COMPUTATION WITH 2-TUPLE LINGUISTIC VARIABLES AND ITS APPLICATION IN MATRIX GAMES

A thesis submitted to

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in

MATHEMATICS

by

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under the supervision of

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DECLARATION

I affirm that the research work presented in this thesis entitled "**Computation with 2-tuple linguistic variables and its application in matrix games**" for the award of the degree of *Doctor of Philosophy in Mathematics* is an authentic record of my own work carried out under the supervision of *Prof. Anjana Gupta*, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

Unless otherwise stated, the research work represented in this thesis is my own research. I have not previously submitted this thesis in part or full to any other university or institute with the purpose of receiving a degree or diploma. This thesis contains no other person's data, graphs or other information, unless specifically acknowledged.

Date : **(Tanya Malhotra)**

CERTIFICATE

This is to certify that the thesis entitled **"Computation with 2-tuple linguistic variables and its application in matrix games"** submitted by **Ms. Tanya Malhotra** in the Department of Applied Mathematics, Delhi Technological University, Delhi, India for the award of degree of *Doctor of Philosophy in Mathematics*, is a record of bonafide research work carried out by her under my supervision.

I have read this thesis and that, in my opinion, it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy.

To the best of my knowledge the work reported in this thesis is original and has not been submitted to any other Institution or University in any form for the award of any degree or diploma.

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Date : **(TANYA MALHOTRA)**

Place : Delhi, India.

Dedicated to

My Parents

Mr. Sanjay Malhotra & Mrs. Nandita Malhotra

&

My Sister

Miss. Ananya Malhotra

Contents

Preface

Making a decision entails selecting from a set of options based on a preliminary analysis, which frequently includes human intervention and uncertainty. Besides, grasping the meaning communicated by information in a qualitative setting is necessary before making a further analysis. One of the most challenging issues is to deal with statements, human thoughts preferences, feelings, and so on because of the inherent character of natural language. Over the years, much work has been taken to account for the ambiguity and imprecision of linguistic information by using the theory of fuzzy set and fuzzy linguistic-based approach. Several computational methods have been created to deal with uncertainty, particularly when it is not of a probabilistic character. Specifically, in 2000, a new model known as the "2-tuple linguistic representation model" arose, which improved numerous linguistic processes for handling complex decision-making issues. It facilitates a continuous representation of the linguistic terms, and henceforth, research concerned with the 2-tuple model is profuse and worthwhile considering in deep.

The introspection of the distinguished literature in the 2-tuple model enabled us to realize that some limitations still persist in the existing uncertain 2-tuple models. This motivated us to improve the existing uncertain models to make their implementation more flexible and consistent in decision-making processes. Therefore, in our study, we have addressed the constraints and challenges associated with the existing 2-tuple model and have worked towards its development to enhance its applicability. Further, we have supported our study by applying the 2-tuple model in the domain of matrix games and decision analysis. This has enabled us to contribute to the researchers worldwide who are working in this field and are also looking for exploration.

The thesis entitled "**Computation with 2-tuple linguistic variable and its application in matrix games**" comprises of six chapters followed by the summary and future scope. The bibliography and the list of publications are provided at the end of the thesis.

The introductory **Chapter 1** presents a short overview of computing with words 2-

tuple based linguistic model as well as its elementary application anticipated in distinct decision-making models followed by its extension. Thus, the current chapter creates a background and motivates this thesis's work. The chapter is based on a review paper, "A systematic review of developments in the 2-tuple linguistic model and its applications in decision analysis," published in **Soft Computing**, Springer (2020).

The **Chapter 2** entitled, "Group operations and properties for 2-tuple linguistic variables with its application" establish the basis for a theory of 2-tuple linguistic groups under the given binary operation in a classical impression. In literature, the concept of fuzzy algebra has been a subject of research for many years and has made significant progress. Nevertheless, the abstract theory of linguistic groups is pristine and yet to be explored. The use of fuzzy linguistic concepts to represent practical situations with qualitative data has shown to be a powerful approach. Several computational techniques have been introduced to alleviate the computation between linguistic terms. Among these computational techniques, the proposal of a 2-tuple linguistic model is a useful tool by easing out the computations and avoiding information loss when applied in some practical decision-making situations. In the study of linguistic information, the aggregation of 2-tuple linguistic labels is a crucial problem. Several computing models existing in the literature are well-suited to deal with this problem. However, it is noted that the existing operational laws are not satisfying the closure property. Moreover, to the best of knowledge, no theory has been developed to support the concept of linguistic groups. For this reason, the foundation of the theory of 2-tuple linguistic groups under a crisp binary operation is a milestone in this direction, overcoming the constraints of the existing operational laws which operate without information loss. The chapter has given a formal methodology to claim that the 2-tuple linguistic term set forms an algebraic structure group. Further, a similarity relation between the linguistic groups is obtained, and some properties of the operational laws, group isomorphic and homomorphic relation, have been discussed in detail. Lastly, the physical meaning of the abstract concept so developed has been showcased in bipolar graphs and matrix games. The chapter is based on a research paper entitled, "Group operations and isomorphic relation with the 2-tuple linguistic variables", published in **Soft Computing**, springer **24**, 18287–18300 (2020) and "Group isomorphic properties with some novel operational laws for 2-tuple linguistic variables and its application in linguistic matrix games" Communicated in *IEEE Transactions on Systems, Man, and Cybernetics: Systems*.

A qualitative decision making problems with linguistic term set where all plausible linguistic descriptors provided by experts have symmetric and uniform distribution has been investigated by several scholars. Obviously, it might not be suitable in practical life decision problems since the experts may prefer linguistic labels distributed non-uniformly and non-symmetrically. Numerous studies have been developed on theoretical and practical applications to handle an unbalanced linguistic context. However, the current unbalanced linguistic computational models are complex and computationally more expensive. Therefore, in **Chapter 3** entitled, "Methodology for unbalanced linguistic terms" we propose a newly constructed methodology to handle a set of unbalanced linguistic terms and further develop a novel 2-tuple linguistic technique for the unbalanced linguistic set. The new 2-tuple unbalanced linguistic model is computationally less complicated and can avoid information loss. Finally, numerical illustrations present the concrete steps of the developed approach and manifest the practicality and flexibility of this model by elucidating a comparative analysis with existing models. The chapter is based on a research paper titled, "A New 2-Tuple Linguistic Approach for Unbalanced Linguistic Term Sets", published in **IEEE Transactions on Fuzzy Systems 29** (8) 2158–2168 (2021).

Chapter 4 entitled, "Matrix games with probabilistic multiplicative unbalanced linguistic information" proposes a novel concept of the probabilistic multiplicative unbalanced linguistic term set considering the probabilities as well as non-uniformity of distinct linguistic labels. Further, based on the proposed concept a unified mechanism to solve a two-person linguistic matrix game having probabilistic multiplicative unbalanced linguistic information is suggested. The proposed approach can be perceived as a convenient technique for multiple criteria decision-making (MCDM) problems. Numerical illustrations are presented to discuss the significance of the proposed methodology. The chapter is based on a research paper titled, "Probabilistic multiplicative unbalanced linguistic term set and its application in matrix games", communicated in *International journal of machine learning and cybernetics, Springer*.

In **Chapter 5** entitled, "Matrix games with interval-valued 2-tuple linguistic information" a 2-player non-cooperative zero-sum interval-valued 2-tuple fuzzy linguistic (IVTFL) matrix game is proposed, and interval-valued linguistic linear programming (IVLLP) methodology is suggested to solve such class of games. A hypothetical example is used to demonstrate the suggested method's applicability in the practical world. The chapter is based on a research paper titled, "Methodology for Interval-Valued Matrix Games with 2-Tuple Fuzzy Linguistic Information", published in **In: Sergeyev Y., Kvasov D. (eds) Numerical Computations: Theory and Algorithms. NUMTA 2019. Lecture Notes in**

Computer Science, Springer, Cham. , **11974**, (2020). https://doi.org/10.1007/978-3-030-40616-5_12.

Chapter 6 entitled, "Interval norm approach for solving two player zero sum matrix games with interval payoffs" present a new approach that gives a unique outlook for solving a two-player zero-sum interval-valued matrix game (ZSIMG) based on the interval matrix norm framework. The methodology presented in this chapter helps obtain an approximated interval game value for the corresponding ZSIMG without undergoing the existing process of solving traditional interval linear mathematical models. The chapter is based on the research paper titled, "Interval norm approach for solving two-player zero-sum matrix games with interval payoffs" **Submitted** in *Computational optimization and application, Springer*.

After chapter 6, we present the summary of the research work carried out in this thesis. In addition, the future scope of the thesis has been discussed briefly.

Finally the thesis ended with the bibliography and list of publications.

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Chapter 1

Introduction

Most real-world problems incorporate uncertainty and erroneous information within their framework of definition such that it is inappropriate to model such problems via numerical values. Under these circumstances, the utility of a fuzzy linguistic-based approach and computing with words (CWWs) processes have provided effectual results. Concretely, because of explicit linguistic computations and high interpretability, the 2-tuple linguistic (2TL) model surpasses. The 2TL model has upgraded several linguistic processes for solving complex decision-making issues. The present introductory chapter[1] *initially provides a brief understanding of a fuzzy linguistic approach and other existing linguistic computational models (LCMs) designated to introduce the necessity of the 2TL framework. The chapter explicitly discusses the linguistic model based on the 2-tuples and its foundation. Its extension to interval-valued 2TL framework, unbalanced LCM, linguistic distribution, and so forth is also explained. Further, it justifies the significant applications of a 2TL model witnessed in varied disciplines, primarily decision analysis and game theory (GT). Henceforth, the fundamentals of this chapter are proficient in providing a platform to motivate the research work carried out in this thesis.*

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[1]The content of chapter is based on the review paper "A systematic review of developments in 2-tuple linguistic model and its applications in decision analysis," *Soft Computing*, Springer (2020). https://doi.org/10.1007/s00500-020-05031-2 (**SCIE, Impact Factor: 3.643**)

Several real-world problems deal with information that is uncertain and vague. The more possibility in a question, the less accurate we can be in our understanding of that problem, and as a result, it should be obstructed apparently. Problems at the molecular level are not addressed in the late nineteenth century. As a result, to achieve a more credible solution and simultaneously quantifying the amount of risk leads to the development of probability theory and more advanced statistical methods in the late 20^{th} century. The question of possibility was challenged by the studies of Max-Black in vagueness persisting in the problem, first in 1937. Later on, it was challenged by L.A. Zadeh [316], in 1965 who introduces the notion of Fuzzy sets. The introduction of the Fuzzy set (FS) theory has shown great alacrity and competence in modeling the uncertainty abiding within the problems in an attempt to challenge the existing probability theory along with the very basis on which it depends, i.e., classical binary logic.

Traditional set theory is based primarily on the idea that either an element belongs to a collection or does not belong to it. There is no significant presence of partial membership of elements of a set. However, FS theory permits us to extend the fundamental concepts of classical set theory and logic in an infinite way that seems hugely sensible. The conviction of binary membership was extended by Zadeh to accommodate varied "degrees of membership" on the unit interval $[0,1]$ where 0 and 1 represent "non-membership" and "full membership" respectively, and the infinite number of values lying in between 0 and 1 represent partial membership. For more details about the contributions made to the theory of fuzzy sets and applications, one can refer [316–323, 328, 329].

Decision-making and decision analysis are considered among the most prominent components of our daily lives and have been studied extensively in the literature. Understanding its complexity, it is observed that most real-world problems are commonly uncertain in several ways. Lack of information can cause the system's future state to be ambiguous. This uncertainty persisting within the problem has been addressed using probabilistic models and statistical tools. However, in several situations, non-deterministic models are required to offer better results. The fuzzy system has evolved as an uncertain system covering each field of real-life problems. Subsequently, over time, a plethora of fuzzy multi-criteria decision-making (MCDM) approaches has been comprehensively studied in the literature. The motivation for developing diverse fuzzy MCDM approaches is because of researchers' and practitioners' desire to reinforce decision-making methods via a recent enhancement in computer technology, scientific computing, and mathematical optimization. In paper [197], the systematic review of fuzzy MCDM methods and their application in different fields from 1994 to 2014 has been given.

1.1 Fuzzy decision making

The research prospected on issues concerning fuzzy logic has its origin with the commencing work of Zadeh in 1965 [316]. This line of research analyzed the FS theory concept starting with the utility of standard boolean sets and progressing to a multi-valued logic [26]. The advancement witnessed in this field was so expeditious that FS theory has been recognized worldwide as a prominent research field within a few years, being studied by several researchers and practitioners around the globe in theoretical and practical aspects [198]. Amid multiple theoretical and practical developments, the theory of FS outshines as a research area of study of decision-making. Its application in modern decision science is not astonishing, given that decision analysis is viewed as an area in which human-originated information is ubiquitous.

The research on fuzzy decision-making (FDM) evolved from a wide number of researchers throughout the world developing concepts suchlike FSs [316], fuzzy environments [27], approximate reasoning [324–326] and applications of FSs in decision systems [341]. Its central argument states that many real-world decisions are made in an framework where the consequences of all plausible actions are not fully understood. Human subjectivity influences decision-making, which is a multistage process. According to the paper [26], a fuzzy decision is an intersection of objectives and demarcations presented inside a multistage mechanism, where human intelligence can manipulate fuzzy concepts and fuzzy answer instructions.

1.2 Linguistic Decision Making

With increasing complexities in the practical decision-making (DM) problems, it is observed that the information persisting within them is uncertain and fragmentary. Also, several aspects of real-world problems require human intervention, where they need to select among different alternatives through rationale and cognitive process. Such decision problems are ill-structured and are not submissive to quantitative characterization. Meanwhile, there may exist some situations where the present information is not quantifiable because of its nature (e.g., when categorizing the quality of research paper, the terms like “Excellent,” “Good,” “Average” can be used), or the cost of computation for quantitative information present in the problems is very high that an "approximate value" is bearable. Therefore, experts often prefer to express the results in a natural language rather than precise numerical value. This leads to the notion of a linguistic variable (LV). LVs are defined as variables whose values are accepted as either words or an ordered blend of words used in a natural or synthetic language, whose meaning is characterized by a semantic principle [324]. For instance, age can be a LV whose values are considered linguistics, such as very young, young, not young, not very young, ..., etc., which is closer to the human subjective thinking process than numbers like 1, 2, 3,..., 100. The formal definition of LV is stated below:

Definition 1.2.1. [324] A LV is described as $(H, T(H), U, G, M(H))$ where H represents the name of LV identical with a real variable name; $T(H)$ represents a term set consisting of the names of LV meanings H. Every term in a set is a fuzzy variable whose meanings lies within the range U, $[U_{\min}, U_{\max}]$; G refers to the syntactic rule generating the names of values in H; and $M(H)$ refers to the semantic rule represented by the membership function (MF) 'μ'.

The linguistic approximation was initiated ideally by L.A. Zadeh [324] and was considered as the most prominent application area of the linguistic variable concept. He presented in the paper that the linguistic-based approach was not associated with the conventional non-mathematical ways of dealing with humanistic systems. Instead, it was represented as an amalgam between the quantitative and qualitative aspects, based on words when numerical characterizations were inappropriate. The progress in the fuzzy linguistic-based approach is remarkable with the growing number of years and considered a core area in different disciplines like decision-making, supply chain, pattern recognition, energy optimization, approximate reasoning, and so forth. The usefulness of linguistic-based information implies a broad perspective in the current specialized literature on linguistic decision analysis and CWWs related methodologies. The broad classification of diverse fields where linguistic modeling (LM) seems considerably logical is mentioned in Figure 1.1.

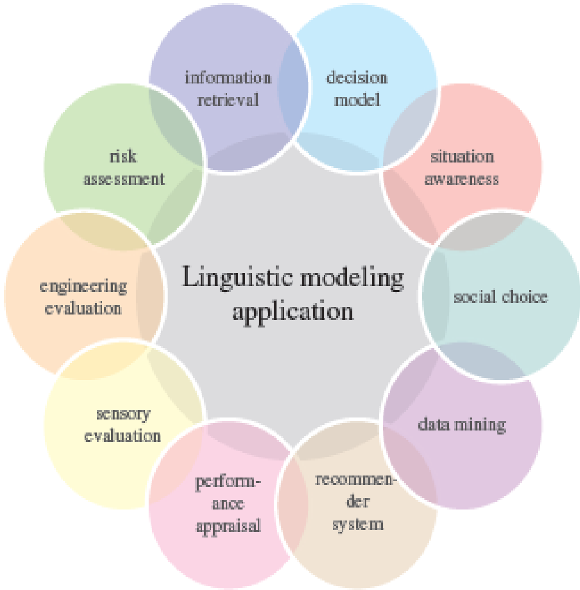

Figure 1.1: Linguistic modeling application

1.2.1 Fuzzy Linguistic Approach

While the rapid development of FS theory to DM problems where only incomplete or uncertain information is available is the subject of substantial research over the last decade, it is believed that it brings about another problem of how to potentially present the decision that could be easily understandable by humans. The solution to the mentioned problem is the evolution of a fuzzy linguistic-based approach that aims to offer excellent results more compatible with human cognitive mechanisms.

Zadeh [327] proposed a fuzzy linguistic-based approach in 1975, which served as an initiatory tool to provide a language for inaccurate estimation of the components involved in the DM process that are either intrinsically fuzzy or are inadequate of precise measurement. Henceforth, it provides many reliable results and, consequently, successfully applied in different research areas. For more details on the fuzzy linguistic approach, one can refer to the papers [20,60,61,88,103,104,261,276,324,327]. Here, Table 1.1 shows a summary of the existing literature primarily centered on the application area of the fuzzy linguistic based approach. In this table, the chronological order of the publication along with a synopsis of the key contributions made by the fuzzy linguistic-based approach is given.

Table 1.1: Summary of applications based on a fuzzy linguistic approach

Title	year	paper	Description	Why Fuzzy Linguistic Approach	Strong/Weak points
"The Concept of a Linguistic Variable and Its Applications to Approximate Reasoning Part I, Part II, Part III"	1975	[324–326]	This paper demonstrates an idea of a linguistic variable and its application in approximate reasoning, which is considered to be a mode of reasoning that is not exact nor very inexact.	Linguistic variables provides a method to estimate characterization of phenomena which are really complex or obscure to be amenable to description in standard quantitative terms.	This approach offers a more realistic framework for human reasoning than classical binary logic. Also, computation with linguistic variables may require the solution of non-linear programs and thereby leads to results that are imprecise to the same degree as the underlying probabilities.
"A linguistic approach to decision making with fuzzy sets"	1980	[261]	This paper proposes a method for fuzzy decision making based on linguistic approximation and truth qualification.	Fuzzy linguistic approach is considered to be the most expropriate in a situation where information is vague or imprecise. Hence, decision expressed in such cases should be linguistic rather numerical.	The proposed approach creates a linguistic evaluation of the choices, and therefore results acquired from this approach can be used in fuzzy decision problems. Decision-makers associated with the choice handle at every level, and as a result, the proposed approach is of great practical importance.
"Fuzzy set based linguistic approach: Theory and application"	1980	[23]	Fuzzy sets and logic are considered to be premise for the linguistic approach and this paper very well explains this approach and further illustrate it with example.	Proposed approach helps in developing models that can able to mimic approximate reasoning.	The linguistic approach can able to handle two major issues: firstly, the proposed methodology relates a name to an unbalanced fuzzy set on the premise of semantic similitude and, secondly, perform mathematical operations with fuzzy numbers.
"The problem of linguistic approximation in clinical decision making"	1988	[76]	This paper explains linguistic approximation method in computerized framework which is the background of medical decision making.	The proposed method is useful in expressing linguistic outputs that are simple and reliable. Moreover, the choices can be completely represented by graphical representation of the suitability sets over a decision space $[0,1]$ to avoid uncertainty.	The linguistic approximation approach is useful for clinically recognized linguistic terms whose meaning is well defined in the medical community. However, some problems still persist with the existing approach. The first concern is regarding a choice of the threshold, which is used to differentiate good and bad approximations. The second concern is about a limited number of terms at the user's disposal.
"Linguistic decision making models"	1992	[61]	This paper proposes the model for decision making problems in which knowledge about gains and true state of nature is given as linguistic rather than numeric.	Linguistic decision models proved to be an edge over other models based on classical decision theory. Linguistic decision models could very well handle representation of the true state of nature, and gains are represented by means of linguistic terms, and the objective of decision-makers are vaguely established.	This article is restricted to single objective decision-making problems.
"A fuzzy linguistic approach generalizing boolean information retrieval: a model and its evaluation"	1993	[21]	In this article, an extension of the weighted boolean retrieval model is being formalized within the fuzzy linguistic framework. In this model, linguistic descriptors replace numeric query weight. This fuzzy linguistic model is explained, and further evaluation is being carried out of its implementation on a boolean information retrieval system.	Proposed approach has an ability to deal with the imprecision and subjectivity characterizing retrieval activity.	The retrieval model defined within the fuzzy framework provides an understandable and convenient means of dealing with qualitative and imprecise criteria. The introduction of linguistic weights in queries has improved the expressive power of the weighted Boolean query language.
"Non-numeric multi-Criteria multi-person decision making"	1993	[306]	The objective of this paper is to describe a multi-expert MCDM strategy for estimating and selecting the alternatives based on a non-numeric scale.	The non-numeric linguistic scale has an ability to deal with the imprecision and thereby provides a more flexible framework for an evaluator.	The central idea of the proposed methodology in using linguistic terms linked with the scores is that it makes it simple for an evaluator to manipulate. The process explained in this paper allows experts for the multi-criteria evaluation of every object, followed by an aggregation of these individual experts in order to obtain an overall object evaluation.

Table 1.1: (Continued)

Title	year	paper	Description	Why Fuzzy Linguistic Approach	Strong/Weak points
"FUZMAR: An approach to aggregating market research data based on fuzzy reasoning"	1994	[308]	This article introduces an approach for validating models that contain variables having linguistic values that are drawn from a linear ordering scale. To develop the models, the methodology provided in the paper relies on fuzzy logic operations, and the principle of lowest entropy to choose between competing models.	In the proposed methodology, linguistic values assumed to give accurate and reliable results. Also, computations made on linguistic values are simpler than numerical ones.	The key idea of the work is to indicate a methodology to use the information provided by market surveys to predict the values of other related variables of interest to market research analysts. The proposed methodology based on the use of Shannon's entropy. Further, an approach to a simple aggregate idea to form a complicated one is also introduced.
"A multiple criteria linguistic decision model for human decision making"	1994	[164]	This article proposes a multiple criteria linguistic decision model (MCLDM) for human decision-making problems. MCLDM is beneficial to appraise a set of alternatives over varied criteria by using linguistic variables and then presents linguistic decision output for selecting the best alternatives.	The proposed approach is beneficial in describing a systematic way of making linguistic decisions within a fuzzy environment, so as to tackle imprecise information well, that cannot be handled by probability theory.	MCLDM method is one of the most powerful methods for managing decision makers engaged in making decisions in a situation where the information available is vague or uncertain. Moreover, MCLDM is useful in performing sensitivity analysis.
"A fuzzy linguistic approach to a multi criteria sequencing problem"	1996	[1]	This paper considers a fuzzy approach to a single machine scheduling problem, where the system's variables are defined with the help of linguistic terms. The fundamental point of the study is to figure out the period of the transforming times, on grouping the jobs in the machine. And lastly, to determine the common expected rate in a near-optimal way.	Fuzzy set theoretic linguistic framework overcomes the difficulty in approximating the parameters which best describe the system under consideration. The proposed approach proves to be beneficial in a situation where parameters such as processing times of tasks are characterized by vagueness. Such vagueness is well handled qualitatively rather numerically.	The advantage of using the proposed approach in many real production situations is to give promising results when the information available is vague. It could able to deal with several scheduling models in which processing times of jobs are not precisely known and henceforth give considerable results.
"Using fuzzy numbers in educational grading system"	1996	[149]	This paper frames out a structural model of a fuzzy academic grading system and further proposes an algorithmic program for it. This study also comprised of a strategy to frame out membership functions of various linguistic values with dissimilar weights.	Observed scores of students, as well as scores of distinct questions in an examination, are vague, thereby fuzzy set theory is beneficial in aggregating scores. Moreover, the linguistic variable is being used to judge student's performance via grade and hence, is beneficial in aggregating scores well.	The proposed approach offers a result that is quite reliable and uses the fuzzy set theory as its basis to provide us with the information needed to develop a new grading method. In the proposed methodology, the grades to be assigned corresponds to the degree of membership so that aggregation of different test scores into a single score is plausible. Further, it provides us means to strengthen the quality of the grading system, thereby giving a clear indication about the student's academic performance.
"Direct approach processes in group decision making (GDM) using linguistic ordered weighted average operators"	1996	[104]	In this paper, the properties and the axiomatic of the linguistic ordered weighted averaging operator (LOWA) by presenting the rationality of its aggregation way is being presented. Further, the use of the LOWA operator is also being introduced in order to solve GDM problems from individual linguistic preference relations.	Proposed approach helps in incorporating high human consistency in decision models.	The focal thought created in the paper is to find out evidence of rationality of the LOWA operator. Also, further indicating its requirement in processes of GDM in a linguistic environment.
"Application of a fuzzy linguistic approach to analyse asian airports competitiveness"	1997	[229]	This paper proposes an analysis of the potential competitiveness of 9 major airports, which is based on the fuzzy linguistic approach and also on the airport experts' point of view.	The proposed approach provides a practical and applicable assessment of airport competitiveness in the East Asia region. Also, it is even more flexible and adaptable to deal with competitiveness associated with designated influencing factors.	The central idea of our proposed approach is that it helps in converting influencing factors to the competitiveness of finite scales of using linguistic variables.

Table 1.1: (Continued)

Title	year	paper	Description	Why Fuzzy Linguistic Approach	Strong/Weak points
"A new method for tool steel materials selection under fuzzy environment"	1997	[43]	This paper provides a new approach in solving the tool steel materials selection problem under the framework of fuzzy, where the significance weights of distinctive criteria and the ranking of different preferences underneath distinctive criteria are assessed in linguistic phrases expressed by way of trapezoidal fuzzy numbers. Further, an example is given to illustrate the newer methodology properly.	The proposed technique is much more proficient than rather a traditional method presented by Wang and Chang in 1995. Henceforth, it helps the designer to choose the most relevant tool steel materials systematically. Also, it makes the execution much faster.	The proposed technique makes use of easy arithmetic operations instead of complex arithmetic operations for aggregation and ranking of fuzzy numbers. As a result, its execution is much quicker.
"Linguistic labels for expressing fuzzy preference relations in fuzzy group decision making"	1998	[199]	This paper proposes improvements to pairwise group decision making based on fuzzy preference relations and then propose the use of linguistic labels for expressing the fuzzy preference relation.	The proposed approach helps represent preferences in linguistic labels rather than numerical, thereby producing the results which are more realistic in presenting imprecise preference relations.	The linguistic labels representation method proposed has helped in improving simplicity and flexibility in the applications of the group decision-making method.
"Multi-criteria Multi-stages linguistic evaluation and ranking of machine tools"	1999	[80]	This paper shows a model for the linguistic evaluation of machine tools parameters and a procedure for alternatives ranking.	The proposed approach is helpful in modeling and ranking linguistically evaluated characteristics of machine tools. It is primarily based on the application of FSs theory and represents an up-gradation to the former approaches by overcoming the inconsistency that appears in the final result.	Suggested methodology for machine tools evaluation and ranking may be used for both machine tools selection and linguistic quantification of an already selected machine tool. By linguistic quantification of machine tools elements rigidity and by setting a significance of these elements for given machining conditions, it is possible to generate a linguistic value of rigidity of machine tool as a whole.
"Linguistic decision analysis: steps for solving decision problems under linguistic information"	2000	[114]	This paper displays a study on the steps to be embraced in linguistic decision analysis under linguistic data.	The linguistic-based approach provides a smoother framework for solving decision-making problems such that it helps in representing information in a more straight forward and adequate manner.	The utility of linguistic models in decision problems is highly beneficial in the decision-making problems where results cannot be expressed in numerical form.
"Extensions of the TOPSIS for group decision-making (GDM) under fuzzy environment"	2000	[41]	This paper aims to present the development of the TOPSIS method for solving the GDM problem under the framework of fuzzy. In this paper, linguistic terms are in a way used to represent the ranking of every attribute and the weight of each criterion, that can be described accordingly with the triangular fuzzy number. Vertex method is also designed to estimate the distance between the two triangular fuzzy ratings. Finally, concerned with the notion of TOPSIS, a closeness coefficient is depicted to select the ranking order of all alternatives.	Crisp values are insufficient to handle practical, real-life situations. Some amount of fuzziness is linked with decision information and GDM problems. Thereby, the utility of a linguistic approach to appraising the weights of all criteria and ranking of every alternative concerning every criterion has provided a more flexible and realistic framework than numerical value.	The proposed approach is useful in solving MCDM problems within a fuzzy framework. The need of linguistic variables in various decision problems is highly beneficial in assessing alternatives concerning criteria and significance weights. Further, with the help of a vertex method distance between the two fuzzy triangular numbers can be estimated. Henceforth, extends the TOPSIS technique to the fuzzy environment.
"A linguistic decision model for personnel management solved with a linguistic bi-objective genetic algorithm"	2001	[117]	This paper exhibits a model for staff selection in the state of uncertainty with the goal that it will both limit the risks emerging from the performance of undertakings by unsuitable personnel and boost the limit of the firm via optimal assignment of workers.	The linguistic formulation is very much useful in a situation where the information available is vague or imprecise. Thereby working in qualitative areas suchlike a personnel management linguistic approach has provided a more flexible and realistic framework than numerical value.	The proposed method has given promising results henceforth; it can be adopted and further implemented to different practical, real-life problems under the same consideration.

The use of linguistic labels in a fuzzy linguistic-based approach is appropriate for modeling practical DM problems where the information available is uncertain. We have to make a suitable choice in selecting linguistic labels for the given term set and their semantics. To achieve the mentioned goal, a significant aspect is to analyze the corresponding granularity of the set. In general, the granularity of the set which is denoted as '$g+1$' describes the degree of distinct linguistic labels in a linguistic term set (LTS), $\mathbf{LT} = \{\ell_i \,|\, i = 0, \ldots, g\}$. The determination of the set's granularity depends on the given linguistic DM problem. Once the set's granularity is fixed, the linguistic descriptors and their associated semantics are established. Varied methodologies existing in the literature to determine these requisites are given in papers [324–326]. Lastly, it is envisioned that the distribution of distinct linguistic labels is defined on a finite ordered scale set exhibiting the following properties [118].

(i) The LTS is ordered: $\ell_i \geq \ell_j$, if $i \geq j$.

(ii) The negation operator is given as $\text{neg}(\ell_i) = \ell_j$ such that $j = g - i$.

(iii) Maximization operator: $\max(\ell_i, \ell_j) = \ell_i$ if $i \geq j$.

(iv) Minimization operator: $\min(\ell_i, \ell_j) = \ell_i$ if $i \leq j$.

For demonstration, we consider a linguistic set **LT** having seven terms specified as, $\mathbf{LT} =$ $\{$ ℓ_0 : "Worst" (W), ℓ_1 : "Very Low" (VL), ℓ_2 : "Low" (L), ℓ_3 : "Medium" (M), ℓ_4 : "High" (H), ℓ_5 : "Very High" (VH), ℓ_6 : "Excellent" (Ex) $\}$.
Here, $\text{neg}(\ell_4) = \ell_2$, $\max(\ell_1, \ell_3) = \ell_3$, $\min(\ell_1, \ell_3) = \ell_1$.

Context-free grammar methodology [142] is another valuable approach in describing the linguistic labels. However, in the current study, we restrict ourselves to the ordered structure approach. The semantics of the stated linguistic descriptors boost the computational process of LVs. Usually, semantics are represented with membership functions (see graphically, Fig 1.2)

1.2.2 Conventional Linguistic Computational Models

It is a well-known fact that some human DM problems are too complex to be handled via traditional quantitative models. Hence, linguistic descriptors camouflage the uncertainty embedded in those problems. For improving the accuracy along with facilitating the methodologies of CWW by handling the problems involving linguistic information, distinct LCMs have been designed within the majored literature suchlike:

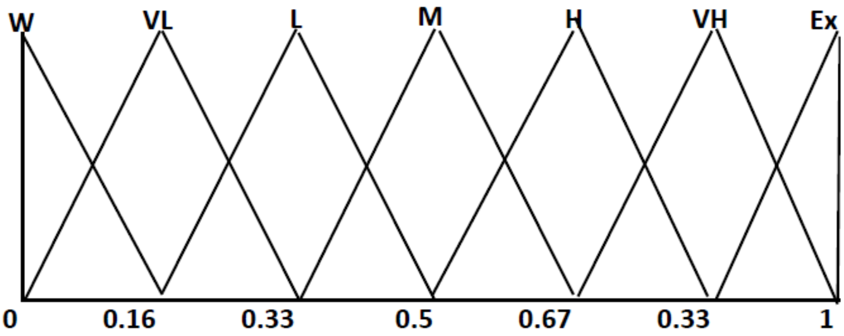

Figure 1.2: A set of seven linguistic terms with their semantics

- **The extension principle-based computational model** [22, 76, 77, 139]. This computational model supports the fuzzy linguistic based approach and performed operations directly on the membership function defined for the linguistic labels by utilizing the fuzzy extension principle [77, 139]. The fuzzy computations applied on $\mathbf{LT^n}$ which symbolizes the 'n' cartesian product of **LT** results in a fuzzy number $F(\mathbb{R})$, that generally do not agree to any linguistic label in the given primary LTS. So, to obtain an information which is interpreted linguistically, it is essential to define a linguistic approximation function as $app_1(.) : F(\mathbb{R}) \rightarrow \mathbf{LT}$ expressing the final result in the original expression domain. However, the approximation process defined has appeared as a primary disadvantage of the model as it causes information loss and inaccuracy in the outcome as presented in Figure 1.3.

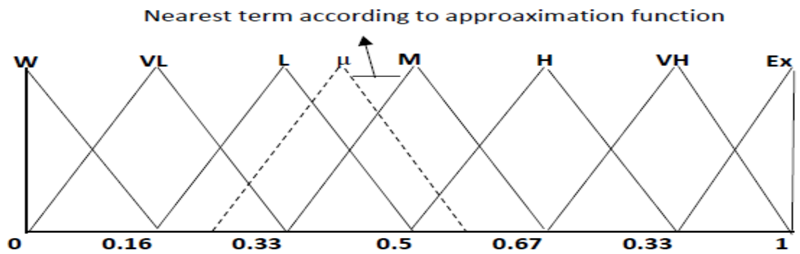

Figure 1.3: A process of linguistic approximation

- **The ordinal scale based computational models** [305]. In this symbolic LCM, the information is represented corresponding to the fuzzy linguistic based approach. It further enhances the use of the collection of linguistic terms' ordered structure, i.e., $\mathbf{LT} = \{\ell_i \,|\, i = 0, \ldots, g\}$ such that $\ell_i < \ell_j \iff i < j$ to facilitate the required symbolic computations. The intermediary results are usually the numerical values, $\alpha \in [0, g]$, which need to be estimated in every step of the mechanism via an approximation function $app_2(.)$. Formally, it can be demonstrated as:

$$LT^n \xrightarrow{C} [0, g] \xrightarrow{app_2(.)} \{0, \dots, g\} \rightarrow LT$$

Here, C represents a linguistic aggregation operator operating over the indices, $\{0, \dots, g\}$ of the LTS, **LT**, $app_2(.)$ represents an approximation function that helps in obtaining an index i, $i \in \{0, \dots, g\}$ which is associated to a linguistic label in **LT** from a value in $[0, g]$. The approximation function mentioned in the process offers inaccurate and distorted information as $app_2(.)$ which is applied on any real value in the interval, $[0, g]$ taking value of the nearest index, $i \in \{0, \dots, g\}$.

.

Different ordinal scale-based computational models having similar computations are presented in [62, 295]. Due to simple adaptation and ease for decision makers [302, 303, 305], symbolic models have been extensively enforced to various problems of DM. However, the lack of precision and information drop caused by the necessity to outright the final results in the primary linguistic domain, i.e., discrete, which is shown as a retranslation step given in Figure 1.4 were the primary concerns of the conventional LCMs. Henceforth, it is a salient requirement to develop more precise resulting LCM for explicitly carrying out CWWs processes.

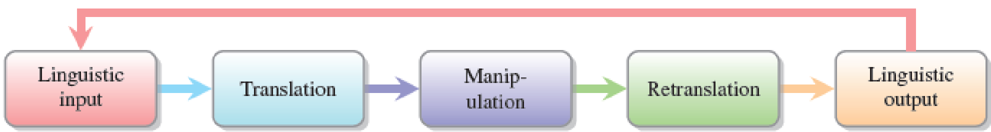

Figure 1.4: Computing with words scheme [187, 303]

1.3 2-tuple Linguistic Model

The traditional symbolic LCMs lack accuracy due to the information loss that occurs primarily due to the prerequisite for enumerating the ultimate results in the primary expression domain. As a result, to overcome the shortcomings of the existing models and facilitate the process of CWWs, a new model with simplified computation and interpretation of results is required. Additionally, the model must be fully capable of representing the linguistic results with accuracy so that it does not belong to the initial LTS, LT. Hence, initiating a linguistic model with continuous representation is the appropriate response in this direction.

The introduction of the 2TL model, which overcame the precision constraint of traditional symbolic models, signaled the start of a new era in the CWW literature. This model's ascendancy lies in the fact that the representation of this model remains continuous in its domain, while it is considered discrete in the other existing conventional models. Therefore, the 2TL model specifies any information in the universe of discourse.

1.3.1 2-tuple Linguistic Model Based on Symbolic Translation

The initiatory 2TL model was proposed by Herrera & Martínez [106] where author presents the linguistic information in terms of a couple of values, known as a 2-tuple. The model is created with the help of a linguistic term and a number that is $(\ell_i,\alpha) \in LT \times [-0.5, 0.5)$, where $\ell_i \in LT$, a linguistic term and $\alpha \in [-0.5, 0.5)$, a crisp value portraying the "symbolic translation".

The central idea behind the foundation of the linguistic model based on 2-tuple is to boost the authenticity and assist the processes of CWW by considering the continuity of the linguistic domain [187]. This model is also referred to as a continuous 2TL model—further, Herrera and Martínez [111], in their paper, has solved a multi-expert MCDM problem that is elucidated in a linguistic context of multi-granularity using those mentioned above linguistic computational techniques. In that paper, the authors have presented a comparison of the linguistic labels, its authenticity, and the consistency of the final results obtained using these computational models. It reveals that the 2TL model is highly efficient in managing the accuracy of linguistic information.

Next, we proceed to primarily discuss the definitions and mathematical formulations of the 2TL model.

Definition 1.3.1. [106] Let $\mathbf{LT} = \{\ell_i \,|\, i = 0, \ldots, g\}$ be defined as the finite and totally ordered LTS having cardinality $g+1$ and let $\beta \in [0, g]$ be the numeric value. Let $i \in \{0, \ldots, g\}$ and $\alpha \in [-0.5, 0.5)$ be two values then α is called a "*symbolic translation*" which means, a precise numeric value representing the translated value from initial result β to the nearest index label i. Here, i is denoted as round(β) while $\alpha = \beta - i$.

This 2TL model is described via transformation function established between numeric values and 2-tuples, as defined above, to carry forward the linguistic computational processes.

Definition 1.3.2. [106] Let $\mathbf{LT} = \{\ell_i \,|\, i = 0 \text{ to } g\}$ be defined as a finite LTS and let $\beta \in [0, g]$ be a numeric value that express the outcome of a symbolic aggregation operation, then the 2-tuple representing the identical information corresponding to β is given below:

$$\Delta : [0, g] \rightarrow \overline{\mathbf{LT}}$$

$$\Delta(\beta) = (\ell_{\text{round}(\beta)}, \beta - i)$$

where $\overline{\mathbf{LT}} \equiv \mathbf{LT} \times [-0.5, 0.5)$, '*round(.)*' is the general rounding operation assigning to crisp value β an integeral value $i \in \{0, 1, \ldots, g\}$ nearest to β, and $\alpha = \beta - i$, is termed as the "symbolic translation".

Remark 1.3.1. [106] Δ mapping as defined above is one-one and onto hence invertible, therefore there exist $\Delta^{-1} : \overline{\mathbf{LT}} \rightarrow [0, g]$ which returns an identical crisp value $\beta \in [0, g] \subset \mathbb{R}$ corresponding to equivalent 2-tuple such that $\Delta^{-1}(\ell_i, \alpha) = \beta = i + \alpha$.

Remark 1.3.2. 2TL term is obtained from a linguistic term ℓ_i by merely including a value 0 as "symbolic translation": $\ell_i \in \mathbf{LT} \Rightarrow (\ell_i, 0) \in \mathbf{LT} \times [-0.5, 0.5)$.

Remark 1.3.3. Linguistic model based on 2-tuple can be utilized from any one of the membership function maintaining the semantics of the given linguistic terms and also upgrade the authenticity of the traditional symbolic ways. Nevertheless, in the paper [106, 111] it was demonstrated that the utilization of the triangular shaped membership functions bring about the most likely outcome with high precision.

Further, Herrera and Martínez [106] presented the comparison of 2TL information by using conventional lexicographic order.

Definition 1.3.3. Let (ℓ_i,α_i) and (ℓ_j,α_j) be two 2TL variables.

- if $i < j \implies (\ell_i,\alpha_i) < (\ell_j,\alpha_j)$.
- if $i = j$ then,
  1. if $\alpha_i = \alpha_j \implies (\ell_i, \alpha_i) = (\ell_j, \alpha_j)$;
  2. if $\alpha_i < \alpha_j \implies (\ell_i, \alpha_i) < (\ell_j, \alpha_j)$;
  3. if $\alpha_i > \alpha_j \implies (\ell_i, \alpha_1) > (\ell_j, \alpha_2)$.

The operation of negation is found to be an elementary notion of LVs. The following definition of negation of 2TL variable is broadened in light of conventional negation operator for LVs.

Definition 1.3.4. [106] Let $\mathbf{LT} = \{\ell_i \,|\, i = 0, \ldots, g\}$ be the finite LTS with cardinality $g+1$, then the 2TL negation operator is given as

$$\text{neg}((\ell_i, \alpha)) = \Delta(g - (\Delta^{-1}(\ell_i, \alpha))).$$

Its worth noting that if $\alpha = 0$, the negation operator is given as $\text{neg}(\ell_i) = \ell_{g-i}$, which is similar to a standard negation of the linguistic terms.

Remark 1.3.4. Based on the property of linearity and monotonicity of the Δ operator, the following observations for the 2TL variables $\tilde{v}_i = (\ell_i, \alpha_i)$ and $\tilde{v}_j = (\ell_j, \alpha_j)$ readily hold:

(i) $\text{neg}(\text{neg}(\tilde{v}_i)) = \tilde{v}_i$;

(ii) $\min\{\text{neg}(\tilde{v}_i), \text{neg}(\tilde{v}_j)\} = \text{neg}(\max\{\tilde{v}_i, \tilde{v}_j\})$;

(iii) $\max\{\text{neg}(\tilde{v}_i), \text{neg}(\tilde{v}_j)\} = \text{neg}(\min\{\tilde{v}_i, \tilde{v}_j\})$; here the operation of max and min for two 2TL variables are considered as the similar operation given in the Definition 1.3.3.

We will now proceed to present the following illustration to demonstrate the above-mentioned concept.

Example 1.3.1. Consider the predefined LTS,

$$\mathbf{LT} = \{\ell_0 : \text{VB}, \ell_1 : \text{B}, \ell_2 : \text{M}, \ell_3 : \text{G}, \ell_4 : \text{VG}\}$$

Note: “Very Bad” (VB), “Bad” (B), “Medium” (M),“ Good ”(G), “Very Good” (VG). Let $\overline{\mathbf{LT}} = \{(\ell_0, 0.4), (\ell_2, 0.243), (\ell_1, 0.259), (\ell_2, -0.4369)\}$ be a set of 2TL variables. Then, if $\beta = 1.5631 \implies i = \text{round}(1.5631) = 2$ and $\alpha = -0.4369$; hence, the corresponding 2TL variable is given as $(\ell_2, -0.4369)$. Contrastingly, by using the Δ^{-1} operator, i.e., $\Delta^{-1}(\ell_2, -0.4369) = 1.5631$, we can transform the given 2TL variable into its original numeric value. Also, based on the classical lexicographic ranking relation, the elements of the set $\overline{\mathbf{LT}}$ can be arranged as $(\ell_0, 0.4) < (\ell_1, 0.259) < (\ell_2, -0.4369) < (\ell_2, 0.243)$. Figure 1.5 presents the 2TL representations.

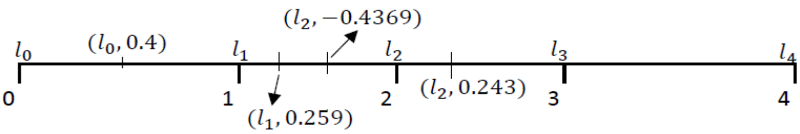

Figure 1.5: Representation of 2TL model proposed by Herrera and Martínez

Now, we will discuss few elementary aggregation operators that are defined for 2TL variables based on the above-mentioned definitions of Δ and Δ^{-1} operators.

(A) *Arithmetic mean (AM) operator*

Definition 1.3.5. [106] Let $X=\{(\ell_1, \alpha_1), (\ell_2, \alpha_2), \ldots, (\ell_k, \alpha_k)\}$ be the 2TL set, the arithmetic mean operator $\overline{X}^e$ is defined as:

$$\overline{X}^e = \Delta\left(\frac{\sum_{i=1}^{k} \Delta^{-1}(\ell_i, \alpha_i)}{k}\right).$$

The $\overline{X}^e$ estimated for the set X is incurred in a precise way without any information loss.

(B) *Weighted average (WA) operator*

In the case of weighted average, considering the nature of the variable x, distinct values say x_i have different importance. As a consequence, each value x_i is associated with a weight w_i, which signifies the variable's nature.

Definition 1.3.6. [106] Let $X=\{(\ell_1, \alpha_1), (\ell_2, \alpha_2), \ldots, (\ell_k, \alpha_k)\}$ be the 2TL and let $W = \{w_i \mid i = 1, 2, \ldots, k\}$ be the set of weights associated with 2TL set, then the WA operator $\overline{X}^w$ for the set X is defined as:

$$\overline{X}^w = \Delta\left(\frac{\sum_{i=1}^{k} \Delta^{-1}(\ell_i, \alpha_i) \cdot w_i}{\sum_{i=1}^{k} w_i}\right)$$

(C) *Ordered weighted average (OWA) operator*

In literature, the weighted aggregation operator was initiated by Yager [303]. In that operator, the weights which are given are not connected with a predetermined value. Instead, it is provided that the weights are connected to a fixed position.

The OWA operator X^{ow} for handling 2TL are defined as:

Definition 1.3.7. [106] Let $X=\{(\ell_1, \alpha_1), (\ell_2, \alpha_2), \ldots, (\ell_k, \alpha_k)\}$ be the 2TL set and let $W = \left\{w_i \,|\, i = 1, 2, \ldots, k,\ \sum_{i=1}^{k} w_i = 1 \text{ and } w_i \in [0,1]\right\}$ be the set of associated weights. Then the 2-tuple OWA operator X^{ow} is computed as:

$$X^{ow} = \Delta\left(\sum_{j=1}^{k} w_j \cdot \beta_j^*\right).$$

where the numerical value β_j^* represents the largest of j^{th} value among all the β_i values.

Furthermore, the generalized version of the 2TL variable and the translation function is proposed by Chen and Tai [40].

Definition 1.3.8. [40] Let $\mathbf{LT} = \{\ell_i \,|\, i = 0, \ldots, g\}$ be the finite LTS with cardinality $g+1$ and let β be the crisp value such that $\beta \in [0,1]$ then, the transformation of β to 2TL variable is procured in the following aspect:

$$\Delta : [0,1] \longrightarrow \overline{\mathbf{LT}}$$

$$\Delta(\beta) = (\ell_i, \alpha) \qquad with \qquad \begin{cases} \ell_i, & i = round(\beta g), \\ \alpha = \beta - i/g, & \alpha \in [-0.5/g, 0.5/g). \end{cases}$$

where $\overline{\mathbf{LT}} \equiv \mathbf{LT} \times [-0.5, 0.5)$, where '*round(.)*' is the general rounding operation, ℓ_i being the nearest index linguistic label to β, and α represents the "symbolic translation".

On the contrary, $\Delta^{-1} : \overline{\mathbf{LT}} \longrightarrow [0,1]$ can also be defined to convert the 2TL variable into an equivalent crisp value β ($\beta \in [0,1]$) in the following aspect:

$$\Delta^{-1}(\ell_i, \alpha) = \beta = i/g + \alpha.$$

Clearly, $\Delta(\frac{i}{g}) = (\ell_i, 0)$ and $\Delta^{-1}(\ell_i, 0) = \frac{i}{g}$.

The negation of 2TL variable (ℓ_i, α) can be given as, $\text{neg}(\ell_i, \alpha) = \Delta(1 - \Delta^{-1}(\ell_i, \alpha))$. Likewise, the ordering relation of the generalized 2TL variables can be defined as given in Definition 1.3.3.

1.4 Extensions of the 2-tuple Linguistic Model

Being the most beneficial model for CWWs processes due to its more straightforward interpretation and competence of expressing any linguistic data in a continuous domain, the 2-tuple model, has many applications. Over the subsequent years, many researchers have proposed variant versions of this model, stating its mathematical formulations, operations, and aggregation operators. This section has brought to light the extensions of the 2TL model and its applications in DM. Below we summarize some of the extensions of the 2TL model individually by highlighting the main results.

1.4.1 Unbalanced linguistic term sets (ULTSs)

It is known that Herrera and Martínez' 2-tuple model is alleged to address only the LTS having the symmetric distribution of linguistic labels. However, some problems are subjected to term sets having uneven distribution of linguistic labels, such type of set is termed as ULTS [112]. There are mainly two categories of ULTS. In the first category, the distribution of the linguistic terms are uneven such that the cardinality of the terms on one side of the intermediate-term is higher than the other, and the corresponding distance between the consecutive terms are not equal (namely, ULTS of type I, see Fig. 1.6). Meanwhile, in the second category, an equal number of the linguistic terms are distributed on the left and right sides of the intermediate-term with unequal spacings (namely, ULTS of the type II, see Fig. 1.7).

Herrera et al. [112] represented a fuzzy linguistic based approach to handle a context of ULTS. In the methodology, Herrera et al. [112] take into consideration a ULTS, namely,

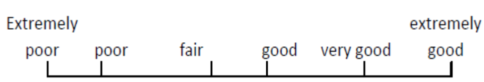

Figure 1.6: Unbalanced linguistic set type I

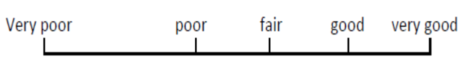

Figure 1.7: Unbalanced linguistic set type II

LT as a set of minimum, maximum, and central label, and the distribution of the leftover labels on either side of the central term, i.e., $\mathbf{LT} = \mathbf{LT}_L \cup \mathbf{LT}_C \cup \mathbf{LT}_R$ such that:

- $\mathbf{LT}_L$ represents *left lateral set* containing all the terms except the central one.
- $\mathbf{LT}_C$ represents *central set* containing just one central term.
- $\mathbf{LT}_R$ represents *right lateral set* containing all the linguistic labels that are higher than the central linguistic label.

Herrera et al. [112] accustomed the conception of linguistic hierarchy to acquire linguistic representations of unbalanced linguistic values in the format of 2-tuple. A linguistic hierarchy (LH) [108] is determined as a collection of levels, where each and every level is represented by a LTS, $\mathbf{LT}^{n(t)} = \left\{\ell_0^{n(t)}, \ell_1^{n(t)}, \ldots, \ell_{n(t)-1}^{n(t)}\right\}$. Each level associated with a linguistic hierarchy is constructed as $l(t, n(t))$, where t is a numeric value indicating the hierarchy level and $n(t)$ indicates the cardinality of the LTS corresponding to the level t. Here, we underline that the linguistic terms utilized in the linguistic hierarchical process are given by triangular-shaped membership functions distributed uniformly in $[0, 1]$.

Usually, we witness that the LTS, $\mathbf{LT}^{n(t+1)}$ of level $t + 1$ can be easily acquired from the previous term set $\mathbf{LT}^{n(t)}$ as

$$l(t, n(t)) \rightarrow l(t + 1, 2 \cdot n(t) - 1).$$

In LH [108], transformation functions are defined between linguistic labels from distinct levels to perform CWWs techniques in a multi granular linguistic context avoiding any information loss.

Definition 1.4.1. [108] For any level t and t', the transformation function is determined as $\mathrm{TF}_{t'}^{t} : l(t, n(t)) \rightarrow l(t', n(t'))$ such that

$$\mathrm{TF}_{t'}^{t}(\ell_i^{n(t)}, \alpha^{n(t)}) = \Delta\left(\frac{\Delta_t^{-1}(\ell_i^{n(t)}, \alpha^{n(t)}) \cdot (n(t') - 1)}{n(t) - 1}\right). \tag{1.4.1}$$

For the most part, any level could be chosen in the LCM characterized for the LH to bring together the multi-granular linguistic data in the model characterized for the LH. However, for the sake of convenience and generalization, we refer 't_m' as the maximum level in the LH, i.e., $l(t_m, n(t_m)) = \mathbf{LT}^{n(t_m)} = \left\{\ell_i^{n(t_m)} \,|\, i = 0, \ldots, n(t_m) - 1\right\}$.

Next, we present below the details of the transformation process to transform any 2-tuple linguistic variable (ℓ_i, α) into the term in $LH = \cup_t l(t, n(t))$:

(i) Representation in the LH: The representation algorithm proposed in the paper utilizes the concept of a LH to address the unbalanced terms. As a consequence, the initial step to bring about CWWs processes to convert the unbalanced terms lying in a set **LT** into their corresponding terms in the LH, $\ell_k^{n(t)} \in LH = \cup_t l(t, n(t))$ by virtue of the transformation function ψ given as:

$$\psi : \overline{\mathbf{LT}} \to LH(\overline{\mathbf{LT}}) \tag{1.4.2}$$

such that $\psi(\ell_i, \alpha) = (\ell_{I(i)}^{G(i)}, \alpha)$, for all $(\ell_i, \alpha) \in \overline{\mathbf{LT}}$.
Here, I and G represents a function assigning every unbalanced label $\ell_i \in \mathbf{LT}$ index of the label symbolizing it in LH and the cardinality of label set of LH where it is presented, respectively.

(ii) Computational phase: The computational model characterized for LH helps in accomplishing the process of CWWs. At first, use the Eq. 1.4.1 to transform $(\ell_{I(i)}^{G(i)}, \alpha)$ (for $i = 0, \ldots, g$) into linguistic 2-tuples, referred as $(\ell_{I'(i)}^{n(t_m)}, \lambda') \in \overline{\mathbf{LT}^{n(t_m)}}$. Without loss of generality, if $G(i) = n(t')$, then Eq. 1.4.3 is obtained. Further, use Herrera and Martínez model over $\overline{\mathbf{LT}^{n(t_m)}}$ to acquire desired result $(\ell_r^{n(t_m)}, \lambda_r') \in \overline{\mathbf{LT}^{n(t_m)}}$.

$$(\ell_j^{n(t_m)}, \lambda') = TF_{t_m}^{t'}(\ell_{I(i)}^{G(i)}, \alpha) \tag{1.4.3}$$

(iii) Re-translation process: A process of re-translation is performed to change the obtained result $(\ell_r^{n(t_m)}, \lambda_r') \in \overline{\mathbf{LT}^{n(t_m)}}$ back into the desired unbalanced term in $\overline{\mathbf{LT}}$, by utilizing the following function, ψ^{-l}, i.e.,

$$\psi^{-1} : LH(\overline{\mathbf{LT}}) \to \overline{\mathbf{LT}}. \tag{1.4.4}$$

such that $\psi^{-1}(\ell_r^{n(t_m)}, \lambda_r') = (\ell_{result}, \lambda_{result}) \in \overline{\mathbf{LT}}$.

The detailed explanation of the methodology to deal with ULTSs is given in the corresponding paper [107].

In sequence with the computational models to address unbalanced linguistic information, Wang and Hao [271] presented an updated formulation of the current 2TL model that is known as a proportional 2TL model. This proportional 2-tuples are subjected to a notion of symbolic proportion and canonical characteristics values (CCV) of linguistic labels. The formal definition is stated below:

Definition 1.4.2. [271] Let $\mathbf{LT} = \{\ell_0, \ell_1, \dots, \ell_g\}$ be a finite ordered set with $g+1$ ordinal terms and $I = [0,1]$ and

$$IL \equiv I \times \mathbf{LT} = \{(\alpha, \ell_i) : \alpha \in [0,1], i = 0, 1, \dots, g\}$$

For a given pair (ℓ_i, ℓ_{i+1}) of two consecutive ordinal terms of LTS, **LT**, any two arbitrary elements (α, ℓ_i), (β, ℓ_{i+1}) of set IL is known a symbolic proportion pair (SPP) and α, β are referred as a pair of symbolic proportions that are corresponding to the pair (ℓ_i, ℓ_{i+1}) if $\alpha + \beta = 1$. A SPP (α, ℓ_i), $(1-\alpha, \ell_{i+1})$ will be denoted by $(\alpha\ell_i, (1-\alpha)\ell_{i+1})$ and the set of all the SPPs is denoted by $\bar{L}$ that is

$$\bar{L} = \{(\alpha\ell_i, (1-\alpha)\ell_{i+1}) : \alpha \in [0,1], i = 0, 1, \dots, g-1\},$$

the set $\bar{L}$ will be termed as ordinal proportional 2TL set which is generated by **LT** and the members of set $\bar{L}$ will be called as ordinal proportional 2-tuples utilization of which represents the ordinal information for CWW.

Remark 1.4.1. We can either use $(0\ell_{i-1}, 1\ell_i)$ or $(1\ell_i, 0\ell_{i+1})$ as representatives of ordinal terms ℓ_i in $\bar{L}$ for $i = 2, \dots, g-1$.

Definition 1.4.3. [271] Let $\mathbf{LT} = \{\ell_0, \ell_1, \dots, \ell_g\}$ be an ordinal LTS and $\bar{L}$ be the set of ordinal proportional 2TL generated by **LT**, then comparison of ordinal data presented by proportional 2-tuples is lugged out as follows: let $(\alpha\ell_i, (1-\alpha)\ell_{i+1}), (\beta\ell_j, (1-\beta)\ell_{j+1}) \in \bar{L}$ be any two SPPs such that

$$\begin{aligned}
&(\alpha\ell_i, (1-\alpha)\ell_{i+1}) < (\beta\ell_j, (1-\beta)\ell_{j+1}) \\
&\Leftrightarrow \alpha i + (1-\alpha)(i+1) < \beta j + (1-\beta)(j+1) \\
&\Leftrightarrow i + (1-\alpha) < j + (1-\beta)
\end{aligned}$$

Thus for any two proportional 2-tuples $(\alpha\ell_i, (1-\alpha)\ell_{i+1}), (\beta\ell_j, (1-\beta)\ell_{j+1})$ we get the following relation :

1. if $i < j$ then,
   (a) if $i = j-1$ and $\alpha = 0, \beta = 1$ then $(\alpha\ell_i, (1-\alpha)\ell_{i+1}), (\beta\ell_j, (1-\beta)\ell_{j+1})$ imitates identical information.
   (b) otherwise $(\alpha\ell_i, (1-\alpha)\ell_{i+1}) < (\beta\ell_j, (1-\beta)\ell_{j+1})$

2. if $i = j$ then,

   (a) for $\alpha = \beta$ we have, $(\alpha\ell_i, (1-\alpha)\ell_{i+1})$,$(\beta\ell_j, (1-\beta)\ell_{j+1})$ imitates identical information.

   (b) for $\alpha < \beta$ we have, $(\alpha\ell_i, (1-\alpha)\ell_{i+1}) > (\beta\ell_j, (1-\beta)\ell_{j+1})$.

   (c) for $\alpha > \beta$ we have, $(\alpha\ell_i, (1-\alpha)\ell_{i+1}) < (\beta\ell_j, (1-\beta)\ell_{j+1})$.

Definition 1.4.4. [271] Proportional 2-tuple negation operator is given as:

$$Neg((\alpha\ell_i, (1-\alpha)\ell_{i+1})) = ((1-\alpha)\ell_{g-i-1}, \alpha\ell_{g-i})$$

where $g+1$ is the cardinality of **LT** , $\mathbf{LT} = \{\ell_k | k = 0, 1, \ldots, g\}$.

Dong et al. [64] extended the current version of the Herrera and Martínez model to manage ULTSs by laying down the idea of a numerical scale. The mathematical formulations of the Numerical scale model is given as follows:

Definition 1.4.5. [64] Let $\mathbf{LT} = \{\ell_i \,|\, i = 0 \text{ to } g\}$ be a finite LTS, and $\mathbb{R}$ has the usual meaning. The function: $NS : \mathbf{LT} \to \mathbb{R}$ is circumscribed as a numerical scale of **LT**, and $NS(\ell_i)$ is termed as numerical index of ℓ_i. If the NS function is defined as strictly monotonically increasing, then it is termed as an ordered numerical scale.

Definition 1.4.6. [64] Numerical scale function $\overline{NS} : \overline{\mathbf{LT}} \to \mathbb{R}$ is defined in the following aspect:

$$\overline{NS}((\ell_i, \alpha)) = \begin{cases} NS(\ell_i) + \alpha \times ((NS(\ell_{i+1})) - NS(\ell_i)), \ \alpha \geq 0; \\ NS(\ell_i) + \alpha \times ((NS(\ell_i)) - NS(\ell_{i-1})), \ \alpha < 0. \end{cases}$$

For simplicity of the notation, NS is used in lieu of $\overline{NS}$.

Proposition 1.4.1. [64] By defining $NS(\ell_i) = i$ for $i \in \{0, \ldots, g\}$ yields the Herrera and Martínez model.

Usually, linear triangular or trapezoidal membership functions are being used to express semantics of the terms belonging to a LTS. However, in the proportional model introduced by Wang and Hao [271] the semantics of linguistic labels are expressed by using trapezoidal fuzzy numbers symmetrical in nature. If the semantics of ℓ_i is defined by $T[b_i - \sigma_i, b_i, c_i, c_i + \sigma_i]$, in the Wang and Hao model [271], the *canonical characteristic values* (CCVs) of ℓ_i is $\frac{b_i + c_i}{2}$, i.e., $CCV(s_i) = \frac{b_i + c_i}{2}$.

Proposition 1.4.2. [64] By defining $NS(\ell_i) = CCV(\ell_i)$ for $i \in \{0, \ldots, g\}$ yields the Wang and Hao model.

Propositions 1.4.1 and 1.4.2 provide a strong relation of the NS to the model of Herrera and Martínez [110] and Wang and Hao [271].

Furthermore, Guo and Huynh [97] suggested a proportional 2TL screening evaluation model and the so-called preference-preserving transformation based on CCV. In that paper, the authors presented a newer screening assessment model that not only refines the standard requirement of equal distance between labels. It also can reflect evaluators' level of confidence in decisions, enriching the knowledge of results, and thus providing decision-makers with more detailed guidance.

Interval judgments thoroughly endorse several decision-making models. As a consequence, Dong et al. [65] developed a new LCM primarily based upon the 2-tuple and intervals, which was termed as an interval version of the 2TL model. Authors have defined the notion of interval NS that converts linguistic terms into interval numbers and has further introduced a generalized inverse operation of interval NS. By continuing the study of the interval 2TL model, Dong and E. Herrera-Viedma [68] introduces a novel consistency-driven methodology in the problem of DM along with the linguistic preference relations. The proposed methodology stipulates a peculiar way out to set the interval NS without requiring the semantics given by interval type-2 fuzzy sets (IT2FSs). Furthermore, Dong [69] proposed a methodology to connect two distinct models designed to address ULTS, i.e., LCM based on hierarchical linguistic concept [112] and the NS model [64]. Moreover, authors have designed a novel CWW methodology relied on the NS model and ULTS to handle unbalanced linguistic information hesitant in nature. Henceforth, the NS model establishes an integrated framework to link the classical linguistic model of 2-tuples, the proportional 2-tuples, and the hierarchical linguistic model.

M. Cai et al. [50] developed a new LCM based on symbolic models. Z. Pei and Li Zheng [233] constructed a series of the normal distribution, which is based on unbalanced linguistic scale sets and has furthermore developed a novel approach for defining unbalanced linguistic information in a format of 3-tuple. Lastly, M. Cai and Z. Gong [51] redefined the concept of ULTS and provided the graph for the representation model.

1.4.2 Linguistic distribution (LD)

According to Wang and Hao model [271], the linguistic information is provided in terms of proportional 2-tuples where symbolic proportions are given to two consecutive ordinal linguistic terms. However, it is envisaged that the proportional 2TL model seems improper to handle decision situations where performances of the alternatives are evaluated via uncertain linguistic judgments as examined in the papers [120, 294, 310, 311]. As a consequence, to overcome the limitations offered by the mentioned model, Zhang et al. [336] has introduced a notion of the distribution assessment in an LTS where symbolic proportions are assigned to all the linguistic terms. In that paper, authors have developed a consensus model to facilitate decision-makers in improving the level of consensus. Followed by discussing the desirable properties of the model as well as measures of consistency and consensus in favor of linguistic preference relations (LPRs) centered upon distribution assessments. Lastly, some operational laws and aggregation operators on account of linguistic distribution assessments are also introduced.

Meanwhile, considering both Wang and Hao's model [271] and Zhang et al.'s model [336] it is envisioned that the symbolic proportions involved in them are summing up to 1. Henceforth, both these models are inadequate to handle incomplete information. So, to overcome this limitation, Guo et al. [98] proposed a new proportional LD model for modeling incomplete linguistic assessments addressed in MADM problems. However, given varying subjective conditions, it found challenging for decision-makers in providing exact symbolic proportions in LD assessments. As a consequence, Dong et al. [72] developed a notion of LD assessments in terms of interval symbolic proportions under a multi-granular unbalanced linguistic context. Furthermore, Zhang et al. [339] introduced a new LCM for solving large-scale group decision making (LGDM) problem to model linguistic information in multi granular context by allowing the maximum information at the first stage and further removal of the initial processes of aggregation as well as managing the required information given by experts along with the utilization of LD assessments to acquire accurate results. Wu et al. [279] presented a novel linguistic GDM model referred to as the maximum support degree model (MSDM) primarily based upon the utilization of LD [336] and hesitant fuzzy linguistic term sets (HFLTSs) [240]. The proposed model aims at providing maximum degree support of the group opinion and further guarantees the accuracy.

Although the HFLTSs and LDs are appropriate enough to represent linguistic preferences in a more realistic form in an uncertain decision environment, however, they are not flexible enough due to some limitations. In the case of the HFLTSs, several successive linguistic labels are utilized to extract the preferences of the decision-makers'. While LDs present certain probabilistic preference information over linguistic terms. However, the elicitation of preferences is not closer to humans' cognitive processes. As a consequence, Wu et al. [285] recently developed a more comprehensive format of linguistic expressions ideally termed as flexible linguistic expression (FLE) to substantially improve the flexibility in constructing complex linguistic expressions as well as eliciting linguistic preferences. Further, in the paper, authors developed a novel linguistic GDM model along with FLEs, termed as FLE-based GDM (FLEGDM). For the detailed explanation of the proposed methodology, one can refer to the corresponding paper [285]. In the sequent, Wu et al. [286] proposed a notion of flexible linguistic preference relations (FLPRs) where linguistic expressions are expressed in a more flexible way to generalize all kinds of linguistic preference relations (LPRs). Further, in that paper, the authors presented a novel methodology to rank alternatives centered upon preference information in FLPRs by envisaging the LD as well as priority approximation (PA) of FLPRs. Lastly, a comparative study for the obtained results of priority vectors is put forward to validate the proposal.

1.4.3 Multi-granularity linguistic term sets (MGLTSs)

The problems of GDM addressing linguistic information incorporates the intricacy and further encompasses the involvement of several experts in the DM process. In the specialized literature, the majority of the proposals designed for solving GDM problems are subjected to situations where experts express linguistic information within the same LTS [106, 150]. However, in many practical, real-life problems, it is envisioned that most of the experts have their distinct cultures, qualifications, knowledge background, and skills. Also, their decisions are taken according to different circumstances. As a consequence, the representation of the information via unique LTS seems inadequate. Thus, in such cases, experts often use LTSs with different cardinalities, commonly referred to as MGLTSs, in order to represent their individual assessment information without any information loss.

The research on the problems of DM assessed with information in a multi-granular context holds considerable importance in practical applications, and several methods have been developed in the literature to solve problems involving MGLTSs. Morente-Molinera et al. [201] presented a systematic review of the recent enhancements in multi-granular linguistic methodologies from 2000 to 2014. In that review, the authors have presented an in-depth analysis of the respective methodologies designed for solving GDM problems as well as more relevant applications. Further, Morente-Molinera et al. [202] proposed a novel methodology for solving MCGDM problems involving a high amount of alternatives. In that paper, fuzzy ontologies reasoning procedures are utilized in order to obtain the alternatives ranking classification, and multi-granular linguistic methodology automatically is being used to improve human-computer communication. The proposed method is highly advantageous and is useful for cases involving a high amount of alternatives. For a deep understanding of the method, one can refer to paper [202].

Tian et al. [262] presented a novel approach to managing hesitant fuzzy linguistic MCGDM problems with multi-granular unbalanced LTS. In that paper, authors have initially developed a signed distance measure, which is considered as the support tool for HFLTS. This measure is primarily centered on the ordinal semantics of linguistic descriptors and the possibility distribution method. For the unification of the multi-granular hesitant unbalanced linguistic information, a signed-distance based transformation function is proposed. Furthermore, a comprehensive consensus measure is presented so that it can robustly measure the consensus degree among decision-makers and also give guidance to members in order to modify their judgments before the selection process. Lastly, a numerical example validates the applicability of the proposed methodology in the practical scenario. The results and an in-depth comparative analysis of the proposed approach with the existing methods [290] demonstrate the possibility and effectiveness of the proposed methodology in tackling MCGDM problems with HFLTSs based on balanced LTSs to some degree. The proposed approach is an upgraded and generalized version of Wu and Xu's [290] method.

1.4.4 Interval-valued 2-tuple Linguistic Model

Several hard decision problems in the literature demand experts to convey their opinions based on a present criterion using specific linguistic or 2TL information from a predefined LTS. In practise, however, it is unlikely that every decision maker will be able to convey his or her choice information clearly and precisely under such constraints. From this point of view, the experts may conclude that the cardinality of the LTS is either too small to properly convey their professional judgements on some characteristics or too large to completely express their professional opinions on other attributes. In such circumstances, Zhang [332] proposed the interval-valued 2TL model, which can be thought of as a standardised interval 2TL model and is well suited to multi-attribute GDM problems in multi-granular linguistic settings. Decision information can be fully articulated and unified under a multi-granular linguistic framework utilising an interval-valued 2TL model.

Definition 1.4.7. [332] Let $\mathbf{LT} = \{\ell_i : i = 0, 1, \ldots, g\}$ be a LTS. An interval-valued 2TL variable comprises of two 2TL terms, denoted by $[(\ell_i, \alpha_i), (\ell_j, \alpha_j)]$ where $i \leq j$. Here ℓ_i and ℓ_j represent the linguistic labels from $\mathbf{LT}$ and α_i and α_j are the symbolic translations.

Analogous to the Δ and Δ^{-1} functions defined in Definition 1.3.8, Zhang [332] also proposed a bijection function to obtain the corresponding 2TL interval from a given numerical interval $\in [0,1] \times [0,1]$ and vice-versa.

Let $[\beta_i, \beta_j] \in [0,1]$ be a numeric interval. The corresponding linguistic interval can be obtained as follows.

$$\Delta([\beta_i, \beta_j]) = [(\ell_i, \alpha_i), (\ell_j, \alpha_j)] \text{ where } \begin{cases} \ell_i & i = \text{round}\,(\beta_i\, g) \\ \alpha_i = \beta_i - \frac{i}{g} & \alpha_i \in \left[\frac{-1}{2g}, \frac{1}{2g}\right) \\ \ell_j & j = \text{round}\,(\beta_j\, g) \\ \alpha_j = \beta_j - \frac{j}{g} & \alpha_j \in \left[\frac{-1}{2g}, \frac{1}{2g}\right). \end{cases}$$

Also, an interval-valued 2TL variable can be transformed into an interval $[\beta_i, \beta_j]$, $\beta_i, \beta_j \in [0,1], \beta_i \leq \beta_j$, as follows.

$$\Delta^{-1}\big([(\ell_i, \alpha_i), (\ell_j, \alpha_j)]\big) = \left[\frac{i}{g} + \alpha_i, \frac{j}{g} + \alpha_j\right] = [\beta_i, \beta_j].$$

Remark 1.4.2. If $\ell_i = \ell_j$, then it implies that the interval 2TL representation $[(\ell_i, \alpha_i), (\ell_j, \alpha_j)] \equiv (\ell_i, [\alpha_i, \alpha_j])$ where $\ell_i \in \mathbf{LT}$ and $[\alpha_i, \alpha_j]$, $(\alpha_i \leq \alpha_j, \alpha_i, \alpha_j \in [\frac{-1}{2g}, \frac{1}{2g}])$ is an interval form of symbolic translation.

Remark 1.4.3. If $\ell_i = \ell_j$ and $\alpha_i = \alpha_j$, then the interval 2TL representation reduces to 2TL variable.

Zhang [332] defined score and accuracy functions to rank interval-valued 2TL variables.

Definition 1.4.8. For an interval-valued 2TL variable $A = [(\ell_i, \alpha_i), (\ell_j, \alpha_j)]$, the 2TL score function is defined as

$$S(A) = (\ell_k, \alpha_k), \text{ where } k = round\Big(\frac{i+j+(\alpha_i+\alpha_j)g}{2}\Big),$$
$$\text{and } \alpha_k = \Big(\frac{i+j+(\alpha_i+\alpha_j)g}{2g}\Big) - \frac{k}{g}.$$

and the accuracy function can be given as:

$$H(A) = \Big(\frac{j}{g}+\alpha_j\Big) - \Big(\frac{i}{g}+\alpha_i\Big).$$

The ranking scheme to compare any two interval-valued 2TL variables is summarized in the subsequent fashion.

Definition 1.4.9. Let A= $[(\ell_i,\alpha_i),(\ell_j,\alpha_j)]$ and B= $[(\ell_{i'},\alpha_{i'}),(\ell_{j'},\alpha_{j'})]$ be two interval-valued 2TL variables.

1. If $S(A) > S(B)$ then A> B.
2. If $S(A) < S(B)$ then A< B.
3. If $S(A) = S(B)$ then,
   (a) if $H(A) > H(B)$ then $A < B$;
   (b) if $H(A) < H(B)$ then $A > B$;
   (c) if $H(A) = H(B)$ then $A = B$, that is, both A and B represent same information.

While comparing $S(A)$, we require to use a ranking scheme of 2TL variables defined previously.

1.4.5 Heterogeneous Information

Nowadays, the DM process has become one of the essential human activity. In many complex real-life decision problems, the attributes can be either quantitative or qualitative. The quantitative attribute values can be given using distinct numerical types suchlike real numbers, fuzzy numbers (triangular or trapezoidal), interval numbers, and so forth. Complex GDM problems often contain heterogeneous information, i.e., the information in a mixed form. However, heterogeneous GDM problems are specified in 3 frameworks. The very first framework is related to different preference formats where opinions of the decision-makers are expressed by different preference relations such as preference orderings, utility functions, multiplicative preference relations, and fuzzy preference relations.

The second framework appears when each expert has different levels of knowledge and background related to the problem, or when the experts have different LTSs to assess preferences, i.e., the multi granular and unbalanced linguistic contexts. The third framework focuses majorly on the heterogeneous expressions of the experts, which are used to express or provide their particular preferences for the attributes of each alternative. It provides information about attributes, which consists of not only crisp or uncertain information, but also interval numbers, fuzzy numbers, and linguistic data [158]. Several authors have proposed different methodologies to solve GDM problems assessing information in a heterogeneous form [158, 171, 190, 338]. For a more detailed study of the methodology, one can refer to [158, 171, 190, 338].

1.5 Contribution of 2-tuple Linguistic Model

Human beings are always engaged in making decisions beneath the linguistic environment to tackle the ill-structured problems successfully. Several methods have developed in the literature for managing linguistic information like method grounded on extension principles and symbols. Both these techniques have certain limitations, which is resolved by the introduction of a 2TL model. The 2TL model has improved the accuracy and facilitated the processes of CWW by managing the linguistic information in the continuous linguistic domain. In the present section, we emphasize the decision models centered upon the 2TL model as well as its extension.

On the broader prospect, MADM and MCGDM problems are two majorly considered decision-making problems. MADM process ideally composes of two-steps: (1) collection of decision information about attribute weights as well as attribute values. In the MADM problem, there exists a finite set of alternatives. The attributes are pointers that measure the given alternatives, and thereby making each attribute as necessary, determine within the procedure of DM. (2) Aggregating the decision information. It is one of the most prominent concepts in the DM problem. As different experts come up with different knowledge areas and preferences. As a consequence, it is the process of merging an entire collection of information into a particular representative value. Hence, aggregation operators have become one of the most prominent tools while merging of information in a DM problem. Henceforth, it has become the most populous studied field among researchers and has drawn a great deal of consideration from practitioners among various disciplines [44, 90, 121, 245, 301]. (Refer Table 1.2 for more clarity of MADM model)

Table 1.2: Summary of MCDM model based on 2-tuple linguistic model

Title	year	paper	Description	Approach/Strategy	Strong/Weak points
"A model based on linguistic 2-tuples for dealing with multi granular hierarchical linguistic contexts in multi-expert decision making (MEDM)"	2001	[108]	This paper provides a technique for dealing with the problem of MEDM, defined within a multi-granular linguistic context so that we can unify the knowledge evaluated in it without losing any information.	Multi granular linguistic framework	The proposed model is based on the 2TL model. The development of different functions leads to the transformation of LTSs of a linguistic hierarchy without losing any information. The proposed functions are further applied to decision models based on multi granular linguistic information.
"Multi granular hierarchical linguistic model for design evaluation based on safety and cost analysis"	2005	[188]	This paper proposes a method dependent on multi-expert MCDM that handle multi granular linguistic data without any information loss, to evaluate different design options for large engineering system is related to safety and cost criteria.	Multi-granular hierarchical linguistic model	This article presents an evaluation approach for the design assessment of complicated engineering systems based on safety and cost analysis. The usage of the linguistic model has proved beneficial than traditional probability models and tools as it could able to handle vagueness and uncertainty well. Thereby linguistic approach provides a more realistic and favorable means in supporting solutions related to such complex decision problems.
"A fuzzy model for design evaluation based on multiple criteria analysis in engineering systems."	2006	[189]	The central idea of the paper is to develop a fuzzy evaluation model depending on multi-criteria decision analysis. The proposed model deals with the information, i.e., both numerical and linguistic, in order to evaluate different design options available for an engineering system following the safety, cost, and technical performance criteria assessed within the distinct utility spaces.	Fuzzy rule based evidential reasoning	In the proposed model, to deal with the evaluation framework where information assessed in distinct expression domains and scales, that is by using the potential of heterogeneous information. Distinct fuzzy transformation functions are presented so that it permits to conduct this heterogeneous information into common utility space by way of 2-tuple linguistic information. Further, an evaluation procedure primarily based on a multi-expert MCDM model to estimate the suitability of distinctive pattern options have been developed.
"A method for multi attribute decision-making (MADM) with incomplete weight information in linguistic setting"	2007	[297]	The central idea of the study is to propose a method to solve MADM problems well equipped with linguistic information, wherein the attribute weight information is inadequately known. In order to determine the attribute weights, an optimization model primarily based on the ideal point of attribute values is also established in the paper.	ideal point of attribute values	The proposed method is well sufficient for solving MADM problems with incomplete information. Moreover, simple optimization models established have provided a simple and exact formula for obtaining the attribute weights. Henceforth, making the proposed method more favorable for practical application.
"Linguistic multi person decision making based on the use of multiple preference relations"	2009	[66]	This paper proposes a linguistic multi-person decision-making model that depends on linguistic preference relations, integrating fuzzy preference relations, distinctive types of multiplicative preference relations, and multi-granular linguistic preference relations. Furthermore, conditions underneath which the proposed decision model satisfy social choice axiom have also been discussed.	Fuzzy majority and 2-tuple ordered weighted averaging (TOWA) operator, linguistic preference relation.	Preference relations are extensively used in decision models. This paper has given considerable contributions to present ongoing research based on linguistic preference relations and TOWA operators. In the proposed model, a set of transformation functions has been developed to relate fuzzy preference relations and distinctive types of multiplicative preference relations with multi-granular linguistic preference relations. Further, the linguistic version of the selection procedure based on the OWA like operator and the fuzzy majority have been discussed. Finally, the internal consistency of the proposed transformation function has been analyzed in detail.

Table 1.2: Continued

Title	year	paper	Description	Approach/Strategy	Strong/Weak points
“Fuzzy multi-criteria decision-making (MCDM) approach for personnel selection"	2010	[75]	This paper proposes a fuzzy MCDM algorithm depending on the principles of fusion of fuzzy information and 2-tuple linguistic model. Further, “technique for order preference by similarity to ideal solution (TOPSIS)" is also being established.	Fuzzy multi criteria decision-making approach	The proposed methodology is pertinent in managing information appraised using both linguistic as well as numerical scales, making problems with multiple information. Also, the proposed method could able to tackle heterogeneous information and thereby permits the use of distinct semantic types of the decision-maker.
“A multi granular linguistic model for management decision making in performance appraisal"	2010	[2]	The paper aims to propose a performance appraisal model, wherein assessments are modeled by utilizing linguistic information described by sets of distinctive reviewers to manage the uncertainty and subjectivity of such assessments.	Multi granular linguistic scheme	The proposed model provides a pretty flexible framework and thereby permits the management group to customizes how to aggregate the individual speculations and how to classify employees. Subsequently, this model suggests an increment of flexibility and an improvement in the cure of information with uncertainty and vagueness in the performance appraisal model.
“An innovative multi-criteria supplier selection based on 2-tuple MULTIMOORA and hybrid data"	2011	[19]	This paper is an extension of multi-criteria decision-making (MCDM) method MULTIMOORA (multi-objective analysis by ratio analysis plus the full multiplicative form) to handle fuzzy supplier selection problem. This extension of MULTIMOORA primarily based on the 2TL method. Thus varied crisp and fuzzy numbers are represented, converted, and mapped into a basic linguistic term set with the help of a 2TL model.	Multi-objective analysis by ratio analysis plus the full multiplicative form (MULTIMOORA)	MULTIMOORA is a pretty effective tool for assessing the sustainability of varied phenomena resulting in an unbiased rating of alternatives. New MCDM method MULTIMOORA-2T proposed in the paper aimed at a combination of hybrid data, specifically real numbers, interval numbers, and linguistic variables. Thus, the proposed strategy able to handle both objective and subjective criteria.
“Uncertain linguistic Bonferroni mean operators and their application to multiple attribute decision making"	2013	[287]	This paper aims to develop two aggregation techniques called the “uncertain linguistic Bonferroni mean (ULBM)" operator and the “uncertain linguistic geometric Bonferroni mean" (ULGBM) operator for aggregating the uncertain linguistic information. The “uncertain linguistic weighted Bonferroni mean" (ULWBM) operator and the “uncertain linguistic weighted geometric Bonferroni mean" (ULWGBM) operator is defined based on which two procedures for MADM under the uncertain linguistic environments are developed.	Bonferroni mean and geometric bonferroni mean operator	The prominent advantage of the proposed operators and approaches over the traditional uncertain linguistic operators and approaches is because these operators accommodate the uncertain linguistic environment. Also, due to the consideration of the interrelationship among the input arguments, therefore making it more feasible and practical.

Table 1.2: Continued

Title	year	paper	Description	Approach/Strategy	Strong/Weak points
“A method based on induced aggregation operators and distance measures to multiple attribute decision making under 2-tuple linguistic environment"	2014	[153]	The objective of this article is to develop the 2-tuple linguistic induced generalized ordered weighted averaging distance (2LIGOWAD) operator, which is considered as newer multi-attribute decision-making (MADM) approach for dealing with 2-tuple linguistic information. The paper consists of some of its essential properties and further by using quasi arithmetic mean this operator is generalized. Finally, the paper concludes with an application of a proposed operator to a group decision-making problem about the selection of strategies.	Induced aggregation operators and distance measures	The 2LIGOWAD operator is capable of dealing with complicated reordering processes that represent a broad spectrum of factors in an uncertain linguistic environment. Consequently, deal in situations where the available information has a high degree of uncertainty. Further, the proposed operator is capable of dealing with complicated attitudinal characters in the decision process by utilizing order inducing variables. The 2LIGOWAD operator is generalized by using the Quasi arithmetic means to obtain the Quasi arithmetic 2LIOWAD (Q2LIOWAD). This approach includes the 2LIGOWAD operator as a special case and a lot of other cases. Thereby, we obtain a more robust formulation of this model.
“New linguistic aggregation operator and its application to multiple attribute decision-making"	2015	[148]	This paper proposes a new linguistic aggregation operator in order to deal with qualitative linguistic information, and further desirable properties like monotonicity, idempotent, commutative, and boundedness have also been studied.	Uncertain linguistic aggregation operator	The proposed aggregation operator depends on the linguistic scale function that helps in converting linguistic variables to the [0,1] and, therefore, proficiently avoids the problem caused by the product of numerical values and linguistic variables, aggregated results belong to initial linguistic expression domain. Uncertain linguistic aggregation operator could able to aggregate linguistic variable with non-equidistant labels.
“Linguistic discriminative aggregation in multi-Criteria decision-making (MCDM)"	2016	[4]	This paper aims to develop and discuss in detail new linguistic aggregation operators termed as linguistic discriminative averaging (LDA) and linguistic discriminative ordered geometric (LDOG) operator along with their properties. Further, ordered variants and generalized variants of the proposed operators have also been investigated.	Aggregation operators	The proposed linguistic aggregation operators are modeled to give weight to a criterion according to the degree of variation in the various evaluations of alternatives against it. Further, these operators are very much useful in discriminating among alternatives in MCDM problems.
“Adaptive linguistic weighted aggregation operators for multi-criteria decision making"	2017	[3]	This paper proposes new aggregation operators for MCDM under linguistic setting, and further its properties are investigated.	Aggregation Operators	The proposed operators are mainly designed for a MCDM environment, where the experts provide the evaluations of the alternatives against criteria in the linguistic terms, and the management decides the primary criteria weights. It is useful in a situation where primary weight information is not available.

Over the years, as the complexity of the socio-economic environment increases [44]. DM processes have become one of the most wide-ranging and prominent application areas and have attracted various practitioners and researchers. Many institutions have turned from single decision-makers to group decision-makers to successfully and proficiently complete the task. A simple GDM problem is specified as a problem where the DM is considered with manifold alternatives and experts who are trying to accomplish a peculiar solution by considering their preferences or point of view. Ideally, the GDM problem composed of two steps: (1.) *Aggregation Phase* A phase that aggregates experts' preferences, and (2.) *Exploitation Phase* A phase that helps in obtaining a solution set of alternatives in the decision problem. The table 1.3 below is arranged in a chronological order of publication and outline briefly about the summary of the existing literature related to a GDM model based on the 2TL model.

Table 1.3: Summary of GDM model based on a 2-tuple linguistic model and its extension

Title	year	paper	Description	Approach/Strategy	Strong/Weak points
"A group decision support approach to evaluate experts for R and D project selection"	2008	[246]	This paper aims to propose a group decision support approach in order to assess experts for R and D project selection wherein the criteria and their attributes are summarized majorly grounded on the experience with the "National Natural Science Foundation of China (NSFC)".	group decision support approach	The aim of the proposed method is to present a feasible expert evaluation approach that can be used for R and D project selection and thereby both GDM approach and group decision support system are required to help aggregating and eliciting group judgements.
"A method for group decision making with multi-granularity linguistic assessment information"	2008	[137]	This paper aims to develop a methodology to solve the GDM problems subjected to multi-granularity linguistic assessment data.	Linguistic assessment information, goal programming	The methodology proposed in the paper able to express the multi-granularity linguistic information in terms of fuzzy numbers. A linear goal programming model is designed to integrate the information of fuzzy assessment and, lastly, to compute the collective ranking values of alternatives without the requirement of information transformation.
"Group decision making based on computing with linguistic variables and an example in information system selection"	2008	[42]	This paper proposes a transformation technique to transform the non-homogeneous linguistic information into a standard linguistic term set. Then, grounded on computing with the 2TL variables, a decision-making model is developed to manage the GDM problems.	2-tuple linguistic variables	The linguistic variables defined in terms of 2-tuple is utilized in the proposed model representing the subjective judgment of every decision-maker. After that, the 2-tuple model and operation method is applied to deal with the aggregation of ranking and weighting among items and criteria effectively. An extended version of TOPSIS within a fuzzy environment is also developed primarily based on CWWs.
"PROMETHEE-MD-2T method for project selection"	2009	[122]	The objective of this paper is to develop two newer multi-criteria 2-tuple GDM methods called "Preference Ranking Organisation Method for Enrichment Evaluation Multi Decision maker 2-Tuple-I and II (PROMETHEE-MD-2T-I and II)". These methods are capable enough to integrate inside their procedure, both quantitative and qualitative information in an uncertain context.	Promethee method	The proposed methods apply to all kinds of decision-making problems with heterogeneous and multi-granular information. Therefore, the applicability of these techniques to real problems provides better outcomes in MCGDM. It offers to the decision-makers an extensive and simpler application of the aggregation operators of the PROMETHEE multi-criteria method.
"Group decision making with incomplete fuzzy linguistic preference relations"	2009	[5]	This paper aims to develop a method to estimate missing preference values while dealing with incomplete fuzzy linguistic preference relations appraised employing a 2TL approach.	Incomplete fuzzy linguistic preference relations	The proposed methodology is useful in estimating missing values in the incomplete 2-tuple fuzzy linguistic preference relations based on the additive consistency property.

Table 1.3: Continued

Title	year	paper	Description	Approach/Strategy	Strong/Weak points
“The unbalanced linguistic aggregation operator in group decision making"	2012	[342]	This paper aims to propose an updated linguistic aggregation operator to handle unbalanced linguistic values in GDM problems and linguistic hierarchies to express unbalanced linguistic values. Finally, the unbalanced linguistic ordered weighted geometric operator to aggregate unbalanced linguistic evaluation values is also presented.	ordered weighted geometric (OWG) operator	The unbalanced linguistic ordered weighted geometric (ULOWG) operator proposed in the paper to solve linguistic group decision-making problems where experts use unbalanced linguistic values to express their evaluation for problems. The proposed operator has several enthralling properties.
“Linguistic power aggregation operators and their application to multiple attribute group decision making"	2012	[298]	This paper aims to propose an updated linguistic aggregation operator based on a power-average (PA) operator. For the unknown weighting vector of the decision-maker, a different approach based on a linguistic power ordered weighted average operator is also proposed. Further, these approaches are extended to solve GDM problems.	Power-average (PA) operator, linguistic weighted average operator	The operators that are developed in the paper can easily relieve the influence of petty arguments on the aggregated outcomes, and thereby the obtained aggregated outcomes are more reasonable. Two approaches have been developed to deal with the GDM problems under linguistic preference values when the weighting vector of the decision-maker is known and when it is unknown. The prominent characteristic of the approaches that have been developed is that they take all the decision arguments and their relationships into account. Further, an extension of the operators to the uncertain linguistic environment has also been developed.
“Some interval-valued 2-tuple linguistic aggregation operators and application in multi-attribute group decision making (MAGDM)"	2013	[331]	This paper proposes a method that deals with MAGDM problems based on interval-valued 2-tuple linguistic information. Some new aggregation operators and their desirable properties have been discussed in detail. Furthermore, an updated methodology to decide the weight vector of the interval-valued 2-tuple aggregation operator primarily based on the idea of the degree of precision has also been presented.	interval-valued 2-tuple linguistic information, degree of precision	Interval-valued 2-tuple linguistic model is suitable in handling the MAGDM problem described under a multi granular linguistic notion as it is capable of fully expressing the information and also able to unify it easily without involving any tedious aggregation steps.
“Multi-attribute group decision-making (MAGDM) with multi-granularity linguistic assessment information: An improved approach based on deviation and TOPSIS"	2013	[165]	This paper aims to propose an updated method concerning GDM problems associated with multi-granularity linguistic assessment information. The computational formula is developed to transform and unify the multi-granularity linguistic comparison matrices. Furthermore, the technique of standard and mean deviation to decide the attribute weights that are not known is applied. Finally, the weights of the decision-makers will be decided by using the extended TOPSIS method.	“TOPSIS (technique for order preference by similarity to an ideal solution) method", standard and mean deviation	The proposed method could able to overcome the limitation of the classical approaches. It can combine a standard and mean deviations as well as the TOPSIS method. Further, the MAGDM approach transforms multi-granularity linguistic information simpler. Along with this, it takes into account the weights of the attributes and decision-makers by the process of aggregating the assessment information and objectively weighing the attributes and decision-makers.

Table 1.3: Continued

Title	year	paper	Description	Approach/Strategy	Strong/Weak points
“Multi-attribute group decision making with aspirations: A case study"	2014	[91]	This paper aims to propose an integrated multi-attribute decision-making approach for problems with consideration of the decision maker's aspirations. By solving the case problem of China Southern Airlines (CAS).	Aspiration utility function	This study presents an integrated decision-making approach for solving multi-attribute group decision-making problems with aspirations. The proposed approach integrates aspirations into the utility theory for MADM. Further, it uses optimization techniques for reducing the gap between group opinion and individual opinions, thereby allowing managers to make more reliable decisions. It also provides a consistency coefficient to managers for checking the group decision making quality. Lastly, it can accommodate complex decision data by incorporating numerical values, interval numbers, linguistic terms, and uncertain linguistic terms.
“Method of multi-criteria group decision-making based on cloud aggregation operators with linguistic information"	2014	[236]	The central idea of the paper is to define cloud operators (such as “cloud weighted arithmetic averaging (CWAA)" operator, “cloud-ordered weighted arithmetic averaging (COWA)" operator, and “cloud hybrid arithmetic (CHA)" operator), followed by its application in the MCGDM problem. Further, in the paper, a section is devoted, which explains the conversion between linguistic variables and clouds.	cloud model, aggregation operator	Traditional methods existing in the literature are not robust enough to convert qualitative concepts to quantitative information in linguistic MCDM problems; neither can they completely reflect the fuzziness and randomness inherent in qualitative concepts. However, the cloud model proposed in the paper overcome these difficulties. The three numerical characteristics of the cloud model render the transformation between this qualitative concept and quantitative information smoothly and effectively.
“Consensus-Based Group Decision Making Under Multi-granular Unbalanced 2-Tuple Linguistic Preference Relations"	2015	[71]	The GDM model is proposed in this study, along with multi-granular unbalanced 2TL preference relations. The transformation function is used to link both balanced and unbalanced multi granular language preference relations. A consensus model is also offered to assist decision-makers in reaching an agreement.	Proportional 2-tuple linguistic model	The proposed multi-granular linguistic GDM model is based on the linguistic model of 2-tuple. The transformation function presented in the paper can be reduced into a traditional transformation function given in [**?**] beneath the context of a multi-granular balanced linguistic environment. The consensus model so proposed sets out a new means to manage both the consistency of an individual and group via a linear programming model. The model also stipulates a novel means by minimizing the loss of information when the consensus has been established.
“A model based on subjective linguistic preference relations for group decision making problems"	2016	[200]	In this paper, a novel definition of preference relation, the so-called “subjective linguistic preference relation", is proposed. These preference relations are based on the concept of subjective evaluations, introduced in the LCM based on discrete fuzzy numbers. Further, an example of a multi-expert DM problem with a hierarchical multi-granular linguistic context is analyzed to illustrate the potential of the proposed method and its advantages concerning other methods.	Subjective linguistic preference relation	The advantage of the proposed model is that it offers experts more flexibility to express their opinions, and also, this model guarantees no loss of information.

Table 1.3: Continued

Title	year	paper	Description	Approach/Strategy	Strong/Weak points
“A method for multi-criteria group decision making with 2-tuple linguistic information based on cloud model"	2017	[333]	To remedy the fault of the existing operators, this study proposes a 2-tuple hybrid ordered weighted geometric (THOWG) operator that synthetically considers the importance of both individual and ordered position. This research also introduces a new cloud formation approach for converting 2TL variables into clouds.	Cloud model	The proposed method has the advantage of allowing new THOWG operators to be developed in order to overcome the restrictions of existing 2-linguistic power aggregation operators. Traditional 2TL operators either overlook or dismiss the importance of the individual or the ordered position. New cloud-generating methods for converting 2TL variables into clouds have also been given. This method incorporates the cloud model's tremendous advantages to handle the randomness of natural languages, resulting in significantly improved decision quality. Finally, various novel cloud algorithms have been devised, including cloud distance, cloud possibility degree, and cloud support degree.
“Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching"	2017	[156]	This paper introduces a personalized individual semantics (PIS) model to tailor individual semantics by a numerical scale model primarily based on intervals and the traditional linguistic model of 2-tuple. Further, to obtain and express the PIS model, a consistency driven optimization-based model is also proposed. Finally, to justify the feasibility and validity of the PIS model, it is applied to a linguistic GDM problem with a consensus reaching process, by defining the notion of the individual linguistic understanding.	Interval numerical scale, 2-tuple linguistic model	The introduction of PIS methodology to model and solve linguistic GDM problems with preference relations could able to improve the management of different meanings of words for different people. The proposed model is based on the numerical interval scale. This is because it could able to handle distinct linguistic representations in a more precise way. Moreover, a novel frame of work developed for handling PIS could able to redesign the phases of CWW, which helps in obtaining tailored linguistic results easier to interpret and understandable by humans.
“The 2-rank consensus reaching model in the multigranular linguistic multiple-attribute Group Decision-Making (MAGDM)"	2017	[335]	In this paper, a 2-rank consensus reaching model with minimum adjustments for the 2-rank MAGDM problem is developed. An optimization model to support a 2-rank consensus rule is also proposed, which is further transformed into a mixed $0-1$ linear programming model. Finally, a comparative analysis of the proposed model with the existing consensus models is also presented.	2-rank consensus reaching model	Preferences of the decision-makers are expressed by using the context of multi granular linguistic term sets. Optimization based model is put forwarded to support the consensus rule. The optimal adjusted uniform linguistic decision matrices are transformed into multi granular linguistic decision matrices, which are used as the references for decision-makers to modify their preferences. Further, an iterative 2-rank consensus reaching process with the minimum adjustments is also developed. Moreover, many useful properties of the proposed model is also discussed.

Table 1.3: Continued

Title	year	paper	Description	Approach/Strategy	Strong/Weak points
“Group decision making based on linguistic distributions and hesitant assessments: Maximizing the support degree with an accuracy constraint"	2018	[279]	In this paper, a novel approach is designed to propose a new linguistic group decision model called the maximum support degree model (MSDM). This model aims to ensure maximum support degree of the group opinion and further guarantying the accuracy of the group opinion. Further, a mixed 0-1 linear programming approach is also presented to solve the MSDM. Finally, the utility and validity of the MSDM in multiple attribute group decision making is proven.	Linguistic distributions and Hesitant fuzzy linguistic term sets (HFLTSs)	The aggregation methodology proposed in this study ensures maximum degree support of the group opinion in linguistic distributions based on GDM problems and further guarantees that the opinion of the group is an HFLTS has a certain degree of accuracy. In the MSDM, linguistic distributions are used to express decision-makers' individual preferences to provide more probabilistic preference information over linguistic terms, and the HFLTS with certain accurate constraint is employed to guarantee the accuracy of the group opinion.

Due to the societal developments and an increase in the complexity of the problems, we come across several decision problems where a group of individuals is having differences in cultural and knowledge background. As a result, decision-makers may represent their respective evaluations using MGLTSs. In literature, several methodologies have been developed to tackle the problems of GDM involving multi-granularity linguistic information. Zhang and Guo [334] proposed a GDM method where decision-makers are required to express their assessment using multi-granularity in an uncertain linguistic environment along with insufficient information of the attributes' weight. The information of the evaluation is transformed into a trapezoidal fuzzy number (TFNs). Optimization models are established to minimize the deviation between the evaluation of each decision-maker and the group's collective evaluation for each alternative. These models of optimization are solved to obtain the group's collective evaluation value of the alternatives. Finally, the closeness coefficient and ranking for each alternative are evaluated. The optimal alternative based on the ranking can be further selected. The method proposed is simple and easy to understand. Henceforth, is well applicable in the GDM problems involving multi-granularity uncertain linguistic and insufficient weight information.

Li [154] introduced the notion of extended LVs to propose a new approach for solving multi-attribute GDM problems under linguistic assessments. Further, Y. Ju [136] proposed an upgraded method to solve MCGDM problems, where both the criteria values and criteria weights take the form of linguistic information, and the information about linguistic criteria weights is partly known or completely unknown. Morente-Molinera et al. [201] proposed a systematic literature review on multi-granular fuzzy linguistic modeling in GDM from the year 2000 and 2014 and further discussed their drawbacks and advantages.

With the developing headway of the 2TL model in DM processes and other disciplines, it had been scrutinized that multi-objective linguistic optimization problem concomitant with the 2TL model was a significant advantage. Multi-objective optimization problems helped manage problems that were engrossed with conflicting objectives. There exist, class of issues, where every objective function was self-governed, thereby, cannot be solved by transforming into multiple objective optimization problems. Numerous methods had been developed in the literature to solve such problems [47, 48]. These methods were primarily based on Tsukamoto's fuzzy reasoning method, where the objective function was converted into a crisp form, and further, the objective function was solved by using any conventional optimization technique. Nevertheless, this methodology endured through a limitation as the resulting output of the problem was attained in crisp form rather than linguistic. To overcome this limitation, Gupta and Muhuri [95] developed a newer method grounded upon the 2TL model in solving multiple linguistic optimization problems.

In certain situations, for instance, the problem of the social network, e-democracy, and so forth decisions are required to be taken by relatively larger group size, categorized as the large scale GDM. By defining a consensus model based on PIS for large scale GDM, Li et al [157] gave the first proposal in this field. For the detailed knowledge of the model, one can refer to the corresponding paper [157].

1.6 Application of the 2-tuple linguistic model and its extension

The 2TL model is prudent over other existing approaches by overcoming the restrictions experienced by conventional methods. After having reviewed the dominant position that the 2TL model holds among variant LCMs as well as other decision models, we move forward to show extensive applications of the model in various disciplines. Table 1.4 shows the application depending on the 2TL model as well as its extension.

Table 1.4: Application based on 2-tuple linguistic model

Title	paper
Supply chain	[155, 309]
Nuclear safeguards	[162, 238]
Selection Process	[19, 122, 246]
Engineering Systems	[6, 189]
Recommender System	[191, 192, 231, 232, 239]
risk evaluation	[46, 163, 230, 312, 340]

The 2TL model is used for segmentation of color images [225]. For this, a set of experts are being provided such that every object of the considering image is assigned degree of every pixel by each expert. The objects are represented by means of fuzzy linguistic labels and then by using the DM model based on 2-tuples an aggregation phase to classify each pixel is applied. Other application areas of 2TL model are social judgement analysis [292], weapon system evaluation [278], cooking recipe recommendation [24, 25], decision-making [71, 118, 156, 166, 168, 170, 218, 247, 291], personalized individual semantics [156, 157], renewable energy [79], energy optimization [96, 203], natural language generation [141], supplier selection problems [269, 273, 289], Sensory evaluation model [194, 195], a cloud-based decision support model [275, 281], emergency management [237], emergency decision-making [337], financial technologies selection [204], analysis of information and communication technology [49], information retrieval system, hotel selection [313], non-coorperative matrix games [248, 249].

1.7 Motivation

Analysis of the practical life problems collectively present aspects that can appertain to different nature, based on these nature's aspects the present problems deal with a different type of information. Generally, the problems present quantitative aspects easily accessible through precise numeric values, nevertheless in some cases, they present qualitative aspects that are complicated to handle by precise numeric values. Linguistic assessments successfully modeled via LVs in the problem presenting qualitative aspects have provided better results. The presence of linguistic information requires CWWs processes to obtain the solution for the real-life problems in a fashion emulating the subjectivity of the human mind. The 2TL and the variant extension of the model have delivered remarkable advancement in the uncertainty domain.

In this thesis, we have analyzed both the theoretical and practical aspects of the 2TL model with its extension. Since the literature of extension of the 2TL model is abundant, therefore, we restricted our study, focusing majorly on the unbalanced LCMs and interval-valued 2TL model. We proposed various algorithms and methodologies for meticulously modeling the intricate decision problems with the broad applications witnessed in matrix games and decision sciences and have further validated the models with appropriate data.

Chapter 2 presents the abstract theory of a 2TL group based on a crisp binary operation in a conventional sense. The LTS utilized in the current theory is a balanced LTS having all linguistic terms distributed symmetrically. A practical application in bipolar graphs and matrix games has been discussed to showcase the physical meaning of the abstract concept so developed.

It is noteworthy that experts cannot always express the qualitative information that is distributed uniformly and symmetrically in real world decision system. In some situations, experts have to use LTSs having the non-uniform distribution of the linguistic labels. Several pioneering studies have been put forward to support the context of an ULTS. However, there exist some weaknesses in the existing unbalanced LCMs. Henceforth, motivated by the discussed concern, a new 2-tuple unbalanced LCM based on distance measure is proposed in Chapter 3 with an application example to demonstrate the performance excellence of the model.

Further, in Chapter 4, we extend the concept of unbalanced LTS to propose the notion of probabilistic multiplicative ULTS, which considers the probability and the non-uniformity of the linguistic labels so that the information is represented in a more flexible form that aligns human thoughts. Game theory is considered an essential domain of decision science. Fuzzy theory, stochastic framework, intervals, and linguistic models have been successfully incorporated in the traditional game theory to model uncertain information in matrix game problems. However, probabilistic multiplicative unbalanced linguistic information has not been employed within the current game models to the best of our knowledge. So, the proposed concept of Chapter 4 is also dedicated to modeling uncertain information in game theory under a probabilistic multiplicative unbalanced linguistic environment.

Chapter 5 is dedicated to modeling uncertain information in interval-valued matrix games by proposing interval-valued linguistic linear programming methodology. We put forth the concept of max-min principle for defining the lower and upper value of the interval linguistic game problem that builds up an analogy with the point based matrix game problem. However, in the absence of pure strategies, we designed a new approach in the chapter for evaluating the optimal strategies and value of the game.

Meanwhile, Chapter 6 introduced a unique outlook for solving interval-valued matrix game problems by developing the notion of interval matrix norm. The methodology devised in the chapter is capable of obtaining an approximated interval game value for the corresponding zero-sum interval matrix game without undergoing the existing process of solving traditional interval linear mathematical models. Additionally, some new matrix game results primarily based on interval norm concept have been developed. Finally, a case study is reported to validate the efficacy of the proposed method.

A summary followed by the future scope of the research work is evinced to conclude the thesis after chapter 6.

Chapter 2

Group operations and properties for 2-tuple linguistic variables with its application

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This chapter[1] [2] is implemented in two segments. The objective of the initial module is to put forward the theory of 2TL groups under the binary operation in the conventional impression. In addition, a similarity relation between the set of all 2TL information and numerical interval, $[-n, n]$, is obtained. The notion of a 2TL bipolar graph is envisioned as a practical application to explicate the appropriateness of the proposed linguistic group isomorphic relation. The next module discusses the catalog of properties of 2TL abstract group isomorphism and homomorphism, along with laying down the concept of linguistic kernels, cosets, normal subgroups, and factor groups. Some new operational laws are also given for a 2TL term set such that the final computed results belong to the original term set. Finally, an equivalent of the fundamental theorem of group homomorphism for a 2TL group is obtained. Consequently, based on the listed linguistic group isomorphic properties and new operational laws, this chapter also gives a constructive study on the

[1]The content of this chapter is based on research paper "Group operations and isomorphic relation with the 2-tuple linguistic variables", *Soft Computing*, springer **24**, 18287–18300 (2020). https://doi.org/10.1007/s00500-020-05367-9 (**SCIE, Impact Factor: 3.643**)

[2]"Group isomorphic properties with some novel operational laws for 2-tuple linguistic variables and its application in linguistic matrix games" (Communicated)

theory of matrix games by introducing linguistic matrix norms. The proposed concept is then taken forward to present a new methodology to solve a two-player zero-sum linguistic matrix game having 2TL information offering approximated linguistic game value without solving any linguistic linear mathematical equations. Lastly, a real-life practical example from the equity market domain is illustrate to exhibit the feasibility and consistency of the developed approach.

2.1 Introduction

The Zadeh's [316] introduction of fuzzy sets is considered to be the most relevant concept in describing the problems of real-life since every object in the physical world possesses some degree of fuzziness. Therefore, fuzzy-based approaches are well-suited to deal with uncertainty and have become a considerable research area with manifold applications in respect of engineering, computer science, and so forth. Several investigations have also been envisioned to "fuzzify" many essential mathematical structures like algebra, topological spaces, groups, etc.

It is a well-known fact that the classical theory of groups is recognized as the study of algebraic structures designed to model and thus contemplate the symmetries of specific objects. In other words, group theory plays a notable role in mathematics to analyze individual objects' symmetries. An object's symmetry can be either discrete or continuous. It is, therefore, imperative to examine each discrete and continuous group. Ideally, group theory evolution begins in 1770 and is extended to the twentieth century. However, the nineteenth century witnessed the significant era of development of group theory. Since then, the scope of group theory has been expanded to geometry, cryptography, particle physics, combinatorics, some areas of analysis, and so forth. Many academics have made significant contributions to the development of theories and perspectives that have enriched the wide-ranging application of group theory based on a crisp set (to mention only a few, one may refer to [81, 143, 144, 205, 241]).

The scope of group theory is further anticipated in terms of fuzzy variables and fuzzy sets [316] widely referred to as the fuzzy group theory. Rosenfeld [242] initially introduced the idea of fuzzy groups to outspread the elementary concepts of classical group theory in an effort to reinforce fuzzy groups. Subsequently, Luca and Termini [82] defined appropriate algebraic properties associated with the class of fuzzy sets as well as making a careful examination of certain new algebraic aspects of the "fuzzy sets" theory, thereby linking it with Brouwerian lattices. Furthermore, Anthony and Sherwood [7] redefined the concept of fuzzy algebraic structures based on the kind of semi-groups defined on the unit interval [0, 1] called as "t-norm" introduced by Schweizer and Sklar [252]. In that paper, authors have stated new results and properties in light of the proposed definition and, consequently, overcomes the limitation possessed by the previously stated fuzzy algebraic concept [82, 242].

Das [83] inspected fuzzy groups to characterize all the fuzzy subgroups of finite cyclic groups and put forward the idea of "level subgroups" of a fuzzy subgroup centered on a concept of "level subset" developed by Zadeh. For a more detailed knowledge about the results and deeper understanding of a concept, a reader can refer to the corresponding paper [83]. Mukherjee and Bhattacharya [206] gave an introduction of a "fuzzy normal subgroup" and "fuzzy coset" and have further established a fuzzy Lagrange's theorem. Thereby, presented the analogs of the basic group-theoretic results in an elementary fashion. In a sequel, Mukherjee, and Bhattacharya [28, 207] further investigated several analogous concepts of classical group theory in terms of fuzzy groups and proved the fuzzy variants of a few prominent group theoretical results. Also, Bhattacharya [29] proposed a more detailed study about the "level subgroups" of a fuzzy subgroup and further examined the possibility of the unique determination of the fuzzy subgroup by the class of "level subgroups" of a fuzzy subgroup. The author also rectified the inaccuracies persisting within the few results of Das [83] and later on initiated to delineate all the fuzzy subgroups of a group of finite order in an effort to generalize the results of Rosenfeld [242] and Das [83].

Sherwood [251] presented the concept of product of fuzzy groups and further discussed the related properties as well. Later on, Ray [243] gave the introduction of the isomorphism of fuzzy groups and have stated an acceptable definition of the fuzzy version of the isomorphism theorems. Further, Choudhary et al. [53] introduced the concept of a fuzzy homomorphism between two groups and to study its effect on the fuzzy subgroup. In a sequel, Chakraborty and Khare [54] proved an analogous version of the Fundamental Theorem of Homomorphism and the Second Isomorphism Theorem for fuzzy homomorphisms. Later, Ajmal [13] catered the highly-awaited solution of the problem of presenting a one-to-one correspondence between the family of fuzzy subgroups of a given group, constituting the kernel of a given homomorphism, and the family of fuzzy subgroups of the homomorphic image of the given group. Analogous to classical normal groups and group homomorphism, Akgül [8] defined "fuzzy normal subgroups," "fuzzy level normal subgroups," and their homomorphism. Subsequently, Makamba [208] introduced the notion of the internal and external direct product of a fuzzy subgroup of a corresponding group G and has additionally established fuzzy isomorphic relation between fuzzy subgroups. Since the literature of fuzzy algebra is profuse and vast, so therefore, readers can also refer to the papers (to mention a few, see [9, 84, 85, 100, 145, 226, 244, 263, 264, 315]) in order to understand the detail concepts of fuzzy algebra.

Although "fuzzy sets" theory and fuzzy algebra have been massively studied and become a robust field of research in severe engineering branches, medical sciences, graph theory, etc., there are persisting some real-life situations where “fuzzy sets" theory is not suitable to handle uncertainty. This happens due to the presence of much enigmatic and imprecise information in intricate problems. As a result, it is quite clumsy and unnatural to express results quantitatively in terms of a fuzzy variable. Therefore, the use of LV and linguistic-based approaches, as developed by Zadeh [324–326], is much suitable and straightforward to handle uncertainty and thus has been studied extensively (see [20, 61, 63, 103, 104, 261, 324–327]). Several DM problems are solved using linguistic information to model experts' preferences, which implies using computing with words (CWWs) processes. To efficiently address linguistic information, Herrera and Martínez [106] proposed a pioneering linguistic computational model, namely “2-tuple linguistic model” so as to authenticate the linguistic computations by overcoming the limitation possessed by the classical symbolic LCMs as discussed in chapter 1.

Since in meaning, a linguistic variable is in sight as a fuzzy set and 2-tuple representation is alleged to be the most effective way of representing linguistic information. Therefore, in this chapter, we link the theory of 2TL set and classical group theory to develop the notion of 2TL groups under binary operations in a classical sense. Consequently, this chapter extend the concept of fuzzy algebra, particularly fuzzy groups in the 2TL scenario, which is viewed as the initial aspect of our chapter.

Since game theory (GT) is found to be the most significant area of research in the subject of operation research (OR) providing a mathematical configuration to study the strategies of rational individuals in a competitive environment. After the groundbreaking work of Von Neumann and Morgenstern in the early $20th$ century, GT has gained a considerable attention in multiple disciplines and wide-ranging fields. It is often viewed that a fundamental problem linked with the ordinary GT is that game players make decisions in a crisp sense. However, within a realistic situations the majority of games always believed in uncertain environments where the knowledge about the payoffs are not completely known by the players. Therefore, the introduction of imprecise matrix game problem to deal with the uncertainty is envisioned from the perspective of stochastic payoff games [92, 254], fuzzy payoff games [30, 31, 55, 77, 219], single linguistic terms [14, 15], and 2-tuple linguistic term [248].

The matrix game methodology given in the various uncertain game models is needed to solve two linear optimization problems for each corresponding player to obtain an optimal solution. However, solutions obtained for these problems will be troublesome for the zero-sum matrix game (ZSMG) problems having a bigger size. Therefore, the current chapter is also dedicated to construct a practical method convenient enough to attain an approximated solution accelerated with the matrix norms of the game matrix having 2TL as payoff value. For this, based on the 2TL group-theoretic properties and new operational laws, we introduce the notion of 2TL matrix norms. Then, we introduce the notion of $1'-$ norm and $\infty'-$ norm of the 2TL payoff matrix. Henceforth, succeeded in obtaining an approximated solution for a zero-sum linguistic matrix game (ZSLMG). We also state certain conditions and present a new outlook of matrix game with 2TL information and its solution procedure.

Although any ZSLMG problem can be solved by utilizing the linguistic linear programming (LLP) method [248], our proposed novel methodology may solve and offer an approximated matrix game value without addressing any system of linear mathematical equations. The proposed method gives us an optimal range of the game value containing an exact solution to the game problem, henceforth giving players an idea about the value of the game at the beginning of the problem itself. Moreover, using the LLP method for a large-scale matrix game problem is quite tedious and time-consuming, whereas linguistic matrix norm methodology offers an approximated game value at a faster rate and reduces the computational cost. Additionally to the straightforwardness of use, this is the most crucial and novel part of our methodology. To the best of our knowledge, it is foreseen that the concept of 2TL matrix norms is not used in imprecise matrix games yet, and the presented study brings about a new viewpoint to the game solution.

The subject matter of the current chapter is presented in the subsequent sections as follows. The forthcoming Section 2.2 presents a methodology claiming that a finite subscript-symmetric linguistic term set (SSLTS), intervals $[-n,n]$ and $[\frac{-1}{2},\frac{1}{2}]$ forms a group under a binary operator '$*$', '$\oplus$' respectively. In addition, we have proved that $\overline{LT} \equiv LT \times [\frac{-1}{2},\frac{1}{2}]$ is a 2TL direct group under a binary operation '$\circ$'. The definition of all these group operations are defined in the following section. Lastly, a result describing an isomorphic relation between the 2TL group, $\overline{LT}$, and a numerical interval, $[-n,n]$ based on the classical definition of isomorphic groups is also presented. To validate the results and showcase the physical meaning of the isomorphic groups, we herein propose an application in terms of bipolar graphs. In this connection, we first developed the concept of 2TL bipolar set (2TLBS) accompanied by the introduction of the notion of 2TL bipolar graphs (2TLBGs) based on a finite SSLTS and after that define its relevant properties. Later on, we define graph isomorphism viewed as a direct consequence of a 2TL group isomorphic relation. Additionally, the properties of a homomorphic and isomorphic relation, the novel concept of cosets, normal subgroups, linguistic version of kernels, followed by the fundamental theorem of linguistic group homomorphism with some new operational laws primarily based on a 2TL term set, is examined in the Section 2.3. Section 2.4 explains the formulation of the linguistic matrix norm approach to solve ZSMG problem with 2TL information along with the demonstration of its applicability in practical DM problem and further presents some comparative analysis with the existing method. Lastly, Section 2.5 provides the concluding remarks of this chapter.

The work presented in this chapter is advantageous in developing the basis for linguistic group theory. Although the fuzzy group theory has been developed and comprehensively studied in the literature, linguistic group theory development is pristine and yet to be explored. Therefore, the ideas and results exhibited in this chapter are a milestone in this direction and also provide a more flexible framework in proving an “algebraic structure groups” for the 2TL model by using SSLTS. Moreover, the chapter’s results depict an analogy with an existing theory of groups based on a crisp set.

2.1.1 Preliminaries

We first aims to review some basic definitions and operational laws related to the subscript symmetric LCM, classical group theory and the matrix norm approach for solving two-person classical ZSMGs followed by the foundations of matrix games with linguistic information required for this chapter.

Subscript-symmetric linguistic computational model

Definition 2.1.1. [299] Assume $LT = \{t_{-n}, \ldots, t_0, \ldots, t_n\}$ be a finite SSLTS such that the following properties are satisfied:

(i) The set LT is termed as an ordered set i.e., $t_p > t_q$ if and only if $p > q$;

(ii) The negation operator of a linguistic variable is defined as: $\text{neg}(t_p) = t_{-p} \in LT$.

Here, t_0 represents a middle linguistic term, and the rest over terms are placed uniformly on each side of the middle term. In particular, t_{-n} and t_n represent the lower and upper bound of the LTS.

The existing 2TL model explained briefly in the section 1.3 is defined over the domain $[0, n]$. However, the present model can be easily specified for the domain $[-n, n]$ by using the above mentioned SSLTS, namely LT. Meanwhile, the notion of Δ and Δ^{-1} operators, ideally formalized by Herrera and Martínez [106] is extended to the set LT and the transformation function facilitating to convert a numeric value x to a corresponding 2TL variable is given below.

Definition 2.1.2. Let $LT = \{t_p \mid p = -n \ldots, 0, 1, \ldots, n\}$ be a finite SSLTS having cardinality $2n+1$ and let $x \in [-n, n]$. Then, the transformation function Δ needed to express the 2TL variable identical to x is identified as.

$$\Delta\colon\ [-n,n] \to LT \times [\frac{-1}{2}, \frac{1}{2}]$$

$$\Delta(x) = (t_p, \alpha) \text{ where } p = \text{round}(x)\ \in\ LT \,\&\, \alpha = x - p\ \in [\frac{-1}{2}, \frac{1}{2}].$$

where round(*) is termed as the general rounding operation, t_p represent the linguistic label nearest to x, and α referred as the "symbolic translation".

From the above function, it has been clearly envisioned that the Δ is a bijective mapping [106] and therefore, its inverse is stipulated by,

$$\Delta^{-1} : LT \times [\frac{-1}{2}, \frac{1}{2}] \to [-n,n] \text{ as } \ \Delta^{-1}(t_p, \alpha) = p + \alpha = x.$$

Furthermore, Herrera & Martínez [106] has established the comparison law for the well-known linguistic model of 2-tuple primarily based upon the classical lexicographic ordering. Hence, correspondingly the following definition can be outlined to state the ranking order.

Definition 2.1.3. Let (t_p, α_p) and (t_q, α_q) be two 2TL variables. Then,

(i) If $p < q$ then $(t_p, \alpha_p) < (t_q, \alpha_q)$.

(ii) If $p = q$, i.e., $t_p = t_q$, then

  (a) if $\alpha_p = \alpha_q$, then $(t_p, \alpha_p) = (t_q, \alpha_q)$;

  (b) if $\alpha_p > \alpha_q$, then $(t_p, \alpha_p) > (t_q, \alpha_q)$;

  (c) if $\alpha_p < \alpha_q$, then $(t_p, \alpha_p) < (t_q, \alpha_q)$.

For a comprehensive and complete elucidation of the theory for the existing 2TL model, one can refer to the introductory chapter 1.

Classical Group Theory

In the following, we review the formal definition of the classical group G and some elementary concepts required for this chapter.

Definition 2.1.4. [99] A group G is defined as a nonempty set along with a group operator '$\bullet \in \{+, \cdot\}$' such that it should satisfy the subsequent characteristics:

(i) $g \bullet h \in G \, \forall \, g, h \in G$;

(ii) Associative property holds trivially;

(iii) There exist an element $e \in G$ such that $g \bullet e = e \bullet g = g \, \forall \, g \in G$;

(iv) For every $g \in G$, there exist an inverse element $f \in G$ such that $g \bullet f = f \bullet g = e$.

It can be noted that the inverse and identity element of a group G is uniquely defined.

Note 2.1.1. *If a group G satisfies an additional condition*

$$g \bullet h = h \bullet g$$

for all $g, h \in G$ then G is referred as abelian group.

In a generic way, if $\bullet = +$ then we write $g \bullet h$ as $g + h \in G \, \forall \, g, h \in G$ and the identity element is 0 while the inverse element of g is $-g$. On the other hand if $\bullet = \cdot$ then we write $g \bullet h$ as $g \cdot h \in G \, \forall \, g, h \in G$ and the identity element is 1 while the inverse element of g is g^{-1}.

Note 2.1.2. *If a group G consists of a finite range of elements, then G is referred to as a finite group; otherwise, we call G an infinite group.*

Note 2.1.3. *The total number of elements present within the group G is referred as order of the group and is denoted by $|G|$.*

Definition 2.1.5. [99] A subset H of G forms a subgroup of G if it satisfies all the group properties under the same binary operation of G and it is denoted by $H \leq G$.

Definition 2.1.6. *[99] A group homomorphism is defined as a mapping $\phi : G \to \overline{G}$ that preserves the operation of the group; i.e., $\phi(g \bullet h) = \phi(g) \# \phi(h) \, \forall \, g, h \in G$ where $\bullet, \# \in \{+, \cdot\}$ are binary operations of group G and $\overline{G}$, respectively.*

Definition 2.1.7. *[99] Group isomorphism is defined as a bijective homomorphic mapping ϕ from a group G to group $\overline{G}$. i.e., $\phi : G \to \overline{G}$ such that $\phi(g \bullet h) = \phi(g) \# \phi(h) \, \forall \, g, h \in G$ where $\bullet, \# \in \{+, \cdot\}$ are binary operations of group G and $\overline{G}$, respectively. When group G is isomorphic to $\overline{G}$, then we say that $G \cong \overline{G}$, i.e., both groups are equivalent and seems out to be abstractly similar in several aspects.*

Definition 2.1.8. [99] Let $G_1, G_2, \ldots, G_n$ be a finite collection of arbitrary groups. The direct product of groups $G_1, G_2, \ldots, G_n$ is defined as a set of n-tuples such that ith component is an element of G_i and the operation is component wise.

Mathematically, direct product of groups is written as:

$$G_1 \times G_2 \times \ldots \times G_n = \left\{ (g_1, g_2, \ldots, g_n) \,|\, g_i \in G_i \right\}$$

where,

$$(g_1, g_2, \ldots, g_n) \bullet (g_1', g_2', \ldots, g_n') = (g_1 \bullet g_1', g_2 \bullet g_2', \ldots, g_n \bullet g_n')$$

and '$\bullet$' is the respective group operations of each G_i .

For a complete knowledge about classical group theory, it is advised to refer any familiar text on abstract algebra, for instance, [99].

Two-player zero-sum matrix games

In literature, a ZSMG engaging only two players is considered the most straightforward game. Out of all the solution methods that have been formulated to solve such matrix games, the linear programming technique is the most prevalent one. However, the linear programming method generally becomes a time consuming method for the bigger size matrix game. Therefore, to overcome this limitation Izgi and Özkaya [133] proposed a novel approach to solve two players' classical zero-sum games by utilizing the concept of matrix norms. The present subsection review some basics of that method.

Definition 2.1.9. [209] Let $A \in \mathbb{R}^{p \times q}$ be a matrix of real numbers, then 1− norm and ∞− norm are successively given in the following manner:

- $||A||_1 = \max_d \sum_c |a_{cd}|$ represents the greatest absolute column sum.

- $||A||_\infty = \max_c \sum_d |a_{cd}|$ represents the greatest absolute row sum.

Throughout the paper, we refer 1− norm and ∞− norm for a matrix of real numbers whereas 1′− norm and ∞′− norm for a matrix with 2TL information.

Lemma 2.1.1. [133] Let the triplet (X, Y, A) be a finite two player ZSMG where X, Y represents a mixed strategy set of the corresponding players' I and II, respectively, $A \in \mathbb{R}^{p\times q}$ is the given payoff matrix and v represents the value of the game. Then,

$\frac{k}{||A||_\infty} \leq v \leq ||A||_1$ for $v > 0$,

$-||A||_1 \leq v \leq \frac{k}{||A||_\infty}$ for $v < 0$.

where $k = \max\limits_{1\leq c\leq p, c\neq l} \sum_{d=1}^{q} v|a_{cd}|$ and $||A||_\infty = \sum_{d=1}^{q} |a_{ld}|$ for fixed l.

Theorem 2.1.1. [133] Let (X, Y, A) be a finite two player ZSMG where $A \in \mathbb{R}^{p\times q}$ is the given payoff matrix and v represents the value of the game. Then,

for $|v| \geq 1$, then $\frac{||B||_\infty}{||A||_\infty} \leq |v| \leq ||A||_1$

for $|v| \leq 1$ and $v \neq 0$, then $\frac{1}{||A||_1} \leq |v| \leq \frac{||A||_\infty}{||B||_\infty}$.

where B is represented as the row-wise induced matrix of A.

Next, result helps in obtaining the largest and smallest elements of the mixed strategy set.

Theorem 2.1.2. [133] Let $A \in \mathbb{R}^{p\times q}$ be the real valued payoff matrix where each entry i.e., $a_{cd} > 0 \forall c, d$. Then, the boundaries for $x_{\max}$ and $x_{\min}$ representing the biggest and smallest elements of the mixed strategy set, respectively, are defined in the following manner,

$$x_{\max} \geq LB \text{ where } LB = \max\left\{\frac{1-\frac{|v|}{||A||_1}}{p-1}, \frac{|v|}{||B||_1}\right\}$$

$$x_{\min} \leq UB \text{ where } UB = \min\left\{\frac{1-\frac{|v|}{||B||_1}}{p-1}, \frac{|v|}{||A||_1}\right\}$$

where B represents the column wise induced matrix of A.

For the justification and deeper discussion of the above-mentioned results one can refer to the corresponding paper [133].

Zero-sum linguistic matrix game

The matrix game problem where the sum of the payoffs corresponding to any given set of strategies is zero is termed as two players zero-sum game [36]. A ZSMG is a particular case of the constant sum game and has subjected to several findings both in the fuzzy as well as conventional set up. However, the game problems with linguistic payoff matrices are pristine and required to be explored.

In the present subsection, we review the basic terminologies and notations related to the ZSMGs within a 2TL framework. The following definitions are taken from paper [248] and can be easily extended to the SSLTS, LT as mentioned already.

Definition 2.1.10. A two-player ZSLMG $\widetilde{G}$ is defined by a quadruplet $(S^n, S^m, \text{LT}, \widetilde{A})$, where S^n and S^m are the strategy sets of player I and II, $\text{LT} = \{\ell_{-g}, \ldots, \ell_0, \ell_1, \ldots, \ell_g\}$, with cardinality $2g+1$, is a SSLTS for both the players, $\widetilde{A}$ is the linguistic payoff matrix of player I against player II, and $neg(\widetilde{A})$ is the payoff matrix for player II.

Since the lexicographic ordering is available in the 2TL variables, one can easily extend the notion of the value of the game to the linguistic matrix game $\widetilde{G}$.

Definition 2.1.11. A matrix game $\widetilde{G}$ with payoff matrix $\widetilde{A} = [\widetilde{a}_{ij}]_{n\times m}$ has the linguistic lower value and the linguistic upper value defined as,

$$\widetilde{v}^- = \max_{i=1,\ldots,n} \min_{j=1,\ldots,m} \widetilde{a}_{ij}, \qquad \widetilde{v}^+ = \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} \widetilde{a}_{ij}.$$

Here, it is considered that $\widetilde{v}^-$ (player I gain floor) is the minimum linguistic payoff that player I is assured to receive while $\widetilde{v}^+$ (player II loss ceiling) is the maximum linguistic loss of player II. The value of the game $\widetilde{G}$ exists if and only if $\widetilde{v}^- = \widetilde{v}^+$. The strategies i^* and j^*, yielding the payoff $\widetilde{a}_{i^*j^*} = \widetilde{v}^- = \widetilde{v}^+$, are optimal for player I and player II, respectively. The pair (i^*, j^*) is also known as the saddle point of the game $\widetilde{G}$.

In the case, where solution set of the game $\widetilde{G}$ does not possess pure strategies. We define the solution set as mixed strategies.

Definition 2.1.12. A mixed strategy is an ordered pair of vectors $(x, y) \in S^n \times S^m$, where

$$S^n = \{(x_1, \ldots, x_n) \;:\; x_i \geq 0,\; i = 1\ldots, n,\; \textstyle\sum_{i=1}^n x_i = 1\};$$
$$S^m = \{(y_1, \ldots, y_m) \;:\; y_j \geq 0,\; j = 1, \ldots, m,\; \textstyle\sum_{j=1}^m y_j = 1\}.$$

Here, x_i is the probability of choosing an arbitrary strategy i by player I and y_j is the probability of selecting an arbitrary strategy j by player II.

2.2 Group operations and isomorphic relation with the 2-tuple linguistic variables

In the present section, we give a comprehensive study to introduce the concept of 2TL groups based upon the binary operator in a crisp sense. To do this, we initially show that $[-n,n]$, LT, the interval $[\frac{-1}{2},\frac{1}{2}]$ forms a group under the binary operation '$*$', '$\oplus$' respectively and further, prove the assertion that the set of all 2TL variables, $LT\times[\frac{-1}{2},\frac{1}{2}]$ is isomorphic to the nonempty set of numerical interval $[-n,n]$, where n is a positive integer. Hence, both the groups are abstractly similar and have identical algebraic properties.

Proposition 2.2.1. Let $[-n,n]$ be any arbitrary nonempty set where n is presumed to be a sufficiently large positive integer. Then $[-n,n]$ is a group concerning binary operation '$*$' defined as:

$$a*b = \begin{cases} a+b+n, & \text{if } a+b<-n \\ a+b, & \text{if } -n\le a+b\le n \\ a+b-n, & \text{if } a+b>n \end{cases}$$

for all $a,b\in[-n,n]$.

Proof. Let a_p, $a_q\in X$ be any arbitrary element. Based on the binary operation '$*$', we infer clearly that, a_p, $a_q\in X$. Therefore, X satisfies the closure property. Now, let a_p, a_q, $a_r\in X$ and let $a_q*a_r=\mu_1$ such that $a_p*(a_q*a_r)=a_p*\mu_1=\rho_1$ therefore,

$$\rho_1 = \begin{cases} a_p+\mu_1+n, & \text{if } a_p+\mu_1<-n \\ a_p+\mu_1, & \text{if } -n\le a_p+\mu_1\le n \\ a_p+\mu_1-n, & \text{if } a_p+\mu_1>n \end{cases} \tag{2.2.1}$$

since $\mu_1=a_q*a_r$ i.e.

$$\mu_1 = \begin{cases} a_q+a_r+n, & \text{if } a_q+a_r<-n \\ a_q+a_r, & \text{if } -n\le a_q+a_r\le n \\ a_q+a_r-n, & \text{if } a_q+a_r>n \end{cases} \tag{2.2.2}$$

By substituting equation 2.2.2 in equation 2.2.1 we obtain following cases defined below:

Case(i) If $a_q + a_r < -n$ then, $\mu_1 = a_q + a_r + n$ therefore,

$$\rho_1 = \begin{cases} a_p + a_q + a_r + 2n, & \text{if } a_p + a_q + a_r < -2n \\ a_p + a_q + a_r + n, & \text{if } -2n \le a_p + a_q + a_r \le 0 \\ a_p + a_q + a_r, & \text{if } a_p + a_q + a_r > 0 \end{cases}$$

Case(ii) If $-n \le a_q + a_r \le n$ then, $\mu_1 = a_q + a_r$ therefore,

$$\rho_1 = \begin{cases} a_p + a_q + a_r + n, & \text{if } a_p + a_q + a_r < -n \\ a_p + a_q + a_r, & \text{if } -n \le a_p + a_q + a_r \le n \\ a_p + a_q + a_r - n, & \text{if } a_p + a_q + a_r > n \end{cases}$$

Case(iii) If $a_q + a_r > n$ then, $\mu_1 = a_q + a_r - n$ therefore,

$$\rho_1 = \begin{cases} a_p + a_q + a_r, & \text{if } a_p + a_q + a_r < 0 \\ a_p + a_q + a_r - n, & \text{if } 0 \le a_p + a_q + a_r \le 2n \\ a_p + a_q + a_r - 2n, & \text{if } a_p + a_q + a_r > 2n \end{cases}$$

Next, we let $a_p * a_q = \mu_2$ such that $(a_p * a_q) * a_r = \mu_2 * a_r = \rho_2$ therefore,

$$\rho_2 = \begin{cases} \mu_2 + a_r + n, & \text{if } \mu_2 + a_r < -n \\ \mu_2 + a_r, & \text{if } -n \le \mu_2 + a_r \le n \\ \mu + a_r - n, & \text{if } \mu_2 + a_r > n \end{cases} \tag{2.2.3}$$

Since $\mu_2 = a_p * a_q$ i.e.,

$$\mu_2 = \begin{cases} a_p + a_q + n, & \text{if } a_p + a_q < -n \\ a_p + a_q, & \text{if } -n \le a_p + a_q \le n \\ a_p + a_q - n, & \text{if } a_p + a_q > n \end{cases} \tag{2.2.4}$$

By substituting equation 2.2.4 in equation 2.2.3 we obtain the following cases which are defined below:

Case(i) If $a_p + a_q < -n$ then, $\mu_2 = a_p + a_q + n$ therefore,

$$\rho_2 = \begin{cases} a_p + a_q + a_r + 2n, & \text{if } a_p + a_q + a_r < -2n \\ a_p + a_q + a_r + n, & \text{if } -2n \le a_p + a_q + a_r \le 0 \\ a_p + a_q + a_r, & \text{if } a_p + a_q + a_r > 0 \end{cases}$$

Case(ii) If $-n \le a_p + a_q \le n$ then, $\mu_2 = a_p + a_q$ therefore,

$$\rho_2 = \begin{cases} a_p + a_q + a_r + n, & \text{if } a_p + a_q + a_r < -n \\ a_p + a_q + a_r, & \text{if } -n \le a_p + a_q + a_r \le n \\ a_p + a_q + a_r - n, & \text{if } a_p + a_q + a_r > n \end{cases}$$

Case(iii) If $a_p + a_q > n$ then, $\mu_2 = a_p + a_q - n$ therefore,

$$\rho_2 = \begin{cases} a_p + a_q + a_r, & \text{if } a_p + a_q + a_r < 0 \\ a_p + a_q + a_r - n, & \text{if } 0 \le a_p + a_q + a_r \le 2n \\ a_p + a_q + a_r - 2n, & \text{if } a_p + a_q + a_r > 2n \end{cases}$$

From all the above cases we infer that $\rho_1 = \rho_2 \Rightarrow a_p * (a_q * a_r) = (a_p * a_q) * a_r$. Hence, X satisfies associative property. Next, 0 is an identity element such that for every $a_p \in [-n, n]$ we have $a_p * 0 = 0 * a_p = a_p$. Finally, for every $a_p \in X = [-n, n]$ their exist $(-a_p) \in X = [-n, n]$ such that $a_p * (-a_p) = (-a_p) * a_p = 0$. Hence, $X = [-n, n]$ is a group. □

Proposition 2.2.2. Let $LT = \{t_{-n}, \ldots, t_0, t_1, \ldots, t_n\}$ be a finite SSLTS. Then LT is a group concerning binary operation '$*$' defined as:

$$t_p * t_q = t_{p*q} = \begin{cases} t_{p+q+n}, & \text{if } p + q < -n \\ t_{p+q}, & \text{if } -n \le p + q \le n \\ t_{p+q-n}, & \text{if } p + q > n \end{cases}$$

for all $t_p, t_q \in LT$.

Proof. We here give a detailed proof. Consider SSLTS $LT = \{t_{-n},\dots,t_0,t_1,\dots,t_n\}$ and let $t_p, t_q \in LT$ be any two arbitrary linguistic term. Based on a binary operation '$*$', we infer that, $t_p * t_q \in LT\ \forall\ t_p, t_q \in LT$. Hence, closure property holds. Next, let t_p, t_q, $t_r \in LT$ be a linguistic term. Let $t_q * t_r = t_{q*r} = \nu_1$ such that $t_p * (t_q * t_r) = t_p * \nu_1 = \tau_1$ therefore,

$$\begin{aligned}\tau_1 &= t_p * \nu_1\\ &= t_p * t_{q*r}\\ &= \begin{cases} t_{p+(q*r)+n}, & \text{if } p+(q*r) < -n\\ t_{p+(q*r)}, & \text{if } -n \le p+(q*r) \le n\\ t_{p+(q*r)-n}, & \text{if } p+(q*r) > n\end{cases}\end{aligned} \tag{2.2.5}$$

since $\nu_1 = t_q * t_r = t_{q*r}$ i.e.,

$$\nu_1 = \begin{cases} t_{q+r+n}, & \text{if } q+r < -n\\ t_{q+r}, & \text{if } -n \le q+r \le n\\ t_{q+r-n}, & \text{if } q+r > n\end{cases} \tag{2.2.6}$$

Substitute equation 2.2.6 in equation 2.2.5 to obtain the following cases defined below:

Case(i) If $q+r < -n$ then, $\nu_1 = t_{q+r+n}$ therefore,

$$\begin{aligned}\tau_1 &= t_p * \nu_1\\ &= t_p * t_{q*r}\\ &= \begin{cases} t_{p+q+r+2n}, & \text{if } p+q+r < -2n\\ t_{p+q+r+n}, & \text{if } -2n \le p+q+r \le 0\\ t_{p+q+r}, & \text{if } p+q+r > 0\end{cases}\end{aligned}$$

Case(ii) If $-n \le q+r \le n$ then, $\nu_1 = t_{q*r} = t_{q+r}$ therefore,

$$\begin{aligned}\tau_1 &= t_p * \nu_1\\ &= t_p * t_{q*r}\\ &= \begin{cases} t_{p+q+r+n}, & \text{if } p+q+r < -n\\ t_{p+q+r}, & \text{if } -n \le p+q+r \le n\\ t_{p+q+r-n}, & \text{if } p+q+r > n\end{cases}\end{aligned}$$

Case(iii) If $q+r>n$ then, $\nu_1 = t_{q*r} = t_{q+r-n}$ therefore,

$$\begin{aligned}\tau_1 &= t_p * \nu_1\\ &= t_p * t_{q*r}\\ &= \begin{cases} t_{p+q+r}, & \text{if } p+q+r<0\\ t_{p+q+r-n}, & \text{if } 0 \le p+q+r \le 2n\\ t_{p+q+r-2n}, & \text{if } p+q+r>2n \end{cases}\end{aligned}$$

Next, let $t_p * t_q = t_{p*q} = \nu_2$ such that $(t_p * t_q) * t_r = \nu_2 * t_r = \tau_2$ therefore,

$$\begin{aligned}\tau_2 &= \nu_2 * t_r\\ &= t_{p*q} * t_r\\ &= \begin{cases} t_{(p*q)+r+n}, & \text{if } (p*q)+r<-n\\ t_{(p*q)+r}, & \text{if } -n \le (p*q)+r \le n\\ t_{(p*q)+r-n}, & \text{if } (p*q)+r>n \end{cases}\end{aligned} \tag{2.2.7}$$

since $\nu_2 = t_p * t_q = t_{p*q}$ i.e.,

$$\nu_2 = \begin{cases} t_{p+q+n}, & \text{if } p+q<-n\\ t_{p+q}, & \text{if } -n \le p+q \le n\\ t_{p+q-n}, & \text{if } p+q>n \end{cases} \tag{2.2.8}$$

By substituting ν_2 from equation 2.2.8 in equation 2.2.7 we obtain the following cases defined below:

Case(i) If $p+q<-n$ then, $\nu_2 = t_{p+q+n}$ therefore,

$$\begin{aligned}\tau_2 &= \nu_2 * t_r\\ &= t_{p+q+n} * t_r\\ &= \begin{cases} t_{p+q+r+2n}, & \text{if } p+q+r<-2n\\ t_{p+q+r+n}, & \text{if } -2n \le p+q+r \le 0\\ t_{p+q+r}, & \text{if } p+q+r>0 \end{cases}\end{aligned}$$

Case(ii) If $-n \leq p+q \leq n$ then, $\nu_2 = t_{p*q} = t_{p+q}$ therefore,

$$\begin{aligned}\tau_2 &= \nu_2 * t_r \\ &= t_{p+q} * t_r \\ &= \begin{cases} t_{p+q+r+n}, & \text{if } p+q+r < -n \\ t_{p+q+r}, & \text{if } -n \leq p+q+r \leq n \\ t_{p+q+r-n}, & \text{if } p+q+r > n \end{cases}\end{aligned}$$

Case(iii) If $p+q > n$ then, $\nu_2 = t_{p*q} = t_{p+q-n}$ therefore,

$$\begin{aligned}\tau_2 &= \nu_2 * t_r \\ &= t_{p+q-n} * t_r \\ &= \begin{cases} t_{p+q+r}, & \text{if } p+q+r < 0 \\ t_{p+q+r-n}, & \text{if } 0 \leq p+q+r \leq 2n \\ t_{p+q+r-2n}, & \text{if } p+q+r > 2n \end{cases}\end{aligned}$$

From all the above cases we infer that $\tau_1 = \tau_2 \Rightarrow t_p * (t_q * t_r) = (t_p * t_q) * t_r$. Hence, LT satisfies associative property. Next, $t_0 \in LT$ is an identity element such that for every $t_p \in LT$ we have $t_p * t_0 = t_0 * t_p = t_p$. Finally, for every $t_p \in LT$ their exist $t_{-p} \in LT$ such that $t_p * t_{-p} = t_{-p} * t_p = 0$. Hence, LT is a group. □

Proposition 2.2.3. Let $H = [\frac{-1}{2}, \frac{1}{2}]$ be a closed and bounded interval. Then H is a group concerning a binary operation '⊕' given by.

$$\alpha_p \oplus \alpha_q = \begin{cases} \alpha_p + \alpha_q, & \text{if} \alpha_p + \alpha_q \in \left(\frac{-1}{2}, \frac{1}{2}\right) \\ \alpha_p + \alpha_q - \text{round}(\alpha_p + \alpha_q), & \text{else} \end{cases}$$

Proof. Let $H = [\frac{-1}{2}, \frac{1}{2}] = \left\{\alpha \mid \frac{-1}{2} \leq \alpha \leq \frac{1}{2}\right\}$ be a nonempty set. Let $\alpha_p,\ \alpha_q \in H$, then followed by the definition clearly, $\alpha_p \oplus \alpha_q \in H$. Next, let $\alpha_p, \alpha_q, \alpha_r \in H$. Consider $\alpha_q \oplus \alpha_r = \mu$

such that

$$\alpha_p \oplus (\alpha_q \oplus \alpha_r) = \alpha_p \oplus \mu = \begin{cases} \alpha_p + \mu, & \text{if } \alpha_p + \mu \in \left(\frac{-1}{2}, \frac{1}{2}\right) \\ \alpha_p + \mu - \text{round}(\alpha_p + \mu), & \text{else} \end{cases} \tag{2.2.9}$$

since,

$$\mu = \alpha_q \oplus \alpha_r = \begin{cases} \alpha_q + \alpha_r, & \text{if } \alpha_q + \alpha_r \in \left(\frac{-1}{2}, \frac{1}{2}\right) \\ \alpha_q + \alpha_r - \text{round}(\alpha_q + \alpha_r), & \text{else} \end{cases}$$

By substituting value of μ in equation 2.2.9 we obtain the following cases define below:

Case(i) If $\alpha_q + \alpha_r \in (\frac{-1}{2}, \frac{1}{2})$ then, $\mu = \alpha_q + \alpha_r$ therefore,

$$\begin{aligned} \alpha_p \oplus (\alpha_q \oplus \alpha_r) &= \alpha_p \oplus \mu \\ &= \begin{cases} \alpha_p + \mu, & \text{if } \alpha_p + \mu \in (\frac{-1}{2}, \frac{1}{2}) \\ \alpha_p + \mu - \text{round}(\alpha_p + \mu), & \text{else} \end{cases} \\ &= \begin{cases} \alpha_p + \alpha_q + \alpha_r, & \text{if} \alpha_p + \alpha_q + \alpha_r \in \left(\frac{-1}{2}, \frac{1}{2}\right) \\ \alpha_p + \alpha_q + \alpha_r - \text{round}(\alpha_p + \alpha_q + \alpha_r), & \text{else} \end{cases} \end{aligned}$$

Case(ii) If $\mu \notin \left(\frac{-1}{2}, \frac{1}{2}\right)$ then, $\mu = \alpha_q + \alpha_r - \text{round}(\alpha_q + \alpha_r)$ therefore,

$$\begin{aligned} \alpha_p \oplus (\alpha_q \oplus \alpha_r) &= \alpha_p \oplus \mu \\ &= \begin{cases} \alpha_p + \mu, & \text{if } \alpha_p + \mu \in \left(\frac{-1}{2}, \frac{1}{2}\right) \\ \alpha_p + \mu - \text{round}(\alpha_p + \mu), & \text{else} \end{cases} \\ &= \begin{cases} \alpha_p + \alpha_q + \alpha_r - \text{round}(\alpha_q + \alpha_r), & \text{if } \alpha_p + \alpha_q + \alpha_r \\ & - \text{round}(\alpha_q + \alpha_r) \in (\frac{-1}{2}, \frac{1}{2}) \\ \alpha_p + \alpha_q + \alpha_r - \text{round}(\alpha_q + \alpha_r) - & \\ \quad \text{round}\left(\alpha_p + \alpha_q + \alpha_r - \text{round}(\alpha_q + \alpha_r)\right), & \text{else} \end{cases} \end{aligned}$$

Similarly, we can have $(\alpha_p \oplus \alpha_q) \oplus \alpha_r$ and $\alpha_p \oplus (\alpha_q \oplus \alpha_r) = (\alpha_p \oplus \alpha_q) \oplus \alpha_r$ and thus associative property holds. Next, $0 \in [\frac{-1}{2}, \frac{1}{2}]$ is an identity element and finally for every $\alpha_p \in [\frac{-1}{2}, \frac{1}{2}]$ their exist $(-\alpha_p) \in [\frac{-1}{2}, \frac{1}{2}]$ such that $\alpha_p \oplus (-\alpha_p) = (-\alpha_p) \oplus \alpha_p = 0$. Hence, H is a group under a binary operation '$\oplus$'. □

It is known that the direct product of groups is itself forms a group [99]. Therefore, in this direction, the underlying theorem gives an elementary proof of the 2TL term set to be a group.

Theorem 2.2.1. Let $LT = \{t_{-n}, \ldots, t_0, t_1, \ldots, t_n\}$ be the SSLTS with odd cardinality $2n+1$ and let $\alpha_p \in [\frac{-1}{2}, \frac{1}{2}]$ be the numeric value representing the value of symbolic translation corresponding a linguistic term t_p. Then $\overline{LT} \equiv LT \times [\frac{-1}{2}, \frac{1}{2}]$ is a direct group under a binary operation '$\circ$' defined as:

$$(t_p, \alpha_p) \circ (t_q, \alpha_q) = (t_p \circ t_q, \alpha_p \circ \alpha_q)$$

such that $t_p \circ t_q = t_p * t_q$ and $\alpha_p \circ \alpha_q = \alpha_p \oplus \alpha_q$
where $t_p, t_q \in LT$ and $\alpha_p, \alpha_q \in [\frac{-1}{2}, \frac{1}{2}]$.

Proof. Let $\overline{LT} = \left\{(t_p, \alpha_p) \,|\, t_p \in LT \text{ and } \alpha_p \in [\frac{-1}{2}, \frac{1}{2}]\right\}$ be a nonempty set of 2TL terms. From the definition $(t_p, \alpha_p) \circ (t_q, \alpha_q) = (t_p \circ t_q, \alpha_p \circ \alpha_q) = (t_p * t_q, \alpha_p \oplus \alpha_q) \in \overline{LT} \, \forall \, (t_p, \alpha_p), (t_q, \alpha_q) \in \overline{LT}$, so closure property holds. Next associative property is followed by using proposition 2.2.2 and 2.2.3 such that $(t_p, \alpha_p) \circ \big((t_q, \alpha_q) \circ (t_r, \alpha_r)\big) = \big((t_p, \alpha_p) \circ (t_q, \alpha_q)\big) \circ (t_r, \alpha_r)$. Next $(t_0, 0) \in \overline{LT}$ is an identity element and finally for every $(t_p, \alpha_p) \in \overline{LT}$ their exist $(t_{-i}, (-\alpha_p)) \in \overline{LT}$ such that

$$(t_p, \alpha_p) \circ (t_{-p}, (-\alpha_p)) = (t_{-p}, (-\alpha_p)) \circ (t_p, \alpha_p) = (t_0, 0)$$

Hence $\overline{LT}$ is a group. □

Next, we present the main result of this section.

Theorem 2.2.2. Let $LT = \{t_{-n}, \ldots, t_0, \ldots, t_{-n}\}$ be the finite SSLTS and let $x \in [-n, n]$ a numeric value supporting the outcome of an operation of symbolic aggregation. Then a 2TL direct group, $\overline{LT}$ and $[-n, n]$ are isomorphic.

Proof. Consider the mapping,

$$\phi : [-n, n] \longrightarrow \overline{LT} \text{ defined by}$$

$$\phi(x_p) = (t_p, \alpha_p)$$

with,

$$\begin{cases} t_p, & p = \text{round}(x_p), \\ \alpha_p = x_p - p, & \alpha_p \in [\frac{-1}{2}, \frac{1}{2}]. \end{cases}$$

Let $x_p, x_q \in [-n, n]$ such that $\phi(x_p)$=$\phi(x_q) \Leftrightarrow (t_p, \alpha_p)$=$(t_q, \alpha_q) \Leftrightarrow (t_p, x_p - p)$=$(t_q, x_q - q)$ $\Leftrightarrow t_p = t_q, x_p - p = x_q - q \Leftrightarrow p = q,\ x_p - p = x_q - q \Leftrightarrow x_p = x_q$. Therefore, ϕ is well-defined and one-one. ϕ is a onto function. This is easy to see that for every $(t_p, \alpha_p) \in \overline{LT}$ their exist $x_p \in [-n, n]$ such that,

$$\phi(x_p) = (t_p, \alpha_p) \text{ where } \begin{cases} t_p, & p = \text{round}(x_p), \\ \alpha_p = x_p - p, & \alpha_p \in [\frac{-1}{2}, \frac{1}{2}]. \end{cases}$$

Next, we proceed to show ϕ is homomorphic. Let $x_p, x_q \in [-n, n]$ such that,

$$\phi(x_p) = (t_p, \alpha_p), \text{ with } \begin{cases} t_p, & p = \text{round}(x_p), \\ \alpha_p = x_p - p, & \alpha_p \in [\frac{-1}{2}, \frac{1}{2}]. \end{cases}$$

$$\phi(x_q) = (t_q, \alpha_q), \text{ with } \begin{cases} t_q, & q = \text{round}(x_q), \\ \alpha_q = x_q - q, & \alpha_q \in [\frac{-1}{2}, \frac{1}{2}]. \end{cases}$$

$$\phi(x_p * x_q) = (t_r, \alpha_r), \text{ with } \begin{cases} t_r, & r = \text{round}(x_p * x_q) \\ \alpha_r = (x_p * x_q) - r, & \alpha_r \in [\frac{-1}{2}, \frac{1}{2}]. \end{cases}$$

We claim $\phi(x_p * x_q) = \phi(x_p) \circ \phi(x_q)$, i.e., to show $t_r = t_p * t_q$ and $\alpha_r = \alpha_p \oplus \alpha_q$, via algorithm.

The algorithm 2.2.1-2.2.4 defines a decision rule to represent the homomorphic relation between the groups, considering all the possible cases such that

$$\begin{aligned} \phi(x_p * x_q) &= (t_r, \alpha_r) \\ &= (t_p * t_q, \alpha_p \oplus \alpha_q) \\ &= (t_p \circ t_q, \alpha_p \circ \alpha_q) \\ &= (t_p, \alpha_p) \circ (t_q, \alpha_q) \\ &= \phi(x_p) \circ \phi(x_q). \end{aligned}$$

Hence, ϕ is an isomorphism.

Case (i)

Algorithm 2.2.1. Require: $p+q<-n$, then, followed by the definition we have, $t_p * t_q = t_{p+q+n}$.

if $p+q+n=r$ **then**
  $t_r = t_{p+q+n}$,
  $\alpha_r = \alpha_p \oplus \alpha_q$.
else
  t_r is represented as t_m, where $m \leftarrow r-1$,
  if $m \in (-n,n)$ **then**
    $t_m = t_{p+q+n}$,
    $\alpha_r = \alpha_p \oplus \alpha_q$.
  end if
  if $m \leq -n$ **then**
    $m = r-1+n$ such that $t_m = t_{p+q+n} = t_r$,
    $\alpha_r = \alpha_p \oplus \alpha_q$.
  end if
end if

Case (ii)

Algorithm 2.2.2. Require: $-n<p+q<n$, then, followed by the definition we have $t_p * t_q = t_{p+q}$.

if $r=p+q$ **then**
  $t_r = t_{p+q}$,
  $\alpha_r = \alpha_p \oplus \alpha_q$.
else
  t_r is represented as t_m, where $m \leftarrow r+1$ or $m \leftarrow r-1$ such that
  $t_r = t_m$,
  $\alpha_r = \alpha_p \oplus \alpha_q$.
end if

Case (iii)

Algorithm 2.2.3. Require: $p+q>n$, then, followed by the definition we have $t_p * t_q = t_{p+q-n}$.

if $p+q-n=r$ **then**
  $t_r = t_{p+q-n}$,
  $\alpha_r = \alpha_p \oplus \alpha_q$.

else
  t_r is represented as t_m, where $m \leftarrow r+1$,
  if $m \in (-n, n)$ **then**
    $t_m = t_{p+q-n}$,
    $\alpha_r = \alpha_p \oplus \alpha_q$.
  end if
  if $m \geq n$ **then**
    $m = r+1-n$ such that $t_m = t_{p+q-n} = t_r$,
    $\alpha_r = \alpha_p \oplus \alpha_q$.
  end if
end if

Case (iv)

Algorithm 2.2.4. **if** $p+q=-n$ or $p+q=n$ **then**
  $r \leftarrow n$ or $-n$ such that $t_r = t_{p+q}$ or $t_r = t_{-(p+q)}$,
  $\alpha_r = \alpha_p \oplus \alpha_q$.
end if

□

It is requisite to view that the aforementioned algorithm guarantees that the mapping $x \longrightarrow (t, \alpha)$ is an operation preserving map under a defined binary operation. Moreover, the time complexity of the proposed algorithm is linear, which has a practical advantage in laying the foundations for 2TL group homomorphism in several real-life problems.

In particular, if there exists an isomorphism of a group $\overline{LT}$ to itself, i.e., $\overline{LT} \longrightarrow \overline{LT}$ then it is termed as an automorphism of a 2TL group.

We now consider the following illustration to understand the working of the algorithm as mentioned above and further combining the result to indicate that the mapping $x \longrightarrow (t, \alpha)$ is an isomorphism defined from $[-n, n]$ to $\overline{LT}$ under binary operation previously stated.

Example 2.2.1. Consider a mapping $\phi : [-3, 3] \longrightarrow \overline{LT}$ such that $\phi(x_p) = (t_p, \alpha_p)$ where $t_p \in \{t_{-3}, t_{-2}, t_{-1}, t_0, t_1, t_2, t_3\}$ be the SSLTS and $\alpha_p \in \left[\frac{-1}{2}, \frac{1}{2}\right]$. Clearly, the defined mapping is bijective. Next, in order to establish isomorphism between the groups we need to preserve the group operation. In this direction, we shall claim $\phi(x_p * x_q) = \phi(x_p) \circ \phi(x_q)$, i.e., to show $t_r = t_p * t_q$ and $\alpha_r = \alpha_p \oplus \alpha_q$.

For this, we use algorithm 2.2.1-2.2.4. Since the order of both the groups are infinite, therefore, we have constructed a MATLAB code based on the aforementioned algorithm to generate numbers randomly to perform further routine computations. We present in Table 2.1 the list of cases to accomplish the result.

Table 2.1: Computations for randomly generated numbers by using Algorithm 2.2.1-2.2.4.

x_p	x_q	$\phi(x_p * x_q)$	$\phi(x_p) \circ \phi(x_q)$
1.8	2.8	$(t_2, -0.4)$	$(t_2, -0.2) \circ (t_3, -0.2) = (t_2, -0.4)$
0.8	−2.4	$(t_{-1}, 0.4)$	$(t_1, -0.2) \circ (t_{-2}, -0.4) = (t_{-1}, 0.4)$
−2.8	−2.4	$(t_{-2}, -0.2)$	$(t_{-3}, 0.2) \circ (t_{-2}, -0.4) =$ $(t_{-2}, -0.2)$
−0.4	2.5	$(t_3, 0.1)$	$(t_0, -0.4) \circ (t_3, -0.5) = (t_3, 0.1)$
2.0	−1.5	$(t_0, 0.5)$	$(t_2, 0) \circ (t_{-2}, 0.5) = (t_0, 0.5)$
1.5	1.5	$(t_1, 0)$	$(t_2, -0.5) \circ (t_2, -0.5) = (t_1, 0)$
−2.4	1.9	$(t_0, -0.5)$	$(t_{-2}, -0.4) \circ (t_2, -0.1) = (t_0, -0.5)$

From table 2.1, we can infer that the mapping is group operation preserving, i.e., $\phi(x_p * x_q) = \phi(x_p) \circ \phi(x_q)$ and therefore, $[-3, 3]$ is isomorphic to $\overline{LT}$.

Remark 2.2.1. It can be noted here that in order to simplify the calculations and understand the result described above, we have taken $n = 3$. However, the procedure is analogous for any integral value of n.

2.2.1 Application of 2-tuple linguistic group in bipolar graphs

In literature, Zhang [343, 344] presented the notion of bipolar fuzzy sets (BFSs). The extent of membership degree for a BFS is depicted as falling inside the range $[-1, , 1]$, which is regarded an extrapolation of FSs. Here, the element with 0 membership degree is considered insignificant to the associated property. In contrast, the elements whose membership degree lying in the range $(0, 1]$ represent the element fulfills the property, unlike the elements whose degree of membership lying within the range $[-1, 0)$ represent the element is somewhat fulfilling the implicit counter-property. Nevertheless, bipolar information plays a significant role in a wide-ranging multiagent decision analysis where both-sided judgemental thinking is involved and therefore, this domain has pervasively inspired several researchers across the globe. Recently, Akram [10–12] has put forward the impression of bipolar fuzzy graphs to amplify the purview of BFSs within the construction of a structure of fuzzy graphs.

The amalgamation of linguistic information with the existing notion of FSs has allegedly played a pivotal role in handling the uncertainty persisting within the problem. Therefore, inspired by the work of Akram [10] in the present section, we bring forth the foundation of

2-tuple linguistic bipolar set (2TLBS) and, after that, discuses the basic algebra on the set. We later make headway to give the application of 2TL groups by developing the notion of 2-tuple linguistic bipolar graphs (2TLBGs) and further establishing the graph isomorphic relation.

The 2-tuple linguistic bipolar set

It is well known that the popularly known 2TL representation is ideally introduced by Herrera and Martínez [106] for the LTS $\mathbf{LT} = \{t_p \,|\, p = 0, 1, \ldots, n\}$ primarily based upon the symbolic translation. Since the traditional 2-tuple model is capable of effectively handling linguistic information without any loss. As a consequence, we will first introduce the notion of 2TLBS based on the 2-tuple model using predefined SSLTS $LT = \{t_p \,|\, p = -n, \ldots, 0, \ldots, n\}$ followed by some basic operations on the set which are required to establish the conception of 2TLBG given in the subsequent part of the section.

Definition 2.2.1. Let $X = [-n, n]$ be any arbitrary nonempty set and let $LT = \{ t_p \,|\, p = -n, \ldots, 0, \ldots, n \}$ be a SSLTS. Then, we define 2TLBS on X as follows:

$$\overline{\mathbf{C}} = \{(x, \phi^+(|x|), \phi^-(-|x|)) \,|\, x \in X\}$$

with $\phi^+ : X \to LT^+ \times \left[\frac{-1}{2}, \frac{1}{2}\right)$ such that $\phi^+(x) = (t_{p^+}, \alpha)$ where $t_{p^+} \in LT^+ = \{t_p \,|\, p = 0, 1, \ldots, n\}$ and $\phi^- : X \to LT^- \times \left(\frac{-1}{2}, \frac{1}{2}\right]$ such that $\phi^-(x) = (t_{p^-}, \alpha)$ where $t_{p^-} \in LT^- = \{t_p \,|\, p = -n, \ldots, -1, 0\}$.

Here, t_{p^+} represents one of the positive linguistic terms from the set LT^+ whereas t_{p^-} represents the counter linguistic term belonging to the LT^- and α indicates a numerical value that captures the uncertainty about a corresponding linguistic term. Consequently, both $\phi^+(x)$ and $\phi^-(x)$ represents the positive and negative 2-tuple representation based on the linguistic term respectively.

For notational simplicity, we use $\overline{\mathbf{C}} = (\phi^+, \phi^-)$ to refer 2TLBS.

Example 2.2.2. Let $X = [-2, 2]$ and let $LT = \{ t_{-2} =$ "Very poor (VP)", $t_{-1} =$ "Poor (P)", $t_0 =$ "Fair (F)", $t_1 =$ "Good (G)", $t_2 =$ "Very Good (VG)" $\}$ be predefined SSLTS. Then,

$$\begin{aligned}\overline{\mathbf{C}} &= \{(-1.2, \phi^+(1.2), \phi^-(-1.2)), (-0.5, \phi^+(0.5), \phi^-(-0.5)), (1.8, \phi^+(1.8), \phi^-(-1.8))\} \\ &= \{(-1.2, (G, 0.2), (P, -0.2)), (-0.5, (G, -0.5), (P, 0.5)), (1.8, (VG, -0.2), (VP, 0.2))\}\end{aligned}$$

is the 2TLBS for given set X.

Definition 2.2.2. Suppose $\overline{\mathbf{C}} = (\phi^+_{\overline{\mathbf{C}}}, \phi^-_{\overline{\mathbf{C}}})$, $\overline{\mathbf{D}} = (\phi^+_{\overline{\mathbf{D}}}, \phi^-_{\overline{\mathbf{D}}})$ are any two 2TLBS defined on set X. Then the following operations for set $\overline{\mathbf{C}}$ and $\overline{\mathbf{D}}$ holds:

1. $\big(\overline{\mathbf{C}} \cup \overline{\mathbf{D}}\big)(x) = \big(\phi^+_{\overline{\mathbf{C}}\cup\overline{\mathbf{D}}}, \phi^-_{\overline{\mathbf{C}}\cup\overline{\mathbf{D}}}\big)$ where $\phi^+_{\overline{\mathbf{C}}\cup\overline{\mathbf{D}}} = \max(\phi^+_{\overline{\mathbf{C}}}, \phi^+_{\overline{\mathbf{D}}})$ and $\phi^-_{\overline{\mathbf{C}}\cup\overline{\mathbf{D}}} = \min(\phi^-_{\overline{\mathbf{C}}}, \phi^-_{\overline{\mathbf{D}}})$.
2. $\big(\overline{\mathbf{C}} \cap \overline{\mathbf{D}}\big)(x) = (\phi^+_{\overline{\mathbf{C}}\cap\overline{\mathbf{D}}}, \phi^-_{\overline{\mathbf{C}}\cap\overline{\mathbf{D}}})$ where $\phi^+_{\overline{\mathbf{C}}\cap\overline{\mathbf{D}}} = \min(\phi^+_{\overline{\mathbf{C}}}, \phi^+_{\overline{\mathbf{D}}})$ and $\phi^-_{\overline{\mathbf{C}}\cap\overline{\mathbf{D}}} = \max(\phi^-_{\overline{\mathbf{C}}}, \phi^-_{\overline{\mathbf{D}}})$.

It is worth noting that the necessary operational laws defined in the above definition are valid as the comparison of two 2TL variables is plausible. However, the other set operations can be easily deduced for 2TLBSs by combining the operations existing for the BFSs and the operations of 2TL variables.

Next, based on the result proved in the former section that the 2TL set $\overline{\mathrm{LT}} \equiv \mathrm{LT} \times [\frac{-1}{2}, \frac{1}{2}]$ is a direct group under a binary operation '$\circ$' we proceed further to review the following definition required for this section.

Definition 2.2.3. Assume X to be a nonempty set and $\overline{\mathbf{C}} = (\phi^+_{\overline{\mathbf{C}}}, \phi^-_{\overline{\mathbf{C}}})$ be any 2TLBS. Then, a mapping $\overline{\mathbf{C}} = (\phi^+_{\overline{\mathbf{C}}}, \phi^-_{\overline{\mathbf{C}}}) : X \times X \to \overline{LT} \times \overline{LT}$ is known as 2TL bipolar relation on set X such that $\phi^+_{\overline{\mathbf{C}}} \in LT^+ \times [\frac{-1}{2}, \frac{1}{2})$ and $\phi^-_{\overline{\mathbf{C}}} \in LT^- \times (\frac{-1}{2}, \frac{1}{2}]$.

Definition 2.2.4. *Let X be any arbitrary set and let $\overline{\mathbf{C}} = (\phi^+_{\overline{\mathbf{C}}}, \phi^-_{\overline{\mathbf{C}}})$ and $\overline{\mathbf{D}} = (\phi^+_{\overline{\mathbf{D}}}, \phi^-_{\overline{\mathbf{D}}})$ be any two 2TLBSs on set X. If $\overline{\mathbf{C}} = (\phi^+_{\overline{\mathbf{C}}}, \phi^-_{\overline{\mathbf{C}}})$ is a 2TL bipolar relation. Then, $\overline{\mathbf{C}}$ can be considered as a 2TL bipolar relation on $\overline{\mathbf{D}}$ satisfying the subsequent condition:*

$$\phi^+_{\overline{\mathbf{C}}}(c, d) \leq \min(\phi^+_{\overline{\mathbf{D}}}(c), \phi^+_{\overline{\mathbf{D}}}(d))$$

and

$$\phi^-_{\overline{\mathbf{C}}}(c, d) \geq \max(\phi^-_{\overline{\mathbf{D}}}(c), \phi^-_{\overline{\mathbf{D}}}(d)) \qquad \textit{for all} \quad c, d \in X.$$

Note 2.2.1. We call 2TL bipolar relation $\overline{\mathbf{C}} = (\phi^+_{\overline{\mathbf{C}}}, \phi^-_{\overline{\mathbf{C}}})$ on set X as symmetric if $\phi^+_{\overline{\mathbf{C}}}(c, d) = \phi^+_{\overline{\mathbf{C}}}(d, c)$ and $\phi^-_{\overline{\mathbf{C}}}(c, d) = \phi^-_{\overline{\mathbf{C}}}(d, c)$ for all $c, d \in X$.

The 2-tuple linguistic bipolar graphs

With the above concept of 2TLBS, we herein propose 2TLBGs and further consider an application of group operations and isomorphic relation already proved for the linguistic term set of 2-tuples to 2TLBGs by developing an isomorphic relation between the graphs.

In the classical graph theory [123], a graph is considered as an ordered pair $T = (V, E)$ where V and E represents a set of vertices and edges of T, respectively. Analogously, we define the 2TLBGs based on SSLTS.

It is noteworthy to take $\mathbf{T}^*$ as a notation for 2TLBG, and T is a notation for crisp graphs throughout this section.

Definition 2.2.5. Let $\overline{\mathbf{C}} = (\phi^+_{\overline{\mathbf{C}}}, \phi^-_{\overline{\mathbf{C}}})$ and $\overline{\mathbf{D}} = (\phi^+_{\overline{\mathbf{D}}}, \phi^-_{\overline{\mathbf{D}}})$ be any two 2TLBSs on set V and $E \subseteq V \times V$, respectively. Then, we define 2TLBG as $\mathbf{T}^* = (\overline{\mathbf{C}}, \overline{\mathbf{D}})$ such that $\phi^+_{\overline{\mathbf{D}}}(\{c, d\}) \leq \min(\phi^+_{\overline{\mathbf{C}}}(c), \phi^+_{\overline{\mathbf{C}}}(d))$ and $\phi^-_{\overline{\mathbf{D}}}(\{c, d\}) \geq \max(\phi^-_{\overline{\mathbf{C}}}(c), \phi^-_{\overline{\mathbf{C}}}(d))$ for all $\{c, d\} \in E$.

Here, we represent $\overline{\mathbf{C}}$ as 2TL bipolar vertex set of V, $\overline{\mathbf{D}}$ as 2TL bipolar edge set E, respectively. Also, we use cd as the notation for an element of edge set E, i.e., $\{c, d\} \equiv cd$.

Example 2.2.3. *Consider a crisp graph $T = (V, E)$ such that $V = \{u, v, w\}$ and $E = \{uv, vw, wu\}$. Suppose $X = [-3, 3]$ and $LT = \{t_p \,|\, i = -3, -2, -1, 0, 1, 2, 3\}$ be predefined SSLTS. Let $\overline{\mathbf{C}} = (\phi^+_{\overline{\mathbf{C}}}, \phi^-_{\overline{\mathbf{C}}})$ and $\overline{\mathbf{D}} = (\phi^+_{\overline{\mathbf{D}}}, \phi^-_{\overline{\mathbf{D}}})$ be any two 2TLBSs defined by*

	u	v	w
$\phi^+_{\overline{\mathbf{C}}}$	$(t_2, 0.4)$	$(t_2, -0.5)$	$(t_0, 0.3)$
$\phi^-_{\overline{\mathbf{C}}}$	$(t_{-2}, -0.4)$	$(t_{-2}, 0.5)$	$(t_0, -0.3)$
	uv	vw	wu
$\phi^+_{\overline{\mathbf{D}}}$	$(t_1, 0.4)$	$(t_0, 0.1)$	$(t_0, 0.2)$
$\phi^-_{\overline{\mathbf{D}}}$	$(t_{-1}, -0.4)$	$(t_0, -0.1)$	$(t_0, -0.2)$

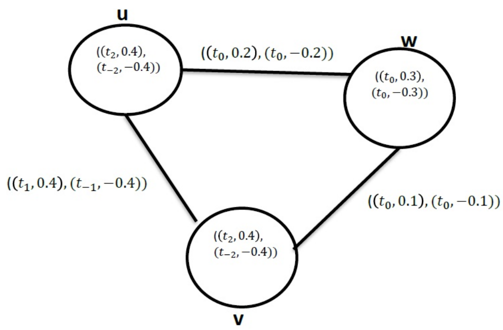


Figure 2.1: A 2-tuple linguistic bipolar graph $\mathbf{G}^*$

From routine calculations, we infer that $\mathbf{T}^* = (\overline{\mathbf{C}}, \overline{\mathbf{D}})$ is a 2TLBG of T (as given in Fig. 2.1).

In the literature of classical graph theory [123], graphs T_1 and T_2 are said to be isomorphic if there is a one-one and onto mapping between the sets of vertices of T_1 and T_2 such that any two arbitrary vertices v_1 and v_2 are adjoined in T_1 if and only if corresponding vertex $f(v_1)$ and $f(v_2)$ in graph T_2 are adjoined. If such an order-preserving bijective map exists between the two graphs, then we call the corresponding graph to be isomorphic, i.e., $T_1 \cong T_2$. On the other hand, graph automorphism is defined as an isomorphism of the graph itself, i.e., a well-defined mapping defined from the vertices of the given graph T to the vertices of T so that the following graph T is isomorphic with itself.

Based on the notion of isomorphism of crisp graphs, we shall now present the following definition to define an isomorphism of 2TLBGs considered to be a direct application of a 2TL group isomorphism.

Definition 2.2.6. *Let* $\mathbf{T_1}^* = (\overline{\mathbf{C_1}}, \overline{\mathbf{D_1}})$ *and* $\mathbf{T_2}^* = (\overline{\mathbf{C_2}}, \overline{\mathbf{D_2}})$ *be any two 2TLBGs. A homomorphism* $f : \mathbf{T_1}^* \to \mathbf{T_2}^*$ *is a mapping* $f : V_1 \to V_2$ *satisfying the subsequent conditions:*

1. $\phi^+_{\overline{\mathbf{C_1}}}(c_1) \leq \phi^+_{\overline{\mathbf{C_2}}}(f(c_1)), \phi^-_{\overline{\mathbf{C_1}}}(c_1) \geq \phi^-_{\overline{\mathbf{C_2}}}(f(c_1))$;
2. $\phi^+_{\overline{\mathbf{D_1}}}(c_1 d_1) \leq \phi^+_{\overline{\mathbf{D_2}}}(f(c_1)f(d_1))$ *and* $\phi^-_{\overline{\mathbf{D_1}}}(c_1 d_1) \geq \phi^-_{\overline{\mathbf{D_2}}}(f(c_1)f(d_1))$.

for all $c_1 \in V_1, c_1 d_1 \in E_1$.

Definition 2.2.7. *Let* $\mathbf{T_1}^* = (\overline{\mathbf{C_1}}, \overline{\mathbf{D_1}})$ *and* $\mathbf{T_2}^* = (\overline{\mathbf{C_2}}, \overline{\mathbf{D_2}})$ *be any two 2TLBGs. An isomorphism* $f : \mathbf{T_1}^* \to \mathbf{T_2}^*$ *is a bijective map* $f : V_1 \to V_2$ *satisfying the following conditions:*

1. $\phi^+_{\overline{\mathbf{C_1}}}(c_1) = \phi^+_{\overline{\mathbf{C_2}}}(f(c_1)), \phi^-_{\overline{\mathbf{C_1}}}(c_1) = \phi^-_{\overline{\mathbf{C_2}}}(f(c_1))$;
2. $\phi^+_{\overline{\mathbf{D_1}}}(c_1 d_1) = \phi^+_{\overline{\mathbf{D_2}}}(f(c_1)f(d_1))$ *and* $\phi^-_{\overline{\mathbf{D_1}}}(c_1 d_1) = \phi^-_{\overline{\mathbf{D_2}}}(f(c_1)f(d_1))$.

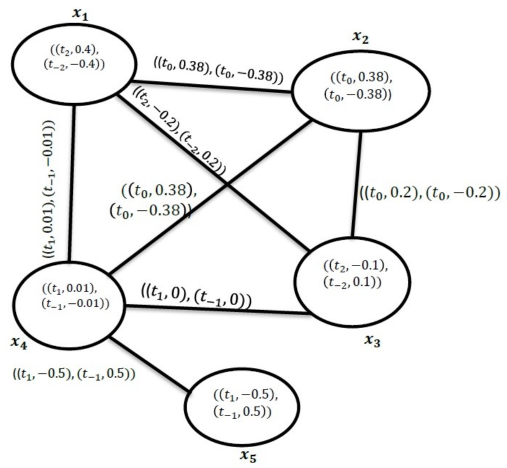
for all $c_1 \in V_1, c_1 d_1 \in E_1$.

We can observe that the above mappings are well-defined since the 2TL set forms a group under a given binary operation, and also, ϕ is an isomorphism. Consequently, the graphical structures of an isomorphic groups are similar.

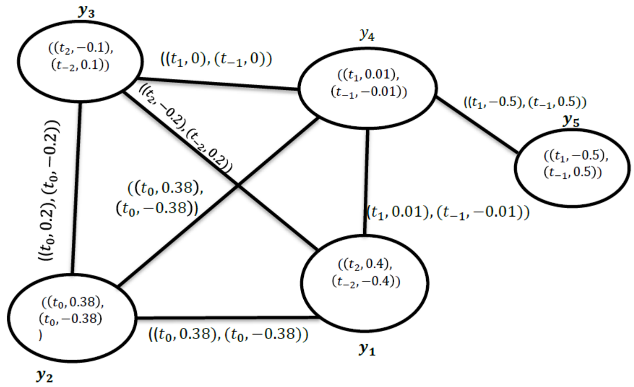
Example 2.2.4. Suppose $X = [-3, 3]$ and $LT = \{t_p \,|\, p = -3, -2, -1, 0, 1, 2, 3\}$ be predefined SSLTS. Consider 2TLBGs $\mathbf{T_1}^*$ and $\mathbf{T_2}^*$ (as shown in Fig. 2.2). Therefore, by making routine computations we can observe that the two graphs are isomorphic, i.e., $\mathbf{T_1}^* \cong \mathbf{T_2}^*$.

2.3 Group isomorphic properties with some novel operational laws for 2-tuple linguistic variables

Definition 2.3.1. The linguistic kernel of a homomorphism $\phi : [-n, n] \to \overline{LT}$ is defined as $Ker(\phi) = \{\beta \in [-n, n] \,|\, \phi(\beta) = (t_0, 0)\}$.



(a) $\mathbf{T_1}^*$



(b) $\mathbf{T_2}^*$

Figure 2.2: A 2-tuple linguistic bipolar graphs $\mathbf{T_1}^*$, $\mathbf{T_2}^*$

The following proposition is beneficial in the sense of knowing the properties of elements under homomorphism.

Proposition 2.3.1. Let $\phi : [-n, n] \to \overline{LT}$ be a 2TL group homomorphism. Then,

(i) $\phi(0) = (t_0, 0)$;

(ii) $\phi(\text{neg}(\beta)) = \text{neg}(\phi(\beta))$;

(iii) $\phi^{(-1)}((t_c, \alpha_c)) = \{\beta \in [-n, n] \mid \phi(\beta) = (t_c, \alpha_c)\} = \beta * Ker(\phi)$.

Proof. Consider the mapping,

$$\phi : [-n, n] \to \overline{LT} \text{ defined by}$$

$$\phi(x_c) = (t_c, \alpha_c)$$

For the sake of convenience, we have $e = 0 \in [-n, n]$ and $\tilde{e} = (t_0, 0) \in \overline{LT}$. Then, we can write $0 = 0 * 0$ such that $\phi(0) = \phi(0 * 0) = \phi(0) \circ \phi(0)$. But $\phi(0) \in \overline{LT} \implies \phi(0) = (t_0, 0)$. Using cancellation law, $\phi(0) = (t_0, 0)$. Next, we need to prove that $\phi(\text{neg}(\beta)) = \text{neg}(\phi(\beta))$. So, for any $\beta \in [-n, n]$ observe that $\phi(\beta) * \phi(\text{neg}(\beta)) = \phi(\beta * (\text{neg}(\beta)) = \phi(\beta * \text{neg}(\beta)) = \phi(0) = (t_0, 0)$ implies that $\phi(\beta) \circ \phi(\text{neg}(\beta)) = (t_0, 0) \equiv 0$ therefore, $\phi(\text{neg}(\beta)) = \text{neg}(\phi(\beta))$. Finally, we need to show that $\phi^{(-1)}((t_c, \alpha_c)) = \{\beta \in [-n, n] \mid \phi(\beta) = (t_c, \alpha_c)\} = \beta \oplus Ker(\phi)$. Consider $x \in \phi^{(-1)}(t_c, \alpha_c) \implies \phi(x) = (t_c, \alpha_c) = \phi(\beta)$. Now, by taking $\phi(\beta)$ to the other side we obtain $\phi(x) * \text{neg}(\phi(\beta)) = 0 \equiv (t_0, 0) \implies \phi(x * \text{neg}(\beta)) = (t_0, 0)$. Therefore, $x * \text{neg}(\beta) \in Ker(\phi) \implies x \in \beta \oplus Ker(\phi)$. Now, consider $k \in Ker(\phi)$ then, we have $\phi(\beta * k) = \phi(\beta) \circ \phi(k) = \phi(\beta) \circ (t_0, 0) = (t_c, \alpha_c) \circ (t_0, 0) = (t_c * t_0, \alpha_c \oplus 0) = (t_c, \alpha_c) \implies \beta * k = \phi^{-1}(t_c, \alpha_c)$. Hence, $\phi^{(-1)}((t_c, \alpha_c)) = \{\beta \in [-n, n] \mid \phi(\beta) = (t_c, \alpha_c)\} = \beta * Ker(\phi)$. □

We will now present the new operational laws for 2TL term set.

Definition 2.3.2. Let LT be a predetermined SSLTS and $\overline{LT} = \{t_c \mid c = -n, \ldots, 0, \ldots, n\}$ be a set of 2TL variables. Then,

(i) $(t_c, \alpha_c) \odot (t_d, \alpha_d) = (t_c \odot t_d, \alpha_c \odot \alpha_d)$

where $t_c \odot t_d = \begin{cases} t_{c \cdot d}, & \text{if } c \cdot d \in \{-n, \ldots, 0, \ldots, n\} \\ t_{\text{round}(\frac{c \cdot d}{n})}, & \text{otherwise.} \end{cases}$ and

$$\alpha_c \odot \alpha_d = \begin{cases} \alpha_c \odot \alpha_d - 0.25, & \alpha_c \odot \alpha_d < 0, \\ \alpha_c \odot \alpha_d + 0.25, & \text{otherwise.} \end{cases}$$

(ii)
$$\begin{aligned} (t_c, \alpha_c) \oslash (t_d, \alpha_d) &= (t_c \oslash t_d, \alpha_c \oslash \alpha_d) \\ &= (t_{\min(c,d)}, \min(\alpha_c, \alpha_d)). \end{aligned}$$

The following property of the above-mentioned algebra can be simply obtained by Definition 2.3.2.

Proposition 2.3.2. Let Γ be the set of all 2TL variables, i.e., $(t_c, \alpha_c) \in \Gamma$ and $(t_d, \alpha_d) \in \Gamma$. Then, $(t_c, \alpha_c) \odot (t_d, \alpha_d) \in \Gamma$ and $(t_c, \alpha_c) \oslash (t_d, \alpha_d) \in \Gamma$.

Proof. The proof of the theorem can be given directly from the definition. □

Proposition 2.3.3. Let $\phi : [-n,n] \to \overline{LT}$ be a 2TL group isomorphism. Then,

(i) $\phi(0) = (t_0, 0)$;

(ii) $\phi(\text{neg}(\beta)) = \text{neg}(\phi(\beta))$;

(iii) For any element β_1 and β_2 in $[-n, n]$, β_1 and β_2 commute iff $\phi(\beta_1)$ and $\phi(\beta_2)$ commute.

(iv) $\phi^{-1} : \overline{LT} \to [-n, n]$ is an isomorphism.

Proof. The proof of the theorem can be stated trivially. □

Definition 2.3.3. Let $G = [-n, n]$ be an additive group and $H = [\frac{-1}{2}, \frac{1}{2}] \subseteq G$. For any $\beta \in G$, the set $\{\beta * h \mid h \in H\}$ is denoted by $\beta * H$ is referred as linguistic left coset of H in G.

$$\beta * H = \{\beta * h \mid h \in H\} = \begin{cases} \beta + h + \frac{n}{2}, & \text{if } \beta + h < \frac{-n}{2}, \\ \beta + h, & \text{if } \frac{-n}{2} \leq \beta + h \leq \frac{n}{2}, \\ \beta + h - \frac{n}{2}, & \text{if } \beta + h > \frac{n}{2}. \end{cases}$$

In the similar fashion, we can state the definition of linguistic right coset as well, i.e., $H * \beta$.

Next, we proceed to discuss the properties of linguistic cosets.

Proposition 2.3.4. Let $G = [-n, n]$ be an additive group and $H = [\frac{-1}{2}, \frac{1}{2}] \subseteq G$. Let β_1 and β_2 be any two arbitrary element of G. Then,

(i) $\beta \in \beta * H$;

(ii) $\beta * H = H \iff \beta \in H$;

(iii) $\beta * H = \beta' * H$ or $\beta * H \cap \beta' * H = \phi$;

(iv) $\beta * H = \beta' * H \iff \beta' * \text{neg}(\beta) \in H$;

(v) $\beta * H = H * \beta' \iff H = \beta * H * \text{neg}(\beta)$;

(vi) $\beta * H$ is a subgroup of $G \iff \beta \in H$.

Proof. To prove property (i), we note that as $0 \in H$, so $\beta = \beta + 0 \in \beta * H \implies \beta \in \beta * H$. Next, to prove (ii) property we initially suppose that $\beta * H = H$ then, $\beta = \beta + 0 \in \beta * H = H$ $\implies \beta \in H$. Conversely, Suppose that $\beta \in H \implies \beta * H \subseteq H$ since closure property holds true. Let $h \in H$ now as $\beta \in H \implies h * \text{neg}(\beta) \in H$. Consider, $h = h * \beta * \text{neg}(\beta) = (h * \text{neg}(\beta)) * \beta \in \beta * H \implies H \subseteq \beta * H$. Therefore, $\beta * H = H$. Next, in order to show

$\beta * H = \beta' * H$ or $\beta * H \cap \beta' * H = \phi$. Let if possible $\beta * H \cap \beta' * H \neq \phi$ then we claim $\beta * H = \beta' * H$. Let $\tilde{x} \in \beta * H \cap \beta' * H$ then their exist $h_1, h_2 \in H$ such that we have $\tilde{x} = \beta * H$ and $\tilde{x} = \beta' * H$. Then, based on the definition '$*$' we have

$$\tilde{x} = \beta * h_1 = \begin{cases} \beta + h_1 + \frac{n}{2}, & \text{if } \beta + h_1 < \frac{-n}{2}, \\ \beta + h_1, & \text{if } \frac{-n}{2} \leq \beta + h_1 \leq \frac{n}{2}, \\ \beta + h_1 - \frac{n}{2}, & \text{if } \beta + h_1 > \frac{n}{2}. \end{cases} \text{ and } \tilde{x} = \beta * h_2 = \begin{cases} \beta + h_2 + \frac{n}{2}, & \text{if } \beta + h_2 < \frac{-n}{2}, \\ \beta + h_2, & \text{if } \frac{-n}{2} \leq \beta + h_2 \leq \frac{n}{2}, \\ \beta + h_2 - \frac{n}{2}, & \text{if } \beta + h_2 > \frac{n}{2}. \end{cases}$$

We now contemplate the following cases:

Case 1: $\beta + h_1 < \frac{-n}{2}$

$\tilde{x} = \beta + h_1 + \frac{n}{2} \implies \beta = \tilde{x} - h_1 - \frac{n}{2}$. Also, $\beta + h_1 + \frac{n}{2} = \beta' + h_2 + \frac{n}{2} \implies \beta = \beta' + h_2 - h_1$. Now, $\beta * H = (\beta' + h_2 - h_1) * H = \beta' * H$ followed by property (ii). Therefore, $\beta * H = \beta' * H$. In the similar fashion the proof of other two cases can also be given, i.e.,

Case 2: $\frac{-n}{2} \leq \beta + h_1 \leq \frac{n}{2}$

Case 3: $\beta + h_1 > \frac{n}{2}$

Next, we claim $\beta * H = \beta' * H \iff \beta' * \text{neg}(\beta) \in H$. Suppose $\beta * H = \beta' * H$. Let $\beta * h_1 = \beta' * h_2$ for some $h_1, h_2 \in H$. Now, followed by the definition if $\beta + h_1 < \frac{-n}{2}$ we have $\beta * h_1 = \beta + h_1 + \frac{n}{2}$ and $\beta' * h_2 = \beta' + h_2 + \frac{n}{2} \implies \beta + h_1 + \frac{n}{2} = \beta' + h_2 + \frac{n}{2} \implies \beta' * \text{neg}(\beta) = h_1 - h_2 \in H \implies \beta' - \beta \in H$. For $\beta + h_1 \in [\frac{-n}{2}, \frac{n}{2}]$ result follows directly. Lastly, for the case where $\beta + h_1 > \frac{n}{2}$ the proof is similar as case (1). Conversely, we suppose that $\beta' * \text{neg}(\beta) \in H$ by using property (ii) we have $(\beta' * \text{neg}(\beta)) * H = H$ therefore, $\beta * H = \beta' * H$. Finally, property (v) and (vi) can be proved trivially.

□

Definition 2.3.4. Let $G = [-n, n]$ be an additive group and $H = [\frac{-1}{2}, \frac{1}{2}]$ be a subgroup of G. Then, H is referred as a linguistic normal subgroup if $\beta * H = H * \beta \ \forall \ \beta \in G$. Here,

$$\beta * H = \begin{cases} \beta + h + \frac{n}{2}, & \text{if } \beta + h < \frac{-n}{2}, \\ \beta + h, & \text{if } \frac{-n}{2} \leq \beta + h \leq \frac{n}{2}, \\ \beta + h - \frac{n}{2}, & \text{if } \beta + h > \frac{n}{2}. \end{cases}$$

Remark 2.3.1. It is noted that to test a subgroup H of G is linguistic normal in G iff $\beta * H * \text{neg}(\beta) \subseteq H \ \forall \beta \in G$ (for proof See property (vi) in previous proposition).

Definition 2.3.5. Let $G = [-n, n]$ be a group and $H = [\frac{-1}{2}, \frac{1}{2}]$ be a linguistic normal subgroup of G. Then the set $G/H = \{\beta * H | \beta \in G\}$ is referred a group under the following group operation

$$(\beta * H) * (\beta' * H) = \beta * \beta' * H$$

where, $\beta * H = \begin{cases} \beta + h + \frac{n}{2}, & \text{if } \beta + h < \frac{-n}{2}, \\ \beta + h, & \text{if } \frac{-n}{2} \leq \beta + h \leq \frac{n}{2}, \\ \beta + h - \frac{n}{2}, & \text{if } \beta + h > \frac{n}{2}. \end{cases}$

Note 2.3.1. It is noted that a mapping $\phi : G \to \overline{LT}$ is a group homomorphism. Then, $ker(\phi)$ is a linguistic normal subgroup of G. Since $K = Ker(\phi) = \{\beta \in [-n, n] | \phi(\beta) = (t_0, 0)\}$. Clearly, $\phi(0) = (t_0, 0) \implies 0 \in K = Ker(\phi) \neq \phi$. Next, by taking $x, y \in Ker(\phi)$ we have $\phi(x) = (t_0, 0)$ and $\phi(y) = (t_0, 0)$. we claim $x * \text{neg}(y) \in Ker(\phi)$. Consider $\phi(x * \text{neg}(y)) = \phi(x) \circ \phi(\text{neg}(y))$ as ϕ is a homomorphism. Therefore, $\phi(x * \text{neg}(y)) = (t_0, 0) \in Ker(\phi)$. Lastly, we are left to show that $Ker(\phi)$ is a normal subgroup of G. For any $\beta \in G$ and $x \in Ker(\phi)$ we have, $\phi(\beta * x * \text{neg}(\beta)) = \phi(\beta) \circ \phi(x) \circ \phi(\text{neg}(\beta)) = \phi(\beta) \circ \phi(x) \circ \text{neg}(\phi(\beta)) = (t_0, 0) \implies \beta * x * \text{neg}(\beta) \in Ker(\phi)$.

We next discuss the main theorem of this section.

Theorem 2.3.1. Let $\phi : G \to \overline{LT}$ is a onto group homomorphism with $Ker(\phi)$, then

$$\frac{[-n, n]}{Ker(\phi)} \approx \overline{LT}$$

Proof. Consider a mapping $f : \frac{[-n,n]}{K} \approx \overline{LT}$ such that $f(\beta * K) = \phi(\beta)\ \forall \beta \in G$ where $K = Ker(\phi)$. We first show that the map is well-defined and one-one. Let $\beta * K = \beta' * K \iff \beta * \text{neg}(\beta') \in K \iff \phi(\beta * \text{neg}(\beta')) = (t_0, 0)$. Since ϕ is a homomorphism, therefore $\phi(\beta) * \text{neg}(\beta') = (t_0, 0) \iff \phi(\beta) = \phi(\beta') \iff f(\beta * K) = f(\beta' * K)$. Next, $f((\beta * K) * (\beta' * K)) = f(\beta * \beta' * K) = \phi(\beta * \beta') = \phi(\beta) \circ \phi(\beta') = f(\beta * K) \circ f(\beta' * K)$ thus f is a homomorphism. Lastly, to show f is onto. Let $(t_c, \alpha_c) \in \overline{LT}$ be any arbitrary element. Since ϕ is an onto homomorphism, so there exist $\beta \in [-n, n]$ such that $\phi(\beta) = (t_c, \alpha_c) \implies f(\beta * K) = (t_c, \alpha_c)$. Therefore, f is onto. Hence proved. □

After discussing about the properties of 2TL group homomorphism and isomorphism we present a new outlook of zero sum imprecise matrix games having 2TL information by introducing the notion of linguistic matrix norms, in the next forthcoming section.

2.4 Methodology for matrix norm to solve zero-sum matrix game with 2-tuple linguistic information

In literature, Izgi and Özkaya [133] initially presented a novel approach to solve classical ZSMG problems by using the idea of matrix norms. However, the 2TL model proposed by Herrera and Martínez [111] provides a competent tool in addressing uncertainty persisting within several organizational decision-making and realistic game-related problems. Therefore, Singh et al. [248] gave a unified mechanism to solve matrix games with 2TL information by adopting the LLP approach, which is a bit time-consuming for bigger size game problems. On the contrary, according to the matrix norm methodology, game value is obtained faster without solving any linear mathematical equations. Therefore, inspired by this idea, we will introduce matrix norms to solve the ZSMG with 2TL information in the present section.

We primarily study the elementary terminologies and concepts associated with the 2TL matrix norm methodology for matrix game based on the group-theoretic properties given in the previous section. Next, we derive the inequalities for obtaining an optimal range that consists of a 2TL game value and further define some results to find boundary conditions for the maximal and minimal elements in the mixed strategy set. Lastly, the implementation of the proposed methodology is given for validation.

2.4.1 A new perspective of a 2-tuple linguistic matrix game with matrix norm

Motivated by the notion of matrix norm given in [209], we can present the analogous definition of matrix norm for 2TL information.

Definition 2.4.1. A 2TL matrix norm is a function $||.||$ defined from the set of all 2TL matrices into the set of all 2TL term set that obeys the subsequent properties:

(i) $||\widetilde{A}|| \geq 0$ and $||\widetilde{A}|| = 0 \iff (t_c, 0) = 0 \forall c$;

(ii) $||\omega\widetilde{A}|| = \omega||\widetilde{A}||$ for $\omega \in [0, 1]$;

(iii) $||\widetilde{A} \circ \widetilde{B}|| \leq ||\widetilde{A}|| \circ ||\widetilde{B}||$ where the binary operation '$\circ$' has the similar meaning defined in the theorem 2.2.1 for the matrices $\widetilde{A}$, $\widetilde{B}$ of the similar size;

(iv) $||\widetilde{A} \odot \widetilde{B}|| \leq ||\widetilde{A}|| \odot ||\widetilde{B}||$ for all matrices of similar size. Here, the binary operation '$\odot$' is taken in respect of definition 2.3.2.

Here, $\widetilde{A}$ represents the payoff matrix with 2TL information.

Definition 2.4.2. Let $LT = \{t_c \,|c = -n, \ldots, 0, \ldots, n\}$ be the finite ordered pre-defined SSLTS such that absolute 2TL variable is given as follows:

$$Abs(t_c, \alpha_c) = \begin{cases} (\text{neg}(t_c), -\alpha_c) & \text{for } c < 0, \\ (t_c, \alpha_c) & \text{otherwise} \end{cases}$$

where neg has a usual meaning.

Definition 2.4.3. Let $\widetilde{A} \in \overline{LT}^{n\times m}$ then $1'-$ norm and $\infty'-$ norm are given as follows:

- $||\widetilde{A}||_1 = \max_d \bigcirc_c(Abs(\tilde{a}_{cd})) = (t_c, \alpha_c)$ = the largest absolute column sum of the 2TL payoff matrix;
- $||\widetilde{A}||_\infty = \max_c \bigcirc_d(Abs(\tilde{a}_{cd})) = (t_c, \alpha_c)$ = the largest absolute row sum of the 2TL payoff matrix.

where $\tilde{a} \in \overline{LT}$.

We observe that in the above definition if $||\widetilde{A}||_1 = (t_c, \alpha_c) \notin \overline{LT}$ or $||\widetilde{A}||_\infty = (t_c, \alpha_c) \notin \overline{LT}$, then $(t_c, \alpha_c) = (t_{c-1}, \alpha_c)$.

Based on the notion of those as mentioned above, the $1'-$ norm and $\infty'-$ norm, we derive the inequalities in the following lemma and further state some new theorem offering new panorama to the solution of the imprecise matrix game and successfully obtain the required inequalities for the 2TL matrix game value.

It is noteworthy that to avoid any confusion; we assume Player I (PI) as a row player and Player II (PII) as a column player of the game matrix with 2TL information. Consequently, it is significant to mention that we define the following results of the matrix game in the respect of the PI, i.e., a row player. However, one may readily use the proposed approach for PII as well.

Lemma 2.4.1. Let $\widetilde{A}$ be a $p \times q$ payoff matrix having 2TL information and $\widetilde{v}$ be the 2TL value of the game of a 2-person ZSLMG. Then,

$\dfrac{\widetilde{h}}{||\widetilde{A}||_\infty} \leq \widetilde{v} \leq ||\widetilde{A}||_1$ for $\widetilde{v} > 0$ i.e., $t_c > 0$ for some c;

$\text{neg}(||\widetilde{A}||_1) \leq \text{neg}(\widetilde{v}) \leq \dfrac{\widetilde{h}}{||\widetilde{A}||_\infty}$ for $\text{neg}(\widetilde{v}) < 0$ i.e., $t_c < 0$ for some $c < 0$.
where $\widetilde{h} = \max\limits_{1 \leq c \leq p, c \neq r} \bigcirc_{d=1}^{q} \widetilde{v} \odot Abs(\tilde{a}_{cd})$ and $||\widetilde{A}||_\infty = \bigcirc_{d=1}^{q} Abs(\tilde{a}_{rd})$ for fixed r and $||\widetilde{A}||_1 = \bigcirc_{c=1}^{p} Abs(\tilde{a}_{ct})$ for fixed t.

Proof. Consider the game matrix $\widetilde{A} = [\tilde{a}_{cd}]_{p \times q}$ with 2TL information where $\tilde{a}_{cd} \in \overline{LT}$.

For the notational simplicity and understanding, we use throughout the proof $\tilde{v}^+ > (t_0, 0)$ as positive 2TL game value $\tilde{v}$ and $\text{neg}(\tilde{v}^+) < (t_0, 0)$ as negative 2TL game value $\tilde{v}$. Also, we represent $Abs(\tilde{a}_{cd})$ as $\tilde{a}_{cd}$.

We now contemplate the following cases:

Case 1 For $\tilde{v}^+ > (t_0, 0)$:

Let $||\widetilde{A}||_\infty = \bigcirc_{d=1}^{q} \tilde{a}_{rd}$ for fixed r. From the definition we obtain

$$\begin{aligned} \bigcirc_{d=1}^{q} \tilde{a}_{rd} &\geq \max \bigcirc_{d=1}^{q} \tilde{a}_{cd} \\ \text{for}\, c = 1, 2, \ldots, p \ \ &\text{and}\ c \neq r \\ \Rightarrow 1 &\geq \frac{\max \bigcirc_{d=1}^{q} \tilde{a}_{cd}}{\bigcirc_{d=1}^{q} \tilde{a}_{rd}} \\ \Rightarrow \tilde{v}^+ &\geq \frac{\widetilde{h}}{\sum_{d=1}^{q} \tilde{a}_{rd}} \\ \text{where, } \widetilde{h} &= \max_{1 \leq c \leq p, c \neq r} \bigcirc_{d=1}^{q} \tilde{v}^+ \odot \tilde{a}_{cd} \end{aligned}$$

Therefore,

$$\frac{\widetilde{h}}{||\widetilde{A}||_\infty} \leq \tilde{v}^+ \tag{2.4.1}$$

Clearly, the 2TL game value is evaluated as $\tilde{v}^+ = \oplus_{c=1}^{p} x_c \tilde{a}_{cd}$ for any fixed d. Here, x_c represents the probability credited to each element of the mixed strategy set of PI. Then, it follows that $\tilde{v}^+ \leq \bigcirc_{c=1}^{p} \tilde{a}_{cd}\ \forall\ d$. After taking max on both sides we obtain,

$$\tilde{v}^+ \leq ||\widetilde{A}||_1 \tag{2.4.2}$$

From equation 2.4.1 and 2.4.2 we obtain $\dfrac{\widetilde{h}}{||\widetilde{A}||_\infty} \leq \tilde{v}^+ \leq ||\widetilde{A}||_1$.

Case 2 For $\text{neg}(\tilde{v}^+) < [0, 0]$:

From case 1 we have $1 \geq \dfrac{\max \bigcirc_{d=1}^{q} \tilde{a}_{cd}}{||\widetilde{A}||_\infty}$. Now, since the present case deals with $\text{neg}(\tilde{v}^+)$. Hence, we obtain the following inequality $\dfrac{\widetilde{h}}{||\widetilde{A}||_\infty} \geq \text{neg}(\tilde{v}^+)$, where

$\widetilde{h} = \max\limits_{1\le c\le p, c\neq r} \bigodot_{d=1}^{q} \text{neg}(\tilde{v}^{+}) \odot \tilde{a}_{cd}$. Next, to obtain the other inequality we consider the relation $\text{neg}(\tilde{v}^{+}) = \oplus_{c=1}^{p} x_c \tilde{a}_{ct}$ for any fixed t. Contrastingly, $\text{neg}(\tilde{a}_{cd}) \le \text{neg}(\tilde{a}_{cd}) x_c$ as $x_c \in [0, 1]$. Therefore, the inequality $\text{neg}(\tilde{v}^{+}) \ge \oplus_{c=1}^{p}(x_c\, \text{neg}(\tilde{a}_{ct}) \ge \bigodot_{c=1}^{p} \text{neg}(\tilde{a}_{ct}) \ge \text{neg}(||\widetilde{A}||_1)$ is valid since $\bigodot_{c=1}^{p} \tilde{a}_{ct} \le \max\limits_{1\le d\le q} \bigodot_{c=1}^{p} \tilde{a}_{ct} = ||\widetilde{A}||_1$. Hence, the result follows.

□

Before we proceed to give the next result, we give the following definition beneficial in the main theorem.

Definition 2.4.4. For $\widetilde{A} \in \overline{LT}^{p\times q}$ a 2TL game matrix, let $||\widetilde{A}||_\infty$ have the sum of absolute values of the s^{th} entries of the row. Then, the matrix $\widetilde{N} \in \overline{LT}^{(p-1)\times q}$ obtained after deleting s^{th} row of the matrix $\widetilde{A}$ is termed as a row-wise induced matrix of $\widetilde{A}$. Analogously, if $||\widetilde{A}||_1$ represents sum of absolute values of the t^{th} column entries. Then, the matrix $\widetilde{N} \in \overline{LT}^{p\times(q-1)}$ obtained after deleting t^{th} column of the matrix $\widetilde{A}$ is termed as column-wise induced matrix of $\widetilde{A}$.

Theorem 2.4.1. Let $\widetilde{A} \in \overline{LT}^{p\times q}$ be a 2TL payoff matrix for the two-person ZSLMG and $\tilde{v}$ be a 2TL game value. Then,

(i) $\dfrac{||\widetilde{N}||_\infty}{||\widetilde{A}||_\infty} \le \tilde{v} \le ||\widetilde{A}||_1$ whenever $\tilde{v} \ge (t_1, 0)$;

(ii) $\dfrac{1}{||\widetilde{A}||_1} \le \text{neg}(\tilde{v}) \le \dfrac{||\widetilde{A}||_\infty}{||\widetilde{N}||_\infty}$ whenever $\text{neg}(\tilde{v}) \le (t_1, 0)$ and $\text{neg}(\tilde{v}) \neq 0$.
where $\widetilde{N}$ is represented as the row-wise induced matrix of $\widetilde{A}$.

Proof. The proof of the theorem can be directly proved with the help of lemma 2.4.1. □

Remark 2.4.1. The critical purpose of the above result is to obtain the boundaries for the 2TL game value. As it is significant to analyze the linguistic game value from the viewpoint of each corresponding player. Consequently, we acquire two distinct inequalities for the single linguistic game value. Since our primary concern is to obtain optimal boundaries for the linguistic game value. Therefore, it suffices to compare each inequality obtained for both the PI and PII, respectively, and thereby choose the best optimal boundaries for the game value such that the original linguistic game value falls within the ambit of optimum range.

It is pointed out that the results offered by the main theorem 2.4.1 are sufficient enough to solve the bigger size linguistic matrix game problem so that we can have an impression about the approximated linguistic game value without explicitly solving the auxiliary pair of linguistic linear mathematical models. Hence, reducing the issue of computational cost and time complexity of the linguistic game problem.

Next, we proceed to establish a result that points out some necessary conditions to obtain the boundaries for the largest and smallest elements in the corresponding mixed strategy set. For simplicity notation, we refer $x_{\max}$ as the largest element and $x_{\min}$ as the smallest element of the mixed strategy set of the game players.

Theorem 2.4.2. Let $\widetilde{A} \in \overline{LT}^{p\times q}$ be a 2TL payoff matrix with all entries positive. Then,

$$x_{\max} \geq \max\left\{ \frac{\bigcirc_{c=1}^{p} \tilde{a}_{cd}}{(p-1).||\widetilde{A}||_1} \mid d = 1, 2, \ldots, q \,\text{and}\, \bigcirc_c \tilde{a}_{cd} \neq ||\widetilde{G}||_1 \right\}$$

$$x_{\min} \leq \frac{1}{p-1} - \frac{\max\left\{ \frac{\bigcirc_{c=1}^{p} \tilde{a}_{cd}}{(p-1).||\widetilde{A}||_1} \mid d = 1, 2, \ldots, q \,\text{and}\, \bigcirc_c \tilde{a}_{cd} \neq ||\widetilde{G}||_1 \right\}}{p-1}$$

where $\tilde{a} > (t_0, 0)$.

Proof. Consider the linguistic payoff matrix $\widetilde{A}^{p\times q}$.

Without loss of generality, we presume that $|||\widetilde{A}||_1 = \tilde{a}_{1r} \circ \tilde{a}_{2r} \circ \ldots \circ \tilde{a}_{pr}$ for any arbitrary fixed r. Then, based on the definition we have $(\tilde{a}_{1r} \circ \tilde{a}_{2r} \circ \ldots \circ \tilde{a}_{pr})x_{\max} \geq \tilde{a}_{1t} \circ \tilde{a}_{2t} \circ \ldots \circ \tilde{a}_{pt}$, where $t = 1, 2, \ldots, r-1, r+1, \ldots, q$.
i.e., we have the following set of inequalities

$$\left.\begin{aligned}
(\tilde{a}_{1r} \circ \tilde{a}_{2r} \circ \ldots \circ \tilde{a}_{pr})x_{\max} &\geq \tilde{a}_{11} \circ \tilde{a}_{21} \circ \ldots \circ \tilde{a}_{p1}, \\
(\tilde{a}_{1r} \circ \tilde{a}_{2r} \circ \ldots \circ \tilde{a}_{pr})x_{\max} &\geq \tilde{a}_{12} \circ \tilde{a}_{22} \circ \ldots \circ \tilde{a}_{p2}, \\
&\vdots \\
(\tilde{a}_{1r} \circ \tilde{a}_{2r} \circ \ldots \circ \tilde{a}_{pr})x_{\max} &\geq \tilde{a}_{1(r-1)} \circ \tilde{a}_{2(r-1)} \circ \ldots \circ \tilde{a}_{p(r-1)} \\
(\tilde{a}_{1r} \circ \tilde{a}_{2r} \circ \ldots \circ \tilde{a}_{pr})x_{\max} &\geq \tilde{a}_{1(r+1)} \circ \tilde{a}_{2(r+1)} \circ \ldots \circ \tilde{a}_{p(r+1)} \\
&\vdots \\
(\tilde{a}_{1r} \circ \tilde{a}_{2r} \circ \ldots \circ \tilde{a}_{pr})x_{\max} &\geq \tilde{a}_{1q} \circ \tilde{a}_{2q} \circ \ldots \circ \tilde{a}_{pq}
\end{aligned}\right\} \text{(q-1)}$$

Since $x_1+x_2+\ldots+x_p = 1 \implies x_1+x_2+\ldots+x_{\max}+x_{\min}+\ldots+x_p = 1 \implies \underbrace{x_1+x_2+\ldots+x_{\max}+\ldots+x_p}_{\text{p-1}} = 1 - x_{\min}$.

$|||\widetilde{A}||_1(p-1)x_{\max} \geq \tilde{a}_{1t} \circ \tilde{a}_{2t} \circ \ldots \circ \tilde{a}_{pt}$, where $t = 1, 2, \ldots, r-1, r+1, \ldots, q$. Now, by taking max on both sides and rearranging the terms we obtain,

$$x_{\max} \geq \max\left\{ \frac{\bigcirc_{c=1}^{p} \tilde{a}_{cd}}{(p-1).||\widetilde{A}||_1} \mid d = 1, 2, \ldots, q \text{ and} \bigcirc_c \tilde{a}_{cd} \neq ||\tilde{A}||_1 \right\} \tag{2.4.3}$$

Next, in order to show the boundaries for $x_{\min}$, we consider the following equation:

$x_1 + x_2 + \ldots + x_{\max} + x_{\min} + \ldots + x_p = 1$

$\Rightarrow \underbrace{x_1 + x_2 + \ldots + x_{\min} + \ldots + x_p}_{\text{p-1}} = 1 - x_{\max}$

$\Rightarrow (p-1)x_{\min} = 1 - x_{\max}$

$\Rightarrow x_{\max} = 1 - (p-1)x_{\min}$.

Now, by using the equation 2.4.3 we obtain

$$1 - (p-1)x_{\min} \geq \max\left\{ \frac{\bigcirc_{c=1}^{p} \tilde{a}_{cd}}{(p-1).||\widetilde{A}||_1} \mid d = 1, 2, \ldots, q \right\},$$

and $\bigcirc_c \tilde{a}_{cd} \neq ||\widetilde{A}||_1$. After rearranging the terms we obtain the desired inequality and hence the theorem. □

Remark 2.4.2. It is significant to point out that our proposed approach helps calculating the approximate value of the 2TL game falling within the optimal boundaries of the game value that we have obtained using theorem 2.4.1 and the elements of the mixed strategy set are selected by taking consideration of the inequalities acquired by using theorem 2.4.2

Remark 2.4.3. It is worthwhile to emphasis that by giving careful consideration to the required bounds obtained for $x_{\max}$ and $x_{\min}$ based on theorem 2.4.2 we arbitrarily select an appropriate value for $x_{\max}$ and $x_{\min}$, respectively. The remaining elements of the mixed strategy set for the players are hereafter decided in a manner that it must satisfy the principal of probability theory, i.e., all the strategies for the player sums up to 1.

Before addressing a numerical example in the subsequent section to illustrate the utilization of a novel approach for the ZSLMG, we provide a solution algorithm for the same to obtain the required optimal solution. It is noted that the subsequent algorithm is given for PI; however, one may use for PII as well.

Consider the game model for PI.

Step 1: Consider absolute value of the payoff matrix with 2TL information such that $Abs(\tilde{a}_{cd}) \in \overline{LT}$.

Step 2: Calculate the boundaries for linguistic game value using theorem 2.4.1.

Step 3: Choose appropriate value for $x_{\max}$ and $x_{\min}$ depending on the related inequality given in theorem 2.4.2.

Step 4: Determine the suitable value for the remaining strategies one by one, arbitrarily.

Step 5: Evaluate approximate value of the linguistic game $\tilde{v}_{app}$ by randomly selecting any column of the given payoff matrix.

Step 6: Compare the $\tilde{v}_{app}$ with the original linguistic game value $\tilde{v}$ obtained by solving linguistic game problem via linguistic linear programming method given in paper [248].

2.4.2 Application to equity market domain

In the present section, we further illustrate the expediency of the proposed methodology by taking a practical, real-life problem of the company's selection problem to invest.

We consider a DM problem involving two competitive players: the first is an investor who aims to invest his money by choosing stocks after carefully analyzing a company's fundamentals, and the second player is Nature. The idea of presenting a game model with Nature is originated from the problem of portfolio choice given in the paper [255]. However, in this particular section, it is presumed that the decision-maker is reactionary, expecting Nature to compete against him to minimize his payoffs.

Example 2.4.1. As we know, investing is a mode to increase wealth over time, but this process also withholds the risk of dropping money, especially while selecting the companies one may wish to invest in. To become a successful investor and avoid costly mistakes, it is significant to perform thorough research in understanding the fundamentals of the companies before investing. Therefore, in this example, we consider the investor as the PI whose only goal is to invest his money in the company to maximize the returns, while nature is regarded as PII who is conflicting against him to formulate the two-player matrix game problem.

Next, after preliminary screening an investor has shortlisted 6 companies as an alternatives set, i.e., $T_c\,(c = 1, 2, 3, 4, 5, 6)$ where he wants to invest in. Since investing decision is tough and is not always easy for a person to invest his hard-earned money without conducting a research. To compete these shortlisted companies so as to choose the most stable of them and also rank them from the viewpoint of their significance degree, it is crucial for an investor to perform the evaluation and selection operation

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
x_1	$(t_0,0)$	$(t_0,0.4)$	$(t_{-2},0.4)$	$(t_{-1},0.4)$	$(t_0,-0.2)$	$(t_{-2},0.2)$	$(t_1,-0.2)$	$(t_2,0.4)$
x_2	$(t_{-3},0.2)$	$(t_3,0)$	$(t_4,0)$	$(t_1,0.2)$	$(t_2,0.2)$	$(t_3,0.2)$	$(t_2,0.4)$	$(t_3,0.2)$
x_3	$(t_1,0)$	$(t_3,0.2)$	$(t_3,0.2)$	$(t_{-1},-0.2)$	$(t_1,-0.2)$	$(t_0,0.2)$	$(t_1,-0.2)$	$(t_3,0.4)$
x_4	$(t_1,0.2)$	$(t_4,0)$	$(t_{-3},-0.2)$	$(t_1,0.4)$	$(t_3,0.2)$	$(t_1,0.2)$	$(t_0,-0.2)$	$(t_4,0)$
x_5	$(t_1,-0.4)$	$(t_{-2},0)$	$(t_{-1},0.2)$	$(t_3,0.4)$	$(t_1,-0.2)$	$(t_1,0.2)$	$(t_1,-0.3)$	$(t_0,-0.4)$
x_6	$(t_4,0)$	$(t_1,0.4)$	$(t_2,0.4)$	$(t_3,0.4)$	$(t_1,0.2)$	$(t_3,-0.4)$	$(t_4,0)$	$(t_3,0.2)$

Table 2.3: 2-tuple linguistic payoff matrix.

primarily based on various essential criteria. Nevertheless, in this case study, we consider eight main criteria: the performance of company (C_1); the market value of company (C_2); the efficiency level of company (C_3); the business model of company (C_4); Employee satisfaction level (C_5); shareholders funds (C_6); Companies future innovation networks (C_7); Companies debt and liabilities assessments (C_8). Since an investor is careful about strategic choice of Nature, so he will select his mixed strategy set as $x = (x_1, x_2, x_3, x_4, x_5, x_6), x_c \geq 0, c = 1, 2, 3, 4, 5, 6, \sum_{c=1}^{6} x_c = 1$ over the alternative set $\{T_c | c = 1, 2, 3, 4, 5, 6\}$. On the contrary, the investor view nature as a non-cooperative player, thus to counter the choice of his mixed strategy, Nature will choose the mixed strategy set as $y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)\, y_d \geq 0, d = 1, 2, 3, 4, 5, 6, 7, 8\ \sum_{d=1}^{8} y_d = 1$, over the criteria set $\{C_d | d = 1, 2, 3, 4, 5, 6, 7, 8\}$.

To deal with this problem more efficiently, an investor consider the linguistic decision matrix given in table 2.3 which is based on the predefined SSLTS $LT = \{ t_{-4} : \text{W}, t_{-3} : \text{EP}, t_{-2} : \text{VP}, t_{-1} : \text{P}, t_0 : \text{M}, t_1 : \text{G}, t_2 : \text{VG}, t_3 : \text{EG}, t_4 : \text{PR} \}$.

Note: "Worst" (W), "Extremely poor" (EP), "Very poor" (VP), "Poor" (P), "Medium" (M), "Good" (G), "Very good" (VG), "Extremely good" (EG), "Perfect" (PR).

We first check the entries of the given payoff matrix. Since some of the entries are negative therefore we take neg of all such entries. Next, we proceed to solve the matrix game problem for row player. In this case, $||\widetilde{A}||_1 = (t_3, 0.4)$, $||\widetilde{A}||_\infty = (t_3, -0.2)$ and corresponding row wise induced matrix of $\widetilde{A}$ is given by $||\widetilde{N}||_\infty = (t_2, -0.1)$ and $||\widetilde{N}||_1 = (t_2, 0.2)$. Then, according to the theorem 2.4.1, the boundaries for the approximated 2TL game value is calculated as follows: $\dfrac{||\widetilde{N}||_\infty}{||\widetilde{A}||_\infty} \leq \tilde{v} \leq ||\widetilde{A}||_1$ whenever $\tilde{v} \geq (t_1, 0)$.

Based on the new operational laws defined for the 2TL information given in the Definition 2.3.2, we have $(t_2, -0.2) \leq \tilde{v} \leq (t_3, 0.4)$. Since Δ^{-1} is monotonically increasing bijective function, therefore we apply Δ^{-1} throughout the equation, i.e., $\Delta^{-1}((t_2, -0.2)) \leq \Delta^{-1}(\tilde{v}) \leq \Delta^{-1}((t_3, 0.4)) \implies 1.8 \leq \Delta^{-1}(\tilde{v}) \leq 3.4$.

Finally, to obtain maximum and minimum element of the strategy set we have the subsequent boundary conditions, i.e.,
$x_{\max} \geq \max\{\frac{(t_2,-0.4)}{(5).(t_3,0.4)}, \frac{(t_1,0)}{(5)(t_3,0.4)}, \frac{(t_3,0.2)}{(5)(t_3,0.4)}, \frac{(t_2,0.2)}{(5)(t_3,0.4)}, \frac{(t_1,0.1)}{(5)(t_3,0.4)}, \frac{(t_3,-0.4)}{(5)(t_3,0.4)}\}$.

Since ϕ^{-1} is a bijective homomorphism, therefore we apply ϕ^{-1} and after simplifying the calculation we obtain $\Rightarrow x_{\max} \geq \max\{0.094, 0.059, 0.1882, 0.1294, 0.065, 0.1529\}$
$\Rightarrow x_{\max} \geq 0.1882$.

Now, $x_{\min} \leq 0.2-0.037$, then the required bound for $x_{\min}$ is $x_{\min} \leq 0.1623$. Based on the remark 2.4.3, we randomly select the value of strategies as $x_{\min} = 0$, $x_1 = 0.20$, $x_2 = 0.15$, $x_3 = 0.25$, $x_4 = 0$, $x_{\max} = 0.4$.

Now, as our primary concern is to find out a 2TL value of the game such that the value must fall within the optimum boundaries we have obtained above. Therefore, one can choose any arbitrary column for evaluating the approximate game value. However, in this scenario we select sixth column of the payoff matrix $\widetilde{A}$ to obtain the approximate game value, i.e., $\tilde{v}_{app} = 0(t_2,-0.2) \oplus 0.20(t_3,0.2) \oplus 0.15(t_0,0.2) \oplus 0.25(t_1,0.2) \oplus 0(t_1,0.2) \oplus 0.4(t_3,-0.4)$, after applying Δ^{-1} on both sides we get $\Delta^{-1}(\tilde{v}_{app}) = 2.01$.

We next solve the matrix game problem for the column player. The required 2TL matrix norms are $||\widetilde{A}^T||_1 = (t_3,-0.2)$, $||\widetilde{A}^T|||_\infty = (t_3,0.4)$ and $||\widetilde{N}^T|||_\infty = (t_2,0.2)$. Here, '$T$' represents transpose of matrix $\widetilde{A}$. Now, based on Theorem 2.4.1 we obtain $\frac{(t_2,0.2)}{(t_3,0.4)} \leq \tilde{w} \leq (t_3,-0.2) \Rightarrow (t_2,0.2) \leq \tilde{w} \leq (t_3,-0.2)$.

Analogously, we can obtain the boundaries for $y_{\max}$ and $y_{\min}$ by using Theorem 2.4.2. Therefore, $y_{\max} \geq \max\{0.05, 0.04, 0.97, 0.06\} \Rightarrow y_{\max} \geq 0.097$ and $y_{\min} \leq 0.129$. Consequently, as mentioned-above the randomly selected strategy values are $y_{\max} = 0.45$, $y_{\min} = 0,\ y_1 = 0.01,\ y_2 = 0,\ y_3 = 0.223,\ y_4 = 0.18,\ y_5 = 0.02,\ y_6 = 0.117$. We choose column 2 to obtain approximated game value for PII, $\Delta^{-1}(\tilde{w}_{app}) = 0\Delta^{-1}((t_3,-0.2)) + 0.01\Delta^{-1}((t_3,0)) + 0\Delta^{-1}((t_4,0)) + 0.223\Delta^{-1}((t_1,0.2)) + 0.18\Delta^{-1}((t_2,0.2)) + 0.02\Delta^{-1}((t_3,0.2)) + 0.117\Delta^{-1}((t_2,0.4)) + 0.45\Delta^{-1}((t_3,0.2)) = 2.3$. On comparing the boundaries of game value for PI and PII, respectively we envisioned that the boundaries of the game value corresponding to the PII is optimum.

2.4.3 Comparison and discussion

The comparison of the proposed methodology is implemented from the viewpoint of an information representation, and its application anticipated in the domain of multiple decision analysis.

Compared with the LLP methodology proposed by Singh and Gupta [248] to solve the ZSMG with 2TL information, the 2TL matrix norm approach allows players to provide approximate linguistic game value without explicitly solving the pair of auxiliary linguistic linear mathematical models as solved below:

For PI	For PII
min $\quad V = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$	max $\quad V = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8$
subject to	subject to
$0X_1 + 2.8X_2 + X_3 + 1.2X_4 + 0.6X_5 + 4X_6 \geq 1,$	$0Y_1 + 0.4Y_2 + 1.6Y_3 + 0.6Y_4 + 0.2Y_5 + 1.8Y_6 + 0.8Y_7 + 2.4Y_8 \leq 1,$
$0.4X_1 + 3X_2 + 3.2X_3 + 4X_4 + 2X_5 + 1.4X_6 \geq 1,$	$2.8Y_1 + 3Y_2 + 4Y_3 + 1.2Y_4 + 2.2Y_5 + 3.2Y_6 + 2.4Y_7 + 3.2Y_8 \leq 1,$
$1.6X_1 + 4X_2 + 3.2X_3 + 3.2X_4 + 0.8X_5 + 2.4X_6 \geq 1,$	$Y_1 + 3.2Y_2 + 3.2Y_3 + 1.2Y_4 + 0.8Y_5 + 0.2Y_6 + 0.8Y_7 + 3.4Y_8 \leq 1,$
$0.6X_1 + 1.2X_2 + 1.2X_3 + 1.4X_4 + 3.4X_5 + 3.4X_6 \geq 1,$	$1.2Y_1 + 4Y_2 + 3.2Y_3 + 1.4Y_4 + 3.2Y_5 + 1.2Y_6 + 0.2Y_7 + 4Y_8 \leq 1,$
$0.2X_1 + 2.2X_2 + 0.8X_3 + 3.2X_4 + 0.8X_5 + 1.2X_6 \geq 1,$	$0.6Y_1 + 2Y_2 + 0.8Y_3 + 3.4Y_4 + 0.8Y_5 + 1.2Y_6 + 0.7Y_7 + 0.4Y_8 \leq 1,$
$1.8X_1 + 3.2X_2 + 0.2X_3 + 1.2X_4 + 1.2X_5 + 2.6X_6 \geq 1,$	$4Y_1 + 1.4Y_2 + 2.4Y_3 + 3.4Y_4 + 1.2Y_5 + 2.6Y_6 + 4Y_7 + 3.2Y_8 \leq 1,$
$0.8X_1 + 2.4X_2 + 0.8X_3 + 0.2X_4 + 0.7X_5 + 4X_6 \geq 1,$	$Y_1, Y_2, Y_3, Y_4, y_5, Y_6, Y_7, Y_8 \geq 0.$
$2.4X_1 + 3.2X_2 + 3.4X_3 + 4X_4 + 0.4X_5 + 3.2X_6 \geq 1,$	
$X_1, X_2, X_3, X_4, X_5, X_6 \geq 0.$	

The optimal solution obtained is (0, 0.093, 0, 0.1781, 0, 0.188) and $V^* = 0.459$ for PI and (0, 0, 0, 0.1286, 0.2117, 0.1187, 0, 0) for PII.

Here, for the investor, the criteria for which $y_d^* = 0$ are of minimal importance. Analogously, the alternatives with $x_c^* = 0$ are not much relevant to the investor to invest his money.

Based on the scores of the alternatives obtained above, the ranking is given as $T_6 > T_4 > T_2 > T_1 = T_3 = T_5$ and alternative T_6 is the recommended company whose stocks can be chosen for investing money.

However, it is envisioned that solving those mentioned above auxiliary linear mathematical equations becomes tedious for large-scale game problems. Finding an approximated solution without solving the linear equations reduces the computational cost and time complexity in such a situation. Henceforth, it is considered the most significant advantage part of our new approach to a certain extent.

In example 2.4.1, the original value of the game is given as $v = 2.1786$ for both the players. Therefore, for the case of row player the absolute error is $|v - v_{app}| = |2.1786 - 2.01| = 0.1$. Also, for the case of column player the absolute error is $|v - w_{app}| = |2.1786 - 2.3| = 0.13$. It is foreseen that the absolute error so obtained is small in both cases. Hence, the proposed method is showing consistency with the methodology proposed in the corresponding paper [248]. Moreover, in this algorithm, the approximated game value evaluated for the row and column player, more precisely maximizing and minimizing player, follows the relation $v_{app} \leq w_{app}$. It is henceforth showing similarity with the existing results (as shown in [248]).

Finally, it is perceived that the proposed game-theoretic approach is utilized to solve multi decision analysis problems, which perhaps seems to be more advantageous in comparison to the methodology given in the literature (See [248]) in the sense that the latter evaluates the weights as an intermediate step by solving linear mathematical equations. However, our proposed approach evaluates the weights directly using the boundary conditions.

2.5 Conclusion

In this chapter, we have developed a formal method that shows the intervals $[-n, n]$ and $[\frac{-1}{2}, \frac{1}{2}]$, and LT forms a group with respect to a binary operation in a crisp sense. Then, we have proved that the 2TL term set i.e., $\overline{LT} \equiv LT \times [\frac{-1}{2}, \frac{1}{2}]$ is a direct group under the defined operation '$\circ$' and later on developed an isomorphic relation between the 2TL direct group and the interval $[-n, n]$. An application in terms of bipolar linguistic graphs is also given to exhibit the effectiveness and practicality of the developed approach. In this connection, we initially defined the concept of 2TLBS, followed by the introduction of 2TLBGs. We have also established the isomorphic relation between the 2TLBGs, which directly follows from the 2TL group isomorphism. This is because when the two groups are isomorphic, they are abstractly similar and have the same graphical structures.

In addition, some of the properties of 2TL group isomorphism and homomorphism are discussed. We introduce the concept of linguistic kernels, cosets, normal subgroups, and factor groups. We also propose some novel algebraic operational laws for the 2TL term set. The results calculated by using the operational laws are closed such that it avoids any information loss. Lastly, based on the proposed algebraic concept fundamental theorem of 2TL group homomorphism is obtained. Next, we demonstrated an application of novel 2TL group isomorphism and its properties in the study of linguistic matrix games by presenting the novel linguistic norm method helpful in solving a two-player ZSLMG problem. For this, we propose a notion of a $1'-$ norm and $\infty'-$ norm that builds up an analogy with the existing notion of matrix norm method defined for real numbers. Some results and theorems are stated and proved to find optimal boundaries for the linguistic game value depending on the norm concept. Furthermore, we exhibit a result facilitating the game players to obtain lower and upper bounds of the mixed strategy set's most prominent and smallest element, respectively. Consequently, the result is based on the normalization concept, which helps provide bounds for the matrix game problem whose game value is unknown. Moreover, the approximated linguistic game value obtained in a row and column player scenario satisfies the generic inequalities, showing the similarity with existing linear programming methods. Finally, a real-life problem of the company's stock selection problem to invest and comparison of results with the existing method show that the proposed methodology demonstrates consistent results and promotes studies in imprecise matrix games.

Nevertheless, the findings predicted in the chapter are described using the SSLTS, the terms of which are uniformly and symmetrically positioned on either side of the middle term. We tend to envision that the research presented in the direction of the 2TL term set is to commence the theory of 2TL groups by utilizing classical group theory based definitions and results. We believe that the key findings proposed in this study will give rise to a new algebraic view of the 2TL model and can be seen as a significant step towards enhancing the importance of classical group theory in the uncertainty domain.

Chapter 3

Methodology for unbalanced linguistic term sets

Several real-world problems employ linguistic-based approaches to handle qualitative data. The set of linguistic terms that are utilized in the problems are mostly alleged to be symmetrically distributed. However, with the advent of time, as the complexity of the problem increases, the equidistant linguistic term set seems improper. Consequently, in such cases, experts often prefer to use the set of the unbalanced linguistic term to direct the appraisal for the problems. In this chapter[1]*, we tend to propose a method that is newly designed to deal with a set of unbalanced linguistic terms. In this direction, we initially propose an algorithm to represent unbalanced linguistic information via a multiplicative linguistic label set that has a global inconsistent linguistic term distribution. Furthermore, in light of the Herrera and Martinez, "2-tuple linguistic model," we develop a novel 2-tuple approach for the unbalanced linguistic set, which is based on the notion of minimum distance measure. Finally, to validate the proposed model in the physical realm and to demonstrate the functioning of the method, a numerical example is being elucidated. The proposed methodology seeks to indicate a reduction in the computation time and also enhances the decision-makers' evaluations.*

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[1]The result of this chapter is based on a research paper "A New 2-Tuple Linguistic Approach for Unbalanced Linguistic Term Sets" *IEEE Transactions on Fuzzy Systems* **29** (8) 2158–2168 (2021). (**SCI, Impact Factor: 12.029**)

3.1 Introduction

The societal development engenders complex situations, consequently increases indeterminacy within the problems. In general, there may exist some intricate problems entailing quantitative aspects that are well handled through precise numerical values. However, some cases entail qualitative aspects as well that are more arduous to be managed through precise and exact values. Therefore, to handle such problems adequately, Zadeh [324–326] initiated a fuzzy linguistic-based approach whereby qualitative information is represented via LVs. The utility of this approach aims to provide more logical and precise results in the problems encapsulating uncertainty, which is of non-probabilistic nature. Hence, on account of successful results, the fuzzy linguistic-based approach has an application of broad-spectrum, which are witnessed in distinct areas suchlike information retrieval system [33, 34, 124–126, 345], web quality [127, 128], aggregation operators [105, 109, 288], and so forth.

The presence of qualitative information in a decision problem insinuates the appropriateness for a LCMs [107, 129, 188, 271]. According to the existing majored literature several LCMs has become widespread, for instance LCMs that are based on membership functions [76, 137], ordinal scales [109, 295, 305], type-2 fuzzy sets [267, 292], granular computing [56, 234]. Followed by the 2-tuple LCM and its extensions [64, 106, 111, 112, 271].

Since a determination of the membership function is difficult in some of the practical applications, as a consequence, symbolic models like Herrera and Martínez [106], 2TL model has gained considerable attention. This is because of the competence of a 2TL model to interpret and evaluate the results in a highly accurate and straightforward manner. Moreover, it provides a way out to handle linguistic information in a continuous range in place of a discrete one without any information loss. Most of these LCMs and their respective enhancements rely upon the term set having a uniform distribution of the linguistic terms. Such type of linguistic term set is termed as a balanced linguistic term set (BLTS). The most commonly referred BLTSs are $\mathbf{LT} = \left\{\ell_i \mid i = 0, 1, \ldots, g\right\}$ and $\text{LT} = \left\{\ell_i \mid i = -g, \ldots, 0, \ldots, g\right\}$, where '$g$' is a positive integer and each linguistic term is equidistant about their respective central cardinal. Apparently, these two BLTS follow the rule of monotonicity stating that any two linguistic terms, namely, ℓ_α and ℓ_β belonging to either set LT_1 or LT_2 are comparable, i.e, $\ell_\alpha \geq \ell_\beta$ iff $\alpha \geq \beta$.

However, there exist problems where information managed with linguistic assessments require LVs appraised in an unevenly distributed LTSs. Such type of sets is termed as unbalanced linguistic term sets (ULTSs) where the distribution of the terms are uneven. It is precisely viewed in the problems that linguistic information distributed unevenly appears due to necessity of assessing the preferences of the linguistic terms having higher granularity on either side of the central term, or maybe in light of the explicit nature of the LVs which are defined in the problems for computations. In compliance with the existing literature, there are primarily 2 classes of ULTSs. In the first class of ULTSs, the distribution of the linguistic terms are uneven such that the cardinality of the terms on one side of the central term is higher than the other and corresponding distance between the consecutive terms are not equal (namely, ULTS of Ist class, see Fig. 1.6). Meanwhile, in the second class of ULTSs, an equal number of the linguistic terms are distributed on both the sides of the central term with unequal spacings (namely, ULTS of the second class, see Fig. 1.7).

Recent years mark the remarkable progress in the LCMs developed to deal with the ULTSs. Several methodologies were introduced in the literature, which ideally concerns with the presentations and applications of both the classes of ULTSs. One of the pioneering studies addressing the linguistic information assessed in ULTS is proposed by Herrera et al. [112]. In that paper, authors have described a representation model to allocate semantics to the linguistic terms, which are diffused unevenly. The semantics of a linguistic term is capable of expressing each linguistic terms via parametric membership function by utilizing the concept of LH. Lastly, in the paper, they have developed a computational model that uses the existing 2-tuple model [106] as its basis for ULTSs to carry out processes of CWWs without information loss. The details of the model is given in section 1.4. Followed by this pioneering work, Wang and Hao [271] presented a new proportional 2TL model having "symbolic proportion" as its basis. In the proposed method, it is mentioned that linguistic information is expressed in a 2-tuple format, which composes two proportional linguistic terms. Hence, the proportional 2TL model deals with ULTS avoiding any information loss. Continuing further with the study of proportional 2-tuple model, Wang and Hao [272] bring forth a unifying link between the framework of Lawry [150] and the traditional 2-tuple framework [106], in parallel with the CWWs, processes for the line of reasoning with linguistic syllogisms mathematically feasible. Zou et al. [342] proposed a novel linguistic aggregation operator elicited from the model of traditional 2-tuple [106] and linguistic hierarchies [108] to express unbalanced linguistic values. Jiang et al. [138] presented the linguistic proportional 2-tuple power average operator aggregating the linguistic values of ULTSs.

Abchir and Truck [16] broadened the traditional 2-tuple model by introducing a different kind of fuzzy partition for ULTS. Bartczuk et al. [35] presented a novel approach, which is a modification of the traditional 2-tuple linguistic model for handling ULTS. In that paper, linguistic information is represented by an extended-term set which composes a pair that carries a linguistic term along with a value representing correction factor. This correction factor is used to describe the relocation of the term corresponding to its respective position in a term set distributed evenly.

Dong et al. [64] developed a new interpretation of the 2TL model constructed to be an extended version of it primarily based on the notion of numerical scale. Moreover, guided by the notion of the "transitive calibration matrix," along with its consistency index Dong et al. [67] introduced a novel scheme for generating individual numerical scales within the analytic hierarchy process (AHP). Further, Dong et al. [69,70], proposed a connection link between the two different models, i.e., LCM based on the linguistic hierarchical concept [112] and the model of numerical scale [64], and lastly demonstrated the equivalence of these models to deal with ULTSs. Continuing with the noticeable advancements in the field of 2-tuples LCMs developed for addressing ULTSs, Dong, and Herrera-Viedma [68] and Dong et al. [65] introduced interval numerical scales. Wang et al. [293] proposed a methodology of normalized numerical scaling to form BLTSs or ULTSs into the unique interval $[0,1]$. M. Cai et al. [50] developed a new LCM based on symbolic models. Z. Pei and Li Zheng [233], constructed a series of a normal distribution which is based upon unbalanced linguistic scale sets and additionally, formulated a novel approach for defining unbalanced linguistic information in a format of 3-tuple. Further, M. Cai and Z. Gong [51], redefined the concept of ULTS and provided the graph for the representation model.

Li et al. [160] proposed personalized individual semantics model by virtue of an interval type numerical scale and the linguistic model of 2-tuple and later on solved problems of group decision making with a process of consensus-reaching. Considering the wide-ranging decision problems existing in an uncertain decision environment providing exact values for symbolic proportions is not always easy for decision-makers. Moreover, there exist situations where linguistic distribution assessments are required to assess in ULTS. Therefore, to fix these issues, Dong et al. [72] developed the notion of unbalanced linguistic distribution assessments in the light of interval symbolic proportions and further presented the application of the proposed methodology.

However, concerning the above LCMs, it is envisioned that the representation of the ULTSs is made by using a set of linguistic terms distributed uniformly. As a consequence, the literal meaning of a ULTS is missing. Thereby in this study, we are taking the theory of ULTS a step forward. In this chapter, we aim to develop a methodology to represent and manage ULTS. The representation of the ULTS is given by the multiplicative unbalanced linguistic scale set initially proposed by Tang et al. [265]. Secondly, we present a novel 2-tuple model for unbalanced linguistic information based upon a notion of minimum distance measures.

Our method is advantageous because of two reasons; firstly, in our approach, a multiplicative linguistic scale set is utilized to represent unbalanced linguistic information where the terms are unevenly distributed. In the existing literature of ULTS, it is mentioned that an evenly distributed linguistic scale set is utilized to express the original unbalanced information. As a result, it leads to the absence of the existence of a formal definition of unbalanced linguistic information within the problem. Henceforth, we able to subjugate the limitation by employing a multiplicative linguistic scale set to substitute the given unbalanced information. In this way, throughout the procedure, a formal definition of ULTS is preserved. Secondly, our proposed method has a more straightforward design. As a consequence, it is computationally less expensive and less complicated in comparison to the other LCMs, suchlike, LCM based on linguistic hierarchical concept [112] and numerical scale [64], and so forth.

The organization of the current chapter is presented as follows: Section 3.2 present a representation algorithm for ULTS and further propose a new 2-tuple unbalanced linguistic computational model (ULCM) to address ULTS. Next, for the sake of completeness, we define basic aggregation operators based on a new 2-tuple ULCM in Section 3.3. To validate the proposed methodology and illustrate the practical relevance of the method in real-world, 2 numerical examples related to the ULTS of first and second types are presented in section 3.4. Further, Subsection 3.4.1 gives an analysis and comparison of the proposed model with other existing works. Lastly, Section 3.5 covers the concluding remarks for the model presented in the chapter.

3.2 Methodology for unbalanced linguistic term set

For reviewing the basics of the ULCMs one can refer to the section 1.4. Here, we proceed to briefly explain the methodology constructed for managing unbalanced linguistic information. For doing this, we design a novel approach to represent semantics to the linguistic terms that belong to a ULTS and finally propose the new 2-tuple ULCM.

3.2.1 Representation algorithm for unbalanced linguistic set

In literature, Tang et al. [265] defined a multiplicative linguistic scale set distributed throughout inconsistently as $LS=\{t_{a^{(-m)}}, t_{a^{-(m-1)}}, \ldots, t_{a^0}, \ldots, t_{a^m}\}$ where $a \in \mathbb{R}$ and $m \in \mathbb{Z}^+$.

Here, before we step forward to present a representation algorithm for ULTS, we state the following definition.

Definition 3.2.1. Let $LS=\{t_{a^{(-n)}}, t_{a^{-(n-1)}}, \ldots, t_{a^0}, \ldots, t_{a^m}\}$ where, $a > 1$, $n, m \in \mathbb{Z}^+$ be a multiplicative ULTS with cardinality $n+m+1$ such that n and m may or may not be equal. Then we call the term t_{a^i} as a plausible value for a LV. The set LS is termed as an ordered set if the following properties are satisfied:

(P1): The set LS is ordered: $t_{a^i} \geq t_{a^j}$ for $i \geq j$;

(P2): A maximization operator: $\max(t_{a^i}, t_{a^j}) = t_{a^i}$ for $i \geq j$;

(P3): A minimization operator: $\min(t_{a^i}, t_{a^j}) = t_{a^i}$ for $i \leq j$.

We now proceed to present the representation algorithm in order to manage unevenly distributed unbalanced linguistic information:

Step 1 Partition the ULTS LS into the following three subsets, i.e, $LS = LS_{left} \cup LS_{center} \cup LS_{right}$:

- LS_{left} defines the set of linguistic terms lying left to the central term.
- LS_{center} defines the set of central term.
- LS_{right} defines the set of linguistic terms lying right to the central term.

Step 2 Set $\#(LS_{left}) = n$ and $\#(LS_{right}) = m$, where $\#(LS_{left})$, $\#(LS_{right})$ are cardinalities of the set LS_{left}, LS_{right} respectively.

Step 3 Representation of the central linguistic label set.

Algorithm 3.2.1. Identify the central linguistic term such that $t_{i_c} \leftarrow t_{a^0}$. For the central linguistic term, $\text{neg}(t_{a^i}) = t_{a^i}$ for $i \in \{-n, -(n-1), \ldots, 0, 1, \ldots, m\}$

Step 4 Represent the left linguistic label set. The decision rule of representation of set LS_{left} is given below.

Algorithm 3.2.2. Representation of set LS_{left}

$i_L \leftarrow \#(LS_{left})$;

while $i_L \geq 1$ **do**

  $t_{i_L} = t_{a^{-i_L}}$;

  $i_L = i_L - 1$.

end

Step 5 Represent the right linguistic label set. The decision rule of representation of set LS_{right} is given below

Algorithm 3.2.3. Representation of set LS_{right}

$i_R \leftarrow 1$;

while $i_R \leq \#(LS_{right})$ **do**

  $t_{i_R} = t_{a^{i_R}}$;

  $i_R = i_R + 1$.

end

Example 3.2.1. To present the working of the proposed representation method, we consider the same example of a grading system evaluation taken from [112] (see Fig. 3.1).

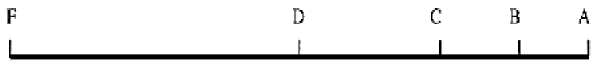

Figure 3.1: Grading system evaluation [112].

We express $S = \{F, D, C, B, A\}$ with $\#(S) = 5$, in the form of set $\{t_{a^{-1}}, t_{a^0}, t_{a^1}, t_{a^2}, t_{a^3}\}$. According to the representation method, partition the set S, i.e, $S_{left} = \{F\}$, $S_{right} = \{C, B, A\}$ and $S_{center} = \{D\}$. Hence, $n = 1$ and $m = 3$. After implementing the decision rule, we obtain

$F \leftarrow t_{a^{-1}}, D \leftarrow t_{a^0}, C \leftarrow t_{a^1}, B \leftarrow t_{a^2}, A \leftarrow t_{a^3}$.

After developing a methodology to represent semantics to the terms of ULTS, we proceed next to develop a novel 2TL model for ULTS to effectuate the process of CWW in a precise way without any information loss.

3.2.2 2-tuple representation for unbalanced linguistic set

In the present subsection, we formalize the new 2-tuple based computational model for the ULTS based on a notion of minimum distance measure. In this framework, the so-called translation function defined converts a numerical value into a 2-tuple represented as (t_λ, γ) where $t_\lambda \in LS$ and $\gamma \in [-0.5, 0.5]$ is referred as a symbolic translation and vice-versa. The 2TL representation discussed in the present section is defined on the similar line of the representation proffered in the paper [106], however, the computation of λ and γ are different. After demarcating the translation function, we present different computational operators suchlike negation operator and comparison operator to manage the unbalanced linguistic information.

Let LS=$\left\{t_{a^{(-n)}}, t_{a^{-(n-1)}}, \ldots, t_{a^0}, \ldots, t_{a^m}\right\}$ where, $a > 1$ and $n, m \in \mathbb{Z}^+$ and n, m may or may not be equal is a multiplicative ULTS whose terms are non uniformly and inconsistently distributed. The numerical value $\beta \in [a^{-n}, a^m]$, where n and m may or may not be equal and $\beta \notin \left\{t_{a^{(-n)}}, t_{a^{-(n-1)}}, \ldots, t_{a^0}, \ldots, t_{a^m}\right\}$ is obtained as a value from a symbolic method aggregating unbalanced linguistic information. Subsequently an approximation function is utilized to represent the index of the outcome in an unbalanced set LS.

We shall now move a step forward to develop a novel approach to represent an unbalanced linguistic information in terms of 2-tuples (t_λ, γ), where $t_\lambda \in LS$ and $\gamma \in [-0.5, 0.5]$:

1. t_λ expresses the linguistic label which is the center of information where λ is of the form a^i for $i \in \left\{-n, -(n-1), \ldots, 0, \ldots, m\right\}$;

2. γ expresses a numerical value computed as a ratio of $(\beta - \lambda)$ and sum of d_L & $|d_R|$ where λ is the index of closest label and d_L & d_R represent nearest left and right distance of λ from β respectively.

Next, we define a translation function giving 2TL value encrypted in the space $LS \times [-0.5, 0.5]$ from a numerical value $\beta \in [a^{-n}, a^m]$.

Definition 3.2.2. Let $LS = \left\{t_{a^{(-n)}}, t_{a^{-(n-1)}}, \ldots, t_{a^0}, \ldots, t_{a^m}\right\}$ where, $a > 1$ and $n, m \in \mathbb{Z}^+$ and n, m may or may not be equal, be a multiplicative ULTS having $n + m + 1$ as cardinality. Let $\beta \in [a^{-n}, a^m]$ be a numerical value that supports symbolic aggregation result. Then the translation function that translates a numeric value β into a corresponding 2-tuple unbalanced linguistic value is shown below:

$$\Omega : [a^{-n}, a^{m}] \longrightarrow LS \times [-0.5, 0.5]$$

$$\Omega(\beta) = (t_{\lambda}, \gamma)$$

in which $\begin{cases} \lambda = a^{i_L} \text{ or } \lambda = a^{i_R} & \text{for } i_L, i_R \in \{-n, \ldots, 0, \ldots, m\}; \\ \gamma = \frac{\beta - \lambda}{d_L + |d_R|} & \text{for } \gamma \in [-0.5, 0.5]. \end{cases}$

The decision rule for the computation of values λ and γ defined above in the definition is given in algorithm 3.2.4.

Algorithm 3.2.4. Input: $LS = \{t_{a^{(-n)}}, t_{a^{-(n-1)}}, \ldots, t_{a^0}, \ldots, t_{a^m}\}$ & $\beta \in [a^{-n}, a^{m}]$, where n and m may or may not be equal and $a > 1$.

Begin:

$i_L \leftarrow \arg\min_i \{\beta - a^i \mid \beta - a^i \geq 0, i \in \{-n, \ldots, 0, \ldots, m\}\}$;

$i_R \leftarrow \arg\max_i \{\beta - a^i \mid \beta - a^i \leq 0, i \in \{-n, \ldots, 0, \ldots, m\}\}$. **Then:**

$\lambda_L \leftarrow a^{i_L}$, and $\lambda_R \leftarrow a^{i_R}$, such that $\beta \in (\lambda_L, \lambda_R)$.

Next, compute:

$d_L \leftarrow \beta - \lambda_L$, and $d_R \leftarrow \beta - \lambda_R$.

Define: $d \leftarrow \min(d_L, |d_R|)$;

if $d = d_L$ **then**

$\lambda \leftarrow \lambda_L$;

$\gamma \leftarrow \frac{\beta - \lambda}{d_L + |d_R|}$.

else

$\lambda \leftarrow \lambda_R$;

$\gamma \leftarrow \frac{\beta - \lambda}{d_L + |d_R|}$.

end if

Note 3.2.1. We can infer from the Algorithm 3.2.4 that the "λ" value is evaluated by virtue of the nearest distance measure from the numerical value "β" whereas "γ" is termed as a value of "symbolic translation." However, according to the algorithm 3.2.4, the range of the value "γ" belongs to $[-0.5, 0.5]$.

Proposition 3.2.1. The symbolic translation $\gamma \in [-0.5, 0.5]$.

Proof. We know, $\gamma = \frac{\beta - \lambda}{d_L + |d_R|}$ where $\lambda = \lambda_R \text{ or } \lambda_L$, and $\beta \in [\lambda_L, \lambda_R] \subseteq [a^{-n}, a^{m}]$. We consider the following two cases:

Case 1: $\lambda = \lambda_L$
In this case, $d_L \leq |d_R| \implies 2d_L \leq |d_R| + d_L \implies \frac{\beta-\lambda_L}{d_L+|d_R|} \leq \frac{1}{2} \implies \gamma \leq \frac{1}{2}$.

Case 2: $\lambda = \lambda_R$
In this case, $|d_R| \leq d_L \implies 2|d_R| \leq d_L + |d_R| \implies \frac{-|d_R|}{d_L+|d_R|} \geq \frac{-1}{2} \implies \frac{\beta-\lambda_R}{d_L+|d_R|} \geq \frac{-1}{2}$
$\implies \gamma \geq \frac{-1}{2}$.

□

We next put forward the concept of numerical scale function based on ULTS. In literature, Dong et al. [64] suggested the notion of a numerical scale based on a BLTS, $\mathbf{LT} = \{\ell_i \,|\, i = 0 \text{ to } g\}$ as given in Definition 1.4.6. Analogously, we provide a definition of a numerical scale based on a multiplicative linguistic scale set, LS.

Definition 3.2.3. Let LS=$\{t_{a^{(-n)}}, t_{a^{-(n-1)}}, \ldots, t_{a^0}, \ldots, t_{a^m}\}$ where, $a > 1$ and $n, m \in \mathbb{Z}^+$ and n, m may or may not be equal is a multiplicative ULTS and $\mathbb{R}$ has the usual meaning in mathematical terms. Then we define a numerical scale function as $NS : LS \rightarrow \mathbb{R}$ such that $NS(t_{a^i}) = a^i$.

Definition 3.2.4. Numerical scale function $\overline{NS} : LS \times [-0.5, 0.5] \rightarrow \mathbb{R}$ is defined as follows:

$$\overline{NS}((t_{a^i}, \gamma)) = \begin{cases} NS(t_{a^i}) + \gamma \times ((NS(t_{a^{i+1}})) - NS(t_{a^i})), \ \gamma \geq 0; \\ NS(t_{a^i}) + \gamma \times ((NS(t_{a^i})) - NS(t_{a^{i-1}})), \ \gamma < 0. \end{cases}$$

Now, we proceed further to give a proposition to obtain our model from a numerical scale.

Proposition 3.2.1. When setting $NS(t_{a^i})$=a^i where $i = -n, -(n-1), \ldots, 0, 1, \ldots, m$, we have $NS((t_{a^i}, \gamma_{a^i})) = \Omega^{-1}((t_{a^i}, \gamma_{a^i}))$, for any $(t_{a^i}, \gamma_{a^i}) \in LS \times [-0.5, 0.5]$.

Proof. Consider the following two cases:

Case 1: $\gamma \geq 0$
In this case, $a^i = a^{i_L}$ and $a^{i+1} = a^{i_R}$ for $i = -n, -(n-1), \ldots, 0, 1, \ldots, m$; $\Omega^{-1}(t_{a^i}, \gamma) = \gamma \times (d_L + |d_R|) + a^i = \gamma \times (\beta - a^i + |(\beta - a^{i+1})|) + a^i$; since $\beta < a^{i+1}$ therefore $\Omega^{-1}(t_{a^i}, \gamma) = \gamma \times (a^{i+1} - a^i) + a^i$. Now, from the definition of numerical scale function, we have $NS((t_{a^i}, \gamma)) = a^i + \gamma \times (a^{i+1} - a^i)$. Hence, $NS((t_{a^i}, \gamma)) = \Omega^{-1}((t_{a^i}, \gamma))$.

Case 2: $\gamma < 0$

In this case, $a^i = a^{i_R}$ and $a^{i-1} = a^{i_L}$ for $i = -n, -(n-1), \ldots, 0, 1, \ldots, m$; $\Omega^{-1}(t_{a^i}, \gamma) = \gamma \times (d_L + |d_R|) + a^i = \gamma \times (\beta - a^{i-1} + |(\beta - a^i)|) + a^i$; since $\beta < a^i$ therefore $\Omega^{-1}(t_{a^i}, \gamma) = \gamma \times (a^i - a^{i-1}) + a^i$. Now, from the definition of numerical scale function, we have $NS((t_{a^i}, \gamma)) = a^i + \gamma \times (a^i - a^{i-1})$. Hence, $NS((t_{a^i}, \gamma)) = \Omega^{-1}((t_{a^i}, \gamma))$.

□

We provide the following example for a better understanding of the concept of new 2-tuple ULCM.

Consider a multiplicative ULTS $LS = \left\{t_{\frac{1}{8}}, t_{\frac{1}{4}}, t_{\frac{1}{2}}, t_1, t_2, t_4, t_8\right\}$, here we have taken $a = 2$ (not specific) and $n = m = 3$. Suppose $\beta = 5.8 \in [\frac{1}{8}, 8]$ be the value obtained as a result of symbolic aggregation. Then, the representation of β in terms of 2-tuple is given as

$$\Omega(5.8) = (t_4, 0.45).$$

The graphical representation of the 2-tuple computation calculated above is given in the Fig. 3.2.

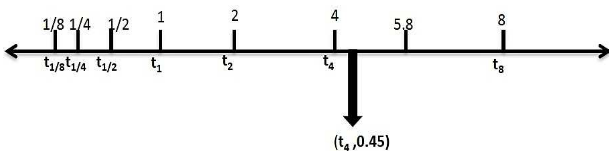

Figure 3.2: Graphical representation of 2-tuple for ULTS

Proposition 3.2.2. Let $LS = \left\{t_{a^{(-n)}}, t_{a^{-(n-1)}}, \ldots, t_{a^0}, \ldots, t_{a^m}\right\}$ be an ULTS and let (t_λ, γ) be a 2TL corresponding to numerical value $\beta \in [a^{-n}, a^m] \subset \mathbb{R}$. Then, there always exist a function Ω^{-1} which returns an equivalent numerical information from a corresponding 2TL information.

Proof. It is trivial to show that a translation function Ω is a bijective. We consider the following function:

$$\Omega^{-1} : LS \times [-0.5, 0.5] \longrightarrow [a^{-n}, a^m]$$

$$\Omega^{-1}(t_\lambda, \gamma) = \gamma \times (d_L + |d_R|) + \lambda = \beta. \square$$

Remark 3.2.1. A linguistic term $t_\alpha \in LS$ can be easily converted into a 2-tuple by adding a value 0 as minimum distance measure, i.e., $t_\alpha \in LS \implies (t_\lambda, 0)$.

Remark 3.2.2. It is noted that if $t_\alpha \in LS$ then t_α is termed as actual unbalanced linguistic term and meanwhile if $t_\alpha \notin LS$ then t_α is termed as virtual unbalanced linguistic term.

Herrera and Martínez [110] has proposed a comparison of a 2TL information based upon classical lexicographic ordering. Analogously, we present a comparison of 2TL information for ULTS in compliance with the existing lexicographic approach.

Definition 3.2.5. Let $(t_{\lambda_i}, \gamma_i)$, $(t_{\lambda_j}, \gamma_j)$ be represented as two 2-tuples. Then, the 2-tuples are compared as follows:

- If $\lambda_i < \lambda_j$ then $(t_{\lambda_i}, \gamma_i) < (t_{\lambda_j}, \gamma_j)$;
- If $\lambda_i = \lambda_j$ then
  1. If $\gamma_i = \gamma_j \implies (t_{\lambda_i}, \gamma_i) = (t_{\lambda_j}, \gamma_j)$;
  2. If $\gamma_i < \gamma_j \implies (t_{\lambda_i}, \gamma_i) < (t_{\lambda_j}, \gamma_j)$;
  3. If $\gamma_i > \gamma_j \implies (t_{\lambda_i}, \gamma_i) > (t_{\lambda_j}, \gamma_j)$.

Definition 3.2.6. Let $LS = \left\{t_{a^{(-n)}}, \ldots, t_{a^0}, t_{a^1}, \ldots, t_{a^m}\right\}$ be an ULTS with cardinality $n + m + 1$. Then we define negation operator for a 2-tuple as:

$$\mathrm{neg}(t_{a^i}, \gamma) = (\mathrm{neg}(t_{a^i}), -\gamma)$$

where the existence of a $\mathrm{neg}(t_{a^i})$ for an unbalanced linguistic term in all the plausible cases is given in Algorithm 3.2.5.

Algorithm 3.2.5. Input: LS=$\left\{t_{a^{(-n)}}, t_{a^{-(n-1)}}, \ldots, t_{a^0}, \ldots, t_{a^m}\right\}$ where n and m may or may not be equal and $a > 1$.

$LS = LS_{left} \cup LS_{center} \cup LS_{right}$.

Begin:

$\#(LS_{left}) \leftarrow n$, and, $\#(LS_{right}) \leftarrow m$ **Then:**

if $n = m$ **then**

  for $i \leftarrow 0$ to n **do**

    $\mathrm{neg}(t_{a^i}) \leftarrow t_{a^{-i}}$;

  end for

end if

```
if n ≠ m then
    if n < m then
        for i ← 0 to m do
            if i ≤ n then
                neg(t_{a^i}) ← t_{a^{-i}};
            else
                neg(t_{a^i}) ← t_{a^{m-i}};
            end if
        end for
    else
        for i ← 0 to n do
            if i ≤ m then
                neg(t_{a^{(-i)}}) ← t_{a^i}; else

                end
                neg(t_{a^{(-i)}}) ← t_{a^{i-n}};
            end if
        end for
    end if
end if
```

Remark 3.2.3. Based on the property of linearity and monotonicity of the Ω operator, it is easily observed that for the case $n = m$, the following holds:

(i) $\text{neg}(\text{neg}(\tilde{v}_1)) = \tilde{v}_1$;

(ii) $\min\{\text{neg}(\tilde{v}_1), \text{neg}(\tilde{v}_2)\} = \text{neg}(\max\{\tilde{v}_1, \tilde{v}_2\})$;

(iii) $\max\{\text{neg}(\tilde{v}_1), \text{neg}(\tilde{v}_2)\} = \text{neg}(\min\{\tilde{v}_1, \tilde{v}_2\})$.

where $\tilde{v}_1 = (t_{\lambda_1}, \gamma_1)$, $\tilde{v}_2 = (t_{\lambda_2}, \gamma_2)$ are two unbalanced linguistic 2-tuple variables and the max and min operations are considered as given in the Definition 3.2.1.

After, discussing the existence of negation operator for ULTS, we wish here, to point out three observations:

- In first place, we consider the case, $n \neq m$, for instance take $n = 2$ and $m = 3$ such that $LS=\{t_{a^{-2}}, t_{a^{-1}}, t_{a^0}, t_{a^1}, t_{a^2}, t_{a^3}\}$. Then according to the Algorithm 3.2.5, it is stated that $\text{neg}(t_{a^1}) = t_{a^{-1}}$, $\text{neg}(t_{a^2}) = t_{a^{-2}}$, $\text{neg}(t_{a^3}) = t_{a^0} = \text{neg}(t_{a^0})$. Henceforth, we may claim that for the case $n \neq m$, $\text{neg}(\text{neg}(t_{a^i})) \neq t_{a^i}$ for $i \in \{-2, -1, 0, 1, 2, 3\}$.

- The case $n \neq m$ can be converted it into the case $n = m$, by adding dummy linguistic variables, 'd_{a^i}' for $i \in \{-n, \ldots, 0, \ldots, m\}$ to the set LS. Hence, proceed likewise as described in Algorithm 3.2.5.

- The dummy linguistic variables which are added to the set LS are given 'no' weights. As a result, in several MCDM problems or linguistic game-theoretic models, we cannot be able to obtain an optimal solution for such a case. This validates our proposed Algorithm 3.2.5.

When we have expressed all the unbalanced linguistic terms and have also defined the linguistic information in terms of 2-tuples, it, therefore, necessitates us to aggregate such information. Henceforth, the following section discusses the aggregating operators for a new 2-tuple ULCM.

3.3 Aggregation operators for 2-tuple model for unbalanced linguistic term set

Several aggregation operators have been developed in literature [106] for the traditional 2TL model based upon "*symbolic translation*." The aggregating operation carried out by these so-called aggregation operators are without any information loss.

Since the primary focus of the aggregation operator is to mitigate value set into a single one that gives an overview of the inputs in a particular way. As a consequence, it plays a significant role in decision problems and further amalgamation of information. However, the information to be considered is not always balanced. Sometimes, unbalanced linguistic information is also involved in the problem. Henceforth, in the present section, we extend these operators for the developed 2-tuple ULCM.

The transformation function Ω and Ω^{-1} discussed in the previous section helps in converting a numerical value into a linguistic value given in the form of 2-tuples and vice-versa.

(A) *Arithmetic mean (AM) operator*

Definition 3.3.1. Let $X = \{(t_{\lambda_1}, \gamma_1), (t_{\lambda_2}, \gamma_2), \ldots, (t_{\lambda_k}, \gamma_k)\}$ be the 2-tuple ULTS, the arithmetic mean operator $\overline{X}^{AM}$ is defined as:

$$\overline{X}^{AM} = \Omega\left(\frac{\sum_{i=1}^{k} \Omega^{-1}(t_{\lambda_i}, \gamma_i)}{k}\right).$$

The functions Ω and Ω^{-1} have their usual meaning, as discussed in the previous section. Moreover, the $\overline{X}^{AM}$ estimated for the set X is incurred in a precise way without any information loss.

(B) *Weighted average (WA) operator*

In the case of weighted average, considering the nature of the variable x, distinct values say x_i have different importance. As a consequence, each value x_i is associated with a weight w_i, which signifies the variable's nature.

In literature, the equivalent operator exists for the Herrera and Martinez' [106] 2TL model. Likewise, we can extend the theory concerning the WA operator for a ULCM based on 2-tuples.

Definition 3.3.2. Let $X=\{(t_{\lambda_1}, \gamma_1), (t_{\lambda_2}, \gamma_2), \ldots, (t_{\lambda_k}, \gamma_k)\}$ be the 2-tuple ULTS and let $W = \{w_i \mid i = 1, 2, \ldots, k\}$ be the set of weights associated with 2TL set, then the WA operator $\overline{X}^{WA}$ for the set X is defined as:

$$\overline{X}^{WA} = \Omega\left(\frac{\sum_{i=1}^{k} \Omega^{-1}(t_{\lambda_i}, \gamma_i) \cdot w_i}{\sum_{i=1}^{k} w_i}\right)$$

(C) *Ordered weighted average (OWA) operator*

In literature, the weighted aggregation operator is initiated by Yager [304]. In that operator, the weights which are given are not connected with a predetermined value. Instead, it is provided that the weights are connected to a fixed position.

The OWA operator X^{OWA} for dealing with 2-tuple ULCM are defined as:

Definition 3.3.3. Let $X=\{(t_{\lambda_1}, \gamma_1), (t_{\lambda_2}, \gamma_2), \ldots, (t_{\lambda_k}, \gamma_k)\}$ be the 2-tuple uLTS and let $W = \left\{w_i \,|\, i = 1, 2, \ldots, k,\ \sum_{i=1}^{k} w_i = 1 \text{ and } w_i \in [0, 1]\right\}$ be the set of associated weights. Then the 2-tuple OWA operator X^{OWA} is computed as:

$$X^{OWA} = \Omega\left(\sum_{j=1}^{k} w_j \cdot \beta_j^*\right).$$

where the numerical value β_j^* represents the largest of j^{th} value among all the β_i values.

3.4 Numerical illustration and comparative analysis

In the present section, we consider 2 numerical examples to exhibit the efficacy of the proposed 2-tuple ULCM in the physical realm.

In the Example 3.4.1, we consider the usual problem of evaluating students' expertise from variant tests to obtain a global evaluation, which is a similar example discussed in Herrera et al. [112]. The ULTS utilized here is of type I, i.e., a set where the distribution of the linguistic terms on the left and right side of the central term is unequal ($m \neq n$).

The Example 3.4.2 is related to the stock market, which is an organized financial market and plays a significant role in raising the economic infrastructure. In the current scenario, it is envisioned that most of the car manufacturing companies in India are undergoing huge losses. Consequently, affecting Indian stock market saliently. Henceforth, the current study provides a comparative analysis of the stock prices for different car manufacturing companies in the financial year 2018. The ULTS utilized here is of type II, i.e., several linguistic terms distributed on the left and right sides of the central term are equal ($m = n$).

Example 3.4.1. Suppose a teacher wants to acquire a maximum evaluation of his pupils who have appeared in 6 different exams equally essential. The exams are completed, and the evaluations are being done using the grading system. The grading scale is considered to be a ULTS, LS, which is given in example 3.2.1. Also, the respective grades obtained by students in each exam are given in table 3.1.

	E1	E2	E3	E4	E5	E6
J. Smith	(D,0)	(C,0)	(B,0)	(C,0)	(C,0)	(C,0)
M. Grant	(A,0)	(D,0)	(D,0)	(C,0)	(B,0)	(A,0)

Table 3.1: 2-tuple linguistic assessment of unbalanced set [112].

Herrera et al [112] has solved the example to obtain global evaluations for 2 students based on the linguistic hierarchical concept for unbalanced linguistic information. The detailed calculations and required steps are provided in the paper [112].

Now, for the similar mission we solve this example by using our methodology. According to our approach, we initially apply the semantic representation algorithm for unbalanced linguistic information. The descriptive information about the working of this algorithm is given in example 3.2.1. From the analysis, we infer that $a = 2$ to be the best value for the base. However, one may take any value of 'a.' Hence, we obtain $LS_{left} : \{F \leftarrow t_{2^{-1}}\}$, $LS_{center} : \{D \leftarrow t_{2^0}\}$, $LS_{right} : \{C \leftarrow t_{2^1}, B \leftarrow t_{2^2}, A \leftarrow t_{2^3}\}$.

Next, to compute the global evaluations of the students by applying arithmetic mean operator defined in section 3.3, we proceed as follows:

$$\begin{aligned}\bar{x}_{J.Smith} &= \Omega\left(\frac{1}{6}\left(\Omega^{-1}(D,0) + 4\times\Omega^{-1}(C,0) + \Omega^{-1}(B,0)\right)\right)\\ &=\Omega\left(\frac{1}{6}\left(\Omega^{-1}(t_{2^0},0) + 4\times\Omega^{-1}(t_{2^1},0) + \Omega^{-1}(t_{2^2},0)\right)\right)\\ &=\Omega(2.167) = (C, 0.0835).\end{aligned}$$

$$\begin{aligned}\bar{x}_{M.Grant} =&\Omega\left(\frac{1}{6}\left(2\times\Omega^{-1}(A,0) + 2\times\Omega^{-1}(D,0) + \Omega^{-1}(B,0)\right.\right.\\ &\left.\left.+\Omega^{-1}(C,0)\right)\right)\\ =&\Omega\left(\frac{1}{6}\left(2\times\Omega^{-1}(t_{2^3},0) + 2\times\Omega^{-1}(t_{2^0},0) + \Omega^{-1}(t_{2^2},0)\right.\right.\\ &\left.\left.+\Omega^{-1}(2^1,0)\right)\right)\\ =&\Omega(4.00) = (B, 0).\end{aligned}$$

Hence, from the above analysis, we observe that M. Grant has scored maximum grades in comparison to J. Smith.

The ranking order of the two students so obtained is similar to the order obtained in the paper [112]. Henceforth, we conclude that our proposed model is consistent with the model of Herrera et al. [112].

Example 3.4.2. In this illustration, we have considered a fundamental real-life decision problem where the computations are made using the type II ULTS discussed in the previous section.

Suppose an investor wants to invest his/her money into a particular car manufacturing company by taking into consideration that which company has a high market value as well as high share growth in the financial year 2018 (FY2018). The alternatives of the Indian car companies that an investor is looking up in particular are:

x_1	x_2	x_3
"Maruti Suzuki India"	"Tata Motors"	"Mahindra & Mahindra"

The factors which affect the share prices of the aforementioned car companies are broadly classified into three classes: (1) Macroeconomic class, (2) Microeconomic class, and (3) Class of linguistic factors. The list of these factors is given in table 3.2.

Macro economic factors	Micro economic factors	Class of linguistic factors
M_1: Gross domestic product (GDP)(%)	m_1: Book value per share (In INR)	ℓ_1: Investors confidence
M_2: Interest rate (%)	m_2: Depreciation (In Rs Cr)	ℓ_2: Risk of Investment
M_3: Inflation rate (%)	m_3: Dividend per share (In Rs)	
M_4: Index of industrial production (IIP)(%)	m_4: Net profit margin (%)	

Table 3.2: List of factors affecting stock prices

Each factor equally affects the price of the stock market. The evaluations of the stock prices for the auto company affected by these factors are assessed in the unbalanced linguistic scale set LS={VL, L, M, H, VH} where VL: "Very low," L: "Low," M: "Moderate," H: "High," VH: "Very high". The Table 3.3 provides the values of different independent factors that affect the stock prices of the auto company for the FY2018. The data given in the Table 3.3 is taken from https://www.moneycontrol.commoneycontrol and https://www.rbi.org.inReserve bank of India.

Apart from the listed quantitative factors, however, there exist certain qualitative factors that also affect the stock prices. So, to incorporate these factors and further obtain accurate results without any information loss, we apply a 2-tuple model for managing unbalanced linguistic information.

Based on the values recorded in Table 3.3 along with the qualitative factors, we obtain the fluctuations of the stock prices for the auto companies. The variations incurred in the values of the stock prices for the companies due to these factors for the FY2018 are evaluated in the unbalanced linguistic scale set (as shown in the Table 3.4).

Factors	x_1	x_2	x_3	
GDP				6.8%
Interest rate				6.4%
Inflation rate				3.48%
IIP				31.4%
Book value per share	8,648.198	294.65	837.08	
Depreciation (In Rs Crs.)	727.80	770.55	431.80	
Increase/Decrease in stocks (In Rs Crs.)	−331.98	−231	98.14	
Net Profit/Loss for the period (In Rs Crs.)	1896.78	353.62	1251.58	
Dividend per share (In Rs)	80.00	0.00	7.50	
Net profit margin (%)	9.68	−1.75	8.98	

Table 3.3: Values of the Independent factors affecting stock prices for the FY2018.

	Factors									
	M_1	M_2	M_3	M_4	m_1	m_2	m_3	m_4	ℓ_1	ℓ_2
x_1	VH	L	VL	VH	VH	VL	VH	VH	VH	VH
x_2	L	H	L	H	L	VH	L	VL	L	H
x_3	H	L	H	L	L	H	L	VH	H	L

Table 3.4: Evaluations of the factors for each company

Next, we apply our technique to deal with unbalanced linguistic information to obtain which company has a higher growth in share market for the FY2018. For this, we follow the similar steps as done in previous example. Henceforth, according to the semantic representation algorithm we obtain $LS = \{t_{2^{-2}} : \text{VL}, t_{2^{-1}} : \text{L}, t_{2^0} : \text{M}, t_{2^1} : \text{H}, t_{2^2} : \text{VH}\}$. Lastly, we convert the unbalanced linguistic evaluations obtained for each alternatives into 2TL representation for further computations.

Now, these 2-tuples are aggregated by utilizing a 2-tuple arithmetic mean operator for ULTS, thereby obtaining the collective values given in the table 3.5.

x_1	x_2	x_3
(H, 0.45)	(M, 0.275)	(M, 0.45)

Table 3.5: Aggregated results

Hence, the solution set of alternatives we obtain as $\{x_1\}$, i.e., “Maruti Suzuki India” has a high share growth for the FY2018. Therefore an investor would likely invest his money into the same company.

3.4.1 Analysis and Comparison

The present section aims to provide a comparative study of our model with the existing works in an effort to validate the efficiency of our proposal.

So far, computational models existing in the literature [72] deal with ULTSs proposed by Herrera et al. [107]. The authors in that paper considered ULTS, $LT = LT_L \cup LT_C \cup LT_R$ where LT_L represents a minimum label set, LT_R represents a maximum label, and LT_C represents a central label set. As a consequence, the methodology depicted could be able to address only the first class of ULTS (See Figure 1.6). In the sequent, methods portrayed in the paper [64, 66, 271] could be able to manage the second class of ULTS (See Figure 1.7). However, in contrast to the existing methodologies, our proposal is well-suited to handle both the classes of ULTS. In our proposal, the ULTS LS is partitioned as $LS = LS_{left} \cup LS_{center} \cup LS_{right}$ where it is no longer necessary for the set LS_{left} and LS_{right} to be represented as minimum, and maximum label set respectively. The details of the representation algorithm are given in Section 3.2. Henceforth, we contemplate our proposed definition to handle any real-world situations involving unbalanced linguistic information.

The example 3.4.1 given in the previous section is from the paper [112]. According to the results obtained in the corresponding paper 'M. Grant' scored maximum in comparison to 'J. Smith' which is analogous to the results shown in this chapter. Therefore, the comparison of our results exhibits the consistency of our proposal.

Furthermore, it is seen in the papers [50, 51, 64, 112, 271] that ULTS is expressed by a symmetrically distributed LTS. So, in our proposal we work on to overcome this limitation and define ULTS as $LS = \left\{t_{a^{(-n)}}, t_{a^{-(n-1)}}, \ldots, t_{a^0}, \ldots, t_{a^m}\right\}$ where $a \in \mathbb{R}$. The terms distributed in the set LS are globally inconsistent. Our proposal is not subjected to any transformation function converting unbalanced linguistic scale set into a balanced set which might be a tedious and time taking job in several large scale decision problems.

Although any group decision problems dealing with unbalanced linguistic information can be tackled by using well-known methodologies provided in the literature, our approach is an improvement to offer acceptable results at a faster rate. Moreover, it is witnessed that the time complexity of our model is linear and further involves less number of computation steps. Consequently, it's computational cost decreases in contrast with the model-based upon linguistic hierarchy [112]. Lastly, the design of our model is much simpler, which is considered to be the most significant component of our proposal.

3.5 Conclusion

In this chapter, we have formulated a novel approach to handle unbalanced linguistic information wherein linguistic information is presupposed to be asymmetrically and non-uniformly distributed. Further, we have designed an algorithm to represent semantics to the unbalanced linguistic terms and afterward developed a 2-tuple model for ULTS based on the concept of minimum distance measure. For the sake of completeness, we provided basic aggregation operators based on Ω and Ω^{-1} functions. The utility of our methodology lies in the fact that it is less expensive and further dispels the complexity instead of other existing methods that are based on linguistic hierarchy [112]. Lastly, two numerical illustrations are provided to validate and demonstrate the usability of the suggested model in the physical realm.

The proposed model is generalized for both the types of ULTSs and also useful for the cases where linguistic information is assessed in the set suchlike $LS = \{$ very low, fair, high, very high $\}$. It is noted here that the distance between each of the consecutive terms assessed in the set is unequal. However, to carry out the computations, one may proceed by adding a dummy variable to the set LS. It is envisioned that the addition of dummy variable into the set LS does not provide us with the optimal solution in the several cases suchlike cases of MCDM problems, fuzzy linguistic-based game-theoretic models, financial problems, and so forth which may lead to a limitation.

Hence, one can work for the further extension of this model to subjugate the limitation associated with it. To complete this model, we give an exhaustive study relevant to the aggregation operators based on our model in the future and further give a comparison. Moreover, we should emphasize that this model has increased the efficacy of several linguistic solving processes and has produced results in less computation time, which, in a way, center the continuity of work.

Chapter 4

Matrix games with Probabilistic multiplicative unbalanced linguistic information

Probabilistic linguistic term sets (PLTSs) are regarded as a suitable tool for enunciating evaluators' complex linguistic perceptions more accurately within the intricate qualitative setting. It is viewed that the prevailing PLTS is primarily based on the predefined additive symmetric LTS. Nevertheless, the representation of probabilistic linguistic information is not explicitly limited to the additive symmetric LTS. Sometimes, experts are more affluent to prefer plausible linguistic labels whose distribution is non-uniform. Henceforth, in the present chapter[1], we initially develop a novel concept of the probabilistic multiplicative ULTS that considers the probabilities of distinct linguistic labels and the non-uniformity of the labels. Afterward, we put forward specific operational laws for the newly constructed probabilistic linguistic label set such that the resultant and corresponding probability information of the obtained linguistic labels is preserved. For the sake of convenience, some elementary aggregation operators beneficial in aggregating probabilistic linguistic information in DM problems are also constructed. Furthermore, based on the proposed concept, this study initiates to design a unified two-person linguistic matrix game model with probabilistic multiplicative unbalanced linguistic information as a parameter and addresses the imprecise information by information measure function. Such a two-player probabilistic unbalanced linguistic matrix game is considered a con-

[1]The content of this chapter is based on research paper "Probabilistic multiplicative unbalanced linguistic term set and its application in matrix games" (**Submitted).**

venient technique for multiple decision analysis. Additionally, the proposed game model involves a re-translation process to convert the output back into the information belonging to the original probabilistic unbalanced linguistic domain without losing the information, thereby escalating the interpretability of the game model when compared with other existing uncertain matrix games methodologies. Finally, we discuss the significance of the proposed methodology and concept to question its validity and usefulness by presenting suitable examples.

4.1 Introduction

The formulation of traditional game theory was done 80 years ago which was introduced ideally by Von Neumann and Morgenstern [268]. Since then, the advancement in game theory has gone beyond limitations and has been conclusively applied in diverse fields like economics, finance, sociology, and so forth. It is anticipated that within the traditional setup of games, the payoffs are well-known with certainty. Nevertheless, getting to the practical world, the assumption of certainty is not meaningful on several occasions. As a consequence, the persisting impreciseness within the problem is modeled via contrasting ways, and one can easily encounter in the devoted literature of uncertain matrix games the existence of Interval payoff games [178,179,220], stochastic payoffs games [92,253,254], and fuzzy payoff games [30, 31, 55, 77, 219] to mention a few.

Although the foundation of fuzzy games has overcome the difficulties envisioned in conventional game-theoretic models, it can still not eradicate vagueness. Moreover, in the fuzzy environment, to provide payoff values accurately, players require the understanding of membership or non-membership function and their shape, which causes hindrance to players in practical scenarios. Thus, to efface vagueness that inheres in strategic communication, Arfi [14, 15] anchored game theory in terms of linguistic fuzzy logic, which is a more relaxed and straightforward approach to express information and payoff opinions reasonably without restraining to precision. Since then, many investigations have been directed on building the more robust endowment of linguistic game-theoretic methodologies. Papers in this direction of research are [248, 249].

Meanwhile, if we consider complicated real-life game problems, then it is hard enough for the players to express their qualitative information by taking simple and unique linguistic terms as envisioned in the paper [248]. For instance, while evaluating the class performance of a student, a teacher may think that "the student's score is good in mathematics, but his score is much better in science", the corresponding linguistic term set in this particular case can be identified as, $LT = \{\text{“}good,\text{”}\,\text{“}better\text{”}\}$. In this instance, all possible linguistic labels have equal importance, i.e., the weights of "good" and "better" are 50% and 50%, respectively. Nevertheless, the score obtained in science is "much better", which means it should be considered as more important then "good". To express such kind of information realistically, Pang et al. [235] developed the notion of probabilistic linguistic term set (PLTS). With the PLTS, the players or decision-makers (DMs) provide assessment using several linguistic terms instead of a single term such that the relevance of each linguistic term is assigned a weight through probability. For the above instance, a PLTS can be fixed as $LT = \{\text{“}good\text{”}(0.2), \text{“}better\text{”}(0.8)\}$, which conveys that the precedence of “good” is 0.2 while the precedence of “better” is 0.8. It testifies that the “better” is more significant than the “good”.

Since PLTS provides us a more extensible and comprehensive way for DMs to elicit hesitant linguistic information, many scholars have been dedicated to examining mathematical operations as well as measures of PLTSs [167, 314]. Pang et al. [235] introduced the study of mathematical operations and aggregation operators for PLTSs. However, the final results obtained by them are exceeding the bounds and are not PLTS. To overcome the drawback so that final results are not exceeding the bounds Gou and Xu [101] proposed new probabilistic linguistic operators. Further, the new probabilistic linguistic weighted arithmetic average (WAA) operator was presented in the paper [348]. Moreover, numerous beneficial DM methodologies have also been developed based on the context of PLTSs [93, 169, 175, 211, 282–284]. For a more detailed understanding of the idea of PLTS and to seek its wide-ranging application, one can also refer to the papers [130, 173, 174, 176, 177, 259, 266, 284, 300, 349].

In most available works about PLTSs, it is generally presumed that DMs will utilize a linguistic label set having a uniform distribution of all plausible linguistic labels to provide their evaluations. Nevertheless, in some possible practical DM situations, DMs may prefer to represent more linguistic information lying on the right side rather than on the left side of the mid-term so that the LVs have non-uniform and non-symmetrical distribution. To remedy, Herrera et al. [112] proposed a methodology to address unbalanced linguistic information by making use of non-uniformly distributed linguistic labels. In addition, a lot of functional studies have been established ideally concerning both theoretical and practical aspects of unbalanced linguistic representation models. For instance, Numerical scale model [64, 69], Proportional 2TL model [271], linguistic 3-tuple model [233], Personalized individual semantic model [159, 160] and so forth. In sequence, one can also refer to the papers [57, 72–74, 132, 274] for a deeper understanding of the literature centered on the context of an ULTS. The methodology proposed in these papers is capable of addressing unbalanced linguistic information.

It is noted that the ULTS can reflect the non-uniformity of distinct linguistic labels, while the PLTS can consist of different linguistic terms associated with different weights by utilizing the probabilities. The PLTS promotes the affluence and flexibility of representing convoluted cognitive-linguistic information. From this viewpoint, the construction of probabilistic ULTS focuses on explicating the non-uniform and asymmetrical distribution of linguistic assessment values along with the probability associated with each unbalanced linguistic label. Henceforth, to efficiently express qualitative expression closed to human ideas to deal with complicated decision problems, Han et al. [130] introduced the notion of probabilistic ULTSs. In that paper, the authors developed a novel computational model centered on Archimedean copula and co-copula to manage unbalanced linguistic information. However, the representation of the information is still given by an additive scale set such that the literal meaning of ULTS is missing.

Therefore, to overcome the concerned issue and be motivated by the work presented in Chapter 3, the current Chapter will introduce the notion of probabilistic multiplicative unbalanced linguistic term set (PM-ULTS), reflecting not just the probabilistic information of LVs but also emphasize the irregular and non-symmetric dispersion of linguistic labels to maintain the literal meaning of ULTSs. This is the pioneer research gap filled by the work presented in this study to model probabilistic unbalanced information. Further, we put forth some new operational laws and elementary aggregation operators to aggregate probabilistic multiplicative unbalanced linguistic terms. The operational laws presented in the chapter can easily sustain the operation results without exceeding the boundaries of multiplicative ULTSs and can keep probabilistic information complete after the operations.

Since it is acclaimed that GT can establish mathematical tools of strategic interrelations among rational DMs. Consequently, in several situations, evaluating the payoffs in terms of crisp values is challenging while it is effortless to collect uncertain information in the form of PLTS. Therefore, recently Mi et al. [214] put forward a two-person ZSMG model taking the probabilistic linguistic information as input and imparts imprecise information by using the triangular membership function. The PLTS defined in that paper is based on predefined symmetric LTS. Thus, to take the theory of probabilistic linguistic matrix games a step forward, we investigate the game model within the context of probabilistic multiplicative unbalanced linguistic circumstances, enhancing the feasibility and flexibility of the game theory scope. This is the second aspect of this Chapter. Later on, working examples about the companies selection problem to invest in and comparative analysis of the proposed method with the existing ones have also been presented to exhibit its effectiveness and universality.

The novelties of the Chapter can be outlined as follows:

(1) The novel operational laws developed for PM-ULTS are closed, i.e., the final result

obtained after the operation does not exceed the bounds of LTSs, thereby overcoming the unreasonableness in logic proficiency. In addition, these laws can efficiently deal with the persisting decision information by expressing it using the PM-ULTSs and can avoid losing the given priority information validly.

(2) We formulate a probabilistic multiplicative unbalanced linguistic linear programming (PMULLP) model to solve a constant sum unbalanced linguistic matrix game problem. Further, the PMULLP model is transformed into a classical linear programming problem (LPP) by defining the information measure of the given probabilistic multiplicative unbalanced linguistic terms. The defined information measure can efficiently avoid losing the information present within the PM-ULTS.

(3) The computational unbalanced linguistic game model proposed in the chapter is applied to solve the MADM problem modeled as a constant sum unbalanced linguistic game problem.

(4) We make some comparative analysis for the proposed method with the existing method to highlight the advantages.

Finally, the chapter is structured in the following manner: Section 4.2 presents some preliminary definitions and concepts about the existing probabilistic linguistic term sets, and conventional two-player ZSMGs. In Section 4.3, a novel concept of PM-ULTSs is proposed, following that the new ranking method, operational laws with its properties, and the aggregation operators are also given. Next, based on the proposed concept, the methodology to solve uncertain linguistic matrix games with probabilistic multiplicative unbalanced linguistic information is presented in Section 4.4. Furthermore, we illustrate an application example about companies' stock selection problem to present the effectiveness of the method in a physical realm, which is followed by the comparison analysis with the existing methods provided in Section 4.5. Lastly, some concluding remarks are specified in Section 4.6.

4.2 Preliminaries

The present section briefly review some elementary definitions and concepts related to our proposal presented in this chapter. Specifically, existing probabilistic linguistic term set and conventional zero-sum matrix games.

4.2.1 Probabilistic linguistic term sets

Pang et al. [235] extended HFLTS by introducing the notion of PLTS where probability is allocated to each linguistic term. The added probability in the set can easily avoid the original information loss and therefore PLTS has gained several attention of the researchers [172, 346, 347].

Definition 4.2.1. [235] Let $LT = \{\ell_{(-\tau)}, \ldots, \ell_0, \ldots, \ell_\tau\}$ be a finitely ordered LTS. Then, a PLTS is defined as follows:

$$LT(p) = \left\{\ell_i^{(k)}(p^{(k)}) \,|\, k = 1 \,\text{to}\, \#LT(p)\right\}$$

where $\ell_i^{(k)} \in LT\,(i = -\tau, \ldots, 0, \ldots, \tau)$, the term $\ell_i^{(k)}(p^{(k)})$ expresses the linguistic term $\ell_i^{(k)}$ associated with the probability $p^{(k)}$, $0 \leq \sum_{k=1}^{\#LT(p)} p^{(k)} \leq 1$, and $\#LT(p)$ represents the number of all given distinct linguistic terms in $LT(p)$.

It is noted that if $\sum_{k=1}^{\#LT(p)} p^{(k)} = 1$ represents complete information of probabilistic distribution of each plausible linguistic terms; if $\sum_{k=1}^{\#LT(p)} p^{(k)} < 1$ represents the existence of partial ignorance which persists in practical decision-making problems and if $\sum_{k=1}^{\#LT(p)} p^{(k)} = 0$ represents complete ignorance.

Next, the details of the normalizing method for PLTS $LT(p)$ with $\sum_{k=1}^{\#LT(p)} p^{(k)} < 1$ is given below:

Definition 4.2.2. [235] Given a PLTS $LT(p)$ with $\sum_{k=1}^{\#LT(p)} p^{(k)} < 1$, the normalized PLTS $\overline{LT}(p)$ is defined as: $\overline{LT}(p) = \left\{\ell_i^{(k)}(\bar{p}^{(k)}) \,|\, k = 1 \,\text{to}\, \#LT(p)\right\}$ where $\bar{p}^{(k)} = \frac{p^{(k)}}{\sum_{k=1}^{\#LT(p)} p^{(k)}}$ for all $k = 1 \,\text{to}\, \#LT(p)$.

4.2.2 A zeros-sum matrix game

We review some elementary definitions from classical matrix GT taken from popular text Barron [36].

Definition 4.2.3. [36] A two-person ZSMG G is described as a triplet (S_1, S_2, A), where S_1 and S_2 termed as a finite strategy set for player I (PI) and player II (PII), respectively, and A is a real-valued payoff matrix of PI against PII while $-A$ is regarded as the real-valued payoff matrix for PII.

Definition 4.2.4. [36] Given matrix $A = [a_{ij}]_{n\times m}$, let v^- and v^+ are the values corresponding to PI and PII, respectively such that: $v^- = \max\limits_{i=1,\ldots,n} \min\limits_{j=1,\ldots,m} a_{ij}$; $v^+ = \min\limits_{j=1,\ldots,m} \max\limits_{i=1,\ldots,n} a_{ij}$.

Definition 4.2.5. [36] Suppose $(x, y) \in S_1 \times S_2$ is considered to be an ordered pair of mixed strategy, such that

$$S_1 = \left\{ (x_1, \ldots, x_n) : x_i \geq 0, i = 1, \ldots, n, \sum_{i=1}^{n} x_i = 1 \right\};$$

$$S_2 = \left\{ (y_1, \ldots, y_m) : y_j \geq 0, j = 1, \ldots, m, \sum_{j=1}^{m} y_j = 1 \right\}.$$

Here, x_i is taken as the probability of PI selecting strategy i and y_j is the probability of PII choosing strategy j.

For more comprehensive overview about crisp zero-sum matrix game, one can refer to the text in [36].

4.3 Probabilistic multiplicative unbalanced linguistic term set

The present section is devoted to the concept of PM-ULTSs. Followed by the introduction of novel comparison method, some operational laws and its properties and the aggregation operators.

4.3.1 The notion of the probabilistic multiplicative unbalanced linguistic term set

In literature, the idea of reflecting the probabilistic information for the evenly distributed linguistic terms is proposed by Pang et al. [235]. Analogously, we define the concept of PM-ULTSs in considering the case where the linguistic terms are unevenly distributed in the following definition:

Definition 4.3.1. Let $LS = \left\{ t_{a^i} | i = -n, \ldots, 0, \ldots, m \right\}$ be an ULTS where n and m may or may not be equal and $a > 1$, a probabilistic multiplicative ULTS is defined as: $LS(p) = \left\{ t_{a^i}^{(k)}(p^{(k)}) | \, k = 1, 2, \ldots, \#LS(p) \right\}$ where $t_{a^i}^{(k)} \in LS$, $p^{(k)} \geq 0$, $\sum_{k=1}^{\#LS(p)} p^{(k)}$ may or may not be equal to 1 and $\#LS(p)$ is the cardinality of the multiplicative unbalanced linguistic terms in $LS(p)$.

Note 4.3.1. The term $t_{a^i}^{(k)}(p^{(k)})$ is interpreted as unbalanced linguistic term t_{a^i} associated with the probability $p^{(k)}$.

It is noted that if $\sum_{k=1}^{\#LS(p)} p^{(k)} < 1$ or $\sum_{k=1}^{\#LS(p)} p^{(k)} > 1$ then the probability information within the set is incomplete whereas if $\sum_{k=1}^{\#LS(p)} p^{(k)} = 1$ then the information is complete.

Remark 4.3.1. For the incomplete PM-ULTS, we perform the normalization process to transform the set into complete PM-ULTS. The normalization of the PM-ULTS can be acquired by using the following steps:

(i) Normalize the probability of PM-ULTS. If $\sum_{k=1}^{\#LS(p)} p^{(k)} \neq 1$, then the given PM-ULTS is defined as $\widetilde{LS}(p) = \left\{ t^k_{a^i}(\bar{p}^k) | \sum_{k=1}^{\#LS(p)} \bar{p}_k = 1 \right\}$ where $\bar{p} = \frac{p^k}{\sum_{k=1}^{\#LS(p)} p^{(k)}}$, $t_{a^i} \in LS$, $p^{(k)} \geq 0$, and $k = 1 \,\text{to}\, \#LS(p)$;

(ii) Normalize the granularity of PM-ULTS. Let $LS_1(p)$ and $LS_2(p)$ are two PM-ULTS such that $\#LS_1(p)$, $\#LS_2(p)$ represent the cardinalities of the set respectively. If $\#LS_1(p) < \#LS_2(p)$, we add $\#LS_2(p) - \#LS_1(p)$ multiplicative unbalanced linguistic terms to the set $LS_1(p)$, so that both PM-ULTSs have equal cardinalities. It is noted that the added multiplicative unbalanced linguistic terms are the least one having the probability as zero in $LS_1(p)$.

Henceforth, the resultant PM-ULTSs are termed as the normalized PM-ULTSs. Hereafter, for the sake of convenience we refer $LS_1(p)$ and $LS_2(p)$ as normalized PM-ULTS.

Unless otherwise mentioned, the PM-ULTS is always presupposed to be a normalized PM-ULTS. Furthermore, it is viewed that in general PM-ULTS are not ordered set, therefore to fix the position of probabilistic multiplicative unbalanced linguistic terms we define ordered PM-ULTSs in the following fashion.

Definition 4.3.2. Given a PM-ULTS $LS(p) = \left\{ t^{(k)}_{a^i}(p^{(k)}) | \, k = 1, 2, \ldots, \#LS(p) \right\}$, then $LS(p)$ is termed as an ordered PM-ULTS if the unbalanced linguistic terms $t^{(k)}_{a^i}(p^{(k)})(k = 1, \ldots, \#LS(p))$ are arranged in a descending order of the values $a^i \cdot p$ for each k.

Next, in literature, Mao et al. [204] designed a possibility degree algorithm to rank series of classic PLTSs which is robust and beneficial approach. On the similar lines, the simplified ranking method to rank PM-ULTSs can also be stated.

Definition 4.3.3. Given ULTS LS. Assume $LS_1(p) = \left\{ t^{(k_1)}_{a^i}(\bar{p_1}^(k_1) | k_1 = 1 \,\text{to}\, \#LS_1(p) \right\}$ and $LS_2(p) = \{ \, t^{(k_2)}_{a^j}(\bar{p_2}^(k_2) | k_2 = 1 \,\text{to}\, \#LS_1(p) \, \}$ for $(i, j = -n, \ldots, 0, \ldots, m)$ are two normalized PM-ULTSs. Then, an unbalanced binary relation $\hat{R}(t_{a^i}^{(k_1)}, t_{a^j}^{(k_2)})$ is defined as:

$$\hat{R}(t_{a^i}^{(k_1)}, t_{a^j}^{(k_2)}) = \begin{cases} \bar{p_1}^{k_1}\bar{p_2}^{k_2}, & \text{if} a^i > a^j \\ \frac{1}{2}\bar{p_1}^{k_1}\bar{p_2}^{k_2}, & \text{if} a^i = a^j \\ 0, & \text{if} a^i < a^j \end{cases} \tag{4.3.1}$$

From Eq. (4.3.1) we can infer that the possibility degree of $t_{a^i}^{(k_1)}$ in $LS_1(p)$ is greater than $t_{a^j}^{(k_2)}$ in $LS_2(p)$. Hence, the possibility degree is given as:

$$P(LS_1(p) \geq LS_2(p)) = \sum_{k_1=1}^{\#LS_1(p)} \sum_{k_2=1}^{\#LS_2(p)} \hat{R}(t_{a^i}^{(k_1)}, t_{a^j}^{(k_2)}) \tag{4.3.2}$$

Mao et al. [204] have defined some desirable properties of the possibility degree for the classic normalized PLTSs. However, it is anticipated that these properties hold true even for the case of normalized PM-ULTSs. Meanwhile, for the case if $P(LS_1(p) \geq LS_2(p)) = P(LS_2(p) \geq LS_1(p)) = 0.5$, then it can be concluded that the classic PLTSs cannot be compared and hence, range value of PLTSs is calculated which represents the degree of dispersion. On the similar grounds, the range value for normalized PM-ULTSs $LS(p)$ is stated as follows:

Definition 4.3.4. Let $LS = \{t_{a^i} | i = -n, \ldots, 0, \ldots, m\}$ be an ULTS and $LS(p) = \{ t_{a^i}^{(k)}(\bar{p}^{(k)} | k_1 = 1 \text{ to} \#LS(p) \}$ be a normalized PM-ULTS. Let $a_i^{(k)}$ $(i = -n, \ldots, 0, 1, \ldots, m)$ be the subscript of unbalanced linguistic term $t_{a^i}^{(k)}$. Let $a^{i^{(-)}} = \min_k \{a^i\}$ and $a^{i^{(+)}} = \max_k \{a^i\}$ be the lower and upper boundaries of the term $t_{a^i}^{(k)}$, $\bar{p}^-$ and $\bar{p}^+$ be the respective probabilities. Then, the range value is given as:

$$R(LS(p)) = |a^{i^{(+)}}\bar{p}^+ - a^{i^{(-)}}\bar{p}^-|. \tag{4.3.3}$$

After combining the possibility degree alongwith the range value, Mao et al. [204] defined the ordered relation for two classic PLTSs. In similar fashion, a simplified proposal of the ordered relation for two normalized PM-ULTS $LS_1(p)$ and $LS_2(p)$ can be stated below:

Definition 4.3.5. Let $LS_1(p)$ and $LS_2(p)$ be any two normalized PM-ULTS. Then,

(i) If $P(LS_1(p) \geq LS_2(p)) > 0.5$, then $LS_1(p) > LS_2(p)$;

(ii) If $P(LS_1(p) \geq LS_2(p)) = 0.5$, then

(a) if $R(LS_1(p)) < R(LS_2(p))$, then $LS_1(p) > LS_2(p)$;

(b) if $R(LS_1(p)) = R(LS_2(p))$, then $LS_1(p) = LS_2(p)$.

Next, based on the analysis mentioned above, Mao et al. [204] states a possibility algorithm to rank given list of classic PLTSs. Analogously, we propose a possibility algorithm to rank list of normalized PM-ULTSs $LS_i(p)$ $(i = 1, 2, \ldots, n)$

Step 1: Calculate the possibility degrees $P_{ij} = P(LS_i(p) \geq LS_j(p))$ $(i, j = 1, 2, \ldots, n)$ using Eq. (4.3.2) and the range value $R(LS_i(p))$ $(i = 1, 2, \ldots, n)$ using Eq. (4.3.3).

Step 2: Calculate the superior index $P_i = \sum_{j=1}^{n} P_{ij}$.

Step 3: The descending values of P_i generates the desired ranking order of $LS_i(p)$ $(i = 1, 2, \ldots, n)$. However, if for some $LS_i(p)$ the value of superior index is equal, then the reordering of $LS_i(p)$ is given by the ascending value of $R(LS_i(p))$.

We proceed to define the negation of PM-ULTS in the following manner.

Definition 4.3.6. Let $LS(p) = \left\{ t_{a^i}^{(k)}(\bar{p}^{(k)}) | k = 1 \,\text{to}\, \#LS(p) \right\}$ be a normalized PM-ULTS. Then, the negation of PM-ULTS is defined as: $\text{neg}(LS(p)) = \{\, \text{neg}(t_{a^i})^{(k)}((\bar{p})^{(k)}) \,|\, k = 1, 2, \ldots, \#LS(p) \,\}$.

Next, for clarity of the ranking algorithm defined, we give the following example:

Example 4.3.1. Consider the list of four PM-ULTS i.e., $LS_1(p) = \{\, t_{1/2^3}(0.1), t_{1/2^2}(0.4), t_{2^0}(0.3), t_2(0.2) \,\}$, $LS_2(p) = \left\{ t_{1/2}(0.5), t_{2^0}(0.5) \right\}$, $LS_3(p) = \left\{ t_{2^2}(0.2), t_{2^3}(0.8) \right\}$, $LS_4(p) = \left\{ t_{2^2}(0.4), t_{2^3}(0.6) \right\}$ based on ULTS $LS = \{\, t_{1/2^3}, t_{1/2^2}, t_{1/2}, t_{2^0}, t_{2^1}, t_{2^2}, t_{2^3} \,\}$. Now, we will apply the proposed ranking algorithm to rank the given list of PM-ULTSs.

Step 1: Calculate the possibility degrees P_{ij} $(i, j = 1, 2, 3, 4)$ using Eq. (4.3.2) and the range value $R(LS_i(p))$ $(i = 1, 2, 3, 4)$ using Eq. (4.3.3) as demonstrated: $P_{11} = 0.5$, $P_{12} = 0.425$, $P_{13} = 0$, $P_{14} = 0$, $P_{21} = 0.575$, $P_{22} = 0.5$, $P_{23} = 0$, $P_{24} = 0$, $P_{31} = 1$, $P_{32} = 1$, $P_{33} = 0.5$, $P_{34} = 0.68$, $P_{41} = 1$, $P_{42} = 1$, $P_{43} = 0.32$, $P_{44} = 0.5$. $R(LS_1(p)) = 0.3875$, $R(LS_2(p)) = 0.25$, $R(LS_3(p)) = 5.6$, $R(LS_4(p)) = 3.2$.

Step 2: The corresponding superior index $P_i = \sum_{j=1}^{4} P_{ij} (i = 1, 2, 3, 4)$ is obtained as: $P_1 = 0.925$, $P_2 = 1.075$, $P_3 = 3.18$, $P_4 = 2.82$.

Step 3: From the values of $P_i(i = 1, 2, 3, 4)$ and $R(LS_i(p))$ $(i = 1, 2, 3, 4)$ obtained in the previous steps, the ranking order is given as: $LS_3(p) \succ LS_4(p) \succ LS_2(p) \succ LS_1(p)$.

4.3.2 Operational laws for probabilistic multiplicative unbalanced linguistic terms

In the present section we will present some new operational laws primarily based on the given PM-ULTSs. For this, we will assume that all PM-ULTSs are normalized and have equal cardinality, i.e., for each PM-ULTSs $LS_1(p_1)$ and $LS_2(p_2)$, we have $\sum_{k=1}^{\#LS_1(p_1)} \bar{p}_1^k = \sum_{k=1}^{\#LS_2(p_2)} \bar{p}_2^k = 1$ and $\#LS_1(p_1) = \#LS_2(p_2)$.

Definition 4.3.7. Let $LS_1(p_1)$ and $LS_2(p_2)$ be any two arbitrary ordered PM-ULTSs, $LS_1(p_1) = \{ t_{a^i}^{(k)}(\bar{p_1}^k) \mid k = 1, 2, \ldots, \#LS_1(p_1) \}$ and $LS_2(p_2) = \left\{ t_{a^j}^{(k)}(\bar{p_2}^k) \mid k = 1, 2, \ldots, \#LS_2(p_2) \right\}$ for $(i, j = -n, \ldots, 0, \ldots, m)$. Then, the operational laws based on minimum distance measures are given as follows:

(1) Addition operation
$LS_1(p_1) \oplus LS_2(p_2) = \left\{ t_{a^i \oplus a^j}^{(k)}(\bar{p}_1^k \oplus \bar{p}_2^k) \mid k = 1, 2, \ldots, \#LS_1(p_1) \right\}$

(2) Multiplication operation
$LS_1(p_1) \otimes LS_2(p_2) = \left\{ t_{a^i \otimes a^j}^{(k)}(\bar{p}_1^k \otimes \bar{p}_2^k) \mid k = 1, 2, \ldots, \#LS_1(p_1) \right\}$.

(3) Scalar multiplication operation
For $\lambda \in (0, 1)$ being any arbitrary scalar, $\lambda \odot LS(p) = \left\{ t_{\lambda \odot a^i}^{(k)}(\bar{p}^k) \mid k = 1, 2, \ldots, \#LS(p) \right\}$.

(4) Power operation
$(LS(p))^\lambda = \left\{ t_{a^{\lambda i}}^{(k)}(\bar{p}^{(k)}) \mid k = 1, 2, \ldots, \#LS(p) \right\}$

Given the two arbitrary PM-ULTSs $LS_1(p_1)$ and $LS_2(p_2)$, the computational values for the aforementioned operational laws can be outlined in the Algorithm 4.3.1, 4.3.2, 4.3.3, and 4.3.4.

Algorithm 4.3.1. Computation of $t_{a^i \oplus a^j}^{(k)}$ and $\bar{p}_1^{(k)} \oplus \bar{p}_2^{(k)}$

Input: $LS_1(p_1) = \left\{ t_{a^i}^{(k)}(\bar{p_1}^k) \mid k = 1, 2, \ldots, \#LS_1(p_1) \right\}$ and $LS_2(p_2) = \{ t_{a^j}^{(k)}(\bar{p_2}^k) \mid k = 1, 2, \ldots, \#LS_2(p_1) \}$, $(i, j = -n, \ldots, 0, 1, \ldots, m)$ where $a > 1$ and n may or may not be equal.

Output: $LS_1(p_1) \oplus LS_2(p_2) = \left\{t^{(k)}_{a^i \oplus a^j}(\bar{p}^k_1 \oplus \bar{p}^k_2) \,|\, k = 1, 2, \ldots, \#LS_1(p_1)\right\}$,

Begin:

for $k \leftarrow 1$ to $\#LS_1(p_1)$ **do**

if $t^{(k)}_{a^i \oplus a^j} \in \left\{t_{a^{-n}}, \ldots, t_{a^0}, \ldots, t_{a^m}\right\}$ **then**

$t^{(k)}_{a^i \oplus a^j} = t^{(k)}_{a^i + a^j}$;

end if

if $t^{(k)}_{a^i \oplus a^j} \notin \left\{t_{a^{-n}}, \ldots, t_{a^0}, \ldots, t_{a^m}\right\}$ **then**

$t^{(k)}_{a^i \oplus a^j} \leftarrow t^{(k)}_d$ where

$$d \leftarrow \begin{cases} a^i + a^j \text{ if } d \in [a^{-n}, a^m]; \\ a^i + a^j - a^m \text{ otherwise.} \end{cases}$$

if $d \in [a^{-n}, a^m]$ or $d \notin [a^{-n}, a^m]$ **then**

$i_L \leftarrow \arg\min_i \left\{d - a^i \,|\, d - a^i \geq 0\right\}$;

$i_R \leftarrow \arg\max_i \left\{d - a^i \,|\, d - a^i \leq 0\right\}$, for $(i = -n, \ldots, 0, \ldots, m)$.

Compute:

$D_{left} \leftarrow d - a^{i_L}$ and $D_{right} \leftarrow d - a^{i_R}$ where $i_L, i_R \in (-n, \ldots, 0, \ldots, m)$

Define: $D = \min(D_{left}, |D_{right}|)$

if $D \leftarrow D_{left}$ **then**

$t^{(k)}_{a^i \oplus a^j} = t^{(k)}_{a^{i_L}}$

else

$t^{(k)}_{a^i \oplus a^j} = t^{(k)}_{a^{i_R}}$

end if

end if

end if

Next compute for $\bar{p}^{(k)}_1 \oplus \bar{p}^{(k)}_2$**:**

$$\bar{p}^{(k)}_1 \oplus \bar{p}^{(k)}_2 \leftarrow p^{(k)}_3, \text{ where } p^{(k)}_3 \leftarrow \begin{cases} \bar{p}^k_1 + \bar{p}^k_2, & \text{if } \bar{p}_1 + \bar{p}_2 \leq 1; \\ \bar{p}^k_1 + \bar{p}^k_2 - 1, & \text{if } \bar{p}_1 + \bar{p}_2 > 1. \end{cases}$$

if $\sum_{k=1}^{\#LS(p_1)} p^{(k)}_3 \leftarrow 1$ **then**

done

else

$p^{(k)}_3 \leftarrow \frac{p^{(k)}_3}{\sum_{k=1}^{\#LS_1(p_1)} p^{(k)}_3}$

end if

end for

Algorithm 4.3.2. Computation of $t^{(k)}_{a^i \otimes a^j}$ **and** $\bar{p}^{(k)}_1 \otimes \bar{p}^{(k)}_2$

Input: $LS_1(p_1) = \left\{t^{(k)}_{a^i}(\bar{p_1}^k) \,|\, k = 1, 2, \ldots, \#LS_1(p_1)\right\}$ and $LS_2(p_2) = \{\, t^{(k)}_{a^j}(\bar{p_2}^k) \,|\, k =$

$1, 2, \ldots, \#LS_2(p_1)\ \}$, $(i, j = -n, \ldots, 0, 1, \ldots, m)$ where $a > 1$ and n may or may not be equal.

Output: $LS_1(p_1) \otimes LS_2(p_2) = \left\{ t^{(k)}_{a^i \otimes a^j}(\bar{p}^k_1 \otimes \bar{p}^k_2) | \, k = 1, 2, \ldots, \#LS_1(p_1) \right\}$,

Begin:

$k \leftarrow 1$

while $k \leq \#LS_1(p_1)$ **do**

$$t^{(k)}_{a^i \otimes a^j} \leftarrow \begin{cases} t^k_{a^{i+j+n}} \text{ if } i+j < -n; \\ t^k_{a^{i+j}} \text{ if } i+j \in \{-n, -(n-1), \ldots, 0, \ldots, m\} \\ t^k_{a^{i+j-m}} \text{ if } i+j > m. \end{cases} \quad \textbf{and}$$

$$\bar{p}^k_1 \otimes \bar{p}^k_2 \leftarrow \bar{p}^{(k)}_3, \bar{p}^{(k)}_3 = \frac{p^{(k)}_3}{\sum_{k=1}^{\#LS_1(p_1)} p^{(k)}_3}$$

end while

Algorithm 4.3.3. Computation of $t^{(k)}_{\lambda \odot a^i}$

Input: $LS(p) = \left\{ t^{(k)}_{a^i}(\bar{p}^k) \, | \, k = 1, 2, \ldots, \#LS(p) \right\}$, where $a > 1$ and n may or may not be equal and $\lambda \in (0, 1)$.

Output: $\lambda \odot LS(p) = \left\{ t^{(k)}_{\lambda \odot a^i}(\bar{p}^k) | \, k = 1, 2, \ldots, \#LS(p) \right\}$

Begin:

for $k \leftarrow 1$ to $\#LS_1(p_1)$ **do**

if $t^{(k)}_{\lambda \odot a^i} \in \{t_{a^{-n}}, \ldots, t_{a^0}, \ldots, t_{a^m}\}$ **then**

$t^{(k)}_{\lambda \odot a^i} = t^{(k)}_{\lambda \cdot a^i}$;

else

if $t^{(k)}_{\lambda \odot a^i} \notin \{t_{a^{-n}}, \ldots, t_{a^0}, \ldots, t_{a^m}\}$ **then**

$t^{(k)}_{\lambda \odot a^i} \leftarrow t^{(k)}_d$ where

$$d \leftarrow \begin{cases} t^k_{\lambda \cdot a^i + a^{-n}} \text{ if } d \notin [a^{-n}, a^m]; \\ t^k_{\lambda \cdot a^i} \text{ if } d \in [a^{-n}, a^m] \end{cases}$$

if $d \in [a^{-n}, a^m]$ or $d \notin [a^{-n}, a^m]$ **then**

$i_L \leftarrow \arg\min_i \{d - a^i \mid d - a^i \geq 0\}$;

$i_R \leftarrow \arg\max_i \{d - a^i \mid d - a^i \leq 0\}$, for $(i = -n, \ldots, 0, \ldots, m)$.

Compute:

$D_{left} \leftarrow d - a^{i_L}$ and $D_{right} \leftarrow d - a^{i_R}$ where $i_L, i_R \in (-n, \ldots, 0, \ldots, m)$

Define: $D = \min(D_{left}, |D_{right}|)$

if $D \leftarrow D_{left}$ **then**

$t^{(k)}_{\lambda \odot a^i} = t^{(k)}_{a^{i_L}}$

else

$t^{(k)}_{\lambda\odot a^i} = t^{(k)}_{a^{i_R}}$

end if

end if

end if

end if

end for

Algorithm 4.3.4. Computation of $t^{(k)}_{a^{\lambda\cdot i}}$

Input: $LS(p) = \left\{t^{(k)}_{a^i}(\bar{p}^k) \,|\, k = 1, 2, \ldots, \#LS(p)\right\}$, where $a > 1$ and n may or may not be equal and $\lambda \in (0, 1)$.

Output: $LS(p)^\lambda = \left\{t^{(k)}_{a^{\lambda\cdot i}}(\bar{p}^k) | \, k = 1, 2, \ldots, \#LS(p)\right\}$,

Begin:

for $k \leftarrow 1$ to $\#LS_1(p_1)$ **do**

if $\lambda \cdot i \in \left\{-n, -(n-1), \ldots, 0, \ldots, m\right\}$ **then**

done

else

if $\lambda \cdot i \notin \left\{-n, -(n-1), \ldots, 0, \ldots, m\right\}$ **then**

$i_L \leftarrow \arg\min_i \left\{(\lambda \cdot i) - a^i \,|\, (\lambda \cdot i) - a^i \ \geq\ 0\right\}$;

$i_R \leftarrow \arg\max_i \left\{(\lambda \cdot i) - a^i \,|\, (\lambda \cdot i) - a^i \ \leq\ 0\right\}$, for $(i = -n, \ldots, 0, \ldots, m)$.

Define: $I = \min(i_L, |i_R|)$

if $I \leftarrow i_L$ **then**

$t^{(k)}_{a^{\lambda\cdot i}} = t^{(k)}_{a^{i_L}}$

else

$t^{(k)}_{a^{\lambda\cdot i}} = t^{(k)}_{a^{i_R}}$

end if

end if

end if

end for

Remark 4.3.2. The final outcomes obtained after using the operational laws should preserve the probability information.

Based on the definition 4.3.7 the subsequent properties of the aforementioned operational laws we can defined as follows.

Theorem 4.3.1. (Closure properties). Let Γ be the set of all PM-ULTSs, i.e., $LS_1(p_1) \in \Gamma$ and $LS_2(p_2) \in \Gamma$. Then,

(i) $LS_1(p_1) \oplus LS_2(p_2) \in \Gamma$;

(ii) $LS_1(p_1) \otimes LS_2(p_2) \in \Gamma$;

(iii) $\lambda \odot LS_1(p_1) \in \Gamma$;

(iv) $(LS_1(p_1))^{\lambda} \in \Gamma$;

(v) $\text{neg}(LS_1(p_1) \oplus LS_2(p_2)) \in \Gamma$;

(vi) $\text{neg}(LS_1(p_1) \otimes LS_2(p_2)) \in \Gamma$;

(vii) $\text{neg}(\lambda \odot LS_1(p_1)) \in \Gamma$;

(viii) $\text{neg}((LS_1(p_1))^{\lambda}) \in \Gamma$.

Proof. The proof of theorem 4.3.1 is trivial and can be followed directly from the definition 4.3.7. □

Theorem 4.3.2. (Commutative law). For any $LS_1(p_1)$ and $LS_2(p_2)$ we have:

(i) $LS_1(p_1) \oplus LS_2(p_2) = LS_2(p_2) \oplus LS_1(p_1)$;

(ii) $LS_1(p_1) \otimes LS_2(p_2) = LS_2(p_2) \otimes LS_1(p_1)$.

Proof. According to the definition 4.3.7, we have

$$
\begin{aligned}
& LS_1(p_1) \oplus LS_2(p_2) \\
&= \left\{ t^{(k)}_{a^i \oplus a^j} (\bar{p}_1^k \oplus \bar{p}_2^k) \,|\, k = 1, 2, \ldots, \#LS_1(p_1) \right\} \\
&= \left\{ t^{(k)}_{a^j \oplus a^i} (\bar{p}_2^k \oplus \bar{p}_1^k) \,|\, k = 1, 2, \ldots, \#LS_1(p_1) \right\} \\
&= LS_2(p_2) \oplus LS_1(p_1).
\end{aligned}
$$

Similarly, we can prove the second part. □

Theorem 4.3.3. (Associative law). Let $LS_1(p_1)$, $LS_2(p_2)$, $LS_3(p_3)$ be any three normalized PM-ULTSs, then

(i) $LS_1(p_1) \oplus LS_2(p_2) \oplus LS_3(p_3) = LS_1(p_1) \oplus (LS_2(p_2) \oplus LS_3(p_3))$;

(ii) $(LS_1(p_1) \otimes LS_2(p_2)) \otimes LS_3(p_3) = LS_1(p_1) \otimes (LS_2(p_2) \otimes LS_3(p_3))$.

Proof. The proof of theorem 4.3.3 is trivial and therefore it is omitted. □

Remark 4.3.3. It is noteworthy to derive the subsequent conclusions based on the aforementioned operational laws of PM-ULTSs.

(i) Linking the similar unbalanced linguistic terms having different probabilities associated together. For instance, consider the terms $t_{2^2}(0.2)$ and $t_{2^2}(0.3)$, then we can connect them into $t_{2^2}(0.5)$ by summing up their corresponding probabilities. Nevertheless, if the sum total of two probabilities is not within the bounds then it can be re-translated by subtracting 1 from the resultant, i.e., if $\sum_{i=1}^{n} p_i > 1$ where n is the number of similar unbalanced linguistic terms, then $\sum_{i=1}^{n} p_i - 1$ would be the resultant value.

(ii) We rank the resultant probabilistic multiplicative unbalanced linguistic terms obtained after employing the operational laws in increasing order of its subscript.

(iii) The operational laws are defined in such a way that the resultant probabilistic information is preserved.

An example can be presented to illustrate the whole mechanism of the above mentioned operational laws and algebra operations.

Example 4.3.2. Let $LS = \{t_{a^i} \,|\, i = -4, \ldots, 0, \ldots, 3\}$ be a ULTS. For convenience, we take $a = 2$. Let $LS_1(p_1) = \{\, t_{1/8}(0.1276), t_1(0.8147), t_4(0.9572) \,\}$ and $LS_2(p_2) = \{\, t_{1/16}(0.9134), t_{1/2}(0.9572), t_8(0.4854) \,\}$. Since the given sets are not normalized, therefore we will firstly normalize the given PM-ULTSs such that $\widetilde{LS}_1(p_1) = \{t_{1/8}(0.0669), t_1(0.4290), t_4(0.5041)\}$ and $\widetilde{LS}_2(p_2) = \{t_{1/16}(0.3877), t_{1/2}(0.4063), t_8(0.2060)\}$, then

(1) $\widetilde{LS}_1(p_1) \oplus \widetilde{LS}_2(p_2)$
$= \{t_{1/8 \oplus 1/16}(0.4546), t_{1 \oplus 1/2}(0.8353), t_{4 \oplus 8}(0.7101)\}$
$= \{t_{0.1875}(0.4546), t_{1.5}(0.8353), t_{12}(0.7101)\}$
$= \{t_{0.1250}(0.2273), t_1(0.41765), t_4(0.3551)\}$.

(2) $\widetilde{LS}_1(p_1) \otimes \widetilde{LS}_2(p_2)$
$= \left\{t_{1/2^3 \otimes 1/2^4}(0.0259), t_{1 \oplus 1/2}(0.1743), t_{2^2 \oplus 2^3}(0.1038)\right\}$
$= \{t_{1/8}(0.0259), t_{1/2}(0.1743), t_{2^2}(0.1038)\}$
$= \{t_{1/8}(0.0852), t_{1/2}(0.5734), t_4(0.3414)\}$.

(3) Take $\lambda = 0.8$.
$0.8 \odot \widetilde{LS}_1(p_1)$
$= 0.8 \odot \{t_{1/8}(0.0669), t_1(0.4290), t_4(0.5041)\}$
$= \{t_{0.1250}(0.0669), t_1(0.4290), t_4(0.5041)\}$.

(4) Take $\lambda = 0.8$.
$(\widetilde{LS}_1(p_1))^{0.8}$
$= \left(\left\{t_{1/2^3}(0.0669), t_{2^0}(0.4290), t_{2^2}(0.5041)\right\}\right)^{0.8}$
$= \left\{t_{1/2^2}(0.0669), t_1(0.4290), t_{2^2}(0.5041)\right\}$.

(5) $\mathrm{neg}(\widetilde{LS}_1(p_1) \oplus \widetilde{LS}_2(p_2))$
$= \left\{\mathrm{neg}(t_{1/2^3})(0.7727), \mathrm{neg}(t_1)(0.58235), \mathrm{neg}(t_4)(0.6449)\right\}$
$= \left\{t_{2^3}(0.7727), t_1(0.58235), t_{1/2^2}(0.6449)\right\}$
$= \left\{t_{1/2^2}(0.3225), t_1(0.2912), t_{2^3}(0.3864)\right\}$.

4.3.3 Aggregation operators for PM-ULTSs

Considering the significance of aggregation operators in practical DM or matrix game problems useful for emulsion of information, we will propose some elementary aggregation operators for PM-ULTSs.

Before we proceed to define the aggregation operators the PM-ULTSs utilized during the operation are always presumed to be a normalized PM-ULTSs having equal cardinalities.

Definition 4.3.8. Let $\left\{LS_i(p_i)|i = 1, 2, \ldots, K\right\}$ be the cluster of normalized PM-ULTSs. The probabilistic multiplicative unbalanced linguistic averaging (PMULA) operator is defined as a mapping $LS_1(p_1) \times LS_2(p_2) \times \ldots \times LS_K(p_K) \to LS(p)$ such that
$PMULA(LS_1(p_1), LS_2(p_2), \ldots, LS_K(p_K))$
$= \frac{1}{K}(LS_1(p_1) \oplus LS_2(p_2) \oplus \ldots \oplus LS_K(p_K))$.

Definition 4.3.9. Let $\left\{LS_i(p_i)|i = 1, 2, \ldots, K\right\}$ be the cluster of normalized PM-ULTSs. Assign a relevant weight vector $W = (w_1, w_2, \ldots, w_K)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{K} w_i = 1$. The probabilistic multiplicative unbalanced linguistic weighted averaging (PMULWA) operator is defined as: $PMULWA(LS_1(p_1), LS_2(p_2), \ldots, LS_K(p_K))$
$= (w_1 \odot LS_1(p_1)) \oplus \ldots \oplus (w_K \odot LS_K(p_K))$
$= (w_1 \odot \left\{t^{(k)}_{a^{i_1}}(\bar{p}^{(k)}_1)|k = 1, \ldots, \#LS_1(p_1)\right\}) \oplus \ldots \oplus (w_K \odot \left\{t^{(k)}_{a^{i_K}}(\bar{p}^{(k)}_K)|k = 1, \ldots, \#LS_1(p_1)\right\})$.

Definition 4.3.10. Let $\left\{LS_i(p_i)|i = 1, 2, \ldots, K\right\}$ be the cluster of normalized PM-ULTSs. Then the probabilistic multiplicative unbalanced linguistic geometric (PMULG) operator is defined as: $PMULG(LS_1(p_1), LS_2(p_2), \ldots, LS_K(p_K))$
$= (LS_1(p_1) \otimes \ldots LS_K(p_K))^{1/K}$
$= (LS_1(p_1))^{1/K} \otimes \ldots (LS_K(p_K))^{1/K}$.

Definition 4.3.11. Let $\{LS_i(p_i)|i = 1, 2, \ldots, K\}$ be the cluster of normalized PM-ULTSs. Assign a relevant weight vector $W = (w_1, w_2, \ldots, w_K)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{K} w_i = 1$. Then the probabilistic multiplicative unbalanced linguistic weighted geometric (PMULWG) operator is defined as: $PMULWG(LS_1(p_1), LS_2(p_2), \ldots, LS_K(p_K))$
$= (LS_1(p_1))^{w_1} \otimes \ldots (LS_K(p_K))^{w_K}$.

Considering the flexibility and interpretability of PLTS to handle complex uncertainties, numerous studies in the literature have been devoted to application and achievements. Recently, Mi et al. [214] introduced a two-person ZSMG model taking probabilistic linguistic information as input. However, real-life problems are not always subjected to uniformly distributed LTSs. Henceforth, the subsequent section is explicitly dedicated to introducing a constant-sum matrix game problem with probabilistic unbalanced linguistic information. It is envisioned that the probabilistic multiplicative unbalanced linguistic information has not been established in the current matrix game models to our optimum knowledge.

4.4 Matrix games with probabilistic multiplicative unbalanced linguistic information

The present section proposes the notion of a constant-sum matrix game problem with probabilistic unbalanced linguistic information. For this, we initially introduce the terminology related to the constant-sum probabilistic unbalanced linguistic matrix game in Section 4.4.1. Then, in Section 4.4.2 probabilistic unbalanced linguistic linear programming methodology is developed to acquire the optimal strategy set in the absence of pure strategy of corresponding players and the probabilistic unbalanced linguistic game value.

4.4.1 Terminology of a constant-sum probabilistic multiplicative unbalanced linguistic information

Due to the perplexity and uncertainty present within the practical DM problems where DMs prefer to present linguistic labels distributed non-uniformly, the probabilistic unbalanced linguistic information is introduced to model approximate payoff values.

Definition 4.4.1. A two-player constant-sum unbalanced linguistic game $\widetilde{G}$ is described as a quadruple $(S_1, S_2, LS, \widetilde{R})$, where S_1 and S_2 represents the strategy set of the player I (PI) and player II (PII), $LS = \{t_{a^i}|i = -n, \ldots, 0, \ldots, m\}$ having $n+m+1$ cardinality is a presupposed ULTS for both the PI and PII, $\widetilde{R} = [LS_{cd}(p)]_{r\times q}$ is the payoff matrix for PI with probabilistic unbalanced linguistic information denoted as:

$$\widetilde{R} = \begin{pmatrix} LS_{11}(p) & LS_{12}(p) & \dots & LS_{1q}(p) \\ LS_{21}(p) & LS_{22}(p) & \dots & LS_{2q}(p) \\ \vdots & \vdots & \ddots & \vdots \\ LS_{r1}(p) & LS_{r2}(p) & \dots & LS_{rq}(p) \end{pmatrix}$$

where $LS_{cd}(p) = \left\{(t_{a^i})_{cd}(p^{(k)}) | k = 1, \dots, \#LS(p)\right\}$, $\#LS_{11}(p) = \#LS_{12}(p) = \dots = \#LS_{rq}(p) = \#LS(p)$. On the contrary $\mathrm{neg}(\widetilde{R})$ is the payoff matrix for PII.

It is worth observing that the probabilistic unbalanced linguistic payoff matrix $\widetilde{R} = [LS_{cd}(p)]_{r\times q}$ for PI is defined with the intention that each $LS_{cd}(p)$ represents response to strategy c of PI, $c = 1, \dots, r$ when PII selects strategy d, $d = 1, \dots, q$, to play the game. Consequently, the corresponding payoff of PI is $LS_{cd}(p)$ while the payoff of PII is $\mathrm{neg}(LS_{cd}(p))$.

A probabilistic unbalanced linguistic payoff implies the importance it holds for a game player to play a particular strategy that can be easily expressed on a multiplicative unbalanced linguistic scale set, i.e., LS.

Since the ranking of PM-ULTSs is possible and is defined in the previous section 4.3, therefore the idea of the classical value of the game can be easily extended to the probabilistic multiplicative unbalanced linguistic matrix game $\widetilde{G}$. Furthermore, the PI and PII are examined from the prospects of the gain floor and the lost ceiling, respectively. Finally, the aim of players are esteemed as follows:

Definition 4.4.2. Let $\widehat{v}$ and $\widehat{\omega}$ are the values corresponding to PI and PII, respectively such that:

$$\widehat{v}^{1-}(p) = \max_{c=1,\dots,r} \min_{d=1,\dots,q} LS_{cd}^{(k)}(p^{(k)});$$
$$\widehat{\omega}^{2-}(p) = \max_{d=1,\dots,q} \min_{c=1,\dots,r} \mathrm{neg}(LS_{cd}^{(k)}(p^{(k)})).$$

Note 4.4.1. From above definition, it is noted that $\widehat{v}(p) = \widehat{v}^{1-}(p)$ and $\widehat{\omega}(p) = \mathrm{neg}(\widehat{\omega}^{2-}(p))$ refers to the lower and upper value of the game $\widetilde{G}$, respectively. The value of the game is defined if $\widehat{v}(p) = \widehat{\omega}(p)$. Meanwhile, the strategies c^* and d^* offering the payoff value $LS_{c^*d^*}(p)$ such that the equality condition holds are optimal pure strategies for PI and PII. Also, the ordered strategy pair (c^*, d^*) is ascertained as the saddle point of the game $\widetilde{G}$.

Note 4.4.2. Obviously from Remark 2.4.3, it is clear that

$$\begin{aligned}\widehat{\omega}(p) &= \text{neg}(\widehat{\omega}^{2-}(p)) \\ &= \text{neg}(\max_{d=1,\dots,q}\min_{c=1,\dots,r}\text{neg}(LS_{cd}^{(k)}(p^{(k)}))) \\ &= \min_{d=1,\dots,q}\max_{c=1,\dots,r} LS_{cd}^{(k)}(p^{(k)}).\end{aligned}$$

Since the above analysis holds true only for the case $n = m$, therefore, we consider the following example having predefined ULTS for $n = m$ to understand the above concept of the matrix game.

Example 4.4.1. Let $LS = \{t_{1/8}, t_{1/4}, t_{1/2}, t_{2^0}, t_2, t_{2^2}, t_{2^3}\}$ be the presumed ULTS. Suppose for PI the probabilistic unbalanced linguistic payoff matrix $\widetilde{R}$ having all the PM-ULTSs as payoff entries is stated in the Table 4.1 and additionally, the payoff matrix for PII i.e., $\text{neg}(\widetilde{R})$ is given in Table 4.2.

	C_1	C_2	C_3
R_1	$\{t_{2^0}(1), t_{2^0}(0), t_{2^0}(0)\}$	$\{t_2(0.6), t_{2^2}(0.4), t_{2^1}(0)\}$	$\{t_{2^2}(0.3), t_{2^3}(0.7), t_{2^2}(0)\}$
R_2	$\{t_{1/8}(0.8), t_{1/4}(0.2), t_{1/4}(0)\}$	$\{t_{1/4}(0.6), t_{1/2}(0.4), t_{1/4}(0)\}$	$\{t_{2^0}(1), t_{2^0}(0), t_{2^0}(0)\}$
R_3	$\{t_{1/8}(0.8), t_{1/4}(0.2), t_{1/4}(0)\}$	$\{t_{2^2}(0.3), t_{2^3}(0.7), t_{2^2}(0)\}$	$\{t_{2^2}(0.3), t_{2^3}(0.7), t_{2^2}(0)\}$

Table 4.1: The comprehensive evaluation payoff matrix $\widetilde{R}$ in form of PM-ULTSs

	C_1	C_2	C_3
R_1	$\{t_{2^0}(1), t_{2^0}(0), t_{2^0}(0)\}$	$\{t_{1/2}(0.6), t_{1/4}(0.4), t_{1/2}(0)\}$	$\{t_{1/4}(0.3), t_{1/8}(0.7), t_{1/4}(0)\}$
R_2	$\{t_8(0.8), t_4(0.2), t_4(0)\}$	$\{t_4(0.6), t_2(0.4), t_4(0)\}$	$\{t_{2^0}(1), t_{2^0}(0), t_{2^0}(0)\}$
R_3	$\{t_8(0.8), t_4(0.2), t_4(0)\}$	$\{t_{1/4}(0.3), t_{1/8}(0.7), t_{1/4}(0)\}$	$\{t_{1/4}(0.3), t_{1/8}(0.7), t_{1/4}(0)\}$

Table 4.2: The comprehensive evaluation payoff matrix $\text{neg}(\widetilde{R})$ in form of PM-ULTSs

From the given payoff matrices, it is anticipated that each payoff entry provided in the form of PM-ULTSs are normalized with equal cardinalities. Hence, we can proceed further. Now, based on the ranking algorithm and definition 4.3.5 given in the previous section, one can order the PM-ULTSs $LS_{cd}(p)$ as well as $\text{neg}(LS_{cd}(p))$. Therefore, the lower and upper values of the game are given as follows:

For gain floor: PI

$$\begin{aligned}\widehat{v}^{1-}(p) &= \widehat{v}(p) \\ &= \max_{c=1,2,3}\min_{d=1,2,3,4} LS_{cd}(p_{cd}) \\ &= \{t_{2^0}(1), t_{2^0}(0), t_{2^0}(0)\}\end{aligned}$$

For loss ceiling: PII

$$\widehat{\omega}^{2-}(p) = \max_{d=1,2,3,4} \min_{c=1,2,3} \text{neg}(LS_{cd}(p_{cd}))$$
$$= \{t_{2^0}(1), t_{2^0}(0), t_{2^0}(0)\}$$

Hence, in this instance (1, 1) is the saddle point and probabilistic unbalanced linguistic value of the game is $\widehat{v}(p) = \widehat{\omega}(p) = \{t_{2^0}(1), t_{2^0}(0), t_{2^0}(0)\}$.

However, it is not always plausible for a probabilistic unbalanced linguistic matrix game to have an equality i.e., $\widehat{v}(p) = \widehat{\omega}(p)$. In general, the following inequality exist for any probabilistic unbalanced linguistic matrix game.

Theorem 4.4.1. Suppose $\widehat{v}(p)$ and $\widehat{\omega}(p)$ be the probabilistic unbalanced linguistic lower and upper values of probabilistic unbalanced linguistic matrix game $\widetilde{G}$ such that both values exist. Then, $\widehat{v}(p) \leq \widehat{\omega}(p)$.

Proof. Without loss of generality, let $\widehat{v}(p)$ and $\widehat{\omega}(p)$ both exist, so that for some column d and fixed row c, we have,

$$\min_{d=1,\ldots,q} LS_{cd}(p) \leq LS_{cd}(p),$$

Now, take max over $c = 1, \ldots, r$ on both sides, such that,

$$\widehat{v}(p) \equiv \max_{c=1,\ldots,r} \min_{d=1,\ldots,q} LS_{cd}(p) \leq \max_{c=1,\ldots,r} LS_{cd}(p)$$

$$\Rightarrow \widehat{v}(p) \leq \max_{c=1,\ldots,r} LS_{cd}(p).$$

Seeing that the above inequality claims for any d. Hence, the subsequent inequality is obtained.

$$\widehat{v}(p) \leq \min_{d=1,\ldots,q} \max_{c=1,\ldots,r} LS_{cd}(p)$$

Hence, $\widehat{v}(p) \leq \widehat{\omega}(p)$. □

Considering the crisp setup of a 2-person ZSMG where the expected payoff is described as the statistical expectation of information present within the payoff matrix. However, probabilistic unbalanced linguistic matrix game has ambiguity and uncertainty within the assessment of the payoff entries such that the veracity of the expected value is incomprehensible. Thus, it is necessary to give the following definition explaining the probabilistic unbalanced linguistic expected payoffs of corresponding players.

Definition 4.4.3. Consider $x^{(k)} = (x_1^{(k)}, \ldots, x_r^{(k)}) \in S_1$ and $y^{(k)} = (y_1^{(k)}, \ldots, y_q^{(k)}) \in S_2$ as a mixed strategy pair associated with degree of importance for corresponding PI and PII, respectively. Then, PI's probabilistic unbalanced linguistic expectation payoff is defined by

$$\widetilde{E}_{\widetilde{R}}(x^{(k)}, y^{(k)}) = \bigoplus_{c=1}^{r} \left(x_c^{(k)} \left(\bigoplus_{d=1}^{q} LS_{cd}(p) \odot y_d^{(k)} \right) \right).$$

On the other hand PII's probabilistic unbalanced linguistic expectation payoff is
$\text{neg}(\widetilde{E}_{\widetilde{R}}(x^{(k)}, y^{(k)}))$.

4.4.2 Methodology for probabilistic multiplicative unbalanced linguistic matrix game

This section majorly discusses the probabilistic multiplicative unbalanced linguistic linear programming (PMULLP) method for matrix game problem having probabilistic multiplicative unbalanced linguistic information.

Based on the presumed multiplicative ULTS, $LS = \left\{ t_{a^i} | i = -n, \ldots, 0, \ldots, n \right\}$, the probabilistic unbalanced linguistic payoff matrix is viewed as:

$$\widetilde{R} = \left(LS_{cd}(p) \right)_{r \times q}.$$

Let $\widehat{v}_{(p)}$ refers to the minimal probabilistic unbalanced linguistic payoff value of the PI and let $x^{(k)} = (x_1^{(k)}, \ldots, x_r^{(k)}) \in S_1$ denote its mixed strategy set associated with certain degree of importance. Now, based on the goal that PI is having, the unbalanced linguistic expectation of PI for the d^{th} strategy of PII is presented as the weighted average of the probabilistic multiplicative unbalanced linguistic variables $LS_{1d}(p), LS_{2d}, \ldots, LS_{rd}$, i.e., $LS_{1d}(p)x_1^{(k)} \oplus LS_{2d}(p)x_2^{(k)} \oplus \ldots \oplus LS_{rd}(p)x_r^{(k)}$. Similarly, $\widehat{\omega}_{(p)}$ can be viewed as the maximal probabilistic unbalanced linguistic payoff value of the PII, $y^{(k)} = (y_1^{(k)}, \ldots, y_q^{(k)}) \in S_2$ be its mixed strategy set, and the unbalanced linguistic expectation of PII against PI's c^{th} strategy is presented as $LS_{c1}(p)y_1^{(k)} \oplus LS_{c2}(p)y_2^{(k)} \oplus \ldots \oplus LS_{cq}(p)y_q^{(k)}$.

Hence, we have the required PMULLP problem for PI:

$$\max \quad \widehat{v}(p) \tag{PMULLP1}$$

$$\text{subject to} \begin{cases} \bigoplus_{c=1}^{r} LS_{cd}(p)x_c \geq \widehat{v}(p), \; d = 1, \ldots, q \\ \sum_{c=1}^{r} x_c = 1; x_c \geq 0, \; c = 1, \ldots, r. \end{cases}$$

where the probabilistic unbalanced linguistic objective value can be outlined as $\widehat{v}(p) = \{ \widetilde{v}^{(k)}(p^{(k)}) | k = 1, \ldots, \#LS(p) \}, \widetilde{v} \in LS$, and, $\sum_{k=1}^{\#LS(p)} p^{(k)} = 1$.

Next, for PII, we have the following system:

$$\min \quad \widehat{\omega}(p) \tag{PMULLP2}$$

$$\text{subject to} \begin{cases} \bigoplus_{d=1}^{q} LS_{cd}(p)y_d \leq \widehat{\omega}(p), \; c = 1, \ldots, r \\ \sum_{d=1}^{q} y_d = 1; y_d \geq 0, \; d = 1, \ldots, q. \end{cases}$$

where the probabilistic unbalanced linguistic objective value can be outlined as $\widehat{\omega}(p) = \left\{ \widetilde{\omega}^{(k)}(p^{(k)}) | k = 1, \ldots, \#LS(p) \right\}, \widetilde{\omega} \in LS$, and, $\sum_{k=1}^{\#LS(p)} p^{(k)} = 1$.

After setting up the mathematical models namely PMULLP1 and PMULLP2 of a probabilistic multiplicative unbalanced linguistic matrix game, it is necessary to describe a unified mechanism to solve these models. Before we proceed to define the main steps of the algorithm we will give the following definition required in the procedure.

Definition 4.4.4. Let $LS = \left\{ t_{a^i} | i = -n, \ldots, 0, \ldots, n \right\}$ be the presumed ULTSs and $LS(p) = \left\{ t_{a^i}(p^{(k)}) | k = 1, \ldots, \#LS(p) \right\}$ be the PM-ULTSs. Then, the information measure of the PM-ULTSs is defined as a mapping:

$$IM : LS(p) \rightarrow \mathbb{R}$$

$$\text{such that } IM(t_{a^i}(p)) = p \cdot a^i.$$

For the sake of convenience while solving the probabilistic unbalanced linguistic game problem, we rewrite $p \cdot a^i$ as $p \cdot \Omega^{-1}(t_{a^i})$. The definition of Ω^{-1} is same as given in the Proposition 3.2.2.

We take into consideration certain assumptions before we conduct the needful steps of the newly designed method i.e.,

- Firstly, we need to normalize the payoff matrix in such a way that each payoff entry has equal cardinality.
- Secondly, the probability of the unbalanced linguistic terms within the set $LS(p)$ should be nonzero.
- Finally, the cardinality of the probabilistic unbalanced linguistic game value of both the players should be same as that of each of the payoff entry in the payoff matrix.

Now we provide the main steps involved in the working process of solving probabilistic unbalanced linguistic matrix game problem:

Step 1: Analyze the original two-player constant sum probabilistic unbalanced linguistic matrix game problem and determine the mixed strategy set $x = (x_1^{(k)}, x_2^{(k)}, \ldots, x_r^{(k)}) \in S_1$ and $y = (y_1^{(k)}, y_2^{(k)}, \ldots, y_q^{(k)})$ of the players I and II, respectively. Based on the ULTS $LS = \{t_{a^i} | i = -n, \ldots, 0, \ldots, n\}$, the PI and PII express their multiplicative unbalanced linguistic evaluation estimates over the strategies $x_c^{(k)} (c = 1, \ldots, r)$ with respect to the strategies $y_d^{(k)} (d = 1, \ldots, q)$. All the different multiplicative unbalanced linguistic evaluation estimates $LS_{cd}^{(k)} (k = 1, \ldots, \#LS(p))$ having the corresponding probability $p^{(k)} (k = 1, \ldots, \#LS(p))$ are represented as PM-ULTS $LS_{cd}(p) = \left\{(t_{a^i})_{cd}^{(k)}(p^{(k)}) | k = 1, \ldots, \#LS_{cd}(p)\right\}$, where $p^{(k)} > 0, k = 1, \ldots, \#LS(p)$, $\#LS_{11}(p) = \#LS_{12}(p) = \ldots = \#LS_{rq}(p) = \#LS(p)$, and $\sum_{k=1}^{\#LS(p)} p^{(k)} = 1$. All the PM-ULTSs $LS_{cd}(p)(c = 1, \ldots, r; d = 1, \ldots, q)$ are recorded within the probabilistic multiplicative unbalanced linguistic payoff matrix $\widetilde{R} = [LS_{cd}(p)]_{r \times q}$.

Step 2: Formulate PMULLP1 and PMULLP2 for PI and PII, respectively.

Step 3: For PI, due to the fact that there exist ordering relation of probabilistic multiplicative unbalanced linguistic term, therefore in IVLLP1, max $\{\widehat{v}(p)\}$ is easily converted into max $\left\{\widetilde{v}^{(k)}(p^{(k)}) | k = 1, \ldots, \#LS(p)\right\}$, which is considered as a multi-objective linguistic linear optimization problem (MOLLOP). Similarly, for PII we have min $\{\widetilde{\omega}(p)\}$ which is also considered as a MOLLOP min $\left\{\widetilde{\omega}^{(k)}(p^{(k)}) | k = 1, \ldots, \#LS(p)\right\}$ where $\widetilde{v}^{(k)}, \widetilde{\omega}^{(k)} \in LS$.

Step 4: Now decompose the multi objective problem PMULLP1 into single objective PMULLP1.1 problem as follows:

For each $k = 1,\ldots,\#LS(p)$, we have

$$\max \quad \widetilde{v}^{(k)}(p^{(k)}) \qquad \text{(PMULLP1.1)}$$

$$\text{subject to}\begin{cases}\bigoplus_{c=1}^{r}(t_{a^i})_{cd}^{(k)}(p^{(k)})x_c^{(k)} \geq \widetilde{v}^{(k)}(p^{(k)}),\ d=1,\ldots,q\\ \sum_{c=1}^{r} x_c^{(k)} = 1;\, x_c \geq 0,\ c=1,\ldots,r.\end{cases}$$

Similarly, for PII we have

For each $k = 1,\ldots,\#LS(p)$

$$\min \quad \widetilde{\omega}^{(k)}(p^{(k)}) \qquad \text{(PMULLP2.1)}$$

$$\text{subject to}\begin{cases}\bigoplus_{d=1}^{q}(t_{a^i})_{cd}^{(k)}(p^{(k)})y_d^{(k)} \leq \widetilde{\omega}^{(k)}(p)^{(k)},\ c=1,\ldots,r\\ \sum_{d=1}^{q} y_d^{(k)} = 1;\, y_d \geq 0,\ d=1,\ldots,q.\end{cases}$$

Step 5: Since Ω^{-1} is determined as monotonically increasing transformation function, henceforth on application of Ω^{-1} on both the sides of the constraint of PMULLP1.1 we obtain the following system of linear optimization problem:

$$\begin{cases}\sum_{c=1}^{r}\Omega^{-1}((t_{a^i})_{cd}^{(k)})\cdot p^{(k)}x_c^{(k)} \geq \Omega^{-1}(\widetilde{v}^{(k)})\cdot p^{(k)},\ d=1,\ldots,q\\ \sum_{c=1}^{r} x_c^{(k)} = 1;\, x_c \geq 0,\ c=1,\ldots,r.\end{cases}$$

Now, following from the definition 4.4.4, we can rewrite the term $\Omega^{-1}((t_{a^i})_{cd}^{(k)})\cdot p^{(k)}$ as $(a^i)_{cd}^{(k)}p^{(k)}(k=1,\ldots,\#LS(p))$. Next, by Assuming that $\Omega^{-1}(\tilde{v}^{(k)})\cdot p^{(k)} = v^{(k)}p^{(k)} > 0$, we make the following substitution as $X_c^{(k)} = \frac{x_c^{(k)}}{v^{(k)}p^{(k)}}(k=1,\ldots,\#LS(p))$, $(c = 1,\ldots,r)$ and $V^{(k)} = \frac{1}{v^{(k)}p^{(k)}}$. Therefore, the transformed system of linear optimization problem is given as follows:

$$\min \quad V^{(k)} = X_1^{(k)} + \ldots + X_r^{(k)} \qquad \text{(PMULLP1.2)}$$

$$\text{subject to}\begin{cases}\sum_{c=1}^{r}(a^i)_{cd}^{(k)}p^{(k)}X_c^{(k)} \geq 1,\ d=1,\ldots,q\\ X_c^{(k)} \geq 0,\ c=1,\ldots,r.\end{cases}$$

Step 6: We repeat the above step for PII and the problem can be constructed in an analogous fashion

$$\begin{cases} \sum_{d=1}^{q} \Omega^{-1}((t_{a^i})_{cd}^{(k)}) \cdot (p^{(k)}) y_d^{(k)} \leq \Omega(\tilde{\omega}^{(k)}) \cdot (p)^{(k)},\ c = 1, \ldots, r \\ \sum_{d=1}^{q} y_d^{(k)} = 1;\ y_d \geq 0,\ d = 1, \ldots, q. \end{cases}$$

Recall that, $\Omega^{-1}((t_{a^i})_{cd}^{(k)}) \cdot p^{(k)}$ as $(a^i)_{cd}^{(k)} p^{(k)} (k = 1, \ldots, \#LS(p))$. Also, by setting $Y_d^{(k)} = \frac{y_d^{(k)}}{\omega^{(k)} p^{(k)}} (k = 1, \ldots, \#LS(p))$,
$(d = 1, \ldots, q)$ and $V^{(k)} = \frac{1}{\omega^{(k)} p^{(k)}}$ since $\Omega^{-1}(\tilde{\omega}^{(k)}) \cdot p^{(k)} = \omega^{(k)} p^{(k)} > 0$. Therefore, the transformed system of linear optimization problem is given as follows:

$$\begin{aligned} \max \quad & V^{(k)} = Y_1^{(k)} + \ldots + Y_q^{(k)} \qquad \text{(PMULLP2.2)} \\ \text{subject to} & \begin{cases} \sum_{d=1}^{q} (a^i)_{cd}^{(k)} p^{(k)} Y_d^{(k)} \leq 1,\ c = 1, \ldots, r \\ Y_d^{(k)} \geq 0,\ d = 1, \ldots, q. \end{cases} \end{aligned}$$

Step 7: After solving the mathematical models PMULLP1.2 and PMULLP2.2, we can obtain the optimal strategies $(x_1^{(k)*}, \ldots, x_r^{(k)*})$ and $(y_1^{(k)*}, \ldots, y_q^{(k)*})$ and the value of the game $p^{(k)} v^{(k)} = p^{(k)} \omega^{(k)} = \frac{1}{V^{(k)*}}$.

Step 8: After representing probabilistic unbalanced linguistic information by information measure function and further obtaining the corresponding efficient game value mentioned in the previous step, we can then apply the retranslation process to retransform the optimal values into probabilistic unbalanced linguistic information. The retranslation mechanism can be performed in the following two steps:

(1) For the value $p^{(k)} v^{(k)*} = p^{(k)} \omega^{(k)*} = \frac{1}{V^{(k)*}}$, if $\frac{1}{V^{(k)*}} \in \{a^i | i = -n, \ldots, 0, \ldots, n\}$ then, $\tilde{v}^{(k)} = \tilde{\omega}^{(k)} = \Omega(\frac{1}{V^{(k)*}})$ $(k = 1, \ldots, \#LS(p))$. If $\frac{1}{V^{(k)*}} \notin \{a^i | i = -n, \ldots, 0, \ldots, n\}$ then either $\frac{1}{V^{(k)*}} \in [a^{-n}, a^n]$ or otherwise $\frac{1}{V^{(k)*}} = \frac{1}{V^{(k)*}} + a^{-n}$. Finally, $\tilde{v}^{(k)} p^{(k)} = \tilde{\omega}^{(k)} p^{(k)} = \Omega(\frac{1}{V^{(k)*}})$.

It is noted that if $\gamma = 0 \implies \Omega(\frac{1}{V^{(k)*}}) = t_\lambda^{(k)}$ and for $\gamma \neq 0$ then Ω has its usual meaning and (t_λ, γ).

(2) Finally, the optimal mixed strategy for PI and PII is represented as $x^* = (x_1^*, \ldots, x_r^*)$ where $x_c^* = \{\ x_c^{(k)} | k = 1, \ldots, \#LS(p)\ \}$, $(c = 1, \ldots, r)$, $y^* = (y_1^*, \ldots, x_q^*)$ where $y_d^* = \left\{ y_d^{(k)} | k = 1, \ldots, \#LS(p) \right\}$, $(d = 1, \ldots, q)$, respectively. The probabilistic unbalanced linguistic game value can be represented as $\widehat{v}(p) = \{\ \widetilde{v}^{(k)}(\theta) | k =$

$1,\ldots,\#LS(p)$ } and $\widehat{\omega}(p)=\left\{\widetilde{\omega}^{(k)}(\theta)|k=1,\ldots,\#LS(p)\right\}$ where $\theta^{(k)}=\begin{cases}1, & \text{if } \gamma=0;\\ \gamma+0.5, & \text{otherwise.}\end{cases}$

It is noteworthy that the linear mathematical models (PMULLP1.2) and (PMULLP2.2) are primal-dual linear optimization models in the crisp setup.

4.5 A numerical illustration and comparison analysis

In the current section, we further illustrate the expediency of PM-ULTS and understand the usefulness and applicability of the two-player constant-sum game having probabilistic multiplicative unbalanced linguistic information with no saddle point by taking a practical, real-life problem of companies selection problem to invest.

Example 4.5.1. The problem statement taken in the current illustration is same as the one presented in the Example 2.4.1.

Here, after preliminary screening an investor has shortlisted 4 companies as an alternatives set, i.e., $A_c\,(c=1,2,3,4)$ where he wants to invest in. Since investing decision is tough and is not always easy for a person to invest his hard-earned money without conducting a research. To compete these shortlisted companies so as to choose the most stable of them and also rank them from the viewpoint of their significance degree, it is crucial for an investor to perform the evaluation and selection operation primarily based on various essential criteria. Nevertheless, in this case study, we consider six main criteria: the performance of company (C_1); the market value of company (C_2); the efficiency level of company (C_3); the business model of company (C_4); Employee satisfaction level (C_5); shareholders funds (C_6). Since an investor is careful about strategic choice of Nature, so he will select his mixed strategy set as $x=(x_1^{(k)},x_2^{(k)},x_3^{(k)},x_4^{(k)}),\,x_c\geq 0,\,c=1,2,3,4,\,\sum_{c=1}^{4}x_c^{(k)}=1$ for each $(k=1,\ldots,\#LS)$ over the alternative set $\left\{A_c|c=1,2,3,4\right\}$. On the contrary, the investor view nature as a non-cooperative player, thus to counter the choice of his mixed strategy, Nature will choose the mixed strategy set as $y=(y_1^{(k)},y_2^{(k)},y_3^{(k)},y_4^{(k)},y_5^{(k)},y_6^{(k)})\,y_d\geq 0,\,d=1,2,3,4,5,6,\,\sum_{d=1}^{6}y_d^{(k)}=1,\,(k=1,\ldots,\#LS)$ over the criteria set $\left\{C_d|d=1,2,3,4,5,6\right\}$.

To deal with this problem by means of PM-ULTSs, an investor utilize the following ULTS to evaluate the companies A_c:

$$LS=\left\{\text{P, VW, W, Avg, G, VG, PR}\right\}$$

$$\max \quad \widehat{v}(p) = \left\{ \widetilde{v}^{(k)}(p^{(k)}) \,|\, k = 1, 2, 3 \right\}$$

subject to

$$\begin{cases} \{Avg(0.5), G(0.25), VG(0.25)\} x_1^{(k)} \oplus \{W(0.5), G(0.25), Avg(0.25)\} x_2^{(k)} \oplus \{W(0.5), VW(0.25), VG(0.25)\} x_3^{(k)} \oplus \{PR(0.25), VG(0.25), G(0.5)\} x_4^{(k)} \geq \widehat{v}(p) \\ \{G(0.5), Avg(0.25), VG(0.25)\} x_1^{(k)} \oplus \{VG(0.33), G(0.33), Avg(0.33)\} x_2^{(k)} \oplus \{W(0.5), G(0.25), Avg(0.25)\} x_3^{(k)} \oplus \{G(0.25), VG(0.5), PR(0.25)\} x_4^{(k)} \geq \widehat{v}(p) \\ \{PR(0.25), VG(0.5), G(0.25)\} x_1^{(k)} \oplus \{Avg(0.5), W(0.25), G(0.25)\} x_2^{(k)} \oplus \{G(0.5), Avg(0.25), VG(0.25)\} x_3^{(k)} \oplus \{PR(0.5), VG(0.25), G(0.25)\} x_4^{(k)} \geq \widehat{v}(p) \\ \{Avg(0.5), G(0.25), VG(0.25)\} x_1^{(k)} \oplus \{Avg(0.5), G(0.25), VG(0.25)\} x_2^{(k)} \oplus \{G(0.25), Avg(0.5), VG(0.25)\} x_3^{(k)} \oplus \{W(0.25), Avg(0.5), G(0.25)\} x_4^{(k)} \geq \widehat{v}(p) \\ \{G(0.5), VG(0.25), PR(0.25)\} x_1^{(k)} \oplus \{W(0.5), VW(0.25), Avg(0.25)\} x_2^{(k)} \oplus \{VG(0.5), PR(0.25), G(0.25)\} x_3^{(k)} \oplus \{W(0.5), Avg(0.25), G(0.25)\} x_4^{(k)} \geq \widehat{v}(p) \\ \{VG(0.25), G(0.25), PR(0.5)\} x_1^{(k)} \oplus \{VW(0.5), W(0.25), Avg(0.25)\} x_2^{(k)} \oplus \{Avg(0.25), G(0.5), VG(0.25)\} x_3^{(k)} \oplus \{W(0.33), Avg(0.33), G(0.33)\} x_4^{(k)} \geq \widehat{v}(p) \\ \sum_{c=1}^{4} x_c^{(k)} = 1; x_c^{(k)} \geq 0, \; c = 1, 2, 3, 4. \end{cases} \tag{M1}$$

Note: Poor (P), Very weak (VW), Weak (W), Average (Avg), Good (G), Very Good (VG), Perfect (PR). Based on the representation algorithm stated in chapter 3, we have

$$LS = \left\{ t_{1/8} : \text{(P)}, t_{1/4} : \text{(VW)}, t_{1/2} : \text{(W)}, t_1 : \text{(Avg)}, t_2 : \text{(G)}, t_4 : \text{(VG)}, t_8 : \text{(PR)} \right\}$$

Next, as we proceed further it is perceived that in the present case study the investor provide his preferences in respect of several multiplicative unbalanced linguistic terms on the response alternatives $A_c \, (c = 1, 2, 3, 4)$ subject to the six criteria $C_d \, (d = 1, 2, 3, 4, 5, 6)$. Since investors studied the overall companies' performance based on the mentioned criteria for the past four financial years (FY), i.e., $2016 - 2020$. Thus, the original multiplicative unbalanced linguistic payoff matrices for the four consecutive years are given in Tables 4.3-4.6. It is noted that the blanks mentioned in the Tables 4.5 and 4.6 represent that for a given FY, the companies evaluation for that particular criteria is very weak. As a result, it can be neglected. By summing up, these four tables directly, the assessment information in Table 4.7 can be exhibited in terms of PM-ULTSs, which is regarded as a payoff matrix for this decision analysis problem modeled as a probabilistic unbalanced linguistic matrix game. To understand the representation of the linguistic evaluation of A_2 concerning C_2, i.e., $\left\{VG(0.5), G(0.25), Avg(0.25)\right\}$ is derived based on the analysis made by the investor for four consecutive years that for two out of four years companies overall assessment is very good (VG), for one out of four years is good (G), and also for one year it is average (Avg). Next, we normalize the payoff matrix, and hence final normalized probabilistic unbalanced linguistic matrix is given in Table 4.8.

After attaining the normalized probabilistic unbalanced linguistic payoff matrix, we can construct the PMULLP problem (See Eqn. (M1)) for PI by using model (PMULLP1).

Next, based on the methodology presented in the previous section, the MOLLOP is decomposed into three single objective PMULLP problem, and further based on the transformation function, the system is converted into three crisp linear optimization problems (See

$$\min \quad V^1 = X_1^1 + X_2^1 + X_3^1 + X_4^1 \text{subject to} \begin{cases} 0.5X_1^1 + 0.0625X_2^1 + 0.25X_3^1 + 2X_4^1 \geq 1 \\ X_1^1 + 0.165X_2^1 + 0.25X_3^1 + 0.5X_4^1 \geq 1 \\ 2X_1^1 + 2X_2^1 + X_3^1 + 4X_4^1 \geq 1 \\ 0.5X_1^1 + 0.5X_2^1 + 0.5X_3^1 + 0.125X_4^1 \geq 1 \\ X_1^1 + 0.25X_2^1 + 0.2X_3^1 + 0.25X_4^1 \geq 1 \\ X_1^1 + 0.125X_2^1 + 0.33X_3^1 + 0.125X_4^1 \geq 1 \\ X_1^1, X_2^1, X_3^1, X_4^1 \geq 0. \end{cases} \tag{M1.1}$$

$$\min \quad V^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 \text{subject to} \begin{cases} 0.5X_1^2 + X_2^2 + 0.0625X_3^2 + X_4^2 \geq 1 \\ 0.25X_1^2 + 0.66X_2^2 + 0.5X_3^2 + 2X_4^2 \geq 1 \\ 2X_1^2 + 0.5X_2^2 + 0.25X_3^2 + X_4^2 \geq 1 \\ 0.5X_1^2 + 0.125X_2^2 + 0.5X_3^2 + 0.5X_4^2 \geq 1 \\ X_1^2 + 0.0625X_2^2 + 2X_3^2 + 0.25X_4^2 \geq 1 \\ 0.5X_1^2 + 0.125X_2^2 + 0.66X_3^2 + 0.5X_4^2 \geq 1 \\ X_1^2, X_2^2, X_3^2, X_4^2 \geq 0. \end{cases} \tag{M1.2}$$

$$\min \quad V^3 = X_1^3 + X_2^3 + X_3^3 + X_4^3 \text{subject to} \begin{cases} X_1^3 + 0.25X_2^3 + X_3^3 + X_4^3 \geq 1 \\ X_1^3 + 0.33X_2^3 + 0.25X_3^3 + 2X_4^3 \geq 1 \\ 0.5X_1^3 + 0.25X_2^3 + X_3^3 + 0.5X_4^3 \geq 1 \\ X_1^3 + 0.5X_2^3 + X_3^3 + 0.5X_4^3 \geq 1 \\ 2X_1^3 + 0.25X_2^3 + 0.5X_3^3 + 0.5X_4^3 \geq 1 \\ 4X_1^3 + 0.25X_2^3 + 0.132X_3^3 + 0.5X_4^3 \geq 1 \\ X_1^3, X_2^3, X_3^3, X_4^3 \geq 0. \end{cases} \tag{M1.3}$$

Eqn. (M1.1), (M1.2), (M1.3)).

The optimal solution obtained for the mentioned problem of PI are $X^{(1)*} = (2, 0, 0, 0)$, $X^{(2)*} = (2, 0, 0.2857, 1.7143)$, $X^{(3)*} = (0.2449, 0, 0.7347, 0.2857)$ and $V^* = (2, 2, 1.2653)$.

Hence, for PI, the optimal mixed strategy is obtained as $x_1^* = \{1, 0, 0.1935\}$, $x_2^* = 0$, $x_3^* = \{0, 0.14285, 0.5806\}$, $x_4^* = \{0, 0.85715, 0.2258\}$. After the re-translation process the optimal probabilistic unbalanced linguistic value is $\widehat{v}(p) = \{t_{1/2}(0.9612), t_{2^0}(0.0387), t_{1/2}(0)\}$.

Similarly, we can formulate the PMULLP (See Eqn. (M2)) problem for PII by using model (PMULLP2).

$$\min \quad \widehat{\omega}(p) = \left\{\widetilde{\omega}^{(k)}(p^{(k)}) | k = 1, 2, 3\right\}$$

subject to

$$\begin{cases} \{Avg(0.5), G(0.25), VG(0.25)\}\, y_1^{(k)} \oplus \{G(0.5), Avg(0.25), VG(0.25)\}\, y_2^{(k)} \oplus \{PR(0.25), VG(0.5), G(0.25)\}\, y_3^{(k)} \oplus \{Avg(0.5), G(0.25), VG(0.25)\}\, y_4^{(k)} \oplus \{G(0.5), VG(0.25), PR(0.25)\}\, y_5^{(k)} \oplus \{VG(0.25), G(0.25), PR(0.5)\}\, y_6^{(k)} \leq \widehat{\omega}(p) \\ \{W(0.5), G(0.25), Avg(0.25)\}\, y_1^{(k)} \oplus \{VG(0.33), G(0.33), Avg(0.33)\}\, y_2^{(k)} \oplus \{Avg(0.5), W(0.25), G(0.25)\}\, y_3^{(k)} \oplus \{Avg(0.5), G(0.25), VG(0.25)\}\, y_4^{(k)} \oplus \{W(0.5), VW(0.25), Avg(0.25)\}\, y_5^{(k)} \oplus \{VW(0.5), W(0.25), Avg(0.25)\}\, y_6^{(k)} \leq \widehat{\omega}(p) \\ \{W(0.5), VW(0.25), VG(0.25)\}\, y_1^{(k)} \oplus \{W(0.5), G(0.25), Avg(0.25)\}\, y_2^{(k)} \oplus \{G(0.5), Avg(0.25), VG(0.25)\}\, y_3^{(k)} \oplus \{G(0.25), Avg(0.5), VG(0.25)\}\, y_4^{(k)} \oplus \{VG(0.5), PR(0.25), G(0.25)\}\, y_5^{(k)} \oplus \{Avg(0.25), G(0.5), VG(0.25)\}\, y_6^{(k)} \leq \widehat{\omega}(p) \\ \{PR(0.25), VG(0.25), G(0.5)\}\, y_1^{(k)} \oplus \{G(0.25), VG(0.5), PR(0.25)\}\, y_2^{(k)} \oplus \{PR(0.5), VG(0.25), G(0.25)\}\, y_3^{(k)} \oplus \{W(0.25), Avg(0.5), G(0.25)\}\, y_4^{(k)} \oplus \{W(0.5), Avg(0.25), G(0.25)\}\, y_5^{(k)} \oplus \{W(0.33), Avg(0.33), G(0.33)\}\, y_6^{(k)} \leq \widehat{\omega}(p) \\ \sum_{d=1}^{6} y_d^{(k)} = 1;\ y_d^{(k)} \geq 0,\ d = 1, 2, 3, 4, 5, 6. \end{cases} \tag{M2}$$

The corresponding three linear optimization problem are also given (See Eqn. (M2.1), (M2.2), (M2.3)). After solving, the optimal solution obtained for the mentioned problem for PII are $Y^{(1)*} = (0.4, 0, 0, 1.6, 0, 0)$, $Y^{(2)*} = (0, 0, 0, 2, 0, 0)$, $Y^{(3)*} = (0, 0.2449, 0.8571, 0, 0.1633, 0)$ and $V^* = (2, 2, 1.2653)$.

Hence, for PII, the optimal mixed strategy is obtained as $y_1^* = \{0.2, 0, 0\}$, $y_2^* = \{0, 0, 0.1935\}$,

$$\max \quad V^1 = Y_1^1+Y_2^1+Y_3^1+Y_4^1+Y_5^1+Y_6^1 \text{subject to} \begin{cases} 0.5Y_1^1+Y_2^1+2Y_3^1+0.5Y_4^1+Y_5^1+Y_6^1 \leq 1 \\ 0.0625Y_1^1+0.165Y_2^1+2Y_3^1+0.5Y_4^1+0.25Y_5^1+0.125Y_6^1 \leq 1 \\ 0.25Y_1^1+0.25Y_2^1+Y_3^1+0.5Y_4^1+0.2Y_5^1+0.33Y_6^1 \leq 1 \\ 2Y_1^1+0.5Y_2^1+4Y_3^1+0.125Y_4^1+0.25Y_5^1+0.125Y_6^1 \leq 1 \\ Y_1^1, Y_2^1, Y_3^1, Y_4^1, Y_5^1, Y_6^1 \geq 0. \end{cases} \tag{M2.1}$$

$$\max \quad V^2 = Y_1^2+Y_2^2+Y_3^2+Y_4^2+Y_5^2+Y_6^2 \text{subject to} \begin{cases} 0.5Y_1^2+0.25Y_2^2+2Y_3^2+0.5Y_4^2+Y_5^2+0.5Y_6^2 \leq 1 \\ Y_1^2+0.66Y_2^2+0.5Y_3^2+0.125Y_4^2+0.0625Y_5^2+0.125Y_6^2 \leq 1 \\ 0.0625Y_1^2+0.5Y_2^2+0.25Y_3^2+0.5Y_4^2+2Y_5^2+0.66Y_6^2 \leq 1 \\ Y_1^2+2Y_2^2+Y_3^2+0.5Y_4^2+0.25Y_5^2+0.5Y_6^2 \leq 1 \\ Y_1^2, Y_2^2, Y_3^2, Y_4^2, Y_5^2, Y_6^2 \geq 0. \end{cases} \tag{M2.2}$$

$$\max \quad V^3 = Y_1^3+Y_2^3+Y_3^3+Y_4^3+Y_5^3+Y_6^3 \text{subject to} \begin{cases} Y_1^3+Y_2^3+0.5Y_3^3+Y_4^3+2Y_5^3+4Y_6^3 \leq 1 \\ 0.25Y_1^3+0.33Y_2^3+0.25Y_3^3+0.5Y_4^3+0.25Y_5^3+0.25Y_6^3 \leq 1 \\ Y_1^3+0.25Y_2^3+Y_3^3+Y_4^3+0.5Y_5^3+0.132Y_6^3 \leq 1 \\ Y_1^3+2Y_2^3+0.5Y_3^3+0.5Y_4^3+0.5Y_5^3+0.5Y_6^3 \leq 1 \\ Y_1^3, Y_2^3, Y_3^3, Y_4^3, Y_5^3, Y_6^3 \geq 0. \end{cases} \tag{M2.3}$$

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	Avg	G	PR	Avg	G	VG
A_2	W	VG	Avg	Avg	W	VW
A_3	W	W	G	G	VG	Avg
A_4	PR	G	PR	W	W	W

Table 4.3: Unbalanced linguistic payoff matrix for the FY2016-2017

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	Avg	Avg	VG	Avg	G	G
A_2	G	G	W	G	W	VW
A_3	VW	G	Avg	Avg	VG	G
A_4	VG	VG	PR	Avg	W	Avg

Table 4.4: Unbalanced linguistic payoff matrix for the FY2017-2018

$y_3^* = \{0, 0, 0.6774\}$, $y_4^* = \{0.8, 1, 0\}$, $y_5^* = \{0, 0, 0.1291\}$, $y_6^* = 0$. After the re-translation process the optimal probabilistic unbalanced linguistic value is $\widehat{\omega}(p) = \{\ t_{1/2}(0.9612),$ $t_{2^0}(0.0387), t_{1/2}(0)\ \}$.

Here, for the investor, the criteria for which $y_d^* = 0$ are of least significance. Analogously, the alternatives with $x_c^* = 0$ are irrelevant to the investor.

Finally, to rank the given alternatives we calculate the total score of the alternatives as follows: $A_1 = 1.1935$, $A_2 = 0$, $A_3 = 0.72345$, $A_4 = 1.08295$. Hence, the four alternatives are ordered as $A_1 > A_4 > A_3 > A_2$, and alternative A_1 is the recommended company for the investor to invest his money.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	G	VG	VG	G	VG	PR
A_2	Avg	Avg	G	Avg	VW	W
A_3	W	W	G	Avg	G	-
A_4	G	VG	VG	G	Avg	G

Table 4.5: Unbalanced linguistic payoff matrix for the FY2018-2019

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	VG	G	G	VG	PR	PR
A_2	W	-	Avg	VG	Avg	Avg
A_3	VG	Avg	VG	VG	PR	VG
A_4	G	PR	G	Avg	G	Avg

Table 4.6: Unbalanced linguistic payoff matrix for the FY2019-2020

It is perceived that the game-theoretic approach to solving multi decision analysis problem having probabilistic multiplicative unbalanced linguistic information is more advantageous in comparison to the methodologies given in literature in the sense that the latter needs prior information of the weights of the given criteria, unlike our developed approach that assesses the weights as an intermediary step. In addition, compared to the game-theoretic models with single linguistic terms [14, 15], linguistic 2-tuple information [248, 249], the probability unbalanced linguistic information allows players not only to provide evaluations on several linguistic terms non-uniformly distributed instead of the single linguistic term which is symmetrically distributed as well as its probability information. Moreover, considering the game models having linguistic 2-tuple information does not handle an incomplete evaluation. As a result, it increases the burden and difficulty level of the players. However, probabilistic linguistic games efficiently handle problems with incomplete information. Also, recently Mi et al. [214] has proposed a methodology to solve matrix game problems in terms of probabilistic linguistic information based on symmetrically LTSs. In that paper, the authors have approximated the probabilistic linguistic information into a classical triangular membership function. Consequently, it leads to information loss. However, we are not converting the probabilistic linguistic information into some approximation function in our proposed method. Instead, we define an information measure of the probabilistic linguistic term by using the bijective function Ω and Ω^{-1}. Thereby, avoids information loss in the final re-translation process.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	$\{Avg(0.5), G(0.25), VG(0.25)\}$	$\{G(0.5), Avg(0.25), VG(0.25)\}$	$\{PR(0.25), VG(0.5), G(0.25)\}$	$\{Avg(0.5), G(0.25), VG(0.25)\}$	$\{G(0.5), VG(0.25), PR(0.25)\}$	$\{VG(0.25), G(0.25), PR(0.5)\}$
A_2	$\{W(0.5), G(0.25), Avg(0.25)\}$	$\{VG(0.25), G(0.25), Avg(0.25)\}$	$\{Avg(0.5), W(0.25), G(0.25)\}$	$\{Avg(0.5), G(0.25), VG(0.25)\}$	$\{W(0.5), VW(0.25), Avg(0.25)\}$	$\{VW(0.5), W(0.25), Avg(0.25)\}$
A_3	$\{W(0.5), VW(0.25), VG(0.25)\}$	$\{W(0.5), G(0.25), Avg(0.25)\}$	$\{G(0.5), Avg(0.25), VG(0.25)\}$	$\{G(0.25), Avg(0.5), VG(0.25)\}$	$\{VG(0.5), PR(0.25), G(0.25)\}$	$\{Avg(0.25), G(0.25), VG(0.25)\}$
A_4	$\{PR(0.25), VG(0.25), G(0.5)\}$	$\{G(0.25), VG(0.5), PR(0.25)\}$	$\{PR(0.5), VG(0.25), G(0.25)\}$	$\{W(0.25), Avg(0.5), G(0.25)\}$	$\{W(0.5), Avg(0.25), G(0.25)\}$	$\{W(0.25), Avg(0.5), G(0.25)\}$

Table 4.7: Comprehensive probabilistic multiplicative unbalanced linguistic payoff matrix

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	$\{Avg(0.5), G(0.25), VG(0.25)\}$	$\{G(0.5), Avg(0.25), VG(0.25)\}$	$\{PR(0.25), VG(0.5), G(0.25)\}$	$\{Avg(0.5), G(0.25), VG(0.25)\}$	$\{G(0.5), VG(0.25), PR(0.25)\}$	$\{VG(0.25), G(0.25), PR(0.5)\}$
A_2	$\{W(0.5), G(0.25), Avg(0.25)\}$	$\{VG(0.33), G(0.33), Avg(0.33)\}$	$\{Avg(0.5), W(0.25), G(0.25)\}$	$\{Avg(0.5), G(0.25), VG(0.25)\}$	$\{W(0.5), VW(0.25), Avg(0.25)\}$	$\{VW(0.5), W(0.25), Avg(0.25)\}$
A_3	$\{W(0.5), VW(0.25), VG(0.25)\}$	$\{W(0.5), G(0.25), Avg(0.25)\}$	$\{G(0.5), Avg(0.25), VG(0.25)\}$	$\{G(0.25), Avg(0.5), VG(0.25)\}$	$\{VG(0.5), PR(0.25), G(0.25)\}$	$\{Avg(0.33), G(0.33), VG(0.33)\}$
A_4	$\{PR(0.25), VG(0.25), G(0.5)\}$	$\{G(0.25), VG(0.5), PR(0.25)\}$	$\{PR(0.5), VG(0.25), G(0.25)\}$	$\{W(0.25), Avg(0.5), G(0.25)\}$	$\{W(0.5), Avg(0.25), G(0.25)\}$	$\{W(0.25), Avg(0.5), G(0.25)\}$

Table 4.8: Comprehensive normalized probabilistic multiplicative unbalanced linguistic payoff matrix

4.6 Conclusion

This chapter introduces the concept of PM-ULTSs, which considers the probability of linguistic variables and the non-uniformity of linguistic labels. After then, some new operational laws for managing probabilistic unbalanced linguistic information are also developed, which could produce valid results and keep the property of operational laws closed. Furthermore, some elementary aggregation operators to aggregate PM-ULTSs have been constructed for further consideration. The need for these operators lies in the fact that they are beneficial in the situation involving information that cannot be expressed with an actual number rather the probabilistic multiplicative unbalanced linguistic information. Finally, we have applied our proposed concept to investigate the two-player matrix game problem having probabilistic unbalanced linguistic information allowing game theory to accept the incomplete linguistic information, which is non-symmetrically distributed as input. The information measure for the probabilistic multiplicative unbalanced linguistic term is calculated to transform the probabilistic linguistic information. The information measure is primarily based on the bijective function Ω and Ω^{-1} such that it avoids the loss of information in the re-translation process. Thus, the linear optimization model for both the players is formulated to obtain optimal game value and corresponding mixed strategies. The calculated game value is further re-translated into the original PM-ULTS to maintain the interpretability of the probabilistic unbalanced linguistic game. Furthermore, the validity and advantages of the probabilistic unbalanced linguistic matrix game were verified and justified by applying it to solve real-life multi-criteria decision analysis problems about the companies selected by a general investor to invest his sum of money from the perspective of the game with Nature.

It is perceived that the proposed probabilistic unbalanced linguistic matrix game methodology perfectly handles complicated decision-analysis problems under the probabilistic unbalanced linguistic environment. In addition, the method is found to be consistent and more acquirable to solve a DM problem as it is capable of generating the optimal weights for the given alternatives in an intermedial step, unlike the other prevailing methodologies where the weights value are presupposed to be well-known a priori. The optimal game value obtained is viewed as a PM-ULTSs that represents the uncertainty more reasonably using several unbalanced linguistic terms instead of single linguistic term such that it closely aligns human thoughts. We believe that the wide-ranging application of our proposed concept make it promising and it is readily apparent that the present study bring about a new view point to the uncertain linguistic matrix game solution and strategic analysis.

Although the practical example of companies' selection problem by a general investor to invest his money is provided to validate the expediency and efficacy of the suggested method, it can be extended to solve several GDM problems. In the future, one can widen the scope of the present study to define unbalanced linguistic bimatrix games, cooperative games in the probabilistic environment.

Chapter 5

Matrix games with interval-valued 2-tuple linguistic information

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We investigate a non-cooperative 2-player zero-sum interval-valued 2-tuple fuzzy linguistic (IVTFL) matrix game in this chapter[1] *and provide a methodology for determining the saddle point and optimal interval-valued linguistic value of the game. In this direction, we created an auxiliary pair of interval-valued linguistic linear programming (IVLLP) problems that are then transformed into traditional interval linear programming (ILP) problems to attain optimal strategy sets for both corresponding players as the solution region that is not only completely feasible but also completely optimal. A hypothetical example is used to demonstrate the suggested method's applicability in the actual world. The transformed ILP problems are solved utilizing the best-worst case (BWC) approach, enhanced-interval linear programming (EILP) method, and linguistic linear programming (LLP) technique of solving interval linguistic matrix game problems to validate the suggested solution scheme, and lastly, the results are compared.*

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[1]The work presented in this chapter comprises the results of a research paper entitled "Methodology for Interval-Valued Matrix Games with 2-Tuple Fuzzy Linguistic Information". In: Sergeyev Y., Kvasov D. (eds) Numerical Computations: Theory and Algorithms. NUMTA 2019. Lecture Notes in Computer Science. Springer, Cham. , **11974**, (2020). https://doi.org/10.1007/978-3-030-40616-5_12. (**Conference Proceedings Citation Index (CPCI)**)

5.1 Introduction

Non-cooperative game theory in its classical set up was introduced by Von-Neumann and Morgenstern [268] in 1944. It asserts that every player is exposed to the game's precisely known information. The prevailing knowledge of the game permits each player to furnish appropriate evaluations to their utility functions corresponding to different pair of strategies. The postulations made for the exact payoffs can be considered as the stringent ideology in the real world scenario which involves uncertain and ambiguous information. Imprecision and uncertainty have been incorporated in game theory by using various frameworks like fuzzy, stochastic etc. Several researchers have contributed significantly in enhancing the literature of fuzzy games [30,31,58,260] and stochastic games [18,146]. However, in the world of uncertainties, it is also challenging for players to express payoffs in terms of membership functions in fuzzy environment or probability distribution functions in stochastic environment. To facilitate the players with effortless choice of payoffs, a new version of matrix games under uncertainty is proposed by Arfi [14, 15] based on linguistic fuzzy logic. To annex a new dimension to the matrix game problems under linguistic environment, Singh et al. [248] defined matrix games with linguistic information and proposed a linguistic linear programming (LLP) approach to solve such class of games. Singh et al. [249] further extended the matrix games to interval-valued 2TL framework to increase the level of uncertainty in game problems and adopted LLP approach to solve it. The authors formulated a pair of auxiliary LLP problems to obtain the linguistic lower and upper bounds of interval linguistic value of the game.

In the present chapter, we extend the work of solving interval linguistic matrix game (ILG) problems one step forward. Here, we propose a mechanism to compare IVTFL variables using the bounds of the intervals and subsequently, define interval linguistic lower value (ILLV) and interval linguistic upper value (ILUV) of the matrix game by introducing the concept of max-min and min-max principle. In the absence of pure strategies, we suggest IVLLP formulation to obtain the interval linguistic value of game with the optimal strategies of both players by transforming it to conventional ILP problem. To validate the proposed methodology, Best Worst Case (BWC) method [17], Enhanced Interval Linear Programming (EILP) method [350] and LLP method [249] are adopted to solve the transformed ILP problems and provide a comparative analysis. The duality principle of IVLLP is also taken into consideration in order to prove the equality of ILLV and ILUV of the game for player I and II, respectively.

The remaining chapter unfolds as follows. In subsequent section 5.2, explains a new approach to compare two IVTFL variables based on the end point approach. In section 5.3 , a zero sum interval-valued 2TL matrix game is defined with its interval linguistic lower and upper values using max-min principle. Section 5.4 discusses interval-valued linguistic linear programming approach to solve the game in absence of pure strategies with a hypothetical illustration. The chapter concludes in Section 5.5.

5.2 Comparison of interval-valued 2-tuple fuzzy linguistic variables

In literature, Zhang [338] defined the comparison of interval-valued 2TL variables using score and accuracy values. It gives a total ordering of the linguistic intervals that does not show analogy with classical numeric intervals [37, 38, 52, 78, 135, 215, 216, 256, 281].

So, here, we present a new comparison scheme of interval-valued 2TL variables. The approach involves the bounds of the intervals that allows to define a partial ordering of the linguistic intervals. Here, we consider the following cases to encompass all possible pair of intervals.

(1) **Case of Disjoint Intervals:** Let $\widetilde{\mu} = [(\ell_{i^{(L)}}, \alpha_{i^{(L)}}), (\ell_{i^{(U)}}, \alpha_{i^{(U)}})]$, $\widetilde{\nu} = [(\ell_{j^{(L)}}, \alpha_{j^{(L)}}), (\ell_{j^{(U)}}, \alpha_{j^{(U)}})]$ be two disjoint IVTFL variables. Then

$$\widetilde{\mu} < \widetilde{\nu} \text{ iff } (\ell_{i^{(U)}}, \alpha_{i^{(U)}}) < (\ell_{j^{(L)}}, \alpha_{j^{(L)}}).$$

(2) **Case of Nested Intervals :** Let $\widetilde{\mu} = [(\ell_{i^{(L)}}, \alpha_{i^{(L)}}), (\ell_{i^{(U)}}, \alpha_{i^{(U)}})]$, $\widetilde{\nu} = [(\ell_{j^{(L)}}, \alpha_{j^{(L)}}), (\ell_{j^{(U)}}, \alpha_{j^{(U)}})]$ be the two IVTFL variables such that one of the following cases occur :

(i) If $i^{(L)} \leq j^{(L)} < j^{(U)} \leq i^{(U)} \Rightarrow (\ell_{i^{(L)}}, \alpha_{i^{(L)}}) \leq (\ell_{j^{(L)}}, \alpha_{j^{(L)}}) < (\ell_{j^{(U)}}, \alpha_{j^{(U)}}) \leq (\ell_{i^{(U)}}, \alpha_{i^{(U)}})$.

(ii) If $i^{(L)} = j^{(L)} = j^{(U)} = i^{(U)} \Rightarrow \alpha_{i^{(L)}} \leq \alpha_{j^{(L)}} < \alpha_{j^{(U)}} \leq \alpha_{i^{(U)}}$ such that $(\ell_{i^{(L)}}, \alpha_{i^{(L)}}) \leq (\ell_{j^{(L)}}, \alpha_{j^{(L)}}) < (\ell_{j^{(U)}}, \alpha_{j^{(U)}}) \leq (\ell_{i^{(U)}}, \alpha_{i^{(U)}})$.

(iii) If $i^{(L)} = j^{(L)} < j^{(U)} < i^{(U)} \Rightarrow \alpha_{i^{(L)}} \leq \alpha_{j^{(L)}} < \alpha_{j^{(U)}} \leq \alpha_{i^{(U)}}$ such that $(\ell_{i^{(L)}}, \alpha_{i^{(L)}}) \leq (\ell_{j^{(L)}}, \alpha_{j^{(L)}}) < (\ell_{j^{(U)}}, \alpha_{j^{(U)}}) \leq (\ell_{i^{(U)}}, \alpha_{i^{(U)}})$.

(iv) If $i^{(L)} = j^{(L)} = j^{(U)} < i^{(U)} \Rightarrow \alpha_{i^{(L)}} \leq \alpha_{j^{(L)}} < \alpha_{j^{(U)}} \leq \alpha_{i^{(U)}}$ such that $(\ell_{i^{(L)}}, \alpha_{i^{(L)}}) \leq (\ell_{j^{(L)}}, \alpha_{j^{(L)}}) < (\ell_{j^{(U)}}, \alpha_{j^{(U)}}) \leq (\ell_{i^{(U)}}, \alpha_{i^{(U)}})$.

(v) If $i^{(L)} \leq j^{(L)} = j^{(U)} \leq i^{(U)} \Rightarrow \alpha_{i^{(L)}} \leq \alpha_{j^{(L)}} < \alpha_{j^{(U)}} \leq \alpha_{i^{(U)}}$ such that $(\ell_{i^{(L)}}, \alpha_{i^{(L)}}) \leq (\ell_{j^{(L)}}, \alpha_{j^{(L)}}) < (\ell_{j^{(U)}}, \alpha_{j^{(U)}}) \leq (\ell_{i^{(U)}}, \alpha_{i^{(U)}})$.

(vi) If $i^{(L)} \leq j^{(L)} < j^{(U)} = i^{(U)} \Rightarrow \alpha_{i^{(L)}} \leq \alpha_{j^{(L)}} < \alpha_{j^{(U)}} \leq \alpha_{i^{(U)}}$ such that $(\ell_{i^{(L)}}, \alpha_{i^{(L)}}) \leq (\ell_{j^{(L)}}, \alpha_{j^{(L)}}) < (\ell_{j^{(U)}}, \alpha_{j^{(U)}}) \leq (\ell_{i^{(U)}}, \alpha_{i^{(U)}})$.

All above cases infer that the linguistic interval $\widetilde{\nu}$ is contained in $\widetilde{\mu}$, denoted as $\widetilde{\nu} \subset \widetilde{\mu}$. It demonstrates the inclusion property of linguistic intervals i.e. the interval $\widetilde{\nu}$ is nested within $\widetilde{\mu}$ and cannot be ordered in respect of values.

(3) **Case of Overlapped Intervals :** Let $\widetilde{\mu} = [(\ell_{i^{(L)}}, \alpha_{i^{(L)}}), (\ell_{i^{(U)}}, \alpha_{i^{(U)}})]$, $\widetilde{\nu} = [(\ell_{j^{(L)}}, \alpha_{j^{(L)}}), (\ell_{j^{(U)}}, \alpha_{j^{(U)}})]$ be two overlapping IVTFL variables such that

$$(\ell_{i^{(L)}}, \alpha_{i^{(L)}}) \leq (\ell_{j^{(L)}}, \alpha_{j^{(L)}}) < (\ell_{i^{(U)}}, \alpha_{i^{(U)}}) \leq (\ell_{j^{(U)}}, \alpha_{j^{(U)}}),$$

then $\widetilde{\mu} < \widetilde{\nu}$.

For instance, consider the predefined LTS, $LT = \{\ell_{-2}$: "Very Bad" (VB), ℓ_{-1} : "Bad" (B), ℓ_0 : "Medium" (M), ℓ_1 : "Good" (G), ℓ_2 : "Very Good" (VG)$\}$. Suppose $S = \{\widetilde{\mu}_1 = [(\ell_{-2}, 0), (\ell_0, 0)], \widetilde{\mu}_2 = [(\ell_{-2}, 0.8), (\ell_{-1}, 0.23)], \widetilde{\mu}_3 = [(\ell_0, 0.05), (\ell_2, -0.5)], \widetilde{\mu}_4 = [(\ell_{-1}, 0), (\ell_1, 0)]\}$ be a set of IVTFL variables using the predefined LTS LT. Here, μ_1 and μ_2 are nested linguistic intervals whereas interval μ_1 is disjoint with μ_3 and overlapping with μ_4, comparing which we obtain that $\mu_1 < \mu_3$, $\mu_1 < \mu_4$ but μ_1 and μ_2 can not be compared. Only the inclusion property can be discussed i.e. $\mu_2 \subset \mu_1$. On the similar grounds, the other pair of intervals can be compared.

In literature, Singh et al. [249] adopted the matrices formulated using lower bounds and upper bounds of the payoff intervals to define interval-valued linguistic (IVL) value of the game. However, the authors suggested LLP approach to solve IVL matrix game in case of mixed strategies. Unlike the existing solution scheme, here, based on the comparison of linguistic intervals defined in the preceding section, the value of the interval fuzzy linguistic game is defined in the light of min-max principle and subsequently, IVLLP problem approach is proposed to solve such games.

5.3 A zero-sum interval-valued linguistic matrix game

Definition 5.3.1. A two-player zero-sum IVL matrix game $\widetilde{G}_{Int}$ is characterized by a quadruplet $(S^n, S^m, LT, \widetilde{A}_{Int})$, where S^n, S^m are strategy sets for player I and II respectively and LT $= \{\ell_{-g}, \ldots, \ell_0, \ldots, \ell_g\}$ is the predefined SSLTS. The matrix $\widetilde{A}_{Int} = \left([\widetilde{a}_{ij}^{(L)}, \widetilde{a}_{ij}^{(U)}]\right)_{n\times m}$; $i = 1, \ldots n$, $j = 1, \ldots, m$ is the interval-valued linguistic payoff matrix of player I in defiance of player II whereas neg $\widetilde{A}_{Int} = \left([neg(\widetilde{a}_{ij}^{(U)}), neg(\widetilde{a}_{ij}^{(L)})]\right)_{n\times m}$ depicts the payoff matrix of player II such that the payoffs of two players sum up to $(\ell_0, 0)$.

Since the comparison of IVTFL variables are proposed in the preceding section, the IVL value of the game can be defined in the subsequent manner.

Definition 5.3.2. For a given IVL matrix game $\widetilde{G}_{Int}$ with payoff matrix $\widetilde{A}_{Int}$, the IVL lower value, $\widetilde{v}_{Int}^{-}$ and IVL upper value, $\widetilde{v}_{Int}^{+}$ of the game is defined as,

$$\widetilde{v}_{Int}^{-} = \max_{i=1,\ldots,n} \min_{j=1,\ldots,m} [\widetilde{a}_{ij}^{(L)}, \widetilde{a}_{ij}^{(U)}],$$
$$\widetilde{v}_{Int}^{+} = \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} [\widetilde{a}_{ij}^{(L)}, \widetilde{a}_{ij}^{(U)}].$$

The IVL value, $\widetilde{v}_{Int}$ of the game exists when $\widetilde{v}_{Int}^{-} = \widetilde{v}_{Int}^{+} = \widetilde{v}_{Int}$.

The strategy set (i^*, j^*) for which these values are equal is called the saddle point of the game and i^*, j^* are optimal strategies of players I and II respectively.

For any IVL matrix game, the following inequality holds.

Theorem 5.3.1. Suppose $\widetilde{v}_{Int}^{-} = [\widetilde{v}^{-(L)}, \widetilde{v}^{-(U)}]$ and $\widetilde{v}_{Int}^{+} = [\widetilde{v}^{+(L)}, \widetilde{v}^{+(U)}]$ be the IVL lower and upper values of an interval linguistic matrix game $\widetilde{G}_{Int}$ such that both values exist. Then, $\widetilde{v}_{Int}^{-} \leq \widetilde{v}_{Int}^{+}$.

Proof. We are given that $\widetilde{v}_{Int}^{-}$ and $\widetilde{v}_{Int}^{+}$ both exist, so for some column j and fixed row i, we have,

$$\min_{j=1,\ldots,m} [\widetilde{a}_{ij}^{(L)}, \widetilde{a}_{ij}^{(U)}] \leq [\widetilde{a}_{ij}^{(L)}, \widetilde{a}_{ij}^{(U)}],$$

By taking max over $i = 1, \ldots, n$ on both sides, we obtain,

$$\widetilde{v}_{Int}^{-} \equiv \max_{i=1,\ldots,n} \min_{j=1,\ldots,m} [\widetilde{a}_{ij}^{(L)}, \widetilde{a}_{ij}^{(U)}] \leq \max_{i=1,\ldots,n} [\widetilde{a}_{ij}^{(L)}, \widetilde{a}_{ij}^{(U)}]$$

$$\Rightarrow \widetilde{v}_{Int}^{-} \leq \max_{i=1,\ldots,n} [\widetilde{a}_{ij}^{(L)}, \widetilde{a}_{ij}^{(U)}].$$

Since the above inequality holds for any j. Hence, we obtain the following inequality.

$$\widetilde{v}_{Int}^{-} \leq \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} [\widetilde{a}_{ij}^{(L)}, \widetilde{a}_{ij}^{(U)}]$$

Hence, $\widetilde{v}_{Int}^{-} \leq \widetilde{v}_{Int}^{+}$. □

Now, we exemplify the above theory using an illustration.

Example 5.3.1. Two firms need to introduce a number of essentially equivalent new products. In the next two months, the companies are planning to launch the products. The payoffs are the companies' share, which it will acquire taking into account the months during which production takes place. The payoffs of the companies appear in the form of IVTFL variables from the set of predefined linguistic terms, LT = $\{\ell_{-2}:$ "Very Low" (VL), $\ell_{-1}:$ "Low" (L), $\ell_0:$ "Fair" (F), $\ell_1:$ "Good" (G), $\ell_2:$ "Very Good" (VG).$\}$. The interval linguistic payoff matrix for PI is given as.

$$\widetilde{A} = \begin{bmatrix} VL & [(VL,0.2),(L,0.4561)] \\ [(G,0.4),VG] & G \end{bmatrix}$$

Here,

$$\begin{aligned} \widetilde{v}_{Int}^{-} &= \max_{i=1,2} \min_{j=1,2} \left\{[\tilde{a}_{ij}^{(L)}, \tilde{a}_{ij}^{(U)}]\right\} \\ &= \max\{\min\{VL,[(VL,0.2),(L,0.4561)]\},\min\{[(G,0.4),VG)],G\}\} \\ &= \max\{VL,G\} = G. \end{aligned}$$

$$\begin{aligned} \text{Also,}\ \widetilde{v}_{Int}^{+} &= \min_{j=1,2} \max_{i=1,2} \left\{[\tilde{a}_{ij}^{(L)}, \tilde{a}_{ij}^{(U)}]\right\} \\ &= \min\{\max\{VL,[(G,0.4),VG)]\},\max\{[(VL,0.2),(L,0.4561)],G\}\} \\ &= \min\{[(G,0.4),VG],G\} = G. \end{aligned}$$

Here, $(2,2)$ is the saddle point and $\widetilde{v}_{Int} = G$ is the IVL value of the matrix games. This shows that in order to maximize the profit both the firms should launch their products in the second month simultaneously.

In the above example, if we replace the entry $[\tilde{a}_{11}^{(L)}, \tilde{a}_{11}^{(U)}]$ as $[F,G]$ and $[\tilde{a}_{21}^{(L)}, \tilde{a}_{21}^{(U)}]$ as VL, then $\widetilde{v}_{Int}^{-} = [(VL,0.2),(L,0.4561)]$, and $\widetilde{v}_{Int}^{+} = [F,G]$, it depicts the absence of pure strategies. The validity of Theorem 1 can also be deduced from the example as in case of pure strategy, the equality holds whereas in another case, $\widetilde{v}_{Int}^{-} < \widetilde{v}_{Int}^{+}$.

To evaluate the strategy sets and optimal value of the game in absence of pure strategy, here we define the IVLLP approach to solve such games.

5.4 Interval-valued linguistic linear programming approach to solve interval linguistic matrix games

Suppose, we have the interval linguistic payoff matrix $\widetilde{A}_{Int}$ using the predefined LTS $LT = \{\ell_{-g}, \ldots, \ell_0, \ldots, \ell_g\}$ as follows.

$$\widetilde{A}_{Int} = \begin{pmatrix} [\tilde{a}_{11}^{(L)}, \tilde{a}_{11}^{(U)}] & [\tilde{a}_{12}^{(L)}, \tilde{a}_{12}^{(U)}] & \ldots & [\tilde{a}_{1m}^{(L)}, \tilde{a}_{1m}^{(U)}] \\ [\tilde{a}_{21}^{(L)}, \tilde{a}_{21}^{(U)}] & [\tilde{a}_{22}^{(L)}, \tilde{a}_{22}^{(U)}] & \ldots & [\tilde{a}_{2m}^{(L)}, \tilde{a}_{2m}^{(U)}] \\ \vdots & \ddots & \vdots & \vdots \\ [\tilde{a}_{n1}^{(L)}, \tilde{a}_{n1}^{(U)}] & [\tilde{a}_{n2}^{(L)}, \tilde{a}_{n2}^{(U)}] & \ldots & [\tilde{a}_{nm}^{(L)}, \tilde{a}_{nm}^{(U)}] \end{pmatrix}$$

where $[\tilde{a}_{ij}^{(L)}, \tilde{a}_{ij}^{(U)}]$ is the payoff of player I on selecting i^{th} strategy when player II selects the j^{th} strategy.

Here, we may assume that each entries of the interval linguistic payoff matrix is either $[\tilde{a}_{ij}^{(L)}, \tilde{a}_{ij}^{(U)}] < 0$ or $[\tilde{a}_{ij}^{(L)}, \tilde{a}_{ij}^{(U)}] > 0$. Let $S^n = \left\{\mathbf{X} = (x_1, x_2, \ldots, x_n) \mid x_i \geq 0, \ \sum_{i=1}^{n} x_i = 1\right\}$ and $S^m = \left\{\mathbf{Y} = (y_1, y_2, \ldots, y_m) \mid y_j \geq 0, \ \sum_{j=1}^{m} y_j = 1\right\}$ be the mixed strategy set for player I and II, respectively. Then, the expected payoff of player I when player II selects j^{th} strategy, is taken as the weighted average of the IVL variables in the j^{th} column i.e. $[\widetilde{a}_{1j}^{(L)}, \widetilde{a}_{1j}^{(U)}]x_1 \oplus \ldots \oplus [\widetilde{a}_{nj}^{(L)}, \widetilde{a}_{nj}^{(U)}]x_n$.

Hence, the required IVLLP problem for player I is given as.

$$\max \quad \widetilde{v}^{-}_{Int} \qquad \text{(IVLLP1)}$$

subject to

$$[\widetilde{a}^{(L)}_{11}, \widetilde{a}^{(U)}_{11}]x_1 \oplus \ldots \oplus [\widetilde{a}^{(L)}_{n1}, \widetilde{a}^{(U)}_{n1}]x_n \geq \widetilde{v}^{-}_{Int}$$
$$[\widetilde{a}^{(L)}_{12}, \widetilde{a}^{(U)}_{12}]x_1 \oplus \ldots \oplus [\widetilde{a}^{(L)}_{n2}, \widetilde{a}^{(U)}_{n2}]x_n \geq \widetilde{v}^{-}_{Int}$$
$$\vdots$$
$$[\widetilde{a}^{(L)}_{1m}, \widetilde{a}^{(U)}_{1m}]x_1 \oplus \ldots \oplus [\widetilde{a}^{(L)}_{nm}, \widetilde{a}^{(U)}_{nm}]x_n \geq \widetilde{v}^{-}_{Int}$$
$$x_1 + x_2 + \ldots + x_n = 1$$
$$x_1, x_2, \ldots, x_n \geq 0.$$

Using the monotonicity of Δ^{-1} operator, the inequality constraints of above IVLLP model can be rewritten as follows,

$$\Delta^{-1}([\widetilde{a}^{(L)}_{11}, \widetilde{a}^{(U)}_{11}])x_1 \oplus \ldots \oplus \Delta^{-1}([\widetilde{a}^{(L)}_{n1}, \widetilde{a}^{(U)}_{n1}])x_n \geq \Delta^{-1}(\widetilde{v}^{-}_{Int})$$
$$\Delta^{-1}([\widetilde{a}^{(L)}_{12}, \widetilde{a}^{(U)}_{12}])x_1 \oplus \ldots \oplus \Delta^{-1}([\widetilde{a}^{(L)}_{n2}, \widetilde{a}^{(U)}_{n2}])x_n \geq \Delta^{-1}(\widetilde{v}^{-}_{Int})$$
$$\vdots$$
$$\Delta^{-1}([\widetilde{a}^{(L)}_{1m}, \widetilde{a}^{(U)}_{1m}])x_1 \oplus \ldots \oplus \Delta^{-1}([\widetilde{a}^{(L)}_{nm}, \widetilde{a}^{(U)}_{nm}])x_n \geq \Delta^{-1}(\widetilde{v}^{-}_{Int})$$

and the objective function max $\widetilde{v}^{-}_{Int} \equiv \max\ \Delta^{-1}(\widetilde{v}^{-}_{Int})$.

By taking $\Delta^{-1}([\widetilde{a}^{(L)}_{ij}, \widetilde{a}^{(U)}_{ij}]) = [a^{(L)}_{ij}, a^{(U)}_{ij}],\ i = 1, \ldots, n,\ j = 1, \ldots, m$, and $\Delta^{-1}(\widetilde{v}^{-}_{Int}) = v_{Int}$, the constraints of model IVLLP1 is given as.

$$[a^{(L)}_{11}, a^{(U)}_{11}]x_1 + \ldots + [a^{(L)}_{n1}, a^{(U)}_{n1}]x_n \geq v_{Int}$$
$$[a^{(L)}_{12}, a^{(U)}_{12}]x_1 + \ldots + [a^{(L)}_{n2}, a^{(U)}_{n2}]x_n \geq v_{Int}$$
$$\vdots$$
$$[a^{(L)}_{1m}, a^{(U)}_{1m}]x_1 + \ldots + [a^{(L)}_{nm}, a^{(U)}_{nm}]x_n \geq v_{Int}$$
$$x_1 + x_2 + \ldots + x_n = 1$$
$$x_1, x_2, \ldots, x_n \geq 0.$$

Here, we assume that $0 \notin \Delta^{-1}(\widetilde{v}^{-}_{Int}) = v_{Int}$.

We set, $X_i = \dfrac{x_i}{v_{Int}}$, $i = 1, \ldots n$, and $V_{Int} = \dfrac{1}{v_{Int}} = \left[\dfrac{1}{v^{(U)}}, \dfrac{1}{v^{(L)}}\right]$. Hence, by making the above substitutions, the model IVLLP1 is transformed into standard ILP problem for player I, given below.

$$\min V_{Int} = \quad X_1 + X_2 + \ldots + X_n \tag{ILP1}$$

subject to

$$[a^{(L)}_{11}, a^{(U)}_{11}]X_1 + \ldots + [a^{(L)}_{n1}, a^{(U)}_{n1}]X_n \geq [1,1]$$

$$[a^{(L)}_{12}, a^{(U)}_{12}]X_1 + \ldots + [a^{(L)}_{n2}, a^{(U)}_{n2}]X_n \geq [1,1]$$

$$\vdots$$

$$[a^{(L)}_{1m}, a^{(U)}_{1m}]X_1 + \ldots + [a^{(L)}_{nm}, a^{(U)}_{nm}]X_n \geq [1,1]$$

$$X_1,\ X_2, \ldots, X_n \geq 0.$$

Analogously, we can formulate an IVLLP problem for player II.

$$\min \widetilde{v}^{+}_{Int} \tag{IVLLP2}$$

subject to

$$[\widetilde{a}^{(L)}_{11}, \widetilde{a}^{(U)}_{11}]y_1 \oplus \ldots \oplus [\widetilde{a}^{(L)}_{1m}, \widetilde{a}^{(U)}_{1m}]y_m \leq \widetilde{v}^{+}_{Int}$$

$$[\widetilde{a}^{(L)}_{21}, \widetilde{a}^{(U)}_{21}]y_1 \oplus \oplus \ldots \oplus [\widetilde{a}^{(L)}_{2m}, \widetilde{a}^{(U)}_{2m}]y_m \leq \widetilde{v}^{+}_{Int}$$

$$\vdots$$

$$[\widetilde{a}^{(L)}_{n1}, \widetilde{a}^{(U)}_{n1}]y_1 \oplus \ldots \oplus [\widetilde{a}^{(L)}_{nm}, \widetilde{a}^{(U)}_{nm}]y_m \leq \widetilde{v}^{+}_{Int}$$

$$y_1 + y_2 + \ldots + y_m = 1$$

$$y_1, y_2, \ldots, y_m \geq 0.$$

Recall $0 \notin v_{Int} = \Delta^{-1}(\widetilde{v}^{-}_{Int})$. If v_{Int} is the value of the interval linguistic game then $v_{Int} = \Delta^{-1}(\widetilde{v}^{+}_{Int})$.

By taking $Y_j = \dfrac{y_j}{v_{Int}}$, $j = 1, \ldots, m$, and as earlier we discussed that $\Delta^{-1}([\widetilde{a}^{(L)}_{ij}, \widetilde{a}^{(U)}_{ij}]) = [a^{(L)}_{ij}, a^{(U)}_{ij}]$, the corresponding model IVLLP2 reduces to the following standard ILP problem for player II.

$$\max V_{Int} = \quad Y_1 + Y_2 + \ldots + Y_m \tag{ILP2}$$

subject to

$$[a_{11}^{(L)}, a_{11}^{(U)}]Y_1 \ldots + [a_{1m}^{(L)}, a_{1m}^{(U)}]Y_m \leq [1,1]$$

$$[a_{21}^{(L)}, a_{21}^{(U)}]Y_1 + \ldots + [a_{2m}^{(L)}, a_{2m}^{(U)}]Y_m \leq [1,1]$$

$$\vdots$$

$$[a_{n1}^{(L)}, a_{n1}^{(U)}]Y_1 + \ldots + [a_{nm}^{(L)}, a_{nm}^{(U)}]Y_m \leq [1,1]$$

$$Y_1, Y_2, \ldots, Y_m \geq 0.$$

Here, models ILP1 and ILP2 can be solved using any existing methods for solving ILP problems to obtain the optimal mixed strategies $\mathbf{X}^*_{\mathbf{Int}} \in S^n$ and $\mathbf{Y}^*_{\mathbf{Int}} \in S^m$ along with the interval linguistic value of the game V^*_{Int}. It is also noteworthy that both models ILP1 and ILP2 form a primal-dual interval linear programs in the crisp set-up.

Example 5.4.1. Consider the zero-sum IVTFL matrix game having interval payoffs defined from the previously defined SSLTS LT $= \{\ell_{-3}$: "Very Low"(VL), ℓ_{-2} : "Low"(L), ℓ_{-1} : "Moderately Low"(ML), ℓ_0 : "Average"(Avg), ℓ_1 : "Moderately High"(MH), ℓ_2 : "High"(H), ℓ_3 : "Very High"$(VH)\}$ with payoff matrix,

$$\widetilde{A}_{Int} = \begin{pmatrix} [(VL,0.2),(L,0)] & [(H,-0.2),(VH,-0.3)] & [(MH,0.3),(H,0.2)] & [(ML,-0.13),(Avg,0)] \\ [(VH,0),(VH,0)] & [(H,0),(VH,-0.2)] & [(L,-0.4),(ML,0)] & [(MH,0),(H,-0.2)] \\ [(MH,-0.28),(MH,-0.28)] & [(L,0),(ML,0)] & [(VH,0),(VH,0)] & [(VH,0),(VH,0)] \end{pmatrix}$$

Let PI's mixed strategies be given as $\mathbf{x}^{(L)} = (x_1^{(L)}, x_2^{(L)}, x_3^{(L)})$, $x_i^{(L)} \geq 0$, $i = 1, \ldots, 3$, $\sum_{i=1}^{3} x_i^{(L)} = 1$, and $\mathbf{x}^{(U)} = (x_1^{(U)}, x_2^{(U)}, x_3^{(U)})$, $x_i^{(U)} \geq 0$, $i = 1, \ldots, 3$, $\sum_{i=1}^{3} x_i^{(U)} = 1$ for the given interval payoff matrix, $\widetilde{A}_{Int}$. In addition, PII's mixed strategies are defined as $\mathbf{y}^{(L)} = (y_1^{(L)}, y_2^{(L)}, y_3^{(L)}, y_4^{(L)})$, $y_j^{(L)} \geq 0$, $j = 1, \ldots, 4$, $\sum_{j=1}^{4} y_j^{(L)} = 1$, and $\mathbf{y}^{(U)} = (y_1^{(U)}, y_2^{(U)}, y_3^{(U)}, y_4^{(U)}, y_5^{(U)})$, $y_j^{(U)} \geq 0$, $j = 1, \ldots, 5$, $\sum_{j=1}^{5} y_j^{(U)} = 1$.

We build models (IVLLP1 and IVLLP2) based on the suggested methodology, which are then translated into classical ILP problems to derive optimal strategy sets for PI and PII, respectively.

Method 1: Best-Worst method

For Player I:

Best-sub model	Worst-sub model
$\min \quad V^{-(U)} = X_1 + X_2 + X_3$	$\min \quad V^{-(L)} = X_1 + X_2 + X_3$
subject to	subject to
$-2.8X_1 + 3X_2 + 2.4X_3 \geq 1,$	$-2X_1 + 3X_2 + 3X_3 \geq 1,$
$1.8X_1 + 2X_2 - 2X_3 \geq 1,$	$2.7X_1 + 2.8X_2 - X_3 \geq 1,$
$1.3X_1 - 2.4X_2 + 3X_3 \geq 1,$	$2.2X_1 - X_2 + 3X_3 \geq 1,$
$-1.13X_1 + X_2 + 3X_3 \geq 1,$	$0X_1 + 1.8X_2 + 3X_3 \geq 1,$
$X_1, X_2, X_3 \geq 0.$	$X_1, X_2, X_3 \geq 0.$

Solving these two problems, we obtain the optimal strategy of player I as $x_1 = [0.3146, 0.3685]$, $x_2 = [0.3149, 0.3289]$, $x_3 = [0.3149, 0.3575]$ with IVL lower value of the game given as, $v_{Int}^{-} = [(\ell_1, -0.33), (\ell_1, 0.43)]$.

For Player II:

Best-sub model	Worst-sub model
$\max \quad V^{+(U)} = Y_1 + Y_2 + Y_3 + Y_4$	$\max \quad V^{+(L)} = Y_1 + Y_2 + Y_3 + Y_4$
subject to	subject to
$-2.8Y_1 + 1.8Y_2 + 1.3Y_3 - 1.13Y_4 \leq 1,$	$-2Y_1 + 2.7Y_2 + 2.2Y_3 + 0Y_4 \leq 1,$
$3Y_1 + 2Y_2 - 2.4Y_3 + Y_4 \leq 1,$	$3Y_1 + 2.8Y_2 - 1Y_3 + 1.8Y_4 \leq 1,$
$2.4Y_1 - 2Y_2 + 3Y_3 + 3Y_4 \leq 1,$	$3Y_1 - 1Y_2 + 3Y_3 + 3Y_4 \leq 1,$
$Y_1, Y_2, Y_3, Y_4 \geq 0.$	$Y_1, Y_2, Y_3, Y_4 \geq 0.$

For player II, the optimal strategy set is $y_1 = [0.2077, 0.2288]$, $y_2 = [0.4004, 0.4422]$, $y_3 = [0.3484, 0.3718]$, $y_4 = 0$ with IVL upper value of the game, $v_{Int}^{+} = [(\ell_1, -0.33), (\ell_1, 0.43)]$.

Method 2: Enhanced interval-valued linear programming method

For Player I:

Sub-problem I	Sub-problem II
$\min \quad V^{-(U)} = X_1^U + X_2^U + X_3^U$	$\min \quad V^{-(L)} = X_1^L + X_2^L + X_3^L$
subject to	subject to
$-2X_1^U + 3X_2^U + 2.4X_3^U \geq 1,$	$-2.8X_1^L + 3X_2^L + 3X_3^L \geq 1,$
$1.8X_1^U + 2X_2^U - X_3^U \geq 1,$	$2.7X_1^L + 2.8X_2^L - 2X_3^L \geq 1,$
$1.3X_1^U - X_2^U + 3X_3^U \geq 1,$	$2.2X_1^L - 2.4X_2^L + 3X_3^L \geq 1,$
$0X_1^U + X_2^U + 3X_3^U \geq 1,$	$-1.13X_1^L + 1.8X_2^L + 3X_3^L \geq 1,$
$X_1^U, X_2^U, X_3^U \geq 0.$	$X_1^L, X_2^L, X_3^L \geq 0.$

For Player II:

Sub-problem I	Sub-problem II
$\max \quad V^{+(U)} = Y_1^U + Y_2^U + Y_3^U + Y_4^U$	$\max \quad V^{+(L)} = Y_1^L + Y_2^L + Y_3^L + Y_4^L$
subject to	subject to
$-2Y_1^U + 1.8Y_2^U + 1.3Y_3^U \leq 1,$	$-2.8Y_1^L + 2.7Y_2^L + 2.2Y_3^L - 1.13Y_4^L \leq 1,$
$3Y_1^U + 2Y_2^U - Y_3^U + Y_4^U \leq 1,$	$3Y_1^L + 2.8Y_2^L - 2.4Y_3^L - 1.8Y_4^L \leq 1,$
$2.4Y_1^U - Y_2^U + 3Y_3^L + 3Y_4^L \leq 1,$	$3Y_1^L - 2Y_2^L + 3Y_3^L + 3Y_4^U \leq 1,$
$Y_1^U, Y_2^U, Y_3^U, Y_4^U \geq 0.$	$Y_1^L, Y_2^L, Y_3^L, Y_4^L \geq 0.$

Solving the above models, the optimal strategies of PI and PII are evaluated as $x_1 = [0.3008, 0.3395]$, $x_2 = [0.2726, 0.3201]$, $x_3 = [0.2813, 0.3196]$ and $y_1 = [0.1455, 0.235]$, $y_2 = [0.3478, 0.4462]$, $y_3 = [0.3102, 0.3589]$, $y_4 = 0$ respectively with $v_{Int} = [(\ell_1, -0.06), (\ell_1, -0.03)]$.

Method 3: Linguistic Linear Programming (LLP) Method

We split our matrix $\widetilde{A}_{Int}$ into linguistic lower matrix and linguistic upper matrix to obtain interval linguistic lower and upper values of the interval linguistic matrix game. The mathematical formulation for this problem is similar to that of BWC.

The optimal strategies obtained in respect of both players and value of the game using various existing methodologies to solve ILP problems, are tabulated below.

For Player I :

Method	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	Optimal Value
BWC	$[0.3146, 0.3685]$	$[0.3149, 0.3289]$	$[0.3149, 0.3575]$	$[(\ell_1, -0.33), (\ell_1, 0.43)]$
EILP	$[0.3008, 0.3395]$	$[0.2726, 0.3201]$	$[0.2813, 0.3196]$	$[(\ell_1, -0.06), (\ell_1, -0.03)]$
LLP	$[0.3146, 0.3685]$	$[0.3149, 0.3289]$	$[0.3149, 0.3575]$	$[(\ell_1, -0.33), (\ell_1, 0.43)]$

For Player II :

Method	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$	Optimal Value
BWC	$[0.2077, 0.2288]$	$[0.4004, 0.4422]$	$[0.3484, 0.3718]$	0	$[(\ell_1, -0.33), (\ell_1, 0.43)]$
EILP	$[0.1455, 0.235]$	$[0.3478, 0.4462]$	$[0.3102, 0.3589]$	0	$[(\ell_1, -0.06), (\ell_1, -0.03)]$
LLP	$[0.2077, 0.2288]$	$[0.4004, 0.4422]$	$[0.3484, 0.3718]$	0	$[(\ell_1, -0.33), (\ell_1, 0.43)]$

Here, the solution region obtained using EILP method is completely optimal and feasible. However, BWC and LLP approach provides a solution region which is completely optimal but may not be feasible. This is because it incorporates some infeasible points within the solution set.

5.5 Conclusion

In this chapter, we have studied the 2-player zero sum IVL matrix game problems. We proposed a new methodology for comparing two IVTFL variables and subsequently, put forward the concept of max-min principle for defining the lower and upper value of the interval linguistic game problem. However, in the absence of pure strategies, we designed a new approach for evaluating the optimal strategies and value of the game. We envision that the proposed method can easily be applied to large scale interval linguistic game problems, manufacturing companies, large scale DM problems where the existing players (or DMs) have conflicting objectives.

Chapter 6

Interval norm approach for solving two player zero sum matrix games with interval payoffs

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The work of this chapter[1]*, present a new approach that gives a unique outlook for solving a two-player zero-sum interval-valued matrix game based on the interval matrix norm framework. The methodology presented in our work is useful in obtaining an approximated interval game value for the corresponding zero-sum interval matrix game without undergoing the existing process of solving traditional interval linear mathematical models. For this, we initially proceed to propose the concept of an interval* $1-$ *norm and* $\infty-$ *norm for the interval matrix game problem. Later on, based on the proposed interval norm concept, some new results have been developed, pointing out some necessary conditions to obtain the interval optimum boundaries for the interval game value if at all exist. Additionally, an efficient method based on the normalization concept is developed to obtain the appropriate bounds for the largest and smallest element lying within the mixed strategy set for the corresponding game players. Lastly, the established method is applied to the practical numerical examples, and, a comparison of results with the traditional methods for imprecise matrix game is provided to demonstrate the consistency and usefulness of the proposed methodology.*

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[1]This chapter is based on a research paper entitled "Interval norm approach for solving two player zero sum matrix games with interval payoffs" (**Submitted**)

6.1 Introduction

Game theory [36] is focused with the analysis of conflicting situations with the goal of capturing strategic behavior. Due to its adaptable character and uses in a wide range of conflicts and problems, different types of mathematical game models have been thoroughly researched and successfully implemented in a variety of fields suchlike economics, finance, computer science, sociology, political science etc. Nevertheless, due to a lack of specific information and a player's imprecise understanding of the situation, it is difficult to evaluate payoffs properly in gaming situations. The players can only make educated guesses about the payoff value. The most important and much debated topic among the scholars is how to deal with uncertainty. Most of these complex scenarios cannot be well represented by crisp data. Henceforth, the need for structured and accurate assessment brings the existence of matrix games into the fuzzy environment (See for reference [30–32, 58, 180–184]) and stochastic environment (See for reference [**?**, 18, 86]).

In almost all the matrix game models defined under fuzzy or stochastic environment, the payoffs are treated as fuzzy numbers or probabilities, with the assumption that their membership functions/ non membership functions or else the probabilistic distribution function have already been explained. However, in unknown situations, it is not always possible for the game players to define the membership functions/ distribution function. In many circumstances, the payoffs can be thought of as interval numbers, meaning that for any fixed strategy, the players' payoffs can fluctuate within a certain range. Therefore, interval-valued matrix game (IMG) formulation have received boundless attention among researchers for managing matrix game problems under uncertainty.

In the literature, many researchers have developed techniques/methods for obtaining the solution of IMG problems. Collins [59] proposed a methodology to model uncertainty involved in matrix games by considering IMG. In that paper, the authors extended classical matrix games' results to fuzzily determined IMGs by utilizing an appropriate fuzzy interval-based ranking method. Liu and Kao [186] developed a method to solve zero-sum IMG concerning two players having payoff matrix with each entry as interval. They constructed a pair of 2-level mathematical programs to obtain the lower and upper bound of the interval game value. Finally, based upon the duality theorem and further applying the technique of variable substitution, two-level mathematical programs are amended into a pair of classical one-level linear mathematical programs. However, the method proposed by Liu and Kao [186] focused mainly on obtaining the bounds (i.e., both lower and upper) of the interval game value of the IMG and did not suggest any particular technique for computing corresponding optimal strategies for game participants. Additionally, the methodology in that paper results in several additional variables and constraints in the derived supplementary linear mathematical programming models, culminating in high computational requirements. Nayak and Pal [222] structured a pair of ILP models for the IMG. Authors separated the interval and chose only the lower bounds of PI's gain-floor and PII's loss-ceiling as objective functions in the solving practise, and then transformed the ILP models into classic linear mathematical models utilising interval based inequality relations [257, 258].

Further, Li [185] modified the inappropriate formulations provided in the paper [221] and pointed out some vital mistakes in the existing approach. Moreover, the author derived the bi-objective mathematical programming models and put forward to solve the model via lexicographic ordering technique to obtain a rational and credible solution of the generic IMG. Later on, Li et al. [178] derived a couple of bi-objective linear mathematical programming models from the established auxiliary ILP models that is primarily based upon the defined interval inequality relations and the fuzzy based ranking index. Here, authors' have used the method of weighted average to solve the bi-objective linear programming mathematical models, unlike the lexicographic method suggested in the paper [185]. Since the methodologies mentioned above are competent to address game problems with complete-information. However, most practical problems are exposed to uncertain situations where players often do not have sufficient information regarding payoffs. In such cases, the payoffs deal with uncertain payoffs that are estimated in the form of intervals from available data. In this context, Dey and Zaman [87] proposed a robust optimization methodology for addressing incomplete-information for two-player zero-sum and nonzero-sum games considering single or multiple interval inflows.

After shedding light on the dedicated literature of IMGs, it is envisaged that most of the developed methodologies for solving two-players IMGs with either complete or incomplete information require the use of ILP methods or programs. However, as the interval game matrix's size increases, the classical methods of obtaining the solution of these problems will become a tedious job. Moreover, in most of the ILP methods studied in the literature (See [224]), it is foreseen that the ILP models are transformed intially into two linear mathematical submodels by using lower bounds and upper bounds of the given interval payoffs to obtain lower and upper bound of the interval value of the IMG separately. As a consequence, the results for the IMG problems obtained by the authors are primarily based upon the bounds of the interval payoffs. Henceforth, in the current study we focuses on proposing a rational method proficient enough to achieve an approximate solution faster without splitting the interval payoffs into lower and upper bounds of the given inputs.

It is also observed that while solving zero-sum IMG problem via existing ILP methods the solution space so obtained may consist of some infeasible points and thereby becomes inefficient in handling interval uncertainties present within the real-world system. To eliminate the infeasibility of the solution space and providing the solution that is both completely feasible and optimal several improvements in the methods and its sub-models have been proposed in the past (See [224]). Nevertheless, the ILP methods increases the computation cost and complexity of the solution space for the scenario where game problem is having large counting of decision variables. So, to avoid the limitation of the existing methods and further to reflect a possibility of attaining a set of potential interval solutions for IMGs proficient enough in meeting a certain level of satisfaction, we proposed a method in this research that uses the concept of $1-$ interval norm and $\infty-$ interval norm for the interval payoff matrix.

In our methodology, based on the novel concept of interval matrix norm, we present a new result stating some necessary conditions to compute optimal interval boundaries for the interval game value without solving any mathematical linear equations. Since the computation of a mixed strategy set for corresponding game players is a significant step within any existing game problems. So, in this regard, we present results imperative to find the largest and smallest elements of the mixed strategy sets for the game players and further establish the $\min-\max/\max-\min$ interval inequality relations. It is also notable to emphasize that the proposed interval norm concept and the proposed results are all based on the modified interval arithmetic operations (See [102]) instead usual interval arithmetic to attain the better solution space for the given IMG problem. Lastly, to highlight the application of the suggested approach some numerical illustration is also given. Further, we provide insight about the comparison of the obtained results with the existing ILP method to demonstrate the consistency and usefulness of the developed methodology.

Although a novel approach of solving a two-player ZSMG centered on the framework of matrix norms in terms of real numbers has already been developed in the literature (See [133]), nevertheless, solving zero-sum IMGs by using interval norms are unfamiliar and needed to be studied. Therefore, the technique adapted in the chapter has made a significant contribution in enriching the theory of interval matrix norms and puts forward a new outlook in IMGs. The proposed method is immensely useful in the condition where the interval game problem involves large decision variables. Consequently, obtaining an approximated interval game value at a faster rate without explicitly solving the interval linear mathematical models that generally increases the complexity of the problem by increasing the constraints number.

Even though the developed methodology helps obtain the solution faster, it is significant to note that our methodology does not endure splitting of intervals into lower and upper bounds persisting in the existing approach. Accordingly, substantiating the viewpoint of logic to be an interval throughout the solving process. Henceforth, facilitating players to understand the structural behavior of the strategies within the range of admissible values.

6.1.1 Basic definitions

We first refresh some foundational definitions and results based on the interval numbers and modified interval arithmetic operations required for this chapter.

Moore [215] initially suggests an interval number and is further studied as a superset of the real line $\mathbb{R}$ by extending the domain of real number.

Definition 6.1.1. [215] An interval number is defined as $\hat{m} = [m^L, m^U]$ where $m^L, m^U \in \mathbb{R}$ and $m^L \leq m^U$. Here, the numbers m^L and m^U are termed as lower and upper bound of the interval $\hat{m}$, respectively. If $m^L = m^U$ then, $\hat{m}$ is degenerate, and the interval $\hat{m}$ evolves into a crisp number.

In addition, an interval number is alternately expressed in terms of $\hat{m} =< c(\hat{m}), w(\hat{m}) >$ where, $c(\hat{m}), w(\hat{m})$ represents mid-point and half-width of an interval $\hat{m}$, respectively. i.e.,

$$c(\hat{m}) = \frac{m^L + m^U}{2}$$

and

$$w(\hat{m}) = \frac{m^U - m^L}{2}$$

Definition 6.1.2. [215] Suppose that $M^L = (m^L_{ij}), M^U = (m^U_{ij}) \in \mathbb{R}^{p\times q}, p, q \in \mathbb{N}$ are two matrices such that $\forall i, j; (m^L_{ij}) \leq (m^U_{ij})$. Thus, an interval matrix is given in the subsequent form:

$$\hat{M} = [M^L, M^U] = \left\{M^L \leq M \leq M^U \mid M \in \mathbb{R}^{p\times q}\right\}.$$

The set of all $p \times q$ interval matrices is termed as $p \times q$ interval matrix spaces which is denoted by $\mathbb{IR}^{p\times q}$.

Definition 6.1.3. [215] An interval number $\hat{m}$ is purely positive if $m^L \geq 0$ and it is purely negative if $m^U \leq 0$.

Further, Farhadsefat et al. [94] introduced interval matrix norms induced by generic point matrix norms within the space of interval matrices. The following definition is taken from [94].

Definition 6.1.4. [94] A function $|||.||| : \mathbb{IR}^{p\times q} \to \mathbb{R}$ is defined as an interval matrix norm in the space $\mathbb{IR}^{p\times q}$ if the subsequent properties holds true:

(i) $|||\hat{A}||| \geq 0$, and $|||\hat{A}||| = 0$ when $\hat{A} = 0$;

(ii) $|||\hat{A} + \hat{B}||| \leq |||\hat{A}||| + |||\hat{B}|||$;

(iii) $|||\alpha\hat{A}||| = |\alpha||||\hat{A}|||$, where $\alpha \in \mathbb{R}$.

For a more detailed knowledge about interval matrix norms, one can refer to the corresponding paper [94].

In literature, Moore [215] proposed an extensive study on interval arithmetic by extending ordinary arithmetic of real numbers. However, due to certain discrepancies existing within the traditional arithmetic operations defined for the set of interval numbers, Ganesan and Veeramani [102] introduced the newest set of interval arithmetic operations. The modified interval arithmetic operations easily overcome the existing discrepancies and follow distributive law.

Definition 6.1.5. [102] Suppose $\hat{x} = [x^L, x^U]$, $\hat{y} = [y^L, y^U] \in \mathbb{IR}$ be any two closed and bounded real intervals. Then, the new modified interval arithmetic operations are specified as follows:

(i) $\hat{x} + \hat{y} = [x^L, x^U] + [y^L, y^U] = [(c(\hat{x}) + c(\hat{y})) - \theta, (c(\hat{x}) + c(\hat{y})) + \theta]$, where $\theta = \dfrac{(y^U + x^U) - (y^L + x^L)}{2}$.

(ii) $\hat{x} - \hat{y} = [x^L, x^U] - [y^L, y^U] = [(c(\hat{x}) - c(\hat{y})) - \theta, (c(\hat{x}) - c(\hat{y})) + \theta]$, where $\theta = \dfrac{(y^U + x^U) - (y^L + x^L)}{2}$.

(iii) $\hat{x}.\hat{y} = [x^L, x^U][y^L, y^U] = [c(\hat{x})c(\hat{y}) - \theta, c(\hat{x})c(\hat{y}) + \theta]$ where, $\theta = \min\left\{(c(\hat{x})c(\hat{y})) - \alpha, \beta - (c(\hat{x})c(\hat{y}))\right\}$. Here, $\alpha = \min(x^L y^L, x^L y^U, x^U y^L, x^U y^U)$ and $\beta = \max(x^L y^L, x^L y^U, x^U y^L, x^U y^U)$.

(iv) $1 \div \hat{x} = \dfrac{1}{\hat{x}} = \dfrac{1}{[x^L, x^U]} = \left[\dfrac{1}{c(\hat{x})} - \theta, \dfrac{1}{c(\hat{x})} + \theta\right]$, where $\theta = \min\left\{\dfrac{1}{x^U}(\dfrac{x^U - x^L}{x^U + x^L}), \dfrac{1}{x^L}(\dfrac{x^U - x^L}{x^U + x^L})\right\}$ and $0 \notin [x^L, x^U]$.

Remark 6.1.1. From (iii) it is evident that for any scalar λ, $\lambda\hat{x} = [\lambda x^L, \lambda x^U]$ whenever $\lambda \geq 0$ and $\lambda\hat{x} = [\lambda x^U, \lambda x^L]$ for $\lambda < 0$.

It is noteworthy that $\odot \in \{\oplus, \ominus, \otimes, \oslash\}$ denotes the set of existing interval arithmetic operations whereas $* \in \{+, -, ., \div\}$ denotes the modified interval arithmetic operations. Moreover, $\hat{x} * \hat{y} \subseteq \hat{x} \odot \hat{y}$ where $*, \odot$ represents the similar meaning mentioned above.

Also, $\mathbb{IR} = \left\{\hat{x} = [x^L, x^U] \mid x^L, x^U \in \mathbb{R} \text{ and } x^L \leq x^U\right\}$ represents a set of proper intervals, while $\overline{\mathbb{IR}} = \{ \hat{x} = [x^L, x^U] \mid x^L, x^U \in \mathbb{R} \text{ and } x^L > x^U \}$ represents a set of improper intervals on the set of real numbers $\mathbb{R}$. In addition, $\mathbb{D} = \mathbb{IR} \cup \overline{\mathbb{IR}}$ denotes a set of all generalized (i.e., both proper and improper) intervals.

Next, we discuss the concept of monadic operator “dual" proposed by Ganesan and Veeramani [102] that we will be incorporating in our zero-sum IMG problem. The monadic operator “dual" is proficient enough to invert the end-points of the interval in a way to express an element-to-element symmetricity between proper and improper intervals in the space $\mathbb{D}$.

Definition 6.1.6. [102] Let $\hat{x} = [x^L, x^U] \in \mathbb{D}$ be any arbitrary closed interval. Then, the dual of an interval $\hat{x}$ is defined as $\text{dual}(\hat{x}) = \text{dual}[x^L, x^U] = [x^U, x^L]$.

Note 6.1.1. For any interval $\hat{x} = [x^L, x^U]$, $\hat{x} - \text{dual}(\hat{x}) = [x^L, x^U] - [x^U, x^L] = [x^L - x^L, x^U - x^U] = [0, 0]$.

Note 6.1.2. For any interval $\hat{x} = [x^L, x^U]$, $\frac{\hat{x}}{\hat{x}} = \frac{\hat{x}}{\text{dual}(\hat{x})} = [x^L, x^U] \times \frac{1}{\text{dual}([x^L, x^U])} = [x^L, x^U] \times \frac{1}{([x^U, x^L])} = [x^L, x^U] \times \left[\frac{1}{x^L}, \frac{1}{x^U}\right] = [1, 1]$.

By incorporating the concept of dual within the modified interval arithmetic operations, it is evident that the new interval arithmetic satisfies the group properties concerning the binary operation addition and multiplication and further, sustaining the distributive law between intervals while preserving the inclusion monotonicity. For a more profound knowledge, one can easily refer to the corresponding paper [102, 223].

6.1.2 Interval comparison

In this subsection, a brief notion of interval comparison to define an interval ordering relation is presented that is beneficial to understand the proposed concept given in the paper. Extensive research on comparing distinct intervals lying within interval space, i.e., $\mathbb{IR}$, is provided in Sengupta et al. [258] and Nayak and Pal [221].

Definition 6.1.1. Let $\hat{x} = [x^L, x^U]$, $\hat{y} = [y^L, y^U] \in \mathbb{IR}$ be any two closed intervals. Let $\text{AI}(\hat{x} \leq \hat{y}) = \dfrac{(c(\hat{y}) - c(\hat{x}))}{(w(\hat{x}) + w(\hat{y}))}$ be an acceptability index function. Then, consider the following cases:

Case 1 (Disjoint subintervals). $\hat{x} < \hat{y}$ whenever $\text{AI}(\hat{x} < \hat{y}) \geq 1$ and $x^U < y^L$, $c(\hat{x}) < c(\hat{y})$.

Case 2 (Partial overlapping Subintervals). $\hat{x} < \hat{y}$ whenever $0 < \text{AI}(\hat{x} < \hat{y}) < 1$ and $y^L < x^U$, $c(\hat{x}) < c(\hat{y})$.

Case 3 (Nested subintervals). $\hat{x} \subset \hat{y}$ whenever $y^L \leq x^L < x^U \leq y^U$.

It is noted that from case 1 and 2, the comparison between the interval is crisply defined. However, in case 3, the comparison between the interval is not crisply defined because interval $\hat{x}$ is completely contained within the interval $\hat{y}$.

Moreover from the case of nested subintervals, it can be certainly viewed that when $y^L < x^L$ then some values within the interval $\hat{y}$ are less than and some are greater than with the possible values in $\hat{x}$. Next, if $y^L = x^L$ and $x^U < y^U$ then every plausible value of $\hat{y}$ is greater than $\hat{x}$. Lastly, if $x^U = y^U$ and $y^L < x^L$ then every plausible value of $\hat{y}$ is lower than $\hat{x}$. Therefore, $\hat{x}$ is nested within $\hat{y}$ but, it is not possible to define an ordering relation between intervals $\hat{x}$ and $\hat{y}$.

6.2 Mathematical formulation of zero-sum interval-valued matrix games

The present section mainly discusses the mathematical formulation of the two-player zero-sum IMG under the pure strategy and the absence of pure strategy.

6.2.1 Zero-sum interval-valued matrix games under pure strategy

Definition 6.2.1. A two-player zero-sum IMG $\widehat{IG}_{Int}$ is expressed by a triplet $(S^p, S^q, \hat{G}_{Int})$, where S^p, S^q represents the strategy sets for PI and PII respectively. The matrix $\hat{G}_{Int} = \left([g^L_{ij}, g^U_{ij}]\right)_{p\times q}$; $i = 1, \ldots p$, $j = 1, \ldots, q$ is the given interval-valued payoff matrix of PI disregard of PII and opposite of each interval input of the payoff matrix $\hat{G}_{Int}$, i.e., $\text{opp}(\hat{G}_{Int}) = (-\text{dual}(\hat{G}_{Int})) = \left([-g^L_{ij}, -g^U_{ij}]\right)_{p\times q}$ expresses the interval payoff matrix of PII such that the addition of the payoffs of two players is $[0, 0]$.

Since the interval comparison is plausible, therefore interval game value can be defined consecutively in the subsequent manner.

Definition 6.2.2. For a given IMG, $\widehat{IG}_{Int}$ having payoff matrix $\hat{G}_{Int}$, the lower interval game value, $\hat{v}^{-}_{Int}$ as well as upper interval game value, $\hat{v}^{+}_{Int}$ is defined as,

$$\hat{v}^{-}_{Int} = \max_{i=1,\ldots,p} \min_{j=1,\ldots,q} [g^{L}_{ij}, g^{U}_{ij}],$$
$$\hat{v}^{+}_{Int} = \min_{j=1,\ldots,q} \max_{i=1,\ldots,p} [g^{L}_{ij}, g^{U}_{ij}].$$

The existence of interval game value, $\hat{v}_{Int}$ is plausible whenever $\hat{v}^{-}_{Int} = \hat{v}^{+}_{Int} = \hat{v}_{Int}$.

The strategy set (i^{*}, j^{*}) corresponding which the value of the game mentioned-above are equal is termed as the saddle point of the interval game, and the index i^{*}, j^{*} represents optimal strategies of the players I and II respectively.

However, it is noteworthy that the above equality does not always hold despite both the values exist. Therefore, the following inequality is summarized.

Theorem 6.2.1. *(see [178]) For any IMG $\hat{G}_{Int}$, the inequality $\hat{v}^{-}_{Int} \leq \hat{v}^{+}_{Int}$ is valid.*

Next, we evaluate the sets of strategy and interval optimal game value in the nonexistence of pure strategy. In this direction, the ILP approach is developed and is considered to be one of the most peculiar method to solve such games primarily based on the new modified interval arithmetic operations (as shown in definition 6.1.5).

6.2.2 Interval-valued linear programming approach to solve interval matrix games

Let us consider the IMG with the following interval payoff matrix $\hat{G}_{Int}$.

$$\hat{G}_{Int} = \begin{pmatrix} [g^{L}_{11}, g^{U}_{11}] & [g^{L}_{12}, g^{U}_{12}] & \cdots & [g^{L}_{1q}, g^{U}_{1q}] \\ [g^{L}_{21}, g^{U}_{21}] & [g^{L}_{22}, g^{U}_{22}] & \cdots & [g^{L}_{2q}, g^{U}_{2q}] \\ \vdots & \ddots & \vdots & \vdots \\ [g^{L}_{p1}, g^{U}_{p1}] & [g^{L}_{p2}, g^{U}_{p2}] & \cdots & [g^{L}_{pq}, \tilde{g}^{U}_{pq}] \end{pmatrix}$$

where $[g^{L}_{ij}, g^{U}_{ij}]$ is the payoff of PI choosing i^{th} strategy when PII chooses the j^{th} strategy.

Throughout the paper, we will attempt to assume that each inflows of the IMG is either $[g^{L}_{ij}, g^{U}_{ij}] < 0$ or $[g^{L}_{ij}, g^{U}_{ij}] > 0$. Let $S^{p} = \left\{X = (x_1, x_2, \ldots, x_p) \mid x_i \geq 0,\ \sum_{i=1}^{p} x_i = 1\right\}$ and $S^{q} = \left\{Y = (y_1, y_2, \ldots, y_q) \mid y_j \geq 0,\ \sum_{j=1}^{q} y_j = 1\right\}$ be the mixed strategy set for the corresponding PI and PII, respectively. Then, the expected payoff of PI whenever PII chooses

j^{th} strategy, is stipulated as follows: $[g_{1j}^L, g_{1j}^U]x_1 + \ldots + [g_{pj}^L, g_{pj}^U]x_p$ where the operation '+' is taken to be as modified interval arithmetic operation defined in definition 6.1.5.

Therefore, the requisite ILP problem for the corresponding game PI is defined as follows.

$$\begin{aligned}
&\max \quad \hat{v}_{Int}^- &&\text{(ILP1)}\\
&[g_{11}^L, g_{11}^U]x_1 + \ldots + [g_{p1}^L, g_{p1}^U]x_p \geq \hat{v}_{Int}^-\\
&[g_{12}^L, g_{12}^U]x_1 + \ldots + [g_{p2}^L, g_{p2}^U]x_p \geq \hat{v}_{Int}^-\\
&\vdots\\
&[g_{1q}^L, g_{1q}^U]x_1 + \ldots + [g_{pq}^L, g_{pq}^U]x_p \geq \hat{v}_{Int}^-\\
&x_1 + x_2 + \ldots + x_p = 1\\
&x_1, x_2, \ldots, x_p \geq 0.
\end{aligned}$$

Since $0 \notin \hat{v}_{Int}^-$.

Therefore, we can set, $\hat{X}_i = \dfrac{x_i}{\hat{v}_{Int}^-}$, $i = 1, \ldots p$, and $\hat{V}_{Int} = \dfrac{1}{\hat{v}_{Int}^-}$. Next, by making the aforementioned substitutions, the model ILP1 is evolved into a generic ILP problem for PI, defined below.

$$\begin{aligned}
&\min V_{Int} = \quad X_1 + X_2 + \ldots + X_p &&\text{(ILP2)}\\
&\text{subject to}\\
&[g_{11}^L, g_{11}^U]X_1 + \ldots + [g_{p1}^L, g_{p1}^U]X_p \geq [1, 1]\\
&[g_{12}^L, g_{12}^U]X_1 + \ldots + [g_{p2}^L, g_{p2}^U]X_p \geq [1, 1]\\
&\vdots\\
&[g_{1q}^L, g_{1q}^U]X_1 + \ldots + [g_{pq}^L, g_{pq}^U]X_p \geq [1, 1]\\
&X_1, X_2, \ldots, X_p \geq 0.
\end{aligned}$$

Analogously, an ILP problem in regards to PII, can be easily formulated as follows.

$$
\begin{aligned}
&\min\ \hat{v}^{+}_{Int} && \text{(ILP3)}\\
&\text{subject to}\\
&[g^L_{11}, g^U_{11}]y_1 + \ldots + [g^L_{1q}, g^U_{1q}]y_q \leq \hat{v}^{+}_{Int}\\
&[g^L_{21}, g^U_{21}]y_1 + \ldots + [g^L_{2q}, g^U_{2q}]y_q \leq \hat{v}^{+}_{Int}\\
&\vdots\\
&[g^L_{p1}, g^U_{p1}]y_1 + \ldots + [g^L_{pq}, g^U_{pq}]y_q \leq \hat{v}^{+}_{Int}\\
&y_1 + y_2 + \ldots + y_q = 1\\
&y_1, y_2, \ldots, y_q \geq 0.
\end{aligned}
$$

Recall $0 \notin \hat{v}^{+}_{Int}$.

By taking $\hat{Y}_j = \dfrac{y_j}{\hat{v}^{+}_{Int}}$, $j = 1, \ldots, q$, we obtain the following generic ILP problem for PII.

$$
\begin{aligned}
&\max\ V_{Int} = \quad Y_1 + Y_2 + \ldots + Y_q && \text{(ILP4)}\\
&\text{subject to}\\
&[g^L_{11}, g^U_{11}]Y_1 \ldots + [g^L_{1q}, g^{(U)}_{1q}]Y_q \leq [1, 1]\\
&[g^L_{21}, g^U_{21}]Y_1 + \ldots + [g^L_{2q}, g^U_{2q}]Y_q \leq [1, 1]\\
&\vdots\\
&[g^L_{p1}, g^U_{p1}]Y_1 + \ldots + [g^L_{pq}, g^U_{pq}]Y_q \leq [1, 1]\\
&Y_1, Y_2, \ldots, Y_q \geq 0.
\end{aligned}
$$

Here, the obtained models ILP1 and ILP4 can be solved easily by utilizing any prevailing methods designed for solving ILP problems (See [224]) to find the optimal mixed strategies $\hat{X}^*_{Int} \in S^p$ and $\hat{Y}^*_{Int} \in S^q$ along with the interval game value $\hat{V}^*_{Int}$. It is worth noticing that both the pair of models ILP1 and ILP2 form an interval version of primal-dual problems within the crisp sense.

However, it is envisioned that the problem of IMG with a large number of variables increases the size of the interval payoff matrix. As a result, solving the ILP problem with the prevailing methods is a tedious job. Henceforth, in the subsequent section, we present a methodology to overcome the existing limitation of the conventional ILP approach and present a new perspective of IMGs within the ambit of interval game theory.

6.3 Proposed Method

The basics of matrix norm method to solve two-player classical ZSMGs is given in section 2.1.1. However, in the present section, we present a novel approach to obtain the required optimum inequalities of the interval game value based on the concept of interval version of 1− norm and ∞− norm. After this, we establish results to find the largest and smallest element of the players' mixed strategy set without solving the pair of auxiliary ILP problems structured for PI and PII (as shown in the previous section).

It is customary to conjectured PI as a row player (maximizing Player) and PII as a column player (minimizing Player). Moreover, we examine this game in terms of the row player, i.e., PI. However, our approach can readily be defined for PII.

In literature, Meyer [209] developed extensive research on matrix norms and have proposed the concept of 1− norm and ∞− norm based on a set of real numbers $\mathbb{R}$. Analogously, we attempt to amplify the notion of point-based 1− norm and ∞− norm into its interval version, unlike the induced interval matrix norm.

Definition 6.3.1. For $\hat{A}_{Int} \in \mathbb{IR}^{p\times q}$, we define interval 1− norm and interval ∞− norm in the following manner:

- $|||\hat{A}_{Int}|||_1 = \max_j \sum_i |[a^L_{ij}, a^U_{ij}]|$ represents the greatest absolute column sum;
- $|||\hat{A}_{Int}|||_\infty = \max_i \sum_j |[a^L_{ij}, a^U_{ij}]|$ represents the greatest absolute row sum.

As a particular instance, it is worth noting that the following result's inequalities are given in a crisp sense. It is customary to posits that each inputs of the interval payoff matrix is crisply comparable (i.e., either the entries are disjoint intervals or overlapped intervals). Consequently, the following lemma points out the interval game inequalities emphasizing the crispness of the suitable interval comparisons.

Lemma 6.3.1. Let $\hat{G}_{Int}$ be a $p \times q$ real interval valued game matrix and $\hat{v} = [v^L, v^U]$ represents the interval game value for IMG. Then,

$$\frac{\hat{h}}{|||\hat{G}_{Int}|||_\infty} \leq \hat{v} \leq |||\hat{G}_{Int}|||_1 \text{ for } \hat{v} > 0,$$

$$-|||\hat{G}_{Int}|||_1 \leq \hat{v} \leq \frac{\hat{h}}{|||\hat{G}_{Int}|||_\infty} \text{ for } \hat{v} < 0.$$

where $\hat{h} = \max\limits_{1\leq i\leq p, i\neq r} \sum_{j=1}^{q} \hat{v}|[g^L_{ij}, g^U_{ij}]|$ and $|||\hat{G}_{Int}|||_\infty = \sum_{j=1}^{q} |[g^L_{rj}, g^U_{rj}]|$ for fixed r and $|||\hat{G}_{Int}|||_1 = \sum_{i=1}^{p} |[g^L_{it}, g^U_{it}]|$ for fixed t.

Proof. Consider the interval valued game matrix $\hat{G}_{Int}$. Throughout the proof, we represent $\hat{v}^+ > [0,0]$ for positive interval game value $\hat{v}$ and $\hat{v}^- < [0,0]$ for negative interval game value $\hat{v}$. We now consider the subsequent cases:

Case 1 For $\hat{v}^+ > [0,0]$:

Let $|||\hat{G}_{Int}|||_\infty = \sum_{j=1}^{q} |[g_{rj}^L, g_{rj}^U]|$ for fixed r. From the definition it follows that

$$\sum_{j=1}^{q} |[g_{rj}^L, g_{rj}^U]| \geq \max \sum_{j=1}^{q} |[g_{ij}^L, g_{ij}^U]|$$

$$\text{for} \, i = 1, 2, \ldots, p \;\text{ and } i \neq r$$

$$\Rightarrow 1 \geq \frac{\max \sum_{j=1}^{q} |[g_{ij}^L, g_{ij}^U]|}{\sum_{j=1}^{q} |[g_{rj}^L, g_{rj}^U]|}$$

$$\Rightarrow \hat{v}^+ \geq \frac{\hat{h}}{\sum_{j=1}^{q} |[g_{rj}^L, g_{rj}^U]|}$$

$$\text{where, } \hat{h} = \max_{1 \leq i \leq p, i \neq r} \sum_{j=1}^{q} \hat{v}^+ |[g_{ij}^L, g_{ij}^U]|$$

Therefore,

$$\frac{\hat{h}}{|||\hat{G}_{Int}|||_\infty} \leq \hat{v}^+ \tag{6.3.1}$$

Clearly, the interval game value is evaluated as $\hat{v}^+ = \sum_{i=1}^{p} \hat{x}_i [g_{ij}^L, g_{ij}^U]$ for any fixed j. Here, $\hat{x}_i$ represents the probability assigned to each element of the mixed strategy set of Player I. Then, it follows that $\hat{v}^+ \leq \sum_{i=1}^{p} |[g_{ij}^L, g_{ij}^U]| \; \forall \; j$. After taking max on both sides we obtain,

$$\hat{v}^+ \leq |||\hat{G}_{Int}|||_1 \tag{6.3.2}$$

From equation 6.3.1 and 6.3.2 we obtain $\dfrac{\hat{h}}{|||\hat{G}_{Int}|||_\infty} \leq \hat{v}^+ \leq |||\hat{G}_{Int}|||_1$.

Case 2 For $\hat{v}^- < [0,0]$:

From case 1 we have $1 \geq \dfrac{\max \sum_{j=1}^{q} |[g_{ij}^L, g_{ij}^U]|}{|||\hat{G}_{Int}|||_\infty}$. Now, since the present case deals with $\hat{v}^-$. Hence, we obtain the following inequality $\dfrac{\hat{h}}{|||\hat{G}_{Int}|||_\infty} \geq \hat{v}^-$, where $\hat{h} = \max\limits_{1 \leq i \leq p, i \neq r} \sum_{j=1}^{q} \hat{v}^- |[g_{ij}^L, g_{ij}^U]|$. Next, to obtain the other inequality we consider the relation $\hat{v}^- = \sum_{i=1}^{p} \hat{x}_i [g_{it}^L, g_{it}^U]$ for any fixed t. Contrastingly, $-|[g_{ij}^L, g_{ij}^U]| \leq -|[g_{ij}^L, g_{ij}^U]| x_i$ as $x_i \in [0,1]$. Therefore, the inequality $\hat{v}^- \geq \sum_{i=1}^{p} (-\hat{x}_i [g_{it}^L, g_{it}^U]) \geq -\sum_{i=1}^{p} |[g_{it}^L, g_{it}^U]| \geq -|||\hat{G}_{Int}|||_1$ is valid since $\sum_{i=1}^{p} |[g_{it}^L, g_{it}^U]| \leq \max\limits_{1 \leq j \leq q} \sum_{i=1}^{p} |[g_{it}^L, g_{it}^U]| = |||\hat{G}_{Int}|||_1$. Hence, the result follows.

□

Before we give the next result, we provide the subsequent definition useful in the main theorem.

Definition 6.3.2. For $\hat{G}_{Int} \in \mathbb{IR}^{p\times q}$ an interval valued game matrix, let $|||\hat{G}_{Int}|||_\infty$ have the sum of absolute values of the s^{th} entries of the row. Then, the matrix $\hat{N}_{Int} \in \mathbb{IR}^{(p-1)\times q}$ obtained after deleting s^{th} row of the matrix $\hat{G}_{Int}$ is termed as a row-wise interval induced matrix of $\hat{G}_{Int}$. Analogously, if $|||\hat{G}_{Int}|||_1$ represents sum of absolute values of the t^{th} column entries. Then, the matrix $\hat{N}_{Int} \in \mathbb{IR}^{(p\times(q-1)}$ obtained after deleting t^{th} column of the matrix $\hat{G}_{Int}$ is termed as column-wise interval-valued induced matrix of $\hat{G}_{Int}$.

Theorem 6.3.1 (Main Theorem)**.** Let $\hat{G}_{Int} \in \mathbb{IR}^{p\times q}$ be an interval payoff matrix for the two-person zero sum IMG and $\hat{v}$ be an interval value of the game. Then,

(i) $\dfrac{|||\hat{N}_{Int}|||_\infty}{|||\hat{G}_{Int}|||_\infty} \leq |\hat{v}| \leq |||\hat{G}_{Int}|||_1$ whenever $|\hat{v}| = |[v^L, v^U]| \geq 1$;

(ii) $\dfrac{1}{|||\hat{G}_{Int}|||_1} \leq |\hat{v}| \leq \dfrac{|||\hat{G}_{Int}|||_\infty}{|||\hat{N}_{Int}|||_\infty}$ whenever $|\hat{v}| = |[v^L, v^U]| \leq 1$ and $|\hat{v}| = |[v^L, v^U]| \neq 0$.
where $\hat{N}_{Int}$ is represented as the row-wise induced matrix of $\hat{G}_{Int}$.

Proof. Consider the interval valued game matrix $\hat{G}_{Int}$. Without loss of generality, we presuppose that $|||\hat{G}_{Int}|||_\infty = \sum_{j=1}^{q} |[g^L_{rj}, g^U_{rj}]|$ for fixed r. From the definition it follows that

$$\sum_{j=1}^{q} |[g^L_{rj}, g^U_{rj}]| \geq \max \sum_{j=1}^{q} |[g^L_{ij}, g^U_{ij}]| \text{ for} i = 1, 2, \ldots, p \text{ and } i \neq r.$$

Therefore, $1 \geq \dfrac{\max \sum_{j=1}^{q} |[g^L_{ij}, g^U_{ij}]|}{\sum_{j=1}^{q} |[g^L_{rj}, g^U_{rj}]|}$, $\hat{v} \leq |||\hat{G}_{Int}|||_1$ for positive interval game value $\hat{v}$ and $-|||\hat{G}_{Int}|||_1 \leq \hat{v}$ for negative interval game value $\hat{v}$.

Case 1 For $\hat{v} > [0, 0]$:

(a) Whenever $\hat{v} \geq 1$, then by using lemma 6.3.1 we have, $\dfrac{\hat{h}}{|||\hat{G}_{Int}|||_\infty} \leq \hat{v} \leq |||\hat{G}_{Int}|||_1$

(b) Whenever $\hat{v} \in (0, 1]$, we assume $\dfrac{1}{\hat{\omega}} = [v^L, v^U] \Rightarrow \hat{\omega} = \dfrac{1}{\hat{v}} \geq 1$. Therefore, by using (a) we obtain $\dfrac{\hat{h}}{|||\hat{G}_{Int}|||_\infty} \leq \hat{\omega} \leq |||\hat{G}_{Int}|||_1$ by making suitable arrangements we get the desired result, i.e., $\dfrac{1}{|||\hat{G}_{Int}|||_1} \leq |\hat{v}| \leq \dfrac{|||\hat{G}_{Int}|||_\infty}{\hat{h}}$.

Case 2 For $\hat{v} < [0, 0]$:

(a) Whenever $\hat{v} \leq -1$, then by using lemma 2.4.1 we have, $-|||\hat{G}_{Int}|||_1 \leq \hat{v} \leq \frac{\hat{h}}{|||\hat{G}_{Int}|||_\infty}$.

(b) Whenever $\hat{v} \in [-1, 0)$, we assume $\frac{1}{\hat{t}} = [v^L, v^U] \Rightarrow \hat{t} = \frac{1}{\hat{v}} \leq -1$. Therefore, by using (a) we obtain $-|||\hat{G}_{Int}|||_1 \leq \hat{t} \leq -\frac{\hat{h}}{|||\hat{G}_{Int}|||_\infty}$ by making suitable arrangements we get the desired result, i.e., $\frac{1}{|||\hat{G}_{Int}|||_1} \leq |\hat{v}| \leq \frac{|||\hat{G}_{Int}|||_\infty}{\hat{h}}$.

Here, $\hat{h}$ has the same definition given in previous lemma 6.3.1.

Now, from the inequalities obtained above and by making certain arrangements the result follows directly.

□

Remark 6.3.1. From the above result, we obtain the boundaries for the interval game value. Since it is requisite to analyze the interval game value from the viewpoint of each player. As a result, we acquire two distinct inequalities for the same interval version of the game value. It is noteworthy that we aim to obtain optimal boundaries for the interval game value. Therefore, it suffices to compare each inequality obtained for both the PI and PII, respectively, and thereby choose the best optimal boundaries for the game value such that the original interval game value falls within the ambit of optimum range.

It is pointed out that the results offered by the main theorem 6.3.1 may not be sufficient to solve the IMG of order 2×2 as they are easily solvable via generic ILP methods. However, it is preferable to utilize these inequalities for an IMG of bigger size so that we can have an impression about the approximated interval game value without explicitly solving the auxiliary pair of interval linear mathematical models, i.e., ILP1 and ILP3. Hence, reducing the computational cost and time complexity of the game problem.

Next, we proceed to establish a result that points out some necessary conditions to obtain the boundaries for the largest and smallest elements in the mixed strategy set. For simplicity notation, we refer $\hat{x}_{\max}$ and $\hat{x}_{\min}$ for the largest and the smallest elements of the mixed strategy set of the players, respectively.

Theorem 6.3.2. Let $\hat{G}_{Int} \in \mathbb{IR}^{p\times q}$ be an IMG with all entries positive of the interval payoff matrix. Then,

$$\hat{x}_{\max} \geq \max\left\{ \frac{\sum_{i=1}^{p} \hat{g}_{ij}}{(p-1).|||\hat{G}|||_1} \mid j = 1, 2, \ldots, q \text{ and} \sum_i \hat{g}_{ij} \neq |||\hat{G}|||_1 \right\}$$

$$\hat{x}_{\min} \leq \frac{1}{p-1} - \frac{\max\left\{ \frac{\sum_{i=1}^{p} \hat{g}_{ij}}{(p-1).|||\hat{G}|||_1} \mid j = 1, 2, \ldots, q \text{ and} \sum_i \hat{g}_{ij} \neq |||\hat{G}|||_1 \right\}}{p-1}$$

where $\hat{g}_{ij} = [g_{ij}^L, g_{ij}^U]$ and $\hat{g} > [0, 0]$.

Proof. Consider the interval payoff matrix $\mathbb{IR}^{p\times q}$

$$\hat{G}_{Int} = \begin{pmatrix} \hat{g}_{11} & \hat{g}_{12} & \cdots & \hat{g}_{1q} \\ \hat{g}_{21} & \hat{g}_{22} & \cdots & \hat{g}_{2q} \\ \vdots & \ddots & \vdots & \vdots \\ \hat{g}_{p1} & \hat{g}_{p2} & \cdots & \hat{g}_{pq} \end{pmatrix}$$

where each entry of payoff matrix $\hat{g}_{ij} = [g_{ij}^L, g_{ij}^U] \in \mathbb{IR}$ and $\hat{g}_{ij} > [0, 0]$.

Without loss of generality, we attempt to presume that $||||\hat{G}_{Int}|||_1 = \hat{g}_{1r} + \hat{g}_{2r} + \ldots + \hat{g}_{pr}$ for any arbitrary fixed r. Then, according to the definition it follows that $\big(\hat{g}_{1r} + \hat{g}_{2r} + \ldots + \hat{g}_{pr}\big)\hat{x}_{\max} \geq \hat{g}_{1t} + \hat{g}_{2t} + \ldots + \hat{g}_{pt}$, where $t = 1, 2, \ldots, r-1, r+1, \ldots, q$.
i.e., we have the following set of inequalities

$$\left.\begin{aligned} \big(\hat{g}_{1r} + \hat{g}_{2r} + \ldots + \hat{g}_{pr}\big)\hat{x}_{\max} &\geq \hat{g}_{11} + \hat{g}_{21} + \ldots + \hat{g}_{p1}, \\ \big(\hat{g}_{1r} + \hat{g}_{2r} + \ldots + \hat{g}_{pr}\big)\hat{x}_{\max} &\geq \hat{g}_{12} + \hat{g}_{22} + \ldots + \hat{g}_{p2}, \\ &\vdots \\ \big(\hat{g}_{1r} + \hat{g}_{2r} + \ldots + \hat{g}_{pr}\big)\hat{x}_{\max} &\geq \hat{g}_{1(r-1)} + \hat{g}_{2(r-1)} + \ldots + \hat{g}_{p(r-1)} \\ \big(\hat{g}_{1r} + \hat{g}_{2r} + \ldots + \hat{g}_{pr}\big)\hat{x}_{\max} &\geq \hat{g}_{1(r+1)} + \hat{g}_{2(r+1)} + \ldots + \hat{g}_{p(r+1)} \\ &\vdots \\ \big(\hat{g}_{1r} + \hat{g}_{2r} + \ldots + \hat{g}_{pr}\big)\hat{x}_{\max} &\geq \hat{g}_{1q} + \hat{g}_{2q} + \ldots + \hat{g}_{pq} \end{aligned}\right\} \text{(q-1)}$$

Since $\hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_p = 1 \implies \hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_{\max} + \hat{x}_{\min} + \ldots + \hat{x}_p = 1 \implies \underbrace{\hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_{\max} + \ldots + \hat{x}_p}_{\text{p-1}} = 1 - \hat{x}_{\min}$.
$||||\hat{G}_{Int}|||_1 (p-1)\hat{x}_{\max} \geq \hat{g}_{1t} + \hat{g}_{2t} + \ldots + \hat{g}_{pt}$, where $t = 1, 2, \ldots, r-1, r+1, \ldots, q$. Now, by taking max on both sides and rearranging the terms we obtain,

$$\hat{x}_{\max} \geq \max\left\{ \frac{\sum_{i=1}^{p} \hat{g}_{ij}}{(p-1).|||\hat{G}|||_1} \mid j = 1, 2, \ldots, q \text{ and} \sum_i \hat{g}_{ij} \neq |||\hat{G}|||_1 \right\} \tag{6.3.3}$$

Next, in order to show the boundaries for $\hat{x}_{\min}$, we consider the following equation:
$\hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_{\max} + \hat{x}_{\min} + \ldots + \hat{x}_p = 1$

$$\Rightarrow \underbrace{\hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_{\min} + \ldots + \hat{x}_p}_{\text{p-1}} = 1 - \hat{x}_{\max}$$

$\Rightarrow (p-1)\hat{x}_{\min} = 1 - \hat{x}_{\max}$

$\Rightarrow \hat{x}_{\max} = 1 - (p-1)\hat{x}_{\min}.$

Now, by using the equation 2.4.3 we obtain

$$1 - (p-1)\hat{x}_{\min} \geq \max\left\{ \frac{\sum_{i=1}^{p} \hat{g}_{ij}}{(p-1).|||\hat{G}|||_1} \mid j = 1, 2, \ldots, q \right\},$$

and $\sum_i \hat{g}_{ij} \neq |||\hat{G}|||_1$. After rearranging the terms we obtain the desired inequality and hence the theorem. □

Remark 6.3.2. It is important to point out that our novel approach helps calculate the approximate interval value of the game that must fall within the optimal boundaries of the game value that we have obtained using theorem 6.3.1. Also, the elements of the mixed strategy set are selected by taking consideration of the inequalities acquired by using theorem 6.3.2

Remark 6.3.3. It is worthwhile to emphasis that by giving careful consideration to the required bounds obtained for $\hat{x}_{\max}$ and $\hat{x}_{\min}$ based on theorem 6.3.2 we arbitrarily select an appropriate value for $\hat{x}_{\max}$ and $\hat{x}_{\min}$, respectively. The remaining elements of the mixed strategy set for the players are hereafter decided in a manner that it must satisfy the principal of probability theory, i.e., all the strategies for the player sums up to 1.

Before we address a numerical example in the subsequent section to illustrate the utilization of a novel approach for the IMGs, we provide a solution algorithm for the zero-sum IMGs to obtain the required optimal solution. It is noted that the subsequent algorithm is given for the row player (i.e., PI); however, one may use for the column player (i.e., PII) as well.

Consider the model ILP1 for Player I.

Step 1 Check each entry of the interval payoff matrix i.e., $\hat{g}_{ij} \in \mathbb{IR}$. If any entry $\hat{g}_{ij} < 0$ then add a suitable constant $\hat{k}_{arb}$ to each entry of $\hat{g}_{ij}$ so that $\hat{G}_{Int}$ becomes a positive matrix.

Step 2 Calculate the boundaries for interval game value using theorem 6.3.1.

Step 3 Choose appropriate value for $\hat{x}_{\max}$ and $\hat{x}_{\min}$ depending on the related inequality given in theorem 6.3.2.

Step 4 Determine the suitable value for the remaining strategies one by one, arbitrarily based on the fact that an interval is a non-decreasing function.

Step 5 Evaluate approximate value of the IMG $\hat{v}_{app}$ by randomly selecting any column of the given payoff matrix.

Step 6 Compare the $\hat{v}_{app}$ with the original interval game value $\hat{v}$ obtained by solving IMG problem via ILP method.

Remark 6.3.4. It is well known that IMGs is formulated as a union of numerous point-based matrix game problem. Therefore, it is worth emphasizing that if for any arbitrary value of $x \in [x^L, x^U] = \hat{x}$, the approximate interval value of the game $\hat{v}_{app}$ is not efficiently computable, consequently giving some absurd relation. Then, we may conclude that our IMG problem possesses a weak optimal solution. On the contrary, for all $x \in [x^L, x^U] = \hat{x}$, if the interval game value is efficiently computable and satisfies the obtained boundary relation, then the given IMGs problem possess a strong optimal solution.

6.4 Application of Interval matrix games

Traditional game theory has been applied to practical decision problems in management, finance, business, economics, and other fields. The subsequent example is derived from the study, and it shows how the results and proposed approach well suited for any zero-sum IMGs when applied to evaluate the impact of the unprecedented COVID-19 pandemic situation on the share price of the telecommunications industry.

Example 6.4.1. (A case study)

The contagious spread of the anomalous COVID-19 pandemic has jeopardized the entire world and transformed the global outlook unexpectedly. This pandemic is not merely a concern of global health but also a rapid global economic recession too. The rapid spread of COVID-19 has made countries impose restrictions on the movement of daily lives. Consequently, it requires people to spend more time staying at home and more data utilization for work as well as leisure, results in a momentous impact on the telecom sector. It is inconceivable to imagine a world without a mobile connection having a high-speed internet facility in the present situation. More than 50 percent of the world's population are connected to mobile connection. The emergence of the 5th generation mobile network (5G) in telecommunications is considered a new kind of communication revolution planned to connect virtually everyone and everything. Therefore, it is going to be a worthwhile and much-needed business in the immediate future.

Based on the significance and impact of the COVID-19 pandemic on the telecommunication industry, we present a case study to envisage how share prices of the telecommunication companies got affected due to this pandemic situation. Here, we consider the list of seven telecommunication companies (i.e., PII) whose share prices got affected due to the pandemic situation (i.e., PI), which is categorized into a fixed number of financial years (FYs). Since there is uncertainty in the company's stock price, a single stock value cannot perfectly model the payoff value for any FY. Henceforth, it is more reasonable and adequate to presuppose that each payoff must belongs to some interval.

For this numerical instance, it is presupposed that the variation of the stock prices for the listed companies can be modeled under the assumption that the pandemic situation (i.e., PI) is a rational "player" that will select an optimal strategy. Suppose that the options (i.e., strategies) for this PI are given as follows: before pandemic situation (FY2019) $\hat{x}_1$, during pandemic situation (FY2020) $\hat{x}_2$ and after pandemic situation (FY2021) $\hat{x}_3$. For the companies , the options (i.e., strategies) are as follows: "Bharti Airtel" $\hat{y}_1$, "Reliance Communication" $\hat{y}_2$, "Vodafone idea limited" $\hat{y}_3$, "Tata communications" $\hat{y}_4$, "Tata teleservice" $\hat{y}_5$, "Suyog telematics" $\hat{y}_6$ and "OnMobile Global" $\hat{y}_7$. In this case, a stock value in the FY for the given company cannot be accurately modeled by a precise value. Hence, a matrix game problem with interval-valued payoffs (i.e., IMG) can appropriately present the overview of the game from both players' perspectives.

Consider the subsequent interval payoff matrix for this situation, in which the normalized percentage of stock value for each company in the supplied FYs is presented in interval format:

$$\hat{G}_{Int} = \begin{array}{c} \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} \hat{y}_1 & \hat{y}_2 & \hat{y}_3 & \hat{y}_4 & \hat{y}_5 & \hat{y}_6 & \hat{y}_7 \end{array} \\ \begin{pmatrix} [2.69,4.86] & [0,0.14] & [0.03,0.23] & [2.6,8.26] & [0.02,0.05] & [1.9,5.24] & [0.26,0.5] \\ [3.81,6.12] & [0,0.04] & [0.02,0.13] & [2.06,12.4] & [0.02,0.09] & [2.47,4.67] & [0.13,0.68] \\ [5,7.43] & [0.02,0.06] & [0.04,0.14] & [9.3,20] & [0.07,0.55] & [3.61,5.49] & [0.5,1.54] \end{pmatrix} \end{array}$$

where the interval $[2.69, 4.86]$ represents that the normalized percentage amount of the stock value for the given company, "Bharti Airtel" is between 2.69% and 4.86% when it is before pandemic situation (i.e., FY2019). Other interval entries in the interval payoff matrix can be annotated analogously.

Now, the decision-maker tries to estimate the expected range of the stock value in the normalized percentage form. Namely, the lower and upper bounds of the interval-type value of the IMG need to be determined to analyze the change in the share prices of the telecommunication companies due to the pandemic situation.

We firstly solve the IMG problem for row player (i.e., PI). In this case, $|||\hat{G}_{Int}|||_1 = [13.96, 40.66]$, $|||\hat{G}_{Int}|||_\infty = [18.54, 30.49]$ and corresponding row wise induced interval matrix of $\hat{G}_{Int}$ is given by $|||\hat{N}_{Int}|||_\infty = [8.51, 23.13]$. Then, according to the theorem 2.4.1, the boundaries for the approximated interval game value is calculated as follows: $\frac{1}{|||\hat{G}_{Int}|||_1} \leq |\hat{v}| \leq \frac{|||\hat{G}_{Int}|||_\infty}{|||\hat{N}_{Int}|||_\infty}$ whenever $|\hat{v}| = |[v^L, v^U]| \leq 1$;

$\Rightarrow \frac{1}{[13.96, 40.66]} \leq [v^L, v^U] \leq \frac{[18.54, 30.49]}{[8.51, 23.13]}$.

Based on the modified interval arithmetic the above mentioned inequality transforms into $[0.02, 0.055] \leq [v^L, v^U] \leq [0.7416, 2.2002]$.

We now calculate the boundaries for $\hat{x}_{\max}$ and $\hat{x}_{\min}$ by utilizing Theorem 2.4.2. Here, $p = 3$ since $\hat{G}_{Int} \in \mathbb{IR}^{3\times 7}$.

$$\hat{x}_{\max} \geq \max\left\{ \frac{[11.5, 18.41]}{(2).[13.96, 32.94]}, \frac{[0.02, 0.24]}{(2)[13.96, 32.94]}, \frac{[0.09, 0.5]}{(2)[13.96, 32.94]}, \frac{[0.11, 0.69]}{(2).[13.96, 32.94]}, \frac{[7.98, 15.4]}{(2).[13.96, 32.94]}, \frac{[0.89, 2.72]}{(2).[13.96, 32.94]} \right\}$$

$\Rightarrow \hat{x}_{\max} \geq \max\{ [0.1748, 0.4622], [0.0003, 0.0053], [0.0014, 0.0112], [0.0017, 0.0153], [0.1213, 0.3767], [0.0135, 0.0633] \}$

$\Rightarrow \hat{x}_{\max} \geq [0.1748, 0.4622]$

$\Rightarrow x^L_{\max} \geq 0.1748$, $x^U_{\max} \geq 0.4622$ and $x^L_{\max} \leq x^U_{\max}$.

Now, $\hat{x}_{\min} \leq 0.5 - [0.0874, 0.2311]$, then the required bound for $\hat{x}_{\min}$ is $\hat{x}_{\min} \leq [0.2689, 0.4126]$. Based on the remark 2.4.3, we randomly select the value of strategies as $\hat{x}_{\min} = 0$, $\hat{x}_1 = [0, 0.4126]$, $\hat{x}_{\max} = [0.1748, 1]$. In addition, it is worth pointing out that the plausible distributions of the interval strategies value do not make any major difference to the interval value of the game as far as the value lies within the optimum range.

Now, since we aim to find out an approximate interval value of the game, the value must fall within the optimum boundaries we have obtained above. Therefore, one can choose any arbitrary column for evaluating the approximate interval game value. However, in this scenario we select first column of the payoff matrix $\hat{G}_{Int}$ to obtain the approximate interval game value, i.e., $\hat{v}_{app} = 0[2.69, 4.86] + [0, 0.4226][3.81, 6.12] + [0.1748, 1][5, 7.43] = [0, 1.473]$.

We next solve the IMG problem for the column player. The required interval matrix norms are $|||\hat{G}^T_{Int}|||_1 = [18.54, 30.49]$, $|||\hat{G}^T_{Int}|||_\infty = [13.96, 40.66]$ and $|||\hat{N}^T_{Int}|||_\infty = [11.5, 18.41]$. Here, '$T$' represents transpose of matrix $\hat{G}_{Int}$. Now, based on Theorem 2.4.1 we obtain $\frac{1}{[18.54,30.49]} \leq |\hat{w}| \leq \frac{[13.96,40.66]}{[11.5,18.41]} \Rightarrow [0.03,0.048] \leq |\hat{w}| \leq [0.7608,2.8852]$.

Analogously, we can obtain the boundaries for $\hat{y}_{\max}$ and $\hat{y}_{\min}$ by using Theorem 2.4.2. Therefore, $\hat{y}_{\max} \geq \max\{$ [0.0413, 0.1273], [0.047, 0.1682] $\} \Rightarrow \hat{y}_{\max} \geq [0.047,0.1682]$ and $\hat{y}_{\min} \leq [0.172,0.1922]$. Analogously, as mentioned-above we randomly select the strategy scenario as $\hat{y}_{\max} = [0.8,1]$, $\hat{y}_{\min} = 0$, $\hat{y}_1 = [0,0.1]$, $\hat{y}_2 = [0,0.001]$, $\hat{y}_3 = 0 = \hat{y}_4$, $\hat{y}_5 = [0,0.02]$.

We choose column 1 to obtain interval approximated game value for Player II, $\hat{w}_{app} = 0+[0,0.1][0,0.14]+[0,0.001][0.03,0.23]+[0,0.02][1.9,5.24]+[0.8,1][0.26,0.5] = [0.2,0.48]$. On comparing the boundaries of interval game value for PI and PII, respectively we envisaged that the boundaries of the interval game value corresponding to the PI is optimum.

The next example is a particular case of zero-sum IMGs where each inputs of the interval payoff matrix is a degenerate interval. The subsequent zero-sum game problem is taken from the corresponding paper [147] where it is solved by using classical linear programming approach. Moreover, the same matrix game problem is also given in the paper [133] where it is solved by using novel norm approach. Here, we are considering the same problem and will present it as an application of our proposed algorithm.

Example 6.4.2. Consider the following payoff matrix G

$$G = \begin{pmatrix} 0.4298 & 0.4298 & 0.9253 & 0.9253 & 0.0936 & 0.5293 \\ 0.4073 & 0.6989 & 0.4073 & 0.4804 & 0.5311 & 0.7425 \\ 0.7208 & 0.5616 & 0.5616 & 0.4726 & 0.7625 & 0.1954 \end{pmatrix}$$

Initially, we solve the game problem from the side of defender (row player). Since the method of computation for obtaining the boundaries for the approximated value of the game is similar to the method provided in the corresponding paper [133]. Therefore, we proceed to apply our novel approach to find out the bounds of the largest and smallest element of the mixed strategy set and the selection of the remaining element of the strategy set are obtained by keeping in mind the remark 6.3.3.

Based on the definition of $1-$ norm and $\infty-$ norm we know that $||G||_1 = 1.8942$, $||G||_\infty = 3.3331$. Also, in this scenario $p = 3$. Therefore, the boundaries for $x^*_{\max}$ and $x^*_{\min}$ can be depicted successively in the similar fashion as given in the previous example.

$$x^*_{\max} \geq \max\left\{\frac{1.5579}{(2)1.8942}, \frac{1.6903}{(2)1.8942}, \frac{1.8783}{(2)1.8942}, \frac{1.3872}{(2)1.8942}, \frac{1.4672}{(2)1.8942}\right\}$$

$x^*_{\max} \geq 0.4958$.

$x^*_{\min} \leq 0.5 - \frac{0.4958}{2}$. Thus, $x^*_{\min} \leq 0.2521$.

After obtaining suitable inequalities for $x^*_{\max}$ and $x^*_{\min}$ we must randomly select the appropriate value for the strategy $x^*_{\max}$ and $x^*_{\min}$. Also, to calculate the approximate value of the game v_{app}, we need to decide the values of the remaining element of the mixed strategy set by keeping in mind the remark 6.3.3.

Therefore, after making the routine computations in this scenario we consecutively select the desired value for the strategies as $x^*_{\min} = 0.2021$, $x^*_1 = 0.3021$ and $x^*_{\max} = 0.4958$. However, it is worthwhile to point out that the plausible distributions of the strategies value do not make any major difference onto the value of the game as far as the value lies within the optimum range. Since we can randomly chose any arbitrary column of the given payoff matrix in order to calculate the approximate game value. Henceforth, in this perspective, we calculate the value of the game by choosing first column i.e., $v_{app} = (0.4298 \times 0.2021) + (0.4073 \times 0.3021) + (0.7208 \times 0.4958) = 0.5673$.

Next, we estimate the approximate value of the game from the side of attacker (column player). So, in this context we first obtain the bounds for $y^*_{\max}$ and $y^*_{\min}$ successively as follows (here, $p = 6$ since $G^T \in \mathbb{R}^{6\times 3}$).

$y^*_{\max} \geq \max\left\{\frac{3.2675}{(5)3.3331}, \frac{3.2745}{(5)3.3331}\right\}$

$y^*_{\max} \geq 0.1965$.

$y^*_{\min} \leq 0.5 - 0.09825$. Thus, $y^*_{\min} \leq 0.4$. Since $y^*_{\min} \leq y^*_{\max}$. Thus, $y^*_{\min} \leq 0.1965$. As in this case we have 6 elements in the strategy set therefore we choose the values as follows: $y^*_{\min} = 0.0714$, $y^*_1 = 0.1104$, $y^*_2 = 0.1524$, $y^*_3 = 0.1211$, $y^*_4 = 0.1448$ and $y^*_{\max} = 0.3999$. Now, in order to calculate the approximate value of the game we prefer to select second column. Therefore, $v_{app} = (0.4073 \times 0.0714) + (0.6989 \times 0.1104) + (0.4073 \times 0.1524) + (0.4804 \times 0.1211) + (0.5311 \times 0.1448) + (0.7425 \times 0.3999) = 0.6003$.

6.4.1 Comparative analysis

In the literature, Li [179] have proposed a linear programming method to solve IMG problems and further provide a comparison of the solution obtained via other existing methods. From the analysis, the author concluded that linear programming approach is peculiar over other methods and offer a more reliable and rational solution. Therefore, in the present section we give a comparison of results obtained by using our novel approach with the ILP method.

We consider the IMG problem (as shown in Example 6.4.1) and solve it using the existing ILP method. For this, we firstly constructed a pair of auxiliary interval linear mathematical models. In light of the models (See ILP1 and ILP3) presented in section 6.2.2, we convert it into the generic ILP models (ILP2 and ILP4). Since we know that several methods exist for solving ILP problems for instance Best-Worst case ILP method [17], Enhanced ILP method [350] and so on (See [224]). However, many of these methods offer a solution space that may be infeasible or may lose some of the optimal points. So, in order to eliminate the infeasible part from the solution space several ameliorated methods have been proposed.

Improved interval linear programming (IILP) method (See [224]) is one of the improved method that sufficiently handles the interval uncertainties and further eliminate infeasible region from the solution space, thereby, providing solution that is both absolutely feasible and optimal. Here, in this scenario, we solve the zero-sum IMG problem after transforming into generic ILP model by using IILP method. Hence, the following are two sub-models for the PI and PII.

For Player I:

sub model 1	sub model 2
$\min \quad V^{-(U)} = X_1 + X_2 + X_3$	$\min \quad V^{-(L)} = X_1 + X_2 + X_3$
subject to	subject to
$4.86X_1 + 6.12X_2 + 7.43X_3 \geq 1,$	$2.69X_1 + 3.81X_2 + 5X_3 \geq 1,$
$0.14X_1 + 0.04X_2 + 0.06X_3 \geq 1,$	$0X_1 + 0X_2 + 0.02X_3 \geq 1,$
$8.26X_1 + 12.4X_2 + 20X_3 \geq 1,$	$0.03X_1 + 0.02X_2 + 0.04X_3 \geq 1,$
$0.05X_1 + 0.09X_2 + 0.55X_3 \geq 1,$	$2.6X_1 + 2.06X_2 + 9.3X_3 \geq 1,$
$5.24X_1 + 4.67X_2 + 5.49X_3 \geq 1,$	$0.02X_1 + 0.02X_2 + 0.07X_3 \geq 1,$
$0.5X_1 + 0.68X_2 + 1.54X_3 \geq 1,$	$1.9X_1 + 2.47X_2 + 3.61X_3 \geq 1,$
$X_1, X_2, X_3 \geq 0.$	$0.26X_1 + 0.13X_2 + 0.5X_3 \geq 1,$
	$X_1, X_2, X_3 \geq 0.$

Solving the above two linear mathematical problems and by making suitable substitutions, the optimal strategy of PI is obtained as follows: $\hat{x}_1 = [0, 0.845]$, $\hat{x}_2 = [0, 0]$, $\hat{x}_3 = [0.1552, 1]$ and the corresponding interval value of the game is $\hat{v} = [0.02, 0.1276]$.

For Player II:

Sub model 1	Sub model 2
max $V^{+(U)} = Y_1+Y_2+Y_3+Y_4+Y_5+Y_6+Y_7$ subject to $4.86Y_1+0.14Y_2+0.23Y_3+8.26Y_4+0.05Y_5+5.24Y_6+0.5Y_7 \le 1,$ $6.12Y_1+0.04Y_2+0.13Y_3+12.4Y_4+0.09Y_5+4.67Y_6+0.68Y_7 \le 1,$ $7.43Y_1+0.06Y_2+0.14Y_3+20Y_4+0.55Y_5+5.49Y_6+1.54Y_7 \le 1,$ $Y_1,Y_2,Y_3,Y_4,Y_5,Y_6,Y_7 \ge 0.$	max $V^{+(L)} = Y_1+Y_2+Y_3+Y_4+Y_5+Y_6+Y_7$ subject to $2.69Y_1+0Y_2+0.03Y_3+2.6Y_4+0.02Y_5+1.9Y_6+0.26Y_7 \le 1,$ $3.81Y_1+0Y_2+0.02Y_3+2.06Y_4+0.02Y_5+2.47Y_6+0.13Y_7 \le 1,$ $5Y_1+0.02Y_2+0.04Y_3+9.3Y_4+0.07Y_5+3.61Y_6+0.5Y_7 \le 1,$ $Y_1,Y_2,Y_3,Y_4,Y_5,Y_6,Y_7 \ge 0.$

Solving the above two linear mathematical problems and by making suitable substitutions, the optimal strategy of Player II is obtained as follows: $\hat{y}_1 = [0,0]$, $\hat{y}_2 = [0.8622,1]$, $\hat{y}_3 = 0$, $\hat{y}_4 = 0$, $\hat{y}_5 = [0,0.1379]$, $\hat{y}_6 = 0$, $\hat{y}_7 = 0$ and the corresponding interval value of the game is $\hat{v} = [0.02,0.1276]$. Based on the values of $\hat{y}_j$ $(j = 1,2,3,4,5,6,7)$, it is observed that "Reliance Communications" have the maximum share price growth among the other listed telecommunication companies. Moreover, the solution space so obtained for the interval game problem is both optimal and feasible.

Now, we analyze the results obtained for the IMG problem by using our new algorithm (See Section 6.3) and existing interval linear programming method (See Section 6.2). We postulate the following points:

(i) The new algorithm is centered on the interval norm methodology, that facilitates the game players to evaluate approximate interval game value without explicitly solving the pair of auxiliary interval linear mathematical models as solved above. Henceforth, it is considered the most significant advantage part of our new approach to a certain extent.

(ii) In example 6.4.1, considering the case of row player, the approximated interval game value obtained by using our algorithm is $\hat{v}_{app} = [0,0.1473]$ and the original optimal game value is $\hat{v} = [0.02,0.1276]$. Therefore, the calculation of the absolute error is given as $|\hat{v} - \text{dual}(\hat{v}_{app})| = |[0.02,0.1276] - ([0.1473,0])| \Rightarrow |v^L - v^L_{app}| = 0.02$ and $|v^U - v^U_{app}| = 0.02$. Analogously, the approximated interval game value obtained for the case of column player is $\hat{w}_{app} = [0.2,0.483]$ and the absolute error calculation in this scenario is $|v^L - v^L_{app}| = 0.18$ and $|v^U - v^U_{app}| = 0.3$. It is envisioned that the better-approximated interval game value is obtained for the case of row player, which is more nearer to the actual interval game value. Henceforth, the proposed method is showing consistent results with the existing methods.

(iii) In the paper, Izgi and Özkaya [133] has proposed a novel method for calculating bounds for the maximum and minimum element of the strategy set based on the original value of the game (See theorem 2.1.2). However, as the matrix size increases, it is not always feasible for the game players to know about the new games' original value. Therefore, to overcome this limitation, we propose a method primarily based on normalization to determine the reasonable bounds for the largest and smallest element of the strategy set for both the players.

(iv) In example 6.4.2, the original value of the game is given as $v = 0.5791$ (See [147]) for both the players. Therefore, for the case of row player the absolute error is $|v - v_{app}| = |0.5791 - 0.5673| = 0.01$. Also, for the case of column player the absolute error is $|v - v_{app}| = |0.5791 - 0.0.6003| = 0.02$. It is foreseen that the absolute error so obtained is quite small in both cases. Hence, the proposed method is showing consistency with the results existing in the corresponding paper [133].

(v) The new algorithm is competent enough to solve large-scale IMG problems without solving mathematical equations, thereby diminishing the computational cost. Also, our new algorithm reduces the time complexity, which is otherwise considered a tedious process for large-scale IMG problems.

(vi) In this algorithm, the approximated interval game value evaluated for the row and column player, more precisely maximizing and minimizing player, follows the relation $\hat{v}_{app} \leq \hat{w}_{app}$. Henceforth, showing similarity with the existing results (as shown in [185]).

6.5 Conclusion

In this chapter, we anticipate a new viewpoint to handle interval uncertainties by solving two-player zero-sum IMG problems via the interval matrix norm method. We propose a notion of a $1-$ interval norm and $\infty-$ interval norm that builds up an analogy with the existing notion of point matrix norm defined for real numbers. Some results and theorems are stated and proved to find optimal boundaries for the interval game value depending on the norm concept. Furthermore, we exhibit a result facilitating the game players to obtain lower and upper bounds of the mixed strategy set's biggest and smallest element, respectively. The result so devised is based on the normalization concept, which is different from the result proposed in the paper (See Theorem 2.1.2), consequently, useful in providing bounds for the IMG problem whose game value is not known priorly. Moreover, the approximated interval game value obtained in a row and column player scenario satisfies the generic interval inequalities, showing the similarity with existing interval linear programming methods. Finally, a case study on how the rapid spread of COVID-19 situation has affected the stock prices of the telecommunication industries and comparison of results with the existing method show that the proposed methodology demonstrates consistent results and is proficient in promoting studies in imprecise matrix games.

Since all the traditional methods developed in the past are well-versed to solve IMGs, the IILP method is the most popular method and has given more accurate results. On the contrary, our proposed method suffices to find an approximate interval game value faster, omitting to solve a linear interval equation system, which will be a tedious process for a bigger size matrix game. It is easily foreseen from the discussion and comparison that the present study is substantially distinct from the existing works of the IMGs. Although the proposed method has shown consistent results with the existing method, some issues need to be fixed. Firstly, in our method, we randomly select an appropriate value for the mixed strategy set elements satisfying the boundary conditions. As a consequence, it is not always a reliable method in a large scale IMG problem involving a large number of decision variables. So, our future work will focus on resolving this issue and providing an improved interval norm method that proficiently finds a better interval solution lying near the exact optimal solution of the IMG. In recent times, some methodologies for addressing an imprecise game problem with multiple interval inputs (See [87]) have been developed based on the ILP approach. So, our research will continue in this direction, thereby extending the proposed method's utility to address imprecise game problems having incomplete information or multiple interval inputs.

Summary and future scope of the work

Many practical situations certainly deal with uncertainty and vagueness and further increase the complexity of the problem. Using probabilistic models to deal with such uncertainty has resulted in unsatisfactory results. There are different computational models based on the linguistic term set to accomplish CWW processes; among them, the 2TL model CWW has produced successful results without any information loss and is additionally applicable in complicated situations. It has been extensively applied in decision-making and many other related areas due to its accuracy and simplicity.

In this thesis, we have focused our comprehensive study primarily on computation with 2TL variables in the field of game theory and decision science. We have analyzed both the abstract and practical aspects of the 2TL model and further worked to design algorithms and various operational laws under the framework of the 2-tuple linguistic variable so that it can be efficiently applied in complicated decision problems. In chapter 2, we laid the foundation of 2TL group theory where we have developed a formal methodology that claims the 2TL term set, i.e., $\overline{LT} \equiv LT \times [\frac{-1}{2}, \frac{1}{2}]$ forms a direct group under the binary operation '$\circ$' and later on developed an isomorphic relation between the 2TL direct group and the interval $[-n, n]$. Subsequently, in the chapter, some of the properties of 2TL group isomorphism and homomorphism are discussed, and the concept of linguistic kernels, cosets, normal subgroups, and factor groups are introduced. We also propose some novel algebraic operational laws for the 2TL term set. Finally, the strength of the proposed abstract concept is presented in terms of bipolar linguistic graphs and linguistic matrix games.

Chapter 3 has formulated a novel approach to handling unbalanced linguistic information wherein linguistic information is presupposed to be asymmetrically and non-uniformly distributed. Further, we have designed an algorithm to represent semantics to the unbalanced linguistic terms and afterward developed a 2TL model for ULTS based on the concept of minimum distance measure. Two numerical illustrations are also provided to validate and demonstrate the usability of the suggested model in the physical realm.

Further, chapter 4 extends the concept of ULTS proposed in the previous chapter to introduce the notion of PM-ULTSs, which considers the probability of linguistic variables and the non-uniformity of linguistic labels. After then, some new operational laws for managing probabilistic unbalanced linguistic information are also developed, which could produce valid results and keep the property of operational laws closed. Furthermore, some elementary aggregation operators to aggregate PM-ULTSs have been constructed for further consideration. Additionally, the proposed concept is applied to investigate the two-player constant sum matrix game problem having probabilistic unbalanced linguistic information allowing game theory to accept the incomplete linguistic information, which is non-symmetrically distributed as input. The successful application of the proposed technique in the real-life decision analysis problem highlights its efficiency and competency.

Chapter 5 is dedicated to studying matrix games under an IVL framework. We have designed a methodology to solve a 2-player zero-sum IVL matrix game problem to acquire the lower and upper value of the interval linguistic game problem. In this connection, We initially develop a novel approach for comparing two IVTFL variables and subsequently put forward a new method for evaluating the optimal strategies and value of the game.

Chapter 6 presents a new viewpoint to handle interval uncertainties by solving two-player zero-sum IMG problems via the interval matrix norm method. We propose a notion of a $1-$ interval norm and $\infty-$ interval norm that builds up an analogy with the existing notion of point matrix norm defined for real numbers. Finally, a case study on how the rapid spread of COVID-19 has affected the stock prices of the telecommunication industries has been presented. Comparing results with the existing method shows that the proposed methodology demonstrates consistent results and is proficient in promoting studies of imprecise matrix games.

Since the inception of the 2TL model, it is foreseen that the model has gained considerable attention among other LCMs. It provides flexibility and eases the experts to incorporate vagueness and uncertainty due to subjective human thinking in decision situations. Although the 2TL framework has ubiquitous applications in multiple fields, there still are some challenges persisting with the model and, hence, there is a scope of advancement. The work presented in this thesis can be extended to explore more complex decision problems having linguistic information. In the future, the dimension of 2-tuple linguistic group theory can be extended to the ULTS thereby, enhancing the importance of classical group theory in the uncertainty domain. To complete the novel 2-tuple unbalanced linguistic computational model, one can give an exhaustive study relevant to the aggregation operators. It is perceived that game theory under a linguistic environment has a great scope. Therefore, the proposed probabilistic unbalanced linguistic matrix game methodology can be furthered to study bi-matrix unbalanced linguistic games and cooperative games. In recent times, some methodologies for addressing an imprecise game problem with multiple interval inputs (See [87]) have been developed based on the classical ILP approach. So, the research can be continued to extend the proposed methods given in the thesis to address imprecise game problems with incomplete information or multiple interval inputs.

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List of Publications

1. **Tanya Malhotra**, Anjana Gupta; *A systematic review of developments in 2-tuple linguistic model and its applications in decision analysis*, Soft Computing, springer, (2020). https://doi.org/10.1007/s00500-020-05031-2. (**SCIE, Impact Factor: 3.643**)

2. **Tanya Malhotra**, Anjana Gupta; *Group operations and isomorphic relation with the 2-tuple linguistic variables*, Soft Computing, Springer, **24**, 18287-–18300 (2020). (**SCIE, Impact Factor: 3.643**)

3. **Tanya Malhotra** and Anjana Gupta; *A New 2-Tuple Linguistic Approach for Unbalanced Linguistic Term Sets*, IEEE Transactions on Fuzzy Systems. **29** (8) 2158–2168 (2021). (**SCI, Impact Factor: 12.029**)

4. **Tanya Malhotra** and Anjana Gupta; *Methodology for Interval-Valued Matrix Games with 2-Tuple Fuzzy Linguistic Information*, In: Sergeyev Y., Kvasov D. (eds) Numerical Computations: Theory and Algorithms. NUMTA 2019. Lecture Notes in Computer Science, **11974**, (2020). https://doi.org/10.1007/978-3-030-40616-5_12. (**Conference Proceedings Citation Index (CPCI)**)

5. **Tanya Malhotra** and Anjana Gupta; *Group isomorphic properties with some novel operational laws for* 2*-tuple linguistic variables and its application in linguistic matrix games*, is **Communicated** in *IEEE Transactions on Systems, Man, and Cybernetics: Systems*

6. **Tanya Malhotra** and Anjana Gupta; Probabilistic multiplicative unbalanced linguistic term set and its application in matrix games, is **Communicated** in *International journal of machine learning and cybernetics*.

7. **Tanya Malhotra** and Anjana Gupta; *Interval norm approach for solving two-player zero-sum matrix games with interval payoffs* is **Communicated** in *Computational optimization and application*.

Papers Presented in Conferences

- Presented a research paper entitled "**Best-Worst case approach For Matrix Games With Interval-Valued 2-tuple Linguistic Information**" in *Internationl Conference on Recent Advances in Pure and Applied Mathematics (ICRAPAM)* held at Delhi Technological University, Delhi on 24 October, 2018.

- Presented a research paper entitled "**Methodology for interval-valued matrix games with 2-tuple fuzzy linguistic information**" in *Numerical Computations: Theory and Algorithms: The 3rd international conference and summer school (NUMTA2019), Italy* held at University of Calabria, Italy during 15th-21st JUNE, 2019.
