A Study on Estimates of Convergence of Certain Approximation Operators

A Thesis
Submitted for the award of degree of
Doctor of Philosophy
in Mathematics

by

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To my beloved parents,

who taught me to be strong and fearless,

and my husband,

who encouraged me to always follow my heart.

Certificate

Department of Applied Mathematics
Delhi Technological University, Delhi

This is to certify that the research work embodied in the thesis entitled "A Study on Estimates of Convergence of Certain Approximation Operators" submitted by Nav Shakti Mishra (2K18/PHD/AM/10) is the result of her original research carried out in the Department of Applied Mathematics, Delhi Technological University, Delhi, for the award of **Doctor of Philosophy** under the supervision of **Prof. Naokant Deo**.

It is further certified that this work is original and has not been submitted in part or fully to any other university or institute for the award of any degree or diploma.

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Date: 06 March 2023
Place: Delhi , India

Prof. Naokant Deo
(Supervisor)

Prof. S. Siva Prasad Kumar
(Head of Department)

Declaration

I declare that the research work in this thesis entitled "**A Study on Estimates of Convergence of Certain Approximation Operators**" for the award of the degree of ***Doctor of Philosophy in Mathematics*** has been carried out by me under the supervision of ***Prof. Naokant Deo***, Department of Applied Mathematics, Delhi Technological University, Delhi, India, and has not been submitted by me earlier in part or full to any other university or institute for the award of any degree or diploma.

I declare that this thesis represents my ideas in my own words and where others' ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission.

Nav Shakti Mishra
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Date: 06 March 2023

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Abstract

This thesis is mainly a study of convergence estimates of various approximation operators. Approximation theory is indeed an old topic in mathematical analysis that remains an appealing field of study with several applications. The findings presented here are related to the approximation of specific classes of linear positive operators. The introductory chapter is a collection of relevant definitions and literature of concepts that are used throughout this thesis.

The second chapter is based on approximation of certain exponential type operators. The first section of this chapter presents the study of convergence estimates of Kantorovich variant of Ismail-May operators. Further, a two variable generalisation of the proposed operators is also discussed. The second section is dedicated to a modification of Ismail-May exponential type operators which preserve functions of exponential growth. The modified operators in general are not of exponential type.

In chapter three, we present a Durrmeyer type construction involving a class of orthogonal polynomials called Apostol-Genocchi polynomials and Păltănea operators with real parameters α, λ and ρ. We establish approximation estimates such as a global approximation theorem and rate of approximation in terms of usual, $r-$th and weighted modulus of continuity. We further study asymptotic formulae such as Voronovskaya theorem and quantitative Voronovskaya theorem. The rate of convergence of the proposed operators for the functions whose derivatives are of bounded variation is also presented.

Inspired by the King's approach, chapter four deals with the preservation of functions of the form t^s, $s \in \mathbb{N} \cup \{0\}$. Followed by some useful lemmas, we determine the rate of convergence of the proposed operators in terms of usual modulus of continuity and Peetre's K- functional. Further, the degree of approximation is also established for the function of bounded variation. We also illustrate via figures and tables that the proposed modification provides better approximation for preservation of test function e_3.

In chapter five, we consider a Kantorovich variant of the operators proposed by Gupta and Holhos (68) using arbitrary sequences which preserves the exponential functions of the form a^{-x}. It is shown that the order of approximation can be made better with appropriate choice of sequences with certain conditions. We therefore provide necessary moments and central moments and some useful lemmas. Further, we present a quantitative asymptotic formula and estimate the error in approximation. Graphical representations are provided in the end with different choices of sequences satisfying the given conditions.

The last chapter summarizes the thesis with a brief conclusion and also discusses the future prospects of this thesis.

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$\mathbb{N}$	the set of natural numbers
$\mathbb{N} \cup \{0\}$	the set of natural number including zero
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$[a, b]$	a closed interval
(a, b)	an open interval
Λ	index set
e_n	denotes the n-th monomials with $e_n : [a, b] \to \mathbb{R}$, $e_n(x) = x^n$, $n \in \mathbb{N}_0$
$(x)_n$	the rising factorial $(x)_n := x\,(x+1)\,(x+2) \ldots (x+n-1)\,,\ (x)_0 = 1$
$\Omega(f, \delta)$	the weighted modulus of continuity
$C[a, b]$	the set of all real-valued continuous function defined on compact interval $[a, b]$
$C^r[a, b]$	the set of all real-valued, r-times continuously differentiable function ($r \in \mathbb{N}$)
$C[0, \infty)$	the set of all continuous functions defined on $[0, \infty)$
$C_B[0, \infty)$	the set of all continuous and bounded functions on $[0, \infty)$
$C_B^r[0, \infty)$	the set of all r-times continuously differentiable functions in $C_B[0, \infty)$ ($r \in \mathbb{N}$)
$B_\rho[0, \infty)$	the set of all functions f defined on $[0, \infty)$ satisfying the condition : $\lvert f(x)\rvert \leq M\rho(x)$, M is a positive constant, and ρ is weight function.
$C_\rho[0, \infty)$	the subspace of all continuous function in $B_\rho[0, \infty)$

Chapter 1

Introduction

"Although this may seem a paradox, all exact science is dominated by the idea of approximation. When a man tells you that he knows the exact truth about anything, you are safe in inferring that he is an inexact man. Every careful measurement in science is always given with the probable error ... every observer admits that he is likely wrong, and knows about how much wrong he is likely to be."

-Bertrand Russell

These words, penned down by renowned philosopher Bertrand Russell are a great thought on how none of our views is entirely accurate; they all have some element of ambiguity and inaccuracy. We can only make an effort to measure or approximate this error and reduce it as much as we can. The concept of approximation dominates every area of research. Humankind has always sought to complete tasks as accurately as possible while minimising errors brought on by procedural, environmental, instrumental, or human factors.

In mathematics, the main focus of theory of approximation is on identifying the best ways to approximate functions with simpler ones and quantifying the errors that are introduced thereby. The foundation of approximation theory was laid on a result first given by Karl Weierstrass (139) in 1885, which states that for every continuous function f on a closed interval $[a, b]$ and any $\epsilon > 0$, there exists a polynomial p of degree n on $[a, b]$ such that

$$|f(x) - p(x)| < \epsilon, \ \forall x \in [a, b]$$

In other words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy.

1.1 Preliminaries

In this section, we recall some definitions and properties regarding approximation operators discussed here that will be of interest to the whole thesis.

1.1.1 Linear positive operators

Definition 1.1.1 *Let X, Y be two linear spaces of real functions. Then the mapping $L : X \to Y$ is a linear operator if:*

$$L(\alpha f + \beta g; x) = \alpha L(f; x) + \beta L(g; x),$$

for all $f, g \in X$ and $\alpha, \beta \in \mathbb{R}$. If for all $f \in X$ and $f \geq 0$, it follows that $L(f; x) \geq 0$, then L is called a positive operator.

Next, we define the modulus of continuity, mainly used to measure quantitatively the uniform continuity of functions.

1.1.2 Usual and higher order modulus of continuity

Definition 1.1.2 *Let $f \in C[a, b]$ and $\delta \geq 0$, then*

$$\omega(f, \delta) = \sup\{|f(x + h) - f(x)| : x, x + h \in [a, b], 0 \leqslant h \leqslant \delta\}.$$

Here ω is known as the usual modulus of continuity or simply first order modulus of continuity which was introduced by H. Lebesgue in 1910.

Some of the error estimates in this thesis are given in terms of the modulus of continuity of higher order. Therefore we now give the definition of ω_r, $r \in \mathbb{N}$, as given in 1981 by L.L. Schumaker (127).

Definition 1.1.3 *Let $f \in C[a, b]$, then for $r \in \mathbb{N}$ and $\delta \geq 0$, the modulus of continuity of order r is defined as:*

$$\omega_r(f, \delta) = \sup\left\{\left|\Delta_h^r f(x)\right| : x, x + rh \in [a, b], 0 \leqslant h \leqslant \delta\right\},$$

where

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^{r-i} \begin{pmatrix} r \\ i \end{pmatrix} f(x+ih)$$

denotes the forward difference with step size h. In particular, for $r = 1$, $\omega(f,\delta)$ is the usual modulus of continuity.

Proposition 1.1.4 *The modulus of continuity of order r verifies the following properties:*

1. $\omega_r(f,.)$ *is a positive, non decreasing and continuous function on* $(0,\infty)$.
2. $\omega_r(f,0) = 0$
3. $\omega_r(f,.)$ *has sub-additive property.*
4. $\omega_{r+1}(f,.) \leq 2\omega_r(f,.)$ *for all* $x \geq 0$.
5. $\omega_r(f,nx) \leq n^r \omega_r(f,x)$ *for all* $n \in \mathbb{N}$ *and* $x > 0$.
6. $\omega_r(f,kx) \leq (1+[k])^r \omega_r(f,x)$ *for all* $k > 0$ *and* $x > 0$, *where* $[\alpha]$ *denotes the integer part of* α.

For $r = 1$, *these properties are valid for the usual modulus of continuity* $\omega(f,.)$.

1.1.3 Ditzian-Totik modulus of smoothness

We recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional (49). Let $\varphi(x) = \sqrt{x(1-x)}$ and $f \in C[0,1]$, then the first order modulus of smoothness is defined as:

$$\omega_\varphi(f,\delta) = \sup_{0\leq h\leq\delta} \left\{ \left| f\left(x + \frac{h\varphi(x)}{2}\right) \right| - \left| f\left(x - \frac{h\varphi(x)}{2}\right) \right|, x \pm \frac{h\varphi(x)}{2} \in [0,1] \right\}. \quad (1.1)$$

Further, the corresponding Peetre's K-functional is given by

$$K_\varphi(f,\delta) = \inf_{g\in W_\varphi} \{\|f-g\| + \delta\|\varphi g'\|\}, \quad \delta > 0 \quad (1.2)$$

where

$$W_\varphi = \{g : \|\varphi g'\| < \infty, \ g \in AC[0,1]\},$$

$AC[0,1]$ denotes the space of all absolutely continuous functions on every interval $[a,b] \subset (0,1)$ and $\|.\|$ is the uniform norm in $C[0,1]$. Moreover, from [(49), p. 11], there exists a constant $C > 0$ such that:

$$K_\varphi(f,\delta) \leq C\omega_\varphi(f,\delta). \quad (1.3)$$

1.1.4 Weighted spaces and corresponding modulus of continuity

Let $B_\rho(I)$ be the space of all functions f defined on the interval $I \in \mathbb{R}$ for which there exist a constant $M > 0$ such that $|f(x)| \leq M\rho(x)$, for every $x \in I$, where ρ is a positive continuous function called weight. In 1974, A.D. Gadjiev (57; 59) introduced the weighted space $C_\rho(I)$, which is the set of all continuous functions f on the interval $I \in \mathbb{R}$ and $f \in B_\rho(I)$. This space is a Banach space, endowed with the norm

$$\|f\|_\rho = \sup_{x\in I} \frac{|f(x)|}{\rho(x)}.$$

For $I = [0, \infty)$, the subspace $C^*_\rho[0, \infty)$ is defined as follows:

$$C^*_\rho[0, \infty) := \{f \in C_\rho[0, \infty) : \lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} = k < +\infty\}.$$

Many authors (8; 75) use the following weighted modulus of continuity $\Omega(f, \delta)$ for $f \in C_B(0, \infty)$:

$$\Omega(f, \delta) = \sup_{x\in[0,\infty),|h|<\delta} \frac{|f(x + h) - f(x)|}{(1 + h^2)(1 + x^2)}.$$

Let us denote by $C^*[0, \infty)$, the Banach space of all real valued continuous functions on $[0, \infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and is finite endowed with the uniform norm. In (31), the following theorem is proved:

Theorem 1.1.5 *If the sequence $A_n : C^*[0, \infty) \to C^*[0, \infty)$ of positive linear operators satisfies the conditions*

$$\lim_{n\to\infty} A_n\left(e^{-kt}; x\right) = e^{-kx},\ k = 0, 1, 2$$

uniformly in $[0, \infty)$, then

$$\lim_{n\to\infty} A_n(f; x) = f(x),$$

uniformly in $[0, \infty)$, for every $f \in C^*[0, \infty)$.

1.1.5 Modulus of continuity for exponential functions

To find rate of convergence of operators satisfying the conditions from the above theorem , we use the following modulus of continuity:

$$\omega^*(f, \delta) = \sup\left\{|f(x) - f(t)| :\ x, t \geqslant 0, \left|e^{-x} - e^{-t}\right| \leqslant \delta\right\}$$

defined for every $\delta \geq 0$ and every function $f \in C^*[0, \infty)$.

Proposition 1.1.6 *The modulus of continuity defined for exponential functions has the following properties:*

1. *$\omega^*(f,\delta)$ can be expressed in terms of usual modulus of continuity, by the relation*

$$\omega^*(f,\delta) = \omega(f^*,\delta)$$

where f^* is the continuous function on $[0,\infty)$ given by:

$$f^*(x) = \begin{cases} f(-\ln(x)), & x \in (0,\infty] \\ \lim\limits_{t\to\infty} f(t), & x = 0 \end{cases}$$

2. *For every $t, x \in [0,1]$ and $M > 0$, we have*

$$\omega^*(f,\delta) \leq \left(1 + e^M\right) \cdot \omega(f,\delta)$$

3. *The defined modulus of continuity ω^* possess the following property:*

$$|f(t) - f(x)| \leq \left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}\right) \omega(f^*,\delta).$$

1.1.6 Lipschitz Spaces

Definition 1.1.7 *For non-negative real numbers a and b, the Lipschitz space (120) is defined as:*

$$Lip_M(\beta) = \left\{ f \in C[0,1] : |f(t) - f(x)| \leq M \frac{|t-x|^\beta}{(ax^2 + bx + t)^{\beta/2}};\ x,t \in (0,1) \right\},$$

where $\beta \in (0,1]$ and M is a positive constant.

To discuss some direct estimates of operators in this thesis, we used Lipschitz-type maximal function of order β defined by Lenze (94) as follows:

$$\varpi_\beta(f;x) = \sup_{t\neq x, t\in[0,1]} \frac{|f(t) - f(x)|}{|t-x|^\beta}, \quad x \in [0,1], \tag{1.4}$$

where $\beta \in (0,1]$.

The definitions we provided above are for functions with a single variable. These definitions are slightly different for a function with two independent variables. In this thesis, we have also studied the bivariate generalization of some operators. Therefore we provide some definitions which will be used accordingly.

1.1.7 Total and partial modulus of continuity

To establish the degree of approximation of bivariate operators in the space of continuous functions on compact set $I^2 = [a,b] \times [a,b]$, the total modulus of continuity for function $f \in C(I^2)$ is defined by:

$$\omega_{total}(f;\delta_1,\delta_2) = \sup\{|f(t_1,t_2) - f(x,y)| : (t_1,t_2),(x,y) \in I^2, |t_1 - x| \leq \delta_1, |t_2 - y| \leq \delta_2\}.$$

Further, the partial moduli of continuity with respect to the independent variables x and y is given as:

$$\omega^{(1)}(f;\delta) = \sup\{|f(x_1,y) - f(x_2,y)| : y \in I, |x_1 - x_2| \leq \delta\},$$

and

$$\omega^{(2)}(f;\delta) = \sup\{|f(x,y_1) - f(x,y_2)| : x \in I, |y_1 - y_2| \leq \delta\}.$$

Both total and partial modulus of continuity for bivariate functions satisfy the properties of usual modulus of continuity and can be studied more in (20).

Next, to establish the rate of convergence of bivariate operators, we define the following Peetre's K- functional.

1.1.8 Peetre's K-functional for bivariate operators

Let $C^2(I^2)$ denote the set of all continuous functions on $I^2 = [0,1] \times [0,1]$, whose first and second derivatives exist and are continuous on the interval I^2. We define the norm:

$$||f||_{C^2(I^2)} = ||f||_{C(I^2)} + \sum_{k=1}^{2}\left(\left\|\frac{\partial^k f}{\partial x^k}\right\|_{C(I^2)} + \left\|\frac{\partial^k f}{\partial y^k}\right\|_{C(I^2)}\right).$$

From Butzer and Berens(32), the Peetre's K-functional for $f \in C^2(I^2)$ is defined as:

$$K(f;\sigma) = \inf_{t \in C^2(I^2)} \left\{||f - t||_{C^2(I^2)} + \sigma||t||_{C^2(I^2)}, \sigma > 0\right\}.$$

For a positive constant M, Peetre's K-functional satisfies the following inequality:

$$K(f;\sigma) \leqslant M\left\{\varpi_2\left(f, \sqrt{\sigma}\right) + \min\{1,\sigma\}\,||f||_{C(I^2)}\right\}$$

where $\varpi_2\left(f, \sqrt{\sigma}\right)$ is the second order modulus of continuity for bivariate functions, defined as:

$$\varpi_2\left(f, \sqrt{\sigma}\right) = \sup\left\{\left|\sum_{i=0}^{2}(-1)^{2-i} f(x + ix_0, y + iy_0)\right| : (x,y), (x + ix_0, y + iy_0) \in I^2, |x_0| \leqslant \sigma, |y_0| \leqslant \sigma\right\}$$

1.2 Historical Background and Literature Review

A simple yet powerful tool for deciding whether a given sequence of linear positive operators on $C[0,1]$ or $C[0,2\pi]$ is an approximation process or not are the Korovkin theorems. These theorems are an abstract results in approximation which gives conditions for uniform approximation of continuous functions on a compact metric space. The Korovkin theorem [(19) pp.218] elegantly says that if $(L_n)_{n\geq 1}$ is an arbitrary sequence of linear positive operators on the space $C[a,b]$, and if

$$\lim_{n\to\infty} L_n(e_i;x) \to e_i \text{ uniformly on } [a,b],$$

for the test functions $e_i(x) = x^i$, $i = 0,1,2$ then

$$\lim_{n\to\infty} L_n(f;x) \to f \text{ uniformly on } [a,b],$$

for each $f \in C[a,b]$.

The above theorem, known as Korovkin's first theorem, was proposed by P.P. Korovkin (93) in 1953. Korovkin's second theorem has a similar statement, but the space $C[0,1]$ is replaced by the space $C[0,2\pi]$, i.e. the space of all 2π periodic real-valued functions on $\mathbb{R}$. The test functions e_i in this case belong to the set $\{1, \cos(x), \sin(x)\}$ for $i = 0,1,2$ respectively. H.Bohmann (30) in 1952 had proved a result similar to Korovkin's first theorem but concerning sequences of linear positive operators on $C[0,1]$ of the form

$$L(f;x) = \sum_{i\in I} f(a_i)\,\phi_i, \quad f \in C[0,1]$$

where $(a_i)_{i\in\Lambda}$ is a finite set of numbers in $[0,1]$ and $\phi_i \in C[0,1]$, $i \in \Lambda$. Therefore, Korovkin's first theorem is also known as Bohman-Korovkin Theorem. An immediate analogue of Korovkin's theorem does not hold if the domain of definition of the function f becomes unbounded and hence requires the function to have some finite limit at infinity. For continuous and unbounded functions on $[0,\infty)$, A.D Gadžiev (57) in 1974 introduced a weighted space $C_\rho[0,\infty)$ defined as the set of all continuous functions f on the interval $[0,\infty)$ for which there exists a positive constant M such that $|f(x)| \leq M\rho(x)$, for every $x \in [0,\infty)$. Here ρ is a positive continuous function called the weight function. The space $C_\rho[0,\infty)$ is a Banach space equipped with the norm

$$\|f\|_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}.$$

The Korovkin theorem by Gadžiev is given as: Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous, strictly increasing and unbounded function. Set $\rho(x) = 1 + \varphi^2(x)$. If the sequence of linear positive operators $L_n : C_\rho[0,\infty) \to C_\rho[0,\infty)$ verifies

$$\lim_{n\to\infty} \left\| L_n(\varphi^i; x) - \varphi^i(x) \right\|_\rho = 0, \;\; i = 0,1,2$$

Then,

$$\lim_{n\to\infty} \| L_n(f; x) - f(x) \|_\rho = 0$$

for every $f \in C_\rho[0,\infty)$ for which $\lim_{n\to\infty} \frac{f(x)}{\rho(x)}$ exists and is finite.

With the application of Korovkin theorems to study the uniform convergence of linear positive operators, advancement in approximation theory began with the development of new linear positive operators, the first and most important of which are the Bernstein polynomials. In 1912, S.N. Bernstein (26) gave an elegant proof of the famous Weierstrass approximation theorem by defining a sequence of polynomials called Bernstein operators on the closed interval $[0,1]$ (extended on $[a,b]$ by simple manipulations). These operators are defined as:

Let f be a bounded function on $[0,1]$. The Bernstein operator of degree n with respect to f is defined as:

$$B_n(f; x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1] \tag{1.5}$$

where

$$b_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}, k = 0,1,2,\cdots,n$$

and $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$ represents the binomial coefficient. It should be noted that $b_{n,k}(x) \in P_n, \; k = 0,1,2,\cdots,n$ where P_n denotes the space of all polynomials of degree at most n, are the so-called Bernstein polynomials.

Proposition 1.2.1 *Some important properties of the Bernstein polynomials are listed as follows:*

1. *The Bernstein polynomials of degree n form a basis for P_n.*
2. *The Bernstein polynomials satisfy symmetry property $b_{n,k}(x) = b_{n,n-k}(1-x), \; k = 0,1,2,\cdots,n$*
3. *The Bernstein polynomials are all positive over $[0,1]$, that is $b_{n,k}(x) \geq 0, \forall x \in [0,1]$.*
4. *Another important property is that the Bernstein polynomials form a partition of unity:*

$$\sum_{k=0}^{n} b_{n,k}(x) = 1.$$

5. *Recursive formula for the Bernstein polynomials is as follows:*

$$b_{n,k}(x) = (1-x)\, b_{n-1,k}(x) + x b_{n-1,k-1}(x).$$

Since the Bernstein operators were only suitable for approximating functions on a compact interval, O. Szász in 1950 (132), and G. Mirakiyan in 1941 presented a generalization of these operators for a continuous function f on the interval $[0,\infty)$ which later came to be known as Szász-Mirakiyan operators. These operators are defined as:

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x)\, f\left(\frac{k}{n}\right), \quad x \in [0,\infty), \tag{1.6}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

In 1957, V. A. Baskakov (25) introduced another sequence of linear positive operators on the interval $[0,\infty)$ called Baskakov operators which are defined as:

$$V_n(f;x) = \sum_{k=0}^{\infty} v_{n,k}(x)\, f\left(\frac{k}{n}\right), \quad x \in [0,\infty), \tag{1.7}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}.$$

To approximate integrable functions on the compact interval $[a,b]$, Kantorovich (88) was the first to define the integral variant of Bernstein operators by replacing the weight function with the average mean of the weight function in the vicinity of the point $\frac{k}{n}$ as:

$$\hat{B}_n(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt.$$

where $b_{n,k}(x)$ is defined in (1.5).

Similarly Szász Kantorovich operators on the unbounded interval $[0,\infty)$ for given basis function $s_{n,k}(x)$ in (1.6) are defined as:

$$\hat{S}_n(f;x) = n\sum_{k=0}^{\infty} s_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t)\,dt. \tag{1.8}$$

For Baskakov operators, the integral variant on the semi real axis is:

$$\hat{V}_n(f;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x) \int_{k/(n-1)}^{(k+1)/(n-1)} f(t)\,dt. \tag{1.9}$$

where $v_{n,k}(x)$ is defined in similar manner as in (1.7). To estimate functions on an unbounded interval, Kantorovich forms of various approximation operators have been defined from time to time. For further reference, one can visit the articles (6; 14; 18; 39; 62; 110).

In 1967, J. L. Durrmeyer (52) gave a more generalised integral modification of Bernstein operators by replacing the values of $f(k/n)$ by an integral over the weight function on the interval $[0,1]$. These so called Bernstein-Durrmeyer operators were first studied by Derrienic (43) and are defined as:

$$\tilde{B}(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x)\int_{0}^{1} b_{n,k}(t)\,f(t)\,dt. \tag{1.10}$$

In 1985, Mazhar and Totik (108) introduced the Szász-Durrmeyer operators as follows:

$$\tilde{S}_n(f;x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_{0}^{\infty} s_{n,k}(t)\,f(t)\,dt. \tag{1.11}$$

In the same year Sahai and Prasad (125) also established the Baskakov-Durrmeyer operators defined as follows:

$$\tilde{V}_n(f;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x)\int_{0}^{\infty} v_{n,k}(t)\,f(t)\,dt. \tag{1.12}$$

where $b_{n,k}(x)$, $s_{n,k}(x)$ and $v_{n,k}(x)$ are same as in (1.5), (1.6) and (1.7) respectively. Durrmeyer type variants of a number of linear positive operators were constructed in subsequent years. One can refer to the articles (5; 15; 38; 61; 84).

With the advancement in approximation theory, researchers were drawn to developing novel approximation operators that had faster convergence rates and were applicable within a variety of functions and spaces. May (106) in 1976 first defined operators of the form:

$$W_\lambda(f;x) = \int_{-\infty}^{\infty} S(\lambda,x,t)\,f(t)\,dt,$$

and termed it as exponential operators provided they satisfy two conditions, first is the homogenous partial differential equation

$$\frac{\partial}{\partial x}S(\lambda,x,t) = \frac{\lambda(t-x)}{q(x)}S(\lambda,x,t), \tag{1.13}$$

where, $S(\lambda,x,t) \geq 0$ is the kernel of these operators and q is a polynomial of at most degree n which is analytic and positive for $x \in (a,b)$ for some a,b such that $-\infty \leq a \leq$

$b \leq +\infty$, while second is the normalization condition

$$W_\lambda(1; x) = \int_{-\infty}^{\infty} S(\lambda, x, t)dt = 1. \tag{1.14}$$

Operators satisfying the above conditions are, for example, the Bernstein operators, Szász Mirakiyan operators, Post-Widder operators, Gauss-Weierstrass operators and Baskakov operators. These well-known operators are thus referred to as exponential operators. Some approximation properties were also studied for polynomials of degree at most 2. A year later, Ismail and May (79) proved that for a polynomial q of any degree, the approximation operator W_λ can be uniquely determined and satisfies the differential equation (1.13) along with the normalisation condition (1.14). As a consequence of this, they recovered some known operators for constant, linear and quadratic polynomials. Further, they gave some new operators for cubic polynomials q such as: for the polynomial $q(x) = x(1 + x)^2$, new exponential operators derived by Ismail and May using the method of bilateral Laplace transform are defined as:

$$R_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}}{k!} (xe^{-x})^k f\left(\frac{k}{n+k}\right), \quad n \in \mathbb{N} \quad x \in (0, 1). \tag{1.15}$$

The convergence properties and the corresponding Bézier variant of these operators were extensively studied in (96). A complete asymptotic expansion for this sequence of operators is also derived in (2). Another exponential operators corresponding to $q(x) = 2x^{3/2}$ is also studied in (1; 97). Sato (126) studied the global behaviour of exponential operators like Bernstein, Szász Mirakiyan, Gauss-Weierstrass etc., in weighted spaces. Totik (136) described their theoretical approximation properties from the point of view of global uniform approximation. For more information on exponential operators, one can refer to the book (70) and the articles cited therein. In this thesis, we have investigated some modifications of exponential operators given by Ismail and May (79) and discussed their convergence properties.

Another approximation operator examined in this thesis are the gamma-type operators. A vital tool among the researchers to study linear positive operators is Euler's gamma function, which for $r > 0$ is defined as follows:

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1} dt.$$

For $(a, b) \in \mathbb{R}$, Miheşan (109) defined a more general linear transform of f, also called the $(a, b)-$gamma transform, as follows:

$$\Gamma(r)^{(a,b)}(f;x) = \frac{1}{\Gamma(r)} \int_0^\infty e^{-t} t^{r-1} f\left(xe^{-bt}\left(\frac{t}{r}\right)^a\right) dt. \tag{1.16}$$

The transform (4.1) reproduces distinct integral operators for different values of a, b and r. The derived operators have been introduced and studied extensively by researchers over the past few decades; for instance, see (21; 72; 95; 103; 116). For $b = 0$, $a = -1$ and $r = n+1$, one can obtain a particular operator first introduced and studied by Lupaş and Müller (104) and also referred to as gamma operators. In this thesis work, we have proposed certain gamma-type operators which possess the property of reproducing polynomial functions of the form t^s, $s \in \mathbb{N}$.

1.3 Improvement in the Order of Approximation

The central idea in approximation theory is to estimate the rate of convergence of the sequence of operators using various convergence methods. These methods aim to improve the rate of convergence of operators, thereby reducing the error induced during the approximation process. Let $E_n(f)$ be the error function for the best uniform approximation of function f by trigonometric or algebraic polynomials T_n of degree n, then

$$E_n(f) = \inf_{T_n} \sup_x |f(x) - T_n(x)|.$$

We expect that smoother the function f is, the faster $E_n(f)$ converges to zero.
If f is r times continuously differentiable on some compact interval, then

$$E_n(f) \leqslant C_r \|f^r\| n^{-r}, \quad n = 1, 2, \cdots$$

For instance, we say that $E_n(f)$ tends to zero at least as fast as $1/n$ whenever f is differentiable i.e. degree of approximation of f is $1/n$ and $1\big/n^2$ when it is twice differentiable, and so on. Estimates of this type of estimating error has a rich history. The first results of this type were given by S.N. Bernstein (27) and later Favard (53) found the best constant C_r. Jackson (82) then refined the above given estimate of $E_n(f)$ by using subtler measures of the smoothness of a function f such as modulus of continuity $\omega(f, \delta)$ for $f \in C(I)$ as: If f is r times continuous differentiable, then

$$E_n(f) \leqslant C_r n^{-r} \omega\left(f^r, n^{-1}\right), \quad n = 1, 2, \cdots.$$

The continuity of f ensures that $\omega(f, \delta) \to 0$ as $\delta \to 0$. Although the linear positive operators are conceptually simpler, easy to construct and study, they lack the rapidity of convergence for sufficiently smooth functions. In the same context, a well-known theorem of Korovkin states that the optimal rate of convergence for any sequence of linear operators is at most $O(n^{-2})$. Thus if we want to have a better order of approximation for smoother functions, we must slacken the positivity condition. Several investigations indicate that even when a sequence or class of linear positive operators is saturated with a certain order of approximation, some carefully chosen linear combinations of its members give better order of approximation of smoother functions.

Our interest in this thesis is to improve the order of approximation of classical and existing operators. For instance, many of the standard operators in approximation theory preserve the test functions e_0 and e_1 for all $n \in \mathbb{N}$, i.e.,

$$L_n(e_0; x) = e_0 \quad L_n(e_1; x) = e_1.$$

An important approach to improve the order of approximation was given by J.P. King in his pioneer work (91). He presented a non-trivial sequence of positive linear operators defined on $C[0,1]$ that preserved the test functions e_0 and e_2. Let $\{r_n(x)\}$ be a sequence of continuous functions defined on $[0,1]$ such that $r_n(x) \in [0,1]$. Then the operators $V_{n,r_n} : C[0,1] \to C[0,1]$ are defined as:

$$V_{n,r_n}(f; x) = \sum_{k=0}^{n} (r_n(x))^k (1 - r_n(x))^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1]$$

where

$$r_n(x) = \begin{cases} x^2, & n = 1 \\ \frac{-1}{2(n-1)} + \sqrt{\left(\frac{n}{n-1}\right)x^2 + \frac{1}{4(n-1)^2}}, & n = 2, 3, \cdots \end{cases}$$

The operators V_{n,r_n} interpolate f at the endpoints 0 and 1 and are not polynomial operators. King also proved that the order of approximation of operators V_{n,r_n} is at least as good as the order of approximation of Bernstein operators for $x \in [0, \frac{1}{3}]$. Inspired by his work, other modifications of well-known operators were constructed as well to fix certain functions and to study their approximation and shape-preserving properties. In (35) Cárdenas-Morales et al. presented a family of sequences of linear Bernstein-type operators $B_{n,\alpha},\ n > 1$, depending on a real parameter $\alpha \geq 0$, and fixing the polynomial function $e_2 + \alpha e_1$. Among other things, the authors prove that if f is convex and increasing on $[0,1]$, then $f(x) \leq B_{n,\alpha}(f; x) < B_n(f; x)$ for every $x \in [0,1]$. Duman and Özarsalan (50) gave a modification of the classical Szász Mirakiyan operators to provide a better error estimation. Ozsarac and Acar (118) presented a new modification of the Baskakov operators, which preserve the functions $e^{\mu t}$ and $e^{2\mu t}, \mu > 0$.

An improvement in the order of approximation can also be made with the help of arbitrary positive sequences under suitable conditions. In 1937, Chlodovsky (36) used arbitrary sequences (b_n) to introduce certain polynomials and generalized the renowned Bernstein operators. For a function f defined on $[0,\infty)$ and bounded on every finite interval $[0,b_n] \subset [0,\infty)$, the classical Bernstein-Chlodovsky operators are defined by

$$C_{n,k}^{[b_n]}(f;x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} f\left(\frac{b_n k}{n}\right), \ x \in [0,b_n]$$

where (b_n) is an increasing sequence of positive reals with the property that

$$\lim_{n\to\infty} b_n = \infty \ , \ \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

When $b_n = 1$ $n \in \mathbb{N}$, the Bernstein-Chlodovsky operators transform into the classical Bernstein operators. The advantage of Chlodovsky operators is that they contain a factor b_n, tending to infinity and having a certain degree of freedom. Therefore, they are generally more efficient in approximating functions than their corresponding classical counterpart. This motivated researchers to work in this direction; we mention here some essential papers, see (4; 11; 128; 131), and references therein. In this thesis, we have used King's approach as well as the sequential approach to present a better modification of various operators, thereby reducing the error and improving the rate of approximation of the considered operators.

1.4 Chapter-wise overview of the thesis

The thesis consists of six chapters, whose contents are described below:

The literature and historical context of some key approximation operators are covered in **Chapter 1**. Along with a brief summary of the chapters this thesis is divided into, we also discuss some preliminary instruments that will be employed subsequently to derive our main results.

Chapter 2 is dedicated to some exponential operators, a concept first studied by C.P. May and later investigated in-depth alongside Mourad E.H. Ismail. It is majorly divided into two sections. First section considers an exponential operator associated with the polynomial $x(1+x)^2$ which are defined as:

$$\tilde{R}_\lambda(f;t) = e^{-\frac{\lambda t}{(1+t)}} \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!} \left(\frac{t}{1+t}\right)^k e^{-\frac{\lambda t}{(1+t)}} f\left(\frac{k}{\lambda}\right), \quad \lambda > 0, t \in (0,\infty).$$

With change of variable $x = \frac{t}{1+t}$, for all $x \in [0,1]$ and $\lambda > 0$, we proposed an integral modification of these operators in Kantorovich sense in the following way:

$$L_{\lambda}(f;x) = e^{-\lambda x}\sum_{k=0}^{\infty}\frac{\lambda(\lambda+k)^k}{k!}(xe^{-x})^k\int_{\frac{k}{\lambda+k}}^{\frac{k+1}{\lambda+k}} f(t)dt. \tag{1.17}$$

We established the rate of convergence of the proposed operators with direct approximation theorem using Ditzian-Totik modulus of smoothness. Local approximation results for Lipschitz class functions and a quantitative Voronovoskaya type theorem in terms of first and second-order modulus of continuity to study the order of approximation. Through the use of illustrated graphs and an error estimation table, the convergence of the operators has also been demonstrated. Additionally, we also proposed a bivariate generalization of the operators (1.17). These operators can be used to approximate integrable two-dimensional functions. Some general approximation properties of these operators are also studied here.

The second section is dedicated to a modification of another exponential operators proposed by Ismail and May (79) associated with the polynomial x^3 and are defined as:

$$\mathcal{B}_n(f;x) = \int_0^{\infty} l_n(x,t)f(t)dt, \quad x\in(0,\infty)$$

where

$$l_n(x,t) = \sqrt{\frac{n}{2\pi}}e^{n/\sigma_n(x)}t^{-3/2}\exp\left(-\frac{nt}{2(\sigma_n(x))^2}-\frac{n}{2t}\right).$$

Here $\sigma_n(x) = \frac{2nx}{2n+Ax^2}$, $A\in\mathbb{R}$, is calculated under the assumption that these operators preserve exponential functions of the form e^{Ax}. It is to be noted that the modified operators are not exponential operators. The proposed operators preserve the exponential functions of the form e^{Ax} and provide a better approximation for functions when $A > 0$. The moments and central moments of the proposed operators are evaluated with the help of moment-generating functions. We have also considered a particular case of preservation of exponential function for $A = -1$, i.e., for e^{-x}. Some approximation results associated with the rate of convergence and order of approximation are also provided, along with some numerical examples and graphical representations.

Chapter 3 is based on a Durrmeyer type construction of certain operators based on a class of orthogonal polynomials called Apostol-Genocchi polynomials. For $f\in C[0,\infty)$, and $\rho > 0$, these operators are defined as follows:

$$\mathcal{M}_n^{\alpha,\lambda}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{\alpha,\lambda}(x)\int_0^{\infty} l_{n,k}^{\rho}(t)f(t)\,dt, \quad x\in[0,\infty)$$

where

$$l_{n,k}^{\rho}(t) = n\rho e^{-n\rho t}\frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)},$$

and

$$s_{n,k}^{\alpha,\lambda}(x) = e^{-nx}\left(\frac{1+\lambda e}{2}\right)^{\alpha}\frac{g_k^{\alpha}(nx;\lambda)}{k!}.$$

Here $g_k^{\alpha}(x;\lambda)$ is the generalized Apostol-Genocchi polynomials of order α, and $l_{n,k}^{\rho}(t)$ is the Păltănea basis.

We first establish a global approximation result for the proposed operators followed by its convergence estimate in terms of usual, r-th and weighted modulus of continuity. We further study the asymptotic type results, such as the Voronovskaya theorem and quantitative Voronovskaya theorem. Additionally, for functions with bounded variation defined on the interval $(0,\infty)$, we estimate the rate of pointwise convergence of the suggested operators. An absolute error table and graphical representations serve as the result's last forms of validation.

Researchers have spent the last few decades studying an extensive array of approximation operators due to the development of the theory of the gamma function. **Chapter 4** therefore focuses mainly on investigating a modification of certain gamma-type operators. We used King's approach (91) to present a modification of certain gamma type operators such that they preserve the test functions $e_s(x) = x^s,\ s \in \mathbb{N}$. The modified operators are defined as:

$$\widetilde{\mathcal{F}}_{n,s}(f;x) = \frac{(2n)!(\wp_{n,s}(x))^{n+1}}{n!(n-1)!}\int_0^{\infty}\frac{t^{n-1}}{(\wp_{n,s}(x)+t)^{2n+1}}f(t)\,dt.$$

where

$$\wp_{n,s}(x) = \left[\frac{(n-s+1)_s}{(n)_s}\right]^{1/s}x,$$

Here $(n)_s$ denotes the rising factorial with $(n)_0 = 1,\ (n)_s = n(n+1)\cdots(n+s-1)$.

We used recursion formulas to establish the moments and central moments of the proposed operators, which are further used to determine the convergence rate of the proposed operators in the sense of the usual modulus of continuity and Peetre's K-functional. Additionally, the degree of approximation is also established for the function of bounded variation. We also illustrate via figures and tables that the proposed modification provides a better approximation for preserving the test function e_3.

The concept of modifying operators via a sequential approach is dealt with in **Chapter 5**. We considered a Kantorovich form of the Baskakov type operators with the help of two arbitrary sequences (r_n) and (s_n), satisfying the conditions $\lim_{n\to\infty} r_n = \infty$ and $\lim_{n\to\infty} \frac{s_n}{n} = 0$ that preserve exponential functions of the form $a^{-x}, a > 1$ with arbitrary order and are defined in the following way:

$$\hat{\mathcal{K}}_n^{[s_n]}(f;\vartheta(n,x)) = r_n \sum_{k=0}^{\infty} p_{n,k}^{[s_n]}(\vartheta(n,x)) \int_{\frac{k}{r_n}}^{\frac{k+1}{r_n}} f(t)\,dt,$$

with

$$\vartheta(n,x) = \frac{\left(\frac{r_n\left(1-a^{-1/r_n}\right)a^x}{\ln(a)}\right)^{s_n/n} - 1}{s_n\left(1-a^{-1/r_n}\right)}.$$

One can also significantly achieve better approximation by appropriate sequences (r_n) and (s_n). We prove an essential lemma which is further helpful to establish certain direct results, including quantitative type asymptotic formula and Voronovskaya theorem. We also provide an estimation of error using the usual modulus of continuity. Further, the results are validated via convergence and error estimation graphs showing a better approximation of the proposed operators.

The final chapter, **Chapter 6**, concludes the thesis with a summary before providing insight into the author's thoughts on the future direction of the research.

We now move on to our first chapter, which explores some important exponential operators based on recent studies by M.E.H. Ismail and C.P. May.

Chapter 2

Operators due to Ismail and May

This chapter discusses exponential operators, a concept first used by C.P. May and later investigated in-depth alongside Mourad E.H. Ismail. By active researchers these operators are referred to as Ismail-May operators. The first section of this chapter is dedicated to the Kantorovich form of exponential operators associated with the polynomial $x(1+x)^2$. Moreover, a two variable generalization of the proposed operators and its approximation properties are also investigated. The second section is focused on another exponential operators associated with the polynomials x^3. These operators are modified in a way that preserves the functions with exponential growth and provides better convergence with change in certain parameters.

2.1 Kantorovich Variant of Ismail-May Operators

2.1.1 Introduction

Ismail and May (79), in continuation of the remarkable work done by May (106), considered some exponential type operators $W_n(f,x) = \int\limits_{-\infty}^{\infty} S(n,x,t)\, f(t)\, dt$ which hold the normalization condition $W_n(1,x) = \int\limits_{-\infty}^{\infty} S(n,x,t)dt = 1$ and satisfy the homogenous partial differential equation:

$$\frac{\partial}{\partial x} S(n,x,t) = \frac{n(t-x)}{q(x)} S(n,x,t), \tag{2.1}$$

where, $S(n,x,t) \geq 0$ is the kernel of these operators, $q(x)$ is analytic and positive for $x \in (a,b)$ for some a,b such that $-\infty \leq a \leq b \leq +\infty$.

Ismail and May (79) proved that for a polynomial q of any degree, the approximation operators W_n is uniquely determined and satisfy the differential equation (2.1) along with the normalisation condition. As a result, they recovered some known operators for constant, linear and quadratic polynomials and further gave some new operators for cubic polynomials. From (123), using the identity

$$e^{nx} = \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}}{k!}(xe^{-x})^k, \tag{2.2}$$

one such operator corresponding to $q(x) = x(1+x)^2$ is defined as:

$$\tilde{R}_n(f;t) = e^{-\frac{nt}{(1+t)}} \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}}{k!}\left(\frac{t}{1+t}\right)^k e^{-\frac{nt}{(1+t)}} f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}, \ t \in (0,\infty).$$

With change of variable $x = \frac{t}{1+t}$, the corresponding approximation operators is given by:

$$R_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}}{k!}(xe^{-x})^k f\left(\frac{k}{n+k}\right), \qquad x \in (0,1). \tag{2.3}$$

Various approximation properties of operators (2.3) has already been studied in (96) along with its Bézier variant and bivariate generalisation.

In the year 1930, Kantorovich (87) introduced integral modification of Bernstein operators in the class of Lebesgue integrable functions on the interval [0,1], to obtain better approximation results as ordinary Bernstein operators were not considered suitable in approximating discontinuous functions of general type; see (99). In the past few decades, several researchers have presented their Kantorovich modification for various linear positive operators and studied their approximation properties. One can refer to articles (7; 18; 39; 47; 62; 80; 83; 86; 113) for more literature in this area.

Inspired by the above stated works, here we define the integral form of operators (2.3) in Kantorovich sense, for $n \in \mathbb{N}$ and $x \in [0,1]$ as follows:

$$L_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{n(n+k)^k}{k!}(xe^{-x})^k \int_{\frac{k}{n+k}}^{\frac{k+1}{n+k}} f(t)dt. \tag{2.4}$$

2.1.2 Auxiliary Results

Before proceeding to our main results, we provide some useful lemmas which will be helpful throughout in proof of our main theorems.

Lemma 2.1.1 *For $x \in [0,1]$ and $n \in \mathbb{N}$, the moments of the operators (2.4) are as follows:*

$$L_n(1;x) = 1;$$

$$L_n(t;x) = \tfrac{(2n-1)}{2(n+1)}x + \tfrac{1}{2n};$$

$$L_n\left(t^2;x\right) = \tfrac{(3n^2-3n+1)}{3(n+1)(n+2)}x^2 + \tfrac{(6n^2-2n-1)}{3n(n+1)^2}x + \tfrac{1}{3n^2};$$

$$L_n\left(t^3;x\right) = \left(\tfrac{4n^3-6n^2+4n-1}{4(n+1)(n+2)(n+3)}\right)x^3 + \left(\tfrac{18n^4+8n^3-21n^2+6n+2}{4n(n+1)^2(n+2)^2}\right)x^2 + \left(\tfrac{14n^3-3n^2-3n-1}{4n^2(n+1)^3}\right)x + \tfrac{1}{4n^3};$$

$$L_n\left(t^4;x\right) = \left(\tfrac{1}{5(n+1)(n+2)(n+3)(n+4)}\right)x^4 - \tfrac{2}{5}\left(\tfrac{n^3+3n^2-3}{n(n+1)^2(n+2)^2(n+3)^2}\right)x^3 + \left(\tfrac{6n^4+24n^3+33n^2+18n+4}{5n^2(n+1)^3(n+2)^3}\right)x^2 - \left(\tfrac{4n^3+6n^2+4n+1}{5n^3(n+1)^4}\right)x + \tfrac{1}{5n^4}.$$

Proof 2.1.2 *Using the identity (2.2) in operators (2.4), the moments can be easily obtained by direct computation.*

Lemma 2.1.3 *For operators (2.4), the central moments are as follows:*

$$L_n((t-x);x) = \tfrac{-3x}{2(n+1)} + \tfrac{1}{2n};$$

$$L_n\left((t-x)^2;x\right) = \tfrac{(-3n+13)}{3(n+1)(n+2)}x^2 + \tfrac{(3n^2-8n-4)}{3n(n+1)^2}x + \tfrac{1}{3n^2};$$

$$L_n\left((t-x)^4;x\right) = \left(\tfrac{15n^2-340n+501}{5(n+1)(n+2)(n+3)(n+4)}\right)x^4 - \left(\tfrac{2(15n^5-210n^4-749n^3+238n^2+1485n+402)}{5n(n+1)^2(n+2)^2(n+3)^2}\right)x^3 + \left(\tfrac{15n^6-150n^5-414n^4+234n^3+1023n^2+558n+124}{5n^2(n+1)^3(n+2)^3}\right)x^2 + \left(\tfrac{25n^4-24n^3-36n^2-24n-6}{5n^3(n+1)^4}\right)x + \tfrac{1}{5n^4}.;$$

Lemma 2.1.4 *Since for $n \in \mathbb{N}$ and $x \in (0,1)$, the maximum values of $x(1-x)$ and $(1-3x)$ are $\frac{1}{4}$ and 1 respectively, therefore we have:*

$$L_n((t-x);x) = \frac{n(1-3x)+1}{2n(n+1)} \leq \frac{1}{2n},$$

and

$$L_n\left((t-x)^2;x\right) \leq \frac{x(1-x)}{n} \leq \frac{1}{4n}.$$

Lemma 2.1.5 *For $f \in C[0,1]$ and $n \in \mathbb{N}$, we have:*

$$|L_n(f;x)| \leq \|f\|.$$

2.1.3 Main Results

For the estimation of the rate of convergence of the proposed operators (2.4)*, we use the definition of Ditzian-Totik modulus of smoothness and Peetre's K-functional given in subsection 1.1.3.*

Theorem 2.1.6 *For $f \in C[0,1]$, $n \in \mathbb{N}$ and $\varphi(x) = \sqrt{x(1-x)}$, we have:*

$$|L_n(f;x) - f(x)| \leq 2K_\varphi\left(f; \frac{1}{\sqrt{2n}\varphi(x)}\right).$$

Proof 2.1.7 *We can write,*

$$g(t) = g(x) + \int_x^t g'(u)du. \tag{2.5}$$

On applying $L_n(.;x)$ on both sides of (2.5), we have:

$$L_n(g;x) = g(x) + L_n\left(\int_x^t g'(u)du; x\right). \tag{2.6}$$

We first, we determine $\int_x^t g'(u)du$ for $x, t \in (0,1)$ as follows:

$$\begin{aligned}
\left|\int_x^t g'(u)du\right| &\leq \|\varphi g'\| \left|\int_x^t \frac{1}{\varphi(u)}du\right| \\
&= \|\varphi g'\| \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\right| \\
&\leq \|\varphi g'\| \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \\
&\leq 2\,\|\varphi g'\|\,|t-x| \left(\frac{1}{\sqrt{x}+\sqrt{t}} + \frac{1}{\sqrt{1-x}+\sqrt{1-t}}\right) \\
&\leq 2\,\|\varphi g'\|\,|t-x| \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right) \\
&\leq 2\sqrt{2}\,\|\varphi g'\|\,\frac{|t-x|}{\varphi(x)}.
\end{aligned} \tag{2.7}$$

In view of (2.6), (2.7) and Cauchy-Schwarz inequality and later using Remark 2.1.4, we get:

$$\begin{aligned}
|L_n(g;x) - g(x)| &\leq \frac{2\sqrt{2}\,\|\varphi g'\|}{\varphi(x)} \sqrt{L_n((t-x)^2;x)} \\
&\leq \frac{\sqrt{2}\,\|\varphi g'\|}{\sqrt{n}\varphi(x)}.
\end{aligned} \tag{2.8}$$

Using well-known triangle inequality and (2.8)*, we have:*

$$\begin{aligned}|L_n(f;x)-f(x)| \leq& |L_n(f-g;x)|+|f(x)-g(x)|+|L_n(g;x)-g(x)|\\ \leq& 2\,\|f-g\|+\frac{\sqrt{2}\,\|\varphi g'\|}{\sqrt{n}\varphi(x)}\\ \leq& 2\left(\|f-g\|+\frac{\|\varphi g'\|}{\sqrt{2n}\varphi(x)}\right). \qquad (2.9)\end{aligned}$$

Taking infimum on the right hand side of (2.9), the result is obtained.

The following theorems establish the rate of convergence of the operators (2.4) for functions in Lipschitz space. For non-negative real numbers a and b, the Lipschitz-type space (120) is defined as:

$$Lip_M(\beta)=\left\{f\in C[0,1]:|f(t)-f(x)|\leq M\frac{|t-x|^{\beta}}{(t+ax^2+bx)^{\beta/2}};\ x,t\in(0,1)\right\},$$

where $\beta\in(0,1]$.

Theorem 2.1.8 *For* $f\in Lip_M(\beta)$ *and* $x\in(0,1]$*, we have:*

$$|L_n(f;x)-f(x)|\leq M\left(\frac{\rho_n^2(x)}{ax^2+bx}\right)^{\beta/2},$$

where $\rho_n^2(x)=L_n((t-x)^2;x)$.

Proof 2.1.9 *First, we estimate the result for* $\beta=1$*. We can write*

$$\begin{aligned}|L_n(f(t);x)-f(x)| \leq& e^{-nx}\sum_{k=0}^{\infty}\frac{n(n+k)^k}{k!}(xe^{-nx})^k\int_{\frac{k}{n+k}}^{\frac{k+1}{n+k}}|f(t)-f(x)|\,dt \qquad (2.10)\\ \leq& Me^{-nx}\sum_{k=0}^{\infty}\frac{n(n+k)^k}{k!}(xe^{-nx})^k\int_{\frac{k}{n+k}}^{\frac{k+1}{n+k}}\frac{|t-x|}{\sqrt{(t+ax^2+bx)}}dt.\end{aligned}$$

Using the relation $\frac{1}{(t+ax^2+bx)}<\frac{1}{(ax^2+bx)}$ *and Cauchy-Schwarz inequality, we get:*

$$\begin{aligned}|L_n(f;x)-f(x)| \leq& \frac{M}{\sqrt{(ax^2+bx)}}e^{-nx}\sum_{k=0}^{\infty}\frac{n(n+k)^k}{k!}(xe^{-nx})^k\int_{\frac{k}{n+k}}^{\frac{k+1}{n+k}}|t-x|\,dt\\ \leq& \frac{M}{\sqrt{(ax^2+bx)}}L_n\left(|t-x|\,;x\right)\\ \leq& M\sqrt{\frac{\rho_n^2(x)}{(ax^2+bx)}}.\end{aligned}$$

Next, we prove this theorem for $0 < \beta < 1$. *Using Hölder's inequality with* $p = \frac{1}{\beta}$ *and* $q = \frac{1}{1-\beta}$ *in (2.10), we have:*

$$\begin{aligned}
|L_n(f;x) - f(x)| \leq & \left\{ e^{-nx} \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}}{k!} (xe^{-nx})^k \left((n+k) \int_{\frac{k}{n+k}}^{\frac{k+1}{n+k}} |f(t) - f(x)|\, dt \right)^{\frac{1}{\beta}} \right\}^{\beta} \\
\leq & M \left\{ e^{-nx} \sum_{k=0}^{\infty} \frac{n(n+k)^{k}}{k!} (xe^{-nx})^k \left(\int_{\frac{k}{n+k}}^{\frac{k+1}{n+k}} \frac{|t-x|}{\sqrt{(t+ax^2+bx)}} dt \right) \right\}^{\beta} \\
\leq & \frac{M}{(ax^2+bx)^{\beta/2}} (L_n(|t-x|;x))^{\beta} = M \left(\frac{\rho_n^2(x)}{ax^2+bx} \right)^{\beta/2},
\end{aligned}$$

and hence the theorem.

In our next theorem, we discuss the direct estimate of the operators (2.4) using Lipschitz-type maximal function of order β *defined by Lenze (94) as follows:*

$$\omega_\beta(f,x) = \sup_{t \neq x, t \in [0,1]} \frac{|f(t) - f(x)|}{|t-x|^{\beta}}, \quad x \in [0,1], \tag{2.11}$$

where $\beta \in (0,1]$.

Theorem 2.1.10 *For* $f \in C[0,1]$, $x \in [0,1]$ *and* $0 < \beta \leq 1$, *we have:*

$$|L_n(f;x) - f(x)| \leq \omega_\beta(f,x) \left(\rho_n^2(x)\right)^{\beta/2}.$$

Proof 2.1.11 *Using (2.11), we have:*

$$\begin{aligned}
|L_n(f;x) - f(x)| \leq & L_n(|f(t) - f(x)|;x) \\
\leq & \omega_\beta(f,x) L_n\left(|t-x|^{\beta};x\right).
\end{aligned}$$

Applying Hölder's inequality:

$$\begin{aligned}
|L_n(f;x) - f(x)| \leq & \omega_\beta(f,x)(L_n((t-x)^2;x))^{\beta/2} \\
\leq & \omega_\beta(f,x)\left(\rho_n^2(x)\right)^{\beta/2}.
\end{aligned}$$

Now, we present an improved quantitative Voronovoskaya theorem for operators (2.4) involving second-order modulus of smoothness which gives a higher order of approximation than the one which is only in terms of first-order modulus of continuity (see (14)). For more on Voronovoskaya theorems, we refer to the readers the following articles (54; 55; 64; 133; 137).

Theorem 2.1.12 *For $n > 0$ and $f \in C^2[0,1]$, we obtain:*

$$\left| n\left[L_n(f;x) - f(x)\right] - \left(\frac{1-3x}{2}\right) f'(x) - \frac{1}{2}x(1-x)f''(x) \right|$$
$$\leq \frac{9}{32}\left\{\frac{68}{9\sqrt{n+1}}\omega_1(f'';h) + \omega_2(f'';h)\right\} + \frac{3}{2(n+1)}\left\{\|f'\|_\infty + \|f''\|_\infty\right\}.$$

Proof 2.1.13 *From [(64), Theorem 3], we have:*

$$\left| L_n(f;x) - f(x) - L_n((t-x);x)f'(x) - \frac{1}{2}L_n((t-x)^2;x)f''(x) \right|$$
$$\leq L_n((t-x)^2;x)\left\{\frac{\left|L_n((t-x)^3;x)\right|}{L_n((t-x)^2;x)}\frac{5}{6h}\omega(f'';h) + \left(\frac{3}{4} + \frac{\left|L_n((t-x)^4;x)\right|}{L_n((t-x)^2;x)}\frac{1}{16h^2}\right)\omega_2(f'';h)\right\}$$

Using Lemma 2.1.3, we obtain:

$$\frac{\left|L_n((t-x)^4;x)\right|}{L_n((t-x)^2;x)} \leq \frac{3(n+2)}{(n+1)^2} \quad \text{and} \quad \frac{\left|L_n((t-x)^3;x)\right|}{L_n((t-x)^2;x)} \leq \frac{17}{2(n+1)}.$$

Therefore the following inequalities hold:

$$\left| L_n(f;x) - f(x) - \left(\frac{-3x}{2(n+1)} + \frac{1}{2n}\right) f'(x) \right.$$
$$\left. - \frac{1}{2}\left(\frac{(-3n+13)}{3(n+1)(n+2)}x^2 + \frac{(3n^2-8n-4)}{3n(n+1)^2}x + \frac{1}{3n^2}\right) f''(x) \right|$$
$$\leq \left(\frac{(-3n+13)}{3(n+1)(n+2)}x^2 + \frac{(3n^2-8n-4)}{3n(n+1)^2}x + \frac{1}{3n^2}\right)$$
$$\times \left\{\frac{17}{2(n+1)}\frac{5}{6h}\omega(f'';h) + \left(\frac{3}{4} + \frac{3(n+2)}{(n+1)^2}\frac{1}{16h^2}\right)\omega_2(f'';h)\right\}.$$

Multiply both sides by n and taking $h = \frac{1}{\sqrt{n+1}}$

$$\left| n\left[L_n(f;x) - f(x)\right] - \left(\frac{1}{2} - \frac{3nx}{2(n+1)}\right) f'(x) \right.$$
$$\left. - \frac{1}{2}\left(\frac{n(-3n+13)}{3(n+1)(n+2)}x^2 + \frac{(3n^2-8n-4)}{3(n+1)^2}x + \frac{1}{3n}\right) f''(x) \right|$$
$$\leq x(1-x)\left\{\frac{85}{12\sqrt{n+1}}\omega(f'';h) + \frac{15}{16}\omega_2(f'';h)\right\}$$
$$\leq \frac{9}{32}\left\{\frac{68}{9\sqrt{n+1}}\omega(f'';h) + \omega_2(f'';h)\right\}.$$

We can write

$$\begin{aligned}
&\left| n\left[L_n(f;x)-f(x)\right]-\left(\frac{1-3x}{2}\right)f'(x)-\frac{1}{2}x(1-x)f''(x)\right| \\
&\leq \left| n\left[L_n(f;x)-f(x)\right]-\left(\frac{1}{2}-\frac{3nx}{2(n+1)}\right)f'(x)\right. \\
&\qquad \left. -\frac{1}{2}\left(\frac{n(-3n+13)}{3(n+1)(n+2)}x^2+\frac{(3n^2-8n-4)}{3(n+1)^2}x+\frac{1}{3n}\right)f''(x)\right| \\
&\qquad +\left|\frac{3x}{2(n+1)}f'(x)+\left(\frac{(11n+3)}{3(n+1)(n+2)}x^2-\frac{7}{6}\frac{(2n+1)}{(n+1)^2}x+\frac{1}{6n}\right)f''(x)\right| \\
&\leq \frac{9}{32}\left\{\frac{68}{9\sqrt{n+1}}\omega(f'';h)+\omega_2(f'';h)\right\}+\frac{3}{2(n+1)}\left\{\|f'\|_\infty+\|f''\|_\infty\right\}.
\end{aligned}$$

Example 2.1.14 *Let* $f(x) = x^2 - 4x + 1$, $x \in [0,1]$. *For* $n = 10, 20, 40, 100$ *the approximation of the function* f *by the proposed operators (2.4) is illustrated in Figure 2.1. In Table 2.1 the absolute error* $\xi_n = |L_n(f;x) - f(x)|$ *of the function* $f(x) = 4x^2 - 4x + 1$ *is computed for certain values of* x *in the interval* $[0,1]$ *and shown graphically in Figure 2.2.We can see that the error tends to zero with increase in parameter* n*. Clearly operators (2.4) converges to the function* $f(x)$ *for different choices of* n*.*

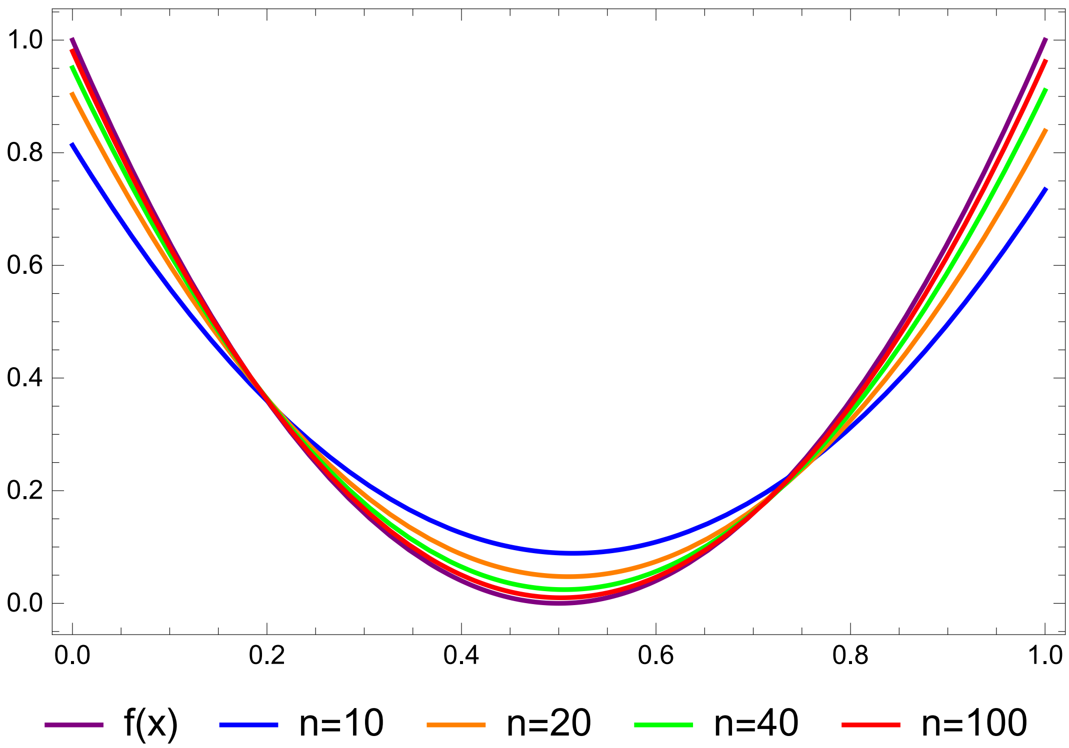

Figure 2.1: The convergence of operators $L_n(f;x)$ to the function $f(x) = 4x^2 - 4x + 1$ for $n = 10, 20, 40, 100$.

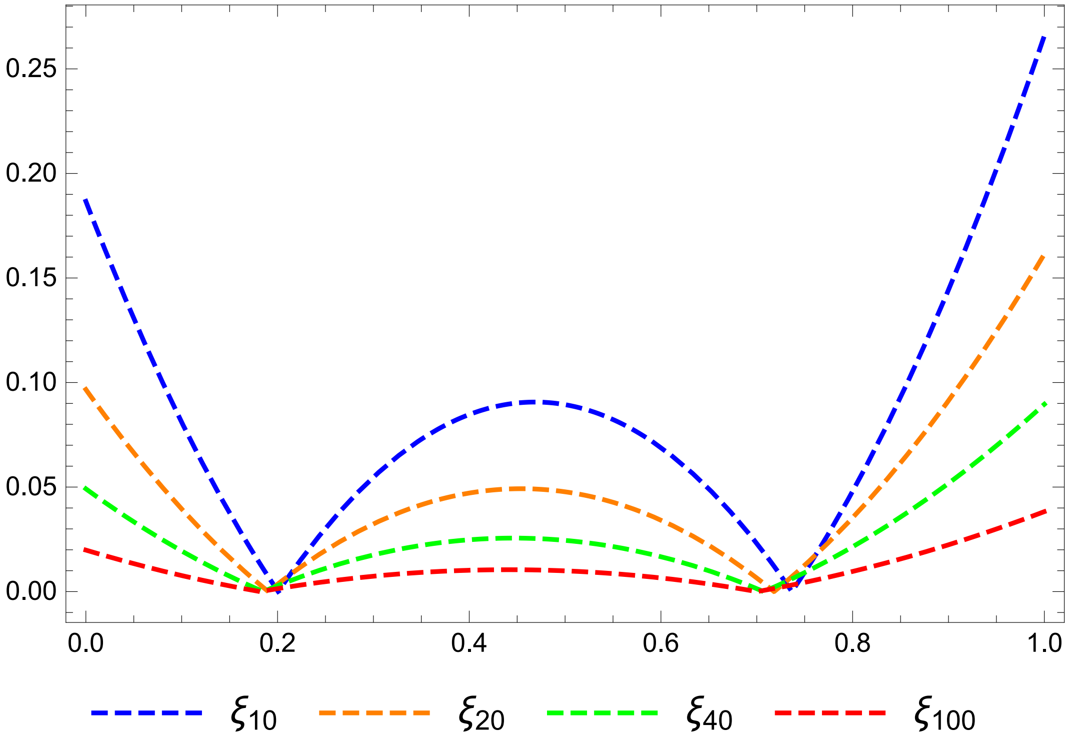

Figure 2.2: Graphical representation of absolute error of operators $L_n(f;x)$ for $f(x) = 4x^2 - 4x + 1$ for $n = 10, 20, 40, 100$.

Table 2.1: Absolute error between the operators (2.4) and function $f(x) = 4x^2 - 4x + 1$ for $n = 10, 20, 40, 100$.

x	ξ_{10}	ξ_{20}	ξ_{40}	ξ_{100}
0.04	0.084701	0.047133	0.024787	0.010207
0.12	0.062832	0.029769	0.014363	0.005604
0.20	0.000477	0.003516	0.002832	0.001408
0.28	0.045715	0.027751	0.015219	0.006425
0.36	0.075747	0.042935	0.022800	0.009445
0.44	0.089616	0.049069	0.025573	0.010470
0.52	0.087324	0.046152	0.023540	0.009497
0.60	0.068871	0.034185	0.016700	0.006529
0.68	0.034255	0.013167	0.005054	0.001564
0.76	0.016521	0.016901	0.011400	0.005396
0.84	0.083460	0.056020	0.032660	0.014354
0.92	0.166560	0.104190	0.058727	0.025307

2.1.4 Bivariate generalisation of Ismail-May-Kantorovich operators

In this section, we study the bivariate generalisation of the operators (2.4). In the past few years, researchers have constructed bivariate of several operators and investigated their approximation properties. Readers can refer to the following books and articles (12; 16; 23; 24; 28; 37; 56; 67; 71; 89) to study work related to this topic . The bivariate extension of the operators (2.4) is defined as:

$$L_{n_1n_2}^{k_1k_2}(f;x,y) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\wp_{n_1n_2}^{k_1k_2}(x,y)\int_{\frac{k_1}{n_1+k_1}}^{\frac{k_1+1}{n_1+k_1}}\int_{\frac{k_2}{n_2+k_2}}^{\frac{k_2+1}{n_2+k_2}} f(t_1,t_2)dt_1dt_2, \tag{2.12}$$

for $(x,y)\in I^2=[0,1]\times[0,1]$ *and* $n_1,n_2>0$, *where the basis functions is considered as:*

$$\wp_{n_1n_2}^{k_1k_2}(x,y) = e^{-(n_1x+n_2y)}\frac{n_1n_2(n_1+k_1)^{k_1}(n_2+k_2)^{k_2}}{k_1!\,k_2!}(xe^{-x})^{k_1}(ye^{-y})^{k_2}.$$

The bivariate generalisation of Ismail-May-Kantorovich operators can be rewritten as:

$$L_{n_1n_2}^{k_1k_2}(f;x,y) = L_{n_1,k_1}(f;x)\times L_{n_2,k_2}(f;y).$$

Lemma 2.1.15 *Let* $e_{rs}(t_1,t_2)=t_1^rt_2^s$, $0\le r+s\le 2$. *For* $(x,y)\in I^2=[0,1]\times[0,1]$ *and* $n_1,n_2\in\mathbb{N}$, *we have:*

$$\begin{aligned}
L_{n_1n_2}^{k_1k_2}(e_{00};x,y) &= 1;\\
L_{n_1n_2}^{k_1k_2}(e_{10};x,y) &= \frac{(2n_1-1)}{2(n_1+1)}x+\frac{1}{2n_1};\\
L_{n_1n_2}^{k_1k_2}(e_{01};x,y) &= \frac{(2n_2-1)}{2(n_2+1)}y+\frac{1}{2n_2};\\
L_{n_1n_2}^{k_1k_2}(e_{20};x,y) &= \frac{(3n_1{}^2-3n_1+1)}{3(n_1+1)(n_1+2)}x^2+\frac{(6n_1{}^2-2n_1-1)}{3n_1(n_1+1)^2}x+\frac{1}{3n_1{}^2};\\
L_{n_1n_2}^{k_1k_2}(e_{02};x,y) &= \frac{(3n_2{}^2-3n_2+1)}{3(n_2+1)(n_2+2)}y^2+\frac{(6n_2{}^2-2n_2-1)}{3n_2(n_2+1)^2}y+\frac{1}{3n_2{}^2}.
\end{aligned}$$

Remark 2.1.16 *Using Lemma 2.1.15, we have:*

$$\begin{aligned}
L_{n_1n_2}^{k_1k_2}((e_{10}-x);x,y) &= \frac{-3x}{2(n_1+1)}+\frac{1}{2n_1};\\
L_{n_1n_2}^{k_1k_2}((e_{01}-y);x,y) &= \frac{-3y}{2(n_2+1)}+\frac{1}{2n_2};\\
L_{n_1n_2}^{k_1k_2}\left((e_{10}-x)^2;x,y\right) &= \frac{(-3n_1+13)}{3(n_1+1)(n_1+2)}x^2+\frac{\left(3n_1{}^2-8n_1-4\right)}{3n_1(n_1+1)^2}x+\frac{1}{3n_1{}^2};\\
L_{n_1n_2}^{k_1k_2}\left((e_{01}-y)^2;x,y\right) &= \frac{(-3n_2+13)}{3(n_2+1)(n_2+2)}y^2+\frac{\left(3n_2{}^2-8n_2-4\right)}{3n_2(n_2+1)^2}y+\frac{1}{3n_2{}^2}.
\end{aligned}$$

Here we give the uniform convergence of Ismail-May-Kantorovich operators for bivariate functions.

Theorem 2.1.17 *For any $f \in C(I^2)$,the space of all continuous bivariate functions on the interval $I^2 = [0,1] \times [0,1]$, we have:*

$$\lim_{n_1,n_2\to\infty} \left\| L_{n_1 n_2}^{k_1 k_2}(f;x,y) - f(x,y) \right\| = 0.$$

Proof 2.1.18 *Using Lemma 2.1.15 and Volkov's Theorem (138), the theorem can be proved easily.*

Next, we establish the degree of approximation of operators (2.12) for $f \in C(I^2)$ in terms of total and partial modulus of continuity. The total modulus of continuity in case of bivariate functions is given by:

$$\omega_{total}(f;\delta_1,\delta_2) = \sup\{|f(t_1,t_2) - f(x,y)| : (t_1,t_2),(x,y) \in I^2, |t_1 - x| \leq \delta_1, |t_2 - y| \leq \delta_2\}.$$

Further, the partial moduli of continuity with respect to x and y is given as:

$$\omega^{(1)}(f;\delta) = \sup\{|f(x_1,y) - f(x_2,y)| : |x_1 - x_2| \leq \delta,\ y \in I\},$$

and

$$\omega^{(2)}(f;\delta) = \sup\{|f(x,y_1) - f(x,y_2)| : x \in I,\ |y_1 - y_2| \leq \delta\}.$$

Both total and partial modulus of continuity for bivariate functions satisfy the properties of usual modulus of continuity and can be studied more in (20).

Theorem 2.1.19 *Let $f \in C(I^2)$, then for all $(x,y) \in I^2$, the inequality:*

(i) $\left| L_{n_1 n_2}^{k_1 k_2}(f;x,y) - f(x,y) \right| \leq 4\omega_{total}(f;\delta_1,\delta_2)$;

(ii) $\left| L_{n_1 n_2}^{k_1 k_2}(f;x,y) - f(x,y) \right| \leq 2\left(\omega^{(1)}(f;\delta_1) + \omega^{(2)}(f;\delta_2)\right)$

holds true, where $\delta_1(x) = L_{n_1}\Big((t_1 - x)^2; x\Big)^{\frac{1}{2}}, \quad \delta_2(y) = L_{n_2}\Big((t_2 - y)^2; y\Big)^{\frac{1}{2}}.$

Proof 2.1.20 *(i) Taking into account monotonicity of ω_{total} and linearity of L_{n_1}, L_{n_2}, we can write*

$$\begin{aligned}
\left| L_{n_1 n_2}^{k_1 k_2}(f;x,y) - f(x,y) \right| &\leq L_{n_1 n_2}^{k_1 k_2}\left(|f(t_1,t_2) - f(x,y)| ; x, y \right) \\
&\leq \omega_{total}(f;\delta_1,\delta_2)\left(L_{n_1}(1;x) + \frac{1}{\delta_1} L_{n_1}\left(|t_1 - x| ; x \right) \right) \\
&\times \left(L_{n_2}(1;y) + \frac{1}{\delta_2} L_{n_2}\left(|t_2 - y| ; y \right) \right).
\end{aligned}$$

Applying Cauchy-Schwarz inequality, we have:

$$\left|L_{n_1n_2}^{k_1k_2}(f;x,y)-f(x,y)\right| \le \omega_{total}(f;\delta_1,\delta_2)\left(L_{n_1}(1;x)+\frac{1}{\delta_1}L_{n_1}\Big((t_1-x)^2;x\Big)^{\frac{1}{2}}\right)$$
$$\times\left(L_{n_2}(1;y)+\frac{1}{\delta_2}L_{n_2}\Big((t_2-y)^2;y\Big)^{\frac{1}{2}}\right)$$

Taking $\delta_1(x)$, $\delta_2(y)$ as in Remark 2.1.16 we get the desired result.

(ii) Considering the partial modulus of continuity of $f(x,y)$ and applying Cauchy-Schwarz inequality, we get:

$$\begin{aligned}\left|L_{n_1n_2}^{k_1k_2}(f;x,y)-f(x,y)\right| &\le L_{n_1n_2}^{k_1k_2}(|f(t_1,t_2)-f(x,y)|;x,y)\\
&\le L_{n_1n_2}^{k_1k_2}(|f(t_1,t_2)-f(x,t_2)|;x,y)\\
&\quad + L_{n_1n_2}^{k_1k_2}(|f(x,t_2)-f(x,y)|;x,y)\\
&\le L_{n_1n_2}^{k_1k_2}\Big(\omega^1(f;|t_1-x|);x,y\Big)\\
&\quad + L_{n_1n_2}^{k_1k_2}\Big(\omega^2(f;|t_2-y|);x,y\Big)\\
&\le \omega^1(f;\delta_{n_1})\left(1+\frac{1}{\delta_{n_1}}L_{n_1}(|t_1-x|;x)\right)\\
&\quad + \omega^2(f;\delta_{n_2})\left(1+\frac{1}{\delta_{n_2}}L_{n_2}(|t_2-y|;x)\right)\\
&\le \omega^1(f;\delta_{n_1})\left(1+\frac{1}{\delta_{n_1}}L_{n_1}\Big((t_1-x)^2;x\Big)^{\frac{1}{2}}\right)\\
&\quad + \omega^2(f;\delta_{n_2})\left(1+\frac{1}{\delta_{n_2}}L_{n_2}\Big((t_2-y)^2;x\Big)^{\frac{1}{2}}\right)\\
&\le 2\Big(\omega^1(f;\delta_1)+\omega^2(f;\delta_2)\Big).\end{aligned}$$

which completes the theorem.

Next we give a Voronovoskaya type estimate for bivariate operators (2.12).

Theorem 2.1.21 *Let $f(x)$ be a twice differentiable function on the interval I^2. Then we have:*

$$\lim_{n\to\infty} n\Big(L_{n,n}^{k_1,k_2}(f;x,y)-f(x,y)\Big) = \frac{(1-3x)}{2}f_x(x,y)+\frac{(1-3y)}{2}f_y(x,y)$$
$$+\,x(1-x)f_{xx}(x,y)+y(1-y)f_{yy}(x,y),$$

uniformly on I^2.

Proof 2.1.22 *Let $(x,y)\in I^2$ be arbitrary. Then by Taylor's theorem we have:*

$$\begin{aligned}f(t_1,t_2) =& f(x,y)+f_x(x,y)(t_1-x)+f_y(x,y)(t_2-y)\\
&+\frac{1}{2}\{f_{xx}(x,y)(t_1-x)^2+2f_{xy}(x,y)(t_1-x)(t_2-y)\\
&+f_{yy}(x,y)(t_2-y)^2\}+\psi(t_1,t_2;x,y)\sqrt{(t_1-x)^4+(t_2-y)^4}\end{aligned} \tag{2.13}$$

for $(t_1, t_2) \in I^2$*, where* $\psi(t_1, t_2; x, y) \in C(I^2)$ *and* $\psi(t_1, t_2; x, y) \to 0$ *as* $(t_1, t_2) \to (x, y)$*. Applying* $L_{n,n}^{k_1,k_2}(f; x, y)$ *on both sides of (2.13), we get:*

$$\begin{aligned} L_{n,n}^{k_1,k_2}(f; x, y) =& f(x, y) + f_x(x, y)L_{n,k}((t_1 - x); x) + f_y(x, y)L_{n,k}((t_2 - y); x) \\ &+ \frac{1}{2}\{f_{xx}(x, y)L_{n,k}((t_1 - x)^2; x) + f_{yy}(x, y)L_{n,k}((t_2 - y)^2; y) \\ &+ 2f_{xy}(x, y)L_{n,n}^{k_1,k_2}((t_1 - x)(t_2 - y); x, y)\} \\ &+ L_{n,n}^{k_1,k_2}\left(\psi(t_1, t_2; x, y)\sqrt{(t_1 - x)^4 + (t_2 - y)^4}; x, y\right). \end{aligned} \tag{2.14}$$

Now, using Hölder's inequality, we have:

$$\begin{aligned} &\left|L_{n,n}^{k_1,k_2}\left(\psi(t_1, t_2; x, y)\sqrt{(t_1 - x)^4 + (t_2 - y)^4}; x, y\right)\right| \\ &\leq \left\{L_{n,n}^{k_1,k_2}(\psi^2(t_1, t_2; x, y); x, y)\right\}^{1/2}\left\{L_{n,n}^{k_1,k_2}((t_1 - x)^4 + (t_2 - y)^4; x, y)\right\}^{1/2} \\ &\leq \left\{L_{n,n}^{k_1,k_2}(\psi^2(t_1, t_2; x, y); x, y)\right\}^{1/2}\left\{L_{n,k}((t_1 - x)^4; x) + L_{n,k}((t_2 - y)^4; y)\right\}^{1/2}. \end{aligned}$$

In view of Theorem 2.1.17, $L_{n,n}^{k_1,k_2}(\psi^2(t_1, t_2; x, y); x, y) \to 0$ *as* $n \to \infty$ *uniformly on* I^2 *and since* $L_{n,k}((t_1 - x)^4; x) = O\left(\frac{1}{n^2}\right)$*,* $L_{n,k}((t_2 - y)^4; x) = O\left(\frac{1}{n^2}\right)$*, therefore we have:*

$$\lim_{n\to\infty} nL_{n,n}^{k_1,k_2}\left(\psi(t_1, t_2; x, y)\sqrt{(t_1 - x)^4 + (t_2 - y)^4}; x, y\right) = 0,$$

uniformly for $(x, y) \in I^2$*. Also using Remark 2.1.16,*

$$\begin{aligned} &\lim_{n\to\infty} nL_{n,k}((t_1 - x); x) = \frac{1 - 3x}{2} \text{ and } \lim_{n\to\infty} n(L_{n,k}(t_2 - y); y) = \frac{1 - 3y}{2}, \\ &\lim_{n\to\infty} n(L_{n,k}(t_1 - x)^2; x) = x(1 - x) \text{ and } \lim_{n\to\infty} n(L_{n,k}(t_2 - y)^2; y) = y(1 - y). \end{aligned}$$

and

$$\lim_{n\to\infty} nL_{n,k}((t_1 - x); x)L_{n,k}((t_2 - y); y) = 0.$$

Combining the above estimates with (2.14), we get the desired result.

The next result establishes the rate of convergence of bivariate operators (2.12) with the help of Peetre's K- functional. Let $C^2(I^2)$ *denote the set of all continuous functions on* $I^2 = [0, 1] \times [0, 1]$*, whose first and second derivatives exist and are continuous on the interval* I^2*. We define the norm:*

$$\|h\|_{C^2(I^2)} = \|h\|_{C(I^2)} + \sum_{k=1}^{2}\left(\left\|\frac{\partial^k h}{\partial x^k}\right\|_{C(I^2)} + \left\|\frac{\partial^k h}{\partial y^k}\right\|_{C(I^2)}\right).$$

From Butzer and Berens (32), the Peetre's K-functional for $h \in C^2(I^2)$ *is defined as:*

$$K(h; \sigma) = \inf_{t\in C^2(I^2)}\left\{\|h - t\|_{C^2(I^2)} + \sigma\|t\|_{C^2(I^2)}, \sigma > 0\right\},$$

and the second order modulus of continuity is given as:

$$\omega^{(c)}\left(h,\ \sqrt{\sigma}\right)=\sup\left\{\left|\sum_{i=0}^{2}(-1)^{2-i}h\left(x+ix_0,y+iy_0\right)\right| : (x,y),(x+ix_0,y+iy_0)\in I^2,|x_0|\le\sigma,|y_0|\le\sigma\right\}.$$

Theorem 2.1.23 *For $f\in C(I^2)$, the following inequality holds true:*

$$\left|\widetilde{L}_{n_1,n_2}^{k_1,k_2}(g;x,y)-g(x,y)\right|\le 4K(f;P_{n_1,n_2}(x,y))+\omega^{(c)}\left(f;\ \sqrt{b_{n_1,n_2}(x,y)}\right),$$

where

$$P_{n_1,n_2}(x,y)=\delta_1(x)+\delta_2(y)+b_{n_1,n_2}(x,y),$$

and

$$b_{n_1,n_2}(x,y)=\left(\frac{1}{2n_1}-\frac{3x}{2(n_1+1)}\right)^2+\left(\frac{1}{2n_2}-\frac{3y}{2(n_2+1)}\right)^2.$$

Proof 2.1.24 *Before proceeding to the proof, we consider the following auxiliary operators:*

$$\widetilde{L}_{n_1,n_2}^{k_1,k_2}(g;x,y)=L_{n_1,n_2}^{k_1,k_2}(g;x,y)-g\left(\frac{1}{2n_1}-\frac{(1-2n_1)x}{2(n_1+1)},\frac{1}{2n_2}-\frac{(1-2n_2)y}{2(n_2+1)}\right)+g(x,y). \quad (2.15)$$

Then, using Remark 2.1.16, we have:

$$\widetilde{L}_{n_1,n_2}^{k_1,k_2}((t_1-x);x,y)=0,\ \ \widetilde{L}_{n_1,n_2}^{k_1,k_2}((t_2-y);x,y)=0. \quad (2.16)$$

Let $h\in C^2(I^2)$ and $t_1,t_2\in I$, using Taylor's theorem, we can write

$$\begin{aligned}h(t_1,t_2)-h(x,y)&=h(t_1,y)-h(x,y)+h(t_1,t_2)-h(t_1,y)\\&=\frac{\partial h(x,y)}{\partial x}(t_1-x)+\int_x^{t_1}(t_1-u)\frac{\partial^2 h(u,y)}{\partial u^2}du\\&+\frac{\partial h(x,y)}{\partial y}(t_2-y)+\int_y^{t_2}(t_2-v)\frac{\partial^2 h(x,v)}{\partial v^2}dv.\end{aligned} \quad (2.17)$$

Applying operators $\widetilde{L}_{n_1,n_2}^{k_1,k_2}$ on both sides of (2.17) *and using (2.16), we get:*

$$\begin{aligned}\widetilde{L}_{n_1,n_2}^{k_1,k_2}(h;x,y)-h(x,y)=&\widetilde{L}_{n_1,n_2}^{k_1,k_2}(\int_x^{t_1}(t_1-u)\frac{\partial^2 h(u,y)}{\partial u^2}du;x,y)\\&+\widetilde{L}_{n_1,n_2}^{k_1,k_2}(\int_y^{t_2}(t_2-v)\frac{\partial^2 h(x,v)}{\partial v^2}dv;x,y)\end{aligned}$$

Hence,

$$\begin{aligned}\left|\widetilde{L}_{n_1,n_2}^{k_1,k_2}(h;x,y)-h(x,y)\right| \leq & L_{n_1,n_2}^{k_1,k_2}\Big(\int_x^{t_1}|t_1-u|\left|\frac{\partial^2 h(u,y)}{\partial u^2}\right|du;x,y\Big)\\ &+\int_x^{\left(\frac{1}{2n_1}-\frac{(1-2n_1)x}{2(n_1+1)}\right)}\left|\frac{1}{2n_1}-\frac{(1-2n_1)\,x}{2(n_1+1)}-u\right|\left|\frac{\partial^2 h(u,y)}{\partial u^2}\right|du\\ &+L_{n_1,n_2}^{k_1,k_2}\Big(\int_y^{t_2}|t_2-v|\left|\frac{\partial^2 h(x,v)}{\partial v^2}\right|dv;x,y\Big)\\ &+\int_y^{\left(\frac{1}{2n_2}-\frac{(1-2n_2)y}{2(n_2+1)}\right)}\left|\frac{1}{2n_2}-\frac{(1-2n_2)\,y}{2(n_2+1)}-v\right|\left|\frac{\partial^2 h(x,v)}{\partial v^2}\right|dv\end{aligned}$$

$$\begin{aligned}&\leq\left\{L_{n_1,n_2}^{k_1,k_2}((t_1-x)^2;x,y)+\left(\frac{1}{2n_1}-\frac{(1-2n_1)\,x}{2(n_1+1)}-x\right)^2\right\}\|h\|_{C^2(I^2)}\\ &\quad+\{L_{n_1,n_2}^{k_1,k_2}((t_2-y)^2;x,y)+\left(\frac{1}{2n_2}-\frac{(1-2n_2)\,y}{2(n_2+1)}-y\right)^2\}\|h\|_{C^2(I^2)}.\\ &\leq\left\{\delta_1(x)+\delta_2(y)+\left(\frac{1}{2n_1}-\frac{3x}{2(n_1+1)}\right)^2+\left(\frac{1}{2n_2}-\frac{3y}{2(n_2+1)}\right)^2\right\}\|h\|_{C^2(I^2)}.\end{aligned}$$

Also

$$\left|\widetilde{L}_{n_1,n_2}^{k_1,k_2}(g;x,y)\right| \leq 3\|g\|_{C(I^2)}. \tag{2.18}$$

Now in view of (2.18), we can write

$$\begin{aligned}\left|\widetilde{L}_{n_1,n_2}^{k_1,k_2}(g;x,y)-g(x,y)\right| \leq & \left|\widetilde{L}_{n_1,n_2}^{k_1,k_2}(g-h;x,y)\right|\\ &+\left|\widetilde{L}_{n_1,n_2}^{k_1,k_2}(h;x,y)-h(x,y)\right|+|h(x,y)-g(x,y)|\\ &+\left|g\left(\frac{1}{2n_1}-\frac{(1-2n_1)\,x}{2(n_1+1)},\frac{1}{2n_2}-\frac{(1-2n_2)\,y}{2(n_2+1)}\right)-g(x,y)\right|\\ &\leq\left(4\|g-h\|_{C(I^2)}+P_{n_1,n_2}(x,y)\|h\|_{C^2(I^2)}\right)\\ &\quad+\omega^{(c)}\left(f;\ \sqrt{b_{n_1,n_2}(x,y)}\right).\end{aligned}$$

Taking the infimum on the right hand side over all $h \in C^2(I^2)$, it follows:

$$\left|\widetilde{L}_{n_1,n_2}^{k_1,k_2}(g;x,y)-g(x,y)\right| \leq 4K(f;P_{n_1,n_2}(x,y))+\omega^{(c)}\left(f;\ \sqrt{b_{n_1,n_2}(x,y)}\right),$$

which completes the proof.

2.2 On the preservation of functions with exponential growth by modified Ismail-May operators

In connection to exponential operators, Ismail-May (79) proved that for a polynomial p of degree $n \in \mathbb{N}$, kernel $S(n,x,t)$ can be uniquely determined. As a consequence of this, besides classifying some classical operators such as Szász-Mirakjan operators, classical Bernstein operators, Post-Widder operators etc. as exponential operators they also constructed some new exponential operators of degree greater than 2. For instance, if $p(x) = 2x^{3/2}$, the newly constructed operators are defined as:

$$\mathcal{L}_n(f;x) = e^{-n\sqrt{x}}\left\{f(0) + n\int_0^\infty e^{-nt/\sqrt{x}}t^{-1/2}\mathcal{I}_1\left(2n\sqrt{t}\right)f(t)\,dt\right\},$$

where $\mathcal{I}_n(y)$ is a first kind modified Bessel function identified as:

$$I_n(y) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(n+k)!}\left(\frac{y}{2}\right)^{n+2k}.$$

These operators were further studied in detail in (41). Again for $p(x) = x(1+x)^2$, the corresponding operators obtained are:

$$\mathcal{R}_n(f;x) = \sum_{k=0}^{\infty} r_{n,k}(x) f\left(\frac{k}{n+k}\right), \quad x \in (0,1),$$

where

$$r_{n,k}(x) = \frac{n(n+k)^{k-1}}{k!}e^{-nx}(xe^{-x})^k,$$

and $n > 0$. These operators were further studied in detail in (96; 110).

Another such operator for $p(x) = x^3$ is defined as:

$$\mathcal{P}_n(f,x) = \int_0^\infty k_n(x,t)f(t)dt, x \in (0,\infty) \tag{2.19}$$

whose kernel is defined as

$$k_n(x,t) = \left(\frac{n}{2\pi}\right)^{1/2} e^{n/x}t^{-3/2}exp\left(-\frac{nt}{2x^2} - \frac{n}{2t}\right).$$

These operators were studied further in detail by Gupta (66). All the three approximation processes cited above are examples of exponential operators as they satisfy the homogenous partial differential equation (1.13) and the normalization condition (1.14). In the past years, there have been several modifications of operators to enhance their convergence and error estimation process, see (40; 47; 116). In 2003, King (91) presented a sequence of linear positive operators which approximated each continuous function on $[0,1]$ while preserving the test function x^2. This remarkable approach has been since applied by many researchers to propose their modifications and fulfil the need to achieve better approximation. For instance, Duman and Özarslan (50) gave a modification of classical Szász operators to provide a better error estimation, Ozsarac and Acar (118) presented a new modification of Baskakov operators which preserve the functions $e^{\mu t}$ and $e^{2\mu t}$, $\mu > 0$, Bodur et al. (29) introduced a general class of Baskakov–Szász–Stancu operators preserving exponential functions, and many more. Readers can refer to these recent articles and books (9; 10; 63; 69; 134) for more such interesting papers related to this approach.

Inspired by the above-mentioned researches, we propose to construct a modification of the operators (2.19) which reproduce exponential functions. It is noteworthy that the resultant modified operators are not exponential operators. We begin with the following form of the operators (2.19), for functions $f \in C(0,\infty)$ we consider

$$\mathcal{B}_n(f;x) = \int_0^\infty l_n(x,t)f(t)dt, \quad x \in (0,\infty) \tag{2.20}$$

where

$$l_n(x,t) = \sqrt{\frac{n}{2\pi}} e^{n/\sigma_n(x)} t^{-3/2} \exp\left(-\frac{nt}{2(\sigma_n(x))^2} - \frac{n}{2t}\right).$$

Using calculation analogous to that given in (66), we can evaluate $\mathcal{B}_n\left(e^{at};x\right)$ as:

$$\mathcal{B}_n\left(e^{at};x\right) = \exp\left(\frac{n}{\sigma_n(x)}\left(1-\sqrt{\frac{n-2a(\sigma_n(x))^2}{n}}\right)\right), \tag{2.21}$$

which is the moment generating function of our proposed operators (2.20), and is used to find the moments and central moments used throughout this section of our chapter.

2.2.1 Preservation for e^{-x}

We begin with our proposed operators (2.20). Assuming that they preserve the exponential function e^{-x}, we can write $\mathcal{B}_n(e^{-t};x) = e^{-x}$ and therefore making use of (2.21), we get

$$e^{-x} = \exp\left(\frac{n}{\sigma_n(x)}\left(1-\sqrt{1+\frac{2(\sigma_n(x))^2}{n}}\right)\right).$$

Comparing exponents on either sides of the above equation and with easy manipulations, we obtain

$$\sigma_n(x) = \frac{2nx}{2n-x^2}.$$

Thus our proposed operators can be rewritten in the following form:

$$\mathcal{B}_n(f;x) = \int_0^\infty l_n(x,t) f(t)dt, \quad x \in (0,\infty) \tag{2.22}$$

where

$$l_n(x,t) = \left(\frac{n}{2\pi}\right)^{1/2} e^{(n/x - x/2)} t^{-3/2} exp\left(-\frac{\left(2n - x^2\right)^2 t}{8nx^2} - \frac{n}{2t}\right).$$

Lemma 2.2.1 *For all $x \in (0,\infty)$ and $n \in \mathbb{N}$, we have*

$$\mathcal{B}_n(e^{\gamma t};x) = \exp\left(\frac{2n - x^2}{2x}\left(1 - \sqrt{1 - \frac{8nx^2\gamma}{(2n - x^2)^2}}\right)\right),$$

which is also the moment generating function of the operators (2.22).

Lemma 2.2.2 *For the operators (2.22), if $e_v(t) = t^v$, $v = 0, 1, 2, ..,$ then the moments are as follows:*

$$\begin{aligned}
\mathcal{B}_n(e_0;x) &= 1,\\
\mathcal{B}_n(e_1;x) &= \sigma_n(x),\\
\mathcal{B}_n(e_2;x) &= \sigma_n^2(x) + \frac{\sigma_n^3(x)}{n},\\
\mathcal{B}_n(e_3;x) &= \sigma_n^3(x) + \frac{3\sigma_n^4(x)}{n} + \frac{3\sigma_n^5(x)}{n^2},\\
\mathcal{B}_n(e_4;x) &= \sigma_n^4(x) + \frac{6\sigma_n^5(x)}{n} + \frac{15\sigma_n^6(x)}{n^2} + \frac{15\sigma_n^7(x)}{n^3}.
\end{aligned}$$

Proof 2.2.3 *In view of moment generating function given in Lemma 2.2.1, the $r^{th}-$ moment of operators (2.22) is given by-*

$$\mathcal{B}_n^{[r]}(x) = \left[\frac{\partial^r}{\partial\gamma^r}\left\{\exp\left(\frac{2n - x^2}{2x}\left(1 - \sqrt{1 - \frac{8nx^2\gamma}{(2n - x^2)^2}}\right)\right)\right\}\right]_{\gamma=0}. \tag{2.23}$$

The expansion of (2.23) in terms of γ calculated using mathematica software is as follows:

$$\begin{aligned}
1 &+ \frac{2nx\gamma}{2n - x^2} + \frac{2n^2\left(2nx^2 - x^4 + 2x^3\right)\gamma^2}{(2n - x^2)^3}\\
&+ \frac{4\left(4n^5x^3 - 4n^4x^5 + 12n^4x^4 + n^3x^7 - 6n^3x^6 + 12n^3x^5\right)\gamma^3}{3(2n - x^2)^5}\\
&+ \frac{2\begin{pmatrix} 8n^7x^4 - 12n^6x^6 + 48n^6x^5 + 6n^5x^8 - 48n^5x^7 \\ +120n^5x^6 - n^4x^{10} + 12n^4x^9 - 60n^4x^8 + 120n^4x^7 \end{pmatrix}\gamma^4}{3(2n - x^2)^7} + O\left(\gamma^5\right).
\end{aligned}$$

Thus the r^{th}-moment of the operators (2.22) can be obtained by evaluating r^{th}-partial differentiation with respect to γ of the above expansion at $\gamma = 0$.

Lemma 2.2.4 *Let $\eta_{n,m}(x) = \mathcal{B}_n((t-x)^m; x), m = 1, 2,$ denote the central moments of operators (2.22), then*

$$\eta_{n,1}(x) = \sigma_n(x) - x,$$
$$\eta_{n,2}(x) = (\sigma_n(x) - x)^2 + \frac{\sigma_n^3(x)}{n}.$$

Proof 2.2.5 *Using the property of change of origin of moment generating functions*

$$e^{-\gamma x}\exp\left(\frac{2n-x^2}{2x}\left(1-\sqrt{1-\frac{8nx^2\gamma}{(2n-x^2)^2}}\right)\right),$$

Expanding this in terms of γ, we get

$$1-\frac{x^3\gamma}{x^2-2n}+\frac{\left(8n^2x^3+2nx^6-x^8\right)\gamma^2}{2\left(2n-x^2\right)^3}$$
$$+\frac{\left(48n^3x^6+96n^3x^5+4n^2x^9-24n^2x^8-4nx^{11}+x^{13}\right)\gamma^3}{6\left(2n-x^2\right)^5}$$
$$+\frac{\begin{pmatrix}384n^5x^6+192n^4x^9+576n^4x^8+1920n^4x^7+8n^3x^{12}\\-192n^3x^{11}-384n^3x^{10}-12n^2x^{14}+48n^2x^{13}+6nx^{16}-x^{18}\end{pmatrix}\gamma^4}{24(2n-x^2)^7}+O\left(\gamma^5\right).$$

The coefficient of $\gamma^m/m!$ in the above expansion is the $m^{th}-$order central moment of operators (2.22) .

Remark 2.2.6 *In view of Lemma 2.2.4, for adequately large n we have:*

1. $\lim_{n\to\infty} n\eta_{n,1}(x) = \frac{x^3}{2}$,
2. $\lim_{n\to\infty} n\eta_{n,2}(x) = x^3$,
3. $\lim_{n\to\infty} n^2\eta_{n,4}(x) = 3x^6$.

In (75), Holhoş defined modulus of continuity for exponential operators as:

$$\omega^*(f,\delta) = \sup_{|e^{-x}-e^{-t}|\le\delta} |f(x) - f(t)|, \quad x, t \ge 0.$$

and provided a quantitative result for sequence of linear positive operators on a class of real-valued continuous functions on $(0,\infty)$. These functions $f(x)$ have finite limit at infinity and are endowed with Chebyshev norm.
The defined modulus of continuity ϖ possess the following property:

$$|f(t) - f(x)| \le \left(1+\frac{(e^{-x}-e^{-t})^2}{\delta^2}\right)\omega^*(f,\delta). \tag{2.24}$$

The result by Holhoş (75) is given as:

Theorem 2.2.7 *(75) If $Q_n : C(0,\infty) \to C(0,\infty)$ satisfy the following inequality for $f_v(t) = e^{-vt}$, $v = 0, 1, 2$*

$$\|Q_n(f_v; x) - f_v(x)\|_\infty = \rho_v(n),$$

then for every $f \in C(0,\infty)$, we have

$$\|Q_n(f; x) - f(x)\|_\infty \leq \rho_0(n)\,\|f\|_\infty + (2 + \rho_0(n))\,\omega^*\left(f, \sqrt{\rho_0(n) + 2\rho_1(n) + \rho_2(n)}\right).$$

Theorem 2.2.8 *The sequence of modified exponential operators $\mathcal{B}_n : C(0,\infty) \to C(0,\infty)$ satisfy the following inequality for $f \in C(0,\infty)$:*

$$\|\mathcal{B}_n(f; x) - f(x)\|_\infty \leqslant 2\omega^*\left(f, \sqrt{\rho_2(n)}\right),$$

where $\rho_2(n)$ tends to zero for adequately large n.

Proof 2.2.9 *Since the operators preserve the constant as well as exponential function e^{-x}, so by Theorem 2.2.7, $\rho_0(n) = 0$ and $\rho_1(n) = 0$. We only need to evaluate $\rho_2(n)$. Next, from Lemma 2.2.1, we have*

$$\mathcal{B}_n\left(e^{-2t}; x\right) = \exp\left(\frac{2n - x^2}{2x}\left(1 - \sqrt{1 + \frac{16nx^2}{(2n - x^2)^2}}\right)\right).$$

Consider a sequence of functions

$$f_n(x) = \exp\left(\frac{2n - x^2}{2x}\left(1 - \sqrt{1 + \frac{16nx^2}{(2n - x^2)^2}}\right)\right) - e^{-2x}.$$

As $f_n(x)$ vanishes at end points of $(0,\infty)$, therefore there exists a point $\vartheta_n \in (0,\infty)$ such that

$$\|f_n\|_\infty = f_n(\vartheta_n).$$

Also the derivative of the sequence of functions vanishes at ϑ_n i.e. $f'_n(\vartheta_n) = 0$. Making use of mathematica software, this gives

$$\frac{2n + \vartheta_n^2}{2\vartheta_n^2}\left(\frac{-1}{\sqrt{1 + \frac{16n\vartheta_n^2}{(2n - \vartheta_n^2)^2}}} + 1\right)\exp\left(\frac{2n - \vartheta_n^2}{2\vartheta_n}\left(1 - \sqrt{1 + \frac{16n\vartheta_n^2}{(2n - \vartheta_n^2)^2}}\right)\right) = 2e^{-2\vartheta_n}.$$

Therefore, we have

$$\begin{aligned}\|f_n\|_\infty = &\exp\left(\frac{2n - \vartheta_n^2}{2\vartheta_n}\left(1 - \sqrt{1 + \frac{16n\vartheta_n^2}{(2n - \vartheta_n^2)^2}}\right)\right)\\ &\times\left(2 - \frac{2n + \vartheta_n^2}{2\vartheta_n^2}\left(\frac{-1}{\sqrt{1 + \frac{16n\vartheta_n^2}{(2n - \vartheta_n^2)^2}}} + 1\right)\right) = \rho_2(n) \to 0,\end{aligned}$$

as $n \to \infty$. Thus in view of Theorem 2.2.7, we get the required result.

Let $C_B(0,\infty)$ denote the space of all real valued continuous and bounded functions equipped with the Chebyshev norm and let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g\in C_B^2((0,\infty))} \{\|f-g\| + \delta\|g''\|, \delta > 0\},$$

where $C_B^2(0,\infty) = \{g \in C_B(0,\infty) : g', g'' \in C_B(0,\infty)\}$.

Theorem 2.2.10 *Let $f \in C_B(0,\infty)$. We define auxiliary operators:*

$$\hat{\mathcal{T}}_n(f;x) = \mathcal{B}_n(f;x) - f(\sigma_n(x)) + f(x), \tag{2.25}$$

then, there exists a constant $R > 0$ such that

$$|\mathcal{B}_n(f;x) - f(x)| \leqslant R\omega_2(f, \sqrt{\delta}) + \omega(f, \sigma_n(x) - x),$$

where

$$\delta = \eta_{n,2}(x) + (\sigma_n(x) - x)^2.$$

Proof 2.2.11 *Let $g \in C_B^2(0,\infty)$ and $x, t \in (0,\infty)$, then by application of Taylor's expansion, we have*

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Using (2.25) and the fact that $\hat{\mathcal{T}}_n((t-x);x) = 0$, we have

$$\begin{aligned}
\left|\hat{\mathcal{T}}_n(g;x) - g(x)\right| &= \left|\hat{\mathcal{T}}_n\left(\int_x^t (t-u)g''(u)du; x\right)\right| \\
&\leqslant \left|\mathcal{B}_n\left(\int_x^t (t-u)g''(u)du, x\right)\right| + \left|\int_x^{\sigma_n(x)} (\sigma_n(x) - u)\, g''(u)du\right| \\
&\leqslant \left(\eta_{n,2}(x) + (\sigma_n(x) - x)^2\right)\|g''\|.
\end{aligned} \tag{2.26}$$

Also, we have

$$|\mathcal{B}_n(f;x)| \leqslant \|f\|. \tag{2.27}$$

Combining (2.25), (2.26) and (2.27), we get

$$\begin{aligned}
|\mathcal{B}_n(f;x) - f(x)| &\leq \left|\hat{\mathcal{T}}_n(f-g);x) - (f-g)(x)\right| + \left|\hat{\mathcal{T}}_n(g;x) - g(x)\right| + |f(x) - f(\sigma_n(x))| \\
&\leqslant 2\|f-g\| + \left(\eta_{n,2}(x) + (\sigma_n(x) - x)^2\right)\|g''\| + \omega(f, \sigma_n(x) - x).
\end{aligned}$$

Taking infimum over all $g \in C_B^2(0,\infty)$ and using the relation given in (44), $K_2(f,\delta) \leqslant C\omega_2\left(f, \sqrt{\delta}\right)$, $\delta > 0$, we get the desired result.

One of the most important pointwise convergence results in the theory of approximation by linear positive operators are Voronovoskaya type theorems and its quantitative form. These theorems are extensively studied for various operators as well as for functions belonging to different spaces. In our next theorem, we present a quantitative Voronovskaya type theorem for the proposed operators (2.22).

Theorem 2.2.12 *Let $f \in C_B(0,\infty)$ with continuous first and second derivative exist. Then for $x \in (0,\infty)$, the following inequality holds:*

$$\left| n\left[\mathcal{B}_n(f;x) - f(x)\right] - \frac{x^3}{2}\left(f'(x) + f''(x)\right)\right|$$
$$\leq |u_n(x)|\,|f'(x)| + |v_n(x)|\,|f''(x)| + \frac{x^3}{2}\varpi\left(f'', \frac{1}{\sqrt{n}}\right)\left(1 + 3x^3e^{-2x}\right).$$

where $u_n(x) = n\eta_{n,1}(x) - \frac{x^3}{2}$ and $v_n(x) = \frac{1}{2}\left(n\eta_{n,2}(x) - x^3\right)n\eta_{n,2}(x) - x^3$.

Proof 2.2.13 *By the Taylor's expansion, we have*

$$f(t) = \sum_{i=0}^{2}(t-x)^i\frac{f^{(i)}(x)}{i!} + \Theta(t,x)(t-x)^2, \tag{2.28}$$

where $\Theta(t,x)$ is a continuous function given by:

$$\Theta(t,x) = \frac{f''(\Im) - f''(x)}{2}, \quad \Im \in (x,t).$$

Applying the operators $\mathcal{B}_n$ to the inequality (2.28), we can write

$$\mathcal{B}_n(f;x) - \sum_{i=0}^{2}\eta_{n,i}(x)\frac{f^{(i)}(x)}{i!} = \mathcal{B}_n\left(\Theta(t,x)(t-x)^2;x\right).$$

Therefore using Remark 2.2.6, we get

$$\left| n\left[\mathcal{B}_n(f;x) - f(x)\right] - \frac{x^3}{2}\left(f'(x) + f''(x)\right)\right|$$
$$\leq \left|n\eta_{n,1}(x) - \frac{x^3}{2}\right||f'(x)| + \frac{1}{2}\left|n\eta_{n,2}(x) - x^3\right||f''(x)| + \left|n\mathcal{B}_n\left(\Theta(t,x)(t-x)^2;x\right)\right|$$
$$\leq |u_n(x)|\,|f'(x)| + |v_n(x)|\,|f''(x)| + \left|n\mathcal{B}_n\left(\Theta(t,x)(t-x)^2;x\right)\right|, \tag{2.29}$$

where $u_n(x) = n\eta_{n,1}(x) - \frac{x^3}{2} \to 0$ and $v_n(x) = \frac{1}{2}\left(n\eta_{n,2}(x) - x^3\right)n\eta_{n,2}(x) - x^3 \to 0$ in accordance with Remark 2.2.6, for adequately large n.
Using the Property 2.24 of modulus of continuity defined by Holhoş (75), we get

$$|\Theta(t,x)| \leq \frac{1}{2}\left(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\right)\omega^*(f'',\delta).$$

Hence, after applying Cauchy-Schwarz inequality to the last part of (2.29), we get

$$n\mathcal{B}_n\left(|\Theta(t,x)|(t-x)^2;x\right) \leq \frac{n}{2}\omega^*(f'',\delta)\,\eta_{n,2}(x) + \frac{n}{2\delta^2}\omega^*(f'',\delta)\sqrt{\mathcal{B}_n\left((e^{-x}-e^{-t})^4;x\right)}\sqrt{\eta_{n,4}(x)}.$$

Choosing $\delta = n^{-1/2}$,

$$n\mathcal{B}_n\left(|\Theta(t,x)|(t-x)^2;x\right) \leq \frac{1}{2}\omega^*\left(f'',\frac{1}{\sqrt{n}}\right)\left[n\eta_{n,2}(x) + \sqrt{n^2\mathcal{B}_n\left((e^{-x}-e^{-t})^4;x\right)}\sqrt{n^2\eta_{n,4}(x)}\right].$$

In view of (2.29) and Remark 2.2.6, we obtain the desired result.

Corollary 2.2.14 *Let* $f \in C_B^2(0,\infty)$*, then for* $x \in (0,\infty)$ *we have*

$$\lim_{n\to\infty} n\left[\mathcal{B}_n(f;x) - f(x)\right] = \frac{x^3}{2}\left[f'(x) + f''(x)\right]$$

Remark 2.2.15 *The convergence of the modified operators* $\mathcal{B}_n$ *in the above theorem takes place for sufficiently large n. Using Lemma 2.2.1 for* $\gamma = -1,-2,-3,-4$ *and mathematica software, we get*

$$\begin{aligned}
&\lim_{n\to\infty} n^2\mathcal{B}_n\left((e^{-x}-e^{-t})^4;x\right)\\
&= \lim_{n\to\infty} n^2\left[\mathcal{B}_n\left(e^{-4t};x\right) - 4e^{-x}\mathcal{B}_n\left(e^{-3t};x\right) + 6e^{-2x}\mathcal{B}_n\left(e^{-2t};x\right)\right.\\
&\quad\left. -4e^{-3x}\mathcal{B}_n(e^{-t};x) + e^{-4x}\right]\\
&= \lim_{n\to\infty} n^2\left[\exp\left(\frac{2n-x^2}{2x}\left(1-\sqrt{1+\frac{32nx^2}{(2n-x^2)^2}}\right)\right) - 4e^{-x}\exp\left(\frac{2n-x^2}{2x}\left(1-\sqrt{1+\frac{24nx^2}{(2n-x^2)^2}}\right)\right)\right.\\
&\left. +6e^{-2x}\exp\left(\frac{2n-x^2}{2x}\left(1-\sqrt{1+\frac{16nx^2}{(2n-x^2)^2}}\right)\right) - 4e^{-3x}\exp\left(\frac{2n-x^2}{2x}\left(1-\sqrt{1+\frac{8nx^2}{(2n-x^2)^2}}\right)\right)\right]\\
&= 3e^{-4x}x^6
\end{aligned}$$

2.2.2 The case of e^{Ax}, $A \in \mathbb{R}$

In this section, we present a more general form of the operators (2.20) that reproduces both constants and exponential functions of the form e^{Ax}, $A \in \mathbb{R}$. We observe that the modified operators possess faster and better rate of convergence as compared to the original operators (2.20) for $A > 0$. To validate our results, we exhibit some graphical representations with the aid of numerical examples and compare the rate of convergence of both original and the modified operators.

Taking into consideration operators (2.20) again and assuming they reproduce functions of the form e^{Ax}, i.e $\mathcal{B}_n(e^{At};x)=e^{Ax}$, we obtain

$$\sigma_n(x)=\frac{2nx}{2n+Ax^2}.$$

Operators (2.20) therefore now take the following form:

$$\mathcal{B}_n^A(f;x)=\int_0^\infty l_n(x,t)f(t)dt,\quad x\in(0,\infty) \tag{2.30}$$

where

$$l_n(x,t)=\left(\frac{n}{2\pi}\right)^{1/2}e^{(n/x+Ax/2)}t^{-3/2}\exp\left(-\frac{\left(2n+Ax^2\right)^2}{8nx^2}-\frac{n}{2t}\right).$$

Lemma 2.2.16 *For all $x\in(0,\infty)$ and $n\in\mathbb{N}$, we have*

$$\mathcal{B}_n^A\left(e^{At};x\right)=\exp\left(\frac{2n+Ax^2}{2x}\left(1-\sqrt{1-\frac{8nAx^2}{(2n+Ax^2)^2}}\right)\right),$$

$$\mathcal{B}_n^A\left(te^{At};x\right)=\frac{2nx}{\sqrt{(2n+Ax^2)^2-8nAx^2}}\exp\left(\frac{2n+Ax^2}{2x}\left(1-\sqrt{1-\frac{8nAx^2}{(2n+Ax^2)^2}}\right)\right),$$

$$\mathcal{B}_n^A\left(t^2e^{At};x\right)=\frac{4n^2}{\left((2n+Ax^2)^2-8nAx^2\right)}\left[\frac{2x^3}{\sqrt{(2n+Ax^2)^2-8nAx^2}}+x^2\right]\times\exp\left(\frac{2n+Ax^2}{2x}\left(1-\sqrt{1-\frac{8nAx^2}{(2n+Ax^2)^2}}\right)\right).$$

Proof 2.2.17 *The quantities $\mathcal{B}_n^A\left(te^{At};x\right)$ and $\mathcal{B}_n^A\left(t^2e^{At};x\right)$ are obtained simply by successively partially differentiating $\mathcal{B}_n^A\left(e^{At};x\right)$ with respect to A on both sides.*

Lemma 2.2.18 *For $e_r(x)=x^r, r\in\mathbb{N}\cup\{0\}$, the operators (2.30) hold the following moments:*

$$\mathcal{B}_n^A(e_0;x)=1,$$

$$\mathcal{B}_n^A(e_1;x)=\frac{2nx}{2n+Ax^2},$$

$$\mathcal{B}_n^A(e_2;x)=\frac{4n^2\left(Ax^4+2nx^2+2x^3\right)}{\left(Ax^2+2n\right)^3},$$

$$\mathcal{B}_n^A(e_3;x)=\frac{8n^3\left(A^2x^7+4Anx^5+6Ax^6+4n^2x^3+12nx^4+12x^5\right)}{\left(2n+Ax^2\right)^5},$$

$$\mathcal{B}_n^A(e_4;x)=\frac{16n^4\begin{pmatrix}A^3x^{10}+6A^2nx^8+12A^2x^9+12An^2x^6+48Anx^7\\+60Ax^8+8n^3x^4+48n^2x^5+120nx^6+120x^7\end{pmatrix}}{\left(2n+Ax^2\right)^7}.$$

Lemma 2.2.19 *Let $\eta_{n,m}^{A}(x) = \mathcal{B}_n^A((t-x)^m; x), m = 1, 2, 4,$ denote the central moments of operators (2.30). Then, it can be verified:*

$$\eta_{n,1}^{A}(x) = -\frac{Ax^3}{2n+Ax^2},$$

$$\eta_{n,2}^{A}(x) = \frac{\left(8n^2x^3+2A^2nx^6+A^3x^8\right)}{\left(2n+Ax^2\right)^3},$$

$$\eta_{n,4}^{A}(x) = \frac{\left(-A^5x^{13}-4A^4nx^{11}-4A^3n^2x^9-24A^2n^2x^8-48An^3x^6+96n^3x^5\right)}{\left(2n+Ax^2\right)^5}.$$

Moreover, in view of Lemma 2.2.19, we note that

$$\frac{\eta_{n,4}^{A}(x)}{\eta_{n,2}^{A}(x)} = \frac{\left(-\frac{A^5x^{13}}{n^5} - \frac{4A^4x^{11}}{n^4} - \frac{4A^2x^8(Ax+6)}{n^3} - \frac{48x^5}{n^2}(Ax-2)\right)}{\left(2+\frac{Ax^2}{n}\right)^2\left(8+\frac{2A^2x^6}{n^2}+\frac{A^3x^8}{n^3}\right)}.$$

Therefore, one can observe that for a fixed $x \in [0, \infty)$ and sufficiently large n, the ratio

$$\frac{\eta_{n,4}^{A}(x)}{\eta_{n,2}^{A}(x)} \to 0$$

with order of convergence $O\left(n^{-2}\right)$.

In order to prove our next theorem, let us define a space $\mathcal{S}$ of all functions having exponential growth of order A endowed with norm:

$$\|f\|_A = \sup_{x\in(0,\infty)} \left|f(x)e^{-Ax}\right| < \infty.$$

Let for some $0 \leq \alpha < 1$, $Lip(\alpha, A)$ be the space containing all those functions f which satisfy $\varpi(f, \delta, A) \leq M\delta^\alpha$, where ϖ is the first order modulus of continuity defined in (48) as:

$$\varpi(f, \delta, A) \leq \sup_{h<\delta, x\in(0,\infty)} |f(x) - f(x+h)| e^{-Ax},$$

and for every positive number $h > 0$ and $k \in \mathbb{N}$ has the following property:

$$\varpi(f, kh, A) \leq k.e^{A(k-1)h}.\omega_1(f, h, A). \tag{2.31}$$

Theorem 2.2.20 *Let $\mathcal{B}_n^A : S \to (0, \infty)$. If $f \in C_B^2(0, \infty) \cap \mathcal{S}$ and $f'' \in Lip(\alpha, A)$, then for fixed $x \in (0, \infty)$ and $n > 2Ax$, we have*

$$\left|\mathcal{B}_n^A(f; x) - f(x) - \eta_{n,1}^{A}(x) f'(x) - \eta_{n,2}^{A}(x)\frac{f''(x)}{2}\right|$$
$$\leq \frac{1}{2}\varpi\left(f'', \sqrt{\frac{\eta_{n,4}^{A}(x)}{\eta_{n,2}^{A}(x)}}, A\right)\left[2e^{2Ax} + M(A, x) + \sqrt{M(2A, x)}\right]\eta_{n,2}^{A}(x),$$

where $M(A, x) = \frac{\left(2+Ax^2\right)^2}{\left(2-Ax^2\right)^3}e^{2Ax}$ is a constant independent of n but dependent on A and x.

Proof 2.2.21 *By Taylor's expansion, we have*

$$f(t) = f(x) + (t-x) f'(x) + \frac{(t-x)^2}{2!} f''(x) + \Theta_2(t,x), \tag{2.32}$$

where

$$\Theta_2(t,x) = \frac{f''(\tau) - f''(x)}{2}(t-x)^2.$$

such that τ lies between x and t and $\Theta_2(t,x)$ is a continuous function which vanishes as t approaches x.

Applying the operator $\mathcal{B}_n^A$ on (2.32) and in view of Lemma 2.2.19, we have:

$$\left|\mathcal{B}_n^A(f;x) - f(x) - \eta_{n,1}^A(x) f'(x) - \eta_{n,2}^A(x) f''(x)\right| \leq \mathcal{B}_n^A(|\Theta_2(t,x)|;x). \tag{2.33}$$

Using the Property 2.31 of exponential modulus of continuity and with calculations we have the relation

$$\mathcal{B}_n^A(|\Theta_2(t,x)|;x) \leq \frac{\varpi(f'',h,A)}{2}\left[\mathcal{B}_n^A\left(\left(e^{2Ax} + e^{At}\right).\left(|t-x|^2 + \frac{|t-x|^3}{h}\right);x\right)\right]. \tag{2.34}$$

Taking x fixed and $n > 2Ax$, we have

$$\begin{aligned}
\mathcal{B}_n^A\left((t-x)^2 e^{At};x\right) &= \left[\left(\frac{2x^3}{\sqrt{(2n+Ax^2)^2 - 8nAx^2}} + x^2\right)\frac{4n^2}{(2n+Ax^2)^2 - 8nAx^2}\right.\\
&\qquad \left. - \frac{4x^2 n}{\sqrt{(2n+Ax^2)^2 - 8nAx^2}} + x^2\right] e^{Ax}\\
&\leq \frac{1}{\left(1 - \frac{8nAx^2}{(2n+Ax^2)^2}\right)^{3/2}}\left[\frac{8n^2x^3}{(2n+Ax^2)^3} + x^2\left(\sqrt{1 - \frac{8nAx^2}{(2n+Ax^2)^2}}\right.\right.\\
&\qquad \left.\left. -4n\left(1 - \frac{8nAx^2}{(2n+Ax^2)^2}\right) + 1\right)\right] e^{Ax}\\
&\leq \frac{1}{\left(1 - \frac{8nAx^2}{(2n+Ax^2)^2}\right)^{3/2}}\left[\frac{8n^2x^3 + 2nA^2x^6 + A^3x^8}{(2n+Ax^2)^3} + x^2\right.\\
&\qquad \left. \times\left(1 - 4n\left(1 - \frac{8nAx^2}{(2n+Ax^2)^2}\right) + 1\right)\right] e^{Ax}\\
&\leq \frac{1}{\left(1 - \frac{8nAx^2}{(2n+Ax^2)^2}\right)^{3/2}}\left[\frac{8n^2x^3 + 2nA^2x^6 + A^3x^8}{(2n+Ax^2)^3} + 2x^2\right] e^{Ax}\\
&\leq \frac{\left(2 + Ax^2\right)^2}{(2 - Ax^2)^3} e^{2Ax}\eta_{n,2}^A(x) \leq M(A,x)\,\eta_{n,2}^A(x),
\end{aligned} \tag{2.35}$$

where

$$M(A,x)=\frac{\left(2+Ax^2\right)^2}{\left(2-Ax^2\right)^3}e^{2Ax}.$$

Moreover using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\mathcal{B}_n^A\left(|t-x|^3e^{At};x\right) &\leq \sqrt{\mathcal{B}_n^A\left((t-x)^2e^{2At};x\right)}.\sqrt{\eta_{n,4}^A(x)}\\ &\leq \sqrt{M(2A,x)\,\eta_{n,2}^A(x)}.\sqrt{\eta_{n,4}^A(x)}. \end{aligned}\tag{2.36}$$

Combining (2.33), (2.35) and (2.36) and substituting $h=\sqrt{\frac{\eta_{n,4}^A(x)}{\eta_{n,2}^A(x)}}$ *in (2.34), we get*

$$\mathcal{B}_n^A\left(|\Theta_2(t,x)|;x\right)\leq\frac{1}{2}\varpi\left(f'',\sqrt{\frac{\eta_{n,4}^A(x)}{\eta_{n,2}^A(x)}},A\right)\left[2e^{2Ax}+M(A,x)+\sqrt{M(2A,x)}\right]\eta_{n,2}^A(x),$$

and hence the theorem.

Remark 2.2.22 *One can easily observe in the above theorem,*

- *Second central moment* $\eta_{n,4}^A(x)$ *of the proposed operators (2.30) is smaller for* $A>0$ *and* $x>\frac{1}{2A}$ *as compared to that of original operators (2.19).*
- *For* $A>0$, *the ratio* $\frac{\eta_{n,4}^A(x)}{\eta_{n,2}^A(x)}$ *is higher for the original operators as compared to that for our modified operators.*
- *In addition, the constant* $M(A,x)$ *which is independent of* n *is also significantly reduced for our modified exponential operators if we take* $A>0$.

Thus judging on the basis of above mentioned rationales, we can say that Theorem 2.2.20 is an improved version of (66, Theorem 4) for $A>0$.

Corollary 2.2.23 *Let* $f,f''\in\mathcal{S}$ *and* $A>0$, *then for any* $x\in(0,\infty)$, *we have*

$$\lim_{n\to\infty}n\left[\mathcal{B}_n^A(f;x)-f(x)\right]=\frac{x^3}{2}\left[-Af'(x)+f''(x)\right].$$

Remark 2.2.24 *The advantage of Corollary 2.2.23 over Corollary 2.2.14 is in the fact that latter is defined for a larger function space* $\mathcal{S}$ *while the former is only for* $(0,\infty)$.

2.2.3 Conclusion

We now conclude that our proposed operators (2.30) is an improved approximation operator which not only preserves constant and exponential functions e^{Ax} and but in fact also provides faster convergence and better approximation for some functions in comparison to the original exponential operators for $A > 0$. Here we have shown properties which are superior to that of original operators and work for a much wider function spaces. To validate our statements, we exhibit some figures based on numerical examples to show a faster rate of convergence of our modified operators and also its comparison with the original operators (2.19) for arbitrarily chosen values of n and $A > 0$.

2.2.4 Graphical Comparisons

Example 2.2.25 *Let $f(x) = 5x(sinh(x))$. Then we have the following graphical representations where our function $f(x)$ is represented in purple color throughout.*

a) *Figure 2.3 exhibits the comparison between the modified operator $\mathcal{B}_n^A$ (Cyan), and the original operator $\mathcal{P}_n$ (Red) for $n = 10, A = 1$.*

b) *Figure 2.4 exhibits the comparison between the modified operator $\mathcal{B}_n^A$ (Magenta) and the original operator $\mathcal{P}_n$ (Blue) for $n = 50, A = 1$.*

c) *Figure 2.5 shows the rate of convergence of the modified operators $\mathcal{B}_n^A$ for $A = 1$ and $n = 10$ (Cyan), $n = 25$ (Orange) and $n = 50$ (Magenta), towards the function $f(x)$. The graph clearly shows faster rate of convergence.*

After analyzing Figure 2.3 and Figure 2.4, it can be easily concluded that operators $\mathcal{B}_n^A$ provide faster rate of convergence and therefore better approximation as compared to the operators $\mathcal{P}_n$ for $A > 0$.

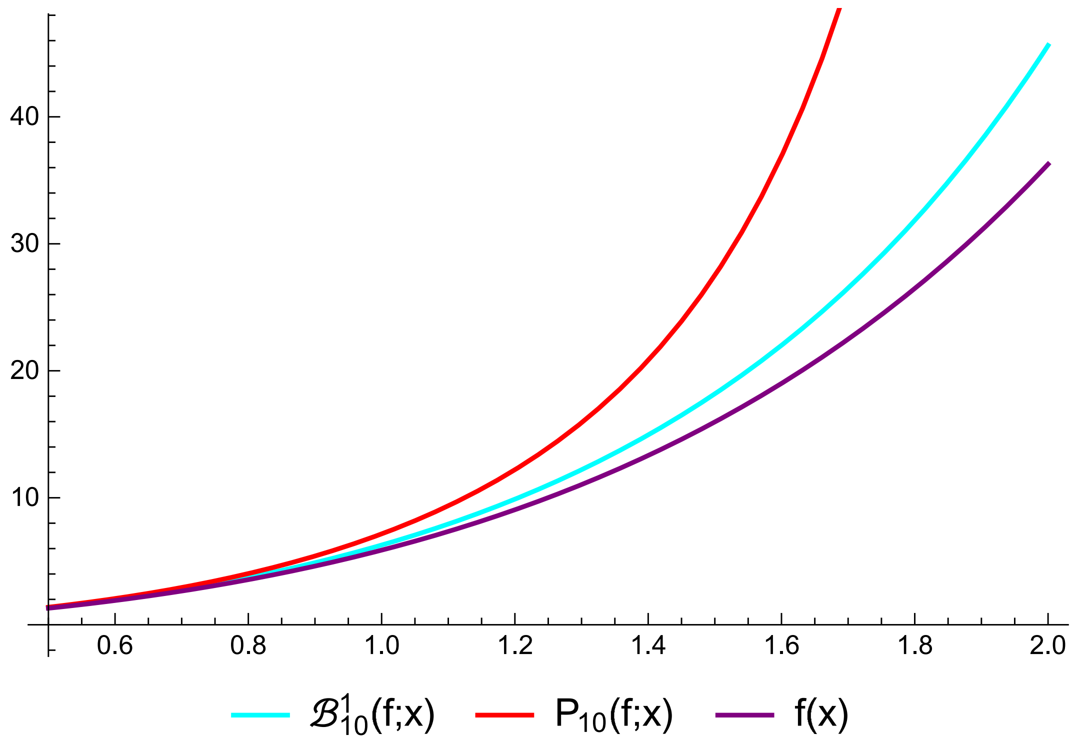

Figure 2.3: Comparison between convergence of operators $\mathcal{B}^A_{10}$ (Cyan), $\mathcal{P}_{10}$ (Red) towards function $f(x)$ (Purple) for $n = 10$ and $A = 1$.

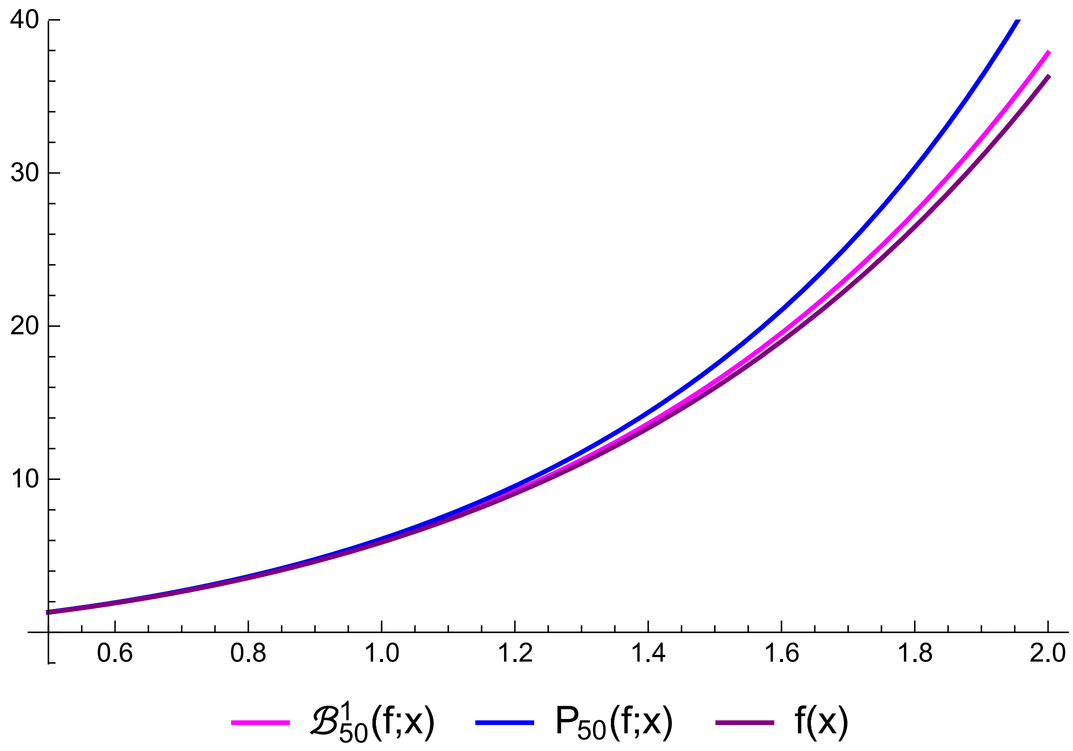

Figure 2.4: Comparison between convergence of operators $\mathcal{B}^A_{50}$ (Magenta), $\mathcal{P}_{50}$ (Blue) towards function $f(x)$ (Purple) for $n = 50$ and $A = 1$.

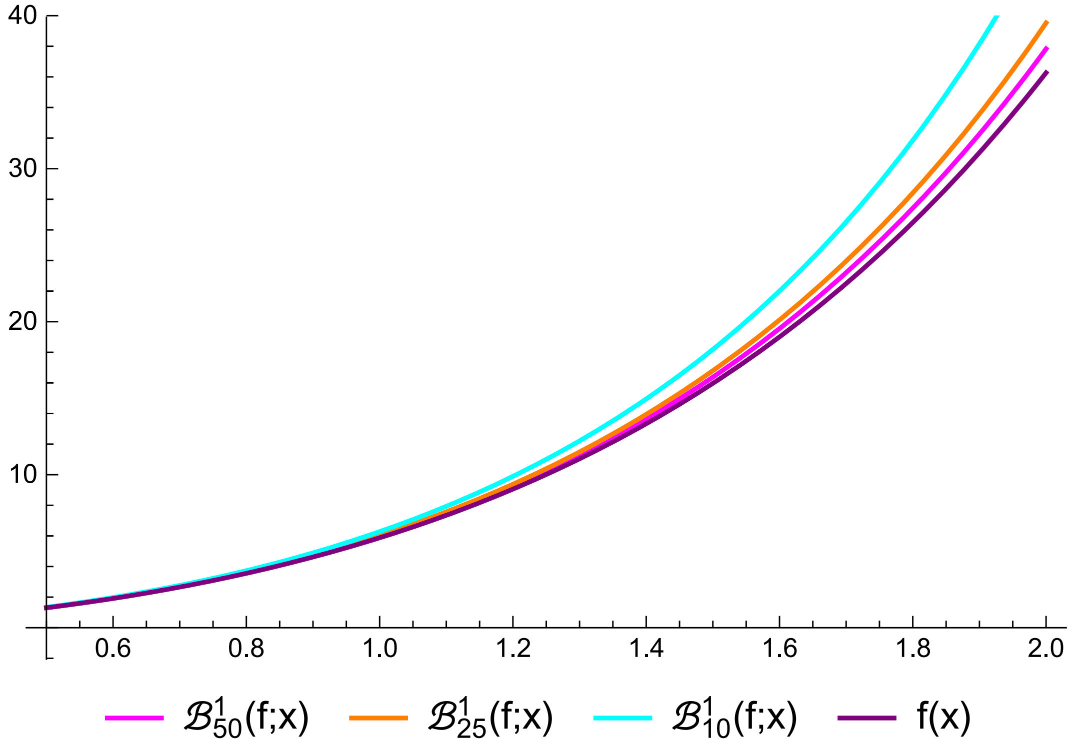

Figure 2.5: Convergence of $\mathcal{B}_n^A(f;x)$ for the function $f(x) = 5x(Sinh(x))$ (Purple) for $n = 10$ (Cyan), $n = 25$ (Orange), $n = 50$ (Magenta) for $A = 1$.

Chapter 3

Approximation by Apostol-Genocchi and Pǎltǎnea operators

P. L. Chebyshev pioneered the study of orthogonal polynomials in the late 19th century, followed by the works of A. A. Markov and T. J. Steiltjes. This chapter deals with the Durrmeyer type construction of operators based on a class of orthogonal polynomials called Apostol-Genocchi polynomials. We used Pǎltǎnea type operators for the proposed Durrmeyer construction. Some approximation results to estimate the convergence rate and order of the approximation is provided. Lastly, we have presented some graphs and an absolute error table to illustrate the convergence and the effect of change in parameter over the proposed operators.

3.1 Introduction

Recently, Prakash et al. (124) proposed the following sequence of linear positive operators:

$$G_n^{\alpha,\lambda}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{\alpha,\lambda}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty), \tag{3.1}$$

where $s_{n,k}^{\alpha,\lambda}(x) = e^{-nx}\left(\frac{1+\lambda e}{2}\right)^{\alpha} \frac{g_k^{\alpha}(nx;\lambda)}{k!}$ and $g_k^{\alpha}(x;\lambda)$ is the generalized Apostol-Genocchi polynomials of order α, which belong to the class of orthogonal polynomials. These polynomials were defined for a complex variable z, $|z| < \pi$ in (101). However, in this study we limit ourselves to a real variable $t \in [0,\infty)$. The generalized Apostol Genocchi polynomial of order α i.e. $g_k^{\alpha}(x;\lambda)$ can be estimated with the help of following generating function:

$$\left(\frac{2t}{1+\lambda e^t}\right)^{\alpha} e^{xt} = \sum_{k=0}^{\infty} g_k^{\alpha}(x;\lambda)\frac{t^k}{k!}. \tag{3.2}$$

The more explicit form of $g_k^{\alpha}(x;\lambda)$ was proposed by Luo and Srivastava in (102). They presented some elementary properties of these polynomials and derived explicit series representation of $g_k^{\alpha}(x;\lambda)$ in terms of hypergeometric function defined by Gauss. The series is given as follows:

$$g_k^{\alpha}(x;\lambda) = 2^{\alpha}\alpha!\binom{k}{\alpha}\sum_{n=0}^{k-\alpha}\binom{k-\alpha}{n}\binom{\alpha+n-1}{n}\frac{\lambda^n}{(1+\lambda)^{\alpha+n}}$$
$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^n(x+j)^{k-n-\alpha}{}_2F_1\left(\alpha+n-k,n;n+1;\frac{j}{x+j}\right),$$

where $\{k,\alpha\}\in\mathbb{N}\cup\{0\}$, $\lambda\in\mathbb{R}/\{-1\}$, $x\in\mathbb{R}$ and ${}_2F_1(a,b;c;t)$ denotes the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = {}_2F_1(b,a;c;t) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{t^k}{k!}.$$

In particular, for $\alpha = 1$ and $\lambda = 1$, these operators reduce to classical Genocchi polynomials which are obtained by the following generating function:

$$\frac{2te^{xt}}{e^t+1} = \sum_{k=0}^{\infty} g_k(x)\frac{t^k}{k!}, \quad |t|<2\pi,\ x\in\mathbb{R},$$

where $g_k(x) = g_k(x;1)$. It can be clearly seen that $g_k(x)$ are the k^{th}-degree polynomials, few terms of which are given as follows:

$$g_1(x) = 1, \quad g_2(x) = 2x-1, \quad g_3(x) = 3x(x-1),$$

$$g_4(x) = 4x^3-6x^2+1, \quad g_5(x) = 5x^4-10x^3+5x, \cdots$$

For the case $x = 0$, one can obtain the so-called Genocchi numbers g_k using the relation:

$$g_k(x) = \sum_{i=0}^{k}\binom{k}{i}g_i x^{k-i}.$$

Genocchi numbers can be defined in many ways depending on the field where they are intended to be applied. They find a wide range of application in numerical analysis, combinatorics, number theory, graph theory etc. Luo (100; 101) defined Apostol-Genocchi polynomials of higher order and also introduced $q-$Apostol-Genocchi polynomials. The relationship of these polynomials with Zeta function was also studied. In the past few decades, a surprising number of papers appeared studying Genocchi numbers, their combinatorial relations, Genocchi polynomials and their generalisations along with their various expansions and integral representations. To the readers, we suggest following articles (34; 42; 115; 121; 129) and references therein.

In the recent past, much work has been dedicated towards the Durrmeyer type modification of linear positive operators. For instance, Dhamija and Deo (46) introduced the Durrmeyer form of Jain operators based on inverse Pólya-Eggenberger distribution. They studied its moments with the aid of Vandermonde convolution formula and analysed other approximation properties. Heilmann and Raşa (73) studied a link between Baskakov–Durrmeyer type operators and their corresponding classical Kantorovich variants. Acu and Radu (15) introduced and studied a class of operators which link α-Bernstein operators and genuine α-Bernstein Durrmeyer operators. To see more work relevant to this area, one may refer to the following articles (13; 45; 61; 65; 85; 111).

Inspired by above stated researches, we considered a Durrmeyer type modification of Apostol-Genocchi operators based on Păltănea basis on positive real line. For $f \in C[0,\infty)$, and $\rho > 0$, the proposed operators are as follows:

$$\mathcal{M}_n^{\alpha,\lambda}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{\alpha,\lambda}(x) \int_0^{\infty} l_{n,k}^{\rho}(t) f(t)\, dt, \quad x \in [0,\infty), \tag{3.3}$$

where $l_{n,k}^{\rho}(t) = n\rho e^{-n\rho t} \frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}$ and $s_{n,k}^{\alpha,\lambda}(x)$ is defined in (3.1).

The outline of this chapter is as follows: We consider a Durrmeyer type construction of Apostol-Genocchi operators based on the basis function due to Păltănea (122) with real parameters α, λ and ρ. We establish approximation estimates such as a global approximation theorem and rate of approximation in terms of usual, $r-$th and weighted modulus of continuity. We further study asymptotic formulae such as Voronovskaya theorem and quantitative Voronovskaya theorem. The last theorem is an application of the proposed operators for the functions whose derivatives are of bounded variation. Moreover, the approximation and the absolute error therein has been shown graphically by varying the values of various parameters using Mathematica software.

3.2 Preliminaries

Before proceeding to our main results, we state some general lemmas which are useful throughout this chapter. In addition, we have used Mathematica software to calculate moments and central moments of the proposed operators (3.3).

Lemma 3.2.1 *For $e_s(t) = t^s$, $s \in \mathbb{N} \cup \{0\}$ and $\rho > 0$, we have*

$$\int_0^\infty l^\rho_{n,k}(t)t^s = \frac{(k\rho + s - 1)!}{(n\rho)^s (k\rho - 1)!} = \frac{(k\rho)_s}{(n\rho)^s}.$$

where the symbol $(\beta)_n = \beta(\beta + 1)(\beta + 2)\ldots(\beta + n - 1)$, $(\beta)_0 = 1$ denotes the rising factorial.

Lemma 3.2.2 *For operators* (3.3)*, the moments are obtained as follows:*

$$\mathcal{M}_n^{\alpha,\lambda}(e_0; x) = 1,$$

$$\mathcal{M}_n^{\alpha,\lambda}(e_1; x) = x + \frac{\alpha}{n(1+\lambda e)},$$

$$\mathcal{M}_n^{\alpha,\lambda}(e_2; x) = x^2 + \frac{x}{n}\left[\frac{1+2\alpha+\lambda e}{(1+\lambda e)} + \frac{1}{\rho}\right] + \frac{1}{n^2}\left[\frac{\alpha^2-2\alpha\lambda e-\alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)}\right],$$

$$\begin{aligned}\mathcal{M}_n^{\alpha,\lambda}(e_3; x) &= x^3 + \frac{x^2}{n}\left[\frac{3(\alpha+\lambda e+1)}{(1+\lambda e)} + \frac{3}{\rho}\right]\\ &+ \frac{x}{n^2}\left[\frac{3\alpha^2-3\alpha\lambda^2e^2-3\alpha\lambda e+3\alpha+\lambda^2e^2+2\lambda e+1}{(1+\lambda e)^2} + \frac{3(2\alpha+\lambda e+1)}{\rho(1+\lambda e)} + \frac{2}{\rho^2}\right],\\ &+ \frac{1}{n^3}\left[\frac{\alpha^3-3\alpha^2\lambda^2e^2-6\alpha^2\lambda e-\alpha\lambda^3e^3-4\alpha\lambda^2e^2-5\alpha\lambda e}{(1+\lambda e)^3} + \frac{2\alpha}{\rho^2(1+\lambda e)} + \frac{3\left(\alpha^2-\alpha\lambda^2e^2-2\alpha\lambda e\right)}{\rho(1+\lambda e)^2}\right]\end{aligned}$$

$$\begin{aligned}\mathcal{M}_n^{\alpha,\lambda}(e_4; x) &= x^4 + \frac{x^3}{n}\left[\frac{2\alpha+3\lambda e+3}{(1+\lambda e)} + \frac{6}{\rho}\right]\\ &+ \frac{x^2}{n^2}\left[\frac{25+12\alpha+6\alpha^2+50\mathrm{e}\lambda+25\mathrm{e}^2\lambda^2-6\alpha\mathrm{e}^2\lambda^2}{(1+\lambda e)^2} + \frac{18(1+\alpha+\mathrm{e}\lambda)}{(1+\lambda e)\rho} + \frac{11}{\rho^2}\right]\\ &+ \frac{x}{n^3}\left[\frac{7+20\alpha+63\alpha^2+2\alpha^3-6\alpha^2e^2\lambda^2-9\alpha^2\lambda e-5\alpha e^3\lambda^3+4\alpha e^2\lambda^2+24\alpha\lambda e+7e^3\lambda^3+21\lambda e}{(1+\lambda e)^3}\right.\\ &\left.+\frac{6\left(3\alpha^2-3\alpha e^2\lambda^2-3\alpha\lambda e+3\alpha+e^2\lambda^2+2\lambda e+1\right)}{(1+\lambda e)^2\rho} + \frac{11(1+2\alpha+\lambda e)}{(1+\lambda e)\rho^2} + \frac{6}{\rho^3}\right]\\ &\frac{1}{n^4}\left[\frac{\alpha^4-6e^2\alpha^3\lambda-12e\alpha^3\lambda-16e^4\alpha\lambda^4+8e^3\alpha\lambda^3-82e^3\alpha\lambda^2-118e^2\alpha\lambda^2-66\alpha+\lambda e}{(1+\lambda e)^4}\right.\\ &+ \frac{6\left(\alpha^3-3e^2\alpha^2\lambda^2-6e\alpha^2\lambda-e^3\alpha\lambda^3-4e^2\alpha\lambda^2-5e\alpha\lambda\right)}{(1+\lambda e)^3\rho}\\ &\left.\frac{11\left(\alpha^2-\alpha e^2\lambda^2-2\alpha\lambda e\right)}{(1+\lambda e)^2\rho^2} + \frac{6\alpha}{(1+\lambda e)\rho^3}\right].\end{aligned}$$

Proof 3.2.3 *In the proposed operators* (3.3)*, we have*

1. *For $s = 0$, in view of Lemma 3.2.1 and generating function* (3.2)*, we have,*

$$\mathcal{M}_n^{\alpha,\lambda}(e_0; x) = 1$$

2. *For $s = 1$, again using Lemma 3.2.1 we have*

$$\mathcal{M}_n^{\alpha,\lambda}(e_1; x) = \frac{e^{-nx}}{n}\left(\frac{1 + \lambda e}{2}\right)^\alpha \sum_{k=0}^\infty \frac{g_k^\alpha(nx; \lambda)}{k!}k. \tag{3.4}$$

On differentiating both sides of (3.2) *with respect to t and taking limits $t \to 1$ and $x \to nx$, we have*

$$\sum_{k=0}^\infty \frac{g_k^\alpha(nx; \lambda)}{k!}k = 2^\alpha e^{nx}\left(\frac{1}{1 + \lambda e}\right)^{1+\alpha}(\alpha + nx(1 + \lambda e)).$$

Making use of this value in (3.4)*, we obtain the first moment.*

3. *Similarly for $s = 2$, we have*

$$\mathcal{M}_n^{\alpha,\lambda}(e_2; x) = \frac{e^{-nx}}{n^2}\left(\frac{1+\lambda e}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{g_k^{\alpha}(nx;\lambda)}{k!} k^2. \tag{3.5}$$

On differentiating both sides of (3.2) *with respect to t and taking limits $t \to 1$ and $x \to nx$, we have*

$$\sum_{k=0}^{\infty} \frac{g_k^{\alpha}(nx;\lambda)}{k!} k(k-1) = 2^{\alpha} e^{nx}\left(\frac{1}{1+\lambda e}\right)^{1+\alpha} \{2ns\alpha(1+\lambda e)$$
$$+ n^2x^2(1+\lambda e)^2 - (\lambda e(3+\lambda e) - \alpha + 1)\Big\}.$$

Combining this with the first order moment and (3.5)*, we obtain the third moment.*

The higher order moments can be calculated in a similar way.

Lemma 3.2.4 *Let us define $\delta_n^{(s)}(x) = \mathcal{M}_n^{\alpha,\lambda}(\Phi_s; x)$, where $\Phi_s(t) = (t-x)^s$ and $s = 1, 2$. Then, from Lemma 3.2.2 we have*

$$\delta_n^{(1)}(x) = \frac{\alpha}{n(1+\lambda e)},$$
$$\delta_n^{(2)}(x) = \frac{x}{n}\left[1 + \frac{1}{\rho}\right] + \frac{1}{n^2}\left[\frac{\alpha^2 - 2\alpha\lambda e - \alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)}\right].$$

Furthermore,

$$\lim_{n\to\infty} n\mathcal{M}_n^{\alpha,\lambda}(\Phi_1; x) = \frac{\alpha}{(1+\lambda e)},$$
$$\lim_{n\to\infty} n\mathcal{M}_n^{\alpha,\lambda}(\Phi_2; x) = \left(1 + \frac{1}{\rho}\right)x,$$
$$\lim_{n\to\infty} n^2\mathcal{M}_n^{\alpha,\lambda}(\Phi_4; x) = \frac{3\left(1 + (6+8\alpha)\rho - (3+8\alpha)\rho^2 + \lambda e\left(1 + 6\rho - 3\rho^2\right)\right)x^2}{(1+\lambda e)\rho^2},$$

and

$$\lim_{n\to\infty} n^3\mathcal{M}_n^{\alpha,\lambda}(\Phi_6; x) = \frac{(1+\rho)(2+\rho+3\alpha\rho+\lambda e(2+\rho))x^5}{(1+\lambda e)^3\rho^4}.$$

Remark 3.2.5 *Since fourth and sixth central moments are too lengthy and unnecessarily space consuming, we are omitting their exact values here. Instead, we choose to write their limiting values, which is useful in the proofs of our main theorems.*

3.3 Main theorems

Theorem 3.3.1 *For any* $f \in C_B[0,\infty)$*, where* $C_B[0,\infty)$ *is the class of all continuous and bounded functions, we have*

$$\mathcal{M}_n^{\alpha,\lambda}(f;x) = f(x),$$

uniformly on any compact subset of $[0,\infty)$.

Proof 3.3.2 *Taking into account Lemma 3.2.2, we can easily see that* $\mathcal{M}_n^{\alpha,\lambda}(e_s;x) \to x^s$ *for each* $s = 0,1,2$ *and hence using the well known Korovkin's theorem (93), operators* $\mathcal{M}_n^{\alpha,\lambda}$ *converge uniformly on each compact subset of* $[0,\infty)$.

3.3.1 Global Approximation

Let us denote $B_\rho[0,\infty)$ the space of all functions f on positive real axis that satisfy the condition $|f(x)| \leq M\left(1+x^2\right)$ where M is a positive constant.

Let $C_\rho[0,\infty)$ be the subspace of $B_\rho[0,\infty)$ containing all continuous f on $[0,\infty)$ endowed with the norm:

$$\|f\|_2 = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

Also, let $C_\rho^l[0,\infty) := \{f \in C_\rho[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} = l, \text{ is finite}\}$.

Theorem 3.3.3 *For each* $f \in C_\rho^l[0,\infty)$*, we have*

$$\lim_{n\to\infty} \left\|\mathcal{M}_n^{\alpha,\lambda}(f;x) - f\right\|_2 = 0.$$

Proof 3.3.4 *The proof of this theorem is based on the application of Korovkin theorem (58) defined on the interval* $[0,\infty)$*. Therefore it would suffice if we prove that:*

$$\lim_{n\to\infty} \left\|\mathcal{M}_n^{\alpha,\lambda}(e_s;x) - e_s\right\|_2 = 0, \qquad s = 0,1,2. \tag{3.6}$$

For $s = 0$*, condition* (3.6) *holds as operators* $\mathcal{M}_n^{\alpha,\lambda}$ *preserve constant functions. Next, we can write*

$$\left\|\mathcal{M}_n^{\alpha,\lambda}(e_1;x) - x\right\|_2 \leq \sup_{x\in[0,\infty)} \frac{\alpha}{n(1+\lambda e)(1+x^2)} \to 0,$$

for adequately large n. Therefore the condition (3.6) *is satisfied for* $s = 1$.
Finally, we write,

$$\left\|\mathcal{M}_n^{\alpha,\lambda}(e_2;x) - x^2\right\|_2 \leq \sup_{x\in[0,\infty)} \frac{1}{(1+x^2)}\left\{\frac{x}{n}\left(1+\frac{1}{\rho}\right) + \frac{1}{n^2}\left(\frac{\alpha^2+2\alpha\lambda e-\alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)}\right)\right\},$$

which suggests,

$$\lim_{n\to\infty} \left\| \mathcal{M}_n^{\alpha,\lambda}(e_2;x) - e_2 \right\|_2 = 0.$$

Hence the theorem.

Next, we denote the r-th order modulus of continuity by $\omega_r(f,\delta)$ and define it as

$$\omega_r(f,\delta) = \sup_{x\in[0,\infty)} \sup_{0\le h\le\delta} \left|\Delta_h^r f(x)\right|.$$

where Δ denotes the forward difference. In particular, the usual modulus of continuity is defined for $r = 1$ and is denoted by $\omega(f,\delta)$. Moreover, we define the norm as $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$.

Also, the Peetre's K-functional for the function $g \in C_B^2[0,\infty)$ is defined as:

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \left\{ \|f-g\| + \delta \|g''\| : g \in C_B^2[0,\infty) \right\},$$

where

$$C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}.$$

The next theorem establishes the degree of approximation of the operators $\mathcal{M}_n^{\alpha,\lambda}$ in terms of the usual and second order modulus of continuity for $f \in C_B[0,\infty)$.

Theorem 3.3.5 *For $\hbar \in C_B^2[0,\infty)$, define the auxiliary operators $\widetilde{Q}_n^{\alpha,\lambda}$ as*

$$\widetilde{Q}_n^{\alpha,\lambda}(\hbar;x) = \mathcal{M}_n^{\alpha,\lambda}(\hbar;x) - \hbar\left(x + \frac{\alpha}{n(1+\lambda e)}\right) + \hbar(x), \tag{3.7}$$

then, there exist a constant $C > 0$ such that

$$|\mathcal{M}_n^{\alpha,\lambda}(\hbar,x) - \hbar(x)| \le C\omega_2(\hbar, \sqrt{\delta}) + \omega(\hbar, \delta_n^{(1)}(x)),$$

where

$$\delta = \delta_n^{(2)}(x) + \left(\frac{\alpha}{n(1+\lambda e)}\right)^2.$$

Proof 3.3.6 *Using Lemma 3.2.2, one can easily observe that $\widetilde{Q}_n^{\alpha,\lambda}((t-x);x) = 0$. Let $f \in C_B^2[0,\infty)$, then by Taylor's expansion we have*

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-u)f''(u)du,$$

Moreover, we can write,

$$\begin{aligned}\left|\widetilde{Q}_n^{\alpha,\lambda}(f;x)-f(x)\right| &= \left|\widetilde{Q}_n^{\alpha,\lambda}\left(\int_x^t (t-u)f''(u)du, x\right)\right|\\ &\le \left|\mathcal{M}_n^{\alpha,\lambda}\left(\int_x^t (t-u)f''(u)du, x\right)\right|\\ &+\left|\int_x^{\left(x+\frac{\alpha}{n(1+\lambda e)}\right)}\left(\left(x+\frac{\alpha}{n(1+\lambda e)}\right)-u\right)f''(u)du\right|\\ &\le \left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right)+\left(\frac{\alpha}{n(1+\lambda e)}\right)^2\right)\|f''\|. \end{aligned} \tag{3.8}$$

Since we know that,

$$\left|\mathcal{M}_n^{\alpha,\lambda}(\hbar;x)\right| \le \|\hbar\|,$$

therefore,

$$\left|\widetilde{Q}_n^{\alpha,\lambda}(\hbar;x)\right| \le \left|\mathcal{M}_n^{\alpha,\lambda}(\hbar;x)\right| + \left|\hbar\left(x+\frac{\alpha}{n(1+\lambda e)}\right)\right| + |\hbar(x)| \le 3\|\hbar\|. \tag{3.9}$$

Finally combining equations (3.7)*,* (3.8) *and* (3.9)*, we get*

$$\begin{aligned}\left|\mathcal{M}_n^{\alpha,\lambda}(\hbar;x)-\hbar(x)\right| &\le \left|\widetilde{Q}_n^{\alpha,\lambda}(\hbar-f);x)-(\hbar-f)(x)\right| + \left|\widetilde{Q}_n^{\alpha,\lambda}(f;x)-f(x)\right|\\ &+\left|\hbar(x)-\hbar\left(x+\frac{\alpha}{n(1+\lambda e)}\right)\right|\\ &\le 4\|\hbar-f\| + \left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right)+\left(\frac{\alpha}{n(1+\lambda e)}\right)^2\right)\|f''\|\\ &+\left|\hbar(x)-\hbar\left(x+\frac{\alpha}{n(1+\lambda e)}\right)\right|\\ &\le C\left\{\|\hbar-f\| + \left(\delta_n^{(2)}(x)+\left(\frac{\alpha}{n(1+\lambda e)}\right)^2\right)\|f''\|\right\}\\ &+\omega\left(\hbar, \frac{\alpha}{n(1+\lambda e)}\right).\end{aligned}$$

Taking infimum over all $f \in C_B^2[0,\infty)$ and using the result $K_2(f,\delta) \le \omega_2\left(f, \sqrt{\delta}\right)$ due to (44), we get the desired outcome.

Theorem 3.3.7 *Let $\hbar \in C_B(0,\infty)$, then for any $r > 0$, $x \in [0,r]$ and adequately large n, we have*

$$\left|\mathcal{M}_n^{\alpha,\lambda}(\hbar;x)-\hbar(x)\right| \le 4H_\hbar\left(1+x^2\right)\frac{D}{n} + 2\omega_{r+1}\left(\hbar, \sqrt{\frac{D}{n}}\right),$$

where D is a positive constant.

Proof 3.3.8 *If $x \in [0, r]$ and $t > r + 1$, then $t - x > 1$. Therefore, we have the following inequality:*

$$|\hbar(t) - \hbar(x)| \leq 4H_{\hbar}\left(1 + x^2\right)(t - x)^2. \tag{3.10}$$

Again for $x \in [0, r]$ and $t \in [0, r + 1]$ and using the well known inequality $\omega(f, \beta\delta) \leq (\beta + 1)\omega(f, \delta)$, $\beta \in (0, \infty)$, one can obtain

$$|\hbar(t) - \hbar(x)| \leq \left(1 + \frac{|t - x|}{\delta}\right)\omega_{r+1}(\hbar, \delta). \tag{3.11}$$

From (3.10) *and* (3.11)*, we can write*

$$|\hbar(t) - \hbar(x)| \leq 4H_{\hbar}\left(1 + x^2\right)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right)\omega_{r+1}(\hbar, \delta).$$

Applying operators $\mathcal{M}_n^{\alpha,\lambda}$ in the above relation and making use of Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\mathcal{M}_n^{\alpha,\lambda}(\hbar; x) - \hbar(x)\right| \leq & 4H_{\hbar}\left(1 + x^2\right)\mathcal{M}_n^{\alpha,\lambda}\left((t - x)^2; x\right) + \left(1 + \frac{1}{\delta}\mathcal{M}_n^{\alpha,\lambda}(|t - x|; x)\right)\omega_{r+1}(\hbar, \delta) \\ & \leq 4H_{\hbar}\left(1 + x^2\right)\mathcal{M}_n^{\alpha,\lambda}\left((t - x)^2; x\right) + 2\omega_{r+1}\left(\hbar, \sqrt{\mathcal{M}_n^{\alpha,\lambda}\left((t - x)^2; x\right)}\right).\end{aligned}$$

Since $\mathcal{M}_n^{\alpha,\lambda}\left((t - x)^2; x\right) \leq \frac{D}{n}$, where D is a positive constant, it follows that for adequately large n, we have

$$\left|\mathcal{M}_n^{\alpha,\lambda}(\hbar; x) - \hbar(x)\right| \leq 4H_{\hbar}\left(1 + x^2\right)\frac{D}{n} + 2\omega_{r+1}\left(\hbar, \sqrt{\frac{D}{n}}\right).$$

which is the required result.

3.3.2 Voronovoskaya Theorems

The weighted modulus of continuity $\Omega(f, \delta)$ due to (8), for each $f \in C_{\rho}^{l}[0, \infty)$ is defined as:

$$\Omega(f, \delta) = \sup_{x \in [0,\infty), |h| < \delta} \frac{|f(x + h) - f(x)|}{(1 + h^2 + x^2 + h^2x^2)}.$$

In the next theorem, we discuss the quantitative Voronovskaya theorem for the proposed operators (3.3) and derive a Voronovoskaya asymptotic formula as a resulting corollary, making use of the following properties of weighted modulus of continuity. For every $f \in C_f^l[0, \infty)$,

1. $\Omega(f, \delta) \to 0 \quad \text{as} \quad \delta \to 0;$

2. $|f(t) - f(x)| \leq \left(1 + (t - x)^2\right)\left(1 + x^2\right)\Omega(f, |t - x|).$

Theorem 3.3.9 *Let $\hbar'' \in C_\rho^l[0,\infty)$ and $x \in [0,\infty)$. Then we have*

$$\Big|\mathcal{M}_n^{\alpha,\lambda}(\hbar;x) - \hbar(x) - \frac{\alpha}{n(1+\lambda e)}\hbar'(x) - \frac{1}{2}\left(\frac{x}{n}\left(1+\frac{1}{\rho}\right) + \frac{1}{n^2}\left(\frac{\alpha^2+2\alpha\lambda e-\alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)}\right)\right)\hbar''(x)\Big| \leq \frac{8\left(1+x^2\right)}{n}\Omega\left(\hbar'', \frac{1}{\sqrt{n}}\right).$$

Proof 3.3.10 *By Taylor's expansion, we may write*

$$\begin{aligned}\mathcal{M}_n^{\alpha,\lambda}(\hbar;x) - \hbar(x) &= \mathcal{M}_n^{\alpha,\lambda}\left((\hbar(t)-\hbar(x));x\right)\\ &= \mathcal{M}_n^{\alpha,\lambda}\left((t-x)\hbar'(x) + \frac{(t-x)^2}{2}\hbar''(x) + \lambda(t,x)(t-x)^2;x\right),\end{aligned}$$

where $\lambda(t,x) = (\hbar''(\varsigma) - \hbar''(x))/2$ is a continuous function which tends to zero at 0 *and ς lies between x and t. Using Lemma 3.2.4, we get*

$$\Big|\mathcal{M}_n^{\alpha,\lambda}(\hbar;x) - \hbar(x) - \frac{\alpha}{n(1+\lambda e)}\hbar'(x) - \frac{1}{2}\left(\frac{x}{n}\left(1+\frac{1}{\rho}\right) + \frac{1}{n^2}\left(\frac{\alpha^2+2\alpha\lambda e-\alpha e^2\lambda^2}{(1+\lambda e)^2} + \frac{\alpha}{\rho(1+\lambda e)}\right)\right)\hbar''(x)\Big| \leq \mathcal{M}_n^{\alpha,\lambda}\left(|\lambda(t,x)|(t-x)^2;x\right).$$

With simple manipulations in Property (2) *of weighted modulus of continuity and using $|\varsigma - x| \leq |t-x|$, we can write*

$$|\lambda(t,x)| \leq 8\left(1+x^2\right)\left(1+\frac{(t-x)^4}{\delta^4}\right)\Omega(\hbar'',\delta),$$

which implies that

$$|\lambda(t,x)|(t-x)^2 \leq 8\left(1+x^2\right)\left((t-x)^2+\frac{(t-x)^6}{\delta^4}\right)\Omega(\hbar'',\delta).$$

Therefore in view of Lemma 3.2.4, we can write

$$\mathcal{M}_n^{\alpha,\lambda}\left(|\lambda(t,x)|(t-x)^2;x\right) \leq 8\left(1+x^2\right)\Omega(\hbar'',\delta)\left\{\delta_n^{(2)}(x) + \frac{1}{\delta^4}\delta_n^{(6)}(x)\right\},$$

as $n \to \infty$.
Choosing $\delta = \frac{1}{\sqrt{n}}$, we get the desired outcome.

Corollary 3.3.11 *Let f be a bounded and integrable function on the interval* $[0,\infty)$ *such that the second derivative of f exists at a fixed point $x \in [0,\infty)$. Then*

$$\lim_{n\to\infty} n(\mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)) = \frac{\alpha}{(1+\lambda e)}f'(x) + x\left(1+\frac{1}{\rho}\right)f''(x).$$

3.3.3 Functions of derivatives of Bounded Variation

Next we estimate the rate of convergence of the operators (3.3) for functions with derivatives of bounded variation defined on $[0,\infty)$.

Let $f \in DBV_\tau[0,\infty)$ be the class of functions whose derivatives are of bounded variation on any finite subinterval of $[0,\infty)$ and satisfy the growth condition $|f(t)| \leq Kt^\tau, \tau > 0, \forall t > 0$ and constant $K > 0$. For such functions, let us represent our proposed operators (3.3) in the following form:

$$\mathcal{M}_n^{\alpha,\lambda}(f;x) = \int_0^\infty \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)f(t)dt, \tag{3.12}$$

where $\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t) = \sum_{k=0}^{\infty} s_{n,k}^{\alpha,\lambda}(x) l_{n,k}^{\rho}(t)$ as defined in (3.3).

Lemma 3.3.12 *For $x \in [0,\infty)$ and adequately large n, we have*

(i) if $0 \leq y < x$, then

$$\vartheta_n(x,y) = \int_0^y \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt \leq \frac{K\delta_n^{(2)}(x)}{n(x-y)^2},$$

(ii) and if $x < z \leq \infty$, then

$$1 - \vartheta_n(x,z) = \int_z^\infty \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt \leq \frac{K\delta_n^{(2)}(x)}{n(z-x)^2}.$$

Proof 3.3.13 (*i*) *Taking into account Lemma 3.2.2 and proposed operators* (3.3)*, we have*

$$\begin{aligned}\vartheta_n(x,z) &= \int_0^y \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt\\ &\leq \int_0^y \mathfrak{q}_n^{\alpha,\rho}(x;t)\left(\frac{x-t}{x-y}\right)^2 dt = \frac{1}{(x-y)^2}\int_0^y (t-x)^2 \mathfrak{q}_n^{\alpha,\rho}(x;t)\,dt\\ &\leq \frac{1}{(x-y)^2}\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2;x\right) \leq \frac{K\delta_n^{(2)}(x)}{n(x-y)^2}\end{aligned}$$

Proof of (*ii*) *is similar to* (*i*).

Theorem 3.3.14 *Consider a function f whose derivative is of bounded variation on every sub-interval of* $[0,\infty)$ *that satisfies the growth condition $|f(t)| \leq Kt^\tau$ for some absolute constant K and $\tau > 0$. If there exists an integer γ, $(2\gamma \geq \tau)$ such that $f(t) \leq O(t^\gamma)$ for every $t > 0$, then for $\gamma > 0$, $x \in [0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}\left|\mathcal{M}_n^{\alpha,\lambda}(f;x)-f(x)\right| &\leq \frac{1}{2}\left(f'(x+)+f'(x-)\right)\delta_n^{(1)}(x)\\&+\sqrt{\frac{K\delta_n^{(2)}(x)}{4n}}\left|f'(x+)-f'(x-)\right|+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}\left(f'_x\right)\\&+\frac{K\delta_n^{(2)}(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{x}{\sqrt{k}}}\left(f'_x\right)+\frac{K\delta_n^{(2)}(x)}{nx^2}\left|f(2x)-f(x)-xf'(x)\right|\\&+\wp(\gamma,\tau,x)+\frac{K\delta_n^{(2)}(x)}{nx^2}\left|f(x)\right|+\sqrt{\frac{K\delta_n^{(2)}(x)}{n}}f'(x+),\end{aligned}$$

where $\bigvee_b^a(f)$ *denotes the total variation of f on any finite subinterval* $[a,b]$ *of* $[0,\infty)$ *and* $\wp(\gamma,\tau,x):=2^\gamma\left(\int_0^\infty(t-x)^{2\tau}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right)^{\frac{\gamma}{2\tau}}$.

Proof 3.3.15 *For* $x\in[0,\infty)$*, we can write for our proposed operators* (3.12) *that:*

$$\begin{aligned}\mathcal{M}_n^{\alpha,\lambda}(f;x)-f(x)&=\int_0^\infty \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\left(f(t)-f(x)\right)dt\\&=\int_0^\infty \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\left(\int_x^t f'(x)\,du\right)dt.\end{aligned}\tag{3.13}$$

Also for any $f\in DBV_\gamma[0,\infty)$*, we have,*

$$\begin{aligned}f'(u)&=\frac{1}{2}\left(f'(x+)+f'(x-)\right)+f'_x(u)+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\operatorname{sgn}(u-x)\\&+\delta_x(u)\left(f'(u)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right),\end{aligned}\tag{3.14}$$

where

$$\delta_x(u)=\begin{cases}1, & u=x,\\0, & u\neq x.\end{cases}$$

It can be easily verified that:

$$\int_0^\infty\left(\int_x^t\left(f'(u)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right)\delta_x(u)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt=0.$$

Now, in view of our proposed operators (3.12)*, we may write,*

$$\begin{aligned}&\int_0^\infty\left(\int_x^t\left(\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\\&=\frac{1}{2}\left(f'(x+)+f'(x-)\right)\mathcal{M}_n^{\alpha,\lambda}((t-x);x).\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_0^\infty \left(\int_x^t \left(\frac{1}{2} \left(f'(x+) - f'(x-) \right) \operatorname{sgn}(u-x) \right) du \right) \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\, dt \\
&\leq \frac{1}{2} \left| f'(x+) - f'(x-) \right| \int_0^\infty |t-x|\, \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\, dt \\
&\leq \frac{1}{2} \left| f'(x+) - f'(x-) \right| \left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2; x \right) \right)^{1/2}. \qquad (3.15)
\end{aligned}$$

Making use of equations (3.14)-(3.15) *and Lemma 3.2.2 in* (3.13)*, we get*

$$\begin{aligned}
\mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x) \leq\ & \frac{1}{2} \left(f'(x+) + f'(x-) \right) \mathcal{M}_n^{\alpha,\lambda}\left((t-x); x \right) \\
&+ \frac{1}{2} \left| f'(x+) - f'(x-) \right| \left(\mathcal{M}_n^{\alpha,\lambda}\left((t-x)^2; x \right) \right)^{1/2} \\
&+ \int_0^\infty \left(\int_x^t f'_x(x)\, du \right) \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\, dt \\
\leq\ & \frac{1}{2} \left(f'(x+) + f'(x-) \right) \mathcal{M}_n^{\alpha,\lambda}\left((t-x); x \right) \\
&+ \sqrt{\frac{K\delta_n^2(x)}{4n}} \left| f'(x+) - f'(x-) \right| \\
&+ \int_0^\infty \left(\int_x^t f'_x(x)\, du \right) \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\, dt. \qquad (3.16)
\end{aligned}$$

Taking absolute values on both sides and rewriting (3.16) *as:*

$$\begin{aligned}
\left| \mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x) \right| \leq\ & \frac{1}{2} \left(f'(x+) + f'(x-) \right) \mathcal{M}_n^{\alpha,\lambda}\left((t-x); x \right) \\
&+ \sqrt{\frac{K\delta_n^2(x)}{4n}} \left| f'(x+) - f'(x-) \right| + P_{n_1}(x) + P_{n_2}(x), \qquad (3.17)
\end{aligned}$$

where

$$P_{n_1}(x) = \left| \int_0^x \left(\int_x^t f'_x(x)\, du \right) \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\, dt \right|,$$

and

$$P_{n_2}(x) = \left| \int_x^\infty \left(\int_x^t f'_x(x)\, du \right) \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\, dt \right|.$$

Integrating by parts after applying Lemma 3.3.12, and taking $y = x - \frac{x}{\sqrt{n}}$*, we obtain*

$$P_{n_1}(x) \leq \int_0^{x-\frac{x}{\sqrt{n}}} \vartheta_n(x;t) \left| f'_x(t) \right| dt + \int_{x-\frac{x}{\sqrt{n}}}^{x} \vartheta_n(x;t) \left| f'_x(t) \right| dt$$

Since $f'_x(x) = 0$ and $\vartheta_n(x;t) \leq 1$, it implies

$$\int_{x-\frac{x}{\sqrt{n}}}^{x} \vartheta_n(x;t) \left|f'_x(t)\right| dt = \int_{x-\frac{x}{\sqrt{n}}}^{x} \vartheta_n(x;t) \left|f'_x(t) - f'_x(x)\right| dt$$

$$\leq \int_{x-\frac{x}{\sqrt{n}}}^{x} \bigvee_{t}^{x} (f'_x)\, dt \leq \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (f'_x)\,.$$

Again using Lemma 3.3.12 and substituting $y = x - \frac{x}{u}$, we have:

$$\int_{0}^{x-\frac{x}{\sqrt{n}}} \vartheta_n(x;t) \left|f'_x(t)\right| dt \leq \frac{K\delta_n^{(2)}(x)}{n} \int_{0}^{x-\frac{x}{\sqrt{n}}} \frac{\left|f'_x(t)\right|}{(x-t)^2} dt$$

$$\leq \frac{K\delta_n^{(2)}(x)}{nx} \int_{1}^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^{x} (f'_x)\, du$$

$$\leq \frac{K\delta_n^{(2)}(x)}{nx} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} (f'_x)$$

Thus, we can write $P_{n_1}(x)$ as:

$$P_{n_1}(x) \leq \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (f'_x) + \frac{K\delta_n^{(2)}(x)}{nx} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x} (f'_x). \tag{3.18}$$

Next, to estimate $P_{n_2}(x)$, we have

$$P_{n_2}(x) \leq \left| \int_{x}^{2x} \left(\int_{x}^{t} f'_x(u)\, du \right) \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\, dt \right| + \left| \int_{2x}^{\infty} \left(\int_{x}^{t} f'_x(u)\, du \right) \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\, dt \right|$$

$$\leq A_n(x) + B_n(x)\,,$$

where

$$A_n(x) = \left| \int_{x}^{2x} \left(\int_{x}^{t} f'_x(u)\, du \right) \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\, dt \right|,$$

and

$$B_n(x) = \left| \int_{2x}^{\infty} \left(\int_{x}^{t} f'_x(u)\, du \right) \mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\, dt \right|.$$

Since $1-\vartheta_n(x,t)\leq 1$, *by putting* $t=x+\frac{x}{u}$ *successively, we have*

$$\begin{aligned}
A_n(x) &= \left|\int_x^{2x} f'_x(u)\,(1-\vartheta_n(x,2x))\,du-\int_x^{2x} f'_x(t)\,(1-\vartheta_n(x,t)\,dt\right|\\
&\leq \frac{K\delta_n^{(2)}(x)}{nx^2}\,|f(2x)-f(x)-xf'(x)|+\int_x^{x+\frac{x}{\sqrt{n}}}\left|f'_x(t)\right||1-\vartheta_n(x,t)|dt\\
&+\int_{x+\frac{x}{\sqrt{n}}}^{2x}\left|f'_x(t)\right||1-\vartheta_n(x,t)|dt\\
&\leq \frac{K\delta_n^{(2)}(x)}{nx^2}\,|f(2x)-f(x)-xf'(x)|+\frac{K\delta_n^{(2)}(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{V_x^t(f'_x)}{(t-x)^2}dt\\
&+\int_x^{x+\frac{x}{\sqrt{n}}}\bigvee_x^t(f'_x)dt\\
&\leq \frac{K\delta_n^{(2)}(x)}{nx^2}\,|f(2x)-f(x)-xf'(x)|+\frac{K\delta_n^{(2)}(x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{\sqrt{k}}}(f'_x)\\
&+\frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}(f'_x)
\end{aligned}$$

Further we estimate the value of $B_n(x)$ *as follows:*

$$\begin{aligned}
B_n(x) &= \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(u)du\right)\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)dt\right|\\
&\leq \int_{2x}^{\infty} t^{\gamma}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt+|f(x)|\int_{2x}^{\infty}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt+\sqrt{\frac{K\delta_n^{(2)}(x)}{n}}f'(x+) \qquad (3.19)
\end{aligned}$$

It is obvious that $t\leq 2(t-x)$ *and* $x\leq t-x$, *when* $t\geq 2x$. *Now applying Hölder's inequality in the first term of* (3.19), *we get*

$$\begin{aligned}
B_n(x) &= 2^{\gamma}\left(\int_0^{\infty}(t-x)^{2\tau}\mathfrak{q}_{n,\rho}^{\alpha,\lambda}(x;t)\,dt\right)^{\frac{\gamma}{2\tau}}+\frac{K\delta_n^{(2)}(x)}{nx^2}\,|f(x)|\\
&+\sqrt{\frac{K\delta_n^{(2)}(x)}{n}}f'(x+)\\
&= \wp(\gamma,\tau,x)+\frac{K\delta_n^{(2)}(x)}{nx^2}\,|f(x)|+\sqrt{\frac{K\delta_n^{(2)}(x)}{n}}f'(x+) \qquad (3.20)
\end{aligned}$$

Finally combining equations (3.18)-(3.20) *and putting values of* $P_{n_1}(x)$ *and* $P_{n_2}(x)$ *in* (3.17) *we get the required result and the theorem is proved.*

Example 3.3.16 *Let* $f(x) = x^4 - 3x^3 + 2x^2 + 1$. *We choose parameters* $\alpha = \lambda = 2$ *and* $\rho = 3$. *For* $n = 10, 50, 100, 200$, *we have the following representations:*

(a) *Figure 3.1 shows the rate of approximation of the operators* $\mathcal{M}_n^{\alpha,\lambda}$ *towards the function* $f(x)$. *Clearly the proposed operators* (3.3) *converge to the function* $f(x)$ *for sufficiently large n.*

(b) *In Figure 3.2, the associated absolute error* $\Theta_n = |\mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)|$ *is represented graphically for* $x \in [0,\infty)$. *It can be observed that error is monotonically decreasing for increasing n.*

(c) *An error estimation table for arbitrary values of x in interval* $[0,\infty)$ *is provided in Table 3.1 which depicts that for higher value of n, the error approaches to zero.*

Therefore it can be concluded that proposed operators (3.3) *provide good approximation for n adequately large.*

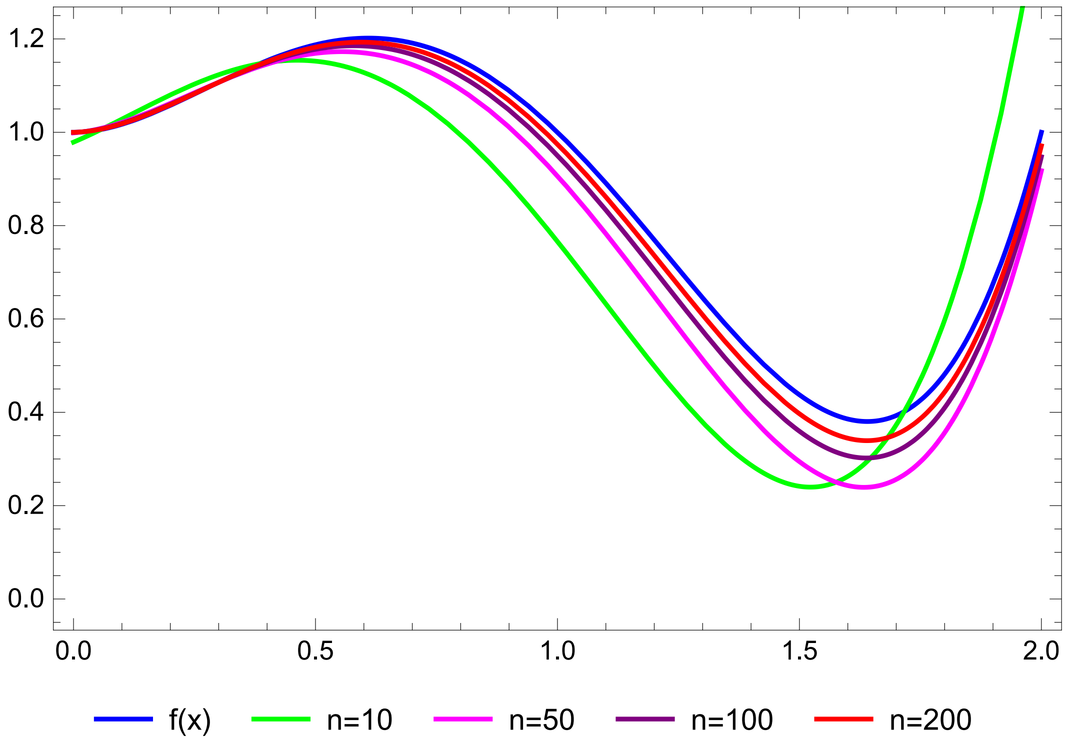

Figure 3.1: Convergence of $\mathcal{M}_n^{\alpha,\lambda}(f;x)$ for the polynomial function $f(x) = x^4-3x^3+2x^2+1$ with parameters $\alpha = \lambda = 2$ $\rho = 3$.

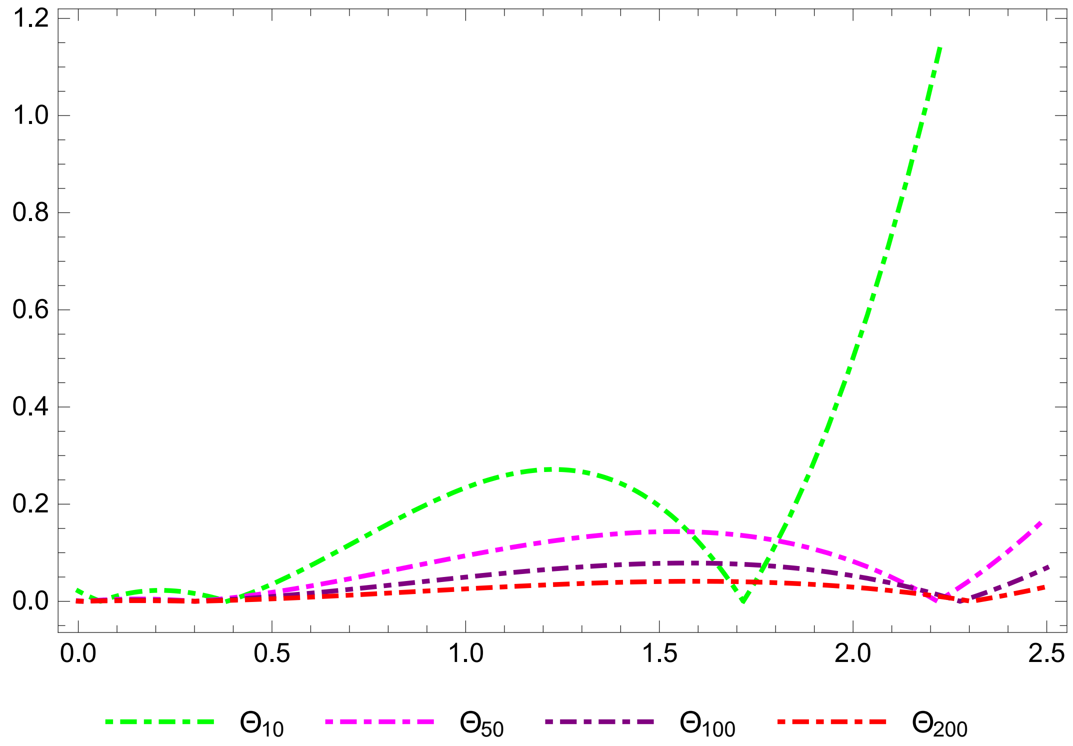

Figure 3.2: Absolute error $\Theta_n = \mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)|$ of the proposed operators for $f(x) = x^4 - 3x^3 + 2x^2 + 1$ with parameters $\alpha = \lambda = 2\ \ \rho = 3$.

Example 3.3.17 *Figure 3.3 illustrates the effect of increase in values of parameter ρ on the rate of convergence of proposed operators $\mathcal{M}_n^{\alpha,\lambda}$ for the function $f(x) = 4x(x-1.1)(x-1.9)$ while keeping the value of α, λ and n fixed. Here we chose $n = 10$ and $\alpha = \lambda = 2$ to show the impact of the parameter ρ clearly. One can easily deduce from the figure that as we increase the value of ρ the rate of convergence gets relatively faster.*

Table 3.1: Table for Absolute error $\Theta_n = |\mathcal{M}_n^{\alpha,\lambda}(f;x) - f(x)|$ of the proposed operators $\mathcal{M}_n^{\alpha,\lambda}$.

x	Θ_{10}	Θ_{50}	Θ_{100}	Θ_{200}
0.4	0.01008	0.00857	0.00459	0.00236
0.8	0.16621	0.06526	0.03455	0.01775
1.2	0.23245	0.12043	0.06478	0.03353
1.6	0.04548	0.12322	0.06985	0.03698
2.0	0.92183	0.02279	0.02434	0.01540
2.4	2.65087	0.23173	0.09718	0.04392
2.8	5.48687	0.69118	0.32014	0.15371
3.2	9.68409	1.40641	0.66997	0.32667
3.6	15.49680	2.42828	1.17209	0.57553
4.0	23.17920	3.80765	1.85192	0.91298

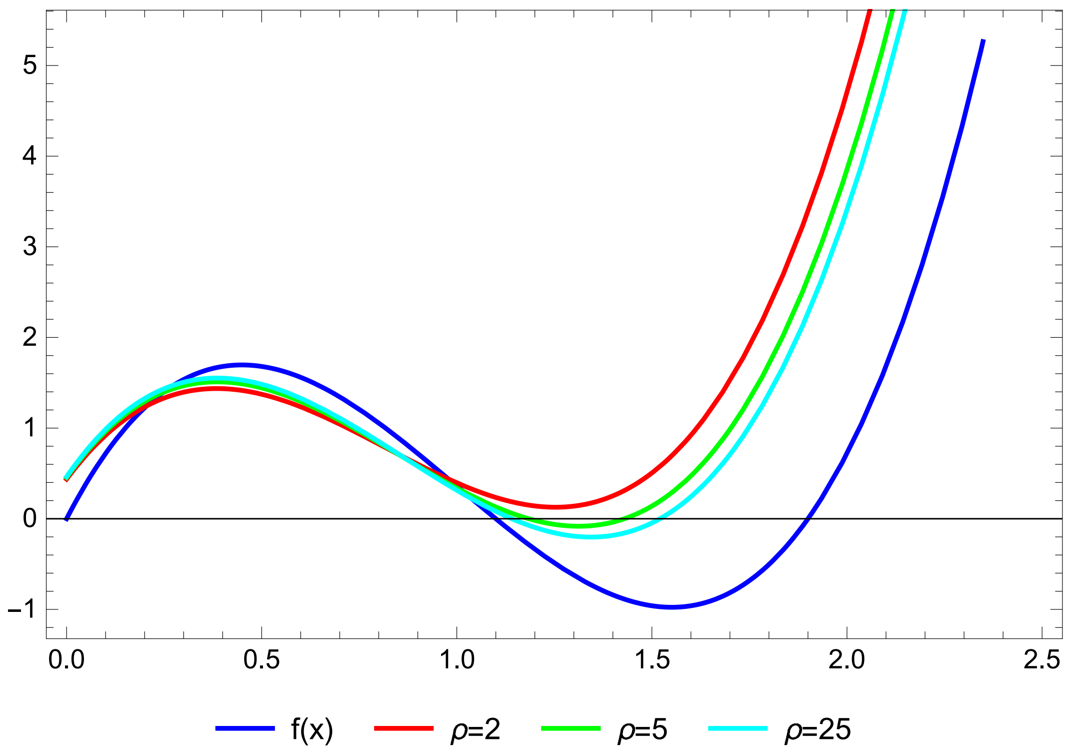

Figure 3.3: Effect of increase in parametric value ρ for given $n = 10, \alpha = \lambda = 2$ on the convergence rate of proposed operators.

Chapter 4

Convergence estimates of certain gamma type operators

Researchers have spent the last few decades studying a large array of approximation operators due to the development of theory of gamma function. This chapter focuses mainly on the investigation of a modification of certain gamma-type operators. These operators preserve the test functions $e_s = t^s, s \in \mathbb{N}$ *and it can be observed that the best approximation is attained while preservation of the test function* e_3*. We have investigated the approximation properties of these operators in the sense of the usual modulus of continuity and Peetre's K-functional. Further, the degree of approximation is also established for the function of bounded variation. The results are validated with some figures and error table.*

4.1 Introduction

A vital tool among the researchers to study linear positive operators is Euler's gamma function, which for $r > 0$ is defined as follows:

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1} dt.$$

For $(a, b) \in \mathbb{R}$, Miheşan (109) defined a more general linear transform of f, also called the $(a, b)-$gamma transform, as follows:

$$\Gamma(r)^{(a,b)}(f; x) = \frac{1}{\Gamma(r)} \int_0^\infty e^{-t} t^{r-1} f\left(x e^{-bt} \left(\frac{t}{r}\right)^a\right) dt. \tag{4.1}$$

The transform (4.1) reproduces distinct integral operators for different values of a, b and r. For $b = 0,\ a = -1$ and $r = n+1$, one can obtain a particular operator first introduced and studied by Lupaş and Müller (104) and also referred to as gamma operators. Let E be the space of all measurable real-valued function f defined on the interval $(0, \infty)$ and locally bounded in any subinterval $[c, d], 0 < c < d < \infty$. Then for any $f \in E$, the gamma operators are described in the following manner:

$$G_n(f; x) = \frac{1}{\Gamma(n+1)} \int_0^\infty e^{-t} t^n f\left(\frac{(n+1)x}{t}\right) dt.$$

The function $f \in E$ must satisfy the following conditions:

- $|f(t)| \leq pe^{c/t}$ as $t \to 0$ for some constants p and c.
- $|f(t)| \leq q\left(1 + t^d\right)$ as $t \to \infty$ for some constants q and d.

It was also verified that for a fixed $x > 0$, $G_n(f; x)$ exists for $n \geq \max\{[c/x] + 1, [d]\}$, where $[c]$ represents the greatest integer function. With change of variables $t = ux$, one can yield an alternate representation of the above operators

$$G_n(f; x) = \frac{x^{n+1}}{\Gamma(n+1)} \int_0^\infty e^{-ux} u^n f\left(\frac{n}{u}\right) du. \tag{4.2}$$

Mazhar (107) introduced another sequence of operators based upon operators (4.2) which are represented by following formula

$$\begin{aligned} F_n(f; x) &= \int_0^\infty g_n(x, u)\, du \int_0^\infty g_{n-1}(u, t) f(t)\, dt \\ &= \frac{(2n)! x^{n+1}}{n!\,(n-1)!} \int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}} f(t)\, dt, \end{aligned} \tag{4.3}$$

for $n > 1,\ x > 0$ and the condition that the integral is convergent for any f.

Using similar techniques due to Mazhar (107), authors in (81; 105) have also proposed their respective gamma type operators. In the past few decades, integral operators has become an extensive area of research and studies have been dedicated towards analyzing their approximation behaviours. We suggest articles by (38; 90; 98; 110; 111; 130).

King (91) proposed a modification of Bernstein operators which preserve the test functions $e_0(t) = 1$ and $e_2(t) = t^2$. Inspired by this approach, many other mathematicians devoted their research to produce modifications of well-known approximation processes fixing certain (polynomial, exponential, or more general) functions and to study their approximation and shape preserving properties. One can refer to the articles (51; 63; 77; 119). In this article, we use a similar approach to present a modification of operators (4.3) such that the modified operator preserves the test function $e_s(x) = x^s,\ s \in \mathbb{N}$. We begin by setting $\wp_{n,s}(x) \in (0, \infty)$ and rewriting operators (4.3) as

$$\widetilde{\mathcal{F}}_{n,s}(f;x) = \frac{(2n)!(\wp_{n,s}(x))^{n+1}}{n!\,(n-1)!}\int_0^\infty \frac{t^{n-1}}{(\wp_{n,s}(x)+t)^{2n+1}} f(t)\,dt. \tag{4.4}$$

Assuming the operators (4.4) preserve the test function $e_s,\ s \in \mathbb{N}$, we can write

$$\begin{aligned}\widetilde{\mathcal{F}}_{n,s}(t^s;x) = x^s &= \frac{(2n)!}{n!\,(n-1)!}\int_0^\infty \frac{(\wp_{n,s}(x))^{n+1}t^{n-1}}{(\wp_{n,s}(x)+t)^{2n+1}}\cdot t^s dt\\ &= \frac{(n+s-1)!\,(n-s)!}{n!\,(n-1)!}(\wp_{n,s}(x))^s\\ &= \frac{(n)_s}{(n-s+1)_s}(\wp_{n,s}(x))^s,\end{aligned}$$

implying

$$\wp_{n,s}(x) = \left[\frac{(n-s+1)_s}{(n)_s}\right]^{1/s} x,$$

where $(n)_s$ denotes the rising factorial with $(n)_0 = 1,\ (n)_s = n(n+1)\cdots(n+s-1)$. Therefore our proposed modified operators take the form:

$$\widetilde{\mathcal{F}}_{n,s}(f;x) = \frac{(2n)!x^{n+1}}{n!\,(n-1)!}\left(\frac{(n-s+1)_s}{(n)_s}\right)^{n+1/s}\int_0^\infty \frac{t^{n-1}}{\left(\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s}x+t\right)^{2n+1}} f(t)\,dt, \tag{4.5}$$

It is noteworthy that as special cases, for $s = 1$ the modified operators resolve into operators (4.3) and for $s = 2$ into operators due to Izgi and Büyükyazici (81).

Followed by some useful lemmas, in this chapter, we determine the rate of convergence of the proposed operators in terms of usual modulus of continuity and Peetre's K-functional. Further, the degree of approximation is also established for the function of bounded variation. We also illustrate via figures and tables that the proposed modification provides better approximation for preservation of test function e_3.

Remark 4.1.1 *For convenience sake, we shall take*

$$\wp_s(n) = \left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s},$$

wherever deemed necessary such that our modified operators (4.5) transform into

$$\widetilde{\mathcal{F}}_{n,s}(f;x) = \frac{(2n)!(\wp_s(n)\,x)^{n+1}}{n!\,(n-1)!}\int_0^\infty \frac{t^{n-1}}{(\wp_s(n)\,x+t)^{2n+1}} f(t)\,dt. \tag{4.6}$$

4.2 Auxiliary Results

We shall require the following lemmas to achieve our main results:

Lemma 4.2.1 *Let $e_m(t) = t^m$, then the $m-th$ order moment with $m \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{N}$ can be determined by the following relation:*

$$\widetilde{\mathcal{F}}_{n,s}(e_m;x) = \left[\frac{(n-s+1)_s}{(n)_s}\right]^{m/s} \frac{(n)_m}{(n-m+1)_m} x^m.$$

Therefore, first few moments are

$\widetilde{\mathcal{F}}_{n,s}(e_0;x) = 1;$

$\widetilde{\mathcal{F}}_{n,s}(e_1;x) = \left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} x;$

$\widetilde{\mathcal{F}}_{n,s}(e_2;x) = \frac{(n+1)}{(n-1)}\left(\frac{(n-s+1)_s}{(n)_s}\right)^{2/s} x^2;$

$\widetilde{\mathcal{F}}_{n,s}(e_3;x) = \frac{(n+2)(n+1)}{(n-2)(n-1)}\left(\frac{(n-s+1)_s}{(n)_s}\right)^{3/s} x^3;$

$\widetilde{\mathcal{F}}_{n,s}(e_4;x) = \frac{(n+3)(n+2)(n+1)}{(n-3)(n-2)(n-1)}\left(\frac{(n-s+1)_s}{(n)_s}\right)^{4/s} x^4.$

Lemma 4.2.2 *Let us denote the central moments of the proposed operators (4.6) as $\xi_v^{\widetilde{\mathcal{F}}_{n,s}}(x) = \widetilde{\mathcal{F}}_{n,s}((t-x)^v;x)$, $v = 1,2,3,\cdots$, then the central moments satisfy the following recursion formula:*

$$[m-n]\,\xi_{v+1}^{\widetilde{\mathcal{F}}_{n,s}}(x) = x\left[n(1-\wp_s(n)) - m(\wp_s(n)+2)\right]\xi_v^{\widetilde{\mathcal{F}}_{n,s}}(x) - mx^2\left[\wp_s(n)+1\right]\xi_{v-1}^{\widetilde{\mathcal{F}}_{n,s}}(x)$$

where $\wp_s(n) = \left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s}$.

Proof 4.2.3 *The proposed operators (4.6) can be rewritten as:*

$$\widetilde{\mathcal{F}}_{n,s}(f;x) = \int_0^\infty \hbar_{n,s}(x,t) f(t)\,dt,$$

where

$$\hbar_{n,s}(x,t) = \frac{(2n)!}{n!\,(n-1)!}\frac{(\wp_s(n)\,x)^{n+1}t^{n-1}}{(\wp_s(n)\,x+t)^{2n+1}}. \tag{4.7}$$

Differentiating (4.7) on both sides with respect to t, we obtain,

$$\frac{\partial}{\partial t}\hbar_{n,s}(x,t) = \frac{(2n)!}{n!\,(n-1)!}\frac{(\wp_s(n)\,x)^{n+1}t^{n-1}}{(\wp_s(n)\,x+t)^{2n+1}}\left(\frac{-(n+2)\,t+\wp_s(n)(n-1)\,x}{t(\wp_s(n)\,x+t)}\right)$$

which can be rewritten as:

$$t\left(\wp_s(n)x+t\right)\frac{\partial}{\partial t}\hbar_{n,s}(x,t)=\hbar_{n,s}(x,t)\left(-(n+2)t+\wp_s(n)(n-1)x\right)$$
$$=\hbar_{n,s}(x,t)\left(-(n+2)(t-x)-x\left(n+2-\wp_s(n)(n-1)\right)\right).$$

Multiplying $(t-x)^{\upsilon}$ *on both sides and integrating with respect to t from* $t=0$ *to* $t=\infty$*, we get,*

$$\int_0^{\infty} t\left(\wp_s(n)x+t\right)\left(\frac{\partial}{\partial t}\hbar_{n,s}(x,t)\right)(t-x)^{\upsilon}dt$$
$$=\int_0^{\infty}\left(-(n+2)(t-x)-x\left(n+2-\wp_s(n)(n-1)\right)\right)\hbar_{n,s}(x,t)(t-x)^{\upsilon}dt \tag{4.8}$$
$$=-(n+2)\int_0^{\infty}\hbar_{n,s}(x,t)(t-x)^{\upsilon+1}dt$$
$$-x\left(n+2-\wp_s(n)(n-1)\right)\int_0^{\infty}\hbar_{n,s}(x,t)(t-x)^{\upsilon}dt$$
$$=-(n+2)\,\xi_{\upsilon+1}^{\widetilde{\mathcal{F}}_{n,s}}(x)-x\left(n+2-\wp_s(n)(n-1)\right)\xi_{\upsilon}^{\widetilde{\mathcal{F}}_{n,s}}(x). \tag{4.9}$$

Moreover, we can write,

$$t\left(\wp_s(n)x+t\right)=(t-x)^2+x\left(\wp_s(n)+2\right)(t-x)+x^2\left(\wp_s(n)+1\right). \tag{4.10}$$

Applying (4.10) in left hand side of (4.8) and further integrating by parts, we have:

$$\int_0^{\infty} t\left(\wp_s(n)x+t\right)\left(\frac{\partial}{\partial t}\hbar_{n,s}(x,t)\right)(t-x)^{\upsilon}dt$$
$$=\int_0^{\infty}\hbar_{n,s}(x,t)(t-x)^{\upsilon+2}dt+x\left(\wp_s(n)+2\right)\int_0^{\infty}\hbar_{n,s}(x,t)(t-x)^{\upsilon+1}dt$$
$$+x^2\left(\wp_s(n)+1\right)\int_0^{\infty}\hbar_{n,s}(x,t)(t-x)^{\upsilon}dt$$
$$=-(\upsilon+2)\,\xi_{\upsilon+1}^{\widetilde{\mathcal{F}}_{n,s}}(x)-(\upsilon+1)\,x\left(\wp_s(n)+2\right)\xi_{\upsilon}^{\widetilde{\mathcal{F}}_{n,s}}(x)-\upsilon x^2\left(\wp_s(n)+1\right)\xi_{\upsilon-1}^{\widetilde{\mathcal{F}}_{n,s}}(x). \tag{4.11}$$

Using expressions (4.9) and (4.11) in relation (4.8), we have:

$$\hbar_{n,s}(x,t)\left(-(n+2)(t-x)-x\left(n+2-\wp_s(n)(n-1)\right)\right)$$
$$=-(\upsilon+2)\,\xi_{\upsilon+1}^{\widetilde{\mathcal{F}}_{n,s}}(x)-(\upsilon+1)\,x\left(\wp_s(n)+2\right)\xi_{\upsilon}^{\widetilde{\mathcal{F}}_{n,s}}(x)-\upsilon x^2\left(\wp_s(n)+1\right)\xi_{\upsilon-1}^{\widetilde{\mathcal{F}}_{n,s}}(x).$$

Simplifying terms on either side, we obtain the required recurrence relation.

In view of the above recursion formula, the first few central moments are

$$\xi_1^{\widetilde{\mathcal{F}}_{n,s}}(x)=\left[\left(\tfrac{(n-s+1)_s}{(n)_s}\right)^{1/s}-1\right]x;$$

$$\xi_2^{\widetilde{\mathcal{F}}_{n,s}}(x)=\left[\tfrac{(n+1)}{(n-1)}\left(\tfrac{(n-s+1)_s}{(n)_s}\right)^{2/s}-2\left(\tfrac{(n-s+1)_s}{(n)_s}\right)^{1/s}+1\right]x^2;$$

$$\xi_4^{\widetilde{\mathcal{F}}_{n,s}}(x)=\left[\tfrac{(n+3)(n+2)(n+1)}{(n-3)(n-2)(n-1)}\left(\tfrac{(n-s+1)_s}{(n)_s}\right)^{4/s}-\tfrac{4(n+2)(n+1)}{(n-2)(n-1)}\left(\tfrac{(n-s+1)_s}{(n)_s}\right)^{3/s}+\tfrac{6(n+1)}{(n-1)}\left(\tfrac{(n-s+1)_s}{(n)_s}\right)^{2/s}\right.$$
$$\left.-4\left(\tfrac{(n-s+1)_s}{(n)_s}\right)^{1/s}+1\right]x^4.$$

Definition 4.2.4 *(33)A Lebesgue point of the function f is a point $x\in\mathbb{R}$ that holds the following condition:*

$$\lim_{\hbar\to 0^+}\frac{1}{\hbar}\int_0^{\hbar}|f(x+t)-f(x)|dt=0.$$

4.3 Main Results

For all $f\in C_B(0,\infty)$ and $\delta>0$, the following inequality holds:

$$|f(t)-f(x)|\le\omega(f,\delta)\left(1+\frac{|t-x|}{\delta}\right)$$

where by $\omega(f,\delta)$ we mean the first order modulus of continuity or usual modulus of continuity as defined in subsection (1.1.2).

Theorem 4.3.1 *For $f\in C_B[0,\infty)$, we have,*

$$\left|\widetilde{\mathcal{F}}_{n,s}(f;x)-f(x)\right|\le\underset{\smile}{M}\omega(f,err(s)),$$

where $\underset{\smile}{M}$ is a positive constant and $err(s)$ is the error function of $s=1,2,3,\cdots$, defined by $err(s)=\sqrt{\xi_2^{\widetilde{\mathcal{F}}_{n,s}}(x)}$.

Proof 4.3.2 *We begin with,*

$$\begin{aligned}\left|\widetilde{\mathcal{F}}_{n,s}(f;x)-f(x)\right|&\le\frac{(2n)!(\wp_s(n)x)^{n+1}}{n!(n-1)!}\int\limits_0^\infty\frac{t^{n-1}}{(\wp_s(n)x+t)^{2n+1}}|f(t)-f(x)|\,dt\\&\le\omega(f,\delta)\frac{(2n)!(\wp_s(n)x)^{n+1}}{n!(n-1)!}\int\limits_0^\infty\frac{t^{n-1}}{(\wp_s(n)x+t)^{2n+1}}\left(1+\frac{|t-x|}{\delta}\right)dt\\&\le\omega(f,\delta)\left(1+\frac{\sqrt{\xi_{n,2}^{\widetilde{\mathcal{F}}_{n,s}}(x)}}{\delta}\right)\\&\le\underset{\smile}{M}\omega\left(f,\sqrt{\xi_{n,2}^{\widetilde{\mathcal{F}}_{n,s}}(x)}\right),\end{aligned}$$

where $\delta=err(s)=\sqrt{\xi_{n,2}^{\widetilde{\mathcal{F}}_{n,s}}(x)}$.

In particular, for $s = 1, 2, 3$, we have

$$\left|\widetilde{\mathcal{F}}_{n,1}(f;x) - f(x)\right| \leq \c{M}\omega\left(f, x\sqrt{\frac{2}{n-1}}\right),$$

$$\left|\widetilde{\mathcal{F}}_{n,2}(f;x) - f(x)\right| \leq \c{M}\omega\left(f, x\sqrt{\frac{2\left(\sqrt{n+1} - \sqrt{n-1}\right)}{\sqrt{n+1}}}\right),$$

$$\left|\widetilde{\mathcal{F}}_{n,3}(f;x) - f(x)\right| \leq \c{M}\omega\left(f, x\sqrt{\left(\frac{(n-2)^2(n+1)}{(n+2)^2(n-1)}\right)^{1/3} - \frac{2(n-2)(n-1)}{(n+2)(n+1)} + 1}\right).$$

Table 4.1: Table for error function err(s)

n	$err(1)$	$err(2)$	$err(3)$	$err(4)$	$err(5)$	$err(6)$
5	0.7071x	0.6058x	0.5774x	0.6057x	0.6756x	x
7	0.5773x	0.5176x	0.5000x	0.5169x	0.5562x	0.6128x
10	0.4714x	0.4369x	0.4264x	0.4359x	0.4606x	0.4957x
50	0.2020x	0.1990x	0.1980x	0.1989x	0.2017x	0.2061x
100	0.1421x	0.1411x	0.1407x	0.1411x	0.1421x	0.1437x
1000	0.0447x	0.0447x	0.0447x	0.0447x	0.0447x	0.0448x

Remark 4.3.3 *From Table 4.1, it can be noted that the error is monotonically decreasing from $s = 1$ to $s = 3$ and then begins to rise again. Therefore we can say that the proposed operators $\widetilde{\mathcal{F}}_{n,s}(f;x)$ provide better approximation until they preserve the test function e_3. This can also be seen from the fact that for $x \in (0,\infty)$ and $n \geq 1$,*

$$\sqrt{\frac{2}{n-1}} \geq \sqrt{\frac{2\left(\sqrt{n+1} - \sqrt{n-1}\right)}{\sqrt{n+1}}} \geq \sqrt{\left(\frac{(n-2)^2(n+1)}{(n+2)^2(n-1)}\right)^{1/3} - \frac{2(n-2)(n-1)}{(n+2)(n+1)} + 1}$$

Also, given Table 4.1, we conclude that for higher–order preservation of test functions such as e_4 or e_5, we cannot achieve better approximation. However, the convergence of the proposed operators occurs in all the cases for an adequately large value of n.

Example 4.3.4 *Consider the polynomial function $f(x) = 8x^3 + 10x + 6$. Below are the figures (figs. 4.1 to 4.6) depicting the convergence of the operators $\widetilde{\mathcal{F}}_{n,s}(f;x)$ towards the function $f(x)$ for $s = 1, 2, 3, 4, 5, 6$, i.e., for the preservation of the test functions e_1, e_2, e_3, e_4, e_5, e_6. It can be observed from the graphs, that the proposed modified operators (4.6) approximate the function $f(x)$ better when they preserve the test function e_3.*

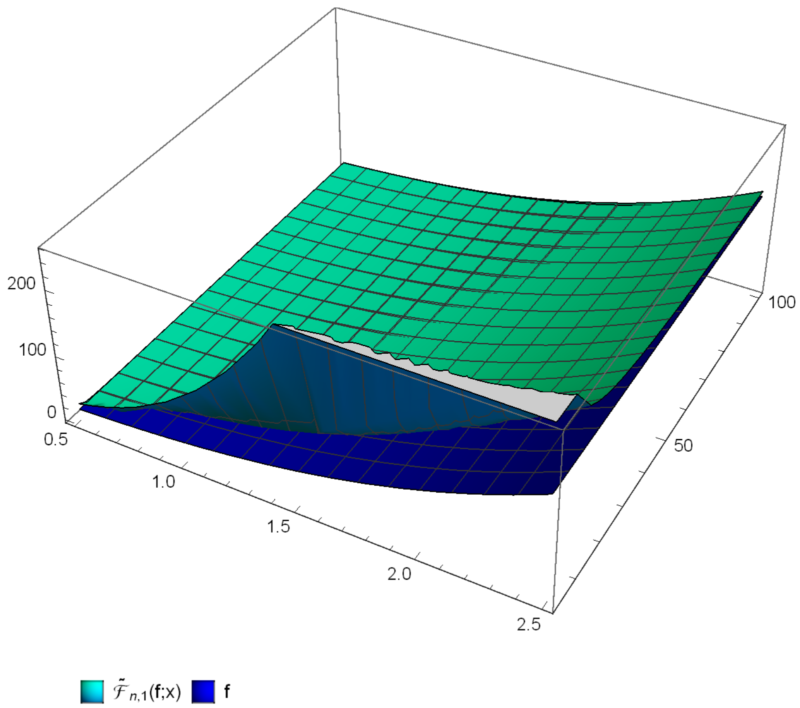

Figure 4.1: Convergence of $\widetilde{\mathcal{F}}_{n,s}(f;x)$ (Cyan) towards $f(x)$ (Blue) while preserving test function e_1.

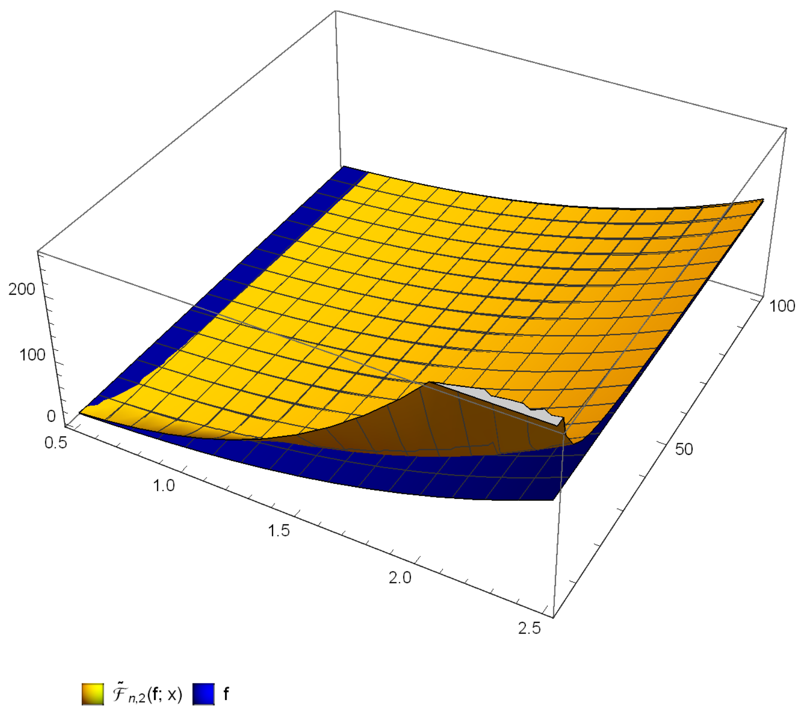

Figure 4.2: Convergence of $\widetilde{\mathcal{F}}_{n,s}(f;x)$ (Yellow) towards $f(x)$ (Blue) while preserving test function e_2.

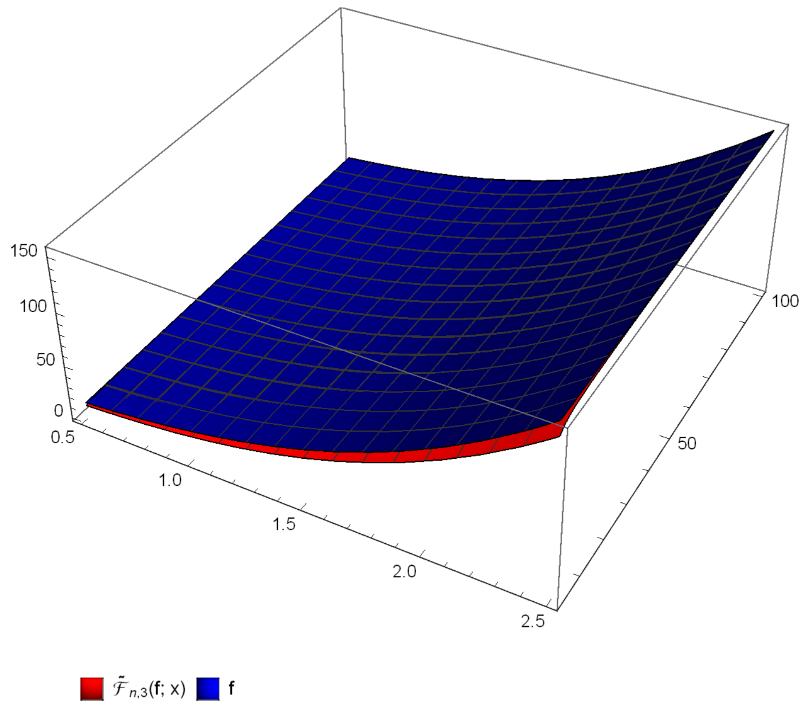

Figure 4.3: Convergence of $\widetilde{\mathcal{F}}_{n,s}(f;x)$ (Red) towards $f(x)$ (Blue) while preserving test function e_3.

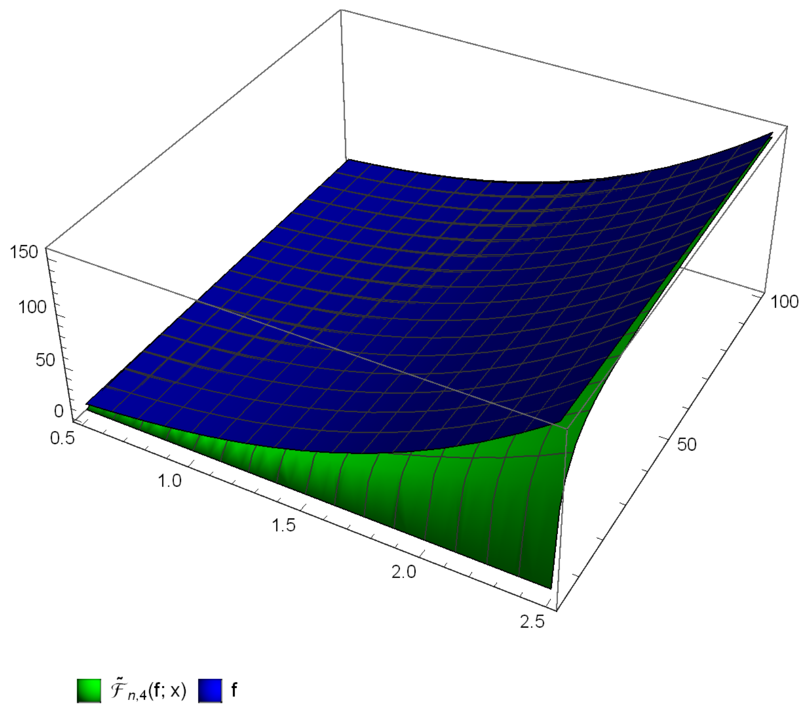

Figure 4.4: Convergence of $\widetilde{\mathcal{F}}_{n,s}(f;x)$ (Green) towards $f(x)$ (Blue) while preserving test function e_4.

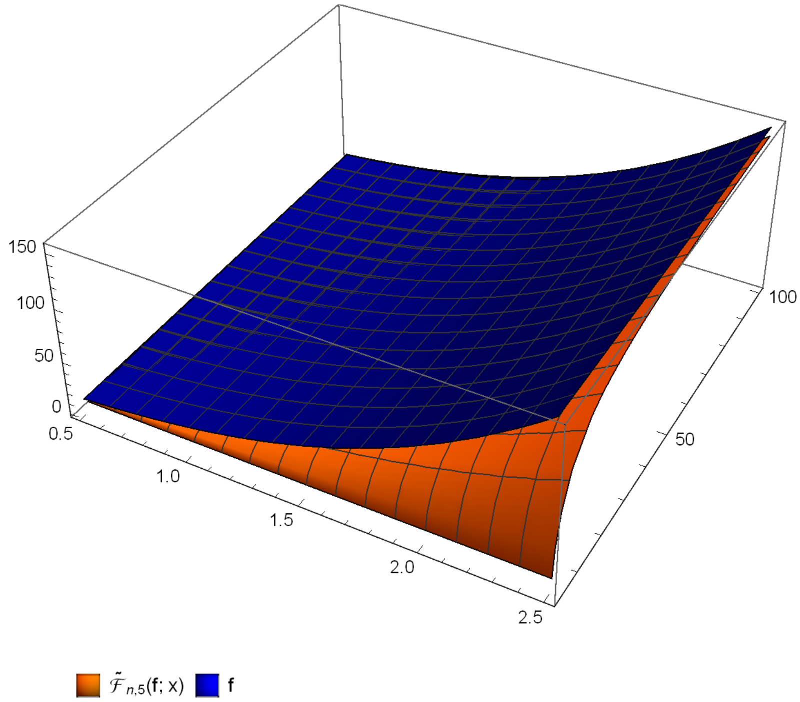

Figure 4.5: Convergence of $\widetilde{\mathcal{F}}_{n,s}(f;x)$ (Orange) towards $f(x)$ (Blue) while preserving test function e_5.

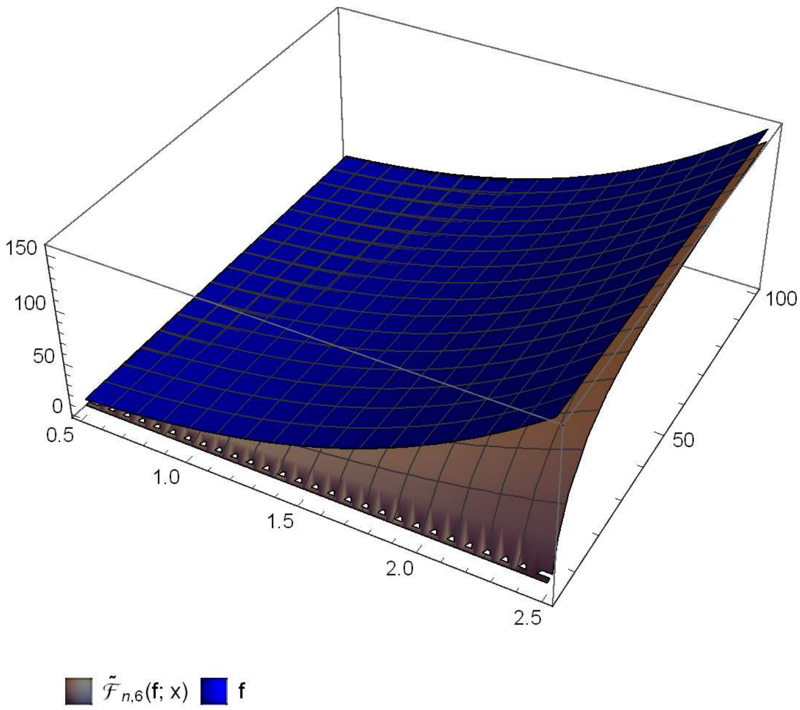

Figure 4.6: Convergence of $\widetilde{\mathcal{F}}_{n,s}(f;x)$ (Grey) towards $f(x)$ (Blue) while preserving test function e_6.

Theorem 4.3.5 *Let $f \in C_B[0,\infty)$, then there exists a constant $C > 0$ such that*

$$\left|\widetilde{\mathcal{F}}_{n,s}(f;x) - f(x)\right| \leq \omega_2\left(f, \sqrt{\delta_{n,s}}\right) + \omega\left(f, \left|\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} - 1\right| x\right),$$

where

$$\delta_{n,s} = \left[\frac{2n}{n-1}\left(\frac{(n-s+1)_s}{(n)_s}\right)^{2/s} - 4\left(\frac{(n-s+1)_r}{(n)_s}\right)^{1/s} + 2\right] x^2.$$

Proof 4.3.6 *We begin by defining an auxiliary sequence of operators,*

$$\mathcal{T}_{n,s}(f;x) = \widetilde{\mathcal{F}}_{n,s}(f;x) - f\left(\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} x\right) + f(x), \tag{4.12}$$

In view of Lemma 4.2.1, one can observe that the above defined operators preserve both constant and linear functions. Let $\mathfrak{h} \in C_B^2(0,\infty)$, where

$$C_B^2(0,\infty) = \{\mathfrak{h} \in C_B[0,\infty) : \mathfrak{h}', \mathfrak{h}'' \in C_B[0,\infty)\}$$

and $x, t \in (0,\infty)$, then by Taylor's expansion we have,

$$\mathfrak{h}(t) = \mathfrak{h}(x) + (t-x)\mathfrak{h}'(x) + \int_x^t (t-u)\mathfrak{h}''(u)du.$$

From (4.12) and the fact that $\mathcal{T}_{n,s}((t-x);x) = 0$, we have

$$\begin{aligned}
\left|\mathcal{T}_{n,s}(\mathfrak{h};x) - \mathfrak{h}(x)\right| &= \left|\mathcal{T}_{n,s}\left(\int_x^t (t-v)\mathfrak{h}''(v)dv; x\right)\right| \\
&\leqslant \left|\widetilde{\mathcal{F}}_{n,s}\left(\int_x^t (t-v)\mathfrak{h}''(v)dv, x\right)\right| \\
&\quad + \left|\int_x^{\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} x} \left(\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} x - v\right)\mathfrak{h}''(v)dv\right| \\
&\leqslant \left(\xi_2^{\widetilde{\mathcal{F}}_{n,s}}(x) + \left(\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} x - x\right)^2\right)\|\mathfrak{h}''\|.
\end{aligned} \tag{4.13}$$

Also, in view of Lemma 4.2.1, we have

$$\left|\widetilde{\mathcal{F}}_{n,s}(f;x)\right| \leqslant \|f\|. \tag{4.14}$$

Combining equations (4.12), (4.13) and (4.14), we get

$$\begin{aligned}
\left|\widetilde{\mathcal{F}}_{n,s}(f;x) - f(x)\right| &\leqslant \left|\mathcal{T}_{n,s}((f-\mathfrak{h});x) - (f-\mathfrak{h})(x)\right| + \left|\mathcal{T}_{n,s}(\mathfrak{h};x) - \mathfrak{h}(x)\right| \\
&\quad + \left|f(x) - f\left(\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} x\right)\right| \\
&\leqslant 4\|f-\mathfrak{h}\| + \left(\xi_2^{\widetilde{\mathcal{F}}_{n,s}}(x) + \left(\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} x - x\right)^2\right)\|\mathfrak{h}''\| \\
&\quad + \omega\left(f, \left|\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} x - x\right|\right).
\end{aligned}$$

Taking infimum over all $\mathfrak{h} \in C_B^2(0,\infty)$ *and using Peetre's K-functional defined in subsection* (1.1.3)*, we have*

$$\left|\widetilde{\mathcal{F}}_{n,s}(f;x) - f(x)\right| \leq \{\|f - \mathfrak{h}\| + \delta_{n,s}\|\mathfrak{h}''\|\} + \omega\left(f, \left|\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} x - x\right|\right)$$
$$\leq K_2(f, \delta_{n,s}) + \omega\left(f, \left|\left(\frac{(n-s+1)_s}{(n)_s}\right)^{1/s} x - x\right|\right).$$

Finally, using Theorem 2.4 (p 177) by Devore and Lorentz (44), we obtain the required result.

4.3.1 Functions of bounded variation

In the next section, we investigate the behaviour of operators (4.6) for functions having bounded variation in the interval $(0,\infty)$. We attempt to derive an estimate of their rate of pointwise convergence with the help of Lebesgue point of functions $f(x)$ and Lebesgue-Stieltjes Integral.

For the proof of this theorem, we begin by considering the equivalent form of the operators (4.6) as follows:

$$\widetilde{\mathcal{F}}_{n,s}(f;x) = \int_0^\infty \hbar_{n,s}(x,t) f(t)dt, \tag{4.15}$$

where

$$\hbar_{n,s}(x,t) = \begin{cases} 0, & t = 0 \\ \frac{(2n)!}{n!(n-1)!}\frac{t^{n-1}(\wp_s(n)x)^{n+1}}{(\wp_s(n)x+t)^{2n+1}}, & 0 < t < \infty. \end{cases}$$

Lemma 4.3.7 *For all* $x \in [0,\infty)$*, if* $0 \leq y < x$*, we have*

$$\vartheta_{n,s}(x,y) = \int_0^y \hbar_{n,s}(x,t)dt$$
$$\leq \frac{\xi_2^{\widetilde{\mathcal{F}}_{n,s}}(x)}{(x-y)^2} \leq \frac{Dx^2}{n(x-y)^2},$$

and if $x < z < \infty$*,*

$$1 - \vartheta_{n,s}(x,z) = \int_0^y \hbar_{n,s}(x,t)dt$$
$$\leq \frac{\xi_2^{\widetilde{\mathcal{F}}_{n,s}}(x)}{(z-x)^2} \leq \frac{Dx^2}{n(z-x)^2}.$$

for sufficiently large n and an arbitrary positive constant $D \geq 2$*.*

Theorem 4.3.8 *Consider a function f of bounded variation on every finite sub-interval of $(0,\infty)$ which satisfies the growth condition:*

$$|f(t)| \leq Rt^{\beta},$$

for some absolute constant R and some $\beta > 0$. Then for any $\epsilon > 0$ and $l \in \mathbb{N}$ such that $(2l \geq \beta)$, and adequately large n, we have

$$\left|\widetilde{\mathcal{F}}_{n,s}(f;x) - f(x)\right| \leq \frac{D}{n}\sum_{i=1}^{n} \bigvee_{x-\frac{x}{\sqrt{i}}}^{x+\frac{x}{\sqrt{i}}}(f) + \varepsilon\int_{x-\delta}^{x+\delta} \hbar_{n,s}(x,t)\,dt + R2^{2l}A(l)\,\frac{x^{2l}}{n^{l}},\ x \in (0,\infty)$$

where $\delta := \frac{x}{\sqrt{n}}$, $\bigvee_{a}^{b}(f)$ represents the total variation of f on any arbitrary finite sub–interval $[a,b]$ and D is an arbitrary positive constant.

Proof 4.3.9 *We can write,*

$$\begin{aligned}
\left|\widetilde{\mathcal{F}}_{n,s}(f;x) - f(x)\right| &= \left|\left[\left(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^{x} + \int_x^{x+\frac{x}{\sqrt{n}}} + \int_{x+\frac{x}{\sqrt{n}}}^{\infty}\right)\hbar_{n,s}(x,t)\left[f(t)-f(x)\right]dt\right]\right| \\
&\leq \left|\int_0^{x-\frac{x}{\sqrt{n}}} \hbar_{n,s}(x,t)\left(f(t)-f(x)\right)dt\right| + \left|\int_{x-\frac{x}{\sqrt{n}}}^{x} \hbar_{n,s}(x,t)\left(f(t)-f(x)\right)dt\right| \\
&+ \left|\int_x^{x+\frac{x}{\sqrt{n}}} \hbar_{n,s}(x,t)\left(f(t)-f(x)\right)dt\right| + \left|\int_{x+\frac{x}{\sqrt{n}}}^{\infty} \hbar_{n,s}(x,t)\left(f(t)-f(x)\right)dt\right| \\
&= \left|J_{1,s}(n,x)\right| + \left|J_{2,s}(n,x)\right| + \left|J_{3,s}(n,x)\right| + \left|J_{4,s}(n,x)\right|. \qquad (4.16)
\end{aligned}$$

We begin by computing the integrals $J_{2,s}(n,x)$ and $J_{3,s}(n,x)$ respectively. Assigning

$$F(t) := \int_t^x |f(y) - f(x)|\,dy,$$

then by recalling the definition (4.2.4) of Lebesgue point, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $0 < x - t \leq \delta$,

$$F(t) \leq \varepsilon(x-t). \qquad (4.17)$$

Let us assume that $\delta := \frac{x}{\sqrt{n}}$. Now integrating $J_{2,s}(n,x)$ by parts and employing (4.17), we obtain:

$$\begin{aligned}
\left|J_{2,s}(n,x)\right| &= \left|-F(x-\delta)\,\hbar_{n,s}(x,x-\delta) + \int_{x-\delta}^{x} F(t)\,\frac{\partial}{\partial t}\hbar_{n,s}(x,t)\,dt\right| \\
&\leq |-F(x-\delta)|\,\hbar_{n,s}(x,x-\delta) + \int_{x-\delta}^{x} |F(t)|\frac{\partial}{\partial t}\hbar_{n,s}(x,t)\,dt \\
&\leq \varepsilon\delta\hbar_{n,s}(x,x-\delta) + \varepsilon\int_{x-\delta}^{x}(x-t)\frac{\partial}{\partial t}\hbar_{n,s}(x,t)\,dt.
\end{aligned}$$

Again integrating by parts,

$$\begin{aligned}\left|J_{2,s}(n,x)\right| &\le \varepsilon\delta\hbar_{n,s}(x,x-\delta)+\varepsilon\left\{-\delta\hbar_{n,s}(x,x-\delta)+\int_{x-\delta}^{x}\hbar_{n,s}(x,t)\,dt\right\}\\ &= \varepsilon\int_{x-\delta}^{x}\hbar_{n,s}(x,t)\,dt. \end{aligned}\tag{4.18}$$

Using analogous approach for $J_{3,s}(n,x)$, we obtain the inequality:

$$\left|J_{3,s}(n,x)\right| \le \varepsilon\int_{x}^{x+\delta}\hbar_{n,s}(x,t)\,dt. \tag{4.19}$$

Next we shall use Lebesgue–Stieltjes integral to estimate the integrals $J_{1,s}(n,x)$ and $J_{4,s}(n,x)$ respectively. We begin by estimating $J_{1,s}(n,x)$ using the following Lebesgue–Stieltjes representation:

$$\begin{aligned}J_{1,s}(n,x) &= \int_0^{x-\frac{x}{\sqrt{n}}}(f(t)-f(x))\,d_t(\vartheta_{n,s}(x,t))\\ &= \left(f\left(x-\frac{x}{\sqrt{n}}\right)-f(x)\right)\vartheta_{n,s}\left(x,x-\frac{x}{\sqrt{n}}\right)-(f(0)-f(x))\,\vartheta_{n,s}(x,0)\\ &\quad-\int_0^{x-\frac{x}{\sqrt{n}}}\vartheta_{n,s}(x,t)\,d_t(f(t)-f(x)).\end{aligned}$$

Since $\left(f\left(x-\frac{x}{\sqrt{n}}\right)-f(x)\right)\le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)$, it follows that,

$$\left|J_{1,s}(n,x)\right| \le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)\left|\vartheta_{n,s}\left(x,x-\frac{x}{\sqrt{n}}\right)\right|+\int_0^{x-\frac{x}{\sqrt{n}}}\left|\vartheta_{n,s}(x,t)\right|d_t\left(-\bigvee_{t}^{x}(f)\right).$$

From Lemma 4.3.7, we see that,

$$\vartheta_{n,s}\left(x,x-\frac{x}{\sqrt{n}}\right)\le\frac{Dx^2}{n}\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2}.$$

Accordingly,

$$\left|J_{1,s}(n,x)\right| \le \frac{Dx^2}{n}\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)+\frac{Dx^2}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{1}{(x-t)^2}d_t\left(-\bigvee_{t}^{x}(f)\right).$$

Using integration by parts in the last integral, we have

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{1}{(x-t)^2}d_t\left(-\bigvee_{t}^{x}(f)\right) &= -\frac{1}{(x-t)^2}\bigvee_{t}^{x}(f)\Big|_0^{x-\frac{x}{\sqrt{n}}}+\int_0^{x-\frac{x}{\sqrt{n}}}\frac{2}{(x-t)^3}\bigvee_{t}^{x}(f)\,dt\\ &= -\frac{n}{x^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)+\frac{1}{x^2}\bigvee_{0}^{x}(f)+\int_0^{x-\frac{x}{\sqrt{n}}}\frac{2}{(x-t)^3}\bigvee_{t}^{x}(f)\,dt.\end{aligned}$$

Substituting $t=x-\frac{x}{\sqrt{\lambda}}$ in the last integral, we have

$$\int_0^{x-\frac{x}{\sqrt{n}}}\frac{2}{(x-t)^3}\bigvee_{t}^{x}(f)\,dt=\frac{1}{x^2}\int_1^{n}\bigvee_{x-\frac{x}{\sqrt{\lambda}}}^{x}(f)\,d\lambda=\frac{1}{x^2}\sum_{i=1}^{n}\bigvee_{x-\frac{x}{\sqrt{i}}}^{x}(f)$$

As a result, we have

$$\begin{aligned}\left|J_{1,s}(n,x)\right| &\le \frac{Dx^2}{n}\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)+\frac{Dx^2}{n}\left(-\frac{n}{x^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)+\frac{1}{x^2}\bigvee_{0}^{x}(f)+\frac{1}{x^2}\sum_{i=1}^{n}\bigvee_{x-\frac{x}{\sqrt{i}}}^{x}(f)\right)\\ &= \frac{D}{n}\left(\bigvee_{0}^{x}(f)+\sum_{i=1}^{n}\bigvee_{x-\frac{x}{\sqrt{i}}}^{x}(f)\right). \qquad (4.20)\end{aligned}$$

Finally to estimate $J_{4,s}(n,x)$, we introduce the following function:

$$\rho_x(t)=\begin{cases} f(t) & 0\le t\le 2x\\ f(2x) & 2x<t<\infty\end{cases}$$

We rewrite $J_{4,s}(n,x)$ as:

$$\begin{aligned}\left|J_{4,s}(n,x)\right| &= \int_{x+\frac{x}{\sqrt{n}}}^{\infty}\rho_x(t)\,d_t\left(\vartheta_{n,s}(x,t)\right)+\int_{2x}^{\infty}(f(t)-f(2x))d_t\left(\vartheta_{n,s}(x,t)\right)\\ &=: J_{4,s}^{(1)}(n,x)+J_{4,s}^{(2)}(n,x)\end{aligned}$$

Next we evaluate the integral $J_{4,s}^{(1)}(n,x)$ as follows:

$$\begin{aligned}J_{4,s}^{(1)}(n,x) = \lim_{a\to\infty}\Bigg\{ & f\left(x+\frac{x}{\sqrt{n}}\right)\left(1-\vartheta_{n,s}\left(x,x+\frac{x}{\sqrt{n}}\right)\right)\\ &+\rho_x(a)\left(\vartheta_{n,s}(x,a)-1\right)+\int_{2x}^{a}f(t)\,d_t\left(\vartheta_{n,s}(x,t)\right)\Bigg\}.\end{aligned}$$

According to Lemma 4.3.7, we obtain

$$\begin{aligned}J_{4,s}^{(1)}(n,x) &= \frac{Dx^2}{n}\lim_{a\to\infty}\left\{\frac{n}{x^2}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f)+\frac{\rho_x(a)}{(a-x)^2}+\int_0^x\frac{1}{(t-x)^2}d_t\left(\bigvee_x^t(\rho_x)\right)\right\}\\ &= \frac{Dx^2}{n}\left\{\frac{n}{x^2}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f)+\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}d_t\left(\bigvee_x^t(f)\right)\right\}. \qquad (4.21)\end{aligned}$$

Integrating by parts the last integral, we have

$$\begin{aligned}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}d_t\left(\bigvee_x^t(f)\right) &= \frac{1}{(t-x)^2}\bigvee_x^t(f)\Bigg|_{x+\frac{x}{\sqrt{n}}}^{2x}+\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{2}{(x-t)^3}\bigvee_x^t(f)\,dt\\ &= \frac{1}{x^2}\bigvee_x^{2x}(f)-\frac{n}{x^2}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f)+\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{2}{(x-t)^3}\bigvee_x^t(f)\,dt. \qquad (4.22)\end{aligned}$$

Substituting $t=x+\frac{x}{\sqrt{\lambda}}$ in (4.22), we get

$$\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{2}{(x-t)^3}\bigvee_x^t(f)\,dt=\frac{1}{x^2}\int_1^n\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f)\,d\lambda=\frac{1}{x^2}\sum_{i=1}^{n}\bigvee_{x}^{x+\frac{x}{\sqrt{i}}}(f) \qquad (4.23)$$

Finally combining (4.21), (4.22) and (4.23), we get

$$J^{(1)}_{4,s}(n,x) \leq \frac{2D}{n}\left\{\bigvee_{x}^{2x}(f) + \sum_{i=1}^{n-1}\bigvee_{x}^{x+\frac{x}{\sqrt{i}}}(f_x)\right\}$$
$$= \frac{D}{n}\sum_{i=1}^{n}\bigvee_{x}^{x+\frac{x}{\sqrt{i}}}(f),$$

where D is an arbitrary positive constant.

This is the required estimate of $\left|J^{(1)}_{4,s}(n,x)\right|$. *Next we proceed to estimate* $J^{(2)}_{4,s}(n,x)$. *It is noteworthy that for every* $t > 0$, *there exists an integer* $l(2l > \beta)$ *such that* $f(t) = O\left(t^{2l}\right)$. *Also for some* $\beta > 0, R > 0$, *and sufficiently large t, function* $f(t)$ *fulfills the growth condition* $|f(t)| \leq Rt^{\beta}$. *Therefore whenever* $t \geq 2x \Rightarrow 2(t-x) \geq t$, *we get*

$$J^{(2)}_{4,s}(n,x) \leq R2^{2l}A(l)\frac{x^{2l}}{n^{l}}.$$

Combining $J^{(1)}_{4,s}(n,x)$ *and* $J^{(2)}_{4,s}(n,x)$, *we get*

$$\left|J_{4,s}(n,x)\right| \leq \frac{D}{n}\sum_{i=1}^{n}\bigvee_{x}^{x+\frac{x}{\sqrt{i}}}(f) + R2^{2l}A(l)\frac{x^{2l}}{n^{l}}. \tag{4.24}$$

Lastly, using equations (4.18), (4.19), (4.20) and (4.24) in (4.16), we have

$$\begin{aligned}\left|\widetilde{\mathcal{F}}_{n,s}(f;x) - f(x)\right| &= \left|J_{1,s}(n,x)\right| + \left|J_{2,s}(n,x)\right| + \left|J_{3,s}(n,x)\right| + \left|J_{4,s}(n,x)\right| \\ &\leq \frac{D}{n}\left(\bigvee_{0}^{x}(f) + \sum_{i=1}^{n}\bigvee_{x-\frac{x}{\sqrt{i}}}^{x}(f)\right) + \varepsilon\int_{x-\delta}^{x+\delta}\hbar_{n,s}(x,t)\,dt \\ &\quad + \frac{D}{n}\sum_{i=1}^{n}\bigvee_{x}^{x+\frac{x}{\sqrt{i}}}(f) + R2^{2l}A(l)\frac{x^{2l}}{n^{l}} \\ &\leq \frac{D}{n}\sum_{i=1}^{n}\bigvee_{x-\frac{x}{\sqrt{i}}}^{x+\frac{x}{\sqrt{i}}}(f) + \varepsilon\int_{x-\delta}^{x+\delta}\hbar_{n,s}(x,t)\,dt + R2^{2l}A(l)\frac{x^{2l}}{n^{l}},\end{aligned}$$

which is the required result and the proof is done.

Chapter 5

Approximation by generalized Baskakov Kantorovich operators of arbitrary order

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This chapter addresses an improved Kantorovich form of the Baskakov type operators using arbitrary sequences. These operators preserve exponential functions of the form $a^{-x},\ a > 1$ *and approximate functions with arbitrary order, i.e. one can achieve significantly better approximation with appropriate choices of sequences. With proof of important lemmas and theorems, we have examined the various approximation properties of the proposed operators. Finally, we have demonstrated the convergence of the proposed operators using arbitrary set of sequences.*

5.1 Introduction

The cornerstone of the theory of approximation of functions is the Bohman-Korovkin criterion (30; 92). It established that for any continuous function f on $[a, b]$, if a sequence of linear positive operators $L_n(e_k; x)$ converges uniformly to $e_k,\ k \in \{0, 1, 2\}$ on $[a, b]$, then the sequence of operators $L_n(f; x)$ converges uniformly to f on $[a, b]$. In (31), an extension of the above Korovkin theorem was given by Boyanov and Veselinov for the exponential functions. Holhoş (76) later extended their work to determine the rate of convergence of functions of exponential type on $(0, \infty)$.

In 1937, I. Chlodovsky (36) used arbitrary sequences (b_n) to introduce certain polynomials and generalize the renowned Bernstein polynomials. For a function f defined on $[0,\infty)$ and bounded on every finite interval $[0,b_n]\subset[0,\infty)$, the classical Bernstein-Chlodovsky operators are defined as:

$$C_{n,k}^{[b_n]}(f;x)=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}f\left(\frac{b_nk}{n}\right),$$

where (b_n) is an increasing sequence of positive reals with the property that

$$\lim_{n\to\infty}b_n=\infty\ ,\ \lim_{n\to\infty}\frac{b_n}{n}=0.$$

When $b_n=1,\ n\in\mathbb{N}$, the Bernstein-Chlodovsky operators transform into the classical Bernstein operators. The advantage of Chlodovsky polynomials is that they contain a factor b_n, tending to infinity, having a certain degree of freedom. Therefore they turn out to be generally more efficient in approximating functions than their corresponding classical counterpart.

Recently Agratini (17) highlighted certain classes of discrete and integral type linear positive operators on unbounded intervals for which the order of approximation can be made arbitrarily small. Also, Gal-Opris (60) considered several kinds of modified Baskakov operators using arbitrary sequences (λ_n), $n\in\mathbb{N}$ for which $\lim_{n\to\infty}\lambda_n=0$ and suggested that the order of approximation can be made as small as possible by appropriately choosing λ_n. Very recently, Gupta-Holhoş (68) considered certain Baskakov operators consisting of two arbitrary positive real sequences (r_n) and (s_n) such that the operators preserved exponential functions of the form a^{-x} and also provided pliability to achieve good approximation for appropriately chosen sequences. These operators are defined as:

$$L_n^{[s_n]}(f;x)=\sum_{k=0}^{\infty}p_{n,k}^{[s_n]}(x)f\left(\frac{k}{r_n}\right),\quad x\geq0,\ n\in\mathbb{N} \tag{5.1}$$

where

$$p_{n,k}^{[s_n]}(x)=(-1)^k\binom{-n/s_n}{k}\frac{(s_nx)^k}{(1+s_nx)^{\frac{n}{s_n}+k}},$$

satisfying the conditions

$$0<s_n\leq\frac{n}{2},\ \lim_{n\to\infty}\frac{s_n}{n}=0,\ \lim_{n\to\infty}r_n=\infty.$$

To study articles related to the above researches, one can see the following articles (22; 40; 97; 112; 114; 117). In this research work, we aim to consider a Kantorovich variant of the operators (5.1) which preserve the exponential functions a^{-x} and also fulfil the need to achieve better rate of approximation for certain functions. For arbitrary sequences (r_n) and (s_n), satisfying the conditions $\lim_{n\to\infty}r_n=\infty$ and $\lim_{n\to\infty}\frac{s_n}{n}=0$, we begin by considering the Kantorovich form of the above operators as follows:

$$\mathcal{K}_n^{[s_n]}(f;x) = r_n \sum_{k=0}^{\infty} p_{n,k}^{[s_n]}(x) \int_{\frac{k}{r_n}}^{\frac{k+1}{r_n}} f(t)\,dt,$$

where $p_{n,k}^{[s_n]}(x)$ is same as defined in (5.1).

Now consider the following operators:

$$\mathcal{K}_n^{[s_n]}(f;\vartheta(n,x)) = \hat{\mathcal{K}}_n^{[s_n]}(f;x) = r_n \sum_{k=0}^{\infty} p_{n,k}^{[s_n]}(\vartheta(n,x)) \int_{\frac{k}{r_n}}^{\frac{k+1}{r_n}} f(t)\,dt, \tag{5.2}$$

Alternately, we can write,

$$\hat{\mathcal{K}}_n^{[s_n]}(f;x) = \mathcal{K}_n^{[s_n]}(f;x) \circ \vartheta(n,x),$$

where $\vartheta(n,x)$ shall be determined later by taking into consideration the property of p-reservation of exponential functions a^{-x}.

For fixed $a > 1$, assuming that the operators (5.2) preserve the exponential function a^{-x}, then

$$\begin{aligned}\mathcal{K}_n^{[s_n]}\left(a^{-t};\vartheta(n,x)\right) = a^{-x} &= r_n \sum_{k=0}^{\infty} p_{n,k}^{[s_n]}(\vartheta(n,x)) \int_{\frac{k}{r_n}}^{\frac{k+1}{r_n}} a^{-t}dt\\ &= \frac{r_n\left(1-a^{-1/r_n}\right)}{\ln(a)} \sum_{k=0}^{\infty} p_{n,k}^{[s_n]}(\vartheta(n,x))a^{-k/r_n}\\ &= \frac{r_n\left(1-a^{-1/r_n}\right)}{\ln(a)}\left(1+\left(1-a^{-1/r_n}\right)s_n\vartheta(n,x)\right)^{-n/s_n},\end{aligned}$$

which after simple manipulations gives

$$\vartheta(n,x) = \frac{\left(\frac{r_n\left(1-a^{-1/r_n}\right)a^x}{\ln(a)}\right)^{s_n/n} - 1}{s_n\left(1-a^{-1/r_n}\right)}.$$

Substituting this in (5.2), our modified operators can be rewritten in the form:

$$\begin{aligned}\hat{\mathcal{K}}_n^{[s_n]}(f;x) =& \frac{r_n}{\left(1-a^{-1/r_n}\right)^{-n/s_n}}\\ &\times \sum_{k=0}^{\infty}(-1)^k\binom{-n/s_n}{k}\left(\left(\frac{r_n\left(1-a^{-1/r_n}\right)a^x}{\ln(a)}\right)^{s_n/n} - 1\right)^k\\ &\times\left(\left(\frac{r_n\left(1-a^{-1/r_n}\right)a^x}{\ln(a)}\right)^{s_n/n} - a^{-1/r_n}\right)^{-\frac{n}{s_n}-k} \int_{\frac{k}{r_n}}^{\frac{k+1}{r_n}} f(t)dt.\end{aligned}$$

Next, we move on to some auxiliary results, which will be used further to prove our main theorems.

5.2 Auxiliary Results

Lemma 5.2.1 *Let us represent the p^{th} moment of the operators (5.2) by $Q_{n,p}(x) = \hat{\mathcal{K}}_n^{[s_n]}\left(e_p; x\right)$, where $e_p(t) = t^p$, $p = 0, 1, 2, 3, 4$. Then*

$$Q_{n,0}(x) = 1;$$

$$Q_{n,1}(x) = \frac{1}{2r_n} + \frac{n\vartheta(n,x)}{r_n};$$

$$Q_{n,2}(x) = \frac{1}{3r_n^2} + \frac{n\vartheta(n,x)(2+s_n\vartheta(n,x))}{r_n^2} + \frac{n^2\vartheta^2(n,x)}{r_n^2};$$

$$Q_{n,3}(x) = \frac{1}{4r_n^3} + \frac{n\vartheta(n,x)\left(7+9s_n\vartheta(n,x)+4s_n{}^2\vartheta_n^2(x)\right)}{2r_n^3} + \frac{3(n\vartheta(n,x))^2(3+2s_n\vartheta(n,x))}{2r_n^3} + \frac{n^3\vartheta^3(n,x)}{r_n^3};$$

$$Q_{n,4}(x) = \frac{1}{5r_n^4} + \frac{(n\vartheta(n,x))^2\left(15+24s_n\vartheta(n,x)+11s_n{}^2\vartheta^2(n,x)\right)}{r_n^4} + \frac{n\vartheta(n,x)\left(6+15s_n\vartheta(n,x)+16s_n{}^2\vartheta^2(n,x)+6s_n{}^3\vartheta^3(n,x)\right)}{r_n^4} + \frac{2(n\vartheta(n,x))^3(4+3s_n\vartheta(n,x))}{r_n^4} + \frac{n^4\vartheta^4(n,x)}{r_n^4}.$$

Proof 5.2.2 *First notice that*

$$\int_{\frac{k}{r_n}}^{\frac{k+1}{r_n}} t^p dt = \frac{1}{p+1}\left(\frac{k}{r_n}\right)^{p+1} \sum_{l=1}^{p+1} \binom{p+1}{l}\left(\frac{1}{k}\right)^l. \tag{5.3}$$

Next, from calculation analogous to (68), for $m = 0, 1, 2, \cdots$, consider the sum

$$\begin{aligned} \rho_m(\vartheta(n,x)) =& r_n \sum_{k=0}^{\infty} p_{n,k}^{[s_n]}(\vartheta(n,x)) \frac{k(k-1)(k-2)\cdots(k-m)}{r_n^{m+2}} \\ =& \frac{n(n+s_n)(n+2s_n)\cdots(n+ms_n)}{r_n^{m+1}}(\vartheta(n,x))^{m+1}. \end{aligned} \tag{5.4}$$

Finally, using the fact that $p_{n,k}^{[s_n]}(\vartheta(n,x)) = 1$, the integral (5.3) and the sum (5.4), we obtain

$$Q_{n,1}(x) = \rho_0(\vartheta(n,x)) + \frac{1}{2r_n},$$

$$Q_{n,2}(x) = \rho_1(\vartheta(n,x)) + \frac{2}{r_n}\rho_0(\vartheta(n,x)) + \frac{1}{3r_n^2},$$

and so on. The rest of the moments can be obtained in a similar way. Hence the proof.

Lemma 5.2.3 *The central moments $\eta_{n,p}^{[s_n]}(x) = \hat{\mathcal{K}}_n^{[s_n]}((t-x)^p; x)$ of the operators (5.2) are given as:*

$$\eta_{n,1}^{[s_n]}(x) = \frac{1}{2r_n} + \frac{n\vartheta(n,x)}{r_n} - x;$$

$$\eta_{n,2}^{[s_n]}(x) = \frac{1}{3r_n^2} - \frac{x}{r_n} + \frac{n\vartheta(n,x)}{r_n}\left(\frac{2}{r_n} + \frac{s_n}{n}\cdot\frac{n\vartheta(n,x)}{r_n}\right) + \left(\frac{n\vartheta(n,x)}{r_n} - x\right)^2;$$

$$\begin{aligned}\eta_{n,4}^{[s_n]}(x) = {} & x^4 - \frac{2x^3}{r_n} + \frac{2x^2}{{r_n}^2} - \frac{x}{{r_n}^3} + \frac{1}{5{r_n}^4} - \frac{4n\vartheta(n,x)x^3}{r_n} + \frac{12n\vartheta(n,x)x^2}{{r_n}^2} + \frac{6\vartheta^2(n,x)n^2x^2}{{r_n}^2} \\ & + \frac{6s_n n\vartheta^2(n,x)x^2}{{r_n}^2} - \frac{14n\vartheta(n,x)x}{{r_n}^3} - \frac{18n^2\vartheta^2(n,x)x}{{r_n}^3} - \frac{4n^3\vartheta^3(n,x)x}{{r_n}^3} \\ & - \frac{18s_n n\vartheta^2(n,x)x}{{r_n}^3} - \frac{12s_n n^2\vartheta^3(n,x)x}{{r_n}^3} - \frac{8s_n^2 n\vartheta^3(n,x)x}{{r_n}^3} + \frac{6n\vartheta(n,x)}{{r_n}^4} \\ & + \frac{15n^2\vartheta^2(n,x)}{{r_n}^4} + \frac{8n^3\vartheta^3(n,x)}{{r_n}^4} + \frac{n^4\vartheta^4(n,x)}{{r_n}^4} + \frac{15s_n n\vartheta^2(n,x)}{{r_n}^4} + \frac{24s_n n^2\vartheta^3(n,x)}{{r_n}^4} \\ & + \frac{6s_n n^3\vartheta^4(n,x)}{{r_n}^4} + \frac{16s_n^2 n\vartheta^3(n,x)}{{r_n}^4} + \frac{11s_n^2 n^2\vartheta^4(n,x)}{{r_n}^4} + \frac{6s_n^3 n\vartheta^4(n,x)}{{r_n}^4}.\end{aligned}$$

Lemma 5.2.4 *Using the notation $x_n \sim y_n$ for the limit $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$, the asymptotic behaviour of the central moments of the operators (5.2) can be expressed in the following way:*

$$\eta_{n,1}^{[s_n]}(x) \sim \frac{x\ln(a)}{2}\left(\frac{s_n x}{n} + \frac{1}{r_n}\right),$$

$$\eta_{n,2}^{[s_n]}(x) \sim x\left(\frac{s_n x}{n} + \frac{1}{r_n}\right),$$

$$\eta_{n,4}^{[s_n]}(x) \sim 3x^2\left(\frac{s_n x}{n} + \frac{1}{r_n}\right)^2,$$

as $n \to \infty$.

Proof 5.2.5 *Observe that*

$$\frac{1}{r_n\left(1 - a^{-1/r_n}\right)} \sim \frac{1}{2r_n} + \frac{1}{\ln(a)}, \tag{5.5}$$

and,

$$\left[\left(\frac{r_n\left(1 - a^{-1/r_n}\right)a^x}{\ln(a)}\right)^{s_n/n} - 1\right] \sim \frac{s_n}{n}\left(x\ln(a) - \frac{\ln(a)}{2r_n} + \frac{s_n}{2n}(x\ln(a))^2\right).$$

Therefore, we have

$$\frac{n\vartheta(n,x)}{r_n} \sim x + \frac{x\ln(a)}{2r_n} - \frac{1}{2r_n} + \frac{s_n}{2n}x^2\ln(a),$$

which implies that,

$$\eta_{n,1}^{[s_n]}(x) = \frac{n\vartheta(n,x)}{r_n} - x + \frac{1}{2r_n} \sim \frac{x\ln(a)}{2}\left(\frac{1}{r_n} + \frac{s_n x}{n}\right).$$

This is our first relation. Moreover, in view of the first central moment, we can say that

$$\frac{n\vartheta(n,x)}{r_n} \to x, \text{as } \ n \to \infty.$$

Using this fact and neglecting higher order terms, we can easily obtain asymptotic behaviours of other higher order central moments.

Lemma 5.2.6 *For every $x \in [0,\infty)$ and $n \geq 1$, we have*

$$\hat{\mathcal{K}}_n^{[s_n]}\left(\left|a^{-t} - a^{-x}\right|^2; x\right) \leq \mathcal{W}_a \cdot \max\left(\frac{s_n}{n}, \frac{1}{r_n}\right),$$

where $\mathcal{W}_a$ is a constant depending on $a > 1$, but independent of x and n.

Proof 5.2.7 *We begin with the fact that $\hat{\mathcal{K}}_n^{[s_n]}$ preserves both constants and exponential functions a^{-x}, therefore we can write,*

$$\hat{\mathcal{K}}_n^{[s_n]}\left(\left|a^{-t}-a^{-x}\right|^2;x\right)=\hat{\mathcal{K}}_n^{[s_n]}\left(a^{-2t};x\right)-a^{-2x}.$$

Now,

$$\begin{aligned}\hat{\mathcal{K}}_n^{[s_n]}\left(a^{-2t};x\right)&=r_n\sum_{k=0}^{\infty}\binom{-n/s_n}{k}\frac{(-s_n\vartheta(n,x))^k}{(1+s_n\vartheta(n,x))^{n/s_n+k}}\int_{\frac{k}{r_n}}^{\frac{k+1}{r_n}}a^{-2t}dt\\&=\frac{r_n\left(1-a^{-2/r_n}\right)}{2\ln(a)}\sum_{k=0}^{\infty}\binom{-n/s_n}{k}\frac{\left(-s_n\vartheta(n,x)\,a^{-2/r_n}\right)^k}{(1+s_n\vartheta(n,x))^{n/s_n+k}}\\&=\frac{r_n\left(1-a^{-2/r_n}\right)}{2\ln(a)}\left[1+s_n\vartheta(n,x)\left(1-a^{-2/r_n}\right)\right]^{-n/s_n}\\&=\frac{r_n\left(1-a^{-2/r_n}\right)}{2\ln(a)}\\&\quad\times\left[1+\left(1+a^{-1/r_n}\right)\left(\left(\frac{r_n\left(1-a^{-1/r_n}\right)a^x}{\ln(a)}\right)^{s_n/n}-1\right)\right]^{-n/s_n}.\end{aligned}$$

For convenience, let us take the notation

$$\lambda_a^{[r_n]}(x):=\frac{r_n\left(1-a^{-1/r_n}\right)a^x}{\ln(a)}.$$

Then, we can write

$$\begin{aligned}\hat{\mathcal{K}}_n^{[s_n]}\left(a^{-2t};x\right)-a^{-2x}=&e^{\ln\left(\frac{r_n\left(1-a^{-2/r_n}\right)}{2\ln(a)}\right)-\frac{n}{s_n}\ln\left(1+\left(1+a^{-1/r_n}\right)\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)}\\&-e^{-2x\ln(a)}.\end{aligned}$$

Next we define a function $\hbar:[0,\infty)\to\mathbb{R}$ such that

$$\begin{aligned}\hbar(x)=&\ln\left(\frac{r_n\left(1-a^{-2/r_n}\right)}{2\ln(a)}\right)-\frac{n}{s_n}\ln\left(1+\left(1+a^{-1/r_n}\right)\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)\\&+2x\ln(a).\end{aligned}$$

Then, in view of inequality $\frac{\beta-\gamma}{\ln(\beta)-\ln(\gamma)}<\beta,\ \ 0<\gamma<\beta$, we obtain

$$\begin{aligned}\hat{\mathcal{K}}_n^{[s_n]}\left(a^{-2t};x\right)-a^{-2x}&\le\hbar(x)\\&\times e^{\ln\left(\frac{r_n\left(1-a^{-2/r_n}\right)}{2\ln(a)}\right)-\frac{n}{s_n}\ln\left(1+\left(1+a^{-1/r_n}\right)\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)}\end{aligned}\tag{5.6}$$

Now to estimate $\hbar(x)$*, we apply* $\ln(1+\beta) \geq \frac{\beta}{1+\beta}$, $\beta \geq 0$ *to the logarithm part of* $\hbar(x)$*. We obtain:*

$$\hbar(x) \leq \ln\left(\frac{r_n\left(1-a^{-2/r_n}\right)}{2\ln(a)}\right) - \frac{n}{s_n}\left[\frac{\left(1+a^{-1/r_n}\right)\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)}{\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)(1+a^{-1/r_n})+1}\right] + 2x\ln(a).$$

Since, $\ln\left(\frac{r_n\left(1-a^{-2/r_n}\right)}{2\ln(a)}\right) \leq 0$*, we can write,*

$$\hbar(x) \leq 2x\ln(a) - \frac{n}{s_n}\left[\frac{\left(1+a^{-1/r_n}\right)\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)}{\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)(1+a^{-1/r_n})+1}\right].$$

For every $\beta \in \mathbb{R}$*, making use of the inequality* $a^\beta - 1 \geq \beta\ln(a)$*, and with further simplifications, we get:*

$$\hbar(x) \leq \frac{x\ln(a)\left(1-a^{-1/r_n}+2\left(1+a^{-1/r_n}\right)\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)}{\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)(1+a^{-1/r_n})+1} - \left[\frac{\left(1+a^{-1/r_n}\right)\ln\left(\frac{r_n\left(1-a^{-1/r_n}\right)}{\ln(a)}\right)}{\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)(1+a^{-1/r_n})+1}\right].$$

By application of the inequalities $1-a^{-\beta} \leq \beta\ln(a)$ *and* $a^\beta - 1 \leq \beta a^\beta \ln(a)$, $\beta \in \mathbb{R}$*, (which are alternate forms of the same inequality), in the first term and in the last term of (5.5), we get:*

$$\hbar(x) \leq \frac{x\ln(a)\left(\frac{\ln(a)}{r_n}+\frac{2s_n}{n}\left(1+a^{-1/r_n}\right)\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}\ln\left(\lambda_a^{[r_n]}(x)\right)\right)}{\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)(1+a^{-1/r_n})+1} + \frac{\frac{\left(1+a^{-1/r_n}\right)\ln(a)}{r_n}}{\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)(1+a^{-1/r_n})+1}.$$

Substituting this bound of $\hbar(x)$ *in (5.6), we obtain:*

$$\hat{\mathcal{K}}_n^{[s_n]}\left(a^{-2t};x\right) - a^{-2x} \leq \frac{\frac{r_n\left(1-a^{-2/r_n}\right)}{2\ln(a)}}{\left(1+(1+a^{-1/r_n})\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)^{n/s_n+1}} \times\left[x\ln(a)\left(\frac{\ln(a)}{r_n}+\frac{2s_n}{n}\left(1+a^{-1/r_n}\right)\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}\ln\left(\lambda_a^{[r_n]}(x)\right)\right)+\frac{\left(1+a^{-1/r_n}\right)\ln(a)}{r_n}\right].$$

Since $\frac{r_n\left(1-a^{-2/r_n}\right)}{2\ln(a)} \leq 1$*, therefore we can now write,*

$$\hat{\mathcal{K}}_n^{[s_n]}\left(a^{-2t};x\right) - a^{-2x} \leq \max\left(\frac{s_n}{n}, \frac{1}{r_n}\right) \tag{5.7}$$

$$\cdot \frac{\left(1+2x\left(1+a^{-1/r_n}\right)\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}\right) x\ln^2(a) + \left(1+a^{-1/r_n}\right)\ln(a)}{\left(1+(1+a^{-1/r_n})\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)^{n/s_n+1}}. \tag{5.8}$$

We observe that

$$\frac{\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}}{\left(1+(1+a^{-1/r_n})\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)} \leq 1.$$

Therefore, the inequality (5.7) *becomes,*

$$\hat{\mathcal{K}}_n^{[s_n]}\left(a^{-2t};x\right) - a^{-2x} \leq \max\left(\frac{s_n}{n}, \frac{1}{r_n}\right) \cdot \left[\frac{x\ln^2(a)\left(1+2x\left(1+a^{-1/r_n}\right)\right) + \left(1+a^{-1/r_n}\right)\ln(a)}{\left(1+(1+a^{-1/r_n})\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)^{n/s_n}}\right]$$

Now, all we need to do is to find an estimate of the upper bound of

$$w(x) = \left[\frac{x\ln^2(a)\left(1+2x\left(1+a^{-1/r_n}\right)\right) + \left(1+a^{-1/r_n}\right)\ln(a)}{\left(1+(1+a^{-1/r_n})\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)^{n/s_n}}\right].$$

Using Bernoulli's inequality,

$$(1+\beta)^{\zeta} \geq \left(1+\frac{\zeta\beta}{2}\right)^2, \ \ \beta \geq 0, \zeta \geq 2$$

and by choosing $\zeta = \frac{n}{s_n} \geq 2$*, we have:*

$$\left(1+\left(1+a^{-1/r_n}\right)\left(\lambda_a^{[r_n]}(x)^{s_n/n}-1\right)\right)^{n/s_n} \geq \left(1+\frac{n}{2s_n}\left(1+a^{-1/r_n}\right) \times\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)^2.$$

Again, using the inequality $a^{\beta}-1 \geq \beta\ln(a)$ *and further simplifications, we have:*

$$\left(1+\frac{n}{2s_n}\left(1+a^{-1/r_n}\right)\left(\left(\lambda_a^{[r_n]}(x)\right)^{s_n/n}-1\right)\right)^2 \geq \left(1+\frac{\left(1+a^{-1/r_n}\right)}{4}x\ln(a)\right)^2$$

Changing the variable by taking $z = \frac{\left(1+a^{-1/r_n}\right)x\ln(a)}{4}$*, and notice that* $1 \leq \left(1+a^{-1/r_n}\right) \leq 2$ *for* $n \geq 1$*, then*

$$w(x) \leq \max_{z\geq 0} \frac{4z\ln(a)+32z^2+2\ln(a)}{(1+z)^2} = \mathcal{W}_a$$

where, $\mathcal{W}_a$ is a constant depending only on a and independent of n and x. Using mathematica software, the value of $\mathcal{W}_a$ can be calculated as:

$$\mathcal{W}_a = \begin{cases} 32, & \ln(a) \leq 16; \\ 2\ln(a), & \ln(a) > 16. \end{cases}$$

5.3 Main Results

Let $C^{\star}[0,\infty)$ represent the Banach space of all real valued continuous functions on the unbounded interval $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and is finite, endowed with the uniform norm $\|f\|_{[0,\infty)} = \sup_{x\in[0,\infty)} |f(x)|$.

For the given space, we estimate the error of approximation $|\hat{\mathcal{K}}_n^{[s_n]}(f;x) - f(x)|$ for uniformly continuous functions in this space using the following modulus of continuity ϖ with argument δ and defined as

$$\varpi(f,\delta) = \sup_{\substack{t,x\geq 0 \\ |a^{-t}-a^{-x}|\leq\delta}} |f(t) - f(x)|.$$

This modulus of continuity is a particular case of (76) and (78) and holds the following property:

$$|f(t) - f(x)| \leq \left(1 + \frac{\left|a^{-t} - a^{-x}\right|}{\delta}\right)\varpi(f,\delta), \quad \delta > 0. \tag{5.9}$$

Theorem 5.3.1 *For arbitrary sequences (s_n), (r_n) and $f \in C^{\star}[0,\infty)$, we have*

$$\left\|\hat{\mathcal{K}}_n^{[s_n]}(f;x) - f(x)\right\|_{[0,\infty)} \leq 2\cdot\varpi\left(f, \sqrt{\mathcal{W}_a \cdot \max\left(\frac{s_n}{n}, \frac{1}{r_n}\right)}\right), \; n \in \mathbb{N},$$

where $\mathcal{W}_a$ is calculated in Lemma 5.2.6.

Proof 5.3.2 *From the fact that operators $\hat{\mathcal{K}}_n^{[s_n]}$ preserve the constant functions, we can write*

$$\left|\hat{\mathcal{K}}_n^{[s_n]}(f;x) - f(x)\right| \leq \hat{\mathcal{K}}_n^{[s_n]}(|f(t) - f(x)|;x).$$

Applying operators $\hat{\mathcal{K}}_n^{[s_n]}$ on both sides of property (5.9), we get

$$\hat{\mathcal{K}}_n^{[s_n]}(|f(t) - f(x)|;x) \leq \left(1 + \frac{\hat{\mathcal{K}}_n^{[s_n]}\left(\left|a^{-t} - a^{-x}\right|;x\right)}{\delta_n}\right)\varpi(f,\delta_n).$$

With the use of Cauchy-Schwarz inequality, we obtain

$$\hat{\mathcal{K}}_n^{[s_n]}\left(|f(t)-f(x)|;x\right)\leq\left(1+\frac{\sqrt{\hat{\mathcal{K}}_n^{[s_n]}\left(|a^{-t}-a^{-x}|^2;x\right)}}{\delta_n}\right)\varpi\left(f,\delta_n\right).$$

Choosing $\delta_n = \sup\limits_{x\in[0,\infty)} \sqrt{\hat{\mathcal{K}}_n^{[s_n]}\left(|a^{-t}-a^{-x}|^2;x\right)}$, *and applying the estimates from Lemma 5.2.6, we obtain the desired result.*

Remark 5.3.3 *Since the rate of approximation depends on the choice of sequences* (r_n) *and* (s_n)*, therefore one can achieve better approximation with appropriate choices of these sequences.*

Theorem 5.3.4 *Let* $f, f^{'}, f^{''} \in C^{\star}[0,\infty)$*, then for any* $x\in[0,\infty)$*, we have*

$$\left|\hat{\mathcal{K}}_n^{[s_n]}(f;x)-f(x)-f'(x)\,\eta_{n,1}^{[s_n]}(x)-\frac{f''(x)}{2}\eta_{n,2}^{[s_n]}(x)\right|$$
$$\leq\left(\frac{\eta_{n,2}^{[s_n]}(x)}{2}+\frac{\sqrt{W_a\cdot\eta_{n,4}^{[s_n]}(x)}}{4}\right)\varpi\left(f'',\sqrt{\max\left(\frac{s_n}{n},\frac{1}{r_n}\right)}\right).$$

Proof 5.3.5 *The Taylor's formula is given as:*

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)}{2}(t-x)^2+\hbar(t,x)(t-x)^2,$$

where $\hbar(t,x)=\frac{f''(\nu)-f''(x)}{2}$ *such that* $\nu\in(t,x)$*. Applying the operator* $\hat{\mathcal{K}}_n^{[s_n]}$ *on either sides of the Taylor's formula, we have*

$$\hat{\mathcal{K}}_n^{[s_n]}(f;x)-f(x)-f'(x)\,\eta_{n,1}^{[s_n]}(x)-\frac{f''(x)}{2}\eta_{n,2}^{[s_n]}(x)=\hat{\mathcal{K}}_n^{[s_n]}\left(\hbar(t,x)(t-x)^2;x\right).$$

Taking absolute value on both sides, we now estimate the upper bound of $\hbar(t,x)$*. By application of the property (5.9) of modulus of continuity, and using the inequality* $|a^{-\nu}-a^{-x}|\leq\left|a^{-t}-a^{-x}\right|$ *as* $\nu\in(t,x)$*, we obtain:*

$$|\hbar(t,x)|=\frac{|f''(\nu)-f''(x)|}{2}\leq\frac{1}{2}\left(1+\frac{|a^{-\nu}-a^{-x}|}{\delta}\right)\varpi\left(f'',\delta\right)$$
$$\leq\frac{1}{2}\left(1+\frac{\left|a^{-t}-a^{-x}\right|}{\delta}\right)\varpi\left(f'',\delta\right)$$

Using Cauchy-Schwarz inequality, we obtain:

$$\hat{\mathcal{K}}_n^{[s_n]}\left(|\hbar(t,x)|(t-x)^2;x\right)\leq\left(\frac{\eta_{n,2}^{[s_n]}(x)}{2}+\frac{1}{2\delta_n}\sqrt{\hat{\mathcal{K}}_n^{[s_n]}\left(|a^{-t}-a^{-x}|^2;x\right)}\cdot\sqrt{\eta_{n,4}^{[s_n]}(x)}\right)$$
$$\times\varpi\left(f'',\delta_n\right)$$

Choosing $\delta_n=\sqrt{\max\left(\frac{s_n}{n},\frac{1}{r_n}\right)}$ *and using Lemma 5.2.6 we obtain the required result.*

Remark 5.3.6 *Taking account of Lemma 5.2.4, for sufficiently large n, we have the following two cases:*

Case 1: ***If*** $\lim_{n\to\infty} \frac{r_n s_n}{n} = l$ ***for some finite*** $l \in [0,\infty)$***, we have***

$$r_n \eta_{n,1}^{[s_n]}(x) \to \tfrac{x\ln(a)}{2}(1+lx)$$

$$r_n \eta_{n,2}^{[s_n]}(x) \to x(1+lx)$$

$$r_n^2 \eta_{n,4}^{[s_n]}(x) \to 3x^2(1+lx)^2$$

Case 2: ***If*** $\lim_{n\to\infty} \frac{r_n s_n}{n} = \infty$***, then we have***

$$\tfrac{n}{s_n}\eta_{n,1}^{[s_n]}(x) \to \tfrac{x^2\ln(a)}{2}$$

$$\tfrac{n}{s_n}\eta_{n,2}^{[s_n]}(x) \to x^2$$

$$\tfrac{n^2}{s_n^2}\eta_{n,4}^{[s_n]}(x) \to 3x^4$$

In view of Theorem 5.3.4, and Remark 5.3.6, we now present a Voronovoskaya type theorem in the form of corollary. The proofs are easy, so we choose to neglect it.

Corollary 5.3.7 *Let* $f,\ f'' \in C^\star[0,\infty)$*. If* $\lim_{n\to\infty} \frac{r_n s_n}{n} = l$*, then for any* $x \in [0,\infty)$*, we have*

$$\lim_{n\to\infty} r_n \left|\hat{\mathcal{K}}_n^{[s_n]}(f;x) - f(x)\right| = f'(x)\cdot\frac{x(1+lx)\ln(a)}{2} + f''(x)\cdot\frac{x(1+lx)}{2}.$$

Corollary 5.3.8 *Let* $f,\ f'' \in C^\star[0,\infty)$*. If* $\lim_{n\to\infty} \frac{r_n s_n}{n} = \infty$*, then for any* $x \in [0,\infty)$*, we have*

$$\lim_{n\to\infty} \frac{n}{s_n}\left|\hat{\mathcal{K}}_n^{[s_n]}(f;x) - f(x)\right| = f'(x)\cdot\frac{x^2\ln(a)}{2} + f''(x)\cdot x^2.$$

5.4 Comparison for arbitrary chosen sequences (r_n) and (s_n)

In this section, our aim is to graphically compare the rate of approximation of our proposed operators for different pairs of arbitrarily chosen sequences (r_n) and (s_n) satisfying the conditions $r_n \to \infty$ and $\frac{s_n}{n} \to 0$ as $n \to \infty$. We shall also compare the error in approximation for our operators for these arbitrary sequences. We want to establish through these graphs that the goodness of approximation of the proposed operators rely on the appropriate choices of sequences (r_n) and (s_n).

Example 5.4.1 *Consider the arbitrary pair of sequences* $r_n = n,\ s_n = \frac{1}{n}$; $r_n = n,\ s_n = \frac{1}{ln(n)},\ n \neq 1$; *and* $r_n = 2^n,\ s_n = \frac{1}{\sqrt{n}}$. *For* $a = 3$ *and the function* $f(x) = 14x^3 - 18x^2 + 4x + 1,\ x \geq 0$, *Figure 5.1 shows the convergence of our proposed operators for* $n = 20$. *Let the error of approximation of the proposed operators be defined and denoted as* $\mathfrak{E}_n^{[s_n]}(f; x) = \left|\hat{\mathcal{K}}_n^{[s_n]}(f; x) - f(x)\right|$. *Then, Figure 5.2 represents the error in approximation for each pair of given sequences. It can be observed from both the figures that the rate of convergence is different for different pair of sequences and the best approximation is achieved for the pair of sequences* $r_n = 2^n,\ s_n = \frac{1}{\sqrt{n}}$. *Therefore, we can conclude that error in approximation can be made as small as one wants by choosing appropriate sequences.*

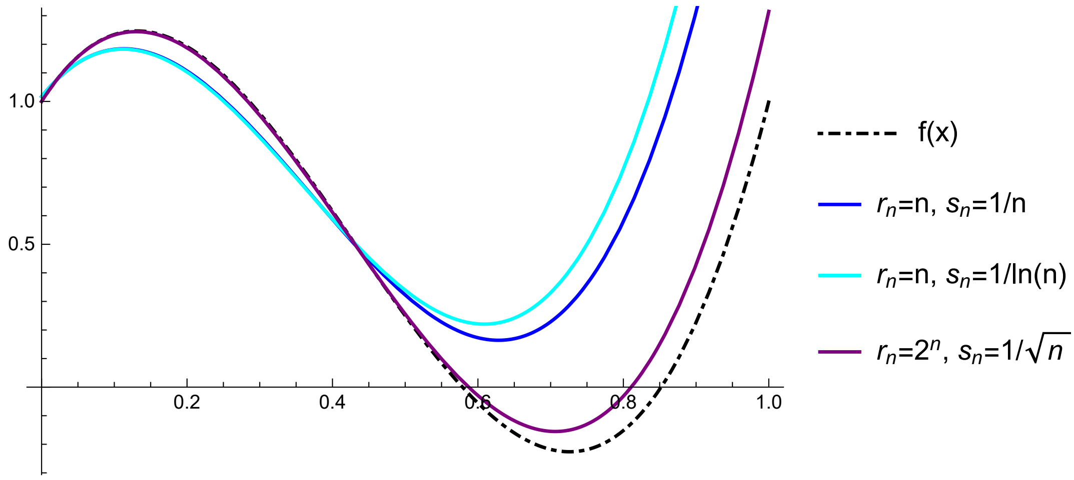

Figure 5.1: Convergence of $\hat{\mathcal{K}}_n^{[s_n]}(f; x)$ towards $f(x)$ for arbitrary sequences (r_n) and (s_n) with $n = 20$ and $a = 3$.

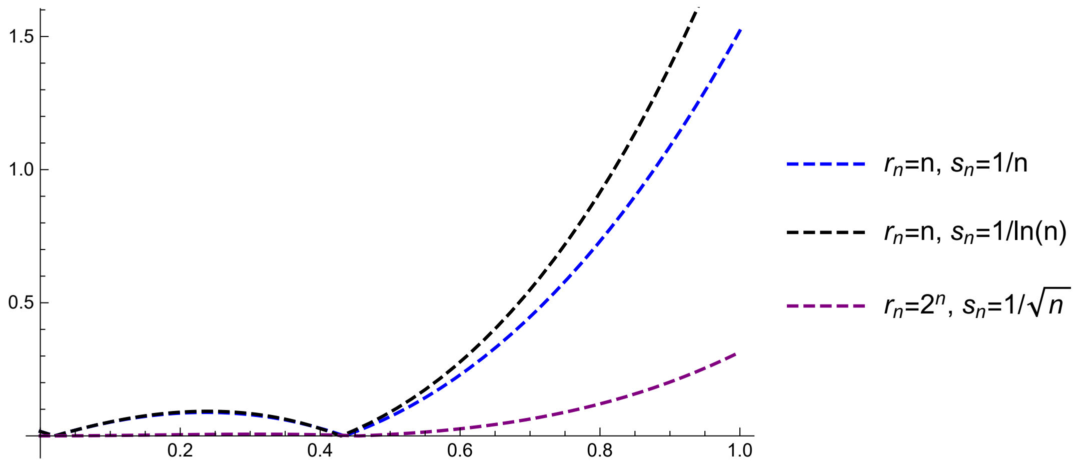

Figure 5.2: Error $\mathfrak{E}_n^{[s_n]}(f; x)$ in convergence for arbitrary sequences (r_n) and (s_n) with $n = 20$ and $a = 3$.

Example 5.4.2 *Let $f(x) = \left(x - x^2\right) a^{2x}$, $x \in [0, \infty)$. Choose $r_n = n$ and $s_n = \frac{1}{n}$ and $a = 3$ as the arbitrary sequences. Figure 5.3 illustrate the convergence of the proposed operators for different values of n. Also $\mathfrak{E}_{20}^{[s_n]}(f;x)$ (Purple), $\mathfrak{E}_{30}^{[s_n]}(f;x)$ (Orange) and $\mathfrak{E}_{50}^{[s_n]}(f;x)$ (Cyan) in Figure 5.4, denote the error in approximation of the function $f(x)$ by the proposed operators (5.2) for $n = 20, 30, 50$.*

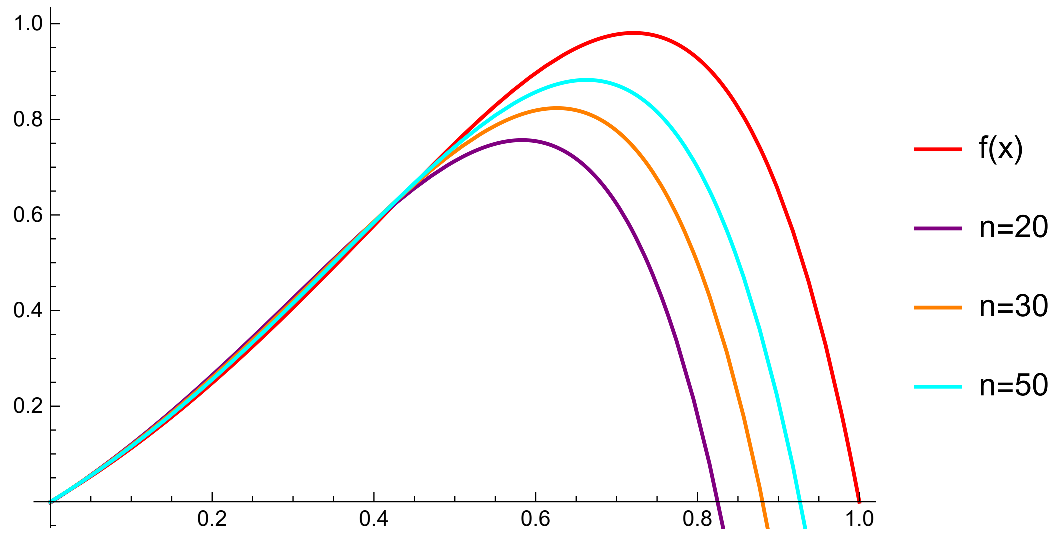

Figure 5.3: Convergence of $\hat{\mathcal{K}}_n^{[s_n]}(f;x)$ towards $f(x)$ for $r_n = n$, $s_n = \frac{1}{n}$ and $a = 3$ for $n = 20, 30, 50$.

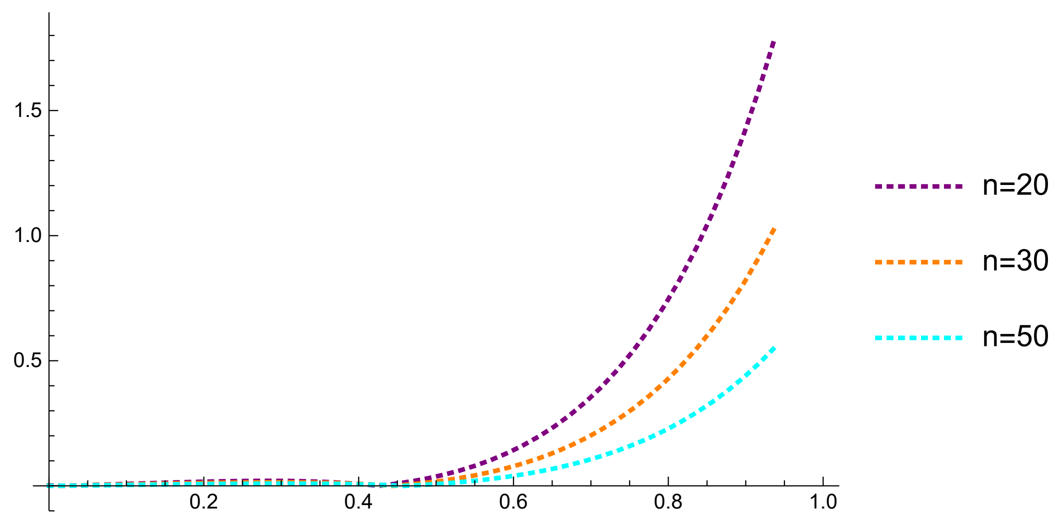

Figure 5.4: Error $\mathfrak{E}_n^{[s_n]}(f;x)$ in convergence of the proposed operators for $r_n = n$, $s_n = \frac{1}{n}$ and $a = 3$ for $n = 20, 30, 50$.

Chapter 6

Conclusion and Future Scope

The aim of this chapter is to present a concluding remarks to our thesis and illustrate some of the prospects that define our current and future endeavours in scientific research.

6.1 Conclusion

This thesis is mainly a study of convergence estimates of various approximation operators. The introductory chapter consists of definitions and literature survey of concepts used throughout this thesis. In the second chapter, we discuss approximation operators of exponential type. The first section of this chapter presents the study of convergence estimates of Kantorovich variant of Ismail-May operators. Further we also propose a two variable generalisation of these operators. In the second section, we present a modification of Ismail-May exponential operators which preserve functions of exponential growth. It must be noted that the modified operators are not of exponential type.

The third chapter presents a Durrmeyer type construction involving a class of orthogonal polynomials called Apostol-Genocchi polynomials and Păltănea operators with real parameters α, λ and ρ. We establish approximation estimates such as a global approximation theorem and rate of approximation in terms of usual, $r-$th and weighted modulus of continuity. We further study asymptotic formulae such as Voronovskaya theorem and quantitative Voronovskaya theorem. The rate of convergence of the proposed operators for the functions whose derivatives are of bounded variation is also presented.

Inspired by the King's approach, the next chapter deals with the preservation of functions of the form t^s, $s \in \mathbb{N} \cup \{0\}$. Followed by some useful lemmas, we determine the rate of convergence of the proposed operators in terms of usual modulus of continuity and Peetre's K- functional. Further, the degree of approximation is also established for the function of bounded variation. We also illustrate via figures and tables that the proposed modification provides better approximation for preservation of test function e_3.

In the final chapter, we consider a Kantorovich variant of the operators proposed by Gupta and Holhos (68) using arbitrary sequences which preserves the exponential functions of the form a^{-x}. It is shown that the order of approximation can be made better with appropriate choice of sequences with certain conditions. We therefore provide necessary moments and central moments and some useful lemmas. Further, we present a quantitative asymptotic formula and estimate the error in approximation. Graphical representations are provided in the end with different choices of sequences satisfying the given conditions.

6.2 Academic future plans

I intend to continue my research in the area of approximation by linear positive operators and hope to share my findings to the mathematical community. Further, my aim shall be to present parametric generalizations of existing operators which enables us to approximate a wider space of functions. In addition to providing modifications to existing operators in this area, I shall strive to study and contribute towards developing new operators.

Another interesting problem in the theory of approximation is to determine which operators present the best approximation. There exists various operators with good convergence rate, some of which are studied in this thesis. As part of my further research, I would like to compare the existing operators through the method of difference of operators to determine the ones which provide better approximation.

In this thesis, we have also studied the modifications of certain exponential operators. The exponential-type operators were introduced four decades ago and since then no new exponential-type operators were introduced although several generalizations of existing exponential-type operators were proposed and studied. In (79), it was established that corresponding to each polynomial $q(x)$, a unique approximation operator can be obtained which satisfies the differential equation (1.13) and normalization condition (1.14). Operators corresponding to constant, linear, quadratic and cubic polynomials were obtained using the method of bilateral Laplace transform. In future, I shall attempt to seek the methods that can determine the exponential operators associated with higher order polynomials.

Very recently, Tyliba and Wachnicki (135) extended the work of Ismail-May (79) with a more general family of operators by introducing a new parameter β. These operators were termed as semi-exponential operators and studied by Herzog (74). Although they admit the normalization condition (1.14) but satisfy a modified homogenous partial differential equation defined as:

$$\frac{\partial}{\partial x} S^{\beta}(\lambda, x, t) = \left(\frac{\lambda(t-x)}{q(x)} - \beta \right) S^{\beta}(\lambda, x, t), \ \beta > 0.$$

The corresponding semi-exponential operators for Bernstein, Baskakov and Ismail-May operators were obtained by Abel et al. (3). Since the area of semi exponential operators is not much explored, I intend to study the approximation properties of these operators such as the complete asymptotic expansion, behaviour of their derivatives through simultaneous approximation and other convergence estimates.

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List of Publications

1. Nav Shakti Mishra, Naokant Deo. Kantorovich variant of Ismail-May Operators. *Iranian Journal of Science and Technology Transaction A: Science, Springer-Verlag*, 44(3), 739–748 (2020). https://doi.org/10.1007/s40995-020-00863-x **(SCIE, Impact Factor 1.553)**

2. Nav Shakti Mishra, Naokant Deo. On the preservation of functions with exponential growth by modified Ismail-May operators. *Mathematical Methods in the Applied Sciences*. 44(11), 9012–9025 (2021). https://doi.org/10.1002/mma.7328 **(SCIE, Impact Factor 3.002)**

3. Nav Shakti Mishra, Naokant Deo. Convergence estimates of certain gamma type operators. *Mathematical Methods in the Applied Sciences*. 45(7), 3802–3816 (2022). https://doi.org/10.1002/mma.8017 **(SCIE, Impact Factor 3.002)**

4. Nav Shakti Mishra, Naokant Deo. Approximation by generalized Baskakov Kantorovich operators of arbitrary order. *Bulletin of the Iranian Mathematical Society*. 48, 3839–3854 (2022). https://doi.org/10.1007/s41980-022-00719-7. **(SCIE, Impact Factor 0.776)**

5. Nav Shakti Mishra, Naokant Deo. Approximation by a composition of Apostol-Genocchi and Păltănea Durrmeyer operators. *Kragujevac Journal of Mathematics*. 48(4), 629–646 (2024). **(ESCI, SCOPUS)**

Papers presented in International Conferences

1. Convergence estimates of bivariate generalization of Kantorovich form of certain exponential operators; *International Conference on Advances in Multi-Disciplinary Sciences and Engineering Research* (ICAMSER-2021), Chitkara University, Himachal Pradesh. July 2-3, 2021

2. Better approximation for functions with exponential growth by modified Ismail–May operators; *Innovations in Applied Science and Engineering* (ICIASE-2022), Dr. B.R.Ambedkar National Institute of Technology, Jalandhar. May 18-19, 2022