A few entanglement criterion for two-qubit and two-qudit systems based on realignment operation

A DISSERTATION

submitted in partial fulfilment of the requirements for the award of degree
of

MASTER OF SCIENCE

in

Applied Mathematics

by

Shweta Kalson
(2K20/MSCMAT/29)

Anchal Singh
(2K20/MSCMAT/03)

Under the Supervision of

Dr. Satyabrata Adhikari

Assistant Professor

Department of Applied Mathematics
Delhi Technological University
(Formerly Delhi College of Engineering)
Bawana Road, Delhi-110042
May 2022

CANDIDATE'S DECLARATION

This dissertation entitled **A few entanglement criterion for two-qubit and two-qudit system based on realignment operation** submitted by us to the Department of Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement of the award of the degree of Master of Science, is original and not copied from any source without proper citation. The work presented in this dissertation has been carried out under the supervision of **Dr. Satyabrata Adhikari**, Department of Applied Mathematics, Delhi Technological University, Delhi. We hereby state that, to the best of our understanding, the work contained in this dissertation has not been submitted earlier, either in part or in full, to this or any other university/institution for the price of any degree or diploma.
The work has been communicated in a SCIE indexed journal with the following details:

Title of the Paper : A few entanglement criterion for two-qubit and two-qudit system based on realignment operation

Author's Name : Shweta Kalson and Anchal Singh

Name of the Journal : Quantum Information Processing [QINP]

Status of Paper : Communicated

Date of paper communication : 29 April 2022

Date of paper acceptance : Awaited

Date of paper publication : Awaited

Shweta Kalson
2K20/MSCMAT/29

Anchal Singh
2K20/MSCMAT/03

CERTIFICATE

I hereby certify that the dissertation entitled **A few entanglement criterion for two-qubit and two-qudit system based on realignment operation** which is submitted by **Shweta Kalson and Anchal Singh**, Department of Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement of the award of the degree of Master of Science, is a record of dissertation work carried out by the students under my supervision. To the best of my knowledge, this work has not been submitted in part of full for any Degree or Diploma to this University or elsewhere.

Place: Delhi
Date: 30 April 2022

Satyabrata Adhikari

SUPERVISOR:

Dr. Satyabrata Adhikari
Assistant Professor (UGC-FRP)
Department of Applied Mathematics
Delhi Technological University
Delhi-110042

ACKNOWLEDGEMENT

We take the opportunity to express our deep sense of gratitude to our supervisor, *Dr. Satyabrata Adhikari*, Assistant Professor (UGC-FRP), DTU, for his constant encouragement, cooperation and invaluable guidance throughout this dissertation. We are thankful to him for giving the facilities to fulfil our work smoothly. We thank the teachers of the department for imparting in us the knowledge and understanding of mathematics. Without their kind efforts we would not have reached this stage. Lastly, we would like to extend our appreciation to all those who could not be mentioned here but played their role to inspire us to accomplish our cherished goal.

Shweta Kalson
2K20/MSCMAT/29

Anchal Singh
2K20/MSCMAT/03

ABSTRACT

It is known that realignment crierion is necessary but not a sufficient criterion even for a two-qubit system. We have derived necessary and sufficient condition based on realignment operation for a particular class of two-qubit system and thus solved this problem partially for two-qubit system. We have shown that the lower bound of the trace norm of realigned form of the particular form of the density matrix exists if and only if the two-qubit state is entangled. The derived necessary and sufficient condition detects two-qubit entangled states, which are not detected by the realignment criterion. Further, we have obtained the upper bound of the minimum singular value of the realigned form of the density matrix for the $d \otimes d$ dimensional separable states. Moreover, we provide the geometrical interpretation of the derived separability criterion for $d \otimes d$ dimensional system. Moreover, we show that our criterion may also detect bound entangled state. Our criterion is beneficial in the sense that it requires to calculate only minimum singular value of the realigned matrix while on the other hand realignment criterion requires all singular values of the realigned matrix. Thus, our criterion has computational advantage over the realignment criterion.

Contents

List of Figures

Chapter 1

Introduction

It is well known that entanglement detection problem [1, 2] is one of the prime problem in quantum information theory. This problem can be considered as NP complete [3] and so all entangled state cannot be detected by just one criterion. Thus, there exist a vast literature in the context of the development of different entanglement detection criterion [4, 5, 6, 7, 8, 9, 10]. Partial transposition (PT) criterion is the first entanglement detection criterion introduced by Peres [11] and later Horodecki et.al [12] proved that the PT criterion is necessary and sufficient for $2 \otimes 2$ and $2 \otimes 3$ dimensional system. Mathematically, If any bipartite state described by the density operator

$$\rho_{AB} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

where A, B, C, D denote the block matrices then the partial transposition operation performed on ρ may be defined as

$$\rho_{AB}^{T_B} = \begin{pmatrix} P^T & Q^T \\ R^T & S^T \end{pmatrix} \tag{1.1}$$

where T_B denote the partial transposition with respect to the subsystem B and T denote the usual transposition operation. In higher dimensional system, PT criterion is only necessary but not sufficient.

Like Partial transposition operation, there also exist another operation that permute the elements of the density matrix in a different way and the resulting operation is known as realignment operation [13, 14]. Realignment operation can be explained as follows: Let us consider a bipartite state

$$\rho_{AB} = \sum_{ijkl} \rho_{ijkl}|ij\rangle\langle kl|$$

in the composite system $H_A^{d_1} \otimes H_B^{d_2}$, where $H_A^{d_1}$ and $H_B^{d_2}$ denoting the Hilbert spaces representing the individual systems A and B of dimension d_1 and d_2 respectively. The realignment operation R when acting on the state ρ_{AB} give the output as

$$R(\rho_{AB}) = \sum_{ijkl} \rho_{kjil}|kj\rangle\langle il| \tag{1.2}$$

In matrix notation, the density matrix ρ_{AB} for $2 \otimes 2$ system is given by

$$\rho_{AB} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \tag{1.3}$$

where

$$\begin{aligned} E &= \begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ \rho_{1,2}^* & \rho_{2,2} \end{pmatrix}, F = \begin{pmatrix} \rho_{1,3} & \rho_{1,4} \\ \rho_{2,3} & \rho_{2,4} \end{pmatrix}, \\ G &= \begin{pmatrix} \rho_{1,3}^* & \rho_{2,3}^* \\ \rho_{1,4}^* & \rho_{2,4}^* \end{pmatrix}, H = \begin{pmatrix} \rho_{3,3} & \rho_{3,4} \\ \rho_{3,4}^* & \rho_{4,4} \end{pmatrix} \end{aligned} \tag{1.4}$$

When realignment operation R performed on ρ_{AB} then the resulting output matrix look like

$$R(\rho_{AB}) = \begin{pmatrix} R(E) \\ R(F) \\ R(G) \\ R(H) \end{pmatrix} \tag{1.5}$$

where

$$\begin{aligned} R(E) &= \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \rho_{1,2}^* & \rho_{2,2} \end{pmatrix}, \\ R(F) &= \begin{pmatrix} \rho_{1,3} & \rho_{1,4} & \rho_{2,3} & \rho_{2,4} \end{pmatrix}, \\ R(G) &= \begin{pmatrix} \rho_{1,3}^* & \rho_{2,3}^* & \rho_{1,4}^* & \rho_{2,4}^* \end{pmatrix}, \\ R(H) &= \begin{pmatrix} \rho_{3,3} & \rho_{3,4} & \rho_{3,4}^* & \rho_{4,4} \end{pmatrix} \end{aligned} \tag{1.6}$$

Motivation of the Thesis:

The motivation of this work is as follows: we know that realignment criterion is necessary but not a sufficient criterion even for a two-qubit system.Firstly, We have derived a necessary and sufficient condition based on realignment operation for a par-

ticular class of two-qubit system and thus solved this problem partially for two-qubit system.Secondly,We have shown that the lower bound of the trace norm of realigned form of the particular form of the density matrix exists if and only if the two-qubit state is entangled. The derived necessary and sufficient condition detects two-qubit entangled states, which are not detected by the realignment criterion.Thirdly,we have obtained the upper bound of the minimum singular value of the realigned form of the density matrix for the $d \otimes d$ dimensional separable states.Lastly, we show that our criterion may also detect bound entangled state.
Our criterion is beneficial in the sense that it requires to calculate only minimum singular value of the realigned matrix while on the other hand realignment criterion requires all singular values of the realigned matrix.

1.1 Few Definitions and Results

Definition 1: Tensor Product

The tensor product of two matrices, $X = [x_{ij}]_{1 \leq i,j \leq m,n}$ and $Y = [y_{kl}]_{1 \leq k,l \leq p,q}$ is given by;

$$X \otimes Y = \begin{pmatrix} x_{11}Y & x_{12}Y & \cdots & x_{1n}Y \\ x_{21}Y & x_{22}Y & \cdots & x_{2n}Y \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1}Y & x_{m2}Y & \cdots & x_{mn}Y \end{pmatrix}_{mq \times np}$$

Definition 2: Separable States

A pure state $|\psi\rangle \in H$ is called separable state if we can find states $\left|\phi^A\right\rangle \in H_A$ and $\left|\phi^B\right\rangle \in H_B$, such that

$$|\psi\rangle = \left|\phi^A\right\rangle \otimes \left|\phi^B\right\rangle$$

If this condition does not hold, then the state $|\psi\rangle$ is known as **Entangled State**.

Definition 3: Density Operator

An operator ρ that satisfies the following conditions is known as a density operator:
(i) $Tr(\rho)$ is equal to one.
(ii) ρ is a positive operator.

Definition 4: Bound Entangled State

Bound entangled states are the states which are entangled and have positive partial transposition.

Definition 5: Singular Values

A Singular value of a real matrix A is the positive square root of an eigenvalue of the symmetric matrix A^TA or AA^T.

$$\sigma_i = \sqrt{\lambda_i} \tag{1.7}$$

where, σ_i are the singular values. λ_i are the eigenvalues.

Result 1: PPT Criteria

If the partial transpose of the density matrix ϱ has no negative eigenvalues,then the state ρ is called positive partial transpose (PPT) state:

$$\varrho^{T_A} \geq 0 \Leftrightarrow \varrho^{T_B} \geq 0 \tag{1.8}$$

We call a matrix negative partial transpose (NPT) if it is not PPT.
For $2 \otimes 2$ and $2 \otimes 3$ system, the state ρ is separable if and only if it is PPT.This condition is only necessary for higher dimensional system that is, if the state ρ is separable then the state is PPT.Conversely, for higher dimensional system if the state is NPT, then it is entangled.

Result 2: Realignment Criteria

If the state ρ_{AB} is separable then $||R(\rho_{AB})||_1 \leq 1$ where $||.||_1$ denotes the trace norm and defined by $||A||_1 = \text{Tr} \sqrt{AA\dagger}$. Since the right hand side of the inequality is unity in the realignment criterion, which does not depend on the state under investigation so it may be called as state independent realignment criterion.

Chapter 2

State dependent realignment criterion

We should note here an important fact that the realignment criterion is necessary but not sufficient. In this respect, we can ask the following question: Can we derive a necessary and sufficient entanglement condition for a two-qubit state using realignment operation? In this work, we have answered this question partially. We derive here the necessary and sufficient condition for the existence of entanglement in a particular class of a two-qubit system using the realignment operation. The derived criterion is state dependent and hence it can be named as state dependent realignment criterion. The derived separability criterion is important in the sense that it can detect a two-qubit entangled state which is not detected by the realignment criterion.
Let us consider a particular class of two-qubit state described by the density operator ϱ_{AB}, given by

$$\varrho_{AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix}, \rho_{11} + \rho_{33} + \rho_{44} = 1 \tag{2.1}$$

We can always choose the state parameter ρ_{11}, ρ_{14}, ρ_{14}^*, ρ_{33} and ρ_{44} in such a way that ϱ_{AB} represent a positive semi-definite matrix. The partial transposition of ϱ_{AB}

is given by

$$\varrho_{AB}^{T_B} = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & 0 & \rho_{14} & 0 \\ 0 & \rho_{14}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix} \tag{2.2}$$

where T_B denote the partial transposition with respect to the system B.
The eigenvalues of $\varrho_{AB}^{T_B}$ are given as

$$\begin{aligned} \lambda_1 &= \rho_{11}, \lambda_2 = \rho_{44} \\ \lambda_3 &= \frac{1}{2}\rho_{33} + \frac{1}{2}\sqrt{\rho_{3,3}^2 + 4|\rho_{14}|^2} \\ \lambda_4 &= \frac{1}{2}\rho_{33} - \frac{1}{2}\sqrt{\rho_{3,3}^2 + 4|\rho_{14}|^2} \end{aligned} \tag{2.3}$$

It can be observed that the eigenvalues λ_1, λ_2 and λ_3 will always be positive. The eigenvalue λ_4 will only be positive when $|\rho_{14}| = 0$. Thus, $\varrho_{AB}^{T_B}$ has negative eigenvalues only if $|\rho_{14}| \neq 0$ and hence for any non-zero state parameter $|\rho_{14}|$, the state ϱ_{AB} represent an entangled state.
After performing realignment operation on the state ρ_{AB}, the state reduces to

$$R(\varrho_{AB}) = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{14} & 0 & 0 \\ 0 & 0 & \rho_{14}^* & 0 \\ \rho_{33} & 0 & 0 & \rho_{44} \end{pmatrix} \tag{2.4}$$

If s_1, s_2, s_3, s_4 denote the singular values of $R(\varrho_{AB})$ then the singular values are given by

$$\begin{aligned} s_1 &= \sqrt{\frac{1}{2}[f(\rho_{11}, \rho_{33}, \rho_{44}) + \sqrt{f(\rho_{11}, \rho_{33}, \rho_{44})^2 - 4(\rho_{11}\rho_{44})^2}]}, \\ s_2 &= \sqrt{\frac{1}{2}[f(\rho_{11}, \rho_{33}, \rho_{44}) - \sqrt{f(\rho_{11}, \rho_{33}, \rho_{44})^2 - 4(\rho_{11}\rho_{44})^2}]}, \\ s_3 &= \frac{|\rho_{1,4}|}{2}, s_4 = \frac{|\rho_{1,4}|}{2} \end{aligned} \tag{2.5}$$

where $f(\rho_{11}, \rho_{33}, \rho_{44}) = \rho_{11}^2 + \rho_{33}^2 + \rho_{44}^2$.
The trace norm of $R(\varrho_{AB})$ can be calculated as

$$\begin{aligned} ||R(\varrho_{AB})||_1 &= s_1 + s_2 + s_3 + s_4 \\ &= s_1 + s_2 + |\rho_{14}| \end{aligned} \tag{2.6}$$

Applying $AM - GM$ inequality on s_1 and s_2, we get

$$s_1 + s_2 \geq 2(s_1 s_2)^{\frac{1}{2}} = 2\sqrt{\rho_{11}\rho_{44}} \tag{2.7}$$

Using (2.7) in (2.6), we get

$$\begin{aligned} ||R(\varrho_{AB})||_1 &\geq 2\sqrt{\rho_{11}\rho_{44}} + |\rho_{14}| \\ &= 2\sqrt{\rho_{11}\rho_{44}} + \sqrt{-\lambda_3\lambda_4} \end{aligned} \tag{2.8}$$

Here, λ_4 denote the minimum eigenvalue of partial transposed matrix of ϱ_{AB} and from partial transposition criterion, we can say that λ_4 will be negative if and only if the state ϱ_{AB} is an entangled state. Since it has also been shown in [15] that all eigenvalues of partial transposed matrix lying in $[-\frac{1}{2}, 1]$ so, $\lambda_4 \in [-\frac{1}{2}, 0]$. Thus, we have the following inequality

$$\lambda_4 < 0 \implies \frac{\rho_{33}}{2} < \frac{1}{2}\sqrt{\rho_{33}^2 + 4|\rho_{14}|^2} \tag{2.9}$$

Using (2.9) in (2.8), we get

$$\begin{aligned} ||R(\varrho_{AB})||_1 \geq\ & 2\sqrt{\rho_{11}\rho_{44}} + \\ & \sqrt{-\rho_{33}(\frac{\rho_{33}}{2} - \frac{1}{2}\sqrt{\rho_{33}^2 + 4|\rho_{14}|^2})} \end{aligned} \tag{2.10}$$

Now, we are in a position to state the following theorem for the particular class of two-qubit state as:

Theorem-1: Let us consider a class of two-qubit state described by the density operator $\varrho_{AB} = \rho_{11}|00\rangle\langle 00| + \rho_{14}|00\rangle\langle 11| + \rho_{33}|10\rangle\langle 10| + \rho_{14}^*|11\rangle\langle 00| + \rho_{44}|11\rangle\langle 11|]$. The state ϱ_{AB} is entangled if and only if

$$\begin{aligned} ||R(\varrho_{AB})||_1 \geq\ & 2\sqrt{\rho_{11}\rho_{44}} + \\ & \sqrt{-\rho_{33}(\frac{\rho_{33}}{2} - \frac{1}{2}\sqrt{\rho_{33}^2 + 4|\rho_{14}|^2})} \end{aligned} \tag{2.11}$$

2.1 A few states which are not detected by realignment criterion but detected by our criterion

In this section, we will discuss about few states which are not detected by realignment criterion but they may be detected either by using Theorem-1 or Corollary-1.

Example-1:

Let us consider a state ρ_1, which is given by [13]

$$\rho_1 = \begin{pmatrix} \frac{5}{8} & 0 & 0 & \frac{1}{32} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ \frac{1}{32} & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{2.12}$$

The partial transposition of the density matrix ρ_1 is given by

$$\rho_1^{T_B} = \begin{pmatrix} \frac{5}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{32} & 0 \\ 0 & \frac{1}{32} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{2.13}$$

The eigenvalues of partial transposed matrix $\rho_1^{T_B}$ are given by

$$\lambda_1^{(1)} = \frac{5}{8}, \lambda_2^{(1)} = \frac{1}{4}, \lambda_3^{(1)} = \frac{1}{32}(2+\sqrt{5})$$
$$, \lambda_4^{(1)} = \frac{1}{32}(2-\sqrt{5}) \tag{2.14}$$

Since one eigenvalue of $\rho_1^{T_B}$ is negative so, the state ρ_1 is an entangled state. The realigned matrix of ρ_1 is given by

$$R(\rho_1) = \begin{pmatrix} \frac{5}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{32} & 0 & 0 \\ 0 & 0 & \frac{1}{32} & 0 \\ \frac{1}{8} & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{2.15}$$

The trace norm of $R(\rho_1)$ are given by

$$||R(\rho_1)||_1 = 0.9464 \tag{2.16}$$

We can observe that $||R(\rho_1)||_1 < 1$ and thus the realignment criterion failed to conclude whether the state ρ_1 is entangled or not. Now, our task is to test for the entanglement of the state ρ_1 using theorem-1.
The LHS and RHS of the inequality in theorem-1 gives

$$\begin{aligned} LHS &= ||R(\rho_1)||_1 = 0.9464 \\ RHS &= 2\sqrt{\rho_{11}\rho_{44}} \\ &+\sqrt{-\rho_{33}(\frac{\rho_{33}}{2} - \frac{1}{2}\sqrt{\rho_{33}^2 + 4|\rho_{14}|^2})} = 0.8207 \end{aligned} \tag{2.17}$$

Therefore, Theorem-1 is verified for the state ρ_1 and thus it can be concluded that ρ_1 is an entangled state.

Example-2:

Let us consider another state described by the state ρ_2

$$\rho_2 = \frac{1}{2}\begin{pmatrix} \frac{7}{6} & 0 & 0 & \frac{1}{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{14} & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{2.18}$$

The partial transposition of the density matrix ρ_2 is given by

$$\rho_2^{T_B} = \frac{1}{2}\begin{pmatrix} \frac{7}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{14} & 0 \\ 0 & \frac{1}{14} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{2.19}$$

The eigenvalues of $\rho_2^{T_B}$ are given by $\lambda_1 = 0.2500$, $\lambda_2 = 0.5833$, $\lambda_3 = 0.1740$, $\lambda_4 = -0.0073$. By partial transposition criterion, the state $\rho_2^{T_B}$ is an entangled state.
The realigned matrix $R(\rho_2)$ is given by

$$R(\rho_2) = \frac{1}{2}\begin{pmatrix} \frac{7}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{14} & 0 & 0 \\ 0 & 0 & \frac{1}{14} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{2.20}$$

The trace norm of the realigned matrix $R(\rho_2)$ is found out to be $||R(\rho_2)||_1 = 0.9213$. Therefore, in this case also realignment criterion does not detect the entangled state ρ_2.

To detect the entangled state ρ_2, let us now use our criterion given in (2.11). The RHS of (2.11) is given by the quantity $2\sqrt{\rho_{11}\rho_{44}}+\sqrt{-\rho_{33}(\frac{\rho_{33}}{2} - \frac{1}{2}\sqrt{\rho_{33}^2 + 4|\rho_{14}|^2})}$ and it can be calculated for the state ρ_2 as 0.795. Thus, our criterion (2.11) is satisfied and hence the state ρ_2 is verified to be an entangled state.

Chapter 3

Separability criterion for $d \otimes d$ dimensional system based on realignment operation

In this section, we will derive few separability criterion for $d \otimes d$ dimensional system using two other existing separability criterion, namely, (i) realignment criterion [13] and (ii) structural physical approximation of partial transposition (SPA-PT) criterion [16]. The derived criterion provides only necessary condition for a state under probe to be a separable state.

3.1 Separability criterion in terms of minimum singular value of realigned matrix

Let us consider an arbitrary $d \otimes d$ dimensional bipartite state shared by two distant partners Alice (A) and Bob(B), which is described by the density operator ρ_{AB}. We can now present the separability criterion given in the following theorem:

Theorem-2: If $R(\rho_{AB})$ denote the realigned form of the density matrix ρ_{AB} and if ρ_{AB} is separable then

$$s_{min}[R(\rho_{AB})] < \frac{1}{d^2} \tag{3.1}$$

where $s_{min}[R(\rho_{AB})]$ denotes the minimum singular value of $R(\rho_{AB})$.
Proof: If $\tilde{\rho}_{AB}$ represents the SPA-PT of $\rho_{AB}^{T_B}$ then all eigenvalues of $\tilde{\rho}_{AB}$ are positive. Thus $\lambda_{min}[\tilde{\rho}_{AB}]$ is also positive. We can now apply $AM-GM$ inequality on the two positive quantities $||R(\rho_{AB})||_1$ and $\lambda_{min}[\tilde{\rho}_{AB}]$ and we get

$$\begin{aligned}
\frac{||R(\rho_{AB})||_1 + \lambda_{min}[]\tilde{\rho}_{AB}]}{2} &\geq [||R(\rho_{AB})||_1 \lambda_{min}[\tilde{\rho}_{AB}]]^{\frac{1}{2}} \\
&= [\sum_{i=1}^{d^2} s_i[R(\rho_{AB})]\lambda_{min}[\tilde{\rho}_{AB}]]^{\frac{1}{2}} \\
&\geq d[s_{min}[R(\rho_{AB})]\lambda_{min}[\tilde{\rho}_{AB}]]^{\frac{1}{2}}
\end{aligned} \tag{3.2}$$

Since, it is given that the state ρ_{AB} is separable so, from the realignment criterion we have

$$||R(\rho_{AB})||_1 \leq 1 \tag{3.3}$$

Using (3.3) and after a little bit of simplification, the inequality (3.2) reduces to

$$[\lambda_{min}[\tilde{\rho}_{AB}]]^2 + 2(1 - 2d^2 s_{min}[R(\rho_{AB})])\lambda_{min}[\tilde{\rho}_{AB}] + 1 \geq 0 \tag{3.4}$$

The expression given in (3.4) represents a quadratic equation in $\lambda_{min}[\tilde{\rho}_{AB}]$ and since it is non-negative so the discriminant of the quadratic expression must be negative. Thus, we have

$$s_{min}[R(\rho_{AB})] < \frac{1}{d^2} \tag{3.5}$$

Hence proved.

Corollary-1: If $\rho_{AB} \in H_A^d \otimes H_B^d$, where H_A^d and H_B^d denoting the Hilbert spaces representing the individual systems A and B of dimension d each respectively and if the inequality (3.1) is violated for any state ρ_{AB} then the state ρ_{AB} is an entangled state.

3.2 Geometrical Interpretation of separability criterion

In this section, we study the geometrical interpretation of the separability condition. To start with, we assume that the state ρ_{AB} is separable. We can then recall the separability condition (3.4) in terms of the non-negativity of the quadratic expression, which can be re-expressed as

$$(x + \frac{A}{2})^2 = y + \frac{A^2 - 4}{4} \tag{3.6}$$

where $x = \lambda_{min}[\tilde{\rho}_{AB}]$, $A = 2(1 - 2d^2 s_{min}[R(\rho_{AB})])$ and $y = [\lambda_{min}[\tilde{\rho}_{AB}]]^2 + 2(1 - 2d^2 s_{min}[R(\rho_{AB})])\lambda_{min}[\tilde{\rho}_{AB}] + 1$. The equation (3.6) represents a parabola with vertex at $(-\frac{A}{2}, \frac{4-A^2}{4})$. Since, the state ρ_{AB} is separable so (3.1) holds and thus the vertex

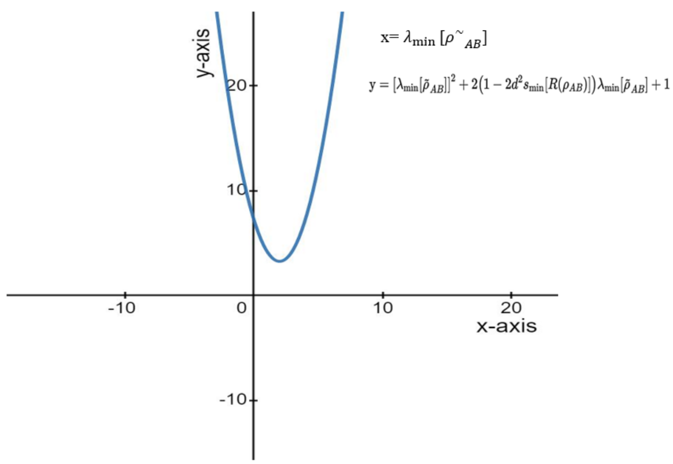

Figure 3.1: Geometrically, the states lying inside the parabola are entangled states while all separable states are lying outside the parabola. But there exist entangled states that are lying even outside the parabola.

of the parabola lie at the first quadrant of $x - y$ plane. Hence, the separability criterion given in theorem-1 can be interpreted as "all separable states are lying outside the parabola". Since the derived separability criterion is only necessary but not sufficient so there exist entangled states that may lie outside the parabola. Further, we may note that the states lying inside the parabola are entangled states and this interpretation may be claimed from corollary-1.

Chapter 4

Detection of NPTES and PPTES

In this section, we will take few examples to verify the result given in Theorem-2 and Corollary-1. To accomplish this task, we first consider negative partial transpose entangled state and then we consider positive partial transpose entangled state or bound entangled state.

Example-3

Let us consider a $3 \otimes 3$ dimensional state described by the density operator ρ_3

$$\rho_3 = \begin{pmatrix} P_1 & Q_1 & R_1 \\ Q_1^\dagger & Q_2 & R_2 \\ R_1^\dagger & R_2^\dagger & R_3 \end{pmatrix}, 0 \leq f \leq 1 \tag{4.1}$$

where,

$$P_1 = \begin{pmatrix} \frac{2+6f}{24} & 0 & 0 \\ 0 & \frac{1-f}{8} & 0 \\ 0 & 0 & \frac{1-f}{8} \end{pmatrix}; Q_1 = \begin{pmatrix} 0 & \frac{9f-1}{24} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; R_1 = \begin{pmatrix} 0 & 0 & \frac{9f-1}{24} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; Q_2 = \begin{pmatrix} \frac{1-f}{8} & 0 & 0 \\ 0 & \frac{2+6f}{24} & 0 \\ 0 & 0 & \frac{1-f}{8} \end{pmatrix};$$

$$R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{9f-1}{24} \\ 0 & 0 & 0 \end{pmatrix}; R_3 = \begin{pmatrix} \frac{1-f}{8} & 0 & 0 \\ 0 & \frac{1-f}{8} & 0 \\ 0 & 0 & \frac{2+6f}{24} \end{pmatrix}$$

The state ρ_3 is separable for $0 \leq f \leq \frac{1}{3}$ and entangled for $\frac{1}{3} < f \leq 1$.
For $3 \otimes 3$ So, the matrix ρ_3 we have,

$$\rho_3 = \begin{pmatrix} \frac{2+6f}{24} & 0 & 0 & 0 & \frac{9f-1}{24} & 0 & 0 & 0 & \frac{9f-1}{24} \\ 0 & \frac{1-f}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-f}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-f}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{9f-1}{24} & 0 & 0 & 0 & \frac{2+6f}{24} & 0 & 0 & 0 & \frac{9f-1}{24} \\ 0 & 0 & 0 & 0 & 0 & \frac{1-f}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-f}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-f}{8} & 0 \\ \frac{9f-1}{24} & 0 & 0 & 0 & \frac{9f-1}{24} & 0 & 0 & 0 & \frac{2+6f}{24} \end{pmatrix}$$

The realigned form of the density matrix ρ_3 is denoted by $R(\rho_3)$ and it is given by

$$R(\rho_3) = \begin{pmatrix} P_1^{(R)} & Q_1^{(R)} & R_1^{(R)} \\ (Q_1^{\dagger})^{(R)} & Q_2^{(R)} & R_2^{(R)} \\ (R_1^{\dagger})^{(R)} & (R_2^{\dagger})^{(R)} & R_3^{(R)} \end{pmatrix}, 0 \leq f \leq 1 \tag{4.2}$$

where,

$$P_1^{(R)} = \begin{pmatrix} \frac{2+6f}{24} & 0 & 0 \\ 0 & \frac{9f-1}{24} & 0 \\ 0 & 0 & \frac{9f-1}{24} \end{pmatrix};\ Q_1^{(R)} = \begin{pmatrix} 0 & \frac{1-f}{8} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ R_1^{(R)} = \begin{pmatrix} 0 & 0 & \frac{1-f}{8} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$Q_2^{(R)} = \begin{pmatrix} \frac{9f-1}{24} & 0 & 0 \\ 0 & \frac{2+6f}{24} & 0 \\ 0 & 0 & \frac{9f-1}{24} \end{pmatrix};\ R_2^{(R)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1-f}{8} \\ 0 & 0 & 0 \end{pmatrix};\ R_3^{(R)} = \begin{pmatrix} \frac{9f-1}{24} & 0 & 0 \\ 0 & \frac{9f-1}{24} & 0 \\ 0 & 0 & \frac{2+6f}{24} \end{pmatrix}$$

The corresponding realigned matrix is

$$R(\rho_3) = \begin{pmatrix} \frac{2+6f}{24} & 0 & 0 & 0 & \frac{1-f}{8} & 0 & 0 & 0 & \frac{1-f}{8} \\ 0 & \frac{9f-1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{9f-1}{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{9f-1}{24} & 0 & 0 & 0 & 0 & 0 \\ \frac{1-f}{8} & 0 & 0 & 0 & \frac{2+6f}{24} & 0 & 0 & 0 & \frac{1-f}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{9f-1}{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{9f-1}{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9f-1}{24} & 0 \\ \frac{1-f}{8} & 0 & 0 & 0 & \frac{1-f}{8} & 0 & 0 & 0 & \frac{2+6f}{24} \end{pmatrix}$$

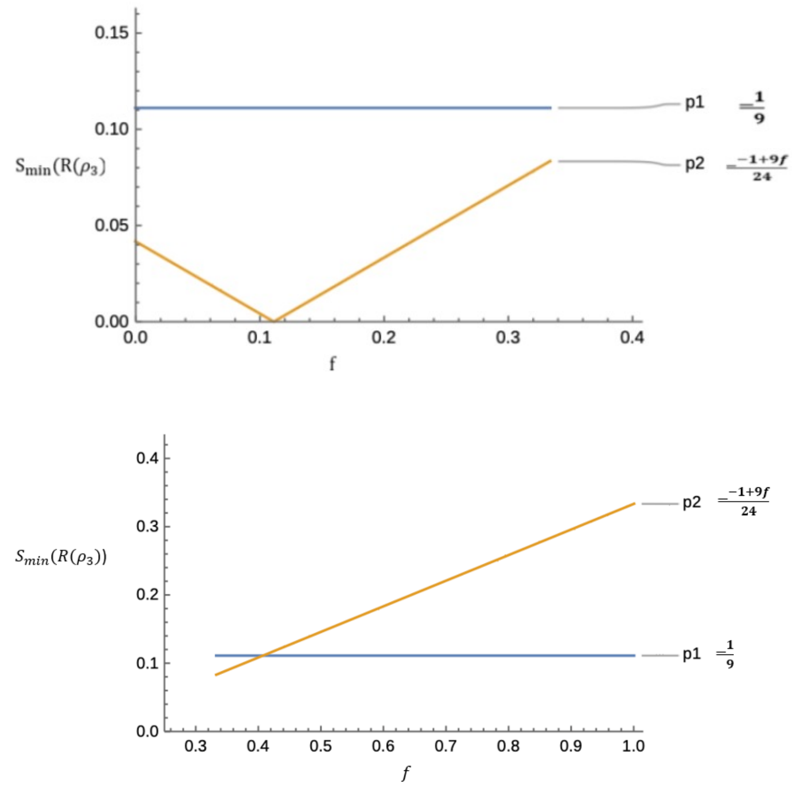

Figure 4.1: (a) In this figure, $s_{min}(R(\rho_3)) \leq \frac{1}{9}$ when $0 \leq f \leq \frac{1}{3}$ and thus ρ_3 represent a separable state (b) In this figure, $s_{min}(R(\rho_3)) > \frac{1}{9}$ when $\frac{11}{27} < f \leq 1$ and thus ρ_3 represent an entangled state $\frac{1}{3} < f \leq 1$

.

Singular values of $R(\rho_3)$ are as follows:

$$s_1(R(\rho_3)) = \frac{1}{3} \tag{4.3}$$

$$s_{2,3,4,5,6,7,8,9}(R(\rho_3)) = \frac{9f-1}{24} \tag{4.4}$$

The minimum singular value of $R(\rho_3)$ is given by

$$s_{min}(R(\rho_3)) = \frac{9f-1}{24} \tag{4.5}$$

Therefore, applying Theorem-1, we find that the state ρ_3 is separable when the state parameter $f \in [0, \frac{1}{3}]$. Further, using corollary-1, we conclude that the state ρ_3 is entangled if $f > \frac{11}{27} \approx 0.4074$.

Example-4

Let us consider another free entangled state (NPTES) which is described by the density operator ρ_β [17]

$$\rho_\beta = \frac{2}{7}|\psi^+\rangle\langle\psi^+| + \frac{\beta}{7}\sigma^+ + \frac{5-\beta}{7}\sigma^-, 4 \leq \beta \leq 5 \tag{4.6}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$, $\sigma^+ = \frac{1}{3}|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|$,
$\sigma^- = \frac{1}{3}|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|$.
The state ρ_β is NPTES for $4 \leq \beta \leq 5$ [17].

$$\rho_\beta = \begin{pmatrix} \frac{2}{21} & 0 & 0 & 0 & \frac{2}{21} & 0 & 0 & 0 & \frac{2}{21} \\ 0 & \frac{\beta}{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5-\beta}{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5-\beta}{21} & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{21} & 0 & 0 & 0 & \frac{2}{21} & 0 & 0 & 0 & \frac{2}{21} \\ 0 & 0 & 0 & 0 & 0 & \frac{\beta}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta}{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5-\beta}{21} & 0 \\ \frac{2}{21} & 0 & 0 & 0 & \frac{2}{21} & 0 & 0 & 0 & \frac{2}{21} \end{pmatrix}$$

The Corresponding Realigned Matrix is:

$$R(\rho_\beta) = \begin{pmatrix} \frac{2}{21} & 0 & 0 & 0 & \frac{\beta}{21} & 0 & 0 & 0 & \frac{5-\beta}{21} \\ 0 & \frac{2}{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{21} & 0 & 0 & 0 & 0 & 0 \\ \frac{5-\beta}{21} & 0 & 0 & 0 & \frac{2}{21} & 0 & 0 & 0 & \frac{\beta}{21} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{21} & 0 \\ \frac{\beta}{21} & 0 & 0 & 0 & \frac{5-\beta}{21} & 0 & 0 & 0 & \frac{2}{21} \end{pmatrix}$$

The minimum singular value of $s_{min}(R(\rho_\beta))$ is given by

$$s_{min}(R(\rho_\beta)) = \frac{1}{21}\sqrt{3\beta^2 - 15\beta + 19} \quad (4.7)$$

It can be easily found that $s_{min}(R(\rho_\beta))$ is always greater than $\frac{1}{9}$ for $4 \leq \beta \leq 5$. Therefore, the state ρ_β violate Theorem-1. Thus using corollary-1, we can say that

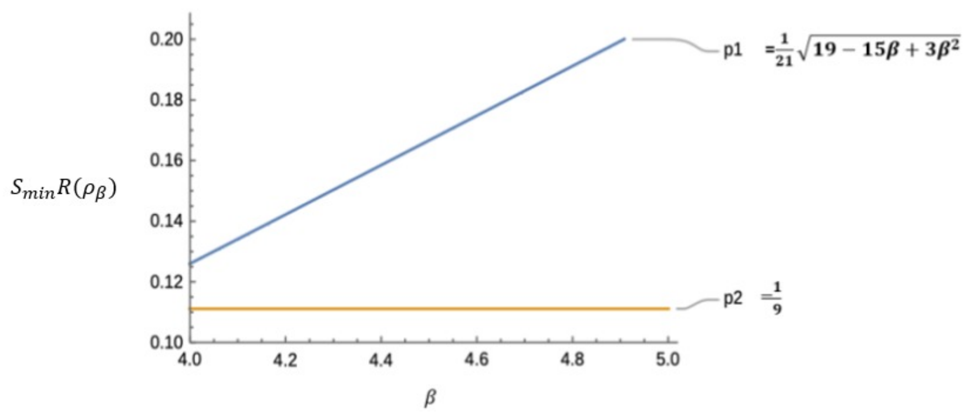

Figure 4.2: Figure shows that the state ρ_β represent an entangled state when $4 \leq \beta \leq 5$

the state ρ_β is entangled, which is indeed true.

Example-5

Let us again recall the state ρ_β given by (4.6). Considering the different range of the state parameter β, the state ρ_β can be re-expressed as

$$\rho_\beta^{(1)} = \frac{2}{7}|\psi^+\rangle\langle\psi^+| + \frac{\beta}{7}\sigma^+ + \frac{5-\beta}{7}\sigma^-, 2 \leq \beta \leq 4 \quad (4.8)$$

In [17], it has been shown that the state $\rho_\beta^{(1)}$ is separable for $2 \leq \beta \leq 3$ and PPTES for $3 < \beta \leq 4$. We can verify this result using our criterion which is based on minimum singular value of the realigned matrix of $\rho_\beta^{(1)}$. The minimum singular value of the realigned matrix $R(\rho_\beta^{(1)})$ is denoted by $s_{min}(R(\rho_\beta^{(1)}))$ and it is given by (4.7). The graph is plotted for minimum singular value $s_{min}(R(\rho_\beta^{(1)}))$ and we find that

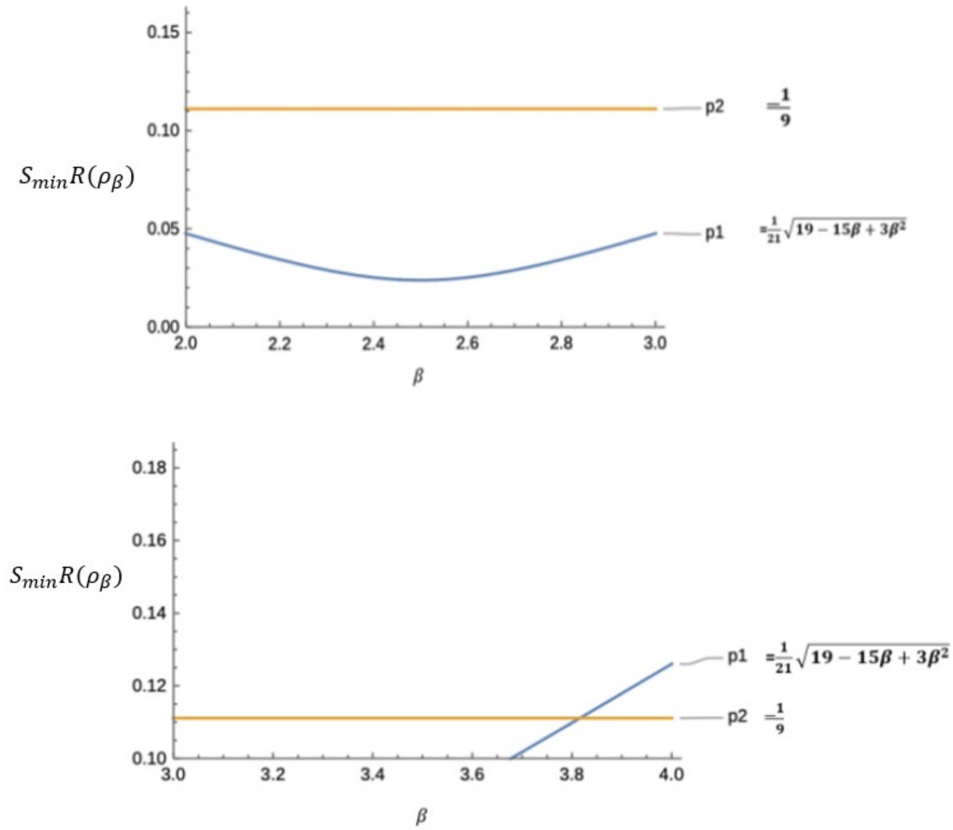

Figure 4.3: (a) Figure shows that the state ρ_β represent a separable state when $2 \leq \beta \leq 3$ (b) Figure shows that the state ρ_β represent a bound entangled state when $3.815 \leq \beta \leq 4$

$s_{min}(R(\rho_\beta^{(1)}))$ satisfies Theorem-1 for $2 \leq \beta \leq 3$. Thus, the state $\rho_\beta^{(1)}$ is again verified to be separable using our criterion given in Theorem-1. Moreover, corollary-1 detects PPTES when $\beta \in [3.815, 4]$.

Chapter 5

Conclusion

To summarize, we have reviewed the realignment criterion which is necessary but not sufficient condition for both lower as well as higher dimensional system. To fill this gap for lower dimensional system, we have considered a particular class of two-qubit state and derived a state dependent criterion based on the realignment operation. Our criterion is state dependent in the sense that the RHS of the inequality derived here as an entanglement criterion depends on the state. We may call it as state dependent realignment criterion. We should note here that the RHS of the inequality appeared in the realignment criterion is unity and thus it does not depend on the state. We have shown that the state dependent realignment criterion is necessary and sufficient for the considered particular class of two-qubit state. It is now open for the possible extension of the state dependent realignment criterion for any arbitrary bipartite two qubit state and arbitrary bipartite higher dimensional system. Moreover, to reduce the computational complexity in the calculation of the trace norm of the realigned matrix of the higher dimensional system which is required in realignment criterion, we provide here another separability condition which is based on the minimum singular value of the realignment of the $d \otimes d$ dimensional density matrix. This separability condition is necessary but not sufficient. We also study the geometrical interpretation of the derived separability condition. We have also shown that our criterion has the capability of detecting the bound entangled state.

Bibliography

[1] R. Horodecki, P. Horodecki, M. Horodecki and Karol Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[2] O. Guhne and G. Toth, Phys. Rep. **474**, 1 (2009).

[3] L. Gurvits, Proc. 35th Ann. ACM Symp. Theory Comput. pp 10-19 (2003); L. Gurvits, J. Comput. Syst. Sci. **69**, 448 (2004); Y. Huang, New J. Phys. 16 033027 (2014).

[4] P. Horodecki, Phys. Lett. A **232**, 333 (1997).

[5] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).

[6] M. A. Nielsen and J. Kempe, Phys. Rev. Lett. **86**, 5184 (2001).

[7] O. Guhne, P. Hyllus, O. Gittsovich, and J. Eisert, Phys. Rev. Lett. **99**, 130504 (2007).

[8] D. Bruss, J. I. Cirac, P. Horodecki, F. Hulpke, B. Kraus, M. Lewenstein, and A. Sanpera, J. Mod. Optics **49**, 1399 (2002).

[9] S. L. Braunstein, S. Ghosh, and S. Severini, Annals of Combinatorics **10**, 291 (2006).

[10] S. Dutta, B. Adhikari, S. Banerjee, and R. Srikanth, Phys. Rev. A **94**, 012306 (2016).

[11] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[12] M. Horodecki, P. Horodecki, R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[13] O. Rudolph, Phys. Rev. A **67**, 032312 (2003).

[14] K. Chen, L.-A. Wu, Quant. Inf. Comp. **3**, 193 (2003).

[15] S. Rana, Phys. Rev. A **87**, 054301 (2013).

[16] P. Horodecki and A. Ekert, Phys. Rev. Lett. **89**, 127902 (2002).

[17] P. Horodecki, M. Horodecki and R. Horodecki, Phys. Rev. Lett. **82**, 1056 (1999).