RADIUS ESTIMATES AND CERTAIN DIFFERENTIAL SUBORDINATIONS FOR ANALYTIC FUNCTIONS

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in

MATHEMATICS

by

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under the supervision of

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DECLARATION

I declare that the research work reported in this thesis titled "**Radius Estimates and certain differential subordinations for analytic functions**" for the award of the degree of *Doctor of Philosophy in Mathematics* has been carried out by me under the supervision of *Prof. S. Sivaprasad Kumar*, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

Unless otherwise indicated, this thesis represents my original research work. This thesis has not been submitted by me earlier in part or full to any other University or Institute for the award of any degree or diploma. It is not intended to include other people's data, graphs, or other information, unless explicitly acknowledged.

Date :

Place : Delhi, India **(Priyanka Goel)**

<u>CERTIFICATE</u>

On the basis of the declaration submitted by **Ms. Priyanka Goel**, I hereby certify that the thesis titled **"Radius Estimates and certain differential subordinations for analytic functions"** submitted to the Department of Applied Mathematics, Delhi Technological University for the award of the degree of *Doctor of Philosophy in Mathematics*, is a record of bonafide research work carried out by her under my supervision.

To the best of my knowledge, the work reported in this thesis is original and has not been submitted to any other institution or university, in any form, for the award of any degree or diploma.

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Date :

Place : Delhi, India. **(PRIYANKA GOEL)**

This thesis is dedicated to my parents and my husband.

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Contents

Preface

Univalent function theory is a fascinating topic of complex analysis and deals with the geometrical aspects of analytic functions due to which it is classified under geometric function theory. Furthermore, the theory of differential subordination plays a crucial role in Univalent function theory as it is used as a tool for establishing various implication results. This thesis chiefly focuses on establishing results pertaining to radius estimates and differential subordination implications for certain classes of analytic functions, which are either well known or introduced and studied here. It comprises of six chapters and concluded with the future scope. The chapter wise arrangement of the content is as follows:

Chapter 1 provides a brief review of the general principles of Univalent function theory. It provides an overview of the relevant literature and mentions some of the remarkable works done by various authors. The concepts and the techniques which serve as prerequisite for the main study are also discussed in this chapter.

Chapter 2 introduces a new subclass of $\mathcal{S}^*$ associated with the modified sigmoid function, given by $\mathcal{S}^*_{SG} = \{f \in \mathcal{A} : zf'(z)/f(z) \prec 2/(1+e^{-z})\}$. To describe the general behavior of the functions in this class, we study its geometrical properties and use these properties to prove certain results. Furthermore, we discuss admissibility conditions for the modified sigmoid function. The first and second order admissibility conditions are simply obtained by the general admissibility criteria given by Miller and Mocanu [61] and for the third order conditions, we derive a new criteria by modifying the previously existing theory.

Chapter 3 presents differential subordination implications, involving the modified sigmoid function and other well known Ma-Minda functions, proved by using three different techniques. Using Miller Mocanu lemma, we prove several first order differential subordination results involving real parameters. Later by using the method of contradiction, we extend these results for complex parameters. Finally, using the idea of admissible functions, we underwent tedious computations to prove differential subordination results upto third order, which has not been done before in the literature.

In **Chapter 4**, we employ some remote properties of Schwarz function in order to find radius estimates for three classes, namely, $G_{\frac{1}{2},\frac{1}{2}}$-the Silverman class, $\Omega = \{f \in \mathcal{A} : |zf'(z) - f(z)| < 1/2,\ z \in \mathbb{D}\}$ and $\mathcal{S}^*_{SG}$-the class of Sigmoid starlike functions. In addition, we prove sufficient conditions in the form of differential inequalities for a general form of the Silverman class. Finally, using the concept of subordination, we develop a number of inclusion relations for the general form of Silverman class and the class Ω, involving various subclasses of starlike functions.

Chapter 5 deals with differential subordination results involving Pythagorean means. In the first part we prove an extremely general result involving the convex weighted harmonic mean of $p(z)$ and $p(z)\Theta(z) + zp'(z)\Phi(z)$, where Θ, Φ are analytic functions. Furthermore, we discuss some special cases of this result. In the next part, a combination of arithmetic, geometric and harmonic mean of $p(z)$ and quantities involving its derivatives has been taken into consideration. We prove certain implications for this combination and use them further for proving starlikeness and univalence criteria. Establishments in this chapter generalize many earlier known results.

In **Chapter 6**, we introduce the following special type of differential subordination implication:

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z) + \alpha} \prec h(z) \quad \Rightarrow p(z) \prec h(z), \tag{0.0.1}$$

where p, Q are analytic with $p(0) = 1$ and $0 \neq \beta, \alpha \in \mathbb{C}$. The differential subordination given above is a general form of Briot-Bouquet differential subordination. As a consequence, we discuss some of the special cases of the aforementioned result. Moreover, we prove some results which are analogous to open door lemma and integral existence theorem. As an application, the outcomes of this chapter have been used to obtain criteria for univalence and starlikeness.

The bibliography and list of publications have been given at the end of the thesis.

List of Symbols

Symbols	Meanings
$\mathbb{C}$	complex plane
$\mathbb{D}$	open unit disk $\{z : |z| < 1\}$
$\mathcal{H}[a,n]$	class of analytic functions in $\mathbb{D}$ of the form $f(z) = a + \sum_{k=n}^{\infty} a_k z^k$
$\mathcal{A}$	subclass of $\mathcal{H}[0,1]$ of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$
$\mathcal{S}$	subclass of $\mathcal{A}$ consisting of univalent functions
$\mathcal{S}^*$	subclass of $\mathcal{S}$ of starlike functions
$\mathcal{C}$	subclass of $\mathcal{S}$ of convex functions
$\mathcal{K}$	class of close-to-convex functions
$\mathcal{P}$	class of functions p, having positive real part in $\mathbb{D}$ and $p(0) = 1$
$\mathscr{K}(z)$	Koebe function
$\omega(z)$	Schwarz function, which is analytic and satisfies $\omega(0) = 0$ and $|\omega(z)| \leq |z|$.
$\varphi(z)$	Ma-Minda function
Π_M	class of Ma-Minda functions
$\prec$	subordination
$\mathcal{S}^*(\varphi)$	class of starlike functions associated with Ma-Minda function
$\mathcal{C}(\varphi)$	class of convex functions associated with Ma-Minda function
$\mathbb{R}$	set of real numbers
$\mathbb{N}$	set of natural numbers

Symbols	Meanings
$\mathcal{S}^*[A,B]$	class of analytic functions satisfying $zf'/f \prec (1+Az)/(1+Bz)$
$\mathcal{S}^*(\alpha)$	class of starlike functions of order α $(0 \le \alpha < 1)$
$\mathcal{C}(\alpha)$	class of convex functions of order α $(0 \le \alpha < 1)$
$\mathcal{M}(\kappa)$	class of analytic functions satisfying $\mathrm{Re}(zf'(z)/f(z)) < \kappa$ $(\kappa > 1)$
$\mathcal{S}^*_L$	class of analytic functions satisfying $zf'/f \prec \sqrt{1+z}$
$\mathcal{S}^*_e$	class of analytic functions satisfying $zf'/f \prec e^z$
$\mathcal{SS}^*(\alpha)$	class of strongly starlike functions of order α
$\mathcal{S}_P$	class of parabolic starlike functions
$k-\mathcal{ST}$	class of k- starlike functions
$\mathcal{S}^*_{SG}$	class of analytic functions satisfying $zf'/f \prec 2/(1+e^{-z})$
Ω	class of normalized analytic functions satisfying $\lvert zf'(z)-f(z)\rvert < 1/2$

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Chapter 1

Introduction

This chapter presents the basic theory of univalent functions, comprising of definitions and results which serve as prerequisites. It covers the elementary parts of univalent function theory along with the past as well as the recent developments done by various authors in the same field, in order of their occurrence. It provides a review of some general principles, underlying the two major concepts of univalent functions and differential subordination. Some of the definitions and theorems are supported by examples and figures.

The study of univalent functions initially emerged in the early 20th century. In 1907, a result of Koebe [43] set out the groundwork for the theory of univalent functions. Further, it led to the development of a conjecture by Bieberbach [10], that stood as a challenge for many years and was finally proved by De Branges [11], using special functions, in the year 1985. In the meantime, several attempts were made to prove this conjecture, which eventually led to the development of many important subclasses of univalent functions and also resulted in the emergence of various new techniques in geometric function theory such as convolution and differential subordination. Now, a good amount of literature exists, which can be found in the text books by Duren [21], Goluzin [26], Graham and Kohr [28], Hayman [31] and Pommerenke [74]. The book by Goodman [27] illustrates a large number of contributions to the topic and comes close to serving as a compendium on univalent functions. A few additional books are by Jenkins [33], Milin [59], Montel [66] and Hallenbeck and MacGregor [30]. *"Univalent functions: The Primer"* by Thomas et. al [94] is a recent book on the subject.

1.1 Basic Definitions

Let $\mathcal{H} = \mathcal{H}(\mathbb{D})$ be the class of functions, analytic in $\mathbb{D} := \mathbb{D}_1$, where $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] := \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots\}.$$

We denote by $\mathcal{A}_n$, the class of functions f of the form

$$f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots,$$

which are analytic in $\mathbb{D}$ and $\mathcal{A} := \mathcal{A}_1$. A function f is said to be univalent in a domain $\mathcal{D} \subset \mathbb{C}$ if it is one-one in $\mathcal{D}$ and a function f is locally univalent at a point z_0 if it is univalent in some neighbourhood of z_0. A necessary and sufficient condition for local univalence is that the derivative of the function never vanishes at the point of local univalence. If a function is univalent in a domain $\mathcal{D}$, its derivative never becomes 0 on $\mathcal{D}$, but the converse may not be true. For instance, we have e^z whose derivative is non-zero on the entire complex plane but it is not univalent in any disk with radius greater than π, which is centered at 0.

Let $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ consisting of univalent functions. In other words,

$\mathcal{S}$ is defined as the class of all functions, normalized with the conditions $f(0) = 0$ and $f'(0) = 1$, which are analytic and univalent in $\mathbb{D}$. Throughout the study, we restrict our domain to be $\mathbb{D}$ and it is justified by the Riemann Mapping Theorem, which ensures the existence of an analytic, univalent function that maps any simply connected domain $\mathcal{D}$ onto $\mathbb{D}$. In addition, the properties of any analytic and univalent function g corresponds with the function $f(z) = (g(z) - g(0))/g'(0)$ and therefore we only consider the functions, normalized by the above conditions. The following example gives a geometrical interpretation of univalent functions.

***Example* 1.** The function $f_1(z) = z + z^2/2$ is a member of $\mathcal{S}$ whereas the function $f_2(z) = z + z^2$ is in $\mathcal{A}$ but not in $\mathcal{S}$ since it is non-univalent. It can be clearly depicted from Figure 1.1 that $f_2(z)$ is non-univalent as some portion of $f_2(\mathbb{D})$ is overlapping.

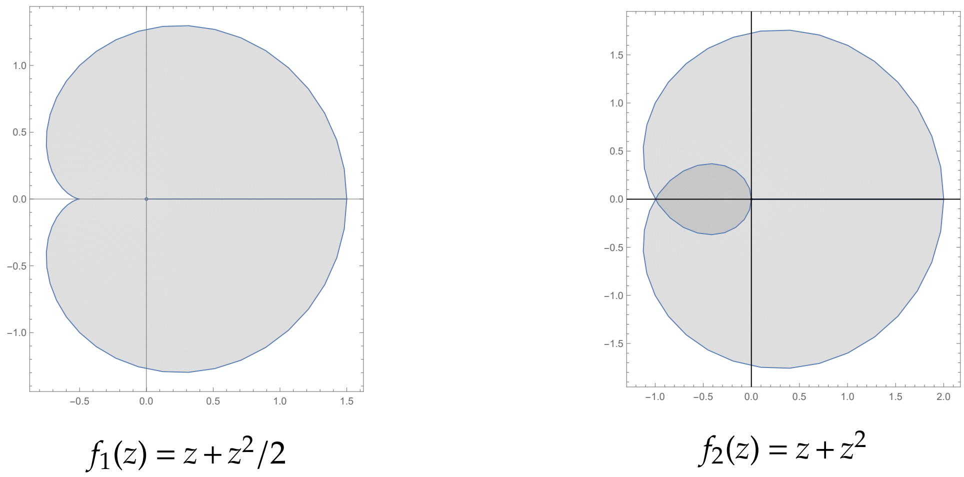


Figure 1.1: Image domain of a univalent and a non univalent function

One of the very important member of class $\mathcal{S}$ is the Koebe function given by $\mathcal{K}(z) = z/(1-z)^2$. In 1907, Koebe [43] proved that every member of $\mathcal{S}$ contains a disk of radius k. In 1916, Bieberbach obtained the value of k as 1/4 and using this result the author proved that for every member f of $\mathcal{S}$, $|a_2| \leq 2$. Based on this result, Bieberbach proposed the following conjecture:

Conjecture 1.1.1 (Bieberbach). [10] Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be a univalent function, then $|a_n| \leq n$, $n = 2, 3, 4, \ldots$. Equality occurs if and only if f is a rotation of the Koebe function.

The extremity of the Koebe function is not only restricted to the coefficient inequality mentioned above but it also acts as an extremal function in geometrical sense. It maps

$\mathbb{D}$ onto the complex plane $\mathbb{C}$, leaving a slit from $-\infty$ to $-1/4$ on the negative side of real axis and this is the largest image domain of all the members of $\mathcal{S}$.

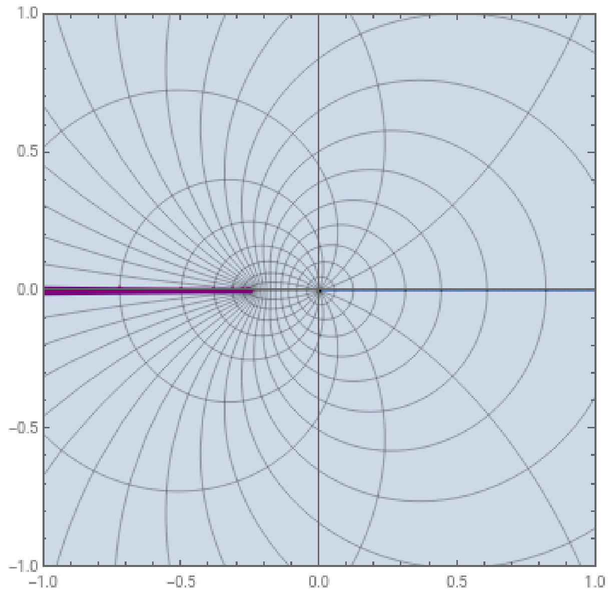


Figure 1.2: Image domain of Koebe function

Subordination: Let f and F be members of $\mathcal{H}$, we say that f is subordinate to F, written as $f \prec F$, if there exists a Schwarz function ω which is analytic in $\mathbb{D}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = F(\omega(z))$. Further, if F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$. Here f is called the subordinate function and F is called the superordinate function. Note that the univalence of superordinate function is not necessary for subordination. We discuss below some properties of subordination:

- Scalar addition or scalar multiplication does not alter the property of subordination. In other words, if $f \prec F$, then $\alpha + \beta f \prec \alpha + \beta F$ for $0 \neq \beta, \alpha \in \mathbb{R}$.

- If $f \prec F$ and $g : f(\mathbb{D}) \to \mathbb{C}$ is an analytic function, then $g \circ f \prec g \circ F$. It directly follows from the definition of subordination which says that there exists a Schwarz function ω such that $f(\mathbb{D}) = F(\omega(\mathbb{D}))$, which is sufficient to conclude that $g(f(\mathbb{D})) = g(F(\omega(\mathbb{D})))$.

- *Lindelöf Principle*: If $f \prec F$, then $|f'(0)| \leq |F'(0)|$ and the image under F of each disk $|z| \leq r < 1$ contains the image under f of the same disk.

- $\max_{z\in\mathbb{D}} |f(z)| \leq \max_{z\in\mathbb{D}} |F(z)|$.

- $\min_{z\in\mathbb{D}} \operatorname{Re} F(z) \leq \operatorname{Re} f(z) \leq \max_{z\in\mathbb{D}} \operatorname{Re} F(z)$.

Moreover, we have Rogosinski's Theorem which gives a coefficient inequality as a consequence of subordination.

Theorem 1.1.2. [21] Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $F(z) = \sum_{n=1}^{\infty} b_n z^n$ be analytic in $\mathbb{D}$ and $f \prec F$, then

$$\sum_{k=1}^{n} |a_k|^2 \le \sum_{k=1}^{n} |b_k|^2, \quad n = 1, 2, \ldots.$$

We end up this section with the following example:

***Example* 2.** Let $f(z) = z/(2-z)$ and $F(z) = z/(1-z)$. By taking $\omega(z) = z/2$, we can write $f(z) = F(\omega(z))$, which implies that $f \prec F$. Also, we may note that F is univalent and so we can apply the other approach. It is clear that $f(0) = F(0) = 0$ and Figure 1.3 shows that $f(\mathbb{D}) \subset F(\mathbb{D})$, which is sufficient to conclude that $f \prec F$.

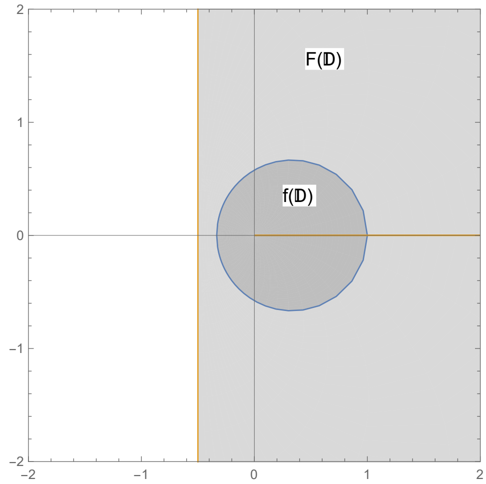


Figure 1.3: Geometrical Interpretation of subordination

1.2 Subclasses of Analytic functions

Carathéodory Class: An analytic function p defined on $\mathbb{D}$, which is of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{1.2.1}$$

such that $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{D})$, is known as the Carathéodory function and the class of all such functions is known as the Carathéodory class, denoted by $\mathcal{P}$. The Möbius function

$$P(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^2 + \cdots$$

plays a central role in the class $\mathcal{P}$. This function maps $\mathbb{D}$ onto the right half plane and maximizes the bound of the coefficient inequalities given as follows:

Lemma 1.2.1 (Carathéodory Lemma). [74] If $p(z)$ is in $\mathcal{P}$ and is given by (1.2.1), then $|c_n| \leq 2$ for each n. Equality occurs when p is a rotation of $P(z)$.

However, both the functions $P(z)$ and the Koebe function are extremal functions of their respective classes but there is a difference in their characters. Upto rotation, Koebe function is the unique solution in many extremal problems whereas infinite number of functions exist in $\mathcal{P}$ for which $|c_n| = 2$ and none of these can be obtained by rotation of $P(z)$. Note that the class $\mathcal{P}$ is convex. It means that if $f_1(z)$, $f_2(z), \ldots, f_n(z)$ are all members of $\mathcal{P}$ and $t_k \geq 0$ $(k = 1, 2, ..n)$ such that $t_1 + t_2 + \cdots + t_n = 1$, then the function

$$f(z) = \sum_{k=1}^{n} t_k f_k(z)$$

is also a member of $\mathcal{P}$. This can also be taken to an infinite sum. Another property of $\mathcal{P}$ says that if $f \in \mathcal{P}$, then $1/f$ is also in $\mathcal{P}$. A general form of class $\mathcal{P}$ has been defined in the following manner: For $-1 \leq B < A \leq 1$, let

$$\mathcal{P}_n[A,B] := \left\{ p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots : p(z) \prec \frac{1+Az}{1+Bz} \right\}.$$

Taking $A = 1 - 2\alpha$ and $B = -1$, the above class reduces to $\mathcal{P}_n(\alpha)$ which can be further reduced to the Carathéodory class $\mathcal{P}$ by taking $\alpha = 0$ and $n = 1$. Although this class has numerous other noteworthy properties yet we present below, a vital outcome which has been oftentimes used to help our primary outcomes.

Lemma 1.2.2. [78] If $p \in \mathcal{P}_n[A,B]$, then for $|z| = r$,

$$\left| p(z) - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}} \right| \leq \frac{(A-B) r^n}{1 - B^2 r^{2n}}.$$

In particular, if $p \in \mathcal{P}_n(\alpha)$, then

$$\left| p(z) - \frac{1 + (1-2\alpha) r^{2n}}{1 - r^{2n}} \right| \leq \frac{2(1-\alpha) r^n}{1 - r^{2n}}.$$

Starlike Functions: A domain $\mathcal{D} \subset \mathbb{C}$ is said to be starlike with respect to a point a_0 in $\mathcal{D}$ if the line segment joining a_0 and any other point of $\mathcal{D}$ lies entirely in $\mathcal{D}$. Further, we call a function f starlike if it maps $\mathbb{D}$ onto a domain which is starlike with respect to the origin. Analytically, we say that a function $f \in \mathcal{A}$ is starlike if and only if $\operatorname{Re}(zf'(z)/f(z)) > 0$ on $\mathbb{D}$. Here, it is assumed that f is univalent because if we consider

f to be starlike and non-univalent, then $f(\mathbb{D})$ must be starlike in multi-sheeted region but presently we are considering only plane regions for starlikeness property. This makes the class of starlike functions, a subclass of $\mathcal{S}$ and we represent it as follows:

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \right\}.$$

Next we see a general form of the above class. A function $f \in \mathcal{A}$ is said to be starlike of order α, $0 \le \alpha < 1$ if and only if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$ on $\mathbb{D}$ and the class of all such functions is denoted by $\mathcal{S}^*(\alpha)$. Note that $\mathcal{S}^*(0) = \mathcal{S}^*$.

Convex Functions: A domain $\mathcal{D} \subset \mathbb{C}$ is said to be convex if the line segment joining any two points of $\mathcal{D}$ lies completely inside $\mathcal{D}$. Further, a function f is called convex if it maps $\mathbb{D}$ onto a convex domain. Note that a convex domain is starlike with respect to each of its points and thus every convex function is starlike. The analytical characterization of a convex function is given by $\operatorname{Re}(1+zf''(z)/f'(z)) > 0$ on $\mathbb{D}$. Clearly, the class of convex functions is contained in $\mathcal{S}$ and is represented as follows:

$$\mathcal{C} := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}.$$

Moreover, a function $f \in \mathcal{A}$ is said to be convex of order $\alpha, 0 \le \alpha < 1$ if and only if $\operatorname{Re}(1+zf''(z)/f'(z)) > \alpha$ on $\mathbb{D}$. The class of all such functions is denoted by $\mathcal{C}(\alpha)$ and $\mathcal{C}(0) = \mathcal{C}$. A comparison between a starlike function and a convex function is illustrated by the following example.

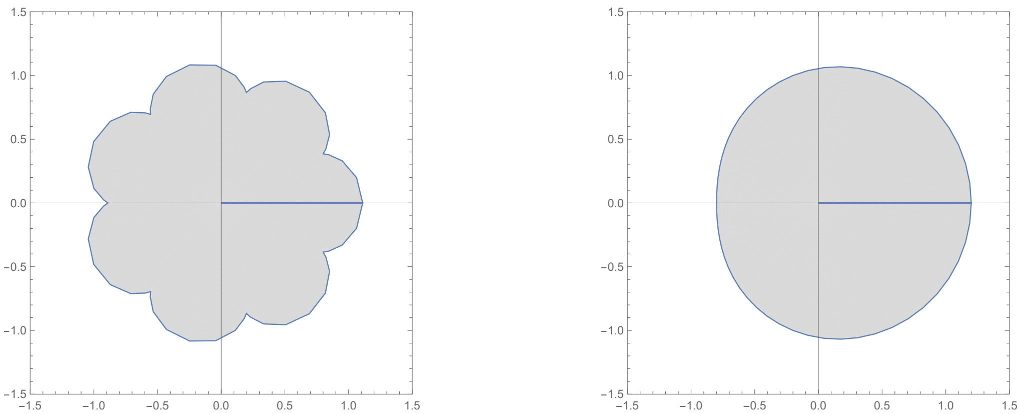


Image of $z + \frac{z^8}{9}$ Image of $z + \frac{z^2}{5}$

Figure 1.4: Starlike and Convex function

***Example* 3.** The function $f_1(z) = z + z^8/9$ is starlike since $f_1(\mathbb{D})$ is starlike with respect to

0 and the image of $f_2(z) = z + z^2/5$ is a convex domain, so f_2 is convex (see Figure 1.4). Also, both the functions are members of $\mathcal{S}$ and thus $f_1 \in \mathcal{S}^*$ and $f_2 \in \mathcal{C}$.

Using the analytical characterization of starlike and convex functions, we have the following representations in terms of $\mathcal{P}$.

Theorem 1.2.3. [21] Let f be analytic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$, then

(a) $f \in \mathcal{S}^* \iff \frac{zf'(z)}{f(z)} \in \mathcal{P}$

(b) $f \in \mathcal{C} \iff 1 + \frac{zf''(z)}{f(z)} \in \mathcal{P}$.

Moreover, we have a two-way communication between the class $\mathcal{S}^*$ and the class $\mathcal{C}$, given by

Theorem 1.2.4 (Alexander's Theorem). [1] Let f be analytic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$, then

$$f \in \mathcal{C} \iff zf'(z) \in \mathcal{S}^*.$$

Close-to-Convex functions: A function $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{D}$ if there exists a convex function g and a real number $\alpha \in (-\pi/2, \pi/2)$ such that

$$\operatorname{Re}\left(\frac{e^{i\alpha} f'(z)}{g'(z)}\right) > 0 \text{ on } \mathbb{D}.$$

The class of all such functions is denoted by $\mathcal{K}$. By Alexander's theorem, if g is a convex function, then $h(z) = zg'(z)$ is starlike. As a result, we can say that a function $f \in \mathcal{K}$ if there exists a starlike function h and a real number $\alpha \in (-\pi/2, \pi/2)$ such that

$$\operatorname{Re}\left(\frac{e^{i\alpha} zf'(z)}{h(z)}\right) > 0 \text{ on } \mathbb{D}.$$

Kaplan [39] proved that every close-to-convex function is univalent in $\mathbb{D}$. In geometrical terms, it means that if $f \in \mathcal{K}$, then the complement of the image of $\mathbb{D}$ under f is the union of non-intersecting half lines.

Strongly Starlike functions: We say that a function $f \in \mathcal{A}$ is strongly starlike of order α $(0 < \alpha \le 1)$ if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad \text{on } \mathbb{D}.$$

The class of all such functions is denoted by $\mathcal{SS}^*(\alpha)$. Geometrically, it means that the image of $\mathbb{D}$ mapped by zf'/f must lie inside the sector, which lies in the right half plane, is symmetric with respect to the real axis and makes an angle $\alpha\pi/2$ with positive real axis. Hence for $\alpha = 1$, the sector becomes the whole right half plane and therefore $\mathcal{SS}^*(1) = \mathcal{S}^*$.

***Example* 4.** The function $f(z) = e^z$ is a strongly starlike function of order $2/\pi$.

Bounded Turning functions: A function $f \in \mathcal{A}$ is said to be a function of bounded turning if $\operatorname{Re} f'(z) > 0$ on $\mathbb{D}$. In other words, f is a function of bounded turning if $f' \in \mathcal{P}$. The class of all such functions is denoted by $\mathcal{R}$. We observe that $\mathcal{R} \subset \mathcal{S}$ since f is a member of $\mathcal{A}$ and $\operatorname{Re} f'(z) > 0$, which is a sufficient condition for a function to be univalent due to the following result.

Theorem 1.2.5 (Noshiro and Warschawski Theorem). [68,102] Suppose that for some real α, $\operatorname{Re} e^{i\alpha} f'(z) > 0$ in a convex domain $\mathcal{D}$, then f is univalent in $\mathcal{D}$.

***Example* 5.** The function $f(z) = z + z^2/2$ is a function of bounded turning.

Typically Real functions: A function $f \in \mathcal{A}$ is said to be typically real if for every $z \in \mathbb{D}$,

$$sign(\operatorname{Im} f(z)) = sign(\operatorname{Im} z).$$

The class of all such functions is denoted by $\mathcal{TR}$ and it can also be represented as follows:

$$\mathcal{TR} = \{f \in \mathcal{A} : (\operatorname{Im} f(z))(\operatorname{Im} z) > 0,\ z \in \mathbb{D}\}.$$

Geometrically, we can say that a function is typically real on $\mathbb{D}$ if it maps the upper half of $\mathbb{D}$ into the upper half plane and the lower half of $\mathbb{D}$ into the lower half plane. Note that the coefficients of a typically real function are always real.

***Example* 6.** The function $f_1(z) = z + z^2/2$ is a member of $\mathcal{TR}$ and the function $f_2(z) = z + iz^2/2$ is not in $\mathcal{TR}$ since for $z = 1/2 - i/10$, $\operatorname{Im} f_2(z)$ is positive.

In addition to the classes listed above, we now present some more classes that are determined by analytical characterizations.

For $\beta > 1$, the class $\mathcal{M}(\beta)$ introduced by Uralegaddi et al. [99] is given by

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} < \beta,\ z \in \mathbb{D} \right\}.$$

A closely related class studied by Ravichandran and Kumar [77] is the class of starlike functions of reciprocal order α $(0 \leq \alpha < 1)$, given by

$$\mathcal{RS}^*(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{f(z)}{zf'(z)} > \alpha \right\}.$$

The class $k-\mathcal{ST}$ of k-starlike functions $(k \geq 0)$ was introduced by Kanas and Wiśniowska [38] and characterized by the condition $\operatorname{Re}(zf'(z)/f(z)) > k|zf'(z)/f(z)-1|$. Further, this class was generalised by Kanas and Răducanu [36] by adding a parameter α, given by

$$\mathcal{ST}(k,\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| + \alpha \right\}.$$

Geometrically, the boundary of the domain $\Omega_{k,\alpha} = \{w \in \mathbb{C} : \operatorname{Re} w > k|w-1| + \alpha\}$ represents an ellipse for $k > 1$, a parabola for $k = 1$ and a hyperbola for $0 < k < 1$.

1.3 Ma Minda Subclasses

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are in $\mathcal{H}$, then the convolution of f and g is given by $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. In 1989, Shanmugam [85] considered the classes

$$\mathcal{S}_g^*(h) = \left\{ f \in \mathcal{A} : \frac{z(f*g)'(z)}{(f*g)(z)} \prec h(z) \right\} \text{ such that } \frac{(f*g)(z)}{z} \neq 0$$

and

$$C_g(h) = \left\{ f \in \mathcal{A} : 1 + \frac{z(f*g)''(z)}{(f*g)'(z)} \prec h(z) \right\} \text{ such that } (f*g)'(z) \neq 0,$$

where h is a convex univalent function with $h(0) = 0$ and $\operatorname{Re} h(z) > 0$ $(z \in \mathbb{D})$. Using the similar concept, in 1992, Ma and Minda [53] gave a general form of various subclasses of starlike and convex functions for which the respective quantities $zf'(z)/f(z)$ and $1 + zf''(z)/f'(z)$ are subordinate to a function with special properties. These subclasses are defined as:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \text{ and } C(\varphi) = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\},$$

where φ satisfies the following properties: (i) φ is analytic and univalent; (ii) φ is symmetric with respect to real axis; (iii) φ has positive real part in $\mathbb{D}$; (iv) φ is starlike with respect to $\varphi(0) = 1$; (v) $\varphi'(0) > 0$. For convenience, let us denote the class of all such functions by Π_M. Several well-known classes can be obtained by specializing φ, such as

- $\mathcal{S}^*\left(\dfrac{1+z}{1-z}\right) = \mathcal{S}^*$ and $\mathcal{C}\left(\dfrac{1+z}{1-z}\right) = \mathcal{C}$.
- $\mathcal{S}^*\left(\dfrac{1+(1-2\alpha)z}{1-z}\right) = \mathcal{S}^*(\alpha)$ and $\mathcal{C}\left(\dfrac{1+(1-2\alpha)z}{1-z}\right) = \mathcal{C}(\alpha), \quad 0 \le \alpha < 1$.
- $\mathcal{S}^*\left(\left(\dfrac{1+z}{1-z}\right)^{\alpha}\right) = \mathcal{SS}^*(\alpha), \quad 0 < \alpha \le 1$.

Over the past decade, many authors came up with different subclasses of $\mathcal{S}^*$, which they defined by taking φ as a particular Ma-Minda function. Some of the classes are listed below:

1. By taking $\varphi(z) = (1+Az)(1+Bz)$, where $-1 \le B < A \le 1$, we obtain the class of Janowski starlike functions denoted by $\mathcal{S}^*[A,B]$, which was introduced by Janowski [32]. The analytical representation of this class is given as follows: For $f \in \mathcal{S}$,
$$f \in \mathcal{S}^*[A,B] \iff \left|\frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2}\right| \le \frac{A-B}{1-B^2}.$$
Geometrically, it means that $f \in \mathcal{S}^*[A,B]$ if $zf'(z)/f(z)$ lies in the disk centered at $(1-AB)/(1-B^2)$ with radius $(A-B)/(1-B^2)$.

2 The class of parabolic starlike functions introduced by Rønning [81] is defined as follows:
$$\mathcal{S}^*_P = \mathcal{S}^*\left(1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right).$$
The class is associated with a parabola and the analytical representation of this class is given by: For $f \in \mathcal{A}$,
$$f \in \mathcal{S}^*_P \text{ if } \operatorname{Re}\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right|.$$

3 By taking $\varphi(z) = \sqrt{1+z}$, Sokół and Stankiewicz [91] defined the class of starlike functions associated with the lemniscate of Bernoulli, denoted by $\mathcal{S}^*_L$.

4 In 2015, by taking $\varphi(z) = e^z$, Mendiratta et. al [58] introduced the class $\mathcal{S}^*_e$.

5 In 2016, Sharma et. al [86] considered $\varphi(z) = 1 + 4z/3 + 2z^2/3$, which maps $\mathbb{D}$ onto a cardioid shaped region. Using this function the authors introduced the class $\mathcal{S}^*_C$.

6 Cho et. al [16] introduced the class $\mathcal{S}^*_S$ by taking $\varphi(z) = 1 + \sin z$.

7 In 2015, Raina and Sokół [76] considered a crescent shaped region by taking $\varphi(z) = z + \sqrt{1+z^2}$ and the class of starlike functions associated with this region is denoted by $\mathcal{S}^*_{\leftmoon}$.

8 Recently, another class associated with a cardioid is defined by taking $\varphi(z) = 1 + ze^z$ (see [45]), denoted by $\mathcal{S}^*_{\wp}$.

9 In [57], the class $\mathcal{S}^*_{RL}$ is defined by taking φ as

$$\varphi_{RL}(z) = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}.$$

Geometrically, the function φ_{RL} represents the left half of the lemniscate of Bernoulli.

10 The class $\mathcal{S}^*_{Ne}$ associated with the Nephroid domain was introduced by Wani and Swaminathan [101] by taking $\varphi(z) = 1 + z - z^3/3$.

11 Kumar and Ravichandran [47] introduced the class $\mathcal{S}^*_R := \mathcal{S}^*(1 + z(k+z)/k(k-z))$ with $k = \sqrt{2}+1$.

For all these classes, authors have dealt with several problems such as finding radius estimates, bounds on coefficient functionals, proving differential subordination results, establishing inclusion relations etc. Our motivation has also led us to introduce the subclass $\mathcal{S}^*_{SG} := \mathcal{S}^*(2/(1+e^{-z}))$ of starlike functions by taking $\varphi(z) = 2/(1+e^{-z})$. In the subsequent chapters, we will discuss this class in detail.

1.4 Differential Subordination

It is known that the study of differential inequalities refers to the idea of determining the behavior of a function from the properties of its derivatives. A differential subordination in complex plane is the generalization of a differential inequality in real line. The concept of differential subordination was introduced by Miller and Mocanu with the remarkable article "Differential subordination and univalent functions" in 1981 [60]. Ever since, there have been hundreds of papers published on the subject and the theory has been extended and applied in a wide range of fields including differential equations, partial differential equations, meromorphic functions, harmonic functions, integral operators, Banach spaces and functions of numerous complex variables. The theory of differential subordination brought a progressive change and pulled in many researchers to use differential subordination techniques for the study of univalent functions (for example [2,4,6,7,14,15,17,19,55,62,67,69] and many more). Miller and Mocanu presented extremely straightforward interpretations for a number of results in univalent function theory that had previously required laborious and lengthy methods. The definition of a differential subordination presented by Miller and Mocanu is as follows:

Definition 1.4.1. [61] Let $\psi : \mathbb{C}^n \times \mathbb{D} \to \mathbb{C}$ and h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the differential subordination

$$\psi(p(z), zp'(z), z^2p''(z), .., z^{n-1}p^{n-1}(z); z) \prec h(z), \tag{1.4.1}$$

then p is called a solution of the differential subordination (1.4.1). The univalent function q is called a dominant of the solutions of the differential subordination if $p \prec q$ for every p satisfying (1.4.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.4.1) is called the best dominant of (1.4.1).

The gist of this whole theory is the following implication

$$\psi(p(z), zp'(z), .., z^{n-1}p^{n-1}(z)) \prec h(z) \Rightarrow p(z) \prec q(z), \qquad z \in \mathbb{D}, \tag{1.4.2}$$

which gives rise to the three types of problems stated as follows:

(i) Given univalent functions h and q, find conditions on ψ so that (1.4.2) holds.

(ii) Given ψ and h, find a dominant q so that (1.4.2) holds. Moreover, find the best dominant.

(iii) Given ψ and dominant q, find the largest class of univalent functions h such that (1.4.2) holds.

We discuss below some important techniques, commonly used for proving our main results.

Miller-Mocanu Theorem: This technique allows us to set the dominant q and then find the appropriate h so that (1.4.2) holds. However, there are other forms of this theorem as well where we can set h and find q. So the Miller-Mocanu theorem, which is stated below, helps us to work out Type 2 and Type 3 problems mentioned above.

Theorem 1.4.1. [61, Theorem 3.4h] Let q be univalent in $\mathbb{D}$ and let θ and ψ be analytic in a domain $\mathcal{D}$ containing $q(\mathbb{D})$ with $\psi(w) \neq 0$, when $w \in q(\mathbb{D})$. Set $Q(z) := zq'(z)\psi(q(z))$ and $h(z) := \theta(q(z)) + Q(z)$. Suppose that either

(i) h is convex, or (ii) Q(z) is starlike.

In addition, assume that

(iii) $\operatorname{Re}\left(\dfrac{zh'(z)}{Q(z)}\right) > 0 \; (z \in \mathbb{D})$.

If p is analytic in $\mathbb{D}$ with $p(0) = q(0)$, $p(\mathbb{D}) \subset \mathcal{D}$ and

$$\theta(p(z)) + zp'(z)\psi(p(z)) \prec \theta(q(z)) + zq'(z)\psi(q(z)), \tag{1.4.3}$$

then $p \prec q$ and q is the best dominant.

Method of contradiction: This is the earliest method of tackling this type of problems and is very popular among researchers even today. This method is primarily based on the following two results:

Lemma 1.4.2 (Jack's Lemma). [82] Let $\omega(z)$ be analytic in $\mathbb{D}$ with $\omega(0) = 0$. Suppose that $|\omega(z)|$ attains its maximum value at a point $z_0 \in \mathbb{D}$, where $|z_0| = r$, then there exists a real number k such that $z_0\omega'(z_0) = k\omega(z_0)$.

Definition 1.4.2. [61] Let Q be the set of functions q that are analytic and univalent in $\overline{\mathbb{D}}\backslash\mathbf{E}(q)$, where

$$\mathbf{E}(q) = \{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty\} \tag{1.4.4}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D}\backslash\mathbf{E}(q)$.

Lemma 1.4.3 (Lemma 2.2d)**.** [61] Let $q \in Q$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{D}$ with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exists $z_0 = r_0 e^{i\theta_0} \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}\backslash\mathbf{E}(q)$ and an $m \geq n \geq 1$ for which $p(\mathbb{D}_{r_0}) \subset q(\mathbb{D})$,

$$(i)\ p(z_0) = q(\zeta_0)\ (ii)\ z_0p'(z_0) = m\zeta_0 q'(\zeta_0) \text{ and } (iii)\ \operatorname{Re}\left(1 + \frac{z_0p''(z_0)}{p'(z_0)}\right) \geq m\operatorname{Re}\left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right).$$

Admissibility Conditions: Despite being introduced many years ago, this method of determining admissibility conditions has only recently been put into practice. This method enables us to set the dominant function q and the function h as well, to determine ψ.

Definition 1.4.3. Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{D}$ that satisfy the following admissibility condition: For $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D} \backslash \mathbf{E}(q)$ and $m \geq n$,

$$\psi(r,s,t;z) \notin \Omega, \text{ whenever } r = q(\zeta),\ s = m\zeta q'(\zeta) \text{ and } \operatorname{Re}\frac{t}{s} + 1 \geq m \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q(\zeta)} + 1\right).$$

Theorem 1.4.4 (Theorem 2.3b)**.** [61] Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a,n]$ satisfies

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,$$

then $p \prec q$.

1.5 Radius Problem

Given a set of functions $\mathscr{M}$ and a property $\mathscr{P}$ which functions may or may not have in $\mathbb{D}$, we aim at finding the largest radius R such that every function in the set $\mathscr{M}$ has the property $\mathscr{P}$ in each disk $|z| < r$ for every $r < R$. We denote this radius by $R_{\mathscr{P}}(\mathscr{M})$. Moreover, we say that this radius is sharp if there exists a function F in $\mathscr{M}$ such that F satisfies property $\mathscr{P}$ if and only if $|z| < R_{\mathscr{P}}(\mathscr{M})$ and such a function is known as the extremal function. This can be explained with the following example:

***Example* 7.** Let f be a typically real function, then f is univalent in the disk $|z| < \sqrt{2} - 1$. Note that for $F(z) = z(1+z^2)/(1-z^2)^2$, the radius is sharp. Thus, we can write $R_{\mathcal{S}}(\mathcal{TR}) = \sqrt{2} - 1$.

Radius problems in particular, have been attempted by many authors for multiple classes of analytic functions (see [3, 13, 54, 78, 90, 100]).

Synopsis of the Thesis

The thesis mainly deals with the differential subordination problems and radius estimations for several subclasses of analytic functions. Moreover, the geometric properties of analytic functions having some special characteristics have been studied in the form of inclusion relations, starlikeness criteria, convexity criteria, conditions of univalence, conditions to be close-to-convex etc. A newly defined subclass of starlike functions, associated with the modified sigmoid function has been extensively examined, resulting to the development of some new concepts. Sufficient conditions as well as radius problems have been addressed for the Silverman class. In addition, differential subordination results involving the classical Pythagorean means and a general version of Briot Bouquet differential subordination have been considered. The thesis is divided into five main chapters and an introduction chapter, which provides a brief review of the general principles of univalent function theory.

In **Chapter 2** titled *"Starlike Functions Associated with Modified Sigmoid Function"*, motivated by the works of [16, 57, 58, 90], we introduce the following subclass of starlike functions

$$\mathcal{S}^*_{SG} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{2}{1+e^{-z}} \right\}.$$

The image domain of $\mathbb{D}$ mapped by $2/(1+e^{-z})$ is denoted by Δ_{SG}. In the first place, we present a structural formula which allows us to construct functions belonging to this class. To describe the general behavior of the functions in this class, we derive growth theorem, distortion theorem and rotation theorem. We derive the bounds of the real part of the modified sigmoid function, which enables us to explore its geometric properties. For the centre lying between these bounds, we attain the maximum radius of the disk that can be inscribed in Δ_{SG} as well as the minimum radius of the disk that can contain Δ_{SG}. We develop criteria to ensure that the class $\mathcal{S}^*_{SG}$ is contained in each of these classes $\mathcal{S}^*(\alpha)$, $\mathcal{M}(\beta)$, $\mathcal{SS}^*(\beta)$, $k-\mathcal{ST}$, $\mathcal{ST}(1,\alpha)$, by establishing conditions on the parameters. Furthermore, we estimate sharp bounds for the first five coefficients of the functions belonging to $\mathcal{S}^*_{SG}$, by using some well known properties of Carathéodory functions. Finally, we discuss the admissibility requirements for the $\mathcal{S}^*_{SG}$ class. The conditions for first and second order differential subordination are obtained by applying general admissibility conditions given by Miller and Mocanu [61]. To accomplish the third order admissibility conditions, results by Antonino and Miller [7] have been modified to entertain Ma-Minda functions, as they fail to satisfy these results. Some important results of this chapter are as follows

1. Let $2/(1+e) < a < 2e/(1+e)$. If

$$r_a = \frac{e-1}{e+1} - |a-1|,$$

then

$$\{w \in \mathbb{C} : |w-a| < r_a\} \subset \Delta_{SG}.$$

2. Let $\Omega \subset \mathbb{C}$, $q \in Q$ and $k \geq m \geq n \geq 2$. The class of admissible functions $\Psi_n[\Omega, q]$

consists of $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ satisfying the admissibility condition

$$\psi(r,s,t,u;z) \notin \Omega \quad \text{when } z \in \mathbb{D},\ r = q(\zeta),\ s = m\zeta q'(\zeta),$$

$$\operatorname{Re}\left(1+\frac{t}{s}\right) \geq m\left(1+\operatorname{Re}\frac{\zeta q''(\zeta)}{q'(\zeta)}\right)$$

and

$$\operatorname{Re}\frac{u}{s} \geq m^2 \operatorname{Re}\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)} + 3m(k-1)\operatorname{Re}\frac{\zeta q''(\zeta)}{q'(\zeta)}$$

for $\zeta \in \partial\mathbb{D}\backslash\mathbf{E}(q)$.

3. Suppose $p \in \mathcal{H}[a,n]$ with $m \geq n \geq 2$. Let $q \in Q(a)$ satisfies

$$\left|\frac{zp'(z)}{q'(\zeta)}\right| \leq m, \quad z \in \mathbb{D},\ \zeta \in \partial\mathbb{D}\backslash\mathbf{E}(q). \tag{1.5.1}$$

If $\Omega \subset \mathbb{C}$ and $\psi \in \Psi_n[\Omega, q]$ with

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \subset \Omega,$$

then $p \prec q$.

This chapter investigates the geometric features of the modified sigmoid function, which is considered to be one of the special functions owing to its importance in mathematical analysis, which eventually inspired researchers from various fields to extend this study.

In **Chapter 3** titled *"Higher Order Differential Subordination involving Modified Sigmoid function"*, inspired by the works of [7,16,46], we prove a number of differential subordination results by using various techniques. In the first part of this chapter, we prove differential subordination results obtained by using the Miller Mocanu lemma and some geometrical aspects of the associated functions. These results can be categorized as follows:

$$1+\beta\frac{zp'(z)}{p^k(z)} \prec \frac{2}{1+e^{-z}} \Rightarrow p(z) \prec \phi(z)\ (k=0,1,2)$$

and

$$p(z)+\beta\frac{zp'(z)}{p^k(z)} \prec \frac{2}{1+e^{-z}} \Rightarrow p(z) \prec \phi(z)\ (k=0,1,2),$$

where $\phi(z)$ is taken as $(1+Az)/(1+Bz)$ or $\sqrt{1+z}$. We consider β to be a real number and find conditions on β so that the implications stated above hold. These results are derived with the help of the Miller Mocanu Lemma and the conditions so obtained are best possible. Further, we prove same results by interchanging the position of $\phi(z)$ with the function $2/(1+e^{-z})$. Next we take β to be complex number and prove similar implications by using the well known method of contradiction. In addition, we find conditions on β, γ and δ ($\beta \in \mathbb{C}$ and δ, $\gamma \in \mathbb{R}^+$) in order to prove the implications that follow:

$$\left.\begin{array}{l} (i)\ 1+\beta(zp'(z))^n \prec \dfrac{2}{1+e^{-z}} \\ (ii)\ 1+\beta\dfrac{zp'(z)}{(p(z))^n} \prec \dfrac{2}{1+e^{-z}} \\ (iii)\ p(z)+\beta\dfrac{zp'(z)}{(p(z))^n} \prec \dfrac{2}{1+e^{-z}} \\ (iv)\ p(z)+\dfrac{zp'(z)}{\delta p(z)+\gamma} \prec \dfrac{2}{1+e^{-z}} \end{array}\right\} \quad \text{implies} \quad p(z) \prec \frac{2}{1+e^{-z}}.$$

These results have been proved by the admissibility conditions of first order, derived in previous chapter. Further, we extend this work by obtaining differential subordination results of second order, which are of the following form:

$$1+\gamma zp'(z)+\beta z^2p''(z) \prec h(z) \quad \Rightarrow \quad p(z) \prec \frac{2}{1+e^{-z}},$$

where we take $h(z)$ as $(1+Az)/(1+Bz)$, e^z, $z+\sqrt{1+z^2}$, $1+\sin z$, $1+ze^z$, $\sqrt{1+z}$, $2/(1+e^{-z})$ and β, $\gamma \in \mathbb{R}^+$. We conclude this chapter with differential subordination implications involving derivatives up to third order, summarized as follows:

$$1+\gamma zp'(z)+\beta z^2p''(z)+\alpha z^3p'''(z) \prec h(z) \quad \Rightarrow \quad p(z) \prec \frac{2}{1+e^{-z}},$$

where α, β, γ are positive real numbers and $h(z)$ is taken as $(1+Az)/(1+Bz)$, e^z, $z+\sqrt{1+z^2}$, $1+\sin z$, $1+ze^z$, $\sqrt{1+z}$ or $2/(1+e^{-z})$. In this chapter, many results have been proved by applying the following two lemmas, which simplified the calculations significantly.

1. Let $r_0 \approx 0.546302$ be the positive root of the equation $r^2+2\cot(1)r-1=0$. Then

$$\left|\log\left(\frac{1+z}{1-z}\right)\right| \geq 1 \ \text{ on } |z|=R \text{ if and only if } \ R \geq r_0.$$

2. For any complex number z, we have

$$|\log(1+z)| \geq 1 \quad \text{if and only if} \quad |z| \geq e-1.$$

This chapter introduces some novel ideas that have enabled numerous researchers to establish higher order differential subordination results.

Chapter 4 titled *"Radius Estimates and Sufficient Conditions for Certain Analytic Functions"* is inspired by the works of [73, 93] , wherein our focus is to compute radius estimates such that the functions belonging to some well known subclasses of starlike functions become functions of the following classes respectively,

$$\Omega = \left\{ f \in \mathcal{A} : |zf'(z) - f(z)| < \frac{1}{2}, \, z \in \mathbb{D} \right\}, \tag{1.5.2}$$

defined by Peng and Zhong [73];

$$G_{\lambda,\alpha} = \left\{ f \in \mathcal{A} : \left| \frac{1 - \alpha + \alpha z f''(z)/f'(z)}{zf'(z)/f(z)} - (1 - \alpha) \right| < \lambda, z \in \mathbb{D} \right\} \quad 0 < \alpha \leq 1, \, \lambda > 0,$$

a general form of Silverman class and $\mathcal{S}^*_{SG}$, the class of Sigmoid starlike functions. These results are proved by utilising some less known properties of Schwarz function as well as the growth theorem and extremal functions of the corresponding classes. Further, we find sufficient conditions for a function to belong to $G_{\lambda,\alpha}$. These sufficient conditions are identified as differential inequalities involving derivatives up to second order. Our proofs of sufficient conditions allow us to construct double integral functions, which belong to $G_{\lambda,\alpha}$. The notions of differential subordination have been used to prove these outcomes. Moreover in this chapter, we prove a general result stating the inclusion relation of $G_{\lambda,\alpha}$ with $\mathcal{S}^*(\varphi)$. Some of the special cases are obtained by assuming particular values of $\varphi(z)$. The following are some of the significant findings in this chapter:

1. If $f \in \Omega$, then $f \in G_{\frac{1}{2},\frac{1}{2}}$ in the disc $|z| < r_0$, where $r_0 \approx 0.430496$ is the smallest positive root of $55r^{12} - 28r^{11} - 854r^{10} + 148r^9 + 2969r^8 - 212r^7 - 4286r^6 + 28r^5 + 2875r^4 + 96r^3 - 888r^2 - 32r + 96 = 0$.

2. Let $f \in \mathcal{A}_n$, $0 \leq \alpha < 1$ and $\lambda > 0$. If

$$\left| zf''(z) - \alpha \left(f'(z) - \frac{f(z)}{z} \right) \right| < \delta, \tag{1.5.3}$$

   where δ is the smallest positive root of $\phi(r) := (1+n)(2\alpha n - \lambda(n+1) - n)r^2 + n(1 - \alpha + n)(2\lambda(n+1) + n + \alpha n^2)r - \lambda n^2(n+1-\alpha)^2$, then $f \in G_{\lambda,\alpha}$.

3. Let $f \in G_{\lambda,\alpha}$ $(\lambda > 0, \, 1/3 < \alpha \leq 1)$. Then $zf'(z)/f(z) \prec 1/(1 \pm cz)$, where $c = \lambda/(3\alpha - 1)$

and the result is sharp.

This chapter discusses several aspects of the Schwarz function that have not been extensively employed. As an application, results of Obradovič and Tuneski [70] and Bulboacă and Tuneski [12] have been acquired as a special case.

In **Chapter 5** titled *"Pythagorean means and Differential Subordination"*, persuaded by the works [14, 19, 24, 37], we prove a differential subordination implication involving the convex weighted harmonic mean of $p(z)$ and $p(z)\Theta(z)+zp'(z)\Phi(z)$, where Θ, Φ are analytic functions. Special cases have been discussed for the functions e^z, $\sqrt{1+z}$ and $((1+z)/(1-z))^\gamma$. Moreover in this chapter, we prove results involving a combination of arithmetic, geometric and harmonic mean of analytic functions. Many previously known results are generalised by this result and a number of conditions are established, which are sufficient for starlikeness, univalence, strongly starlikness etc. Some of the important outcomes of this chapter are as follows:

1. Let $t\in[0,1]$ and $\Theta,\Phi\in\mathcal{H}$ with $\Theta(0)=1$. By $\mathcal{H}(t;\Theta,\Phi)$, we mean the subclass of $\mathcal{H}$ of all functions f such that

$$H_{t;\Theta,\Phi,f}(z) := \begin{cases} \dfrac{P_{0;\Theta,\Phi,f}(z)P_{1;\Theta,\Phi,f}(z)}{P_{1-t;\Theta,\Phi,f}(z)} & P_{1-t;\Theta,\Phi,f}(z)\neq 0, \\ \lim_{\mathbb{D}\ni\zeta\to z}\dfrac{P_{0;\Theta,\Phi,f}(\zeta)P_{1;\Theta,\Phi,f}(\zeta)}{P_{1-t;\Theta,\Phi,f}(\zeta)} & P_{1-t;\Theta,\Phi,f}(z)=0, \end{cases} \tag{1.5.4}$$

is an analytic function in $\mathbb{D}$, where

$$P_{t;\Theta,\Phi,f}(z) := (1-t+t\Theta(z))f(z)+t\Phi(z)zf'(z), \quad z\in\mathbb{D}$$

and define $H_{t;\Theta,\Phi,0}\equiv 0$.

2. Let $\delta\in[0,1]$, $h\in\mathcal{Q}$ with $0\in\overline{h(\mathbb{D})}$ and $\Theta,\Phi\in\mathcal{H}$ be such that $\Theta(0)=1$, $\operatorname{Re}\Phi(z)>0$ $(z\in\mathbb{D})$ and

$$\operatorname{Re}\left(\Phi(z)+\frac{h(\zeta)}{\zeta h'(\zeta)}(\Theta(z)-1)\right)>0, \quad z\in\mathbb{D},\ \zeta\in\partial\mathbb{D}. \tag{1.5.5}$$

If $p\in\mathcal{H}(\delta;\Theta,\Phi)$, $p(0)=h(0)$ and $H_{\delta;\Theta,\Phi,p}\prec h$, then $p\prec h$. This result has generalised some existing results.

3. We obtain conditions on all the parameters so that the following implication

holds.

$$\mathrm{Re}\left(\gamma(p(z))^{\delta}+(1-\gamma)\frac{(p(z))^{\mu}\left(p(z)+\dfrac{zp'(z)}{p(z)}\right)^{1-\mu}}{1+\rho\dfrac{zp'(z)}{p^{2}(z)}}\right)>\beta\Rightarrow \mathrm{Re}\,p(z)>\alpha.$$

With the development of a general type of harmonic mean, this chapter demonstrates a differential subordination implication with the assistance of mathematical properties of harmonic mean. Special cases are obtained for the exponential function, lemniscate of Bernoulli and strongly starlike functions. Our primary conclusions reduce to the results of [19,37,50,51,63–65,84] for specific values of the concerned parameters.

Chapter 6 titled *"On a Briot-Bouquet type Differential Subordination"* entails a general form of Briot-Bouquet differential subordination and derives an analogue to the Open door lemma applying to it. This result take different forms, when applied to particular functions. Special cases of this result have been discussed for the functions e^z, $\sqrt{1+z}$ and $((1+z)/(1-z))^{\gamma}$. It has likewise been shown for the Janowski starlike functions that the same consequence holds, by laying out conditions on the parameters involved. In addition, we demonstrate differential subordination implications for the solutions of a given differential equation. There is one noteworthy aspect that these results are proved using the following integral representation of $p(z)$, which is the sole solution of the associated differential equation:

$$p(z)=z^{\beta}f^{\alpha}(z)\left(\beta\int_0^z f^{\alpha-1}(t)f'(t)t^{\beta}dt\right)^{-1}-\frac{\alpha}{\beta}.$$

Some other integral representations of $p(z)$ have also been obtained depending on $Q(z)$. As an application, two special cases have been discussed for $(1-z)/(1+z)$ and $1/(1+z)$. Using these two results, we obtain a number of sufficient conditions for univalence, starlikeness and $F-$starlikeness. This chapter's key findings include the following:

1. Let $\alpha,\beta\in\mathbb{C}$ with $\beta\neq 0$ and h be a function, convex in $\mathbb{D}$. Suppose $Q\in\mathcal{H}[1,n]$ be a function for which following conditions hold:

   (i) $\mathrm{Re}\left(\dfrac{1}{\beta h(z)+\alpha}\right)>0 \quad (z\in\mathbb{D}).$

   (ii) $\mathrm{Re}\left(\dfrac{1}{\beta h(\zeta)+\alpha}+(Q(z)-1)\dfrac{h(\zeta)}{\zeta h'(\zeta)}\right)>0 \quad (z\in\mathbb{D},\ \zeta\in h^{-1}(p(D))),$

where $D = \{z \in \mathbb{D} : p(z) = h(\zeta) \text{ for some } \zeta \in \partial\mathbb{D}\}$. Let p be analytic in $\mathbb{D}$ and $p(0) = h(0)$ with

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z) + \alpha} \prec h(z),$$

then $p(z) \prec h(z)$.

2. Let $p(z)$ be analytic in $\mathbb{D}$ with $p(0) = 1$ and $Q(z) \in \mathcal{P}$. Suppose that $\alpha \geq 0, \beta > 0$ and p satisfies

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z) + \alpha} = 1,$$

then $\operatorname{Re} p(z) > 0$.

This chapter establishes sufficient conditions in the form of differential equations that suffice to imply a differential subordination, which eventually helped us to infer sufficient conditions for starlikeness and univalence.

Chapter 2

Starlike Functions Associated with Modified Sigmoid Function

In this chapter, we introduce a new subclass of starlike functions, given by

$$\mathcal{S}^*_{SG} = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{2}{1+e^{-z}} =: G(z) \right\},$$

where $G(z)$ *is the modified sigmoid function. We study some geometric properties of* $\mathcal{S}^*_{SG}$ *and use them to obtain several inclusion relations involving other subclasses of starlike functions. Further, we obtain sharp bounds of first five coefficients and derive structural formula for* $\mathcal{S}^*_{SG}$*. In addition, we establish growth, rotation and distortion theorems for* $\mathcal{S}^*_{SG}$*. Finally, we deduce the admissibility conditions for second and third order differential subordination associated with the modified sigmoid function.*

2.1 Introduction

The study of Ma Minda functions has been a topic of great interest and is considerably improved upon by numerous authors over the years. Various researchers have explored problems such as coefficient estimates, radius estimates, differential subordinations, inclusion relations etc. for the subclasses of starlike functions associated with particular Ma Minda functions, which has elevated the theory in a wide range of ways. With the aim of extending this theory for a special function, we consider the sigmoid function, given by $g(z) = 1/(1+e^{-z})$. In order to obtain the normalized form of $g(z)$, we define the modified sigmoid function as $G(z) = 2/(1+e^{-z})$. The modified sigmoid function maps $\mathbb{D}$ onto a domain $\Delta_{SG} := \{w \in \mathbb{C} : |\log(w/(2-w))| < 1\}$, which is symmetric about the real axis. Moreover, G is convex and hence, starlike with respect to $G(0) = 1$. Also G has positive real part in $\mathbb{D}$ and $G'(0) > 0$. Hence, G falls under the category of Ma-Minda functions, or we can say that $G \in \Pi_M$. So, the classes $\mathcal{S}^*(G)$ and $\mathcal{C}(G)$ naturally become the subclasses of $\mathcal{S}^*$ and $\mathcal{C}$, respectively. As mentioned in Chapter 1, there are several forms of $\mathcal{S}^*(\varphi)$, which have been investigated for their geometric properties. For instance, Rønning [81] investigated the class $\mathcal{S}^*_P$ of parabolic starlike functions. Sokół [90] considered the class $\mathcal{S}^*_L = \mathcal{S}^*(\sqrt{1+z})$ and obtained several radius estimates. Mendiratta et al. [58] studied $\mathcal{S}^*_e = \mathcal{S}^*(e^z)$ and proved number of inclusion relations, coefficient estimates and differential subordination results for this class. Afterward, Mendiratta et al. [57] accomplished similar work for the class $\mathcal{S}^*_{RL}$, associated with the left-half of the lemniscate of Bernoulli. In the year 2018, Cho et al. [16] examined the class $\mathcal{S}^*_S = \mathcal{S}^*(1+\sin z)$ and proved a variety of results. Motivated by these works, we introduce

$$\mathcal{S}^*_{SG} = \mathcal{S}^*(G) \qquad \text{and} \qquad \mathcal{C}_{SG} = \mathcal{C}(G).$$

Analytically, a function $f \in \mathcal{S}^*_{SG}$ if and only if $zf'(z)/f(z)$ lies in the region Δ_{SG}. From this definition, we have the following representation formula: A function f is in $\mathcal{S}^*_{SG}$ if and only if there exists an analytic function ϕ, satisfying $\phi(z) \prec G(z) = 2/(1+e^{-z})$ such that

$$f(z) = z\exp\left(\int_0^z \frac{\phi(t)-1}{t}dt\right). \tag{2.1.1}$$

Here are some examples of the functions which belong to the class $\mathcal{S}^*_{SG}$: Let

$$\phi_1(z) = 1 + \frac{z}{4},\ \phi_2(z) = \frac{4+2z}{4+z},\ \phi_3(z) = \frac{7+ze^z}{7}\ \text{ and }\ \phi_4(z) = 1 + \frac{z\sin z}{3}.$$

Since $G(z)$ is univalent in $\mathbb{D}$, $\phi_i(0) = G(0)$ $(i = 1,2,3,4)$ and $\phi_i(\mathbb{D}) \subset G(\mathbb{D})$, it is easy to deduce that $\phi_i \prec G$. Thus, the functions in the class $\mathcal{S}^*_{SG}$ corresponding to each of the $\phi_i's$ are obtained by using the representation formula given by (2.1.1), respectively as follows:

$$f_1(z) = ze^{z/4},\ f_2(z) = z + \frac{z^2}{4},\ f_3(z) = z\exp\left(\frac{e^z-1}{7}\right)\ \text{ and }\ f_4(z) = z\exp\left(\frac{1-\cos z}{3}\right).$$

In particular, if we take $\phi(z) = G(z) = 2/(1+e^{-z})$, then we obtain the following function

$$f_{SG}(z) = z\exp\left(\int_0^z \frac{e^t-1}{t(e^t+1)}dt\right) = z + \frac{z^2}{2} + \frac{z^3}{8} + \frac{z^4}{144} - \frac{5z^5}{1152} + \dots, \tag{2.1.2}$$

which plays the role of an extremal function for many problems in the class $\mathcal{S}^*_{SG}$. Utilising some subordination results proved by Ma and Minda [53], it follows that if $f \in \mathcal{S}^*_{SG}$, then $f(z)/z \prec f_{SG}(z)/z$ and therefore, we obtain the following result.

Theorem 2.1.1. Let $f \in \mathcal{S}^*_{SG}$ and $f_{SG}(z)$ be the extremal function given by (2.1.2). Then, the following holds whenever $|z| = r < 1$:

(*i*) Growth Theorem: $-f_{SG}(-r) \leq |f(z)| \leq f_{SG}(r)$. In particular, $f(\mathbb{D})$ contains $\Delta := \{w : |w| < -f_{SG}(-1) \approx 0.614535\}$.

(*ii*) Rotation Theorem: $\left|\arg(f(z)/z)\right| \leq \max_{|z|=r} \arg(f_{SG}(z)/z)$.

(*iii*) Distortion Theorem: $f'_{SG}(-r) \leq |f'(z)| \leq f'_{SG}(r)$. Equality holds for some $z \neq 0$ if and only if f is a rotation of f_{SG}.

In the subsequent sections, we establish many inclusion relations and find coefficient estimates. Also, we establish a result which ascertains the largest disk that can be inscribed inside Δ_{SG} as well as the smallest disk that contains Δ_{SG}. Some of the fundamental results from the theory of admissible functions are also discussed with regard to the modified sigmoid function.

2.2 About $\mathcal{S}^*_{SG}$

Before proving our main results, we present the bounds of the real part of $2/(1+e^{-z})$ as follows.

Lemma 2.2.1. The function $G(z) = 2/(1+e^{-z})$ satisfies

$$\min_{|z|=r} \operatorname{Re} G(z) = G(-r) \qquad \text{and} \qquad \max_{|z|=r} \operatorname{Re} G(z) = G(r),$$

whenever $0 < r < 1$.

Proof. Let $G(z) = 2/(1+e^{-z})$. Then a boundary point of $G(\mathbb{D}_{r_0})$, where $\mathbb{D}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$, can be written in the form $G(r_0e^{i\theta}) = 2/(1+e^{-r_0e^{i\theta}})$. The outward normal at the point $G(\zeta)$ is given by $\zeta G'(\zeta)$ where $|\zeta| = r_0$. Thus for $\zeta = r_0e^{i\theta}$, we have

$$\zeta G'(\zeta) = \frac{2r_0e^{i\theta}e^{-r_0e^{i\theta}}}{(1+e^{-r_0e^{i\theta}})^2}.$$

Since we need to find the bounds of the real part, it is sufficient to find the points, at which the imaginary part of the normal is constant. The imaginary part of $\zeta G'(\zeta)$ is given by

$$h(\theta) = \delta(2e^{-r_0\cos\theta}r_0\sin(\theta - r_0\sin\theta) + 4e^{-2r_0\cos\theta}r_0\sin\theta + 2e^{-3r_0\cos\theta}r_0\sin(\theta + r_0\sin\theta)),$$

where

$$\delta = \frac{1}{|1+e^{-r_0e^{i\theta}}|^4}.$$

A simple computation yields that $h(\theta) = 0$ for $\theta = 0$ and $\theta = \pi$. It is easy to check that the maximum value is obtained at $\theta = 0$ and the minimum value is obtained at $\theta = \pi$. Since r_0 is arbitrary, it follows that

$$\min_{|z|=r} \operatorname{Re} G(z) = \frac{2}{1+e^{r}} \qquad \text{and} \qquad \max_{|z|=r} \operatorname{Re} G(z) = \frac{2}{1+e^{-r}}.$$

□

In the following lemma, we find the radius of the largest disk that can be inscribed in the domain Δ_{SG} with centre $(a,0)$, where a lies between the bounds of the real part of $2/(1+e^{-z})$.

Lemma 2.2.2. Let $2/(1+e) < a < 2e/(1+e)$. If

$$r_a = \frac{e-1}{e+1} - |a-1|,$$

then

$$\{w \in \mathbb{C} : |w-a| < r_a\} \subset \Delta_{SG}. \tag{2.2.1}$$

Proof. Let $\phi(z) = 2/(1+e^{-z})$. Then a boundary point of the domain $\phi(\mathbb{D})$ can be represented as

$$\phi(e^{i\theta}) = \frac{2(1+e^{-\cos\theta}\cos(\sin\theta))}{1+e^{-2\cos\theta}+2e^{-\cos\theta}\cos(\sin\theta)} + i\frac{2e^{-\cos\theta}\sin(\sin\theta)}{1+e^{-2\cos\theta}+2e^{-\cos\theta}\cos(\sin\theta)},$$

where $-\pi \le \theta \le \pi$. Now, consider

$$h(\theta) = \frac{4-4a((1+e^{-\cos\theta}\cos(\sin\theta)))}{1+e^{-2\cos\theta}+2e^{-\cos\theta}\cos(\sin\theta)} + a^2, \tag{2.2.2}$$

which is the square of the distance of $\phi(e^{i\theta})$ from $(a,0)$. Since $h(\theta) = h(-\theta)$, it is sufficient to consider the interval $0 \le \theta \le \pi$. A computation indicates that $h(\theta)$ is a decreasing function, whenever $2/(1+e) < a \le e/(e+1)$. Therefore,

$$r_a = \min_{0\le\theta\le\pi} \sqrt{h(\theta)} = \sqrt{h(\pi)} = a - \frac{2}{1+e}.$$

If $e/(e+1) < a \le (e+2)/(e+1)$, then the graph of $h(\theta)$ reveals that it is increasing for $\theta \in [0,\theta_a]$ and decreasing for $\theta \in [\theta_a,\pi]$ where θ_a is a root of $h'(\theta)$, whose value depends on the value of a. Hence, the minimum of $h(\theta)$ in this case is attained either at 0 or π. Also,

$$h(\pi) - h(0) = \frac{4(a-1)(e-1)}{e+1}.$$

Hence,

$$\min_{0\le\theta\le\pi} h(\theta) = \begin{cases} h(\pi), & a < 1 \\ h(0), & a > 1. \end{cases}$$

In the end, let us assume $(e+2)/(e+1) < a < 2e/(e+1)$. For this range of a, we find that $h(\theta)$ is an increasing function and therefore,

$$r_a = \min_{0\le\theta\le\pi} \sqrt{h(\theta)} = \sqrt{h(0)} = \frac{2e}{1+e} - a.$$

Combining the above three cases based upon the decreasing and increasing nature of the function $h(\theta)$, we have the following two cases:

(i) For $2/(1+e) < a \leq 1$, the minimum of $h(\theta)$ is attained at π. Therefore,

$$\min_{0\leq\theta\leq\pi} h(\theta) = h(\pi)$$

and $r_a = a - 2/(1+e)$.

(ii) For $1 \leq a < 2e/(1+e)$, the minimum of $h(\theta)$ is attained at 0. Therefore,

$$\min_{0\leq\theta\leq\pi} h(\theta) = h(0)$$

and $r_a = 2e/(1+e) - a$.

Upon fusing the above two cases, we have $r_a = (e-1)/(e+1) - |a-1|$, whenever $2/(1+e) < a < 2e/(e+1)$. □

Remark 1. On the similar lines of the proof of Lemma 2.2.2, we conclude that

$$\Delta_{SG} \subset \{w \in \mathbb{C} : |w-a| < R_a\},$$

where R_a is given by

$$R_a = \begin{cases} \dfrac{2e}{e+1} - a, & \dfrac{2}{e+1} < a \leq \dfrac{e}{e+1} \\ \sqrt{h(\theta_a)}, & \dfrac{e}{e+1} < a \leq \dfrac{e+2}{e+1} \\ a - \dfrac{2}{1+e}, & \dfrac{e+2}{e+1} < a < \dfrac{2e}{e+1}, \end{cases}$$

$h(\theta)$ is given by (2.2.2) and θ_a is a root of $h'(\theta)$, whose value depends on a.

2.3 Inclusion Relations

The following result uses a variety of subclasses of analytic functions, which we have already pointed out in Chapter 1.

Theorem 2.3.1. The class $\mathcal{S}^*_{SG}$ satisfies the following inclusion relations:

(*i*) $\mathcal{S}^*_{SG} \subset \mathcal{S}^*(\alpha)$, whenever $0 \leq \alpha \leq 2/(1+e)$.

(*ii*) $\mathcal{S}^*_{SG} \subset \mathcal{RS}^*(1/\beta) \subset \mathcal{M}(\beta)$, whenever $\beta \geq 2e/(1+e)$.

(iii) Let $s_0 \approx 1.94549$ be the smallest root of the equation $\cos t + e^{\cos t}\cos(\sin t - t) = 0$ and $h(z) = \arg(2/(1+e^{-z}))$. Then $\mathcal{S}^*_{SG} \subset \mathcal{SS}^*(\beta)$, whenever $\beta \geq 2h(e^{is_0})/\pi \approx 0.353914$.

(iv) If $k > 1$ and $0 \leq \alpha < 1$, then $\mathcal{ST}(k,\alpha) \subset \mathcal{S}^*_{SG}$, whenever $k \geq (2e - \alpha(1+e))/(e-1)$. In particular, $k - \mathcal{ST} \subset \mathcal{S}^*_{SG}$, whenever $k \geq 2e/(e-1)$.

(v) If $0 \leq \alpha < 1$, then $\mathcal{S}^*_{SG} \subset \mathcal{ST}(1,\alpha)$, whenever $\alpha \leq (3-e)/(1+e)$.

The result is sharp.

Proof. (i) and (ii) follows as under:

Let $f \in \mathcal{S}^*_{SG}$, then $zf'(z)/f(z) \prec 2/(1+e^{-z})$. By Lemma 2.2.1, it is easy to deduce that

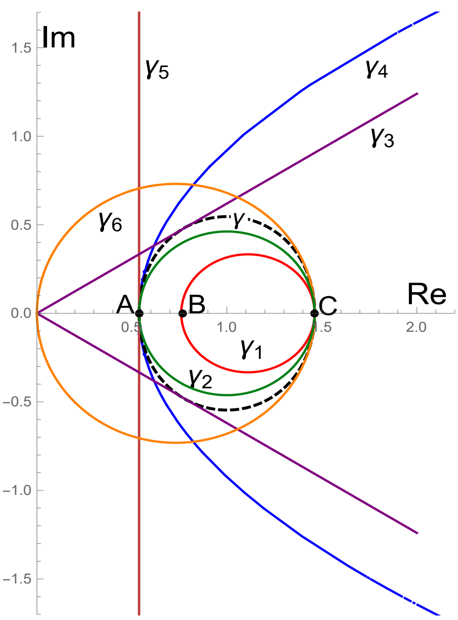
$$\min_{|z|=1} \operatorname{Re} \frac{2}{1+e^{-z}} < \operatorname{Re} \frac{zf'(z)}{f(z)} < \max_{|z|=1} \operatorname{Re} \frac{2}{1+e^{-z}},$$

which implies

$$\frac{2}{1+e} < \operatorname{Re} \frac{zf'(z)}{f(z)} < \frac{2e}{1+e}.$$

Thus, $f \in \mathcal{S}^*(2/(1+e)) \cap \mathcal{M}(2e/(1+e))$. Now, we consider

$$\operatorname{Re} \frac{f(z)}{zf'(z)} > \min_{|z|=1} \operatorname{Re} \frac{1+e^{-z}}{2} = \frac{1}{2}\left(1 + \min_{|z|=1} \operatorname{Re} e^{-z}\right) = \frac{1+e}{2e}. \tag{2.3.1}$$



$$\gamma : \left|\log \frac{w}{2-w}\right| = 1$$
$$\gamma_1 : \operatorname{Re} w = \frac{2e}{e-1}|w-1|$$
$$\gamma_2 : |w-1| = \frac{e-1}{e+1}$$
$$\gamma_3 : |\arg w| = \frac{89}{500}\pi$$
$$\gamma_4 : \operatorname{Re} w - |w-1| = \frac{3-e}{1+e}$$
$$\gamma_5 : \operatorname{Re} w = \frac{2}{1+e}$$
$$\gamma_6 : \left|w - \frac{e}{1+e}\right| = \frac{e}{1+e}$$
$$A = \frac{2}{1+e}$$
$$B = \frac{2e}{3e-1}$$
$$C = \frac{2e}{e+1}$$

Figure 2.1: Boundary curves of best dominants and subordinants of $2/(1+e^{-z})$

From equation (2.3.1), it follows that $f \in \mathcal{RS}^*(\beta)$, whenever $\beta \leq (1+e)/2e$. Equivalently, $f \in \mathcal{RS}^*(1/\beta)$, whenever $\beta \geq 2e/(1+e)$. We know that $f \in \mathcal{RS}^*(1/\beta)$ if and only if

$$\left|\frac{zf'(z)}{f(z)} - \frac{\beta}{2}\right| < \frac{\beta}{2}, \tag{2.3.2}$$

which implies that $\operatorname{Re} zf'(z)/f(z) < \beta$. Using this fact, we have

$$\mathcal{S}^*_{SG} \subset \mathcal{RS}^*(1/\beta) \subset \mathcal{M}(\beta), \text{ whenever } \beta \geq 2e/(1+e).$$

Note that the class $\mathcal{RS}^*(\beta)$ is equivalent to the class $\mathcal{S}^*_M$ considered by Sokół [90], for the case when $M = 1/2\beta$. Thus $f \in \mathcal{RS}^*(1/\beta)$ is equivalent to saying that Δ_{SG} is contained in the disk given in (2.3.2), whenever $\beta \geq 2e/(1+e)$.

(iii) If $f \in \mathcal{S}^*_{SG}$, then

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &< \max_{|z|=1} \arg\frac{2}{1+e^{-z}} \\ &= \max_{0\leq\theta<2\pi} \arctan\left(\frac{\sin(\sin\theta)}{e^{\cos\theta}+\cos(\sin\theta)}\right) \\ &= \max_{0\leq\theta<2\pi} h(e^{i\theta}).\end{aligned}$$

Putting $h'(e^{i\theta}) = 0$ reduces to $\cos\theta + e^{\cos\theta}\cos(\sin\theta - \theta) = 0$ and $s_0 \approx 1.94549$ is a root of this equation. Now we observe that $h''(e^{is_0}) < 0$ and hence $\max_{0\leq\theta<2\pi} h(e^{i\theta}) = h(e^{is_0}) \approx 0.555926$. Thus $f \in \mathcal{SS}^*(2h(e^{is_0})/\pi)$.

(iv) Consider the domain $\Omega_{k,\alpha} = \{w \in \mathbb{C} : Re w > k|w-1| + \alpha\}$ whose boundary $\partial\Omega_{k,\alpha}$ represents an ellipse for $k > 1$ and is given by:

$$\frac{(x-\lambda)^2}{a^2} + \frac{(y-\delta)^2}{b^2} = 1,$$

with

$$\lambda = \frac{k^2-\alpha}{k^2-1}, \ \delta = 0, \ a = \left|\frac{k(\alpha-1)}{k^2-1}\right| \text{ and } b = \left|\frac{(\alpha-1)}{\sqrt{k^2-1}}\right|.$$

For the ellipse $\Omega_{k,\alpha}$ to lie inside Δ_{SG}, $\lambda + a$ should not exceed $2e/(1+e)$ which yields the inequality $k \geq (2e - \alpha(e+1))/(e-1)$. Taking $\alpha = 0$, it follows that $k - \mathcal{ST} \subset \mathcal{S}^*_{SG}$, whenever $k \geq 2e/(e-1)$.

(v) Taking $k = 1$ in part (iv), we observe that the boundary of the domain $\Omega_{1,\alpha} = \{w \in \mathbb{C} : \operatorname{Re} w > |w-1| + \alpha\}$ represents a parabola. Now $\Delta_{SG} \subset \Omega_{1,\alpha}$, provided $\operatorname{Re} w - |w-1| > \alpha$, where $w = 2/(1+e^{-z})$. Upon taking $z = e^{i\theta}$, we have

$$\frac{2(1+e^{-\cos\theta}\cos(\sin\theta)) - \sqrt{1+e^{-4\cos\theta} - 2e^{-2\cos\theta} + 4e^{-2\cos\theta}\sin^2(\sin\theta)}}{1+e^{-2\cos\theta}+2e^{-\cos\theta}\cos(\sin\theta)} > \alpha.$$

A calculation shows that the expression on the left hand side attains its minimum at $\theta = \pi$ and is equal to $(3-e)/(1+e)$. Thus $\mathcal{S}^*_{SG} \subset \mathcal{ST}(1,\alpha)$, whenever $\alpha \leq (3-e)/(1+e)$.

The sharpness of all the above relations can we verified by Fig 2.1. □

Theorem 2.3.2. Let $-1 < B < A \leq 1$, then $\mathcal{S}^*[A,B] \subset \mathcal{S}^*_{SG}$ if either of the following conditions hold:

(*i*) $2(1-B) \leq (1-A)(1+e)$ and $2(1-B^2) < (1-AB)(1+e) \leq (1-B^2)(1+e)$.

(*ii*) $(1+A)(1+e) \leq 2e(1+B)$ and $(1-B^2)(1+e) \leq (1-AB)(1+e) < 2e(1-B^2)$.

Proof. Let $f \in \mathcal{S}^*[A,B]$, then $zf'(z)/f(z) \in \mathcal{P}[A,B]$. Using Lemma 1.2.2, we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2},$$

which represents a disk. To show that this disk is contained in Δ_{SG}, it is sufficient to show that this disk is contained in the disk given in (2.2.1). Let $a = (1-AB)/(1-B^2)$, then using Lemma 2.2.2, we have

$$r_a = \frac{e-1}{e+1} - \left|\frac{1-AB}{1-B^2} - 1\right|.$$

Now, it suffices to show that $(A-B)/(1-B^2) \leq r_a$. If part (i) holds, then by multiplying the first inequality by $(1+B)/(1+e)(1-B^2)$, we obtain the required result for $2/(1+e) < a \leq 1$. Similarly for part (ii), the result follows when $1 \leq a < 2e/(1+e)$. □

2.4 Coefficient Bounds

In this section, we find sharp bounds on the first five coefficients of the functions belonging to the class $\mathcal{S}^*_{SG}$.

Theorem 2.4.1. If $f(z) = z + a_2z^2 + a_3z^3 + \ldots \in \mathcal{S}^*_{SG}$, then (*i*) $|a_2| \leq 1/2$, (*ii*) $|a_3| \leq 1/4$, (*iii*) $|a_4| \leq 1/6$ and (*iv*) $|a_5| \leq 1/8$. These bounds are sharp.

Proof. Let $f \in \mathcal{S}^*_{SG}$, then there exists a Schwarz function $\omega(z) = \sum_{k=1}^{\infty} w_k z^k$ such that

$$\frac{zf'(z)}{f(z)} = \frac{2}{1+e^{-\omega(z)}}. \tag{2.4.1}$$

Suppose $\omega(z)$ is taken as $\omega(z) = (p(z)-1)/(p(z)+1)$, where $p(z) = 1 + c_1z + c_2z^2 + \ldots \in \mathcal{P}$. Then by substituting $\omega(z)$, $p(z)$ and $f(z)$ in (2.4.1) and comparing the coefficients, we obtain a_i's in terms of c_i's as follows:

$$a_2 = \frac{1}{4}c_1, \quad a_3 = \frac{1}{8}\left(c_2 - \frac{c_1^2}{4}\right), \quad a_4 = \frac{1}{48}\left(\frac{7}{24}c_1^3 - \frac{5}{2}c_1c_2 + 4c_3\right)$$

and

$$a_5 = -\frac{1}{16}\left(\frac{17}{1152}c_1^4 - \frac{7}{24}c_1^2c_2 + \frac{3}{8}c_2^2 + \frac{2}{3}c_1c_3 - c_4\right).$$

(i) Using Lemma 1.2.1, we have $|c_1| \leq 2$, which further implies that $|a_2| \leq 1/2$.

(ii) For a_3, we use the inequality $|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}$ given by Ma and Minda [53], which yields $|a_3| \leq 1/4$.

(iii) For a_4, first we rewrite (2.4.1) as follows:

$$zf'(z) = (2f(z) - zf'(z))e^{\omega(z)}. \tag{2.4.2}$$

By substituting $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $\omega(z) = \sum_{k=1}^{\infty} w_k z^k$ in (2.4.2) and comparing the coefficient of z^4, we get

$$6a_4 = w_3 + \frac{3}{4}w_1w_2 + \frac{1}{24}w_1^3.$$

Using [75, Lemma 2], it follows that $|6a_4| \leq 1$ and hence the result.

(iv) For a_5, the result follows by applying [79, Lemma 2.1] with $\gamma = 17/1152$, $a = 3/8$, $\alpha = 1/3$ and $\beta = 7/36$.

The extremal functions for the initial coefficients $a_n (n = 2, 3, 4, 5)$ are of the form:

$$f_n(z) = z\exp\int_0^z \frac{e^{t^{n-1}} - 1}{t(e^{t^{n-1}} + 1)}dt,$$

obtained by taking $\omega(z) = z^{n-1}$ in (2.4.1). □

***Example* 8.** (i) Let $f(z) = z + a_2z^2$. Then $f \in \mathcal{S}^*_{SG}$ if and only if $|a_2| \leq \sqrt{(e-1)/2e}$.

(ii) Let $f(z) = z/(1-bz)^2$. Then $f \in \mathcal{S}^*_{SG}$ if and only if $|b| \leq \sqrt{(e-1)/(1+3e)}$.

Proof. (i) It is known that $f \in \mathcal{S}^*_{SG}$ if and only if $zf'(z)/f(z)$ lies in the domain Δ_{SG}. Since $\mathcal{S}^*_{SG} \subset \mathcal{S}^*$, it follows that $|a_2| \leq 1/2$. We observe that $w(z) = zf'(z)/f(z) = (1+2a_2z)/(1+a_2z)$ maps $\mathbb{D}$ onto the disk

$$\left| w - \frac{1-2|a_2|^2}{1-|a_2|^2} \right| < \frac{|a_2|}{1-|a_2|^2}. \tag{2.4.3}$$

Thus $f \in \mathcal{S}^*_{SG}$ if and only if this disk is contained in Δ_{SG}. Now, it is sufficient to prove that the disk given by (2.4.3) is contained in the disk (2.2.1). Since $(1-2|a_2|^2)/(1-|a_2|^2) \leq 1$, then by using Lemma 2.2.2, we deduce that the disk mapped by $w(z) = zf'(z)/f(z)$ is contained in Δ_{SG} if and only if

$$\frac{2}{1+e} \leq \frac{1-2|a_2|^2}{1-|a_2|^2} \quad \text{and} \quad \frac{|a_2|}{1-|a_2|^2} \leq \frac{1-2|a_2|^2}{1-|a_2|^2} - \frac{2}{1+e}.$$

The above two inequalities yield

$$|a_2| \leq \sqrt{\frac{e-1}{2e}} \quad \text{and} \quad |a_2| \leq \frac{e-1}{2e}.$$

Hence, $f \in \mathcal{S}^*_{SG}$ if and only if

$$|a_2| \leq \sqrt{\frac{e-1}{2e}}.$$

(ii) We know that $f(z) = z/(1-bz)^2 = z + 2bz^2 + 3b^2z^3 + \ldots \in \mathcal{S}^*_{SG}$. As $\mathcal{S}^*_{SG} \subset \mathcal{S}^*$, we have $|b| \leq 1$. Now, we observe that $w(z) = zf'(z)/f(z) = (1+bz)/(1-bz)$ maps $\mathbb{D}$ onto the disk

$$\left| w - \frac{1+|b|^2}{1-|b|^2} \right| \leq \frac{2|b|}{1-|b|^2}.$$

Since $(1+|b|^2)/(1-|b|^2) \geq 1$, then by Lemma 2.2.2, it follows that Δ_{SG} contains the above disk if and only if

$$\frac{1+|b|^2}{1-|b|^2} \leq \frac{2e}{1+e} \quad \text{and} \quad \frac{2|b|}{1-|b|^2} \leq \frac{2e}{1+e} - \frac{1+|b|^2}{1-|b|^2}.$$

The above two inequalities yield

$$|b| \leq \sqrt{\frac{e-1}{1+3e}} \quad \text{and} \quad |b| \leq \frac{e-1}{1+3e}.$$

Hence, $f \in \mathcal{S}^*_{SG}$ if and only if

$$|b| \leq \sqrt{\frac{e-1}{1+3e}}.$$

□

2.5 Admissibility conditions

The monograph "Differential subordination and univalent functions" by Miller and Mocanu in the year 1981 is considered to be a master piece. It started a revolution by inciting researchers towards using it for the study of univalent functions. Problems related to differential subordination were considered by many authors, for example, see [2, 9, 15, 46, 85, 92]. In Chapter 2 of this book, the authors have provided a set of lemmas concluded by the admissibility conditions for second order differential subordination. A good deal of work has been done using these results. In 2018, Madaan et al. [55] investigated the class of admissible functions associated with lemniscate of Bernoulli and proved several first and second order differential subordination implications. Recently, Naz et al. [67] have established a number of generalised first order differential subordination implications by defining the admissibility conditions for exponential function. In [4], Ali et al. modified the concept of second order differential subordination by introducing β-admissible functions. In the year 2011, Antonino and Miller [7] extended the concept of differential subordination and admissibility conditions for third order by introducing the following definitions and results.

Definition 2.5.1. [7] Let $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ and h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the third order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{2.5.1}$$

then p is called the solution of the differential subordination (2.5.1). The univalent function q is called a dominant of the solutions of the differential subordination if $p \prec q$ for every p satisfying (2.5.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (2.5.1) is called the best dominant of (2.5.1).

Definition 2.5.2. [7] Let Q be the set of functions q that are analytic and univalent on $\overline{\mathbb{D}}\backslash\mathbf{E}(q)$, where

$$\mathbf{E}(q) = \{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty\}$$

and are such that $\min|q'(\zeta)| = \rho > 0$ for $\zeta \in \partial\mathbb{D}\backslash\mathbf{E}(q)$. The subclass of Q for which $q(0) = a$ is denoted by $Q(a)$.

Lemma 2.5.1. [7] Let $z_0 \in \mathbb{D}$ and $r_0 = |z_0|$. Let $f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots$ be continuous on $\overline{\mathbb{D}}_{r_0}$ and analytic on $\mathbb{D} \cup \{z_0\}$ with $f(z) \neq 0$ and $n \geq 2$. If $|f(z_0)| = \max\{|f(z)| : z \in \overline{\mathbb{D}}_{r_0}\}$ and $|f'(z_0)| = \max\{|f'(z)| : z \in \overline{\mathbb{D}}_{r_0}\}$, then there exist real constants m, k, l such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = m, \; \frac{z_0 f''(z_0)}{f'(z_0)} + 1 = k \text{ and } \operatorname{Re} \frac{z_0 f'''(z_0)}{f''(z_0)} + 2 = l,$$

where $l \geq k \geq m \geq n \geq 2$.

Definition 2.5.3. [7] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $n \geq 2$. The class of admissible operators $\Psi_n[\Omega, q]$ consists of those $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t, u; z) \notin \Omega, \text{ where } r = q(\zeta), \; s = n\zeta q'(\zeta),$$

$$\operatorname{Re}\left(1 + \frac{t}{s}\right) \geq n\left(1 + \operatorname{Re} \frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \text{ and } \operatorname{Re} \frac{u}{s} \geq n^2 \operatorname{Re} \frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}$$

for $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}\backslash\mathbf{E}(q)$.

Lemma 2.5.2. [7] Let $p \in \mathcal{H}[a, n]$ with $n \geq 2$, and let $q \in Q(a)$ such that it satisfies

$$\operatorname{Re} \frac{\zeta q''(\zeta)}{q'(\zeta)} \geq 0 \quad \text{and} \quad \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq n,$$

when $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}\backslash\mathbf{E}(q)$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, a]$ and

$$\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \subset \Omega,$$

then $p \prec q$.

Now we consider the function $q(z) = 2/(1 + e^{-z})$ and define the admissible class $\Psi[\Omega, q]$, where Ω is any subset of $\mathbb{C}$. It is known that q is analytic and univalent on $\overline{\mathbb{D}}$, $q(0) = 1$ and q maps $\mathbb{D}$ onto $\Delta_{SG} = \{w \in \mathbb{C} : |\log(w/(2 - w))| < 1\}$. Since $\mathbf{E}(q) = \phi$, $\zeta \in \partial\mathbb{D}\backslash\mathbf{E}(q)$ if and only if $\zeta = e^{i\theta}(0 \le \theta < 2\pi)$. Now let us consider

$$|q'(\zeta)| = \frac{2e^{-\cos\theta}}{1 + e^{-2\cos\theta} + 2e^{-\cos\theta}\cos(\sin\theta)} =: d(\theta) \tag{2.5.2}$$

whose minimum value is $2e/(1+e)^2$. Clearly $\min|q'(\zeta)| > 0$ and thus $q \in Q(1)$ and the class $\Psi[\Omega, q]$ is well defined. For $|\zeta| = 1$, $q(\zeta) \in q(\partial\mathbb{D}) = \{w \in \mathbb{C} : |\log(w/(2-w))| = 1\} = \partial\Delta_{SG}$. Thus we have $|\log(q(\zeta)/(2-q(\zeta)))| = 1$ and so $\log(q(\zeta)/(2-q(\zeta))) = e^{i\theta}$ $(0 \le \theta < 2\pi)$, which further implies that $q(\zeta) = 2/(1+e^{-e^{i\theta}})$. Therefore $\zeta q'(\zeta) = 2e^{i\theta}e^{-e^{i\theta}}/(1+e^{-e^{i\theta}})^2$ and

$$\frac{\zeta q''(\zeta)}{q'(\zeta)} = e^{i\theta}\left[1 - \frac{2}{1+e^{-e^{i\theta}}}\right] = \frac{e^{i\theta}(e^{-e^{i\theta}} - 1)}{e^{-e^{i\theta}} + 1}. \tag{2.5.3}$$

On comparing real part on either side of (2.5.3), we have

$$\operatorname{Re}\frac{\zeta q''(\zeta)}{q'(\zeta)} = \frac{e^{-2\cos\theta}\cos\theta - \cos\theta + 2e^{-2\cos\theta}\sin\theta\sin(\sin\theta)}{1 + e^{-2\cos\theta} + 2e^{-\cos\theta}\cos(\sin\theta)} =: g(\theta). \tag{2.5.4}$$

Note that the function $g(\theta)$ given by (2.5.4) attains its minimum at $\theta = 0$ and thus the minimum value is $g(0) = (1-e)/(1+e)$. Further, the class $\Psi[\Omega, \frac{2}{1+e^{-z}}]$ is defined to be the class of all the functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ which satisfy the following conditions:

$$\psi(r, s, t; z) \notin \Omega,$$

whenever

$$r = q(\zeta) = \frac{2}{1+e^{-e^{i\theta}}};\ s = m\zeta q'(\zeta) = \frac{2me^{i\theta}e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^2};\ \operatorname{Re}\left(1 + \frac{t}{s}\right) \ge m(1 + g(\theta)), \tag{2.5.5}$$

where $z \in \mathbb{D}$, $0 \le \theta < 2\pi$ and $m \ge 1$. If $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$, then the admissibility conditions (2.5.5) reduce to

$$\psi\left(\frac{2}{1+e^{-e^{i\theta}}}, \frac{2me^{i\theta}e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^2}; z\right) \notin \Omega \quad (z \in \mathbb{D},\ 0 \le \theta < 2\pi,\ m \ge 1).$$

We obtain the following result as a special case of Theorem 1.4.4, when $q(z) = 2/(1+e^{-z})$, which is required to prove many of our subsequent results.

Theorem 2.5.3. Let $\psi \in \Psi[\Omega, 2/(1+e^{-z})]$. If $p \in \mathcal{H}[1, n]$ satisfies

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$$

then $p(z) \prec 2/(1+e^{-z})$.

Note that Ma-Minda functions do not satisfy the conditions of the third order differential subordination result of [7] and hence, we are unable to establish third order differential subordination results meant for Ma-Minda functions. To overcome this limitation, without loss of generality, the conditions are altered to accommodate Ma-Minda functions in the following results:

Lemma 2.5.4. Let $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{D}$ with $p(z) \not\equiv a$ and $n \geq 2$, and let $q \in Q(a)$. If there exist points $z_0 = r_0 e^{i\theta_0} \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D} \backslash \mathbf{E}(q)$ and $m \geq n$ such that $p(z_0) = q(\zeta_0)$, $p(\overline{\mathbb{D}}_{r_0}) \subset q(\mathbb{D})$,

$$\left| \frac{zp'(z)}{q'(\zeta)} \right| \leq m$$

when $z \in \overline{\mathbb{D}}_{r_0}$ and $\zeta \in \partial\mathbb{D} \backslash \mathbf{E}(q)$, then there exists a real number $k \geq m \geq n \geq 2$ such that

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0),$$

$$\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \left[\operatorname{Re} \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1 \right]$$

and

$$\operatorname{Re} \frac{z_0^2 p'''(z_0)}{p'(z_0)} \geq m^2 \operatorname{Re} \frac{\zeta_0 q'''(\zeta_0)}{q'(\zeta_0)} + 3m(k-1) \operatorname{Re} \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}.$$

To validate this result, we show the existence of an example as follows: Let $p(z) = 1 + z^2$ and $q(z) = 2/(1+e^{-z})$, then all the conditions of Lemma 2.5.4 are satisfied for $z_0 = \sqrt{(e-1)/(e+1)}$ and $\zeta_0 = 1$. Consequently, the admissibility conditions given in Definition 2.5.3 and Lemma 2.5.2 are restated as follows:

Definition 2.5.4. Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $k \geq m \geq n \geq 2$. The class of admissible operators $\Psi_n[\Omega, q]$ consists of those $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r,s,t,u;z) \notin \Omega \quad \text{when } z \in \mathbb{D},\ r = q(\zeta),\ s = m\zeta q'(\zeta),$$

$$\operatorname{Re}\left(1 + \frac{t}{s}\right) \geq m \left(1 + \operatorname{Re} \frac{\zeta q''(\zeta)}{q'(\zeta)}\right)$$

and

$$\operatorname{Re} \frac{u}{s} \geq m^2 \operatorname{Re} \frac{\zeta^2 q'''(\zeta)}{q'(\zeta)} + 3m(k-1) \operatorname{Re} \frac{\zeta q''(\zeta)}{q'(\zeta)}$$

for $\zeta \in \partial\mathbb{D} \backslash \mathbf{E}(q)$.

Theorem 2.5.5. Let $p \in \mathcal{H}[a,n]$ with $m \geq n \geq 2$ and let $q \in Q(a)$ such that it satisfies

$$\left|\frac{zp'(z)}{q'(\zeta)}\right| \leq m,$$

when $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}\backslash\mathbf{E}(q)$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega,q]$ and

$$\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z \in \mathbb{D}) \subset \Omega,$$

then $p \prec q$.

Let $q(z) = 2/(1+e^{-z})$, we have

$$\zeta^2\frac{q'''(\zeta)}{q'(\zeta)} = e^{2i\theta}\left[1 - \frac{6e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^2}\right].$$

On comparing real parts on either sides, we obtain

$$\operatorname{Re}\zeta^2\frac{q'''(\zeta)}{q'(\zeta)} = \cos 2\theta - \frac{N(\theta)}{D(\theta)} =: h(\theta),$$

where

$$N(\theta) = 6e^{-\cos\theta}(\cos(2\theta - \sin\theta) + 2e^{-\cos\theta}\cos 2\theta + e^{-2\cos\theta}\cos(2\theta + \sin\theta))$$

and

$$D(\theta) \;=\; 1 + e^{-4\cos\theta} + 4e^{-2\cos\theta} + 2e^{-2\cos\theta}\cos(2\sin\theta) + 4e^{-\cos\theta}\cos(\sin\theta)(1 + e^{-2\cos\theta}).$$

The minimum value of $h(\theta)$ is attained at $\theta_0 \approx 0.651068$ and we denote it by

$$\rho := h(\theta_0) \approx -0.406669. \tag{2.5.6}$$

Thus $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ belongs to $\Psi[\Omega, \frac{2}{1+e^{-z}}]$, provided ψ satisfies:

$$\psi(r,s,t;z) \notin \Omega \quad \text{whenever} \quad r = q(\zeta) = \frac{2}{1+e^{-e^{i\theta}}},\ s = m\zeta q'(\zeta) = \frac{2me^{i\theta}e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^2},$$

$$\operatorname{Re}\left(1 + \frac{t}{s}\right) \geq m(1 + g(\theta)) \quad \text{and} \quad \operatorname{Re}\frac{u}{s} \geq (m^2h(\theta) + 3m(k-1)g(\theta)),$$

where $z \in \mathbb{D}$, $0 \leq \theta < 2\pi$, $k \geq m \geq 2$ and $g(\theta)$ is given by (2.5.4). In view of the above conditions, we obtain the following special case of Theorem 2.5.5. The applications of the following lemma are enlisted in the next chapter.

Lemma 2.5.6. Let $p \in \mathcal{H}[1,n]$ with $m \geq n \geq 2$ such that for $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$ it satisfies

$$\left|\frac{zp'(z)(1+e^{-\zeta})^2}{e^{-\zeta}}\right| \leq m.$$

If Ω is a set in $\mathbb{C}$ and $\psi \in \Psi[\Omega, \frac{2}{1+e^{-z}}]$, then

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z \in \mathbb{D}) \subset \Omega \quad \Rightarrow \quad p(z) \prec \frac{2}{1+e^{-z}}.$$

Henceforth r, s and t are as defined in Definition 2.5.4. We now list below a few illustrations derived from Theorem 2.5.3.

***Example* 9.** Let $h : \mathbb{D} \to \mathbb{C}$ be given by

$$h(z) = M\frac{Mz+a}{M+\bar{a}z}, \quad M > 0,\ |a| < M.$$

Then $\Omega = h(\mathbb{D}) = \{w \in \mathbb{C} : |w| < M\}$. Now suppose $\psi(a,b,c;z) = a + \sigma b$, where $\sigma = (M+2+Me)(1+e)/2$, then

$$\begin{aligned}
|\psi(r,s,t;z)| &= |r + s\sigma| \\
&= \left|\frac{2}{1+e^{-e^{i\theta}}} + \frac{2m\sigma e^{i\theta}e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^2}\right| \\
&= \left|\frac{2}{1+e^{-e^{i\theta}}}\right|\left|1 + \frac{m\sigma e^{i\theta}e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})}\right| \\
&\geq \frac{2}{1+e}\left(m\sigma\left|\frac{e^{i\theta}e^{-e^{i\theta}}}{1+e^{-e^{i\theta}}}\right| - 1\right).
\end{aligned}$$

Since $m \geq 1$, we have

$$|\psi(r,s,t;z)| \geq \frac{2}{1+e}\left(\sigma\left(\frac{1}{1+e}\right) - 1\right) \geq M.$$

Clearly, $\psi(r,s,t;z) \notin \Omega$ and thus by using Theorem 2.5.3, we have $p(z) \prec 2/(1+e^{-z})$ whenever

$$p(z) + \frac{(M+2+Me)(1+e)}{2}zp'(z) \prec M\frac{Mz+a}{M+\bar{a}z}.$$

***Example* 10.** Let $h:\mathbb{D}\to\mathbb{C}$ be given by

$$h(z)=\frac{1-(1-2\alpha)z}{1+z},\quad \alpha>1.$$

Then $\Omega=h(\mathbb{D})=\{w\in\mathbb{C}:\operatorname{Re}w<\alpha\}$. Let $\psi(a,b,c;z)=\sigma(1+c/b)$, where $\sigma=\alpha(1+e)/2$. Since

$$\operatorname{Re}\psi(r,s,t;z) = \sigma\operatorname{Re}\left(1+\frac{t}{s}\right)\geq\sigma m(1+g(\theta))\geq\sigma m(1+g(0))=m\alpha\geq\alpha,$$

$\psi(r,s,t;z)\notin\Omega$. Now by using Theorem 2.5.3, we have $p(z)\prec 2/(1+e^{-z})$ whenever

$$\frac{\alpha(1+e)}{2}\left(1+\frac{zp''(z)}{p'(z)}\right)\prec\frac{1-(1-2\alpha)z}{1+z}.$$

***Example* 11.** Let $h:\mathbb{D}\to\mathbb{C}$ be given by $h(z)=1+z$. Then $\Omega=h(\mathbb{D})=\{w\in\mathbb{C}:|w-1|<1\}$. For $\psi(a,b,c;z)=1+\sigma b/a$, where $\sigma=1+e$, we have

$$|\psi(r,s,t;z)-1| = \left|\sigma\frac{s}{r}\right|=\sigma m\left|\frac{e^{i\theta}e^{-e^{i\theta}}}{1+e^{-e^{i\theta}}}\right|\geq\frac{\sigma m}{(1+e)}=m\geq 1,$$

which implies $\psi(r,s,t;z)\notin\Omega$. Then by an application of Theorem 2.5.3, we have $p(z)\prec 2/(1+e^{-z})$ whenever

$$1+(1+e)\frac{zp'(z)}{p(z)}\prec 1+z.$$

***Example* 12.** Let $h:\mathbb{D}\to\mathbb{C}$ be given by $h(z)=1+z/e$. Then $\Omega=h(\mathbb{D})=\{w\in\mathbb{C}:|w-1|<1/e\}$. Suppose $\psi(a,b,c;z)=1+2b/a^2$, then

$$|\psi(r,s,t;z)-1| = \left|\frac{2s}{r^2}\right|=m\left|e^{i\theta}e^{-e^{i\theta}}\right|=me^{-cos\theta}\geq\frac{m}{e}\geq\frac{1}{e}.$$

Clearly $\psi(r,s,t;z)\notin\Omega$ and thus by using Theorem 2.5.3, we have $p(z)\prec 2/(1+e^{-z})$ whenever

$$1+\frac{2zp'(z)}{p^2(z)}\prec 1+\frac{z}{e}.$$

***Example* 13.** Let $h(z)=4ez/(1+e)^3$. Then $\Omega=h(\mathbb{D})=\{w:|w|<4e/(1+e)^3\}$ and let $\psi:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ be given as $\psi(a,b,c;z)=b+c$. For ψ to be in $\Psi[\Omega,\Delta_{SG}]$, we must have

$\psi(r,s,t;z) \notin \Omega$. Now, let us consider

$$
\begin{aligned}
|\psi(r,s,t;z)| &= \left| \frac{2me^{i\theta}e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^2} + t \right| \\
&= \left| \frac{2me^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^2} \right| \left| 1 + \frac{t}{s} \right| \\
&\geq \left| \frac{2me^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^2} \right| \operatorname{Re}\left(1 + \frac{t}{s}\right) \\
&\geq \left| \frac{2me^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^2} \right| (1+g(\theta)) \\
&= \left(\frac{2me^{-\cos\theta}}{1+e^{-2\cos\theta}+2e^{-\cos\theta}\cos(\sin\theta)} \right)(1+g(\theta)).
\end{aligned}
$$

Since $m \geq 1$, we have

$$
\begin{aligned}
|\psi(r,s,t;z)| &\geq \left(\frac{2e^{-\cos\theta}}{1+e^{-2\cos\theta}+2e^{-\cos\theta}\cos(\sin\theta)} \right)(1+g(\theta)) \\
&\geq \frac{4e}{(1+e)^3}.
\end{aligned}
$$

Thus we have $\psi(r,s,t;z) \notin \Omega$. Hence $p(z) \prec 2/(1+e^{-z})$ follows from Theorem 2.5.3, whenever

$$
zp'(z) + z^2p''(z) \prec \frac{4ez}{(1+e)^3}.
$$

Concluding Remarks

Geometric properties of the class $\mathcal{S}^*_{SG}$ have been investigated in this chapter, which are later applied to establish inclusion relations of $\mathcal{S}^*_{SG}$ with other well known subclasses of starlike functions. A magnificent pictorial representation of these inclusion relations has been provided, which validates the sharpness of these results. The sharp coefficient bounds are obtained up to fifth coefficient. The most significant contribution of this chapter is the general third order admissibility criterion for Ma Minda functions which has been obtained by modifying certain results of [7], as none of the Ma Minda functions satify their hypothesis.This study has greatly inspired researchers from various domains to extend it, a few them are mentioned here [23,41,45,56,93,101]. Further investigation of the class $\mathcal{S}^*_{SG}$ has been dealt in the next chapter.

Chapter 3

Higher Order Differential Subordination involving Modified Sigmoid function

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In this chapter, we primarily concentrate on differential subordination problems, where we find conditions on the admissible function. We determine sharp bounds on $\beta \in \mathbb{R}$ *so that various first order differential subordinations such as* $1+\beta zp'(z)/p^k(z) \prec 2/(1+e^{-z})$, $p(z)+\beta zp'(z)/p^k(z) \prec 2/(1+e^{-z})$ *imply* $p(z) \prec (1+Az)/(1+Bz)$ $(-1 \leq A < B \leq 1)$ *or* $\sqrt{1+z}$ *and also when the position of dominants is interchanged. These results are also considered for the case when* β *is a complex number. Additionally, using the idea of admissible functions presented in the preceding chapter, we derive many higher order differential subordination results pertaining to the modified sigmoid function and other well known Ma-Minda functions.*

3.1 Introduction

Differential subordination is an important tool by which various geometrical aspects of univalent functions can be studied. By employing the technique of differential subordination, a large number of research articles on this subject have been published in the literature till now. A few of them are listed below:

1. Tuneski [96] and Tuneski et al. [97] derived conditions for functions in $\mathcal{A}$ to be in $\mathcal{S}^*[A,B]$.

2. In 2007, Ali et al. [6] established conditions on β so that for $k=0,1,2$, the following differential subordination implication holds:

$$1+\beta\frac{zp'(z)}{(p(z))^k} \prec \frac{1+Dz}{1+Ez}, \Rightarrow p(z) \prec \frac{1+Az}{1+Bz} \quad -1\leq B<A\leq 1 \text{ and } -1\leq E<D\leq 1$$

3. In 2013, Kumar et al. [46] obtained sufficient conditions on β for $1+\beta(zp'(z))/(p(z))^k \prec (1+Dz)/(1+Ez)$, where $k=0,1,2$ to imply that $p(z)\prec\sqrt{1+z}$, where $-1\leq E<D\leq 1$.

4. Kumar and Ravichandran [48] obtained sharp bounds on β so that $p(z)\prec e^z$, whenever $1+\beta zp'(z)/(p(z))^k \prec (1+Az)/(1+Bz)$ or $\sqrt{1+z}$ $(k=0,2)$ and many more functions.

5. Lately, Cho et al. [15] obtained sharp bounds on β so as to prove that $1+\beta zp'(z)/(p(z))^k \prec 1+z/(1-\alpha z^2)$ implies $p(z)\prec e^z$ or $\sqrt{1+z}$ and several other functions.

Motivated by these works, we find sharp bounds on $\beta\in\mathbb{R}$ so that the following differential subordination implications hold:

$$1+\beta\frac{zp'(z)}{(p(z))^k} \prec \frac{2}{1+e^{-z}} \qquad \Rightarrow \qquad p(z)\prec\phi_0(z) \qquad (k=0,1,2), \tag{3.1.1}$$

where ϕ_0 is either $(1+Az)/(1+Bz)$ or $\sqrt{1+z}$. Also, conditions on β are obtained in order to prove the implication formed by interchanging the functions $2/(1+e^{-z})$ and $\phi_0(z)$ in (3.1.1). Further, these results are also discussed for the case when β is a complex number. In the articles mentioned above, the authors have mainly used the Miller Mocanu Lemma to prove their results. In the past few years, some authors studied similar problems with a new technique namely, admissibility conditions (see [55,67]).

We use the same technique to prove our results in the third section. The results stated below, are needed to support our primary outcomes in addition to those, mentioned in the previous chapters.

Lemma 3.1.1. [61, Corollary 3.4a] Let h be analytic in $\mathbb{D}$. Let ϕ be analytic in a domain $\mathcal{D}$ containing $h(\mathbb{D})$ and suppose $\operatorname{Re}\phi[h(z)] > 0$ and either h is convex, or $H(z) = zh'(z)\phi[h(z)]$ is starlike. If p is analytic in $\mathbb{D}$ with $p(0) = h(0)$, $p(\mathbb{D}) \subset \mathcal{D}$ and

$$p(z) + zp'(z)\phi[p(z)] \prec h(z),$$

then

$$p(z) \prec h(z).$$

Lemma 3.1.2. [61, Theorem 3.1a] Let h be convex in $\mathbb{D}$ and let $P : \mathbb{D} \to \mathbb{C}$ with $\operatorname{Re} P(z) > 0$. If p is analytic in $\mathbb{D}$, then

$$p(z) + P(z)p(z) \prec h(z) \Rightarrow p(z) \prec h(z).$$

Lemma 3.1.3. Let $r_0 \approx 0.546302$ be the positive root of the equation $r^2 + 2\cot(1)r - 1 = 0$. Then

$$\left|\log\left(\frac{1+z}{1-z}\right)\right| \geq 1 \text{ on } |z| = R \text{ if and only if } R \geq r_0.$$

Proof. Let $z = re^{i\theta}$ be a boundary point of the disk $|z| < r$, where $0 < r \leq 1$. Then

$$\left|\log\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)\right| \geq 1$$

if and only if

$$\left|\log\left|\frac{1-r^2+i2r\sin\theta}{1+r^2-2r\cos\theta}\right| + i\arctan\left(\frac{2r\sin\theta}{1-r^2}\right)\right| \geq 1,$$

which holds if and only if

$$\log^2\sqrt{\frac{(1-r^2)^2+2r\sin\theta}{(1+r^2-2r\cos\theta)^2}} + \left(\arctan\left(\frac{2r\sin\theta}{1-r^2}\right)\right)^2 \geq 1.$$

The expression on the left hand side attains its minimum at $\theta = \pi/2$ and therefore, the above inequality holds if and only if

$$\left(\arctan\left(\frac{2r}{1-r^2}\right)\right)^2 \geq 1,$$

which is true if and only if

$$r \geq (-\cot(1) + \sqrt{1+\cot^2(1)}) \approx 0.546302.$$

□

Lemma 3.1.4. For any complex number z, we have

$$|\log(1+z)| \geq 1 \quad \text{if and only if} \quad |z| \geq e-1.$$

Proof. Let $z = re^{i\theta}$ be a boundary point on the disk $|z| < r$. Then

$$|\log(1+re^{i\theta})| \geq 1$$

if and only if

$$\left|\log|1+r\cos\theta+ir\sin\theta| + i\arctan\left(\frac{r\sin\theta}{1+r\cos\theta}\right)\right| \geq 1,$$

which holds, if and only if

$$\log^2\sqrt{1+r^2+2r\cos\theta} + \left(\arctan\left(\frac{r\sin\theta}{1+r\cos\theta}\right)\right)^2 \geq 1.$$

The expression on the left hand side attains its minimum at $\theta = 0$ and therefore the above inequality holds if and only if

$$\log(1+r) \geq 1,$$

which is true if and only if $r \geq e-1$. □

3.2 Applications of Miller Mocanu Lemma

In this section, we present first order differential subordination results proved by applying the Miller Mocanu lemma.

Theorem 3.2.1. Suppose p is analytic in $\mathbb{D}$ and $p(0) = 1$ such that it satisfies $1+\beta zp'(z) \prec 2/(1+e^{-z})$ and

$$g(z) = \int_0^z \frac{e^t-1}{t(e^t+1)}dt, \tag{3.2.1}$$

then the following holds:

(*i*) $p(z) \prec (1+Az)/(1+Bz)$, whenever $\beta \geq \max\{(1+|B|)g(1)/(A-B), \operatorname{Im} g(i)(1+B^2)/(A-B)\}$, where $-1 < B < A \leq 1$.

(*ii*) $p(z) \prec \sqrt{1+z}$, whenever $\beta \geq g(1)/(\sqrt{2}-1)$.

These bounds are sharp.

Proof. The differential equation

$$1+\beta z q'_{\beta}(z) = \frac{2}{1+e^{-z}} = h(z)$$

has a solution $q_{\beta} : \overline{\mathbb{D}} \to \mathbb{C}$ defined as

$$q_{\beta}(z) = 1 + \frac{1}{\beta}\left[\frac{z}{2} - \frac{z^3}{72} + \frac{z^5}{1200} - \frac{17z^7}{282240}\cdots\right].$$

Now we apply Lemma 1.4.1 with $\theta(w) = 1$ and $\psi(w) = \beta$ and thus the function $Q : \mathbb{D} \to \mathbb{C}$ becomes

$$Q(z) = zq'_{\beta}(z)\psi(q_{\beta}(z)) = \beta z q'_{\beta}(z) = \frac{e^z-1}{e^z+1}.$$

A calculation shows that, for $z \in \mathbb{D}$,

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\frac{2ze^z}{e^{2z}-1} > 0$$

and hence Q is starlike in $\mathbb{D}$. Clearly $h(z) = 1+Q(z)$ and so $\operatorname{Re}(zh'(z)/Q(z)) > 0$ on $\mathbb{D}$. Hence, by Lemma 1.4.1, $1+\beta z p'(z) \prec 1+\beta z q'_{\beta}(z)$ implies $p(z) \prec q_{\beta}(z)$. Now, we only need to show that $q_{\beta}(z) \prec \phi_0(z)$ in each of the parts. Note that if $q_{\beta}(z) \prec \phi_0(z)$,

$$\phi_0(-1) < q_{\beta}(-1) < q_{\beta}(1) < \phi_0(1) \tag{3.2.2}$$

and

$$\operatorname{Im} q_{\beta}(i) < \operatorname{Im} \phi_0(i). \tag{3.2.3}$$

Also, $q_{\beta}(z)$ is analytic and maps $\mathbb{D}$ onto a domain which is convex and symmetric with respect to real axis. Due to the geometry of the respective functions, it is easy to conclude that the conditions given in (3.2.2) and (3.2.3) are necessary as well as sufficient in case of $\phi_0(z) = (1+Az)/(1+Bz)$. Whereas in case of $\phi_0(z) = \sqrt{1+z}$, the condition (3.2.2) alone is necessary as well as sufficient.

(i) Let $\phi_0(z) = (1+Az)/(1+Bz)$. Then (3.2.2) and (3.2.3) give the following three inequalities:

$$q_\beta(-1) \geq \frac{1-A}{1-B}, \quad \text{whenever} \quad \beta \geq g(1)\frac{1-B}{A-B} = \beta_1$$

$$q_\beta(1) \leq \frac{1+A}{1+B}, \quad \text{whenever} \quad \beta \geq g(1)\frac{1+B}{A-B} = \beta_2$$

and

$$\operatorname{Im} q_\beta(i) \leq \operatorname{Im} \frac{1+Ai}{1+Bi}, \quad \text{whenever} \quad \beta \geq \operatorname{Im} g(i)\frac{1+B^2}{A-B} = \beta_3,$$

where $g(1) \approx 0.486889$ and $\operatorname{Im} g(i) \approx 0.514788$. We observe that

$$\max\{\beta_1, \beta_2\} = \beta_0 = \begin{cases} \beta_1, & \textit{if} \quad B < 0 \\ \beta_2, & \textit{if} \quad B > 0. \end{cases}$$

Therefore, $q_\beta(z) \prec (1+Az)/(1+Bz)$, whenever $\beta \geq \max\{\beta_0, \beta_3\}$.

(ii) Let $\phi_0(z) = \sqrt{1+z}$. From (3.2.2), we have the following two inequalities:

$$q_\beta(-1) \geq 0, \quad \text{whenever} \quad \beta \geq g(1) = \beta_1$$

and

$$q_\beta(1) \leq \sqrt{2}, \quad \text{whenever} \quad \beta \geq \frac{g(1)}{\sqrt{2}-1} = \beta_2,$$

where $g(1) \approx 0.486889$. Since $\max\{\beta_1, \beta_2\} = \beta_2$, then $q_\beta(z) \prec \sqrt{1+z}$, whenever $\beta \geq g(1)/(\sqrt{2}-1)$.

□

By taking $p(z) = zf'(z)/f(z)$ in the above theorem, we obtain the following result.

Corollary 3.2.2. Suppose a function $f \in \mathcal{A}$ satisfies the subordination

$$1 + \beta \frac{zf'(z)}{f(z)}\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) \prec \frac{2}{1+e^{-z}}$$

and $g(z)$ be defined by (3.2.1). Then

(*i*) $f \in \mathcal{S}^*[A,B]$, whenever $\beta \geq \max\{(1+|B|)g(1)/(A-B), \operatorname{Im} g(i)(1+B^2)/(A-B)\}$.

(*ii*) $f \in \mathcal{S}^*_L$, whenever $\beta \geq g(1)/(\sqrt{2}-1)$.

Theorem 3.2.3. Suppose p is analytic in $\mathbb{D}$ and $p(0) = 1$ such that it satisfies $1 + \beta zp'(z)/p(z) \prec 2/(1+e^{-z})$ and $g(z)$ be defined by (3.2.1), then the following holds:

(*i*) $p(z) \prec (1+Az)/(1+Bz)$, whenever $\beta \geq \beta_{max}$, where $-1 < B < A < 1$ and

$$\beta_{\max} = \max\left\{\frac{-g(1)}{\log\left(\dfrac{1-A}{1-B}\right)}, \frac{g(1)}{\log\left(\dfrac{1+A}{1+B}\right)}, \frac{\operatorname{Im} g(i)}{\arcsin\left(\dfrac{A-B}{1+B^2}\right)}\right\}. \tag{3.2.4}$$

(*ii*) $p(z) \prec \sqrt{1+z}$, whenever $\beta \geq g(1)/\log(\sqrt{2})$.

These bounds are sharp.

Proof. The differential equation

$$1+\beta\frac{zq_\beta'(z)}{q_\beta(z)} = \frac{2}{1+e^{-z}} = h(z)$$

has a solution $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined as

$$q_\beta(z) = \exp\left[\frac{1}{\beta}\left(\frac{z}{2} - \frac{z^3}{72} + \frac{z^5}{1200} - \frac{17z^7}{282240}\cdots\right)\right].$$

With $\theta(w) = 1$ and $\psi(w) = \beta/w$ in Lemma 1.4.1, the function $Q : \mathbb{D} \to \mathbb{C}$ becomes

$$Q(z) = zq_\beta'(z)\phi(q_\beta(z)) = \beta\frac{zq_\beta'(z)}{q_\beta(z)} = \frac{e^z-1}{e^z+1}.$$

On the similar lines of the proof of Theorem 3.2.1, the following parts are proved.

(i) Let $\phi_0(z) = (1+Az)/(1+Bz)$. Then (3.2.2) and (3.2.3) give the following three inequalities:

$$q_\beta(-1) \geq \frac{1-A}{1-B}, \quad \text{whenever} \quad \beta \geq \frac{-g(1)}{\log((1-A)/(1-B))} = \beta_1,$$

$$q_\beta(1) \leq \frac{1+A}{1+B}, \quad \text{whenever} \quad \beta \geq \frac{g(1)}{\log((1+A)/(1+B))} = \beta_2$$

and

$$\operatorname{Im} q_\beta(i) \leq \operatorname{Im}\frac{1+Ai}{1+Bi}, \quad \text{whenever} \quad \beta \geq \frac{\operatorname{Im} g(i)}{\arcsin\left(\dfrac{A-B}{1+B^2}\right)} = \beta_3.$$

Let $\beta_{\max} = \max\{\beta_1, \beta_2, \beta_3\}$. Then $q_\beta(z) \prec (1+Az)/(1+Bz)$, whenever $\beta \geq \beta_{\max}$.

(ii) Let $\phi_0(z) = \sqrt{1+z}$. Then (3.2.2) gives the following two inequalities:

$$q_\beta(-1) \geq 0 \quad \text{for every } \beta$$

and

$$q_\beta(1) \leq \sqrt{2}, \quad \text{whenever} \quad \beta \geq g(1)/\log(\sqrt{2}).$$

Thus $q_\beta(z) \prec \sqrt{1+z}$, whenever $\beta \geq g(1)/\log(\sqrt{2})$.

□

The following result is obtained by taking $p(z) = zf'(z)/f(z)$ in the above theorem.

Corollary 3.2.4. Suppose a function $f \in \mathcal{A}$ satisfies the subordination

$$1 + \beta\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) \prec \frac{2}{1+e^{-z}}$$

and $g(z)$ be defined by (3.2.1). Then

(*i*) $f \in \mathcal{S}^*[A,B]$, whenever $\beta \geq \beta_{\max}$, where $\beta_{\max}$ is given in (3.2.4).

(*ii*) $f \in \mathcal{S}^*_L$, whenever $\beta \geq g(1)/\log(\sqrt{2})$.

Theorem 3.2.5. Suppose p is analytic in $\mathbb{D}$ and $p(0) = 1$ such that it satisfies $1 + \beta zp'(z)/p^2(z) \prec 2/(1+e^{-z})$ and $g(z)$ be defined by (3.2.1). Then the following are true:

(*i*) $p(z) \prec (1+Az)/(1+Bz)$, whenever $\beta \geq (1+|A|)g(1)/(A-B)$, where $-1 < B < A < 1$.

(*ii*) $p(z) \prec \sqrt{1+z}$, whenever $\beta \geq g(1)\sqrt{2}/(\sqrt{2}-1)$.

These bounds are sharp.

Proof. The differential equation

$$1 + \beta \frac{zq'_\beta(z)}{q^2_\beta(z)} = \frac{2}{1+e^{-z}} = h(z)$$

has a solution $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined as

$$q_\beta(z) = \frac{1}{1 - \dfrac{1}{\beta}\left[\dfrac{z}{2} - \dfrac{z^3}{72} + \dfrac{z^5}{1200} - \dfrac{17z^7}{282240} \cdots\right]}.$$

Taking $\theta(w) = 1$ and $\psi(w) = \beta/w^2$ in Lemma 1.4.1, the function $Q : \mathbb{D} \to \mathbb{C}$ becomes

$$Q(z) = zq_\beta'(z)\phi(q_\beta(z)) = \beta\frac{zq_\beta'(z)}{q_\beta^2(z)} = \frac{e^z - 1}{e^z + 1}.$$

A calculation shows that $\operatorname{Re}(zQ'(z)/Q(z)) = \operatorname{Re}(2ze^z/(e^{2z} - 1)) > 0$, $z \in \mathbb{D}$ and hence Q is starlike in $\mathbb{D}$. Since $h(z) = 1 + Q(z)$, it is clear that $\operatorname{Re}(zh'(z)/Q(z)) > 0$ on $\mathbb{D}$. Thus, as a consequence of Lemma 1.4.1, we have $p(z) \prec q_\beta(z)$. Now, we only need to show that $q_\beta(z) \prec \phi_0(z)$ in each of the parts. The geometry of the respective functions indicate that the condition (3.2.2) is necessary as well as sufficient for $q_\beta(z) \prec \phi_0(z)$.

(i) Let $\phi_0(z) = (1 + Az)/(1 + Bz)$. Then (3.2.2) gives the following two inequalities:

$$q_\beta(-1) \geq \frac{1 - A}{1 - B}, \quad \text{whenever} \quad \beta \geq g(1)\frac{1 - A}{A - B} = \beta_1$$

and

$$q_\beta(1) \leq \frac{1 + A}{1 + B}, \quad \text{whenever} \quad \beta \geq g(1)\frac{1 + A}{A - B} = \beta_2,$$

where $g(1) \approx 0.486889$. We observe that

$$\max\{\beta_1, \beta_2\} = \begin{cases} \beta_1, & \text{if} \quad A < 0 \\ \beta_2, & \text{if} \quad A > 0 \end{cases}$$

Therefore, $q_\beta(z) \prec (1 + Az)/(1 + Bz)$, whenever $\beta \geq g(1)(1 + |A|)/(A - B)$.

(ii) Let $\phi_0(z) = \sqrt{1 + z}$. Then (3.2.2) gives the following two inequalities:

$$q_\beta(-1) \geq 0 \quad \text{for} \quad \begin{cases} \beta < -g(1), & if \quad \beta < 0 \\ \text{any} \quad \beta, & if \quad \beta > 0 \end{cases}$$

and

$$q_\beta(1) \leq \sqrt{2}, \quad \text{whenever} \quad \beta \geq \frac{g(1)\sqrt{2}}{\sqrt{2} - 1}.$$

Since both the conditions hold for the case when $\beta \geq g(1)\sqrt{2}/(\sqrt{2}-1)$, we have $q_\beta(z) \prec \sqrt{1+z}$ for such β.

□

The following result is obtained by taking $p(z) = zf'(z)/f(z)$ in the above theorem.

Corollary 3.2.6. Suppose a function $f \in \mathcal{A}$ satisfies the subordination

$$1+\beta\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right) \prec \frac{2}{1+e^{-z}}$$

and $g(z)$ be defined by (3.2.1). Then

(*i*) $f \in \mathcal{S}^*[A,B]$, whenever $\beta \geq (1+|A|)g(1)/(A-B)$.

(*ii*) $f \in \mathcal{S}^*_L$, whenever $\beta \geq g(1)\sqrt{2}/(\sqrt{2}-1)$.

Further, we prove similar results by using a different technique, where we consider $\beta \in \mathbb{C}$.

Theorem 3.2.7. Let p be a function analytic in $\mathbb{D}$ such that $p(0) = 1$ and

$$1+\beta\frac{zp'(z)}{p^k(z)} \prec \frac{2}{1+e^{-z}} \qquad \text{for } k = 0, 1 \text{ and } 2.$$

Let $r_0 \approx 0.546302$ be the positive root of the equation $r^2 + 2\cot(1)r - 1 = 0$. Then each of the following holds:

(*i*) $p(z) \prec (1+Az)/(1+Bz)$, whenever $|\beta| \geq r_0(1+|B|)^{2-k}(1+|A|)^k/(A-B)$.

(*ii*) $p(z) \prec \sqrt{1+z}$, whenever $|\beta| \geq 2^{(k+3)/2} r_0$.

Proof. (i) Let $q(z) = (1+Az)/(1+Bz)$. Then the function $Q(z)$ given by

$$Q(z) = \beta\frac{zq'(z)}{q^k(z)} = \frac{\beta z(A-B)}{(1+Bz)^{2-k}(1+Az)^k}$$

is starlike in $\mathbb{D}$ (see [46]). Therefore, if the subordination

$$1+\beta\frac{zp'(z)}{p^k(z)} \prec 1+\beta\frac{zq'(z)}{q^k(z)}$$

holds, then $p(z) \prec q(z)$ using Lemma 1.4.1. To prove the desired result, it suffices to show that

$$\frac{2}{1+e^{-z}} \prec 1 + \frac{\beta z q'(z)}{q^k(z)} = 1 + \frac{\beta z(A-B)}{(1+Bz)^{2-k}(1+Az)^k} = h(z).$$

Let $w = \phi(z) = 2/(1+e^{-z})$. Then $\phi^{-1}(w) = \log(w/(2-w))$. The subordination $\phi(z) \prec h(z)$ is equivalent to $z \prec \phi^{-1}(h(z))$. Thus, we only need to show that $|\phi^{-1}(h(e^{it}))| \geq 1$. Taking $z = e^{it}$ $(0 \leq t < 2\pi)$, we have

$$|\phi^{-1}(h(e^{it}))| = \left| \log \left(\frac{1 + \dfrac{\beta e^{it}(A-B)}{(1+Be^{it})^{2-k}(1+Ae^{it})^k}}{1 - \dfrac{\beta e^{it}(A-B)}{(1+Be^{it})^{2-k}(1+Ae^{it})^k}} \right) \right| \geq 1.$$

By Lemma 3.1.3, it follows that the above inequality holds whenever

$$\left| \frac{\beta e^{it}(A-B)}{(1+Be^{it})^{2-k}(1+Ae^{it})^k} \right| \geq r_0,$$

which is true if

$$|\beta| \geq \frac{r_0(1+|B|)^{2-k}(1+|A|)^k}{A-B}.$$

(ii) Let $q(z) = \sqrt{1+z}$. Then, the function $Q(z)$ given by

$$Q(z) = \beta \frac{zq'(z)}{q^k(z)} = \frac{\beta z}{2(1+z)^{(k+1)/2}}$$

is starlike in $\mathbb{D}$ [46]. Therefore, if the subordination

$$1 + \beta \frac{zp'(z)}{p^k(z)} \prec 1 + \beta \frac{zq'(z)}{q^k(z)}$$

holds, then $p(z) \prec q(z)$ by Lemma 1.4.1. To prove the desired result, it is enough to show that

$$\frac{2}{1+e^{-z}} \prec 1 + \frac{\beta z q'(z)}{q^k(z)} = 1 + \frac{\beta z}{2(1+z)^{(k+1)/2}} = h(z).$$

On the similar lines of proof of part (i), we need to show that

$$|\phi^{-1}(h(e^{it}))| = \left| \log \left(\frac{1 + \dfrac{\beta e^{it}}{2(1+e^{it})^{(k+1)/2}}}{1 - \dfrac{\beta e^{it}}{2(1+e^{it})^{(k+1)/2}}} \right) \right| \geq 1.$$

Using Lemma 3.1.3, it follows that the above inequality holds, whenever

$$\left|\frac{\beta e^{it}}{2(1+e^{it})^{(k+1)/2}}\right| \geq r_0,$$

which is true, if

$$|\beta| \geq 2^{(k+3)/2} r_0.$$

□

Corollary 3.2.8. Let r_0 be the positive root of the equation $r^2 + 2\cot(1)r - 1 = 0$. If a function $f \in \mathcal{A}$ satisfies the subordination

$$1 + \beta \frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{2}{1+e^{-z}},$$

then

(*i*) $f \in \mathcal{S}^*[A,B]$, whenever $|\beta| \geq r_0(1+|B|)^2/(A-B)$. In particular, f is starlike for $|\beta| \geq 2r_0$.

(*ii*) $f \in \mathcal{S}^*_L$, whenever $|\beta| \geq 2\sqrt{2}r_0$.

Corollary 3.2.9. Let r_0 be the positive root of the equation $r^2 + 2\cot(1)r - 1 = 0$. If a function $f \in \mathcal{A}$ satisfies the subordination

$$1 + \beta\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{2}{1+e^{-z}},$$

then

(*i*) $f \in \mathcal{S}^*[A,B]$, whenever $|\beta| \geq r_0(1+|B|)(1+|A|)/(A-B)$. In particular, f is starlike for $|\beta| \geq 2r_0$.

(*ii*) $f \in \mathcal{S}^*_L$, whenever $|\beta| \geq 4r_0$.

Corollary 3.2.10. Let r_0 be the positive root of the equation $r^2 + 2\cot(1)r - 1 = 0$. If a function $f \in \mathcal{A}$ satisfies the subordination

$$1 + \beta\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{2}{1+e^{-z}},$$

then

(*i*) $f \in \mathcal{S}^*[A,B]$, whenever $|\beta| \geq r_0(1+|A|)^2/(A-B)$. In particular, f is starlike for $|\beta| \geq 2r_0$.

(*ii*) $f \in \mathcal{S}^*_L$, whenever $|\beta| \geq 4\sqrt{2}r_0$.

In the following results, we find sharp bounds on $\beta \in \mathbb{R}$ so that the following differential subordination implications hold:

$$1+\beta\frac{zp'(z)}{p^k(z)} \prec \phi_0(z) \qquad \Rightarrow \qquad p(z) \prec \frac{2}{1+e^{-z}} \qquad (k=0,1,2)$$

where ϕ_0 is taken as $(1+Az)/(1+Bz)$ and $\sqrt{1+z}$.

Theorem 3.2.11. Let p be a function analytic in $\mathbb{D}$ with $p(0)=1$. If any of the following conditions is true:

(*i*) $1+\beta zp'(z) \prec (1+Az)/(1+Bz)$, where $\beta \geq (A-B)\log(1-|B|)(1+e)/((1-e)|B|)$ and $-1 < B < A \leq 1$ $(B \neq 0)$

(*ii*) $1+\beta zp'(z) \prec \sqrt{1+z}$, where $\beta \geq 2(1-\log 2)(e+1)/(e-1)$,

then $p(z) \prec 2/(1+e^{-z})$.

Proof. (i) The differential equation

$$1+\beta z q'_\beta(z) = \frac{1+Az}{1+Bz} = h(z)$$

has a solution $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined as

$$q_\beta(z) = 1+\frac{(A-B)\log(1+Bz)}{B\beta}.$$

In order to apply Lemma 1.4.1, we take $\theta(w)=1$ and $\psi(w)=\beta$ and hence the function $Q : \mathbb{D} \to \mathbb{C}$ becomes

$$Q(z) = zq'_\beta(z)\phi(q_\beta(z)) = \beta z q'_\beta(z) = \frac{(A-B)z}{(1+Bz)}.$$

A calculation shows that, for $z \in \mathbb{D}$

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\frac{1}{1+Bz} > 0$$

and therefore Q is starlike in $\mathbb{D}$. It is easy to note that $h(z) = 1 + Q(z)$ and so $\operatorname{Re}(zh'(z)/Q(z)) > 0$ on $\mathbb{D}$. By Lemma 1.4.1, $1 + \beta zp'(z) \prec 1 + \beta zq'_\beta(z)$ implies $p(z) \prec q_\beta(z)$. Now, we only need to show that $q_\beta(z) \prec 2/(1+e^{-z}) = \phi(z)$. Geometrically, it can be concluded that the condition (3.2.2) is necessary as well as sufficient and yields the following two inequalities:

$$q_\beta(-1) \geq \frac{2}{1+e}, \quad \text{whenever} \quad \beta \geq \frac{(1+e)(A-B)\log(1-B)}{(1-e)B} = \beta_1$$

and

$$q_\beta(1) \leq \frac{2e}{1+e}, \quad \text{whenever} \quad \beta \geq \frac{(e+1)(A-B)\log(1+B)}{(e-1)B} = \beta_2.$$

We observe that

$$\max\{\beta_1, \beta_2\} = \begin{cases} \beta_1, & \text{if} \quad B > 0 \\ \beta_2, & \text{if} \quad B < 0. \end{cases}$$

Therefore $q_\beta(z) \prec 2/(1+e^{-z})$, whenever $\beta \geq (1+e)(A-B)\log(1-|B|)/(1-e)|B|$.

(ii) The differential equation

$$1 + \beta z q'_\beta(z) = \sqrt{1+z} = h(z)$$

has a solution $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined as

$$q_\beta(z) = 1 + \frac{2}{\beta}\Big(\sqrt{1+z} - \log(\sqrt{1+z}+1) + \log 2 - 1\Big).$$

Taking $\theta(w) = 1$ and $\psi(w) = \beta$ in Lemma 1.4.1, the function $Q : \mathbb{D} \to \mathbb{C}$ becomes

$$Q(z) = zq'_\beta(z)\phi(q_\beta(z)) = \beta z q'_\beta(z) = \sqrt{1+z} - 1.$$

A calculation shows that $Q(z)$ is starlike and $\operatorname{Re}(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$. Proceeding in the same manner as in part (i), we have the following two inequalities:

$$q_\beta(-1) \geq \frac{2}{1+e}, \text{ whenever } \beta \geq \frac{2(e+1)(1-\log 2)}{(e-1)} = \beta_1$$

and

$$q_\beta(1) \leq \frac{2e}{1+e}, \text{ whenever } \beta \geq \frac{e+1}{e-1}\Big(2\sqrt{2} - 2\log(\sqrt{2}+1) + 2\log 2 - 2\Big) = \beta_2.$$

We observe that

$$\max\{\beta_1,\beta_2\} = \beta_1.$$

Thus $q_\beta(z) \prec 2/(1+e^{-z})$, whenever $\beta \geq (e+1)(2-2\log 2)/(e-1)$.

$\square$

Corollary 3.2.12. Let $f \in \mathcal{A}$ and

$$\Phi_\beta(z) = 1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right).$$

Then $f \in \mathcal{S}^*_{SG}$, if any of the following conditions hold:

(*i*) $\Phi_\beta(z) \prec (1+Az)/(1+Bz)$, where $\beta \geq (A-B)\log(1-|B|)(1+e)/((1-e)|B|)$ and $-1 < B < A \leq 1$ $(B \neq 0)$.

(*ii*) $\Phi_\beta(z) \prec \sqrt{1+z}$, where $\beta \geq (e+1)(2-2\log 2)/(e-1)$.

Theorem 3.2.13. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$. If any of the following conditions hold:

(*i*) $1+\beta zp'(z)/p(z) \prec (1+Az)/(1+Bz)$ $(-1 < B < A \leq 1)$ and $\beta \geq \beta_{\max}$, where

$$\beta_{\max} = \begin{cases} \dfrac{(A-B)\log(1-B)}{\log(2/(1+e))B} & \text{if} \quad B < (1-e)/(1+e) \\[2ex] \dfrac{(A-B)\log(1+B)}{\log(2e/(e+1))B} & \text{if} \quad B \geq (1-e)/(1+e), \end{cases}$$

(*ii*) $1+\beta zp'(z)/p(z) \prec \sqrt{1+z}$, where $\beta \geq (2\sqrt{2}-2\log(\sqrt{2}+1)+2\log 2-2)/\log(2e/(1+e))$,

then $p(z) \prec 2/(1+e^{-z})$.

Proof. (i) The differential equation

$$1+\beta\frac{zq'_\beta(z)}{q_\beta(z)} = \frac{1+Az}{1+Bz} = h(z)$$

has a solution $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined as

$$q_\beta(z) = \exp\left(\frac{(A-B)\log(1+Bz)}{B\beta}\right).$$

Taking $\theta(w) = 1$ and $\psi(w) = \beta/w$ in Lemma 1.4.1, the function $Q : \mathbb{D} \to \mathbb{C}$ becomes

$$Q(z) = zq'_\beta(z)\phi(q_\beta(z)) = \frac{\beta z q'_\beta(z)}{q_\beta(z)} = \frac{(A-B)z}{(1+Bz)}.$$

Proceeding in the same manner as in Theorem 3.2.11 (i), we have the following two conditions:

$$q_\beta(-1) \geq \frac{2}{1+e}, \quad \text{whenever} \quad \beta \geq \frac{(A-B)\log(1-B)}{\log(2/(1+e))B} = \beta_1$$

and

$$q_\beta(1) \leq \frac{2e}{1+e}, \quad \text{whenever} \quad \beta \geq \frac{(A-B)\log(1+B)}{\log(2e/(e+1))B} = \beta_2.$$

We observe that

$$\max\{\beta_1, \beta_2\} = \beta_{max} = \begin{cases} \beta_1, & \text{if} \quad B < (1-e)/(1+e) \\ \beta_2, & \text{if} \quad B \geq (1-e)/(1+e). \end{cases}$$

Thus $q_\beta(z) \prec 2/(1+e^{-z})$, whenever $\beta \geq \beta_{\max}$.

(ii) The differential equation

$$1 + \beta \frac{zq'_\beta(z)}{q_\beta(z)} = \sqrt{1+z} = h(z)$$

has a solution $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined as

$$q_\beta(z) = \exp\left(\frac{2}{\beta}\left(\sqrt{1+z} - \log(\sqrt{1+z}+1) + \log 2 - 1\right)\right).$$

We apply Lemma 1.4.1 for $\theta(w) = 1$, $\psi(w) = \beta/w$ and the function $Q : \mathbb{D} \to \mathbb{C}$ given by

$$Q(z) = zq'_\beta(z)\phi(q_\beta(z)) = \beta \frac{zq'_\beta(z)}{q_\beta(z)} = \sqrt{1+z} - 1.$$

Proceeding in the same manner as in Theorem 3.2.11 (ii), we have the following two conditions:

$$q_\beta(-1) \geq \frac{2}{1+e}, \quad \text{whenever} \quad \beta \geq \frac{2\log 2 - 2}{\log(2/(1+e))} = \beta_1$$

and

$$q_\beta(1) \leq \frac{2e}{1+e}, \quad \text{whenever} \quad \beta \geq \frac{2\sqrt{2} - 2\log(\sqrt{2}+1) + 2\log 2 - 2}{\log(2e/(1+e))} = \beta_2.$$

Clearly,

$$\max\{\beta_1, \beta_2\} = \beta_2.$$

Thus $q_\beta(z) \prec 2/(1+e^{-z})$, whenever $\beta \geq (2\sqrt{2} - 2\log(\sqrt{2}+1) + 2\log 2 - 2)/\log(2e/(1+e))$.

□

Corollary 3.2.14. Let $f \in \mathcal{A}$ and

$$\Phi_\beta(z) = 1 + \beta\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right).$$

Then $f \in \mathcal{S}^*_{SG}$, if any of the following conditions hold:

(*i*) $\Phi_\beta(z) \prec (1+Az)/(1+Bz)$ and $\beta \geq \beta_{\max}$, where $-1 < B < A \leq 1$ and

$$\beta_{\max} = \begin{cases} \dfrac{(A-B)\log(1-B)}{\log(2/(1+e))B}, & \text{if} \quad B < (1-e)/(1+e) \\[2ex] \dfrac{(A-B)\log(1+B)}{\log(2e/(e+1))B}, & \text{if} \quad B \geq (1-e)/(1+e). \end{cases}$$

(*ii*) $\Phi_\beta(z) \prec \sqrt{1+z}$, where $\beta \geq (2\sqrt{2} - 2\log(\sqrt{2}+1) + 2\log 2 - 2)/\log(2e/(1+e))$.

Theorem 3.2.15. Let p be a function analytic in $\mathbb{D}$ such that $p(0) = 1$. If any of the following conditions hold:

(*i*) $1 + \beta zp'(z)/p^2(z) \prec (1+Az)/(1+Bz)$ and $\beta \geq \beta_{\max}$, where $-1 < B < A \leq 1$ and

$$\beta_{\max} = \max\left\{\frac{2(A-B)\log(1-B)}{(1-e)B}, \frac{2e(A-B)\log(1+B)}{(e-1)B}\right\}$$

(*ii*) $1 + \beta zp'(z)/p^2(z) \prec \sqrt{1+z}$, where $\beta \geq 4e\left(\sqrt{2} - \log(\sqrt{2}+1) + \log 2 - 1\right)/(e-1)$,

then $p(z) \prec 2/(1+e^{-z})$.

Proof. (i) The differential equation

$$1+\beta\frac{zq_\beta'(z)}{q_\beta^2(z)}=\frac{1+Az}{1+Bz}=h(z)$$

has a solution $q_\beta:\overline{\mathbb{D}}\to\mathbb{C}$ defined as

$$q_\beta(z)=\frac{1}{1-\dfrac{(A-B)\log(1+Bz)}{B\beta}}.$$

We take $\theta(w)=1$ and $\psi(w)=\beta/w^2$ and use Lemma 1.4.1, so that the function $Q:\mathbb{D}\to\mathbb{C}$ becomes

$$Q(z)=zq_\beta'(z)\phi(q_\beta(z))=\beta\frac{zq_\beta'(z)}{q_\beta^2(z)}=\frac{(A-B)z}{(1+Bz)}.$$

Proceeding in the same manner as in Theorem 3.2.11 (i), we have the following two conditions:

$$q_\beta(-1)\geq\frac{2}{1+e},\quad \text{whenever}\quad \beta\geq\frac{2(A-B)\log(1-B)}{(1-e)B}=\beta_1$$

and

$$q_\beta(1)\leq\frac{2e}{1+e},\quad \text{whenever}\quad \beta\geq\frac{2e(A-B)\log(1+B)}{(e-1)B}=\beta_2.$$

Let $B_0\approx 0.796615$ be the root of the equation $\log(1-B)+e\log(1+B)=0$. Then

$$\max\{\beta_1,\beta_2\}=\beta_{\max}=\begin{cases}\beta_1, & \text{if}\quad B<B_0\\ \beta_2, & \text{if}\quad B\geq B_0.\end{cases}\tag{3.2.5}$$

Thus $q_\beta(z)\prec 2/(1+e^{-z})$, whenever $\beta\geq\beta_{\max}$.

(ii) The differential equation

$$1+\beta\frac{zq_\beta'(z)}{q_\beta^2(z)}=\sqrt{1+z}=h(z)$$

has a solution $q_\beta:\overline{\mathbb{D}}\to\mathbb{C}$ defined as

$$q_\beta(z)=\frac{1}{1-\dfrac{2}{\beta}\left(\sqrt{1+z}-\log(\sqrt{1+z}+1)+\log 2-1\right)}.$$

We apply Lemma 1.4.1 with $\theta(w) = 1$, $\psi(w) = \beta/w^2$ and

$$Q(z) = zq'_\beta(z)\phi(q_\beta(z)) = \beta\frac{zq'_\beta(z)}{q^2_\beta(z)} = \sqrt{1+z}-1.$$

Proceeding in the same manner as in Theorem 3.2.11 (ii), we have the following two conditions:

$$q_\beta(-1) \geq \frac{2}{1+e}, \quad \text{whenever} \quad \beta \geq \frac{4(\log 2-1)}{1-e} = \beta_1$$

and

$$q_\beta(1) \leq \frac{2e}{1+e}, \quad \text{whenever} \quad \beta \geq \frac{4e\left(\sqrt{2}-\log(\sqrt{2}+1)+\log 2-1\right)}{e-1} = \beta_2.$$

It can be easily concluded that $\max\{\beta_1,\beta_2\} = \beta_2$. Therefore, we have $q_\beta(z) \prec 2/(1+e^{-z})$, whenever $\beta \geq 4e\left(\sqrt{2}-\log(\sqrt{2}+1)+\log 2-1\right)/(e-1)$. □

Corollary 3.2.16. Let $f \in \mathcal{A}$ and

$$\Phi_\beta(z) = 1+\beta\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right).$$

Then $f \in \mathcal{S}^*_{SG}$, if any of the following conditions hold:

(*i*) $\Phi_\beta(z) \prec (1+Az)/(1+Bz)$ and $\beta \geq \beta_{\max}$, where $\beta_{\max}$ is given by equation (3.2.5) and $-1 < B < A \leq 1$.

(*ii*) $\Phi_\beta(z) \prec \sqrt{1+z}$, where $\beta \geq 4e\left(\sqrt{2}-\log(\sqrt{2}+1)+\log 2-1\right)/(e-1)$.

Again, we extend the preceding three results for $\beta \in \mathbb{C}$.

Theorem 3.2.17. Let p be a function analytic in $\mathbb{D}$ such that $p(0) = 1$. If any of the following conditions holds for $k = 0,1,2$:

(*i*) $1+\beta zp'(z)/p^k(z) \prec (1+Az)/(1+Bz)$, where $|\beta| \geq (A-B)e^{k-1}(1+e)^{2-k}/(1-|B|)2^{1-k}$ and $-1 < B < A \leq 1$

(*ii*) $1+\beta zp'(z)/p^k(z) \prec \sqrt{1+z}$, where $|\beta| \geq (\sqrt{2}+1)e^{k-1}(1+e)^{2-k}$,

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $q(z) = 2/(1+e^{-z})$. Then the function $Q : \mathbb{D} \to \mathbb{C}$ is given by

$$Q(z) = \beta\frac{zq'(z)}{q^k(z)} = \frac{2^{1-k}\beta ze^{-z}}{(1+e^{-z})^{2-k}}.$$

It is easy to see that $Q(z)$ is starlike in $\mathbb{D}$. By Lemma 1.4.1, if the subordination

$$1+\beta\frac{zp'(z)}{p^k(z)} \prec 1+\beta\frac{zq'(z)}{q^k(z)}$$

holds, then $p(z) \prec q(z)$.

(i) To prove the required result, we need to prove that

$$\frac{1+Az}{1+Bz} \prec 1+\frac{\beta zq'(z)}{q^k(z)} = 1+\frac{2^{1-k}\beta z e^{-z}}{(1+e^{-z})^{2-k}} = h(z).$$

Let $w=\phi(z)=(1+Az)/(1+Bz)$. Then $\phi^{-1}(w)=(w-1)/(A-Bw)$. The subordination $\phi(z)\prec h(z)$ is equivalent to $z\prec\phi^{-1}(h(z))$. Thus it suffices to show $|\phi^{-1}(h(e^{it}))|\geq 1$. Taking $z=e^{it}$ $(0\leq t<2\pi)$, we have

$$\begin{aligned}|\phi^{-1}(h(e^{it}))| &\geq \frac{2^{1-k}|\beta|}{(A-B)e^{\cos t}(1+e^{-2\cos t}+2e^{-\cos t}\cos(\sin t))^{\frac{2-k}{2}}+2^{1-k}|B\beta|}\\ &= a(t).\end{aligned}$$

A computation shows that $\min_{0\leq t<2\pi} a(t)$ is attained at $t=0$. Thus

$$a(0)=\frac{2^{1-k}|\beta|}{(A-B)e(1+e^{-1})^{2-k}+2^{1-k}|B\beta|}\geq 1,$$

whenever

$$|\beta|\geq\frac{(A-B)e(1+e^{-1})^{2-k}}{(1-|B|)2^{1-k}}.$$

(ii) Here,we need to prove that $\sqrt{1+z}\prec h(z)$. Let $w=\phi(z)=\sqrt{1+z}$. Then $\phi^{-1}(w)=w^2-1$. To prove the required result, it suffices to show that $|\phi^{-1}(h(e^{it}))|\geq 1$. Taking $z=e^{it}$ $(0\leq t<2\pi)$, we have

$$|\phi^{-1}(h(e^{it}))|=\left|\left(1+\frac{2^{1-k}\beta e^{it}e^{-e^{it}}}{(1+e^{-e^{it}})^{2-k}}\right)^2-1\right|\geq 1,$$

whenever

$$\left|1+\frac{2^{1-k}\beta e^{it}e^{-e^{it}}}{(1+e^{-e^{it}})^{2-k}}\right|\geq\sqrt{2}.$$

The above inequality is true, whenever

$$\frac{2^{1-k}|\beta|e^{-\cos t}}{(1+e^{-2\cos t}+2e^{-\cos t}\cos(\sin t))^{(2-k)/2}}\geq\sqrt{2}+1.$$

The function on the left hand side attains its minimum at $t = 0$ and is equal to $2^{1-k}|\beta|e^{-1}/(1+e^{-1})^{2-k}$. Thus $\phi(z) \prec h(z)$, whenever $|\beta| \geq 2^{k-1}(\sqrt{2}+1)e(1+e^{-1})^{2-k}$. $\square$

Corollary 3.2.18. Let $f \in \mathcal{A}$ and

$$\Phi_\beta(z) = 1 + \beta\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right).$$

Then $f \in \mathcal{S}^*_{SG}$, if any of the following conditions hold:

(*i*) $\Phi_\beta(z) \prec (1+Az)/(1+Bz)$, where $|\beta| \geq (A-B)(1+e)^2/2e(1-|B|)$ and $-1 < B < A \leq 1$.

(*ii*) $\Phi_\beta(z) \prec \sqrt{1+z}$, where $|\beta| \geq (\sqrt{2}+1)(1+e)^2/2e$.

Corollary 3.2.19. Let $f \in \mathcal{A}$ and

$$\Phi_\beta(z) = 1 + \beta\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right).$$

Then $f \in \mathcal{S}^*_{SG}$, if any of the following conditions hold:

(*i*) $\Phi_\beta(z) \prec (1+Az)/(1+Bz)$, where $|\beta| \geq (A-B)(1+e)/(1-|B|)$ and $-1 < B < A \leq 1$.

(*ii*) $\Phi_\beta(z) \prec \sqrt{1+z}$, where $|\beta| \geq (\sqrt{2}+1)(1+e)$.

Corollary 3.2.20. Let $f \in \mathcal{A}$ and

$$\Phi_\beta(z) = 1 + \beta\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right).$$

Then $f \in \mathcal{S}^*_{SG}$, if any of the following conditions hold:

(*i*) $\Phi_\beta(z) \prec (1+Az)/(1+Bz)$, where $|\beta| \geq 2(A-B)e/(1-|B|)$ and $-1 < B < A \leq 1$.

(*ii*) $\Phi_\beta(z) \prec \sqrt{1+z}$, where $|\beta| \geq 2(\sqrt{2}+1)e$.

Now using Lemma 3.1.2, we obtain the following subordination results pertaining to the class $\mathcal{S}^*_{SG}$. In particular when $h(z) = 2/(1+e^{-z})$ and $P(z) = \beta$, we have:

Theorem 3.2.21. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ such that it satisfies

$$p(z) + \beta zp'(z) \prec \frac{2}{1+e^{-z}}, \qquad \operatorname{Re}\beta > 0.$$

Then $p(z) \prec 2/(1+e^{-z})$.

By taking $p(z) = zf'(z)/f(z)$ in Theorem 3.2.21, we obtain the following result.

Corollary 3.2.22. If a function $f \in \mathcal{A}$ satisfies the subordination

$$\frac{zf'(z)}{f(z)}\left[1+\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right] \prec \frac{2}{1+e^{-z}},$$

then $f \in \mathcal{S}^*_{SG}$, whenever $\operatorname{Re}\beta > 0$.

By taking $h(z) = 2/(1+e^{-z})$ and $P(z) = \beta/p(z)$ in Lemma 3.1.2, we obtain the following result:

Theorem 3.2.23. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ satisfying

$$p(z)+\beta\frac{zp'(z)}{p(z)} \prec \frac{2}{1+e^{-z}}$$

and $\kappa(t)$ be defined as

$$\kappa(t) = \frac{-\sin(\sin t)}{e^{\cos t}+\cos(\sin t)}.$$

Then $p(z) \prec 2/(1+e^{-z})$ whenever $\operatorname{Re}\beta > \kappa(-s_0)|\operatorname{Im}\beta|$, where $s_0 \approx 1.94549$ is the smallest root of the equation $\cos t + e^{\cos t}\cos(\sin t - t) = 0$.

Proof. Let $h(z) = 2/(1+e^{-z})$. Then $h(0) = 1$ and $h(\mathbb{D}) = \{w \in \mathbb{C} : |\log(w/(2-w))| < 1\}$ is a convex set. Therefore, h is a convex function. By taking $\phi(w) = \beta/w$ in Lemma 3.1.1 , the function $H(z)$ becomes

$$H(z) = zh'(z)\phi(h(z)) = \frac{\beta z}{(1+e^z)}.$$

A calculation shows that, for $z \in \mathbb{D}$,

$$\operatorname{Re}\frac{zH'(z)}{H(z)} = 1-\operatorname{Re}\left(\frac{ze^z}{1+e^z}\right) > 0$$

and hence H is starlike in $\mathbb{D}$. Also,

$$\operatorname{Re}\phi[h(z)] = \frac{1}{2}\operatorname{Re}\beta(1+e^{-z}) > 0,$$

whenever

$$\operatorname{Re}\beta\cdot\operatorname{Re}(1+e^{-z}) - \operatorname{Im}\beta\cdot\operatorname{Im}(1+e^{-z}) > 0.$$

The above inequality holds, for $z = e^{it}$ $(0 \le t < 2\pi)$, whenever

$$\frac{\operatorname{Re}\beta}{\operatorname{Im}\beta} > \frac{-\sin(\sin t)}{e^{\cos t} + \cos(\sin t)} = \kappa(t) \qquad \text{for the case when } \operatorname{Im}\beta > 0$$

and

$$\frac{\operatorname{Re}\beta}{\operatorname{Im}\beta} < \frac{-\sin(\sin t)}{e^{\cos t} + \cos(\sin t)} = \kappa(t) \qquad \text{for the case when } \operatorname{Im}\beta < 0.$$

The function $\kappa(t)$ attains its maximum at $t = -s_0 \approx -1.94549$, where $\kappa(-s_0) \approx 0.621289$ and minimum at $t = s_0 \approx 1.94549$, where $\kappa(s_0) \approx -0.621289$. So it is sufficient to take $\operatorname{Re}\beta > \kappa(-s_0)|\operatorname{Im}\beta|$. The result follows, by applying Lemma 3.1.1. □

Taking $p(z) = zf'(z)/f(z)$ in Theorem 3.2.23, we obtain the following result.

Corollary 3.2.24. If a function $f \in \mathcal{A}$ satisfies the subordination

$$\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{2}{1+e^{-z}},$$

then f is a member of $\mathcal{S}^*_{SG}$, whenever $\operatorname{Re}\beta > \kappa(-s_0)|\operatorname{Im}\beta|$, where $\kappa(-s_0)$ is as defined in Theorem 3.2.23.

The next result is obtained by taking $h(z) = 2/(1+e^{-z})$ and $P(z) = \beta/(p(z))^2$ in Lemma 3.1.2.

Theorem 3.2.25. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ satisfying

$$p(z) + \beta\frac{zp'(z)}{p^2(z)} \prec \frac{2}{1+e^{-z}}$$

and $\kappa(t)$ be defined as

$$\kappa(t) = \frac{-\left(e^{-2\cos t}\sin(2\sin t) + 2e^{-\cos t}\sin(\sin t)\right)}{1 + e^{-2\cos t}\cos(2\sin t) + 2e^{-\cos t}\cos(\sin t)}.$$

Then $p(z) \prec 2/(1+e^{-z})$, whenever $\operatorname{Re}\beta > \kappa(-s_0)|\operatorname{Im}\beta|$, where s_0 is given in Theorem 3.2.23.

Proof. Let $h(z) = 2/(1+e^{-z})$, which is a convex function. Taking $\phi(w) = \beta/w^2$ in Lemma 3.1.1, the function $H(z)$ becomes

$$H(z) = zh'(z)\phi(h(z)) = \frac{\beta z}{2e^z}.$$

A calculation shows that for $z \in \mathbb{D}$,

$$\operatorname{Re}\frac{zH'(z)}{H(z)} = 1 - \operatorname{Re} z > 0$$

and hence H is starlike in $\mathbb{D}$. Also,

$$\operatorname{Re}\phi[h(z)] = \frac{1}{4}\operatorname{Re}(\beta(1+e^{-z})^2) > 0,$$

whenever

$$\operatorname{Re}\beta \cdot \operatorname{Re}(1+e^{-z})^2 - \operatorname{Im}\beta \cdot \operatorname{Im}(1+e^{-z})^2 > 0.$$

The above inequality holds, for $z = e^{it}$ $(0 \leq t < 2\pi)$, whenever

$$\frac{\operatorname{Re}\beta}{\operatorname{Im}\beta} > \frac{-\left(e^{-2\cos t}\sin(2\sin t) + 2e^{-\cos t}\sin(\sin t)\right)}{1 + e^{-2\cos t}\cos(2\sin t) + 2e^{-\cos t}\cos(\sin t)} = \kappa(t)$$

for the case when $\operatorname{Im}\beta > 0$ and

$$\frac{\operatorname{Re}\beta}{\operatorname{Im}\beta} < \frac{-\left(e^{-2\cos t}\sin(2\sin t) + 2e^{-\cos t}\sin(\sin t)\right)}{1 + e^{-2\cos t}\cos(2\sin t) + 2e^{-\cos t}\cos(\sin t)} = \kappa(t)$$

for the case when $\operatorname{Im}\beta < 0$. The function $\kappa(t)$ attains its maximum at $t = -s_0$ and minimum at $t = s_0$. The approximate value of g at $-s_0$ and s_0 are 2.02374 and −2.02374 respectively. So it is sufficient to take $\operatorname{Re}\beta > \kappa(-s_0)|\operatorname{Im}\beta|$. Thus the result follows, by applying Lemma 3.1.1. □

Corollary 3.2.26. Suppose a function $f \in \mathcal{A}$ satisfies the subordination

$$\frac{zf'(z)}{f(z)} + \beta\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{2}{1+e^{-z}}.$$

Then f is a member of $\mathcal{S}^*_{SG}$, whenever $\operatorname{Re}\beta > \kappa(-s_0)|\operatorname{Im}\beta|$, where $\kappa(-s_0)$ is as defined in Theorem 3.2.25.

In the subsequent sections, we establish various differential subordination implications for first, second and third order respectively. We use the admissibility conditions for the modified sigmoid function, which are derived in the previous Chapter.

3.3 Applications of Admissibility Conditions

First Order Differential Subordination

In this section, we find conditions on β, γ and δ, where β is a complex number and $\delta, \gamma \in \mathbb{R}^+$, so that each of the following implications holds:

$$\left.\begin{array}{l} (i)\ 1+\beta(zp'(z))^n \prec \dfrac{2}{1+e^{-z}} \\ (ii)\ 1+\beta\dfrac{zp'(z)}{(p(z))^n} \prec \dfrac{2}{1+e^{-z}} \\ (iii)\ p(z)+\beta\dfrac{zp'(z)}{(p(z))^n} \prec \dfrac{2}{1+e^{-z}} \\ (iv)\ p(z)+\dfrac{zp'(z)}{\delta p(z)+\gamma} \prec \dfrac{2}{1+e^{-z}} \end{array}\right\} \text{ implies } \quad p(z) \prec \frac{2}{1+e^{-z}}.$$

Theorem 3.3.1. Let n be a positive integer and $\beta \in \mathbb{C}$ be such that $|\beta| \geq r_0(1+e)^{2n}/(2e)^n$, where r_0 is as given in Lemma 3.1.3. If $p(z)$ is analytic in $\mathbb{D}$ such that $p(0) = 1$ and satisfies the subordination

$$1+\beta(zp'(z))^n \prec \frac{2}{1+e^{-z}},$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = 2/(1+e^{-z})$. Then $\Omega = h(\mathbb{D}) = \Delta_{SG}$. Let $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be given as $\psi(a,b;z) = 1+\beta b^n$. In order to show that $\psi \in \Psi[\Omega, \Delta_{SG}]$, we refer to the admissibility conditions given in the previous chapter and deduce that it is sufficient to prove $\psi(r,s;z) \notin \Omega$. Now

$$\begin{aligned} |\beta s^n| &= \left|\beta\left(\frac{2me^{i\theta}e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^2}\right)^n\right| \\ &= |\beta|\left(\frac{2me^{-\cos\theta}}{1+e^{-2\cos\theta}+2e^{-\cos\theta}\cos(\sin\theta)}\right)^n \\ &\geq |\beta|\left(\frac{2me}{(1+e)^2}\right)^n. \end{aligned}$$

Since $m \geq 1$, we have

$$|\beta s^n| \geq \frac{(2e)^n|\beta|}{(1+e)^{2n}} \geq r_0. \tag{3.3.1}$$

Also

$$\left|\log\left(\frac{\psi(r,s;z)}{2-\psi(r,s;z)}\right)\right| = \left|\log\left(\frac{1+\beta s^n}{1-\beta s^n}\right)\right|.$$

Now using Lemma 3.1.3 and equation (3.3.1), we have

$$\left|\log\left(\frac{1+\beta s^n}{1-\beta s^n}\right)\right| \geq 1.$$

Thus $\psi \in \Psi[\Omega, \Delta_{SG}]$ and the result follows at once from Theorem 2.5.3. □

By taking $p(z) = zf'(z)/f(z)$ we have the following result:

Corollary 3.3.2. Let n be a positive integer and $\beta \in \mathbb{C}$ be such that $|\beta| \geq r_0(1+e)^{2n}/(2e)^n$. Suppose f be a function in $\mathcal{A}$ such that

$$1+\beta\left(\frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{zf'(z)}{f(z)}\right)^n \prec \frac{2}{1+e^{-z}},$$

then $f \in \mathcal{S}^*_{SG}$.

Theorem 3.3.3. Let n be a positive integer and $\beta \in \mathbb{C}$ be such that $|\beta| \geq r_0(2e)^{n-1}(1+e)^{2-n}$, where r_0 is as given in Lemma 3.1.3. If $p(z)$ is analytic in $\mathbb{D}$ such that $p(0) = 1$ and satisfies the subordination

$$1+\beta\frac{zp'(z)}{(p(z))^n} \prec \frac{2}{1+e^{-z}},$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = 2/(1+e^{-z})$. Then $\Omega = h(\mathbb{D}) = \Delta_{SG}$ and suppose $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be given as $\psi(a,b;z) = 1+\beta b/a^n$. Note that $\psi \in \Psi[\Omega, \Delta_{SG}]$, provided $\psi(r,s;z) \notin \Omega$. Consider

$$\begin{aligned}\left|\frac{\beta s}{r^n}\right| &= \left|\frac{2^{1-n} m\beta e^{i\theta} e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^{2-n}}\right| \\ &= \frac{2^{1-n} m|\beta| e^{-\cos\theta}}{(1+e^{-2\cos\theta}+2e^{-\cos\theta}\cos(\sin\theta))^{\frac{2-n}{2}}} \\ &\geq |\beta|\frac{2^{1-n} m e^{-1}}{(1+e^{-1})^{2-n}}.\end{aligned}$$

Since $m \geq 1$, we have

$$\left|\frac{\beta s}{r^n}\right| \geq |\beta|\frac{2^{1-n} e^{-1}}{(1+e^{-1})^{2-n}} \geq r_0. \tag{3.3.2}$$

Now we consider

$$\left|\log\left(\frac{\psi(r,s;z)}{2-\psi(r,s;z)}\right)\right| = \left|\log\left(\frac{1+\beta s/r^n}{1-\beta s/r^n}\right)\right|.$$

Using Lemma 3.1.3 and equation (3.3.2), we have

$$\left|\log\left(\frac{1+\beta s/r^n}{1-\beta s/r^n}\right)\right| \geq 1.$$

Thus we have $\psi \in \Psi[\Omega, \Delta_{SG}]$ and the result follows by Theorem 2.5.3. □

Taking $p(z) = zf'(z)/f(z)$ in Theorem 3.3.3, we obtain the following result.

Corollary 3.3.4. Let $\beta \in \mathbb{C}$ be such that $|\beta| \geq r_0(2e)^{n-1}(1+e)^{2-n}$ and n be a positive integer. Suppose f be a function in $\mathcal{A}$ such that

$$1+\beta\left(\frac{zf'(z)}{f(z)}\right)^{-n}\left(\frac{z^2f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{zf'(z)}{f(z)}\right) \prec \frac{2}{1+e^{-z}},$$

then $f \in \mathcal{S}^*_{SG}$.

Theorem 3.3.5. Let $\beta \in \mathbb{C}$ be such that $|\beta| \geq 2^n\left(\sqrt{\frac{1-\cos 1}{1+\cos 1}}\right)\frac{(1+e)^{2-n}}{e^{1-n}}$ and n be a positive integer. If $p(z)$ is analytic in $\mathbb{D}$ such that $p(0) = 1$ and satisfies the subordination

$$p(z) + \beta\frac{zp'(z)}{(p(z))^n} \prec \frac{2}{1+e^{-z}},$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = 2/(1+e^{-z})$ and so $\Omega = h(\mathbb{D}) = \Delta_{SG}$. If $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ is given by $\psi(a,b;z) = a + \beta b/a^n$, then $\psi \in \Psi[\Omega, \Delta_{SG}]$ provided $\psi(r,s;z) \notin \Omega$. Referring to Lemma 2.2.2, it is sufficient to show that $\psi(r,s;z)$ lies outside the smallest disk containing Δ_{SG}. We observe that

$$\begin{aligned}
|\psi(r,s;z) - 1| &= \left|r + \frac{\beta s}{r^n} - 1\right| \\
&= \left|\frac{1-e^{-e^{i\theta}}}{1+e^{-e^{i\theta}}} + \frac{\beta m 2^{1-n} e^{i\theta} e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^{2-n}}\right| \\
&\geq \left|\frac{\beta m 2^{1-n} e^{i\theta} e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})^{2-n}}\right| - \left|\frac{1-e^{-e^{i\theta}}}{1+e^{-e^{i\theta}}}\right| \\
&\geq \frac{|\beta| m 2^{1-n} e^{-\cos\theta}}{(1+e^{-2\cos\theta} + 2e^{-cos\theta}\cos(\sin\theta))^{\frac{2-n}{2}}} - \sqrt{\frac{e^{2\cos\theta} + 1 - 2e^{\cos\theta}\cos(\sin\theta)}{e^{2\cos\theta} + 1 + 2e^{\cos\theta}\cos(\sin\theta)}} \\
&\geq \frac{|\beta| 2^{1-n} e^{-1}}{(1+e^{-1})^{2-n}} - \sqrt{\frac{1-\cos 1}{1+\cos 1}} \\
&\geq \sqrt{\frac{1-\cos 1}{1+\cos 1}}.
\end{aligned}$$

Thus we have $\psi \in \Psi[\Omega, \Delta_{SG}]$ and the result follows by Theorem 2.5.3. □

Taking $p(z) = zf'(z)/f(z)$ in Theorem 3.3.5, we obtain the following result.

Corollary 3.3.6. Let $\beta \in \mathbb{C}$ be such that $|\beta| \geq 2^n \left(\sqrt{\frac{1-\cos 1}{1+\cos 1}} \right) \frac{(1+e)^{2-n}}{e^{1-n}}$ and n be a positive integer. Suppose f be a function in $\mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} + \beta \left(\frac{zf'(z)}{f(z)} \right)^{-n} \left(\frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)} \right)^2 + \frac{zf'(z)}{f(z)} \right) \prec \frac{2}{1+e^{-z}},$$

then $f \in \mathcal{S}^*_{SG}$.

Theorem 3.3.7. Let δ, $\gamma > 0$ be such that $(2\delta + \gamma + \gamma e)(1+e)\sqrt{1-\cos 1} \leq \sqrt{1+\cos 1}$. If $p(z)$ is analytic in $\mathbb{D}$ such that $p(0) = 1$ and satisfies the subordination

$$p(z) + \frac{zp'(z)}{\delta p(z) + \gamma} \prec \frac{2}{1+e^{-z}},$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = 2/(1+e^{-z})$. Then $\Omega = h(\mathbb{D}) = \Delta_{SG} = \{w : |\log(w/(2-w))| < 1\}$ and let $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be given as $\psi(a,b;z) = a + b/(\delta a + \gamma)$. For ψ to be in $\Psi[\Omega, \Delta_{SG}]$, we must have $\psi(r,s;z) \notin \Omega$. Now, let us consider

$$\begin{aligned}
|\psi(r,s;z) - 1| &= \left| r + \frac{s}{\delta r + \gamma} - 1 \right| \\
&= \left| \frac{1 - e^{-e^{i\theta}}}{1 + e^{-e^{i\theta}}} + \frac{2me^{i\theta} e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})(2\delta + \gamma(1+e^{-e^{i\theta}}))} \right| \\
&\geq \left| \frac{2me^{i\theta} e^{-e^{i\theta}}}{(1+e^{-e^{i\theta}})(2\delta + \gamma(1+e^{-e^{i\theta}}))} \right| - \left| \frac{1 - e^{-e^{i\theta}}}{1 + e^{-e^{i\theta}}} \right| \\
&= \frac{2me^{-\cos\theta}}{\sqrt{(1 + e^{-2\cos\theta} + 2e^{-cos\theta} \cos(\sin\theta))A(\theta)}} - \sqrt{\frac{e^{2\cos\theta} + 1 - 2e^{\cos\theta}\cos(\sin\theta)}{e^{2\cos\theta} + 1 + 2e^{\cos\theta}\cos(\sin\theta)}},
\end{aligned}$$

where $A(\theta) = (4\delta^2 + \gamma^2 + 4\delta\gamma + \gamma^2 e^{-2\cos\theta} + 2(2\delta + \gamma)\gamma e^{-\cos\theta} \cos(\sin\theta))$. Since $m \geq 1$, we have

$$\begin{aligned}
|\psi(r,s:z) - 1| &\geq \frac{2e^{-1}}{(1+e^{-1})(2\delta + \gamma + \gamma e)} - \sqrt{\frac{1-\cos 1}{1+\cos 1}} \\
&\geq \sqrt{\frac{1-\cos 1}{1+\cos 1}}.
\end{aligned}$$

Thus we have $\psi \in \Psi[\Omega, \Delta_{SG}]$ and the result follows as an application of Theorem 2.5.3.

□

Taking $p(z) = zf'(z)/f(z)$ in Theorem 3.3.7, we have the following result.

Corollary 3.3.8. Let δ, $\gamma > 0$ be such that $(2\delta+\gamma+\gamma e)(1+e)\sqrt{1-\cos 1} \leq \sqrt{1+\cos 1}$. Suppose f be a function in $\mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} + \left(\delta\frac{zf'(z)}{f(z)} + \gamma\right)^{-1}\left(\frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{zf'(z)}{f(z)}\right) \prec \frac{2}{1+e^{-z}},$$

then $f \in \mathcal{S}^*_{SG}$.

Second Order Differential Subordination

We consider the following differential subordination implication

$$1 + \gamma z p'(z) + \beta z^2 p''(z) \prec h(z) \quad \Rightarrow \quad p(z) \prec \frac{2}{1+e^{-z}}. \tag{3.3.3}$$

Taking $h(z)$ as any of $(1+Az)/(1+Bz)$, $\sqrt{1+z}, e^z, z+\sqrt{1+z^2}, 1+\sin z, 1+ze^z$ and $2/(1+e^{-z})$, sufficient conditions on positive real numbers β and γ are obtained so that the implication (3.3.3) holds. The proofs involve functions $d(\theta)$ and $g(\theta)$ which are given by (2.5.2) and (2.5.4) respectively.

Theorem 3.3.9. Let β, $\gamma > 0$ and $2e(\gamma(1+e)+\beta(1-e))(1-B^2) \geq (1+e)^3(1+|B|)(A-B)$, where $-1 \leq B < A \leq 1$. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ and

$$1 + \gamma z p'(z) + \beta z^2 p''(z) \prec \frac{1+Az}{1+Bz},$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = (1+Az)/(1+Bz)$ so that

$$h(\mathbb{D}) = \Omega = \left\{ w : \left| w - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \right\}.$$

Let $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c;z) = 1 + \gamma b + \beta c$. We know that ψ belongs to

$\Psi[\Omega,\Delta_{SG}]$, if $\psi(r,s,t;z)\notin\Omega$ for $z\in\mathbb{D}$. We consider

$$\begin{aligned}\left|\psi(r,s,t;z)-\frac{1-AB}{1-B^2}\right| &= \left|1+\gamma s+\beta t-\frac{1-AB}{1-B^2}\right|\\ &\geq \left|\gamma s+\beta t\right|-\frac{|B|(A-B)}{1-B^2}\\ &\geq |\gamma s|\operatorname{Re}\left(1+\frac{\beta}{\gamma}\frac{t}{s}\right)-\frac{|B|(A-B)}{1-B^2}\\ &\geq 2m\gamma d(\theta)\operatorname{Re}\left(1+\frac{\beta}{\gamma}(mg(\theta)+m-1)\right)-\frac{|B|(A-B)}{1-B^2}.\end{aligned}$$

Since $m\geq 1$, we have

$$\begin{aligned}\left|\psi(r,s,t;z)-\frac{1-AB}{1-B^2}\right| &\geq 2\gamma d(\theta)\operatorname{Re}\left(1+\frac{\beta}{\gamma}g(\theta)\right)-\frac{|B|(A-B)}{1-B^2}\\ &\geq \frac{2e}{(1+e)^2}\left(\gamma+\beta\left(\frac{1-e}{1+e}\right)\right)-\frac{|B|(A-B)}{1-B^2}\\ &\geq \frac{A-B}{1-B^2}.\end{aligned}$$

Therefore, $\psi\in\Psi[\Omega,\Delta_{SG}]$ and by Theorem 2.5.3, it follows that $p(z)\prec 2/(1+e^{-z})$. □

Theorem 3.3.10. Let $\gamma>\beta>0$ and $4e(\gamma(1+e)+\beta(1-e))(e(\gamma(1+e)+\beta(1-e))-(1+e)^3)\geq(1+e)^6$. Let p be a function analytic in $\mathbb{D}$ with $p(0)=1$ and

$$1+\gamma zp'(z)+\beta z^2p''(z)\prec\sqrt{1+z},$$

then $p(z)\prec 2/(1+e^{-z})$.

Proof. Let $h(z)=\sqrt{1+z}$ and thus $\Omega=\left\{w:\left|w^2-1\right|<1\right\}$. Let $\psi:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ be defined as $\psi(a,b,c;z)=1+\gamma b+\beta c$. Then $\psi\in\Psi[\Omega,\Delta_{SG}]$, provided $\psi(r,s,t;z)\notin\Omega$ for $z\in\mathbb{D}$. We observe that

$$\begin{aligned}\left|(\psi(r,s,t;z))^2-1\right| &= \left|(1+\gamma s+\beta t)^2-1\right|\\ &\geq |\gamma s+\beta t|(|\gamma s+\beta t|-2)\\ &\geq |\gamma s|\operatorname{Re}\left(1+\frac{\beta}{\gamma}\frac{t}{s}\right)\left(|\gamma s|\operatorname{Re}\left(1+\frac{\beta}{\gamma}\frac{t}{s}\right)-2\right).\end{aligned}$$

On similar lines of proof of Theorem 3.3.9 and using the inequality $m \geq 1$, we have

$$\begin{aligned}\left|(\psi(r,s,t;z))^2-1\right| &\geq \frac{2e}{(1+e)^2}\left(\gamma+\beta\left(\frac{1-e}{1+e}\right)\right)\left(\frac{2e}{(1+e)^2}\left(\gamma+\beta\left(\frac{1-e}{1+e}\right)\right)-2\right)\\ &= \frac{4e(\gamma(1+e)+\beta(1-e))(e(\gamma(1+e)+\beta(1-e))-(1+e)^3)}{(1+e)^6}\\ &\geq 1.\end{aligned}$$

Clearly $\psi \in \Psi[\Omega, \Delta_{SG}]$ and therefore, $p(z) \prec 2/(1+e^{-z})$ as an application of Theorem 2.5.3. □

Theorem 3.3.11. Let $\beta, \gamma > 0$ and $2e(\gamma(1+e)+\beta(1-e)) \geq (e-1)(e+1)^3$. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ and

$$1+\gamma z p'(z)+\beta z^2 p''(z) \prec e^z,$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = e^z$, so that $\Omega = h(\mathbb{D}) = \left\{w : \left|\log w\right| < 1\right\}$. Now suppose $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c;z) = 1+\gamma b+\beta c$. For ψ to be in $\Psi[\Omega, \Delta_{SG}]$, we must have $\psi(r,s,t;z) \notin \Omega$ for $z \in \mathbb{D}$. First we consider

$$\begin{aligned}|\gamma s+\beta t| &= \gamma|s|\left|1+\frac{\beta}{\gamma}\frac{t}{s}\right|\\ &\geq \gamma|s|\operatorname{Re}\left(1+\frac{\beta}{\gamma}\frac{t}{s}\right)\\ &\geq 2m\gamma d(\theta)\left(1+\frac{\beta}{\gamma}(mg(\theta)+m-1)\right).\end{aligned}$$

Since $m \geq 1$, we have

$$\begin{aligned}\left|\gamma s+\beta t\right| &\geq 2d(\theta)(\gamma+\beta(g(\theta)))\\ &\geq \frac{2e}{(1+e)^3}(\gamma(1+e)+\beta(1-e))\\ &\geq e-1. \end{aligned} \tag{3.3.4}$$

Note that

$$\left|\log(\psi(r,s,t;z))\right| = \left|\log(1+\gamma s+\beta t)\right|.$$

Using Lemma 3.1.4 and equation (3.3.4), we have

$$\left|\log(1+\gamma s+\beta t)\right| \geq 1,$$

which implies $\psi \in \Psi[\Omega, \Delta_{SG}]$. Hence $p(z) \prec 2/(1+e^{-z})$ using Theorem 2.5.3. □

Theorem 3.3.12. Let $\beta, \gamma > 0$ and $2e(\gamma(1+e)+\beta(1-e)) \geq \sqrt{2}(e+1)^3$. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ and

$$1+\gamma z p'(z)+\beta z^2 p''(z) \prec z+\sqrt{1+z^2},$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = z+\sqrt{1+z^2}$ and therefore $\Omega = \left\{w : \left|w^2-1\right| < 2|w|\right\}$. Let $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c;z) = 1+\gamma b+\beta c$. In order to show that $\psi \in \Psi[\Omega, \Delta_{SG}]$, we must have $\psi(r,s,t;z) \notin \Omega$ for $z \in \mathbb{D}$. Referring to the graph of $z+\sqrt{1+z^2}$ in [87], it is easy to observe that Ω is formed by the circles

$$C_1 : |z-1| = \sqrt{2} \quad \text{and} \quad C_2 : |z+1| = \sqrt{2}.$$

Clearly, Ω includes the disk enclosed by C_1 and excludes the part of the disk enclosed by C_2, which intersects with that of C_1. We see that

$$\left|(\psi(r,s,t;z))-1\right| = \left|\gamma s+\beta t\right|.$$

On the similar lines of proof of Theorem 3.3.11, we have

$$\begin{aligned} \left|\gamma s+\beta t\right| &\geq 2d(\theta)(\gamma+\beta(g(\theta))) \\ &\geq \frac{2e}{(1+e)^3}(\gamma(1+e)+\beta(1-e)) \\ &\geq \sqrt{2}. \end{aligned}$$

Thus we can say that $\psi(r,s,t;z)$ lies outside the circle C_1 which is sufficient to conclude that $\psi(r,s,t;z) \notin \Omega$. Therefore, $\psi \in \Psi[\Omega, \Delta_{SG}]$ and thus $p(z) \prec 2/(1+e^{-z})$ by using Theorem 2.5.3. □

Theorem 3.3.13. Let $\beta, \gamma > 0$ and $2e(\gamma(1+e)+\beta(1-e)) \geq \sinh 1(e+1)^3$. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ and

$$1+\gamma zp'(z)+\beta z^2p''(z) \prec 1+\sin z,$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = 1+\sin z$, so $\Omega = \{w : |\arcsin(w-1)| < 1\}$. Let $\psi : \mathbb{C}^3\times\mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c;z) = 1+\gamma b+\beta c$. We know that ψ belongs to $\Psi[\Omega,\Delta_{SG}]$, provided $\psi(r,s,t;z) \notin \Omega$ for $z \in \mathbb{D}$. Referring to [16, Lemma 3.3], we may observe that the disk $\{w \in \mathbb{C} : |w-1| < \sinh 1\}$ is the smallest disk containing Ω. So, we consider

$$\left|(\psi(r,s,t;z))-1\right| = \left|\gamma s+\beta t\right|.$$

On the similar lines of proof of Theorem 3.3.11, we have

$$\begin{aligned}\left|\gamma s+\beta t\right| &\geq 2d(\theta)\left(\gamma+\beta(g(\theta))\right)\\ &\geq \frac{2e}{(1+e)^3}\left(\gamma(1+e)+\beta(1-e)\right)\\ &\geq \sinh 1.\end{aligned}$$

Clearly, $\psi(r,s,t;z)$ lies outside the disk $\{w \in \mathbb{C} : |w-1| < \sinh 1\}$ which is sufficient to conclude that $\psi(r,s,t;z) \notin \Omega$. Thus $\psi \in \Psi[\Omega,\Delta_{SG}]$ and the result follows from Theorem 2.5.3. □

Theorem 3.3.14. Let β, $\gamma > 0$ and $2(\gamma(1+e)+\beta(1-e)) \geq (e+1)^3$. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ and

$$1+\gamma zp'(z)+\beta z^2p''(z) \prec 1+ze^z,$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = 1+ze^z$ and Ω be the image of $\mathbb{D}$ mapped by $1+ze^z$. Let $\psi : \mathbb{C}^3\times\mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c;z) = 1+\gamma b+\beta c$. For ψ to be in $\Psi[\Omega,\Delta_{SG}]$, we must have $\psi(r,s,t;z) \notin \Omega$ for $z \in \mathbb{D}$. It can be verified that the disk $\{w \in \mathbb{C} : |w-1| < e\}$ is the smallest disk containing Ω. Now, we consider

$$\left|(\psi(r,s,t;z))-1\right| = \left|\gamma s+\beta t\right|.$$

On the similar lines of proof of Theorem 3.3.11, we have

$$\begin{aligned} \left|\gamma s+\beta t\right| &\geq 2d(\theta)\left(\gamma+\beta(g(\theta))\right) \\ &\geq \frac{2e}{(1+e)^3}\left(\gamma(1+e)+\beta(1-e)\right) \\ &\geq e. \end{aligned}$$

Thus we can say that $\psi(r,s,t;z)$ lies outside the disk $\{w \in \mathbb{C} : |w-1| < e\}$ which is sufficient to conclude that $\psi(r,s,t;z) \notin \Omega$. So $\psi \in \Psi[\Omega, \Delta_{SG}]$ and thus, $p(z) \prec 2/(1+e^{-z})$ using Theorem 2.5.3. □

Theorem 3.3.15. Let $\beta, \gamma > 0$ and $2e(\gamma(1+e)+\beta(1-e)) \geq r_0(1+e)^3$, where r_0 is as given in Lemma 3.1.3. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ and

$$1+\gamma z p'(z)+\beta z^2 p''(z) \prec \frac{2}{1+e^{-z}},$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = 2/(1+e^{-z})$ so that $\Omega = \Delta_{SG} = \{w : |\log(w/(2-w))| < 1\}$. Let $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c;z) = 1+\gamma b+\beta c$. For ψ to be in $\Psi[\Delta_{SG}, \Delta_{SG}]$, we must have $\psi(r,s,t;z) \notin \Delta_{SG}$ for $z \in \mathbb{D}$. On the similar lines of Theorem 3.3.11, we have

$$\begin{aligned} \left|\gamma s+\beta t\right| &\geq 2d(\theta)\left(\gamma+\beta(g(\theta))\right) \\ &\geq \frac{2e}{(1+e)^3}\left(\gamma(1+e)+\beta(1-e)\right) \\ &\geq r_0. \end{aligned} \tag{3.3.5}$$

Now, we consider

$$\left|\log\left(\frac{\psi(r,s,t;z)}{2-\psi(r,s,t;z)}\right)\right| = \left|\log\left(\frac{1+\gamma s+\beta t}{1-(\gamma s+\beta t)}\right)\right|.$$

Using Lemma 3.1.3 and equation (3.3.5), we have

$$\left|\log\left(\frac{1+\gamma s+\beta t}{1-(\gamma s+\beta t)}\right)\right| \geq 1,$$

which implies that $\psi \in \Psi[\Delta_{SG}, \Delta_{SG}]$. Hence $p(z) \prec 2/(1+e^{-z})$ using Theorem 2.5.3. □

By taking $p(z) = zf'(z)/f(z)$ in Theorems 3.3.9-3.3.15, we obtain the following result.

Corollary 3.3.16. Let β, $\gamma > 0$ and f be a function in $\mathcal{A}$. Suppose

$$\begin{aligned}\Phi_f(z) &= 1+\gamma\left(\frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{zf'(z)}{f(z)}\right) + \beta\left(\frac{z^3 f^{(3)}(z)}{f(z)} + \frac{2z^2 f''(z)}{f(z)}\right. \\ &\quad \left. +2\left(\frac{zf'(z)}{f(z)}\right)^3 - 2\left(\frac{zf'(z)}{f(z)}\right)^2 - \frac{3z^3 f'(z) f''(z)}{f(z)^2}\right).\end{aligned}$$

Then $f \in \mathcal{S}^*_{SG}$ if any of the following conditions hold:

(*i*) $\Phi_f(z) \prec (1+Az)/(1+Bz)$ and $2e(\gamma(1+e)+\beta(1-e))(1-B^2) \geq (1+e)^3(1+|B|)(A-B)$, where $-1 \leq B < A \leq 1$.

(*ii*) $\Phi_f(z) \prec \sqrt{1+z}$ and $4e(\gamma(1+e)+\beta(1-e))(e(\gamma(1+e)+\beta(1-e))-(1+e)^3) \geq (1+e)^6$.

(*iii*) $\Phi_f(z) \prec e^z$ and $2e(\gamma(1+e)+\beta(1-e)) \geq (e-1)(e+1)^3$.

(*iv*) $\Phi_f(z) \prec z+\sqrt{1+z^2}$ and $2e(\gamma(1+e)+\beta(1-e)) \geq \sqrt{2}(e+1)^3$.

(*v*) $\Phi_f(z) \prec 1+\sin z$ and $2e(\gamma(1+e)+\beta(1-e)) \geq \sinh 1(e+1)^3$.

(*vi*) $\Phi_f(z) \prec 1+ze^z$ and $2(\gamma(1+e)+\beta(1-e)) \geq (e+1)^3$.

(*vii*) $\Phi_f(z) \prec 2/(1+e^{-z})$ and $2e(\gamma(1+e)+\beta(1-e)) \geq r_0(1+e)^3$, where $r_0 \approx 0.546302$ is the positive root of the equation $r^2+2\cot(1)r-1=0$.

Third Order Differential Subordination

In this section, we find sufficient conditions on positive real numbers α, β and γ so that the following third order differential subordination implication holds:

$$1+\gamma zp'(z)+\beta z^2 p''(z)+\alpha z^3 p'''(z) \prec h(z) \quad \Rightarrow \quad p(z) \prec \frac{2}{1+e^{-z}}.$$

Here $h(z)$ is chosen to be any amongst $(1+Az)/(1+Bz)$, $\sqrt{1+z}, e^z, z+\sqrt{1+z^2}, 1+\sin z, 1+ze^z$ and $2/(1+e^{-z})$. The conditions so obtained involve the constants r, s, t, u, m and k, which are given by Definition 2.5.4 for $f(z)=\log(p(z)/(2-p(z)))$ and ρ is as defined in (2.5.6).

Theorem 3.3.17. Let α, β, $\gamma > 0$ and $2e(\gamma(1+e)+\beta(1-e)+\alpha m^2(1+e)\rho+3\alpha m(k-1)(1-e))(1-B^2) \geq (1+e)^3(1+|B|)(A-B)$, where $-1 \leq B < A \leq 1$. Let p be a function analytic in $\mathbb{D}$ with $p(0)=1$ and

$$1+\gamma zp'(z)+\beta z^2 p''(z)+\alpha z^3 p'''(z) \prec \frac{1+Az}{1+Bz},$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = (1+Az)/(1+Bz)$ for $z \in \mathbb{D}$ so that

$$\Omega = \left\{w : \left|w - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}\right\}.$$

Let $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c,d;z) = 1+\gamma b+\beta c+\alpha d$. For ψ to be in $\Psi[\Omega,\Delta_{SG}]$, we must have $\psi(r,s,t,u;z) \notin \Omega$ for $z \in \mathbb{D}$. Now, let us consider

$$\begin{aligned}
\left|\psi(r,s,t,u;z) - \frac{1-AB}{1-B^2}\right| &= \left|1+\gamma s+\beta t+\alpha u - \frac{1-AB}{1-B^2}\right| \\
&\geq \left|\gamma s+\beta t+\alpha u\right| - \frac{|B|(A-B)}{1-B^2} \\
&\geq |\gamma s|\,\mathrm{Re}\left(1+\frac{\beta}{\gamma}\frac{t}{s}+\frac{\alpha}{\gamma}\frac{u}{s}\right) - \frac{|B|(A-B)}{1-B^2} \\
&\geq 2m\gamma d(\theta)\Big(1+\frac{\beta}{\gamma}(mg(\theta)+m-1)+\frac{\alpha}{\gamma}(m^2h(\theta) \\
&\quad +3m(k-1)g(\theta))\Big) - \frac{|B|(A-B)}{1-B^2}.
\end{aligned}$$

Since $m \geq 1$ and $\mathrm{Re}(1+g(\theta)) > 0$, we have

$$\begin{aligned}
\left|\psi(r,s,t,u;z) - \frac{1-AB}{1-B^2}\right| &\geq 2\gamma d(\theta)\left(1+\frac{\beta}{\gamma}g(\theta)+\frac{\alpha}{\gamma}(m^2h(\theta)+3m(k-1)g(\theta))\right) - \frac{|B|(A-B)}{1-B^2} \\
&\geq \frac{2e}{(1+e)^2}\left(\gamma+\beta\left(\frac{1-e}{1+e}\right)+\alpha m^2\rho+3m\alpha(k-1)\left(\frac{1-e}{1+e}\right)\right) - \frac{|B|(A-B)}{1-B^2} \\
&\geq \frac{A-B}{1-B^2}.
\end{aligned}$$

It follows that $\psi \in \Psi[\Omega,\Delta_{SG}]$ and hence $p(z) \prec 2/(1+e^{-z})$ using Lemma 2.5.6. □

Theorem 3.3.18. Let α, β and γ be positive real numbers and $4e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3m\alpha(k-1)(1-e))(e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3m\alpha(k-1)(1-e))-(1+e)^3) \geq (1+e)^6$. Let p be a function analytic in $\mathbb{D}$ with $p(0)=1$ and

$$1+\gamma zp'(z)+\beta z^2p''(z)+\alpha z^3p'''(z) \prec \sqrt{1+z},$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = \sqrt{1+z}$ so that $\Omega = h(\mathbb{D}) = \left\{w : \left|w^2-1\right| < 1\right\}$. Now suppose $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c,d;z) = 1+\gamma b+\beta c+\alpha d$. Note that $\psi \in \Psi[\Omega,\Delta_{SG}]$, if $\psi(r,s,t,u;z) \notin \Omega$

for $z \in \mathbb{D}$. We consider

$$\begin{aligned}\left|(\psi(r,s,t,u;z))^2-1\right| &= \left|(1+\gamma s+\beta t+\alpha u)^2-1\right| \\ &\geq |\gamma s+\beta t+\alpha u|(|\gamma s+\beta t+\alpha u|-2) \\ &\geq |\gamma s|\operatorname{Re}\left(1+\frac{\beta t}{\gamma s}+\frac{\alpha u}{\gamma s}\right)\left(|\gamma s|\operatorname{Re}\left(1+\frac{\beta t}{\gamma s}+\frac{\alpha u}{\gamma s}\right)-2\right).\end{aligned}$$

On similar lines of proof of Theorem 3.3.17 and using the inequality $m \geq 1$, we have

$$\begin{aligned}\left|(\psi(r,s,t,u;z))^2-1\right| &\geq \frac{2e}{(1+e)^2}\left(\gamma+\beta\left(\frac{1-e}{1+e}\right)+\alpha m^2\rho+3\alpha m(k-1)\left(\frac{1-e}{1+e}\right)\right)\left(\frac{2e}{(1+e)^2}\left(\gamma+\right.\right. \\ &\quad \left.\left.\beta\left(\frac{1-e}{1+e}\right)+\alpha m^2\rho+3\alpha m(k-1)\left(\frac{1-e}{1+e}\right)\right)-2\right) \\ &= \frac{4e}{(1+e)^6}(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e))(e(\gamma(1 \\ &\quad +e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e))-(1+e)^3) \\ &\geq 1,\end{aligned}$$

which implies $\psi \in \Psi[\Omega,\Delta_{SG}]$. Hence $p(z) \prec 2/(1+e^{-z})$ by Lemma 2.5.6. □

Theorem 3.3.19. Let α, $\beta, \gamma > 0$ and $2e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3m\alpha(k-1)(1-e)) \geq (e-1)(e+1)^3$. Let p be a function analytic in $\mathbb{D}$ with $p(0)=1$ and

$$1+\gamma zp'(z)+\beta z^2p''(z)+\alpha z^3p'''(z) \prec e^z,$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z)=e^z$ so that $\Omega = \left\{w : \left|\log w\right| < 1\right\}$. Let $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c,d;z) = 1+\gamma b+\beta c+\alpha d$. For ψ to be in $\Psi[\Omega,\Delta_{SG}]$, we must have $\psi(r,s,t,u;z) \notin \Omega$ for $z \in \mathbb{D}$. First, let us consider

$$\begin{aligned}|\gamma s+\beta t+\alpha u| &= \gamma|s|\left|1+\frac{\beta}{\gamma}\frac{t}{s}+\frac{\alpha}{\gamma}\frac{u}{s}\right| \\ &\geq \gamma|s|\operatorname{Re}\left(1+\frac{\beta}{\gamma}\frac{t}{s}+\frac{\alpha}{\gamma}\frac{u}{s}\right) \\ &\geq 2m\gamma d(\theta)\left(1+\frac{\beta}{\gamma}(mg(\theta)+m-1)+\frac{\alpha}{\gamma}(m^2h(\theta)+3m(k-1)g(\theta))\right).\end{aligned}$$

Since $m \geq 1$ and $\operatorname{Re}(1+g(\theta)) > 0$, we have

$$\begin{aligned} \left|\gamma s+\beta t+\alpha u\right| &\geq 2d(\theta)\left(\gamma+\beta g(\theta)+\alpha(m^2h(\theta)+3m(k-1)g(\theta))\right) \\ &\geq \frac{2e}{(1+e)^3}\left(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)\right) \\ &\geq e-1. \end{aligned} \tag{3.3.6}$$

Now we consider

$$\left|\log(1+\psi(r,s,t,u;z))\right| = \left|\log(1+\gamma s+\beta t+\alpha u)\right|.$$

Using Lemma 3.1.4 and equation (3.3.6), we have

$$\left|\log(1+\gamma s+\beta t+\alpha u)\right| \geq 1.$$

Therefore, $\psi \in \Psi[\Omega, \Delta_{SG}]$. Hence $p(z) \prec 2/(1+e^{-z})$. □

Theorem 3.3.20. Let α, $\beta, \gamma > 0$ and $2e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)) \geq \sqrt{2}(e+1)^3$. Let p be a function analytic in $\mathbb{D}$ with $p(0)=1$ and

$$1+\gamma zp'(z)+\beta z^2p''(z)+\alpha z^3p'''(z) \prec z+\sqrt{1+z^2},$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = z+\sqrt{1+z^2}$ so that $\Omega = \left\{w : \left|w^2-1\right| < 2|w|\right\}$. Let $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ be given as $\psi(a,b,c,d;z) = 1+\gamma b+\beta c+\alpha d$. We know that ψ belongs to $\Psi[\Omega, \Delta_{SG}]$, provided $\psi(r,s,t,u;z) \notin \Omega$ for $z \in \mathbb{D}$. We observe that

$$\left|(\psi(r,s,t,u;z))-1\right| = \left|\gamma s+\beta t+\alpha u\right|.$$

On the similar lines of proof of Theorem 3.3.19, we have

$$\begin{aligned} \left|\gamma s+\beta t+\alpha u\right| &\geq 2d(\theta)\left(\gamma+\beta g(\theta)+\alpha(m^2h(\theta)+3m(k-1)g(\theta))\right) \\ &\geq \frac{2e}{(1+e)^3}\left(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)\right) \\ &\geq \sqrt{2}. \end{aligned}$$

As done in the Theorem 3.3.12, the above condition is sufficient to conclude that $\psi(r,s,t,u;z)$ lies outside the circle C_1 and hence $\psi(r,s,t,u;z) \notin \Omega$. So $\psi \in \Psi[\Omega, \Delta_{SG}]$, which further implies $p(z) \prec 2/(1+e^{-z})$ using Lemma 2.5.6. □

Theorem 3.3.21. Let α, $\beta, \gamma > 0$ and $2e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e) \geq \sinh 1(e+1)^3$. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ and

$$1+\gamma zp'(z)+\beta z^2p''(z)+z^3p'''(z) \prec 1+\sin z,$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = 1+\sin z$ and thus $\Omega = \{w : |\arcsin(w-1)| < 1\}$. Let $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c,d;z) = 1+\gamma b+\beta c+\alpha d$. For ψ to be in $\Psi[\Omega,\Delta_{SG}]$, we must have $\psi(r,s,t,u;z) \notin \Omega$ for $z \in \mathbb{D}$. Referring to [16, Lemma 3.3], it is easy to observe that the disk $\{w \in \mathbb{C} : |w-1| < \sinh 1\}$ is the smallest disk that contains Ω. Let us consider

$$\left|(\psi(r,s,t,u;z))-1\right| = \left|\gamma s+\beta t+\alpha u\right|.$$

On the similar lines of proof of Theorem 3.3.19, we have

$$\begin{aligned}\left|\gamma s+\beta t+\alpha u\right| &\geq 2d(\theta)\left(\gamma+\beta g(\theta)+\alpha(m^2h(\theta)+3m(k-1)g(\theta))\right)\\ &\geq \frac{2e}{(1+e)^3}\left(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)\right)\\ &\geq \sinh 1.\end{aligned}$$

Thus we can say that $\psi(r,s,t,u;z)$ lies outside the disk $\{w \in \mathbb{C} : |w-1| < \sinh 1\}$ and so $\psi(r,s,t,u;z) \notin \Omega$. Clearly, $\psi \in \Psi[\Omega,\Delta_{SG}]$ and it follows from Lemma 2.5.6, $p(z) \prec 2/(1+e^{-z})$. □

Theorem 3.3.22. Let $\alpha, \beta, \gamma > 0$ and $2(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)) \geq (e+1)^3$. Let p be a function analytic in $\mathbb{D}$ with $p(0) = 1$ and

$$1+\gamma zp'(z)+\beta z^2p''(z)+\alpha z^3p'''(z) \prec 1+ze^z,$$

then $p(z) \prec 2/(1+e^{-z})$.

Proof. Let $h(z) = 1+ze^z$ for $z \in \mathbb{D}$ and Ω be the image of $\mathbb{D}$ mapped by $1+ze^z$. Let $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b,c,d;z) = 1+\gamma b+\beta c+\alpha d$. For ψ to be in $\Psi[\Omega,\Delta_{SG}]$, we must have $\psi(r,s,t,u;z) \notin \Omega$ for $z \in \mathbb{D}$. It can be easily verified that the disk $\{w \in \mathbb{C} :$

$|w-1|<e\}$ is the smallest disk that contains Ω. We observe that

$$\left|(\psi(r,s,t,u;z))-1\right| = \left|\gamma s+\beta t+\alpha u\right|.$$

On the similar lines of proof of Theorem 3.3.19, we have

$$\begin{aligned}
\left|\gamma s+\beta t+\alpha u\right| &\geq 2d(\theta)\Big(\gamma+\beta g(\theta)+\alpha(m^2h(\theta)+3m(k-1)g(\theta))\Big) \\
&\geq \frac{2e}{(1+e)^3}\Big(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)\Big) \\
&\geq e.
\end{aligned}$$

Thus, $\psi(r,s,t,u;z)$ lies outside the disk $\{w\in\mathbb{C}:|w-1|<e\}$ which implies that $\psi(r,s,t,u;z)\notin\Omega$. Hence $\psi\in\Psi[\Omega,\Delta_{SG}]$ and by Lemma 2.5.6, we have $p(z)\prec 2/(1+e^{-z})$. □

Theorem 3.3.23. Let $\alpha,\beta,\gamma>0$ and $2e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e))\geq(1+e)^3r_0$, where r_0 is as given in Lemma 3.1.3. Let p be a function analytic in $\mathbb{D}$ with $p(0)=1$ and

$$1+\gamma zp'(z)+\beta z^2p''(z)+\alpha z^3p'''(z)\prec\frac{2}{1+e^{-z}},$$

then $p(z)\prec 2/(1+e^{-z})$.

Proof. Let $h(z)=2/(1+e^{-z})$ so that $\Omega=\Delta_{SG}$. Let $\psi:\mathbb{C}^4\times\mathbb{D}\to\mathbb{C}$ be defined as $\psi(a,b,c,d;z)=1+\gamma b+\beta c+\alpha d$. For ψ to be in $\Psi[\Delta_{SG},\Delta_{SG}]$, we must have $\psi(r,s,t,u;z)\notin\Delta_{SG}$ for $z\in\mathbb{D}$. On the similar lines of Theorem 3.3.19, we have

$$\begin{aligned}
\left|\gamma s+\beta t+\alpha u\right| &\geq 2d(\theta)\Big(\gamma+\beta g(\theta)+\alpha(m^2h(\theta)+3m(k-1)g(\theta))\Big) \\
&\geq \frac{2e}{(1+e)^3}\Big(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)\Big) \\
&\geq r_0. \qquad (3.3.7)
\end{aligned}$$

Now, we consider

$$\left|\log\left(\frac{\psi(r,s,t,u;z)}{2-\psi(r,s,t,u;z)}\right)\right| = \left|\log\left(\frac{1+\gamma s+\beta t+\alpha u}{1-(\gamma s+\beta t+\alpha u)}\right)\right|.$$

By Lemma 3.1.3 and equation (3.3.7), the above quantity is greater than or equal to 1 and hence $\psi\in\Psi[\Delta_{SG},\Delta_{SG}]$, which further implies that $p(z)\prec 2/(1+e^{-z})$. □

Taking $p(z)=zf'(z)/f(z)$ in Theorem 3.3.17-3.3.23, we obtain the following result.

Corollary 3.3.24. Let $\alpha, \beta, \gamma > 0$ and f be a function in $\mathcal{A}$. Suppose

$$\begin{aligned}\chi_f(z) &:= 1+\gamma\frac{zf'(z)}{f(z)}+(\gamma+2\beta)\left(\frac{z^2f''(z)}{f(z)}-\left(\frac{zf'(z)}{f(z)}\right)^2\right)\\ &+(\beta+3\alpha)\left(2\left(\frac{zf'(z)}{f(z)}\right)^3-\frac{3z^3f'(z)f''(z)}{f(z)^2}+\frac{3z^3f^{(3)}(z)}{f(z)}\right)+\alpha\left(\frac{z^4f^{(4)}(z)}{f(z)}\right.\\ &\left.-\frac{3z^4f''(z)^2}{f(z)^2}-\left(\frac{6zf'(z)}{f(z)}\right)^4-\frac{4z^4f^{(3)}(z)f'(z)}{f(z)^2}+\frac{12z^4f'(z)^2f''(z)}{f(z)^3}\right).\end{aligned}$$

Then $f \in \mathcal{S}^*_{SG}$, if any of the following conditions hold:

(*i*) $\chi_f(z) \prec (1+Az)/(1+Bz)$ and $2e(\gamma(1+e)+\beta(1-e)+\alpha m^2(1+e)\rho+3\alpha m(k-1)(1-e))(1-B^2) \geq (1+e)^3(1+|B|)(A-B)$, where $-1 \leq B < A \leq 1$.

(*ii*) $\chi_f(z) \prec \sqrt{1+z}$ and $4e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3m\alpha(k-1)(1-e))(e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3m\alpha(k-1)(1-e))-(1+e)^3) \geq (1+e)^6$.

(*iii*) $\chi_f(z) \prec e^z$ and $2e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3m\alpha(k-1)(1-e)) \geq (e-1)(e+1)^3$.

(*iv*) $\chi_f(z) \prec z+\sqrt{1+z^2}$ and $2e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)) \geq \sqrt{2}(e+1)^3$.

(*v*) $\chi_f(z) \prec 1+\sin z$ and $2e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)) \geq \sinh 1(e+1)^3$.

(*vi*) $\chi_f(z) \prec 1+ze^z$ and $2(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)) \geq (e+1)^3$.

(*vii*) $\chi_f(z) \prec 2/(1+e^{-z})$ and $2e(\gamma(1+e)+\beta(1-e)+\alpha m^2\rho(1+e)+3\alpha m(k-1)(1-e)) \geq (1+e)^3 r_0$, where $r_0 \approx 0.546302$ is the positive root of the equation $r^2+2\cot(1)r-1=0$.

Concluding Remarks

An attempt has been made to establish certain third order differential subordination implications pertaining to the modified sigmoid function by applying the general third order admissibility criteria meant for Ma-Minda functions, which is first of its kind, in the literature. The class of Sigmoid starlike functions has been explored extensively for differential subordination problems. Since radius problems related to $\mathcal{S}^*_{SG}$ have not been explored so far, therefore in the next chapter, we will proceed to explore the radius problems related to the class $\mathcal{S}^*_{SG}$ and other classes as well.

Chapter 4

Radius Estimates and Sufficient Conditions for Certain Class of Analytic Functions

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In this chapter, we mainly focus on finding radius estimates for the Silverman class, the class of sigmoid starlike functions $\mathcal{S}^*_{SG}$ *and the class* Ω*, given by*

$$\Omega = \left\{ f \in \mathcal{A} : |zf'(z) - f(z)| < \frac{1}{2},\ z \in \mathbb{D} \right\}. \tag{4.0.1}$$

Further, we consider a general form of the Silverman class, introduced by Tuneski and Irmak [98] as

$$G_{\lambda,\alpha} = \left\{ f \in \mathcal{A} : \left| \frac{1-\alpha+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)} - (1-\alpha) \right| < \lambda, z \in \mathbb{D} \right\} \quad 0 < \alpha \leq 1,\ \lambda > 0 \tag{4.0.2}$$

and derive some sufficient conditions for the above class in the form of differential inequalities. Moreover, we deduce certain inclusion relations involving the above classes and some other well known subclasses of starlike functions.

4.1 Introduction

Radius problems have been one of the topics of major interest in geometric function theory. So far, a variety of techniques have come into existence, which help us to solve radius problems. Our analysis revealed that one or two properties of the Schwarz functions are applied for the majority of radius problems. These properties are given as follows:

Lemma 4.1.1 (Schwarz-Pick Lemma)**.** [44] Let ω be a function analytic on $\mathbb{D}$ such that $|\omega(z)| \leq 1$ and $\omega(0) = 0$, then for all $z \in \mathbb{D}$

$$|\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2}.$$

Lemma 4.1.2. [20] Let $\omega : \mathbb{D} \to \mathbb{D}$ be analytic, then for all $z \in \mathbb{D}$

$$|\omega'(z)| \leq \begin{cases} 1, & |z| \leq \sqrt{2} - 1, \\ \dfrac{(1 + r^2)^2}{4r(1 - r^2)}, & |z| \geq \sqrt{2} - 1. \end{cases}$$

Note that both of the above inequalities provide the upper bound on the Schwarz function's derivative. For the lower bound, we refer to [20] in which Dieudonné has proved a number of inequalities relating to derivatives of Schwarz function. In 1999, Silverman [88] introduced the following class

$$G_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b,\, z \in \mathbb{D} \right\}, \quad b > 0.$$

and established conditions on b for which the class G_b is contained in $\mathcal{S}^*$ and further in the class $\mathcal{S}^*(\alpha)$. In addition, the author estimated the largest radius for which every starlike function of order $1/2$ belongs to G_b. In 2006, the class G_b was generalised by Tuneski and Irmak [98] in the form given by (4.0.2). By taking $\alpha = \lambda = 1/2$, $G_{\lambda,\alpha}$ reduces to the class G_b with $b = 2\lambda$. In 2017, Peng and Zhong [73] introduced a new subclass Ω of $\mathcal{A}$ given by (4.0.1). For this class, the authors proved that $f \in \Omega$ if and only if

$$f(z) = z + \frac{1}{2} z \int_0^z \upsilon(\zeta) d\zeta, \tag{4.1.1}$$

where υ is analytic in $\mathbb{D}$ and $|\upsilon(z)| \leq 1$, $z \in \mathbb{D}$. Moreover, the authors proved its inclusion in $\mathcal{S}^*$, estimated radius of convexity and discussed many other properties of Ω. In

2019, Peng and Obradović [72] estimated logarithmic and inverse coefficient bounds for Ω, proved Robertson's 1/2 conjecture and 1/2 theorem and other results related to Hadamard product and coefficient multipliers. Later in this year, Swaminathan and Wani [93] defined a new class $\Omega_n = \{f \in \mathcal{A}_n : |zf'(z) - f(z)| < 1/2,\ z \in \mathbb{D}\}$. They obtained sufficient conditions for Ω_n, proved inclusion properties of Ω and derived sharp radii estimates for different subclasses of $\mathcal{S}^*$. Motivated by their work, we consider similar problems for the class $G_{\lambda,\alpha}$. We establish sufficient conditions for functions to be in $G_{\lambda,\alpha}$, involving double integrals and utilize these conditions to construct functions in $G_{\lambda,\alpha}$. We also obtain radius estimates for the Silverman class, the class Ω and $\mathcal{S}^*_{SG}$ involving other well known subclasses of Ma-Minda functions. Further by using the concept of subordination, we prove several inclusion relations amongst $G_{\lambda,\alpha}$, Ω and other well known subclasses of $\mathcal{S}^*$.

4.2 Radius Estimates

In the three subsections that follow, we find radius estimates for the original form of Silverman class given by $G_{\frac{1}{2},\frac{1}{2}}$, the class Ω given by (4.0.1) and the class $\mathcal{S}^*_{SG}$ respectively.

4.2.1 The Silverman class, $G_{\frac{1}{2},\frac{1}{2}}$

We begin with the following result, which is interesting to look at in terms of computations and moreover it makes use of rare aspects of Schwarz functions.

Theorem 4.2.1. If $f \in \Omega$, then $f \in G_{\frac{1}{2},\frac{1}{2}}$ in the disc $|z| < r_0$, where $r_0 \approx 0.430496$ is the smallest positive root of $55r^{12} - 28r^{11} - 854r^{10} + 148r^9 + 2969r^8 - 212r^7 - 4286r^6 + 28r^5 + 2875r^4 + 96r^3 - 888r^2 - 32r + 96 = 0$.

Proof. Let $f \in \Omega$, then f can be written in the from (4.1.1). Now if we let $\omega(z) = \int_0^z v(\zeta)d\zeta$, then clearly $\omega(z)$ and $\omega'(z)$ are analytic in $\mathbb{D}$ and we can write f as

$$f(z) = z + \frac{1}{2}z\omega(z). \tag{4.2.1}$$

Now by using the properties of v, we have

$$|\omega(z)| = \left|\int_0^z v(\zeta)d\zeta\right| \leq \int_0^z |v(\zeta)|d\zeta \leq |z|$$

and

$$|\omega'(z)| = |v(z)| \leq 1.$$

Using Schwarz-Pick Lemma, we have for $z \in \mathbb{D}$,

$$|\omega''(z)| \leq \frac{1-|\omega'(z)|^2}{1-|z|^2}. \tag{4.2.2}$$

By using certain results of Dieudonné [20], we have the following inequalities

$$|\omega'(z)| \geq \frac{(|\omega(z)|-r^2)(1+|\omega(z)|)}{r(1-r^2)} \tag{4.2.3}$$

and

$$|z\omega'(z)-\omega(z)| \leq \frac{r^2-|\omega(z)|^2}{1-r^2} \tag{4.2.4}$$

on $|z| = r$, where $|\omega(z)| \leq r$. In view of (4.2.1), we obtain

$$\begin{aligned} \left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}-1\right| &= \left|\frac{z\Big(z(\omega(z)+2)\omega''(z)-z\omega'(z)^2+(\omega(z)+2)\omega'(z)\Big)}{(z\omega'(z)+\omega(z)+2)^2}\right| \\ &\leq \frac{r((2+|z\omega'(z)-\omega(z)|)|\omega'(z)|)+r|\omega''(z)|(2+|\omega(z)|)}{(2(1-|\omega(z)|)-|z\omega'(z)-\omega(z)|)^2}. \end{aligned}$$

Using the inequalities (4.2.2), (4.2.3) and (4.2.4), we get

$$\begin{aligned} \left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}-1\right| &\leq \frac{r}{\left(2(1-|\omega(z)|)-\left(\frac{r^2-|\omega(z)|^2}{1-r^2}\right)\right)^2}\Bigg[\left(2+\left(\frac{r^2-|\omega(z)|^2}{1-r^2}\right)\right)\left(\frac{1-|\omega(z)|^2}{1-r^2}\right) \\ &\quad +r\left(\frac{1-\left(\frac{(|\omega(z)|-r^2)(1+|\omega(z)|)}{r(1-r^2)}\right)^2}{1-r^2}\right)(2+|\omega(z)|)\Bigg]. \end{aligned}$$

Writing $|\omega(z)| = \omega$, the above inequality becomes

$$\begin{aligned} \left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}-1\right| &\leq \frac{1}{(1-r^2)\left(2r^2\omega-3r^2+\omega^2-2\omega+2\right)^2}(r^6\omega+2r^6-r^5\omega^2+r^5-r^4\omega^3 \\ &\quad -4r^4\omega^2-7r^4\omega-6r^4-r^3\omega^4+4r^3\omega^2-3r^3+2r^2\omega^4+8r^2\omega^3+10r^2\omega^2 \\ &\quad +5r^2\omega+2r^2+r\omega^4-3r\omega^2+2r-\omega^5-4\omega^4-5\omega^3-2\omega^2). \end{aligned}$$

For f to be in $G_{\frac{1}{2},\frac{1}{2}}$, it suffices to show that

$$\frac{1}{(1-r^2)(2r^2\omega-3r^2+\omega^2-2\omega+2)^2}(r^6\omega+2r^6-r^5\omega^2+r^5-r^4\omega^3-4r^4\omega^2-7r^4\omega-6r^4$$
$$-r^3\omega^4+4r^3\omega^2-3r^3+2r^2\omega^4+8r^2\omega^3+10r^2\omega^2+5r^2\omega+2r^2+r\omega^4-3r\omega^2+2r-\omega^5$$
$$-4\omega^4-5\omega^3-2\omega^2)<1,$$

which is equivalent to

$$\begin{aligned}\Phi(\omega,r) \;:=\; & \omega^5+\left(r^3-3r^2-r+5\right)\omega^4+\left(1-3r^4\right)\omega^3+(-4r^6+r^5+22r^4-4r^3-32r^2\\ & +3r+10)\omega^2+\left(11r^6-25r^4+23r^2-8\right)\omega-11r^6-r^5+27r^4+3r^3-18r^2\\ & -2r+4>0.\end{aligned}$$

We may note that $\omega=|\omega(z)|\leq|z|=r$, so we have $0\leq\omega\leq r$. Let us write

$$\begin{aligned}A &= -4r^6+r^5+22r^4-4r^3-32r^2+3r+10,\\ B &= 11r^6-25r^4+23r^2-8 \quad \text{and}\\ C &= -11r^6-r^5+27r^4+3r^3-18r^2-2r+4,\end{aligned}$$

then $B^2-4AC<0$, whenever $r<r_1\approx 0.430496$. Also $A>0$, whenever $r<r_2\approx 0.565244$. Thus

$$(-4r^6+r^5+22r^4-4r^3-32r^2+3r+10)\omega^2+\left(11r^6-25r^4+23r^2-8\right)\omega-11r^6-r^5$$
$$+27r^4+3r^3-18r^2-2r+4>0,$$

whenever $r<\min\{r_1,r_2\}=r_1$. Next we observe that coefficients of ω^5 and ω^4 are always positive and coefficient of ω^3 is positive for the range $0\leq r<r_3=(1/3)^{1/4}\approx 0.759836$. It can be easily concluded that

$$\Phi(\omega,r)>0 \quad \text{whenever} \quad r<r_0=\min\{r_1,r_2,r_3\}=r_1.$$

Hence the result. □

Theorem 4.2.2. Let $f\in\mathcal{S}^*(\varphi_i)$ $(i=1,2,3)$, then $f\in G_{\frac{1}{2},\frac{1}{2}}$ in the disk $|z|<r_i$ $(i=1,2,3)$ for the following cases:

(i) $\varphi_1(z)=e^z$ and $r_1\approx 0.537561$ is the smallest positive root of $e^r(1+r^2)^2-4(1-r^2)=0$.

(ii) $\varphi_2(z) = \sqrt{1+z}$ and $r_2 \approx 0.429874$ is the smallest positive root of $(1+r^2)^2 - 4(1-r)^{3/2}(1-r^2) = 0$.

(iii) $\varphi_3(z) = 2/(1+e^{-z})$ and $r_3 \approx 0.683447$ is the smallest positive root of $e^r(1+r^2)^2 - 8(1-r^2) = 0$.

Proof. Let $f \in \mathcal{S}^*(\varphi_i)$, then we have $zf'(z)/f(z) \prec \varphi_i(z)$. Thus there exists a Schwarz function ω with $\omega(0) = 0$ and $|\omega(z)| \leq |z|$ such that

$$\frac{zf'(z)}{f(z)} = \varphi_i(\omega(z)),$$

which further implies

$$\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 = z\omega'(z)\frac{\varphi_i'(\omega(z))}{\varphi_i^2(\omega(z))}.$$

For f to be in $G_{\frac{1}{2},\frac{1}{2}}$, it is sufficient to show that $|z\omega'(z)\varphi_i'(\omega(z))/\varphi_i^2(\omega(z))| < 1$.

(i) Let $\varphi_1(z) = e^z$, then by using Lemma 4.1.2, we have

$$\left|z\omega'(z)\frac{\varphi_1'(\omega(z))}{\varphi_1^2(\omega(z))}\right| = \left|\frac{z\omega'(z)}{e^{\omega(z)}}\right| \leq \frac{e^r(1+r^2)^2}{4(1-r^2)},$$

which is less than 1 provided $r < r_1$.

(ii) Let $\varphi_2(z) = \sqrt{1+z}$. By Lemma 4.1.2 we have for $r < r_2$,

$$\left|z\omega'(z)\frac{\varphi_2'(\omega(z))}{\varphi_2^2(\omega(z))}\right| = \left|\frac{z\omega'(z)}{(1+\omega(z))^{3/2}}\right| \leq \frac{(1+r^2)^2}{4(1-r)^{3/2}(1-r^2)} < 1.$$

(iii) Let $\varphi_3(z) = 2/(1+e^{-z})$, then by using Lemma 4.1.2, we obtain

$$\left|z\omega'(z)\frac{\varphi_3'(\omega(z))}{\varphi_3^2(\omega(z))}\right| = \left|\frac{z\omega'(z)}{2e^{\omega(z)}}\right| \leq \frac{e^r(1+r^2)^2}{8(1-r^2)},$$

which is less than 1 whenever $r < r_3$.

□

4.2.2 The class Ω

This section includes a number of sharp radius estimates that have been determined for the class Ω.

Theorem 4.2.3. If $f \in \mathcal{S}^*_e$, then $f \in \Omega$ in the disc $|z| < \rho_e$, where $\rho_e \approx 0.476813$ is the smallest positive root of $2(e^r - 1)f_e(r) - 1 = 0$ and

$$f_e(z) = z\exp\left(\int_0^z \frac{e^t - 1}{t}dt\right) = z + z^2 + \frac{3z^3}{4} + \frac{17z^4}{36} + \frac{19z^5}{72} + \cdots. \tag{4.2.5}$$

Moreover, this estimate is sharp.

Proof. Let $f \in \mathcal{S}^*_e$. Then $zf'(z)/f(z) \prec e^z$, which further implies that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{|z|=r} |e^{re^{i\theta}} - 1| = e^r - 1.$$

We apply [58, Theorem 2.7] on f and observe that $|f(z)| \leq f_e(r)$ $(|z| = r)$, where f_e is given by (4.2.5). So

$$\left|zf'(z) - f(z)\right| = |f(z)|\left|\frac{zf'(z)}{f(z)} - 1\right| \leq f_e(r)(e^r - 1) \quad \text{on } |z| = r.$$

Taking $|z| < \rho_e$, we have $|zf'(z) - f(z)| < 1/2$ and the result is sharp for the function f_e. □

Theorem 4.2.4. Let $f \in \mathcal{S}^*_{☾}$, then $f \in \Omega$ in the disc $|z| < \rho_{☾}$, where $\rho_{☾} \approx 0.485894$ is the smallest positive root of $2(r + \sqrt{1 + r^2} - 1)f_{☾}(r) - 1 = 0$ and

$$f_{☾}(z) = z\exp\left(\int_0^z \frac{t + \sqrt{1 + t^2} - 1}{t}dt\right) = z + z^2 + \frac{3z^3}{4} + \frac{5z^4}{12} + \frac{z^5}{6} + \cdots. \tag{4.2.6}$$

This result is sharp.

Proof. Let $f \in \mathcal{S}^*_{☾}$. Then $zf'(z)/f(z) \prec z + \sqrt{1 + z^2}$, which is sufficient to say that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{|z|=r} |e^{i\theta} + \sqrt{1 + r^2e^{2i\theta}} - 1| = r + \sqrt{1 + r^2} - 1.$$

By using [76, Theorem 1], we obtain $|f(z)| \leq f_{☾}(r)$ $(|z| = r)$, where $f_{☾}$ is given by (4.2.6).

So on $|z| = r$, we have

$$\left|zf'(z) - f(z)\right| = |f(z)|\left|\frac{zf'(z)}{f(z)} - 1\right| \le f_{\leftmoon}(r)(r + \sqrt{1+r^2} - 1).$$

The above quantity is less that 1/2, provided $r < \rho_{\leftmoon}$. For the function $f_{\leftmoon}$, the inequality holds only in the disk $|z| < \rho_{\leftmoon}$, therefore the result is sharp. □

Theorem 4.2.5. If $f \in \mathcal{S}^*_{SG}$, then $f \in \Omega$ in the disc $|z| < \rho_{SG}$, where $\rho_{SG} \approx 0.799269$ is the smallest positive root of $2\tan(r/2) f_{SG}(r) - 1 = 0$ with

$$f_{SG}(z) = z\exp\left(\int_0^z \frac{e^t - 1}{t(e^t+1)} dt\right) = z + \frac{z^2}{2} + \frac{z^3}{8} + \frac{z^4}{144} - \frac{5z^5}{1152} + \cdots. \tag{4.2.7}$$

Proof. Let $f \in \mathcal{S}^*_{SG}$, so we have $zf'(z)/f(z) \prec 2/(1+e^{-z})$. Therefore

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \max_{|z|=r}\left|\frac{e^{re^{i\theta}} - 1}{e^{re^{i\theta}} - 1}\right| = \tan(r/2).$$

Applying Theorem 2.1.1, we have $|f(z)| \le f_{SG}(r)$ $(|z| = r)$, where f_{SG} is given by (4.2.7). Therefore on $|z| = r$,

$$\left|zf'(z) - f(z)\right| = |f(z)|\left|\frac{zf'(z)}{f(z)} - 1\right| \le f_{SG}(r)\tan(r/2) < 1/2,$$

whenever $r < \rho_{SG}$. Hence the result. □

Theorem 4.2.6. If $f \in \mathcal{S}^*_S$, then $f \in \Omega$ in the disc $|z| < \rho_S$, where $\rho_S \approx 0.531721$ is the smallest positive root of $2 f_S(r)\sinh 1 - 1 = 0$ and

$$f_S(z) = z\exp\left(\int_0^z \frac{\sin t}{t} dt\right) = z + \frac{z^2}{2} + \frac{z^3}{8} + \frac{z^4}{144} - \frac{5z^5}{1152} + \cdots. \tag{4.2.8}$$

Proof. Let $f \in \mathcal{S}^*_S$. Then $zf'(z)/f(z) \prec 1 + \sin z$, which further implies that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \max_{|z|=r}|\sin re^{i\theta}| = \sinh r.$$

Now by applying the growth theorem on f (see [16]), we have $|f(z)| \le f_S(r)$ $(|z| = r)$, where f_S is given by (4.2.8). Therefore

$$\left|zf'(z) - f(z)\right| = |f(z)|\left|\frac{zf'(z)}{f(z)} - 1\right| \le f_S(r)\sinh r \quad \text{on } |z| = r.$$

It follows that $|zf'(z) - f(z)| \le f_S(r)\sinh 1 < 1/2$, provided $r < \rho_S$. Hence the result. □

Theorem 4.2.7. If $f \in \mathcal{S}^*_{\wp}$, then $f \in \Omega$ in the disc $|z| < \rho_{\wp}$, where $\rho_{\wp} \approx 0.43384$ is the smallest positive root of $2r^2 e^{e^r + r - 1} - 1 = 0$. This result is sharp.

Proof. Let $f \in \mathcal{S}^*_{\wp}$, then we have $zf'(z)/f(z) \prec 1 + ze^z$. This further implies

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{|z|=r} |ze^z| = \max_{0 \leq \theta < 2\pi} |re^{i\theta} e^{re^{i\theta}}| = re^r.$$

Now by using the growth theorem given for $\mathcal{S}^*_{\wp}$ in [45], we get $|f(z)| \leq re^{e^r - 1}$ on $|z| = r$. Finally we have

$$\left|zf'(z) - f(z)\right| = |f(z)| \left|\frac{zf'(z)}{f(z)} - 1\right| \leq re^{e^r - 1}(re^r) \quad \text{on } |z| = r.$$

For $r < \rho_{\wp}$, we have $|zf'(z) - f(z)| < 1/2$ and thus $f \in \Omega$. The result is sharp as for the function $f_{\wp}(z) = ze^{e^z - 1}$, the inequality holds only in the disk $|z| < \rho_{\wp}$. □

Theorem 4.2.8. If $f \in \mathcal{S}^*_{RL}$, then $f \in \Omega$ in the disc $|z| < \rho_{RL}$, where $\rho_{RL} \approx 0.768$ is the smallest positive root of $2(\varphi_{RL}(-r) - 1) f_{RL}(r) - 1 = 0$, where

$$\varphi_{RL}(z) = \sqrt{2} - \left(\sqrt{2} - 1\right) \sqrt{\frac{1 - z}{1 + 2\left(\sqrt{2} - 1\right) z}} \tag{4.2.9}$$

and

$$f_{RL}(z) = z \left(\frac{\sqrt{1 - z} + \sqrt{1 + 2(\sqrt{2} - 1) z}}{2} \right)^{2(\sqrt{2} - 1)} \exp(\eta(z)) \tag{4.2.10}$$

with

$$\eta(z) = \sqrt{2(\sqrt{2} - 1)} \tan^{-1}\left(\sqrt{2(\sqrt{2} - 1)} \left(\frac{\sqrt{1 + 2(\sqrt{2} - 1) z} - \sqrt{1 - z}}{\sqrt{1 + 2(\sqrt{2} - 1) z} + 2(\sqrt{2} - 1)\sqrt{1 - z}} \right) \right).$$

Proof. Let $f \in \mathcal{S}^*_{RL}$, then $zf'(z)/f(z) \prec \varphi_{RL}(z)$, given by (4.2.9). So on $|z| = r$,

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{0 \leq \theta < 2\pi} |\varphi_{RL}(re^{i\theta}) - 1| = \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1 + r}{1 - 2(\sqrt{2} - 1) r}} - 1.$$

By using [57, Theorem 2.2(ii)], we get $|f(z)| \leq |f_{RL}(r)|$ on $|z| = r$, where f_{RL} is given by (4.2.10). Therefore

$$\left|zf'(z)-f(z)\right| = |f(z)|\left|\frac{zf'(z)}{f(z)}-1\right| \leq (\varphi_{RL}(-r)-1)f_{RL}(r) < \frac{1}{2},$$

provided $r < \rho_{RL}$. Hence the result. □

Theorem 4.2.9. If $f \in \mathcal{S}^*_L$, then $f \in \Omega$ in the disc $|z| < \rho_L$, where $\rho_L \approx 0.734453$ is the positive root of $8r(1-\sqrt{1-r})\exp(2\sqrt{1+r}-2)-(1+\sqrt{1+r})^2 = 0$.

Proof. Let $f \in \mathcal{S}^*_L$, then it implies that $zf'(z)/f(z) \prec \sqrt{1+z}$. Thus on $|z| = r$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \max_{|z|=r}|\sqrt{1+z}-1| = \max_{0\leq\theta<2\pi}|\sqrt{1+re^{i\theta}}-1| = 1-\sqrt{1-r}.$$

Applying the growth theorem on f, we obtain

$$|f(z)| \leq \frac{4r\exp(2\sqrt{1+r}-2)}{(1+\sqrt{1+r})^2}, \quad \text{on } |z| = r.$$

We observe that

$$|zf'(z)-f(z)| = |f(z)|\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{4r(1-\sqrt{1-r})\exp(2\sqrt{1+r}-2)}{(1+\sqrt{1+r})^2},$$

which is less that $1/2$, provided $r < \rho_L$. Therefore $f \in \Omega$. □

Theorem 4.2.10. If $f \in \mathcal{S}^*_{Ne}$, then $f \in \Omega$ in the disc $|z| < \rho_{Ne}$, where $\rho_{Ne} \approx 0.524752$ is the positive root of

$$2r\left(r+\frac{r^3}{3}\right)\exp\left(r-\frac{r^3}{9}\right) = 0.$$

Proof. If $f \in \mathcal{S}^*_{Ne}$, then $zf'(z)/f(z) \prec 1+z-z^3/3$. We know that on $|z| = r$

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \max_{|z|=r}\left|z-\frac{z^3}{3}\right| = \max_{0\leq\theta<2\pi}\left|re^{i\theta}-\frac{r^2e^{3i\theta}}{3}\right| = r+\frac{r^3}{3}.$$

The growth theorem for the class $\mathcal{S}^*_{Ne}$ (see [101]) implies that for any $f \in \mathcal{S}^*_{Ne}$, $|f(z)| \leq |f_{Ne}(r)|$ on $|z| = r$, where

$$f_{Ne}(z) = z\exp\left(z-\frac{z^3}{9}\right).$$

Using the above inequalities, we get

$$\left|zf'(z)-f(z)\right| = |f(z)|\left|\frac{zf'(z)}{f(z)}-1\right| \leq r\left(r+\frac{r^3}{3}\right)\exp\left(r-\frac{r^3}{9}\right).$$

For $r < \rho_{Ne}$, we have $|zf'(z) - f(z)| < 1/2$ and thus $f \in \Omega$. □

Theorem 4.2.11. If $f \in \mathcal{S}^*_C$, then $f \in \Omega$ in the disc $|z| < \rho_C$, where $\rho_C \approx 0.411914$ is the positive root of

$$2re^{\frac{r^2}{3}+\frac{4r}{3}}\left(\frac{2r^2}{3}+\frac{4r}{3}\right)-1=0.$$

The result is sharp.

Proof. Let $f \in \mathcal{S}^*_C$. Then we have $zf'(z)/f(z) \prec 1 + 4z/3 + 2z^2/3$, which gives

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{4z}{3}+\frac{2z^2}{3}\right|.$$

Taking $z = re^{i\theta}$ $(0 \le \theta < 2\pi)$, we obtain

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \left|\frac{4re^{i\theta}}{3}+\frac{2r^2e^{2i\theta}}{3}\right| = \frac{2}{3}\sqrt{4r^2+r^2+4r^3\cos\theta} \le \frac{4r}{3}+\frac{2r^2}{3}.$$

Now by using the growth theorem for $\mathcal{S}^*_C$ (refer to [86]), we obtain $|f(z)| \le |f_C(r)|$ on $|z| = r$, where

$$f_C(z) = z\exp\left(\frac{4z}{3}+\frac{z^2}{3}\right). \tag{4.2.11}$$

We observe that

$$\left|zf'(z)-f(z)\right| = |f(z)|\left|\frac{zf'(z)}{f(z)}-1\right| \le re^{\frac{r^2}{3}+\frac{4r}{3}}\left(\frac{2r^2}{3}+\frac{4r}{3}\right) < \frac{1}{2},$$

provided $r < \rho_C$. We may note that for $f_C(z)$, the inequality $|zf'(z) - f(z)| < 1/2$ holds only in the disk $|z| < \rho_C$ and thus the result is sharp. □

4.2.3 The class $\mathcal{S}^*_{SG}$

In this section, we obtain sharp radii of Sigmoid starlikeness for functions in $\mathcal{S}^*(\varphi)$, for different choices of φ.

Theorem 4.2.12. The $\mathcal{S}^*_{SG,n}$-radius of the class $\mathcal{S}^*_n[A,B]$ is given by

(*i*) $R_{\mathcal{S}^*_{SG,n}}(\mathcal{S}^*_n[A,B]) = \min\left\{1, \left(\frac{e-1}{A(1+e)-2B}\right)^{\frac{1}{n}}\right\}$, when $0 \le B < A \le 1$.

(*ii*) $R_{\mathcal{S}^*_{SG,n}}(\mathcal{S}^*_n[A,B]) = \min\left\{1, \left(\frac{e-1}{A(1+e)-2Be}\right)^{\frac{1}{n}}\right\}$, when $-1 \le B < A \le 1$ with $B \le 0$.

In particular for the class $\mathcal{S}^*$, we have $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*) = (e-1)/(3e+1)$.

Proof. Let $f \in \mathcal{S}^*_n[A,B]$. Using Lemma 1.2.2, we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-ABr^{2n}}{1-B^2r^{2n}}\right| \leq \frac{(A-B)r^n}{1-B^2r^{2n}}. \tag{4.2.12}$$

(i) If $0 \leq B < A \leq 1$, then

$$a := \frac{1-ABr^{2n}}{1-B^2r^{2n}} \leq 1.$$

Further by Lemma 2.2.2 and equation (4.2.12), we see that $f \in \mathcal{S}^*_{SG,n}$ if

$$\frac{(A-B)r^n}{1-B^2r^{2n}} \leq \frac{1-ABr^{2n}}{1-B^2r^{2n}} - \frac{2}{1+e},$$

which upon simplification, yields

$$r \leq \left(\frac{e-1}{A(1+e)-2B}\right)^{\frac{1}{n}}.$$

The result is sharp due to the function $f_{A,B}(z)$, given by

$$f_{A,B}(z) = \begin{cases} z(1+Bz^n)^{\frac{A-B}{nB}}; & B \neq 0, \\ z\exp\left(\frac{Az^n}{n}\right); & B = 0. \end{cases}$$

(ii) If $-1 \leq B < 0 < A \leq 1$, then

$$a := \frac{1-ABr^{2n}}{1-B^2r^{2n}} \geq 1.$$

Therefore, by Lemma 2.2.2 and equation (4.2.12), we see that $f \in \mathcal{S}^*_{SG,n}$ if

$$\frac{(A-B)r^n}{1-B^2r^{2n}} \leq \frac{2e}{1+e} - \frac{1-ABr^{2n}}{1-B^2r^{2n}},$$

which upon simplification, yields

$$r \leq \left(\frac{e-1}{A(1+e)-2Be}\right)^{1/n}.$$

Hence, the result follows with sharpness due to $f_{A,B}(z)$. □

Corollary 4.2.13. The sharp $\mathcal{S}^*_{SG}$ radius for $\mathcal{S}^*(\alpha)$ is $(e-1)/(1+3e-2\alpha(1+e))$, $0 \leq \alpha < 1$. The bound is sharp for $\mathcal{K}_\alpha(z) = z/(1-z)^{2(1-\alpha)}$.

Corollary 4.2.14. The sharp $\mathcal{S}^*_{SG}$ radius for $\mathcal{S}^*$ is $(e-1)/(1+3e)$. The bound is sharp for $\mathscr{K}(z) = z/(1-z)^2$.

Before we proceed to our next result, we need to recall the following classes: For $0 \leq \alpha < 1$, Kargar et al. [40] defined the class $\mathcal{BS}^*(\alpha) := \{f \in \mathcal{A} : zf'(z)/f(z) \prec 1 + z/(1-\alpha z^2)\}$ associated with the Booth lemniscate. In [42], Khatter et al. generalised $\mathcal{S}^*_L$ and $\mathcal{S}^*_e$ to $\mathcal{S}^*_L(\alpha) := \mathcal{S}^*(\alpha + (1-\alpha)\sqrt{1+z})$ and $\mathcal{S}^*_{\alpha,e} := \mathcal{S}^*(\alpha + (1-\alpha)e^z)$ respectively, for $\alpha \in [0,1)$.

Theorem 4.2.15. The radius estimates of Sigmoid starlikeness, for the classes $\mathcal{BS}^*(\alpha)$, $\mathcal{S}^*_L(\alpha)$ and $\mathcal{S}^*_{\alpha,e}$ are given by

(*i*) $R_{\mathcal{S}^*_{SG}}(\mathcal{BS}^*(\alpha)) = \rho_{\mathcal{BS}}(\alpha) := 2(e-1)/((1+e)+\sqrt{(1+e)^2+4\alpha(e-1)^2})$, where $\alpha \in [0,1)$.

(*ii*) $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_L(\alpha)) = \rho_L(\alpha) := ((e-1)(3+e-2\alpha(1+e)))/((1-\alpha)^2(1+e)^2)$, where $\alpha \in [0,(3+e)/2(1+e))$. In particular, $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_L) = ((e-1)(3+e))/(1+e)^2$.

(*iii*) $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_{\alpha,e}) = \rho_e(\alpha) := \log(2e-\alpha(1+e))/(1+e)(1-\alpha)$, where $\alpha \in [0,(e(1+e)-2e)/((1+e)(e-1)))$. In particular, $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_e) = \log(2e/(1+e))$.

All estimates are sharp.

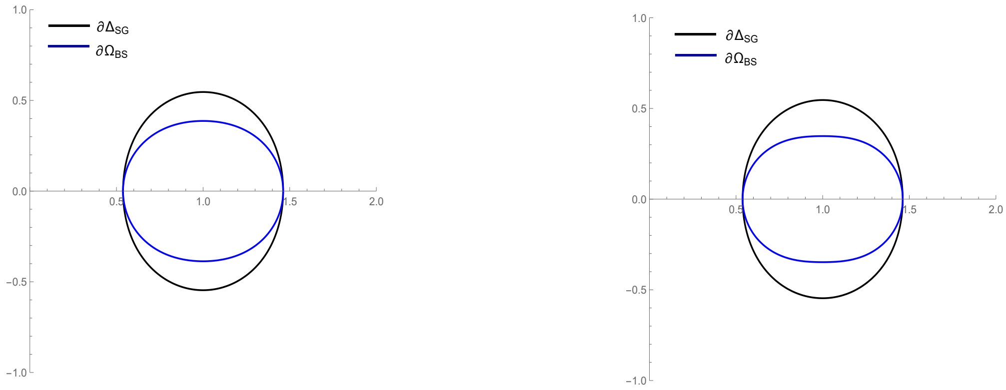
Proof. (i) Let $f \in \mathcal{BS}^*(\alpha)$. Then $zf'(z)/f(z) \prec 1 + z/(1-\alpha z^2)$ and thus

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{z}{1-\alpha z^2}\right| \leq \frac{r}{1-\alpha r^2} \quad \text{on } |z| = r.$$

Using Lemma 2.2.2, it can be said that the above disk lies in Δ_{SG} if $r/(1-\alpha r^2) \leq (e-1)/(e+1)$, which holds if $r \leq \rho_{\mathcal{BS}}(\alpha)$. Sharpness holds for the function

$$f_{\mathcal{BS}}(z) = \begin{cases} z\left(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{1/(2\sqrt{\alpha})}, & \alpha \in (0,1) \\ ze^z, & \alpha = 0. \end{cases}$$

It can be verified with the following graph that $zf'_{\mathcal{BS}}(z)/f_{\mathcal{BS}}(z)$ touches the boundary of Δ_{SG} at the points $\pm 2(e-1)/((1+e)+\sqrt{(1+e)^2+4\alpha(e-1)^2})$. Note that the domain Ω_{BS} denotes the image of $\mathbb{D}$ mapped by the function $1+z/(1-\alpha z^2)$.



$\alpha = 0.5$ $\alpha = 0.9$

Figure 4.1: Sharpness of $R_{\mathcal{S}^*_{SG}}(\mathcal{BS}^*(\alpha))$

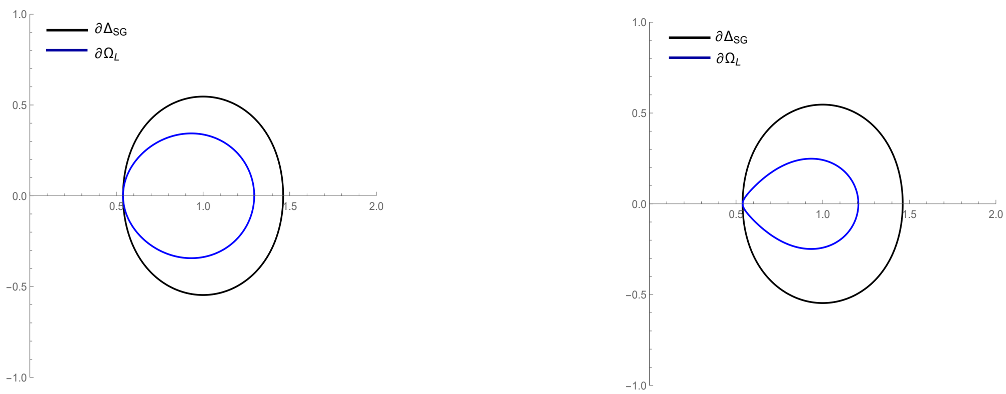
(ii) Let $f \in \mathcal{S}^*_L(\alpha)$, then $zf'(z)/f(z) \prec \alpha + (1-\alpha)\sqrt{1+z}$ and therefore on $|z| = r$

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = |(1-\alpha)(1-\sqrt{1+z})| \leq (1-\alpha)(1-\sqrt{1-r}).$$

By Lemma 2.2.2, it is clear that for the above disk to lie in Δ_{SG}, we need $(1-\alpha)(1-\sqrt{1-r}) \leq (e-1)/(e+1)$, which upon simplification yields $r \leq ((e-1)(3+e-2\alpha(1+e)))/((1-\alpha)^2(1+e)^2)$. Note that for the function

$$f_L(z) = z + (1-\alpha)z^2 + \frac{1}{16}(1-\alpha)(1-2\alpha)z^3 + \cdots,$$

the result is sharp. The sharpness of this result can be verified by the following graph, where Ω_L denotes the image of $\mathbb{D}$ mapped by $\alpha + (1-\alpha)\sqrt{1+z}$.



$\alpha = 0.1$ $\alpha = 0.5$

Figure 4.2: Sharpness of $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_L(\alpha))$

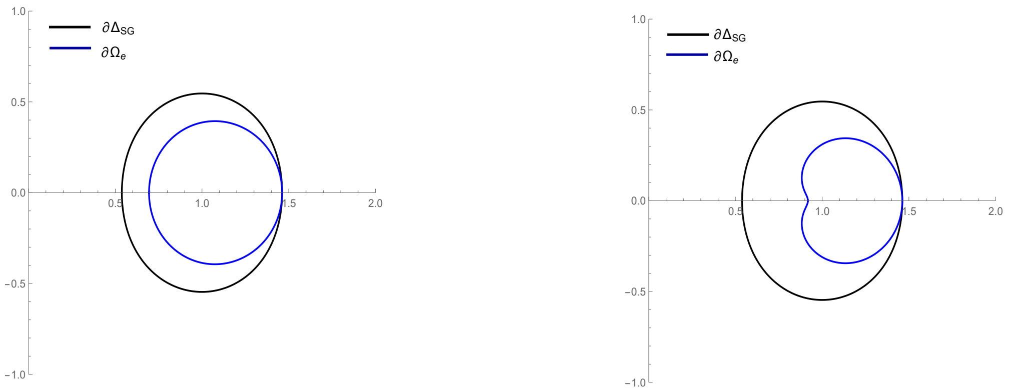
(iii) Let $f \in \mathcal{S}^*_{\alpha,e}$, then $zf'(z)/f(z) \prec \alpha + (1-\alpha)e^z$. So on $|z| = r$

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = (1-\alpha)|e^z - 1| \leq (1-\alpha)(e^r - 1).$$

By Lemma 2.2.2, $f \in \mathcal{S}^*_{SG}$ if $(1-\alpha)(e^r-1) \leq (e-1)/(e+1)$, which is equivalent to $r \leq \rho_e(\alpha)$. The result is sharp for the function

$$f_e(z) = z + (1-\alpha)z^2 + \frac{1}{4}(1-\alpha)(3-2\alpha)z^3 + \cdots$$

and is validated by the following graph. The image of $\mathbb{D}$ mapped by $\alpha + (1-\alpha)e^z$ is denoted by Ω_e.



$\alpha = 0.1$ $\qquad$ $\alpha = 0.9$

Figure 4.3: Sharpness of $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_{\alpha,e})$

□

Theorem 4.2.16. The $\mathcal{S}^*_{SG}$ radius for the classes $\mathcal{S}^*_{RL}$, $\mathcal{S}^*_{C}$ and $\mathcal{S}^*_{R}$ is given by:

(i) $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_{RL}) =: \tilde{\rho}_{RL} = \left(4\sqrt{2} - 7e - 5\right)(e-1)/(32\sqrt{2} - 7e^2 + 6e\left(4\sqrt{2}-5\right) - 47) \approx 0.738309$

(ii) $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_{C}) =: \tilde{\rho}_{C} = -1 + \sqrt{(-1+5e)/(2+2e)} \approx 0.301221$

(iii) $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_{R}) =: \tilde{\rho}_{R} = \left(\sqrt{\left(2\sqrt{2}+3\right)(2e^2-1)} - \left(\sqrt{2}+1\right)e\right)/(1+e) \approx 0.645131.$

Proof. (i) Let $f \in \mathcal{S}^*_{RL}$. Then $zf'(z)/f(z) \prec \varphi_{RL}$, where φ_{RL} is given by (4.2.9). Thus on $|z| = r$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \left(\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}\right).$$

By Lemma 2.2.2, f is in $\mathcal{S}^*_{SG}$ if

$$1-\left(\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}\right)\leq\frac{e-1}{e+1},$$

which is equivalent to $r \leq \tilde{\rho}_{RL}$. This result is sharp for the function f_{RL}, given by (4.2.10).

(ii) Let $f \in \mathcal{S}^*_C$, then we have $zf'(z)/f(z) \prec 1+4z/3+2z^2/3$. Therefore on $|z|=r$, we get

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{2(r^2+2r)}{3},$$

which if not exceeds $(e-1)/(e+1)$, implies that f lies in $\mathcal{S}^*_{SG}$, by Lemma 2.2.2. Solving this, we get $r \leq \tilde{\rho}_C$. The sharpness of this result can be verified by the function $f_C(z)$, given by (4.2.11). Clearly $f_C \in \mathcal{S}^*_C$ and moreover $zf_C'(z)/f_C(z)$ touches the boundary of Δ_{SG} at the point $z_0 = -1+\sqrt{(-1+5e/(2+2e)}$, as shown in the Fig 4.4.

(iii) Let $f \in \mathcal{S}^*_R$, then

$$\frac{zf'(z)}{f(z)}\prec 1+\frac{z(k+z)}{k(k-z)},\text{ where } k=\sqrt{2}+1.$$

Thus on $|z|=r$,

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{r(k+r)}{k(k-r)}.$$

In view of Lemma 2.2.2, $f \in \mathcal{S}^*_{SG}$ if $r(k+r)/k(k-r) \leq (e-1)/(e+1)$. Solving this inequality, we obtain $r \leq \tilde{\rho}_R$. The equality of the radius estimate holds for the the function

$$f_R(z)=\frac{k^2z}{(k-z)^2}e^{-z/k}\quad k=\sqrt{2}+1.$$

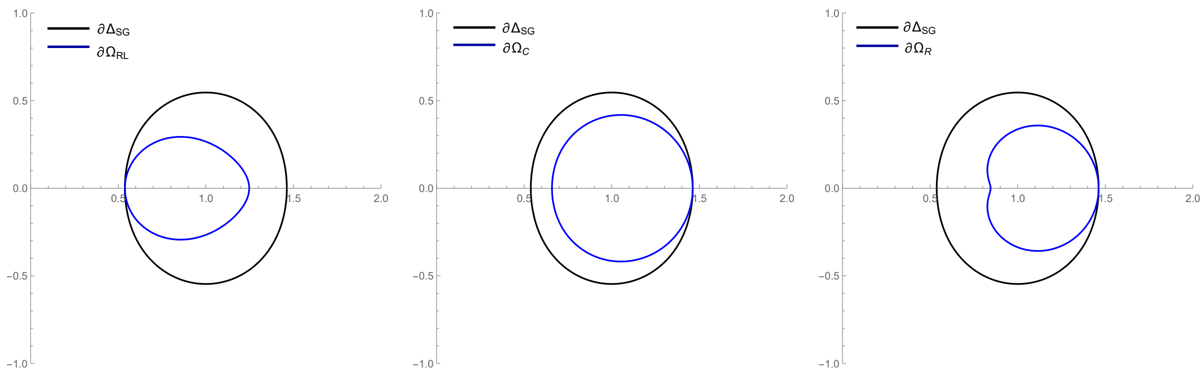


Figure 4.4: Sharpness of $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_{RL})$, $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_C)$ and $R_{\mathcal{S}^*_{SG}}(\mathcal{S}^*_R)$

□

Theorem 4.2.17. Let $f \in \mathcal{S}^*_{SG}$, then each of the following holds:

(*i*) If $2/(1+e) \leq \alpha < 1$, then f is starlike of order α in the disk $|z| < r(\alpha)$, where $r(\alpha) = \log(2/\alpha - 1)$.

(*ii*) If $1 < \beta \leq 2e/(1+e)$, then f is starlike of reciprocal order $1/\beta$ in the disk $|z| < r(\beta)$, where $r(\beta) = \log(\beta/(2-\beta))$. Further, $f \in \mathcal{M}(\beta)$ in this disk.

Proof. (i) Let $f \in \mathcal{S}^*_{SG}$, then

$$\frac{zf'(z)}{f(z)} \prec \frac{2}{1+e^{-z}}.$$

Therefore by Lemma 2.2.1, we have

$$\frac{2}{1+e^r} \leq \mathrm{Re}\,\frac{zf'(z)}{f(z)} \leq \frac{2e^r}{1+e^r} \qquad (|z| = r < 1),$$

which yields the following inequality:

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha, \quad \text{whenever} \quad \frac{2}{1+e^r} > \alpha.$$

This inequality holds, provided $r < \log(2/\alpha - 1)$.

(ii) Similarly,

$$\mathrm{Re}\,\frac{f(z)}{zf'(z)} > \frac{1}{\beta}, \quad \text{whenever} \quad \frac{1+e^r}{2e^r} > \frac{1}{\beta}.$$

This inequality holds for $r < \log(\beta/(2-\beta))$. From Theorem 2.3.1, we have $\mathcal{RS}^*(1/\beta) \subset \mathcal{M}(\beta)$. Thus, $f \in \mathcal{M}(\beta)$ in $|z| < r(\beta)$.

Hence the result. □

4.3 Sufficient Conditions for $G_{\lambda,\alpha}$

Now we present sufficient conditions that have been deduced for the class $G_{\lambda,\alpha}$, in the form of differential inequalities. The idea of differential subordination has been used for proving results in this section.

Theorem 4.3.1. Let $f \in \mathcal{A}_n$, $0 \leq \alpha < 1$ and $\lambda > 0$. If

$$\left| zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| < \delta, \tag{4.3.1}$$

where δ is the smallest positive root of

$$\phi(r) := (1+n)(2\alpha n - \lambda(n+1) - n)r^2 + n(1-\alpha+n)(2\lambda(n+1) + n + \alpha n^2)r - \lambda n^2(n+1-\alpha)^2, \tag{4.3.2}$$

then $f \in G_{\lambda,\alpha}$.

Proof. From (4.3.1), we have

$$zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right) \prec \delta z, \quad z \in \mathbb{D}.$$

Let $P(z) = f'(z) - f(z)/z$, then $P(0) = 0$ and

$$(1-\alpha)P(z) + zP'(z) = zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right) \prec \delta z.$$

Now applying [61, Theorem 3.1b] for $h(z) = \delta z/(1-\alpha)$ and $\gamma = 1-\alpha$, we obtain

$$P(z) \prec \frac{\delta z}{n+1-\alpha},$$

which is equivalent to

$$f'(z) - \frac{f(z)}{z} \prec \frac{\delta z}{n+1-\alpha}. \tag{4.3.3}$$

Now let us suppose $p(z) = f(z)/z$, then from (4.3.3)

$$zp'(z) = f'(z) - \frac{f(z)}{z} \prec \frac{\delta z}{n+1-\alpha}. \tag{4.3.4}$$

Now by using [61, Lemma 8.2a], we get

$$p(z) = \frac{f(z)}{z} \prec 1 + \frac{\delta z}{n(n+1-\alpha)},$$

which further yields the following inequality

$$1 - \frac{\delta}{n(n+1-\alpha)} < \left|\frac{f(z)}{z}\right| < 1 + \frac{\delta}{n(n+1-\alpha)}. \tag{4.3.5}$$

From (4.3.4), it is clear that

$$\left|f'(z) - \frac{f(z)}{z}\right| < \frac{\delta}{n+1-\alpha}, \tag{4.3.6}$$

which further implies

$$|f'(z)| > \left|\frac{f(z)}{z}\right| - \frac{\delta}{n+1-\alpha}, \tag{4.3.7}$$

From (4.3.5) and (4.3.7), we may conclude that

$$|f'(z)| > 1 - \frac{\delta(n+1)}{n(n+1-\alpha)}. \tag{4.3.8}$$

From (4.3.1), we have

$$\left| f'(z)\left(\frac{zf''(z)}{f'(z)}\right) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| < \delta. \tag{4.3.9}$$

Now from (4.3.8) and (4.3.9), we observe that

$$\left(1 - \frac{\delta(n+1)}{n(n+1-\alpha)}\right)\left|\frac{zf''(z)}{f'(z)}\right| < |f'(z)|\left|\frac{zf''(z)}{f'(z)}\right| < \delta + \alpha\left|f'(z) - \frac{f(z)}{z}\right|. \tag{4.3.10}$$

Here we may note that δ is the smaller of the two roots of $\phi(r)$, which is given by (4.3.2). So, we get

$$\delta = \frac{n(n+1-\alpha)(n+\alpha n^2+2\lambda(n+1) - \sqrt{n^2+\alpha^2 n^4+2\alpha n^3+8\alpha\lambda n+12\alpha\lambda n^2+4\alpha\lambda n^3})}{2(n+1)(\lambda(n+1)+n-2\alpha n)}.$$

Since

$$\begin{aligned}(n+\alpha n^2+2\lambda(n+1) - \sqrt{n^2+\alpha^2 n^4+2\alpha n^3+8\alpha\lambda n+12\alpha\lambda n^2+4\alpha\lambda n^3})(n+\alpha n^2+2\lambda(n+1)\\ +\sqrt{n^2+\alpha^2 n^4+2\alpha n^3+8\alpha\lambda n+12\alpha\lambda n^2+4\alpha\lambda n^3}) = 4\lambda(n+1)(\lambda(n+1)+n-2\alpha n),\end{aligned}$$

we have

$$\begin{aligned}\delta &= \frac{2\lambda n(n+1-\alpha)}{n+\alpha n^2+2\lambda(n+1)+\sqrt{n^2+\alpha^2 n^4+2\alpha n^3+8\alpha\lambda n+12\alpha\lambda n^2+4\alpha\lambda n^3}}\\ &\leq \frac{2\lambda n(n+1-\alpha)}{2\lambda(n+1)}.\end{aligned}$$

Therefore

$$1 - \frac{\delta(n+1)}{n(n+1-\alpha)} > 0$$

and thus (4.3.10) implies

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{\delta + \dfrac{\alpha\delta}{n+1-\alpha}}{1 - \dfrac{\delta(n+1)}{n(n+1-\alpha)}} = \frac{n(n+1)\delta}{n(n+1-\alpha)-\delta(n+1)}. \tag{4.3.11}$$

Now let us consider the following inequality

$$\begin{aligned}
\left(1-\frac{\delta(n+1)}{n(n+1-\alpha)}\right)&\left|\alpha\frac{f(z)f''(z)}{(f'(z))^2}-(1-\alpha)+(1-\alpha)\frac{f(z)}{zf'(z)}\right| \\
&< |f'(z)|\left|\alpha\frac{f(z)f''(z)}{(f'(z))^2}-(1-\alpha)+(1-\alpha)\frac{f(z)}{zf'(z)}\right| \\
&= \left|\alpha\frac{f(z)f''(z)}{f'(z)}-(1-\alpha)\left(f'(z)-\frac{f(z)}{z}\right)\right| \\
&< \alpha\left|\frac{f(z)}{z}\right|\left|\frac{zf''(z)}{f'(z)}\right|+(1-\alpha)\left|f'(z)-\frac{f(z)}{z}\right|.
\end{aligned}$$

Using (4.3.5), (4.3.6) and (4.3.11) in the above inequality, we get

$$\begin{aligned}
&\left(1-\frac{\delta(n+1)}{n(n+1-\alpha)}\right)\left|\alpha\frac{f(z)f''(z)}{(f'(z))^2}-(1-\alpha)+(1-\alpha)\frac{f(z)}{zf'(z)}\right| \\
&\quad< \alpha\left(1+\frac{\delta}{n(n+1-\alpha)}\right)\left(\frac{n(n+1)\delta}{n(n+1-\alpha)-\delta(n+1)}\right)+(1-\alpha)\left(\frac{\delta}{n+1-\alpha}\right)=:\tau,
\end{aligned}$$

which implies

$$\begin{aligned}
\left|\alpha\frac{f(z)f''(z)}{(f'(z))^2}-(1-\alpha)+(1-\alpha)\frac{f(z)}{zf'(z)}\right| &< \left(\frac{n(n+1-\alpha)}{n(n+1-\alpha)-\delta(n+1)}\right)\tau \\
&= \lambda.
\end{aligned}$$

Thus we have

$$\left|\frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)}-(1-\alpha)\right| < \lambda$$

and the result follows. □

Corollary 4.3.2. Let $0 \le \alpha < 1$, $\lambda > 0$ and $g \in \mathcal{H}$. If $|g(z)| < \delta$, where δ is the smallest positive root of $\phi(r) := (1+n)(2\alpha n - \lambda(n+1)-n)r^2 + n(1-\alpha+n)(2\lambda(n+1)+n+\alpha n^2)r - \lambda n^2(n+1-\alpha)^2$, then

$$f(z) = z + z^{n+1}\int_0^1\int_0^1 g(rsz)r^{n-\alpha}s^{n-1}drds$$

is in $G_{\lambda,\alpha}$.

Proof. Suppose that $f(z)$ satisfies the following differential equation

$$zf''(z)-\alpha\left(f'(z)-\frac{f(z)}{z}\right)=z^n g(z). \tag{4.3.12}$$

Let

$$H(z)=f'(z)-\frac{f(z)}{z},$$

then from (4.3.12), we have

$$(1-\alpha)H(z)+zH'(z)=z^n g(z).$$

Now applying [61, Theorem 3.1b], we obtain the solution of the above differential equation, given by

$$H(z)=\frac{1}{z^{1-\alpha}}\int_0^z g(t)t^{n-\alpha}dt.$$

Now if we substitute $t=rz$ in the above equation, then

$$H(z)=z^n\int_0^1 g(rz)r^{n-\alpha}dr,$$

Taking $h(z)=f(z)/z$, we have

$$zh'(z)=f'(z)-\frac{f(z)}{z}=H(z).$$

Now by using [61, Lemma 8.2a], we obtain

$$h(z)=1+\int_0^z\frac{H(t)}{t}dt.$$

Substituting $t=sz$ yields

$$\begin{aligned} h(z) &= 1+\int_0^1\frac{H(sz)}{s}ds \\ &= 1+\int_0^1\left(\frac{(sz)^n}{s}\int_0^1 g(rsz)r^{n-\alpha}dr\right)ds \\ &= 1+z^n\int_0^1\int_0^1 g(rsz)r^{n-\alpha}s^{n-1}drds. \end{aligned}$$

Thus

$$f(z)=z+z^{n+1}\int_0^1\int_0^1 g(rsz)r^{n-\alpha}s^{n-1}drds.$$

Now using Theorem 4.3.1 along with the fact that $|g(z)| < \delta$, we have $f \in G_{\lambda,\alpha}$. □

Corollary 4.3.3. Let $f \in \mathcal{A}$ satisfies

$$\left| zf''(z) - \frac{1}{2}\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{3}{8}(5 - \sqrt{21}), \tag{4.3.13}$$

then $z(zf'(z)/f(z))$ is univalent in $\mathbb{D}$.

Proof. If we take $n = 1$, $\alpha = 1/2$ and $\delta = 3(5 - \sqrt{21})/8$ in Theorem 4.3.1, then (4.3.13) implies that $f \in G_{\frac{1}{4},\frac{1}{2}}$. We know that $G_{\frac{1}{4},\frac{1}{2}} = G_{\frac{1}{2}}$ and thus by using [70, Theorem 2], the result follows. □

Theorem 4.3.4. Let $f \in \mathcal{A}_n$, $0 \le \alpha < 1$ and $\lambda > 0$. If

$$|zf''(z) - \alpha(f'(z) - 1)| < \frac{\delta(n+1)(n-\alpha)}{\alpha + (n+1)(n-\alpha)} \quad z \in \mathbb{D}, \tag{4.3.14}$$

where δ is the smallest positive root of $\phi(r) := (1+n)(2\alpha n - \lambda(n+1) - n)r^2 + n(1-\alpha+n)(2\lambda(n+1) + n + \alpha n^2)r - \lambda n^2(n+1-\alpha)^2$, then $f \in G_{\lambda,\alpha}$.

Proof. From (4.3.14), we have for $z \in \mathbb{D}$

$$zf''(z) - \alpha(f'(z) - 1) \prec \frac{\delta(n+1)(n-\alpha)z}{\alpha + (n+1)(n-\alpha)}.$$

Let $P(z) = f'(z) - (1+\alpha)f(z)/z$, then

$$P(z) + zP'(z) = zf''(z) - \alpha f'(z) \prec \frac{\delta(n+1)(n-\alpha)z}{\alpha + (n+1)(n-\alpha)} - \alpha.$$

Using Lemma [61, Theorem 3.1b], we have

$$P(z) \prec \frac{\delta(n-\alpha)z}{\alpha + (n+1)(n-\alpha)} - \alpha,$$

which further implies

$$f'(z) - (1+\alpha)\frac{f(z)}{z} \prec \frac{\delta(n-\alpha)z}{\alpha + (n+1)(n-\alpha)} - \alpha.$$

Now let us take

$$p(z) = \frac{f(z)}{z} - 1 \quad \text{and} \quad q(z) = \frac{\delta z}{\alpha + (n+1)(n-\alpha)}.$$

It is easy to observe that $q(0)=0$, $q'(0)\neq 0$ and $\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)=1>\frac{\alpha}{n}$. Next, we observe

$$zp'(z)-\alpha p(z)=f'(z)-(1+\alpha)\frac{f(z)}{z}+\alpha\prec\frac{\delta(n-\alpha)z}{\alpha+(n+1)(n-\alpha)}=nzq'(z)-\alpha q(z).$$

Then by using Lemma [61, Lemma 8.2a], we obtain

$$\frac{f(z)}{z}-1=p(z)\prec q(z)=\frac{\delta z}{\alpha+(n+1)(n-\alpha)},$$

which implies

$$\left|\frac{f(z)}{z}-1\right|<\frac{\delta}{\alpha+(n+1)(n-\alpha)}. \tag{4.3.15}$$

Finally from (4.3.14) and (4.3.15), we have

$$\begin{aligned}\left|zf''(z)-\alpha\left(f'(z)-\frac{f(z)}{z}\right)\right| &\leq \left|zf''(z)-\alpha(f'(z)-1)\right|+\alpha\left|\frac{f(z)}{z}-1\right|\\ &< \frac{\delta(n+1)(n-\alpha)}{\alpha+(n+1)(n-\alpha)}+\frac{\alpha\delta}{\alpha+(n+1)(n-\alpha)}\\ &= \delta.\end{aligned}$$

Applying Theorem 4.3.1, the result follows. □

Corollary 4.3.5. Let $0\leq\alpha<1$, $\lambda>0$ and $g\in\mathcal{H}$. If

$$|g(z)|<\frac{\delta(n+1)(n-\alpha)}{\alpha+(n+1)(n-\alpha)},\quad z\in\mathbb{D},$$

where δ is the smallest positive root of $\phi(r):=(1+n)(2\alpha n-\lambda(n+1)-n)r^2+n(1-\alpha+n)(2\lambda(n+1)+n+\alpha n^2)r-\lambda n^2(n+1-\alpha)^2$, then

$$f(z)=z+z^{n+1}\int_0^1\int_0^1 g(rsz)r^{n-1-\alpha}s^n\,drds$$

is in $G_{\lambda,\alpha}$.

Proof. Suppose $f\in\mathcal{A}_n$ satisfies

$$zf''(z)-\alpha(f'(z)-1)=z^ng(z).$$

Taking $H(z)=f'(z)-1$, the above equation reduces to

$$zH'(z)-\alpha H(z)=z^ng(z).$$

By using [61, Theorem 3.1b], we obtain the solution of the above differential equation as follows

$$H(z) = z^{\alpha} \int_0^z g(t) t^{n-\alpha-1} dt.$$

Taking $t = rz$, it reduces to

$$H(z) = z^{n} \int_0^1 g(rz) r^{n-\alpha-1} dr$$

and thus

$$f(z) = z + z^{n+1} \int_0^1 \int_0^1 g(rsz) r^{n-1-\alpha} s^n drds.$$

By Theorem 4.3.4, the result follows. □

4.4 Inclusion Relations

A collection of inclusion relations comprising all the classes, previously discussed in this chapter, are provided in this section.

Theorem 4.4.1. Let $f \in G_{\lambda,\alpha}$ $(\lambda > 0,\ 1/3 < \alpha \leq 1)$. Then $zf'(z)/f(z) \prec 1/(1 \pm cz)$, where $c = \lambda/(3\alpha - 1)$ and the result is sharp.

Proof. Let $p(z) = zf'(z)/f(z) = 1/(1 + c\omega(z))$. Then

$$\begin{aligned} \left| \frac{1 - \alpha + \alpha z f''(z)/f'(z)}{zf'(z)/f(z)} - (1-\alpha) \right| &= \left| \frac{1-2\alpha}{p(z)} + \frac{\alpha z p'(z)}{p^2(z)} + 2\alpha - 1 \right| \\ &= \left| (1-2\alpha)c\omega(z) - \alpha c z \omega'(z) \right|. \end{aligned}$$

Now we show that $|\omega(z)| < 1$ for $z \in \mathbb{D}$. Suppose on contrary there exists a point $z_0 \in \mathbb{D}$ such that $|\omega(z_0)| = 1$ and $z_0\omega'(z_0) = k\omega(z_0) (k \geq 1)$. Then

$$\begin{aligned} \left| \frac{1-2\alpha}{p(z_0)} + \frac{\alpha z p'(z_0)}{p^2(z_0)} + 2\alpha - 1 \right| &= |\omega(z_0) c (1 - \alpha(k+2))| \\ &= \left| \frac{\lambda}{1-3\alpha} (1 - \alpha(k+2)) \right| \\ &> \lambda, \end{aligned}$$

which is a contradiction to the assumption that $f \in G_{\lambda,\alpha}$. For the function $f(z) =$

$z/(1 \pm cz)$, we obtain that $zf'(z)/f(z) = 1/(1 \pm cz)$ and

$$\left| \frac{1-2\alpha}{p(z)} + \frac{\alpha z p'(z)}{p^2(z)} + 2\alpha - 1 \right| = \lambda.$$

□

Remark 2. For $\alpha = 1/2$, $G_{\lambda,\alpha}$ reduces to the class G_b defined by Silverman and the above result reduces to [70, Theorem 1] with $b = 2\lambda$.

Remark 3. For $\alpha = 1$, $G_{\lambda,\alpha}$ reduces to the class $G_{\lambda,1}$ defined by Tuneski and the above result reduces to [12, Theorem 3.1] with $h(z) = \lambda z$.

Theorem 4.4.2. Let $\lambda > 0$ and $1/3 < \alpha < 1$ be such that $\lambda < (2-\sqrt{3})(3\alpha - 1)$. Then $G_{\lambda,\alpha} \subset \Omega$.

Proof. Let $f \in G_{\lambda,\alpha}$, then Theorem 4.4.1 implies that

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1+cz} =: \varphi_0(z), \quad \text{where } c = \frac{\lambda}{3\alpha - 1}.$$

By the structural formula, we know that $f \in \mathcal{S}^*(\varphi_0)$ if and only if there exists a function $\varphi(z) \prec \varphi_0(z)$ such that

$$f(z) = z \exp \int_0^z \frac{\varphi(t) - 1}{t} dt.$$

Taking $\varphi(z) = \varphi_0(z)$, we obtain the extremal function for the class $\mathcal{S}^*(\varphi_0)$, given by $\tilde{f}_0(z) = z/(1+cz)$. Then by the growth theorem, we have $|f(z)| \leq \tilde{f}_0(r)$ on $|z| = r$. Hence

$$|zf'(z) - f(z)| = |f(z)| \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq |\tilde{f}_0(1)| \left| \frac{-cz}{1+cz} \right| \leq \frac{c}{(1-c)^2}.$$

We have $c = \lambda/(3\alpha - 1) < 2 - \sqrt{3}$, which further implies that

$$|zf'(z) - f(z)| \leq \frac{c}{(1-c)^2} < \frac{1}{2}$$

and the proof is complete. □

Lemma 4.4.3. Let $\lambda > 0$, $1/3 < \alpha < 1$ and $\varphi \in \Pi_M$ with $\varphi(\mathbb{D}) = \Delta$. Then $G_{\lambda,\alpha} \subset \mathcal{S}^*(\varphi)$, whenever $(1 + r_1)\lambda < (3\alpha - 1)r_1$, where r_1 is the radius of the largest disk contained in Δ and centered at 1.

Proof. Let $f \in G_{\lambda,\alpha}$. Then from the proof of Theorem 4.4.2, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{c}{1-c}, \quad \text{with } c = \frac{\lambda}{3\alpha - 1}.$$

Since $(1+r_1)\lambda < (3\alpha - 1)r_1$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{c}{1-c} = \frac{\lambda}{3\alpha - \lambda - 1} < r_1.$$

Therefore $zf'(z)/f(z)$ lies in Δ and hence $f \in \mathcal{S}^*(\varphi)$. □

φ	$\mathcal{S}^*(\varphi)$	r_1	Reference
$\dfrac{2}{1+e^{-z}}$	$\mathcal{S}^*_{SG}$	$\dfrac{e-1}{e+1}$	Lemma 2.2.2
e^z	$\mathcal{S}^*_e$	$1-\dfrac{1}{e}$	[58]
$1+\sin z$	$\mathcal{S}^*_S$	$\sin 1$	[16]
$\sqrt{1+z}$	$\mathcal{S}^*_L$	$\sqrt{2}-1$	[90]
$1+z-\dfrac{z^3}{3}$	$\mathcal{S}^*_{Ne}$	$\dfrac{2}{3}$	[101]
$1+\dfrac{4}{3}z+\dfrac{2}{3}z^2$	$\mathcal{S}^*_C$	$\dfrac{2}{3}$	[86]
$z+\sqrt{1+z^2}$	$\mathcal{S}^*_{\leftmoon}$	$2-\sqrt{2}$	[76]
$1+ze^z$	$\mathcal{S}^*_{\wp}$	$\dfrac{1}{e}$	[45]

Table 4.1: Radii of the largest disk contained in the image domain of Ma-Minda functions

Theorem 4.4.4. The class $G_{\lambda,\alpha}$ $(\lambda > 0, 1/3 < \alpha < 1)$ satisfies the following inclusion relations:

(i) $G_{\lambda,\alpha} \subset \mathcal{S}^*_{SG}$, whenever $2\lambda e < (e-1)(3\alpha-1)$

(ii) $G_{\lambda,\alpha} \subset \mathcal{S}^*_{e}$, whenever $(2e-1)\lambda < (e-1)(3\alpha-1)$

(iii) $G_{\lambda,\alpha} \subset \mathcal{S}^*_{S}$, whenever $(1+\sin(1))\lambda e < (1+\sin(1))(3\alpha-1)$

(iv) $G_{\lambda,\alpha} \subset \mathcal{S}^*_{L}$, whenever $\sqrt{2}\lambda < (\sqrt{2}-1)(3\alpha-1)$

(v) $G_{\lambda,\alpha} \subset \mathcal{S}^*_{Ne}$, whenever $5\lambda < 2(3\alpha-1)$

(vi) $G_{\lambda,\alpha} \subset \mathcal{S}^*_{C}$, whenever $5\lambda < 2(3\alpha-1)$

(vii) $G_{\lambda,\alpha} \subset \mathcal{S}^*_{\mathbb{C}}$, whenever $(3-\sqrt{2})\lambda < (2-\sqrt{2})(3\alpha-1)$

(viii) $G_{\lambda,\alpha} \subset \mathcal{S}^*_{\wp}$, whenever $(e+1)\lambda < (3\alpha-1)$.

Proof. For different choices of φ with respective values of r_1(refer to Table 4.1), we apply Lemma 4.4.3 and the result follows directly.

□

Concluding Remarks

In this chapter, we have exploited some rare features of Schwarz function to prove our results, involving some complex computations. This new approach would be of great help to explore other classes in the similar direction. In addition, some differential inequalities are proposed as sufficient conditions for the generalised Silverman class, and many more remain to be explored. As we have studied different techniques of differential subordination and applied them effectively to obtain our desired results, we now move forward to study some more general forms of differential subordination, in the subsequent chapter.

Chapter 5

Pythagorean means and Differential Subordination

For $0 \le \alpha \le 1$, *let* $H_\alpha(x,y)$ *be the convex weighted harmonic mean of* x *and* y. *We establish differential subordination implications of the form*

$$H_\alpha(p(z), p(z)\Theta(z) + zp'(z)\Phi(z)) \prec h(z) \Rightarrow p(z) \prec h(z),$$

where Φ, Θ *are analytic functions and* h *is a univalent function satisfying some special properties. Further, we prove differential subordination implications involving a combination of arithmetic, geometric and harmonic mean of the quantities* $p(z)$ *and* $p(z) + zp'(z)/p(z)$. *As an application, we generalize many existing results and obtain sufficient conditions for starlikeness and univalence.*

5.1 Introduction

The three classical Pythagorean means are the arithmetic mean, geometric mean and harmonic mean. Pythagorean means, however, have applications in different domains and are strongly related to the study of univalent function theory. This idea is used in this chapter to prove differential subordination results. Let $\alpha \in [0,1]$ and x, y be any two numbers, then the convex weighted arithmetic mean of x and y is given by

$$A_\alpha(x,y) = (1-\alpha)x + \alpha y.$$

Similarly, the convex weighted geometric mean is defined as

$$G_\alpha(x,y) = x^\alpha y^{1-\alpha}$$

and the convex weighted harmonic mean of x and y is given by

$$H_\alpha(x,y) = \frac{xy}{\alpha y + (1-\alpha)x}.$$

Next we present the classes $\mathcal{M}_\alpha(\beta)$ and $\mathcal{L}_\alpha(\beta)$, which are defined by using the convex arithmetic mean and convex geometric mean of the quantities $zf'(z)/f(z)$ and $1 + zf''(z)/f'(z)$ respectively. The class of α-convex functions of order β is defined as

$$\mathcal{M}_\alpha(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{zf'(z)}\right)\right) > \beta \right\}$$

and the class of α-starlike functions of order β is defined as

$$\mathcal{L}_\alpha(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{\alpha} > \beta \right\},$$

where $z \in \mathbb{D}$, α is any real number and $0 \leq \beta < 1$. In 1996, Kanas et. al [35] introduced and studied differential subordinations involving geometric mean of $p(z)$ and $p(z) + zp'(z)$. Later in 2011, the authors in [49] proved several differential subordination results associated with arithmetic as well as geometric mean of certain analytic functions. In [19], Crişan and Kanas considered a combination of arithmetic and geometric mean for which they established the following implication:

$$\gamma(p(z))^{\delta}+(1-\gamma)(p(z))^{\mu}\left(p(z)+\frac{zp'(z)}{p(z)}\right)^{1-\mu}\prec h(z)\Rightarrow p(z)\prec h(z). \tag{5.1.1}$$

This result, for different choices of $h(z)$, is applied to prove some univalence and starlikeness criteria. A particular form of this expression, for different choices of $h(z)$ has been worked upon by Kanas and Tudor [37]. Recently, Gavriş [24] melded all the three pythagorean means in one expression and proved an implication similar to (5.1.1) for a specific choice of $h(z)$. In the present investigation, we extend the results of [24] by proving a similar implication for another choice of $h(z)$. This result generalised a number of earlier known results and applied to obtain several criteria for starlikeness and strongly starlikeness. In 2014, Cho et al. [14] established conditions on an analytic function $\Phi(z)$ so that the geometric mean of $p(z)$ and $p(z)+zp'(z)\Phi(z)$ is subordinate to $h(z)$ leads $p(z)$ to be subordinate to $h(z)$, where h is a univalent function. Later in the same year, Chojnacka and Lecko [17] proved a similar result for harmonic mean. The arithmetic mean is already covered by Miller and Mocanu in [61]. In our study, we prove some general differential subordination results involving the harmonic mean of the quantities $p(z)$ and $p(z)\Theta(z)+zp'(z)\Phi(z)$, where $\Theta(z)$ and $\Phi(z)$ are analytic functions. As an application, we establish differential subordination results for different subclasses of $\mathcal{S}^*$ as well as the sufficient conditions for starlikeness and univalence, which generalize many well known results. We now define the following class which is mainly required for our upcoming results.

Definition 5.1.1. Let $t\in[0,1]$ and $\Theta,\Phi\in\mathcal{H}$ with $\Theta(0)=1$. By $\mathcal{H}(t;\Theta,\Phi)$, we mean the subclass of $\mathcal{H}$ of all functions f such that

$$H_{t;\Theta,\Phi,f}(z):=\begin{cases}\dfrac{P_{0;\Theta,\Phi,f}(z)P_{1;\Theta,\Phi,f}(z)}{P_{1-t;\Theta,\Phi,f}(z)} & P_{1-t;\Theta,\Phi,f}(z)\neq 0,\\ \lim_{\mathbb{D}\ni\zeta\to z}\dfrac{P_{0;\Theta,\Phi,f}(\zeta)P_{1;\Theta,\Phi,f}(\zeta)}{P_{1-t;\Theta,\Phi,f}(\zeta)} & P_{1-t;\Theta,\Phi,f}(z)=0,\end{cases} \tag{5.1.2}$$

is an analytic function in $\mathbb{D}$, where

$$P_{t;\Theta,\Phi,f}(z):=(1-t+t\Theta(z))f(z)+t\Phi(z)zf'(z),\quad z\in\mathbb{D}$$

and define $H_{t;\Theta,\Phi,0}\equiv 0$.

The class $\mathcal{Q}$ is defined to be the class of convex functions h with the following properties:

1. $h(\mathbb{D})$ is bounded by finitely many smooth arcs which form corners at their end points (including corners at infinity),
2. $E(h)$ is the set of all points $\zeta \in \partial\mathbb{D}$ which corresponds to corners $h(\zeta)$ of $\partial h(\mathbb{D})$,
3. $h'(\zeta) \neq 0$ exists at every $\zeta \in \partial\mathbb{D}\backslash E(h)$.

5.2 A general form of Harmonic Mean

We start this section with the following result, in which we prove a differential subordination implication associated with $H_{t;\Theta,\Phi,f}(z)$, given by (5.1.2) and later in this section, we discuss several applications of this result.

Lemma 5.2.1. Let $\delta \in [0,1]$, $h \in \mathcal{Q}$ with $0 \in \overline{h(\mathbb{D})}$ and $\Theta, \Phi \in \mathcal{H}$ be such that $\Theta(0) = 1$, $\operatorname{Re}\Phi(z) > 0$ $(z \in \mathbb{D})$ and

$$\operatorname{Re}\left(\Phi(z) + \frac{h(\zeta)}{\zeta h'(\zeta)}(\Theta(z) - 1)\right) > 0, \quad z \in \mathbb{D},\ \zeta \in \partial\mathbb{D}. \tag{5.2.1}$$

If $p \in \mathcal{H}(\delta;\Theta,\Phi)$, $p(0) = h(0)$ and $H_{\delta;\Theta,\Phi,p} \prec h$, then $p \prec h$.

Proof. From (5.1.2) it is easy to see that $H_{0;\Theta,\Phi,p} = p$, therefore the implication given in the hypothesis holds when $\delta = 0$. Now let us take $\delta \in (0,1]$. If $p = p(0) \in \mathcal{H}(\delta;\Theta,\Phi)$, then using the fact that $\Theta(0) = 1$ and from (5.1.2), we have $p(0) \in h(\mathbb{D})$ and thus in this case, the implication holds obviously. Now let $p \in \mathcal{H}(\delta;\Theta,\Phi)$ be a nonconstant function and define

$$x := p(z_0) \quad \text{and} \quad y := p(z_0)\Theta(z_0) + z_0p'(z_0)\Phi(z_0). \tag{5.2.2}$$

Since $h \in \mathcal{Q}$, $h'(\zeta_0) \neq 0$ exists. On the contrary, we assume that p is not subordinate to h. Then by using [14, Lemma 2.2] and [17, Lemma 2.3], there exists $z_0 \in \mathbb{D}\backslash\{0\}$ and $\zeta_0 \in \partial\mathbb{D}\backslash E(h)$ such that

$$p(\mathbb{D}_{|z_0|}) \subset h(\mathbb{D}), \quad p(z_0) = h(\zeta_0) \tag{5.2.3}$$

and

$$z_0p'(z_0) = m\zeta_0h'(\zeta_0) \quad \text{for some } m \geq 1. \tag{5.2.4}$$

Using (5.2.3) and (5.2.4) in (5.2.2), we have

$$x = h(\zeta_0) \quad \text{and} \quad y = h(\zeta_0)\Theta(z_0) + m\zeta_0h'(\zeta_0)\Phi(z_0).$$

Let $\mathbb{P}$ be an open half plane, which supports the convex domain $h(\mathbb{D})$ at $h(\zeta_0)$. So

$$x = h(\zeta_0) \in \overline{\mathbb{P}} \quad \text{and} \quad h(\mathbb{D}) \cap \mathbb{P} = \phi. \tag{5.2.5}$$

We observe that

$$y = h(\zeta_0)\Theta(z_0) + m\zeta_0 h'(\zeta_0)\Phi(z_0) = h(\zeta_0) + \zeta_0 h'(\zeta_0)\Psi(z_0), \tag{5.2.6}$$

where

$$\Psi(z) := m\Phi(z_0) + \frac{h(\zeta_0)}{\zeta_0 h'(\zeta_0)}(\Theta(z_0) - 1). \tag{5.2.7}$$

Clearly (5.2.1) together with the fact that $m \geq 1$ implies $\operatorname{Re}\Psi(z) > 0$. Using this along with (5.2.6), we can say that $y \in \mathbb{P}$. For $x, y \in \mathbb{P}$ and $\delta \in (0,1]$, it implies from [17, Lemma 2.1] that the harmonic mean of x and y, $H_{\delta;\Theta,\Phi,p}(z_0) \in \overline{\mathbb{P}}$ provided $y + \delta(x - y) \neq 0$. Taking into account (5.2.5), it follows that $H_{\delta;\Theta,\Phi,p}(z_0) \notin h(\mathbb{D})$, which contradicts the hypothesis and thus the result holds in this case.

For the case when $y + \delta(x - y) = 0$, we have $P_{1-\delta;\Theta,\Phi,p}(z_0) = 0$. By Definition 5.1.1, we know that the limit

$$H_{\delta;\Theta,\Phi,p}(z_0) = \lim_{\mathbb{D} \ni \zeta \to z_0} \frac{P_{0;\Theta,\Phi,p}(\zeta)P_{1;\Theta,\Phi,p}(\zeta)}{P_{1-\delta;\Theta,\Phi,p}(\zeta)}$$

is finite and so $P_{0;\Theta,\Phi,p}(z_0)P_{1;\Theta,\Phi,p}(z_0) = xy = 0$. First let us suppose that $x = 0$, which implies $(1-\delta)y = 0$. Since $x = 0$, y reduces to $y = m\zeta_0 h'(\zeta_0)\Phi(z_0)$. Clearly y can not be zero as $h'(\zeta_0) \neq 0$ and $\operatorname{Re}\Phi(z) > 0$. Hence $x = 0$ if and only if $\delta = 1$. Now we observe from Definition 5.1.1,

$$H_{1;\Theta,\Phi,p}(z_0) = P_{1;\Theta,\Phi,p}(z_0) = y \in \mathbb{P},$$

which further implies $H_{1;\Theta,\Phi,p}(z_0) \notin h(\mathbb{D})$. This is a contradiction to the hypothesis and the result follows. Next let us suppose that $y = 0$, then we have $\delta x = 0$. Since $\delta \in (0,1]$, it follows that $x = 0$ and thus y becomes $y = m\zeta_0 h'(\zeta_0)\Phi(z_0)$, which can never be equal to 0. Therefore such a case is never possible. This completes the proof. □

Remark 4. If we take $t = 1/2$, $\Theta(z) = 1$ and $\Phi(z) = 1$ in $H_{t;\Theta,\Phi,p}(z)$, from Definition 5.1.1 we can say that it reduces to

$$H_{1/2;1,1,p}(z) = \frac{2p(z)(p(z) + zp'(z))}{2p(z) + zp'(z)} =: P(z).$$

Further if we take $h(z) = (1+z)/(1-z)$ in Lemma 5.2.1, it reduces to a result of Kanas and Tudor [37, Theorem 2.1].

Remark 5. If $h(z) = ((1+z)/(1-z))^{\gamma}$, where $\gamma \in (0,1]$, then for $t = 1/2$, $\Theta(z) = 1$ and $\Phi(z) = 1$, Lemma 5.2.1 reduces to [37, Theorem 2.6].

Theorem 5.2.2. Let $\delta \in [0,1]$, $h \in \mathcal{Q}$ with $0 \in \overline{h(\mathbb{D})}$ and $\Theta, \Phi \in \mathcal{H}$ are such that $\Theta(0) = 1$ and

$$\operatorname{Re}\Phi(z) \geq 5|\Theta(z)-1| - \operatorname{Re}(\Theta(z)-1), \quad z \in \mathbb{D}. \tag{5.2.8}$$

If $p \in \mathcal{H}(\delta;\Theta,\Phi)$, $p(0) = h(0)$ and $H_{\delta;\Theta,\Phi,p} \prec h$, then $p \prec h$.

Proof. In view of Lemma 5.2.1, it is sufficient to show that Θ, Φ and h satisfies (5.2.1). Since h is convex and $h(0) = 1$, we can say that $h_1 := h - 1 \in \mathcal{C}$. Using Marx Strohhäcker theorem [61], we have $\operatorname{Re}(\zeta h_1'(\zeta)/h_1(\zeta)) > 1/2$, which is equivalent to

$$\left|\frac{h_1(\zeta)}{\zeta h_1'(\zeta)} - 1\right| \leq 1,$$

which means

$$\left|\frac{h(\zeta)}{\zeta h'(\zeta)} - 1\right| \leq 1 + \frac{1}{|h'(\zeta)|}. \tag{5.2.9}$$

Since $h_1 \in \mathcal{C}$, we have $|h_1'(z)| \geq 1/(1+r)^2$ on $|z| = r$ [27, Theorem.9, pp 118]. We know that $\zeta \in \partial\mathbb{D}$, so we have $|h'(\zeta)| = |h_1'(\zeta)| \geq 1/4$. Thus (5.2.9) reduces to

$$\left|\frac{h(\zeta)}{\zeta h'(\zeta)} - 1\right| \leq 5. \tag{5.2.10}$$

Note that if X, $Y \in \mathbb{C}$ and $|X-1| \leq K$, then

$$\operatorname{Re}(X.Y) = \operatorname{Re} Y + \operatorname{Re} Y(X-1)) \geq \operatorname{Re} Y - |Y|K.$$

Applying this inequality to (5.2.1) and using (5.2.10), we get

$$\operatorname{Re}\left(\Phi(z) + \frac{h(\zeta)}{\zeta h'(\zeta)}(\Theta(z)-1)\right) \geq \operatorname{Re}\Phi(z) + \operatorname{Re}(\Theta(z)-1) - 5|\Theta(z)-1|.$$

From (5.2.7) and (5.2.8), we conclude that $\operatorname{Re}\Psi(z) > 0$ and (5.2.1) holds. Therefore the result follows by Lemma 5.2.1. □

Corollary 5.2.3. Let $\delta \in [0,1]$, $h \in \mathcal{Q}$ with $0 \in \overline{h(\mathbb{D})}$ and $\Theta \in \mathcal{H}$ be a bounded function such that $\Theta(0) = 1$ and $|\Theta(z)| \leq M$ $(z \in \mathbb{D})$ for some $M > 0$. Suppose $\Phi \in \mathcal{H}$ be such that

$$\operatorname{Re}\Phi(z) \geq 6(M+1). \tag{5.2.11}$$

If $p \in \mathcal{H}(\delta;\Theta,\Phi)$, $p(0) = h(0)$ and $H_{\delta;\Theta,\Phi,p} \prec h$, then $p \prec h$.

Proof. We know that $-\operatorname{Re}(\Theta(z)-1) \leq |\Theta(z)-1|$, which gives

$$5|\Theta(z)-1| - \operatorname{Re}(\Theta(z)-1) \leq 6|\Theta(z)-1| \leq 6(|\Theta(z)|+1) \leq 6(M+1).$$

Clearly (5.2.11) is sufficient for (5.2.8) to hold true. Thus the result follows as an application of Theorem 5.2.2. □

Theorem 5.2.4. Let $\delta \in [0,1]$ and $\Theta, \Phi \in \mathcal{H}$ be such that $\Theta(0) = 1$ and

$$\operatorname{Re}\Phi(z) > 2(|\Theta(z)-1| - \operatorname{Re}(\Theta(z)-1)), \quad z \in \mathbb{D}. \tag{5.2.12}$$

If $p \in \mathcal{H}(\delta;\Theta,\Phi)$, $p(0) = 1$ and $H_{\delta;\Theta,\Phi,p} \prec \sqrt{1+z}$, then $p \prec \sqrt{1+z}$.

Proof. We apply Lemma 5.2.1 with $h(z) = \sqrt{1+z}$, then (5.2.1) reduces to

$$\operatorname{Re}\left(2(\Theta(z)-1)\left(1+\frac{1}{\zeta}\right)+\Phi(z)\right) \geq 2\operatorname{Re}(\Theta(z)-1) - 2|\Theta(z)-1| + \operatorname{Re}\Phi(z),$$

which is greater than 0 in view of (5.2.12). Thus the result follows due to an application of Lemma 5.2.1. □

Theorem 5.2.5. Let $\delta \in [0,1]$, $\gamma \in (0,1]$ and $\Theta, \Phi \in \mathcal{H}$, where Θ has real coefficients with $\Theta(0) = 1, \Theta'(0) > 0$ and $\operatorname{Re}\Phi(z) > 0$. Suppose $p \in \mathcal{H}(\delta;\Theta,\Phi)$ be such that p has real coefficients with $p(0) = 1$ and $p'(0) > 0$, then

$$H_{\delta;\Theta,\Phi,p} \prec \left(\frac{1+z}{1-z}\right)^{\gamma} \quad \Rightarrow \quad p(z) \prec \left(\frac{1+z}{1-z}\right)^{\gamma}. \tag{5.2.13}$$

Proof. Let $h(z) = ((1+z)/(1-z))^{\gamma}$. We need to show that

$$\operatorname{Re}(\Theta(z)-1)\frac{h(\zeta)}{\zeta h'(\zeta)} = \operatorname{Re}(\Theta(z)-1)\frac{1-\zeta^2}{2\gamma\zeta} > 0.$$

Let $\Theta(z) = 1 + a_1 z + a_2 z^2 + \cdots$ and define $R(z) = \Theta(z) - 1$, then $R(0) = 0$. Since Θ has real coefficients and $\Theta'(0) > 0$, it is typically real and it is easy to conclude that $R(z)$ is typically real. We know that $\zeta \in h^{-1}(p(D))$, where $D := \{z \in \mathbb{D} : p(z) = h(\zeta) \text{ for some } \zeta \in \mathbb{D}\}$. Clearly $h(z) = ((1+z)/(1-z))^{\gamma}$ is typically real and conditions on p also ensures that it is typically real, so we have $sign(\operatorname{Im} z) = sign(\operatorname{Im} \zeta)$. Now we consider

$$\begin{aligned}\operatorname{Re}\left((\Theta(z)-1)\tfrac{1-\zeta^2}{2\gamma\zeta}\right) &= \frac{1}{2\gamma}\Big(\operatorname{Re}(\Theta(z)-1)\operatorname{Re}\left(\tfrac{1-\zeta^2}{\zeta}\right) \\ &\quad -\operatorname{Im}(\Theta(z)-1)\operatorname{Im}\left(\tfrac{1-\zeta^2}{\zeta}\right)\Big) \\ &= -\frac{1}{2\gamma}\operatorname{Im}(\Theta(z)-1)\operatorname{Im}\left(\frac{1-\zeta^2}{\zeta}\right).\end{aligned}$$

Taking $\zeta = e^{i\theta}$ $(0 \le \theta < 2\pi)$, we have

$$-\operatorname{Im}\left(\frac{1-\zeta^2}{\zeta}\right) = 2\sin\theta \begin{cases} > 0, & \theta \in (0,\pi), \\ < 0, & \theta \in (\pi, 2\pi). \end{cases}$$

Since $\Theta(z)-1$ is typically real, $sign(\operatorname{Im}(\Theta(z)-1)) = sign(\operatorname{Im} z)$. Thus we have

$$\operatorname{Re}\left((\Theta(z)-1)\frac{1-\zeta^2}{2\gamma\zeta}\right) = 2\sin\theta\operatorname{Im}(\Theta(z)-1) \ge 0.$$

Lemma 5.2.1 along with the fact that $\operatorname{Re}\Phi(z) > 0$ implies that (5.2.13) follows and hence the result. □

Theorem 5.2.6. Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*$ with $g(z) \neq zf'(z)$ and

$$\operatorname{Re}\left(\frac{2zf'(z)}{g(z)} - \frac{2z(f'(z))^2}{3g(z)f'(z)+zf''(z)g(z)-zg'(z)f'(z)}\right) > 0, \tag{5.2.14}$$

then $f \in \mathcal{K}$.

Proof. Let $p(z) = zf'(z)/g(z)$, then

$$\frac{2zf'(z)}{g(z)} - \frac{2z(f'(z))^2}{3g(z)f'(z)+zf''(z)g(z)-zg'(z)f'(z)} = \frac{2p(z)(p(z)+zp'(z))}{2p(z)+zp'(z)}.$$

From (5.2.14), we have

$$\operatorname{Re}\left(\frac{2p(z)(p(z)+zp'(z))}{2p(z)+zp'(z)}\right) > 0,$$

or equivalently,

$$\frac{2p(z)(p(z)+zp'(z))}{2p(z)+zp'(z)} \prec \frac{1+z}{1-z}.$$

By applying Lemma 5.2.1 with $\Theta(z) = \Phi(z) = 1$, $t = 1/2$ and $h(z) = (1+z)/(1-z)$, we get

$p(z) \prec (1+z)/(1-z)$, which is equivalent to

$$\operatorname{Re} p(z) = \operatorname{Re} \frac{zf'(z)}{g(z)} > 0.$$

Hence the result. □

So far in this direction, authors have proved differential subordination implications of the form

$$\psi(p(z), zp'(z)) \prec h(z) \Rightarrow p(z) \prec h(z)$$

and the above theorem generalizes many such results in case of harmonic mean. Henceforth, we consider differential subordination implications of the form

$$\psi(p(z), zp'(z)) \prec h(z) \Rightarrow p(z) \prec q(z)$$

for different choices of h, q and ψ. We enlist below a few examples.

***Example* 14.** Let $p(z) = 1 + a_1 z + a_2 z^2 + \cdots$ be analytic in $\mathbb{D}$ with $p(z) \not\equiv 1$. Then $p(z) \prec e^z$, whenever

$$\frac{2p(z)(p(z)+zp'(z))}{2p(z)+zp'(z)} \prec \varphi_i(z)\ (i = 1,2,..,5),$$

where $\varphi_1(z) = \sqrt{1+z}$, $\varphi_2(z) = 2/(1+e^{-z})$, $\varphi_3(z) = z + \sqrt{1+z^2}$, $\varphi_4(z) = 1 + \sin z$ and $\varphi_5(z) = 1 + 4z/3 + 2z^2/3$.

Proof. Let $q(z) = e^z$ and $\Omega_i = \varphi_i(\mathbb{D})$ $(i = 1,2..,5)$. Suppose that $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be a function defined by $\psi(a,b;z) = 2a(a+b)/(2a+b)$. Using the admissibility conditions for e^z given by Naz et al. [67], it is sufficient to prove that $\psi \in \Psi[\Omega_i, e^z]$, or equivalently, $\psi(r,s;z) \notin \Omega$, where $r = e^{e^{i\theta}}$ and $s = me^{i\theta}e^{e^{i\theta}} = mre^{i\theta}$ $(-\pi \le \theta < \pi)$. We observe that

$$\psi(r,s;z) = \frac{2r(r+s)}{2r+s} = \frac{2r(r+mre^{i\theta})}{2r+mre^{i\theta}} = 2e^{e^{i\theta}}\left(1 - \frac{1}{2+me^{i\theta}}\right).$$

Therefore

$$\begin{aligned} \operatorname{Re}\psi(r,s;z) &= \frac{2e^{\cos\theta}\left(\left(m^2+3m\cos\theta+2\right)\cos(\sin\theta) - m\sin\theta\sin(\sin\theta)\right)}{m^2+4m\cos(\theta)+4} \\ &=: a(\theta), \end{aligned}$$

which at $\theta = 0$ becomes $a(0) = 2e(m^2+3m+2)/(5+4m)$. Since $m \ge 1$, we have $a(0) \ge 4e/3$. Thus it is easy to conclude that $\psi(r,s,z) \notin \Omega_i$ $(i = 1,2,..,5)$ and the result follows at once. □

***Example* 15.** Let $p(z) = 1 + a_1z + a_2z^2 + \cdots$ be analytic in $\mathbb{D}$ with $p(z) \not\equiv 1$. Then $p(z) \prec \sqrt{1+z}$, whenever
$$\frac{2p(z)(p(z)+zp'(z))}{2p(z)+zp'(z)} \prec \frac{2}{1+e^{-z}}.$$

Proof. For $\varphi(z) = 2/(1+e^{-z})$, we have $\Omega = \varphi(\mathbb{D}) = \Delta_{SG}$ and $q(z) = \sqrt{1+z}$. Now let $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(a,b;z) = 2a(a+b)/(2a+b)$. Then by using the admissibility conditions for $\sqrt{1+z}$ given by Madaan et al. [55], it suffices to show that $\psi \in \Psi[\Omega, \sqrt{1+z}]$, which is equivalent to, $\psi(r,s;z) \notin \Omega$, where $r = \sqrt{2\cos 2\theta} e^{i\theta}$ and $s = me^{3i\theta}/(2\sqrt{2\cos 2\theta}) = me^{2i\theta}/(2r)$ $(-\pi/4 \leq \theta \leq \pi/4)$. We observe that

$$\psi(r,s;z) = \frac{2r(r+s)}{2r+s} = \frac{2r\left(r + \dfrac{me^{2i\theta}}{2r}\right)}{2r + \dfrac{me^{2i\theta}}{2r}} = 2\sqrt{2\cos 2\theta}e^{i\theta}\left(\frac{m+4\cos 2\theta}{m+8\cos 2\theta}\right).$$

Therefore
$$\operatorname{Re}\psi(r,s;z) = 2\sqrt{2\cos 2\theta}\cos\theta\left(\frac{m+4\cos 2\theta}{m+8\cos 2\theta}\right),$$
which is an increasing function of m. So for $m \geq 1$, we have

$$\operatorname{Re}\psi(r,s;z) \geq 2\sqrt{2\cos 2\theta}\cos\theta\left(\frac{1+4\cos 2\theta}{1+8\cos 2\theta}\right) =: a(\theta),$$

which at $\theta = 0$ becomes $a(0) = 10\sqrt{2}/9 \approx 1.57$. Since $\max \operatorname{Re}(2/(1+e^{-z})) \leq 2e/(1+e) \approx 1.46$, it is easy to conclude that $\psi(r,s,z) \notin \Omega$ and the result follows. □

***Example* 16.** Let $p(z) = 1 + a_1z + a_2z^2 + \cdots$ be analytic in $\mathbb{D}$ with $p(z) \not\equiv 1$. Then $p(z) \prec 2/(1+e^{-z})$, whenever
$$\frac{2p(z)(p(z)+zp'(z))}{2p(z)+zp'(z)} \prec \sqrt{1+z}.$$

Proof. Let us suppose $q(z) = 2/(1+e^{-z})$ and $\Omega = \varphi(\mathbb{D})$, with $\varphi(z) = \sqrt{1+z}$. Also, let $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ is a function given by $\psi(a,b;z) = 2a(a+b)/(2a+b)$. By applying the admissibility conditions for $2/(1+e^{-z})$ (refer to Chapter 2), it is sufficient to prove that $\psi \in \Psi[\Omega, 2/(1+e^{-z})]$, which means $\psi(r,s;z) \notin \Omega$, with $r = 2/(1+e^{-e^{i\theta}})$ and $s = (2me^{i\theta}e^{-e^{i\theta}})/(1+e^{-e^{i\theta}})^2 = (mre^{i\theta}e^{-e^{i\theta}})/(1+e^{-e^{i\theta}})$ $(-\pi \leq \theta < \pi)$. We observe that

$$\psi(r,s;z) = \frac{2r(r+s)}{2r+s} = \frac{4}{1+e^{-e^{i\theta}}}\left(\frac{1+e^{-e^{i\theta}}+me^{i\theta}e^{-e^{i\theta}}}{2+2e^{-e^{i\theta}}+me^{i\theta}e^{-e^{i\theta}}}\right).$$

Then $\operatorname{Re}\psi(r,s;z) = N(\theta)/D(\theta) =: a(\theta)$, where

$$\begin{aligned} N(\theta) &= m^2e^{\cos\theta} + m^2\cos(\sin\theta) + 5me^{\cos\theta}\cos\theta + 3me^{2\cos\theta}\cos(\theta - \sin\theta) \\ &\quad + m\cos(\theta - \sin\theta) + 2m\cos(\theta + \sin\theta) + me^{\cos\theta}\cos(\theta - 2\sin\theta) + 4e^{\cos\theta} \\ &\quad + 2e^{3\cos\theta} + 6e^{2\cos\theta}\cos(\sin\theta) + 2\cos(\sin\theta) + 2e^{\cos\theta}\cos(2\sin\theta) \end{aligned}$$

and

$$\begin{aligned} D(\theta) &= (1 + e^{2\cos\theta} + 2e^{\cos\theta}\cos(\sin\theta))(4 + m^2 + 4e^{2\cos\theta} + 4m\cos\theta \\ &\quad + 8e^{\cos\theta}\cos(\sin\theta) + 4me^{\cos\theta}\cos(\theta - \sin\theta)). \end{aligned}$$

We observe that $a(0) = 4e(1+e+m)/(1+e)(2+2e+m)$ and $m \geq 1$, so we have $a(0) \geq 4e(2+e)/(1+e)(3+2e) \approx 1.635$. Thus it is easy to conclude that $\psi(r,s;z) \notin \Omega$ and the result follows. □

Theorem 5.2.7. Let $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$, $L := k+2$, $M := 2(A+B) + k(2A-B)$, $N := 2AB$, $G := 2(E-D)$, $H := 2AE(k+2) - 2BE(k-1) - AD(k+2) + BD(k-4)$, $I := 2A^2E(k+2) - 2ABE(k-2) - ABD(k+4) + B^2D(k-2)$ and $J := 2A^2BE - 2AB^2D$ with

$$2E(1+A) - D(1+B)) > 0. \tag{5.2.15}$$

In addition, assume

$$GH + HI - 3GJ + IJ + 12GJ \geq 4|GI + HJ| \quad (k \geq 1) \tag{5.2.16}$$

and

$$3 + 2AB + D(B+1)(A(2B+3) + B + 2) \geq 2E(A+1)(A(B+2)+1) + |4A+B|. \tag{5.2.17}$$

Further, let $p(z) = 1 + a_1z + a_2z^2 + \cdots$ be analytic in $\mathbb{D}$ with $p(z) \not\equiv 1$. Then

$$\frac{2p(z)(p(z)+zp'(z))}{2p(z)+zp'(z)} \prec \frac{1+Dz}{1+Ez} \quad \Rightarrow \quad p(z) \prec \frac{1+Az}{1+Bz}.$$

Proof. Define $P(z)$ by

$$P(z) := \frac{2p(z)(p(z)+zp'(z))}{2p(z)+zp'(z)} \quad \text{and} \quad \omega(z) := \frac{p(z)-1}{A-Bp(z)},$$

or equivalently $p(z) = (1+A\omega(z))/(1+B\omega(z))$. Then $\omega(z)$ is clearly meromorphic in $\mathbb{D}$ and $\omega(0) = 0$. We need to show that $|\omega(z)| < 1$ in $\mathbb{D}$. We have

$$P(z) = \frac{2\left(\dfrac{1+A\omega(z)}{1+B\omega(z)}\right)\left(\dfrac{1+A\omega(z)}{1+B\omega(z)} + \dfrac{(A-B)z\omega'(z)}{(1+B\omega(z))^2}\right)}{2\left(\dfrac{1+A\omega(z)}{1+B\omega(z)}\right) + \dfrac{(A-B)z\omega'(z)}{(1+B\omega(z))^2}}.$$

Therefore

$$\frac{P(z)-1}{D-EP(z)} = -\frac{(A-B)(2\omega(z)(1+A\omega(z))(1+B\omega(z)) + (1+2A\omega(z)-B\omega(z))z\omega'(z))}{2(1+A\omega(z))(1+B\omega(z))\Phi_1(z) + (A-B)\Phi_2(z)z\omega'(z)},$$

where

$$\Phi_1(z) = E(1+A\omega(z)) - D(1+B\omega(z)) \text{ and } \Phi_2(z) = 2E(1+A\omega(z)) - D(1+B\omega(z)).$$

On the contrary if there exists a point $z_0 \in \mathbb{D}$ such that $\max_{|z|\le|z_0|} |\omega(z)| = |\omega(z_0)| = 1$, then by [82, Lemma 1.3], there exists $k \ge 1$ such that $z_0\omega'(z_0) = k\omega(z_0)$. Let $\omega(z_0) = e^{i\theta}$, then we have

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| = (A-B)\left|\frac{L+Me^{i\theta}+Ne^{2i\theta}}{G+He^{i\theta}+Ie^{2i\theta}+Je^{3i\theta}}\right|.$$

We observe that

$$|L+Me^{i\theta}+Ne^{2i\theta}|^2 = L^2+M^2+N^2-2LN+2(L+N)M\cos\theta+4LN\cos^2\theta.$$

Choose $t := \cos\theta \in [-1,1]$. Since

$$\min\{at^2+bt+c : -1 \le t \le 1\} = \begin{cases} \dfrac{4ac-b^2}{4a}, & \text{if } a > 0 \text{ and } |b| < 2a \\ a-|b|+c, & \text{otherwise,} \end{cases}$$

we have $|L+Me^{i\theta}+Ne^{2i\theta}|^2 \ge (L-|M|+N)^2$. Next we consider

$$\begin{aligned} |G+He^{i\theta}+Ie^{2i\theta}+Je^{3i\theta}|^2 &= G^2+H^2+I^2+J^2-2GI-2HJ+(2GH+2HI-6GJ \\ &\quad +2IJ)\cos\theta+(4GI+4HJ)\cos^2\theta+8GJ\cos^3\theta, \end{aligned}$$

which is an increasing function of $t = \cos\theta \in [-1,1]$ in view of (5.2.16). Thus we have

$|G+He^{i\theta}+Ie^{2i\theta}+Je^{3i\theta}|^2 \leq (G+H+I+J)^2$. Therefore

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right|^2 \geq \left(\frac{L-|M|+N}{G+H+I+J}\right)^2 =: \psi(k),$$

which in view of (5.2.15) is an increasing function of k. So we have $\psi(k) \geq \psi(1)$ and therefore

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| \geq \frac{3+2AB-|4A+B|}{2E(A+1)(A(B+2)+1)-D(B+1)(A(2B+3)+B+2)},$$

which in view of (5.2.17) is greater than or equal to 1. This contradicts that $P(z) \prec (1+Dz)(1+Ez)$ and that completes the proof. □

Note: The fact that the equations (5.2.15), (5.2.16) and (5.2.17) hold simultaneously is validated by the following set of values: $A=3/8$, $B=0$, $D=1$, $E=123/128$.
In the results that follow, we consider a combination of harmonic mean, geometric mean and arithmetic mean of $p(z)$ and $p(z)+zp'(z)/p(z)$.

5.3 Combination of all Pythagorean means

In this section, we prove certain implications involving a combination of the three classical means, in order to generalize many previously known results.

Theorem 5.3.1. Let $\gamma \in [0,1]$, $\alpha \in [0,1)$, $\delta \in [1,2]$, $\beta \in [0,1)$ and $\rho \in [0,1]$ be such that $\rho \geq \alpha(1+2\alpha)$, whenever $\alpha \in [0,1/2]$. Also, let p be an analytic function with $p(0)=1$ and

$$\operatorname{Re}\left(\gamma(p(z))^{\delta}+(1-\gamma)\frac{p(z)+\dfrac{zp'(z)}{p(z)}}{1+\rho\dfrac{zp'(z)}{p^2(z)}}\right) > \beta,$$

for $\beta \geq \gamma\alpha+(1-\gamma)\beta_0$, where β_0 is given as follows:

$$\beta_0 = \begin{cases} \alpha\dfrac{(1+\alpha)(1-2\alpha)}{\rho(1-\alpha)-2\alpha^2}, & \text{if } I_1 \text{ holds} \\ \alpha & \text{if } (\sim I_1)\wedge I_2 \text{ holds} \\ \alpha+\dfrac{\rho(1-\rho)(1-\alpha)(2(1-\alpha)+\rho)}{16\alpha(2\alpha^2-\rho(1-\alpha))} & \text{if } (\sim I_1)\wedge(\sim I_2)\wedge I_3 \text{ holds} \\ \alpha+\dfrac{\rho(1-\rho)}{16\alpha(1-\alpha)}\left(\dfrac{2\alpha^2-\rho(1-\alpha)}{2(1-\alpha)+\rho}\right) & \text{if } (\sim I_1)\wedge(\sim I_2)\wedge(\sim I_3) \text{ holds,} \end{cases} \tag{5.3.1}$$

with

$$I_1 : 0 \leq \alpha \leq 1/2, \;\; I_2 : \alpha^2 - x^2 + \rho m y > 0 \;\text{ and }\; I_3 : x^2 \geq \tfrac{(2\alpha^2-\rho(1-\alpha))(1-\alpha)}{2(1-\alpha)+\rho}, \tag{5.3.2}$$

provided $x > 0$ and $my \leq -\frac{(1-\alpha)^2+x^2}{2(1-\alpha)}$. Then $\operatorname{Re} p(z) > \alpha$.

Proof. If we let $q(z) = (1+(1-2\alpha)z)/(1-z)$, then it suffices to show that $p \prec q$. On the contrary, let us suppose $p \not\prec q$. Then by Lemma 1.4.3 and [61, Lemma 2.2f], there exist $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D}\backslash\{1\}$ and $m \geq 1$ such that

$$p(z_0) = q(\zeta_0) = \alpha + ix \quad \text{and} \quad z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) =: my \leq -\frac{(1-\alpha)^2 + x^2}{2(1-\alpha)}. \tag{5.3.3}$$

Consequently, we have

$$\gamma(p(z_0))^\delta + (1-\gamma)\frac{p(z_0) + \dfrac{z_0 p'(z_0)}{p(z_0)}}{1 + \rho\dfrac{z_0 p'(z_0)}{p^2(z_0)}} = \gamma(q(\zeta_0))^\delta + (1-\gamma)\frac{q(\zeta_0) + \dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1 + \rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}. \tag{5.3.4}$$

Now let $E(\delta) = (q(\zeta_0))^\delta$ and $L = \{E(\delta) : \delta \in [1,2]\}$. Geometrically, L represents an arc of the logarithmic spiral joining the points $E(1) = q(\zeta_0)$ and $E(2) = (q(\zeta_0))^2$. We know that L cuts each radial halfline at an angle, which is constant. Clearly, $\arg(q(\zeta_0))^\delta$ is an increasing function of δ and thus, L is in the closed halfplane containing the origin and bounded by the line $\operatorname{Re} z = \operatorname{Re} E(1)$. As a result, we have

$$\operatorname{Re}(q(\zeta_0))^\delta \leq \alpha, \quad \delta \in [1,2]. \tag{5.3.5}$$

We observe that

$$\frac{q(\zeta_0) + \dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1 + \rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}} = q(\zeta_0) + (1-\rho)\frac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0) + \rho m\zeta_0 q'(\zeta_0)/q(\zeta_0)}. \tag{5.3.6}$$

Using (5.3.3) and (5.3.6), we have

$$\operatorname{Re}\left(\frac{q(\zeta_0)+\dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1+\rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}\right) = \operatorname{Re}\left(\alpha+ix+(1-\rho)\frac{my}{\alpha+ix+\frac{\rho my}{\alpha+ix}}\right) = \alpha+(1-\rho)\frac{my\alpha(\alpha^2+x^2+\rho my)}{(\alpha^2-x^2+\rho my)^2+4\alpha^2x^2}. \tag{5.3.7}$$

Since $my<0$, we may observe that

$$\operatorname{Re}\left(\frac{q(\zeta_0)+\dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1+\rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}\right) \le \alpha+(1-\rho)\frac{my\alpha(\alpha^2-x^2+\rho my)}{(\alpha^2-x^2+\rho my)^2+4\alpha^2x^2}. \tag{5.3.8}$$

Case (i) $0\le\alpha\le 1/2$.

From hypothesis, we have $\rho\ge\alpha(1+2\alpha)\ge 2\alpha^2/(1-\alpha)$ and from (5.3.3), we have $my\le -\frac{(1-\alpha)^2+x^2}{2(1-\alpha)}$. Therefore

$$\alpha^2-x^2+\rho my\le\frac{2\alpha^2-\rho(1-\alpha)}{2}-x^2\frac{2(1-\alpha)+\rho}{2(1-\alpha)}\le 0.$$

Now using the fact that $\alpha^2x^2>0$ and that $my\le-\frac{(1-\alpha)^2+x^2}{2(1-\alpha)}\le-\frac{1-\alpha}{2}$, we obtain from (5.3.8),

$$\begin{aligned}\operatorname{Re}\left(\frac{q(\zeta_0)+\dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1+\rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}\right) &\le \alpha+(1-\rho)\frac{my\alpha(\alpha^2-x^2+\rho my)}{(\alpha^2-x^2+\rho my)^2}\\ &= \alpha+(1-\rho)\frac{\alpha}{\frac{\alpha^2-x^2}{my}+\rho}\\ &\le \alpha+(1-\rho)\frac{\alpha}{\rho-\frac{2\alpha^2}{1-\alpha}}\\ &= \alpha\frac{(1+\alpha)(1-2\alpha)}{\rho(1-\alpha)-2\alpha^2}.\end{aligned}$$

Case (ii) $1/2<\alpha<1$ and $\alpha^2-x^2+\rho my>0$.

Since $\alpha>1/2$, we have $2\alpha^2/(1-\alpha)>1$ and therefore, $\rho<2\alpha^2/(1-\alpha)$. Also, since $my<0$

and $\alpha^2 - x^2 + \rho m y > 0$, we have

$$\operatorname{Re}\left(\frac{q(\zeta_0)+\dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1+\rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}\right) \leq \alpha + (1-\rho)\frac{my\alpha(\alpha^2-x^2+\rho m y)}{(\alpha^2-x^2+\rho m y)^2+4\alpha^2x^2} \leq \alpha.$$

Case (iii) $1/2 < \alpha < 1$, $\alpha^2 - x^2 + \rho m y \leq 0$ and $x^2 \geq \frac{(2\alpha^2-\rho(1-\alpha))(1-\alpha)}{2(1-\alpha)+\rho}$. From (5.3.7), we have

$$\begin{aligned}
\operatorname{Re}\left(\frac{q(\zeta_0)+\dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1+\rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}\right) &\leq \alpha + (1-\rho)my\frac{\alpha(\alpha^2+x^2+\rho m y)}{(\alpha^2-x^2+\rho m y)^2+4\alpha^2x^2} \\
&\leq \alpha + (1-\rho)\frac{\alpha\rho m^2y^2}{(\alpha^2-x^2+\rho m y)^2+4\alpha^2x^2} \\
&\leq \alpha + (1-\rho)\frac{\alpha\rho m^2y^2}{4\alpha^2x^2} \\
&= \alpha + (1-\rho)\frac{\rho m^2y^2}{4\alpha x^2}. \qquad (5.3.9)
\end{aligned}$$

Using the inequality $my \leq -\frac{(1-\alpha)^2+x^2}{2(1-\alpha)} \leq -\frac{1-\alpha}{2}$ and the condition on x^2 in (5.3.9), we obtain

$$\begin{aligned}
\operatorname{Re}\left(\frac{q(\zeta_0)+\dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1+\rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}\right) &\leq \alpha + (1-\rho)\frac{\rho}{4\alpha}\left(\frac{1-\alpha}{2}\right)^2\left(\frac{2(1-\alpha)+\rho}{(2\alpha^2-\rho(1-\alpha))(1-\alpha)}\right) \\
&= \alpha + \frac{\rho(1-\rho)(1-\alpha)(2(1-\alpha)+\rho)}{16\alpha(2\alpha^2-\rho(1-\alpha))}.
\end{aligned}$$

Case (iv) $1/2 < \alpha < 1$, $\alpha^2 - x^2 + \rho m y \leq 0$ and $x^2 < \frac{(2\alpha^2-\rho(1-\alpha))(1-\alpha)}{2(1-\alpha)+\rho}$.
Considering $my \leq -\frac{(1-\alpha)^2+x^2}{2(1-\alpha)} \leq -\frac{x^2}{2(1-\alpha)}$ and using it in (5.3.9) along with the condition on x^2, we get

$$\begin{aligned}
\operatorname{Re}\left(\frac{q(\zeta_0)+\dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1+\rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}\right) &\leq \alpha + (1-\rho)\frac{\rho}{4\alpha x^2}\left(\frac{-x^2}{2(1-\alpha)}\right)^2 \\
&= \alpha + \frac{\rho(1-\rho)x^2}{16\alpha(1-\alpha)^2} \\
&\leq \alpha + \frac{\rho(1-\rho)}{16\alpha(1-\alpha)}\left(\frac{2\alpha^2-\rho(1-\alpha)}{2(1-\alpha)+\rho}\right).
\end{aligned}$$

Combining all the cases we obtain

$$\operatorname{Re}\left(\frac{q(\zeta_0)+\dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1+\rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}\right)\leq \beta_0, \tag{5.3.10}$$

where β_0 is given by (5.3.1). From (5.3.4), (5.3.5) and (5.3.10), we have

$$\operatorname{Re}\left(\gamma(p(z_0))^{\delta}+(1-\gamma)\frac{p(z_0)+\dfrac{z_0 p'(z_0)}{p(z_0)}}{1+\rho\dfrac{z_0 p'(z_0)}{p^2(z_0)}}\right)\leq \gamma\alpha+(1-\gamma)\beta_0,$$

which contradicts the hypothesis. Hence the result follows. □

Remark 6. If we take $\gamma = 0$ and $\alpha \in [0,1/2]$, the above result reduces to a result of Kanas [34, Theorem 2.4]. Infact, we extended this result for $\alpha \in [0,1)$.

Remark 7. By taking $\gamma = 0$, $\rho = 1/2$ and $\alpha = 0$, we obtain a result of Kanas and Tudor [37, Theorem 2.3]

Theorem 5.3.2. Let $\gamma \in [0,1]$, $\alpha \in [0,1)$, $\mu \in [0,1]$, $\delta \in [1,2]$, $\beta \in [0,1)$ and $\rho \in [0,1]$ be such that $\rho \geq \alpha(1+2\alpha)$, whenever $\alpha \in [0,1/2]$. Also, let p be an analytic function with $p(0) = 1$ and

$$\operatorname{Re}\left(\gamma(p(z))^{\delta}+(1-\gamma)\frac{(p(z))^{\mu}\left(p(z)+\dfrac{zp'(z)}{p(z)}\right)^{1-\mu}}{1+\rho\dfrac{zp'(z)}{p^2(z)}}\right)>\beta,$$

for $\beta \geq \gamma\alpha+(1-\gamma)\beta_1$, where β_1 is defined as follows:

$$\beta_1=\begin{cases}\alpha & I_4 \text{ holds}\\ \alpha+\dfrac{2\alpha\rho^2(1-2\alpha)}{(\rho-2(1-\alpha))^2(\rho(1-\alpha)-2\alpha^2)}, & I_1\wedge(\sim I_4) \text{ holds}\\ \alpha+\dfrac{\alpha\rho(1-\alpha)(4\alpha^2-\rho(1-\alpha))}{4(2\alpha^2-\rho(1-\alpha))^2} & (\sim I_1)\wedge I_2\wedge(\sim I_4) \text{ holds}\\ \alpha+\dfrac{\alpha\rho(1-\alpha)(2(1-\alpha)+\rho)}{2(4\alpha^2(1-\alpha)+\rho(2\alpha-1))} & (\sim I_1)\wedge(\sim I_2)\wedge I_3\wedge(\sim I_4) \text{ holds}\\ \alpha+\dfrac{\alpha\rho(2\alpha^2-\rho(1-\alpha))}{2(4\alpha^2(1-\alpha)+\rho(2\alpha-1))} & (\sim I_1)\wedge(\sim I_2)\wedge(\sim I_3)\wedge(\sim I_4) \text{ holds,}\end{cases}$$

with I_1, I_2 and I_3 given by (5.3.2) and $I_4 : \alpha^2+x^2+\rho m y \leq 0$, provided $x > 0$ and $my \leq -\frac{(1-\alpha)^2+x^2}{2(1-\alpha)}$. Then $\operatorname{Re} p(z) > \alpha$.

Proof. We proceed as done in Theorem 5.3.1 to show that $p \prec q$, where $q(z) = (1+(1-$

$2\alpha)z)/(1-z)$. For if $p \not\prec q$, then there exists $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D}$ and $m \geq 1$ such that

$$\gamma(p(z_0))^{\delta} + (1-\gamma)\frac{(p(z_0))^{\mu}\left(p(z_0) + \dfrac{z_0 p'(z_0)}{p(z_0)}\right)^{1-\mu}}{1 + \rho\dfrac{z_0 p'(z_0)}{p^2(z_0)}}$$
$$= \gamma(q(\zeta_0))^{\delta} + (1-\gamma)\frac{(q(\zeta_0))^{\mu}\left(q(\zeta_0) + \dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right)^{1-\mu}}{1 + \rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}.$$

From the proof of Theorem 5.3.1, we have

$$\operatorname{Re}(q(\zeta_0))^{\delta} \leq \alpha. \tag{5.3.11}$$

Now we set

$$E(\mu) = \frac{(q(\zeta_0))^{\mu}\left(q(\zeta_0) + \dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right)^{1-\mu}}{1 + \rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}} = \frac{q(\zeta_0)\left(1 + \dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}\right)^{1-\mu}}{1 + \rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}$$

and let $L = \{E(\mu) : \mu \in [0,1]\}$, which gives

$$E(0) = \frac{q(\zeta_0) + \dfrac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}}{1 + \rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}} \quad \text{and} \quad E(1) = \frac{q(\zeta_0)}{1 + \rho\dfrac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}}.$$

It was shown in the proof of Theorem 5.3.1 that $\operatorname{Re} E(0) \leq \beta_0$, where β_0 is given by (5.3.1). Now we consider $\operatorname{Re} E(1))$, which by using the conditions (5.3.3) becomes

$$\begin{aligned}
\operatorname{Re} E(1) &= \operatorname{Re}\left(\frac{\alpha + ix}{1 + \dfrac{\rho m y}{(\alpha + ix)^2}}\right) \\
&= \frac{\alpha(\alpha^4 + x^4 + 2\alpha^2 x^2 + \alpha^2 \rho m y - 3x^2 \rho m y)}{(\alpha^2 - x^2 + \rho m y)^2 + 4\alpha^2 x^2} \\
&= \alpha - \frac{\alpha \rho m y(\alpha^2 + x^2 + \rho m y)}{(\alpha^2 - x^2 + \rho m y)^2 + 4\alpha^2 x^2}.
\end{aligned}$$

It is easy to note that $\operatorname{Re} E(1) \leq \alpha$, whenever $\alpha^2+x^2+\rho m y \leq 0$. So now let us assume that $\alpha^2+x^2+\rho m y > 0$ for the rest of the cases. Before we start our first case, we observe that

$$\begin{aligned} -\alpha\rho m y\left(\frac{\alpha^2+x^2+\rho m y}{(\alpha^2-x^2+\rho m y)^2+4\alpha^2x^2}\right) &\leq -\alpha\rho m y\left(\frac{\alpha^2+x^2}{(\alpha^2-x^2+\rho m y)^2+4\alpha^2x^2}\right) \\ &\leq \frac{-\alpha\rho m y(\alpha^2+x^2)}{(\alpha^2+x^2+\rho m y)^2} \\ &= \frac{-\alpha\rho(\alpha^2+x^2)}{m y\left(\frac{\alpha^2+x^2}{m y}+\rho\right)^2}, \end{aligned} \tag{5.3.12}$$

since $my < 0$ and $\alpha^2+x^2+\rho m y > 0$.

Case (i) $0 \leq \alpha \leq 1/2$.

From (5.3.3), we have $my \leq -\frac{(1-\alpha)^2+x^2}{2(1-\alpha)}$, which further implies

$$\alpha^2+x^2+\rho m y \leq \alpha^2 - \frac{\rho(1-\alpha)}{2} + x^2\left(1-\frac{\rho}{2(1-\alpha)}\right). \tag{5.3.13}$$

The inequality $\alpha \leq 1/2$ implies $1-\rho/2(1-\alpha) > 0$. Also from hypothesis, we have $\rho \geq \alpha(1+2\alpha) \geq 2\alpha^2/(1-\alpha)$, which is sufficient to conclude that

$$\alpha^2+x^2+\rho m y \leq x^2\left(1-\frac{\rho}{2(1-\alpha)}\right).$$

Thus

$$\frac{\alpha^2+x^2}{my}+\rho \geq \rho-2(1-\alpha). \tag{5.3.14}$$

We know that $\alpha^2+x^2+\rho m y > 0$, so from (5.3.13) we obtain

$$x^2 \geq \frac{(\rho(1-\alpha)-2\alpha^2)(1-\alpha)}{2(1-\alpha)-\rho}. \tag{5.3.15}$$

Using (5.3.14) and (5.3.15) in (5.3.12), we get

$$\begin{aligned} -\alpha\rho m y\left(\frac{\alpha^2+x^2+\rho m y}{(\alpha^2-x^2+\rho m y)^2+4\alpha^2x^2}\right) &\leq \frac{2\alpha\rho(1-\alpha)(\alpha^2+x^2)}{x^2(\rho-2(1-\alpha))^2} \\ &= \frac{2\alpha\rho(1-\alpha)\left(\frac{\alpha^2}{x^2}+1\right)}{(\rho-2(1-\alpha))^2} \\ &\leq \frac{2\alpha(1-2\alpha)\rho^2}{(\rho-2(1-\alpha))^2(\rho(1-\alpha)-2\alpha^2)}. \end{aligned}$$

Case (ii) $1/2 < \alpha < 1$ and $\alpha^2 - x^2 + \rho m y > 0$

Using the above condition, we obtain

$$\alpha^2 + x^2 + \rho m y \quad < \quad 2(\alpha^2 + \rho m y) \le 2\left(\alpha^2 - \frac{\rho(1-\alpha)}{2}\right).$$

Thus we have

$$\frac{\alpha^2 + x^2}{my} + \rho \ge \frac{-2\left(2\alpha^2 - \rho(1-\alpha)\right)}{1-\alpha}. \tag{5.3.16}$$

Next we observe that

$$\alpha^2 + x^2 \quad \le \quad 2\alpha^2 + \rho m y \le 2\alpha^2 - \frac{\rho(1-\alpha)}{2}. \tag{5.3.17}$$

Using (5.3.16) and (5.3.17) in (5.3.12), we get

$$\begin{aligned} -\alpha\rho m y\left(\frac{\alpha^2 + x^2 + \rho m y}{(\alpha^2 - x^2 + \rho m y)^2 + 4\alpha^2 x^2}\right) &\le \frac{2\alpha\rho\left(2\alpha^2 - \frac{\rho(1-\alpha)}{2}\right)}{4(1-\alpha)\left(\frac{2\alpha^2 - \rho(1-\alpha)}{1-\alpha}\right)^2} \\ &= \frac{\alpha\rho(1-\alpha)(4\alpha^2 - \rho(1-\alpha))}{4(2\alpha^2 - \rho(1-\alpha))^2}. \end{aligned}$$

Case (iii) $1/2 < \alpha < 1$, $\alpha^2 - x^2 + \rho m y \le 0$ and $x^2 \ge \frac{(2\alpha^2 - \rho(1-\alpha))(1-\alpha)}{2(1-\alpha)+\rho}$.

Since $my < 0$, $\alpha^2 + x^2 + \rho m y \le \alpha^2 + x^2$, which further implies

$$\frac{\alpha^2 + x^2}{my} + \rho \ge \frac{-2(\alpha^2 + x^2)}{1-\alpha}. \tag{5.3.18}$$

Now by using (5.3.18) and the condition $my \le -(1-\alpha)/2$ in (5.3.12), we obtain

$$-\alpha\rho m y\left(\frac{\alpha^2 + x^2 + \rho m y}{(\alpha^2 - x^2 + \rho m y)^2 + 4\alpha^2 x^2}\right) \le \frac{\alpha\rho(1-\alpha)}{2(\alpha^2 + x^2)},$$

which after applying the condition on x^2 becomes

$$-\alpha\rho m y\left(\frac{\alpha^2 + x^2 + \rho m y}{(\alpha^2 - x^2 + \rho m y)^2 + 4\alpha^2 x^2}\right) \le \frac{\alpha\rho(1-\alpha)(2(1-\alpha)+\rho)}{2(4\alpha^2(1-\alpha) + \rho(2\alpha - 1))}.$$

Case (iv) $1/2 < \alpha < 1$, $\alpha^2 - x^2 + \rho m y \le 0$ and $x^2 < \frac{(2\alpha^2 - \rho(1-\alpha))(1-\alpha)}{2(1-\alpha)+\rho}$.

Proceeding similarly as done in case (iii) and replacing the condition $my \le -(1-\alpha)/2$

by $my \le -x^2/2(1-\alpha)$, we get

$$-\alpha\rho my\left(\frac{\alpha^2+x^2+\rho my}{(\alpha^2-x^2+\rho my)^2+4\alpha^2x^2}\right) \le \frac{\alpha\rho}{2\left(\frac{\alpha^2}{x^2}+1\right)(1-\alpha)},$$

which by using the condition on x^2 becomes

$$\begin{aligned}-\alpha\rho my\left(\frac{\alpha^2+x^2+\rho my}{(\alpha^2-x^2+\rho my)^2+4\alpha^2x^2}\right) &\le \frac{\alpha\rho}{2\left(\frac{\alpha^2}{x^2}+1\right)(1-\alpha)}\\ &\le \frac{\alpha\rho(2\alpha^2-\rho(1-\alpha))}{2(4\alpha^2(1-\alpha)+\rho(2\alpha-1))}.\end{aligned}$$

Combining all the cases, we obtain that $E(1) \le \beta_1$. We observe that L represents an arc of logarithmic spiral with end points $E(0)$ and $E(1)$, such that it cuts every radial halfline at a constant angle. Also

$$\arg E(\mu) = \arg(q(\zeta_0)) + (1-\mu)\left(1+\frac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}\right) - \arg\left(1+\rho\frac{m\zeta_0 q'(\zeta_0)}{q^2(\zeta_0)}\right)$$

is a decreasing function of μ and thus, $\arg E(1) \le \arg E(\mu) \le \arg E(0)$, $\mu \in [0,1]$. So we may conclude that L lies in the closed halfplane containing the origin and determined by the line $\operatorname{Re} z = \operatorname{Re} E(1)$, which means

$$\operatorname{Re} E(\mu) \le \beta_1. \tag{5.3.19}$$

From (5.3.11) and (5.3.19), we obtain

$$\operatorname{Re}\left(\gamma(p(z))^\delta + (1-\gamma)\frac{(p(z))^\mu\left(p(z)+\frac{zp'(z)}{p(z)}\right)^{1-\mu}}{1+\rho\frac{zp'(z)}{p^2(z)}}\right) \le \gamma\alpha + (1-\gamma)\beta_1 \le \beta,$$

which contradicts the hypothesis and hence the result follows. □

Remark 8. Taking $\rho = 0$, we obtain the result [19, Theorem 2.3]

Remark 9. By taking $\mu = 0$, $\gamma = 0$, $\rho = 1/2$ and $\alpha = 0$, we obtain the result [37, Theorem 2.3].

Remark 10. For $\gamma = 0$, $\rho = 0$ and $\alpha = 0$, we get a result of Lewandowski et al. [51].

Remark 11. For $\delta = 1$, $\rho = 0$, $\mu = 0$ and $\alpha = 0$, we obtain a result of Sakaguchi [84].

By taking $p(z) = zf'(z)/f(z)$ in Theorem 5.3.2, we obtain the following result.

Corollary 5.3.3. Let $\gamma \in [0,1]$, $\alpha \in [0,1)$, $\mu \in [0,1]$, $\delta \in [1,2]$, $\beta \in [0,1)$ and $\rho \in [0,1]$ be such that $\rho \geq \alpha(1+2\alpha)$, whenever $\alpha \in [0,1/2]$. Also, let $f \in \mathcal{A}$ satisfies

$$\operatorname{Re}\left(\gamma\left(\frac{zf'(z)}{f(z)}\right)^{\delta} + (1-\gamma)\frac{\left(\frac{zf'(z)}{f(z)}\right)^{1+\mu}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\mu}}{\rho\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\rho)\frac{zf'(z)}{f(z)}}\right) > \beta,$$

where β is as defined in Theorem 5.3.2. Then f is starlike of order α.

Remark 12. If we take $\delta = 1$, $\mu = 0$ and $\rho = 0$ in the above result, then it implies that the class of γ-convex functions of order β is included in the class of starlike functions of order α. Further by taking $\alpha = 0$, it reduces to the well known result which states that every γ-convex function is starlike(see [63–65]).

Remark 13. By taking $\delta = 1, \rho = 0$ and $\alpha = 0$ in the above result, we obtain the well known result which states that a μ-starlike function is starlike [50, 51].

By taking $p(z) = f'(z)$ in Theorem 5.3.2, we obtain the following result.

Corollary 5.3.4. Let $\gamma \in [0,1]$, $\alpha \in [0,1)$, $\mu \in [0,1]$, $\delta \in [1,2]$, $\beta \in [0,1)$ and $\rho \in [0,1]$ be such that $\rho \geq \alpha(1+2\alpha)$, whenever $\alpha \in [0,1/2]$. Also, let $f \in \mathcal{A}$ satisfies

$$\operatorname{Re}\left(\gamma(f'(z))^{\delta} + (1-\gamma)\frac{(f'(z))^{1+\mu}\left(f'(z)+\frac{zf''(z)}{f'(z)}\right)^{1-\mu}}{f'(z)+\rho\frac{zf''(z)}{f'(z)}}\right) > \beta,$$

where β is as defined in Theorem 5.3.2. Then $\operatorname{Re}(f'(z)) > \alpha$ and therefore f is univalent in $\mathbb{D}$.

By taking $p(z) = f(z)/z$ in Theorem 5.3.2, we obtain the following result.

Corollary 5.3.5. Let $\gamma \in [0,1]$, $\alpha \in [0,1)$, $\mu \in [0,1]$, $\delta \in [1,2]$, $\beta \in [0,1)$ and $\rho \in [0,1]$ be such that $\rho \geq \alpha(1+2\alpha)$, whenever $\alpha \in [0,1/2]$. Also, let $f \in \mathcal{A}$ satisfies

$$\operatorname{Re}\left(\gamma\left(\frac{f(z)}{z}\right)^{\delta} + (1-\gamma)\frac{\left(\frac{f(z)}{z}\right)^{1+\mu}\left(\frac{f(z)}{z}+\frac{zf'(z)}{f(z)}-1\right)^{1-\mu}}{\frac{f(z)}{z}+\rho\frac{zf'(z)}{f(z)}-1}\right) > \beta,$$

where β is as defined in Theorem 5.3.2. Then $\operatorname{Re}(f(z)/z) > \alpha$.

Concluding Remarks

With the construction of a general form of harmonic mean, a differential subordination implication is proved in this chapter. The main result is proved with the help of geometric properties of harmonic mean. Another main result of this chapter extends several works of differential subordination related to arithmetic, geometric or harmonic mean. The proofs involve complex computations and some geometrical concepts from different areas. As a matter of scope, the problems attempted in this chapter can be generalised in different ways. In this chapter, we demonstrate several differential subordination results using a variety of mathematical concepts. As we move forward, we use integral operators to derive a special type of differential subordination, known as Briot-Bouquet type differential subordination, and study it in detail.

Chapter 6

On a Briot-Bouquet type Differential Subordination

We prove some results that are analogous to open door lemma and integral existence theorem. Using the integral representations of the solution of a differential equation, we prove sufficient conditions for univalence and starlikeness. Further, we introduce and study the following special type of differential subordination implication:

$$p(z)Q(z)+\frac{zp'(z)}{\beta p(z)+\alpha}\prec h(z)\quad\Rightarrow p(z)\prec h(z), \tag{6.0.1}$$

which involves generalization of the Briot-Bouquet differential subordination, where $Q(z)$ is analytic and $0\neq\beta,\alpha\in\mathbb{C}$. In addition, we discuss some special cases by taking several functions in place of $h(z)$.

6.1 Introduction

Integral operators play a major role in the field of differential subordination. Authors like Goluzin [25], Robinson [80] and Hallenbeck and Rusheweyh [29] enlightened us by interlinking of integral operators and differential subordinations. The relationship between a differential subordination and its integral analogue allows us to obtain subordination results for an integral operator in a simplified manner. For instance in [52], it was proved that the Libera integral operator preserves some special classes of univalent functions. Later, Miller and Mocanu [61] provided a much simpler proof of this result, by means of differential subordination. In a similar way, Parvathvam [71] and Ali et al. [5] considered Bernardi integral operator and proved interesting results by transforming it into Briot-Bouquet type differential equation. The corresponding first order differential subordination, known as Briot-Bouquet differential subordination, defined as

$$p(z)+\frac{zp'(z)}{\beta p(z)+\alpha}\prec h(z) \tag{6.1.1}$$

was extensively studied by Miller and Mocanu [61]. Many implication results were proved later associating (6.1.1). Previously, Ruscheweyh and Singh [83] considered Briot-Bouquet differential subordination in a more particular form given by

$$p(z)+\frac{zp'(z)}{\beta p(z)+\alpha}\prec\frac{1+z}{1-z}$$

with $\alpha\geq 0$ and $\beta>0$. Later it was generalised to the form given by (6.1.1), in which $h(z)$ is taken to be a univalent function and $\alpha,\beta\neq 0$ are extended to complex numbers. This particular differential subordination has vast number of applications in the univalent function theory, see [18, 22, 62, 69, 89] and the references therein. It is known that the Briot-Bouquet differential subordination is obtained from the Bernardi integral operator. Similarly, the general form of the Bernardi integral operator given by

$$F(z)=I[f,g]=\left(\frac{\alpha+\beta}{g^{\alpha}(z)}\int_0^z g'(t)g^{\alpha-1}(t)f^{\beta}(t)dt\right)^{1/\beta} \tag{6.1.2}$$

with appropriate choice of p and h yields a different type of differential subordination, which we introduce here:

$$p(z)Q(z)+\frac{zp'(z)}{\beta p(z)+\alpha}\prec h(z)\quad (z\in\mathbb{D}), \tag{6.1.3}$$

where α, $\beta \in \mathbb{C}$ with $\beta \neq 0$ and Q is an analytic function such that

$$g(z) = z \exp \int_0^z \frac{Q(t) - 1}{t} dt. \tag{6.1.4}$$

The expression (6.1.3) is clearly a generalization of the Briot-Bouquet differential subordination as it is evident when we choose $Q(z) = 1$. In the present investigation, we find conditions on α, β and $Q(z)$ so that the implication (6.0.1) holds. Further, we examine some special cases concerning this result. For the special cases, we need to recall some of the subclasses of $\mathcal{S}^*$, which are listed in Chapter 1. Currently, we need to learn about a couple of more classes, defined as follows.

- In [18], Coman defined that a function $f \in \mathcal{A}$ is said to be almost strongly starlike of order $\alpha, \alpha \in (0,1]$, with respect to the function $g \in \mathcal{S}^*(1-\alpha)$ if

$$\frac{g(z)f'(z)}{g'(z)f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \text{ or equivalently, } \left|\arg \frac{g(z)f'(z)}{g'(z)f(z)}\right| < \alpha\frac{\pi}{2}$$

  and concluded that such functions are starlike and hence univalent.

- Recently, Antonino and Miller [8] defined the class of F-starlike functions, denoted by $\mathcal{FS}^*$ as

$$\mathcal{FS}^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{F(z)f'(z)}{F'(z)f(z)}\right) > 0 \right\},$$

  where F is fixed univalent function on the closed unit disk $\mathbb{D}$, with at most a single pole on $\partial\mathbb{D}$ and $F(0) = 0$.

Moreover, we explore differential equations and find conditions that suffice to imply a differential subordination. These results are analogous to open door lemma and integral existence theorem. We derive other similar results and find sufficient conditions for starlikeness and univalence as applications of our results. This work has been carried out for a particular type of differential subordination, given by (6.0.1). The basic definitions and results associated with the theory of differential subordination have been already covered in Chapter 1.

6.2 Analogues of Open Door Lemma

In this section, we obtain conditions on the variable coefficients of certain differential equations so that their solution is subordinate to a specific function.

Theorem 6.2.1. Let $p(z)$ be analytic in $\mathbb{D}$ with $p(0) = 1$ and $Q(z) \in \mathcal{P}$. Suppose that $\alpha \geq 0, \beta > 0$ and p satisfies

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z) + \alpha} = 1, \tag{6.2.1}$$

then $\operatorname{Re} p(z) > 0$.

Proof. Let us define the analytic function g by

$$g(z) = z \exp \int_0^z \frac{Q(t) - 1}{t} dt.$$

Then one can verify that the function $p(z)$, given by

$$p(z) = \frac{g^\alpha(z) z^\beta}{\beta} \left(\int_0^z g^{\alpha-1}(t) g'(t) t^\beta \right)^{-1} - \frac{\alpha}{\beta} \tag{6.2.2}$$

is analytic in $\mathbb{D}$ and $p \in \mathcal{H}[1,n]$. The logarithmic differentiation of (6.2.2) reveals that p is a solution of the differential equation (6.2.1). We now use [61, Theorem 2.3i] to prove that $\operatorname{Re} p(z) > 0$. Let $\Omega = \{1\}$ and $\psi(r,s;z) = rQ(z) + s/(\beta r + \alpha)$, then (6.2.1) can be written as

$$\{\psi(p(z), zp'(z); z) | z \in \mathbb{D}\} \subset \Omega.$$

In view of [61, Theorem 2.3i], it is sufficient to show that $\psi \in \Psi_n[\Omega, 1]$, which means admissibility conditions defined for the class $\Psi_n[\Omega, 1]$ are satisfied by ψ or equivalently:

$$\psi(\rho i, \sigma; z) = \frac{\sigma}{\beta \rho i + \alpha} + Q(z)\rho i \neq 1, \tag{6.2.3}$$

where $\rho \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1+\rho^2)$, $z \in \mathbb{D}$ and $n \geq 1$. Suppose on the contrary if we assume (6.2.3) is false, then there exist some values ρ_0, σ_0 and z_0 such that

$$\frac{\sigma_0}{\beta \rho_0 i + \alpha} + i\rho_0 Q(z_0) = 1. \tag{6.2.4}$$

If we let $Q(z) = S(z) + iT(z)$, then (6.2.4) yields

$$\frac{\sigma_0 \alpha}{\alpha^2 + \beta^2 \rho_0^2} - \rho_0 T(z_0) = 1 \quad \text{and} \quad -\frac{\beta \sigma_0 \rho_0}{\alpha^2 + \beta^2 \rho_0^2} + \rho_0 S(z_0) = 0. \tag{6.2.5}$$

Since $\sigma_0 < 0$ and $\alpha \geq 0$, we have $\rho_0 \neq 0$. Therefore from (6.2.5), we deduce that

$$\operatorname{Re} Q(z_0) = S(z_0) = \frac{\beta \sigma_0}{\alpha^2 + \beta^2 \rho_0^2} \leq -\frac{\beta n (1 + \rho_0^2)}{2(\alpha^2 + \beta^2 \rho_0^2)} < 0,$$

which contradicts the hypothesis and hence it follows that $\operatorname{Re} p(z) > 0$. □

Corollary 6.2.2. Let $f \in \mathcal{A}$ be such that $f(z)/z \neq 0$ in $\mathbb{D}$ and $f \in \mathcal{R}$. Then $\operatorname{Re} f(z)/z > 0$.

Proof. Let $Q(z) = f'(z)$ then $Q \in \mathcal{P}$ as $f \in \mathcal{R}$. Choose $p(z) = z/f(z)$, $\beta = 1$ and $\alpha = 0$, then clearly $p(0) = 1$ and $p(z)$ satisfies (6.2.1). Now by an application of Theorem 6.2.1, it follows that $\operatorname{Re} p(z) > 0$ and hence the result. □

Here below we consider an analogue of integral existence theorem:

Theorem 6.2.3. Let φ, $\phi \in \mathcal{H}[1,n]$, with $\varphi(z)\phi(z) \neq 0$ in $\mathbb{D}$. Let λ, η, γ and δ be complex numbers with $\eta \neq 0$, $\lambda + \delta = \eta + \gamma = 1$ and α, β be non negative real numbers with $\beta \neq 0$. Let $g \in \mathcal{A}_n$ and suppose that

$$Q(z) = \lambda \frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec \frac{1+z}{1-z}. \tag{6.2.6}$$

If F is defined by

$$F(z) = \left(\frac{\alpha + \beta}{\phi^{\beta}(z) g^{\lambda\alpha}(z) \varphi^{\alpha}(z) z^{\delta\alpha + \beta\gamma}} \int_0^z g^{\lambda\alpha}(t) \varphi^{\alpha}(t) t^{\beta + \delta\alpha - 1} Q(t)\, dt \right)^{\frac{1}{\eta\beta}}, \tag{6.2.7}$$

then $F \in \mathcal{A}_n$, $F(z)/z \neq 0$ and for $z \in \mathbb{D}$,

$$\operatorname{Re} \left(\frac{\eta \dfrac{zF'(z)}{F(z)} + \dfrac{z\phi'(z)}{\phi(z)} + \gamma}{\lambda \dfrac{zg'(z)}{g(z)} + \dfrac{z\varphi'(z)}{\varphi(z)} + \delta} \right) > 0. \tag{6.2.8}$$

Proof. Let $p(z)$ be given by

$$p(z) = \frac{1}{\beta} g^{\lambda\alpha}(z) \varphi^{\alpha}(z) z^{\beta + \delta\alpha} \left(\int_0^z g^{\lambda\alpha}(t) \varphi^{\alpha}(t) t^{\beta + \delta\alpha - 1} Q(t)\, dt \right)^{-1} - \frac{\alpha}{\beta}. \tag{6.2.9}$$

Then

$$p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$$

is analytic in $\mathbb{D}$ and $p \in \mathcal{H}[1,n]$. The logarithmic differentiation of (6.2.9) shows that $p(z)$ satisfies (6.2.1) with $Q(z)$ as given in (6.2.6). From hypothesis, we have $Q(z) \prec (1+z)/(1-z)$. Thus the hypothesis of the Theorem. 6.2.1 is fulfilled by p as well as Q

and hence it follows that $\operatorname{Re} p(z) > 0$. Using (6.2.7) and (6.2.9), we get

$$F(z) = \left(\frac{(\alpha+\beta) z^{\beta(1-\gamma)}}{\phi^{\beta}(z)(\beta p(z)+\alpha)} \right)^{\frac{1}{\eta\beta}} = z \left(\frac{\alpha+\beta}{\phi^{\beta}(z)(\beta p(z)+\alpha)} \right)^{\frac{1}{\eta\beta}}. \tag{6.2.10}$$

Clearly the expression in the bracket is analytic and non-zero, so we deduce that $F \in \mathcal{A}_n$ and $F(z)/z \neq 0$. By differentiating (6.2.10) logarithmically, we obtain

$$\eta \frac{zF'(z)}{F(z)} = 1 - \gamma - \frac{z\phi'(z)}{\phi(z)} - \frac{zp'(z)}{\beta p(z)+\gamma},$$

which by using (6.2.1) simplifies to

$$\eta \frac{zF'(z)}{F(z)} + \frac{z\phi'(z)}{\phi(z)} + \gamma = p(z)Q(z).$$

Therefore

$$\operatorname{Re} \frac{1}{Q(z)} \left(\eta \frac{zF'(z)}{F(z)} + \frac{z\phi'(z)}{\phi(z)} + \gamma \right) > 0,$$

which is equivalent to (6.2.8) and hence completes the proof. □

We now obtain the following special case of the above theorem when $\delta = \gamma = 0$, $\lambda = \eta = 1$ and $\varphi(z) = 1 = \phi(z)$.

Corollary 6.2.4. Let $g \in \mathcal{S}^*$, $\alpha \geq 0$, $\beta > 0$ and F is defined as

$$F(z) = \frac{\alpha+\beta}{g^{\alpha}(z)} \int_0^z g'(t) g^{\alpha-1}(t) t^{\beta} dt,$$

then F is almost strongly starlike with respect to g and hence univalent.

Theorem 6.2.5. Let $f, g \in \mathcal{A}_n$ and $\phi \in \mathcal{H}[1,n]$ with $\phi(0) = 1$ and $\phi(z) \neq 0$ in $\mathbb{D}$. Let β, α and σ be complex numbers such that $\beta \neq 0$ and $\operatorname{Re}(\beta h(z) + \alpha) > 0$, where h is a convex function with $h(0) = 1$ in $\mathbb{D}$. Suppose

$$\beta \frac{zf'(z)}{f(z)} + \sigma \frac{zg'(z)}{g(z)} \prec \beta h(z) + \sigma. \tag{6.2.11}$$

If F is defined by

$$F(z) = \left(\frac{\beta+\alpha}{z^{\alpha}\phi(z)} \int_0^z f^{\beta}(t) g^{\sigma}(t) t^{\alpha-\sigma-1} dt \right)^{\frac{1}{\beta}}, \tag{6.2.12}$$

then $F \in \mathcal{A}_n$ and

$$\frac{zF'(z)}{F(z)} + \frac{1}{\beta} \frac{z\phi'(z)}{\phi(z)} \prec h(z). \tag{6.2.13}$$

Proof. Let $p(z)$ be given by

$$p(z) = \frac{1}{\beta} f^{\beta}(z) g^{\sigma}(z) z^{\alpha-\sigma} \left(\int_0^z f^{\beta}(t) g^{\sigma}(t) t^{\alpha-\sigma-1} dt \right)^{-1} - \frac{\alpha}{\beta}. \tag{6.2.14}$$

Then $p(z)$ is analytic in $\mathbb{D}$ and $p(0) = 1$. Upon logarithmic differentiation of (6.2.14), we deduce that $p(z)$ satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z) + \alpha} = \frac{zf'(z)}{f(z)} + \frac{\sigma}{\beta} \left(\frac{zg'(z)}{g(z)} - 1 \right). \tag{6.2.15}$$

From (6.2.11) and (6.2.15), it can be concluded that

$$p(z) + \frac{zp'(z)}{\beta p(z) + \alpha} \prec h(z).$$

By applying [61, Theorem 3.2a], we obtain $p(z) \prec h(z)$. Substituting (6.2.12) in (6.2.14), we obtain

$$p(z) = \frac{1}{\beta} \left(\frac{(\alpha+\beta) f^{\beta}(z) g^{\sigma}(z)}{F^{\beta}(z) \phi(z) z^{\sigma}} - \alpha \right).$$

Differentiating logarithmically the following equation

$$\beta p(z) + \alpha = \frac{(\alpha+\beta) f^{\beta}(z) g^{\sigma}(z)}{F^{\beta}(z) \phi(z) z^{\sigma}}$$

and using (6.2.15), we obtain

$$p(z) = \frac{zF'(z)}{F(z)} + \frac{1}{\beta} \frac{z\phi'(z)}{\phi(z)}$$

and thus (6.2.13) follows. □

We now derive another result analogous to open door lemma as follows:

Lemma 6.2.6. Let n be a positive integer and α, β be non-negative real numbers with $\beta \neq 0$. Let $Q \in \mathcal{H}[1,n]$ satisfy

$$Q(z) \prec 1 + z + \frac{nz}{\beta + \alpha(1+z)} \equiv h(z). \tag{6.2.16}$$

If $p \in \mathcal{H}[1,n]$ satisfies the differential equation

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z) + \alpha} = 1, \tag{6.2.17}$$

then $p(z) \prec 1/(1+z)$.

Proof. Let us set $q(z) = 1/(1+z)$, then

$$h(z) = \frac{1}{q(z)} - \frac{nzq'(z)}{q(z)(\beta q(z)+\alpha)}.$$

If $g(z) = z/(\beta+\alpha(1+z))$, then we have

$$\operatorname{Re}\frac{zg'(z)}{g(z)} = 1 - \operatorname{Re}\frac{\alpha z}{\beta+\alpha(1+z)} > 1 - \frac{\alpha}{\beta+2\alpha} = \frac{\beta+\alpha}{\beta+2\alpha} > 0.$$

Since g is starlike and

$$\begin{aligned}\operatorname{Re}\frac{zh'(z)}{g(z)} &= \operatorname{Re}\left((\beta+\alpha(1+z))\left(1+\frac{n(\beta+\alpha)}{(\beta+\alpha(1+z))^2}\right)\right)\\ &= \beta+\alpha(1+\operatorname{Re} z)+n(\beta+\alpha)\operatorname{Re}\left(\frac{1}{\beta+\alpha(1+z)}\right)\\ &\geq \beta+\frac{n(\beta+\alpha)}{\beta+2\alpha}\\ &> 0,\end{aligned}$$

we deduce that h is close to convex and hence univalent in $\mathbb{D}$. Now we consider the boundary curve of h defined as

$$w = h(e^{i\theta}) = u(\theta)+iv(\theta), \quad \theta \in (-\pi,\pi).$$

Suppose

$$\begin{aligned} r(\theta) &= |e^{-i\theta}h(e^{i\theta})|\\ &= \left|e^{-i\theta}+1+\frac{n}{\beta+\alpha(1+e^{i\theta})}\right|\\ &= \left|1+\cos\theta+\frac{n(\beta+\alpha(1+\cos\theta))}{d(\theta)} - i\left(\sin\theta+\frac{n\alpha\sin\theta}{d(\theta)}\right)\right|, \end{aligned} \qquad (6.2.18)$$

where $d(\theta) = (\beta+\alpha(1+\cos\theta))^2+(\alpha\sin\theta)^2$. After simplifying (6.2.18), we have

$$r(\theta) = \sqrt{2(1+\cos\theta)+\frac{n^2+2n(2\alpha+\beta)(1+\cos\theta)}{d(\theta)}}.$$

By using (6.2.16) and (6.2.17), we deduce that

$$Q(z) = \frac{1}{p(z)} - \frac{zp'(z)}{p(z)(\beta p(z)+\alpha)} \prec h(z). \tag{6.2.19}$$

On the contrary, if $p \not\prec q$ then by Lemma 1.4.3, there exist points $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D}$ and $m \geq n$, such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$. From (6.2.19), we have

$$Q(z_0) = \frac{1}{q(\zeta_0)} - \frac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)(\beta q(\zeta_0)+\alpha)} = 1 + \zeta_0 + \frac{m\zeta_0}{\beta + \alpha(1+\zeta_0)}.$$

For $\zeta_0 = e^{i\theta}$, we have

$$|Q(z_0)| = \sqrt{2(1+\cos\theta) + \frac{m^2 + 2m(2\alpha+\beta)(1+\cos\theta)}{d(\theta)}} \geq r(\theta) \quad \theta \in (-\pi, \pi),$$

where $r(\theta)$ is given in (6.2.18). This implies that $Q(z_0) \notin h(\mathbb{D})$, which is a contradiction and thus $p(z) \prec 1/(1+z)$. □

Theorem 6.2.7. Let n be a positive integer and α, β be non negative real numbers with $\beta \neq 0$. Let $f \in \mathcal{A}_n$ and $F = I_{\alpha,\beta}[f]$ be defined as

$$I_{\alpha,\beta}[f] = \left(\frac{\alpha+\beta}{f^{\alpha}(z)} \int_0^z f^{\alpha-1}(t) f'(t) t^{\beta} dt \right)^{1/\beta}. \tag{6.2.20}$$

If

$$\frac{zf'(z)}{f(z)} \prec 1 + z + \frac{nz}{\beta + \alpha(1+z)},$$

then $|zf'(z)/f(z)| < 2|zF'(z)/F(z)|$.

Proof. Let $f \in \mathcal{A}$ satisfy (6.2.20) and define

$$p(z) = z^{\beta} f^{\alpha}(z) \left(\beta \int_0^z f^{\alpha-1}(t) f'(t) t^{\beta} dt \right)^{-1} - \frac{\alpha}{\beta}.$$

By the series expansion, it is easy to verify that p is well defined and $p \in \mathcal{H}[1,n]$. If we let $Q(z) = zf'(z)/f(z)$, then it is easy to show that p satisfies (6.2.17). Hence by Lemma 6.2.6, we deduce that $p(z) \prec 1/(1+z)$. Since $p(z) \neq 0$, we can define the analytic function $F \in \mathcal{A}_n$ by

$$F(z) = z \left(\frac{\alpha+\beta}{\beta p(z) + \alpha} \right)^{1/\beta}.$$

A simple calculation shows that this function coincides with the function given in (6.2.20). So we obtain

$$\frac{F(z)Q(z)}{zF'(z)} = \frac{1}{p(z)} \prec 1+z,$$

which further implies $\left|\frac{zf'(z)/f(z)}{zF'(z)/F(z)} - 1\right| < 1$ and hence the result follows at once. □

Theorem 6.2.8. Let n be a positive integer and α, β be non-negative real numbers with $\beta \neq 0$. Let $f \in \mathcal{A}$ satisfies

$$\frac{f(z)}{zf'(z)} - \left(\frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)}{f(z)}\right)\left(\beta\frac{zf'(z)}{f(z)} + \alpha\right)^{-1} \prec 1+z+\frac{nz}{\beta+\alpha(1+z)}.$$

then $zf'(z)/f(z) \prec 1/(1+z)$.

Proof. Let $p(z) = zf'(z)/f(z)$. Then

$$Q(z) = \frac{f(z)}{zf'(z)} - \left(\frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)}{f(z)}\right)\left(\beta\frac{zf'(z)}{f(z)} + \alpha\right)^{-1}$$

satisfies the following differential equation

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z)+\alpha} = 1.$$

Now applying Lemma 6.2.6, we obtain $p(z) \prec 1/(1+z)$ and that completes the proof. □

If we take $\beta = 1$, $\alpha = 0$ and $n = 1$ in the above theorem, we obtain the following corollary, which is a particular case of a result of Tuneski [95, Theorem 2.5].

Corollary 6.2.9. Let $f \in \mathcal{A}$ satisfies

$$\left|\frac{f(z)f''(z)}{(f'(z))^2}\right| < 2,$$

then $zf'(z)/f(z) \prec 1/(1+z)$.

Theorem 6.2.10. Let n be a positive integer and α, β be non-negative real numbers with $\beta \neq 0$. Let $f \in \mathcal{A}$ satisfies

$$\frac{f(z)}{z} + \left(\frac{zf'(z)}{f(z)} - 1\right)\left(\frac{\beta z}{f(z)} + \alpha\right)^{-1} \prec 1+z+\frac{nz}{\beta+\alpha(1+z)},$$

then $f(z)/z \prec 1+z$.

Proof. Taking $p(z) = z/f(z)$ and

$$Q(z) = \frac{f(z)}{z} + \left(\frac{zf'(z)}{f(z)} - 1\right)\left(\frac{\beta z}{f(z)} + \alpha\right)^{-1},$$

the proof goes similarly as that of Theorem 6.2.8 and the result follows at once. □

By taking $\beta = 1$, $\alpha = 0$ and $n = 1$ in Theorem 6.2.10, we obtain the following result.

Corollary 6.2.11. Let $f \in \mathcal{A}$ be such that $f'(z) \prec 1 + 2z$, then $f(z)/z \prec 1 + z$, or equivalently

$$|f'(z) - 1| < 2 \Rightarrow |f(z)/z - 1| < 1.$$

We now prove the following lemma in order to derive some sufficient conditions for starlikeness:

Lemma 6.2.12. Let n be a positive integer and α, β be non negative integers with $\beta \neq 0$. Suppose that either $\alpha < \beta < 3\alpha$ or $\beta < \alpha < 3\beta$ and $Q \in \mathcal{H}[1,n]$ satisfies

$$Q(z) \prec \frac{1+z}{1-z} + \frac{2nz}{(1-z)((\alpha+\beta) + (\alpha-\beta)z)} = h(z). \tag{6.2.21}$$

If $p \in \mathcal{H}[1,n]$ satisfies the differential equation

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z) + \alpha} = 1, \tag{6.2.22}$$

then $p(z) \prec (1-z)/(1+z)$.

Proof. Let us set $q(z) = (1-z)/(1+z)$, then

$$h(z) = \frac{1}{q(z)} - \frac{nzq'(z)}{q(z)(\beta q(z) + \alpha)}.$$

We know that the Koebe function $k(z) = z/(1-z)^2$ is starlike and

$$\begin{aligned} \operatorname{Re}\frac{zh'(z)}{k(z)} &= \operatorname{Re}\left((1-z)^2\left(\frac{2}{(1-z)^2} + \frac{2n((\alpha+\beta)+(\alpha-\beta)z^2)}{(1-z)^2((\alpha+\beta)+(\alpha-\beta)z)^2}\right)\right) \\ &= \operatorname{Re}\left(2 + \frac{2n((\alpha+\beta)+(\alpha-\beta)z^2)}{((\alpha+\beta)+(\alpha-\beta)z)^2}\right) \\ &= 2 + 2n\operatorname{Re}\left(\frac{(\alpha+\beta)+(\alpha-\beta)z^2}{((\alpha+\beta)+(\alpha-\beta)z)^2}\right). \end{aligned}$$

Now taking $\alpha+\beta=a$, $\alpha-\beta=b$ and $z=e^{i\theta}$ for $\theta\in(-\pi,\pi)$, we have

$$\begin{aligned}\left|\arg\left(\frac{a+be^{2i\theta}}{(a+be^{i\theta})^2}\right)\right| &= |\arg(a+be^{2i\theta})-2\arg(a+be^{i\theta})|\\ &= \left|\arctan\left(\frac{b\sin 2\theta}{a+b\cos 2\theta}\right)-2\arctan\left(\frac{b\sin\theta}{a+b\cos\theta}\right)\right|\\ &\le \left|\arctan\left(\frac{b\sin 2\theta}{a+b\cos 2\theta}\right)\right|+2\left|\arctan\left(\frac{b\sin\theta}{a+b\cos\theta}\right)\right|.\end{aligned}$$

Since $\arctan x$ is an increasing function in $(-\pi,\pi)$, we now find the maximum of $g(x):=b\sin x/(a+b\cos x)$ for $x\in(-\pi,\pi)$. Clearly, $a>0$ and after some elementary calculations, we deduce that $g(x)$ attains its maximum at $x=\pi-\arccos(b/a)$ when $b>0$ and at $x=-\arccos(-b/a)$ when $b<0$, $b/\sqrt{a^2-b^2}$ and $-b/\sqrt{a^2-b^2}$ are the corresponding maximum values. Thus we have

$$\left|\arg\left(\frac{a+be^{2i\theta}}{(a+be^{i\theta})^2}\right)\right|\le 3\arctan\left(\frac{|b|}{\sqrt{a^2-b^2}}\right).$$

For the case when $\beta<\alpha<3\beta$, we observe that

$$3\arctan\left(\frac{|b|}{\sqrt{a^2-b^2}}\right)=3\arctan\left(\frac{\alpha-\beta}{2\sqrt{\alpha\beta}}\right)<3\arctan\left(\frac{1}{3}\sqrt{\frac{\alpha}{\beta}}\right)<3\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{2}.$$

The other case can also be verified in the similar way and it can be concluded that

$$\left|\arg\left(\frac{(\alpha+\beta)+(\alpha-\beta)z^2}{((\alpha+\beta)+(\alpha-\beta)z)^2}\right)\right|<\frac{\pi}{2},$$

which further implies that $\mathrm{Re}(zh'(z)/k(z))>0$. So h is close to convex and hence univalent in $\mathbb{D}$. Now we consider the boundary curve of h defined as

$$h(e^{i\theta})=u(\theta)+iv(\theta),\quad \theta\in(-\pi,\pi).$$

Since $e^{i\theta}$ is a boundary point, without loss of generality we may assume that $(1+e^{i\theta})/(1-e^{i\theta})=i\gamma$ and thus

$$r(\theta)\;=\;|h(e^{i\theta})|=\left|i\gamma+\frac{2ne^{i\theta}}{(1-e^{2i\theta})(\beta/i\gamma+\alpha)}\right|=\left|i\gamma-\frac{n}{\sin\theta(\beta/\gamma+i\alpha)}\right|.$$

So

$$r(\theta) = \sqrt{\gamma^2 + \frac{n^2 + 2n\alpha\gamma\sin\theta}{\sin^2\theta((\beta/\gamma)^2 + \alpha^2)}} = \sqrt{\gamma^2 + \frac{n^2 + 4n\alpha\gamma^2/(1+\gamma^2)}{\sin^2\theta((\beta/\gamma)^2 + \alpha^2)}}. \tag{6.2.23}$$

From (6.2.21) and (6.2.22), we deduce that

$$Q(z) = \frac{1}{p(z)} - \frac{zp'(z)}{p(z)(\beta p(z) + \alpha)} \prec h(z). \tag{6.2.24}$$

On the contrary if p is not subordinate to q, then by Lemma 1.4.3, there exist points $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D}$ and $m \geq n$, such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$. From (6.2.24), we have

$$Q(z_0) = \frac{1}{q(\zeta_0)} - \frac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)(\beta q(\zeta_0) + \alpha)} = \frac{1+\zeta_0}{1-\zeta_0} + \frac{2m\zeta_0}{(1-\zeta_0)((\beta+\alpha) + (\alpha-\beta)\zeta_0)}.$$

For $\zeta_0 = e^{i\theta}$, we have

$$|Q(z_0)| = \sqrt{\gamma^2 + \frac{m^2 + 4m\alpha\gamma^2/(1+\gamma^2)}{\sin^2\theta((\beta/\gamma)^2 + \alpha^2)}} \geq r(\theta) \quad \theta \in (-\pi, \pi),$$

where $r(\theta)$ is given in (6.2.23). This implies that $Q(z_0) \notin h(\mathbb{D})$, which is a contradiction and thus we have $p(z) \prec (1-z)/(1+z)$. □

Theorem 6.2.13. Let n be a positive integer and α, β be non negative real numbers with $\beta \neq 0$, either $\alpha < \beta < 3\alpha$ or $\beta < \alpha < 3\beta$. Let $f \in \mathcal{A}_n$ and $F = A_{\alpha,\beta}[f]$ is as defined in (6.2.20). If

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} + \frac{2nz}{(1-z)((\alpha+\beta) + (\alpha-\beta)z)},$$

then $\operatorname{Re}\left(\frac{zF'(z)/F(z)}{zf'(z)/f(z)}\right) > 0$.

The proof of Theorem 6.2.13 follows by an application of Lemma 6.2.12, similar to that of Theorem 6.2.7 and therefore it is omitted here.

Theorem 6.2.14. Let n be a positive integer and α, β be non-negative real numbers with $\beta \neq 0$. Suppose that either $\alpha < \beta < 3\alpha$ or $\beta < \alpha < 3\beta$ and $f \in \mathcal{A}$ satisfies

$$\Theta(f) \prec \frac{1+z}{1-z} + \frac{2nz}{(1-z)((\alpha+\beta) + (\alpha-\beta)z)},$$

where

$$\Theta(f)=\frac{f(z)}{zf'(z)}-\left(\frac{zf''(z)}{f'(z)}+1-\frac{zf'(z)}{f(z)}\right)\left(\beta\frac{zf'(z)}{f(z)}+\alpha\right)^{-1},$$

then $zf'(z)/f(z)\prec(1-z)/(1+z)$.

The proof is omitted here as it is much akin to Theorem 6.2.8 and can be easily done by using Lemma 6.2.12.

Theorem 6.2.15. Let n be a positive integer and α,β be non-negative real numbers with $\beta\neq 0$. Suppose that either $\alpha<\beta<3\alpha$ or $\beta<\alpha<3\beta$ and $f\in\mathcal{A}$ satisfies

$$\frac{f(z)}{z}+\left(\frac{zf'(z)}{f(z)}-1\right)\left(\frac{\beta z}{f(z)}+\alpha\right)^{-1}\prec\frac{1+z}{1-z}+\frac{2nz}{(1-z)((\alpha+\beta)+(\alpha-\beta)z)},$$

then $f(z)/z\prec(1+z)/(1-z)$.

Proof. Taking $p(z)=z/f(z)$ and

$$Q(z)=\frac{f(z)}{z}+\left(\frac{zf'(z)}{f(z)}-1\right)\left(\frac{\beta z}{f(z)}+\alpha\right)^{-1},$$

the result follows by an application of Lemma 6.2.12. □

By taking $\beta=1$, $\alpha=0$ and $n=1$, we obtain the following result

Corollary 6.2.16. Let $f\in\mathcal{A}$ be such that

$$f'(z)\prec\frac{1+z}{1-z}+\frac{2z}{(1-z)^2},$$

then

$$\frac{f(z)}{z}\prec\frac{1+z}{1-z}.$$

6.3 Generalised Briot-Bouquet type Differential Subordination

We present here all implication results pertaining to the proposed generalised Briot-Bouquet differential subordination. We begin with the following result:

Theorem 6.3.1. Let h be convex in $\mathbb{D}$ and $\alpha,\beta\in\mathbb{C}$ with $\beta\neq 0$. If $Q\in\mathcal{H}[1,n]$ be such that the following conditions hold:

(*i*) $\operatorname{Re}\left(\frac{1}{\beta h(z)+\alpha}\right) > 0 \quad (z \in \mathbb{D})$.

(*ii*) $\operatorname{Re}\left(\frac{1}{\beta h(\zeta)+\alpha} + (Q(z)-1)\frac{h(\zeta)}{\zeta h'(\zeta)}\right) > 0 \quad (z \in \mathbb{D},\ \zeta \in h^{-1}(p(D)))$,

where $D = \{z \in \mathbb{D} : p(z) = h(\zeta) \text{ for some } \zeta \in \partial\mathbb{D}\}$. If p is analytic in $\mathbb{D}$ with $p(0) = h(0)$ and

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z)+\alpha} \prec h(z), \tag{6.3.1}$$

then $p(z) \prec h(z)$.

Proof. Let us suppose p is not subordinate to h. Then by Lemma 1.4.3, there exists $z_0 \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D}$ and $m \geq 1$ such that $p(z_0) = h(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 h'(\zeta_0)$ and therefore we have

$$\psi_0 := \psi(p(z_0), z_0 p'(z_0)) = \psi(h(\zeta_0), m\zeta_0 h'(\zeta_0)) = h(\zeta_0)Q(z_0) + \frac{m\zeta_0 h'(\zeta_0)}{\beta h(\zeta_0)+\alpha},$$

which yields

$$\operatorname{Re}\frac{\psi_0 - h(\zeta_0)}{\zeta_0 h'(\zeta_0)} = \operatorname{Re}\left((Q(z_0)-1)\frac{h(\zeta_0)}{\zeta_0 h'(\zeta_0)} + \frac{m}{\beta h(\zeta_0)+\alpha}\right).$$

Using the fact that $m \geq 1$ together with (i) and (ii), we have

$$\operatorname{Re}\frac{\psi_0 - h(\zeta_0)}{\zeta_0 h'(\zeta_0)} \quad \geq \quad \operatorname{Re}\left((Q(z_0)-1)\frac{h(\zeta_0)}{\zeta_0 h'(\zeta_0)} + \frac{1}{\beta h(\zeta_0)+\alpha}\right) > 0,$$

which implies

$$\left|\arg\frac{\psi_0 - h(\zeta_0)}{\zeta_0 h'(\zeta_0)}\right| < \frac{\pi}{2}.$$

Since $h(\mathbb{D})$ is convex, $h(\zeta_0) \in h(\partial\mathbb{D})$ and $\zeta_0 h'(\zeta_0)$ is the outward normal to $h(\partial\mathbb{D})$ at $h(\zeta_0)$, we conclude that $\psi_0 \notin h(\mathbb{D})$, which contradicts (6.3.1) and hence $p(z) \prec h(z)$. □

Remark 14. If we take $Q(z) = 1$ in Theorem 6.3.1, it reduces to [61, Theorem 3.2a].

Corollary 6.3.2. Let h be convex in $\mathbb{D}$ and $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$. If $Q \in \mathcal{H}[1,n]$ and p is analytic in $\mathbb{D}$ with $p(0) = h(0) = (k-1)/4$, where $k \geq 1$, be such that

$$\operatorname{Re}\left(\frac{1}{\beta h(\zeta)+\alpha}\right) > k|Q(z)-1| - \operatorname{Re}(Q(z)-1) \quad (z \in \mathbb{D},\ \zeta \in \partial\mathbb{D}), \tag{6.3.2}$$

then

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z)+\alpha} \prec h(z) \quad \Rightarrow \quad p(z) \prec h(z).$$

Proof. Since $k \geq 1$, from (6.3.2) it is clear that for $\zeta \in \partial\mathbb{D}$,

$$\mathrm{Re}\left(\frac{1}{\beta h(\zeta)+\alpha}\right) > 0. \tag{6.3.3}$$

Since h is convex, the above inequality holds on $\mathbb{D}$ as well. Also, we can say that $\tilde{h}(z) := h(z) - h(0) \in \mathcal{C}$. Using Marx Strohhäcker theorem [61], we have $\mathrm{Re}(\zeta\tilde{h}'(\zeta)/\tilde{h}(\zeta)) > 1/2$, or equivalently

$$\left|\frac{\tilde{h}(\zeta)}{\zeta\tilde{h}'(\zeta)} - 1\right| \leq 1,$$

which further implies

$$\left|\frac{h(\zeta)}{\zeta h'(\zeta)} - 1\right| \leq 1 + \frac{|h(0)|}{|h'(\zeta)|}. \tag{6.3.4}$$

Since $\tilde{h} \in \mathcal{C}$, we have $|\tilde{h}'(z)| \geq 1/(1+r)^2$ on $|z| = r$ [27, Theorem 9, p 118]. We know that $\zeta \in \partial\mathbb{D}$, so we have $|h'(\zeta)| = |\tilde{h}'(\zeta)| \geq 1/4$. Thus (6.3.4) reduces to

$$\left|\frac{h(\zeta)}{\zeta h'(\zeta)} - 1\right| \leq k.$$

Note that if $X,\ Y \in \mathbb{C}$ and $|X-1| \leq K$, then

$$\mathrm{Re}(X \cdot Y) = \mathrm{Re}\, Y + \mathrm{Re}\, Y(X-1)) \geq \mathrm{Re}\, Y - |Y|K.$$

Using this inequality, we can say that

$$\mathrm{Re}\left((Q(z)-1)\frac{h(\zeta)}{\zeta h'(\zeta)} + \frac{1}{\beta h(\zeta)+\alpha}\right) \;\geq\; \mathrm{Re}(Q(z)-1) - k|Q(z)-1| + \mathrm{Re}\left(\frac{1}{\beta h(\zeta)+\alpha}\right),$$

which by using (6.3.2) implies

$$\mathrm{Re}\left((Q(z)-1)\frac{h(\zeta)}{\zeta h'(\zeta)} + \frac{1}{\beta h(\zeta)+\alpha}\right) > 0. \tag{6.3.5}$$

From (6.3.3) and (6.3.5), we may conclude that the conditions (i) and (ii) of Theorem 6.3.1 are satisfied and as its application, the result follows. □

Corollary 6.3.3. Let $Q \in \mathcal{H}[1,1]$ be a function such that $|Q(z)| \leq M$ $(z \in \mathbb{D})$ for some $M > 0$ and $\alpha,\ \beta \in \mathbb{C}$ with $\beta \neq 0$. Suppose p is analytic and h is convex in $\mathbb{D}$ with $p(0) = h(0) = 1$ such that

$$\mathrm{Re}\left(\frac{1}{\beta h(\zeta)+\alpha}\right) > 6(M+1) \quad (\zeta \in \partial\mathbb{D}), \tag{6.3.6}$$

then

$$p(z)Q(z)+\frac{zp'(z)}{\beta p(z)+\alpha}\prec h(z) \quad\Rightarrow\quad p(z)\prec h(z).$$

Proof. We know that $-\operatorname{Re}(Q(z)-1)\leq |Q(z)-1|$ and since $p(0)=1$, in view of Corollary 6.3.2, we have $k=5$. Thus

$$k|Q(z)-1|-\operatorname{Re}(Q(z)-1)\leq 6|Q(z)-1|\leq 6(|Q(z)|+1)\leq 6(M+1). \qquad (6.3.7)$$

Since (6.3.2) holds due to (6.3.6) and (6.3.7) and therefore the result follows from Corollary 6.3.2. □

Corollary 6.3.4. Let h be convex in $\mathbb{D}$ with $h(0)=1$ and $\alpha,\beta\in\mathbb{C}$ with $\beta\neq 0$. Let $g\in\mathcal{A}$ be defined as

$$g(z)=z\exp\int_0^z \frac{Q(t)-1}{t}dt, \qquad (6.3.8)$$

such that h and Q satisfy the conditions (i) and (ii) of Theorem 6.3.1. If $f\in\mathcal{A}$ and F is given by

$$F(z)=I[f,g]=\left(\frac{\alpha+\beta}{g^{\alpha}(z)}\int_0^z g'(t)g^{\alpha-1}(t)f^{\beta}(t)dt\right)^{1/\beta}, \qquad (6.3.9)$$

then

$$\frac{zf'(z)}{f(z)}\prec h(z) \quad\Rightarrow\quad \frac{zF'(z)/F(z)}{zg'(z)/g(z)}\prec h(z).$$

Proof. From (6.3.8), we have $Q(z)=zg'(z)/g(z)$ and let us suppose $p(z)=zF'(z)/(Q(z)F(z))$. Then by differentiating (6.3.9) and appropriately replacing the expressions, we have

$$p(z)Q(z)+\frac{zp'(z)}{\beta p(z)+\alpha}=\frac{zf'(z)}{f(z)}.$$

Since $zf'(z)/f(z)\prec h(z)$, the result now follows from Theorem 6.3.1. □

If we take $h(z)=((1+z)/(1-z))^{\gamma}$ with $\gamma\in(0,1]$ in Corollary 6.3.4, we obtain the following result.

Corollary 6.3.5. Let $f\in\mathcal{A}$ and g,Q and F are as defined in (6.3.8) and (6.3.9) respectively such that $\operatorname{Re}(Q(z)-1)>1-\gamma$. Then

$$f\in\mathcal{SS}^*(\gamma) \;\Rightarrow\; F \text{ is almost strongly starlike of order } \gamma \text{ w.r.t the function } g.$$

If we take $h(z)=(1+z)/(1-z)$ in Corollary 6.3.4, we obtain the following result.

Corollary 6.3.6. Let $f \in \mathcal{A}$ and g, Q and F are as defined in (6.3.8) and (6.3.9) respectively such that $f \in \mathcal{S}^*$ and g is univalent on $\overline{\mathbb{D}}$, then F is a g-starlike function.

Now we enlist some of the special cases of Theorem 6.3.1 here below:

Corollary 6.3.7. Let $Q \in \mathcal{H}[1,1]$ and α, β be non-negative real numbers with $\beta \neq 0$ such that $|Q(z) - 1| < 1/(\beta e + \alpha)$ on $\mathbb{D}$. Suppose p is analytic in $\mathbb{D}$ with $p(0) = 1$, then

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z) + \alpha} \prec e^z \quad \Rightarrow \quad p(z) \prec e^z.$$

Proof. If we take $h(z) = e^z$, then for $\alpha \geq 0$ and $\beta > 0$, we have

$$\operatorname{Re}\left(\frac{1}{\beta e^z + \alpha}\right) \geq \frac{1}{\beta e + \alpha} > 0. \tag{6.3.10}$$

Further, we observe for $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$,

$$\operatorname{Re}\left(\frac{Q(z) - 1}{\zeta} + \frac{1}{\beta e^z + \alpha}\right) \geq -|Q(z) - 1| + \frac{1}{\beta e + \alpha} > 0. \tag{6.3.11}$$

From (6.3.10) and (6.3.11), we may conclude that both the conditions of Theorem 6.3.1 are satisfied and thus the result now follows from Theorem 6.3.1. □

Corollary 6.3.8. Let $Q \in \mathcal{H}[1,1]$ and $\alpha \geq 0$, $\beta > 0$ be such that

$$|Q(z) - 1| < \operatorname{Re}(Q(z) - 1) + \frac{1}{2(\sqrt{2}\beta + \alpha)}. \tag{6.3.12}$$

Suppose p is analytic in $\mathbb{D}$ with $p(0) = 1$, then

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z) + \alpha} \prec \sqrt{1+z} \quad \Rightarrow \quad p(z) \prec \sqrt{1+z}.$$

Proof. Let $h(z) = \sqrt{1+z}$, then condition (i) of Theorem 6.3.1 is satisfied clearly as for $\alpha \geq 0$ and $\beta > 0$,

$$\operatorname{Re}\left(\frac{1}{\beta\sqrt{1+z} + \alpha}\right) \geq \frac{1}{\beta\sqrt{2} + \alpha} > 0, \quad z \in \mathbb{D}.$$

Now for $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$, we have from (6.3.12)

$$\begin{aligned} \operatorname{Re}\left(2(Q(z) - 1)\left(1 + \frac{1}{\zeta}\right) + \frac{1}{\beta\sqrt{1+z} + \alpha}\right) &\geq 2\operatorname{Re}(Q(z) - 1) - 2|Q(z) - 1| + \frac{1}{\sqrt{2}\beta + \alpha} \\ &> 0, \end{aligned}$$

which implies that condition (ii) of Theorem 6.3.1 is satisfied. Thus the result follows at once from Theorem 6.3.1. ◻

Corollary 6.3.9. Let $Q \in \mathcal{H}[1,1]$ be such that Q has real coefficients with $Q'(0) > 0$ and α, β be non-negative real numbers with $\beta \neq 0$. Suppose p has real coefficients and is analytic with $p(0) = 1$ and $p'(0) > 0$, then for $0 < \gamma \leq 1$,

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z) + \alpha} \prec \left(\frac{1+z}{1-z}\right)^{\gamma} \quad \Rightarrow \quad p(z) \prec \left(\frac{1+z}{1-z}\right)^{\gamma}.$$

Proof. Let $h(z) = ((1+z)/(1-z))^{\gamma}$. Then condition (i) of Theorem 6.3.1 clearly holds. For condition (ii) to hold, we need to show that

$$\operatorname{Re}\left((Q(z)-1)\frac{1-\zeta^2}{2\gamma\zeta} + \frac{1}{\beta\left(\frac{1+\zeta}{1-\zeta}\right)^{\gamma} + \alpha}\right) > 0 \quad (z \in \mathbb{D},\ \zeta \in \partial\mathbb{D}).$$

Let $Q(z) = 1 + a_1 z + a_2 z^2 + \cdots$ and define $R(z) = Q(z) - 1$, then $R(0) = 0$. Since Q has real coefficients and $Q'(0) > 0$, it is typically real and it is easy to conclude that $R(z)$ is typically real. We know that $\zeta \in h^{-1}(p(D))$, where $D := \{z \in \mathbb{D} : p(z) = h(\zeta) \text{ for some } \zeta \in \mathbb{D}\}$. Clearly $h(z) = ((1+z)/(1-z))^{\gamma}$ is typically real and conditions on p also ensures that it is typically real, so we have $sign(\operatorname{Im} z) = sign(\operatorname{Im} \zeta)$. Now we consider

$$\begin{aligned} \operatorname{Re}\left((Q(z)-1)\tfrac{1-\zeta^2}{2\gamma\zeta}\right) &= \frac{1}{2\gamma}\left(\operatorname{Re}(Q(z)-1)\operatorname{Re}\left(\tfrac{1-\zeta^2}{\zeta}\right) - \operatorname{Im}(Q(z)-1)\operatorname{Im}\left(\tfrac{1-\zeta^2}{\zeta}\right)\right) \\ &= -\frac{1}{2\gamma}\operatorname{Im}(Q(z)-1)\operatorname{Im}\left(\frac{1-\zeta^2}{\zeta}\right). \end{aligned}$$

Taking $\zeta = e^{i\theta}$ $(0 \leq \theta < 2\pi)$, we have

$$-\operatorname{Im}\left(\frac{1-\zeta^2}{\zeta}\right) = 2\sin\theta \begin{cases} > 0, & \theta \in (0,\pi), \\ < 0, & \theta \in (\pi, 2\pi). \end{cases}$$

Since $Q(z) - 1$ is typically real, $sign(\operatorname{Im}(Q(z)-1)) = sign(\operatorname{Im} z)$. Thus we have

$$\operatorname{Re}\left((Q(z)-1)\frac{1-\zeta^2}{2\gamma\zeta}\right) = 2\sin\theta \operatorname{Im}(Q(z)-1) \geq 0.$$

Also for $\alpha \geq 0$ and $\beta > 0$, we have $\operatorname{Re}\left(\beta\left(\frac{1+z}{1-z}\right)^{\gamma} + \alpha\right) > 0$ and therefore

$$\operatorname{Re}\left((Q(z)-1)\frac{1-\zeta^2}{2\gamma\zeta} + \frac{1}{\beta\left(\frac{1+\zeta}{1-\zeta}\right)^{\gamma} + \alpha}\right) > 0.$$

Therefore the result follows at once from Theorem 6.3.1. □

Now we use a different technique to prove the next two results, which demonstrates the similar implication for Janowski functions.

Theorem 6.3.10. Let $p(z)$ be analytic in $\mathbb{D}$ with $p(0) = 1$ and $Q \in \mathcal{H}[1, n]$ be a function such that $|Q(z)| < M$ for some $M > 0$. Let $-1 \le B < A < 1$ and $-1 < E < D \le 1$ satisfy

$$(A-B)(1-A)(1+E) > (1+|A|)(\beta+\alpha+|\beta A+\alpha B|)((1+D)(1-B)+M(1+E)(1-A)) \quad (6.3.13)$$

for some α and β, where $\alpha+\beta > 0$. If

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z)+\alpha} \prec \frac{1+Dz}{1+Ez},$$

then $p(z) \prec (1+Az)/(1+Bz)$.

Proof. Let us define $P(z)$ and $\omega(z)$ as

$$P(z) := p(z)Q(z) + \frac{zp'(z)}{\beta p(z)+\alpha}, \qquad \omega(z) := \frac{p(z)-1}{A-Bp(z)}.$$

Then $\omega(z)$ is meromorphic in $\mathbb{D}$ and $\omega(0) = 0$. By the definition of $P(z)$ and $\omega(z)$, we have

$$P(z) = \frac{1+A\omega(z)}{1+B\omega(z)}Q(z) + \frac{(A-B)z\omega'(z)}{(1+B\omega(z))[\beta(1+A\omega(z))+\alpha(1+B\omega(z))]}.$$

Now we need to show that $|\omega(z)| < 1$ in $\mathbb{D}$. On the contrary, let us assume that there exists a point $z_0 \in \mathbb{D}$ such that

$$\max_{|z|\le|z_0|} |\omega(z)| = |\omega(z_0)| = 1.$$

Then by [82, Lemma 1.3], there exists $k \ge 1$ such that $z_0\omega'(z_0) = k\omega(z_0)$. Now by taking $\omega(z_0) = e^{i\theta}$ $(0 \le \theta < 2\pi)$, we have

$$\begin{aligned} |P(z_0)| &= \left|\frac{1+A\omega(z_0)}{1+B\omega(z_0)}Q(z_0) + \frac{(A-B)k\omega(z_0)}{(1+B\omega(z_0))(\beta(1+A\omega(z_0))+\alpha(1+B\omega(z_0)))}\right| \\ &\ge \left|\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\right|\left(\left|\frac{(A-B)k}{(1+Ae^{i\theta})(\beta(1+Ae^{i\theta})+\alpha(1+Be^{i\theta}))}\right| - |Q(z_0)|\right). \end{aligned}$$

We know that for $p > 0$, $|p+qe^{i\theta}|^2 = p^2+q^2+2pq\cos\theta$ attains its maximum at $\theta = 0$ if

$q > 0$ and at $\theta = \pi$ if $q \leq 0$. So, $\max_{0\leq\theta<2\pi} |p + qe^{i\theta}|^2 = (p + |q|)^2$. Thus

$$|P(z_0)| \geq \frac{1-A}{1-B}\left(\frac{(A-B)k}{(1+|A|)(\beta+\alpha+|\beta A+\alpha B|)} - M\right).$$

Clearly the expression on the right hand side is an increasing function of k and attains its minimum at $k = 1$. So

$$|P(z_0)| \geq \frac{1-A}{1-B}\left(\frac{(A-B)}{(1+|A|)(\beta+\alpha+|\beta A+\alpha B|)} - M\right).$$

Now from (6.3.13), we have

$$|P(z_0)| > \frac{1+D}{1+E},$$

which contradicts the fact that $P(z) \prec (1 + Dz)/(1 + Ez)$ and hence completes the proof.

□

Using Theorem 6.3.10, we obtain the following corollary.

Corollary 6.3.11. Let $p(z)$ be analytic in $\mathbb{D}$ with $p(0) = 1$ and Q be a function such that

$$z \exp \int_0^z \frac{Q(t)-1}{t} dt \in \mathcal{S}^*(\varphi).$$

Let $-1 \leq B < A < 1$, $-1 < E < D \leq 1$ and α, β be such that $\alpha + \beta > 0$. Then

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z)+\alpha} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz},$$

whenever any of the following cases hold:

(*i*) $\varphi(z) = e^z$ and $(A-B)(1-A)(1+E) > (1+|A|)(\beta+\alpha+|\beta A+\alpha B|)((1+D)(1-B)+e(1+E)(1-A))$

(*ii*) $\varphi(z) = \sqrt{1+z}$ and $(A-B)(1-A)(1+E) > (1+|A|)(\beta+\alpha+|\beta A+\alpha B|)((1+D)(1-B)+\sqrt{2}(1+E)(1-A))$

(*iii*) $\varphi(z) = 2/(1+e^{-z})$ and $(A-B)(1-A)(1+E)(1+e) > (1+|A|)(\beta+\alpha+|\beta A+\alpha B|)((1+D)(1-B)(1+e)+2e(1+E)(1-A))$

(*iv*) $\varphi(z) = 1+ze^z$ and $(A-B)(1-A)(1+E) > (1+|A|)(\beta+\alpha+|\beta A+\alpha B|)((1+D)(1-B)+(1+e)(1+E)(1-A))$

(*v*) $\varphi(z) = z+\sqrt{1+z^2}$ and $(A-B)(1-A)(1+E) > (1+|A|)(\beta+\alpha+|\beta A+\alpha B|)((1+D)(1-B)+(1+\sqrt{2})(1+E)(1-A))$.

Since the proof of Theorem 6.3.10 loses its validity when $A = 1$ or $E = -1$, the theorem does not reduce to the case when $P(z)$ and $p(z)$, both are subordinate to $(1+az)/(1-z)$ for $0 \le a \le 1$. This case, we handle in the following result with a different approach.

Theorem 6.3.12. Let $\alpha \ge 0$, $\beta > 0$, $0 \le a \le 1$. Assume $p(z)$ has real coefficients and is analytic in $\mathbb{D}$ with $p(0) = 1$ and $p'(0) > 0$. Also, let $Q(z) \in \mathcal{H}[1,1]$ has real coefficients such that $Q'(0) > 0$ and $\operatorname{Re} Q(z) < 1$. If p satisfies

$$p(z)Q(z) + \frac{zp'(z)}{\beta p(z)+\alpha} \prec \frac{1+az}{1-z}, \tag{6.3.14}$$

then $p(z) \prec (1+az)/(1-z)$.

Proof. We need to show that $p(z) \prec q(z) = (1+az)/(1-z)$. For if $p \not\prec q$, then using Lemma 1.4.3, there exists a point $z_0 = r_0 e^{i\theta_0} \in \mathbb{D}$, $\zeta_0 \in \partial\mathbb{D}\backslash\{1\}$ and $m \ge 1$ such that $p(z_0) = q(\zeta_0)$, $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ and $p(\mathbb{D}_{r_0}) \subset q(\mathbb{D})$. Since ζ_0 is a boundary point, we may assume that $\zeta_0 = e^{i\theta}$ for $\theta \in [0, 2\pi)$. Then

$$\begin{aligned} p(z_0)Q(z_0) + \frac{z_0 p'(z_0)}{\beta p(z_0)+\alpha} &= q(\zeta_0)Q(z_0) + \frac{m\zeta_0 q'(\zeta_0)}{\beta q(\zeta_0)+\alpha} \\ &= \left(\frac{1-a}{2} + i\frac{1+a}{2}\cot\theta/2\right)Q(z_0) - \frac{m(a+1)e^{i\theta}}{(1-e^{i\theta})((\beta+\alpha)+(\beta a-\alpha)e^{i\theta})}. \end{aligned}$$

Taking $Q(z_0) = u + iv$, we have

$$\begin{aligned} \operatorname{Re}\left(p(z_0)Q(z_0) + \frac{z_0 p'(z_0)}{\beta p(z_0)+\alpha}\right) &= \frac{(1-a)u}{2} - \frac{(1+a)\cot(\theta/2)v}{2} \\ &\quad - \frac{m(1+a)(\beta(1-a)+2\alpha)(1-\cos\theta)}{(\beta(1-a)+2\alpha)^2(1-\cos\theta)^2+(\beta(a+1)\sin\theta)^2}. \end{aligned}$$

Since Q has real coefficients and $Q'(0) > 0$, it is clear that $Q(z)$ is typically real and thus

$$sign(v) = sign(\operatorname{Im}(Q(z_0))) = sign(\operatorname{Im}(z_0)). \tag{6.3.15}$$

We also have $p'(0) > 0$ with p having real coefficients and $q(z) = (1+az)/(1-z)$ is typically real, which implies

$$sign(\operatorname{Im}(z_0)) = sign((\operatorname{Im}(p(z_0)))) = sign((\operatorname{Im}(q(\zeta_0)))) = sign(\operatorname{Im}(\zeta_0)) = sign(\sin\theta). \tag{6.3.16}$$

From (6.3.15) and (6.3.16), we obtain $sign(v) = sign(\sin\theta)$, which is sufficient to conclude that $v\cot\theta/2 > 0$ for $\theta \in [0, 2\pi)$. Also we know that $\alpha \ge 0$, $\beta > 0$, $m \ge 1$ and $0 \le a \le 1$.

Therefore

$$\operatorname{Re}\left(p(z_0)Q(z_0)+\frac{z_0p'(z_0)}{\beta p(z_0)+\alpha}\right)<\frac{(1-a)u}{2}<\frac{1-a}{2},$$

which contradicts (6.3.14) and hence the result. □

Concluding Remarks

The work carried out in this chapter is not an extension of any earlier work but deals with new ideas and therefore there will be no parallel comparisons. As a matter of future scope, these ideas of transforming differential subordinations into integral operators and vice versa can be extended for other similar subordination cases too. Note that the lemmas obtained in this chapter are not confined to finding some starlikeness or univalence criteria alone, but also paves way in establishing some exceptional implication results.

Future Scope

- The class $\mathcal{S}^*_{SG}$, a subclass of starlike functions has been introduced and studied extensively in context of coefficient, radius and subordination problems. Inspite of that, there still remains a great deal of content to be explored such as coefficient estimation of higher order, estimation of Fekete-Szegö functional, Hankel determinants of order 2, 3 and 4 and many other determinants. One could perhaps attempt to solve other problems like majorization, convolution and inclusion relations involving the classes that weren't taken into account. Moreover, similar subclasses of Ma-Minda functions can be handled in the same manner as we have done for the class $\mathcal{S}^*_{SG}$.

- We have a number of well known subclasses of starlike functions associated with various Ma- Minda functions for which differential subordination results have been attempted up to second order in the past and can be extended up to third order, by using the admissibility criteria, established by us in Chapter 2. Further, variations in the expressions on the left of the differential subordination can be incorporated to establish certain differential subordination implication results. Another way to extend this work is to take complex coefficients instead of real coefficients.

- It has been observed that Schwarz function has played a substantial role in a variety of problems such as radius problems, coefficient problems, majorization problem etc. and despite having a large number of properties, only a few of the properties are really put to use in deriving the results so far. In Chapter 4, some results are proved by using some of the rarely used properties of Schwarz functions which gives a model to use the same techniques for other classes. Depending on the differential subordinations, sufficient conditions are obtained for Silverman class. By taking other forms of differential subordination, more sufficient conditions can be established.

- The convex weighted harmonic mean of the quantities $p(z)$ and $p(z) + zp'(z)/p(z)$ has been generalised by taking variable coefficients, which can also be done for the other forms of mean such as arithmetic or geometric mean. Further, the results are applied to some specified functions. The work can be extended by incorporating parameters and determining conditions for different functions to prevail. A combination of the three means has also been considered and an equivalent differential subordination result has been proved for the Carathéodory function, which can be replaced by other well known functions. Moreover, it is also possible to carry out parallel analysis for different forms of mean by substituting the quantities $p(z)$ and $p(z)+zp'(z)/p(z)$ by some other quantities involving $p(z)$ and its derivatives.

- Briot Bouquet differential subordination is one of the most studied forms of differential subordinations because of its importance in terms of applications. It is generalised in one way and can be more generalised in many different ways. The analytical approach of this work has led to the creation of an analogy of the Open Door Lemma and Integral Existence Theorem, which can serve as an example for exploring other integral operators that may be employed to do the similar work. Additionally, it is possible to extend these ideas to other subordination cases that involve transforming differential subordinations into integral operators.

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List of Publications

1. **Priyanka Goel** and S. Sivaprasad Kumar, *Certain class of starlike functions associated with modified sigmoid function*, Bulletin of Malaysian Mathematical Sciences Society **43** (2020), no. 1, 957–991. (**SCIE, Impact Factor: 1.397**)

2. S. Sivaprasad Kumar and **Priyanka Goel**, *Starlike functions and higher order differential subordinations*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas **114**, 192 (2020). (**SCIE, Impact Factor: 2.276**)

3. S. Sivaprasad Kumar and **Priyanka Goel**, *Application of Pythagorean means and Differential Subordination*, Bull. Belg. Math. Soc. Simon Stevin **29** (2022), no. 4, doi:10.36045/j.bbms.210605.(**SCIE, Impact Factor: 0.647**)

4. S. Sivaprasad Kumar and **Priyanka Goel**, *On a Generalized Briot-Bouquet type Differential Subordination*, Accepted in the Rocky Mountain Journal of Mathematics (**SCIE, Impact Factor: 0.568**)

5. S. Sivaprasad Kumar and **Priyanka Goel**, *Sufficient conditions and radius problems for the Silverman class*, Accepted in the Ukrainian Mathematical Journal **(SCIE, Impact Factor: 0.464)**

6. S. Sivaprasad Kumar and **Priyanka Goel**, *Radius constants for Sigmoid starlike functions*, arXiv preprint arXiv:2208.01241 **(Under Review)**

7. **Priyanka Goel** and S. Sivaprasad Kumar, *On Sharp Bounds Of Certain Close-To-Convex Functions*, arXiv:1907.13385 **(Communicated)**
