram Singh class: $\mathcal{S}^*[\alpha,0]$ and $\mathcal{C}[\alpha,0]$, where $0<\alpha\leq 1$ with the Assumption 6.1.1. The radius of strongly starlikeness is also obtained.

6.2 Starlikeness and Convexity of Wright and Mittag-Leffler functions

We deal here with two special functions.

6.2.1 Wright functions

Let us consider the generalized Bessel function

$$\Phi(\kappa,\delta,z)=\sum_{n\geq 0}\frac{z^n}{n!\Gamma(n\kappa+\delta)},$$

where $\kappa>-1$ and $z,\delta\in\mathbb{C}$, named after E. M. Wright. The function Φ is entire for $\kappa>-1$. From [27, Lemma 1, p. 100], we have the following Hadamard factorization

$$\Gamma(\delta)\Phi(\kappa,\delta,-z^2)=\prod_{n\geq 1}\left(1-\frac{z^2}{\zeta^2_{\kappa,\delta,n}}\right), \tag{6.2.1}$$

where $\kappa,\delta>0$ and $\zeta_{\kappa,\delta,n}$ is the n-th positive root of $\Phi(\kappa,\delta,-z^2)$ and satisfies the following relationship:

$$\breve{\zeta}_{\kappa,\delta,n}<\zeta_{\kappa,\delta,n}<\breve{\zeta}_{\kappa,\delta,n+1}<\zeta_{\kappa,\delta,n+1},\quad (n\geq 1) \tag{6.2.2}$$

where $\breve{\zeta}_{\kappa,\delta,n}$ is the n-th positive root of the derivative of the function $\Psi_{\kappa,\delta}(z)=z^{\delta}\Phi(\kappa,\delta,-z^2)$. Since

$$\Phi(\kappa,\delta,-z^2)\notin\mathscr{A},$$

so we consider the following normalized Wright functions:

$$\left.\begin{aligned} f_{\kappa,\delta}(z)&=\left[z^{\delta}\Gamma(\delta)\Phi(\kappa,\delta,-z^2)\right]^{1/\delta},\\ g_{\kappa,\delta}(z)&=z\Gamma(\delta)\Phi(\kappa,\delta,-z^2),\\ h_{\kappa,\delta}(z)&=z\Gamma(\delta)\Phi(\kappa,\delta,-z).\end{aligned}\right\} \tag{6.2.3}$$

For simplicity, we write $W_{\kappa,\delta}(z):=\Phi(\kappa,\delta,-z^2)$. We denote $\mathscr{S}^*(\phi)$-radius by $R[\mathscr{S}^*(\phi)]$.

Theorem 6.2.1. Let $\kappa,\delta>0$ and $\alpha\in(0,1]$ such that the largest disk $\{w:|w-1|<\alpha\}\subseteq\phi(\mathbb{D})$. Then

$$R[\mathscr{S}^*(\phi)]=R[\mathscr{S}^*(1+\alpha z)],$$

where $\phi(-1)=1-\alpha$ and $\mathscr{S}^*(1+\alpha z)$-radii for the functions $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ defined in (6.2.3) are the smallest positive roots of the following equations respectively:

(i) $rW'_{\kappa,\delta}(r)+\delta\alpha W_{\kappa,\delta}(r)=0;$

(ii) $rW'_{\kappa,\delta}(r)+\alpha W_{\kappa,\delta}(r)=0;$

(iii) $\sqrt{r}W'_{\kappa,\delta}(\sqrt{r})+2\alpha W_{\kappa,\delta}(\sqrt{r})=0.$

Proof. Using (6.2.1), we obtain the following by the logarithmic differentiation of (6.2.3):

$$\begin{aligned} \frac{zf'_{\kappa,\delta}(z)}{f_{\kappa,\delta}(z)} &= 1+\frac{1}{\delta}\frac{zW'_{\kappa,\delta}(z)}{W_{\kappa,\delta}(z)} = 1-\frac{1}{\delta}\sum_{n\geq 1}\frac{2z^2}{\zeta^2_{\kappa,\delta,n}-z^2} \\ \frac{zg'_{\kappa,\delta}(z)}{g_{\kappa,\delta}(z)} &= 1+\frac{zW'_{\kappa,\delta}(z)}{W_{\kappa,\delta}(z)} = 1-\sum_{n\geq 1}\frac{2z^2}{\zeta^2_{\kappa,\delta,n}-z^2} \\ \frac{zh'_{\kappa,\delta}(z)}{h_{\kappa,\delta}(z)} &= 1+\frac{1}{2}\frac{\sqrt{z}W'_{\kappa,\delta}(\sqrt{z})}{W_{\kappa,\delta}(\sqrt{z})} = 1-\sum_{n\geq 1}\frac{z}{\zeta^2_{\kappa,\delta,n}-z}. \end{aligned} \tag{6.2.4}$$

Now using the fact that $||x|-|y|| \leq |x-y|$ and $|z| = r < \zeta_{\kappa,\delta,1}$, we see that $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ belong to $\mathscr{S}^*(1+\alpha z)$ whenever

$$\begin{aligned} \left|\frac{zf'_{\kappa,\delta}(z)}{f_{\kappa,\delta}(z)}-1\right| &\leq \frac{1}{\delta}\sum_{n\geq 1}\frac{2r^2}{\zeta^2_{\kappa,\delta,n}-r^2} \leq \alpha \\ \left|\frac{zg'_{\kappa,\delta}(z)}{g_{\kappa,\delta}(z)}-1\right| &\leq \sum_{n\geq 1}\frac{2r^2}{\zeta^2_{\kappa,\delta,n}-r^2} \leq \alpha \\ \left|\frac{zh'_{\kappa,\delta}(z)}{h_{\kappa,\delta}(z)}-1\right| &\leq \sum_{n\geq 1}\frac{r}{\zeta^2_{\kappa,\delta,n}-r} \leq \alpha. \end{aligned} \tag{6.2.5}$$

The first part of each of the inequalities in (6.2.5) becomes equality when $z = r$. Now consider the following continuous functions:

$$T_f(r) = \frac{1}{\delta}\sum_{n\geq 1}\frac{2r^2}{\zeta^2_{\kappa,\delta,n}-r^2}-\alpha, \quad T_g(r) = \sum_{n\geq 1}\frac{2r^2}{\zeta^2_{\kappa,\delta,n}-r^2}-\alpha$$

and

$$T_h(r) = \sum_{n\geq 1}\frac{r}{\zeta^2_{\kappa,\delta,n}-r}-\alpha.$$

Note that T_f, T_g are increasing in $(0,\zeta_{\kappa,\delta,1})$ and T_h is increasing in $(0,\zeta^2_{\kappa,\delta,1})$. Since $\lim_{r\to 0} T_f(r) = \lim_{r\to 0} T_g(r) = \lim_{r\to 0} T_h(r) = -\alpha < 0$ and $\lim_{r\to\zeta_{\kappa,\delta,1}} T_f(r) = \lim_{r\to\zeta_{\kappa,\delta,1}} T_g(r) = \infty$, $\lim_{r\to\zeta^2_{\kappa,\delta,1}} T_h(r) = \infty$. Therefore, the required $\mathscr{S}^*(1+\alpha z)$-radii for the functions $f_{\kappa,\delta}$ and $g_{\kappa,\delta}$ are the unique positive roots of the equations $T_f(r) = 0$, $T_g(r) = 0$ and for $h_{\kappa,\delta}$ given by $T_h(r) = 0$, respectively in $(0,\zeta_{\kappa,\delta,1})$ and $(0,\zeta^2_{\kappa,\delta,1})$, which can be written as in the statement using (6.2.4). Let $r_{\alpha,f}$, $r_{\alpha,g}$ and $r_{\alpha,h}$ be the roots of $T_f(r) = 0$, $T_g(r) = 0$ and $T_h(r) = 0$ respectively. Then from (6.2.4), we see that

$$\frac{r_{\alpha,f}f'_{\kappa,\delta}(r_{\alpha,f})}{f_{\kappa,\delta}(r_{\alpha,f})} = \frac{r_{\alpha,g}g'_{\kappa,\delta}(r_{\alpha,g})}{g_{\kappa,\delta}(r_{\alpha,g})} = \frac{r_{\alpha,h}h'_{\kappa,\delta}(r_{\alpha,h})}{h_{\kappa,\delta}(r_{\alpha,h})} = 1-\alpha.$$

Then $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ belong to $\mathscr{S}^*(1+\alpha z)$ in $|z| < r_{\alpha,f}$, $r_{\alpha,g}$ and $r_{\alpha,h}$, respectively. Now let $\alpha \in (0,1]$ such that $w_\alpha := \{w : |w-1| < \alpha\}$ is the maximal disk inside $\phi(\mathbb{D})$. Since a function $f_1(z) \in \mathscr{S}^*(\phi)$ if and only if $e^{-it}f_1(e^{it}z) \in \mathscr{S}^*(\phi)$ for all $t \in \mathbb{R}$. Therefore, using (6.2.4) with $z = r_{\alpha,f}$, $r_{\alpha,g}$ and $r_{\alpha,h}$ along with $\phi(-1) = 1-\alpha$, the maximality of the disk w_α implies that $F_{\kappa,\delta}(|z| \leq r)$, $G_{\kappa,\delta}(|z| \leq r)$ and $H_{\kappa,\delta}(|z| \leq r)$ do not lie inside $\phi(\mathbb{D})$ for $r \geq r_{\alpha,f}$, $r_{\alpha,g}$ and $r_{\alpha,h}$ respectively, where $F_{\kappa,\delta}(z) = zf'_{\kappa,\delta}(z)/f_{\kappa,\delta}(z)$, $G_{\kappa,\delta}(z) = zg'_{\kappa,\delta}(z)/g_{\kappa,\delta}(z)$ and $H_{\kappa,\delta}(z) = zh'_{\kappa,\delta}(z)/h_{\kappa,\delta}(z)$ (with some suitable rotation). Hence, $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ belong to $\mathscr{S}^*(\phi)$ in $|z|<r_{\alpha,f}$, $r_{\alpha,g}$ and $r_{\alpha,h}$, respectively and the radii are sharp. □

Remark 6.2.1. From [27, Theorem 1], we see that the equations of Theorem 6.2.1 yields the radius of starlikeness of order $\gamma := 1-\alpha$ for $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$.

We denote $\mathscr{C}(\phi)$-radius by $R[\mathscr{C}(\phi)]$. For our next result, we need the following:

Lemma 6.2.1. Let $\kappa,\delta>0$. Then $\mathscr{C}(1+\alpha z)$-radii for the functions $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ given by (6.2.3) are the smallest positive roots of the following equations respectively:

(i) $\dfrac{r\Psi''_{\kappa,\delta}(r)}{\Psi'_{\kappa,\delta}(r)}+\left(\dfrac{1}{\delta}-1\right)\dfrac{r\Psi'_{\kappa,\delta}(r)}{\Psi_{\kappa,\delta}(r)}+\alpha=0;$

(ii) $rg''_{\kappa,\delta}(r)+\alpha g'_{\kappa,\delta}(r)=0;$

(iii) $rh''_{\kappa,\delta}(z)+\alpha h'_{\kappa,\delta}(r)=0,$

where α is the radius of the disk $\{w:|w-1|\leq\alpha\}$.

Proof. From (6.2.1), (6.2.3) and using the Hadamard representation (see [27, Eq. 7]),

$$\Gamma(\delta)\Psi'_{\kappa,\delta}(z)=\delta z^{\delta-1}\prod_{n\geq1}\left(1-\frac{z^2}{\xi^2_{\kappa,\delta,n}}\right),$$

we have

$$\begin{aligned}1+\frac{zf''_{\kappa,\delta}(z)}{f'_{\kappa,\delta}(z)}&=1+\frac{z\Psi''_{\kappa,\delta}(z)}{\Psi'_{\kappa,\delta}(z)}+\left(\frac{1}{\delta}-1\right)\frac{z\Psi'_{\kappa,\delta}(z)}{\Psi_{\kappa,\delta}(z)}\\&=1-\sum_{n\geq1}\frac{2z^2}{\xi^2_{\kappa,\delta,n}-z^2}-\left(\frac{1}{\delta}-1\right)\sum_{n\geq1}\frac{2z^2}{\zeta^2_{\kappa,\delta,n}-z^2}\end{aligned}$$

and for $\delta>1$, using the following inequality of [46]:

$$\left|\frac{z}{y-z}-\lambda\frac{z}{x-z}\right|\leq\frac{|z|}{y-|z|}-\lambda\frac{|z|}{x-|z|},\quad (x>y>r\geq|z|) \tag{6.2.6}$$

with $\lambda=1-1/\delta$, we get

$$\begin{aligned}\left|\frac{zf''_{\kappa,\delta}(z)}{f'_{\kappa,\delta}(z)}\right|&=\left|\sum_{n\geq1}\frac{2z^2}{\xi^2_{\kappa,\delta,n}-z^2}-\left(1-\frac{1}{\delta}\right)\sum_{n\geq1}\frac{2z^2}{\zeta^2_{\kappa,\delta,n}-z^2}\right|\\&\leq\sum_{n\geq1}\frac{2r^2}{\xi^2_{\kappa,\delta,n}-r^2}-\left(1-\frac{1}{\delta}\right)\sum_{n\geq1}\frac{2r^2}{\zeta^2_{\kappa,\delta,n}-r^2}\\&=-\frac{rf''_{\kappa,\delta}(r)}{f'_{\kappa,\delta}(r)}=-\frac{r\Psi''_{\kappa,\delta}(r)}{\Psi'_{\kappa,\delta}(r)}-\left(\frac{1}{\delta}-1\right)\frac{r\Psi'_{\kappa,\delta}(r)}{\Psi_{\kappa,\delta}(r)}\end{aligned}$$

Since $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ belong to the Laguerre-Pólya class $\mathscr{LP}$, which is closed under differentiation, their derivatives $g'_{\kappa,\delta}$ and $h'_{\kappa,\delta}$ also belong to $\mathscr{LP}$ and the zeros are real. Thus assuming $\tau_{\kappa,\delta,n}$ and

$\eta_{\kappa,\delta,n}$ are the positive zeros of $g'_{\kappa,\delta}$ and $h'_{\kappa,\delta}$, respectively, we have the following representations:

$$g'_{\kappa,\delta}(z)=\prod_{n\geq 1}\left(1-\frac{z^2}{\tau^2_{\kappa,\delta,n}}\right) \quad \text{and} \quad h'_{\kappa,\delta}(z)=\prod_{n\geq 1}\left(1-\frac{z}{\eta_{\kappa,\delta,n}}\right),$$

which yield

$$1+\frac{zg''_{\kappa,\delta}(z)}{g'_{\kappa,\delta}(z)}=1-\sum_{n\geq 1}\frac{2z^2}{\tau^2_{\kappa,\delta,n}-z^2} \quad \text{and} \quad 1+\frac{zh''_{\kappa,\delta}(z)}{h'_{\kappa,\delta}(z)}=1-\sum_{n\geq 1}\frac{z}{\eta_{\kappa,\delta,n}-z}.$$

Now using the inequality $||x|-|y||\leq |x-y|$ and the relation (6.2.2), we see that $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ belong to $\mathscr{C}(1+\alpha z)$ whenever

$$\left|\frac{zf''_{\kappa,\delta}(z)}{f'_{\kappa,\delta}(z)}\right|\leq \sum_{n\geq 1}\frac{2r^2}{\breve{\zeta}^2_{\kappa,\delta,n}-r^2}+\left(\frac{1}{\delta}-1\right)\sum_{n\geq 1}\frac{2r^2}{\zeta^2_{\kappa,\delta,n}-r^2}\leq \alpha,$$

where $\delta>0, |z|=r<\breve{\zeta}_{\kappa,\delta,1}$,

$$\left|\frac{zg''_{\kappa,\delta}(z)}{g'_{\kappa,\delta}(z)}\right|\leq \sum_{n\geq 1}\frac{2r^2}{\tau^2_{\kappa,\delta,n}-r^2}\leq \alpha \quad (|z|=r<\tau_{\kappa,\delta,1})$$

and

$$\left|\frac{zh''_{\kappa,\delta}(z)}{h'_{\kappa,\delta}(z)}\right|\leq \sum_{n\geq 1}\frac{r}{\eta_{\kappa,\delta,n}-r}\leq \alpha \quad (|z|=r<\eta_{\kappa,\delta,1}),$$

respectively. Now further proceeding as in Theorem 6.2.1, the result follows at once. □

The proof of Theorem 6.2.2 is similar to Theorem 6.2.1, and hence it is skipped here.

Theorem 6.2.2. Let $\kappa,\delta>0$ and $\alpha\in(0,1]$ such that the largest disk $\{w:|w-1|<\alpha\}\subseteq\phi(\mathbb{D})$. Then

$$R[\mathscr{C}(\phi)]=R[\mathscr{C}(1+\alpha z)]$$

for the functions $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ given by (6.2.3), where $\phi(-1)=1-\alpha$ and $R[\mathscr{C}(1+\alpha z)]$ is given by Lemma 6.2.1.

Remark 6.2.2. From [27, Theorem 5], we see that equations of Lemma 6.2.1 yields the radius of starlikeness of order $\gamma:=1-\alpha$ for $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$.

6.2.2 Mittag-Leffler functions

In 1971, Prabhakar [132] introduced the following function

$$M(\mu,\nu,a,z):=\sum_{n\geq 0}\frac{(a)_n z^n}{n!\Gamma(\mu n+\nu)},$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol and $\mu, \nu, a > 0$. The functions $M(\mu,\nu,1,z)$ and $M(\mu,1,1,z)$ were introduced and studied by Wiman and Mittag-Leffler, respectively. Now let us consider the set $W_b = A(W_c) \cup B(W_c)$, where

$$W_c := \left\{ \left(\frac{1}{\mu}, \nu \right) : 1 < \mu < 2, \nu \in [\mu-1, 1] \cup [\mu, 2] \right\}$$

and denote by W_i, the smallest set containing W_b and invariant under the transformations A, B and C mapping the set $\{(\frac{1}{\mu}, \nu) : \mu > 1, \nu > 0\}$ into itself and are defined as:

$$\begin{aligned} &A : (\tfrac{1}{\mu}, \nu) \to (\tfrac{1}{2\mu}, \nu), \quad B : (\tfrac{1}{\mu}, \nu) \to (\tfrac{1}{2\mu}, \mu+\nu), \\ &C : (\tfrac{1}{\mu}, \nu) \to \begin{cases} (\tfrac{1}{\mu}, \nu-1), & \text{if } \nu > 1; \\ (\tfrac{1}{\mu}, \nu), & \text{if } 0 < \nu \leq 1. \end{cases} \end{aligned}$$

Kumar and Pathan [88] proved that if $(\frac{1}{\mu}, \nu) \in W_i$ and $a > 0$, then all zeros of $M(\mu,\nu,a,z)$ are real and negative. From [25, Lemma 1, p. 121], we see that if $(\frac{1}{\mu}, \nu) \in W_i$ and $a > 0$, then the function $M(\mu,\nu,a,-z^2)$ has infinitely many zeros, which are all real and have the following representation:

$$\Gamma(\nu) M(\mu,\nu,a,-z^2) = \prod_{n \geq 1} \left(1 - \frac{z^2}{\lambda^2_{\mu,\nu,a,n}} \right),$$

where $\lambda_{\mu,\nu,a,n}$ is the n-th positive zero of $M(\mu,\nu,a,-z^2)$ and satisfy the interlacing relation

$$\xi_{\mu,\nu,a,n} < \lambda_{\mu,\nu,a,n} < \xi_{\mu,\nu,a,n+1} < \lambda_{\mu,\nu,a,n+1} \quad (n \geq 1),$$

where $\xi_{\mu,\nu,a,n}$ is the n-th positive zero of the derivative of $z^\nu M(\mu,\nu,a,-z^2)$. Since $M(\mu,\nu,a,-z^2) \notin \mathscr{A}$, therefore we consider the following normalized forms (belong to the Laguerre-Pólya class):

$$\left. \begin{aligned} f_{\mu,\nu,a}(z) &= \left[z^\nu \Gamma(\nu) M(\mu,\nu,a,-z^2) \right]^{1/\nu}, \\ g_{\mu,\nu,a}(z) &= z\Gamma(\nu) M(\mu,\nu,a,-z^2), \\ h_{\mu,\nu,a}(z) &= z\Gamma(\nu) M(\mu,\nu,a,-z). \end{aligned} \right\} \tag{6.2.7}$$

For simplicity, write $L(\mu,\nu,a,z) := M(\mu,\nu,a,-z^2)$. Now proceeding similarly as in Section 6.2.1, we obtain the following results:

Theorem 6.2.3. Let $(\frac{1}{\mu}, \nu) \in W_i$, $a > 0$ and $\alpha \in (0,1]$ such that the largest disk $\{w : |w-1| < \alpha\} \subseteq \phi(\mathbb{D})$. Then $\mathscr{S}^*(\phi)$-radii for the functions $f_{\mu,\nu,a}$, $g_{\mu,\nu,a}$ and $h_{\mu,\nu,a}$ given by (6.2.7) are the smallest positive roots of the following equations respectively:

(i) $rL'_{\mu,\nu,a}(r) + \nu\alpha L_{\mu,\nu,a}(r) = 0;$

(ii) $rL'_{\mu,\nu,a}(r) + \alpha L_{\mu,\nu,a}(r) = 0;$

(iii) $\sqrt{r} L'_{\mu,\nu,a}(\sqrt{r}) + 2\alpha L_{\mu,\nu,a}(\sqrt{r}) = 0,$

where $\phi(-1) = 1 - \alpha$.

Theorem 6.2.4. Let $(\frac{1}{\mu},\nu) \in W_i$, $a>0$ and $\alpha \in (0,1]$ such that the largest disk $\{w : |w-1| < \alpha\} \subseteq \phi(\mathbb{D})$. Then $\mathscr{C}(\phi)$-radii for the functions $f_{\mu,\nu,a}$, $g_{\mu,\nu,a}$ and $h_{\mu,\nu,a}$ given by (6.2.7) are the smallest positive roots of the following equations respectively:

(i) $rf''_{\mu,\nu,a}(r) + \alpha f'_{\mu,\nu,a}(r) = 0;$

(ii) $rg''_{\mu,\nu,a}(r) + \alpha g'_{\mu,\nu,a}(r) = 0;$

(iii) $rh''_{\mu,\nu,a}(r) + \alpha h'_{\mu,\nu,a}(r) = 0,$

where $\phi(-1) = 1-\alpha$

6.3 Starlikeness and Convexity of Legendre polynomials

The Legendre polynomials P_n are the solutions of the Legendre differential equation

$$((1-z^2)P_n'(z))' + n(n+1)P_n(z) = 0,$$

where $n \in \mathbb{Z}^+$ and using Rodrigues formula, P_n can be represented in the form:

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n(z^2-1)^n}{dz^n}$$

and it also satisfies the geometric condition $P_n(-z) = (-1)^n P_n(z)$. Moreover, the odd degree Legendre polynomials $P_{2n-1}(z)$ have only real roots which satisfy

$$0 = z_0 < z_1 < \cdots < z_{n-1} \quad \text{or} \quad -z_1 > \cdots > -z_{n-1}. \tag{6.3.1}$$

Thus the normalized form is as follows:

$$\mathscr{P}_{2n-1}(z) := \frac{P_{2n-1}(z)}{P'_{2n-1}(0)} = z + \sum_{k=2}^{2n-1} a_k z^k = a_{2n-1} z \prod_{k=1}^{n-1}(z^2 - z_k^2). \tag{6.3.2}$$

Theorem 6.3.1. $R[\mathscr{C}(\phi)] = R[\mathscr{C}(1+\alpha z)]$ for the normalized Legendre polynomial of odd degree as defined in (6.3.2) is given by the smallest positive root $r(\mathscr{P}_{2n-1})$ of the equation

$$r\mathscr{P}''_{2n-1}(r) + \alpha \mathscr{P}'_{2n-1}(r) = 0,$$

where α is the radius of the largest disk $\{w : |w-1| < \alpha\}$ inside $\phi(\mathbb{D})$ such that $\phi(-1) = 1-\alpha$.

Proof. From (6.3.2), we obtain after the logarithmic differentiation

$$1 + \frac{z\mathscr{P}''_{2n-1}(z)}{\mathscr{P}'_{2n-1}(z)} = \frac{z\mathscr{P}'_{2n-1}(z)}{\mathscr{P}_{2n-1}(z)} - \frac{\sum_{k=1}^{n-1} \frac{4z_k^2 z^2}{(z_k^2 - z^2)^2}}{\frac{z\mathscr{P}'_{2n-1}(z)}{\mathscr{P}_{2n-1}(z)}} = 1 - \sum_{k=1}^{n-1} \frac{2z^2}{z_k^2 - z^2} - \frac{\sum_{k=1}^{n-1} \frac{4z_k^2 z^2}{(z_k^2 - z^2)^2}}{1 - \sum_{k=1}^{n-1} \frac{2z^2}{z_k^2 - z^2}},$$

which implies, after using the inequality $||x|-|y|| \leq |x-y|$ and (6.3.1) for $|z|=r<z_1$

$$\left|\left(1+\frac{z\mathscr{P}''_{2n-1}(z)}{\mathscr{P}'_{2n-1}(z)}\right)-1\right| \leq \sum_{k=1}^{n-1}\frac{2r^2}{z_k^2-r^2}+\frac{\sum_{k=1}^{n-1}\frac{4z_k^2r^2}{(z_k^2-r^2)^2}}{1-\sum_{k=1}^{n-1}\frac{2r^2}{z_k^2-r^2}}=-\frac{r\mathscr{P}''_{2n-1}(r)}{\mathscr{P}'_{2n-1}(r)}. \tag{6.3.3}$$

Now let α be the largest such that $\{w:|w-1|\leq\alpha\}\subseteq\phi(\mathbb{D})$. Then from (6.3.3), we see that

$$\mathscr{P}_{2n-1}\in\mathscr{C}(1+\alpha z)\subseteq\mathscr{C}(\phi),$$

whenever

$$r\mathscr{P}''_{2n-1}(r)+\alpha\mathscr{P}'_{2n-1}(r)\geq 0,$$

which holds in $|z|=r\leq r(\mathscr{P}_{2n-1})$. Sharpness of the radius $r(\mathscr{P}_{2n-1})$ follows from the suitable rotation of $\mathscr{P}_{2n-1}$. □

Now we have the starlike case:

Theorem 6.3.2 (Legendre polynomials P_n)**.** The $\mathscr{S}^*(\phi)$-radius $r_\phi(\mathscr{P}_{2n-1})\in(0,z_1)$ of the normalized odd degree Legendre polynomial is the smallest positive root of the following equation:

$$r\mathscr{P}'_{2n-1}(r)-(1-\alpha)\mathscr{P}'_{2n-1}(r)=0,$$

where α is the radius of the largest disk $\{w:|w-1|<\alpha\}$ inside $\phi(\mathbb{D})$ such that $\phi(-1)=1-\alpha$.

Proof. From (6.3.2), after logarithmic differentiation, we obtain

$$\frac{z\mathscr{P}'_{2n-1}(z)}{\mathscr{P}_{2n-1}(z)}=1-\sum_{k=1}^{n-1}\frac{2z^2}{z_k^2-z^2}. \tag{6.3.4}$$

Now applying Assumption 6.1.1 on (6.3.4), we have $\mathscr{P}_{2n-1}\in\mathscr{S}^*(\phi)$ whenever

$$\left|\frac{z\mathscr{P}'_{2n-1}(z)}{\mathscr{P}_{2n-1}(z)}-1\right|\leq\sum_{k=1}^{n-1}\frac{2r^2}{z_k^2-r^2}\leq\alpha, \tag{6.3.5}$$

where $|z|=r<z_1$ and z_k satisfies the condition given in (6.3.1). Now let us consider the strictly increasing continuous function

$$T(r):=\sum_{k=1}^{n-1}\frac{2r^2}{z_k^2-r^2}-\alpha,\quad r\in(0,z_1).$$

We have to show that $T(r)\leq 0$ in $|z|\leq r<z_1$ so that (6.3.5) holds. Since $\lim_{r\to 0}T(r)<0$, $\lim_{r\to z_1}T(r)>0$ and $T'(r)>0$, there exists a unique positive root $r_\phi(\mathscr{P}_{2n-1})\in(0,z_1)$ of $T(r)$ such that $\mathscr{P}_{2n-1}\in\mathscr{S}^*(\phi)$ in $|z|<r_\phi(\mathscr{P}_{2n-1})$. □

6.4 Starlikeness and Convexity of Lommel functions

The Lommel function $\mathscr{L}_{u,v}$ of the first kind is a particular solution of the second-order inhomogeneous Bessel differential equation

$$z^2 w''(z) + z w'(z) + (z^2 - v^2) w(z) = z^{u+1},$$

where $u \pm v \notin \mathbb{Z}^-$ and is given by

$$\mathscr{L}_{u,v} = \frac{z^{u+1}}{(u-v+1)(u+v+1)} {}_1F_2\left(1; \frac{u-v+3}{2}, \frac{u+v+3}{2}; -\frac{z^2}{4}\right),$$

where $\frac{1}{2}(-u \pm v - 3) \notin \mathbb{N}$ and ${}_1F_2$ is a hypergeometric function. Since it is not normalized, therefore we consider the following three normalized functions involving $\mathscr{L}_{u,v}$:

$$\left.\begin{aligned} f_{u,v}(z) &= ((u-v+1)(u+v+1)\mathscr{L}_{u,v}(z))^{\frac{1}{u+1}}, \\ g_{u,v}(z) &= (u-v+1)(u+v+1) z^{-u} \mathscr{L}_{u,v}(z), \\ h_{u,v}(z) &= (u-v+1)(u+v+1) z^{\frac{1-u}{2}} \mathscr{L}_{u,v}(\sqrt{z}). \end{aligned}\right\} \tag{6.4.1}$$

Authors in [6, 21] and [28] proved the radius of starlikeness and convexity for the following normalized functions expressed in terms of $\mathscr{L}_{u-\frac{1}{2},\frac{1}{2}}$:

$$f_{u-\frac{1}{2},\frac{1}{2}}(z), \quad g_{u-\frac{1}{2},\frac{1}{2}}(z) \quad \text{and} \quad h_{u-\frac{1}{2},\frac{1}{2}}(z), \tag{6.4.2}$$

where $0 \neq u \in (-1,1)$. Now we find $R[\mathscr{C}(\phi)]$ of the functions defined in (6.4.2). For simplicity, we write these as f_u, g_u and h_u, respectively and $\mathscr{L}_{u-\frac{1}{2},\frac{1}{2}} = \mathscr{L}_u$.

For the convenience of notations, functions defined in (6.4.2) are written as f_u, g_u and h_u, respectively.

Theorem 6.4.1. The $\mathscr{C}(\phi)$-radii for the functions f_u, g_u and h_u given by (6.4.2) are the smallest positive roots of the following equations respectively:

(*i*) $r f_u''(r) + \alpha f_u'(r) = 0;$

(*ii*) $r g_u''(r) + \alpha g_u'(r) = 0;$

(*iii*) $r h_u''(r) + \alpha h_u'(r) = 0,$

where α is the radius of the largest disk $\{w : |w-1| < \alpha\}$ inside $\phi(\mathbb{D})$ such that $\phi(-1) = 1 - \alpha$.

Proof. We begin with the first part. From (6.4.1), we have

$$1 + \frac{z f_u''(z)}{f_u'(z)} = 1 + \frac{z \mathscr{L}_u''(z)}{\mathscr{L}_u'(z)} + \left(\frac{1}{u+\frac{1}{2}} - 1\right) \frac{z \mathscr{L}_u'(z)}{\mathscr{L}_u(z)}. \tag{6.4.3}$$

Also using the result [28, Lemma 1], we have

$$\mathscr{L}_u(z) = \frac{z^{u+\frac{1}{2}}}{u(u+1)}\Phi_0(z) = \frac{z^{u+\frac{1}{2}}}{u(u+1)}\prod_{n\geq 1}\left(1-\frac{z^2}{\tau_{u,n}^2}\right),$$

where $\Phi_k(z) := {}_1F_2\left(1; \frac{u-k+2}{2}, \frac{u-k+3}{2}; -\frac{z^2}{4}\right)$ with conditions as mentioned in [28, Lemma 1], and from the proof of [28, Theorem 3], we see that the entire function $\frac{u(u+1)}{u+\frac{1}{2}} z^{-u+\frac{1}{2}}\mathscr{L}_u'(z)$ is of order $1/2$ and thus has the following Hadamard factorization:

$$\mathscr{L}_u'(z) = \frac{u+\frac{1}{2}}{u(u+1)} z^{u-\frac{1}{2}}\prod_{n\geq 1}\left(1-\frac{z^2}{\breve{\tau}_{u,n}^2}\right),$$

where $\tau_{u,n}$ and $\breve{\tau}_{u,n}$ are the n-th positive zeros of $\mathscr{L}_u$ and $\mathscr{L}_u'$, respectively and interlace for $0 \neq u \in (-1,1)$ (see [28, Theorem 1]). Now we can rewrite (6.4.3) as follows:

$$1+\frac{zf_u''(z)}{f_u'(z)} = 1-\left(\frac{1}{u+\dfrac{1}{2}}-1\right)\sum_{n\geq 1}\frac{2z^2}{\tau_{u,n}^2-z^2}-\sum_{n\geq 1}\frac{2z^2}{\breve{\tau}_{u,n}^2-z^2}.$$

Let us now consider the case $u \in (0,1/2]$. Then using the inequality $||x|-|y|| \leq |x-y|$ for $|z| = r < \breve{\tau}_{u,1} < \tau_{u,1}$ we get

$$\left|\frac{zf_u''(z)}{f_u'(z)}\right| \leq \left(\frac{1}{u+\dfrac{1}{2}}-1\right)\sum_{n\geq 1}\frac{2r^2}{\tau_{u,n}^2-r^2}+\sum_{n\geq 1}\frac{2r^2}{\breve{\tau}_{u,n}^2-r^2} = -\frac{rf_u''(r)}{f_u'(r)} \tag{6.4.4}$$

and for the case $u \in (1/2,1)$, using the inequality (6.2.6) with $\lambda = 1-1/(u+1/2)$, we also get

$$\left|\frac{zf_u''(z)}{f_u'(z)}\right| \leq -\frac{rf_u''(r)}{f_u'(r)}, \tag{6.4.5}$$

which is same as (6.4.4). When $u \in (-1,0)$, then we proceed similarly substituting u by $u-1$, Φ_0 by Φ_1, where Φ_1 belongs to the Laguerre-Pólya class $\mathscr{LP}$ and the n-th positive zeros $\xi_{u,n}$ and $\breve{\xi}_{u,n}$ of Φ_1 and its derivative Φ_1', respectively interlace. Finally, replacing u by $u+1$, we obtain the required inequality.

For $0 \neq u \in (-1,1)$, the Hadamard factorization for the entire functions g_u' and h_u' of order $1/2$ [28, Theorem 3] is given by

$$g_u'(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{\gamma_{u,n}^2}\right) \quad \text{and} \quad h_u'(z) = \prod_{n\geq 1}\left(1-\frac{z}{\delta_{u,n}^2}\right), \tag{6.4.6}$$

where $\gamma_{u,n}$ and $\delta_{u,n}$ are n-th positive zeros of g_u' and h_u', respectively and $\gamma_{u,1}, \delta_{u,1} < \tau_{u,1}$. Now from

(6.4.1) and (6.4.6), we have

$$1+\frac{zg_u''(z)}{g_u'(z)}=\frac{1}{2}-u+z\frac{\left(\frac{3}{2}-u\right)\mathscr{L}_u'(z)+z\mathscr{L}_u''(z)}{\left(\frac{1}{2}-u\right)\mathscr{L}_u(z)+z\mathscr{L}_u'(z)}=1-\sum_{n\geq 1}\frac{2z^2}{\gamma_{u,n}^2-z^2}$$

$$\begin{aligned}1+\frac{zh_u''(z)}{h_u'(z)}&=\frac{1}{2}\left(\frac{3}{2}-u+\sqrt{z}\frac{\left(\frac{5}{2}-u\right)\mathscr{L}_u'(\sqrt{z})+\sqrt{z}\mathscr{L}_u''(\sqrt{z})}{\left(\frac{3}{2}-u\right)\mathscr{L}_u(\sqrt{z})+\sqrt{z}\mathscr{L}_u'(\sqrt{z})}\right)\\&=1-\sum_{n\geq 1}\frac{z}{\delta_{u,n}^2-z}.\end{aligned}\tag{6.4.7}$$

Using the inequality $||x|-|y||\leq|x-y|$ in (6.4.7) for $|z|=r<\gamma_{u,1}$ and $|z|=r<\delta_{u,1}$, we get

$$\begin{aligned}\left|\frac{zg_u''(z)}{g_u'(z)}\right|&\leq\sum_{n\geq 1}\frac{2r^2}{\gamma_{u,n}^2-r^2}=-\frac{rg_u''(r)}{g_u'(r)}\\\left|\frac{zh_u''(z)}{h_u'(z)}\right|&\leq\sum_{n\geq 1}\frac{r}{\delta_{u,n}^2-r}=-\frac{rh_u''(r)}{h_u'(r)}.\end{aligned}\tag{6.4.8}$$

Now let α be the largest such that $\{w:|w-1|\leq\alpha\}\subseteq\phi(\mathbb{D})$. Then from (6.4.4), (6.4.5), (6.4.7) and (6.4.8), we see that f_u, g_u and h_u belong to $\mathscr{C}(1+\alpha z)\subseteq\mathscr{C}(\phi)$, whenever the following inequalities

$$-\frac{rf_u''(r)}{f_u'(r)}\leq\alpha,\quad -\frac{rg_u''(r)}{g_u'(r)}\leq\alpha\quad\text{and}\quad-\frac{rh_u''(r)}{h_u'(r)}\leq\alpha$$

hold. Further proceeding with the similar method as in Theorem 6.2.1, we obtain the desired equations. The sharpness of the radii follows with the suitable rotations of the functions f_u, g_u and h_u. □

Similarly, we also get the following result:

Theorem 6.4.2 (Lommel function $\mathscr{L}_{u,v}$)**.** Let $0\neq u\in(-1,1)$ and write $\mathscr{L}_{u-\frac{1}{2},\frac{1}{2}}(z)=:\mathscr{L}_u(z)$. Then the $\mathscr{S}^*(\phi)$-radii $r_\phi(f_u)$, $r_\phi(g_u)$ and $r_\phi(h_u)$ of the functions f_u, g_u and h_u given by (6.4.2) are the smallest positive root of the following equations, respectively:

$$(i)\ \begin{cases}2r\mathscr{L}_u'(r)-(2u+1)(1-\alpha)\mathscr{L}_u(r)=0, & \mathit{for}\quad u\in(-\frac{1}{2},1)\\2r\mathscr{L}_u'(r)-(2u+1)(1+\alpha)\mathscr{L}_u(r)=0, & \mathit{for}\quad u\in(-1,-\frac{1}{2});\end{cases}$$

(ii) $2r\mathscr{L}_u'(r)-(2u+1-2\alpha)\mathscr{L}_u(r)=0;$

(iii) $2\sqrt{r}\mathscr{L}_u'(\sqrt{r})-(2u+1-4\alpha)\mathscr{L}_u(\sqrt{r})=0,$

where α is the radius of the largest disk $\{w:|w-1|<\alpha\}$ inside $\phi(\mathbb{D})$ such that $\phi(-1)=1-\alpha$.

6.5 Starlikeness and Convexity of Struve functions

The Struve function $\mathbf{H}_\beta$ of first kind is a particular solution of the second-order inhomogeneous Bessel differential equation

$$z^2 w''(z) + zw'(z) + (z^2 - \beta^2)w(z) = \frac{4\left(\frac{z}{2}\right)^{\beta+1}}{\sqrt{\pi}\Gamma\left(\beta+\frac{1}{2}\right)}$$

and have the following form:

$$\mathbf{H}_\beta(z) := \frac{\left(\frac{z}{2}\right)^{\beta+1}}{\sqrt{\frac{\pi}{4}}\Gamma\left(\beta+\frac{1}{2}\right)}\,{}_1F_2\left(1;\frac{3}{2},\beta+\frac{3}{2};-\frac{z^2}{4}\right),$$

where $-\beta-\frac{3}{2} \notin \mathbb{N}$ and ${}_1F_2$ is a hypergeometric function. Since it is not normalized, so we consider the following three normalized functions involving $\mathbf{H}_\beta$:

$$\left.\begin{aligned} U_\beta(z) &= \left(\sqrt{\pi}2^\beta\left(\beta+\frac{3}{2}\right)\mathbf{H}_\beta(z)\right)^{\frac{1}{\beta+1}}, \\ V_\beta(z) &= \sqrt{\pi}2^\beta z^{-\beta}\Gamma\left(\beta+\frac{3}{2}\right)\mathbf{H}_\beta(z), \\ W_\beta(z) &= \sqrt{\pi}2^\beta z^{\frac{1-\beta}{2}}\Gamma\left(\beta+\frac{3}{2}\right)\mathbf{H}_\beta(\sqrt{z}). \end{aligned}\right\} \tag{6.5.1}$$

Moreover, for $|\beta| \leq 1/2$, $\mathbf{H}_\beta$ (see [24, Lemma 1]) and $\mathbf{H}'_\beta$ have the Hadamard factorizations [28, Theorem 4] given by

$$\mathbf{H}_\beta(z) = \frac{z^{\beta+1}}{\sqrt{\pi}2^\beta\Gamma(\beta+\frac{3}{2})}\prod_{n\geq 1}\left(1-\frac{z^2}{z_{\beta,n}^2}\right)$$

and

$$\mathbf{H}'_\beta(z) = \frac{(\beta+1)z^\beta}{\sqrt{\pi}2^\beta\Gamma(\beta+\frac{3}{2})}\prod_{n\geq 1}\left(1-\frac{z^2}{\breve{z}_{\beta,n}^2}\right) \tag{6.5.2}$$

where $z_{\beta,n}$ and $\breve{z}_{\beta,n}$ are the n-th positive zeros of $\mathbf{H}_\beta$ and $\mathbf{H}'_\beta$, respectively and interlace [28, Theorem 2]. Thus from (6.5.2) with logarithmic differentiation, we obtain respectively

$$\frac{z\mathbf{H}'_\beta(z)}{\mathbf{H}_\beta(z)} = (\beta+1) - \sum_{n\geq 1}\frac{2z^2}{z_{\beta,n}^2 - z^2}$$

and

$$1 + \frac{z\mathbf{H}''_\beta(z)}{\mathbf{H}'_\beta(z)} = (\beta+1) - \sum_{n\geq 1}\frac{2z^2}{\breve{z}_{\beta,n}^2 - z^2}. \tag{6.5.3}$$

Also for $|\beta| \leq 1/2$, the Hadamard factorization for the entire functions V'_β and W'_β of order $1/2$ [28, Theorem 4] is given by

$$V'_\beta(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{\eta^2_{\beta,n}}\right) \quad \text{and} \quad W'_\beta(z) = \prod_{n\geq 1}\left(1-\frac{z}{\sigma^2_{\beta,n}}\right), \tag{6.5.4}$$

where $\eta_{\beta,n}$ and $\sigma_{\beta,n}$ are n-th positive zeros of V'_β and W'_β, respectively. V'_β and W'_β belong to the Laguerre-Pólya class and zeros satisfy $\eta_{\beta,1}, \sigma_{\beta,1} < z_{\beta,1}$. Now proceeding as in Theorem 6.4.1 using (6.5.1), (6.5.2), (6.5.3) and (6.5.4), we obtain the following result:

Theorem 6.5.1 (Starlikeness of Struve function $\mathbf{H}_\beta$)**.** Let $|\beta| \leq 1/2$. Then the $\mathcal{S}^*(\phi)$-radii $r_\phi(U_\beta)$, $r_\phi(V_\beta)$ and $r_\phi(W_\beta)$ of the functions U_β, V_β and W_β as given by (6.5.1) are the smallest positive root of the following equations, respectively:

(i) $r\mathbf{H}'_\beta(r) - (1-\alpha)(\beta+1)\mathbf{H}_\beta(r) = 0;$

(ii) $r\mathbf{H}'_\beta(r) - ((1+\beta)-\alpha)\mathbf{H}'_\beta(r) = 0;$

(iii) $\sqrt{r}\mathbf{H}'_\beta(\sqrt{r}) - (1+\beta-2\alpha)\mathbf{H}_\beta(\sqrt{r}) = 0,$

where α is the radius of the largest disk $\{w : |w-1| < \alpha\}$ inside $\phi(\mathbb{D})$ such that $\phi(-1) = 1-\alpha$.

Theorem 6.5.2 (Convexity of Struve function)**.** Let $|\beta| \leq 1/2$. Then $\mathcal{C}(\phi)$-radii for the functions U_β, V_β and W_β given by (6.5.1) are the smallest positive roots of the following equations respectively:

(i) $rU''_\beta(r) + \alpha U'_\beta(r) = 0;$

(ii) $rV''_\beta(r) + \alpha V'_\beta(r) = 0;$

(iii) $rW''_\beta(r) + \alpha W'_\beta(r) = 0,$

where α is the radius of the largest disk $\{w : |w-1| < \alpha\}$ inside $\phi(\mathbb{D})$ such that $\phi(-1) = 1-\alpha$.

6.6 On Ramanujan type entire functions

Ismail and Zhang [68] defined the following entire function of growth order zero for $\beta > 0$, called Ramanujan type entire function

$$A_p^{(\beta)}(c,z) = \sum_{n\geq 0} \frac{(c;p)_n p^{\beta n^2}}{(p;p)_n} z^n,$$

where $\beta > 0$, $0 < p < 1$, $c \in \mathbb{C}$, $(c;p)_0 = 1$ and $(c;p)_k = \prod_{j=0}^{k-1}(1-cp^j)$ for $k \geq 1$, which is the generalization of both the Ramanujan entire function $A_p(z)$ and Stieltjes-Wigert polynomial $S_n(z;p)$ defined as (see [67, 135]):

$$A_p(-z) = A_p^{(1)}(0,z) = \sum_{n=0}^{\infty} \frac{p^{n^2}}{(p;p)_n} z^n$$

and

$$A_p^{(1/2)}(p^{-n},z) = \sum_{m=0}^{\infty} \frac{(p^{-n};p)_m p^{m^2/2}}{(p;p)_m} z^m = (p;p)_n S_n(zp^{(1/2)-n};p).$$

Since $A_p^{(\beta)}(c,z) \notin \mathscr{A}$, therefore consider the following three normalized functions in $\mathscr{A}$:

$$\left.\begin{aligned} f_{\beta,p,c}(z) &:= \left(z^\beta A_p^{(\beta)}(-c,-z^2)\right)^{1/\beta} \\ g_{\beta,p,c}(z) &:= zA_p^{(\beta)}(-c,-z^2) \\ h_{\beta,p,c}(z) &:= zA_p^{(\beta)}(-c,-z), \end{aligned}\right\} \tag{6.6.1}$$

where $\beta > 0$, $c \geq 0$ and $0 < p < 1$. From [45, Lemma 2.1, p. 4-5], we see that the function

$$z \to \Psi_{\beta,p,c}(z) := A_p^{(\beta)}(-c,-z^2)$$

has infinitely many zeros (all are positive) for $\beta > 0$, $c \geq 0$ and $0 < p < 1$. Let $\psi_{\beta,p,n}(c)$ be the n-th positive zero of $\Psi_{\beta,p,c}(z)$. Then it has the following Weierstrass decomposition:

$$\Psi_{\beta,p,c}(z) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\psi^2_{\beta,p,n}(c)}\right). \tag{6.6.2}$$

Moreover, the n-th positive zero $\Xi_{\beta,p,n}(c)$ of the derivative of the following function

$$\Phi_{\beta,p,c}(z) := z^\beta \Psi_{\beta,p,c}(z) \tag{6.6.3}$$

interlace with $\psi_{\beta,p,n}(c)$ and satisfy the relation

$$\Xi_{\beta,p,n}(c) < \psi_{\beta,p,n}(c) < \Xi_{\beta,p,n+1}(c) < \psi_{\beta,p,n+1}(c)$$

for $n \geq 1$. Now using (6.6.1) and (6.6.2), we have

$$\begin{aligned} \frac{zf'_{\beta,p,c}(z)}{f_{\beta,p,c}(z)} &= 1 + \frac{1}{\beta}\frac{z\Psi'_{\beta,p,c}(z)}{\Psi_{\beta,p,c}(z)} = 1 - \frac{1}{\beta}\sum_{n\geq 1}\frac{2z^2}{\psi^2_{\beta,p,n}(c) - z^2};\ (c > 0) \\ \frac{zg'_{\beta,p,c}(z)}{g_{\beta,p,c}(z)} &= 1 + \frac{z\Psi'_{\beta,p,c}(z)}{\Psi_{\beta,p,c}(z)} = 1 - \sum_{n\geq 1}\frac{2z^2}{\psi^2_{\beta,p,n}(c) - z^2}; \\ \frac{zh'_{\beta,p,c}(z)}{h_{\beta,p,c}(z)} &= 1 + \frac{1}{2}\frac{\sqrt{z}\Psi'_{\beta,p,c}(\sqrt{z})}{\Psi_{\beta,p,c}(\sqrt{z})} = 1 - \sum_{n\geq 1}\frac{z}{\psi^2_{\beta,p,n}(c) - z}, \end{aligned}$$

where $\beta > 0, c \geq 0$ and $0 < p < 1$. Also, using (6.6.3) and the infinite product representation of Φ' [45, p. 14-15, Also see Eq. 4.6], we have

$$\begin{aligned} 1 + \frac{zf''_{\beta,p,c}(z)}{f'_{\beta,p,c}(z)} &= 1 + \frac{z\Phi''_{\beta,p,c}(z)}{\Phi'_{\beta,p,c}(z)} + \left(\frac{1}{\beta} - 1\right)\frac{z\Phi'_{\beta,p,c}(z)}{\Phi_{\beta,p,c}(z)} \\ &= 1 - \sum_{n\geq 1}\frac{2z^2}{\Xi^2_{\beta,p,n}(c) - z^2} - \left(\frac{1}{\beta} - 1\right)\sum_{n\geq 1}\frac{2z^2}{\psi^2_{\beta,p,n}(c) - z^2}. \end{aligned}$$

As $(z\Psi_{\beta,p,c}(z))'$ and $h'_{\beta,p,c}(z)$ belong to $\mathscr{L}\mathscr{P}$. So suppose $\gamma_{\beta,p,n}(c)$ be the positive zeros of $g'_{\beta,p,c}(z)$ (growth order is same as $\Psi_{\beta,p,c}(z)$) and $\delta_{\beta,p,n}(c)$ be the positive zeros of $h'_{\beta,p,c}(z)$. Thus using their infinite product representations, we have

$$1+\frac{zg''_{\beta,p,c}(z)}{g'_{\beta,p,c}(z)}=1-\sum_{n\geq 1}\frac{2z^2}{\gamma^2_{\beta,p,n}(c)-z^2}$$
$$1+\frac{zh''_{\beta,p,c}(z)}{h'_{\beta,p,c}(z)}=1-\sum_{n\geq 1}\frac{z}{\delta^2_{\beta,p,n}(c)-z}.$$

Now proceeding similarly as done in the above sections, we obtain the following results:

Theorem 6.6.1. Let $\beta>0$, $c\geq 0$ and $0<p<1$. Then $\mathscr{S}^*(\phi)$-radii for the functions $f_{\beta,p,c}(z)$, $g_{\beta,p,c}(z)$ and $h_{\beta,p,c}(z)$ given by (6.6.1) are the smallest positive roots of the following equations respectively:

(i) $r\Psi'_{\beta,p,c}(r)+\beta\alpha\Psi_{\beta,p,c}(r)=0$;

(ii) $r\Psi'_{\beta,p,c}(r)+\alpha\Psi_{\beta,p,c}(r)=0$;

(iii) $\sqrt{r}\Psi'_{\beta,p,c}(\sqrt{r})+2\alpha\Psi_{\beta,p,c}(\sqrt{r})=0$,

where α is the radius of the largest disk $\{w:|w-1|<\alpha\}$ inside $\phi(\mathbb{D})$ such that $\phi(-1)=1-\alpha$.

Theorem 6.6.2. Let $\beta>0$, $c\geq 0$ and $0<p<1$. Then $\mathscr{C}(\phi)$-radii for the functions $f_{\beta,p,c}(z)$, $g_{\beta,p,c}(z)$ and $h_{\beta,p,c}(z)$ given by (6.6.1) are the smallest positive roots of the following equations respectively:

(i) $\dfrac{r\Phi''_{\beta,p,c}(r)}{\Phi'_{\beta,p,c}(r)}+\left(\dfrac{1}{\beta}-1\right)\dfrac{r\Phi'_{\beta,p,c}(r)}{\Phi_{\beta,p,c}(r)}+\alpha=0$;

(ii) $rg''_{\beta,p,c}(r)+\alpha g'_{\beta,p,c}(r)=0$;

(iii) $rh''_{\beta,p,c}(r)+\alpha h'_{\beta,p,c}(r)=0$,

where α is the radius of the largest disk $\{w:|w-1|<\alpha\}$ inside $\phi(\mathbb{D})$ such that $\phi(-1)=1-\alpha$.

6.7 Applications and Further Results

6.7.1 Applications to special cases

In the following result, we consider the Caratheódory functions ϕ associated with some well-known classes as well as some recently introduced in [72, 73, 108, 109, 123]:

Corollary 6.7.1. If α be the radius of the largest disk $\{w:|w-1|<\alpha\}$ inside $\phi(\mathbb{D})$, where

1. $\alpha=\min\left\{\left|1-\dfrac{1+D}{1+E}\right|,\left|1-\dfrac{1-D}{1-E}\right|\right\}=\dfrac{D-E}{1+|E|}$ when $\phi(z)=\dfrac{1+Dz}{1+Ez}$, where $-1\leq E<D\leq 1$;
2. $\alpha=\sqrt{2-2\sqrt{2}+\sqrt{-2+2\sqrt{2}}}$ when $\phi(z)=\sqrt{2}-(\sqrt{2}-1)\sqrt{\dfrac{1-z}{1+2(\sqrt{2}-1)z}}$;
3. $\alpha=\sqrt{2}-1$ when $\phi(z)=\sqrt{1+z}$;
4. $\alpha=1-1/e$ when $\phi(z)=e^z$;

5. $\alpha = 2-\sqrt{2}$ when $\phi(z) = z+\sqrt{1+z^2}$;
6. $\alpha = 1/e$ when $\phi(z) = 1+ze^z$;
7. $\alpha = \dfrac{e-1}{e+1}$ when $\phi(z) = \dfrac{2}{1+e^{-z}}$;
8. $\alpha = \sin 1$ when $\phi(z) = 1+\sin z$;
9. for the domains bounded by the conic sections

$$\Omega_\kappa := \{w = u+iv : u^2 > \kappa^2(u-1)^2+\kappa^2 v^2; \kappa \in [0,\infty)\},$$

we have

$$\alpha = \frac{1}{\kappa+1},$$

where the boundary curve of Ω_κ for fixed κ is represented by the imaginary axis ($\kappa = 0$), the right branch of a hyperbola ($0 < \kappa < 1$), a parabola ($\kappa = 1$) and an ellipse ($\kappa > 1$). The univalent Carathéodory functions mapping $\mathbb{D}$ onto Ω_κ is given by

$$\phi(z) := \phi_\kappa(z) = \begin{cases} \dfrac{1+z}{1-z} & \text{for} \quad \kappa = 0; \\ 1+\dfrac{2}{1-\kappa^2}\sinh^2(A(\kappa) arctanh\sqrt{z}) & \text{for} \quad \kappa \in (0,1); \\ 1+\dfrac{2}{\pi^2}\log^2\dfrac{1+\sqrt{z}}{1-\sqrt{z}} & \text{for} \quad \kappa = 1; \\ 1+\dfrac{2}{\kappa^2-1}\sin^2\left(\dfrac{\pi}{2K(t)}F\left(\dfrac{\sqrt{z}}{\sqrt{t}},t\right)\right) & \text{for} \quad \kappa > 1, \end{cases}$$

where $A(\kappa) = (2/\pi)\arccos(\kappa)$ and

$$F(w,t) = \int_0^w \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

is the Legender elliptic integral of the first kind, $K(t) = F(1,t)$ and $t \in (0,1)$ is choosen such that $\kappa = \cosh(\pi K'(t)/2K(t))$.

Then Theorems 6.2.1, 6.2.2, 6.2.3, 6.2.4, 6.3.1, 6.4.1, 6.5.2, 6.6.1 and 6.6.2 hold true for the above choices of ϕ respectively.

Here, in the above corollary for the Janowski functions at (i), we use its inverse representation $|(w-1)/(D-Ew)| < 1$ for the sharpness (also see [106]). Whereas for the Lemniscate of Bernoulli at (iii), we use the fact that if $|w-1| \leq \sqrt{2}-1$, then $|w+1| \leq \sqrt{2}+1$, which implies $|w^2-1| \leq 1$.

6.7.2 Radius of Strongly Starlikeness

To prove our next result, we need the following lemma:

Lemma 6.7.1. [50] If $|z| \leq r < 1$ and $|z_k| = R > r$, then we have

$$\left|\frac{z}{z-z_k}+\frac{r^2}{R^2-r^2}\right| \leq \frac{Rr}{R^2-r^2}.$$

Here $0<\varepsilon\leq 1$ in what follows. The class of strongly starlike functions is given by:

$$\mathcal{S}^*\left(\left(\frac{1+z}{1-z}\right)^{\varepsilon}\right):=\left\{f\in\mathcal{A}:\left|\arg\frac{zf'(z)}{f(z)}\right|<\varepsilon\right\}.$$

Theorem 6.7.2 (Wright functions)**.** Let $\kappa,\delta>0$. Then $\mathcal{S}^*\left(\left(\frac{1+z}{1-z}\right)^{\varepsilon}\right)$-radii for the functions $f_{\kappa,\delta}$, $g_{\kappa,\delta}$ and $h_{\kappa,\delta}$ are the unique positive roots of the following equations:

$$(i)\ \ \frac{2}{\delta}\textstyle\sum_{n\geq 1}\left(\dfrac{\zeta_{\kappa,\delta,n}^2 r^2}{\zeta_{\kappa,\delta,n}^4-r^4}+\sin\left(\dfrac{\pi\varepsilon}{2}\right)\dfrac{r^4}{\zeta_{\kappa,\delta,n}^4-r^4}\right)-\sin\left(\dfrac{\pi\varepsilon}{2}\right)=0;$$

$$(ii)\ \ 2\textstyle\sum_{n\geq 1}\left(\dfrac{\zeta_{\kappa,\delta,n}^2 r^2}{\zeta_{\kappa,\delta,n}^4-r^4}+\sin\left(\dfrac{\pi\varepsilon}{2}\right)\dfrac{r^4}{\zeta_{\kappa,\delta,n}^4-r^4}\right)-\sin\left(\dfrac{\pi\varepsilon}{2}\right)=0;$$

$$(iii)\ \ \textstyle\sum_{n\geq 1}\left(\dfrac{\zeta_{\kappa,\delta,n}^2 r}{\zeta_{\kappa,\delta,n}^4-r^2}+\sin\left(\dfrac{\pi\varepsilon}{2}\right)\dfrac{r^2}{\zeta_{\kappa,\delta,n}^4-r^2}\right)-\sin\left(\dfrac{\pi\varepsilon}{2}\right)=0$$

in $(0,\zeta_{\kappa,\delta,1})$, $(0,\zeta_{\kappa,\delta,1})$ and $(0,\zeta_{\kappa,\delta,1}^2)$ respectively.

Proof. We prove the first part and the rest all follow in a similar manner. From (6.2.4) and using Lemma 6.7.1, we see that

$$\frac{zf'_{\kappa,\delta}(z)}{f_{\kappa,\delta}(z)}=1+\frac{1}{\delta}\frac{zW'_{\kappa,\delta}(z)}{W_{\kappa,\delta}(z)}=1-\frac{1}{\delta}\sum_{n\geq 1}\frac{2z^2}{\zeta_{\kappa,\delta,n}^2-z^2},$$

which implies

$$\begin{aligned}\left|\frac{zf'_{\kappa,\delta}(z)}{f_{\kappa,\delta}(z)}-\left(1-\frac{1}{\delta}\sum_{n\geq 1}\frac{2r^4}{\zeta_{\kappa,\delta,n}^4-r^4}\right)\right|&\leq\frac{1}{\delta}\sum_{n\geq 1}\left|\frac{2z^2}{z^2-\zeta_{\kappa,\delta,n}^2}+\frac{2r^4}{\zeta_{\kappa,\delta,n}^4}-r^4\right|\\&\leq\frac{2}{\delta}\sum_{n\geq 1}\frac{\zeta_{\kappa,\delta,n}^2 r^2}{\zeta_{\kappa,\delta,n}^4-r^4}\end{aligned}\tag{6.7.1}$$

for $|z|\leq r<\zeta_{\kappa,\delta,1}$. Define

$$a:=\left(1-\frac{1}{\delta}\sum_{n\geq 1}\frac{2r^4}{\zeta_{\kappa,\delta,n}^4-r^4}\right)\quad\text{and}\quad R_a:=\frac{2}{\delta}\sum_{n\geq 1}\frac{\zeta_{\kappa,\delta,n}^2 r^2}{\zeta_{\kappa,\delta,n}^4-r^4}.$$

Now from Lemma [50, Lemma 3.1, p. 307], we see that the disk $|w-a|\leq R_a$ in (6.7.1) is contained in the sector $|\arg w|\leq\pi\varepsilon/2$, whenever

$$\frac{2}{\delta}\sum_{n\geq 1}\frac{\zeta_{\kappa,\delta,n}^2 r^2}{\zeta_{\kappa,\delta,n}^4-r^4}\leq\left(\left(1-\frac{1}{\delta}\sum_{n\geq 1}\frac{2r^4}{\zeta_{\kappa,\delta,n}^4-r^4}\right)\right)\sin\left(\frac{\pi\varepsilon}{2}\right)\tag{6.7.2}$$

holds. Let us now define

$$T(r):=\frac{2}{\delta}\sum_{n\geq 1}\left(\frac{\zeta_{\kappa,\delta,n}^2 r^2}{\zeta_{\kappa,\delta,n}^4-r^4}+\sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^4}{\zeta_{\kappa,\delta,n}^4-r^4}\right)-\sin\left(\frac{\pi\varepsilon}{2}\right).$$

Then a simple calculation shows that $T'(r) \geq 0$. Also $\lim_{r\to 0} T(r) < 0$ and $\lim_{r\to \zeta_{\kappa,\delta,1}} T(r) > 0$. Thus (6.7.2) holds in $|z| \leq r_0$, where r_0 is the unique positive root of $T(r) = 0$ in $(0, \zeta_{\kappa,\delta,1})$. This completes the proof. □

Reasoning along the same lines as of Theorem 6.7.2, the following results hold. So, the proofs are omitted here.

Theorem 6.7.3 (Lommel functions)**.** The $\mathcal{S}^*\left(\left(\frac{1+z}{1-z}\right)^{\varepsilon}\right)$-radii for the functions f_u, g_u and h_u are the unique positive roots of the following equations:

$$(i)\;\; \frac{2}{u+\frac{1}{2}}\sum_{n\geq 1}\left(\frac{\tau_{u,n}^2 r^2}{\tau_{u,n}^4 - r^4} + \sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^4}{\tau_{u,n}^4 - r^4}\right) - \sin\left(\frac{\pi\varepsilon}{2}\right) = 0;$$

$$(ii)\;\; 2\sum_{n\geq 1}\left(\frac{\tau_{u,n}^2 r^2}{\tau_{u,n}^4 - r^4} + \sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^4}{\tau_{u,n}^4 - r^4}\right) - \sin\left(\frac{\pi\varepsilon}{2}\right) = 0;$$

$$(iii)\;\; \sum_{n\geq 1}\left(\frac{\tau_{u,n}^2 r}{\tau_{u,n}^4 - r^2} + \sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^2}{\tau_{u,n}^4 - r^2}\right) - \sin\left(\frac{\pi\varepsilon}{2}\right) = 0$$

in $(0, \tau_{u,1})$, $(0, \tau_{u,1})$ and $(0, \tau_{u,1}^2)$, respectively.

Theorem 6.7.4 (Struve functions)**.** Let $|\beta| \leq 1/2$. Then $\mathcal{S}^*\left(\left(\frac{1+z}{1-z}\right)^{\varepsilon}\right)$-radii for the functions U_β, V_β and W_β are the unique positive roots of the following equations:

$$(i)\;\; \frac{2}{\beta+1}\sum_{n\geq 1}\left(\frac{z_{\beta,n}^2 r^2}{z_{\beta,n}^4 - r^4} + \sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^4}{z_{\beta,n}^4 - r^4}\right) - \sin\left(\frac{\pi\varepsilon}{2}\right) = 0;$$

$$(ii)\;\; 2\sum_{n\geq 1}\left(\frac{z_{\beta,n}^2 r^2}{z_{\beta,n}^4 - r^4} + \sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^4}{z_{\beta,n}^4 - r^4}\right) - \sin\left(\frac{\pi\varepsilon}{2}\right) = 0;$$

$$(iii)\;\; \sum_{n\geq 1}\left(\frac{z_{\beta,n}^2 r}{z_{\beta,n}^4 - r^2} + \sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^2}{z_{\beta,n}^4 - r^2}\right) - \sin\left(\frac{\pi\varepsilon}{2}\right) = 0$$

in $(0, z_{\beta,1})$, $(0, z_{\beta,1})$ and $(0, z_{\beta,1}^2)$, respectively.

Theorem 6.7.5 (Mittag-Leffler functions)**.** Let $(\frac{1}{\mu}, \nu) \in W_i$ and $a > 0$. Then $\mathcal{S}^*\left(\left(\frac{1+z}{1-z}\right)^{\varepsilon}\right)$-radii for the functions $f_{\mu,\nu,a}$, $g_{\mu,\nu,a}$ and $h_{\mu,\nu,a}$ are the unique positive roots of the following equations:

$$(i)\;\; \frac{2}{\nu}\sum_{n\geq 1}\left(\frac{\lambda_{\mu,\nu,a,n}^2 r^2}{\lambda_{\mu,\nu,a,n}^4 - r^4} + \sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^4}{\lambda_{\mu,\nu,a,n}^4 - r^4}\right) - \sin\left(\frac{\pi\varepsilon}{2}\right) = 0;$$

$$(ii)\;\; 2\sum_{n\geq 1}\left(\frac{\lambda_{\mu,\nu,a,n}^2 r^2}{\lambda_{\mu,\nu,a,n}^4 - r^4} + \sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^4}{\lambda_{\mu,\nu,a,n}^4 - r^4}\right) - \sin\left(\frac{\pi\varepsilon}{2}\right) = 0;$$

$$(iii)\;\; \sum_{n\geq 1}\left(\frac{\lambda_{\mu,\nu,a,n}^2 r}{\lambda_{\mu,\nu,a,n}^4 - r^2} + \sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^2}{\lambda_{\mu,\nu,a,n}^4 - r^2}\right) - \sin\left(\frac{\pi\varepsilon}{2}\right) = 0$$

in $(0, \lambda_{\mu,\nu,a,1})$, $(0, \lambda_{\mu,\nu,a,1})$ and $(0, \lambda_{\mu,\nu,a,1}^2)$, respectively.

Theorem 6.7.6 (Ramanujan type entire functions)**.** Let $\beta > 0$, $c \geq 0$ and $0 < p < 1$. Then $\mathscr{S}^*\left(\left(\frac{1+z}{1-z}\right)^{\varepsilon}\right)$-radii for the functions $f_{\beta,p,c}(z)$, $g_{\beta,p,c}(z)$ and $h_{\beta,p,c}(z)$ are the unique positive roots of the following equations:

$$(i)\ \ \frac{2}{\beta}\sum_{n\geq 1}\left(\frac{\psi^2_{\beta,p,n}(c)r^2}{\psi^4_{\beta,p,n}(c)-r^4}+\sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^4}{\psi^4_{\beta,p,n}(c)-r^4}\right)-\sin\left(\frac{\pi\varepsilon}{2}\right)=0;$$

$$(ii)\ \ 2\sum_{n\geq 1}\left(\frac{\psi^2_{\beta,p,n}(c)r^2}{\psi^4_{\beta,p,n}(c)-r^4}+\sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^4}{\psi^4_{\beta,p,n}(c)-r^4}\right)-\sin\left(\frac{\pi\varepsilon}{2}\right)=0;$$

$$(iii)\ \ \sum_{n\geq 1}\left(\frac{\psi^2_{\beta,p,n}(c)r}{\psi^4_{\beta,p,n}(c)-r^2}+\sin\left(\frac{\pi\varepsilon}{2}\right)\frac{r^2}{\psi^4_{\beta,p,n}(c)-r^2}\right)-\sin\left(\frac{\pi\varepsilon}{2}\right)=0$$

in $(0,\psi_{\beta,p,1}(c))$, $(0,\psi_{\beta,p,1}(c))$ and $(0,\psi^2_{\beta,p,1}(c))$, respectively.

Highlights of the chapter

We answered the Problem 6.1.1. Further, we observe that using our technique, $\mathscr{S}^*(\phi)$ and $\mathscr{C}(\phi)$-radii for the normalized forms of q-Bessel [20] and q-Mittag-Leffler [167] functions can be achieved. For the q-forms of the other special functions, these radii problems are still open. Also note that if we consider the series representation of the normalized special functions, then combining the technique used in this chapter with the methodology used in [19, 23], we can get the explicit conditions on the relevant parameters for the $\mathscr{S}^*(\phi)$ and $\mathscr{C}(\phi)$-radii, which we shall study separately.

Conclusion and Future Scope

The investigation of properties of starlike and convex functions emerged soon after the Bieberbach conjecture on univalent functions. Goodman [59] and Ronning [143] started investigating the properties like uniform starlikeness and uniformly convexity. Then the paper by Sokół [158] and, Kanas and Wiśniowska [73] gave insights into subclasses of starlike and convex functions associated with the Lemniscate of Bernoulli and Conic domains, respectively. But in 1992, a unified treatment of subclasses of starlike and convex functions was given by Ma and Minda [102]. Since then several fascinating articles in view of the Ma-Minda class, appeared concerning several type of radius and coefficient problems. The main obective of the present study is to solve classical as well as problems of current interest for the general Ma-Minda classes.

In chapter 2, we study a Ma-Minda class of starlike functions associated with the cardioid domain given by $\mathcal{S}^*_{\wp} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + ze^z =: \wp(z) \right\}$, where $\wp$ maps the unit disk onto a cardioid domain. Since the properties of such functions depend on the geometry of the domain and it's explicit mathematical formula. Therefore, we considered cardioid domain $\wp(\mathbb{D})$, which is starlike in shape having a cusp, and the function $\wp$ is transcendental and doesn't have inverse representation. We find the radius of convexity of $\wp(z)$ and establish the inclusion relations between the class $\mathcal{S}^*_{\wp}$ and some well-known classes. Further, we derive sharp radius constants and coefficient related results for the class $\mathcal{S}^*_{\wp}$. As a future scope, open problem concerning the sharp coefficient bounds and a conjecture on third hankel determinant are posed.

In chapter 3, we introduced a new class of analytic functions, and initiated a systematic study of non-univalent functions given by

$$\mathcal{F}(\psi) := \left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \psi(z),\ \psi(0) = 0 \right\},$$

where ψ is univalent. Thus, we started developing a parallel theory reminiscent of Ma-Minda classes. We here mainly derived growth and distortion theorems, and obtained the Koebe domain for such functions. Further, we also analyzed the geometry of the image of the functions $f(z)/z$. The problems related to the coefficient and subordinations are left for future attempt.

In chapter 4, motivated by the work of MacGregor, Campbell and Szegö, we consider several classical problems which have been of great interest in GFT. We prove several sharp majorization theorems, a product theorem, convolution conditions for necessary and sufficient conditions (also for the class $\mathscr{F}(\psi)$) and some radius problems related to the Ma-Minda classes. The application of majorzation can be studied in exploring the radius problems in the theory of harmonic univalent functions. In fact, exploring the concept of majorization in higher dimensions is interesting for future research. In 1961, Goluzin [61] obtained the set of extremal functions $f(z) = z/(1-xz)^2$, $|x| = 1$ for the problem of maximization of the quantity $\Re\Phi(\log(f(z)/z))$ or $|\Phi(\log(f(z)/z))|$ over the class $\mathscr{S}^*$, where Φ is a non-constant entire function. In 1973, MacGregor [105] proved the result for the class $\mathscr{S}^*(\alpha) := \{f \in \mathscr{A} : \Re(zf'(z)/f(z)) > \alpha, \alpha \in [0,1)\}$. Later on Barnard [30] discussed this for Bounded starlike functions. We establish the same for the Ma-Minda classes. Finally, we establish the sharp Bohr phenomenon for the Ma-Minda class of starlike functions and associated subordinating families.

In chapter 5, we continue to explore the idea of Bohr phenomenon along with the Bohr-Rogosinski phenomenon. We establish several fundamental results related to power series to dervie the Bohr-Rogosinski radius for the Ma-Minda classes and associated subordinating families. We further investigate the phenomenon for some classes of harmonic functions, which are quite natural generalization of certain subclasses of univalent functions. We introduce a class of harmonic functions with some conditions on their coefficients to study its application to both the classes of harmonic as well as classes of univalent functions. We also study a generalization of the Bohr phenomenon with the approach of sequence of non-negative functions for the Ma-Minda classes. In future, the ideas developed in this chapter can be used to study the Bohr phenomenon for the classes of harmonic univalent functions whose sharp coefficient bounds are yet to be known.

The interaction of special functions and their geometrical properties is evident in geometric function theory. But special attention has been given to them, particularly, under what conditions special functions are starlike and convex or what is the largest radius for which they will be starlike or convex or uniform convex. In chapter 6, we mainly study certain known normalized special functions, namely Wright and Mittag-Leffler functions, Legendre polynomials, Lommel and Struve functions, and Ramanujan type entire functions for their Ma-Minda starlikeness and convexity. We aim to generalize the known theory and provide simple proof with some simple geometric conditions, which are easy to apply. The ideas developed in this chapter can be used to derived Ma-Minda radii of starlikeness and convexity for some other form of special functions as a future scope.

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