

**DETECTION AND CLASSIFICATION OF BIPARTITE AND
MULTIPARTITE ENTANGLED STATES**

A thesis submitted to

DELHI TECHNOLOGICAL UNIVERSITY

in partial fulfillment of the requirements for the award of the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

By

ANU KUMARI

Under the Supervision of

Dr. Satyabrata Adhikari



DEPARTMENT OF APPLIED MATHEMATICS
DELHI TECHNOLOGICAL UNIVERSITY
(Formerly Delhi College of Engineering)
BAWANA ROAD, DELHI-110 042, INDIA.

October, 2022

Enroll. No. : 2K18/PHD/AM/05

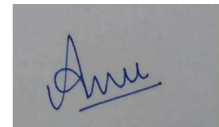
© DELHI TECHNOLOGICAL UNIVERSITY, DELHI, 2022
ALL RIGHTS RESERVED.

DECLARATION

I declare that the research work reported in this thesis entitled “**Detection and Classification of Bipartite and Multipartite Entangled States**” for the award of the degree of *Doctor of Philosophy in Mathematics* has been carried out by me under the supervision of *Dr. Satyabrata Adhikari*, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

The research work embodied in this thesis, except where otherwise indicated, is my original research. This thesis has not been submitted by me earlier in part or full to any other University or Institute for the award of any degree or diploma. This thesis does not contain other person’s data, graphs, or other information unless specifically acknowledged.

Date : 28.10.2022



(Anu Kumari)

CERTIFICATE

On the basis of a declaration submitted by **Ms. Anu Kumari**, Ph.D. scholar, I hereby certify that the thesis titled “**Detection and Classification of Bipartite and Multipartite Entangled States**” submitted to the Department of Applied Mathematics, Delhi Technological University, Delhi, India for the award of the degree of *Doctor of Philosophy in Mathematics*, is a record of bonafide research work carried out by her under my supervision.

I have read this thesis and, in my opinion, it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy.

To the best of my knowledge, the work reported in this thesis is original and has not been submitted to any other Institution or University in any form for the award of any Degree or Diploma.

Satyabrata Adhikari

(Dr. Satyabrata Adhikari)

Supervisor and Assistant Professor
Department of Applied Mathematics
Delhi Technological University
Delhi



(Dr. S. Sivaprasad Kumar)

Head and Professor
Department of Applied Mathematics
Delhi Technological University
Delhi

ACKNOWLEDGEMENTS

It would not have been possible to write this doctoral thesis without the kind support and help of many individuals around me, to only some of whom it is possible to give a particular mention here. I would like to extend my sincere thanks to all of them.

First and foremost, praises and thanks to the GOD, the almighty, for His showers of blessings throughout my research work to complete the research successfully.

I would like to express my special gratitude and thanks to my supervisor Dr. Satyabrata Adhikari, Assistant Professor, Department of Applied Mathematics, Delhi Technological University (DTU), Delhi for the continuous support, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. It is indeed a great privilege and honor for me to work under his supervision. I am extremely grateful for what he has offered me. Without his guidance and persistent help, this thesis would not have been possible.

Besides my supervisor, I would like to thank Prof. S. Sivaprasad Kumar, Head, Department of Applied Mathematics, DTU for providing me the necessary facilities in the Department.

My sincere thanks also goes to Prof. Naokant Deo, DRC chairman, Department of Applied Mathematics, DTU for his everlasting support and guidance. I am grateful to Prof. H. C. Taneja, Professor, Department of Applied Mathematics, DTU and all faculty members of the Department of Applied Mathematics, DTU for their constant support and encouragement.

I also wish to extend my gratitude to Prof. Tabish Qureshi, Professor and Hony. Director, Centre for Theoretical Physics, Jamia Millia Islamia University, Delhi, India and Prof. Sibasish Ghosh, Institute of Mathematical Sciences, Chennai, India for their valuable guidance from the beginning of my Ph.D. research work.

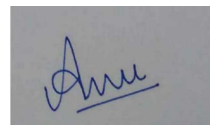
I would like to acknowledge the academic and technical support of the Delhi Technological University, Delhi, and its staff, particularly the Academic-PG section for their all kinds of support. The library facilities of the University have been indispensable. I also thank everyone in the Department of Applied Mathematics particularly Mr. Akhil, Ms. Nisha, Mr. Anurag, Mr. Narendra and Ms. Anju for their support and assistance.

I am extremely grateful to my family and relatives for their love, prayers, caring, and sacrifices for educating me and preparing me for my future. Special thanks to my parents and husband, they were the pillar of strength and support for me in this entire journey.

I place my acknowledge to Ms. Anuma Garg and Ms. Shrutika Aggarwal for valuable discussions on the subject matter and for their support. I also place on record, my sense of gratitude

to one and all who, directly or indirectly, have lent their helping hand in this journey.

My acknowledgements would not be complete without thanking the Council of Scientific and Industrial Research (CSIR), Ministry of Science and Technology, Government of India, for providing fellowship (JRF and SRF) that made my Ph.D. work possible.



(ANU KUMARI)

Date : 28.10.2022

Place : DTU, Delhi, India.

Dedicated
to
My Family

Contents

Declaration page	i
Certificate page	iii
Acknowledgements	v
Preface	xii
List of figures	xv
List of tables	xvi
1 General Introduction	1
1.1 Basics of Linear Algebra	2
1.2 Quantum states	9
1.2.1 Qubit	9
1.2.2 Pure and mixed state	10
1.2.3 Geometrical interpretation of a qubit	10
1.2.4 Bloch vectors for d -dimensional quantum system	11
1.2.5 Representing a d -dimensional quantum state in terms of different basis	12
1.2.6 An illustration: A two-qutrit system	15
1.2.7 Quantum Measurements	21
1.3 EPR Paradox	24
1.4 Quantum entanglement	27
1.4.1 Bipartite and tripartite system	28
1.5 Entanglement Detection	35
1.5.1 Entanglement detection criterion that may not be implementable in an experiment	36
1.5.2 Experimentally implementable entanglement detection schemes	40
1.6 Classification of entanglement	49
2 Detection of entanglement via structural physical approximation	53

2.1	Introduction	54
2.2	Preliminary Results	57
2.3	SPA-PT of an arbitrary qutrit-qubit quantum state	58
2.4	Criteria for the detection of bipartite Negative Partial Transpose Entangled states in arbitrary dimension	62
2.4.1	Lower and Upper bound of the minimum eigenvalue of SPA-PT of a bipartite mixed quantum state in arbitrary dimension	62
2.4.2	Criteria for the detection of $d_1 \otimes d_2$ dimensional bipartite NPTES	64
2.5	Illustrations	66
2.5.1	Qubit-Qubit system	66
2.5.2	Qutrit-Qubit system	69
2.6	Conclusion	72
3	Classification of multipartite states through witness operator	73
3.1	Introduction	74
3.2	Derivation of the inequality required for the construction of classification witness operator	79
3.2.1	Correlation tensor for the canonical form of three-qubit pure state	80
3.2.2	Inequality for the construction of classification witness operator	82
3.3	Construction of classification witness operator	84
3.3.1	Classification witness operator for the classification of states contained in subclass-I and subclass-II	85
3.3.2	Classification witness operator for the classification of states contained in subclass-I and subclass-III	89
3.3.3	Classification of states confined in subclass-IV with state parameters $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and subclass-I	94
3.4	Examples	96
3.5	Classification witness operator for the classification of states contained in subclass-II and subclass-III	98
3.6	Conclusion	101
4	Classification of Three-qubit States using l_1-norm of Coherence	103
4.1	Introduction	104
4.2	l_1 -norm of coherence of the tensor product of bipartite states	106
4.2.1	Coherence-based inequality for the detection of a particular form of three-qubit biseparable states	112
4.2.2	Example	113
4.3	Detection of general three-qubit biseparable states and separable states	114
4.3.1	Coherence-based inequality for the classification of general three-qubit biseparable states	115
4.3.2	Coherence-based inequality for the classification of three-qubit mixed separable states	117

4.3.3	Illustrations	118
4.4	An idea to generalize the results to a four-qubit system	119
4.5	An idea to generalize the results to a three-qutrit system	121
4.6	Conclusion	122
5	Classification of Three-qubit States using SPA-PT	123
5.1	Introduction	124
5.2	Studying the Effect of Partial Transposition Operation on one of the qubit of a Three-Qubit System	125
5.3	Structural physical approximation of partial transposition (SPA-PT) of a single qubit in a three-qubit system	129
5.3.1	When Structural Physical Approximation Map will be Completely Positive?	130
5.3.2	Determination of the matrix elements of the density matrix after SPA-PT operation	131
5.4	Necessary condition for the separability (either in the form of a full separability or biseparability) of a three-qubit state	133
5.5	A Few Examples	134
5.5.1	Genuine Entangled States	134
5.5.2	Biseparable states	136
5.5.3	Separable States	137
5.5.4	Genuine/Biseparable/Separable	138
5.6	Conclusion	139
6	Detection and quantification of entanglement	141
6.1	Measures of entanglement: Concurrence and Negativity	143
6.2	Structured Negativity	145
6.3	Relation between negativity and structured negativity	149
6.3.1	Examples	151
6.4	Conclusion	155
7	Conclusion and Future Scope	157
7.1	Conclusion	157
7.2	Future Scope	161
	Bibliography	164
	List of Publications	181

Preface

The theory of quantum entanglement is a burgeoning field of quantum information theory. The fact that measurements on spatially separated systems cannot always be described by a locally causal theory is one of the most remarkable features in quantum physics. Although the basic understanding of the bipartite system has been established over the last few decades, still there are many unanswered questions about the entanglement properties of the multipartite system. It is particularly unclear how several concepts from the bipartite case can be generalized to multipartite scenarios in a meaningful way. The detection and classification of multipartite entanglement are discussed in this thesis. The differences and similarities between the bipartite and multipartite situations are discussed, various possible generalizations are presented, and results in several areas are obtained.

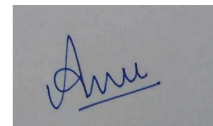
The detection and classification of entanglement properties in a multi-qubit system is a topic of great interest. This topic has been extensively studied, and as a result, we discovered various approaches for detecting and classifying multi-qubit, in particular three-qubit entangled states. The emphasis of this work is on a formalism of methods for the detection and classification of bipartite as well as multipartite quantum systems. We have used the method of structural physical approximation of partially transposed matrix (SPA-PT) for the detection of entangled states in arbitrary dimensional bipartite quantum systems. Also, we have proposed criteria for the classification of all possible stochastic local operations and classical communication (SLOCC) inequivalent classes of a pure and mixed three-qubit state using the SPA-PT map. To quantify entanglement, we have defined a new measure of entanglement based on the method of SPA-PT, which we named as "structured negativity". We have shown that this measure can be used to quantify entanglement for negative partial transposed entangled states (NPTES). Since the methods for detection, classification and quantification of entanglement, defined in this thesis are based on SPA-PT, they may be realized in an experiment.

Three-qubit systems have two types of SLOCC inequivalent genuine entangled classes. These classes are known as the GHZ class and the W class. The GHZ class has been proven to be very useful for various quantum information processing tasks such as quantum teleportation, controlled quantum teleportation, and so on. We divide pure three-qubit states from the GHZ class into four distinct subclasses namely S_1 , S_2 , S_3 , S_4 and shown that three-qubit states either belong to S_2 or S_3 or S_4 may be more efficient than the three-qubit state belonging to S_1 . Moreover, we have constructed various witness operators that can classify the subclasses and demonstrated that the constructed witness operator can be decomposed into Pauli matrices and thus experimentally realized.

Also, we have used coherence to detect and classify the entanglement property of three-qubit states. We have obtained the necessary conditions to discriminate biseparable states from other classes of the three-qubit system. Moreover, we have also obtained the necessary condition that may classify the separable states from other classes of three-qubit entangled states. Since there are only three types of states in a three-qubit system, so if we found that the detected state is neither a separable nor a biseparable state, we may conclude that the given three-qubit state is a genuine entangled state.

Date : 28.10.2022

Place : Delhi, India



(ANU KUMARI)

List of Figures

1.1	Bloch sphere	11
1.2	Experimental setup for the Bell inequality	41
1.3	Hahn Banach Theorem	44
2.1	Plot of concurrence (C) versus α . The dotted line shows the upper bound and the solid line represent the lower bound of the concurrence of the qutrit-qubit state described by the density matrix $\rho_{AB}^{(2)}$	72
3.1	Classification of different subclasses of GHZ class of states described by the four sets S_1, S_2, S_3, S_4	76
6.1	The negativity, structured negativity, and lower bound of concurrence for the two-qubit Werner state coincide with each other. Therefore, equality in (6.3.11) has been achieved for a family of Werner state. . . .	152
6.2	The negativity, structured negativity and lower bound of concurrence coincide for the state ρ_{MEMS} for $C \geq \frac{2}{3}$	153
6.3	The negativity, structured negativity and lower bound of concurrence coincides for the state ρ_{MEMS} for $C < \frac{2}{3}$	153
6.4	The structured negativity of the state ρ_a is greater than the negativity as well as the lower bound of concurrence.	154
6.5	The negativity and structured negativity coincide for the state ρ_α . In this case, $C_{lb}(\rho_\alpha)$ is greater than the structured negativity.	155

List of Tables

2.1	Table verifying (2.5.12) for different values of the state parameters (a, b, f)	68
2.2	Table verifying (2.5.14) for different values of the state parameters (a, b, f)	69
2.3	Table verifying (2.5.13) for differnt values of the state parameters (a, b, f)	69
3.1	Range of the parameter p for which the classification witness operator H_4 classifies the given GHZ state within the subclass-III	98
5.1	Table verifying Result-1 for ρ_{G_3} for differnt values of the state parameters $(\lambda_0, \lambda_1, \lambda_2)$	135
5.2	Table verifying Result-4 for ρ_{B_2} different values of the state parameters $(\lambda_0, \lambda_1, \lambda_2)$	137

Chapter 1

General Introduction

The introductory chapter consists of basic definitions, basic concepts of linear algebra, and the preliminaries of the results obtained in the literature. We then provide a brief review of the theory of bipartite and multipartite entanglement. In the theory of bipartite entanglement, we have discussed various entanglement detection criteria for the two-qubit system as well as the higher dimensional bipartite quantum system. In the case of multipartite entanglement, we have specifically considered a three-qubit system and studied different schemes for the classification of different classes that may exist in the tripartite entangled system.

The study of data storage and transmission through a noisy channel is known as information theory. It focuses on the quantification of data for communication purposes. It is concerned with determining how much information is contained in a message. Since information is related to uncertainty or randomness, so, to quantify the amount of information, we need to determine the amount of uncertainty. The amount of uncertainty in the value of a random variable or the outcome of a random process can be measured in entropy and thus, entropy can be considered an important metric in information theory. For example, determining the result of a fair coin flip (with two equally likely outcomes) offers less information (lower entropy) than determining the result of a dice roll (with six equally likely outcomes).

Nowadays, we can split the whole concept of information theory into two parts: (i) classical information theory and (ii) quantum information theory. Classical information theory is the mathematical theory of information-processing tasks utilizing laws of classical physics whereas, quantum information theory is the study of how such tasks can be done using quantum mechanical systems. Quantum Information Theory is concerned with the use of quantum-mechanical features of physical systems to facilitate effective data storage and transfer. Quantum and classical information theories differ significantly due to the underlying features of quantum physics that do not exist in classical physics. For instance, quantum entanglement is one such feature in quantum physics that has no classical analog.

1.1 Basics of Linear Algebra

Let us first recapitulate some basic concepts of linear algebra that would be needed to discuss the topics of quantum information theory. The basic objects of linear algebra are vector spaces. In quantum mechanics, we generally consider the vector space over the field of complex numbers C . The elements of vector spaces are known as vectors. They can be represented by column vectors.

I. Linearly independent and dependent vectors: A set of vectors $V = \{v_1, v_2, \dots, v_n\}$ is said to be linearly independent if there exist scalars a_1, a_2, \dots, a_n such that

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0 \implies a_i = 0 \forall i \quad (1.1.1)$$

otherwise, the set V is said to be linearly dependent.

II. Basis: A basis β of a vector space is the spanning set $\beta = \{v_1, v_2, \dots, v_n\}$ such that every vector in the vector space can be expressed as a linear combination of vectors present in the basis set such that the set β is linearly independent. Let v be any vector in the vector space, then v can be expressed as $v = \sum_i \alpha_i v_i, \alpha \in C$. The number of elements in a basis set is the dimension of the vector space.

III. Linear space: A linear space (or vector space) over the field F is a set X which satisfies the following properties:

(i) $x + y \in X \forall x, y \in X$

(ii) $x + y = y + x \forall x, y \in X$

(iii) $(x + y) + z = x + (y + z) \forall x, y, z \in X$

(iv) There is a zero element (additive identity), denoted by, 0 in X such that $x + 0 = x, \forall x \in X$

(v) For every $x \in X, \exists$ additive inverse $-x \in X$ such that $x + (-x) = 0 = -x + x$

(vi) For every $\alpha \in F$ and $x \in X, \alpha x \in X$

(vii) For every $\alpha, \beta \in F$ and $x, y \in X$:

(a) $\alpha(\beta x) = (\alpha\beta)x$

(b) $(\alpha + \beta)x = \alpha x + \beta x$

(c) $\alpha(x + y) = \alpha x + \alpha y$

(viii) There is a unit element (multiplicative identity), denoted by 1 , in F such that $1.x = x, \forall x \in X$.

IV. Complete linear space: A normed linear space is said to be complete if every Cauchy sequence converges in the space.

V. Norm: Consider a vector space X defined over a field F . A norm $\|\cdot\|$ on X is defined as a function

$$f : X \rightarrow F \tag{1.1.2}$$

which satisfies the following properties:

1. $\|\mathbf{x}\| \geq 0, \forall \mathbf{x} \in X$.

2. $\|\mathbf{x}\| = 0 \iff \mathbf{x} = 0$.

3. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|, \forall \mathbf{x} \in X, \forall \alpha \in F$.

4. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \forall \mathbf{x}, \mathbf{y} \in X$.

VI. Normed linear space: A normed linear space is a linear vector space over the field of real or complex numbers, on which a norm is defined.

For example, the set $C[a, b]$ of all continuous functions $f(x)$, $x \in [a, b]$, is a normed linear space with respect to norm defined as

$$\|f\|_p = \left[\int_a^b |f(x)|^p \right]^{\frac{1}{p}}, \quad p \geq 1 \quad (1.1.3)$$

VII. Banach Space: A complete normed linear space is known as a Banach space. For instance, The set of all real valued continuous functions $x = x(t)$ defined on the set $[a, b]$, denoted by $C[a, b]$, is a Banach space with respect to the norm defined as

$$\|x\| = \sup_{t \in [a, b]} |x(t)|, \quad x = x(t) \in C[a, b] \quad (1.1.4)$$

VIII. Inner Product: Inner product on a complex linear space X may be defined as a map:

$$\langle \cdot, \cdot \rangle: X \times X \rightarrow C \quad (1.1.5)$$

If we take three arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in X$ and two scalars $\alpha, \beta \in C$, then the following properties holds

(a) $\langle \mathbf{a}, \alpha \mathbf{b} + \beta \mathbf{c} \rangle = \alpha^* \langle \mathbf{a}, \mathbf{b} \rangle + \beta^* \langle \mathbf{a}, \mathbf{c} \rangle$, where α^* and β^* denote the complex conjugate of α and β respectively.

(b) $\langle \mathbf{b}, \mathbf{a} \rangle = \overline{\langle \mathbf{a}, \mathbf{b} \rangle}$

(c) $\langle \mathbf{a}, \mathbf{a} \rangle \geq 0$

(d) $\langle \mathbf{a}, \mathbf{a} \rangle = 0$ if and only if $\mathbf{a} = 0$.

If $X(F)$ is a vector space with inner product $\langle \cdot, \cdot \rangle$ defined on X . Then, a norm can be defined from inner product as

$$\|a\| = \sqrt{\langle a, a \rangle}, \quad \forall a \in X \quad (1.1.6)$$

IX. Inner product space: Let X be a linear space over the field of complex number C . Then, X is called an inner product space if for every pair of elements $x, y \in X$, \exists a complex number denoted by $\langle x, y \rangle$ and is called the inner product of x and y . For example, the n dimensional space R_n or C_n forms an inner product space. Let $x = (\xi_1, \xi_2, \dots, \xi_n)$ and $y = (v_1, v_2, \dots, v_n)$ be two elements of C_n . The inner product of x and y is denoted by $\langle x, y \rangle$ and is given by the formula

$$\langle x, y \rangle = \sum_{i=1}^n \xi_i v_i^* \quad (1.1.7)$$

With the help of this inner product, the length of $x = (\xi_1, \xi_2, \dots, \xi_n)$ can be written as

$$\|x\| = \left(\sum_{i=1}^n |\xi_i|^2 \right)^{\frac{1}{2}} = \langle x, x \rangle^{\frac{1}{2}} \quad (1.1.8)$$

X. Hilbert Space: Hilbert space is a normed linear space which is complete with respect to the norm derived by the inner product.

(i) A Hilbert space is an inner product space and is complete.

(ii) Inner product spaces are normed linear spaces.

(iii) Hilbert spaces are Banach spaces.

(iv) If for an inner product space, the scalar field is the set of real numbers, then we obtain real inner product space and corresponding real Hilbert space.

(v) In real Hilbert space, we simply have $\langle x, y \rangle = \langle y, x \rangle$.

(vi) If for an inner product space, the scalar field is the set of complex numbers, then we obtain complex inner product space and corresponding complex Hilbert space.

XI. Orthogonal vectors: In a vector space, two vectors are said to be orthogonal if their inner product is zero.

XII. Unit vector: If $\|a\| = 1$, then a is known as unit vector.

XIII. Orthonormal vector: A set of vectors $\{v_1, v_2, \dots, v_n\}$ is said to be orthonormal if each vector v_i in the set is a unit vector and distinct vectors v_i and v_j are orthogonal i.e. $\langle v_i | v_j \rangle = \delta_{ij}$, $i, j \in \{1, 2, \dots, n\}$.

XIV. Linear operator: Let X and Y be two vector spaces over the field F . An operator $T : X \rightarrow Y$ is said to be linear if

$$T\left(\sum_i c_i v_i\right) = \sum_i c_i T v_i, \quad v_i \in X, \quad c_i \in F \quad (1.1.9)$$

A linear operator on a vector space X is a linear transformation from X into X . Linear operators may be expressed in terms of matrices.

XV. Matrix representation: Consider two vector spaces X and Y of dimension m and n respectively. Then a linear operator from X to Y may be expressed by a $(m \times n)$ matrix. Let $\beta_1 = \{v_1, v_2, \dots, v_m\}$ be basis of X and $\beta_2 = \{w_1, w_2, \dots, w_n\}$ be basis of Y , then

$$T(v_j) = \sum_{i=1}^m a_{ij} w_i, \quad j \in \{1, 2, \dots, m\} \quad (1.1.10)$$

Then, the matrix of T with respect to the basis β_1 and β_2 can be represented as $A = [a_{ij}]_{m,n}$.

XVI. Eigenvalue and eigenvector: Let X be a vector space over the field of complex numbers C and $T : X \rightarrow X$ be a linear operator. An eigenvector of a linear operator T is a non zero vector $x \in X$ such that $Tx = \lambda x$, $\lambda \in C$ is known as the eigenvalue of the operator T corresponding to eigenvector x . Eigenvalues are the solutions of the characteristic equation, which is given by $\det|T - \lambda I| = 0$ where \det denote the determinant and I denote the identity operator.

XVII. Transpose of a matrix: The transpose of a matrix can be defined by interchanging the rows of the matrix into columns or by transforming the columns of the matrix into rows. For a given matrix A , the transpose of the matrix A is represented by A^T .

XVIII. Conjugate transpose of a matrix: For a linear operator A , the conjugate transpose of A is obtained by applying transposition operation on A and applying complex conjugate on each entry. For a given matrix A , the conjugate transpose of the matrix A is represented A^\dagger .

XIX. Normal operator: An operator A is said to be normal if and only if

$$AA^\dagger = A^\dagger A \quad (1.1.11)$$

XX. Orthogonal operator: An operator A is said to be orthogonal if and only if

$$AA^T = A^T A = I \quad (1.1.12)$$

XXI. Unitary operator: An operator A is said to be unitary if and only if

$$AA^\dagger = A^\dagger A = I \quad (1.1.13)$$

XXII. Hermitian operator: An operator H is said to be Hermitian if and only if

$$H^\dagger = H \quad (1.1.14)$$

The eigenvalues of a Hermitian operators are real numbers.

For example, Pauli Matrices are Hermitian matrices and it can be represented as

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.1.15)$$

XXIII. Properties of Pauli matrices:

(i) Pauli matrices are Hermitian and unitary.

(ii) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$, where I is the identity matrix.

(iii) $\det(\sigma_i) = -1$, $i=x,y,z$

(iv) $\text{Tr}(\sigma_i) = 0$, $i=x,y,z$

(v) Eigenvalues of Pauli matrices are ± 1 .

(vi) $[\sigma_x, \sigma_y] = 2i\sigma_z$, $[\sigma_y, \sigma_z] = 2i\sigma_x$, $[\sigma_z, \sigma_x] = 2i\sigma_y$, where $[\cdot]$ denote the commutation relation.

(vii) $\{\sigma_i, \sigma_j\} = 0$, $i \neq j$, $i, j \in \{x, y, z\}$, where $\{\cdot\}$ denote anti-commutation relation.

XXIV. Partial transpose of a matrix: Given $A = [A_{i,j}]_{i,j=1}^n \in M_n(M_k)$, its partial transpose A^τ is given by,

$$A^\tau = [A_{i,j}^*]_{i,j=1}^n \quad (1.1.16)$$

Here $M_n(M_k)$ is the set of $n \times n$ block matrices with each block in M_k .

A state ρ_d in $d \otimes d$ dimensional quantum system is represented by a $d^2 \times d^2$ density matrix. This $d^2 \times d^2$ density matrix may be divided into d submatrices of order $d \times d$. The partial transposition of the state ρ_d may be calculated by taking the transposition of each of the submatrices. For example, let us consider the density matrix ρ_2^{AB} of a two-qubit composite system AB . It is represented by a 4×4 matrix with 4 submatrices of size 2×2 . The partial transposition of ρ_2^{AB} can be calculated by taking the transposition of each of the submatrices.

$$\rho_2^{AB} = \left(\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array} \right) \xrightarrow{T_B} \left(\begin{array}{cc|cc} a_{11} & a_{21} & a_{13} & a_{23} \\ a_{12} & a_{22} & a_{14} & a_{24} \\ \hline a_{31} & a_{41} & a_{33} & a_{43} \\ a_{32} & a_{42} & a_{34} & a_{44} \end{array} \right) = (\rho_2^{AB})^{T_B} \quad (1.1.17)$$

where T_B denote the partial transposition with respect to the subsystem B .

XXV. Positive Partial Transpose: A bipartite density matrix ρ has a **positive partial transpose** (or the matrix is **PPT**) if its partial transposition has no negative eigenvalues i.e. it is positive semidefinite.

XXVI. Non Positive Partial Transpose: A bipartite density matrix ρ has a **non positive partial transpose** (or the matrix is **non PPT**) if its partial transposition has atleast one negative eigenvalues.

XXVII. Realignment operation on a matrix: For a matrix $X = (x_{ij}) \in \mathbf{C}^{m \times n}$, the vector

$vec(X)$ is defined as

$$vec(X) = (x_{11}, \dots, x_{m1}, x_{12}, \dots, x_{m2}, \dots, x_{1n}, \dots, x_{mn})^T \quad (1.1.18)$$

where T denote the transposition operator.

Let Y be an $m \times m$ block matrix with $n \times n$ subblocks $Y_{i,j}, i, j = 1, \dots, m$. Then the realignment matrix of Y is defined as

$$R(Y) = (vec(Y_{1,1}), \dots, vec(Y_{m,1}), \dots, vec(Y_{1,m}), \dots, vec(Y_{m,m}))^T \quad (1.1.19)$$

A state in $d \otimes d$ dimensional system may be represented by $d^2 \times d^2$ density matrix and this matrix may be further divided in d submatrices of order $d \times d$. The realigned matrix may be obtained by expressing the element of each submatrix in row. For example for a two-qubit density matrix ρ , the realigned matrix ρ^R may be obtained as

$$\rho = \left(\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array} \right) \xrightarrow{R} \left(\begin{array}{cccc} a_{11} & a_{12} & a_{21} & a_{22} \\ a_{13} & a_{14} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{41} & a_{42} \\ a_{33} & a_{34} & a_{43} & a_{44} \end{array} \right) = \rho^R \quad (1.1.20)$$

XXVIII. Inner product of two operators: For two operators A and B in finite dimensional Hilbert space(\mathbf{H}), inner product of A and B may be defined as

$$\langle A, B \rangle = Tr[A^\dagger B] \quad (1.1.21)$$

where A^\dagger denotes the Hermitian conjugate of A and the corresponding norm may be defined as

$$\|A\| = \sqrt{\langle A, A \rangle}, \quad A \in \mathbf{H} \quad (1.1.22)$$

XXIX. Positive and Completely positive map: Let $\gamma: H_1 \rightarrow H_2$ be a linear map. γ is said to be a positive map if it maps positive operators of H_1 to positive operators of H_2 , i.e.

$$\gamma(A) \geq 0, \quad \forall A \geq 0 \quad (1.1.23)$$

A positive map γ is said to be a completely positive if the map defined by,

$$\gamma \otimes I_d : H_1 \otimes M_d \rightarrow H_2 \otimes M_d \quad (1.1.24)$$

is also positive for $d=2,3,4,\dots$ where I_d is the identity matrix of the matrix space M_d of $d \times d$ matrices.

XXX. Dual space of a Vector Space: For a vector space $V(F)$, then the collection of all linear functionals on V forms a vector space over and it is called the dual space of V .

$$V^* = L(V, F) \quad (1.1.25)$$

XXXI. Dual basis of a Vector Space: Let $\beta = \{a_1, a_2, \dots, a_n\}$ be a basis for the vector space V . Then there exist basis $\beta^* = \{f_1, f_2, \dots, f_n\}$ of V^* that satisfies

$$f_i(a_j) = \delta_{ij}, \quad (1 \leq i, j \leq n) \quad (1.1.26)$$

β^* is known as dual basis of β .

1.2 Quantum states

Quantum states can be considered as vectors in a Hilbert space (H). The notation used in quantum mechanics for a vector belong to a vector space is given by

$$|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{pmatrix} \quad (1.2.1)$$

where ψ is the label for the vector and the notation $|\cdot\rangle$ denotes the object is a vector. This notation is also known as a ket vector. Here, $z_i \in \mathbb{C}, i \in \{1, 2, \dots, n\}$ denote the component of the $|\psi\rangle$. Some standard notations related to the description of a quantum mechanical system are given below

(i) z^* denote the complex conjugate of z .

- (ii) $\langle \psi |$ is the dual of $|\psi\rangle$. It is known as bra notation of vector ψ .
- (iii) $\langle \psi | \phi \rangle$ denote the inner product of two vectors $|\psi\rangle$ and $|\phi\rangle$.
- (iv) $|\psi\rangle \otimes |\phi\rangle$ denote the tensor product of two vector $|\psi\rangle$ and $|\phi\rangle$.

1.2.1 Qubit

Quantum bits are used to define quantum states. Qubits, or quantum bits, are the most fundamental unit of quantum information, just as bits are the most fundamental unit in classical information theory. In a two-dimensional Hilbert space, a qubit can be written as a linear combination of two classical bits $|0\rangle$ and $|1\rangle$. Mathematically, a qubit can be expressed as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1 \quad (1.2.2)$$

where α and β are complex numbers. $|\alpha|^2 + |\beta|^2 = 1$ represent the normalization condition. Geometrically, the normalization condition represents all the quantum states $|\psi\rangle$ that lie on the boundary of the sphere.

1.2.2 Pure and mixed state

A quantum system can be found in any of the following two forms: pure states and mixed states. A pure state is always represented by a vector in a Hilbert space while a mixed state can be expressed as a convex combination of more than one pure state. A mixed state is denoted by an operator ρ which is associated with some ensemble $\{p_i, |\psi_i\rangle\}$. Mathematically, a mixed state ρ can be expressed in terms of the ensemble $\{p_i, |\psi_i\rangle\}$ as

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|, \quad \sum_i p_i = 1, \quad 0 \leq p_i \leq 1, \quad i = 1, 2, \dots \quad (1.2.3)$$

An operator ρ which represents a mixed state is popularly known as a density operator. An operator ρ is said to be the density operator if it satisfies the following properties:

$$(i) \quad Tr[\rho] = 1 \quad (1.2.4)$$

$$(ii) \quad \rho \geq 0 \quad \text{i.e. } \rho \text{ has non-negative eigenvalues} \quad (1.2.5)$$

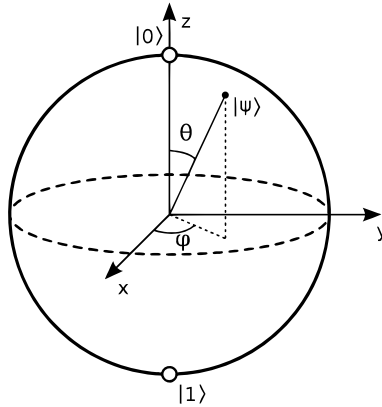


Figure 1.1: Bloch sphere

In terms of $Tr[\rho^2]$, the pure state and the mixed state may be classified as

- (i) ρ is a pure state if and only if $Tr(\rho^2) = 1$.
- (ii) ρ is a mixed state if and only if $Tr(\rho^2) < 1$.

1.2.3 Geometrical interpretation of a qubit

Geometrically, a two-level quantum mechanical system can be represented by a Bloch sphere. A two-level quantum mechanical system is described by four parameters in a complex plane. But the normalization condition reduces one parameter and therefore the system finally is described by three parameters which can be represented in a Bloch sphere [1, 2]. The state lies on the surface of the sphere are pure states of the form

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi \quad (1.2.6)$$

The states lying inside the sphere are mixed states given by

$$\rho^{(2)} = \frac{1}{2}(I + \vec{a} \cdot \vec{\sigma}) \quad (1.2.7)$$

where $\vec{a} = (a_x, a_y, a_z) \in R^3$ is a vector in three dimensional space and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denotes the Pauli matrices. The center of the sphere represent a maximally mixed state which is described by the identity matrix.

1.2.4 Bloch vectors for d -dimensional quantum system

Generalizing the idea of two-dimensional system to d -dimensional system, we call the quantum state lying in d -dimensional Hilbert space as qudit. Any matrix basis that can be used for Bloch vector decomposition in case of qudits must satisfy the following two properties:

- (i) The set of basis elements must contain I and $d - 1$ traceless matrices $\{A_i\}$ of order $d \times d$.
- (ii) The matrices $\{A_i\}$ must be orthogonal.

The Bloch vector expansion of the density matrix $\rho^{(d)}$ may be expressed as

$$\rho^{(d)} = \frac{1}{d}I + \vec{b} \cdot \vec{\Lambda} \quad (1.2.8)$$

where $\vec{b} \cdot \vec{\Lambda}$ is a linear combination of $\vec{b} \in R^{d^2-1}$ and the element of basis matrices $\{A_i\}$. The coefficient b_i may be calculated as $b_i = \langle \Lambda_i \rangle = Tr[\rho^{(d)} \Lambda_i]$. The vector \vec{b} is known as the Bloch vector. Unlike the two-qubit case, the map induced is not bijective this means that it is not necessary that every point on the Bloch sphere in $d^2 - 1$ dimension corresponds to a physical state [3].

The geometric properties of the Bloch sphere in higher dimensions are rather complex as distinct matrix basis induces a different Bloch vector in a Bloch hypersphere, however, every point on the Bloch hypersphere may not correspond to a physical state but all distinct hyper balls are isomorphic as they correspond to the density matrix ρ .

1.2.5 Representing a d -dimensional quantum state in terms of different basis

Since there are various basis such as the generalized Gell-Mann matrix basis, the polarization operator basis, Weyl operator basis so it is important to know which matrix basis is ideal for a certain goal, like calculating the entanglement degree [4]. Here we will define the generalized Gell-Mann matrix basis, the polarization operator basis, Weyl operator basis, and then we will compare these three bases for a particular qutrit-qutrit example.

I. The generalized Gell-Mann matrix (GGM) basis

The GGM [3] are the extensions of Pauli matrices (the basis for a two-qubit system)

and the Gell-Mann matrices (the basis for a two-qutrit system). They are the generators of special unitary matrices of order d ($SU(d)$). They are defined by three different operators as

1. $\frac{d(d-1)}{2}$ symmetric GGM

$$\Lambda_s^{jk} = |j\rangle\langle k| + |k\rangle\langle j|, \quad 1 \leq j < k \leq d \quad (1.2.9)$$

2. $\frac{d(d-1)}{2}$ antisymmetric GGM

$$\Lambda_a^{jk} = -i(|j\rangle\langle k| - |k\rangle\langle j|), \quad 1 \leq j < k \leq d \quad (1.2.10)$$

3. $d - 1$ diagonal GGM

$$\Lambda^l = \sqrt{\frac{2}{l(l+1)}} \left(\sum_{j=1}^l |j\rangle\langle j| - l|l+1\rangle\langle l+1| \right), \quad 1 \leq l \leq d-1 \quad (1.2.11)$$

Thus, there are $d^2 - 1$ hermitian and traceless GGM. Also, they are orthogonal and form a basis for a density matrix of order d .

Decomposition of standard matrix basis into GGM basis

The standard matrix basis is formed by using $d \times d$ density matrices such that only one entry has value 1 and all other entries have value 0. These matrices form an orthonormal basis for the Hilbert-Schmidt space. In terms of operators, they can be expressed as

$$|j\rangle\langle k|, \quad j, k = 1, \dots, d \quad (1.2.12)$$

In general, the Bloch vector expansion of the density matrix using the GGM basis is shown below:

$$\rho_{GGM} = \frac{1}{d}I + \vec{b} \cdot \vec{\sigma} \quad (1.2.13)$$

where $\vec{b} = (\{b_s^{jk}\}, \{b_a^{jk}\}, \{b^l\})$ and the components are given by, $b_s^{jk} = Tr[\Lambda_s^{jk} \rho_{GGM}]$, $b_a^{jk} = Tr[\Lambda_a^{jk} \rho_{GGM}]$ and $b^l = Tr[\Lambda^l \rho_{GGM}]$ for $1 \leq j < k \leq d$ and $1 \leq l \leq d-1$. All Bloch vectors are contained in a hypersphere of a radius $|\vec{b}| \leq \sqrt{(d-1)/2d}$ [4].

The elements of a density matrix in standard matrix basis may be decomposed into

GGM basis as

$$|j\rangle\langle k| = \begin{cases} \frac{1}{2}(\Lambda_s^{jk} + i\Lambda_a^{jk}) & j < k \\ \frac{1}{2}(\Lambda_s^{kj} - i\Lambda_a^{kj}) & j > k \\ \sqrt{\frac{j-1}{2j}}\Lambda^{j-1} + \sum_{n=0}^{d-j-1} \frac{1}{\sqrt{2(j+n)(j+n+1)}}\Lambda^{j+n} + \frac{1}{d}I & j = k \end{cases} \quad (1.2.14)$$

II. The polarization operator basis (POB)

For a $d \times d$ density matrix, POB may be defined as

$$T_{LM} = \sqrt{\frac{2L+1}{2s+1}} \sum_{k,l=1}^d C_{sm_l,LM}^{sm_k} |k\rangle\langle l| \quad (1.2.15)$$

where $s = \frac{d-1}{2}$, $L = 0, 1, \dots, 2s$, $M = -L, -L+1, \dots, L-1, L$, $m_1 = s$, $m_2 = s-1, \dots, m_d = -s$. The coefficients $C_{sm_l,LM}^{sm_k}$ are Clebsch-Gordan coefficients $C_{j_1 m_1, j_2 m_2}^{j m}$ of the angular momentum theory and explicit expression are given in [5].

The polarization operator is a multiple of the Identity matrix for $L=M=0$ [5]. Except for T_{00} , all polarization operators are traceless and orthogonal but may or may not be Hermitian. Therefore these d^2 polarization operators form an orthonormal matrix basis of d dimensional Hilbert Schmidt space.

Any $d \times d$ density matrix ρ can be decomposed into a Bloch vector using the POB, and it often takes the following form:

$$\rho_{POB} = \frac{1}{d}I + \sum_{L=1}^{2s} \sum_{M=-L}^L b_{LM} T_{LM} = \frac{1}{d}I + \vec{b} \cdot \vec{T} \quad (1.2.16)$$

where $\vec{b} = (b_{1,-1}, b_{1,0}, b_{1,1}, b_{2,-2}, b_{2,-1}, b_{2,0}, \dots, b_{LM})$ is the Bloch vector and b_{LM} is given by $b_{LM} = \text{Tr}[T_{LM}^\dagger \rho]$. In general, the POB is not hermitian so, the components b_{LM} are complex. All Bloch vectors are contained in a hypersphere with a radius $|\vec{b}| \leq \sqrt{\frac{d-1}{d}}$.

Decomposition of standard matrix basis into POB

The standard matrix basis can be transformed to POB using the following relation

$$|i\rangle\langle j| = \sum_L \sum_M \sqrt{\frac{2L+1}{2s+1}} C_{sm_j,LM}^{sm_i} T_{LM} \quad (1.2.17)$$

III. Weyl operator basis (WOB)

For d -dimensional Hilbert-Schmidt space, d^2 Weyl's operators may be defined by

$$W_{nm} = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}kn} |k\rangle \langle (k+m) \bmod d|, \quad n, m = 0, 1, \dots, d-1 \quad (1.2.18)$$

The operators W_{nm} given in (1.2.18) are all unitary operators and form an orthonormal basis of the Hilbert spaces [6–8]

$$\text{Tr}[W_{nm}^\dagger W_{l_j}] = d\delta_{nl}\delta_{mj} \quad (1.2.19)$$

The operator $W_{00} = I$. Any $d \times d$ density matrix ρ can be decomposed into a Bloch vector using WOB as

$$\rho_{WOB} = \frac{1}{d}I + \sum_{n,m=0}^{d-1} b_{nm}W_{nm} = \frac{1}{d}I + \vec{b} \cdot \vec{W}, \quad n, m = 0, 1, \dots, d-1 \quad (1.2.20)$$

Assuming $b_{00} = 0$, the other components of the Bloch vector \vec{b} are given by $b_{nm} = \text{Tr}[W_{nm}\rho_{WOB}]$. In general, Weyl operators are not Hermitian so, b_{nm} are complex numbers and b_{nm}^* satisfies $b_{nm}^* = e^{-\frac{2\pi i}{d}nm}b_{-n-m}$. Bloch vectors are contained in a hypersphere with radius $|\vec{b}| \leq \frac{\sqrt{d-1}}{d}$ [4].

Decomposition of standard matrix basis into WOB

The elements of $d \times d$ density matrix may be expressed in terms of WOB as

$$|j\rangle \langle k| = \frac{1}{d} \sum_{i=0}^{d-1} e^{\frac{-2\pi i}{d}lj} W_{l(k-j) \bmod d} \quad (1.2.21)$$

In particular, for a two-dimensional system, the GGB, WOB, and POB can be expressed in terms of Pauli matrices. The corresponding relationship between the Pauli matrices and the basis GGM, WOB, and POB are given by

$$\{W_{00}, W_{01}, W_{10}, W_{11}\} = \{I, \sigma_x, i\sigma_y, \sigma_z\}, \quad (1.2.22)$$

$$\{I, \lambda_s^{12}, \lambda_a^{12}, \lambda^1\} = \{I, \sigma_x, \sigma_y, \sigma_z\}, \quad (1.2.23)$$

$$\{T_{0,0}, T_{1,1}, T_{1,0}, T_{1,-1}\} = \left\{ \frac{1}{\sqrt{2}}I, -\sigma_+, \frac{1}{\sqrt{2}}\sigma_z, \sigma_- \right\} \quad (1.2.24)$$

where $\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

1.2.6 An illustration: A two-qutrit system

Let us consider a two-qutrit isotropic state described by the density operator ρ_β . In standard basis, the state ρ_β may be defined as

$$\rho_\beta = \beta |\phi_+\rangle\langle\phi_+| + \frac{1-\beta}{9} I_9, \quad 0 \leq \beta \leq 1 \quad (1.2.25)$$

where I_9 denotes the identity matrix of order 9 and the state $|\phi_+\rangle$ represents a Bell state in a two-qutrit system and may be expressed as

$$|\phi_+\rangle = \frac{1}{\sqrt{3}}(|11\rangle + |22\rangle + |33\rangle) \quad (1.2.26)$$

I. Expansion of ρ_β in terms of GGM basis

We are now in a position to express a two-qutrit system in terms of GGM basis. For a single-qutrit system, the GGM basis can be expressed as [9, 10]

1. Symmetric GGM

$$\lambda_s^{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_s^{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_s^{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.2.27)$$

2. Antisymmetric GGM

$$\lambda_a^{12} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_a^{13} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_a^{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad (1.2.28)$$

3. Diagonal GGM

$$\lambda^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (1.2.29)$$

Let us first write the Bell state $|\phi_+\rangle\langle\phi_+|$ in the standard basis as

$$\begin{aligned} |\phi_+\rangle\langle\phi_+| &= \frac{1}{3}(|11\rangle\langle 11| + |11\rangle\langle 22| + |11\rangle\langle 33| \\ &+ |22\rangle\langle 11| + |22\rangle\langle 22| + |22\rangle\langle 33| \\ &+ |33\rangle\langle 11| + |33\rangle\langle 22| + |33\rangle\langle 33|) \\ &= A + B \end{aligned} \quad (1.2.30)$$

where A and B may be expressed as

$$\begin{aligned} A &= \frac{1}{3}(|11\rangle\langle 22| + |11\rangle\langle 33| + |22\rangle\langle 33|) \\ &+ (|22\rangle\langle 11| + |33\rangle\langle 11| + |33\rangle\langle 22|) \\ B &= \frac{1}{3}(|11\rangle\langle 11| + |22\rangle\langle 22| + |33\rangle\langle 33|) \end{aligned} \quad (1.2.31)$$

For a qutrit system, the GGM basis is related to the standard basis in the following way

$$\begin{aligned} \Lambda_s^{12} &= |1\rangle\langle 2| + |2\rangle\langle 1|, \Lambda_s^{13} = |1\rangle\langle 3| + |3\rangle\langle 1|, \Lambda_s^{23} = |2\rangle\langle 3| + |3\rangle\langle 2| \\ \Lambda_a^{12} &= -i|1\rangle\langle 2| + i|2\rangle\langle 1|, \Lambda_a^{13} = -i|1\rangle\langle 3| + i|3\rangle\langle 1|, \Lambda_a^{23} = -i|2\rangle\langle 3| + i|3\rangle\langle 2| \\ \Lambda^1 &= |1\rangle\langle 1| - |2\rangle\langle 2|, \Lambda^2 = \frac{1}{\sqrt{3}}(|1\rangle\langle 1| + |2\rangle\langle 2| - 2|3\rangle\langle 3|) \end{aligned} \quad (1.2.32)$$

The matrices A and B can be re-expressed in terms of GGM basis as

$$\begin{aligned} A &= \frac{1}{6}[\Lambda_s^{12} \otimes \Lambda_s^{12} + \Lambda_s^{13} \otimes \Lambda_s^{13} + \Lambda_s^{23} \otimes \Lambda_s^{23} - \Lambda_a^{12} \otimes \Lambda_a^{12} - \Lambda_a^{13} \otimes \Lambda_a^{13} - \Lambda_a^{23} \otimes \Lambda_a^{23}] \\ B &= \frac{1}{3}[\frac{1}{2}\Lambda^1 \otimes \Lambda^1 + \frac{1}{2}\Lambda^2 \otimes \Lambda^2 + \frac{1}{3}I \otimes I] \end{aligned} \quad (1.2.33)$$

Therefore, the Bell state (1.2.30), then can be re-expressed in terms of GGM basis as

$$\begin{aligned} |\phi_+\rangle\langle\phi_+| &= \frac{1}{6}[\Lambda_s^{12} \otimes \Lambda_s^{12} + \Lambda_s^{13} \otimes \Lambda_s^{13} + \Lambda_s^{23} \otimes \Lambda_s^{23} - \Lambda_a^{12} \otimes \Lambda_a^{12} - \Lambda_a^{13} \otimes \Lambda_a^{13} \\ &- \Lambda_a^{23} \otimes \Lambda_a^{23} + \Lambda^1 \otimes \Lambda^1 + \Lambda^2 \otimes \Lambda^2] + \frac{1}{9}I \otimes I \end{aligned} \quad (1.2.34)$$

Thus in terms of GGB, the two-qutrit isotropic state (1.2.25) may be expressed as

$$\rho_\beta = \frac{\beta}{6}\Lambda + \frac{1}{9}I \otimes I \quad (1.2.35)$$

where $\Lambda = \Lambda_s^{12} \otimes \Lambda_s^{12} + \Lambda_s^{13} \otimes \Lambda_s^{13} + \Lambda_s^{23} \otimes \Lambda_s^{23} - \Lambda_a^{12} \otimes \Lambda_a^{12} - \Lambda_a^{13} \otimes \Lambda_a^{13} - \Lambda_a^{23} \otimes \Lambda_a^{23} + \Lambda^1 \otimes \Lambda^1 + \Lambda^2 \otimes \Lambda^2$

II. Expansion of ρ_β in terms of POB

Let us first write the polarizing operators for a three-dimensional system. For a qutrit system, we have $s = 1$ and nine polarizing operators are denoted by $T_{L,M}$, $L = 0, 1, 2$ and $M = -L, \dots, L$ and they are given by

$$T_{0,0} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T_{1,1} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, T_{1,0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (1.2.36)$$

$$T_{1,-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, T_{2,2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T_{2,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.2.37)$$

$$T_{2,0} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T_{2,-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, T_{2,-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (1.2.38)$$

To express the state ρ_β defined in (1.2.25) in terms of POB, we first need to express Bell state (1.2.26) in terms of POB. In the standard basis, Bell state may be written as

$$|\phi_+\rangle\langle\phi_+| = \frac{1}{3} \left[\sum_{i,j=1}^3 |i\rangle\langle j| \otimes |i\rangle\langle j| \right] \quad (1.2.39)$$

Using (1.2.17), the Bell state (1.2.39) may be re-expressed as

$$|\phi_+\rangle\langle\phi_+| = \frac{1}{3} \left[\sum_L \sum_M \sqrt{\frac{2L+1}{3}} C_{sm_j, LM}^{sm_i} T_{LM} \otimes \sum_{L'} \sum_{M'} \sqrt{\frac{2L'+1}{3}} C_{sm_j, L'M'}^{sm_i} T_{L'M'} \right] \quad (1.2.40)$$

Since $M = m_i - m_j$ hold good so \sum_M now reduces to $\sum_{i,j}$. Thus, the above equation (1.2.40) reduces to

$$|\phi_+\rangle\langle\phi_+| = \frac{1}{3} \left[\sum_{L,L'=1}^3 \sqrt{\frac{(2L+1)(2L'+1)}{9}} \left(\sum_{i,j=1}^3 C_{sm_j, LM}^{sm_i} C_{sm_j, L'M'}^{sm_i} \right) T_{LM} \otimes T_{L'M'} \right] \quad (1.2.41)$$

The sum rule for the Clebsch-Gordan coefficients [5] is given by

$$\sum_{\alpha,\gamma} C_{\alpha\alpha,b\beta}^{d\gamma} C_{\alpha\alpha,b'\beta'}^{d\gamma} = \frac{2d+1}{2b+1} \delta_{bb'} \delta_{\beta\beta'} \quad (1.2.42)$$

Using the sum rule for the Clebsch-Gordan coefficients, the equation (1.2.41) can now be purely expressed in terms of POB as

$$\begin{aligned} |\phi_+\rangle\langle\phi_+| &= \frac{1}{3} \left[\sum_{L,L'}^3 \sqrt{\frac{(2L+1)(2L'+1)}{9}} \cdot \frac{3}{2L+1} \delta_{LL'} T_{LM} \otimes T_{L'M} \right] \\ &= \frac{1}{3} \left[\sum_{L,L'=1}^3 \frac{\sqrt{(2L+1)(2L'+1)}}{2L+1} \delta_{LL'} T_{LM} \otimes T_{L'M} \right] \\ &= \frac{1}{3} \left[\sum_{L=1}^3 T_{LM} \otimes T_{LM} \right] \end{aligned} \quad (1.2.43)$$

Using (1.2.15)

$$|\phi_+\rangle\langle\phi_+| = \frac{1}{9} I \otimes I + \frac{1}{3} \left(\sum_{L,M \neq 0,0} T_{LM} \otimes T_{LM} \right) \quad (1.2.44)$$

Using POB Bloch vector notation, the two-qudit state (1.2.25), may be written as

$$\rho_\beta = \frac{1}{9} I \otimes I + \frac{\beta}{3} \left(\sum_{L,M \neq 0,0} T_{LM} \otimes T_{LM} \right) \quad (1.2.45)$$

III. Expansion of ρ_β in terms of WOB

To express the state ρ_β defined in (1.2.25) in terms of WOB, let us first write Weyl's operator for a qutrit system. For a qutrit system ($d = 3$), the Weyl operator may be expressed as

$$W_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, W_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, W_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (1.2.46)$$

$$W_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}, W_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \\ e^{-\frac{2\pi i}{3}} & 0 & 0 \end{pmatrix}, W_{12} = \begin{pmatrix} 0 & 0 & 1 \\ e^{\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{3}} & 0 \end{pmatrix} \quad (1.2.47)$$

$$W_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}, W_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} & 0 & 0 \end{pmatrix}, W_{22} = \begin{pmatrix} 0 & 0 & 1 \\ e^{-\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \end{pmatrix} \quad (1.2.48)$$

Let us start with the expression of Bell state (1.2.26) on standard basis as

$$|\phi_+\rangle\langle\phi_+| = \frac{1}{3} \sum_{j,k=1}^3 |j\rangle\langle k| \otimes |j\rangle\langle k| \quad (1.2.49)$$

Equation (1.2.49), in terms of Weyl operators, can be written as

$$\begin{aligned} |\phi_+\rangle\langle\phi_+| &= \frac{1}{3} \sum_{j,k=0}^2 \left[\frac{1}{3} \sum_{l=0}^2 e^{-\frac{2\pi i}{3}lj} W_{l(k-j) \bmod 3} \otimes \frac{1}{3} \sum_{l'=0}^2 e^{-\frac{2\pi i}{3}l'j} W_{l'(k-j) \bmod 3} \right] \\ &= \frac{1}{27} \sum_{j,k=0}^2 \sum_{l,l'=0}^2 e^{-\frac{2\pi i}{3}j(l+l')} W_{l(k-j) \bmod 3} \otimes W_{l'(k-j) \bmod 3} \end{aligned} \quad (1.2.50)$$

Substituting $m = (k-j) \bmod 3$, $m = 0, 1, 2$. The above equation may be re-written as

$$\begin{aligned} |\phi_+\rangle\langle\phi_+| &= \frac{1}{27} \sum_{m,k=0}^2 \sum_{l,l'=0}^2 e^{-\frac{2\pi i}{3}(k-m)(l+l')} W_{lm} \otimes W_{l'm} \\ &= \frac{1}{27} \left[\sum_{m,k=0}^2 1 \cdot W_{0m} \otimes W_{0m} + \sum_{m,k=0}^2 \sum_{l,l',l+l'=d} 1 \cdot W_{lm} \otimes W_{l'm} \right. \\ &\quad \left. + \sum_m \sum_{l,l' \neq 0, l+l'=d}^2 \left(\sum_k e^{-\frac{2\pi i}{3}(k-m)(l+l')} \right) W_{lm} \otimes W_{l'm} \right] \\ &= \frac{1}{9} \sum_m W_{0m} \otimes W_{0m} + \frac{1}{9} \sum_m \sum_{l,l',l+l'=d} W_{lm} \otimes W_{l'm} \\ &\quad + \frac{1}{27} \left[\sum_m \sum_{l,l',l+l'=d}^2 \left(\sum_k e^{-\frac{2\pi i}{3}(k-m)(l+l')} \right) W_{lm} \otimes W_{l'm} \right] \end{aligned} \quad (1.2.51)$$

Using the equivalence

$$\sum_{n=0}^{d-1} e^{\frac{2\pi i}{d}nx} = \begin{cases} d & \text{if } x = 0 \\ 0 & \text{if } x \neq 0, \end{cases} \quad x \in \mathbb{Z} \quad (1.2.52)$$

We are now in a position to express the Bell state (1.2.26) in terms of WOB. Using the notation $-l \pmod{3} = (3-l)$ and $W_{00} = I$, the Bell state (1.2.26) in terms of WOB may be expressed as

$$|\phi_+\rangle\langle\phi_+| = \frac{1}{9} I \otimes I + \frac{1}{9} \sum_{l,m=0}^2 W_{l,m} \otimes W_{-l,m}, \quad (l,m) \neq (0,0) \quad (1.2.53)$$

The two-qutrit state ρ_β in terms of WOB may be expressed as

$$\rho_\beta = \frac{1}{9}I \otimes I + \frac{\beta}{9} \sum_{l,m=0}^2 W_{l,m} \otimes W_{-l,m} \quad (1.2.54)$$

1.2.7 Quantum Measurements

By what process does measurement is carried out on a quantum system? This may be discussed in the following way: When a measurement is performed on a quantum system by a measurement apparatus, the quantum state collapses probabilistically into one of a number of eigenstates of the observable. Can this behavior itself be explained by quantum mechanics? These are a few interesting and puzzling issue that has not been settled. It is known as the problem of measurement.

If O denotes the observable and $|s\rangle$ is the corresponding eigenstate, then the measurement process may be described as

$$O|s\rangle = \lambda|s\rangle \quad (1.2.55)$$

where λ denote the eigenvalue of O .

A measurement apparatus is used to calculate the eigenvalue λ . After the measurement, the uncertainty of the observable O vanishes at least for the instant, when the measurement is performed [11]. One of the most fundamental concepts in quantum physics is the uncertainty principle. The uncertainty principle may be expressed in different ways such as Robertson uncertainty relation [12], Heisenberg uncertainty relation for joint measurements [13, 14] and Noise-disturbance uncertainty relation [15, 16]. It is widely accepted that any version of the uncertainty principle has a logical connection to the restriction on measuring a system without disturbing it, such as a position measurement often disturbs the momentum.

Let us now explain the idea of measurement when it is performed on a quantum system $|\phi\rangle$. In quantum mechanics, measurements are described by measurement operators $\{M_m\}_{m=1}^n$ where m denote the measurement outcome [2]. These operators act on the state space of the system being measured. These measurement operators satisfy the completeness relation

$$\sum_m M_m^\dagger M_m = I \quad (1.2.56)$$

where I denotes the identity operator. Consider $|\phi\rangle$ to be the state of the quantum system before measurement then the state defining the system after the measurement is given by,

$$|\phi_m\rangle = \frac{M_m|\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}} \quad (1.2.57)$$

The probability of getting measurement outcome m is given by

$$p(m) = \langle\phi|M_m^\dagger M_m|\phi\rangle \quad (1.2.58)$$

Projective measurements: A Projective measurement is described by an observable M that may be decomposed via Spectral decomposition as

$$M = \sum_m m P_m \quad (1.2.59)$$

where P_m are projectors on the eigenspace of M corresponding to eigenvalue m . The projector P_m satisfies the following properties:

- (i) P_m is Hermitian i.e. $P_m^\dagger = P_m$.
- (ii) Idempotent property holds for P_m i.e. $P_m^2 = P_m$.
- (iii) P_m has eigenvalues 0 and 1.

When the projective measurement described by the projectors P_m performed on the state $|\phi\rangle$, the state reduces to

$$|\phi_m^P\rangle = \frac{P_m|\phi\rangle}{\sqrt{p(m)}} \quad (1.2.60)$$

where $p(m)$ denotes the probability of getting outcome m and it is given by

$$p(m) = \langle\phi|P_m|\phi\rangle \quad (1.2.61)$$

For projective measurement P , the average value of the measurement is given by

$$E(P) = \sum_m m p(m) = \langle\phi|P|\phi\rangle = \langle P \rangle_\phi \quad (1.2.62)$$

The standard deviation ($\Delta(P)$) associated with the observations of P may be expressed as

$$[\Delta(P)]_\phi = \sqrt{\langle (P - \langle P \rangle_\phi)^2 \rangle_\phi} = \sqrt{\langle P^2 \rangle_\phi - \langle P \rangle_\phi^2} \quad (1.2.63)$$

von Neumann measurement: Consider a given quantum system described by the density operator ρ written in the computation basis $\{|n\rangle, n = 0, 1, 2, \dots\}$. Then the set of projection operators that projects any state onto the basis states may be defined by $P_n = |n\rangle\langle n|$. After the measurement, the state may be expressed as

$$\rho^V = \frac{P_n \rho P_n}{p(n)} \quad (1.2.64)$$

where $p(n)$ denotes the probability of the system being in state $|n\rangle$ and is defined as

$$p(n) = \langle n | \rho | n \rangle = \text{Tr}[P_n \rho P_n] = \text{Tr}[(P_n)^2 \rho] = \text{Tr}[P_n(\rho)] \quad (1.2.65)$$

Let us now understand the concept of von Neumann measurement by considering a single qubit state $|\psi\rangle$, which is given by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1 \quad (1.2.66)$$

$|\alpha|^2 + |\beta|^2 = 1$ denote the normalization condition. Since the state $|\psi\rangle$ is defined on a two-dimensional Hilbert space spanned by $\{|0\rangle, |1\rangle\}$ so, the two measurement operators defined on this state space are given by $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$. It can be easily shown that the projection operator P_0 and P_1 satisfy the following properties:

(i) $P_i^2 = P_i, i = 0, 1$.

(ii) $P_0^\dagger P_0 + P_1^\dagger P_1 = P_0 + P_1 = I$, where I denote the identity operator.

Let us suppose that the state $|\psi\rangle$ is being measured with the measurement operator P_0 and P_1 . When P_0 is performed on $|\psi\rangle$, the state after the measurement reduces to $\frac{P_0|\psi\rangle}{|\alpha|} = \frac{\alpha}{|\alpha|}|0\rangle$. This output may be obtained with probability $p(0)$ which is given by

$$p(0) = \langle \psi | P_0^\dagger P_0 | \psi \rangle = \langle \psi | P_0 | \psi \rangle = |\alpha|^2 \quad (1.2.67)$$

Similarly, when P_1 is performed on $|\psi\rangle$, the state after the measurement reduces to $\frac{P_1|\psi\rangle}{|\beta|} = \frac{\beta}{|\beta|}|1\rangle$. This output may be obtained with probability $p(1)$ which is given by

$$p(1) = \langle\psi|P_1^\dagger P_1|\psi\rangle = \langle\psi|P_1|\psi\rangle = |\beta|^2 \quad (1.2.68)$$

Positive operator-valued measurement (POVM): Let M_m be the measurement operators which is acting on a quantum system $|\psi\rangle$. Then, the set of operators $\{E_m = M_m^\dagger M_m\}$ is known as POVM. The operators E_m are known as POVM elements. Therefore, we are now in a position to define POVM. POVM may be defined as a set of operators $\{E_m\}$ which satisfies the following two properties:

- (i) each operator E_m ($m = 0, 1, 2, \dots$) are positive.
- (ii) $\sum_m E_m = I$, where I is the Identity operator.

In POVM, the probability of getting outcome m is $p(m) = \langle\psi|E_m|\psi\rangle$. If the measurement operators and the POVM elements coincide in any measurement then the measurement is a projective measurement.

For instance, suppose Alice send Charlie have one of the two states from the set $\{|\psi_1\rangle, |\psi_2\rangle\}$ where $|\psi_1\rangle = |1\rangle$ and $|\psi_2\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$. Charlie doesn't know whether he is given $|\psi_1\rangle$ or $|\psi_2\rangle$, so, he performs POVM to determine the received state. To perform POVM, we first need to define the POVM elements. In this case, the POVM elements can be written in the following form

$$E_1 = \frac{\sqrt{2}}{1 + \sqrt{2}}|0\rangle\langle 0| \quad (1.2.69)$$

$$E_2 = \frac{\sqrt{2}}{1 + \sqrt{2}} \frac{(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)}{2} \quad (1.2.70)$$

$$E_3 = I - E_1 - E_2 \quad (1.2.71)$$

where I is 2×2 identity matrix. Then the set $\{E_1, E_2, E_3\}$ form a POVM. Suppose Charlie is given $|\psi_1\rangle$ and he performs the measurement using $\{E_1, E_2, E_3\}$. There is a zero probability that he will get measurement outcome E_1 . If the result of the measurement outcome comes out to be E_1 , then Charlie can conclude that he was given $|\psi_2\rangle$. Also, if the measurement outcome comes out to be E_2 , the state Charlie has must be $|\psi_1\rangle$. If the measurement outcome is E_3 , then Charlie can infer nothing about the identity of the state but whenever Charlie identifies the state he can never make a mistake in identifying the state.

1.3 EPR Paradox

To explain the EPR paradox, let us discuss the following experiment: Suppose two spin 1/2 particles are ejected in opposite directions each time a button is pressed, and they finally pass through a detector that determines the spin of the particle along a specific axis. The detectors can be rotated which makes it possible to measure the spin along any axis, particularly for x and z axis. After a number of tests, the following observation is made:

Observation-1 (O1): To measure spin along the z-axis, both detectors are configured. Half of the time, the particles on the right have spin down and the particles on the left have up spin, and the other half of the time, the particles on the right are up spin and the particles on the left are down spin.

This implies that we may determine the z-component of the spin of the particle on the right by measuring the z-component of the spin of the particle on the left without actually interacting with the particle on the right. Suppose L denotes the particle which moves to the left and R denotes the particles which move to the right. After performing the experiment and knowing that L has spin up, we know that R has spin down. Since the measurement on L couldn't possibly affect R it means that R was spin down even before the measurement on L was performed.

Let us consider the spin state of the particles before entering the detectors which may be denoted by $|\psi_1\rangle$

$$|\psi_1\rangle = U_z^L D_z^R \quad (1.3.1)$$

where U_z and D_z denote up spin and down spin respectively, along the z-axis. Here

$$U_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, D_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.3.2)$$

Similarly, if L has a down spin, then R has an up spin and the spin state of the particles may be written as

$$|\psi_2\rangle = D_z^L U_z^R \quad (1.3.3)$$

From O1, half of the time, the particles are in state $|\psi_1\rangle$ and in the other half of the time, the particles are in state $|\psi_2\rangle$. After repeating the experiment many more times

we can make another observation that can be stated as follows:

Observation-2 (O2): The two particles never have spins that are opposite to one another when both detectors are configured to monitor spin in the x-direction.

The reason is that if the particles were initially in the state $|\psi_1\rangle$, then the probability that measurement would detect both spins up in the x-direction is given by

$$|\langle U_x^L U_x^R | \psi_1 \rangle|^2 \quad (1.3.4)$$

where U_x and D_x are given by

$$U_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, D_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.3.5)$$

Thus

$$|\langle U_x^L U_x^R | \psi_1 \rangle|^2 = |\langle U_x^L U_x^R | U_z^L D_z^R \rangle|^2 = |\langle U_x^L | U_z^L \rangle \langle U_x^R | D_z^R \rangle|^2 = \frac{1}{4} \quad (1.3.6)$$

If the particles were initially in state $|\psi_2\rangle$, then the probability that both particles have up spin in the x-direction would be

$$|\langle U_x^L U_x^R | \psi_2 \rangle|^2 = |\langle U_x^L U_x^R | D_z^L U_z^R \rangle|^2 = |\langle U_x^L | D_z^L \rangle \langle U_x^R | U_z^R \rangle|^2 = \frac{1}{4} \quad (1.3.7)$$

The probability that both the particles with spin up along the x-axis may be defined as the average of the two probabilities that the particles are in state $|\psi_1\rangle$ and the probability that the particle is in the $|\psi_2\rangle$

$$\frac{1}{2} |\langle U_x^L U_x^R | \psi_1 \rangle|^2 + \frac{1}{2} |\langle U_x^L U_x^R | \psi_2 \rangle|^2 = \frac{1}{4} \quad (1.3.8)$$

From O1 and O2, there is no experiment that satisfies both conclusions. EPR pointed out that these outcomes can be obtained for certain quantum mechanical states. Suppose that before reaching the detectors, two particles are in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_1\rangle - |\psi_2\rangle) = \frac{1}{\sqrt{2}} (U_z^L D_z^R - D_z^L U_z^R) \quad (1.3.9)$$

Then, along the z-axis, the probability of L with up spin and R with down spin may be expressed as

$$|\langle U_z^L D_z^R | \psi \rangle|^2 = \frac{1}{2} |\langle U_z^L D_z^R | \psi_1 \rangle - \langle U_z^L D_z^R | \psi_2 \rangle|^2 = \frac{1}{2} \quad (1.3.10)$$

Similarly, the probability of L with a down spin and R with an up spin is given by

$$|\langle D_z^L U_z^R | \psi \rangle|^2 = \frac{1}{2} |\langle U_z^L D_z^R | \psi_1 \rangle - \langle U_z^L D_z^R | \psi_2 \rangle|^2 = \frac{1}{2} \quad (1.3.11)$$

These two probabilities are the same as we got in case *O1*.

The probability of both L and R with spin up along the x-axis is given by

$$|\langle U_x^L U_x^R | \psi \rangle|^2 = 0 \quad (1.3.12)$$

which is the same as the result of *O2*.

If *O1* and *O2* both are true, then this would mean that before the measurement, R was neither in state D_z^R nor in state U_z^R . So, if we measure that L has spin up along the z-axis, at that very instant we know that R has spin down without actually performing a measurement on R. Also, before the measurement, R was neither spin up nor spin down. So, initially, R was not in the spin down direction which means it must have jumped in a down spin as soon as we found that L has an up spin even when L and R are very far away at the time of measurement. But EPR argued that how could a measurement on L influence the state of R even when L and R are spatially separated? This is known as the EPR paradox. Since this paradox is not explained by the theory of quantum mechanics, so, it makes Einstein, Podolsky, and Rosen believe that the quantum theory is not complete. Many experiments have been performed to solve the EPR paradox but the success came in 1964 when J S Bell put forward his theory in terms of an inequality. The inequality derived by Bell is popularly known as Bell's inequality. An experiment has been performed by Alain Aspect et.al. [17, 18] to show that the correlation that may exist between linear polarization of pair of photons violates Bell's inequality. This indicates the fact that the correlation exists between the linear polarization of two photons may be non-local in nature.

1.4 Quantum entanglement

The term entanglement was first coined by Schrodinger [19] and at the same time, Einstein, Podolsky, and Rosen recognized this feature as a special feature of quantum mechanics that has no classical analog and called it a "spooky action at a distance" [20]. Quantum entanglement has a very complex structure that is very fragile with respect to the environment. In an entangled system, the subsystems are distributed to the regions which are far apart from one another. The amount of entanglement in the system cannot increase on average when a measurement is performed on one subsystem. Let us now understand this complex structure of entanglement that may exist in the bipartite and tripartite systems.

1.4.1 Bipartite and tripartite system

I. Bipartite system

Let us consider two individual systems A and B consisting of the state vectors residing in the Hilbert spaces H_A and H_B respectively. The composite system of two individual systems can be considered a bipartite system and it is described in the Hilbert space H_{AB} . The two individual systems A and B can be referred to as the subsystems of the composite system. A composite Hilbert space H_{AB} , may be defined as the tensor product of two subsystems A and B , and it is given by

$$H_{AB} \equiv H_A^{d_A} \otimes H_B^{d_B} \quad (1.4.1)$$

where d_A and d_B are the dimensions of individual subsystems A and B .

If $|\psi_i\rangle_{AB}$ denote the pure state in the Hilbert space H_{AB} then any state in the composite Hilbert space is described by the density operator ρ_{AB}

$$\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1 \quad (1.4.2)$$

Let us consider a bipartite system described by a density matrix ρ_{AB} . The subsystem A or B may be extracted from the composite system AB by using the mathematical formulation known as the partial trace. The partial trace with respect to subsystem A

or B may be defined as

$$\begin{aligned}\rho^A &= Tr_B(\rho_{AB}) \\ \rho^B &= Tr_A(\rho_{AB})\end{aligned}\quad (1.4.3)$$

Here, the operators ρ^A and ρ^B are positive semi-definite operators and fulfill the condition $Tr(\rho^A) = 1$ and $Tr(\rho^B) = 1$. Thus, the operators ρ^A and ρ^B satisfy the condition of a density matrix, and hence, the operators ρ^A and ρ^B are called reduced density matrices.

(a) Entanglement in bipartite system:

If ρ^{AB} represents the density matrix of a composite system described in the Hilbert space $H_A \otimes H_B$, then it defines a product state if there exist subsystems ρ^A and ρ^B such that

$$\rho^{AB} = \rho^A \otimes \rho^B \quad (1.4.4)$$

More generally, we can say that the state is separable if it can be written as the convex combination of product states

$$\rho^{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1 \quad (1.4.5)$$

where, p_i are the convex weights and $\rho_i^A \otimes \rho_i^B$ represent the product states.

If the state ρ^{AB} is not expressed in the above form (1.4.5), then the state ρ^{AB} is said to be an entangled state. These states cannot be generated via local operation and classical communication (LOCC). For instance, a bipartite two-qubit pure state $|\phi\rangle_{AB} = |0\rangle_A \otimes \frac{|0\rangle_B + |1\rangle_B}{\sqrt{2}}$ represent a pure product state in a two-qubit system.

(b) Schmidt Decomposition: Consider a pure state $|\psi\rangle_{AB}$ of a composite system AB. Then there exist orthonormal states $|i_A\rangle$ and $|i_B\rangle$ for subsystems A and B respectively such that the state $|\psi\rangle_{AB}$ may be written as

$$|\psi\rangle_{AB} = \sum_i \lambda_i |i_A\rangle |i_B\rangle, \quad \sum_i \lambda_i^2 = 1 \quad (1.4.6)$$

where $\lambda_i \geq 0$ are known as Schmidt coefficients.

In particular, when all Schmidt coefficients, except one of the coefficient, are zero, then the state $|\psi\rangle_{AB}$ reduces to a pure product state. When at least two Schmidt co-

efficients are non-zero, then the pure state $|\psi\rangle_{AB}$ is said to be entangled. In this way, Schmidt's decomposition of any pure state may help us to check whether the given pure state is entangled or separable.

(c) Entanglement measures ($\eta(\rho_{AB})$): Entanglement measures quantifies the amount of entanglement in a quantum state $\rho_{AB} = \sum_k p_k (\rho_k)_{AB}$, where p_k denote the classical probability. $\eta(\rho_{AB})$ can be defined as a non-negative real-valued function that satisfies the following properties, some of which are not satisfied by all entanglement measure [21]:

- (i) $\eta(\rho_{AB})$ vanishes for the separable state ρ .
- (ii) A bipartite entanglement measure $\eta(\rho_{AB})$ defines a mapping from the set of density matrices to the set of positive real numbers:

$$\rho \rightarrow \eta(\rho) \in R^+ \quad (1.4.7)$$

A normalization condition is generally used, for example, for two qudit maximally entangled state of the form

$$|\psi\rangle_d = \frac{|0,0\rangle + |1,1\rangle + \dots + |d-1,d-1\rangle}{\sqrt{d}} \quad (1.4.8)$$

The value is given by $\eta(|\psi\rangle_d) = \log(d)$.

- (iii) $\eta(\rho_{AB})$ does not increase on average under local operation and classical communication (LOCC)

$$\eta(\rho) = \sum_i p_i \eta\left(\frac{K_i \rho K_i^\dagger}{\text{Tr}[K_i \rho K_i^\dagger]}\right) \quad (1.4.9)$$

where the K_i are the Kraus operators.

- (iv) For pure states $|\psi\rangle\langle\psi|$, $\eta(\rho)$ reduces to entropy of entanglement i.e.

$$\eta(|\psi\rangle\langle\psi|) = (S \text{Tr}_B)(|\psi\rangle\langle\psi|) \quad (1.4.10)$$

Any function η satisfying the first three conditions is known as an entanglement monotone. If η satisfies the property (i),(ii), and (iv) and also it does not increase under deterministic LOCC transformations, then η is known as an entanglement measure.

(i) von Neumann entropy: Consider a quantum state described by a density operator ρ_{AB} , then the entropy of entanglement is a way to quantify the amount of entanglement

in a pure state. The entropy of entanglement of ρ_{AB} may be defined as

$$E(\rho_{AB}) = V(\rho_A) = V(\rho_B) \quad (1.4.11)$$

where $V(\rho) = -Tr[\rho \log_2 \rho]$ is the von Neumann entropy of ρ . Here $\rho_A = Tr_B(\rho_{AB})$ and $\rho_B = Tr_A(\rho_{AB})$ are the reduced matrices obtained by tracing out subsystem A and B respectively.

(ii) Entanglement of formation: The entanglement of formation of a bipartite mixed state ρ_{AB} may be defined as the convex roof of the von Neumann entropy

$$\eta_F(\rho_{AB}) = \inf_{\{p_k, |\phi_k\rangle\}} \sum_k p_k V[(\rho_A)_k] \quad (1.4.12)$$

where ρ_A is the reduced state of ρ_{AB} . Physically, it may be understood as the minimum number of singlets needed to create one copy of the state [22].

(iii) Concurrence: For two-qubit pure state $|\psi\rangle_{AB}^{2\otimes 2}$, entanglement can be quantified using an entanglement measure known as concurrence [23–25], which is defined as

$$C(|\psi\rangle_{AB}^{2\otimes 2}) = \sqrt{2(1 - Tr(\rho_A^2))} \quad (1.4.13)$$

where ρ_A is the reduced state of $|\psi\rangle_{AB}^{2\otimes 2}$. It may also be generalized to $d \otimes d$ dimensional bipartite pure states as

$$C(|\psi\rangle_{AB}^{d_1 \otimes d_2}) = \sqrt{2v_{d_1} v_{d_2} [1 - Tr(\rho_A^2)]} \quad (1.4.14)$$

where ρ_A is the reduced state of $|\psi\rangle_{AB}^{d_1 \otimes d_2}$ and v_{d_1} and v_{d_2} are positive constants.

Concurrence for the two-qubit mixed state described by the density operator ρ_{AB} , may be defined as,

$$C(\rho_{AB}) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}) \quad (1.4.15)$$

where λ_i 's are the eigenvalues of $\rho_{AB} \widetilde{\rho_{AB}}$ and arranged in descending order. Here, $\widetilde{\rho_{AB}} = (\sigma_y \otimes \sigma_y) \rho_{AB} (\sigma_y \otimes \sigma_y)$.

The closed formula for the concurrence of $d \otimes d$ dimensional mixed bipartite system is still not known.

(iv) Negativity: Negativity [26] is a popular measure of entanglement used for quantification of entanglement in $d \otimes d$ dimensional bipartite system. For a given density

matrix ρ_{AB} , it is defined as the sum of negative eigenvalues of the partially transposed matrix $\rho_{AB}^{T_B}$. It may be expressed as [27]

$$N(\rho_{AB}) = \frac{\|\rho_{AB}^{T_B}\|_1 - 1}{d - 1} \quad (1.4.16)$$

where $\|\cdot\|$ denotes the trace norm.

II. Tripartite system:

In the case of a tripartite system, the structure of entanglement is more complex due to the existence of several inequivalent classes of entanglement. For a tripartite system, all states are divided into three inequivalent classes under stochastic local operations and classical communication (SLOCC). These classes are: (i) class of fully separable states (ii) class of biseparable states and (iii) class of genuine entangled states.

(i) Class of fully separable states: Fully separable (FS) three-qubit pure states may be expressed as

$$|\phi\rangle_{ABC}^{FS} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C \quad (1.4.17)$$

A fully separable mixed state described by the density operator ρ^{FS} can be expressed as

$$\rho^{FS} = \sum_i p_i |\phi_i^{FS}\rangle \langle \phi_i^{FS}|, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1 \quad (1.4.18)$$

where each $|\phi_i^{FS}\rangle$ is a fully separable pure state.

(ii) Class of biseparable states: Three-qubit pure biseparable states may be expressed as

$$|\phi\rangle_{ABC}^{BS} = |a\rangle_A \otimes |d\rangle_{BC} \quad (1.4.19)$$

Here $|d\rangle$ denotes the state which represents an entangled state between the subsystem B and C . In the class of biseparable state, two of the three qubits are grouped as one party. Thus, there may exist three possibilities that can be obtained by taking two parties at one time. Hence, we can find three kinds of biseparable states in this class.

One kind of biseparable state is given in (1.4.19) and the other two may be defined as

$$|\phi\rangle_{ABC}^{BS} = |a_1\rangle_B \otimes |d_1\rangle_{AC} \quad (1.4.20)$$

$$|\phi\rangle_{ABC}^{BS} = |a_2\rangle_C \otimes |d_2\rangle_{AB} \quad (1.4.21)$$

A mixed biseparable state can be expressed as a convex combination of different biseparable pure states. Mathematically, biseparable mixed states may be expressed as [22]

$$\rho^{BS} = \sum_i p_i |\phi_i^{BS}\rangle\langle\phi_i^{BS}| \quad (1.4.22)$$

We should note that the state $|\phi_i^{BS}\rangle$ and $|\phi_j^{BS}\rangle$ ($i \neq j$) may not belong to the same partition. For instance, let us consider a three-qubit state described by the density operator ρ_1 as

$$\rho_1 = q[|0\rangle_A\langle 0| \otimes |\phi^+\rangle_{BC}\langle\phi^+|] + (1-q)[|1\rangle_B\langle 1| \otimes |\phi^-\rangle_{AC}\langle\phi^-|] \quad (1.4.23)$$

where $0 \leq q \leq 1$ and the Bell states $|\phi^+\rangle_{BC}$ and $|\phi^-\rangle_{AC}$ are given by

$$|\phi^+\rangle_{BC} = \frac{1}{\sqrt{2}}(|00\rangle_{BC} + |11\rangle_{BC}), \quad |\phi^-\rangle_{AC} = \frac{1}{\sqrt{2}}(|00\rangle_{AC} - |11\rangle_{AC}) \quad (1.4.24)$$

Identifying $|\phi_1^{BS}\rangle\langle\phi_1^{BS}| = |0\rangle_A\langle 0| \otimes |\phi^+\rangle_{BC}\langle\phi^+|$, a biseparable state in A-BC partition and $|\phi_2^{BS}\rangle\langle\phi_2^{BS}| = |1\rangle_B\langle 1| \otimes |\phi^-\rangle_{AC}\langle\phi^-|$, a biseparable state in another bipartition cut B-AC, we can re-express ρ_1 as

$$\rho_1 = q|\phi_1^{BS}\rangle\langle\phi_1^{BS}| + (1-q)|\phi_2^{BS}\rangle\langle\phi_2^{BS}| \quad (1.4.25)$$

(iii) Class of genuinely entangled state: A three-qubit state is a genuine entangled state if it is neither separable nor biseparable. Therefore, in a genuine three-qubit entangled state, all three qubits must be correlated to each other. For a three-qubit system, there are two SLOCC inequivalent genuine entangled states namely, GHZ states and W states.

Any three-qubit pure state can be written as,

$$|\psi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \lambda_i \geq 0, \sum_{i=0}^4 \lambda_i^2 = 1 \quad (1.4.26)$$

where, $\theta \in [0, \pi]$ and $\{|0\rangle, |1\rangle\}$ denotes the computational basis.

A three-qubit state $|\psi\rangle_{ABC}$ given in (1.4.26) can be classified as a three-qubit state of GHZ type. A three-qubit W state may be expressed as

$$|\psi\rangle_W = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle \quad (1.4.27)$$

To distinguish between GHZ class of states and the W class of states, we need to define a quantity, popularly known as tangle.

Tangle: Tangle [28] is an entanglement measure used to characterize the three-qubit entanglement of the state. Consider a three-qubit pure state defined by $|\phi\rangle_{ABC} \in H_A \otimes H_B \otimes H_C$. In computational basis, the state $|\phi\rangle_{ABC}$ can be expressed as

$$|\phi\rangle_{ABC} = a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle + e|100\rangle + f|101\rangle + g|110\rangle + h|111\rangle \quad (1.4.28)$$

The state $|\phi\rangle_{ABC}$ is normalised if $|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 + |f|^2 + |g|^2 + |h|^2 = 1$.

The tangle of a pure state $|\phi\rangle_{ABC}$ may be denoted by $\tau_3(|\phi\rangle_{ABC})$ and defined as

$$\tau_3(|\phi\rangle_{ABC}) = 4|d_1 - 2d_2 + 4d_3| \quad (1.4.29)$$

where,

$$d_1 = a^2 h^2 + b^2 g^2 + c^2 f^2 + e^2 d^2 \quad (1.4.30)$$

$$d_2 = ahde + ahfc + ahgb + defc + degb + fcgb$$

$$d_3 = agfd + hbce \quad (1.4.31)$$

The three-qubit mixed state can be written as a convex combination of pure three-qubit states

$$\rho_{ABC} = \sum_j p_j \pi_j, \quad \pi_j = \frac{|\phi_j\rangle_{ABC} \langle \phi_j|}{\langle \phi_j | \phi_j \rangle_{ABC}} \quad (1.4.32)$$

The tangle of a three-qubit mixed state may be obtained by the convex roof extension method. Therefore, the tangle for a mixed three-qubit state described by the density operator ρ_{ABC} is given by

$$\tau_3(\rho_{ABC}) = \min_{\{p_j, \pi_j\}} \sum_j p_j \tau_3(\pi_j) \quad (1.4.33)$$

Three- π : Three- π [29] is another measure of entanglement for a three-qubit system. For three-qubit pure states, it may be defined as

$$\pi = \frac{\pi_a + \pi_b + \pi_c}{3} \quad (1.4.34)$$

where, π_a , π_b and π_c may be defined as

$$\pi_a = N_{A(BC)}^2 - N_{AB}^2 - N_{AC}^2 \quad (1.4.35)$$

$$\pi_b = N_{B(AC)}^2 - N_{BA}^2 - N_{BC}^2 \quad (1.4.36)$$

$$\pi_c = N_{C(AB)}^2 - N_{CA}^2 - N_{CB}^2 \quad (1.4.37)$$

Here, $N_{ij} = \frac{\|\rho^{T_i}\| - 1}{2}$, $i, j = \{A, B, C\}$, where $\|\cdot\|$ represent the trace norm and N represent the negativity of the bipartite system and $N_{i(jk)} = 2\sqrt{\det([\text{tr}_{jk}(\rho_{ijk})])}$.

1.5 Entanglement Detection

Quantum entanglement plays a significant role in the development of quantum information theory. The first revolutionary idea of using entanglement as a resource in quantum teleportation was proposed by Bennett et.al [30]. Many experimental proposals were put forward to realize quantum teleportation [31–35]. Afterward, researchers applied the concept of entanglement as a resource in many other quantum information processing tasks such as quantum superdense coding [36–39], remote state preparation [40–43], entanglement swapping [44–47] and quantum cryptography [48–51]. Therefore, entanglement can be considered a vital ingredient in quantum information theory without which the above-mentioned quantum information processing task cannot be performed. Thus, it is necessary to create entangled states to get a better result than classical states in performing quantum information processing tasks. Mere creation of entanglement is not sufficient to fulfill our demand because it is not known whether the generated state is entangled or not.

Creation and detection of entanglement are thus one of the important problems in quantum information theory. Hence, efficient physically realizable methods for the detection, classification, and quantification of quantum entanglement are of great importance. A lot of progress has been achieved in the creation of entangled states experimentally [52–57] but due to the presence of noise in the environment the generated state need not be entangled. Thus, it is important to check whether the state generated in an experiment is entangled or not. The ultimate goal of the detection of entanglement in a given state is to characterize entanglement in a quantitative way and identify the states in which entanglement is maximum. The first step towards this problem is the detection of entanglement in a given state. In the literature, there are many criteria that may help in the detection of entangled states. We can divide these criteria for the detection of entanglement into two categories C1 and C2, which may be given by

C1. Theoretical criteria that may not be realized in an experiment

C2. Criteria that may be realized in an experiment.

1.5.1 Entanglement detection criterion that may not be implementable in an experiment

I. The Positive Partial Transposition (PPT) criteria The first necessary criteria for the detection of entanglement was given by Peres [58]. A bipartite $N \otimes M$ dimensional quantum state described by the density matrix ρ_{AB} , may be expressed as,

$$\rho_{AB} = \sum_{i,j} \sum_{k,l} \rho_{ij,kl} |i\rangle\langle j| \otimes |k\rangle\langle l| \quad (1.5.1)$$

The partial transposition of ρ_{AB} with respect to subsystem A may be expressed as

$$\rho_{AB}^{T_A} = \sum_{i,j} \sum_{k,l} \rho_{ij,kl} |i\rangle\langle l| \otimes |k\rangle\langle j| \quad (1.5.2)$$

Similarly, one can define partial transposition of the density matrix ρ_{AB} with respect to subsystem B.

PPT Criteria [58]: If a bipartite quantum state described by density operator ρ_{AB} is separable, then $\rho_{AB}^{T_A}$ (or $\rho_{AB}^{T_B}$) is a positive semidefinite operator.

Later, Horodecki showed that, if ρ_{AB} defines a separable quantum state in $2 \otimes 2$ or $2 \otimes 3$ dimension, then $\rho_{AB}^{T_A} \geq 0$ and vice-versa. Thus the condition is necessary and sufficient for $2 \otimes 2$ or $2 \otimes 3$ dimensional quantum system but only necessary condition for $d_1 \otimes d_2$ (except $2 \otimes 2$ or $2 \otimes 3$ or $3 \otimes 2$) dimensional system [59, 60].

PPT criterion is an important criterion in the sense that this criterion provides (i) necessary and sufficient conditions for $2 \otimes 2$ and $2 \otimes 3$ dimensional system and (ii) it is based on the spectrum of the partially transposed matrix. This criterion tells us that if the spectrum of the partially transposed matrix has at least one negative eigenvalue, then the given state is entangled.

For instance, consider the two-qubit Werner state

$$\rho_W = \begin{pmatrix} \frac{1-F}{4} & 0 & 0 & 0 \\ 0 & \frac{1+F}{4} & -\frac{F}{2} & 0 \\ 0 & -\frac{F}{2} & \frac{1+F}{4} & 0 \\ 0 & 0 & 0 & \frac{1-F}{4} \end{pmatrix}, \quad 0 \leq F \leq 1 \quad (1.5.3)$$

Then, the partial transpose $\rho_W^{T_B}$ of the density matrix ρ_W may be calculated as

$$\rho_W^{T_B} = \begin{pmatrix} \frac{1-F}{4} & 0 & 0 & -\frac{F}{2} \\ 0 & \frac{1+F}{4} & 0 & 0 \\ 0 & 0 & \frac{1+F}{4} & 0 \\ -\frac{F}{2} & 0 & 0 & \frac{1-F}{4} \end{pmatrix}, \quad 0 \leq F \leq 1 \quad (1.5.4)$$

The spectrum of $\rho_W^{T_B}$ is given by $\{\frac{1-3F}{4}, \frac{1+F}{4}, \frac{1+F}{4}, \frac{1+F}{4}\}$. Thus, the minimum eigenvalue $\frac{1-3F}{4}$ of $\rho_W^{T_B}$ is negative for $\frac{1}{3} < F \leq 1$. Hence using PPT criteria, the state ρ_W is entangled for $\frac{1}{3} < F \leq 1$.

The PPT criteria define only a necessary condition for a higher dimensional quantum system. to illustrate this, consider a two-qutrit state $\rho_{a,b,c}$ defined by

$$\rho_{a,b,c} = \begin{pmatrix} a & 0 & 0 & 0 & b & 0 & 0 & 0 & b \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & a & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \quad (1.5.5)$$

Here, $a = \frac{1+\sqrt{5}}{3+9\sqrt{5}}$, $b = \frac{-2}{3+9\sqrt{5}}$ and $c = \frac{-1+\sqrt{5}}{3+9\sqrt{5}}$. The spectrum of $\rho_{a,b,c}^{T_B}$ is given by $\{\frac{3+\sqrt{5}}{3(1+3\sqrt{5})}, \frac{2\sqrt{5}}{3(1+3\sqrt{5})}, \frac{2\sqrt{5}}{3(1+3\sqrt{5})}, \frac{1+\sqrt{5}}{3(1+3\sqrt{5})}, \frac{1+\sqrt{5}}{3(1+3\sqrt{5})}, \frac{-1+\sqrt{5}}{3(1+3\sqrt{5})}, \frac{-1+\sqrt{5}}{3(1+3\sqrt{5})}, 0, 0\}$. It is clear that all eigenvalues of $\rho_{a,b,c}^{T_B}$ are positive. But it is known that the state $\rho_{a,b,c}$ is an entangled state [61].

This kind of entangled state which is positive under partial transposition operation is known as positive partial transpose entangled states (PPTES) or bound entangled states. The PPTES exist in the higher dimensional system and they are not detected by PPT criteria. For the detection of PPTES, one may use another criterion such as range criteria and realignment criteria.

II. Realignment Criteria [65, 66] Let us consider any bipartite state described by the density operator ρ ,

$$\rho = \sum_{ijkl} a_{ij,kl} |ij\rangle\langle kl| \in H_A^{d_1} \otimes H_B^{d_2} \quad (1.5.6)$$

where $H_A^{d_1}$ and $H_B^{d_2}$ denote the Hilbert spaces of dimension d_1 and d_2 of the subsystems A and B respectively. The realigned matrix ρ^R of the density matrix ρ may be expressed as

$$\rho^R = \sum_{ijkl} a_{ik,jl} |ik\rangle\langle jl| \quad (1.5.7)$$

The realignment operation can be used to detect entanglement in higher dimensional systems. Especially, it is useful to detect PPTES, which are not detected by PPT criteria. The criteria based on the realignment operation is called realignment criteria.

Realignment Criteria: If the state described by a density operator ρ is separable then $\|\rho^R\| \leq 1$, where $\|\cdot\|$ denote the trace norm defined as $\|H\| = \text{Tr}(\sqrt{HH^\dagger})$.

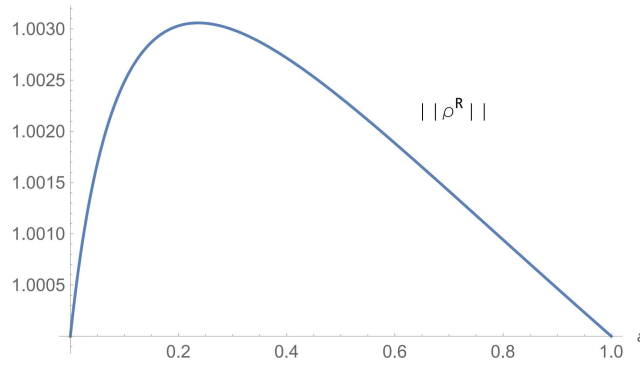
To illustrate the realignment criteria let us consider a two-qutrit state defined in [67].

$$\rho_a = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}, 0 \leq a \leq 1 \quad (1.5.8)$$

Then, the realigned matrix of the density matrix ρ_a may be calculated as

$$\rho_a^R = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}, 0 \leq a \leq 1 \quad (1.5.9)$$

It can be seen from the figure given below that $\|\rho_a^R\| > 1$ for $0 < a < 1$. Thus the state described by the density matrix ρ_a is entangled for $0 < a < 1$.



III. Reduction Criteria [64] Similar to PPT criteria, another entanglement detection criteria can be formulated using a positive but not completely positive map. Like the partial transposition map, there exists another positive but not completely positive map which is known as the reduction map. Reduction map on a single qubit system X may be defined as

$$\Lambda^R(X) = \text{Tr}(X) \cdot I - X \quad (1.5.10)$$

Reduction criteria: If a bipartite state described by a density operator ρ_{AB} is separable then $(I_A \otimes \Lambda^R)(\rho_{AB}) = \rho_A \otimes I - \rho_{AB} \geq 0$, where Λ denote any positive map.

A reduction map is a so-called decomposable map and may be expressed as

$$\Lambda^R = P_1 + P_2 \circ T \quad (1.5.11)$$

where $\{P_1, P_2\}$ are the completely positive maps and T denotes the transposition operator. The reduction criteria is weaker than the PPT criteria because $I_A \otimes \Lambda^R$ can never detect entanglement unless $I_A \otimes T$ detects it. It provides a necessary and sufficient condition for separability in the case of a two-qubit quantum system [22].

For illustration, consider a two-qutrit isotropic state defined as

$$\rho_\alpha = \alpha |\phi_+\rangle \langle \phi_+| + \frac{1-\alpha}{9} I, \quad \alpha \in \mathbb{R}, \quad 0 \leq \alpha \leq 1 \quad (1.5.12)$$

where $|\phi_+\rangle$ is the maximally entangled state and it is given by

$$|\phi_+\rangle = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle) \quad (1.5.13)$$

After applying reduction map on the density matrix ρ_α , the matrix $(\rho_\alpha)_A \otimes I - \rho_\alpha$ becomes

$$\begin{pmatrix} -\frac{2}{9}(-1+\alpha) & 0 & 0 & 0 & -\frac{\alpha}{3} & 0 & 0 & 0 & -\frac{\alpha}{3} \\ 0 & \frac{2+\alpha}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2+\alpha}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2+\alpha}{9} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\alpha}{3} & 0 & 0 & 0 & -\frac{2}{9}(-1+\alpha) & 0 & 0 & 0 & -\frac{\alpha}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{2+\alpha}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2+\alpha}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2+\alpha}{9} & 0 \\ -\frac{\alpha}{3} & 0 & 0 & 0 & -\frac{\alpha}{3} & 0 & 0 & 0 & -\frac{2}{9}(-1+\alpha) \end{pmatrix} \quad (1.5.14)$$

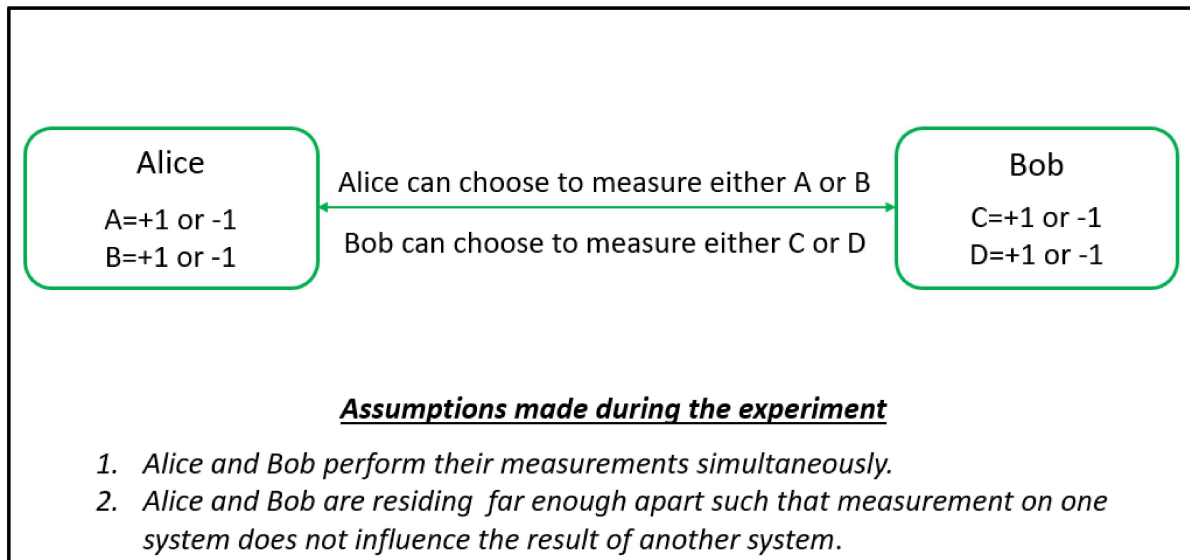
The minimum eigenvalue of the resultant matrix $(\rho_\alpha)_A \otimes I - \rho_\alpha$ is $-\frac{2}{9}(-1+4\alpha)$ and it is negative for $0.25 < \alpha \leq 1$. Thus, using reduction criteria, we can conclude that the state ρ_α is entangled for $0.25 < \alpha \leq 1$.

IV. Range Criteria [62,63] Range criteria can be considered one of the earliest criteria for the detection of entangled states. Sometimes, it may detect those higher dimensional entangled states which are not detected by either PPT criteria or realignment criteria. It asserts that if a state described by a density operator ρ is separable, then there exist a set of product vectors $|a_i b_i\rangle$ such that the set $\{|a_i b_i\rangle\}$ spans the range of density matrix ρ and the set $\{|a_i^* b_i\rangle\}$ spans the range of ρ^{TA} . This criterion detects many entangled states which are not detected by PPT criteria. But the drawback of this criteria is that it cannot be used when the state is affected by noise because then the density matrix ρ and its partially transposed matrix ρ^A will usually have full rank and thus the condition for range criteria is automatically satisfied.

1.5.2 Experimentally implementable entanglement detection schemes

All the criteria defined above for the detection of entangled states depend on a positive but not completely positive map. It is known that positive but not completely positive maps do not correspond to a physically realizable operator whereas a completely positive map does. Thus, the above-defined entanglement detection criteria may not be realized in an experiment. We will now discuss some of the important experimentally realizable criteria for the detection of entangled states. The first tool for the detection of entanglement experimentally was given by John Bell in 1964 [68]

Figure 1.2: Experimental setup for the Bell inequality



in the form of Bell's inequality. The objective of this study is to quantitatively express the Einstein-Podolsky-Rosen paradox.

I. Bell Inequality [2,68] The basic idea behind Bell's inequality is that if measurement is performed by both the parties say, Alice and Bob on their individual subsystems of a bipartite composite system with the assumption that measurement result exists locally before the measurement then it is feasible to obtain bounds on certain quantities composed from the correlation terms of two subsystems of a composite system.

We are now in a position to explain the following experiment described in Figure 1.2. The assumptions made during the experiment are: the assumption of realism and the assumption of locality. Let us assume that Charlie prepares two particles. After finishing the preparation, he sends one particle each to Alice and Bob. As soon as Alice receives her particle, she measures it. Imagine that she has access to two separate measuring apparatuses, and she may select to perform one of the two measurements. Alice chooses the measurement randomly, where the physical properties of the measurements may be described by P_A and P_B respectively. For simplicity, suppose that each of the measurements can have one of the two possible outcomes, +1 or -1. Assume that the attribute P_A of Alice's particle has value A, where A is thought to be an objective feature of Alice's particle that is only disclosed by the measurement, and B stands for the value obtained by measuring the property P_B . Similarly, Bob is able to measure one of the two physical properties, P_C or P_D , indicating that the property has an objectively existing value C or D, where each takes a value of either

+1 or -1. Bob doesn't know in advance which property he will measure, Once he receives the particle, he chooses the measurement randomly. It is assumed that Alice and Bob perform their measurement simultaneously so, Alice's measurement doesn't affect Bob's measurement and vice-versa. We now take the random variables A, B, C , and D in such a way that the expression constructed from them is nonfactorizable. Keeping in mind this fact, we may consider the expression $AC - BC + BD + AD$. This expression may be re-expressed as

$$AC - BC + BD + AD = (A - B)C + (A + B)D \quad (1.5.15)$$

since, $A, B = \pm 1$ thus either $(A - B)C = 0$ or $(A + B)D = 0$. In either case, we have

$$AC - BC + BD + AD = \pm 2 \quad (1.5.16)$$

Consider

$$E(AC - BC + BD + AD) = \sum_{abcd} p(a, b, c, d)(ac - bc + bd + ad) \quad (1.5.17)$$

where $p(a, b, c, d)$ denotes the probability of the system before the measurement, and $E(.)$ denotes the mean of the quantity $(.)$. Thus,

$$- \sum_{abcd} p(a, b, c, d) \times 2 \leq E(AC - BC + BD + AD) \leq \sum_{abcd} p(a, b, c, d) \times 2 \quad (1.5.18)$$

Since $\sum_{abcd} p(a, b, c, d) = 1$, so the inequality may be re-written as

$$-2 \leq E(AC - BC + BD + AD) \leq 2 \quad (1.5.19)$$

Further, we have

$$\begin{aligned} E(AC - BC + BD + AD) &= \sum_{abcd} p(a, b, c, d)ac - \sum_{abcd} p(a, b, c, d)bc \\ &+ \sum_{abcd} p(a, b, c, d)bd + \sum_{abcd} p(a, b, c, d)ad \\ &= E(AC) - E(BC) + E(BD) + E(AD) \end{aligned} \quad (1.5.20)$$

Comparing equations (1.5.19) and (1.5.20), the following inequality is obtained,

$$-2 \leq E(AC) - E(BC) + E(BD) + E(AD) \leq 2 \quad (1.5.21)$$

This inequality is known as Bell's inequality. If Bell's inequality given in (1.5.21) is violated by any quantum state, then the state will violate the assumption of local realism. Thus, the state may be considered non-local and thus entangled. All separable states satisfy Bell's inequality but there may also exist entangled states which satisfy Bell's inequality. For higher dimensional system Bell's inequality have been developed in two forms: (i) Clauser-Horne type inequality for two qutrit system and (ii) CHSH type inequality for two arbitrary d-dimensional systems which is now known as Collins-Gisin-Linden-MasserPopescu (CGLMP) inequalities. The tightness of the CGLMP inequality was demonstrated in [69]. In a two-qubit system, maximally entangled states violate Bell's inequality maximally whereas, in a higher dimensional system, there may exist non-maximally entangled states that violate Bell's inequality maximally. [69]. Acin et.al studied nonlocality in the case of a two-qudit system up to $d = 8$ and found that there exists a non-maximally entangled state in which the violation of CGLMP inequalities is more than the maximally entangled state [70].

Furthermore, we should note that the expression involved in the derivation of Bell's inequality is not unique and can be modified in such a way that the resulting expression would not be factorizable. Otherwise, the inequality derived from the expression will not take part in the detection of an entangled state.

II. Witness Operator or Entanglement Witnesses [71, 72] Witness operators are the fundamental tool for the detection of entanglement experimentally. It can be used to detect entangled states in bipartite as well as multipartite quantum systems. Entanglement witnesses are Hermitian operators and have at least one negative eigenvalue.

An observable \mathbf{W} is an entanglement witness if it satisfies the following two conditions:

$$\begin{aligned} H_1 : Tr(\mathbf{W}\rho_s) &\geq 0, \text{ for all separable states } \rho_s. \\ H_2 : Tr(\mathbf{W}\rho_e) &< 0, \text{ for at least one entangled state } \rho_e. \end{aligned} \quad (1.5.22)$$

Thus, if $Tr(\mathbf{W}\rho) < 0$, then we say that the state ρ is detected by an entanglement witness \mathbf{W} . The first example of witness operator can be considered as $W_B = 2I - \mathbf{B}$ where \mathbf{B} denote the Bell operator which may be defined as

$$\mathbf{B} = A \otimes C + B \otimes C + B \otimes D - A \otimes D \quad (1.5.23)$$

Indeed it can be shown that W_B represents a witness operator.

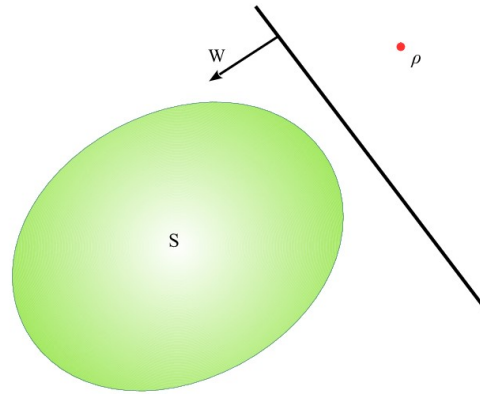


Figure 1.3: Hahn Banach Theorem

Geometrically, entanglement witness \mathbf{W} represents a hyperplane that separates at least one entangled state from the set of separable states. This can be seen as the consequence of the Hahn Banach theorem of functional analysis which states that for a convex and closed set S in a finite-dimensional Banach space, there exist a hyperplane that separates ρ from S , when ρ is a point in the Banach space that does not belong to S .

Let us define three sets S_0, S_1 , and S_2 which are defined by

$$\begin{aligned}
 S_0 &= \{\rho : \text{Tr}(W\rho) = 0\} \\
 S_1 &= \{\rho : \text{Tr}(W\rho) > 0\} \\
 S_2 &= \{\rho : \text{Tr}(W\rho) < 0\}
 \end{aligned} \tag{1.5.24}$$

The set S_0 contains all those states ρ which are lying on the hyperplane \mathbf{W} . In this case, the hyperplane \mathbf{W} touches the set S_0 and thus the hyperplane \mathbf{W} can be considered as a tangent plane to the set S_0 . The set S_1 contains all those states ρ which are either separable states or the states which are not detected by the witness \mathbf{W} . The set S_2 denotes the set of entangled states which are detected by \mathbf{W} .

If W_1 and W_2 are two entanglement witness then W_1 is said to be finer than W_2 if W_1 detects all entangled states which are detected by W_2 . An entanglement witness W_{opt} is known as the optimal entanglement witness (OEW) if there exists no entanglement witness which is finer than W_{opt} i.e. if W_{opt} forms a tangent plane to the set of separable states S . Entanglement witnesses may be further classified into two classes as decomposable entanglement witnesses (DEW) and non-decomposable entanglement witnesses (NDEW). Decomposable entanglement witnesses may be expressed

as

$$W_D = aP + (1 - a)Q^{T_B}, \quad a \in [0, 1] \quad (1.5.25)$$

where P and Q are positive semidefinite operators. The operators which cannot be expressed in the form (1.5.25) are non-decomposable entanglement witnesses. DEW cannot detect PPTES. Optimal DEW, W may be expressed as $W = Q^{T_B}$ for some positive semidefinite operator Q which contains no product vectors in its range. Also, for an optimal DEW, W , the partial transposition W^{T_B} is not a witness operator. An entanglement witness is non-decomposable if and only if it detects PPTES. Every entanglement witness, however, corresponds to a positive but not completely positive map via the Choi-Jamiolkowski isomorphism [73, 74].

III. Structural Physical Approximation [75] Another method for the experimental detection of entanglement is the method of structural physical approximation (SPA). The concept of SPA has been introduced in [75]. In the SPA method, a positive but not completely positive map is approximated by a completely positive map. The method is useful in the sense that it transforms a non-physical operator to a physical operator and hence makes it experimentally realizable [76]. A lot of progress has already been done in both theoretical as well as experimental aspects of SPA [77–84].

The main idea behind SPA is how much proportion of white noise needs to be added to a non-physical operator Λ , such that the approximated operator $\tilde{\Lambda}$ will be completely positive. Mixing the map Λ with the white noise transforms any density matrix into a maximally mixed state, thus resulting map can have no negative values. For any non-physical operator Λ , the approximated map $\tilde{\Lambda}$ can be written as a convex combination of the operator Λ and the depolarizing map D that may be expressed as

$$\tilde{\Lambda} = (1 - p)\Lambda + pD, \quad 0 \leq p \leq 1 \quad (1.5.26)$$

If a d -dimensional system is described by a density operator ρ and Λ be any positive operator then the approximation of Λ denoted by $\tilde{\Lambda}$, which may be evolved as

$$\tilde{\Lambda}(\rho) = (1 - p)\Lambda(\rho) + p\frac{I_d}{d}, \quad 0 \leq p \leq 1 \quad (1.5.27)$$

where we have used the fact that $D[\rho] = \frac{I_d}{d}$ and I_d denote the d -dimensional identity matrix.

In particular, let us consider the positive but not completely positive transposition map T , which is defined in d -dimensional system. The SPA of T may be expressed as

$$\tilde{T} = \frac{1}{d+1}T + \frac{d}{d+1}D \quad (1.5.28)$$

\tilde{T} is completely positive for $p \geq \frac{d}{d+1}$ [84]. In particular, the SPA of the transposition map of a qutrit system has been realized in experiments through prepare and measure strategy [84].

In 2008 [77], a conjecture has been posed that SPA leads to separable states. Various examples were given in support of this conjecture [77, 78]. For instance, let us take a transposition map acting on one of the subsystems of a composite bipartite system that may lead to entanglement breaking. Further, assume a $d \otimes d$ dimensional maximally entangled state of the form

$$P_+ = |\phi_+\rangle\langle\phi_+|, \quad |\phi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle \quad (1.5.29)$$

In particular, taking $\Lambda = T$ in (1.5.27), it reduces to

$$\tilde{T}(\rho) = p \frac{I}{d} + (1-p)T(\rho), \quad 0 < p < 1 \quad (1.5.30)$$

where p takes the minimum value for which \tilde{T} becomes a completely positive map. The corresponding witness operator \tilde{E}_T acting on the maximally entangled state P_+ , may be expressed as

$$\tilde{E}_T(P_+) = (id \otimes \tilde{T})(P_+) = \frac{p}{d^2} I \otimes I + \frac{1-p}{d} \left[\sum_{i=1}^d |ii\rangle\langle ii| + \sum_{i \neq j, i, j=1}^d |ij\rangle\langle ji| \right] = \rho_W \quad (1.5.31)$$

Then, \tilde{E}_T is positive and thus \tilde{T} is completely positive when

$$p \geq \frac{d}{d+1} = p_{min} \quad (1.5.32)$$

Using the fact that PPT criteria is necessary and sufficient for the detection of $d \otimes d$ -dimensional Werner state described by the density operator ρ_W [77] and from (1.5.31), we have

$$(\tilde{E}_T)^\Gamma = \frac{p}{d^2} I \otimes I + (1-p)P_+ \quad (1.5.33)$$

where Γ denotes the partial transposition. The operator $(\widetilde{E_T})^\Gamma$ is non-negative for $0 < p < 1$ and thus, $\widetilde{E_T}$ represent a separable state. Hence, the SPA of the transposition map is entanglement breaking. Later, this SPA conjecture was disproved by Kye and Ha [79–82].

Now we are in a position to discuss the separability criteria of $d \otimes d$ dimensional system using the structural physical approximation of partial transposition (SPA-PT). If we have some prior information about the arbitrary dimensional bipartite system described by the density matrix ρ , then the eigenvalues of SPA-PT of ρ can be used to detect whether the state ρ is entangled or not. Using this idea, criteria have been proposed for the detection of negative partial transpose entangled states (NPTES) in the arbitrary dimensional bipartite system. Horodecki et.al. [75] have studied SPA-PT map extensively to develop the entanglement detection criteria for $d \otimes d$ dimensional system.

To understand SPA-PT in $d \otimes d$ dimensional system, let us start our discussion with the partial transposition operation which is a purely mathematical operation and may not be directly implementable in an experiment. The partial transposition criteria which is based on the partial transposition operation may be stated as: If a state described by the density operator $\rho^{d \otimes d}$ is separable then

$$(id \otimes T)\rho \geq 0 \quad (1.5.34)$$

where T denotes the transposition operation. Since the map, $id \otimes T$ is not a completely positive map and thus cannot be implemented in a laboratory. This means that the partial transposition operation is a non-physical operation and thus, to make it a physically realizable operator, we approximate it by the method of structural physical approximation. The structural physical approximation of the map $id \otimes T$, where T denotes the transposition operation for $d \otimes d$ dimensional quantum system may be expressed as [85]

$$(\widetilde{id \otimes T}) = (1 - q)(id \otimes T) + qD \otimes D \quad (1.5.35)$$

where $D(.) = \frac{I}{d}$ denote the depolarizing channel. When the approximated map $(\widetilde{id \otimes T})$ acts on the bipartite $d \otimes d$ dimensional density matrix $\rho^{d \otimes d}$, (1.5.35) reduces to

$$(\widetilde{id \otimes T})(\rho^{d \otimes d}) = (1 - q_{min})(id \otimes T)(\rho^{d \otimes d}) + \frac{q_{min}}{d^2}(I \otimes I) \quad (1.5.36)$$

where q_{min} denotes the minimum value for which $(\widetilde{id \otimes T})(\rho^{d \otimes d})$ becomes a positive semidefinite matrix. $(\widetilde{id \otimes T})(\rho^{d \otimes d})$ becomes a positive semidefinite matrix when

$$q \geq \frac{d^4 \lambda}{d^4 \lambda + 1} \equiv q_{min} \quad (1.5.37)$$

where $-\lambda (\lambda > 0)$ denote the most negative eigenvalue obtained when the induced map $[(id \otimes id) \otimes (id \otimes T)]$ acts on the maximally entangled state $\frac{1}{d} \sum_{i=1}^{d^2} |i\rangle|i\rangle$. Using (1.5.37) and $\lambda = \frac{1}{d}$, the approximated map (1.5.36) may be re-expressed as

$$[\widetilde{id \otimes T}](\rho^{d \otimes d}) = \frac{1}{d^3 + 1} (id \otimes T)(\rho^{d \otimes d}) + \frac{d}{d^3 + 1} (id \otimes id) \quad (1.5.38)$$

The physically realizable PPT criteria for $d \otimes d$ dimensional system described by the density operator ρ obtained by applying the SPA-PT, which may be read as: If ρ is separable then [76]

$$\lambda_{min}[\widetilde{id \otimes T}](\rho^{d \otimes d}) \geq \frac{d}{d^3 + 1} \quad (1.5.39)$$

where $\lambda_{min}[\widetilde{id \otimes T}](\rho)$ denote the minimum eigenvalue of SPA-PT of ρ .

If any $d \otimes d$ dimensional quantum state ρ violate inequality (1.5.39), then the state ρ is an entangled state.

Let us now generalize the separability criteria for $d_1 \otimes d_2$ dimensional system using the SPA-PT map. To do this, consider a $d_1 \otimes d_2$ dimensional system described by a density operator $\rho^{d_1 \otimes d_2}$. Here we assume that $d_1 = \min\{d_1, d_2\}$. The SPA-PT for $d_1 \otimes d_2$ dimensional system may be defined as

$$(\widetilde{id \otimes T}) = (1 - p)(id \otimes T) + pD_1 \otimes D_2 \quad (1.5.40)$$

where

$$p = \frac{\lambda d_1^3 d_2}{1 + \lambda d_1^3 d_2}, \quad \lambda = -\min_{Q>0} Tr[Q(id \otimes T)P_+^{d_1}] \quad (1.5.41)$$

$$P_+^{d_1} = |\phi_+\rangle\langle\phi_+|, \quad |\phi_+\rangle = \frac{1}{\sqrt{d_1}} \sum_{i=1}^{d_1} |ii\rangle \quad (1.5.42)$$

$D_i(\cdot) = \frac{I_i}{d_i}$ for $i=1,2$ denote the depolarizing channel acting on the individual subsystem and the parameter p denote the minimum value for which $(\widetilde{id \otimes T})$ is completely positive. Thus, the separability criteria for $d_1 \otimes d_2$ dimensional system in terms of the

minimum eigenvalue of SPA-PT map may be stated as: If $\rho^{d_1 \otimes d_2}$ is a separable state in $d_1 \otimes d_2$ system, then

$$\lambda_{\min}[\widetilde{(id \otimes T)}(\rho^{d_1 \otimes d_2})] \geq \frac{\lambda d_1 d_2}{1 + \lambda d_1^3 d_2}, \quad \lambda = -\min_{Q>0} Tr[Q(id \otimes T)P_+^{d_1}] \quad (1.5.43)$$

If the inequality (1.5.43) is violated then the state $\rho^{d_1 \otimes d_2}$ represent an entangled state. From (1.5.43), it is clear that to determine whether any arbitrary dimensional quantum state is entangled or not, we need to find the minimum eigenvalue of the SPA-PT of the state under consideration. But it would be a tedious task to determine the matrix elements of the SPA-PT of the higher dimensional quantum system described by the density operator $\rho^{d_1 \otimes d_2}$. Thus, we have considered a particular bipartite composite quantum system such as a qutrit-qubit system described by the density matrix $\rho^{3 \otimes 2}$ and then calculated the matrix elements of the SPA-PT of $\rho^{3 \otimes 2}$ [86]. Therefore, by constructing the general form of the matrix of SPA-PT of $\rho^{3 \otimes 2}$, it would be possible to calculate $\lambda_{\min}[\widetilde{(id \otimes T)}(\rho^{3 \otimes 2})]$ at least for the qutrit-qubit system.

SPA-PT method not only helps in the detection of entanglement but also has many applications such as to estimate (i) optimal singlet fraction [87], (ii) entanglement negativity [88] of the two-qubit density matrix in an experiment, (iii) concurrence of a two-qubit system using 4 moments [89].

1.6 Classification of entanglement

Another important problem in quantum information theory is the problem of classification of multi-qubit quantum state [90], especially when the number of qubits is either equal to three or more than three. In the case of a two-qubit system, the classification of entanglement is not that important as compared to the system where three or more than three qubits are involved because a two-qubit system can be classified as either separable or entangled state [91]. Therefore, when we increase the number of qubits, the complicacy in the structure of the multi-qubit system will also increase, and thus, we may have different classes of entangled states. These classes of entangled states need to be characterized on the basis of the entanglement present in the different subsystems and accordingly, we can use that class of entangled states in appropriate quantum information processing tasks. In particular, three-qubit states may be classified into six inequivalent classes under stochastic local operation and classical

communication (SLOCC) as one fully separable state, three biseparable states, and two genuinely entangled states. The fully separable and biseparable states may be expressed in the form as [92, 93]

$$\rho_{sep}^{ABC} = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C, \sum_i p_i = 1 \quad (1.6.1)$$

$$\rho_{bisep} = p_1 \rho_{bisep}^{A-BC} + p_2 \rho_{bisep}^{B-AC} + p_3 \rho_{bisep}^{C-AB}, \sum_i p_i = 1 \quad (1.6.2)$$

where,

$$\begin{aligned} \rho_{bisep}^{A-BC} &= \sum_i |a_i\rangle_A \langle a_i| \otimes |\phi_i\rangle_{BC} \langle \phi_i| \\ \rho_{bisep}^{B-AC} &= \sum_i |b_i\rangle_B \langle b_i| \otimes |\phi_i\rangle_{AC} \langle \phi_i| \\ \rho_{bisep}^{C-AB} &= \sum_i |c_i\rangle_C \langle c_i| \otimes |\phi_i\rangle_{AB} \langle \phi_i| \end{aligned}$$

Here, $|a_i\rangle$, $|b_i\rangle$ and $|c_i\rangle$ are (unnormalized) states of systems A, B and C, respectively and $|\phi_i\rangle$ are states of two systems in equations (4.1.2). If any three-qubit state is neither fully separable nor biseparable state, then the state is said to be a genuine entangled state. There are two classes of genuine three-qubit entangled states namely, W class of states and GHZ class of states.

The five-parameter canonical form of three-qubit pure state $|\psi\rangle_{ABC}$ shared between three distant partners A, B, and C is given by [94]

$$|\psi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \quad (1.6.3)$$

with $0 \leq \lambda_i \leq 1 (i = 0, 1, 2, 3, 4)$ and $0 \leq \theta \leq \pi$. The normalization condition of the state (1.6.3) is given by

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1. \quad (1.6.4)$$

Equation of the type (1.6.3) represents a GHZ state whereas W vectors can be written as

$$|\psi\rangle_W = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle \quad (1.6.5)$$

The tangle for three-qubit state $|\psi\rangle_{ABC}$ is given by

$$\tau_\psi = 4\lambda_0^2\lambda_4^2 \quad (1.6.6)$$

If at least one of the λ_0 or λ_4 is zero then the tangle of the given three-qubit state is zero and if none of the λ_0 or λ_4 is zero, then the tangle of the state under investigation is non-zero. Thus, if the tangle is non-zero for any three-qubit state, then the state belongs to the class of GHZ states and if the tangle is zero for any three-qubit state then the state can be either separable or biseparable or W state in three qubit system. Thus, the tangle can be used to classify GHZ class of states from the set of separable states, biseparable states and W class of states [95]. Now, it is important to further classify separable states, biseparable states, and W states. In this context, all possible six SLOCC inequivalent three-qubit pure states can be distinguished with the help of observables [96]. An experiment has been carried out using an NMR quantum information processor, to classify these six SLOCC inequivalent classes [97]. Classification of genuine three-qubit mixed states has been studied by constructing GHZ and W witness operator in [98]. Sabin et al. [99] have studied the classification of a three-qubit system based on reduced two-qubit entanglement. Classification of four qubit pure states has been studied in [100–102]. For multi-qubit pure system, classification has been studied in [103–106]. The complete classification of multipartite systems is still not known due to the complicated nature of multipartite systems.

Although there are various methods for the detection and classification of entanglement which exists in the literature, most of them suffer from a serious drawback that they cannot be implemented in an experimental setup. To overcome this defect, we have used the method of witness operator and SPA-PT method to classify a three-qubit quantum system.

Chapter 2

Detection of entanglement via structural physical approximation

Detection of entanglement is an important problem in quantum information theory and thus various entanglement detection schemes have been proposed to detect entanglement in a mixed bipartite quantum system. In this chapter¹, we have started with the study of SPA-PT map acting on the qutrit-qubit system described by the density operator $\rho^{2\otimes 3}$ to detect entanglement in the given system. Here, it is shown how to apply SPA-PT on $\rho^{2\otimes 3}$ and thus, obtained explicitly the matrix elements of $\widetilde{\rho^{2\otimes 3}}$, where $\widetilde{\rho^{2\otimes 3}}$ denote the SPA-PT of $\rho^{2\otimes 3}$. Using the matrix elements of $\widetilde{\rho^{2\otimes 3}}$, one can find the minimum eigenvalue of the approximated matrix, and using the minimum eigenvalue one can check whether the state under investigation is entangled or not. But the entanglement detection procedure based on the minimum eigenvalue of SPA-PT of the given density matrix will become very tedious when we increase the dimension of the bipartite quantum system. Thus, to check entanglement in an arbitrary dimensional bipartite system, we have proposed three different criteria based on SPA-PT. Also, we have provided a lower and upper bound of concurrence of arbitrary dimensional mixed bipartite NPTES.

¹This chapter is based on a published research paper "Detection of a mixed bipartite entangled state in arbitrary dimension via a structural physical approximation of partial transposition, *Physical Review A* **100**, 052323 (2019)".

2.1 Introduction

Entanglement [90, 107] is a crucial element that enhanced the power of quantum computation [108]. This non-local feature of quantum mechanics plays an important role in the field of quantum information theory. Thus, it is important to check whether the quantum state under investigation is entangled or not. If in an experiment, one intends to generate an entangled state, then it is not necessary that the generated state at the output is entangled. This happens due to the presence of noise and thus the problem of detection of entanglement will become an important issue. Several criteria had already been proposed for solving the entanglement detection problem but yet partial result has been achieved [109–118].

The first entanglement detection criteria i.e. the PPT criteria provides a necessary and sufficient condition for separability in $2 \otimes 2$ and $2 \otimes 3$ dimensional system. But for $d_1 \otimes d_2$ (except $2 \otimes 2$ and $2 \otimes 3$) dimensional system, it provides only a necessary condition due to the existence of PPTES. The other weakness of PPT criteria is that it depends on the spectrum of partial transposition of the density matrix. Partial transposition is a positive but not completely positive map and hence, it may not be implemented in a laboratory. This defect can be rectified by using SPA-PT which transforms a positive but not completely positive operator into a completely positive operator and thus, makes the operator experimentally realizable.

H. T. Lim et.al. have studied the SPA-PT of the single qutrit system and shown that it can be implemented in the laboratory with linear optical elements [84]. Also, the SPA-PT method has been applied to a two-qubit system to study the entanglement detection problem [87]. Thus, one may ask that the SPA-PT method works well in a two-qubit system and can be implemented in an experiment but can we use this method to detect entanglement in the higher dimensional system also? To answer this question one first need to determine the density matrix of SPA-PT of any arbitrary dimensional quantum system. Once the density matrix is obtained, one may calculate the minimum eigenvalue of the SPA-PT of the given density matrix and this eigenvalue can be used in the detection of entanglement. But this procedure is very tedious.

In this chapter, we have considered $d_1 \otimes d_2$ dimensional NPTES and derived three different criteria for the detection of entanglement employing the SPA-PT method. It is known that the average fidelity between two quantum states can be realized in an experiment [119]. Thus, we have expressed our proposed criteria to detect NPTES,

in terms of the average fidelity between two quantum states, and hence the detection criteria can be implemented in an experiment. Among three criteria, two of them are given in terms of the concurrence of the given state so it is essential to find out the concurrence. As far as we know, there does not exist any procedure to calculate the actual value of the concurrence of the given mixed state in arbitrary dimension but in the literature, there exists a lower bound of the concurrence [120]. In the present work also, we have provided a lower and upper bound of the concurrence that can be realized in an experiment.

Let us consider a $d_1 \otimes d_2$ dimensional mixed quantum system and perform the partial transposition operation on the second subsystem. Since the partial transposition map $id_A \otimes \mathbf{T}$, where \mathbf{T} denotes the positive transposition map and id_A represent the identity operator of the individual subsystem A , is not a physical map so we consider the approximated map $\widetilde{id_A \otimes \mathbf{T}}$, which is a completely positive map corresponds to a quantum channel that can be experimentally implementable [85]. The approximated map can be expressed as [76]

$$\widetilde{id_A \otimes \mathbf{T}} = (1 - q^*)(id_A \otimes \mathbf{T}) + \frac{q^*}{d_1 d_2} I_A \otimes I_B \quad (2.1.1)$$

I_B represents the identity operator of the individual subsystem B and q^* denote the minimum value of q for which $\widetilde{id_A \otimes \mathbf{T}}$ become a positive semi-definite operator. Since $\widetilde{id_A \otimes \mathbf{T}}$ is a completely positive map so when it operate on a density matrix ρ^{in} , it gives another density matrix $\tilde{\rho}^{out}$ at the output, i.e. $(\widetilde{id_A \otimes \mathbf{T}})\rho^{in} = \tilde{\rho}^{out}$. The minimum eigenvalue of the density matrix $\tilde{\rho}^{out}$ is important in the sense that there exists a critical value of the minimum eigenvalue $\lambda_{min}(\tilde{\rho}^{out})$ below which the state described by the density matrix ρ^{in} is entangled. For instance, $2 \otimes 2$ dimensional quantum state is entangled when the minimum eigenvalue $\lambda_{min}(\tilde{\rho}^{out})$ is less than $\frac{2}{9}$. So the analysis of minimum eigenvalue of $\tilde{\rho}^{out}$ is needed.

Using the idea of SPA, a positive map Λ can be transformed to a completely positive map $\tilde{\Lambda}$ and the SPA map $\tilde{\Lambda}$ may be expressed as

$$\tilde{\Lambda} = (1 - p)\Lambda + pD, \quad p \in [0, 1] \quad (2.1.2)$$

where p denotes the minimum value for which $\tilde{\Lambda}$ represents a completely positive operator. $D[\rho]$ is the depolarizing operator which may be defined as $D[\rho] = \frac{I_d}{d}$, I_d is the d -dimensional identity matrix. For linear maps of the form $id \otimes \Lambda$, the SPA may be

expressed as

$$\widetilde{id \otimes \Lambda} = (1-p)I \otimes \Lambda + pD \otimes D, \quad p \in [0,1] \quad (2.1.3)$$

where p denote the minimum value for which $\widetilde{id \otimes \Lambda}$ is positive. Lim et al. have demonstrated the experimental realization of SPA-PT for a two-qubit system using single-photon polarization qubits and linear optical devices. The corresponding decomposition of SPA-PT for a two-qubit system ρ_{AB} , may be expressed as [83, 87]

$$(\widetilde{id \otimes T})[\rho_{AB}] = \frac{1}{3}(id \otimes \tilde{T})[\rho_{AB}] + \frac{2}{3}(\tilde{\theta} \otimes D)[\rho_{AB}] \quad (2.1.4)$$

where \tilde{T} denote the SPA to the transposition map which may be expressed as

$$\tilde{T}[\rho_{AB}] = \sum_{k=1}^4 Tr[M_k \rho_{AB}] |v_k\rangle \langle v_k| \quad (2.1.5)$$

where $M_k = \frac{|v_k^*\rangle \langle v_k^*|}{2}$ for $k = 1, 2, 3, 4$. The basis $|v_1\rangle$, $|v_2\rangle$, $|v_3\rangle$ and $|v_4\rangle$ may be expressed as

$$\begin{aligned} |v_1\rangle &= \frac{1}{\sqrt{1+|b_1|^2}}(|0\rangle + b_1|1\rangle) \\ |v_2\rangle &= \frac{1}{\sqrt{1+|b_2|^2}}(|0\rangle - b_2|1\rangle) \\ |v_3\rangle &= \frac{1}{\sqrt{1+|b_2|^2}}(|0\rangle + b_2|1\rangle) \\ |v_4\rangle &= \frac{1}{\sqrt{1+|b_1|^2}}(|0\rangle - b_1|1\rangle) \end{aligned} \quad (2.1.6)$$

where $b_1 = \frac{ie^{\frac{2\Pi}{3}i}}{i+e^{-\frac{2\Pi}{3}i}}$ and $b_2 = \frac{ie^{\frac{2\Pi}{3}i}}{i-e^{-\frac{2\Pi}{3}i}}$. In (2.1.4), $\tilde{\theta}$ denote the SPA to the inversion map where inversion map may be defined as $\theta[\rho_{AB}] = -\rho_{AB}$. The inversion map $\tilde{\theta}$ may be expressed as

$$\tilde{\theta}[\rho_{AB}] = \sum_{k=1}^4 Tr[M_k \rho_{AB}] \sigma_y |v_k\rangle \langle v_k| \sigma_y \quad (2.1.7)$$

where σ_y is defined in (1.1.15)

2.2 Preliminary Results

Result-1: For any two hermitian (n,n) matrices X and Y , we have

$$\lambda_{\min}(X)Tr(Y) \leq Tr(XY) \leq \lambda_{\max}(X)Tr(Y) \quad (2.2.1)$$

Proof: It is known that for any hermitian (n,n) matrices X and Y , the following inequality holds [121]

$$\sum_{i=1}^n \lambda_i(X)\lambda_{n-i+1}(Y) \leq Tr(XY) \leq \sum_{i=1}^n \lambda_i(X)\lambda_i(Y) \quad (2.2.2)$$

where $\lambda_{\min} = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n = \lambda_{\max}$.

In the LHS of inequality (2.2.2), if we replace all eigenvalues of X by its minimum eigenvalue, and in the RHS, if we replace all eigenvalues of X by its maximum, then the inequality (2.2.2) reduces to (2.2.1). Hence proved.

Result-2: If W represents the witness operator that detects the entangled quantum state described by the density operator ρ_{AB} and $C(\rho_{AB})$ denote the concurrence of the state ρ_{AB} then the lower bound of concurrence is given by [122]

$$C(\rho_{AB}) \geq -Tr[W\rho_{AB}], \quad Tr[W] = 1 \quad (2.2.3)$$

Result-3: If ρ_{AB} denote the density operator of a bipartite quantum state in any arbitrary dimension then all eigenvalues of $\rho_{AB}^{T_B}$ lying within the interval $[\frac{-1}{2}, 1]$ [123].

Result-4: If any arbitrary two-qubit density operator ρ_{AB} is given by

$$\rho_{AB} = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{12}^* & e_{22} & e_{23} & e_{24} \\ e_{13}^* & e_{23}^* & e_{33} & e_{34} \\ e_{14}^* & e_{24}^* & e_{34}^* & e_{44} \end{pmatrix}, \quad \sum_{i=1}^4 e_{ii} = 1 \quad (2.2.4)$$

where $(*)$ denotes the complex conjugate, then the SPA-PT of ρ_{AB} is given by [87]

$$\begin{aligned}\widetilde{\rho}_{AB} &= \left[\frac{1}{3}(I \otimes \widetilde{T}) + \frac{2}{3}(\widetilde{\Theta} \otimes \widetilde{D}) \right] \rho_{AB} \\ &= \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12}^* & E_{22} & E_{23} & E_{24} \\ E_{13}^* & E_{23}^* & E_{33} & E_{34} \\ E_{14}^* & E_{24}^* & E_{34}^* & E_{44} \end{pmatrix}\end{aligned}\quad (2.2.5)$$

where

$$\begin{aligned}E_{11} &= \frac{1}{9}(2 + e_{11}), E_{12} = \frac{1}{9}e_{12}^*, E_{13} = \frac{1}{9}e_{13}, \\ E_{14} &= \frac{1}{9}e_{23}, E_{22} = \frac{1}{9}(2 + e_{22}), E_{23} = \frac{1}{9}e_{14}, \\ E_{24} &= \frac{1}{9}e_{24}, E_{33} = \frac{1}{9}(2 + e_{33}), E_{34} = \frac{1}{9}e_{34}^*, \\ E_{44} &= \frac{1}{9}(2 + e_{44})\end{aligned}\quad (2.2.6)$$

2.3 SPA-PT of an arbitrary qutrit-qubit quantum state

Using the idea of SPA-PT on a two-qubit system, in this section, we will obtain the SPA-PT of the general density matrix of a qutrit-qubit quantum state.

To achieve our goal, let us consider an arbitrary qutrit-qubit quantum state described by the density operator in the computational basis as

$$\rho_{AB} = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ t_{12}^* & t_{22} & t_{23} & t_{24} & t_{25} & t_{26} \\ t_{13}^* & t_{23}^* & t_{33} & t_{34} & t_{35} & t_{36} \\ t_{14}^* & t_{24}^* & t_{34}^* & t_{44} & t_{45} & t_{46} \\ t_{15}^* & t_{25}^* & t_{35}^* & t_{45}^* & t_{55} & t_{56} \\ t_{16}^* & t_{26}^* & t_{36}^* & t_{46}^* & t_{56}^* & t_{66} \end{pmatrix}, \sum_{i=1}^6 t_{ii} = 1 \quad (2.3.1)$$

where $(*)$ denotes the complex conjugate.

The decomposition of SPA-PT for a qutrit-qubit quantum state ρ_{AB} is given by

$$\widetilde{\rho}_{AB} = \left[\frac{1}{4}(id \otimes \widetilde{T}) + \frac{3}{4}(\widetilde{\Theta} \otimes \widetilde{D}) \right] \rho_{AB} \quad (2.3.2)$$

The operator $\tilde{T}[\cdot]$ denote the SPA of partial transposition of (\cdot) and it is given by

$$\tilde{T}[\cdot] = \sum_{k=1}^9 \text{tr}[M_k(\cdot)] |v_k\rangle\langle v_k|, \quad M_k = \frac{1}{3} |v_k^*\rangle\langle v_k^*| \quad (2.3.3)$$

Moreover, the remaining two operators $\tilde{\Theta}$ and \tilde{D} denote the SPA of inversion map Θ and depolarization map D respectively and they can be defined as

$$\begin{aligned} \tilde{\Theta}[\cdot] &= \sum_{i=1}^9 (\text{tr}[M_i(\cdot)] \lambda_a |v_k\rangle\langle v_k| \lambda_a) \\ &= \lambda_a \tilde{T}[\cdot] \lambda_a \end{aligned} \quad (2.3.4)$$

$$\tilde{D}[\cdot] = \frac{1}{4} \sum_{i=0,x,y,z} \sigma_i[\cdot] \sigma_i \quad (2.3.5)$$

where $\sigma_0 = I$, $\sigma_i (i = x, y, z)$ denote the Pauli matrices and

$$\begin{aligned} \lambda_a &= a\lambda_a^{(01)} + b\lambda_a^{(12)} + c\lambda_a^{(02)}, \quad a, b, c \in \mathbb{R} \\ \lambda_a^{(01)} &= -i|0\rangle\langle 1| + i|1\rangle\langle 0| \\ \lambda_a^{(12)} &= -i|1\rangle\langle 2| + i|2\rangle\langle 1| \\ \lambda_a^{(02)} &= -i|0\rangle\langle 2| + i|2\rangle\langle 0| \end{aligned} \quad (2.3.6)$$

and the vectors $|v_i\rangle$, $i \in \{1, 2, \dots, 9\}$ may be given by

$$\begin{aligned} |v_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + w|1\rangle), |v_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + w^2|1\rangle), |v_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |v_4\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + w|2\rangle), |v_5\rangle = \frac{1}{\sqrt{2}}(|0\rangle + w^2|2\rangle), |v_6\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle) \\ |v_7\rangle &= \frac{1}{\sqrt{2}}(w|0\rangle + |2\rangle), |v_8\rangle = \frac{1}{\sqrt{2}}(w^2|0\rangle + |2\rangle), |v_9\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle) \end{aligned} \quad (2.3.7)$$

where $\{1, w, w^2\}$ are cube roots of unity.

Let $\tilde{\rho}_{AB}$ denote the SPA-PT of ρ_{AB} . The density operator for $\tilde{\rho}_{AB}$ is given by

$$\tilde{\rho}_{AB} = \begin{pmatrix} \tilde{t}_{11} & \tilde{t}_{12} & \tilde{t}_{13} & \tilde{t}_{14} & \tilde{t}_{15} & \tilde{t}_{16} \\ \tilde{t}_{12}^* & \tilde{t}_{22} & \tilde{t}_{23} & \tilde{t}_{24} & \tilde{t}_{25} & \tilde{t}_{26} \\ \tilde{t}_{13}^* & \tilde{t}_{23}^* & \tilde{t}_{33} & \tilde{t}_{34} & \tilde{t}_{35} & \tilde{t}_{36} \\ \tilde{t}_{14}^* & \tilde{t}_{24}^* & \tilde{t}_{34}^* & \tilde{t}_{44} & \tilde{t}_{45} & \tilde{t}_{46} \\ \tilde{t}_{15}^* & \tilde{t}_{25}^* & \tilde{t}_{35}^* & \tilde{t}_{45}^* & \tilde{t}_{55} & \tilde{t}_{56} \\ \tilde{t}_{16}^* & \tilde{t}_{26}^* & \tilde{t}_{36}^* & \tilde{t}_{46}^* & \tilde{t}_{56}^* & \tilde{t}_{66} \end{pmatrix}, \sum_{i=1}^6 \tilde{t}_{ii} = 1 \quad (2.3.8)$$

$$\begin{aligned} \tilde{t}_{11} &= \frac{3}{32}[(a^2 + c^2) + a^2(t_{33} + t_{44}) + c^2(t_{55} + t_{66}) + ac(t_{35} + t_{35}^* + t_{46} + t_{46}^*)] \\ &+ \frac{1}{4}\left[\frac{2}{3}t_{11} + \frac{1}{3}t_{22}\right] \end{aligned} \quad (2.3.9)$$

$$\begin{aligned} \tilde{t}_{13} &= \frac{3}{32}[bc(1 + t_{55} + t_{66}) - a^2(t_{13} + t_{24}) - ac(t_{15} + t_{26}) + ab(t_{35}^* + t_{46}^*)] \\ &+ \frac{1}{4}\left[\frac{2}{3}t_{13} + \frac{1}{3}t_{24}\right] \end{aligned} \quad (2.3.10)$$

$$\begin{aligned} \tilde{t}_{15} &= \frac{3}{32}[-ab(1 + t_{33} + t_{44}) - ac(t_{13} + t_{24}) - c^2(t_{15} + t_{26}) - bc(t_{35} + t_{46})] \\ &+ \frac{1}{4}\left[\frac{2}{3}t_{15} + \frac{1}{3}t_{26}\right] \end{aligned} \quad (2.3.11)$$

$$\begin{aligned} \tilde{t}_{22} &= \frac{3}{32}[(a^2 + c^2) + a^2(t_{33} + t_{44}) + c^2(t_{55} + t_{66}) + ac(t_{35} + t_{35}^* + t_{46} + t_{46}^*)] \\ &+ \frac{1}{4}\left[\frac{1}{3}t_{11} + \frac{2}{3}t_{22}\right] \end{aligned} \quad (2.3.12)$$

$$\begin{aligned} \tilde{t}_{24} &= \frac{3}{32}[bc(1 + t_{55} + t_{66}) - a^2(t_{13} + t_{24}) - ac(t_{15} + t_{26}) + ab(t_{35}^* + t_{46}^*)] \\ &+ \frac{1}{4}\left[\frac{1}{3}t_{13} + \frac{2}{3}t_{24}\right] \end{aligned} \quad (2.3.13)$$

$$\begin{aligned} \tilde{t}_{26} &= \frac{3}{32}[-ab(1 + t_{33} + t_{44}) - ac(t_{13} + t_{24}) - c^2(t_{15} + t_{26}) - bc(t_{35} + t_{46})] \\ &+ \frac{1}{4}\left[\frac{1}{3}t_{15} + \frac{2}{3}t_{26}\right] \end{aligned} \quad (2.3.14)$$

$$\begin{aligned}\tilde{t}_{33} &= \frac{3}{32}[a^2 + b^2 + a^2(t_{11} + t_{22}) + b^2(t_{55} + t_{66}) - ab(t_{15} + t_{15}^*) - ab(t_{26} + t_{26}^*)] \\ &+ \frac{1}{4}\left[\frac{2}{3}t_{33} + \frac{1}{3}t_{44}\right]\end{aligned}\quad (2.3.15)$$

$$\begin{aligned}\tilde{t}_{35} &= \frac{3}{32}[ac(1 + t_{11} + t_{22}) - bc(t_{15} + t_{26}) + ab(t_{13}^* + t_{24}^*) - b^2(t_{35} + t_{46})] \\ &+ \frac{1}{4}\left[\frac{2}{3}t_{35} + \frac{1}{3}t_{46}\right]\end{aligned}\quad (2.3.16)$$

$$\begin{aligned}\tilde{t}_{44} &= \frac{3}{32}[a^2 + b^2 + a^2(t_{11} + t_{22}) + b^2(t_{55} + t_{66}) - ab(t_{15} + t_{15}^*) - ab(t_{26} + t_{26}^*)] \\ &+ \frac{1}{4}\left[\frac{1}{3}t_{33} + \frac{2}{3}t_{44}\right]\end{aligned}\quad (2.3.17)$$

$$\begin{aligned}\tilde{t}_{46} &= \frac{3}{32}[ac(1 + t_{11} + t_{22}) - bc(t_{15} - t_{26}) + ab(t_{13}^* + t_{24}^*) - b^2(t_{35} + t_{46})] \\ &+ \frac{1}{4}\left[\frac{1}{3}t_{35} + \frac{2}{3}t_{46}\right]\end{aligned}\quad (2.3.18)$$

$$\begin{aligned}\tilde{t}_{55} &= \frac{3}{32}[(b^2 + c^2) + c^2(t_{11} + t_{22}) + b^2(t_{33} + t_{44}) + bc(t_{13} + t_{13}^* + t_{24} + t_{24}^*)] \\ &+ \frac{1}{4}\left[\frac{2}{3}t_{55} + \frac{1}{3}t_{66}\right]\end{aligned}\quad (2.3.19)$$

$$\begin{aligned}\tilde{t}_{66} &= \frac{3}{32}[(b^2 + c^2) + c^2(t_{11} + t_{22}) + b^2(t_{33} + t_{44}) + bc(t_{13} + t_{13}^* + t_{24} + t_{24}^*)] \\ &+ \frac{1}{4}\left[\frac{1}{3}t_{55} + \frac{2}{3}t_{66}\right]\end{aligned}\quad (2.3.20)$$

$$\begin{aligned}\tilde{t}_{12} &= \frac{1}{12}t_{12}^*, & \tilde{t}_{14} &= \frac{1}{12}t_{23}, & \tilde{t}_{16} &= \frac{1}{12}t_{25}, & \tilde{t}_{23} &= \frac{1}{12}t_{14}, & \tilde{t}_{25} &= \frac{1}{12}t_{16}, \\ \tilde{t}_{34} &= \frac{1}{12}t_{34}^*, & \tilde{t}_{36} &= \frac{1}{12}t_{45}, & \tilde{t}_{45} &= \frac{1}{12}t_{36}, & \tilde{t}_{56} &= \frac{1}{12}t_{56}^*\end{aligned}\quad (2.3.21)$$

The value of the parameters a, b, c can be chosen in such a way that $Tr(\tilde{\rho}_{AB}) = 1$.

2.4 Criteria for the detection of bipartite Negative Partial Transpose Entangled states in arbitrary dimension

In this section, we derive the criteria for the detection of bipartite NPTEs in arbitrary dimensions employing the method of SPA-PT. In the first part, we will derive the lower and upper bound of the minimum eigenvalue of SPA-PT of the bipartite mixed quantum state in arbitrary dimension, and then using this lower and upper bound in the second part, we will derive the criterion for detection of NPTEs in arbitrary dimensional bipartite systems.

2.4.1 Lower and Upper bound of the minimum eigenvalue of SPA-PT of a bipartite mixed quantum state in arbitrary dimension

Let us consider a bipartite mixed quantum state in arbitrary dimension described by the density operator ρ_{AB} . If $\tilde{\rho}_{AB}$ denote the SPA-PT of ρ_{AB} and Q be any positive semi-definite operator such that $Tr(Q) = 1$ then the quantity $Tr[(\tilde{\rho}_{AB} + Q^{T_B})\rho_{AB}]$, where T_B denote the partial transposition with respect to the subsystem B, may be expressed as

$$\begin{aligned} Tr[(\tilde{\rho}_{AB} + Q^{T_B})\rho_{AB}] &= Tr[\tilde{\rho}_{AB}\rho_{AB} + Q^{T_B}\rho_{AB}] \\ &= Tr[\tilde{\rho}_{AB}\rho_{AB}] + Tr[Q^{T_B}\rho_{AB}] \\ &= Tr[\tilde{\rho}_{AB}\rho_{AB}] + Tr[Q\rho_{AB}^{T_B}] \end{aligned} \quad (2.4.1)$$

Taking $X = \tilde{\rho}_{AB}$ and $Y = \rho_{AB}$ in the LHS part of (2.2.1), we get

$$\lambda_{min}(\tilde{\rho}_{AB}) \leq Tr(\tilde{\rho}_{AB}\rho_{AB}) \quad (2.4.2)$$

Similarly, applying Result-1 with $X = \rho_{AB}^{T_B}$ and $Y = Q$ in (2.2.1), we obtain

$$\lambda_{min}(\rho_{AB}^{T_B}) \leq Tr[Q\rho_{AB}^{T_B}] = Tr[Q^{T_B}\rho_{AB}] \quad (2.4.3)$$

Adding (2.4.2) and (2.4.3), we get

$$\begin{aligned} \lambda_{min}(\tilde{\rho}_{AB}) + \lambda_{min}(\rho_{AB}^{T_B}) &\leq Tr[\tilde{\rho}_{AB}\rho_{AB}] + Tr[Q^{T_B}\rho_{AB}] \\ \Rightarrow \lambda_{min}(\rho_{AB}^{T_B}) &\leq G \end{aligned} \quad (2.4.4)$$

where $G = Tr[\tilde{\rho}_{AB}\rho_{AB}] + Tr[Q^{T_B}\rho_{AB}] - \lambda_{min}(\tilde{\rho}_{AB})$.

Therefore, the above inequality gives the upper bound on the minimum eigenvalue of $\rho_{AB}^{T_B}$.

A bipartite density operator ρ_{AB} in any arbitrary dimension, represent a NPTES if and only if $\lambda_{min}(\rho_{AB}^{T_B})$ is negative. Further, if we assume that NPTES described by the density operator ρ_{AB} detected by the witness operator $W = Q^{T_B}$ then we have

$$\lambda_{min}(\tilde{\rho}_{AB}) \geq Tr[\tilde{\rho}_{AB}\rho_{AB}] + Tr[W\rho_{AB}] \quad (2.4.5)$$

Again, inequality (2.4.4) can be re-expressed as

$$\begin{aligned} \lambda_{min}(\tilde{\rho}_{AB}) &\leq -\lambda_{min}(\rho_{AB}^{T_B}) + Tr[\tilde{\rho}_{AB}\rho_{AB}] \\ &+ Tr[Q^{T_B}\rho_{AB}] \end{aligned} \quad (2.4.6)$$

Using Result-3 in (2.4.6) and $W = Q^{T_B}$, the inequality (2.4.6) reduces to

$$\lambda_{min}(\tilde{\rho}_{AB}) \leq \frac{1}{2} + Tr[\tilde{\rho}_{AB}\rho_{AB}] + Tr[W\rho_{AB}] \quad (2.4.7)$$

Combining inequalities (2.4.5) and (2.4.7), we get the lower bound (L) and upper bound (U) of the minimum eigenvalue of the SPA-PT of ρ_{AB} and they are given by

$$L \leq \lambda_{min}(\tilde{\rho}_{AB}) \leq U \quad (2.4.8)$$

where

$$L = Tr[\tilde{\rho}_{AB}\rho_{AB}] + Tr[W\rho_{AB}] \quad (2.4.9)$$

$$U = \frac{1}{2} + Tr[\tilde{\rho}_{AB}\rho_{AB}] + Tr[W\rho_{AB}] \quad (2.4.10)$$

We note that the lower bound given by L can be negative also but since the minimum eigenvalue $\lambda_{min}(\tilde{\rho}_{AB})$ of the positive semi-definite operator $\tilde{\rho}_{AB}$ is always positive so the inequality (2.4.8) can be re-expressed as

$$\max\{L, 0\} \leq \lambda_{min}(\tilde{\rho}_{AB}) \leq U \quad (2.4.11)$$

where L and U are given by (2.4.9) and (2.4.10) respectively.

2.4.2 Criteria for the detection of $d_1 \otimes d_2$ dimensional bipartite NPTES

The entanglement of a bipartite mixed quantum state ρ_{AB} in $d_1 \otimes d_2$ dimension can be detected by computing the value of L . To see this, let us consider the second term of L , which is given by $Tr(W\rho_{AB}) = Tr(Q^{T_B}\rho_{AB})$. Since the witness operator has been constructed by taking the partial transpose of a positive semi-definite operator so it is not physically realizable and so we would like to approximate the witness operator W in such a way that it would become a completely positive operator. Therefore, if \tilde{W} is the approximation of the witness operator W then it can be expressed as

$$\tilde{W} = pW + \frac{1-p}{d_1d_2}I, 0 \leq p \leq 1, \quad (2.4.12)$$

The value of the parameter p should be chosen in such a way that \tilde{W} becomes a positive semi-definite operator. Further, we note that $Tr(\tilde{W}) = 1$. Thus, the operator \tilde{W} represents a quantum state.

Criterion-1: A bipartite $d_1 \otimes d_2$ dimensional quantum state described by the density operator ρ_{AB} represent a NPTES iff

$$Tr(\tilde{W}\rho_{AB}) < \frac{1-p}{d_1d_2} = R \quad (2.4.13)$$

Proof: Using (2.4.12), the relation between $Tr(W\rho)$ and $Tr(\tilde{W}\rho)$ can be established as

$$Tr(W\rho_{AB}) = \frac{1}{p}[Tr(\tilde{W}\rho_{AB}) - \frac{1-p}{d_1d_2}] \quad (2.4.14)$$

Since W denote the witness operator that detect the NPTES ρ_{AB} so $Tr(W\rho_{AB}) < 0$ and hence proved the required criterion.

Since \tilde{W} have all the properties of a quantum state so $Tr(\tilde{W}\rho_{AB})$ can be considered same as the average fidelity between two mixed quantum state \tilde{W} and ρ_{AB} and therefore, it is given by [119]

$$Tr(\tilde{W}\rho_{AB}) = F_{avg}(\tilde{W}, \rho_{AB}) \quad (2.4.15)$$

It is also known that the average fidelity $F_{avg}(\tilde{W}, \rho_{AB})$ can be estimated experimentally by Hong-Ou-Mandel interferometry and thus equation (2.4.13) can be re-expressed

as

$$F_{avg}(\tilde{W}, \rho_{AB}) < \frac{1-p}{d_1 d_2} \quad (2.4.16)$$

Criterion-2: The bipartite state ρ_{AB} of any arbitrary dimension is NPTES iff

$$\lambda_{min}(\tilde{\rho}_{AB}) \geq F_{avg}(\rho_{AB}, \tilde{\rho}_{AB}) - C(\rho_{AB}) \quad (2.4.17)$$

where $C(\rho_{AB})$ denote the concurrence of the density operator ρ_{AB} . The lower and upper bound of $C(\rho_{AB})$ is given by

$$\frac{1-p}{p d_1 d_2} - \frac{F_{avg}(\tilde{W}, \rho_{AB})}{p} \leq C(\rho_{AB}) \leq F_{avg}(\rho_{AB}, \tilde{\rho}_{AB}) \quad (2.4.18)$$

Proof: Let us first recall the lower bound L of the minimum eigenvalue $\lambda_{min}(\tilde{\rho}_{AB})$ and using Result-2 in (2.4.9), we get

$$L \geq Tr[\tilde{\rho}_{AB}\rho_{AB}] - C(\rho_{AB}) \quad (2.4.19)$$

The RHS of the inequality (2.4.19) may take a non-negative or negative value. Consider the case when $Tr[\tilde{\rho}_{AB}\rho_{AB}] - C(\rho_{AB})$ is non-negative i.e.

$$C(\rho_{AB}) \leq Tr[\tilde{\rho}_{AB}\rho_{AB}] = F_{avg}(\rho_{AB}, \tilde{\rho}_{AB}) \quad (2.4.20)$$

Then using (2.4.19), we can re-write (2.4.5) as

$$\lambda_{min}(\tilde{\rho}_{AB}) \geq F_{avg}(\rho_{AB}, \tilde{\rho}_{AB}) - C(\rho_{AB}) \quad (2.4.21)$$

Moreover, using Result-2, (2.4.14) and (2.4.15), we get

$$C(\rho_{AB}) \geq \frac{1}{p} \left[\frac{1-p}{d_1 d_2} - F_{avg}(\tilde{W}, \rho_{AB}) \right] \quad (2.4.22)$$

Combining (2.4.20) and (2.4.22), we get

$$\frac{1-p}{p d_1 d_2} - \frac{F_{avg}(\tilde{W}, \rho_{AB})}{p} \leq C(\rho_{AB}) \leq F_{avg}(\rho_{AB}, \tilde{\rho}_{AB}) \quad (2.4.23)$$

Hence proved.

Since the average fidelity $F_{avg}(\tilde{W}, \rho_{AB})$ and $F_{avg}(\rho_{AB}, \tilde{\rho}_{AB})$ can be estimated experimen-

tally so the lower bound and upper bound of $C(\rho_{AB})$ can be estimated experimentally.

Criterion-3: The bipartite state ρ_{AB} of any arbitrary dimension is NPTES iff

$$U^{ent} < \frac{1}{2} \quad (2.4.24)$$

where $U^{ent} = \frac{1}{2} + F_{avg}(\rho_{AB}, \widetilde{\rho_{AB}}) - C(\rho_{AB})$ and $C(\rho_{AB})$ is given by

$$C(\rho_{AB}) > F_{avg}(\rho_{AB}, \widetilde{\rho_{AB}}) \quad (2.4.25)$$

Proof: Let us recall the upper bound U of the minimum eigenvalue $\lambda_{min}(\widetilde{\rho_{AB}})$ and using Result-2 in (2.4.10), we get

$$U \geq \frac{1}{2} + Tr[\widetilde{\rho_{AB}}\rho_{AB}] - C(\rho_{AB}) = U^{ent} \quad (2.4.26)$$

If the quantity $Tr[\widetilde{\rho_{AB}}\rho_{AB}] - C(\rho_{AB})$ is non-negative then $U \geq \frac{1}{2}$. Also if the state is separable i.e. if $C(\rho_{AB}) = 0$ then the value of U again comes out to be greater than $\frac{1}{2}$. So it would be difficult to detect the entangled state when $Tr[\widetilde{\rho_{AB}}\rho_{AB}] - C(\rho_{AB}) \geq 0$. If we now consider the case when $Tr[\widetilde{\rho_{AB}}\rho_{AB}] - C(\rho_{AB}) < 0$, which indeed may be the case, then we can obtain $U^{ent} < \frac{1}{2}$. Thus, we can infer when $U^{ent} < \frac{1}{2}$ for $C(\rho_{AB}) > Tr[\widetilde{\rho_{AB}}\rho_{AB}] = F_{avg}(\rho_{AB}, \widetilde{\rho_{AB}})$, the state is NPTES. Hence the criterion.

2.5 Illustrations

In this section, we will verify our entanglement detection criteria given in the previous section by taking the example of a class of qubit-qubit system and qutrit-qubit system.

2.5.1 Qubit-Qubit system

Let us consider a large class of two-qubit systems described by the density operator $\rho_{AB}^{(1)}$ given as [124]

$$\rho_{AB}^{(1)} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & f & 0 \\ 0 & f^* & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, a+b = \frac{1}{2} \quad (2.5.1)$$

where $(*)$ denotes the complex conjugate.

The two-qubit density matrix of the form (2.5.1) has been studied by many authors [125–127] as this form of density matrix has high entanglement. In particular, Ishizaka and Hiroshima [128] have studied such density matrix and maximized the entanglement for a fixed set of eigenvalues with one of the eigenvalues being zero.

The density operator $\rho_{AB}^{(1)}$ is positive semi-definite only when

$$|f| \leq b \quad (2.5.2)$$

The partial transpose of $\rho_{AB}^{(1)}$ is given by

$$(\rho_{AB}^{(1)})^{T_B} = \begin{pmatrix} a & 0 & 0 & f \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ f^* & 0 & 0 & a \end{pmatrix} \quad (2.5.3)$$

$(\rho_{AB}^{(1)})^{T_B}$ has negative eigenvalue if

$$|f| > a \quad (2.5.4)$$

Therefore, the state $\rho_{AB}^{(1)}$ is an entangled state iff

$$a < |f| \leq b \quad \text{and} \quad a + b = \frac{1}{2} \quad (2.5.5)$$

The concurrence of the state $\rho_{AB}^{(1)}$ is given by

$$C(\rho_{AB}^{(1)}) = |f| - a \quad (2.5.6)$$

Let us now assume that if $\rho_{AB}^{(1)}$ is an entangled state then it is detected by the witness operator $W^{(1)}$. The witness operator $W^{(1)}$ can be expressed as

$$W^{(1)} = |\psi\rangle\langle\psi|^{T_B} - \frac{1}{\sqrt{1+|k|^2}}(k|00\rangle + |11\rangle) \quad (2.5.7)$$

where $k = \frac{-f}{|f|}$.

Since it is not possible to implement partial transposition operation experimentally so we use Result-4 to obtain the SPA-PT of $\rho_{AB}^{(1)}$ and $W^{(1)}$. SPA-PT of $\rho_{AB}^{(1)}$ and $W^{(1)}$ are

Table 2.1: Table verifying (2.5.12) for different values of the state parameters (a, b, f)

State parameter (a, b, f)	$F_{avg}(\tilde{W}^{(1)}, \rho_{AB}^{(1)})$	Criterion-1	Nature of $\rho_{AB}^{(1)}$
(0.05, 0.45, 0.4+0.1i)	0.04589	satisfied	Entangled
(0.1, 0.4, 0.25+0.25i)	0.08214	satisfied	Entangled
(0.15, 0.35, 0.24+0.2i)	0.11253	satisfied	Entangled
(0.2, 0.3, 0.27+0.13i)	0.13344	satisfied	Entangled

given in the following form

$$\tilde{\rho}_{AB}^{(1)} = \begin{pmatrix} \frac{2+a}{9} & 0 & 0 & \frac{f}{9} \\ 0 & \frac{2+b}{9} & 0 & 0 \\ 0 & 0 & \frac{2+b}{9} & 0 \\ \frac{f^*}{9} & 0 & 0 & \frac{2+a}{9} \end{pmatrix} \quad (2.5.8)$$

$$\tilde{W}^{(1)} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{k}{6} & 0 \\ 0 & \frac{k^*}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \quad (2.5.9)$$

Also, the average fidelities between two pair of mixed quantum state $(\rho_{AB}^{(1)}, \tilde{\rho}_{AB}^{(1)})$ and $(\tilde{W}^{(1)}, \rho_{AB}^{(1)})$ respectively, are given by

$$F_{avg}(\rho_{AB}, \tilde{\rho}_{AB}^{(1)}) = \frac{2a}{9}(2+a) + \frac{2b}{9}(2+b) \quad (2.5.10)$$

$$F_{avg}(\tilde{W}^{(1)}, \rho_{AB}^{(1)}) = \frac{2a+b-|f|}{3} \quad (2.5.11)$$

Now we are in a position to discuss criterion-1, criterion-2, and criterion-3 for a large class of qubit-qubit systems described by the density operator $\rho_{AB}^{(1)}$.

Criterion-1 for the density operator $\rho_{AB}^{(1)}$ takes the form as

$$F_{avg}(\tilde{W}^{(1)}, \rho_{AB}^{(1)}) < \frac{1}{6} \quad (2.5.12)$$

The satisfaction of the above criterion is given in Table 2.1. Next, let us illustrate

Table 2.2: Table verifying (2.5.14) for different values of the state parameters (a, b, f)

State parameter (a, b, f)	$F_{avg}(\tilde{W}^{(1)}, \rho_{AB}^{(1)})$	$F_{avg}(\tilde{\rho}_{AB}^{(1)}, \rho_{AB}^{(1)})$	$C(\rho_{AB}^{(1)})$
(0.05, 0.45, 0.2+0.2i)	0.08905	0.26777	0.23284
(0.1, 0.4, 0.25+0.25i)	0.08215	0.26	0.25355
(0.15, 0.35, 0.24+0.2i)	0.11253	0.25444	0.16241
(0.2, 0.3, 0.27+0.13i)	0.13344	0.25111	0.09966

Table 2.3: Table verifying (2.5.13) for different values of the state parameters (a, b, f)

State parameter (a, b, f)	$\lambda_{min}(\tilde{\rho}_{AB}^{(1)})$	Criterion-2	Nature of $\rho_{AB}^{(1)}$
(0.05, 0.45, 0.2+0.2i)	0.19635	satisfied	Entangled
(0.1, 0.4, 0.25+0.25i)	0.19405	satisfied	Entangled
(0.15, 0.35, 0.24+0.2i)	0.20417	satisfied	Entangled
(0.2, 0.3, 0.27+0.13i)	0.21114	satisfied	Entangled

criterion-2 for the density operator $\rho_{AB}^{(1)}$. Criterion-2 can be re-written for $\rho_{AB}^{(1)}$ as

$$\lambda_{min}(\tilde{\rho}_{AB}^{(1)}) \geq F_{avg}(\rho_{AB}^{(1)}, \tilde{\rho}_{AB}^{(1)}) - C(\rho_{AB}^{(1)}) \quad (2.5.13)$$

where $C(\rho_{AB})$ is given by

$$\frac{1}{2} - 3F_{avg}(\tilde{W}^{(1)}, \rho_{AB}^{(1)}) \leq C(\rho_{AB}^{(1)}) \leq F_{avg}(\rho_{AB}^{(1)}, \tilde{\rho}_{AB}^{(1)}) \quad (2.5.14)$$

We can now verify (2.5.13) and (2.5.14) by taking different values of the parameters of the state $\rho_{AB}^{(1)}$. The newly derived lower and upper bound of the concurrence $C(\rho_{AB}^{(1)})$ and the criterion-2 can be verified by Table 2.2 and Table 2.3 respectively.

2.5.2 Qutrit-Qubit system

It is known that certain qutrit-qubit entangled state shows the property of time-invariant entanglement under collective dephasing. Further, it has been observed that all dimension of Hilbert space studied so far does not exhibit simultaneously two important properties such as time-invariant entanglement and freezing dynamics. The qutrit-qubit system given in the example below is important in the sense that it exhibits time-invariant entanglement, as well as freezing dynamics of entanglement under collective dephasing [129, 130].

Let us now consider an example of a qutrit-qubit quantum state described by the

density operator $\rho_{AB}^{(2)}$,

$$\rho_{AB}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2} & 0 & 0 & \frac{\alpha}{2} & 0 \\ 0 & 0 & \frac{1-\alpha}{2} & 0 & 0 & \frac{1-\alpha}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2} & 0 & 0 & \frac{\alpha}{2} & 0 \\ 0 & 0 & \frac{1-\alpha}{2} & 0 & 0 & \frac{1-\alpha}{2} \end{pmatrix}, 0 \leq \alpha \leq 1 \quad (2.5.15)$$

It has been found that the state $\rho_{AB}^{(2)}$ is entangled for $0 \leq \alpha \leq 1$ [129]. This result can be verified by the witness operator method. The witness operator that detects the quantum state $\rho_{AB}^{(2)}$ as an entangled state is given by

$$W^{(2)} = |\chi\rangle^{TB}\langle\chi|, |\chi\rangle = \frac{1}{\sqrt{1+|\kappa|^2}}(-\kappa|11\rangle + |20\rangle) \quad (2.5.16)$$

where $\kappa = \frac{\alpha + \sqrt{4 - 8\alpha + 5\alpha^2}}{2(1-\alpha)}$.

The operator $W^{(2)}$ has been constructed based on the idea of partial transposition but since it is not practically implementable operation so we use our criteria for the detection of entanglement, which can be implemented in the laboratory.

To implement our first criteria, we will use the SPA-PT of $W^{(2)}$, which is given by

$$\tilde{W}^{(2)} = \begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 & -r\kappa \\ 0 & 0 & 0 & \frac{1}{8} + r|\kappa|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} + r & 0 \\ 0 & 0 & -r\kappa^* & 0 & 0 & \frac{1}{8} \end{pmatrix} \quad (2.5.17)$$

where $r = \frac{1}{4(1+|\kappa|^2)}$.

Therefore, criterion-1 for the density operator $\rho_{AB}^{(2)}$ reduces to

$$F_{avg}(\tilde{W}^{(2)}, \rho_{AB}^{(2)}) - \frac{1}{8} < 0 \quad (2.5.18)$$

where $F_{avg}(\tilde{W}^{(2)}, \rho_{AB}^{(2)})$ is given by

$$F_{avg}(\tilde{W}^{(2)}, \rho_{AB}^{(2)}) = \frac{1 - 4r\alpha}{8} \quad (2.5.19)$$

Since the parameters α and r lying in $[0, 1]$ so $F_{avg}(\tilde{W}^{(2)}, \rho_{AB}^{(2)})$ is always less than zero in this range of parameters. Hence the state $\rho_{AB}^{(2)}$ is entangled for $0 \leq \alpha \leq 1$.

Criterion 2 not only detects the entanglement but also provides the lower and upper bound of the measure of entanglement characterized by concurrence. Therefore, our next task is to calculate the lower and upper bound of the concurrence of the state described by the density operator $\rho_{AB}^{(2)}$. To achieve our goal, we use (2.3.8, 2.3.9-2.3.21) to obtain the SPA-PT of $\rho_{AB}^{(2)}$, which can be expressed as

$$\tilde{\rho}_{AB}^{(2)} = \begin{pmatrix} \tilde{t}_{11} & 0 & \tilde{t}_{13} & 0 & \tilde{t}_{15} & \tilde{t}_{16} \\ 0 & \tilde{t}_{22} & 0 & \tilde{t}_{24} & 0 & \tilde{t}_{26} \\ \tilde{t}_{13}^* & 0 & \tilde{t}_{33} & 0 & \tilde{t}_{35} & 0 \\ 0 & \tilde{t}_{24}^* & 0 & \tilde{t}_{44} & \tilde{t}_{45} & \tilde{t}_{46} \\ \tilde{t}_{15}^* & 0 & \tilde{t}_{35}^* & \tilde{t}_{45}^* & \tilde{t}_{55} & 0 \\ \tilde{t}_{16}^* & \tilde{t}_{26}^* & 0 & \tilde{t}_{46}^* & 0 & \tilde{t}_{66} \end{pmatrix}, 0 \leq \alpha \leq 1 \quad (2.5.20)$$

where

$$\begin{aligned} \tilde{t}_{11} &= \frac{54}{384} + \frac{7\alpha}{384}, \quad \tilde{t}_{13} = \frac{9}{128}, \quad \tilde{t}_{15} = -\frac{9}{128} + \frac{3\alpha}{128}, \\ \tilde{t}_{16} &= \frac{\alpha}{24}, \quad \tilde{t}_{22} = \frac{9}{64} + \frac{23\alpha}{384}, \quad \tilde{t}_{24} = \frac{9}{128}, \\ \tilde{t}_{26} &= -\frac{9}{128} + \frac{3\alpha}{128}, \quad \tilde{t}_{33} = \frac{77}{384} - \frac{23\alpha}{384}, \quad \tilde{t}_{35} = \frac{3}{64} + \frac{3\alpha}{128}, \\ \tilde{t}_{44} &= \frac{61}{384} - \frac{7\alpha}{384}, \quad \tilde{t}_{45} = \frac{1-\alpha}{24}, \quad \tilde{t}_{46} = \frac{3}{64} + \frac{3\alpha}{128}, \\ \tilde{t}_{55} &= \frac{61}{384} + \frac{\alpha}{24}, \quad \tilde{t}_{66} = \frac{77}{384} - \frac{\alpha}{24} \end{aligned} \quad (2.5.21)$$

The lower and upper bound of the concurrence $C(\rho_{AB}^{(2)})$ is given by

$$\frac{1}{2} - 4F_{avg}(\tilde{W}^{(2)}, \rho_{AB}^{(2)}) \leq C(\rho_{AB}^{(2)}) \leq F_{avg}(\rho_{AB}^{(2)}, \tilde{\rho}_{AB}^{(2)}) \quad (2.5.22)$$

where $F_{avg}(\rho_{AB}^{(2)}, \tilde{\rho}_{AB}^{(2)})$ is given by

$$F_{avg}(\rho_{AB}^{(2)}, \tilde{\rho}_{AB}^{(2)}) = \frac{78\alpha^2 - 78\alpha + 154}{768} \quad (2.5.23)$$

The lower and upper bound of the concurrence $C(\rho_{AB}^{(2)})$ is shown in Figure 2.1.

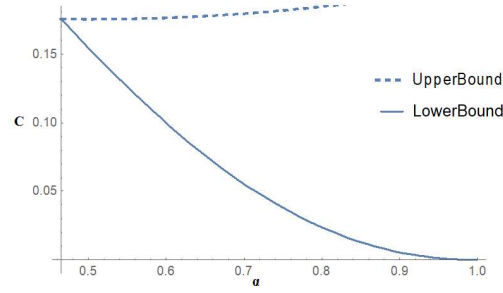


Figure 2.1: Plot of concurrence (C) versus α . The dotted line shows the upper bound and the solid line represent the lower bound of the concurrence of the qutrit-qubit state described by the density matrix $\rho_{AB}^{(2)}$

2.6 Conclusion

In this chapter, we have discussed the method of SPA-PT on a two-qubit system. Using the idea for SPA-PT on a two-qubit system, we perform a SPA map on the qutrit-qubit system and then explicitly calculated the elements of the SPA-PT of the qutrit-qubit system. We have obtained the criteria based on SPA-PT for the detection of NPTES in arbitrary dimensional Hilbert space. The first criterion detects NPTES if and only if the average fidelity of two quantum states described by the density matrix ρ_{AB} and SPA-PT of the witness operator \tilde{W} is less than a quantity R . The quantity R depends on (i) the dimension of the composite Hilbert space and (ii) the parameter that makes the SPA-PT of the witness operator a positive semi-definite. The second criterion tells us that the given state is NPTES if and only if the minimum eigenvalue of the SPA-PT of the given state is greater or equal to the difference between the concurrence of the given state and the average fidelity between the given state and its SPA-PT. Since it would not be possible to find out the exact value of the concurrence of $d_1 \otimes d_2$ dimensional mixed bipartite state so we have derived the lower and upper bound of the concurrence of the given mixed state which may be realized in an experiment also. The third criterion also dealt with the detection of entanglement. Moreover, using the elements of a qutrit qubit system one can find the minimum eigenvalue of the SPA-PT in the qutrit-qubit system. Then, by analyzing the minimum eigenvalue, one can check whether the given qutrit-qubit composite system is entangled or not.

Chapter 3

Classification of multipartite states through witness operator

GHZ class of states can be considered an important class of three-qubit pure states in the sense that, it can serve as an efficient quantum channel for quantum communication purposes. In this chapter¹, we have considered this important class of three-qubit pure states, divide it into different subclasses according to their utility in quantum information processing task and then studied the classification of different subclasses of GHZ class of pure states. We have started with the definition of different subclasses of the canonical form of a pure three-qubit GHZ state and then constructed a classification witness operator to classify these subclasses. The defined subclasses may be denoted as S_1, S_2, S_3, S_4 . The motivation for the classification of the pure GHZ class of states into four subclasses is that the GHZ state belonging to S_1 may be more efficient than the three-qubit GHZ state belonging to S_2 or S_3 or S_4 and vice-versa with respect to some quantum information processing task. We have constructed different witness operators that can classify the subclasses $S_i, i = 2, 3, 4$ from S_1 and further, shown that the constructed witness operator can be decomposed into Pauli matrices. The decomposition of the witness operator into Pauli matrices may make it possible to realize the witness operators in an experiment.

¹Chapter 3 is based on a research paper entitled “Classification witness operator for the classification of different subclasses of three-qubit GHZ class, *Quantum Information processing* **20**, 316 (2021)”

3.1 Introduction

Entanglement is a purely quantum mechanical phenomenon that plays a vital role in the advancement of quantum information theory. The two basic problems of quantum information theory are: (i) detection of n-qubit entangled states and (ii) classification of n-qubit entangled states. The complexity of the system will increase as the number of qubits in the system increases and thus, the difficulty level of the above-mentioned two basic problems also increases. Hence, we restrict ourselves here, to study the classification of three-qubit entangled states. Three-qubit states may be classified as one separable, three biseparable, and two genuine entangled states [95]. These six classes of three-qubit pure state can be considered inequivalent classes under SLOCC. The two SLOCC inequivalent genuine entangled classes are GHZ class and W class. In the literature, it has been shown that there exist observables that can be used to distinguish the above-mentioned six inequivalent classes of three-qubit pure states [96]. The experiment using an NMR quantum information processor has been carried out to classify six inequivalent classes under SLOCC [97]. Monogamy score can also be used to classify pure tripartite systems [131].

The first result for the classification of mixed three-qubit states was given by Acin et. al [98]. They have classified mixed three-qubit states through the method of constructing the witness operator. Sabin et. al. [99] have studied the classification of pure as well as mixed three-qubit entanglement based on reduced two-qubit entanglement.

The classification of the system containing more than three qubits has also been studied in the literature. The classification of different classes of four qubit pure states has been studied in [100–102]. The number of different classes of an n-qubit system increases when we increase the number of qubits and the discrimination of these different classes of the n-qubit system has been studied in [103–106].

In this chapter, we are focusing on the classification of different subclasses of GHZ class. To define different subclasses of GHZ class, let us consider the five-parameter canonical form of three-qubit pure state $|\psi\rangle_{ABC}$ shared between three distant partners A , B and C . The three-qubit pure state $|\psi\rangle_{ABC}$ is given by [94]

$$|\psi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \quad (3.1.1)$$

where $\lambda_i \in R$ and $0 \leq \lambda_i \leq 1 (i = 0, 1, 2, 3, 4)$; $\theta (0 \leq \theta \leq \pi)$ denote the phase parameter. The normalization condition of the state (3.1.1) is given by

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1. \quad (3.1.2)$$

The three-tangle τ_ψ for a pure three-qubit state $|\psi\rangle_{ABC}$ may be defined as [96, 132]

$$\tau_\psi = 4\lambda_0^2\lambda_4^2 \quad (3.1.3)$$

The tangle $\tau_\psi \neq 0$ for GHZ class and $\tau_\psi = 0$ for W class of states. To define the subclasses of GHZ class, we make the following assumption: (i) the state parameters λ_0 and λ_4 are not equal to zero, and (ii) the phase factor $\theta = 0$.

It may be noted that similar calculations can be performed by taking $\theta \neq 0$ also. We are now in a position to divide the three-qubit pure GHZ class of states given in (3.1.1) into four subclasses as:

Subclass-I :

$S_1 = \{|\psi_S\rangle\}$, where

$$|\psi_S\rangle = \lambda_0|000\rangle + \lambda_4|111\rangle \quad (3.1.4)$$

Subclass-II :

$S_2 = \{|\psi_{\lambda_1}\rangle, |\psi_{\lambda_2}\rangle, |\psi_{\lambda_3}\rangle\}$, where

$$|\psi_{\lambda_1}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_4|111\rangle,$$

$$|\psi_{\lambda_2}\rangle = \lambda_0|000\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle,$$

$$|\psi_{\lambda_3}\rangle = \lambda_0|000\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \quad (3.1.5)$$

Subclass-III :

$S_3 = \{|\psi_{\lambda_1,\lambda_2}\rangle, |\psi_{\lambda_1,\lambda_3}\rangle, |\psi_{\lambda_2,\lambda_3}\rangle\}$, where

$$|\psi_{\lambda_1,\lambda_2}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle,$$

$$|\psi_{\lambda_1,\lambda_3}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,$$

$$|\psi_{\lambda_2,\lambda_3}\rangle = \lambda_0|000\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \quad (3.1.6)$$

Subclass-IV (S_4) :

$S_4 = \{|\psi_{\lambda_1, \lambda_2, \lambda_3}\rangle\}$, where

$$|\psi_{\lambda_1, \lambda_2, \lambda_3}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \quad (3.1.7)$$

Different subclasses of GHZ class of states are distributed in four different sets S_1, S_2, S_3, S_4 . Classification of these subclasses can be diagrammatically shown in Figure 3.1. In Figure 3.1, the outermost circle represents GHZ states belonging to subclass IV, the second outermost circle represents the GHZ states belonging to subclass III, the third outermost circle represents the GHZ states belonging to subclass II and the innermost circle represents the standard GHZ class of states belonging to subclass-I. We should note here that these subclasses are not inequivalent under SLOCC. To transform a state from one subclass to another, we need to perform local quantum operations that depend on the state which is to be transformed. So it is necessary to know the state or at least the subclass in which the state belongs.

The motivation of this chapter is divided into three parts: **M1, M2** and **M3**.

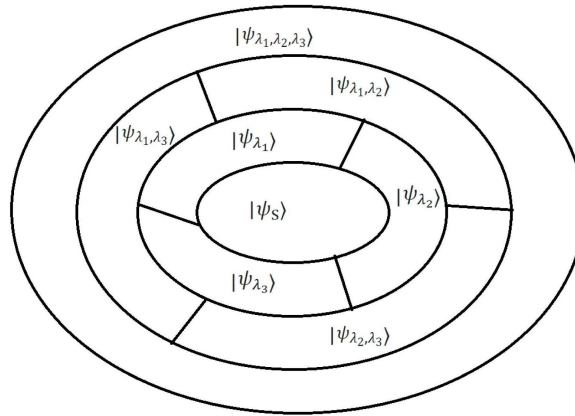


Figure 3.1: Classification of different subclasses of GHZ class of states described by the four sets S_1, S_2, S_3, S_4

M1: Comparing the teleportation fidelities

Let us consider the teleportation scheme introduced by Lee et.al. [133]. According to this teleportation scheme, a single-qubit measurement has been performed either on the qubit A or qubit B or qubit C of the pure three-qubit state. After the measurement, the pure three-qubit state reduces to a two-qubit state at the output. Then the resulting two-qubit state can be used as a resource state for quantum teleportation. The efficiency of the resource state is provided by teleportation fidelity. In particular,

if the single-qubit measurement is performed on either qubit A or qubit B or qubit C of the state $|\psi_S\rangle \in S_1$ then the corresponding maximal teleportation fidelities are given by [133]

$$F_A^{(|\psi_S\rangle)} = F_B^{(|\psi_S\rangle)} = F_C^{(|\psi_S\rangle)} = \frac{2(1 + \lambda_0\lambda_4)}{3} \quad (3.1.8)$$

where $F_i^{(|\psi_S\rangle)}$, $i = A, B, C$ denote the maximal teleportation fidelity when measurement is performed on qubit i . In a similar fashion, if the single-qubit measurement is performed on the state $|\psi_{\lambda'_1}\rangle \in S_2$, then the corresponding maximal teleportation fidelities are given by [133]

$$\begin{aligned} F_A^{(|\psi_{\lambda'_1}\rangle)} &= \frac{2(1 + \lambda'_4 \sqrt{(\lambda'_0)^2 + (\lambda'_1)^2})}{3} \\ F_B^{(|\psi_{\lambda'_1}\rangle)} &= F_C^{(|\psi_{\lambda'_1}\rangle)} = \frac{2(1 + \lambda'_0\lambda'_4)}{3} \end{aligned} \quad (3.1.9)$$

Again, if the single-qubit measurement is performed on the state $|\psi_{\lambda''_1}\rangle \in S_3$, then the corresponding maximal teleportation fidelities are

$$\begin{aligned} F_A^{(|\psi_{\lambda''_{1,2}}\rangle)} &= \frac{2(1 + \lambda''_4 \sqrt{(\lambda''_0)^2 + (\lambda''_1)^2})}{3} \\ F_B^{(|\psi_{\lambda''_{1,2}}\rangle)} &= \frac{2(1 + \lambda''_0 \sqrt{(\lambda''_2)^2 + (\lambda''_4)^2})}{3} \\ F_C^{(|\psi_{\lambda''_{1,2}}\rangle)} &= \frac{2(1 + \lambda''_0\lambda''_4)}{3} \end{aligned} \quad (3.1.10)$$

and if the single-qubit measurement is performed on the state $|\psi_{\lambda'''_1}\rangle \in S_4$, then the corresponding maximal teleportation fidelities are

$$\begin{aligned} F_A^{(|\psi_{\lambda'''_{1,2,3}}\rangle)} &= \frac{2(1 + \sqrt{y})}{3} \\ F_B^{(|\psi_{\lambda'''_{1,2,3}}\rangle)} &= \frac{2(1 + \lambda'''_0 \sqrt{(\lambda'''_2)^2 + (\lambda'''_4)^2})}{3} \\ F_C^{(|\psi_{\lambda'''_{1,2,3}}\rangle)} &= \frac{2(1 + \lambda'''_0 \sqrt{(\lambda'''_3)^2 + (\lambda'''_4)^2})}{3} \end{aligned} \quad (3.1.11)$$

where

$$y = (\lambda'''_0)^2(\lambda'''_4)^2 + (\lambda'''_1)^2(\lambda'''_4)^2 + (\lambda'''_2)^2(\lambda'''_3)^2 - 4\lambda'''_1\lambda'''_2\lambda'''_3\lambda'''_4 \quad (3.1.12)$$

It can be easily seen that there exist state parameters $\lambda_0''', \lambda_1''', \lambda_2''', \lambda_3''', \lambda_4''', \lambda_0'', \lambda_1'', \lambda_2'', \lambda_4'', \lambda_0', \lambda_1', \lambda_4', \lambda_0$ and λ_4 such that the inequalities

$$\begin{aligned}
F_A^{(|\psi_{\lambda_{1,2}}'''\rangle\rangle)} &\geq F_A^{(|\psi_{\lambda_1}'\rangle\rangle)}, F_A^{(|\psi_{\lambda_{1,2,3}}'''\rangle\rangle)} \geq F_A^{(|\psi_{\lambda_{1,2}}'''\rangle\rangle)} \\
F_A^{(|\psi_{\lambda_{1,2,3}}'''\rangle\rangle)} &\geq F_A^{(|\psi_{\lambda_1}'\rangle\rangle)}, F_A^{(|\psi_{\lambda_1}'\rangle\rangle)} \geq F_A^{(|\psi_S\rangle\rangle)} \\
F_A^{(|\psi_{\lambda_{1,2}}'''\rangle\rangle)} &\geq F_A^{(|\psi_S\rangle\rangle)}, F_A^{(|\psi_{\lambda_{1,2,3}}'''\rangle\rangle)} \geq F_A^{(|\psi_S\rangle\rangle)}
\end{aligned} \tag{3.1.13}$$

holds.

In this way, we can compare the teleportation fidelities of the GHZ states belonging to different subclasses. We can conclude from (3.1.13) that the pure three-qubit state $|\psi_{\lambda_1}'\rangle \in S_2$ is more efficient than $|\psi_S\rangle \in S_1$ in the teleportation scheme [133]. In the same way, we can say that the states belonging to subclass S_3 are more efficient than the states belonging to S_2 or S_1 . Also, it can be observed that the states belonging to any of the defined subclasses are GHZ states. Thus it is necessary to discriminate the pure three-qubit states belonging to different subclasses of GHZ class.

M2: Comparing the concurrence of the reduced two-qubit state

We can compare the entanglement between the reduced two-qubit mixed states obtained after tracing out either subsystem A or subsystem B or subsystem C in the following way: If we have GHZ state belonging to subclass S_1 , then after tracing out one qubit, the concurrence of the resulting two-qubit system will become zero, that is, $C_{AB} = C_{AC} = C_{BC} = 0$. Thus, after tracing out one subsystem, the remaining two-qubit state will become a separable state. Now, if we consider GHZ state belonging to subclass S_2 , then we have exactly one of the concurrences either C_{AB} or C_{AC} or C_{BC} of the two-qubit mixed reduced system is non-zero. Thus, if we require any two-qubit entangled state in some quantum information processing protocol, then we can obtain it by tracing out one qubit from a three-qubit GHZ state belonging to subclass S_2 . For example, if we need any two-qubit shared entangled state between Alice and Bob, then we can use three-qubit GHZ state ($|\psi_{ABC}\rangle = \lambda_0|000\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$), lying in subclass S_2 . It is possible, since, the concurrence of the reduced state $\rho_{AB} = Tr_C(|\psi\rangle_{ABC}\langle\psi|)$ is not equal to zero. But this type of situation will not arise in the case of a three-qubit GHZ state belonging to subclass S_1 . Not only the subclass S_2 , but we can use other subclasses such as S_3 and S_4 to get the entangled mixed two-qubit state.

M3: Comparing the three-tangle of the three-qubit state

We can see changes in the tangle in these subclasses as follows:

For a GHZ state belonging to S_1 , we have only two parameters λ_0 and λ_4 . But for the GHZ state belonging to S_2 have parameters λ'_0 , λ'_1 and λ'_4 . Due to the normalization condition, the values of parameters get distributed. Thus, using normalization condition we get, $\lambda_0\lambda_4 > \lambda'_0\lambda'_4$. From (3.1.3), we can say that the tangle of the three-qubit GHZ state belonging to S_1 will be more than the tangle of the GHZ state belonging to S_2 . Again, if we compare the tangle of the three-qubit GHZ state belonging to subclass S_3 and subclass S_2 , then we will find that the tangle of the GHZ state belonging to subclass S_2 will be more than the tangle of the GHZ state belonging to S_3 . In this way, we can conclude that

$$\tau^{(2)} \geq \tau^{(3)} \geq \tau^{(4)} \geq \tau^{(5)} \quad (3.1.14)$$

where, $\tau^{(2)}$ is the tangle of the GHZ state belonging to subclass S_1 , $\tau^{(3)}$ is the tangle of the GHZ state belonging to subclass S_2 , $\tau^{(4)}$ is the tangle of the GHZ state belonging to subclass S_3 and $\tau^{(5)}$ is the tangle of the GHZ state belonging to subclass S_4 .

Therefore, **M1**, **M2** and **M3** provide us sufficient motivation to classify different subclasses of GHZ class of states.

3.2 Derivation of the inequality required for the construction of classification witness operator

In this section, we will construct the Hermitian matrices from the component of the correlation tensor and then use its minimum and maximum eigenvalues to derive the required inequality for the construction of the classification witness operator.

To start with, let us consider any arbitrary three-qubit state described by the density operator ρ . The three-qubit state ρ may be expressed as

$$\begin{aligned} \rho = & \frac{1}{8} [I \otimes I \otimes I + \vec{l} \cdot \vec{\sigma} \otimes I \otimes I + I \otimes \vec{m} \cdot \vec{\sigma} \otimes I + I \otimes I \otimes \vec{n} \cdot \vec{\sigma} + \vec{u} \cdot \vec{\sigma} \otimes \vec{v} \cdot \vec{\sigma} \otimes I + \vec{u} \cdot \vec{\sigma} \otimes I \otimes \vec{w} \cdot \vec{\sigma} \\ & + I \otimes \vec{v} \cdot \vec{\sigma} \otimes \vec{w} \cdot \vec{\sigma} + \sum_{i,j,k=x,y,z} t_{ijk} \sigma_i \otimes \sigma_j \otimes \sigma_k] \end{aligned} \quad (3.2.1)$$

where

$$\begin{aligned}
l_i &= \text{Tr}(\rho(\sigma_i \otimes I \otimes I)), \\
m_i &= \text{Tr}(\rho(I \otimes \sigma_i \otimes I)) \\
n_i &= \text{Tr}(\rho(I \otimes I \otimes \sigma_i)), \\
u_i v_i &= \text{Tr}(\rho(\sigma_i \otimes \sigma_i \otimes I)) \\
u_i w_i &= \text{Tr}(\rho(\sigma_i \otimes I \otimes \sigma_i)), \\
v_i w_i &= \text{Tr}(\rho(I \otimes \sigma_i \otimes \sigma_i)), (i = x, y, z)
\end{aligned} \tag{3.2.2}$$

The correlation coefficient of the state ρ can be obtained as

$$t_{ijk} = \text{Tr}(\rho(\sigma_i \otimes \sigma_j \otimes \sigma_k)), (i, j, k = x, y, z) \tag{3.2.3}$$

Then the correlation tensor \vec{T} can be defined as $\vec{T} = (T_x, T_y, T_z)$, where

$$T_x = \begin{pmatrix} t_{xxx} & t_{xyx} & t_{xzx} \\ t_{xxy} & t_{xyy} & t_{xzy} \\ t_{xxz} & t_{xyz} & t_{xzz} \end{pmatrix} \tag{3.2.4}$$

and

$$T_y = \begin{pmatrix} t_{yxx} & t_{yyx} & t_{yzx} \\ t_{yxy} & t_{yyy} & t_{yzy} \\ t_{yxz} & t_{yyz} & t_{yzz} \end{pmatrix} \tag{3.2.5}$$

and

$$T_z = \begin{pmatrix} t_{zxx} & t_{zyx} & t_{zzx} \\ t_{zxy} & t_{zyy} & t_{zzy} \\ t_{zxz} & t_{zyz} & t_{zzz} \end{pmatrix} \tag{3.2.6}$$

3.2.1 Correlation tensor for the canonical form of three-qubit pure state

Let us consider a three-qubit pure state described by the density operator ρ_ψ

$$\rho_\psi = |\psi\rangle_{ABC}\langle\psi| \tag{3.2.7}$$

where $|\psi\rangle_{ABC}$ is given by (3.1.1).

The components T_x, T_y and T_z of the correlation tensor \vec{T} for the state ρ_ψ is given by

$$T_x = \begin{pmatrix} 2\lambda_0\lambda_4 & 0 & 2\lambda_0\lambda_2 \\ 0 & -2\lambda_0\lambda_4 & 0 \\ 2\lambda_0\lambda_3 & 0 & 2\lambda_0\lambda_1\cos\theta \end{pmatrix} \quad (3.2.8)$$

$$T_y = \begin{pmatrix} 0 & -2\lambda_0\lambda_4 & 0 \\ -2\lambda_0\lambda_4 & 0 & -2\lambda_0\lambda_2 \\ 0 & -2\lambda_0\lambda_3 & 2\lambda_0\lambda_1\sin\theta \end{pmatrix} \quad (3.2.9)$$

$$T_z = \begin{pmatrix} t_{zxx} & t_{zyx} & t_{zzx} \\ t_{zxy} & t_{zyy} & t_{zzy} \\ t_{zxx} & t_{zyz} & t_{zzz} \end{pmatrix} \quad (3.2.10)$$

where $t_{zxx} = -2(\lambda_2\lambda_3 + \lambda_1\lambda_4\cos\theta)$, $t_{zyx} = 2\lambda_1\lambda_4\sin\theta$, $t_{zzx} = 2(\lambda_3\lambda_4 - \lambda_1\lambda_2\cos\theta)$, $t_{zxy} = 2\lambda_1\lambda_4\sin\theta$, $t_{zyy} = 2(\lambda_1\lambda_4\cos\theta - \lambda_2\lambda_3)$, $t_{zzy} = 2\lambda_1\lambda_2\sin\theta$, $t_{zxx} = 2(\lambda_2\lambda_4 - \lambda_1\lambda_3\cos\theta)$, $t_{zyz} = 2\lambda_1\lambda_3\sin\theta$, $t_{zzz} = \lambda_0^2 - \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_4^2$.

The Hermitian matrices can be constructed from T_x and T_y as

$$T_x^T T_x = \begin{pmatrix} a_x & 0 & b_x \\ 0 & c_x & 0 \\ b_x & 0 & d_x \end{pmatrix} \quad (3.2.11)$$

where $a_x = 4\lambda_0^2(\lambda_4^2 + \lambda_3^2)$, $b_x = 4\lambda_0^2(\lambda_2\lambda_4 + \lambda_1\lambda_3\cos\theta)$, $c_x = 4\lambda_0^2\lambda_4^2$, $d_x = 4\lambda_0^2(\lambda_2^2 + \lambda_1^2\cos^2\theta)$.
and

$$T_y^T T_y = \begin{pmatrix} a_y & 0 & b_y \\ 0 & c_y & d_y \\ b_y & d_y & e_y \end{pmatrix} \quad (3.2.12)$$

where $a_y = 4\lambda_0^2\lambda_4^2$, $b_y = 4\lambda_0^2\lambda_2\lambda_4$, $c_y = 4\lambda_0^2(\lambda_4^2 + \lambda_3^2)$, $d_y = -4\lambda_0^2\lambda_1\lambda_3\sin\theta$, $e_y = 4\lambda_0^2(\lambda_2^2 + \lambda_1^2\sin^2\theta)$.

The superscript T refers to the simple matrix transposition operation.

3.2.2 Inequality for the construction of classification witness operator

Let us recall the canonical form of a three-qubit state $|\psi\rangle_{ABC}$ given in (3.1.1). The invariants with respect to the state $|\psi\rangle_{ABC}$ under local unitary transformations are given by [132, 134]

$$\begin{aligned}\lambda_0\lambda_4 &= \frac{\sqrt{\tau_\psi}}{2} \\ \lambda_0\lambda_2 &= \frac{C_{AC}}{2} \\ \lambda_0\lambda_3 &= \frac{C_{AB}}{2} \\ |\lambda_2\lambda_3 - e^{i\varphi}\lambda_1\lambda_4| &= \frac{C_{BC}}{2}\end{aligned}\quad (3.2.13)$$

Here τ_ψ denote the three-tangle of the state $|\psi\rangle_{ABC}$ whereas C_{AB} , C_{AC} and C_{BC} represent the partial concurrences between the pairs (A, B) , (A, C) and (B, C) respectively. Furthermore, the other invariants of three-qubit states under local unitary transformations have been studied in [135] and the invariants are given by

$$\begin{aligned}I_1 &= \langle \psi | \psi \rangle_{ABC} \\ I_2 &= \text{tr}(\rho_C^2) = 2(\lambda_1\lambda_2 + \lambda_3\lambda_4)^2 \\ I_3 &= \text{tr}(\rho_B^2) = 2(\lambda_1\lambda_3 + \lambda_2\lambda_4)^2 \\ I_4 &= \text{tr}(\rho_A^2) = 2\lambda_0^2\lambda_1^2 \\ I_5 &= \frac{1}{4}\tau_\psi^2 = 4\lambda_0^4\lambda_4^4\end{aligned}\quad (3.2.14)$$

where $\rho_A = \text{Tr}_{BC}(|\psi\rangle_{ABC}\langle\psi|)$, $\rho_B = \text{Tr}_{AC}(|\psi\rangle_{ABC}\langle\psi|)$, $\rho_C = \text{Tr}_{AB}(|\psi\rangle_{ABC}\langle\psi|)$ denote reduced density matrices of a single qubit.

Further, recalling the Hermitian matrices $T_x^T T_x$ and $T_y^T T_y$ from (3.2.11) and (3.2.12), we calculate the traces of the Hermitian matrices as

$$\text{Tr}(T_x^T T_x) = 8\lambda_0^2\lambda_4^2 + 4\lambda_0^2\lambda_3^2 + 4\lambda_0^2\lambda_2^2 + 4\lambda_0^2\lambda_1^2\cos^2\theta \quad (3.2.15)$$

and

$$\text{Tr}(T_y^T T_y) = 8\lambda_0^2\lambda_4^2 + 4\lambda_0^2\lambda_3^2 + 4\lambda_0^2\lambda_2^2 + 4\lambda_0^2\lambda_1^2\sin^2\theta \quad (3.2.16)$$

Adding (3.2.15) and (3.2.16), we get

$$\text{Tr}[(T_x^T T_x) + (T_y^T T_y)] = 16\lambda_0^2 \lambda_4^2 + 8\lambda_0^2 \lambda_3^2 + 8\lambda_0^2 \lambda_2^2 + 4\lambda_0^2 \lambda_1^2 \quad (3.2.17)$$

The expression for $\text{Tr}[(T_x^T T_x) + (T_y^T T_y)]$ can be re-expressed in terms of invariants such as three-tangle and partial concurrences as

$$\text{Tr}[(T_x^T T_x) + (T_y^T T_y)] = 4\tau_\psi + 2C_{AB}^2 + 2C_{AC}^2 + 4\lambda_0^2 \lambda_1^2 \quad (3.2.18)$$

In terms of expectation of the operators, the expression (3.2.18) can further be written as

$$\begin{aligned} \text{Tr}[(T_x^T T_x) + (T_y^T T_y)] &= (\langle O_1 \rangle_{\psi_{ABC}})^2 + \frac{1}{2}(\langle O_3 \rangle_{\psi_{ABC}})^2 \\ &+ \frac{1}{2}(\langle O_2 \rangle_{\psi_{ABC}})^2 + 4\lambda_0^2 \lambda_1^2 \end{aligned} \quad (3.2.19)$$

$$\begin{aligned} \text{Tr}[(T_x^T T_x) + (T_y^T T_y)] &\leq [\langle O_1 \rangle_{\psi_{ABC}} + \frac{1}{\sqrt{2}}\langle O_3 \rangle_{\psi_{ABC}} \\ &+ \frac{1}{\sqrt{2}}\langle O_2 \rangle_{\psi_{ABC}} + 2\lambda_0 \lambda_1]^2 \end{aligned} \quad (3.2.20)$$

where

$$\begin{aligned} O_1 &= 2(\sigma_x \otimes \sigma_x \otimes \sigma_x) \\ O_2 &= 2(\sigma_x \otimes \sigma_z \otimes \sigma_x) \\ O_3 &= 2(\sigma_x \otimes \sigma_x \otimes \sigma_z) \end{aligned} \quad (3.2.21)$$

The expectation values of the operator O_1 , O_2 , O_3 may be written in terms of invariants as [96],

$$\begin{aligned} \langle O_1 \rangle &= 4\lambda_0 \lambda_4 = 2\sqrt{\tau_\psi} \\ \langle O_2 \rangle &= 4\lambda_0 \lambda_2 = \frac{C_{AC}}{2} \\ \langle O_3 \rangle &= 4\lambda_0 \lambda_3 = \frac{C_{AB}}{2} \end{aligned} \quad (3.2.22)$$

Using (3.2.20), the upper bound (U) and the lower bound (L) of $\text{Tr}[(T_x^T T_x) + (T_y^T T_y)]$ is

given by

$$L \leq Tr[(T_x^T T_x) + (T_y^T T_y)] \leq U \quad (3.2.23)$$

where $L = \mu_{max}(T_x^T T_x) + \mu_{min}(T_y^T T_y)$ and $U = (4\lambda_0\lambda_4 + 2\sqrt{2}\lambda_0\lambda_3 + 2\sqrt{2}\lambda_0\lambda_2 + 2\lambda_0\lambda_1)^2$.

The lower bound L can be obtained using Weyl's result [136]. $\mu_{max}(T_x^T T_x)$ and $\mu_{min}(T_y^T T_y)$ denote the maximum and minimum eigenvalue of $T_x^T T_x$ and $T_y^T T_y$ respectively.

Thus, equation (3.2.23) can be re-written as

$$\begin{aligned} & [\mu_{max}(T_x^T T_x) + \mu_{min}(T_y^T T_y)]^{\frac{1}{2}} \leq \\ & 4\lambda_0\lambda_4 + 2\sqrt{2}\lambda_0\lambda_3 + 2\sqrt{2}\lambda_0\lambda_2 + 2\lambda_0\lambda_1 \end{aligned} \quad (3.2.24)$$

If $0 \leq \mu_{max}(T_x^T T_x) + \mu_{min}(T_y^T T_y) \leq 1$ then the inequality (3.2.24) reduces to

$$\begin{aligned} & \mu_{max}(T_x^T T_x) + \mu_{min}(T_y^T T_y) \leq \\ & 4\lambda_0\lambda_4 + 2\sqrt{2}\lambda_0\lambda_3 + 2\sqrt{2}\lambda_0\lambda_2 + 2\lambda_0\lambda_1 \end{aligned} \quad (3.2.25)$$

The derived inequality (3.2.25) will be useful in constructing the Hermitian operators for the classification of states that lies within the subclasses of GHZ class.

3.3 Construction of classification witness operator

Let $|\chi\rangle \in S_1$ and $|\omega\rangle$ be any state belong to either S_2 or S_3 or S_4 . Then, the Hermitian operator H is said to be the classification witness operator if

$$\begin{aligned} (a) & Tr(H|\chi\rangle\langle\chi|) \geq 0, \forall |\chi\rangle \in S_1 \\ (b) & Tr(H|\omega\rangle\langle\omega|) < 0, \text{ for at least one } |\omega\rangle \in S_i, \quad (i = 2, 3, 4) \end{aligned} \quad (3.3.1)$$

If the above condition holds then the classification witness operator H classifies the states between (i) subclass-I and subclass-II (ii) subclass-I and subclass-III (iii) subclass-I and subclass-IV.

3.3.1 Classification witness operator for the classification of states contained in subclass-I and subclass-II

We are now in a position to construct the classification witness operator that can classify the states residing in subclass-I and subclass-II.

(I) Classification of states confined in subclass-II with state parameters λ_0 , λ_1 and λ_4 and subclass-I:

The GHZ class of state within subclass-II with state parameters λ_0 , λ_1 , and λ_4 is given by

$$|\psi_{\lambda_1}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_4|111\rangle \quad (3.3.2)$$

with the normalization condition $\lambda_0^2 + \lambda_1^2 + \lambda_4^2 = 1$.

In particular, for $\lambda_1 = 0$, the state $|\psi_{\lambda_1}\rangle$ reduces to $|\psi_{\lambda_1=0}\rangle \in S_1$ where

$$|\psi_{\lambda_1=0}\rangle = \lambda_0|000\rangle + \lambda_4|111\rangle, \lambda_0^2 + \lambda_4^2 = 1 \quad (3.3.3)$$

The Hermitian matrices $T_x^T T_x$ and $T_y^T T_y$ for the state $\rho_{\lambda_1} = |\psi_{\lambda_1}\rangle\langle\psi_{\lambda_1}|$ is given by

$$T_x^T T_x = \begin{pmatrix} 4\lambda_0^2\lambda_4^2 & 0 & 0 \\ 0 & 4\lambda_0^2\lambda_4^2 & 0 \\ 0 & 0 & 4\lambda_0^2\lambda_1^2 \end{pmatrix} \quad (3.3.4)$$

$$T_y^T T_y = \begin{pmatrix} 4\lambda_0^2\lambda_4^2 & 0 & 0 \\ 0 & 4\lambda_0^2\lambda_4^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.3.5)$$

The expression for $Tr[(T_x^T T_x) + (T_y^T T_y)]$ is given by

$$Tr[(T_x^T T_x) + (T_y^T T_y)] = 16\lambda_0^2\lambda_4^2 + 4\lambda_0^2\lambda_1^2 \quad (3.3.6)$$

The maximum eigenvalue of $T_x^T T_x$ is given by

$$\mu_{max}(T_x^T T_x) = \max\{4\lambda_0^2\lambda_1^2, 4\lambda_0^2\lambda_4^2\} \quad (3.3.7)$$

The minimum eigenvalue of $T_y^T T_y$ is given by

$$\mu_{\min}(T_y^T T_y) = 0 \quad (3.3.8)$$

It can be easily observed that in this case $0 \leq \mu_{\max}(T_x^T T_x) + \mu_{\min}(T_y^T T_y) \leq 1$ holds.

Since $\mu_{\max}(T_x^T T_x)$ depends on the value of the two parameters λ_1 and λ_4 so we will investigate two cases independently.

Case-I: $\lambda_4 > \lambda_1$

If $\lambda_4 > \lambda_1$ then $\mu_{\max}(T_x^T T_x) = 4\lambda_0^2 \lambda_4^2$.

The inequality (3.2.25) then can be re-expressed in terms of the expectation value of the operators O_1 as

$$-2\lambda_0 \lambda_1 \leq \langle O_1 \rangle_{\psi_{\lambda_1}} - \frac{\langle O_1 \rangle_{\psi_{\lambda_1}}^2}{4} \quad (3.3.9)$$

If $\lambda_1 = 0$ then the R.H.S of the inequality (3.3.9) is always positive. Further, it can be observed that since $0 \leq \langle O_1 \rangle_{\psi_{\lambda_1}} \leq 1$ so the R.H.S of the inequality (3.3.9) still positive even for $\lambda_1 \neq 0$. Thus the R.H.S of the inequality is positive for every state belonging to S_2 . Hence, for $\lambda_4 > \lambda_1$, it is not possible to make a distinction between the class of states $|\psi_{\lambda_1}\rangle \in S_2$ and $|\psi_{\lambda_1=0}\rangle \in S_1$ using the inequality (3.3.9).

Case-II: $\lambda_4 < \lambda_1$

If $\lambda_4 < \lambda_1$ then $\mu_{\max}(T_x^T T_x) = 4\lambda_0^2 \lambda_1^2$.

The inequality (3.2.25) can be re-written as

$$-2\lambda_0 \lambda_1 \leq \langle O_1 \rangle_{\psi_{\lambda_1}} - 4\lambda_0^2 \lambda_1^2 \quad (3.3.10)$$

We can now define a Hermitian operator H_1 as

$$H_1 = O_1 - \frac{1}{4} \langle O_4 \rangle_{\psi_{\lambda_1}}^2 I \quad (3.3.11)$$

where,

$$O_4 = 2(\sigma_x \otimes I \otimes I) \quad (3.3.12)$$

The expectation value of the operator O_4 , in terms of invariants may be written as,

$$\langle O_4 \rangle = 4\lambda_0 \lambda_1 = 4\sqrt{\frac{I_4}{\sqrt{2}}} \quad (3.3.13)$$

Therefore, the inequality (3.3.10) can be re-formulated as

$$-2\lambda_0\lambda_1 \leq \langle H_1 \rangle_{\psi_{\lambda_1}} \quad (3.3.14)$$

If $\lambda_1 = 0$ then $\langle H_1 \rangle_{\psi_{\lambda_1}} \geq 0$ for all states $|\psi_{\lambda_1=0}\rangle \in S_1$.

For $\lambda_1 \neq 0$, we can calculate $\langle H_1 \rangle_{\psi_{\lambda_1}} = Tr(H_1\rho_{\lambda_1})$ which is given by

$$Tr(H_1\rho_{\lambda_1}) = 4\lambda_0(\lambda_4 - \lambda_0\lambda_1^2) \quad (3.3.15)$$

It can be easily shown that there exist state parameters $\lambda_0, \lambda_1, \lambda_4$ for which $\lambda_4 - \lambda_0\lambda_1^2 < 0$ and thus $Tr(H_1\rho_{\lambda_1}) < 0$. For instance, if we take $\lambda_0 = 0.4$, $\lambda_1 = 0.911043$ and $\lambda_4 = 0.1$, Then $Tr(H_1\rho_{\lambda_1}) = -0.3712$, which is negative.

Thus the Hermitian operator H_1 discriminate the class $|\psi_{\lambda_1}\rangle \in S_2$ from $|\psi_{\lambda_1=0}\rangle \in S_1$.

(II) Classification of states confined in subclass-II with state parameters $\lambda_0, \lambda_i (i = 2, 3)$ and λ_4 and subclass-I:

The GHZ class of state within subclass-II with state parameters $(\lambda_0, \lambda_2, \lambda_4)$ and $(\lambda_0, \lambda_3, \lambda_4)$ are given by

$$|\psi_{\lambda_2}\rangle = \lambda_0|000\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle \quad (3.3.16)$$

with $\lambda_0^2 + \lambda_2^2 + \lambda_4^2 = 1$ and

$$|\psi_{\lambda_3}\rangle = \lambda_0|000\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \quad (3.3.17)$$

with $\lambda_0^2 + \lambda_3^2 + \lambda_4^2 = 1$.

The Hermitian matrices $T_x^T T_x$ and $T_y^T T_y$ for the state $\rho_{\lambda_2} = |\psi_{\lambda_2}\rangle\langle\psi_{\lambda_2}|$ may be given by

$$T_x^T T_x = T_y^T T_y = \begin{pmatrix} 4\lambda_0^2\lambda_4^2 & 0 & 4\lambda_0^2\lambda_2\lambda_4 \\ 0 & 4\lambda_0^2\lambda_4^2 & 0 \\ 4\lambda_0^2\lambda_2\lambda_4 & 0 & 4\lambda_0^2\lambda_2^2 \end{pmatrix} \quad (3.3.18)$$

The Hermitian matrices $T_x^T T_x$ and $T_y^T T_y$ for the state $\rho_{\lambda_3} = |\psi_{\lambda_3}\rangle\langle\psi_{\lambda_3}|$ is given by

$$T_x^T T_x = \begin{pmatrix} 4\lambda_0^2(\lambda_3^2 + \lambda_4^2) & 0 & 0 \\ 0 & 4\lambda_0^2\lambda_4^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.3.19)$$

$$T_y^T T_y = \begin{pmatrix} 4\lambda_0^2\lambda_4^2 & 0 & 0 \\ 0 & 4\lambda_0^2(\lambda_3^2 + \lambda_4^2) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.3.20)$$

The maximum eigenvalue of $T_x^T T_x$ for the state $|\psi_{\lambda_i}\rangle (i = 2, 3)$ is given by

$$\mu_{max}(T_x^T T_x) = 4\lambda_0^2(\lambda_i^2 + \lambda_4^2), i = 2, 3 \quad (3.3.21)$$

The minimum eigenvalue of $T_y^T T_y$ for the state $|\psi_{\lambda_i}\rangle (i = 2, 3)$ is given by

$$\mu_{min}(T_y^T T_y) = 0 \quad (3.3.22)$$

For the state either described by the density operator $\rho_{\lambda_2} = |\psi_{\lambda_2}\rangle\langle\psi_{\lambda_2}|$ or $\rho_{\lambda_3} = |\psi_{\lambda_3}\rangle\langle\psi_{\lambda_3}|$, the expression of $Tr[(T_x^T T_x) + (T_y^T T_y)]$ is given by

$$Tr[(T_x^T T_x) + (T_y^T T_y)] = 16\lambda_0^2\lambda_4^2 + 8\lambda_0^2\lambda_i^2, \quad (i = 2, 3) \quad (3.3.23)$$

The inequality (3.2.25) then can be re-expressed in terms of the expectation value of the operators O_1 and O_4 as

$$-2\sqrt{2}\lambda_0\lambda_i \leq \langle O_1 \rangle_{\psi_{\lambda_i}} - \frac{1}{4}\langle O_1 \rangle_{\psi_{\lambda_i}}^2 - \frac{1}{4}\langle O_i \rangle_{\psi_{\lambda_i}}^2, (i = 2, 3) \quad (3.3.24)$$

We can now define classification witness operators $H_i, (i = 2, 3)$ as

$$H_i = O_1 - \frac{1}{4}[\langle O_i \rangle_{\psi_{\lambda_i}}^2 + \langle O_1 \rangle_{\psi_{\lambda_i}}^2]I \quad (3.3.25)$$

Therefore, the inequality (3.3.24) can be re-formulated as

$$-2\sqrt{2}\lambda_0\lambda_i \leq \langle H_i \rangle_{\psi_{\lambda_i}}, i = 2, 3 \quad (3.3.26)$$

If $\lambda_i = 0, (i = 2, 3)$ then $\langle H_i \rangle_{\psi_{\lambda_i}} \geq 0$ for all states $|\psi_{\lambda_i=0}\rangle (i = 2, 3) \in S_1$.

For $\lambda_i \neq 0, (i = 2, 3)$, we can calculate $Tr(H_i \rho_{\lambda_i})(i = 2, 3)$ which is given by

$$Tr(H_i \rho_{\lambda_i}) = 4\lambda_0 \lambda_4 (1 - \lambda_0 \lambda_4) - 4\lambda_0^2 \lambda_i^2, i = 2, 3 \quad (3.3.27)$$

It can be easily shown that there exist state parameters $\lambda_0, \lambda_i (i = 2, 3), \lambda_4$ for which $Tr(H_i \rho_{\lambda_i}) < 0$. For instance, if we take $\lambda_0 = 0.4, \lambda_i = 0.894427 (i = 2, 3)$ and $\lambda_4 = 0.2$, we get $Tr[H_i \rho_{\lambda_i}] = -0.2176$. Therefore, the classification witness operator $H_i (i = 2, 3)$ classify the class of states $\psi_{\lambda_i} (i = 2, 3) \in S_2$ given in (3.3.16) from the class $|\psi_{\lambda_i=0}\rangle (i = 2, 3) \in S_1$.

3.3.2 Classification witness operator for the classification of states contained in subclass-I and subclass-III

In this subsection, we will construct a classification witness operator to discriminate subclass-I from subclasses of GHZ class spanned by four basis states.

(I) Classification of states confined in subclass-III with state parameters $\lambda_0, \lambda_2, \lambda_3$ and λ_4 and subclass-I:

The GHZ class of state within subclass-III with state parameters $\lambda_0, \lambda_2, \lambda_3$, and λ_4 is given by

$$|\psi_{\lambda_2, \lambda_3}\rangle = \lambda_0 |000\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle \quad (3.3.28)$$

with $\lambda_0^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1$.

The Hermitian matrices $T_x^T T_x$ and $T_y^T T_y$ for the state $\rho_{\lambda_2, \lambda_3} = |\psi_{\lambda_2, \lambda_3}\rangle \langle \psi_{\lambda_2, \lambda_3}|$ is given by,

$$T_x^T T_x = \begin{pmatrix} 4\lambda_0^2(\lambda_3^2 + \lambda_4^2) & 0 & 4\lambda_0^2 \lambda_2 \lambda_4 \\ 0 & 4\lambda_0^2 \lambda_4^2 & 0 \\ 4\lambda_0^2 \lambda_2 \lambda_4 & 0 & 4\lambda_0^2 \lambda_2^2 \end{pmatrix} \quad (3.3.29)$$

$$T_y^T T_y = \begin{pmatrix} 4\lambda_0^2 \lambda_4^2 & 0 & 4\lambda_0^2 \lambda_2 \lambda_4 \\ 0 & 4\lambda_0^2(\lambda_3^2 + \lambda_4^2) & 0 \\ 4\lambda_0^2 \lambda_2 \lambda_4 & 0 & 4\lambda_0^2 \lambda_2^2 \end{pmatrix} \quad (3.3.30)$$

The maximum eigenvalue of $T_x^T T_x$ is given by

$$\mu_{\max}(T_x^T T_x) = \max\{u, v_1\} \quad (3.3.31)$$

where $u = 4\lambda_0^2\lambda_4^2$ and $v_1 = 2\lambda_0^2(1 - \lambda_0^2 + \sqrt{-4\lambda_2^2\lambda_3^2 + (1 - \lambda_0^2)^2})$.

The minimum eigenvalue of $T_y^T T_y$ is given by

$$\mu_{\min}(T_y^T T_y) = 0 \quad (3.3.32)$$

The expression for $Tr[(T_x^T T_x) + (T_y^T T_y)]$ is given by

$$Tr[(T_x^T T_x) + (T_y^T T_y)] = 16\lambda_0^2\lambda_4^2 + 8\lambda_0^2\lambda_2^2 + 8\lambda_0^2\lambda_3^2 \quad (3.3.33)$$

Case-I: If $\mu_{\max}(T_x^T T_x) = u = 4\lambda_0^2\lambda_4^2$ and $\mu_{\min}(T_y^T T_y) = 0$. The inequality (3.2.25) then can be re-expressed in terms of the expectation value of the operators O_1 as

$$-2\sqrt{2}\lambda_0(\lambda_2 + \lambda_3) \leq \langle O_1 \rangle_{\psi_{\lambda_2, \lambda_3}} - \frac{\langle O_1 \rangle_{\psi_{\lambda_2, \lambda_3}}^2}{4} \quad (3.3.34)$$

If $-2\sqrt{2}\lambda_0(\lambda_2 + \lambda_3) = 0$, then RHS of inequality (3.3.34) is always positive for every state $|\psi\rangle$. Thus, in this case, it is not possible to discriminate between the class of states $|\psi_{\lambda_2=0, \lambda_3=0}\rangle \in S_1$ and the class of states $|\psi_{\lambda_2, \lambda_3}\rangle \in S_3$.

Case-II: If $\mu_{\max}(T_x^T T_x) = v_1$ and $\mu_{\min}(T_y^T T_y) = 0$ then the inequality (3.2.25) reduces to

$$-2\sqrt{2}\lambda_0(\lambda_2 + \lambda_3) \leq \langle O_1 \rangle_{\psi_{\lambda_2, \lambda_3}} - P_1 \quad (3.3.35)$$

where,

$$\begin{aligned} P_1 &= 2\lambda_0^2(1 - \lambda_0^2 + \sqrt{-4\lambda_2^2\lambda_3^2 + (1 - \lambda_0^2)^2}) \\ &= 2\langle O_5 \rangle_{\psi_{\lambda_2, \lambda_3}}(1 - \langle O_5 \rangle_{\psi_{\lambda_2, \lambda_3}} \\ &\quad + \sqrt{-\frac{1}{4}\langle O_6 \rangle_{\psi_{\lambda_2, \lambda_3}}^2 + (1 - \langle O_5 \rangle_{\psi_{\lambda_2, \lambda_3}})^2}) \end{aligned} \quad (3.3.36)$$

where,

$$\begin{aligned} O_5 &= \frac{1}{8}(I + \sigma_z \otimes I + \sigma_z \otimes I + \sigma_z) \\ O_6 &= 2(I \otimes \sigma_y \otimes \sigma_y) \end{aligned} \quad (3.3.37)$$

The expectation values of the operators O_5 and O_6 , in terms of invariants may be written as,

$$\begin{aligned}\langle O_5 \rangle &= \lambda_0^2 = \frac{C_{AC}C_{AB}}{2C_{BC}} - \frac{1}{C_{BC}} \sqrt{\frac{2I_4 I_5}{\tau_\psi}} \\ \langle O_6 \rangle &= 4(\lambda_2 \lambda_3 - \lambda_1 \lambda_4) = 2C_{BC}\end{aligned}\quad (3.3.38)$$

We can now define a Hermitian operator H_5 as

$$H_4 = O_1 - P_1 I \quad (3.3.39)$$

Therefore, the inequality (3.3.35) can be re-formulated as

$$-2\sqrt{2}\lambda_0(\lambda_2 + \lambda_3) \leq \langle H_4 \rangle_{\psi_{\lambda_2, \lambda_3}} \quad (3.3.40)$$

If $\lambda_2 + \lambda_3 = 0$ then $\langle H_4 \rangle_{\psi_{\lambda_2, \lambda_3}} \geq 0$ for all states $|\psi_{\lambda_2=0, \lambda_3=0}\rangle$.

For $-2\sqrt{2}\lambda_0(\lambda_2 + \lambda_3) \neq 0$, we can calculate $Tr(H_4 \rho_{\lambda_2, \lambda_3})$, which is given by

$$Tr(H_4 \rho_{\lambda_2, \lambda_3}) = 4\lambda_0 \lambda_4 - 2\lambda_0^2 (1 - \lambda_0^2 + \sqrt{T_1}) \quad (3.3.41)$$

where,

$$T_1 = \lambda_2^4 - 2\lambda_2^2 \lambda_3^2 + 2\lambda_2^2 \lambda_4^2 + (\lambda_3^2 + \lambda_4^2)^2 \quad (3.3.42)$$

It can be easily shown that there exist state parameters $(\lambda_0, \lambda_2, \lambda_3, \lambda_4)$ for which $Tr(H_4 \rho_{\lambda_2, \lambda_3}) < 0$. For instance, if we take $\lambda_0 = 0.35$, $\lambda_2 = 0.3$, $\lambda_3 = 0.864581$ and $\lambda_4 = 0.2$, we get $Tr[H_4 \rho_{\lambda_2, \lambda_3}] = -0.108386$. Therefore, the classification operator H_4 classify the class of states $\rho_{\lambda_2, \lambda_3} \in S_3$ and the class of states $\rho_{\lambda_2=0, \lambda_3=0} \in S_1$.

(II) Classification of states confined in subclass-III with state parameters $\lambda_0, \lambda_1, \lambda_i (i = 2, 3)$ and λ_4 and subclass-I:

The GHZ class of state within subclass-III with state parameters $(\lambda_0, \lambda_1, \lambda_2, \lambda_4)$ and $(\lambda_0, \lambda_1, \lambda_3, \lambda_4)$ are given by

$$|\psi_{\lambda_1, \lambda_2}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle \quad (3.3.43)$$

with $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_4^2 = 1$.

$$|\psi_{\lambda_1, \lambda_3}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \quad (3.3.44)$$

with $\lambda_0^2 + \lambda_1^2 + \lambda_3^2 + \lambda_4^2 = 1$.

The Hermitian matrices $T_x^T T_x$ and $T_y^T T_y$ for the state $\rho_{\lambda_1, \lambda_2} = |\psi_{\lambda_1, \lambda_2}\rangle\langle\psi_{\lambda_1, \lambda_2}|$ is given by

$$T_x^T T_x = \begin{pmatrix} 4\lambda_0^2\lambda_4^2 & 0 & 4\lambda_0^2\lambda_2\lambda_4 \\ 0 & 4\lambda_0^2\lambda_4^2 & 0 \\ 4\lambda_0^2\lambda_2\lambda_4 & 0 & 4\lambda_0^2(\lambda_1^2 + \lambda_2^2) \end{pmatrix} \quad (3.3.45)$$

$$T_y^T T_y = \begin{pmatrix} 4\lambda_0^2\lambda_4^2 & 0 & 4\lambda_0^2\lambda_2\lambda_4 \\ 0 & 4\lambda_0^2\lambda_4^2 & 0 \\ 4\lambda_0^2\lambda_2\lambda_4 & 0 & 4\lambda_0^2\lambda_2^2 \end{pmatrix} \quad (3.3.46)$$

The Hermitian matrices $T_x^T T_x$ and $T_y^T T_y$ for the state $\rho_{\lambda_{1,3}} = |\psi_{\lambda_{1,3}}\rangle\langle\psi_{\lambda_{1,3}}|$ is given by

$$T_x^T T_x = \begin{pmatrix} 4\lambda_0^2(\lambda_3^2 + \lambda_4^2) & 0 & 4\lambda_0^2\lambda_1\lambda_3 \\ 0 & 4\lambda_0^2\lambda_4^2 & 0 \\ 4\lambda_0^2\lambda_1\lambda_3 & 0 & 4\lambda_0^2\lambda_1^2 \end{pmatrix} \quad (3.3.47)$$

$$T_y^T T_y = \begin{pmatrix} 4\lambda_0^2\lambda_4^2 & 0 & 0 \\ 0 & 4\lambda_0^2(\lambda_3^2 + \lambda_4^2) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.3.48)$$

The maximum eigenvalue of $T_x^T T_x$ is given by

$$\mu_{\max}(T_x^T T_x) = \max\{u, v_i\}, i = 2, 3 \quad (3.3.49)$$

where $v_i = 2\lambda_0^2(\lambda_1^2 + \lambda_i^2 + \lambda_4^2 + \sqrt{(\lambda_1^2 + \lambda_i^2 + \lambda_4^2)^2 - 4\lambda_1^2\lambda_4^2})$, $i=2,3$.

The minimum eigenvalue of $T_y^T T_y$ is given by

$$\mu_{\min}(T_y^T T_y) = 0 \quad (3.3.50)$$

The expression for $Tr[(T_x^T T_x) + (T_y^T T_y)]$ is given by

$$Tr[(T_x^T T_x) + (T_y^T T_y)] = 16\lambda_0^2\lambda_4^2 + 8\lambda_0^2\lambda_i^2 + 4\lambda_0^2\lambda_1^2, \quad (i = 2, 3) \quad (3.3.51)$$

Case-I: If $\mu_{max}(T_x^T T_x) = 4\lambda_0^2\lambda_4^2$ and $\mu_{min}(T_y^T T_y) = 0$. The inequality (3.2.25) then can be re-expressed in terms of the expectation value of the operators O_1 as

$$-2\lambda_0(\sqrt{2}\lambda_i + \lambda_1) \leq \langle O_1 \rangle_{\psi_{\lambda_1, \lambda_i}} - \frac{(\langle O_1 \rangle_{\psi_{\lambda_1, \lambda_i}})^2}{4}, \quad i = 2, 3 \quad (3.3.52)$$

If $\sqrt{2}\lambda_i + \lambda_1 = 0 (i = 2, 3)$, then RHS of inequality (3.3.52) is always positive. Thus, the R.H.S of the inequality is positive for every state $|\psi\rangle$. But since $0 \leq \langle O_1 \rangle_{\psi_{\lambda_1, \lambda_i}} \leq 1$ so the R.H.S of the inequality (3.3.52) still positive even for $\sqrt{2}\lambda_i + \lambda_1 \neq 0, (i = 2, 3)$. Thus it is not possible to differentiate between the class of states $|\psi_{\lambda_1=0, \lambda_i=0}\rangle \in S_1 (i = 2, 3)$ and $|\psi_{\lambda_1, \lambda_i}\rangle \in S_3 (i = 2, 3)$, using the inequality (3.3.52) for this case.

Case-II: If $\mu_{max}(T_x^T T_x) = 2\lambda_0^2(\lambda_1^2 + \lambda_i^2 + \lambda_4^2 + \sqrt{(\lambda_1^2 + \lambda_i^2 + \lambda_4^2)^2 - 4\lambda_1^2\lambda_4^2})$, $i=2,3$ and $\mu_{min}(T_y^T T_y) = 0$.

Then the inequality (3.2.25) can be re-written as

$$-2\lambda_0(\sqrt{2}\lambda_i + \lambda_1) \leq \langle O_1 \rangle_{\psi_{\lambda_1, \lambda_i}} - P_i, \quad i = 2, 3 \quad (3.3.53)$$

where, for $i=2,3$

$$P_i = 2\lambda_0^2(\lambda_1^2 + \lambda_i^2 + \lambda_4^2 + \sqrt{(\lambda_1^2 + \lambda_i^2 + \lambda_4^2)^2 - 4\lambda_1^2\lambda_4^2}) \quad (3.3.54)$$

$P_i (i = 2, 3)$ can also be re-expressed in terms of the expectation values of the operators O_1, O_4 and $O_i (i = 2, 3)$ as

$$P_i = q + \sqrt{q^2 - \frac{1}{16} \langle O_1 \rangle_{\psi_{\lambda_1, \lambda_i}}^2 \langle O_4 \rangle_{\psi_{\lambda_1, \lambda_i}}^2} \quad (3.3.55)$$

where $q = \frac{1}{8} [\langle O_1 \rangle_{\psi_{\lambda_1, \lambda_i}}^2 + \langle O_i \rangle_{\psi_{\lambda_1, \lambda_i}}^2 + \langle O_4 \rangle_{\psi_{\lambda_1, \lambda_i}}^2]$ We can now define an Hermitian operator $H_k, k = 5, 6$ as

$$H_k = O_1 - P_i I, \quad k = 5, 6, i = 2, 3, i = k \quad (3.3.56)$$

Therefore, the inequality (3.3.53) can be re-formulated as

$$-2\lambda_0(\sqrt{2}\lambda_i + \lambda_1) \leq \langle H_k \rangle_{\psi_{\lambda_1, \lambda_i}}, \quad i = 2, 3, k = 5, 6 \quad (3.3.57)$$

If $\sqrt{2}\lambda_i + \lambda_1 = 0, i = 2, 3$ then $\langle H_k \rangle_{\psi_{\lambda_1, \lambda_i}} \geq 0, k = 5, 6$ for all states $|\psi_{\lambda_1=0, \lambda_i=0}\rangle \in S_1 (i = 2, 3)$. For $\sqrt{2}\lambda_i + \lambda_1 \neq 0, i = 2, 3$, we can calculate $Tr(H_k \rho_{\lambda_1, \lambda_i}), (i = 2, 3, k = 5, 6)$, which is given by

$$Tr(H_k \rho_{\lambda_1, \lambda_i}) = 4\lambda_0\lambda_4 - 2\lambda_0^2(\lambda_1^2 + \lambda_i^2 + \lambda_4^2 + \sqrt{T_i}), i = 2, 3, k = 5, 6 \quad (3.3.58)$$

where

$$T_i = \lambda_1^4 + 2\lambda_1^2(\lambda_i^2 - \lambda_4^2) + (\lambda_i^2 + \lambda_4^2)^2, i = 2, 3 \quad (3.3.59)$$

It can be easily shown that there exist state parameters $(\lambda_0, \lambda_1, \lambda_i, \lambda_4), (i = 2, 3)$ for which $Tr(H_k \rho_{\lambda_1, \lambda_i}) < 0$ for $k=5,6$. For instance, if we take $\lambda_0 = 0.5, \lambda_1 = 0.83666, \lambda_i = 0.2, (i = 2, 3)$ and $\lambda_4 = 0.1$, we get $Tr[H_k \rho_{\lambda_1, \lambda_i}] = -0.540548, (k = 5, 6)$. Thus, the Hermitian operator $H_k, k=\{5,6\}$ serves as a classification witness operator and classify the class of states described by the density operator $\rho_{\lambda_1, \lambda_i}, (i = 2, 3) \in S_3$ and the class of states $\rho_{\lambda_1=0, \lambda_i=0}, (i = 2, 3) \in S_1$.

3.3.3 Classification of states confined in subclass-IV with state parameters $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and subclass-I

The GHZ class of state within subclass-IV with state parameters $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is given by

$$|\psi_{\lambda_1, \lambda_2, \lambda_3}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \quad (3.3.60)$$

with $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1$.

The Hermitian matrices $T_x^T T_x$ and $T_y^T T_y$ for the state $\rho_{\lambda_1, \lambda_2, \lambda_3} = |\psi_{\lambda_1, \lambda_2, \lambda_3}\rangle \langle \psi_{\lambda_1, \lambda_2, \lambda_3}|$ is given by

$$T_x^T T_x = \begin{pmatrix} 4\lambda_0^2(\lambda_3^2 + \lambda_4^2) & 0 & 4\lambda_0^2(\lambda_1\lambda_3 + \lambda_2\lambda_4) \\ 0 & 4\lambda_0^2\lambda_4^2 & 0 \\ 4\lambda_0^2(\lambda_1\lambda_3 + \lambda_2\lambda_4) & 0 & 4\lambda_0^2(\lambda_1^2 + \lambda_2^2) \end{pmatrix} \quad (3.3.61)$$

$$T_y^T T_y = \begin{pmatrix} 4\lambda_0^2\lambda_4^2 & 0 & 4\lambda_0^2\lambda_2\lambda_4 \\ 0 & 4\lambda_0^2(\lambda_3^2 + \lambda_4^2) & 0 \\ 4\lambda_0^2\lambda_2\lambda_4 & 0 & 4\lambda_0^2\lambda_2^2 \end{pmatrix} \quad (3.3.62)$$

The maximum eigenvalue of $T_x^T T_x$ is given by

$$\mu_{max}(T_x^T T_x) = \max\{u, v_4\} \quad (3.3.63)$$

where $v_4 = 2\lambda_0^2(k + \sqrt{-4(\lambda_2\lambda_3 - \lambda_1\lambda_4)^2 + k^2})$, $k = 1 - \lambda_0^2$.

The minimum eigenvalue of $T_y^T T_y$ is given by

$$\mu_{min}(T_y^T T_y) = 0 \quad (3.3.64)$$

The expression for $Tr[(T_x^T T_x) + (T_y^T T_y)]$ is given by

$$Tr[(T_x^T T_x) + (T_y^T T_y)] = 16\lambda_0^2\lambda_4^2 + 8\lambda_0^2\lambda_3^2 + 8\lambda_0^2\lambda_2^2 + 4\lambda_0^2\lambda_1^2 \quad (3.3.65)$$

Case-I: If $\mu_{max}(T_x^T T_x) = u = 4\lambda_0^2\lambda_4^2$ and $\mu_{min}(T_y^T T_y) = 0$. The inequality (3.2.25) then can be re-expressed in terms of the expectation value of the operators O_1 as

$$-2\lambda_0(\sqrt{2}\lambda_2 + \sqrt{2}\lambda_3 + \lambda_1) \leq \langle O_1 \rangle_{\psi_{\lambda_1, \lambda_2, \lambda_3}} - \frac{\langle O_1 \rangle_{\psi_{\lambda_1, \lambda_2, \lambda_3}}^2}{4} \quad (3.3.66)$$

If $\sqrt{2}\lambda_2 + \sqrt{2}\lambda_3 + \lambda_1 = 0$, then RHS of inequality (3.3.66) is always positive irrespective of the values of the state parameter $(\lambda_1, \lambda_2, \lambda_3)$. Thus it is not possible to differentiate between the class of states $|\psi_{\lambda_1, \lambda_2, \lambda_3}\rangle \in S_4$ and the class of states $|\psi_{\lambda_1=0, \lambda_2=0, \lambda_3=0}\rangle \in S_1$.

Case-II: $\mu_{max}(T_x^T T_x) = 2\lambda_0^2(1 - \lambda_0^2 + \sqrt{-4(\lambda_2\lambda_3 - \lambda_1\lambda_4)^2 + (1 - \lambda_0^2)^2})$ and $\mu_{min}(T_y^T T_y) = 0$. Then the inequality (3.2.25) can be re-written as

$$-2\lambda_0(\sqrt{2}\lambda_2 + \sqrt{2}\lambda_3 + \lambda_1) \leq \langle O_1 \rangle_{\psi_{\lambda_1, \lambda_2, \lambda_3}} - P_4 \quad (3.3.67)$$

where,

$$\begin{aligned}
P_4 &= 2\lambda_0^2(1 - \lambda_0^2 + \sqrt{-4(\lambda_2\lambda_3 - \lambda_1\lambda_4)^2 + (1 - \lambda_0^2)^2}) \\
&= 2\langle O_5 \rangle_{\psi_{\lambda_1, \lambda_2, \lambda_3}} (1 - \langle O_5 \rangle_{\psi_{\lambda_1, \lambda_2, \lambda_3}} + \sqrt{-\frac{\langle O_6 \rangle_{\psi_{\lambda_1, \lambda_2, \lambda_3}}^2}{4} + (1 - \langle O_5 \rangle_{\psi_{\lambda_1, \lambda_2, \lambda_3}})^2})^2 \quad (3.3.68)
\end{aligned}$$

We can now define a Hermitian operator H_7 as

$$H_7 = O_1 - P_4 I \quad (3.3.69)$$

Therefore, the inequality (3.3.67) can be re-formulated as

$$-2\lambda_0(\sqrt{2}\lambda_2 + \sqrt{2}\lambda_3 + \lambda_1) \leq \langle H_7 \rangle_{\psi_{\lambda_1, \lambda_2, \lambda_3}} \quad (3.3.70)$$

If $-2\lambda_0(\sqrt{2}\lambda_2 + \sqrt{2}\lambda_3 + \lambda_1) = 0$ then $\langle H_7 \rangle_{\psi_{\lambda_1, \lambda_2, \lambda_3}} \geq 0$ for all states $|\psi_{\lambda_1=0, \lambda_2=0, \lambda_3=0}\rangle \in S_1$. For $-2\lambda_0(\sqrt{2}\lambda_2 + \sqrt{2}\lambda_3 + \lambda_1) \neq 0$, we can calculate $Tr(H_7\rho_{\lambda_1, \lambda_2, \lambda_3})$ which is given by

$$\begin{aligned}
Tr(H_7\rho_{\lambda_1, \lambda_2, \lambda_3}) &= 4\lambda_0\lambda_4 - 2\lambda_0^2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \\
&\quad + \sqrt{T_4} \quad (3.3.71)
\end{aligned}$$

where,

$$\begin{aligned}
T_4 &= \lambda_1^4 + \lambda_2^4 + \lambda_3^4 + \lambda_4^4 + 8\lambda_1\lambda_2\lambda_3\lambda_4 - 2\lambda_2^2\lambda_3^2 + \\
&\quad 2\lambda_2^2\lambda_4^2 + 2\lambda_1^2\lambda_2^2 + 2\lambda_1^2\lambda_3^2 - 2\lambda_1^2\lambda_4^2 \\
&\quad + 2\lambda_3^2\lambda_4^2 \quad (3.3.72)
\end{aligned}$$

It can be easily shown that there exist state parameters $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for which $Tr(H_7\rho_{\lambda_1, \lambda_2, \lambda_3}) < 0$. For instance, if we take $\lambda_0 = 0.6$, $\lambda_1 = 0.785812$, $\lambda_2 = 0.1$, $\lambda_3 = 0.05$ and $\lambda_4 = 0.1$, we get $Tr[H_7\rho_{\lambda_{123}}] = -0.303798$. Thus, the classification witness operator H_7 classify the class of states described by the density operator $\rho_{\lambda_1, \lambda_2, \lambda_3} \in S_4$ and the class of states described by $\rho_{\lambda_1=0, \lambda_2=0, \lambda_3=0} \in S_1$

3.4 Examples

In this section, we have provided a few examples of three-qubit states for which we construct classification witness operators.

Example-1: The three-qubit maximal slice state is given by [137],

$$|MS\rangle = \frac{1}{\sqrt{2}}(|000\rangle + c|110\rangle + d|111\rangle), \quad c^2 + d^2 = 1 \quad (3.4.1)$$

Let us consider the classification witness operators H_1 , H_2 and H_3 . The classification witness operator H_3 for (3.4.1) is now reduces to

$$W_{MS} = O_1 - I \quad (3.4.2)$$

The expectation value of W_{MS} with respect to the state $|MS\rangle$ can be evaluated as

$$Tr(W_{MS}\rho_{MS}) = (2d - 1) \quad (3.4.3)$$

Therefore, we can verify that $Tr(W_{MS}|MS\rangle\langle MS|) < 0$ for the state parameter $d < \frac{1}{2}$. Further, it is easy to verify that the expectation value of the witness operator H_3 is positive for all states belonging to subclass-I. Since the given state is detected by the classification witness operator H_3 so state (3.4.1) belongs to subclass-II. To investigate the form of the given state lying within subclass II, we need to further classify it from the other classes of states belonging to subclass II. We can check that in the same range of the state parameter d i.e. for $d < \frac{1}{2}$, the value of $Tr(H_1\rho_{MS})$ and $Tr(H_2\rho_{MS})$ are non-negative. Thus, we can say that the classification witness operators H_3 discriminate the maximal slice state from subclass-I, and also it detects the state in the form (3.3.17).

Example-2: Let us consider another three-qubit state defined as

$$|\phi\rangle = \sqrt{p}|G\rangle - \sqrt{1-p}|K\rangle \quad (3.4.4)$$

where,

$$\begin{aligned} |G\rangle &= a|000\rangle + b|111\rangle, \quad a^2 + b^2 = 1 \\ |K\rangle &= c|110\rangle + d|101\rangle, \quad c^2 + d^2 = 1 \end{aligned} \quad (3.4.5)$$

Now our task is to construct a classification witness operator that may distinguish it from the state belonging to subclass-I and also detect the form of the given state that belongs to a particular class within subclass-III. To accomplish our task, let us consider classification witness operators H_4 , H_5 and H_6 given in (3.3.39) and (3.3.56).

Table 3.1: Range of the parameter p for which the classification witness operator H_4 classifies the given GHZ state within the subclass-III

State parameter (a, c)	p	$Tr[H_5\rho]$	$Tr[H_4\rho]$	$Tr[H_6\rho]$
(0.8, 0.3)	(.291,.3)	> 0	< 0	> 0
(0.9, 0.4)	(.548,.57)	> 0	< 0	> 0
(0.91, 0.8)	(.4,.51)	> 0	< 0	> 0
(0.85, 0.35)	(.43,.45)	> 0	< 0	> 0
(0.88, 0.8)	(.25,.385)	> 0	< 0	> 0
(0.78, 0.3)	(.208,.22)	> 0	< 0	> 0
(0.95, 0.4)	(.69,.7)	> 0	< 0	> 0
(0.83, 0.45)	(.26,.31)	> 0	< 0	> 0

We find that the expectation value of the witness operator H_4 is positive for all states belonging to subclass-I but it gives a negative value for some states belonging to subclass-III. Hence the state (3.4.4) belongs to subclass-III. Moreover, we have investigated this classification problem within subclass-III by constructing a table below. It shows that the expectation value of classification witness operator H_4 is negative for some range of the state parameter p while the expectation value of other classification witness operators H_5 and H_6 gives positive values for the same range of the state parameters. This means that the given state (3.4.4) belongs to subclass-III and it takes the form (3.3.28). In Table 3.1, we have found the range of p where the witness operator H_4 detects the GHZ state given in the example whereas H_5 and H_6 do not detect the given GHZ state.

3.5 Classification witness operator for the classification of states contained in subclass-II and subclass-III

The GHZ class of state within subclass-III with state parameters $(\lambda_0, \lambda_1, \lambda_2, \lambda_4)$ is given by

$$|\psi_{\lambda_1, \lambda_2}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle \quad (3.5.1)$$

with $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_4^2 = 1$.

The Hermitian matrices $T_x^T T_x$ and $T_y^T T_y$ for the state $\rho_{\lambda_1, \lambda_2} = |\psi_{\lambda_1, \lambda_2}\rangle\langle\psi_{\lambda_1, \lambda_2}|$ is given by

$$T_x^T T_x = \begin{pmatrix} 4\lambda_0^2 \lambda_4^2 & 0 & 4\lambda_0^2 \lambda_2 \lambda_4 \\ 0 & 4\lambda_0^2 \lambda_4^2 & 0 \\ 4\lambda_0^2 \lambda_2 \lambda_4 & 0 & 4\lambda_0^2 (\lambda_1^2 + \lambda_2^2) \end{pmatrix} \quad (3.5.2)$$

$$T_y^T T_y = \begin{pmatrix} 4\lambda_0^2 \lambda_4^2 & 0 & 4\lambda_0^2 \lambda_2 \lambda_4 \\ 0 & 4\lambda_0^2 \lambda_4^2 & 0 \\ 4\lambda_0^2 \lambda_2 \lambda_4 & 0 & 4\lambda_0^2 \lambda_2^2 \end{pmatrix} \quad (3.5.3)$$

The Hermitian matrices $T_x^T T_x$ and $T_y^T T_y$ for the state $\rho_{\lambda_{1,3}} = |\psi_{\lambda_{1,3}}\rangle\langle\psi_{\lambda_{1,3}}|$ is given by

$$T_x^T T_x = \begin{pmatrix} 4\lambda_0^2 (\lambda_3^2 + \lambda_4^2) & 0 & 4\lambda_0^2 \lambda_1 \lambda_3 \\ 0 & 4\lambda_0^2 \lambda_4^2 & 0 \\ 4\lambda_0^2 \lambda_1 \lambda_3 & 0 & 4\lambda_0^2 \lambda_1^2 \end{pmatrix} \quad (3.5.4)$$

$$T_y^T T_y = \begin{pmatrix} 4\lambda_0^2 \lambda_4^2 & 0 & 0 \\ 0 & 4\lambda_0^2 (\lambda_3^2 + \lambda_4^2) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.5.5)$$

The maximum eigenvalue of $T_x^T T_x$ is given by

$$\mu_{\max}(T_x^T T_x) = \max\{u, v_i\}, i = 2, 3 \quad (3.5.6)$$

where $v_i = 2\lambda_0^2 (\lambda_1^2 + \lambda_i^2 + \lambda_4^2 + \sqrt{(\lambda_1^2 + \lambda_i^2 + \lambda_4^2)^2 - 4\lambda_1^2 \lambda_4^2})$, $i=2,3$.

The minimum eigenvalue of $T_y^T T_y$ is given by

$$\mu_{\min}(T_y^T T_y) = 0 \quad (3.5.7)$$

The expression for $Tr[(T_x^T T_x) + (T_y^T T_y)]$ is given by

$$Tr[(T_x^T T_x) + (T_y^T T_y)] = 16\lambda_0^2 \lambda_4^2 + 8\lambda_0^2 \lambda_2^2 + 4\lambda_0^2 \lambda_1^2 \quad (3.5.8)$$

Case-I: If $\mu_{\max}(T_x^T T_x) = 4\lambda_0^2 \lambda_4^2$ and $\mu_{\min}(T_y^T T_y) = 0$. The inequality (3.2.25) then can

be re-expressed in terms of the expectation value of the operators O_1 as

$$-2\lambda_0(\sqrt{2}\lambda_2 + \lambda_1) \leq \langle O_1 \rangle_{\psi_{\lambda_1, \lambda_2}} - \frac{\langle O_1 \rangle_{\psi_{\lambda_1, \lambda_2}}^2}{4} \quad (3.5.9)$$

If $\lambda_2 = 0$, then above inequality becomes,

$$-2\lambda_0\lambda_1 \leq \langle O_1 \rangle_{\psi_{\lambda_1, \lambda_2}} - \frac{\langle O_1 \rangle_{\psi_{\lambda_1, \lambda_2}}^2}{4} \quad (3.5.10)$$

The R.H.S of the inequality is positive for every state $|\psi\rangle$. Thus it is not possible to differentiate between the class of states $|\psi_{\lambda_1, \lambda_2=0}\rangle \in S_2$ and $|\psi_{\lambda_1, \lambda_2}\rangle \in S_3$, using the inequality (3.5.10) for this case.

Case-II: If $\mu_{\max}(T_x^T T_x) = 2\lambda_0^2(\lambda_1^2 + \lambda_2^2 + \lambda_4^2 + \sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_4^2)^2 - 4\lambda_1^2\lambda_4^2})$ and $\mu_{\min}(T_y^T T_y) = 0$, then the inequality (3.2.25) can be re-written as

$$-2\lambda_0(\sqrt{2}\lambda_2 + \lambda_1) \leq \langle O_1 \rangle_{\psi_{\lambda_1, \lambda_i}} - P_2 \quad (3.5.11)$$

We can now define a Hermitian operator H_8 , as

$$H_8 = O_1 - P_2 I + \frac{O_4}{2} \quad (3.5.12)$$

Therefore, the inequality (3.5.11) can be re-formulated as

$$-2\sqrt{2}\lambda_0\lambda_2 \leq \langle H_8 \rangle_{\psi_{\lambda_1, \lambda_2}} \quad (3.5.13)$$

If $\lambda_2 = 0$ then $\langle H_8 \rangle_{\psi_{\lambda_1, \lambda_2}} \geq 0$, for all states $|\psi_{\lambda_1=0, \lambda_2=0}\rangle \in S_1$ and $|\psi_{\lambda_1, \lambda_2=0}\rangle \in S_2$.

For $\lambda_2 \neq 0$, then there exist state parameters $(\lambda_0, \lambda_1, \lambda_2, \lambda_4)$ for which $Tr(H_8 \rho_{\lambda_1, \lambda_2}) < 0$. For instance, if we take $\lambda_0 = 0.01$, $\lambda_1 = 0.948631$, $\lambda_2 = 0.3$ and $\lambda_4 = 0.1$, we get $Tr[H_8 \rho_{\lambda_1, \lambda_2}] = -0.129027$. Thus, the Hermitian operator H_8 serves as a classification witness operator and classify GHZ class of states described by the density operator $\rho_{\lambda_1, \lambda_2} \in S_3$ and the GHZ of states $\rho_{\lambda_1=0, \lambda_2=0} \in S_1$ or $\rho_{\lambda_1, \lambda_i=0}, (i = 2, 3) \in S_2$.

Similarly, we can construct witness operator that can classify GHZ states belonging to subclass-III and subclass-IV.

3.6 Conclusion

To summarize, we have defined systematically different subclasses of the pure three-qubit GHZ class. The subclass-I denoted by S_1 contain the states of the form $\lambda_0|000\rangle + \lambda_1|111\rangle$. In particular, if $\lambda_0 = \lambda_1 = \frac{1}{\sqrt{2}}$ then the three-qubit state reduces to standard GHZ state and it is known that this state is very useful in various quantum information processing task. In this chapter, it has been shown that there exist states either belong to subclass-II (denoted by S_2) or subclass-III (denoted by S_3) or subclass-IV (denoted by S_4), that may be more useful in some teleportation scheme in comparison to the states belong to S_1 . Also, we found that the tangle of the GHZ states belonging to subclass S_1 is more as compared to GHZ states belonging to subclass S_2 . This observation gives the motivation to discriminate the states belonging to $S_i, i = 2, 3, 4$ from the family of states belonging to S_1 . We have prescribed the method for the construction of the witness operator to study the classification of the states belonging to $S_i, i = 2, 3, 4$.

Chapter 4

Classification of Three-qubit States using l_1 -norm of Coherence

In this chapter¹, we have studied the problem of classification of three-qubit states using l_1 -norm of coherence. In the literature, the problem of classification of three-qubit entangled states has been studied through three-qubit entanglement measures such as tangle. But since tangle is non-zero for only GHZ class of states and zero for the set of separable states, biseparable states, and W class of states, so, it cannot be used to classify the states belonging to separable, biseparable, or W class of states. To overcome this issue, we have used l_1 -norm of coherence for the detection and classification of three-qubit entangled states. In this chapter, we have derived coherence based inequalities that may be satisfied by biseparable states. The inequality satisfied by the biseparable states is denoted by I_1 . The violation of I_1 indicates that the state under investigation is either separable or genuine entangled. Furthermore, we have derived an expression E_1 which is only satisfied by the separable states. If this expression is not satisfied, then the state may be either biseparable or a genuine entangled state. Thus if there exists a three-qubit state which violates the inequality I_1 and also does not satisfy E_1 , then the state under probe is a genuine entangled state.

¹This chapter is based on the research paper entitled “Detection and Classification of Three-qubit States Using l_1 -norm of Coherence”.

4.1 Introduction

Quantum entanglement and coherence are two fundamental features that arise from the superposition principle of quantum mechanics. One notable difference between these two features is that coherence may exist in a single system while more than one system or more than one degree of freedom is required for entanglement. Secondly, the non-zero off-diagonal elements in the density matrix signify the presence of coherence in the quantum system while it may not ensure the existence of entanglement in the given composite quantum system. Both quantum entanglement and quantum coherence can be used as a resource [67, 138]. M. Hillery has shown that coherence may act as a useful resource in the Deutsch-Jozsa algorithm [139]. On the other hand, quantum entanglement has many application in quantum information processing tasks, notably, quantum teleportation [30], quantum superdense coding [36], quantum remote state preparation [40], quantum cryptography [50] etc. Extensive research has already been carried out to understand the non-classical feature of bipartite system [21, 23, 26, 58, 64, 67, 140]. Eventually, when we increase the number of parties in the system, the complications in the shared system between the parties also increase. Therefore, it is indispensable to understand the entanglement properties of the shared multipartite system.

In this chapter, we will consider the problem of classification of entanglement and we have studied this problem specifically in the case of a tripartite system. We will derive coherence-based inequalities to classify a three-qubit entangled system.

Let us start our discussion on the tripartite system by assuming that the Hilbert spaces H_A , H_B and H_C , which describe each subsystem A , B , C of a tripartite system, is spanned by the computational basis states $|0\rangle$ and $|1\rangle$ respectively.

Any three-qubit state may be classified as fully separable states, biseparable states, or genuinely entangled states [95]. If the three-qubit state is shared by three distant parties Alice, Bob and Charlie then the shared state may be either in the form of a fully separable or biseparable state or a genuine entangled state. The fully separable and biseparable state may be expressed in the form as [92, 118]

$$\rho_{sep}^{ABC} = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C, \sum_i p_i = 1 \quad (4.1.1)$$

$$\rho_{bisep} = p_1 \rho_{bisep}^{A-BC} + p_2 \rho_{bisep}^{B-AC} + p_3 \rho_{bisep}^{C-AB}, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1 \quad (4.1.2)$$

where,

$$\begin{aligned} \rho_{bisep}^{A-BC} &= \sum_i |a_i\rangle_A \langle a_i| \otimes |\phi_i\rangle_{BC} \langle \phi_i| \\ \rho_{bisep}^{B-AC} &= \sum_i |b_i\rangle_B \langle b_i| \otimes |\phi_i\rangle_{AC} \langle \phi_i| \\ \rho_{bisep}^{C-AB} &= \sum_i |c_i\rangle_C \langle c_i| \otimes |\phi_i\rangle_{AB} \langle \phi_i| \end{aligned}$$

Here, $|a_i\rangle$, $|b_i\rangle$ and $|c_i\rangle$ are (unnormalized) states of systems A, B, and C, respectively, and $|\phi_i\rangle$ represent the states of two systems jointly. The density operators ρ_i^x , $x = A, B, C$ lying on the Hilbert spaces of dimension 2.

It has been observed that the three-qubit biseparable states are important and useful in many contexts. The importance of biseparable states comes from the fact that they may be used as unextendible biseparable bases (UBB) which are proved to be useful to construct genuinely entangled subspaces [141]. Instead of a genuine tripartite entangled state, it has been shown that a biseparable state is enough to use in the controlled quantum teleportation protocol as a resource state [142]. Barasinski et.al. [143] have analyzed the fidelity of the controlled quantum teleportation via mixed biseparable state and have concluded that a statistical mixture of biseparable states can be suitable for the perfect controlled quantum teleportation. Further, it has been shown that there exists a special class of biseparable state i.e. a non-maximally entangled mixed biseparable X state, which can be useful as a resource state for the attainment of high fidelity in controlled quantum teleportation [144]. Recently, it is shown that the biseparable states can also be used in obtaining the non-zero conference key [145].

Since the three-qubit biseparable state has potential applications in quantum information theory, it is crucial to classify the three-qubit biseparable state. There are a lot of earlier works in this line of research but we mention here a few of them. Eggeling and Werner [146] provided the necessary and sufficient criteria in terms of the projection parameters to detect the biseparable state. But this result is true for only one bipartition cut $A - BC$. In [147], it has been shown that the witness operator may be constructed to distinguish fully separable state, biseparable state, and genuine entangled states for a multipartite system. An n-partite inequality is presented in [148], whose violation by a state implies that the state under investigation is not a biseparable

ble state. A non-linear entanglement witness operator has been constructed to identify all three types of three-qubit states [149]. Using entanglement measures, biseparability in mixed three-qubit systems has also been analyzed in detail in [150, 151]. The necessary and sufficient condition for the detection of permutationally invariant three-qubit biseparable states has been studied in [152]. The multipartite biseparable entangled states under any bipartite partitions can be detected using linear contraction methods [153]. A set of Bell inequalities was introduced in [154], which can distinguish separable, biseparable, and genuinely entangled pure three-qubit states. The method of construction of biseparable state is given in [155] and the classification of three-qubit pure states has been studied in [96]. Recently, it has been shown that a particular class of three-qubit GHZ and W class of states can be discriminated using coherence-based inequality [156].

In this chapter, we will derive the coherence-based inequality which can be used to detect the given three-qubit state as a biseparable and separable state. To accomplish this task, we first find out the formula for the l_1 -norm of coherence of the tensor product of two quantum states, which are subsystems of an n-partite system. Then, we use the derived formula to establish the required inequality for the classification of biseparable and separable states.

4.2 l_1 -norm of coherence of the tensor product of bipartite states

The coherence can be measured by different measures such as distance measure, relative entropy of coherence, and l_p ($p = 1, 2$)-norms. Any function C which maps the set of all states to the non-negative real numbers and satisfies the following properties is a valid measure of coherence [157]:

(C_1) It vanishes for incoherent states i.e.

$$C(\rho) = 0, \text{ whenever } \rho \text{ is an incoherent state} \quad (4.2.1)$$

(C_2) Any valid coherence measure C , should not increase under incoherent operations.

(C₃) Monotonicity under incoherent completely positive and trace-preserving maps i.e

$$C(\rho) \geq C(\phi_{ICPTP}(\rho)) \quad (4.2.2)$$

for all incoherent completely positive and trace-preserving maps ϕ_{ICPTP} .

(C₄) Monotonicity under selective measurements on average i.e.

$$C(\rho) \geq \sum_n p_n C(\rho_n) \quad \forall \{K_n\} \quad (4.2.3)$$

where $\rho_n = \frac{K_n \rho K_n^\dagger}{p_n}$ represent the state corresponding to the outcome n and the probability of occurrence of the state ρ_n is given by $p_n = \text{Tr}[K_n \rho K_n^\dagger]$. $\{K_n\}$ are the Kraus operators such that $\sum_n K_n^\dagger K_n = I$ and $K_n \mathbf{I} K_n^\dagger \subset \mathbf{I}$, I denote the identity operator and \mathbf{I} represent the set of incoherent states.

(C₅) A valid coherence monotone should satisfy the convexity property i.e.

$$\sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n), \quad 0 \leq p_n \leq 1, \sum_n p_n = 1 \quad (4.2.4)$$

for any set of states ρ_n . Here $\rho_n = \frac{K_n \rho K_n^\dagger}{p_n}$ and $p_n = \text{Tr}[K_n \rho K_n^\dagger]$. $\{K_n\}$ are the Kraus operators.

In this chapter, we will use the l_1 norm of coherence which is defined as

$$C_{l_1}(\rho) = \sum_{i,j,i \neq j} |\rho_{ij}| \quad (4.2.5)$$

where ρ_{ij} denotes the complex numbers corresponding to the ij -th entry of the density matrix ρ . l_1 -norm of coherence of a state depends on the choice of basis in which the given state is expressed and thus from now on, we are considering only computational basis to describe the density matrix of the given state. l_1 -norm of coherence satisfies all the properties (C₁-C₅) and thus, it can be considered as a valid coherence monotone. Therefore, it may serve as a useful measure of coherence [157].

Now, we are in a position to state the result on the l_1 -norm of coherence of the tensor product of two quantum states which are the subsystems of an n -partite system.

Result-1: If the density operators $\rho_{A_1, A_2, \dots, A_M}$ and $\rho_{A_{M+1}, A_{M+2}, \dots, A_N}$ denote the subsystem of a n -partite system, then the l_1 -norm of coherence of the tensor product of

$\rho_{A_1, A_2, \dots, A_M}$ and $\rho_{A_{M+1}, A_{M+2}, \dots, A_N}$ is given by

$$\begin{aligned} C_{l_1}(\rho_{A_1, A_2, \dots, A_M} \otimes \rho_{A_{M+1}, A_{M+2}, \dots, A_N}) &= C_{l_1}(\rho_{A_1, A_2, \dots, A_M}) + C_{l_1}(\rho_{A_{M+1}, A_{M+2}, \dots, A_N}) \\ &+ C_{l_1}(\rho_{A_1, A_2, \dots, A_M}) \cdot C_{l_1}(\rho_{A_{M+1}, A_{M+2}, \dots, A_N}) \end{aligned} \quad (4.2.6)$$

Proof: Consider two quantum states described by the density matrices $\rho_{A_1, A_2, \dots, A_M}$ and $\rho_{A_{M+1}, A_{M+2}, \dots, A_N}$. The matrix representation of the density operators $\rho_{A_1, A_2, \dots, A_M}$ and $\rho_{A_{M+1}, A_{M+2}, \dots, A_N}$ are given by

$$\rho_{A_1, A_2, \dots, A_M} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m,1} & a_{m,2} & \cdot & \cdot & \cdot & a_{m,m} \end{pmatrix} \quad (4.2.7)$$

where, $a_{1,1} + a_{2,2} + \cdot + \cdot + \cdot + a_{m,m} = 1$,

$$\rho_{A_{M+1}, A_{M+2}, \dots, A_N} = \begin{pmatrix} a_{m+1,m+1} & a_{m+1,m+2} & \cdot & \cdot & \cdot & a_{m+1,n} \\ a_{m+2,m+1} & a_{m+2,m+2} & \cdot & \cdot & \cdot & a_{m+2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,m+1} & a_{n,m+2} & \cdot & \cdot & \cdot & a_{n,n} \end{pmatrix} \quad (4.2.8)$$

where, $a_{m+1,m+1} + a_{m+2,m+2} + \dots + a_{n,n} = 1$ and $a_{j,i}$ denotes the complex conjugate of $a_{i,j}$.

Then, the l_1 norm of coherence of the density matrices $\rho_{A_1, A_2, \dots, A_M}$ and $\rho_{A_{M+1}, A_{M+2}, \dots, A_N}$ are given by

$$C_{l_1}(A_1, A_2, \dots, A_M) = \sum_{i,j=1, i \neq j}^m |a_{i,j}| \quad (4.2.9)$$

and

$$C_{l_1}(\rho_{A_{M+1}, A_{M+2}, \dots, A_N}) = \sum_{i,j=m+1, i \neq j}^n |a_{i,j}| \quad (4.2.10)$$

Then,

$$\begin{aligned}
C_{l_1}(\rho_{A_1, A_2, \dots, A_M} \otimes \rho_{A_{M+1}, A_{M+2}, \dots, A_N}) &= |a_{1,1}| \left(\sum_{i,j=m+1, i \neq j}^n |a_{i,j}| \right) + |a_{1,2}| \left(\sum_{i,j=m+1}^n |a_{i,j}| \right) \\
&+ \dots + |a_{1,m}| \left(\sum_{i,j=m+1}^n |a_{i,j}| \right) + |a_{2,1}| \left(\sum_{i,j=m+1}^n |a_{i,j}| \right) \\
&+ |a_{2,2}| \left(\sum_{i,j=m+1, i \neq j}^n |a_{i,j}| \right) + \dots + |a_{2,n}| \left(\sum_{i,j=m+1}^n |a_{i,j}| \right) \\
&+ \dots + |a_{m,1}| \left(\sum_{i,j=m+1}^n |a_{i,j}| \right) + \dots + |a_{m,m}| \left(\sum_{i,j=m+1, i \neq j}^n |a_{i,j}| \right)
\end{aligned} \tag{4.2.11}$$

Simplifying the above equation, we get

$$\begin{aligned}
C_{l_1}(\rho_{A_1, A_2, \dots, A_M} \otimes \rho_{A_{M+1}, A_{M+2}, \dots, A_N}) &= \sum_{i,j=m+1, i \neq j}^n |a_{i,j}| [|a_{1,1}| + |a_{2,2}| + \dots + |a_{m,m}|] \\
&+ \sum_{i,j=m+1}^n |a_{i,j}| [|a_{1,2}| + |a_{1,3}| + \dots + |a_{1,m}| + |a_{2,1}| \\
&+ |a_{2,3}| + \dots + |a_{2,n}| + \dots + |a_{m,1}| + \dots + |a_{m,m-1}|]
\end{aligned} \tag{4.2.12}$$

Using normalization condition of $\rho_{A_1, A_2, \dots, A_M}$ and $\rho_{A_{M+1}, A_{M+2}, \dots, A_N}$, we get

$$\begin{aligned}
C_{l_1}(\rho_{A_1, A_2, \dots, A_M} \otimes \rho_{A_{M+1}, A_{M+2}, \dots, A_N}) &= \sum_{i,j=m+1, i \neq j}^n |a_{i,j}| + \left[\sum_{i,j=m+1}^n |a_{i,j}| \right] \cdot \left[\sum_{i,j=m+1, i \neq j}^n |a_{i,j}| \right] \\
&= \sum_{i,j=m+1, i \neq j}^n |a_{i,j}| + \left[1 + \sum_{i,j=m+1, i \neq j}^n |a_{i,j}| \right] \times \\
&\quad \left[\sum_{i,j=m+1, i \neq j}^n |a_{i,j}| \right]
\end{aligned} \tag{4.2.13}$$

From equations (4.2.9) and (4.2.10), we get,

$$\begin{aligned}
C_{l_1}(\rho_{A_1, A_2, \dots, A_M} \otimes \rho_{A_{M+1}, A_{M+2}, \dots, A_N}) &= C_{l_1}(\rho_{A_1, A_2, \dots, A_M}) \\
&+ [1 + C_{l_1}(\rho_{A_{M+1}, A_{M+2}, \dots, A_N})] \cdot [C_{l_1}(\rho_{A_1, A_2, \dots, A_M})]
\end{aligned}$$

Thus, the l_1 -norm of coherence of the tensor product of an m -qubit and an $(n-m)$ qubit,

which are a part of an n-partite quantum system, is given by

$$\begin{aligned}
& C_{l_1}(\rho_{A_1, A_2, \dots, A_M} \otimes \rho_{A_{M+1}, A_{M+2}, \dots, A_N}) \\
&= C_{l_1}(\rho_{A_1, A_2, \dots, A_M}) + C_{l_1}(\rho_{A_{M+1}, A_{M+2}, \dots, A_N}) \\
&+ C_{l_1}(\rho_{A_1, A_2, \dots, A_M}) \cdot C_{l_1}(\rho_{A_{M+1}, A_{M+2}, \dots, A_N})
\end{aligned} \tag{4.2.14}$$

Corollary-1: For any two single qubit quantum states described by the density operators ρ_1 and ρ_2 , the l_1 norm of coherence of the tensor product of ρ_1 and ρ_2 is given by

$$C_{l_1}(\rho_1 \otimes \rho_2) = C_{l_1}(\rho_1) + C_{l_1}(\rho_2) + C_{l_1}(\rho_1) \cdot C_{l_1}(\rho_2) \tag{4.2.15}$$

Proof: Any two single-qubit quantum states described by the density matrices ρ_1 and ρ_2 are given by

$$\rho_i = \begin{pmatrix} a_i & b_i \\ b_i^* & d_i \end{pmatrix}, a_i + d_i = 1, \quad i = 1, 2 \tag{4.2.16}$$

The l_1 norm of coherence of the density matrices ρ_1 and ρ_2 are given by

$$C_{l_1}(\rho_i) = 2|b_i|, \quad i = 1, 2 \tag{4.2.17}$$

The tensor product of ρ_1 and ρ_2 may be defined as

$$\begin{aligned}
\rho_1 \otimes \rho_2 &= \begin{pmatrix} a_1 & b_1 \\ b_1^* & d_1 \end{pmatrix} \otimes \begin{pmatrix} a_2 & b_2 \\ b_2^* & d_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1 a_2 & a_1 b_2 & b_1 a_2 & b_1 b_2 \\ a_1 b_2^* & a_1 d_2 & b_1 b_2^* & b_1 d_2 \\ b_1^* a_2 & b_1^* b_2 & d_1 a_2 & d_1 b_2 \\ b_1^* b_2^* & b_1^* d_2 & d_1 b_2^* & d_1 d_2 \end{pmatrix}
\end{aligned} \tag{4.2.18}$$

The l_1 -norm of coherence of the tensor product $\rho_1 \otimes \rho_2$ is given by

$$\begin{aligned}
C_{l_1}(\rho_1 \otimes \rho_2) &= a_1|b_2| + |b_1|a_2 + |b_1||b_2| + a_1|b_2| \\
&+ |b_1||b_2| + |b_1|d_2 + |b_1|a_2 + |b_1||b_2| \\
&+ d_1|b_2| + |b_1||b_2| + |b_1|d_2 + d_1|b_2| \\
&= 2(a_1 + d_1)|b_2| + 2(a_2 + d_2)|b_1| + 4|b_1| \cdot |b_2| \\
&= 2|b_1| + 2|b_2| + 4|b_1| \cdot |b_2|
\end{aligned} \tag{4.2.19}$$

Using equation (4.2.17), the equation (4.2.19) reduces to

$$C_{l_1}(\rho_1 \otimes \rho_2) = C_{l_1}(\rho_1) + C_{l_1}(\rho_2) + C_{l_1}(\rho_1) \cdot C_{l_1}(\rho_2) \tag{4.2.20}$$

Hence proved.

Corollary-2: If the three-qubit biseparable system described either by the density operator $\rho_{A-BC} \equiv \rho_A \otimes \rho_{BC}$ or $\rho_{B-AC} \equiv \rho_B \otimes \rho_{AC}$ or $\rho_{C-AB} \equiv \rho_C \otimes \rho_{AB}$, then the l_1 -norm of coherence for the density operator ρ_{A-BC} , ρ_{B-AC} and ρ_{C-AB} are given by

$$C_{l_1}(\rho_{A-BC}) = C_{l_1}(\rho_A) + C_{l_1}(\rho_{BC}) + C_{l_1}(\rho_A) \cdot C_{l_1}(\rho_{BC}) \tag{4.2.21}$$

$$C_{l_1}(\rho_{B-AC}) = C_{l_1}(\rho_B) + C_{l_1}(\rho_{AC}) + C_{l_1}(\rho_B) \cdot C_{l_1}(\rho_{AC}) \tag{4.2.22}$$

$$C_{l_1}(\rho_{C-AB}) = C_{l_1}(\rho_C) + C_{l_1}(\rho_{AB}) + C_{l_1}(\rho_C) \cdot C_{l_1}(\rho_{AB}) \tag{4.2.23}$$

1. If the equality given by (4.2.21) is violated by any three-qubit state, then the state under investigation is not a biseparable state of the form $\rho_A \otimes \rho_{BC}$.
2. If the equality given by (4.2.22) is violated by any three-qubit state, then the state under investigation is not a biseparable state of the form $\rho_B \otimes \rho_{AC}$.
3. If any three-qubit state does not satisfy the equality given by (4.2.23), then the given state is not a biseparable state of the form $\rho_C \otimes \rho_{AB}$.

Corollary-3: If the three-qubit system represent the separable system described by the density operator $\rho_{A-B-C} \equiv \rho_A \otimes \rho_B \otimes \rho_C$, then the l_1 -norm of coherence for the

density operator ρ_{A-B-C} is given by

$$\begin{aligned}
C_{l_1}(\rho_{A-B-C}) &= C_{l_1}(\rho_A) + C_{l_1}(\rho_B) + C_{l_1}(\rho_C) \\
&+ C_{l_1}(\rho_A) \cdot C_{l_1}(\rho_B) + C_{l_1}(\rho_A) \cdot C_{l_1}(\rho_C) \\
&+ C_{l_1}(\rho_B) \cdot C_{l_1}(\rho_C) \\
&+ C_{l_1}(\rho_A) \cdot C_{l_1}(\rho_B) \cdot C_{l_1}(\rho_C)
\end{aligned} \tag{4.2.24}$$

If the equality given by (4.2.24) is violated by any three-qubit state, then the state under probe is not a separable state.

4.2.1 Coherence-based inequality for the detection of a particular form of three-qubit biseparable states

In this subsection, we deduce coherence based inequality for the detection of three-qubit biseparable state of the form $\rho^{i-jk}(i \neq j \neq k; i, j, k = A, B, C)$. To verify this inequality, we need only the information on the density matrix elements of the given three-qubit system under investigation.

Result-2: If the three-qubit state described by the density operator $\rho^{A-BC} = \sum_i p_i \rho_A^i \otimes \rho_{BC}^i$ is biseparable such that the l_1 -norm of coherence of at least one of the reduced system is non zero, then the l_1 norm of coherence of the biseparable state satisfies

$$C_{l_1}(\rho^{A-BC}) \leq \sum_i p_i \left(\frac{X_i^2}{4} + X_i \right) \tag{4.2.25}$$

where

$$X_i = C_{l_1}(\rho_A^i) + C_{l_1}(\rho_{BC}^i), \quad i = 1, 2, 3, \dots \tag{4.2.26}$$

Proof: Let us consider a biseparable state for $A - BC$ partition. The biseparable state in this partition is given by

$$\rho^{A-BC} = \sum_i p_i \rho_A^i \otimes \rho_{BC}^i \tag{4.2.27}$$

Then, l_1 norm of coherence of ρ^{A-BC} is given by

$$\begin{aligned}
 C_{l_1}(\rho^{A-BC}) &= C_{l_1}[\sum_i p_i \rho_A^i \otimes \rho_{BC}^i] \\
 &\leq \sum_i p_i C_{l_1}[\rho_A^i \otimes \rho_{BC}^i] \\
 &= \sum_i p_i (C_{l_1}(\rho_A^i) + C_{l_1}(\rho_{BC}^i) + C_{l_1}(\rho_A^i) \cdot C_{l_1}(\rho_{BC}^i)) \quad (4.2.28)
 \end{aligned}$$

We have used the convexity property of the l_1 -norm of coherence in the second step.

The last step follows from Corollary-1.

Now, Arithmetic mean (AM) and Geometric mean (GM) of $C_{l_1}(\rho_A^i)$ and $C_{l_1}(\rho_{BC}^i)$ are given by

$$\frac{C_{l_1}(\rho_A^i) + C_{l_1}(\rho_{BC}^i)}{2} \quad \text{and} \quad \sqrt{C_{l_1}(\rho_A^i) \cdot C_{l_1}(\rho_{BC}^i)} \quad (4.2.29)$$

Using AM-GM inequality [136] on $C_{l_1}(\rho_A^i)$ and $C_{l_1}(\rho_{BC}^i)$, we get

$$\frac{(C_{l_1}(\rho_A^i) + C_{l_1}(\rho_{BC}^i))^2}{4} \geq C_{l_1}(\rho_A^i) \cdot C_{l_1}(\rho_{BC}^i) \quad (4.2.30)$$

From (4.2.28), we get

$$\begin{aligned}
 C_{l_1}(\rho^{A-BC}) &\leq \sum_i p_i (C_{l_1}(\rho_A^i) + C_{l_1}(\rho_{BC}^i) + \frac{(C_{l_1}(\rho_A^i) + C_{l_1}(\rho_{BC}^i))^2}{4}) \\
 &= \sum_i p_i (X_i + \frac{X_i^2}{4}) \quad (4.2.31)
 \end{aligned}$$

where $X_i = C_{l_1}(\rho_A^i) + C_{l_1}(\rho_{BC}^i)$ for $i=1,2,3,\dots$

Hence proved.

4.2.2 Example

Example-1: Let us consider a biseparable state described by the density operator

$\rho = |\psi\rangle_{ABC}\langle\psi|$, where $|\psi\rangle_{ABC}$ is given by

$$|\psi\rangle_{ABC} = \lambda_0|101\rangle + \lambda_1|110\rangle + \lambda_2|111\rangle \quad (4.2.32)$$

where $\lambda_0, \lambda_1, \lambda_2 \in R^+$ and $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$. Let us assume that $\lambda_0 \geq \lambda_1 \geq \lambda_2$.

The value of $C_{l_1}(\rho_A)$ and $C_{l_1}(\rho_{BC})$ is given by

$$C_{l_1}(\rho_A) = 0, \quad C_{l_1}(\rho_{BC}) = 2(\lambda_0\lambda_1 + \lambda_1\lambda_2 + \lambda_0\lambda_2) \quad (4.2.33)$$

For the state $|\psi\rangle_{ABC}$ given in (4.2.32), we can calculate $C_{l_1}(\rho^{A-BC})$ as

$$C_{l_1}(\rho^{A-BC}) = 2(\lambda_0\lambda_1 + \lambda_1\lambda_2 + \lambda_0\lambda_2) \quad (4.2.34)$$

Using (4.2.33) and (4.2.34), it can be shown that the inequality (4.2.31) is satisfied.

Example-2: Consider a $|W\rangle_{ABC}$ state of the form

$$|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|100\rangle_{ABC} + |010\rangle_{ABC} + |001\rangle_{ABC}) \quad (4.2.35)$$

The l_1 -norm of coherence of the $|W\rangle_{ABC}$ state is given by $C_{l_1}(|W\rangle_{ABC}) = 2$.

The reduced single-qubit state may be expressed as

$$\rho_i^W = \text{Tr}_{jk}(|W\rangle_{ABC}) = \frac{1}{3}(2|0\rangle_A\langle 0| + |1\rangle_A\langle 1|), \quad i \neq j \neq k, i, j, k \in \{A, B, C\} \quad (4.2.36)$$

Since the single-qubit density operators ρ_A^W , ρ_B^W and ρ_C^W does not contain any off-diagonal elements so the l_1 -norm of coherence for these single qubit states is given by $C_{l_1}(\rho_A^W) = C_{l_1}(\rho_B^W) = C_{l_1}(\rho_C^W) = 0$. Therefore, it can be easily shown that the equality condition given in (4.2.24) is not maintained for the state (4.2.35). Further, it can be shown that the set of equality conditions given by (4.2.21), (4.2.22) and (4.2.23) are not satisfied by the state (4.2.35). Thus, the state (4.2.35) is neither a fully separable state of the form $\rho_A^W \otimes \rho_B^W \otimes \rho_C^W$ nor a biseparable state of the form $\rho_A^W \otimes \rho_{BC}^W$ or $\rho_B^W \otimes \rho_{CA}^W$ or $\rho_C^W \otimes \rho_{AB}^W$.

4.3 Detection of general three-qubit biseparable states and separable states

A mixed state is said to be fully separable if it can be written as the convex combination of fully separable pure states. A mixed state is said to be biseparable if it is not fully separable and it can be written as a convex combination of biseparable pure states.

Let us recall the three-qubit mixed biseparable state given in (4.1.2) and re-write it as

$$\sigma_{bisep} = p_1 \sigma_{bisep}^{A-BC} + p_2 \sigma_{bisep}^{B-CA} + p_3 \sigma_{bisep}^{C-AB}, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1 \quad (4.3.1)$$

4.3.1 Coherence-based inequality for the classification of general three-qubit biseparable states

Classification of three-qubit mixed states has been studied by constructing the witness operator [98]. In this section, we will study the classification of three-qubit mixed biseparable states using coherence-based inequality.

Result-3: If the three-qubit mixed state described by the density operator in (4.3.1) is biseparable then it satisfies the inequality

$$1 + C_{l_1}(p_1 \sigma_1^{A-BC} + p_2 \sigma_2^{B-CA} + p_3 \sigma_3^{C-AB}) \leq \frac{1}{4} \sum_{i=1}^3 p_i (X_i + 2)^2 \quad (4.3.2)$$

where,

$$\begin{aligned} X_1 &= C_{l_1}(\sigma_1^A) + C_{l_1}(\sigma_1^{BC}) \\ X_2 &= C_{l_1}(\sigma_2^B) + C_{l_1}(\sigma_2^{CA}) \\ X_3 &= C_{l_1}(\sigma_3^C) + C_{l_1}(\sigma_3^{AB}) \end{aligned} \quad (4.3.3)$$

Proof: Let us consider a mixed three-qubit biseparable state whose density matrix is given by $p_1 \sigma_1^{A-BC} + p_2 \sigma_2^{B-CA} + p_3 \sigma_3^{C-AB}$. Using l_1 -norm of coherence of $p_1 \sigma_1^{A-BC} + p_2 \sigma_2^{B-CA} + p_3 \sigma_3^{C-AB}$ and the convexity property of l_1 -norm of coherence, we have,

$$\begin{aligned} C_{l_1}(p_1 \sigma_1^{A-BC} + p_2 \sigma_2^{B-CA} + p_3 \sigma_3^{C-AB}) &\leq p_1 C_{l_1}(\sigma_1^{A-BC}) + p_2 C_{l_1}(\sigma_2^{B-CA}) \\ &\quad + p_3 C_{l_1}(\sigma_3^{C-AB}) \end{aligned} \quad (4.3.4)$$

Using the relation (4.2.21) or other related coherence relation like (4.2.22) or (4.2.23),

we get

$$\begin{aligned}
C_{l_1}(p_1\sigma_1^{A-BC} + p_2\sigma_2^{B-CA} + p_3\sigma_3^{C-AB}) &\leq p_1[C_{l_1}(\sigma_1^A + C_{l_1}(\sigma_1^{BC}) + C_{l_1}(\sigma_1^A) \cdot C_{l_1}(\sigma_1^{BC}))] \\
&+ p_2[C_{l_1}(\sigma_2^B + C_{l_1}(\sigma_2^{CA}) + C_{l_1}(\sigma_2^B) \cdot C_{l_1}(\sigma_2^{CA}))] \\
&+ p_3[C_{l_1}(\sigma_3^C + C_{l_1}(\sigma_3^{AB}) + C_{l_1}(\sigma_3^C) \cdot C_{l_1}(\sigma_3^{AB}))]
\end{aligned} \tag{4.3.5}$$

Recalling expressions of AM and GM of $C_{l_1}(\sigma_i^A)$ and $C_{l_1}(\sigma_i^{BC})$ from (4.2.29) and applying $AM \geq GM$ on $C_{l_1}(\sigma_i^A)$ and $C_{l_1}(\sigma_i^{BC})$, inequality (4.3.5) reduces to,

$$\begin{aligned}
C_{l_1}(p_1\sigma_1^{A-BC} + p_2\sigma_2^{B-CA} + p_3\sigma_3^{C-AB}) &\leq p_1[C_{l_1}(\sigma_1^A) + C_{l_1}(\sigma_1^{BC}) + \frac{1}{4}(C_{l_1}(\sigma_1^A) + C_{l_1}(\sigma_1^{BC}))^2] \\
&+ p_2[C_{l_1}(\sigma_2^B) + C_{l_1}(\sigma_2^{CA}) + \frac{1}{4}(C_{l_1}(\sigma_2^B) + C_{l_1}(\sigma_2^{CA}))^2] \\
&+ p_3[C_{l_1}(\sigma_3^C) + C_{l_1}(\sigma_3^{AB}) + \frac{1}{4}(C_{l_1}(\sigma_3^C) + C_{l_1}(\sigma_3^{AB}))^2]
\end{aligned} \tag{4.3.6}$$

Thus, we have

$$C_{l_1}(p_1\sigma_1^{A-BC} + p_2\sigma_2^{B-CA} + p_3\sigma_3^{C-AB}) \leq p_1(X_1 + \frac{X_1^2}{4}) + p_2(X_2 + \frac{X_2^2}{4}) + p_3(X_3 + \frac{X_3^2}{4}) \tag{4.3.7}$$

Therefore,

$$\begin{aligned}
C_{l_1}(p_1\sigma_1^{A-BC} + p_2\sigma_2^{B-CA} + p_3\sigma_3^{C-AB}) &\leq p_1(X_1 + \frac{X_1^2}{4}) + p_2(X_2 + \frac{X_2^2}{4}) + p_3(X_3 + \frac{X_3^2}{4}) \\
&= \frac{1}{4}[p_1(X_1 + 2)^2 + p_2(X_2 + 2)^2 + p_3(X_3 + 2)^2] - 1
\end{aligned} \tag{4.3.8}$$

where X_1, X_2 and X_3 are given by (4.4.3).

Simplifying (4.3.8), we get

$$1 + C_{l_1}(p_1\sigma_1^{A-BC} + p_2\sigma_2^{B-CA} + p_3\sigma_3^{C-AB}) \leq \frac{1}{4} \sum_{i=1}^3 p_i(X_i + 2)^2 \tag{4.3.9}$$

which is the required result. Hence proved.

Corollary-4: If any three-qubit mixed state violates the inequality (4.3.2), then the given state is not a biseparable state.

4.3.2 Coherence-based inequality for the classification of three-qubit mixed separable states

In this subsection, we will discuss how to classify the given state as a separable state which belongs to the set of mixed three-qubit states using the l_1 -norm of coherence of a single qubit. Let us consider a mixed separable state, which can be expressed as

$$\sigma^{A-B-C} = \sum_i p_i \sigma_i^A \otimes \sigma_i^B \otimes \sigma_i^C \quad (4.3.10)$$

The l_1 -norm of the coherence of the state (4.3.10) is given by

$$\begin{aligned} C_{l_1}(\sigma^{A-B-C}) &= C_{l_1}\left(\sum_i p_i \sigma_i^A \otimes \sigma_i^B \otimes \sigma_i^C\right) \\ &\leq \sum_i p_i C_{l_1}(\sigma_i^A \otimes \sigma_i^B \otimes \sigma_i^C) \\ &= \sum_i p_i \left[\sum_{x=A,B,C} C_{l_1}(\sigma_i^x) + \sum_{x \neq y, x,y=A,B,C} C_{l_1}(\sigma_i^x) C_{l_1}(\sigma_i^y) \right. \\ &\quad \left. + C_{l_1}(\sigma_i^A) C_{l_1}(\sigma_i^B) C_{l_1}(\sigma_i^C) \right] \end{aligned} \quad (4.3.11)$$

The inequality in the second step follows from the convexity property of l_1 -norm of coherence.

We are now in a position to state the condition for separability for a mixed three-qubit system. The statement for separability is as follows:

Result-4: If the three-qubit mixed state described by the density operator $\sigma^{A-B-C} = \sum_i p_i \sigma_i^A \otimes \sigma_i^B \otimes \sigma_i^C$ is separable then it satisfies the inequality

$$\begin{aligned} C_{l_1}(\sigma^{A-B-C}) &\leq \sum_i p_i \left[\sum_{x=A,B,C} C_{l_1}(\sigma_i^x) + \sum_{x \neq y, x,y=A,B,C} C_{l_1}(\sigma_i^x) C_{l_1}(\sigma_i^y) \right. \\ &\quad \left. + C_{l_1}(\sigma_i^A) C_{l_1}(\sigma_i^B) C_{l_1}(\sigma_i^C) \right] \end{aligned} \quad (4.3.12)$$

Corollary-5: If any three-qubit mixed state violates the inequality (4.3.12) then the given state is not a separable state.

4.3.3 Illustrations

Example-1: Consider a mixed three-qubit biseparable state described by the density matrix ρ_1 , which is given by

$$\rho_1 = q|0\rangle^A\langle 0| \otimes |\phi^+\rangle^{BC}\langle\phi^+| + (1-q)|1\rangle^B\langle 1| \otimes |\phi^-\rangle^{AC}\langle\phi^-| \quad (4.3.13)$$

where $0 \leq q \leq 1$ and the Bell states $|\phi^+\rangle^{BC}$ and $|\phi^-\rangle^{AC}$ are given by

$$\begin{aligned} |\phi^+\rangle^{BC} &= \frac{1}{\sqrt{2}}(|00\rangle^{BC} + |11\rangle^{BC}) \\ |\phi^-\rangle^{AC} &= \frac{1}{\sqrt{2}}(|00\rangle^{AC} - |11\rangle^{AC}) \end{aligned} \quad (4.3.14)$$

Comparing equation (4.3.13) with general mixed three-qubit biseparable state, we have $p_1 = q$, $p_2 = 1 - q$ and $p_3 = 0$. l_1 -norm of coherence for the given state, defined in (4.3.13), is given by

$$C_{l_1}(\rho_1) = 1 \quad (4.3.15)$$

For the given state ρ_1 , $C_{l_1}(\rho_1^A) = C_{l_1}(\rho_1^B) = C_{l_1}(\rho_1^C) = 0$, $C_{l_1}(\rho_1^{BC}) = 1$, $C_{l_1}(\rho_1^{AC}) = 1$ and $C_{l_1}(\rho_1^{AB}) = 0$. The quantity X'_i , $i=1,2,3$ can be calculated as

$$\begin{aligned} X_1 &= C_{l_1}(\rho_1^A) + C_{l_1}(\rho_1^{BC}) = 1 \\ X_2 &= C_{l_1}(\rho_1^B) + C_{l_1}(\rho_1^{AC}) = 1 \\ X_3 &= C_{l_1}(\rho_1^C) + C_{l_1}(\rho_1^{AB}) = 0 \end{aligned} \quad (4.3.16)$$

Substituting values of p'_i , X'_i and l_1 -norm of coherence of ρ_1 , we can see that equation (4.3.2) is satisfied for the state (4.3.13).

Example-2: Let us consider the mixed three-qubit state described by the density operator ρ^{ABC} as

$$\rho^{ABC} = q|GHZ\rangle\langle GHZ| + (1-q)|W\rangle\langle W| \quad (4.3.17)$$

where $0 \leq q \leq 1$ and the three-qubit states $|GHZ\rangle$ and $|W\rangle$ are given by

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad (4.3.18)$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \quad (4.3.19)$$

Comparing equation (4.3.17) with general three-qubit mixed biseparable state, we get $p_1 = q$, $p_2 = 1 - q$ and $p_3 = 0$.

The l_1 -norm of coherence of the state (4.3.17) is given by

$$C_{l_1}(\rho^{ABC}) = 3 \quad (4.3.20)$$

For the given state ρ^{ABC} , $C_{l_1}(\rho^A) = C_{l_1}(\rho^B) = C_{l_1}(\rho^C) = 0$ and $C_{l_1}(\rho^{BC}) = C_{l_1}(\rho^{AC}) = C_{l_1}(\rho^{AB}) = \frac{2}{3}$. Therefore, the quantities X'_i 's, $i=1,2,3$ can be written as,

$$X_1 = X_2 = X_3 = C_{l_1}(\rho_1^i) + C_{l_1}(\rho_1^{jk}) = \frac{2}{3}, \quad i \neq j \neq k, \quad i, j, k \in \{A, B, C\} \quad (4.3.21)$$

Substituting values of p'_i 's, X'_i 's and $C_{l_1}(\rho^{ABC})$ in equation (4.3.2), we can observe that the inequality given in Result-3 is violated for any q . Thus, the given state ρ^{ABC} is not a biseparable state. Moreover, a simple calculation also shows that the inequality (4.3.11) is violated for any q . So we can infer that the given state (4.3.17) is not a separable state. Thus, we find that the given state ρ^{ABC} is neither a biseparable state nor a separable state. Hence, we may conclude that the given state is a genuine mixed three-qubit entangled state.

4.4 An idea to generalize the results to a four-qubit system

The results we have obtained in this chapter can be generalized to the multipartite system also. For instance, if we consider a four-qubit system, Result-3 for mixed biseparable states may be re-stated as:

If the four-qubit mixed state described by the density operator

$$\rho = p_1 \sigma_1^{A-BCD} + p_2 \sigma_2^{B-CAD} + p_3 \sigma_3^{C-ABC} + p_4 \sigma_4^{D-ABC} \quad (4.4.1)$$

is biseparable then it satisfies the inequality

$$1 + C_{l_1}(p_1\sigma_1^{A-BCD} + p_2\sigma_2^{B-CAD} + p_3\sigma_3^{C-ABD} + p_4\sigma_4^{D-ABC}) \leq \frac{1}{4} \sum_{i=1}^4 p_i(X_i + 2)^2 \quad (4.4.2)$$

where,

$$\begin{aligned} X_1 &= C_{l_1}(\sigma_1^A) + C_{l_1}(\sigma_1^{BCD}) \\ X_2 &= C_{l_1}(\sigma_2^B) + C_{l_1}(\sigma_2^{CAD}) \\ X_3 &= C_{l_1}(\sigma_3^C) + C_{l_1}(\sigma_3^{ABD}) \\ X_4 &= C_{l_1}(\sigma_4^D) + C_{l_1}(\sigma_4^{ABC}) \end{aligned} \quad (4.4.3)$$

To verify this result, let us consider a mixed biseparable state in a four-qubit system described by the density matrix ρ_{ABCD} , which is given by

$$\rho_{ABCD} = \frac{1}{2}[|0\rangle^A\langle 0|] \otimes |\phi^+\rangle^{BCD}\langle\phi^+| + \frac{1}{2}[|1\rangle^B\langle 1|] \otimes |\phi^-\rangle^{ACD}\langle\phi^-| \quad (4.4.4)$$

where the states $|\phi^+\rangle^{BCD}$ and $|\phi^-\rangle^{ACD}$ are given by

$$\begin{aligned} |\phi^+\rangle^{BCD} &= \frac{1}{\sqrt{2}}(|100\rangle^{BCD} + |010\rangle^{BCD}) \\ |\phi^-\rangle^{ACD} &= \frac{1}{\sqrt{2}}(|100\rangle^{ACD} - |010\rangle^{ACD}) \end{aligned} \quad (4.4.5)$$

Comparing equation (4.4.4) with general mixed four qubit biseparable state (4.4.1), we have $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$, $p_3 = 0$ and $p_4 = 0$. l_1 -norm of coherence for the given state, defined in (4.4.4), is given by

$$C_{l_1}(\rho_{ABCD}) = 1 \quad (4.4.6)$$

For the given state ρ_1 , $C_{l_1}(\rho_{ABCD}^A) = C_{l_1}(\rho_{ABCD}^B) = C_{l_1}(\rho_{ABCD}^C) = C_{l_1}(\rho_{ABCD}^D) = 0$, $C_{l_1}(\rho_{ABCD}^{BCD}) = 1$, $C_{l_1}(\rho_{ABCD}^{ACD}) = 1$, $C_{l_1}(\rho_{ABCD}^{ABD}) = 1$ and $C_{l_1}(\rho_{ABCD}^{ABC}) = 0$. The quantity X_i 's, $i=1,2,3,4$ can be calculated as

$$\begin{aligned} X_1 &= C_{l_1}(\rho_{ABCD}^A) + C_{l_1}(\rho_{ABCD}^{BCD}) = 1 \\ X_2 &= C_{l_1}(\rho_{ABCD}^B) + C_{l_1}(\rho_{ABCD}^{CAD}) = 1 \\ X_3 &= C_{l_1}(\rho_{ABCD}^C) + C_{l_1}(\rho_{ABCD}^{ABD}) = 0 \\ X_4 &= C_{l_1}(\rho_{ABCD}^D) + C_{l_1}(\rho_{ABCD}^{ABC}) = 0 \end{aligned} \quad (4.4.7)$$

Substituting the values of p'_i , X'_i and l_1 -norm of coherence of ρ_{ABCD} in (4.4.2), it can be easily seen that (4.4.2) is satisfied for the state defined in (4.4.4).

Moreover, Result-4 for four-qubit mixed separable states may be re-stated as:

If the four-qubit mixed state described by the density operator $\sigma^{A-B-C-D} = \sum_i p_i \sigma_i^A \otimes \sigma_i^B \otimes \sigma_i^C \otimes \sigma_i^D$ is separable then it satisfies the inequality

$$\begin{aligned} C_{l_1}(\sigma^{A-B-C-D}) \leq & \sum_i p_i \left[\sum_{x=A,B,C,D} C_{l_1}(\sigma_i^x) + \sum_{x \neq y, x, y=A,B,C,D} C_{l_1}(\sigma_i^x) C_{l_1}(\sigma_i^y) \right. \\ & + \sum_{x \neq y \neq z, x, y, z=A,B,C,D} C_{l_1}(\sigma_i^x) C_{l_1}(\sigma_i^y) C_{l_1}(\sigma_i^z) \\ & \left. + C_{l_1}(\sigma_i^A) C_{l_1}(\sigma_i^B) C_{l_1}(\sigma_i^C) C_{l_1}(\sigma_i^D) \right] \end{aligned} \quad (4.4.8)$$

To verify the above result for a four-qubit mixed separable state, let us consider a separable state in a four-qubit system, described by the density operator ρ_2 ,

$$\begin{aligned} \rho_2 = & \frac{1}{4} |0000\rangle^{ABCD} \langle 0000| + \frac{1}{4} |0011\rangle^{ABCD} \langle 0011| \\ & + \frac{1}{4} |1000\rangle^{ABCD} \langle 1000| + \frac{1}{4} |1111\rangle^{ABCD} \langle 1111| \end{aligned} \quad (4.4.9)$$

Then, for the state defined in (4.4.9), $C_{l_1}(\rho_2) = 0$ and $C_{l_1}(\rho_2^A) = C_{l_1}(\rho_2^B) = C_{l_1}(\rho_2^C) = C_{l_1}(\rho_2^D) = 0$, thus we can say that the equation (4.4.8) is verified for the state ρ_2 .

4.5 An idea to generalize the results to a three-qutrit system

The results, we have obtained in this chapter can be generalized to higher dimensional quantum systems. For example, let us consider a three-qutrit state described by the density operator $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle$ is given by,

$$|\psi\rangle = |0\rangle_A \otimes \frac{1}{\sqrt{3}} (|12\rangle_{BC} + |01\rangle_{BC} + |20\rangle_{BC}) \quad (4.5.1)$$

Comparing equation (4.5.1) with general mixed three-qutrit biseparable state, we have $p_1 = 1$, $p_2 = 0$ and $p_3 = 0$. l_1 -norm of coherence for the given state, defined in (4.5.1), is given by

$$C_{l_1}(\rho) = 2 \quad (4.5.2)$$

For the given state ρ , $C_{l_1}(\rho^A) = C_{l_1}(\rho^B) = C_{l_1}(\rho^C) = 0$, $C_{l_1}(\rho^{BC}) = 2$, $C_{l_1}(\rho^{AC}) = 0$ and $C_{l_1}(\rho^{AB}) = 0$. The quantity X'_i , $i=1,2,3$ can be calculated as

$$\begin{aligned} X_1 &= C_{l_1}(\rho^A) + C_{l_1}(\rho^{BC}) = 2 \\ X_2 &= C_{l_1}(\rho^B) + C_{l_1}(\rho^{AC}) = 0 \\ X_3 &= C_{l_1}(\rho^C) + C_{l_1}(\rho^{AB}) = 0 \end{aligned} \quad (4.5.3)$$

Substituting values of p'_i , X'_i and l_1 -norm of coherence of ρ , we can see that equation (4.3.2) is satisfied for the state (4.5.1). Thus, the given states is a biseparable states. Hence, the result obtained in this chapter may be generalized to higher dimensional systems.

4.6 Conclusion

In this chapter, we have used l_1 -norm of coherence for the classification of three-qubit entangled states. Coherence is a basic phenomenon that arises from the superposition principle of quantum mechanics. It can be measured by different measures such as distance measure, relative entropy of coherence, and l_p norms. l_1 -norm is a valid coherence monotone, serves as a useful measure of coherence [157] and also, we have shown that it may be useful in classifying the multi-qubit system. We have calculated the l_1 -norm of coherence for the tensor product m – qubit \otimes $(n - m)$ – qubit. In order to obtain criteria for the classification of three-qubit states, we have obtained inequalities for biseparable states and separable states in terms of l_1 -norm of coherence. Further, we have shown that if the obtained inequality is violated by any three-qubit state then the state under investigation is neither a biseparable state nor a separable state. Since, for a three-qubit system, we have only three categories of state and if we find that the given state is neither a separable state nor a biseparable state then we can conclude that the state is a genuinely entangled state. The results obtained in this chapter are supported by examples. At the end of this chapter, we have provided an idea to generalize the obtained results to higher dimensional and multi-qubit systems.

Chapter 5

Classification of Three-qubit States using SPA-PT

In this chapter¹, we have studied the problem of classification of three-qubit states using a structural physical approximation of partial transposition (SPA-PT). We have exploited the concept of SPA-PT so that the classification of a three-qubit entangled state may be realized in an experiment. To study the classification problem of the three-qubit system, we have constructed a SPA-PT map for the three-qubit quantum system and then the matrix elements of the density matrix describing the SPA-PT of a three-qubit system have been calculated. We have proposed criteria for the classification of all possible SLOCC inequivalent classes of pure as well as mixed three-qubit states through the SPA-PT map.

¹This chapter is based on the research papers entitled “Structural physical approximation of partial transposition makes possible to distinguish SLOCC inequivalent classes of three-qubit system, *European Physical Journal D* **76** 73 (2022)”

5.1 Introduction

Quantum entanglement [20] is a physical phenomenon in which the state of each particle in the group cannot be described independently of the state of the others, even when the particles are separated by a great distance. The entangled quantum system in $d_1 \otimes d_2$ dimensional Hilbert space may be useful in various quantum information processing tasks such as quantum teleportation [30], remote state preparation [40], entanglement swapping [158], secret sharing [159] and quantum repeater [160]. We require entangled states to perform quantum information processing tasks in an efficient way, but the process of generation of entangled states is not an easy task. Even if we generate a quantum state in an experiment, we can ask two more questions: (i) whether the generated multiparticle state is an entangled state or not. (ii) If we found that the multiparticle state is an entangled state, then what type of entangled state it is? For instance, in the case of a three-qubit system, if we know that the three-qubit entangled state is generated at the output, then it is necessary to know whether the generated three-qubit entangled state belongs to a biseparable state or genuine entangled state. Further, if we know that the generated three-qubit entangled state is a genuine entangled state, then it is important to classify it further as two SLOCC inequivalent classes i.e GHZ class and W class. The classification of three-qubit genuine entanglement into these subclasses is important from the quantum communication point of view. Now, we can answer the above questions, if we are able to proceed a little bit further toward the problem of "classification of multi-qubit entangled states". The classification problem starts with a three-qubit system. A three-qubit pure system can be classified as one fully separable state, three biseparable states, and two genuinely entangled states. Genuine entangled states have entanglement in all the subsystems whereas biseparable states have entanglement in two of the three subsystems. The classification of genuine entangled states and biseparable states is equally important as they have their own merits. One of the possible merits of genuine entangled states and biseparable states is the following: genuine entangled states have potential applications in quantum communication [142] whereas biseparable states are useful in obtaining the non-zero conference key [145]. In chapter 4, we have defined a way to classify a three-qubit system using l_1 -norm of coherence but that method may not be realized physically. To overcome this issue, in this chapter, we have used the method of SPA-PT for the classification of a three-qubit

system.

5.2 Studying the Effect of Partial Transposition Operation on one of the qubit of a Three-Qubit System

Here, we will study the effect of partial transposition operation on any one of the qubit of a three-qubit system shared between Alice, Bob, and Charlie. Let us assume that any three-qubit state is described by the density operator ρ_{ABC} . If the entries of the three-qubit state ρ_{ABC} are represented by the 2×2 block matrices then it is given by

$$\rho_{ABC} = \begin{pmatrix} A & B & C & D \\ B^* & E & F & G \\ C^* & F^* & H & I \\ D^* & G^* & I^* & J \end{pmatrix} \quad (5.2.1)$$

where $A, B, C, D, E, F, G, H, I, J$ denote the 2×2 block matrices.

When the partial transposition operation acts on the first qubit A of the state ρ_{ABC} , the state transformed as

$$\rho_{ABC} \rightarrow \rho_{ABC}^{T_A} \equiv [T \otimes id \otimes id](\rho_{ABC}) \quad (5.2.2)$$

The partial transposition with respect to the second and third qubit respectively reduces the state ρ_{ABC} to

$$\rho_{ABC} \rightarrow \rho_{ABC}^{T_B} \equiv [id \otimes T \otimes id](\rho_{ABC}) \quad (5.2.3)$$

$$\rho_{ABC} \rightarrow \rho_{ABC}^{T_C} \equiv [id \otimes id \otimes T](\rho_{ABC}) \quad (5.2.4)$$

The partial transposed states $\rho_{ABC}^{T_A}$, $\rho_{ABC}^{T_B}$, $\rho_{ABC}^{T_C}$ can be expressed in terms of block matrices as

$$\rho_{ABC}^{T_A} = \begin{pmatrix} A & B & C^* & F^* \\ B^* & E & D^* & G^* \\ C & D & H & I \\ F & G & I^* & J \end{pmatrix} \quad (5.2.5)$$

$$\rho_{ABC}^{T_B} = \begin{pmatrix} A & B^* & C & F \\ B & E & D & G \\ C^* & D^* & H & I^* \\ F^* & G^* & I & J \end{pmatrix} \quad (5.2.6)$$

$$\rho_{ABC}^{T_C} = \begin{pmatrix} A^* & B^* & C^* & D^* \\ B & E^* & F^* & G^* \\ C & F & H^* & I^* \\ D & G & I & J^* \end{pmatrix} \quad (5.2.7)$$

It is well known that the partial transposition criterion is necessary and sufficient for $2 \otimes 2$ and $2 \otimes 3$ system while it is only necessary condition for the system $m \otimes n$, ($m \geq 2, n \geq 3$, If $m = 2$ then $n \neq 3$) and for the multipartite system also.

We now consider the simplest tripartite system i.e. $2 \otimes 2 \otimes 2$ quantum system to classify its different SLOCC inequivalent classes through partial transposition operation on any one of the single qubit of the given three-qubit system.

(i) Let us choose a particular form of an arbitrary state lying in the *GHZ*-class, which is given by

$$|GHZ\rangle_{ABC} = \alpha|000\rangle_{ABC} + \beta|111\rangle_{ABC}, \quad |\alpha|^2 + |\beta|^2 = 1 \quad (5.2.8)$$

The density operator $\rho_{GHZ} = |GHZ\rangle_{ABC}\langle GHZ|$ can be expressed as

$$\rho^{GHZ} = \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ B_1^* & E_1 & F_1 & G_1 \\ C_1^* & F_1^* & H_1 & I_1 \\ D_1^* & G_1^* & I_1^* & J_1 \end{pmatrix} \quad (5.2.9)$$

where $A_1 = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & 0 \end{pmatrix}$, $D_1 = \begin{pmatrix} 0 & \alpha\beta^* \\ 0 & 0 \end{pmatrix}$, $J_1 = \begin{pmatrix} 0 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$ and all other 2×2 block matrices are null matrices.

If we apply partial transposition operation on the qubit *A* of the state described by the density operator $(\rho_{ABC}^{GHZ})^{T_A}$ then the partially transposed state $(\rho_{ABC}^{GHZ})^{T_A}$ at the output can be obtained by the prescription given in (5.2.5). The eigenvalues of $(\rho_{ABC}^{GHZ})^{T_A}$ are given by $\{0, 0, 0, 0, |\alpha|^2, |\beta|^2, |\alpha||\beta|, -|\alpha||\beta|\}$. Thus, $(\rho_{ABC}^{GHZ})^{T_A}$ has one negative eigen-

value. The minimum eigenvalue of $(\rho_{ABC}^{GHZ})^{T_A}$ is given by

$$\lambda_{min}(\alpha, \beta) = -|\alpha||\beta| \quad (5.2.10)$$

Since the minimum eigenvalue of $(\rho_{ABC}^{GHZ})^{T_A}$ is negative so the state ρ_{ABC}^{GHZ} under investigation is an entangled state for all non-zero values of the state parameter α and β . The most negative eigenvalue is important for more than one reason, which will be clear in the later stage. The most negative eigenvalue can be obtained for $|\alpha| = \frac{1}{\sqrt{2}}$ and $|\beta| = \frac{1}{\sqrt{2}}$. Therefore, the minimum most eigenvalue of $(\rho_{ABC}^{GHZ})^{T_A}$ is given by

$$\lambda_{min}[(\rho_{ABC}^{GHZ})^{T_A}] = -\frac{1}{2} \quad (5.2.11)$$

For the same value of α and β i.e. for $|\alpha| = \frac{1}{\sqrt{2}}$ and $|\beta| = \frac{1}{\sqrt{2}}$, we can obtain the maximum value of tangle τ which is given by $\tau = 1$.

Proceeding in a similar way, we can obtain the minimum most eigenvalue of $\rho_{GHZ}^{T_B}$ and $\rho_{GHZ}^{T_C}$ and they are given by

$$\lambda_{min}[(\rho_{ABC}^{GHZ})^{T_B}] = \lambda_{min}[(\rho_{ABC}^{GHZ})^{T_C}] = -\frac{1}{2} \quad (5.2.12)$$

Since the minimum most eigenvalues of the partially transposed state with respect to the qubits A , B , and C are the same so we denote it by $\lambda_{min}[\rho_{ABC}^{GHZ}]$. Thus, we have $\lambda_{min}[(\rho_{ABC}^{GHZ})^{T_A}] = \lambda_{min}[(\rho_{ABC}^{GHZ})^{T_B}] = \lambda_{min}[(\rho_{ABC}^{GHZ})^{T_C}] \equiv \lambda_{min}[\rho_{ABC}^{GHZ}]$.

(ii) Next, we will choose a particular form of an arbitrary state belonging to the W -class, which is given by

$$|W\rangle_{ABC} = \lambda_0|001\rangle_{ABC} + \lambda_1|010\rangle_{ABC} + \lambda_2|100\rangle_{ABC} \quad (5.2.13)$$

where the state parameters $\lambda_i (i = 0, 1, 2)$ are real numbers satisfying $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$. We may note here that one may choose a general form of W class of states for detailed analysis but may face difficulty in finding the analytical form of eigenvalues. Thus, we restrict ourselves to studying one of the particular forms defined in (5.2.13). The

density operator ρ_{ABC}^W can be expressed as

$$\rho_{ABC}^W = \begin{pmatrix} A & B & C & D \\ B^* & E & F & G \\ C^* & F^* & H & I \\ D^* & G^* & I^* & J \end{pmatrix} \quad (5.2.14)$$

where,

$$\begin{aligned} A &= \begin{pmatrix} 0 & 0 \\ 0 & \lambda_0^2 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ \lambda_0 \lambda_1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ \lambda_0 \lambda_2 & 0 \end{pmatrix}, \\ E &= \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} \lambda_1 \lambda_2 & 0 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} \lambda_2^2 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (5.2.15)$$

D, G, I, and J are zero matrices.

The eigenvalues of $(\rho_{ABC}^W)^{T_A}$ can be calculated as $\{0, 0, 0, 0, \lambda_0^2 + \lambda_1^2, \lambda_2^2, \lambda_2 \sqrt{\lambda_0^2 + \lambda_1^2}, -\lambda_2 \sqrt{\lambda_0^2 + \lambda_1^2}\}$. Thus, $(\rho_{ABC}^W)^{T_A}$ has one negative eigenvalue irrespective of the sign of the real parameters $\lambda_0, \lambda_1, \lambda_2$. The minimum eigenvalue of $(\rho_{ABC}^W)^{T_A}$ is given by

$$\begin{aligned} \lambda_{min}(\lambda_0, \lambda_1, \lambda_2) &= -|\lambda_2| \sqrt{\lambda_0^2 + \lambda_1^2} \\ &= -|\lambda_2| \sqrt{1 - \lambda_2^2} \end{aligned} \quad (5.2.16)$$

In this case also, we find that the minimum eigenvalue of $(\rho_{ABC}^W)^{T_A}$ is negative so the state described by the density operator ρ_{ABC}^W is an entangled state for all non-zero values of the state parameter $\lambda_i (i = 0, 1, 2)$.

The most negative eigenvalue can be obtained for $|\lambda_2| = \frac{7}{10}$ and for any value of λ_0 and λ_1 satisfying $\lambda_0^2 + \lambda_1^2 = 0.51$. Thus, the minimum most eigenvalue of $(\rho_{ABC}^W)^{T_A}$ is given by

$$\lambda[(\rho_{ABC}^W)^{T_A}] = -0.4999 \quad (5.2.17)$$

Proceeding in a similar way, the eigenvalues of $(\rho_{ABC}^W)^{T_B}$ and $(\rho_{ABC}^W)^{T_C}$ can be calculated as $\{0, 0, 0, 0, \lambda_1^2, \lambda_0^2 + \lambda_2^2, \lambda_1 \sqrt{\lambda_0^2 + \lambda_2^2}, -\lambda_1 \sqrt{\lambda_0^2 + \lambda_2^2}\}$ and $\{0, 0, 0, 0, \lambda_0^2, \lambda_1^2 + \lambda_2^2, \lambda_0 \sqrt{\lambda_1^2 + \lambda_2^2}, -\lambda_0 \sqrt{\lambda_1^2 + \lambda_2^2}\}$ respectively. Following the same procedure, we can ob-

tain the minimum most eigenvalue of $(\rho_{ABC}^W)^{T_B}$ and $(\rho_{ABC}^W)^{T_C}$ respectively as

$$\lambda_{\min}[(\rho_{ABC}^W)^{T_B}] = \lambda_{\min}[(\rho_{ABC}^W)^{T_C}] = -0.4999 \quad (5.2.18)$$

Since the minimum most eigenvalue of the partial transposed state with respect to the qubits A , B and C are same so we can denote it by $\lambda_{\min}[\rho_{ABC}^W]$. Thus, we have $\lambda_{\min}^{\rho_{ABC}^W} = \lambda_{\min}^{\rho_{ABC}^W} = \lambda_{\min}^{\rho_{ABC}^W} = \lambda_{\min}^{\rho_{ABC}^W}$.

5.3 Structural physical approximation of partial transposition (SPA-PT) of a single qubit in a three-qubit system

Let us consider a map that may be defined as the convex combination of the depolarizing map and the partial transposition map. We can mix the depolarizing map with the partial transposition map with respect to any one of the qubits in such a way that the resulting map is completely positive. Let us now start our discussion on SPA-PT of the three-qubit when the partial transposition operation has been performed with respect to the qubit A . Therefore, the newly constructed SPA-PT map, when partial transposition is taken with respect to the qubit A is denoted by $[T \otimes \widetilde{id} \otimes id]$.

After applying the SPA-PT map on the qubit A of the three-qubit state ρ_{ABC} , the state transformed as

$$[T \otimes \widetilde{id} \otimes id]\rho_{ABC} \equiv \widetilde{\rho_{ABC}^{T_A}} = \frac{p_A}{8}(I \otimes I \otimes I) + (1 - p_A)[T \otimes id \otimes id](\rho_{ABC}) \quad (5.3.1)$$

where $0 \leq p_A \leq 1$.

In a similar way, SPA-PT with respect to the qubit B and C respectively transformed the state ρ_{ABC} as

$$[id \otimes \widetilde{T} \otimes id]\rho_{ABC} \equiv \widetilde{\rho_{ABC}^{T_B}} = \frac{p_B}{8}(I \otimes I \otimes I) + (1 - p_B)[id \otimes T \otimes id](\rho_{ABC}) \quad (5.3.2)$$

$$[id \otimes id \otimes \widetilde{T}]\rho_{ABC} \equiv \widetilde{\rho_{ABC}^{T_C}} = \frac{p_C}{8}(I \otimes I \otimes I) + (1 - p_C)[id \otimes id \otimes T](\rho_{ABC}) \quad (5.3.3)$$

where $0 \leq p_B, p_C \leq 1$.

5.3.1 When Structural Physical Approximation Map will be Completely Positive?

In this section, we derive the condition for which the SPA-PT map is completely positive. To start deducing the condition, we first consider the approximation of partial transposition operation with respect to the qubit A . In a similar fashion, one can deduce the same condition by approximating the partial transposition with respect to the other two qubits B and C respectively.

We say that the SPA-PT map with respect to the qubit A is positive if $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}) \geq 0$ holds. Therefore, using (5.3.1), we can write the expression of minimum eigenvalue of the operator $\widetilde{\rho_{ABC}^{T_A}}$ as

$$\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}) = \lambda_{\min}\left[\frac{p_A}{8}(I \otimes I \otimes I) + (1 - p_A)(T \otimes id \otimes id)\rho_{ABC}\right] \quad (5.3.4)$$

Further, the R.H.S of (5.3.4) can be reduced using Weyl's inequality as

$$\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}) \geq \frac{p_A}{8}\lambda_{\min}(I \otimes I \otimes I) + (1 - p_A)\lambda_{\min}[(T \otimes id \otimes id)\rho_{ABC}] \quad (5.3.5)$$

If $\lambda_{\min}[(T \otimes id \otimes id)\rho_{ABC}] \equiv \lambda_{\min}(\rho_{ABC}^{T_A}) \geq 0$, then the above inequality (5.3.5) reduces to

$$\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}) \geq \frac{p_A}{8} \quad (5.3.6)$$

Now, our task is to find out the minimum value of p_A for which the operator $(T \otimes id \otimes id)$ will be completely positive. Since the partial transposition operator is not a completely positive operator so the induced map $[(id \otimes id \otimes id) \otimes (T \otimes id \otimes id)]$ generates at least one negative eigenvalue. The most negative eigenvalue generated when the induced map $[(id \otimes id \otimes id) \otimes (T \otimes id \otimes id)]$ is applying on the state $[(I \otimes I \otimes I) \otimes |GHZ\rangle_{ABC}]$, where $|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Thus, if we suitably choose the minimum value of p_A for which the positive eigenvalues of the maximally mixed three-qubit state generated by the depolarizing map dominate over the minimum most negative eigenvalue generated by the induced map then we can make the operator $(T \otimes id \otimes id)$ completely positive. Therefore, the map $(T \otimes id \otimes id)$ is completely positive and hence physically implementable when

$$p_A \geq \frac{4}{5} \quad (5.3.7)$$

In a similar way, it can be shown that if we take the partial transposition with respect to system B and C then the SPA-PT map will be completely positive when

$$p_B \geq \frac{4}{5} \quad (5.3.8)$$

$$p_C \geq \frac{4}{5} \quad (5.3.9)$$

5.3.2 Determination of the matrix elements of the density matrix after SPA-PT operation

In this section, we study how the entries of the approximated partial transposed density matrix denoted by $\widetilde{\rho_{ABC}^{T_A}}$ are related with the entries of the original matrix described by the density matrix ρ_{ABC} . If we have an arbitrary three-qubit state described by the density operator ρ_{ABC} then after the application of SPA-PT operation with respect to qubit A on it, the density matrix has been changed and changes to $\widetilde{\rho_{ABC}^{T_A}}$. As a consequence, the elements of the matrix $\widetilde{\rho_{ABC}^{T_A}}$ can be expressed in terms of the matrix elements of ρ_{ABC} . Thus, the determination of the matrix elements of the density matrix $\widetilde{\rho_{ABC}^{T_A}}$ is important because the entanglement properties of ρ_{ABC} can be studied using the matrix elements of $\widetilde{\rho_{ABC}^{T_A}}$. To start with, we consider an arbitrary three-qubit quantum state described by the density matrix ρ_{ABC} , which is given by

$$\rho_{ABC} = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} & t_{17} & t_{18} \\ t_{12}^* & t_{22} & t_{23} & t_{24} & t_{25} & t_{26} & t_{27} & t_{28} \\ t_{13}^* & t_{23}^* & t_{33} & t_{34} & t_{35} & t_{36} & t_{37} & t_{38} \\ t_{14}^* & t_{24}^* & t_{34}^* & t_{44} & t_{45} & t_{46} & t_{47} & t_{48} \\ t_{15}^* & t_{25}^* & t_{35}^* & t_{45}^* & t_{55} & t_{56} & t_{57} & t_{58} \\ t_{16}^* & t_{26}^* & t_{36}^* & t_{46}^* & t_{56}^* & t_{66} & t_{67} & t_{68} \\ t_{17}^* & t_{27}^* & t_{37}^* & t_{47}^* & t_{57}^* & t_{67}^* & t_{77} & t_{78} \\ t_{18}^* & t_{28}^* & t_{38}^* & t_{48}^* & t_{58}^* & t_{68}^* & t_{78}^* & t_{88} \end{pmatrix}, \sum_{i=1}^8 t_{ii} = 1 \quad (5.3.10)$$

where $(*)$ denotes the complex conjugate.

The SPA-PT with respect to qubit A of a three-qubit quantum state ρ_{ABC} is given by

$$\widetilde{\rho_{ABC}^{T_A}} = \left[\frac{1}{10}(I \otimes I \otimes I) + \frac{1}{5}(T \otimes I \otimes I)\rho_{ABC} \right] \quad (5.3.11)$$

where T denotes the transposition operator acting on qubit A .

The matrix representation of $\widetilde{\rho_{ABC}^{T_A}}$ is given by

$$\widetilde{\rho_{ABC}^{T_A}} = \begin{pmatrix} \tilde{t}_{11} & \tilde{t}_{12} & \tilde{t}_{13} & \tilde{t}_{14} & \tilde{t}_{15} & \tilde{t}_{16} & \tilde{t}_{17} & \tilde{t}_{18} \\ \tilde{t}_{12}^* & \tilde{t}_{22} & \tilde{t}_{23} & \tilde{t}_{24} & \tilde{t}_{25} & \tilde{t}_{26} & \tilde{t}_{27} & \tilde{t}_{28} \\ \tilde{t}_{13}^* & \tilde{t}_{23}^* & \tilde{t}_{33} & \tilde{t}_{34} & \tilde{t}_{35} & \tilde{t}_{36} & \tilde{t}_{37} & \tilde{t}_{38} \\ \tilde{t}_{14}^* & \tilde{t}_{24}^* & \tilde{t}_{34}^* & \tilde{t}_{44} & \tilde{t}_{45} & \tilde{t}_{46} & \tilde{t}_{47} & \tilde{t}_{48} \\ \tilde{t}_{15}^* & \tilde{t}_{25}^* & \tilde{t}_{35}^* & \tilde{t}_{45}^* & \tilde{t}_{55} & \tilde{t}_{56} & \tilde{t}_{57} & \tilde{t}_{58} \\ \tilde{t}_{16}^* & \tilde{t}_{26}^* & \tilde{t}_{36}^* & \tilde{t}_{46}^* & \tilde{t}_{56}^* & \tilde{t}_{66} & \tilde{t}_{67} & \tilde{t}_{68} \\ \tilde{t}_{17}^* & \tilde{t}_{27}^* & \tilde{t}_{37}^* & \tilde{t}_{47}^* & \tilde{t}_{57}^* & \tilde{t}_{67}^* & \tilde{t}_{77} & \tilde{t}_{78} \\ \tilde{t}_{18}^* & \tilde{t}_{28}^* & \tilde{t}_{38}^* & \tilde{t}_{48}^* & \tilde{t}_{58}^* & \tilde{t}_{68}^* & \tilde{t}_{78}^* & \tilde{t}_{88} \end{pmatrix}, \sum_{i=1}^8 \tilde{t}_{ii} = 1 \quad (5.3.12)$$

where the entries of the density matrix $\widetilde{\rho_{ABC}^{T_A}}$ are given by,

$$\begin{aligned} \tilde{t}_{11} &= \frac{1}{10} + \frac{t_{11}}{5}, \tilde{t}_{12} = \frac{t_{12}}{5}, \tilde{t}_{13} = \frac{t_{13}}{5}, \tilde{t}_{14} = \frac{t_{14}}{5}, \tilde{t}_{15} = \frac{t_{15}^*}{5}, \tilde{t}_{16} = \frac{t_{25}^*}{5}, \tilde{t}_{17} = \frac{t_{35}^*}{5}, \tilde{t}_{18} = \frac{t_{45}^*}{5} \\ \tilde{t}_{22} &= \frac{1}{10} + \frac{t_{22}}{5}, \tilde{t}_{23} = \frac{t_{23}}{5}, \tilde{t}_{24} = \frac{t_{24}}{5}, \tilde{t}_{25} = \frac{t_{16}^*}{5}, \tilde{t}_{26} = \frac{t_{26}^*}{5}, \tilde{t}_{27} = \frac{t_{36}^*}{5}, \tilde{t}_{28} = \frac{t_{46}^*}{5}, \tilde{t}_{33} = \frac{1}{10} + \frac{t_{33}}{5} \\ \tilde{t}_{34} &= \frac{t_{34}}{5}, \tilde{t}_{35} = \frac{t_{17}^*}{5}, \tilde{t}_{36} = \frac{t_{27}^*}{5}, \tilde{t}_{37} = \frac{t_{37}^*}{5}, \tilde{t}_{38} = \frac{t_{47}^*}{5}, \tilde{t}_{44} = \frac{1}{10} + \frac{t_{44}}{5}, \tilde{t}_{45} = \frac{t_{18}^*}{5}, \tilde{t}_{46} = \frac{t_{28}^*}{5} \\ \tilde{t}_{47} &= \frac{t_{38}^*}{5}, \tilde{t}_{48} = \frac{t_{48}^*}{5}, \tilde{t}_{55} = \frac{1}{10} + \frac{t_{55}}{5}, \tilde{t}_{56} = \frac{t_{56}}{5}, \tilde{t}_{57} = \frac{t_{57}}{5}, \tilde{t}_{58} = \frac{t_{58}}{5}, \tilde{t}_{66} = \frac{1}{10} + \frac{t_{66}}{5}, \tilde{t}_{67} = \frac{t_{67}}{5} \\ \tilde{t}_{68} &= \frac{t_{68}}{5}, \tilde{t}_{77} = \frac{1}{10} + \frac{t_{77}}{5}, \tilde{t}_{78} = \frac{t_{78}}{5}, \tilde{t}_{88} = \frac{1}{10} + \frac{t_{88}}{5} \end{aligned} \quad (5.3.13)$$

Following the same procedure, one can determine the matrix elements of the density matrix resulting from the application of completely positive maps $[id \otimes \widetilde{T} \otimes id]\rho_{ABC}$ and $[id \otimes id \otimes \widetilde{T}]\rho_{ABC}$ respectively.

In the next section, we will show that the minimum eigenvalue of $\widetilde{\rho_{ABC}^{T_A}}$, $\widetilde{\rho_{ABC}^{T_B}}$ and $\widetilde{\rho_{ABC}^{T_C}}$ is the entity that may detect whether the given three-qubit state ρ_{ABC} possess the property of entanglement or not, so, it is very essential to extract the information about the entries of the matrix $\widetilde{\rho_{ABC}^{T_A}}$, $\widetilde{\rho_{ABC}^{T_B}}$ and $\widetilde{\rho_{ABC}^{T_C}}$. Thus the matrix elements given by (5.3.13) play a vital role in detecting the entanglement of a three-qubit system when the SPA-PT operation is performed with respect to the system A .

5.4 Necessary condition for the separability (either in the form of a full separability or biseparability) of a three-qubit state

In this section, we will derive the necessary condition for the full separability and biseparability of a three-qubit state. Thus, if any three-qubit state violates the necessary condition then we can infer that the given three-qubit state is a genuine entangled state.

To move forward in this direction, we consider any three-qubit state shared between three distant parties *Alice*(A), *Bob*(B), and *Charlie*(C) and ask whether the shared state is entangled or not. To detect the entanglement in a three-qubit system, one may follow the partial transposition criterion and thus apply partial transposition operation on any one of the qubits of the given three-qubit system. To overcome the difficulty of the real implementation of the partial transposition map in an experiment, we approximate the partial transposition operation by the method of SPA. We have already shown in the previous section that the SPA-PT map can serve as a completely positive map and thus can be implemented in a real experimental setup. Now we are in a position to give the statement of a necessary condition of the separability and biseparability of a three-qubit state.

Theorem-1: If the state described by the density operator ρ_{ABC} denoting either a separable state of the form $A - B - C$ or a biseparable state of the form $A - BC$ then the following inequality is satisfied

$$\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}) \geq \frac{1}{10} \quad (5.4.1)$$

Proof: The required inequality (5.4.1) follows from (5.3.6) and (5.3.7).

Theorem-2: If the state described by the density operator ρ_{ABC} denoting either a separable state of the form $A - B - C$ or a biseparable state of the form $B - AC$ then the following inequality is satisfied

$$\lambda_{\min}(\widetilde{\rho_{ABC}^{T_B}}) \geq \frac{1}{10} \quad (5.4.2)$$

Theorem-3: If the state described by the density operator ρ_{ABC} denoting either a separable state of the form $A - B - C$ or a biseparable state of the form $C - AB$ then

the following inequality is satisfied

$$\lambda_{\min}(\widetilde{\rho_{ABC}^{T_C}}) \geq \frac{1}{10} \quad (5.4.3)$$

Let us now provide a few results that may help in classifying the given three-qubit state ρ_{ABC} as either a separable state, a biseparable state, or a genuine entangled state. To do this task, we assume that $\lambda_{\min}(\widetilde{\rho_{ABC}}) = \max\{\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}, \lambda_{\min}(\widetilde{\rho_{ABC}^{T_B}}, \lambda_{\min}(\widetilde{\rho_{ABC}^{T_C}})\}$.

Result-1: If $\lambda_{\min}(\widetilde{\rho_{ABC}}) < \frac{1}{10}$, then ρ_{ABC} is a genuine entangled state.

Result-2: If $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}) \geq \frac{1}{10}$ and either $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_B}}) < \frac{1}{10}$ or $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_C}}) < \frac{1}{10}$ or both $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_B}}, \lambda_{\min}(\widetilde{\rho_{ABC}^{T_C}}) < \frac{1}{10}$ holds, then ρ_{ABC} is biseparable in $A - BC$ cut.

Result-3: If $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_B}}) \geq \frac{1}{10}$ and either $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}) < \frac{1}{10}$ or $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_C}}) < \frac{1}{10}$ or both $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}, \lambda_{\min}(\widetilde{\rho_{ABC}^{T_C}}) < \frac{1}{10}$ holds, then ρ_{ABC} is biseparable in $B - AC$ cut.

Result-4: If $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_C}}) \geq \frac{1}{10}$ and either $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}) < \frac{1}{10}$ or $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_B}}) < \frac{1}{10}$ or both $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}, \lambda_{\min}(\widetilde{\rho_{ABC}^{T_B}}) < \frac{1}{10}$ holds, then ρ_{ABC} is biseparable in $C - AB$ cut.

Result-5: If $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_A}}) \geq \frac{1}{10}$, $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_B}}) \geq \frac{1}{10}$ and $\lambda_{\min}(\widetilde{\rho_{ABC}^{T_C}}) \geq \frac{1}{10}$ holds, then ρ_{ABC} is a fully separable state.

5.5 A Few Examples

In this section, we discuss a few examples of three-qubit genuine entangled states and three-qubit biseparable states that can be detected by the results given in the previous section.

5.5.1 Genuine Entangled States

Example-1: Let us consider the state $|\psi_{G_1}\rangle$ described by the density operator $\rho_{G_1} = |\psi_{G_1}\rangle\langle\psi_{G_1}|$, where $|\psi_{G_1}\rangle = \alpha|000\rangle + \beta|111\rangle$, $|\alpha|^2 + |\beta|^2 = 1$. We now proceed to calculate the minimum eigenvalue of $\widetilde{\rho_{G_1}^{T_A}}$, $\widetilde{\rho_{G_1}^{T_B}}$ and $\widetilde{\rho_{G_1}^{T_C}}$.

The eigenvalues are given by

$$\begin{aligned} \lambda_{\min}(\widetilde{\rho_{G_1}^{T_A}}) &= \lambda_{\min}(\widetilde{\rho_{G_1}^{T_B}}) = \lambda_{\min}(\widetilde{\rho_{G_1}^{T_C}}) \equiv \lambda_{\min}(\widetilde{\rho_{G_1}}) \\ &= \frac{1 - 2\alpha\beta}{10} \end{aligned} \quad (5.5.1)$$

Table 5.1: Table verifying Result-1 for ρ_{G_3} for different values of the state parameters $(\lambda_0, \lambda_1, \lambda_2)$

State parameter $(\lambda_0, \lambda_1, \lambda_2)$	Minimum eigenvalue of SPA-PT state w.r.t qubit A and C $\lambda_{min}(\widetilde{\rho_{G_3}^{T_A}}), \lambda_{min}(\widetilde{\rho_{G_3}^{T_C}})$	Minimum eigenvalue of SPA-PT state w.r.t qubit B $\lambda_{min}(\widetilde{\rho_{G_3}^{T_B}})$	$\lambda_{min}(\widetilde{\rho_{G_3}})$
(0.7, 0.1, 0.707107)	0.00101	1.295×10^{-18}	0.00101
(0.3, 0.4, 0.866)	0.048	0.0134	0.048
(0.7, 0.3, 0.648)	0.0093	0.0013	0.0093
(0.1, 0.2, 0.9747)	0.0805	0.056	0.0805
(0.2, 0.4, 0.8944)	0.0642	0.02	0.0642

Thus, we can easily find that $\lambda_{min}(\widetilde{\rho_{G_1}}) < \frac{1}{10}$ when $\alpha\beta > 0$. Thus it satisfies Result-1 and hence, we can say that the state $|\psi_{G_1}\rangle$ represent genuine entangled state when $\alpha\beta > 0$.

Example-2: Let us consider a pure three-qubit state which is given by

$$|\psi_{G_2}\rangle = \frac{1}{\sqrt{5}}[|000\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle] \quad (5.5.2)$$

For the given state $|\psi_{G_2}\rangle$, we have

$$\begin{aligned} \lambda_{min}(|\psi_{G_2}\rangle^{T_A}\langle\psi_{G_2}|) &= 0.030718, \lambda_{min}(|\psi_{G_2}\rangle^{T_B}\langle\psi_{G_2}|) = \\ \lambda_{min}(|\psi_{G_2}\rangle^{T_C}\langle\psi_{G_2}|) &= 0.0434315 \end{aligned} \quad (5.5.3)$$

Therefore, we find that $\lambda_{min}(|\psi_{G_2}\rangle\langle\psi_{G_2}|) = \max\{0.030718, 0.0434315\} = 0.0434315$. Thus, $\lambda_{min}(|\psi_{G_2}\rangle\langle\psi_{G_2}|) < \frac{1}{10}$. Hence, the given state $|\psi_{G_2}\rangle\langle\psi_{G_2}|$ is a genuine entangled state.

Example-3: Let us take another state defined by $\rho_{G_3} = |\psi_{G_3}\rangle\langle\psi_{G_3}|$, where $|\psi_{G_3}\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_2|111\rangle, 0 \leq \lambda_i \leq 1, \sum_{i=0}^2 \lambda_i^2 = 1$. For the given state ρ_{G_3} , we can easily verify that for all values of λ_0, λ_1 and λ_2 lying between 0 and 1, we have, $\lambda_{min}(\widetilde{\rho_{G_3}}) < \frac{1}{10}$. Further, we have calculated the values of $\lambda_{min}(\widetilde{\rho_{G_3}})$ by taking some values of λ_0, λ_1 and λ_2 and those values are tabulated in the Table 5.1 for the verification of our result. Thus from Result-1, the given state ρ_{G_3} is a genuine three-qubit entangled state.

Example-4 Consider the state defined by $\rho_{GHZ,W} = q|GHZ\rangle\langle GHZ| + (1-q)|W\rangle\langle W|$, $0 \leq q \leq 1$, where $|GHZ\rangle = \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle]$ and $|W\rangle = \frac{1}{\sqrt{3}}[|001\rangle + |010\rangle + |100\rangle]$. For the given state described by the density operator $\rho_{GHZ,W}$, the minimum eigenvalues are given by

$$\begin{aligned}\lambda_{\min}(\widetilde{\rho_{GHZ,W}^{T_A}}) &= \lambda_{\min}(\widetilde{\rho_{GHZ,W}^{T_B}}) = \lambda_{\min}(\widetilde{\rho_{GHZ,W}^{T_C}}) \\ &= \min\{Q_1, Q_2\}\end{aligned}\quad (5.5.4)$$

where $Q_1 = \frac{1}{30}(4 - q - \sqrt{1 - 2q + 10q^2})$, and $Q_2 = \frac{1}{60}(6 + 3q - \sqrt{32 - 64q + 41q^2})$. It can be easily seen that $\min\{Q_1, Q_2\} < \frac{1}{10}$ and thus, we have $\lambda_{\min}(\widetilde{\rho_{GHZ,W}}) < \frac{1}{10}$. Hence from Result-1, We can say that the given state $\rho_{GHZ,W}$ is a genuine entangled state for all $q \in [0, 1]$.

5.5.2 Biseparable states

Example-1: Consider the state defined by the density matrix $\rho_{B_1} = q|0\rangle\langle 0| \otimes |\phi^+\rangle\langle \phi^+| + (1-q)|1\rangle\langle 1| \otimes |\phi^-\rangle\langle \phi^-|$, $0 \leq q \leq 1$, where $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}[|00\rangle \pm |11\rangle]$. For the given state ρ_{B_1} , the minimum eigenvalues of the partial transposed state are given by $\lambda_{\min}(\widetilde{\rho_{B_1}^{T_A}}) = \frac{1}{10}$, $\lambda_{\min}(\widetilde{\rho_{B_1}^{T_B}}) = \lambda_{\min}(\widetilde{\rho_{B_1}^{T_C}}) = \min\{\frac{q}{10}, \frac{1-q}{10}\}$. When the state parameter q satisfies the inequality $0 \leq q \leq 1$, we observe that the minimum eigenvalue satisfy

$$\lambda_{\min}(\widetilde{\rho_{B_1}^{T_A}}) = \frac{1}{10}, \quad \lambda_{\min}(\widetilde{\rho_{B_1}^{T_B}}) = \lambda_{\min}(\widetilde{\rho_{B_1}^{T_C}}) < \frac{1}{10}\quad (5.5.5)$$

Therefore, we can infer from Result-2, that the given state ρ_{B_1} is biseparable in $A - BC$ cut.

Example-2 Let us take a pure state, which is defined by $\rho_{B_2} = |\psi\rangle_{B_2}\langle \psi|$, where $|\psi\rangle_{B_2} = \lambda_0|001\rangle + \lambda_1|101\rangle + \lambda_2|111\rangle$, $\sum_{i=0}^2 \lambda_i^2 = 1$, $0 \leq \lambda_i \leq 1, (i = 0, 1, 2)$. We find that for the given state described by the density operator ρ_{B_2} that for all values of $\lambda_0, \lambda_1, \lambda_2 \in [0, 1]$, we have, $\lambda_{\min}(\widetilde{\rho_{B_2}^{T_A}}) < \frac{1}{10}$, $\lambda_{\min}(\widetilde{\rho_{B_2}^{T_B}}) < \frac{1}{10}$ and $\lambda_{\min}(\widetilde{\rho_{B_2}^{T_C}}) \geq \frac{1}{10}$. We have constructed Table 5.2 to clarify our result. Thus using Result-4, we can conclude that the given state ρ_{B_2} is biseparable in $AB - C$ cut.

Table 5.2: Table verifying Result-4 for ρ_{B_2} different values of the state parameters $(\lambda_0, \lambda_1, \lambda_2)$

State parameter $(\lambda_0, \lambda_1, \lambda_2)$	Minimum eigenvalue of SPA-PT state w.r.t qubit A and B $\lambda_{\min}(\widetilde{\rho}_{B_2}^{T_A}), \lambda_{\min}(\widetilde{\rho}_{B_2}^{T_B})$	Minimum eigenvalue of SPA-PT state w.r.t qubit C $\lambda_{\min}(\widetilde{\rho}_{B_2}^{T_C})$	$\lambda_{\min}(\widetilde{\rho}_{B_2})$
(0.1, 0.4, 0.911)	0.0818	0.1	0.1
(0.2, 0.4, 0.8944)	0.0642	0.1	0.1
(0.6, 0.1, 0.7937)	0.00475	0.1	0.1
(0.5, 0.4, 0.7681)	0.0232	0.1	0.1

5.5.3 Separable States

Example-1: Let us consider the state also known as Kay state [161, 162], which is defined by

$$\rho_{S_1} = \frac{1}{8+8a} \begin{pmatrix} 4+a & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & a & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & a & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & a & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & a & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & a & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & a & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 4+a \end{pmatrix}, \quad a \geq 2 \quad (5.5.6)$$

The state ρ_{S_1} is a fully separable for $a \geq 4$ and is a PPT entangled state in all possible partitions for $2 \leq a < 2\sqrt{2}$ [161].

For the given state ρ_{S_1} , we find that $\lambda_{\min}(\widetilde{\rho}_{S_1}^{T_A}) = \lambda_{\min}(\widetilde{\rho}_{S_1}^{T_B}) = \lambda_{\min}(\widetilde{\rho}_{S_1}^{T_C}) = \frac{2+5a}{40(1+a)}$. Thus, $\lambda_{\min}(\widetilde{\rho}_{S_1}) = \frac{2+5a}{40(1+a)}$. Hence from Result-5, we conclude that the given state ρ_{S_1} is fully separable for $a \geq 4$.

Example-2: Consider another state defined by $\rho_{S_2} = (1-\alpha)|GHZ\rangle\langle GHZ| + \frac{\alpha}{8}I_8$, $0 \leq \alpha \leq 1$, where $|GHZ\rangle = \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle]$. For the given state ρ_{S_2} , the minimum eigenvalues of $\rho_{S_2}^{T_A}$, $\rho_{S_2}^{T_B}$ and $\rho_{S_2}^{T_C}$ are given by $\lambda_{\min}(\widetilde{\rho}_{S_2}^{T_A}) = \lambda_{\min}(\widetilde{\rho}_{S_2}^{T_B}) = \lambda_{\min}(\widetilde{\rho}_{S_2}^{T_C}) = \frac{\alpha+4}{40}$. Thus, $\lambda_{\min}(\widetilde{\rho}_{S_2}) = \frac{\alpha+4}{40}$. For $0.8 < \alpha \leq 1$, we can check that $\lambda_{\min}(\widetilde{\rho}_{S_2}) > 0.1$. From Result-5, we can say that the given state ρ_{S_2} is fully separable for the state parameter α satisfying $0.8 < \alpha \leq 1$.

Example-3: Let us consider the state described by the density operator $\rho_{S_3} = q|\psi\rangle\langle\psi| + (1-q)|111\rangle\langle 111|$, $0 \leq q \leq 1$, where $|\psi\rangle = \frac{1}{\sqrt{2}}[|001\rangle + |101\rangle]$. The minimum eigenvalues of the partial transposed states are given by $\lambda_{\min}(\widetilde{\rho}_{S_3}^{T_A}) = \lambda_{\min}(\widetilde{\rho}_{S_3}^{T_B}) = \lambda_{\min}(\widetilde{\rho}_{S_3}^{T_C}) = \frac{1}{10}$. Thus, $\lambda_{\min}(\widetilde{\rho}_{S_3}) = \frac{1}{10}$. Therefore, Result-5 tells us that ρ_{S_3} is a fully separable state.

5.5.4 Genuine/Biseparable/Separable

Example-1 Let us consider the state defined by $\rho_1 = q|000\rangle\langle 000| + (1-q)|GHZ\rangle\langle GHZ|$, $0 \leq q \leq 1$, where $|GHZ\rangle = \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle]$. For the given state ρ_1 , we find that $\lambda_{\min}(\widetilde{\rho}_1^{T_A}) = \lambda_{\min}(\widetilde{\rho}_1^{T_B}) = \lambda_{\min}(\widetilde{\rho}_1^{T_C}) = \frac{q}{10}$. Thus,

$$\begin{aligned}\lambda_{\min}(\widetilde{\rho}_1) &= \frac{q}{10}, 0 \leq q < 1 \\ &= \frac{1}{10}, q = 1\end{aligned}\quad (5.5.7)$$

Hence, ρ_1 is a genuine entangled state for $0 \leq q < 1$ and fully separable state for $q = 1$.

Example-2: Let us consider a mixed state, which is a convex combination of GHZ , W and \widetilde{W} state, and it is defined as [163]

$$\begin{aligned}\rho_2 &= q_1|GHZ\rangle\langle GHZ| + q_2|W\rangle\langle W| \\ &+ (1 - q_1 - q_2)|\widetilde{W}\rangle\langle \widetilde{W}|, 0 \leq q_1, q_2 \leq 1\end{aligned}\quad (5.5.8)$$

where,

$$\begin{aligned}|GHZ\rangle &= \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle] \\ |W\rangle &= \frac{1}{\sqrt{3}}[|001\rangle + |010\rangle + |100\rangle] \\ |\widetilde{W}\rangle &= \frac{1}{\sqrt{3}}[|110\rangle + |101\rangle + |011\rangle]\end{aligned}\quad (5.5.9)$$

The minimum eigenvalue of SPA-PT of the state ρ_2 is given by $\lambda_{\min}(\widetilde{\rho}_2^{T_A}) = \lambda_{\min}(\widetilde{\rho}_2^{T_B}) = \lambda_{\min}(\widetilde{\rho}_2^{T_C}) = \frac{1}{30}(4 - q_1 - \sqrt{1 - 2q_1 + 10q_1^2 - 4q_2 + 4q_1q_2 + 4q_2^2})$. If the state parameters q_1 lying in the range $0.25 \leq q_1 \leq 1$ and $q_2 = \frac{1-q_1}{n}$, where n denote a positive integer, then within this range of parameters, the minimum eigenvalues of $\widetilde{\rho}_2^{T_A}$, $\widetilde{\rho}_2^{T_B}$ and $\widetilde{\rho}_2^{T_C}$ satisfying the inequality given by $\lambda_{\min}(\widetilde{\rho}_2^{T_A}) = \lambda_{\min}(\widetilde{\rho}_2^{T_B}) = \lambda_{\min}(\widetilde{\rho}_2^{T_C}) < \frac{1}{10}$. Thus, applying

Result-1, we find that ρ_2 is a genuine entangled state.

Now, our task is to classify two inequivalent classes of genuine entangled states by considering different cases.

Case-I: If the state parameters q_1 lying in the range $0.25 \leq q_1 \leq 0.6269$ and $q_2 = \frac{1-q_1}{n}$ then it has been shown that the three tangle of ρ_2 is zero [163]. Therefore, we find a sub-region in which not only the three-tangle of ρ_2 vanishes but also the state ρ_2 is genuinely entangled. Hence, we can conclude that the state described by the density operator ρ_2 represent a *W* class of state when the state parameters q_1 and q_2 satisfying $0.25 \leq q_1 \leq 0.629$, $q_2 = \frac{1-q_1}{n}$.

Case-II: If the state parameters q_1 lying in the range $0.6269 \leq q_1 \leq 1$ and $q_2 = \frac{1-q_1}{n}$ then the three tangle of genuine entangled state ρ_2 is non-zero. Therefore, the state ρ_2 represent a *GHZ* class of state when the state parameters q_1 and q_2 satisfying $0.6269 < q_1 \leq 1$, $q_2 = \frac{1-q_1}{n}$.

5.6 Conclusion

To conclude, we have used the SPA-PT map to classify a three-qubit system as six SLOCC inequivalent classes. We have started our study by investigating the effect of partial transposition operation on one qubit of a three-qubit system. We have provided a matrix representation of three-qubit partially transposed states, in terms of 2×2 block matrices, when partial transposition operation is performed with respect to the first qubit or the second qubit, or the third qubit. Then, we studied the application of the SPA-PT map on a three-qubit system and explicitly calculated the matrix elements of the matrix corresponding to the SPA-PT of a three-qubit state. Later, we proposed different criteria for the classification of all possible SLOCC inequivalent classes of pure as well as mixed three-qubit states. Since our classification criterion is based on the method of the SPA-PT map, so, we can realize it in an experiment. Thus, using our experimental-friendly criterion, one can classify all possible SLOCC inequivalent classes in a three-qubit system.

Chapter 6

Detection and quantification of entanglement

In this chapter,¹ we have studied the problem of detection and quantification of entangled state ρ in $d \otimes d$ dimensional bipartite quantum system by defining a physically realizable quantity, which we named as structured negativity ($N_S(\rho)$). The defined quantity ($N_S(\rho)$) depends on the minimum eigenvalue of the SPA-PT of the given density matrix ρ and the dimension of the system d . Later, we proved that the introduced quantity ($N_S(\rho)$) satisfies the properties of a valid entanglement monotone. Thereafter, we established a relationship between negativity and structured negativity. We conjecture that the negativity and structured negativity of ρ coincide when the number of negative eigenvalues of the partially transposed matrix ρ^{T_B} is $\frac{d(d-1)}{2}$.

¹This chapter is based on a research paper “Structured Negativity: A physically realizable measure of entanglement based on structural physical approximation, *Annals of Physics* **446** 169113 (2022)”

Quantification of entanglement is one of the crucial tasks in quantum information theory. The importance of this problem can be understood if we consider a simple instance in which we study the relationship between the amount of entanglement present in the shared arbitrary dimensional bipartite resource state and the fidelity of teleportation [164]. The quantification problem has already been studied for a two-qubit system, bipartite higher dimensional, and multi-qubit system but still there exist a few problems in the higher dimensional bipartite mixed system that need to be addressed. There exist various entanglement measures such as concurrence [23–25], negativity [26], relative entropy of entanglement [165], the geometric measure of entanglement [166] that can quantify the amount of entanglement in a two-qubit as well as higher dimensional bipartite pure and mixed state. Now, the question arises whether the entanglement measures already existed in the literature can quantify the amount of entanglement for any arbitrary dimensional bipartite system practically.

In the case of a two-qubit state, entanglement of formation [24] can be measured without prior state reconstruction [89]. Also, it has been shown that a single observable is not sufficient to determine the entanglement of a given unknown pure two-qubit state [167], nevertheless, the amount of entanglement in a pure two-qubit state can be determined experimentally with a minimum of two copies of the state [168]. The situation will become more complex when we will consider the problem of quantification of entanglement for a higher dimensional bipartite system. For higher dimensional bipartite pure state, there exist some measures of entanglement such as generalized concurrence [169], negativity [27], the geometric measure of entanglement that may quantify the amount of entanglement in the given pure state but on the contrary, we have a handful of measures of entanglement which work for the higher dimensional bipartite mixed state. This is due to the fact that till today we don't have any closed formula for the concurrence of higher dimensional bipartite mixed state. Secondly, an easily computable measure of entanglement, namely, negativity may be used to quantify the amount of entanglement in higher dimensional bipartite pure as well as mixed states but the problem with this measure is that it depends on the negative eigenvalues of the non-physical partial transposition operation. Thus, negativity does not correspond to a completely positive map, and hence, difficult to implement it in the laboratory. Recently, the generalized geometric measure of entanglement has been defined for multipartite mixed states [170] but it is not yet clear whether it can be a realizable quantity in an experiment or not.

Another way of quantification of entanglement is by using witness operators [22, 171]

that can be employed for any arbitrary $d_1 \otimes d_2$ dimensional system. Although the witness operators are physically realizable, in general, it is not easy to construct a witness operator for the detection of an entangled state. Thus, we should look for another way of measuring entanglement in a given $d_1 \otimes d_2$ dimensional system.

Horodecki [89] has proposed a protocol based on the SPA-PT map, which directly measures the concurrence of $2 \otimes 2$ system using four moments only but to calculate the expectation of these four moments, the method needs at most 20 copies of the state. This method is efficient in comparison to quantum state tomography with respect to the estimation of state parameters but on the other hand, this method will show its inefficiency if we compare the number of copies required in the above-mentioned methods. This motivates us to define a new measure of entanglement using the structural physical approximation of partial transposition.

In this chapter, we define a physically realizable measure of entanglement which is based on the minimum eigenvalue of the structural physical approximation of partial transposition operation.

6.1 Measures of entanglement: Concurrence and Negativity

Concurrence: A very popular measure for the quantification of bipartite quantum correlations is the concurrence [23–25]. For any two-qubit pure state $|\psi\rangle_{AB}^{2 \otimes 2}$, it is defined as,

$$C(|\psi\rangle_{AB}^{2 \otimes 2}) = \sqrt{2(1 - \text{Tr}(\rho_A^2))} \quad (6.1.1)$$

where ρ_A is the reduced state of $|\psi\rangle_{AB}^{2 \otimes 2}$.

Concurrence for the two-qubit mixed state described by the density operator ρ , may be defined as,

$$C(\rho) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}) \quad (6.1.2)$$

where λ_i 's are the eigenvalues of $\rho \tilde{\rho}$, arranged in descending order. Here, $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho (\sigma_y \otimes \sigma_y)$.

Let $|\psi\rangle_{AB}^{d_1 \otimes d_2}$ be any vector in $d_1 \otimes d_2$ dimensional system. Then, the definition of concurrence for $d_1 \otimes d_2$ dimensional bipartite pure quantum system may be generalized

as [169],

$$C(|\psi\rangle_{AB}^{d_1 \otimes d_2}) = \sqrt{2v_{d_1}v_{d_2}[1 - \text{Tr}(\rho_A^2)]} \quad (6.1.3)$$

where ρ_A is the reduced state of $|\psi\rangle_{AB}^{d_1 \otimes d_2}$ and v_{d_1} and v_{d_2} are positive constants. Except for the two-qubit system, we don't have any closed formula for the concurrence of bipartite $d_1 \otimes d_2$ dimensional mixed state. For higher dimensional mixed states, we can only estimate the amount of entanglement through the lower bound of the concurrence [86, 122, 172].

Negativity: Negativity is another measure of entanglement based on the negative eigenvalues of the partially transposed matrix. It was first introduced as an entanglement measure by Vidal and Werner [26]. The negativity for $d \otimes d$ dimensional system described by the density operator ρ_{AB} may be defined as [27],

$$N(\rho_{AB}) = \frac{\|\rho_{AB}^{T_B}\|_1 - 1}{d - 1} = \frac{2}{d - 1} \sum_{\lambda_i < 0} |\lambda_i(\rho_{AB}^{T_B})| \quad (6.1.4)$$

where $\|\cdot\|_1$ denotes trace norm and ρ^{T_B} is the partial transposition of the density matrix ρ_{AB} with respect to the subsystem B . In the entanglement measure, negativity is useful in comparison to concurrence because unlike concurrence, it depends on the negative eigenvalues of the partially transposed state and thus the exact value of negativity can be calculated very easily even for higher dimensional quantum systems. Although negativity can be calculated exactly in theory for any arbitrary dimensional system it cannot be implemented in the laboratory. The reason behind this is that the partial transposition operation represents a positive but not a completely positive map. Recently, Bartkiewicz et.al. have studied the problem of experimental implementation of the entanglement measure negativity for a two-qubit system and found a way to calculate the negativity in an experiment by using three experimentally accessible moments of the partially transposed density matrix [173]. For a two-qubit system, a feasible scheme for experimental detection and quantification of entanglement was introduced using PPT criteria [174].

To get rid of the difficulty of implementing negativity in an experiment, we have defined a new measure of entanglement and named it "structured negativity". We called the introduced measure "structured negativity" because this measure is based on the method of SPA-PT.

6.2 Structured Negativity

For the sake of completeness, let us recapitulate a few important points about SPA-PT. We start with the partial transposition (PT) map denoted by $id \otimes T$, which corresponds to a positive but not completely positive map, and thus it cannot be implemented in the laboratory. Therefore, we apply SPA-PT operation to implement the PT map in the laboratory.

For $d \otimes d$ dimensional system described by the density operator ρ , the SPA-PT of the state ρ denoted as $\tilde{\rho}$ and it may be expressed as [76],

$$\tilde{\rho} = \frac{d}{d^3 + 1} I \otimes I + \frac{1}{d^3 + 1} [id \otimes T](\rho) \quad (6.2.1)$$

where $I \otimes I$ denote the identity matrix in $d \otimes d$ dimensional system.

The state ρ is PPT if and only if [76]

$$\lambda_{min}(\tilde{\rho}) \geq \frac{d}{d^3 + 1} \quad (6.2.2)$$

where $\lambda_{min}(\tilde{\rho})$ denote the minimum eigenvalue of $\tilde{\rho}$. Otherwise, the state ρ is entangled.

Now we are in a position to define a new measure of entanglement using the separability criteria given in (6.2.2). We may term this new measure of entanglement as structured negativity and it is denoted by $N_S(\rho)$. Therefore, for $d \otimes d$ system, the structured negativity may be defined as,

$$N_S(\rho) = K \cdot \max\left\{\frac{d}{d^3 + 1} - \lambda_{min}(\tilde{\rho}), 0\right\} \quad (6.2.3)$$

where $K = d(d^3 + 1)$.

Lemma 1: Any $d \otimes d$ system which is described by the density operator $\rho = \sum_k p_k \rho_k$, it's SPA-PT is given by,

$$\tilde{\rho} = \sum_k p_k \tilde{\rho}_k \quad (6.2.4)$$

Proof: Consider a bipartite state $\rho = \sum_k p_k \rho_k$ in $d \otimes d$ dimensional system, then its SPA-PT may be written as

$$\begin{aligned}
\tilde{\rho} &= \frac{d}{d^3+1}(I \otimes I) + \frac{1}{d^3+1}(id \otimes T)\rho \\
&= \frac{d}{d^3+1}(I \otimes I) + \frac{1}{d^3+1}(id \otimes T) \sum_k p_k \rho_k \\
&= \frac{d}{d^3+1}(I \otimes I) + \sum_k p_k \frac{1}{d^3+1}(id \otimes T)\rho_k \\
&= \sum_k p_k \frac{d}{d^3+1}(I \otimes I) + \sum_k p_k \frac{1}{d^3+1}(id \otimes T)\rho_k \\
&= \sum_k p_k \left[\frac{d}{d^3+1}(I \otimes I) + \frac{1}{d^3+1}(id \otimes T)\rho_k \right] \\
&= \sum_k p_k \tilde{\rho}_k
\end{aligned} \tag{6.2.5}$$

To show $N_S(\rho)$, a valid measure of entanglement, we need to show that it satisfies a few properties.

P1: $N_S(\rho)$ vanishes if ρ is separable.

Proof: In $d \otimes d$ dimensional quantum system, if ρ is separable then $\lambda_{\min}(\tilde{\rho}) \geq \frac{d}{d^3+1}$ [76].

Thus, $\max\{\frac{d}{d^3+1} - \lambda_{\min}(\tilde{\rho}), 0\} = 0$. Hence, $N_S(\rho) = 0$.

P2: $N_S(\rho)$ is invariant under local unitary transformation.

Proof: $N_S(\rho)$ is invariant under a local change of basis since eigenvalues does not change under local change of basis [175, 176]. Thus, $N_S(\rho) = N_S(U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger)$ where U_A and U_B denotes the unitaries acting on the subsystem A and B respectively.

P3: $N_S(\rho)$ satisfies convexity property i.e.

$$N_S\left(\sum_k p_k \rho_k\right) \leq \sum_k p_k N_S(\rho_k) \tag{6.2.6}$$

Proof: Let us consider a quantum state described by the density operator $\rho = \sum_k p_k \rho_k$, ($0 \leq p_k \leq 1$). The structured negativity of ρ is given by

$$N_S(\rho) = N_S\left(\sum_k p_k \rho_k\right) = K\left[\frac{d}{d^3+1} - \lambda_{\min}(\tilde{\rho})\right] \tag{6.2.7}$$

Using Lemma 1, RHS of equation (6.2.7) may be re-expressed as

$$N_S(\sum_k p_k \rho_k) = K[\frac{d}{d^3+1} - \lambda_{\min}(p_1 \tilde{\rho}_1 + \sum_{k \neq 1} p_k \tilde{\rho}_k)] \quad (6.2.8)$$

Using Weyl's inequality in (6.2.8), we get

$$N_S(\sum_k p_k \rho_k) \leq K[\frac{d}{d^3+1} - (\lambda_{\min}(p_1 \tilde{\rho}_1) + \lambda_{\min}(\sum_{k \neq 1} p_k \tilde{\rho}_k))] \quad (6.2.9)$$

Using Weyl's inequality repeatedly (k-1) times, equation (6.2.9) may be re-written as,

$$\begin{aligned} N_S(\sum_k p_k \rho_k) &\leq K[\frac{d}{d^3+1} - \sum_k \lambda_{\min}(p_k \tilde{\rho}_k)] \\ &= K[\sum_k p_k (\frac{d}{d^3+1} - \lambda_{\min}(\tilde{\rho}_k))] \\ &= \sum_k p_k \cdot K \cdot (\frac{d}{d^3+1} - \lambda_{\min}(\tilde{\rho}_k)) \\ &= \sum_k p_k N_S(\rho_k) \end{aligned} \quad (6.2.10)$$

P4: $N_S(\rho)$ does not increase on average under LOCC [21] i.e.

$$N_S(\rho) \geq \sum_i p_i N_S[(K_i \otimes I) \rho (K_i^\dagger \otimes I)] \quad (6.2.11)$$

where K_i are the Kraus operators.

Proof: Let us consider an entangled state described by a density operator ρ . Now, consider the right-hand side of (6.2.11) that can be expressed as

$$\begin{aligned} \sum_i p_i N_S[(K_i \otimes I) \rho (K_i^\dagger \otimes I)] &= K \sum_i p_i [\frac{d}{d^3+1} - \lambda_{\min}(\widetilde{(K_i \otimes I) \rho (K_i^\dagger \otimes I)})] \\ &= K[\frac{d}{d^3+1} - \sum_i p_i \lambda_{\min}(\widetilde{(K_i \otimes I) \rho (K_i^\dagger \otimes I)})] \\ &\geq K[\frac{d}{d^3+1} - \sum_i \lambda_{\min}(\widetilde{(K_i \otimes I) \rho (K_i^\dagger \otimes I)})] \end{aligned} \quad (6.2.12)$$

The first step follows from the definition of structured negativity given in (6.2.3). In the third step, the inequality follows from the fact $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$.

Using Weyl's inequality, (6.2.12) may be expressed as

$$\begin{aligned}
\sum_i p_i N_S[(K_i \otimes I)\rho(K_i^\dagger \otimes I)] &\geq K\left[\frac{d}{d^3+1} - \lambda_{\min}\left[\sum_i ((K_i \otimes I)\widetilde{\rho}(K_i^\dagger \otimes I))\right]\right] \\
&= K\left[\frac{d}{d^3+1} - \lambda_{\min}\left[\sum_i \left(\frac{d}{d^3+1} I \otimes I\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{1}{d^3+1} (id \otimes T)((K_i \otimes I)\rho(K_i^\dagger \otimes I))\right)\right]\right] \quad (6.2.13)
\end{aligned}$$

In (6.2.13), the second step follows from the definition (6.2.1) of SPA-PT of $(K_i \otimes I)\rho(K_i^\dagger \otimes I)$. Let us assume that the entangled state described by the density operator ρ may be evolved as $\rho' = \sum_{i=1}^m ((K_i \otimes I)\rho(K_i^\dagger \otimes I))$, where m denote the number of Kraus operators.

Then, (6.2.13) can be re-expressed as

$$\begin{aligned}
\sum_i p_i N_S[(K_i \otimes I)\rho(K_i^\dagger \otimes I)] &\geq K\left[\frac{d}{d^3+1} - \lambda_{\min}\left[\frac{md}{d^3+1} I \otimes I + \frac{1}{d^3+1} (id \otimes T)\rho'\right]\right] \\
&= K\left[\frac{d}{d^3+1} - \lambda_{\min}\left[\frac{md}{d^3+1} I \otimes I + \frac{1}{d^3+1} (\rho'^{T_B})\right]\right] \\
&\geq K\left[\frac{d}{d^3+1} - \lambda_{\min}[(\tilde{\rho}') + \frac{(m-1)d}{d^3+1} I \otimes I]\right] \quad (6.2.14)
\end{aligned}$$

Using the upper bound of Weyl's inequality, (6.2.14) may be expressed as

$$\begin{aligned}
&\sum_i p_i N_S[(K_i \otimes I)\rho(K_i^\dagger \otimes I)] \\
&\geq K\left[\frac{d}{d^3+1} - \lambda_{\min}(\tilde{\rho}') - \frac{(m-1)d}{d^3+1}\right] \quad (6.2.15)
\end{aligned}$$

To show $N_S(\rho) \geq \sum_i p_i N_S[(K_i \otimes I)\rho(K_i^\dagger \otimes I)]$, it is sufficient to show that $N_S(\rho) - \sum_i p_i N_S[(K_i \otimes I)\rho(K_i^\dagger \otimes I)] \geq 0$.

Using (6.2.3) and (6.2.15), we have

$$\begin{aligned}
&N_S(\rho) - \sum_i p_i N_S[(K_i \otimes I)\rho(K_i^\dagger \otimes I)] \\
&\geq K\left[\lambda_{\min}(\tilde{\rho}') + \frac{(m-1)d}{d^3+1} - \lambda_{\min}(\tilde{\rho})\right] \quad (6.2.16)
\end{aligned}$$

For an entangled state ρ , we have

$$\lambda_{\min}(\tilde{\rho}) < \frac{d}{d^3+1} \quad (6.2.17)$$

where $\tilde{\rho}$ denote the SPA-PT of ρ .

Using (6.2.17), the inequality (6.2.16) reduces to

$$\begin{aligned} & N_S(\rho) - \sum_i p_i N_S[(K_i \otimes I)\rho(K_i^\dagger \otimes I)] \\ & \geq K[\lambda_{\min}(\tilde{\rho}') + \frac{(m-2)d}{d^3+1}] \end{aligned} \quad (6.2.18)$$

Since $m \geq 2$ so, $\lambda_{\min}(\tilde{\rho}') + \frac{(m-2)d}{d^3+1} \geq 0$. Thus

$$\begin{aligned} & N_S(\rho) - \sum_i p_i N_S[(K_i \otimes I)\rho(K_i^\dagger \otimes I)] \geq 0 \\ \implies & N_S(\rho) \geq \sum_i p_i N_S[(K_i \otimes I)\rho(K_i^\dagger \otimes I)] \end{aligned} \quad (6.2.19)$$

Hence proved.

6.3 Relation between negativity and structured negativity

In this section, we derive the relationship between the negativity and structured negativity of a given quantum state ρ .

Result-1: For any quantum state ρ in $d \otimes d$ dimensional system, the relation between negativity and structured negativity is given by

$$N(\rho) \leq 2(1 - \frac{1}{d})N_S(\rho) \quad (6.3.1)$$

Proof: For any two Hermitian matrices $A, B \in M_n$, Weyl's inequality may be defined as [136, 177, 178]

$$\lambda_k(A+B) \leq \lambda_{k+j}(A) + \lambda_{n-j}(B), \quad k = \{1, 2, \dots, n\}, \quad j = \{0, 1, \dots, n-1\} \quad (6.3.2)$$

where eigenvalues of the matrices A , B , and $A+B$ are arranged in increasing order.

For $k=1$, Weyl's inequality reduces to

$$\lambda_{\min}(A+B) \leq \lambda_{1+j}(A) + \lambda_{n-j}(B) \quad (6.3.3)$$

for $j = 0, 1, \dots, n-1$.

If $\tilde{\rho}$ denote the SPA-PT of $\rho \in M_{d^2}$, then by taking $A = \frac{1}{d^3+1}(id \otimes T)\rho \equiv \frac{1}{d^3+1}\rho^{T_B}$ and

$B = \frac{d}{d^3+1}I \otimes I$, (6.3.3) reduces to

$$\lambda_{\min}(\tilde{\rho}) \leq \lambda_{1+j}\left(\frac{1}{d^3+1}\rho^{T_B}\right) + \frac{d}{d^3+1}\lambda_{d^2-j}(I \otimes I) \quad (6.3.4)$$

for $j = 0, 1 \dots n-1$.

Let us suppose that $(I \otimes T)\rho$ has $q(\leq (d-1)^2)$ number of negative eigenvalues, if ρ is an entangled state.

Putting $j = 0, 1 \dots q-1$ in (6.3.4) and adding, we get

$$q \cdot \lambda_{\min}(\tilde{\rho}) \leq \frac{1}{d^3+1} \sum_{i=1, \lambda_i < 0}^q \lambda_i(\rho^{T_B}) + \frac{dq}{d^3+1} \quad (6.3.5)$$

Exploiting the definition of negativity given in (6.1.4), the above equation (6.3.5) reduces to

$$\lambda_{\min}(\tilde{\rho}) \leq -\frac{(d-1)N(\rho)}{2q(d^3+1)} + \frac{d}{d^3+1} \quad (6.3.6)$$

If ρ is an entangled state, then the minimum eigenvalue of $\tilde{\rho}$ (given in (6.2.3)) may be expressed as,

$$\lambda_{\min}(\tilde{\rho}) = \frac{d^2 - N_S(\rho)}{d(d^3+1)} \quad (6.3.7)$$

Substituting value of $\lambda_{\min}(\tilde{\rho})$ in (6.3.6), we get

$$\frac{d^2 - N_S(\rho)}{d(d^3+1)} \leq -\frac{(d-1)N(\rho)}{2q(d^3+1)} + \frac{d}{d^3+1} \quad (6.3.8)$$

After simplification, we get the required relation,

$$N_S(\rho) \geq \frac{d(d-1)}{2q}N(\rho) \quad (6.3.9)$$

Thus, the structured negativity of a given quantum state is always greater than or equal to $\frac{d(d-1)}{2q}$ times to its negativity. The number of negative eigenvalues of ρ^{T_B} for a $d \otimes d$ dimensional system are at most $(d-1)^2$. Substituting $q \leq (d-1)^2$, (6.3.9) becomes,

$$N_S(\rho) \geq \frac{d}{2(d-1)}N(\rho) \quad (6.3.10)$$

This implies,

$$N(\rho) \leq 2\left(1 - \frac{1}{d}\right)N_S(\rho) \quad (6.3.11)$$

Hence proved.

Remark: For sufficiently large d , we have $1 - \frac{1}{d} \approx 1$. Thus for higher dimensional system, the inequality (6.3.11) reduces to $N(\rho) \leq 2N_S(\rho)$.

6.3.1 Examples

As we have mentioned in Sec 6.1, there does not exist any closed formula for concurrence of arbitrary dimensional bipartite mixed state, nevertheless, there exists a lower bound of the concurrence for arbitrary $d \otimes d$ dimensional system [86, 122, 172]. One such lower bound of concurrence obtained in [172] which may be given by

$$\begin{aligned} C(\rho) &\geq \sqrt{\frac{2}{d(d-1)}}(\max(\|\rho^{T_B}\|, \|R(\rho)\|) - 1) \\ &\equiv C_{lb}(\rho) \end{aligned} \quad (6.3.12)$$

where $C(\rho)$ and C_{lb} respectively denote the concurrence and lower bound of the concurrence of the state ρ in an arbitrary $d \otimes d$ dimensional system. $\|\cdot\|$ denote the trace norm of (\cdot) . $R(\rho)$ and ρ^{T_B} respectively denote the realignment operation and partial transposition operation with respect to subsystem B .

In this subsection, we will provide a few examples by which we can show that Result-1 is indeed true. For the given state ρ , we have compared three measures of entanglement such as negativity ($N(\rho)$), structured negativity ($N_S(\rho)$) and the lower bound of the concurrence ($C_{lb}(\rho)$) and found that structured negativity is always greater than or equals negativity. In a few cases, structured negativity is greater than the lower bound of concurrence. Also, we have observed that for $d \otimes d$ dimensional system, negativity and structured negativity coincides when the number of negative eigenvalues of the partially transposed matrix are $\frac{d(d-1)}{2}$ i.e equality holds in (6.3.9), when $q = \frac{d(d-1)}{2}$, and strict inequality holds when $q \neq \frac{d(d-1)}{2}$.

Example 1: Two-qubit Werner state

Let us consider a two-qubit Werner state defined as [64, 178]

$$\rho_W = F|\psi^-\rangle\langle\psi^-| + (1-F)\frac{I}{4} \quad (6.3.13)$$

where

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \quad (6.3.14)$$

The family of Werner states is the only states invariant under the transformation [64]

$$\rho_W \longrightarrow U \otimes U \rho_W U^\dagger \otimes U^\dagger \quad (6.3.15)$$

where U is a unitary transformation. The state is entangled for $\frac{1}{3} < F \leq 1$. Negativity of ρ_W is $\frac{3F-1}{2}$ and structured negativity, $N_S(\rho_W) = 18[\frac{2}{9} + \frac{1}{12}(-3 + F)]$. From Fig-1, Result-1 is verified.

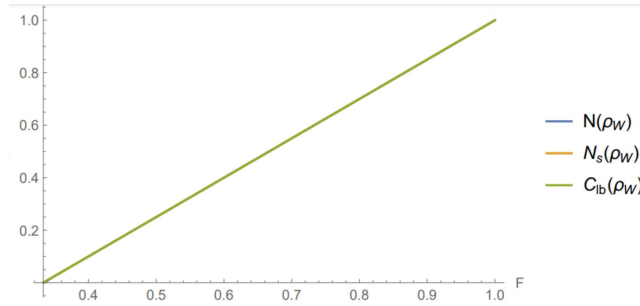


Figure 6.1: The negativity, structured negativity, and lower bound of concurrence for the two-qubit Werner state coincide with each other. Therefore, equality in (6.3.11) has been achieved for a family of Werner state.

Example 2: Two-qubit MEMS state

Consider a maximally entangled mixed state (MEMS) introduced by Munro et.al. [127, 179]

$$\rho_{MEMS} = \begin{pmatrix} h(C) & 0 & 0 & \frac{C}{2} \\ 0 & 1 - 2h(C) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{C}{2} & 0 & 0 & h(C) \end{pmatrix} \quad (6.3.16)$$

where,

$$h(C) = \begin{cases} \frac{C}{2} & C \geq \frac{2}{3} \\ \frac{1}{3} & C < \frac{2}{3} \end{cases} \quad (6.3.17)$$

where C denotes the concurrence of ρ_{MEMS} .

For $C \geq \frac{2}{3}$, $N(\rho_{MEMS}) = -1 + C + \sqrt{1 - 2C + 2C^2}$, $N_S(\rho_{MEMS}) = 18(\frac{2}{9} + \frac{1}{18}(-5 + C + \sqrt{1 - 2C + 2C^2}))$.

From Fig-2, it can be seen that negativity, structured negativity and $C_{lb}(\rho_{MEMS})$ of the state ρ_{MEMS} coincide for $C \geq \frac{2}{3}$. Also for $C < \frac{2}{3}$ negativity and structured negativity of

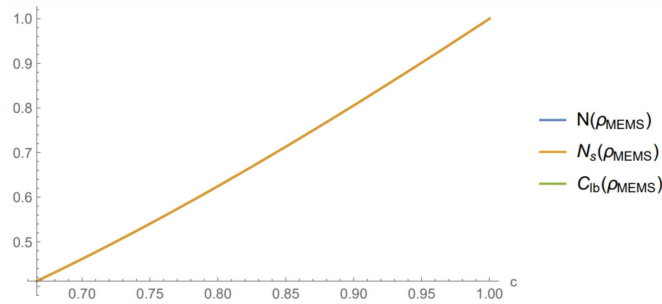


Figure 6.2: The negativity, structured negativity and lower bound of concurrence coincide for the state ρ_{MEMS} for $C \geq \frac{2}{3}$.

ρ_{MEMS} are given by: $N(\rho_{MEMS}) = \frac{1}{3}(-1 + \sqrt{1 + 9C^2})$ and $N_S(\rho_{MEMS}) = 18(\frac{2}{9} + \frac{1}{54}(-13 + \sqrt{1 + 9C^2}))$. From Fig-3, it can be seen that negativity is the same as the structured negativity for $\rho_{MEMS}(C < \frac{2}{3})$.

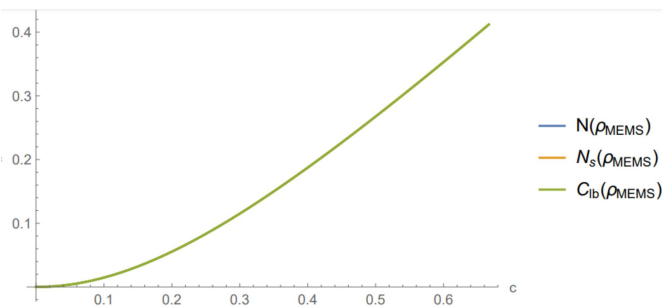


Figure 6.3: The negativity, structured negativity and lower bound of concurrence coincides for the state ρ_{MEMS} for $C < \frac{2}{3}$.

Example 3: Two-qutrit state described by the density operator ρ_a

Consider a two-qutrit state defined in [123], which is described by the density operator

$$\rho_a = \frac{1}{5 + 2a^2} \sum_{i=1}^3 |\psi_i\rangle\langle\psi_i|, \quad \frac{1}{\sqrt{2}} \leq a \leq 1 \quad (6.3.18)$$

where, $|\psi_i\rangle = |0i\rangle - a|i0\rangle$, for $i = \{1, 2\}$ and

$$|\psi_3\rangle = \sum_{i=0}^{n-1} |ii\rangle.$$

For the state ρ_a the negativity $N(\rho_a)$ and the structured negativity $N_S(\rho_a)$ can be calculated as

$$N(\rho_a) = \frac{1}{5+2a^2} - \frac{1-2\sqrt{a}}{5+2a^2} - \frac{1+a^2-\sqrt{5-2a^2+a^4}}{5+2a^2} \quad (6.3.19)$$

$$N_S(\rho_a) = 84 \left[\frac{3}{28} - \frac{7+3a^2}{14(5+2a^2)} \right] \quad (6.3.20)$$

The comparison between $N(\rho_a)$, $N_S(\rho_a)$, $C_{lb}(\rho_a)$ for the two-qutrit state ρ_a has been studied in Fig-4.

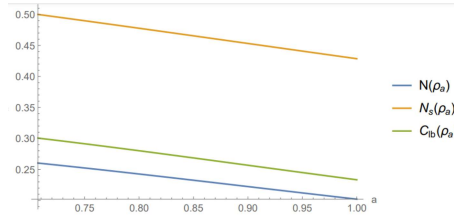


Figure 6.4: The structured negativity of the state ρ_a is greater than the negativity as well as the lower bound of concurrence.

Example 4: Two-qutrit α state

Consider a two-qutrit state defined by [180],

$$\rho_\alpha = \frac{2}{7} |\psi^+\rangle \langle \psi^+| + \frac{\alpha}{7} \sigma_+ + \frac{5-\alpha}{7} \sigma_-, \quad 2 \leq \alpha \leq 5 \quad (6.3.21)$$

where,

$$\begin{aligned} |\psi^+\rangle &= \frac{1}{\sqrt{3}} [|00\rangle + |11\rangle + |22\rangle] \\ \sigma_+ &= \frac{1}{3} (|01\rangle \langle 01| + |12\rangle \langle 12| + |20\rangle \langle 20|) \\ \sigma_- &= \frac{1}{3} (|10\rangle \langle 10| + |21\rangle \langle 21| + |02\rangle \langle 02|) \end{aligned} \quad (6.3.22)$$

The given state ρ_α is NPTES for $4 \leq \alpha \leq 5$.

For the state ρ_α , In this example, first consider ρ_α for $4 \leq \alpha \leq 5$. The SPA-PT of ρ_α is

given by, $N(\rho_\alpha)$ and $N_S(\rho_\alpha)$ may be calculated as

$$N(\rho_\alpha) = -\frac{1}{14}[-5 + \sqrt{41 - 20\alpha + 4\alpha^2}] \quad (6.3.23)$$

$$N_S(\rho_\alpha) = 84\left[\frac{3}{28} - \frac{-131 + \sqrt{41 - 20\alpha + 4\alpha^2}}{1176}\right] \quad (6.3.24)$$

The comparison between $N(\rho_\alpha)$, $N_S(\rho_\alpha)$ and $C_{lb}(\rho_\alpha)$ for the state ρ_α has been studied in fig-5.

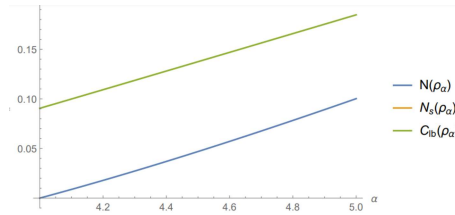


Figure 6.5: The negativity and structured negativity coincide for the state ρ_α . In this case, $C_{lb}(\rho_\alpha)$ is greater than the structured negativity.

6.4 Conclusion

In this chapter, we have defined an entanglement measure based on the minimum eigenvalue of the SPA-PT of the arbitrary dimensional bipartite quantum system. We named the introduced measure structured negativity because it is based on the SPA-PT method and proved that the defined measure satisfies the properties of a valid entanglement measure. Since SPA-PT is a completely positive map, so, the proposed measure of entanglement may be realized in an experiment. The introduced measure of entanglement provides an advantage over the existing measure, negativity as negativity depends on the sum of the negative eigenvalues of the non-physical partial transposition operation whereas structured negativity depends on the minimum eigenvalue of SPA of the partially transposed matrix. We have established a relation between negativity and structured negativity and found that the negativity and structured negativity coincides for a large number of two-qubit systems. Thus, we conjecture that the negativity and structured negativity coincides when $q = \frac{d(d-1)}{2}$. In the end, we have compared negativity, structured negativity, and the lower bound of concurrence C_{lb} and shown that the structured negativity is always greater than or equal to the negativity.

Chapter 7

Conclusion and Future Scope

7.1 Conclusion

This thesis as a whole may be considered as an extensive review of the detection and classification of arbitrary dimensional bipartite and three-qubit systems, where not only detailed references to the literature have been furnished, but also new and original results have been obtained. We have analyzed the concepts of entanglement detection and classification using methods that may be realized experimentally.

In chapter 2, we have explored the methods that exist in the literature and that deal with the detection of entanglement in the bipartite system. Although different criteria are available in the literature for the detection of entanglement, most of them are not physically realizable such as partial transposition (PT) criteria, realignment criteria, range criteria, reduction criteria, etc. Thus, they cannot be implemented in a laboratory. In this chapter, we have proposed three different criteria for the detection of bipartite NPTES using SPA-PT. Since SPA-PT may be realized in an experiment and thus, the criteria defined in chapter 2 may be realized physically. Moreover, since the closed formula for concurrence is still not known for higher dimensional mixed bipartite systems, so we have derived the lower and upper bound of the concurrence that may be implemented in the laboratory. The first criterion detects NPTES if and only if the average fidelity of two quantum states, which are described by the density matrix ρ_{AB} and SPA-PT of the witness operator \tilde{W} is less than a quantity R . The quantity R depends on (i) the dimension of the composite Hilbert space and (ii) the parameter that makes the SPA-PT of the witness operator positive semi-definite. The second criterion tells us that the given state is NPTES if and only if the minimum eigenvalue

of the SPA-PT of the given state is greater or equal to the difference between the concurrence of the given state and the average fidelity between the given state and its SPA-PT. The lower bound of the concurrence is given in terms of the average fidelity of the state ρ_{AB} and SPA-PT of the witness operator \tilde{W} while the upper bound is expressed in terms of the average fidelity of the state ρ_{AB} and its SPA-PT. The third criterion detects a bipartite NPTES ρ_{AB} if and only if $U^{ent} < \frac{1}{2}$ where the quantity U^{ent} depends on the average fidelity of the state ρ_{AB} , its SPA-PT and the concurrence of the state ρ_{AB} . The obtained bounds depend on the average fidelity of two quantum states and the average fidelity can be realized in an experiment. Thus, all the entanglement detection criteria obtained in chapter 2, may be realized experimentally. Furthermore, we have defined a way to apply the SPA-PT map on the qutrit-qubit system and explicitly calculated the matrix elements of the density matrix describing the SPA-PT of the qutrit-qubit system.

In chapter 3, we considered a pure three-qubit quantum system and studied the problem of classification of a particular type of pure genuine entangled states popularly known as GHZ class of states. It is known that in a three-qubit system, there are two SLOCC inequivalent classes of genuinely entangled states namely GHZ class of states and the W class of states. In general, GHZ class of states can serve as an important class of state for quantum communication purposes. To classify GHZ class into further subclasses, we have defined the classification witness operator for the classification of the canonical form of a pure three-qubit GHZ state. Firstly, we have classified the pure three-qubit GHZ class of states into four subclasses denoted by $S_1, S_2, S_3,$ and S_4 . The motivation for the classification of the pure GHZ class of states into four subclasses is that the GHZ state belonging to S_1 may be more efficient than the three-qubit GHZ state belonging to S_2 or S_3 or S_4 and vice-versa with respect to some quantum information processing task. For instance, if we consider a GHZ state belonging to subclass S_1 and a GHZ state belonging to another subclass S_2 then it can be shown that the tangle of the GHZ state belonging to subclass S_1 will be greater than the tangle of the GHZ state belonging to subclass S_2 . If any quantum information processing task depends on tangle then the GHZ state belonging to subclass S_1 will be beneficial in comparison to the GHZ state belonging to S_2 . On the other hand, we found that the maximal teleportation fidelity of the GHZ states belonging to subclass S_2 is greater than the GHZ state belonging to subclass S_1 . Therefore, GHZ states belonging to subclass S_2 are more suitable shared resource states in teleportation than the GHZ state belonging to subclass S_1 . Thus, GHZ states belonging to different

subclasses have their own importance, and hence if we can detect the subclass of the GHZ state under investigation, then we can use that GHZ state in various quantum information processing tasks accordingly, to get optimal output. Now, in order to identify different subclasses of GHZ class of states, we have constructed different witness operators that can classify the subclasses $S_i, i = 2, 3, 4$ from S_1 . Moreover, we have shown that the constructed witness operator can be decomposed into Pauli matrices and hence can be realized experimentally.

In chapter 4, we studied the problem of classification of a three-qubit system using l_1 -norm of coherence. In the literature, the problem of classification of a three-qubit system has been studied using entanglement measures such as tangle and also through the construction of the observables. In a three-qubit system, all the states are classified into six SLOCC inequivalent classes one fully separable state, three biseparable states, and two genuinely entangled states (W class of states and GHZ class of states). While genuine entangled states have entanglement in all the subsystems, biseparable states have entanglement in two of the three subsystems. Biseparable states and genuine entangled states have their own importance and thus, their classification may play a vital role in quantum communication. The classification method based on tangle is not very useful because it cannot discriminate the set of separable states, biseparable states, and W class of states. It happens because the tangle is zero for the set of separable states, biseparable states, and W class of states. In chapter 4, we have used l_1 -norm of coherence to fill this gap for the classification of three-qubit entangled states. Coherence is a basic phenomenon that arises from the superposition principle of quantum mechanics. It can be measured by different measures such as distance measure, relative entropy of coherence, and l_p -norms. l_1 -norm is a valid coherence monotone and serves as a useful measure of coherence. In chapter 4, we have calculated the l_1 -norm of coherence of the tensor product of two quantum systems. In order to obtain criteria for the classification of three-qubit states, we have obtained inequalities for biseparable states and separable states in terms of l_1 -norm of coherence. Further, we have shown that if the obtained inequality is violated by any three-qubit state then the state under investigation is neither a biseparable state nor a separable state. Since, for a three-qubit system, we have only three categories of state and if we find that the given state is neither a separable state nor a biseparable state then we can conclude that the state is a genuinely entangled state. At the end of this chapter, we have provided an idea to generalize the obtained results to higher dimensional and multi-qubit systems.

In chapter 5, we studied the problem of classification of a three-qubit system using SPA-PT. The methods available in the literature are either incapable of classifying all possible SLOCC inequivalent classes or are not physically realizable. The method defined in chapter 4 for the classification of the three-qubit system is basis dependent and may not be realized physically. To overcome this issue, we have exploited the concept of SPA-PT for the classification of a three-qubit entangled state. Thus in the first part of this chapter, we have taken into account the SPA-PT operation to detect the three-qubit entangled state. To start our study, we have applied partial transposition operation on one of the qubit of the three-qubit system and then studied the entanglement properties of the three-qubit system, which is under investigation. We have defined a way to construct a SPA-PT map for a three-qubit quantum system. With the help of this constructed SPA-PT map, we have provided explicitly the matrix elements of the density matrix describing the SPA-PT of a three-qubit system. We have proposed criteria for the classification of all possible SLOCC inequivalent classes of pure as well as mixed three-qubit states through the SPA-PT map. Thus, the criteria defined in this chapter may classify three-qubit pure and mixed states physically.

In chapter 6, we studied the problem of detection and quantification of entangled states in a bipartite quantum system. Measures of entanglement which by definition quantify entangled states may be used for the detection of entanglement. Various entanglement measures exist in the literature such as concurrence, negativity, relative entropy of entanglement, and geometric measure of entanglement that can quantify the amount of entanglement in a two-qubit as well as higher dimensional bipartite pure and mixed state. Out of these, concurrence and negativity are the most popular ones. Although the closed formula for concurrence of mixed two-qubit state can be found in the literature the closed formula does not exist for the concurrence to measure the amount of entanglement in bipartite higher dimensional mixed states. Thus only the lower bound of concurrence is known which helps to estimate the amount of entanglement in the higher dimensional mixed system. On the other hand, negativity is defined for pure as well mixed states in a higher dimensional quantum system. But the problem with negativity is that it is not physically realizable as it depends on the negative eigenvalues of the partially transposed matrix and partial transposition operation corresponds to a non-physical operator. We can overcome this defect by using the structural physical approximation of partial transposition, which by definition transforms a non-physical operator into a physical operator. In this chapter, we have defined a new measure of entanglement using the concept of SPA-PT, which

we named "structured negativity". We have shown that it satisfies the properties of a valid entanglement monotone. Since the new measure of entanglement depends on the minimum eigenvalue of the structural physical approximation of the partially transposed matrix, so, it may be realized in an experiment. Moreover, we have obtained a relationship between negativity and structured negativity. Also, we have compared negativity, structured negativity, and the lower bound of the concurrence obtained by Chen et al. [172] and found that structured negativity may be a better entanglement measure than negativity and in a few cases, it may perform better than the lower bound of concurrence.

7.2 Future Scope

In the literature, various schemes exist for the detection and classification of entangled states in higher dimensional bipartite and multipartite systems but most of them cannot be implemented in the experiment. In the present thesis, we have focussed to overcome this problem and have presented solutions to a few of them, but still, there is a future scope of the thesis, which is discussed below:

(1) Future scope on the detection of entanglement in higher dimensional bipartite and multipartite quantum system:

(1a) In chapter 1, we studied the problem of detection of entanglement using the SPA-PT method and obtained criteria for the detection of NPTEs in an arbitrary dimensional quantum system. In the same way, one may think about the possibility of the development of entanglement detection criteria for PPTES in arbitrary dimensional bipartite systems using SPA-PT.

(1b) In chapter 6, we have studied the problem of quantification of bipartite entanglement in arbitrary dimension. To quantify entanglement in an arbitrary dimensional quantum system, we have introduced a new measure of entanglement. The introduced measure is based on SPA-PT and thus the entanglement may be quantified in an experiment. It would be interesting to quantify PPTES using the SPA-PT method.

(1c) The construction of an efficient entanglement witness for the detection of entangled states, is very difficult with the increase in the size of the system (dimension and number of qubits). In spite of the difficulty, Zhao et al. [181] have constructed witness operators for the detection of multipartite GHZ-like states. In the future, it may be possible to develop some method based on the procedure discussed in this thesis for the construction of the efficient witness operator that may detect the class of higher

dimensional multipartite W class of states.

(1d) In a multipartite system, the detection of Werner states has been studied by constructing a dimension-free entanglement witness using semi-definite programming [182]. It is important to develop such methods for the detection of entanglement as they can be used for the detection of entanglement in higher dimensional multipartite systems. It opens the door for exploring the problem of detection of entanglement using dimension-free witness operators. One may think of constructing such witnesses for other entangled states that exist in a higher dimensional bipartite and multipartite quantum system.

(2) Future scope on the classification of entanglement in the three-qubit and multiqubit system:

(2a) In chapter 3, we have considered three-qubit GHZ class of states and divide it into four subclasses S_1, S_2, S_3 and S_4 . We found that all these four subclasses are important in some quantum information processing tasks. Thus, it is necessary to discriminate them, and hence, we have defined the classification witness operator to classify S_1 from $S_i, i = 2, 3, 4$. But still there is a scope to construct classification witness operator that may help to distinguish S_i from $S_j, (i < j, i, j \in \{2, 3, 4\})$.

(2b) In this thesis, we have studied subclasses of three-qubit GHZ class and along the same line of research, one may think of defining different subclasses of the three-qubit W class of states. In this case also, one may investigate whether the subclasses of the W class of states are SLOCC inequivalent or not. It may happen that the states belonging to these subclasses may perform well in quantum communication and thus their classification may also be important. Hence, there is a scope to construct a classification witness operator for the classification of different subclasses of the W class of states.

(2c) In chapter 5, we have proposed criteria for the classification of a three-qubit system based on SPA-PT. The obtained criteria are based on the minimum eigenvalue of the maximally entangled three-qubit state. But when we extend these results to a multi-qubit system, the following problem may arise: In n -qubit system, where $n > 3$, it is very difficult to find maximally entangled states and thus it becomes more difficult to find the minimum eigenvalue of the maximally entangled state to obtain the condition for classification of a three-qubit system using the method of SPA-PT. But in spite of these difficulties, one may obtain the condition for the classification of a four-qubit system because, in a four-qubit system, maximally entangled states are known [183].

(2d) In chapter 4, we studied the detection and classification of a three-qubit system

using l_1 -norm of coherence. Our method for the classification of a three-qubit system using l_1 -norm of coherence is very general and this classification method can be extended to n -qubit system, where $n \geq 4$. In the present thesis, we have obtained few results on the classification of the four-qubit system but still, there is a scope to work on it in the future. We have classified separable, biseparable, and genuine three-qubit entangled states but we have not explored the possibility of the classification of two SLOCC inequivalent classes of genuine entangled states, viz, W class and GHZ class. Thus, there is a scope to investigate the classification of GHZ and W class using l_1 -norm of coherence.

(2e) Another problem that arises in a multi-qubit system is that the number of SLOCC inequivalent classes increases with the number of increases in the qubits. Thus, the classification problem becomes more complicated with the increase in the number of qubits. It would be interesting to work on finding the number of different SLOCC inequivalent classes of n -qubit system ($n \geq 5$).

The field of quantum information theory is definitely worth following over the next decade since everything points toward significant advances. Detection and classification of entanglement are important problems in quantum information theory. To ensure maximum benefit from future surveys, we still need to improve our theoretical understanding of quantum systems, and this thesis is one of many steps toward achieving this.

References

- [1] F. Bloch, *Nuclear Induction*, Phys. Rev. **70**, (1946) 460-474.
- [2] M. A. Nielsen, I. L. Chuang, *Quantum Computation and Quantum information*, Cambridge University Press (2000).
- [3] G. Kimura, *The Bloch vector for N-level systems*, Phys. Lett. A **314**, (2003) 339-349.
- [4] R. A. Bertlmann and P. Krammer, *Bloch vectors for qudits*, J. Phys. A: Math. Theor. **41**, (2008) 235303(1-21).
- [5] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory Of Angular Momentum* (Singapore: World Scientific) (1988).
- [6] H. Narnhofer, *Entanglement reflected in Wigner functions*, J. Phys. A: Math. Gen. **39**,(2006) 7051-7064.
- [7] B. Baumgartner, B. C. Hiesmayr, and H. Narnhofer, *State space for two qutrits has a phase space structure in its core*, Phys. Rev. A **74**, (2006) 032327(1-14).
- [8] R. F. Werner, *All teleportation and dense coding schemes*, J. Phys. A: Math. Gen. **34**, (2001) 7081-7094.
- [9] C. M. Caves and G. J. Milburn, *Qutrit entanglement*, Opt. Commun. **179**, (2000) 439-446.
- [10] R. A. Bertlmann, K. Durstberger, B. C. Hiesmayr and P. Krammer, *Optimal entanglement witnesses for qubits and qutrits*, Phys. Rev. A **72**, (2005) 052331(1-9).
- [11] J. Greensite, *Lecture Notes on Quantum Mechanics* (2003).
- [12] H. P. Robertson, *The Uncertainty Principle*, Phys. Rev. **34**, (1929) 163-164.

- [13] E. Arthurs and J.L. Kelly, JR. *On the Simultaneous Measurement of a Pair of Conjugate Observables*, Bell Syst. Tech. J. **44**, (1965) 725-729.
- [14] E. Arthurs and M. S. Goodman, *Quantum Correlations: A Generalized Heisenberg Uncertainty Relation*, Phys. Rev. Lett. **60**, (1988) 2447-2449.
- [15] W. Heisenberg, in *Quantum Theory and Measurement*, edited by J.A. Wheeler and W.H. Zurek. Princeton University Press, Princeton, (1983), 62-84, originally published in Z. Phys. **43**, (1927) 172-198.
- [16] W. Heisenberg, *The Physical Principles of the Quantum Theory*, Dover, New York, (1967).
- [17] A Aspect, J Dalibard and G. Roger, *Experimental Test of Bell's Inequalities Using Time-Varying Analyzers*, Phys. Rev. Lett. **49**, (1982) 1804-1807.
- [18] A. Aspect, P. Grangier and G. Roger, *Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities*, Phys. Rev. Lett. **49**, (1982) 91-94.
- [19] E. Schrödinger, *The Present Status of Quantum Mechanics*, Naturwiss. **23**, (1935) 807(1-26).
- [20] A. Einstein, B. Podolsky and N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev. **47** (1935) 777-780.
- [21] M. B. Plenio and S. Virmani, *An introduction to entanglement measures*, Quant. Inf. Comput. **7** (2007) 1-51.
- [22] O. Gühne and G. Tóth, *Entanglement detection*, Physics Reports **474** (2009) 1-75.
- [23] S. Hill and W. K. Wootters, *Entanglement of a Pair of Quantum Bits*, Phys. Rev. Lett. **78** (1997) 5022(1-4).
- [24] W.K. Wootters, *Entanglement of Formation of an Arbitrary State of Two Qubits*, Phys. Rev. Lett. **80** (1998) 2245-2248.
- [25] W.K. Wootters, *Entanglement of formation and concurrence*, Quant. Inf. Comp. **1** (2001) 27-44.

- [26] G. Vidal and R. F. Werner, *Computable measure of entanglement*, Phys. Rev. A **65** (2002) 032314(1-11).
- [27] S. Lee, *Convex-roof extended negativity as an entanglement measure for bipartite quantum systems*, Phys. Rev. A **68** (2003) 062304(1-5).
- [28] C. Eltschka, A. Osterloh, J. Siewert and A. Uhlmann, *Three-tangle for mixtures of generalized GHZ and generalized W states*, New Journal of Physics **10** (2008) 043014(1-10).
- [29] Y. C. Ou and H. Fan, *Monogamy inequality in terms of negativity for three-qubit states*, Phys. Rev. A **75**, (2007) 062308(1-5).
- [30] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels*, Phys. Rev. Lett. **70**, (1993) 1895-1899.
- [31] D. Bouwmeester, J. W. Pan, K. Mattle, M. Eibl, H. Weinfurter and A. Zeilinger, *Experimental quantum teleportation*, Nature **390**, (1997) 575-579.
- [32] D. Boschi, S. Branca, F. D. Martini, L. Hardy, and S. Popescu, *Experimental Realization of Teleporting an Unknown Pure Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels*, Phys. Rev. Lett. **80**, (1998) 1121-1125.
- [33] Y. H. Kim, S. P. Kulik and Y. Shih, *Quantum Teleportation of a Polarization State with a Complete Bell State Measurement*, Phys. Rev. Lett. **86**, (2001) 1370-1373.
- [34] H. Yonezawa, S. L. Braunstein and A. Furusawa, *Experimental Demonstration of Quantum Teleportation of Broadband Squeezing*, Phys. Rev. Lett. **99**, (2007) 110503(1-4).
- [35] X. M. Hu, C. Zhang, B. H. Liu, Y. Cai, X. J. Ye, Y. Guo, W. B. Xing, C. X. Huang, Y. F. Huang, C. F. Li, and G. C. Guo, *Experimental High-Dimensional Quantum Teleportation*, Phys. Rev. Lett. **125**, (2020) 230501(1-6).
- [36] C. H. Bennett and S. Wiesner, *Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states*, Phys. Rev. Lett. **69**, (1992) 2881-2884.
- [37] A. Harrow, P. Hayden and D. Leung, *Superdense Coding of Quantum States*, Phys. Rev. Lett. **92**, (2004) 187901(1-4).

- [38] J. T. Barreiro, T. C. Wei and P. G. Kwiat, *Beating the channel capacity limit for linear photonic superdense coding*, Nat. Phys. **4**, (2008) 282-286.
- [39] B. P. Williams, R. J. Sadler, and T. S. Humble, *Superdense Coding over Optical Fiber Links with Complete Bell-State Measurements*, Phys. Rev. Lett. **118**, (2017) 050501(1-5).
- [40] A. K. Pati, *Minimum classical bit for remote preparation and measurement of a qubit*, Phys. Rev. A **63** (2000) 014302(1-3).
- [41] C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, *Remote State Preparation*, Phys. Rev. Lett. **87** (2001) 077902(1-4).
- [42] H. K. Lo, *Classical-communication cost in distributed quantum-information processing: A generalization of quantum-communication complexity*, Phys. Rev. A **62**, (2000) 012313(1-7).
- [43] D. W. Berry and B. C. Sanders, *Optimal Remote State Preparation*, Phys. Rev. Lett. **90** (2003) 057901(1-4).
- [44] M. Zukowski, A. Zeilinger, M. A. Horne and A. K. Ekert, *"Event-Ready-Detectors" Bell Experiment via Entanglement Swapping*, Phys. Rev. Lett. **71**, (1993) 4287-4290.
- [45] S. Bose, V. Vedral and P. L. Knight, *Multiparticle generalization of entanglement swapping*, Phys. Rev. A **57** (1998) 822-829.
- [46] D. Bouwmeester, J.W. Pan, H. Weinfurter and A. Zeilinger, *Experimental Quantum Teleportation of Qubits and Entanglement Swapping*, Epistemological and Experimental Perspectives on Quantum Physics. Vienna Circle Institute Yearbook **7**. Springer, Dordrecht [1999].
- [47] X. Su, C. Tian, X. Deng, Q. Li, C. Xie, and K. Peng, *Quantum Entanglement Swapping between Two Multipartite Entangled States*, Phys. Rev. Lett. **117** (2016) 240503(1-5).
- [48] C. H. Bennett, and G. Brassard, *Quantum Cryptography: Public Key Distribution and Coin Tossing*, Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India, (IEEE, New York) (1984) 175-179.

- [49] A. K. Ekert, *Quantum cryptography based on Bell's theorem*, Phys. Rev. Lett. **67**, (1991) 661-663.
- [50] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, *Quantum cryptography*, Rev. Mod. Phys. **74**, (2002) 145-195.
- [51] Y. A. Chen, A. N. Zhang, Z. Zhao, X. Q. Zhou, C. Y. Lu, C. Z. Peng, T. Yang, and J. W. Pan, *Experimental Quantum Secret Sharing and Third-Man Quantum Cryptography*, Phys. Rev. Lett. **95**, (2005) 200502(1-4).
- [52] W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, *Violation of Bell Inequalities by Photons More Than 10 km Apart*, Phys. Rev. Lett. **81**, (1998) 3563-3566.
- [53] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter and A. Zeilinger, *Violation of Bell's Inequality under Strict Einstein Locality Conditions*, Phys. Rev. Lett. **81**, (1998) 5059-5043.
- [54] E. Hagley, X. Maître, G. Nogues, C. Wunderlich, M. Brune, J. M. Raimond, and S. Haroche, *Generation of Einstein-Podolsky-Rosen Pairs of Atoms*, Phys. Rev. Lett. **79**, (1997) 1-5.
- [55] C. A. Sackett, D. Kielpinski, B. E. King, C. Langer, V. Meyer, C. J. Myatt, M. Rowe, Q. A. Turchette, W. M. Itano, D. J. Wineland and C. Monroe, *Experimental entanglement of four particles*, Nature **404** (2000) 256-259.
- [56] A. Furusawa, J.L. Sorensen, S.L. Braunstein, C.A. Fuchs, H.J. Kimble, E.S. Polzik, *Unconditional quantum teleportation*, Science **282**, (1998) 706-709.
- [57] J.C. Howell, A.L. Linares and D. Bouwmeester, *Experimental Violation of a Spin-1 Bell Inequality Using Maximally Entangled Four-Photon States*, Phys. Rev. Lett. **88**, (2002) 030401(1-4).
- [58] A. Peres, *Separability Criterion for Density Matrices*, Phys. Rev. Lett. **77**, (1996) 1413-1415.
- [59] P. Horodecki, *Separability criterion and inseparable mixed states with positive partial transposition*, Phys. Lett. A **232**, (1997) 333-339.
- [60] M. Horodecki, P. Horodecki and R. Horodecki, *Separability of mixed states: necessary and sufficient conditions*, Phys. Lett. A **223**, (1996) 1-8.

- [61] B. Bhattacharya, S. Goswami, R. Mundra, N. Ganguly, I. Chakrabarty, S. Bhattacharya, A. S. Majumdar, *Generating and detecting bound entanglement in two-qutrits using a family of indecomposable positive maps*, J. Phys. Commun. **5** (2021) 065008(1-12).
- [62] C.H. Bennett, D.P. DiVincenzo, T. Mor, P.W. Shor, J.A. Smolin and B.M. Terhal, *Unextendible Product Bases and Bound Entanglement*, Phys. Rev. Lett. **82**, (1999) 5385-5388.
- [63] D. Bruß and A. Peres, *Construction of quantum states with bound entanglement*, Phys. Rev. A **61**, (2000) 030301(1-2).
- [64] M. Horodecki and P. Horodecki, *Reduction criterion of separability and limits for a class of distillation protocols*, Phys. Rev. A **59**, (1999) 4206-4216.
- [65] K. Chen and L.-A. Wu, *A matrix realignment method for recognizing entanglement*, Quantum Inf. Comput. **3**, (2003) 193-202.
- [66] O. Rudolph, *Further Results on the Cross Norm Criterion for Separability*, Quantum Inf. Process. **4**, (2005) 219-239.
- [67] P. Horodecki, Phys. Lett. A **232** 333 (1997).
- [68] J. S. Bell, *On the Einstein Podolsky Rosen paradox*, Physics **1**, (1964) 195-200.
- [69] J. L. Chen, C. Wu, L. C. Kwek, C. H. Oh and M. L. Ge, *Violating Bell inequalities maximally for two d-dimensional systems*, Phys. Rev. A **74**, (2006) 032106(1-5).
- [70] A. Acín, T. Durt, N. Gisin, and J. I. Latorre, *Quantum nonlocality in two three-level systems*, Phys. Rev. A **65**, (2002) 052325(1-8).
- [71] B.M. Terhal, *Bell inequalities and the separability criterion*, Phys. Lett. A, **271**, (2000) 319-326.
- [72] M. Lewenstein, B. Kraus, J.I. Cirac, P. Horodecki, *Optimization of entanglement witnesses*, Phys. Rev. A **62**, (2000) 052310(1-16).
- [73] M.D. Choi, *Completely Positive Linear Maps on Complex Matrices*, Linear Algebra and its Applications **10** (1975) 285-290.
- [74] A. Jamiołkowski, *Linear Transformations Which Preserve Trace and Positive Semidefiniteness of Operators*, Rep. Math. Phys. **3** (1972) 275-278.

- [75] P. Horodecki, *From limits of quantum operations to multicopy entanglement witnesses and state-spectrum estimation*, Phys. Rev. A **68** (2003) 052101(1-6).
- [76] P. Horodecki and A. Ekert, *Method for Direct Detection of Quantum Entanglement*, Phys. Rev. Lett. **89** (2002) 127902(1-4).
- [77] J. K. Korbicz, M. L. Almeida, J. Bae, M. Lewenstein, and A. Acín, *Structural approximations to positive maps and entanglement-breaking channels*, Phys. Rev. A **78** (2008) 062105(1-17).
- [78] R. Augusiak, J. Bae, J. Tura and M. Lewenstein, *Checking the optimality of entanglement witnesses: an application to structural physical approximations*, J. Phys. A: Math. Theor. **47** (2014) 065301(1-10).
- [79] K. C. Ha and S. H. Kye, *Separable States with Unique Decompositions*, Comm. Math. Phys. **328**, (2014) 131-153.
- [80] K.C. Ha and S.H. Kye, *The structural physical approximations and optimal entanglement witnesses*, J. Math. Phys **53** (2012) 102204(1-7).
- [81] L.O. Hansen and J. Myrheim, *Visualizing extremal positive maps in unital and trace-preserving form*, Phys. Rev. A **92** (2015) 042306(1-15).
- [82] Erling Størmer, *Separable states and the structural physical approximation of a positive map*, J.Fun. Anal. **264** (2013) 2197-2205.
- [83] H.T. Lim, Y.S. Ra, Y.S. Kim, J. Bae, and Y.H. Kim, *Experimental implementation of the universal transpose operation using the structural physical approximation*, Phys. Rev. A **83** (2011) 020301(1-4).
- [84] H.T. Lim, Y.S. Kim, Y.S. Ra, J. Bae, and Y.H. Kim, *Experimental realization of an approximate transpose operation for qutrit systems using a structural physical approximation*, Phys. Rev. A **86** (2012) 042334(1-7).
- [85] J. Bae, *Designing Quantum Information Processing via Structural Physical Approximation*, Rep. Prog. Phys. **80** (2017) 104001(1-26).
- [86] A. Kumari and S. Adhikari, *Detection of a mixed bipartite entangled state in arbitrary dimension via a structural physical approximation of partial transposition*, Phys. Rev. A **100** (2019) 052323(1-8).

- [87] S. Adhikari, *Structural physical approximation for the realization of the optimal singlet fraction with two measurements*, Phys. Rev. A **97** (2018) 042344(1-6).
- [88] S. Adhikari, *Estimation of entanglement negativity of a two-qubit quantum system with two measurements*, Eur. Phys. Lett. **124**, (2018) 40006(1-7).
- [89] P. Horodecki, *Measuring Quantum Entanglement without Prior State Reconstruction*, Phys. Rev. Lett. **90** (2003) 167901(1-4).
- [90] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, *Quantum entanglement*, J. Mod. Phys **81** (2009) 865-942.
- [91] B. M. Terhal and P. Horodecki, *Schmidt number for density matrices*, Phys. Rev. A **61** (2000) R040301(1-4).
- [92] W. Dur, J. I. Cirac and R. Tarrach, *Separability and Distillability of Multiparticle Quantum Systems*, Phys. Rev. Lett. **83** (1999) 3562-3565.
- [93] G. Toth, O. Guhne, *Entanglement detection in the stabilizer formalism* Phys. Rev. A **72** (2005) 022340(1-14).
- [94] A. Acin, A. Andrianov, L. Costa, E. Jane, J. I. Lattore, and R. Tarrach, *Generalized Schmidt Decomposition and Classification of Three-Quantum-Bit States*, Phys. Rev. Lett. **85** (2000) 1560-1563.
- [95] W. Dur, G. Vidal, and J. I. Cirac, *Three qubits can be entangled in two inequivalent ways*, Phys. Rev. A **62**, 062314 (2000).
- [96] C. Datta, S. Adhikari, A. Das, and P. Agrawal, *Distinguishing different classes of entanglement of three-qubit pure states*, Eur. Phys. J. D **72** (2018) 157(1-7).
- [97] A. Singh, H. Singh, K. Dorai, and Arvind, *Experimental classification of entanglement in arbitrary three-qubit pure states on an NMR quantum information processor*, Phys. Rev. A **98** (2018) 032301(1-9).
- [98] A. Acin, D. Bruss, M. Lewenstein and A. Sanpera, *Classification of Mixed Three-Qubit States*, Phys. Rev. Lett. **87**, (2001) 040401(1-4).
- [99] C. Sabin and G. Garcia-Alcaine, *A classification of entanglement in three-qubit systems*, Eur. Phys. J. D **48** (2008) 435-442.

- [100] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, *Four qubits can be entangled in nine different ways*, Phys. Rev. A **65** (2002) 052112(1-5).
- [101] O. Viehmann, C. Eltschka, and J. Siewert, *Polynomial invariants for discrimination and classification of four-qubit entanglement*, Phys. Rev. A **83** (2011) 052330 (1-4).
- [102] S. M. Zangi, J-L Li, and C-F Qiao, *Entanglement classification of four-partite states under the SLOCC*, J. Phys. A: Math. Theor. **50** (2017) 325301(1-10).
- [103] A. Miyake, *Classification of multipartite entangled states by multidimensional determinants*, Phys. Rev. A **67** (2003) 012108(1-10).
- [104] A. Miyake and M. Wadati, *Multipartite Entanglement and Hyperdeterminants*, Quant. Info. Comp. **2**(Special) (2002) 540-555.
- [105] X. Li and D. Li, *Classification of General n-Qubit States under Stochastic Local Operations and Classical Communication in Terms of the Rank of Coefficient Matrix*, Phys. Rev. Lett. **108** (2012) 180502(1-5).
- [106] L. Chen and Y-X Chen, *Classification of GHZ-type, W-type, and GHZ-W-type multiqubit entanglement*, Phys. Rev. A **74** (2006) 062310(1-17).
- [107] M. Piani, S. Gharibian, G. Adesso, J. Calsamiglia, P. Horodecki, and A. Winter, *All Nonclassical Correlations Can Be Activated into Distillable Entanglement* Phys. Rev. Lett. **106** (2011) 220403(1-4).
- [108] R. Jozsa, and N. Linden, *On the Role of Entanglement in Quantum-Computational Speed-Up*, Proc. R. Soc. Lond. A **459** (2003) 2011-2032.
- [109] R. Augusiak, M. Demianowicz, P. Horodecki, *Universal observable detecting all two-qubit entanglement and determinant-based separability tests*, Phys. Rev. A **77** (2008) 030301(R)(1-4).
- [110] G. Lima, E. S. Gmez, A. Vargas, R. O. Vianna, C. Saavedra, *Fast entanglement detection for unknown states of two spatial qutrits*, Phys. Rev. A **82** (2010) 012302(1-8).
- [111] D. Chruscinski, F. A. Wudarski, *Geometry of entanglement witnesses for two qutrits*, Open Sys. Inf. Dyn. **18** (2011) 375-387.

- [112] N. Ganguly, S. Adhikari, *Witness for edge states and its characteristics*, Phys. Rev. A **80** (2009) 032331(1-7).
- [113] P. Horodecki, R. Augusiak, M. Demianowicz, *General construction of noiseless networks detecting entanglement with the help of linear maps*, Phys. Rev. A **74** (2006) 052323(1-6).
- [114] G. Toth, O. Gühne, *Detecting Genuine Multipartite Entanglement with Two Local Measurements*, Phys. Rev. Lett. **94** (2005) 060501(1-4).
- [115] M. Li, S. M. Fei, X. L. Jost, and H. Fan, *Genuine multipartite entanglement detection and lower bound of multipartite concurrence*, Phys. Rev. A **92** (2015) 062338(1-6).
- [116] S. Q. Shen, T. R. Xu, S. M. Fei, X. L. Jost, and M. Li, *Optimization of ultrafine entanglement witnesses*, Phys. Rev. A **97** (2018) 032343(1-4).
- [117] J. Hou and X. Qi, *Constructing entanglement witnesses for infinite-dimensional systems*, Phys. Rev. A **81** (2010) 062351(1-7).
- [118] G. Toth, *Entanglement witnesses in spin models*, Phys. Rev. A **71** (2005) 010301(R)(1-4).
- [119] C. J. Kwong, S. Felicetti, L. C. Kwek, J. Bae, *Entanglement Detection with Single Hong-Ou-Mandel Interferometry* quant-ph/arXiv1606.00427 (2016) 1-8.
- [120] F. Mintert, M. Kus, A. Buchleitner, *Concurrence of Mixed Bipartite Quantum States in Arbitrary Dimensions*, Phys. Rev. Lett. **92** (2004) 167902(1-4).
- [121] J. B. Lasserre, *A trace inequality for matrix product*, IEEE Trans. on Automatic Control **40** (1995) 1500-1501.
- [122] F. Mintert, *Concurrence via entanglement witnesses*, Phys. Rev. A **75** (2007) 052302(1-4).
- [123] S. Rana, *Negative eigenvalues of partial transposition of arbitrary bipartite states*, Phys. Rev. A **87** (2013) 054301(1-4).
- [124] K. M. O. Connor, and W. K. Wootters, *Entangled rings*, Phys. Rev. A **63** (2001) 052302(1-9).

- [125] D. Bruss, and C. Macchiavello, *On the Entanglement Structure in Quantum Cloning*, Found. Phys. **33** (2003) 1617-1628.
- [126] F. Verstraete, K. Audenaert, and B. D. Moor, *Maximally entangled mixed states of two qubits*, Phys. Rev. A **64** (2001) 012316(1-6).
- [127] W. J. Munro, D. F. V. James, A. G. White, and P. G. Kwiat, *Maximizing the entanglement of two mixed qubits*, Phys. Rev. A **64** (2001) 030302(1-4).
- [128] S. Ishizaka, and T. Hiroshima, *Maximally entangled mixed states under nonlocal unitary operations in two qubits*, Phys. Rev. A **62** (2000) 022310(1-4).
- [129] M. Ali, *Qubit-Qutrit ($2 \otimes 3$) quantum systems: An investigation of some quantum correlations under collective dephasing*, Braz. J. Phys **50** (2020) 124-135.
- [130] G. Karpat, and Z. Gedik, *Correlation dynamics of qubit-qutrit systems in a classical dephasing environment*, Phys. Lett. A **375** (2011) 4166-4171.
- [131] M. N. Bera, R. Prabhu, A. Sen(De), and U. Sen, *Characterization of tripartite quantum states with vanishing monogamy score*, Phys. Rev. A **86** (2012) 012319(1-7).
- [132] G. Torun and A. Yildiz, *Canonical operators and the optimal concentration of three-qubit Greenberger-Horne-Zeilinger states*, Phys. Rev. A **89** (2014) 032320(1-7).
- [133] S. Lee, J. Joo and J. Kim, *Entanglement of three-qubit pure states in terms of teleportation capability*, Phys. Rev. A **72** (2005) 024302(1-4).
- [134] S. Adhikari, *Probabilistic Teleportation of a Single Qubit: Unearthing New W-Class of States*, Journal of Experimental and Theoretical Physics **131** (2020) 375-384.
- [135] A. Sudbery, *On local invariants of pure three-qubit states*, J. Phys. A: Math. Gen. **34** (2001) 643-652.
- [136] R. A. Horn, and C. R. Johnson, *Matrix analysis*, (Cambridge University Press, Cambridge, 1999).
- [137] S. Ghose, N. Sinclair, S. Debnath, P. Rungta and R. Stock, *Tripartite Entanglement versus Tripartite Nonlocality in Three-Qubit Greenberger-Horne-Zeilinger-Class States*, Phys. Rev. Lett. **102** (2009) 250404(1-4).

- [138] A. Streltsov, G. Adesso, M. B. Plenio, *Colloquium: Quantum coherence as a resource*, Rev. Mod. Phys. **89** (2017) 041003(1-34).
- [139] M. Hillery, *Coherence as a resource in decision problems: The Deutsch-Jozsa algorithm and a variation*, Phys. Rev. A **93** (2016) 012111(1-6).
- [140] Y. Zhang, X. Zhao, Z. F. Zheng, L. Yu, Q. P. Su and C. P. Yang, *Universal controlled-phase gate with cat-state qubits in circuit QED*, Phys. Rev. A **96** (2017) 052317(1-9).
- [141] S. Agrawal, S. Halder and Manik Banik, *Genuinely entangled subspace with all-encompassing distillable entanglement across every bipartition*, Phys. Rev. A **99** (2019) 032335(1-10).
- [142] A. Barasinski, A. Cernoch and Karel Lemr, *Demonstration of Controlled Quantum Teleportation for Discrete Variables on Linear Optical Devices*, Phys. Rev. Lett. **122** (2019) 170501(1-6).
- [143] A. Barasinski, I. I. Arkhipov, and J. Svozilik, *Localizable entanglement as a necessary resource of controlled quantum teleportation*, Sci. Rep. 8 (2018) 15209(1-10).
- [144] K. G. Paulson and P. K. Panigrahi, *Tripartite non-maximally-entangled mixed states as a resource for optimally controlled quantum teleportation fidelity*, Phys. Rev. A **100** (2019) 052325(1-7).
- [145] G. Carrara, H. Kampermann, D. Brußand G. Murta, *Genuine multipartite entanglement is not a precondition for secure conference key agreement*, Phys. Rev. Res. **3** (2021) 013264(1-9).
- [146] T. Eggeling and R. F. Werner, *Separability properties of tripartite states with $U \otimes U \otimes U$ symmetry*, Phys. Rev. A **63** (2001)042111(1-15).
- [147] M-J Zhao, Z-X Wang and S. M. Fei, *Multiqubits entanglement based on optimal witness*, Rep. Math. Phys. **63** (2009) 409-416.
- [148] J-D Bancal, N. Gisin, Y-C Liang and S. Pironio, *Device-Independent Witnesses of Genuine Multipartite Entanglement*, Phys. Rev. Lett. **106** (2011) 250404(1-4).
- [149] M-J Zhao, T-G Zhang, X. L. Jost and S-M Fei, *Identification of three-qubit entanglement*, Phys. Rev. A **87** (2013) 012316(1-5).

- [150] R. Lohmayer, A. Osterloh, J. Siewert and A. Uhlmann, *Entangled Three-Qubit States without Concurrence and Three-Tangle*, Phys. Rev. Lett. **97** (2006) 260502(1-4).
- [151] Szilard Szalay, *Separability criteria for mixed three-qubit states*, Phys. Rev. A **83** (2011) 062337(1-17).
- [152] L. Novo, T. Moroder, and O. Gühne, *Genuine multiparticle entanglement of permutationally invariant states*, Phys. Rev. A **88** (2013) 012305(1-8).
- [153] S-Q Shen, M-Y Wang, M. Li and S-M Fei, *Separability criteria based on the realignment of density matrices and reduced density matrices*, Phys. Rev. A **92** (2015) 042332(1-5).
- [154] A. Das, C. Datta, P. Agrawal, *New Bell inequalities for three-qubit pure states*, Phys. Lett. A **381** (2017) 3928-3933.
- [155] K. H. Han and S-H Kye, *Construction of three-qubit biseparable states distinguishing kinds of entanglement in a partial separability classification*, Phys. Rev. A **99** (2019) 032304(1-3).
- [156] P. Kairon, M. Singh and S Adhikari, *Coherence-based inequality for the discrimination of three-qubit GHZ and W class*, Quant. Inf. Process **21** (2022) 173(1-13).
- [157] T. Baumgratz, M. Cramer and M. B. Plenio, *Quantifying Coherence*, Phys. Rev. Lett. **113** (2014) 140401(1-5).
- [158] J. W. Pan, D. Bouwmeester, H. Weinfurter, and A. Zeilinger, *Experimental Entanglement Swapping: Entangling Photons That Never Interacted*, Phys. Rev. Lett. **80** (1998) 3891-3894.
- [159] M. Hillery, V. Bužek, and A. Berthiaume, *Quantum secret sharing*, Phys. Rev. A **59** (1999) 1829-1834.
- [160] Z.D. Li, R. Zhang, X. F. Yin, L. Z. Liu, Y. Hu, Y. Q. Fang, Y. Y. Fei, X. Jiang, J. Zhang, L. Li, N. L. Liu, F. Xu, Y. A. Chen and J. W. Pan, *Experimental quantum repeater without quantum memory*, Nature Photonics **13** (2019) 644-648.
- [161] A. Kay, *Optimal detection of entanglement in Greenberger-Horne-Zeilinger states*, Phys. Rev. A **83** (2011) 020303(R)(1-4).

- [162] Y. A. Kourbolagh and M. Azhdargalam, *Entanglement witnesses for families of the bound entangled three-qubit states*, Int. J. Quant. Inf. **16** (2018) 1850025(1-6).
- [163] E. Jung, M. R. Hwang, D. Park and J. W. Son, *Three-tangle for rank-three mixed states: Mixture of Greenberger-Horne-Zeilinger, W, and flipped-W states*, Phys. Rev. A **79** (2009) 024306(1-4).
- [164] S. K. Sazim, S. Adhikari, S. Banerjee, T. Pramanik, *Quantification of Entanglement of teleportation in Arbitrary Dimensions*, Quan. Inf. Process. **13** (2014) 863-880.
- [165] V. Vedral, M. B. Plenio, M. A. Rippin and P. L. Knight, *Quantifying Entanglement*, Phys. Rev. Lett. **78** (1997) 2275-2279.
- [166] T.C. Wei and P.M. Goldbart, *Geometric measure of entanglement and applications to bipartite and multipartite quantum states*, Phys. Rev. A **68** (2003) 042307(1-12).
- [167] J. M. G. Sancho and S. F. Huelga, *Measuring the entanglement of bipartite pure states*, Phys. Rev. A **61** (2000) 042303(1-7).
- [168] S.P. Walborn, P.H.S. Ribeiro, L. Davidovich, F. Mintert and A. Buchleitner, *Experimental determination of entanglement with a single measurement*, Nature **440** (2006) 1022-1024.
- [169] P. Rungta, V. Buzek, C.M. Caves, M. Hillery, and G.J. Milburn, *Universal state inversion and concurrence in arbitrary dimensions*, Phys. Rev. A **64** (2001) 042315(1-13).
- [170] T. Das, S.S. Roy, S. Bagchi, A. Misra, A. Sen(De), and U. Sen, *Generalized geometric measure of entanglement for multiparty mixed states*, Phys. Rev. A **94** (2016) 022336(1-8).
- [171] F.G.S.L. Brandao, *Quantifying entanglement with witness operators*, Phys. Rev. A **72** (2005) 022310(1-15).
- [172] K Chen, S. Albeverio, and S.M. Fei, *Concurrence of Arbitrary Dimensional Bipartite Quantum States*, Phys. Rev. Lett. **95** (2005) 040504(1-4).

- [173] K. Bartkiewicz, J. Beran, K. Lemr, M. Norek and A. Miranowicz, *Quantifying entanglement of a two-qubit system via measurable and invariant moments of its partially transposed density matrix*, Phys. Rev. A **91** (2015) 022323(1-10).
- [174] K. Bartkiewicz, G. Chimczak and K. Lemr, *Direct method for measuring and witnessing quantum entanglement of arbitrary two-qubit states*, Phys. Rev. A **95** (2017) 022331(1-7).
- [175] M. Ziman, P. Stelmachovic and V. Buzek, *On the local unitary equivalence of states of multi-partite systems*, Fortschr. Phys. **49** (2001) 1123-1131.
- [176] A. Garg and S. Adhikari, *Teleportation Criteria Based on Maximum Eigenvalue of the Shared $d \otimes d$ Dimensional Mixed State: Beyond Singlet Fraction*, Int J Theor Phys **60** (2021) 1038-1052.
- [177] A. Knutson and T. Tao, *Honeycombs and sums of Hermitian matrices*, Not. Am. Math. Soc. **48**, 175 (2001).
- [178] S. Adhikari and S. Banerjee, *Operational meaning of discord in terms of teleportation fidelity*, Phys. Rev. A **86** (2012) 062313(1-5).
- [179] S. Adhikari, A.S. Majumdar, S. Roy, B. Ghosh, N. Nayak, *Teleportation via maximally and non-maximally entangled mixed states*, Quant. Inf. Compt. **10** (2010) 0398-0419.
- [180] P. Horodecki, M. Horodecki and R. Horodecki, *Bound Entanglement Can Be Activated*, Phys. Rev. Lett. **82** (1999) 1056-1059.
- [181] Q. Zhao, G. Wang, X. Yuan and X. Ma, *Efficient and robust detection of multipartite Greenberger-Horne-Zeilinger-like states*, Phys. Rev. A, **99** (2019) 052349(1-7).
- [182] F. Huber, I. Klep, V. Magron and J. Volcic, *Dimension-Free Entanglement Detection in Multipartite Werner States*, Commun. Math. Phys., (2022) DOI: <https://doi.org/10.1007/s00220-022-04485-9>.
- [183] G. Gour and N. R. Wallach, *All maximally entangled four-qubit states*, J. Math. Phys. **51** (2010) 112201(1-24).

- [184] C. Eltschka and J. Siewert, *Optimal class-specific witnesses for three-qubit entanglement from Greenberger-Horne-Zeilinger symmetry*, Quant. Inf. Comput. **13** (2013) 210-220.

List of Publications

1. **Anu Kumari** and Satyabrata Adhikari; *Detection of mixed bipartite entangled state in arbitrary dimension via structural physical approximation of partial transposition*, Physical Review A **100**, (2019), 052323(1-8). **Impact Factor (2.971)**
2. **Anu Kumari** and Satyabrata Adhikari; *Classification witness operator for the classification of different subclasses of three-qubit GHZ class*, Quantum Information Processing **20** (2021) 316(1-25). **Impact Factor (1.965)**
3. **Anu Kumari** and Satyabrata Adhikari; *Structural physical approximation of partial transposition makes possible to distinguish SLOCC inequivalent classes of three-qubit system*, European Physical Journal D **76** (2022) 73(1-9). **Impact Factor (1.611)**
4. **Anu Kumari** and Satyabrata Adhikari; *Structured Negativity: A physically realizable measure of entanglement based on structural physical approximation*, Annals of Physics **446** (2022) 169113(1-12). **Impact Factor (3.036)**
5. **Anu Kumari** and Satyabrata Adhikari; *Detection and Classification of Three-qubit States Using l_1 Norm of Coherence*
Status: Communicated
