COEFFICIENT ESTIMATES AND SPECIAL DIFFERENTIAL SUBORDINATIONS OF CERTAIN ANALYTIC FUNCTIONS

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DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

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under the supervision of

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DECLARATION

I declare that the research work reported in this thesis entitled "**Coefficient Estimates and Special Differential Subordinations of Certain Analytic Functions**" for the award of the degree of *Doctor of Philosophy in Mathematics* has been carried out by me under the supervision of *Prof. S. Sivaprasad Kumar*, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

Unless otherwise indicated, this thesis represents my original research work. This thesis has not been submitted by me earlier in part or full to any other University or Institute for the award of any degree or diploma. It is not intended to include other people's data, graphs, or other information, unless explicitly acknowledged.

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CERTIFICATE

On the basis of declaration submitted by **Ms. Shagun Banga**, student of Ph.D., I hereby certify that the thesis titled **"Coefficient Estimates and Special Differential Subordinations of Certain Analytic Functions"** submitted to the Department of Applied Mathematics, Delhi Technological University, Delhi, India for the award of the degree of *Doctor of Philosophy in Mathematics*, is a record of bonafide research work carried out by her under my supervision.

I have read this thesis and that, in my opinion, it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy.

To the best of my knowledge the work reported in this thesis is original and has not been submitted to any other Institution or University in any form for the award of any Degree or Diploma.

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Date : **(SHAGUN BANGA)**

Place : Delhi, India.

Dedicated to

My Parents

&

My Husband

Contents

Preface

The thesis entitled **"Coefficient Estimates and Special Differential Subordinations of Certain Analytic Functions"** is divided into 6 chapters. After chapter 6, the thesis is concluded with the future scope. The aim at the beginning of each chapter provides a brief summary of the research work done and concluding remarks at the end of each chapter gives the highlights of that chapter. We enlist below chapter wise outline of the research work.

Chapter 1 titled *"Introduction"* provides a quick overview of the topic. It covers fundamental concepts, some essential definitions, terminologies and ideas that are required further to achieve the objectives.

Chapter 2 titled *"First Order Differential Subordinations"* deals with certain differential subordination implications involving certain parameters. These implications are achieved by finding conditions on the parameters. Furthermore, sufficient conditions for normalized analytic function f to belong to various sub-classes of starlike functions are obtained as an application of the derived results.

Chapter 3 titled *"Certain Exact Differential Subordinations"* instigates the concept of exact differential subordinations, which is analogous to first order exact differential equations on the real line. Mainly, this chapter involves two special type of exact differential subordinations to study the newly introduced concept and obtain the dominant and best dominant for these differential subordinations. Certain applications to univalent functions are appended to this chapter.

Chapter 4 titled *"A Special Type of Ma-Minda Function"* deals with the extensive study on Ma-Minda functions based on its deep rooted conditions and it's geometrical aspects. Following which, a special type of Ma-Minda function Φ is introduced and the classes $\mathcal{S}^*(\Phi)$ and $\mathcal{C}(\Phi)$ are defined. A newly defined subclass of starlike functions involving a special type of Ma-Minda function $1-\log(1+z)$ studied here for obtaining inclusion and radius results. In addition, majorization and Bloch function norm related results are discussed.

Chapter 5 titled *"Coefficient Estimates of Certain Analytic Functions"*, establishes bounds of various initial coefficients, certain Hankel determinants for functions in both type of classes, involving Ma-Minda function and the special type of Ma-Minda function. Studied the special cases for each of the classes for certain coefficient estimates. The bounds obtained are all sharp, among which finding the sharp third Hankel determinant for functions in the class associated with lemniscate of Bernoulli is the key feature of this chapter, which was open until now.

Chapter 6 titled *"A Novel Subclass of Starlike Functions"* deals with defining a new subclass $\mathscr{S}^*_\alpha$ of starlike functions involving a real part and modulus of certain expressions, combined by way of an inequality. It is also inferred that this class reduces to a class $\mathcal{S}^*(q_\alpha)$, involving subordination and the subordinating function q_α, which is well-known in the literature with certain interesting properties. Certain inclusion and radius results are deduced for functions in the classes $\mathscr{S}^*_\alpha$ and $\mathcal{S}^*(q_\alpha)$. Furthermore, various sharp coefficient estimates are obtained for functions in $\mathcal{S}^*(q_\alpha)$.

Finally, the bibliography and list of author's publications have been given at the end of the thesis.

List of Symbols

Symbols	Meanings
$\mathbb{C}$	complex plane
$\mathbb{D}$	open unit disk $\{z : \|z\| < 1\}$
$\mathcal{H}[a,n]$	class of analytic functions in $\mathbb{D}$ of the form $f(z) = a + \sum_{k=n}^{\infty} a_k z^k$
$\mathcal{A}$	subclass of $\mathcal{H}[0,1]$ of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$
$\mathcal{S}$	subclass of $\mathcal{A}$ consisting of univalent functions
$\mathcal{S}^*$	subclass of $\mathcal{S}$ of starlike functions
$\mathcal{C}$	subclass of $\mathcal{S}$ of convex functions
$\mathcal{K}$	class of close-to-convex functions
$\mathcal{P}$	class of functions p, having positive real part in $\mathbb{D}$ and $p(0) = 1$
$k(z)$	Koebe function
$\omega(z)$	Schwarz function, which is analytic and satisfies $\omega(0) = 0$ and $\|\omega(z)\| \leq \|z\|$.
$\phi(z)$	Ma-Minda function of the form $\phi(z) = 1 + \sum_{i=1}^{\infty} B_i z^i$ $(B_1 > 0)$
$\Phi(z)$	Special type of Ma-Minda function of the form $\Phi(z) = 1 + \sum_{i=1}^{\infty} C_i z^i$ $(C_1 < 0)$
$\mathscr{M}$	class of Ma-Minda functions
$\widetilde{\mathscr{M}}_{\mathbf{A}}$	class of non-Ma-Minda functions of type-**A**
$\mathscr{M}^{\circ}$	class of special type of Ma-Minda functions
$\prec$	subordination
$\mathcal{S}^*(\phi)$	class of starlike functions associated with Ma-Minda function
$\mathcal{S}^*(\Phi)$	class of starlike functions associated with special type of Ma-Minda function
$\mathcal{C}(\phi)$	class of convex functions associated with Ma-Minda function

Symbols	**Meanings**
$\mathcal{C}(\Phi)$	class of convex functions associated with special-type of Ma-Minda function
$\mathbb{R}$	set of real numbers
$\mathbb{N}$	set of natural numbers
$\mathcal{S}^*((1+Az)/(1+Bz))$	class of Janowski starlike functions
$\mathcal{S}^*(\nu)$	class of starlike functions of order ν $(0 \leq \nu < 1)$
$\mathcal{C}(\nu)$	class of convex functions of order ν $(0 \leq \nu < 1)$
$\mathcal{M}(\kappa)$	class of analytic functions satisfying $\mathrm{Re}(zf'(z)/f(z)) < \kappa$ $(\kappa > 1)$
$\mathcal{N}(\kappa)$	class of analytic functions satisfying $\mathrm{Re}(1+zf''(z)/f'(z)) < \kappa$ $(\kappa > 1)$
$\mathcal{SL}^*$	class of analytic functions satisfying $zf'/f \prec \sqrt{1+z}$
$\mathcal{S}^*_e$	class of analytic functions satisfying $zf'/f \prec e^z$
$\mathcal{SS}^*(\eta)$	class of strongly starlike functions of order η
$\mathcal{S}^*(q_c)$	class of analytic functions satisfying $zf'/f \prec \sqrt{1+cz}$ $(0 < c \leq 1)$
$\mathcal{S}_P$	class of parabolic starlike functions
$k-\mathcal{ST}$	class of k- starlike functions
$\rho-CV$	class of ρ- convex functions $(\rho \in \mathbb{R})$
$H_q(n)$	q^{th} Hankel determinant $(q, n \in \mathbb{N})$
$\psi(z)$	a function $1-\log(1+z)$
$\mathcal{S}^*_l$	class of analytic functions satisfying $zf'/f \prec 1-\log(1+z)$

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Chapter 1

Introduction

This chapter presents a brief review of the past and present developments in the field of Geometric Function Theory. As a result, the current chapter establishes a context, explains the purpose of thesis work, and offers instruments for achieving the objectives. The aim, therefore, is to describe the important definitions, concepts and techniques that we will need in our upcoming chapters.

1.1 Preliminaries

The theory of univalent functions traces its roots back to the early twentieth century and has emerged as one of the most popular branches of Complex Analysis. There is still much research being done in this area today. Since many remarkable properties of univalent functions can be reliably derived from simple geometric considerations, therefore the subject falls under the geometric function theory (GFT). This theory encompasses a vast topic for which substantial literature now exists, notably those are Duren [21], Goluzin [24], Pommerenke [83], Hayman [33], Graham and Kohr [26] and Thomas et al. [110]. The textbook by Goodman [25] entails an encyclopedia of univalent function theory and presents a wide range of results in the field. In fact, a paper by Koebe [44] in 1907 laid down the foundation stone for the topic and few years later in 1914, it was progressed by Gronwall's proof of area theorem. Eventually the Bieberbach conjecture which was proposed in 1916 is the turning point and root cause for the vast literature in the field at present. A comprehensive bibliography and topic reference on the subject have been compiled by Bernardi [10]. The books written by Hallenbeck and MacGregor [30] and Jenkins [37] are some additional resources on univalent functions.

A single valued function $f(z)$, which is analytic except at most one simple pole is said to be univalent in a domain D if it is one-one in D or if it takes no more than one value in D. That is, whenever $f(z_1) = f(z_2)$ then $z_1 = z_2$ for z_1 and z_2 in D. Analytically, a univalent function has a non-zero derivative everywhere in its domain and geometrically, it maps simple curves onto simple curves. Our interest lies in the functions that are analytic and univalent in open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Since the celebrated **Riemann Mapping Theorem** ensures that any simply connected domain $D \neq \mathbb{C}$ is conformally equivalent to the unit disk $\mathbb{D}$, we restrict the domain of these functions to be $\mathbb{D}$. The properties of univalent functions defined on simply connected domains D and unit disk $\mathbb{D}$ are therefore equivalent and are expressed through the Riemann mapping from D onto $\mathbb{D}$. The selection of $\mathbb{D}$ as the domain of functions is also crucial due to the fact that the only functions that are analytic and univalent in $\mathbb{C}$ are of the form $az + b$ $(a \neq 0)$.

Let $\mathcal{H}[a,n]$ denote the class of analytic functions f defined on $\mathbb{D}$, having the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, where n is a positive integer and a, $a_i \in \mathbb{C}$ for all $i \geq n$ with $a_n \neq 0$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}[0,1]$ consisting of analytic functions, which are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Therefore, the Taylor series of such functions will be of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1.1.1}$$

Note that the analytic and univalent nature of a function do not get altered by its translation, rotation or stretching (shrinking). Therefore, we generally consider the subclass $\mathcal{S}$ of $\mathcal{A}$, consisting of normalized analytic univalent functions of the form (1.1.1). In fact, the class $\mathcal{S}$ is compact due to the normalization whereas the space of analytic and univalent functions is not compact in $\mathbb{D}$.

Let us now consider the well-known Koebe function

$$k(z) := \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots nz^n + \cdots \in \mathcal{S}, \tag{1.1.2}$$

which maps the unit disk onto the entire complex plane minus the slit on the negative real axis from $-\infty$ to $-1/4$ and is the largest function in $\mathcal{S}$.

1.2 Certain Classes of Analytic Functions

Let us recall certain subclasses of $\mathcal{S}$ that are governed by geometric conditions. A domain D is said to be starlike with respect to ζ_0, an interior point of D, if each ray initiating from ζ_0 intersects the interior of D in a domain that is either a line segment or a ray. A domain is said to be convex if it is starlike with respect to each of its interior point. That is, in simple words, a domain D is starlike if every point of D is visible from ζ_0 whereas it is convex, if all points of the domain are visible from each of its point. The character of a function is decided by a domain to which it maps the unit disk. Accordingly, if f maps unit disk onto a starlike domain with respect to ζ_0, then it is said to be starlike with respect ζ_0 and if f maps $\mathbb{D}$ onto a convex domain, then it is called a convex function. If $\zeta_0 = 0$, then we simply say f is a starlike function. The function $f(z) = z\exp(z)$, which maps $\mathbb{D}$ onto a cardioid shape domain is an example of a starlike function and $g(z) = \exp(z)$ is an example of a convex function. The sub-classes of $\mathcal{S}$ consisting of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Note that every convex function f is starlike with respect to every point in the domain $f(\mathbb{D})$, hence every convex function is a starlike function but the converse need not be true. The well-known Koebe function $k(z)$, given by (1.1.2) is an example of a starlike function which is not convex. Moreover, $k(z)$ is a starlike function with respect to each $\zeta_0 > -1/4$.

Closely connected to the classes $\mathcal{S}^*$ and $\mathcal{C}$ is the Carathéodory class $\mathcal{P}$, consisting of analytic functions p of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{1.2.1}$$

having positive real part in $\mathbb{D}$. The functions in class $\mathcal{P}$ need not be univalent. For instance, $1 + z^n$ $(n \geq 1)$ belongs to $\mathcal{P}$ but is not univalent for $n \geq 2$. The univalent Möbius function

$$\xi(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots,$$

works in $\mathcal{P}$ as Koebe function works in class $\mathcal{S}$, therefore $\xi(z)$ is the largest function in $\mathcal{P}$, which maps the unit disk onto the right half plane. The function $\xi(z)$ maximizes $|p_n|$ in $\mathcal{P}$, however there exist an unlimited number of additional functions with $p_n = 2$ for $n \geq 2$, none of which can be produced by rotation of the other.

The relation of class $\mathcal{P}$ to the theory of univalent functions is studied independently by Noshiro [76] and Warschawski [113], given as :

Theorem A. (**Noshiro and Warschawski Theorem**) Suppose $\mathrm{Re}(e^{i\alpha} f'(z)) > 0$ holds for all z in a convex domain D for some real α. Then $f(z)$ is univalent in D.

This result serves as a sufficient condition for univalency of analytic functions. The analytic characterization of starlike and convex functions in terms of functions having positive real part is given, respectively as follows:
We have $f \in \mathcal{S}^*$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{D}),$$

equivalently, $zf'(z)/f(z) \in \mathcal{P}$. A function $f \in \mathcal{C}$ if and only if $1 + zf''(z)/f'(z) \in \mathcal{P}$, that is

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

Starlikeness and convexity are inherited properties in the sense that every starlike (convex) function maps each disk $|z| < r < 1$ onto a starlike (convex) domain. There is a two way bridge between the classes $\mathcal{S}^*$ and $\mathcal{C}$, given by the following analytic connection, first noticed by Alexander [2] in 1915.

Theorem B. (**Alexander's Theorem**) Let $f \in \mathcal{A}$. Then $f \in \mathcal{C}$ if and only if $zf'(z) \in \mathcal{S}^*$.

As an interesting example for this result, we consider $f(z) = z/(1-z) \in \mathcal{C}$, which maps $\mathbb{D}$ conformally onto the half plane $\mathrm{Re}(w) > -1/2$. Therefore $zf'(z) = k(z) = z/(1-z)^2 \in \mathcal{S}^*$.

We now discuss one of the remarkable concepts for analytic functions in univalent function theory, called subordination. The subordination of two functions of a complex variable is a natural generalization of inequality of functions of a real variable. This concept has historical roots in 1909 paper by Lindelöf [57], but a few years later is credited to Littlewood [58, 59] and Rogosinski [93, 94] for introducing the term and formulating its basic characteristics. This concept asserts $f(z)$ is subordinate to $g(z)$, denoted by $f(z) \prec g(z)$, if $f(z) = g(\omega(z))$, where ω is a Schwarz function, that is, $\omega(0) = 0$ and $|\omega(z)| < 1$. The essential case is when the subordinating function is univalent. Let g be univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. As a consequence, many subclasses of $\mathcal{S}$ in the theory of univalent functions have been characterized by utilizing the idea of subordination (see [22, 34, 48, 65, 100]).

Let g and h be analytic functions of the form, respectively

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad \text{and} \quad h(z) = z + \sum_{n=2}^{\infty} h_n z^n. \tag{1.2.2}$$

Then the convolution of $g(z)$ and $h(z)$ is defined as

$$(g * h)(z) = z + \sum_{n=2}^{\infty} g_n h_n \, z^n.$$

In the honor of Hadamard, it is also known as Hadamard product of g and h. Using convolution and subordination concepts, we introduce the following fundamental class and point out that many well-known classes are special cases of this class.

$$\mathcal{A}(g, h, \varphi) = \left\{ f \in \mathcal{A} : \frac{(f * g)(z)}{(f * h)(z)} \prec \varphi(z), \varphi \text{ is analytic univalent in } \mathbb{D} \text{ and } \varphi(0) = 1 \right\}. \tag{1.2.3}$$

In 1985, Padmanabhan and Parvatham [81], considered the class $\mathcal{A}(K_a * \tilde{g}, K_a * \tilde{h}, \varphi)$, where $K_a(z) = z/(1-z)^a$ $(a \in \mathbb{R})$, $\tilde{g}(z) := z/(1-z)^2$ and $\tilde{h}(z) := z/(1-z)$ by imposing additional conditions on φ, namely it is convex and $\mathrm{Re}\,\varphi > 0$. Later in the year 1989, Shanmugam [96] extended $\mathcal{A}(K_a * \tilde{g}, K_a * \tilde{h}, \varphi)$ to $\mathcal{A}(g * \tilde{g}, g * \tilde{h}, \varphi)$ by considering a more general g in place of $K_a(z)$. In 1992, Ma and Minda [61] further tweaked the conditions on φ, which we shall denote by ϕ to introduce their immensely important classes, namely

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\} \text{ and } \mathcal{C}(\phi) = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\}, \tag{1.2.4}$$

where $\phi(z)$ has a positive real part in $\mathbb{D}$, $\phi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to 1 with $\phi'(0) > 0$, in addition to the properties of φ given in (1.2.3). Let the Taylor series expansion of such $\phi(z)$ be of the form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{1.2.5}$$

Note that $\mathcal{S}^*(\phi) := \mathcal{A}(\tilde{g}, \tilde{h}, \phi)$ and $\mathcal{C}(\phi) := \mathcal{A}(\acute{g}, \tilde{g}, \phi)$ whenever f is univalent and $\acute{g}(z) := (z + z^2)/(1 - z)^3$. Ever since Ma-Minda introduced the class $\mathcal{S}^*(\phi)$, defined in (1.2.4), a good many subclasses of starlike functions emerged out. To list a few, we have for $\phi(z) = (1 + z)/(1 - z)$, the above classes (1.2.4), respectively reduces to $\mathcal{S}^*$, the class of starlike univalent functions and $\mathcal{C}$, the class of convex univalent functions. Over the years, the classes defined in (1.2.4) were studied extensively for different choices of ϕ. Prominently, for $\phi(z) := (1 + Az)/(1 + Bz)$, $\mathcal{S}^*[A, B] := \mathcal{S}^*((1 + Az)/(1 + Bz))$, is the class of Janowski starlike functions [34], where $-1 \leq B < A \leq 1$. For $A = 1$ and $B = -1$, this class reduces to the class of normalized starlike functions, $\mathcal{S}^*[1, -1] := \mathcal{S}^*((1 + z)/(1 - z))$ and for $A = 1 - 2\nu$ $(0 \leq \nu < 1)$ and $B = -1$, this is the class of starlike functions of order ν, $\mathcal{S}^*(\nu) := \mathcal{S}^*((1 + (1 - 2\nu)z)/(1 - z))$, defined as

$$\mathcal{S}^*(\nu) := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \nu \right\}.$$

Note that $\mathcal{S}^* := \mathcal{S}^*(0)$. Another interesting class is the class of starlike functions f of reciprocal order ν, which satisfies the condition:

$$Re\left(\frac{f(z)}{zf'(z)} \right) > \nu \quad (z \in \mathbb{D}).$$

The class of convex functions of order ν $(0 \leq \nu < 1)$ is defined as

$$\mathcal{C}(\nu) := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} > \nu \right\}.$$

Note that $\mathcal{C} := \mathcal{C}(0)$. Robertson [90] introduced the classes $\mathcal{S}^*(\nu)$ and $\mathcal{C}(\nu)$ and functions in either of these fail to be univalent for $\nu < 0$. Moreover, the generalization of the result by Alexander, given in Theorem B, is defined as: $f \in \mathcal{C}(\nu)$ if and only if $zf'(z) \in \mathcal{S}^*(\nu)$. Let

$\mathcal{M}(\kappa)$ and $\mathcal{N}(\kappa)$ be the sub-classes of $\mathcal{S}$ consisting of the functions f which respectively, satisfy the following

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \kappa \quad \text{and} \quad \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \kappa \quad (\kappa > 1; z \in \mathbb{D}).$$

Uralegaddi et al. [111] investigated the class $\mathcal{N}(\kappa)$ for $1 < \kappa < 4/3$. For $\phi(z) := \sqrt{1+z}$, Sokół and Stankiewicz [105] introduced the class of analytic functions associated with lemniscate of Bernoulli, $\mathcal{SL}^* := \mathcal{S}^*(\sqrt{1+z})$. Functions satisfying $|\log(zf'(z)/f(z))| < 1$, belong to the class $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$, introduced by Mendiratta et al. [65]. For $\phi(z) := ((1+z)/(1-z))^\eta$, $\mathcal{S}^*(\phi)$ reduces to the class of strongly starlike functions of order η [11] and is given by

$$\mathcal{SS}^*(\eta) := \{f \in \mathcal{S} : |\arg(zf'(z)/f(z))| < \eta\pi/2\}, \quad (0 < \eta \leq 1),$$

or equivalently $\mathcal{SS}^*(\eta) := \{f \in \mathcal{S} : zf'/f \prec ((1+z)/(1-z))^\eta\}$. The specialty of this class lies in the fact that it helps to study the function in terms of argument estimates. For $\eta = 1$, it simply reduces to the class $\mathcal{S}^*$. For $\phi(z) := ((1+cz)/(1-z))^{(\eta_1+\eta_2)/2}$, Liu and Srivastava [60] introduced the class

$$\mathcal{SS}^*(\eta_1, \eta_2) = \{f \in \mathcal{S} : -\eta_2\pi/2 < \arg(zf'(z)/f(z)) < \eta_1\pi/2\} \quad (0 < \eta_1, \eta_2 \leq 1), \qquad (1.2.6)$$

or equivalently

$$\mathcal{SS}^*(\eta_1, \eta_2) = \{f \in \mathcal{S} : zf'/f \prec ((1+cz)/(1-z))^{(\eta_1+\eta_2)/2}\}, \qquad (1.2.7)$$

where $\eta = (\eta_1 - \eta_2)/(\eta_1 + \eta_2)$ and $c = e^{\eta\pi i}$. Since the functions belong to the class $\mathcal{SS}^*(\eta_1, \eta_2)$, given in (1.2.6), map the unit disk onto sectors which are not symmetric with respect to the real axis whenever $\eta_1 \neq \eta_2$, we call the subordinating function $((1+cz)/(1-z))^{(\eta_1+\eta_2)/2}$ as an *oblique sector* function. Note that $\mathcal{SS}^*(\eta) := \mathcal{SS}^*(\eta, \eta)$. Sokół [102] introduced the class $\mathcal{S}^*(q_c)$, where $q_c = \sqrt{1+cz}$ $(0 < c \leq 1)$. MacGregor [62] studied the class $\mathcal{R} = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > 0\}$. Let $\mathcal{S}_P := \mathcal{S}^*(\phi_{PAR}(z))$, introduced by Rønning [95], the class of parabolic starlike functions, where

$$\phi_{PAR}(z) := 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad \operatorname{Im}\sqrt{z} \geq 0.$$

Consider the class $k-\mathcal{ST}$ $(k \geq 0)$ of $k-$ starlike functions, which was introduced by

Kanas and Wisniowska [39] as follows:

$$k-\mathcal{ST}=\left\{f\in\mathcal{S}:\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>k\left|\frac{zf'(z)}{f(z)}-1\right|\right\}\quad(z\in\mathbb{D}),$$

for $k=1$, the above class coincides with $\mathcal{S}_P$ and characterized by the condition $\operatorname{Re}(zf'(z)/f(z))>\left|zf'(z)/f(z)-1\right|$ $(z\in\mathbb{D})$. Further, $k-\mathcal{ST}$ was generalized by adding a parameter α and defined by a condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>k\left|\frac{zf'(z)}{f(z)}-1\right|+\alpha\quad(z\in\mathbb{D}),$$

denote by $\mathcal{ST}(k,\alpha)$. Let us consider the domain $\Omega_{k,\alpha}=\{w\in\mathbb{C}:\operatorname{Re}w>|w-1|+\alpha\}$, whose boundary represents an ellipse for $k>0$, a parabola for $k=1$ and a hyperbola for $0<k<1$.

In 1952, Kaplan [40] introduced an interesting subclass of $\mathcal{S}$ containing $\mathcal{S}^*$, the class of close-to-convex functions. An analytic function f is said to be close-to-convex if for some convex function g, we have

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right)>0\quad(z\in\mathbb{D}).$$

The class of such functions is denoted by $\mathcal{K}$. It is worth noting that g does not have to be normalized and f does not have to be univalent from the start. Eventually, every close-to-convex function is proved to be univalent. An equivalent condition for function to be close to convex, in terms of starlike functions is given by:

An analytic function f is close-to-convex if there exists a starlike function h such that $\operatorname{Re}(zf'(z)/h(z))>0$, $(z\in\mathbb{D})$. By these conditions, one can infer that every convex as well as starlike function is close-to-convex. The chain of inclusion can be now given as:

$$\mathcal{C}\subset\mathcal{S}^*\subset\mathcal{K}\subset\mathcal{S}.$$

Mocanu [69] unified the class of convex and starlike functions by considering their combination and introduced a class of $\rho-$ convex functions of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, which are analytic in $\mathbb{D}$ with $f(z)f'(z)/z\neq0$ and satisfies

$$\operatorname{Re}\left(\rho\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\rho)\frac{zf'(z)}{f(z)}\right)>0.$$

The set of all such functions is denoted by $\rho-CV$. These functions are starlike and univalent for all real ρ. Rogosinski [92] introduced the class of typically real functions which are analytic with real coefficients in the power series expansion (1.1.1), satisfying

$$sign(\operatorname{Im} f(z)) = sign(\operatorname{Im} z),$$

for every non real z in $\mathbb{D}$. This class is denoted by TR. In picturesque language such typically real functions are those which map the upper (lower) half of $\mathbb{D}$ to the upper (lower) half of the image domain.

1.3 Coefficient Estimates

The extremal and maximal properties of the Koebe function, belonging to $\mathcal{S}$, laid the foundation for **Coefficient Conjecture** in the theory of univalent functions, which was proposed by Bieberbach in 1916, stated as follows:
Bieberbach Conjecture: Let $f \in \mathcal{S}$ be of the form (1.1.1), then $|a_n| \leq n$ $(n \geq 2)$. The equality holds for all n only if f is the Koebe function or one of its rotations.

In the same year, Bieberbach proved $|a_2| \leq 2$ for functions of the form (1.1.1), in $\mathcal{S}$, using Gronwall Area Theorem [28]. Bieberbach came up with the conjecture largely due to this result. Few years later only, in 1923, Löwner solved this problem for $n = 3$ with devising an inventive way by considering a class of functions described by a differential equation that are dense in $\mathcal{S}$. This method not only answered the Bieberbach problem in the case $n = 3$, but was also utilised to determine the sharp bounds for the coefficients of the inverse function of $f \in \mathcal{S}$ (see [33]). Even more importantly, de Branges employed Löwner's theory in his famous proof of the Bieberbach Conjecture. In 1925, Littlewood proved that $|a_n| < en$ for all n, where $e \approx 2.718$. Later on in 1955, Garabedian and Schiffer gave a proof for $|a_4| \leq 4$ using variational method. In 1968 and 1969, Pederson and Ozawa independently proved that $|a_6| \leq 6$. Then in 1972, Pederson and Schiffer used the Garabedian-Schiffer inequality to establish $|a_5| \leq 5$. This well-known conjecture stood as a challenge for many mathematicians until it was proved affirmatively by Louis de Branges in 1985, which is given in the second edition of the book "Multivalent Function" by Hayman [33]. He proved the conjecture using certain inequalities of special functions. The various attempts used to solve this conjecture resulted in the development of several new techniques such as convolution, subordination and others. Moreover, the proof of this conjecture and the result itself both play a key role in various discoveries in the subject. Covering

Theorem by Koebe is among one of these, which is stated as below:

Theorem C. (Koebe-One-Quarter Theorem). The range of every function in $\mathcal{S}$ contains the disk $\{w : |w| < 1/4\}$.

Basically, this Theorem is a sharp version of the result proved by Koebe in the year 1907, which states that there exist a constant $k > 0$ for which the disk $|w| < k$ is contained in the range of every function in $\mathcal{S}$. Later, Bieberbach applied the bound of the second coefficient for functions in $\mathcal{S}$ to obtain the value of k to be $1/4$. In fact, Bieberbach Conjecture, Koebe-One Quarter Theorem and Area Theorem account for the first sharp results for univalent functions. Bieberbach Conjecture also leads to some geometrical aspects of functions in $\mathcal{S}$ such as distortion and growth theorems.

For each $n \geq 2$, Zalcman conjectured the following coefficient inequality for the class $\mathcal{S}$:

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2.$$

The above inequality also implies the Bieberbach Conjecture $|a_n| \leq n$ (see [12]). This paved a way to the famous coefficient estimate problem in univalent function theory.

Coefficient Problem: If f belongs to some particular class, then finding the bounds of coefficients of f is called the coefficient problem.

In this direction, many authors have obtained initial coefficient bounds, logarithmic and inverse coefficient bounds for functions in various subclasses of $\mathcal{S}$, one may refer to [87, 88, 99, 103, 117]. There are certain coefficient results pertaining to the concept of subordinations, for instance, the Rogosinski result [93], which is as follows:

$$\sum_{k=1}^{n} |g_k|^2 \leq \sum_{k=1}^{n} |h_k|^2, \quad n = 1, 2, \cdots,$$

holds whenever $g \prec h$, where g and h are defined by (1.2.2). Many authors have made remarkable contributions related to the coefficient bounds for functions in Carathéodory class, some of which are listed below:

Lemma A. (**Carathéodory Lemma**.) Let $p \in \mathcal{P}$ of the form $p(z) = 1 + p_1 z + p_2 z^2 + p^3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n$, then

$$|p_n| \leq 2 \quad (n \geq 1).$$

The inequality is sharp.

Ma and Minda [61] gave the following result:

Lemma B. *[61]* Let $p \in \mathcal{P}$ be of the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then

$$|p_2 - v p_1^2| \leq \begin{cases} -4v + 2, & v \leq 0; \\ 2, & 0 \leq v \leq 1; \\ 4v - 2, & v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p(z) = (1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if $p(z) = (1+\alpha)(1+z)/(2(1-z)) + (1-\alpha)(1-z)/(2(1+z))(0 \leq \alpha \leq 1)$ or one of its rotations. If $v = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$. Though the above upper bound is sharp for $0 < v < 1$, still it can be improved as follows:

$$|p_2 - v p_1^2| + v|p_1|^2 \leq 2 \quad (0 < v \leq 1/2) \text{ and } |p_2 - v p_1^2| + (1-v)|p_1|^2 \leq 2 \quad (1/2 \leq v < 1). \quad (1.3.1)$$

In 1958, Grenander and Szegö [27] obtained the following sharp inequalities.

Lemma C. Let $p \in \mathcal{P}$ with coefficients p_n as above, then

$$|p_3 - 2p_1 p_2 + p_1^3| \leq 2 \text{ and } |p_1^4 - 3p_1^2 p_2 + p_2^2 + 2p_1 p_3 - p_4| \leq 2. \quad (1.3.2)$$

In 1982, Libera and Złotkiewicz [56] gave the formulae of p_2, p_3 and recently in 2018, Kwon et al. [52] obtained p_4 in terms of p_1, where $p_i's$ are the coefficients of Carathéodory functions.

Lemma D. Let $p \in \mathcal{P}$ and of the form $1 + \sum_{n=1}^{\infty} p_n z^n$. Then

$$2p_2 = p_1^2 + \gamma(4 - p_1^2), \quad (1.3.3)$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta, \quad (1.3.4)$$

$$8p_4 = p_1^4 + (4-p_1^2)\gamma(p_1^2(\gamma^2-3\gamma+3)+4\gamma) - 4(4-p_1^2)(1-|\gamma|^2)(p_1(\gamma-1)\eta+\overline{\gamma}\eta^2-(1-|\eta|^2)\rho), \tag{1.3.5}$$

for some ρ, γ and η such that $|\rho| \le 1$, $|\gamma| \le 1$ and $|\eta| \le 1$.

In 2015, Ravichandran and Verma [87] established another important coefficient bound for functions in the class $\mathcal{P}$.

Lemma E. [87] Let a, b, c and d satisfy the inequalities $0 < c < 1$, $0 < d < 1$ and

$$8d(1-d)((cb-2a)^2 + (c(d+c)-b)^2) + c(1-c)(b-2dc)^2 \le 4c^2(1-c)^2 d(1-d).$$

If $p = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}$, then

$$|ap_1^4 + dp_2^2 + 2cp_1p_3 - (3/2)bp_1^2p_2 - p_4| \le 2.$$

Bieberbach Conjecture holds for functions in $\mathcal{S}^*$ as well, which was proved by Nevanlinna. For functions in C, it is known that $|a_n| \le 1$. Following these coefficient bounds results, many authors obtained initial coefficient estimates for functions in various subclasses of $\mathcal{S}$, see [56, 87, 88, 103].

Coefficient bounds do not limit to the estimation of initial coefficients, it also includes finding bounds, wherein coefficients are involved such as Fekete-Szegö functional and other Hankel determinants for certain analytic functions [16, 80, 118].

Hankel Determinants

For a function $f \in \mathcal{A}$, the q^{th} Hankel determinant, where $q, n \in \mathbb{N}$ is defined as follows:

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

initially considered in [82] and has been studied by several authors. It also plays an important role in the study of singularities (see [19]). Noor [75] studied the rate of growth of $H_q(n)$ as $n \to \infty$ for functions in $\mathcal{S}$ with bounded boundary.

Note that the Hankel determinant $H_2(1) := a_3 - a_2^2$ coincides with the famous Fekete-Szegö functional. In the year 1983, Bieberbach [25] estimated the bound of $|H_2(1)|$ for functions in the class $\mathcal{S}$. In 1933, Fekete and Szegö, obtained the sharp bounds of $|a_3 - \mu a_2^2|$ for $f \in \mathcal{S}$, μ is real and the problem of finding sharp bounds for non-linear functional $|a_3 - \mu a_2^2|$ is known as Fekete-Szegö functional bounds. Numerous papers have been published on Fekete-Szegö inequality for various subclasses of analytic functions, to name a few, one can refer the paper by Ma and Minda [61] and the paper by Keogh and Merkes [43]. A brief history of Fekete-Szegö problems for the class of starlike, convex, and close to convex functions is available in a paper by Srivastava et al. [107]. The computation of the upper bound of $|H_q(n)|$ for several subclasses of $\mathcal{S}$ has always been a trendy problem in the field of geometric function theory. Recently, Zaprawa [114] obtained the upper bound of $|H_2(n)|$ for the class of typically real functions. The computation for the bound of $|H_2(2)|$, where $H_2(2) := a_2a_4 - a_3^2$ requires the formulae of p_2 and p_3 in terms of p_1, where $p_i's$ are the coefficients of the functions in the Carathéodory class $\mathcal{P}$. In the past, many authors obtained bounds of $|H_2(2)|$ for the subclasses of analytic functions (see [9, 32, 36, 55]). Janteng et al. [35, 36] deduced the following bounds:

$$|H_2(2)| \leq \begin{cases} 1 & \text{for} \quad f \in \mathcal{S}^*, \\ 1/8 & \text{for} \quad f \in \mathcal{C}, \\ 4/9 & \text{for} \quad f \in \mathcal{R}. \end{cases}$$

Another type of second Hankel determinant is

$$H_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = a_3a_5 - a_4^2.$$

Zaprawa [116] investigated this Hankel determinant $H_2(3)$ for several classes of univalent functions. The estimate of the upper bound of the third order Hankel determinant, which is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2), \tag{1.3.6}$$

requires the sharp bounds of the initial coefficients (a_2, a_3, a_4 and a_5), Fekete-Szegö functional, second Hankel determinant $H_2(2)$ and the quantity $L := |a_4 - a_2a_3|$. The first article on $H_3(1)$ came in 2010, where Babalola [8] established the upper bound of $|H_3(1)|$ for $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ by estimating each term in (1.3.6). Later, it was in 2017 when these upper bounds were improved by Zaprawa [115] as:

$$|H_3(1)| \leq \begin{cases} 1 & \text{for} \quad f \in \mathcal{S}^*, \\ 49/540 & \text{for} \quad f \in \mathcal{C}, \\ 41/60 & \text{for} \quad f \in \mathcal{R}. \end{cases}$$

He claimed these to be not sharp yet. Until now the upper bound of $|H_3(1)|$ was obtained using triangle inequality in (1.3.6), as follows:

$$|H_3(1)| \leq |a_3||H_2(2)| + |a_4||L| + |a_5||H_2(1)|,$$

(see [8, 46, 88, 115]). The third Hankel determinant problem became interesting only after the well-known formula of expressing p_4 in terms of p_1 which was recently obtained by Kwon et al. [52], where $p_i's$ are defined in (1.2.1). Generally, it serves as a basic method to obtain sharp bounds in the case of third order Hankel determinant. This technique is employed by us in chapter 5 for achieving the sharp bounds of $|H_3(1)|$ and $|H_2(3)|$ for functions in $\mathcal{SL}^*$. In fact, following the result of Kwon et al. [52], in 2019, the authors [53] were also able to derive the bound $|H_3(1)| \leq 8/9$ for $f \in \mathcal{S}^*$, which is an improvement over the earlier bounds. In the similar direction, recently, Kowalczyk et al. [45] and Lecko et al. [54] obtained the sharp bounds for functions in $\mathcal{C}$ and $\mathcal{S}^*(1/2)$, respectively, as $|H_3(1) \leq 4/135$ and $|H_3(1)| \leq 1/9$.

It is also worth noting that recently, the bound of third order Hankel determinant for functions in $\mathcal{S}^*$ was further improved to 5/9 by Zaprawa et al. [118], by expressing the coefficients of $f \in \mathcal{S}^*$ in terms of the corresponding coefficients of the Schwarz function. This approach differs from the usual one, wherein the coefficients of $f \in \mathcal{S}^*$ are expressed by the corresponding coefficients of Carathéodory function.

1.4 Differential Subordinations

The concept of differential subordination helps one in finding the character of an analytic function provided prior knowledge of the necessary geometric details related to the function or its derivative are known. The study of differential subordination began in 1981 with the remarkable work "Differential subordination and univalent functions" by Miller and Mocanu [66]. While working in the field of univalent functions, they began exploration in the development of this field. It all started with a 1974 paper written jointly with Maxwell O. Reade, which has the result as: Let α be real and p be analaytic in $\mathbb{D}$, then $\mathrm{Re}\left(p(z)+\alpha\frac{zp'(z)}{p(z)}\right)>0 \Rightarrow \mathrm{Re}\,p(z)>0$. This result deals with finding the properties of a function's range from the known range of a combination of the function's derivatives. A differential subordination in the complex plane is analogous to that of a differential inequality on the real line. On real line $f'(x)>0$ implies f is an increasing function. There are various differential implications in the theory of complex-valued functions, where a function's characterization can be found from a differential condition. One such example is Noshiro-Warschawski, which provides the criterion for univalence of analytic functions based on a differential condition. There has been a dispersion of differential implications, until the recent invention of the theory of differential subordination. Miller and Mocanu [67] discussed the general theory of differential subordinations of first, second and third orders and is enriched with various results related to it. This monograph furnishes the definition of differential subordination as follows:

Definition 1.4.1. (Differential Subordination) Let $\psi:\mathbb{C}^2\times\mathbb{D}\to\mathbb{C}$ and let h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfy the (first order) **differential subordination**

$$\psi(p(z),zp'(z);z)\prec h(z), \tag{1.4.1}$$

then p is called a **solution** of the differential subordination.

The univalent function q is called a **dominant** of the solution of the differential subordination, or more simply a dominant, if $p\prec q$ for all p satisfying (1.4.1). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for all dominant q of (1.4.1) is called the **best dominant**. Note that best dominant is unique up to the rotation of $\mathbb{D}$. The first important results of first-order differential subordinations were dealt by Goluzin [23] and by Robinson [91]. This theory ushered a great change, attracting a slew of researchers to apply this technique to the study of univalent functions. Several additional applications or extensions of the theory began to emerge, and hundreds of publications on the subject have emerged in the literature since then, to list a few [3, 13, 14, 50, 98, 106].

Fascinatingly, the consistent study in this topic led authors to introduce various special types of differential subordinations. For instance, the following first-order differential subordination

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z),$$

is called the Briot-Bouquet differential subordination, the name of which is inspired by the Briot-Bouquet type of differential equation of the form $q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z)$. This differential subordination has a huge number of applications in univalent function theory. Following which, recently Kanas and Kowalczyk [38] discussed the general case of the Briot-Bouquet differential subordination involved in the Bernoulli differential equation. The authors named this special type as Briot-Bouquet-Bernoulli differential subordination. Likewise, in chapter 3, we also introduce another special type of first-order differential subordination, namely exact differential subordinations.

Summary of the Thesis

In the present investigation, we chiefly deal with the differential subordination implications, coefficient estimates, radius and inclusion results for certain classes associated with Ma-Minda functions or a special type of Ma-Minda functions. In this direction, second and third chapters deal with the differential subordination implications and their applications associated with Ma-Minda functions. In addition, we find sufficient conditions for analytic functions to belong to various subclasses of $\mathcal{S}^*$ in the second chapter and in the third chapter, we introduce a new concept of exact differential subordinations and obtain the best dominants of the solution of this special type of differential subordinations. In chapter 4, we examine and study the conditions imposed on Ma-Minda functions and their significance. As a consequence, introduce a special type of Ma-Minda function by changing the orientation of Ma-Minda function and study a newly defined class involving it for certain radius and inclusion results. The chapter 5 deals with the estimation of sharp coefficient bounds for functions in fundamental class, which specializes several well-known classes of analytic functions, defined with the help of convolution and subordination. Consequently, many of the bounds of our class reduce to already known bounds of some well-known classes as a special case. The sharp bound of third order Hankel determinant for functions in the class $\mathcal{SL}^*$ is the key feature of this chapter. The last chapter deals with a newly defined class of analytic functions by considering an inequality involving real part and modulus of certain expressions of functions. This class implies another subclass of $\mathcal{S}^*$, defined by the subordination having a well-known subordinating function. Furthermore, derive radius and inclusion results in addition to the coefficient problems for functions in these classes, which are under study.

Chapter 2

First Order Differential Subordinations

In this chapter, we establish sufficient conditions for normalized analytic functions to belong to certain subclasses of starlike functions as an application of our derived differential subordination implication results, namely, $\psi(p) := p^{\lambda}(z)(\alpha + \beta p(z) + \gamma/p(z) + \delta z p'(z)/p^{j}(z)) \prec h(z)$ $(j = 1, 2)$ *implies* $p \prec q$*, where* $h := \psi(q)$ *and* q *belongs to the class* $\mathcal{P}$*. This is achieved by finding conditions on* α*,* β*,* γ*,* δ *and* λ*. In addition, we deduce the results related to argument estimation of* h*, when* $q(z) = ((1+z)/(1-z))^{\eta}$ $(0 < \eta \leq 1)$*.*

2.1 Introduction

The differential subordination implication results have a wide range of applications in the theory of differential subordinations as it provides the sufficient conditions for starlikeness or univalency of certain analytic functions. In this direction, the Lemma 2.2.1, given by Miller and Mocanu [67] is the source of inspiration and is the backbone of many such results. Authors, namely, Sharma and Ravichandran [98] and Ahuja et al. [1] established certain applications of first order differential subordinations. Recently, the authors Ravichandran and Kumar [85] obtained some sufficient conditions for analytic functions p in $\mathbb{D}$ to satisfy certain differential subordination and in particular to have a positive real part.

Motivated by this, we investigate certain differential subordination implications by finding conditions on the parameters involved in it, in this chapter. We deal with the following two admissible classes of analytic functions:

$$p^{\lambda}(z)\left(\alpha+\beta p(z)+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p(z)}\right)\prec h(z)$$

and

$$p^{\lambda}(z)\left(\alpha+\beta p(z)+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^{2}(z)}\right)\prec h(z), \tag{2.1.1}$$

which implies $p(z)\prec q(z)$, where h is univalent and $q(z)\in\mathcal{P}$. Several authors, namely, Cho and Kim [13, 14], Cho et al. [18], and Liu and Srivastava [60] have evolved the concept of finding conditions on the parameters involved in the first order differential subordinations in order to prove $p(z)\prec((1+z)/(1-z))^{\eta}$, $(0<\eta\leq 1)$ by satisfying the condition $|\arg(\psi(p(z),zp'(z))|\leq\arg(h(z))$. As a consequence, we derive results related to argument estimation of function h when q is taken to be $((1+z)/(1-z))^{\eta}$.

Before we proceed to the main results, an important remark is presented below, which is used in almost all of the proofs.

Remark 1. (i) Let the function f be analytic in a bounded domain D and continuous on $\overline{D}$. (ii) Suppose that $u(x,y)$ is a real part of non-constant analytic function f on a bounded domain D and $u(x,y)$ is bounded below (above) in D. If either of the conditions (i), (ii) holds, then the real part of an analytic function f attains its minimum (maximum) value, on the boundary of D, by Minimum (Maximum) Modulus Theorem.

We provide below some examples justifying Remark 1, which are associated with a complex variable.

***Example* 1.** (i) Consider the function $f(z) = z/(1-z)$, $(z \in \mathbb{D})$. Now, to obtain the maximum/minimum value of the real part of $f(z)$ on the bounded domain $\mathbb{D}$, either of the above two conditions of Remark 1 should hold. Clearly, (i) fails as the function $f(z)$ is not continuous on $\overline{\mathbb{D}}$. If we take $z = x + iy$ $(x^2 + y^2 < 1)$, then

$$\operatorname{Re} f(z) = \frac{x - x^2 - y^2}{(1-x)^2 + y^2} =: \tilde{f}(x, y).$$

Intuitively, the function $\tilde{f}(x, y)$ is unbounded when x tends to 1 and y tends or equal to 0. Consider the following two cases where $\tilde{f}(x, y)$ is unbounded in $\mathbb{D}$.

Case 1: Let $y = 0$ in $\tilde{f}(x, y)$, then we get

$$\tilde{f}(x, 0) =: \tilde{f}(x) = x/(1-x) \quad (-1 < x < 1),$$

clearly $\tilde{f}(x)$ is unbounded when x tends to 1. Let us now assume $x = 1 - h$, $(h \to 0)$ in $\tilde{f}(x)$ and we obtain

$$\lim_{h \to 0} \tilde{f}(1-h) = \lim_{h \to 0} \frac{1-h}{h} \to +\infty.$$

This shows that $\tilde{f}(x, y)$ is not bounded above in $\mathbb{D}$.

Case 2: Let $x = 1 - h$ in $\tilde{f}(x, y)$, $(h \to 0)$, we get

$$\lim_{h \to 0} \tilde{f}(1-h, y) = \lim_{h \to 0} \frac{-h - h^2 + 2h - y^2}{h^2 + y^2} \to -1, \quad ((1-h)^2 + y^2 < 1).$$

Therefore, $\tilde{f}(x, y)$ is bounded below here.

We conclude that the real part of $f(z)$ is bounded below but not bounded above in $\mathbb{D}$. Thus, evaluating $f(z)$ on $z = e^{i\theta}$ guarantees to have minimum value of real part of $f(z)$. For $z = e^{i\theta}$ $(-\pi \le \theta < \pi)$, we obtain

$$\operatorname{Re} f(e^{i\theta}) = -1/2.$$

We infer that the minimum value of real part of $f(z)$ is $-1/2$ and maximum can not be obtained by evaluating $\operatorname{Re} f$ on the boundary of $\mathbb{D}$ as it fails to be bounded above.

(ii) Consider the Koebe function $k(z) := z/(1-z)^2$, clearly $k(\mathbb{D}) = \mathbb{C} - (-\infty, -1/4]$. So, the real part of a Koebe function is unbounded. We can also verify this by taking

$z = x + iy$ ($x^2 + y^2 < 1$) and then we have

$$\mathrm{Re}\,k(z) = \frac{x - 2x^2 + x^3 - 2y^2 + xy^2}{(1-x)^2 + y^2} =: f(x,y).$$

It is trivial to say that the real part of $k(z)$ is unbounded if z tends to 1. Equivalently, we can say $f(x,y)$ is unbounded when x tends to 1 and y tends to or equal to 0. This is a complex plane so x can tend to 1 from all the directions, unlike on the real plane, where it can only tend along the real axis. Now, consider the following two cases:

Case 1: Let $y = 0$ in $f(x,y)$, then we have

$$f(x,0) = \frac{x - 2x^2 + x^3}{(1-x)^2} =: f(x).$$

Furthermore, we assume $x = 1 - h$, $(h \to 0)$ in $f(x)$ and we deduce

$$\lim_{h\to 0} f(1-h) = \lim_{h\to 0} \frac{1-h}{h^2} \to +\infty.$$

Thus, $f(x,y)$ is not bounded above in $\mathbb{D}$.

Case 2: Let $x = 1 - h$ in $f(x,y)$ $(h \to 0)$, we get

$$\lim_{h\to 0} f(1-h,y) = \frac{-1}{y^2} \to -\infty \quad ((1-h)^2 + y^2 < 1),$$

whenever $y \to 0$. This shows that $f(x,y)$ is not bounded below either in $\mathbb{D}$.

We conclude that the real part of $k(z)$ is neither bounded below nor above in $\mathbb{D}$. Therefore, evaluating $k(z)$ at $z = e^{i\theta}$ does not guarantee to yield maximum or minimum value of k in $\mathbb{D}$. For if, we consider $z = e^{i\theta}$ $(-\pi \le \theta < \pi)$, then we get

$$\mathrm{Re}(z/(1-z)^2) = -1/(4\sin^2\theta) =: g(\theta).$$

We calculate that the maximum value of $g(\theta)$ is $-1/4$, attained at $\theta = \pi/2$. But this does not imply that $\max \mathrm{Re}\,k(z) = -1/4$, as clearly we observe that the real part of $k(z)$ is unbounded. As a result, evaluating the real part of $k(z)$ on the boundary of $\mathbb{D}$ does not ensure that the maximum value is obtained. This is due to the failure of both the conditions (i) and (ii) in Remark 1 for the Koebe function.

In summary, the Remark 1 concludes that the minimum (maximum) value of the real part of analytic function f is guaranteed, by evaluating f on the boundary of its domain, if either of the two conditions of the Remark is satisfied.

2.2 Differential Subordination Implications

Miller and Mocanu [67] gave the following result which is required to prove our main findings.

Lemma 2.2.1. [67] Let q be univalent in the unit disk $\mathbb{D}$ and θ, φ be analytic in a domain D containing $q(\mathbb{D})$ with $\varphi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that (i) $Q(z)$ is starlike univalent in $\mathbb{D}$, and (ii) $\mathrm{Re}(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$. If p is analytic in $\mathbb{D}$ with $p(0) = q(0)$, $p(\mathbb{D}) \subset D$ and satisfies

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) \tag{2.2.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Before commencing with the main results, let us define $\mathcal{P}_0$, the class of all analytic functions p, which do not vanish anywhere in $\mathbb{D}$, with the normalization $p(0) = 1$.

Theorem 2.2.2. Let λ be a real number and α, β, γ and $\delta (\neq 0)$ be complex numbers. Suppose $q \in \mathcal{P}_0$ be univalent in $\mathbb{D}$ and satisfy the following conditions in $\mathbb{D}$.

(1) $Q(z) := \delta zq'(z)q^{\lambda-1}(z)$ be starlike (univalent).

(2) $\mathrm{Re}\left(\frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}q(z) + \frac{\gamma(\lambda-1)}{\delta q(z)} + (\lambda-1)\frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right) > 0.$

If $p \in \mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec q^{\lambda}(z)\left(\alpha + \beta q(z) + \frac{\gamma}{q(z)} + \delta\frac{zq'(z)}{q(z)}\right),$$

then $p(z) \prec q(z)$. Furthermore, q is the best dominant.

Proof. Let $\theta(\omega) = \omega^{\lambda}(\alpha + \beta\omega + \gamma/\omega)$, $\omega \neq 0$ and $\varphi(\omega) = \delta\omega^{\lambda-1}$. We observe that $\theta(\omega)$ and $\varphi(\omega)$ are analytic in $\mathbb{C} - \{0\}$. Furthermore, $\varphi(\omega) \neq 0$. Let $Q(z)$ and $h(z)$ be given by

$$Q(z) := zq'(z)\varphi(q(z)) = \delta zq'(z)q^{\lambda-1}(z)$$

and

$$h(z) := \theta(q(z)) + Q(z) = q^{\lambda}(z)\left(\alpha + \beta q(z) + \frac{\gamma}{q(z)} + \delta\frac{zq'(z)}{q(z)}\right).$$

Then $\mathrm{Re}(zh'(z)/Q(z))$ reduces to

$$\mathrm{Re}\,H(z) =: \mathrm{Re}\left(\frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}q(z) + \frac{\gamma(\lambda-1)}{\delta q(z)} + (\lambda-1)\frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right), \qquad (2.2.2)$$

which is greater than 0 from condition (2). On substituting $q(z)$ as $p(z)$ in $\theta(q(z)) + Q(z)$, we obtain the required subordination (2.2.1). Also, $Q(z)$ is given to be starlike. Now, an application of Lemma 2.2.1, produces the result. □

The next main result deals with another type of differential subordination implication involving (2.1.1).

Theorem 2.2.3. Let α, β, γ and $\delta(\neq 0)$ be complex numbers and λ be a real number. Let $q \in \mathcal{P}_0$ be univalent in $\mathbb{D}$ and satisfy the following conditions for $z \in \mathbb{D}$.

(1) $Q(z) := \delta zq'(z)q^{\lambda-2}(z)$ be starlike (univalent).

(2) $\mathrm{Re}\left(\frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}q(z) + \frac{\beta(\lambda+1)}{\delta}q^2(z) + (\lambda-2)\frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right) > 0.$

If $p \in \mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^2(z)}\right) \prec q^{\lambda}(z)\left(\alpha + \beta q(z) + \frac{\gamma}{q(z)} + \delta\frac{zq'(z)}{q^2(z)}\right),$$

then $p(z) \prec q(z)$. Furthermore, q is the best dominant.

Proof. Let $\omega \neq 0$, $\theta(\omega)$ be as defined in proof of Theorem 2.2.2 and $\varphi(\omega) = \delta\omega^{\lambda-2}$. Then we have θ and φ are analytic in $\mathbb{C} - \{0\}$ and $\varphi(\omega) \neq 0$. We also have $Q(z) = zq'(z)\varphi(q(z)) = \delta zq'(z)q^{\lambda-2}$. Let $h(z)$ be given by

$$h(z) := \theta(q(z)) + Q(z) = q^{\lambda}(z)\left(\alpha + \beta q(z) + \frac{\gamma}{q(z)} + \delta\frac{zq'(z)}{q^2(z)}\right).$$

A simple calculation yields $zh'(z)/Q(z)$ reduces to

$$H(z) := \frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}q(z) + \frac{\beta(\lambda+1)}{\delta}q^2(z) + (\lambda-2)\frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right), \qquad (2.2.3)$$

clearly $\mathrm{Re}\,H(z) > 0$, given in condition (2). On substituting $p(z)$ in place of $q(z)$ in $\theta(q(z)) + Q(z)$, we deduce equation (2.2.1). Also $Q(z)$ is starlike in $\mathbb{D}$. Now the result follows immediately from Lemma 2.2.1. □

Remark 2. Let $\lambda = \alpha = \gamma = \beta = 0$ and $\delta = 1$, then Theorem 2.2.3 reduces to the result obtained by Ravichandran and Kumar [85].

2.3 Special Cases

Taking α,β,γ and δ to be real numbers in Theorem 2.2.2 and Theorem 2.2.3, we obtain the following results. It is pertinent to mention here that, in view of part (ii) of Remark 1, we impose certain conditions on the parameters involved in the following results to satisfy Lemma 2.2.1. We begin this section by taking $q(z) = (1+(1-2\nu)z)/(1-z)$ in Theorem 2.2.2 and Theorem 2.2.3.

Corollary 2.3.1. Suppose $0 \le \nu < 1$ and $p \in \mathcal{P}_0$.

(a) Let $0 \le \lambda \le 1$ and $\gamma\delta \le 0$. (i) For $0 \le \nu \le 1/2$, let $(\alpha+\nu\beta)/\delta \ge \nu/(2(1-\nu))$ and $1+2(1-\nu)\beta/\delta > 0$. (ii) For $1/2 \le \nu < 1$, let $\alpha+\nu\beta/\delta \ge (1-\nu)/(2\nu)$ and $2\beta/\delta > (\nu-1)/\nu^2$. If p satisfies

$$p^{\lambda}(z)\left(\alpha+\beta p(z)+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p(z)}\right) \prec \left(\frac{1+(1-2\nu)z}{1-z}\right)^{\lambda}\left(\alpha+\beta\left(\frac{1+(1-2\nu)z}{1-z}\right)\right. \\ \left.+\frac{\gamma(1-z)}{1+(1-2\nu)z}+\delta\left(\frac{(1-2\nu)z}{1+(1-2\nu)z}+\frac{z}{1-z}\right)\right),$$

then $\operatorname{Re} p(z) > \nu$.

(b) Let $0 \le \nu < 1$ and $1 \le \lambda \le 2$. (i) For $0 \le \nu \le 1/2$, let $1+2(1-\nu)\alpha/\delta > 0$ and $(\gamma+\nu\alpha)/\delta \ge \nu/(2(1-\nu))$. (ii) For $1/2 \le \nu < 1$, let $2\alpha/\delta > (\nu-1)/\nu^2$ and $(\gamma+\nu\alpha)/\delta \ge (1-\nu)/(2\nu)$. If p satisfies

$$p^{\lambda}(z)\left(\alpha+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right) \prec \left(\frac{1+(1-2\nu)z}{1-z}\right)^{\lambda}\left(\alpha+\frac{\gamma(1-z)}{1+(1-2\nu)z}\right. \\ \left.+\frac{2\delta(1-\nu)z}{(1+(1-2\nu z))^2}\right),$$

then we have $\operatorname{Re} p(z) > \nu$.

Proof. (a) Let $q(z) = (1+(1-2\nu)z)/(1-z)$ in Theorem 2.2.2. Furthermore, we calculate $Q(z) = 2(1-\nu)z/((1-z)^{1+\lambda}(1+(1-2\nu)z)^{1-\lambda})$ and

$$\frac{zQ'(z)}{Q(z)} = 1+(1+\lambda)\left(\frac{z}{1-z}\right)-(1-\lambda)\left(\frac{(1-2\nu)z}{1+(1-2\nu)z}\right).$$

A calculation gives

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = \left(\frac{1-\lambda}{2}\right)\left(\frac{\nu(1-\nu)}{\nu^2+(1-2\nu)\cos^2(\theta/2)}\right),$$

for $z = e^{i\theta}$, where $-\pi \le \theta < \pi$. Since $1-\lambda$, $\nu(1-\nu)$, $\nu^2+(1-2\nu)\cos^2(\theta/2) \ge 0$, it follows that Q is starlike (univalent) in $\mathbb{D}$. Also, from equation (2.2.2), we have

$$H(z) := \frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}\left(\frac{1+(1-2\nu)z}{1-z}\right) + \frac{\gamma(\lambda-1)}{\delta}\left(\frac{1-z}{1+(1-2\nu)z}\right)$$
$$+(\lambda-1)\left(\frac{(1-2\nu)z}{1+(1-2\nu)z} + \frac{z}{1-z}\right) + \left(1+\frac{2z}{1-z}\right)$$

and a simple calculation yields

$$\operatorname{Re} H(z) = 1 + \frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}\nu + \frac{\gamma(\lambda-1)}{\delta}A(\theta) - \frac{\lambda+1}{2} + (\lambda-1)(1-2\nu)B(\theta) =: L(\theta),$$

where $z = e^{i\theta}$ $(-\pi \le \theta < \pi)$, $A(\theta) = (\nu\sin^2(\theta/2))/(\nu^2 + (1-2\nu)\cos^2(\theta/2))$ and $B(\theta) = (1-2\nu+\cos\theta)/(2(1-2\nu+\cos\theta+2\nu(\nu-\cos\theta))$. Also, considering Remark 1, we need to have

$$\gamma/\delta < 1/2 \qquad \text{and} \qquad 1 + 2\beta(1-\nu)/\delta > 0, \tag{2.3.1}$$

which is trivial as $\gamma\delta \le 0$ and $2\beta/\delta > (\nu-1)/\nu^2$ imply these conditions (2.3.1), respectively. To complete the proof, it is enough to show that $L(\theta) \ge 0$. For this, we consider the following cases:

Case 1: Consider $0 \le \nu \le 1/2$, clearly $(1-2\nu) \ge 0$. We calculate $\min_{-\pi\le\theta<\pi} A(\theta) = 0$ and $\max_{-\pi\le\theta<\pi} B(\theta) = 1/(2(1-\nu))$, which are attained at $\theta = 0$ by the second derivative test. Since $0 \le \lambda \le 1$ and $\gamma\delta \le 0$, we obtain

$$L(\theta) \ge \lambda\left(\frac{\alpha}{\delta} + \frac{\beta\nu}{\delta} + \frac{1-2\nu}{2(1-\nu)} - \frac{1}{2}\right) + \frac{1}{2} + \frac{\beta\nu}{\delta} - \frac{1-2\nu}{2(1-\nu)} \ge 0,$$

whenever $(\alpha+\beta\nu)/\delta \ge \nu/(2(1-\nu))$.

Case 2: Consider $1/2 \le \nu < 1$ then $(1-2\nu) \le 0$. By taking the range of λ and $\gamma\delta$ into consideration, we calculate $\min_{-\pi<\theta\le\pi} B(\theta) = -1/(2\nu)$, attained at $\theta = \pi$ by the second derivative test and the minimum value of $A(\theta)$ is as calculated in case 1. On further computation, we get

$$L(\theta) \ge \lambda\left(\frac{\alpha}{\delta} + \frac{\beta\nu}{\delta} - \frac{1-2\nu}{2\nu} - \frac{1}{2}\right) + \frac{1}{2} + \frac{\beta\nu}{\delta} + \frac{1-2\nu}{2\nu} \ge 0,$$

whenever $(\alpha+\beta\nu)/\delta \ge (1-\nu)/(2\nu)$ and $\beta/\delta > (\nu-1)/(2\nu^2)$. With this, we complete the proof for part (a).

(b) For this result, we further assume $\beta = 0$ and let $q(z) = (1+(1-2\nu)z)/(1-z)$ for

$0 \le \nu < 1$ in Theorem 2.2.3, then we obtain the function $Q(z)$ as follows:

$$Q(z) = \frac{2(1-\alpha)z}{(1-z)^{\lambda}(1+(1-2\nu)z)^{2-\lambda}}.$$

Proceeding on the similar lines as in part (a), we have $Q(z)$ starlike (univalent) in $\mathbb{D}$ for $1 \le \lambda \le 2$. If we take $z = e^{i\theta}$ $(-\pi < \theta \le \pi)$, we have

$$\begin{aligned} &\mathrm{Re}\left(\frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}q(z) + (\lambda-2)\frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)}\right) \\ &= 1 + \frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}\nu + (\lambda-2)(1-2\nu)B(\theta) - \frac{\lambda}{2} =: S(\theta), \end{aligned}$$

where $B(\theta)$ is as defined in the proof of part (a). According to Remark 1, we consider the inequality $1 + 2\alpha(1-\nu)/\delta > 0$. Now, to apply Theorem 2.2.3, we find the range of parameters such that $S(\theta) \ge 0$. For this, we consider two cases:

Case 1: Consider $0 \le \nu \le 1/2$, then $(1-2\nu) \ge 0$. Since $1 \le \lambda \le 2$, we take into account the maximum value of $B(\theta)$ and we deduce

$$S(\theta) \ge \lambda\left(\frac{\gamma}{\delta} + \frac{\alpha\nu}{\delta} + \frac{1-2\nu}{2(1-\nu)} - \frac{1}{2}\right) + 1 - \frac{\gamma}{\delta} - \frac{1-2\nu}{1-\nu}.$$

It suffices to find the conditions on the parameters for which $S(\theta) \ge 0$. For this, either (i) let $\gamma/\delta + \alpha\nu/\delta + (1-2\nu/(2(1-\nu)) - 1/2 = 0$, further computation yields $S(\theta) \ge 0$, or (ii) take

$$\lambda \ge \frac{\frac{\gamma}{\delta} + \frac{1-2\nu}{1-\nu} - 1}{\frac{\gamma}{\delta} + \frac{\alpha\nu}{\delta} + \frac{1-2\nu}{2(1-\nu)} - \frac{1}{2}}, \tag{2.3.2}$$

which is a valid expression only if $(\gamma + \alpha\nu)/\delta > \nu/(2(1-\nu))$. Also, as we have $1 \le \lambda$, the inequality (2.3.2) holds if $1 + 2\alpha(1-\nu)/\delta > 0$.

Case 2: Consider $1/2 \le \nu < 1$, then $(1-2\nu) \le 0$. Therefore, we take into account the minimum value of $B(\theta)$ and we deduce

$$S(\theta) \ge \lambda\left(\frac{\gamma}{\delta} + \frac{\alpha\nu}{\delta} - \frac{1-2\nu}{2\nu} - \frac{1}{2}\right) + 1 - \frac{\gamma}{\delta} + \frac{1-2\nu}{\nu}.$$

Proceeding as in the case 1, whenever $(\gamma + \alpha\nu)/\delta \ge (1-\nu)/(2\nu)$ and $\alpha/\delta > -1/(2(1-\nu))$, we have $S(\theta) \ge 0$.

Considering both the cases, we take $\alpha/\delta \ge \max(-1/(2(1-\nu)); (\nu-1)/2\nu^2) = (\nu-1)/(2\nu^2)$, which completes the proof. □

We obtain the relation between the class of starlike functions of reciprocal order ν and $\mathcal{N}(\kappa)$ in the following corollary.

Corollary 2.3.2. Let $f \in \mathcal{A}$. A function f is starlike of reciprocal order ν if

(a) $f \in \mathcal{N}((3-\nu)/2\nu)$, $0 < \nu \leq 3/4$.

(b) $f \in \mathcal{N}(\nu/2(1-\nu))$, $3/4 \leq \nu < 1$.

Proof. Let $p(z) = f(z)/(zf'(z))$, $\beta = \lambda = 0$, $\gamma = -\delta = 1$ and $\alpha = \min((\nu-1)/(2\nu); \nu/(2(\nu-1)))$ in Corollary 2.3.1(a), we get, if f satisfies

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+2(\nu-1)z}{1+(1-2\nu)z} - \frac{z}{1-z} =: S(z),$$

then $\operatorname{Re}(f(z)/(zf'(z))) > \nu$. To complete the proof it suffices to show $\operatorname{Re} S(z) < (3-\nu)/(2\nu)$ for $0 < \nu \leq 3/4$ and $\operatorname{Re} S(z) < \nu/(2(1-\nu))$ for $3/4 \leq \nu < 1$. Since $S(z)$ is bounded above, evaluating it on the boundary of $\mathbb{D}$, we obtain

$$\operatorname{Re} S(e^{i\theta}) = \frac{1-6\nu+4\nu^2+\cos\theta}{-2+4\nu-4\nu^2-2\cos\theta+4\nu\cos\theta} + \frac{1}{2} =: g(\theta).$$

A calculation shows $g''(\theta)_{\theta=\pi} = (1-\nu)(4\nu-3)/(4\nu^3)$ and $g''(\theta)_{\theta=0} = \nu(4\nu-3)/(4(\nu-1)^3)$.
(a) For $0 < \nu \leq 3/4$, $\max_{-\pi<\theta\leq\pi} g(\theta) = (3-\nu)/(2\nu)$, attained at $\theta = \pi$ and we obtain

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3-\nu}{2\nu},$$

equivalently, $f \in \mathcal{N}((3-\nu)/(2\nu))$. This completes the proof for part (a).

(b) For $3/4 \leq \nu < 1$, $\max_{-\pi\leq\theta<\pi} g(\theta) = \nu/(2(1-\nu))$, attained at $\theta = 0$ and we obtain

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{\nu}{(2(1-\nu))},$$

equivalently, $f \in \mathcal{N}(\nu/2(1-\nu))$. This completes the proof for part (b). □

By taking $\nu = 0$ and $\nu = 1/2$, respectively in Corollary 2.3.1(a), we get the following results:

Corollary 2.3.3. Let $0 \leq \lambda \leq 1$, $1 + 2\beta/\delta > 0$, $\gamma\delta \leq 0$ and $\alpha\delta \geq 0$. If $p \in \mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\lambda}\left(\alpha + \beta\left(\frac{1+z}{1-z}\right) + \gamma\left(\frac{1-z}{1+z}\right) + \frac{2\delta z}{1-z^2}\right),$$

then $\operatorname{Re} p(z) > 0$.

Corollary 2.3.4. Let $1+\beta/\delta>0$, $\gamma\delta\leq 0$, $-1+(2\alpha+\beta)/\delta\geq 0$ and $0\leq\lambda\leq 1$. If $p\in\mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha+\beta p(z)+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p(z)}\right)\prec\left(\frac{1}{1-z}\right)^{\lambda}\left(\alpha+\beta\frac{1}{1-z}+\gamma(1-z)+\frac{\delta z}{1-z}\right),$$

then $\operatorname{Re}p(z)>1/2$.

By taking $\nu=0$ and $\nu=1/2$, respectively in Corollary 2.3.1(b), we have the following results:

Corollary 2.3.5. Let $1+2\alpha/\delta>0$, $\gamma\delta\geq 0$ and $1\leq\lambda\leq 2$. If $p\in\mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right)\prec\left(\frac{1+z}{1-z}\right)^{\lambda}\left(\alpha+\gamma\frac{1-z}{1+z}+\frac{2\delta z}{(1+z)^2}\right),$$

then $\operatorname{Re}p(z)>0$.

Corollary 2.3.6. Let $1+\alpha/\delta>0$, $(2\gamma+\alpha)/\delta\geq 1$ and $1\leq\lambda\leq 2$. If $p\in\mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right)\prec\left(\frac{1}{1-z}\right)^{\lambda}(\alpha+\gamma(1-z)+\delta z),$$

then $\operatorname{Re}p(z)>1/2$.

Remark 3. The Corollary 2.3.3 with $\alpha=\gamma=0$ and $\lambda=1$ and Corollary 2.3.5 with $\lambda=2$, $\gamma=0$ independently yield the result of Nunokawa et al. [77, Theorem 1, p. 1386].

Corollary 2.3.7. If $p\in\mathcal{P}_0$ satisfies:

$$p(z)+\frac{zp'(z)}{p(z)}\prec\mathcal{R}(z),$$

where $\mathcal{R}$ is the open door mapping, then $p(z)\prec(1+z)/(1-z)$.

Proof. Let $\lambda=\gamma=\alpha=0$ and $\beta=\delta=1$ in Corollary 2.3.3 or by assuming $\lambda=\alpha=\delta=1$ and $\gamma=0$ in Corollary 2.3.5, we get

$$p(z)+\frac{zp'(z)}{p(z)}\prec\frac{1+z}{1-z}+\frac{2z}{1-z^2}=:\mathcal{R}(z).$$

This completes the proof. □

Remark 4. The above Corollary is the result obtained by Nunokawa et al. [77].

Considering oblique sector function q in the above Theorem 2.2.2, which is defined as

$$q(z)=\left(\frac{1+cz}{1-z}\right)^{\frac{\eta_1+\eta_2}{2}}\qquad(0<\eta_1,\eta_2\leq 1),$$

where c and η are as defined in the equation (1.2.7). We have $\operatorname{Re}(zQ'(z)/Q(z))>0$ for given η and λ from [60, Theorem 2, p. 711], where Q is as defined in the condition (1) of Theorem 2.2.2. Therefore, $Q(z)$ is starlike(univalent) in $\mathbb{D}$. Let

$$H(z)=\left(\frac{\alpha\lambda}{\delta}+\frac{\beta(\lambda+1)}{\delta}\left(\frac{1+cz}{1-z}\right)^{\frac{\eta_1+\eta_2}{2}}+\frac{\gamma(\lambda-1)}{\delta}\left(\frac{1-z}{1+cz}\right)^{\frac{\eta_1+\eta_2}{2}}+\frac{zQ'(z)}{Q(z)}\right),$$

from condition (2) of Theorem 2.2.2 and in view of the fact that $q\in\mathcal{P}$, which implies $1/q(z)\in\mathcal{P}$, we obtain

$$\operatorname{Re}H(z)>\alpha\lambda/\delta\geq 0,$$

provided $\beta(\lambda+1)/\delta$, $\gamma(\lambda-1)/\delta$, $\alpha\lambda/\delta\geq 0$. Therefore, both the conditions of Theorem 2.2.2 get satisfied and we obtain the result as follows:

Corollary 2.3.8. Let $\alpha\lambda/\delta\geq 0$, $\beta(\lambda+1)/\delta\geq 0$, $\gamma(\lambda-1)/\delta\geq 0$ and $|\lambda|\leq 2/(\eta_1+\eta_2)$, where $0<\eta_1,\eta_2\leq 1$. If $p\in\mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha+\beta p(z)+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p(z)}\right)\prec\left(\frac{1+cz}{1-z}\right)^{\frac{(\eta_1+\eta_2)\lambda}{2}}\left(\alpha+\beta\left(\frac{1+cz}{1-z}\right)^{\frac{\eta_1+\eta_2}{2}}+\gamma\left(\frac{1-z}{1+cz}\right)^{\frac{\eta_1+\eta_2}{2}}\right.$$
$$\left.+\frac{\eta_1+\eta_2}{2}\left(\frac{(1+c)z}{(1+cz)(1-z)}\right)\right),$$

then $p(z)\prec((1+cz)/(1-z))^{(\eta_1+\eta_2)/2}$.

Remark 5. By taking $\alpha=\gamma=0$ and $\delta=1$, Corollary 2.3.8 is the result obtained in [60].

Letting $\eta_1=\eta_2$ and $c=1$ in the Corollary 2.3.8, we deduce the result as follows:

Corollary 2.3.9. Let $\alpha\lambda/\delta\geq 0$, $\beta(\lambda+1)/\delta\geq 0$, $\gamma(\lambda-1)/\delta\geq 0$, $0<\eta\leq 1$ and $|\lambda|\leq 1/\eta$. If $p\in\mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha+\beta p(z)+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p(z)}\right)\prec\left(\frac{1+z}{1-z}\right)^{\eta\lambda}\left(\alpha+\beta\left(\frac{1+z}{1-z}\right)^{\eta}+\gamma\left(\frac{1-z}{1+z}\right)^{\eta}+\frac{2\delta\eta z}{1-z^2}\right)$$
$$=:h(z),\tag{2.3.3}$$

then $p(z) \prec ((1+z)/(1-z))^{\eta}$.

By taking $\lambda = 1$ and $\gamma = 0$ in Corollary 2.3.9, we have the following result.

Corollary 2.3.10. Let $\alpha\delta, \beta\delta \geq 0$. If $p \in \mathcal{P}_0$ and

$$\alpha p(z) + \beta p(z)^2 + \delta z p'(z) \prec \alpha\left(\frac{1+z}{1-z}\right)^{\eta} + \beta\left(\frac{1+z}{1-z}\right)^{2\eta} + \frac{2\delta\eta z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\eta},$$

then $|\arg p(z)| < \eta\pi/2$.

Remark 6. Corollary 2.3.10 is the result obtained by Ravichandran and Kumar [85] for $\alpha\delta, \beta\delta > 0$.

By taking $\eta = 1$ in the Corollary 2.3.9, we get the following corollary.

Corollary 2.3.11. Let $\alpha\lambda/\delta \geq 0, \beta(\lambda+1)/\delta \geq 0, \gamma(\lambda-1)/\delta \geq 0$ and $|\lambda| \leq 1$. If $p \in \mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\lambda}\left(\alpha + \beta\left(\frac{1+z}{1-z}\right) + \gamma\left(\frac{1-z}{1+z}\right) + \frac{2\delta z}{1-z^2}\right),$$

then $\operatorname{Re} p(z) > 0$.

We assume $\beta = 0$ in Theorem 2.2.3 to obtain the following result. .

Corollary 2.3.12. Let $\alpha\lambda/\delta \geq 0$, $\gamma(\lambda-1)/\delta \geq 0$, $0 < \eta \leq 1$ and $-1 < \eta\lambda \leq 2$. If $p \in \mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^2(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\eta\lambda}\left(\alpha + \gamma\left(\frac{1-z}{1+z}\right)^{\eta} + \frac{2\delta\eta z}{(1-z)^{1-\eta}(1+z)^{1+\eta}}\right), \quad (2.3.4)$$

then $p(z) \prec ((1+z)/(1-z))^{\eta}$.

Proof. Let $q(z) = ((1+z)/(1-z))^{\eta}$ in Theorem 2.2.3, then $Q(z)$ is given by

$$Q(z) = \frac{2\delta\eta z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\eta(\lambda-1)}$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{2z^2}{1-z^2} + \frac{2\eta(\lambda-1)z}{1-z^2},$$

which is bounded below for $-1 < \eta\lambda \leq 2$, so we evaluate it on the boundary of $\mathbb{D}$ and we obtain $\operatorname{Re}(zQ'(z)/Q(z)) \geq 0$. Thus $Q(z)$ is starlike (univalent) in $\mathbb{D}$. Also, equation (2.2.3) gives

$$\operatorname{Re} H(z) = \operatorname{Re}\left(1 + \frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}\left(\frac{1+z}{1-z}\right)^{\eta} + \frac{2z(z+(\lambda-1)\eta)}{1-z^2}\right).$$

A calculation shows that for $z = e^{i\theta}$ $(-\pi \le \theta < \pi)$, we have

$$\operatorname{Re} H(e^{i\theta}) = \frac{\gamma(\lambda-1)}{\delta} \ge 0,$$

when $\alpha\lambda/\delta, \gamma(\lambda-1)/\delta \ge 0$. □

By taking $\eta = 1$ in Corollary 2.3.12, the result follows as:

Corollary 2.3.13. Let $\alpha\lambda/\delta, \gamma(\lambda-1)/\delta \ge 0$ and $-1 < \lambda \le 2$. If $p \in \mathcal{P}_0$ satisfies

$$p^{\lambda}(z)\left(\alpha + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^2(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\lambda}\left(\alpha + \gamma\left(\frac{1-z}{1+z}\right) + \frac{2\delta z}{(1+z)^2}\right),$$

then $\operatorname{Re} p(z) > 0$.

Now, we deal with the argument based results when $q(z) = ((1+z)/(1-z))^{\eta}$ in Theorem 2.2.2 and Theorem 2.2.3. The argument estimation of the function h, when it assumes the expression given on the right side of the subordination (2.3.3), gives the reformulation of Corollary 2.3.9 for the case when $\gamma = 0$ and $0 \le \lambda \le 1/\eta$, which we state below:

Corollary 2.3.14. Let $\alpha, \beta \ge 0$ and $\delta > 0$. If $p \in \mathcal{P}_0$ satisfies

$$\left|\arg\left(p^{\lambda}(z)\left(\alpha + \beta p(z) + \delta\frac{zp'(z)}{p(z)}\right)\right)\right| < \frac{\pi}{2}\zeta,$$

where (i) $\zeta = \eta\lambda$, whenever $0 \le \eta \le 1/2$, (ii) $\zeta = \eta\lambda + 1/2$, whenever $1/2 \le \eta \le 1$ and $\delta\eta \ge \alpha$, (iii) $\zeta = \eta\lambda + 2/\pi \tan^{-1}(\delta\eta/\alpha)$, whenever $1/2 \le \eta \le 1$ and $\alpha \ge \delta\eta$, then

$$|\arg p(z)| < \frac{\pi}{2}\eta.$$

Proof. Here, $h(z)$ is defined as

$$h(z) := \left(\frac{1+z}{1-z}\right)^{\eta\lambda}\left(\alpha + \beta\left(\frac{1+z}{1-z}\right)^{\eta} + \frac{2\delta\eta z}{1-z^2}\right).$$

We evaluate the function $h(z)$ at $z = e^{i\theta}$ so as to obtain the minimum value of $\arg h(e^{i\theta})$

in order to complete the proof. In view of this, let us consider

$$\begin{aligned} h(e^{i\theta}) &= \left(i\cot\frac{\theta}{2}\right)^{\eta\lambda}\left(\alpha+\beta\left(i\cot\frac{\theta}{2}\right)^{\eta}+i\frac{\delta\eta}{\sin\theta}\right) \\ &= \left|\cot\frac{\theta}{2}\right|^{\eta\lambda} e^{\pm i\eta\lambda\pi/2}\left(\alpha+\beta\left|\cot\frac{\theta}{2}\right|^{\eta} e^{\pm i\eta\pi/2}+i\frac{\delta\eta}{\sin\theta}\right). \end{aligned}$$

Note that the '+' sign comes for $0<\theta<\pi$ and the '−' sign comes for $-\pi<\theta<0$. We also notice that the real and imaginary parts of $h(e^{i\theta})$ are, respectively, an even and odd function of θ. Without loss of generality, we consider $0<\theta<\pi$. Then, we have the following

$$\begin{aligned} \arg h(e^{i\theta}) &= \frac{\pi}{2}\eta\lambda+\arg\left(\alpha+\beta\left|\cot\frac{\theta}{2}\right|^{n} e^{i\eta\pi/2}+i\frac{\delta\eta}{\sin\theta}\right) \\ &= \frac{\pi}{2}\eta\lambda+\tan^{-1}\left(\frac{\beta|\cot(\theta/2)|^{\eta}\sin(\eta\pi/2)+\delta\eta/\sin(\theta)}{\alpha+\beta|\cot(\theta/2)|^{\eta}\cos(\eta\pi/2)}\right) \\ &\geq \frac{\pi}{2}\eta\lambda+\tan^{-1}\left(\frac{\beta s^{\eta}\sin(\eta\pi/2)+\delta\eta}{\alpha+\beta s^{\eta}\cos(\eta\pi/2)}\right), \end{aligned}$$

for $\alpha,\beta\geq 0$, $\delta>0$ and where $s=|\cot\theta/2|$ such that $s_1\leq s\leq s_2$, with $s_1\to 0$ and $s_2\to\infty$. To complete the proof, it suffices to show that $\arg(he^{i\theta})\geq\zeta\pi/2$, equivalently which implies $|\arg(p^{\lambda}(z)(\alpha+\beta p(z)+\delta zp'(z)/p(z)))|<\arg(h(z))$, which will yield $p(z)\prec((1+z)/(1-z))^{\eta}$ as $p^{\lambda}(z)(\alpha+\beta p(z)+\delta zp'(z)/p(z))\prec h(z)$ implies $p(z)\prec((1+z)/(1-z))^{\eta}$ from Corollary 2.3.9. Let

$$g(s):=\frac{\beta s^{\eta}\sin(\eta\pi/2)+\delta\eta}{\alpha+\beta s^{\eta}\cos(\eta\pi/2)}$$

and we consider two cases:

Case 1: Let $0<\eta\leq 1/2$, then

$$g(s)\geq\frac{\beta s^{\eta}\sin(\eta\pi/2)}{\alpha+\beta s^{\eta}\cos(\eta\pi/2)}\geq\frac{\beta s_1^{\eta}\sin(\eta\pi/2)}{\alpha+\beta s_1^{\eta}\cos(\eta\pi/2)}\approx 0.$$

This completes the proof for the mentioned range of η in this case.

Case 2: Let $1/2\leq\eta\leq 1$, then

$$g(s)\geq\frac{\beta s^{\eta}\cos(\eta\pi/2)+\delta\eta}{\alpha+\beta s^{\eta}\cos(\eta\pi/2)}=:l(s).$$

Now, to compute the minimum value of $l(s)$, there arises two sub-cases, either $\alpha\leq\delta\eta$ or $\alpha\geq\delta\eta$. Firstly, let us consider $\delta\eta\geq\alpha$, then upon simplification, we arrive at that $l(s)$

attains its minimum value at $s = s_2$ and we obtain

$$l(s) \geq \frac{\beta s_2^\eta \sin(\eta\pi/2) + \delta\eta}{\alpha + \beta s_2^\eta \cos(\eta\pi/2)} \approx 1.$$

Secondly, consider $\alpha \geq \delta\eta$, which yields that $l(s)$ attains its minimum value at $s = s_1$ and we obtain

$$l(s) \geq \frac{\beta s_1^\eta \sin(\eta\pi/2) + \delta\eta}{\alpha + \beta s_1^\eta \cos(\eta\pi/2)} \approx \frac{\delta\eta}{\alpha}.$$

This completes the proof for the mentioned range of η in this case. Therefore, both the cases yield the desired condition $\arg h(e^{i\theta}) \geq \zeta\pi/2$. Hence the result. □

Remark 7. For $1/2 \leq \eta \leq 1$ and by taking $\beta = 0$ in Corollary 2.3.14, we get the result obtained by Cho et al. [18], when restricting the range of η.

In the following result, we derive the argument relation between $p^\lambda(z)(\alpha + \gamma/p(z) + \delta z p'(z)/p^2(z))$ and $h(z)$, when h assumes the expression defined by the right side of the subordination (2.3.4), such that (2.3.4) holds for $0 < \eta \leq 1$. We further assume $-1 < \lambda \leq 0$.

Corollary 2.3.15. Let $-1 < \lambda \leq 0$, $\alpha \geq -\delta > 0$ and $\gamma \geq 0$. If $p \in \mathcal{P}_0$ satisfies

$$\left|\arg\left(p^\lambda(z)\left(\alpha + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^2(z)}\right)\right)\right| < \frac{\pi}{2}\zeta, \tag{2.3.5}$$

where (i) $\zeta = -\eta\lambda$ whenever $0 < \eta \leq 1/2$, (ii) $\zeta = -\eta\lambda + (2/\pi)\tan^{-1}(-\delta\eta\cos(\eta\pi/2)/\alpha)$ whenever $1/2 \leq \eta \leq 1$, then

$$|\arg p(z)| < \frac{\pi}{2}\eta.$$

Proof. A calculation shows that

$$\begin{aligned} h(e^{i\theta}) &= (i\cot\theta/2)^{\eta\lambda}\left(\alpha + \gamma(i\cot\theta/2)^{-\eta} + i\frac{\delta\eta}{\sin\theta}(i\cot\theta/2)^{-\eta}\right) \\ &= |\cot\theta/2|^{\eta\lambda} e^{\pm i\eta\lambda\pi/2}\Big(\alpha + \gamma|\cot\theta/2|^{-\eta} e^{\mp i\eta\pi/2} \\ &\quad + \frac{\delta\eta}{\sin\theta}|\cot\theta/2|^{-\eta} e^{i\pi/2(1\mp\eta)}\Big). \end{aligned}$$

Note that '+' sign comes for $0 < \theta < \pi$ and the '−' sign comes for $-\pi < \theta < 0$. Also, we observe that the real and imaginary part of $h(e^{i\theta})$ is an even and odd function of

θ, respectively. So we consider here $-\pi<\theta<0$, without loss of generality. Then, we have the following

$$\begin{aligned}\arg h(e^{i\theta}) &= -\frac{\pi}{2}\eta\lambda+\tan^{-1}\left(\frac{\gamma|\cot(\theta/2)|^{-\eta}\sin(\eta\pi/2)+\frac{\delta\eta}{\sin\theta}\cos(\eta\pi/2)}{\alpha+\gamma|\cot(\theta/2)|^{-\eta}\cos(\eta\pi/2)-\frac{\delta\eta}{\sin\theta}\sin(\eta\pi/2)}\right)\\ &\geq -\frac{\pi}{2}\eta\lambda+\tan^{-1}\left(\frac{\gamma s^{-\eta}\sin(\eta\pi/2)-\delta\eta\cos(\eta\pi/2)}{\alpha+\gamma s^{-\eta}\cos(\eta\pi/2)}\right),\end{aligned}$$

whenever $\alpha\geq-\delta>,\gamma\geq 0$ and where $s=|\cot(\theta/2)|$ such that $s_1\leq s\leq s_2$, with $s_1\to 0$ and $s_2\to\infty$. Let $g(s):=\left(\frac{\gamma s^{-\eta}\sin(\eta\pi/2)-\delta\eta\cos(\eta\pi/2)}{\alpha+\gamma s^{-\eta}\cos(\eta\pi/2)}\right)$. Now, there arises two cases:

Case 1: Let $0<\eta\leq 1/2$, then

$$g(s)\geq\left(\frac{\gamma s^{-\eta}\sin(\eta\pi/2)}{\alpha+\gamma s^{-\eta}\cos(\eta\pi/2)}\right)\geq\left(\frac{\gamma s_2^{-\eta}\sin(\eta\pi/2)}{\alpha+\gamma s_2^{-\eta}\cos(\eta\pi/2)}\right)\approx 0.$$

Case 2: Let $1/2\leq\eta\leq 1$, then

$$g(s)\geq\left(\frac{\gamma s^{-\eta}\cos(\eta\pi/2)+\delta\eta\cos(\eta\pi/2)}{\alpha+\gamma s^{-\eta}\cos(\eta\pi/2)}\right)=:l(s).$$

Since $\alpha\geq-\delta$, then

$$l(s)\geq\left(\frac{\gamma s_2^{-\eta}\cos(\eta\pi/2)+\delta\eta\cos(\eta\pi/2)}{\alpha+\gamma s_2^{-\eta}\cos(\eta\pi/2)}\right)\approx\frac{-\delta\eta\cos(\eta\pi/2)}{\alpha}.$$

Therefore, from both the cases, we get $\arg(h(e^{i\theta}))\geq\zeta\pi/2$, where ζ is as given in the hypothesis. We observe that condition (2.3.5) concludes that the subordination (2.3.4) holds. Also, the hypothesis of Corollary 2.3.12 gets satisfied, as a result we get $p(z)\prec((1+z)/(1-z))^{\eta}$, equivalently $|\arg(p(z))|<\eta\pi/2$. This completes the proof. □

For the following couple of results, take $q(z)=e^{\mu z}$ in Theorem 2.2.2 and Theorem 2.2.3.

Corollary 2.3.16. Let $\mu(\neq 0)$ be a real number such that $|\mu|\leq 1$ and $p(z)\in\mathcal{P}_0$.

(a) Let $\delta>0$, $|\lambda\mu|\leq 1$ and $\alpha\lambda\geq-(A+B)$, where we have $A=\beta(\lambda+1)/e^{|\mu|}\geq 0$ and $B=\gamma(\lambda-1)/e^{|\mu|}\geq 0$. If p satisfies

$$p^{\lambda}(z)\left(\alpha+\beta p(z)+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p(z)}\right)\prec e^{\lambda\mu z}\left(\alpha+\beta e^{\mu z}+\gamma e^{-\mu z}+\delta\mu z\right),$$

then $p(z)\prec e^{\mu z}$.

(b) Let $|(\lambda-1)\mu|\leq 1$, $\alpha\lambda/\delta\geq 0$ and $\gamma(\lambda-1)/\delta\geq 0$.

If p satisfies

$$p^{\lambda}(z)\left(\alpha+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right)\prec e^{\mu\lambda z}\left(\alpha+\frac{\gamma}{e^{\mu z}}+\delta\frac{\mu z}{e^{\mu z}}\right),$$

then $p(z)\prec e^{\mu z}$.

Proof. (a) Let $q(z)=e^{\mu z}$ in Theorem 2.2.2. Then we have $Q(z)=\delta\mu z e^{\lambda\mu z}$, furthermore by taking $z=x+iy$ $(x^2+y^2<1)$, we get

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right)=1+\lambda\mu x.$$

A simple computation shows that $Q(z)$ is starlike (univalent) in $\mathbb{D}$ whenever $|\lambda\mu|\le 1$. As a result of the fact $\operatorname{Re} e^{\mu z}, \operatorname{Re} e^{-\mu z}>1/e^{|\mu|}$, we deduce the following

$$\begin{aligned}
&\operatorname{Re}\left(\frac{\alpha\lambda}{\delta}+\frac{\beta(\lambda+1)}{\delta}q(z)+\frac{\gamma(\lambda-1)}{\delta}\frac{1}{q(z)}+(\lambda-1)\frac{zq'(z)}{q(z)}+\left(1+\frac{zq''(z)}{q'(z)}\right)\right)\\
&=\operatorname{Re}\left(1+\lambda\mu z+\frac{\alpha\lambda}{\delta}+\frac{\beta(\lambda+1)}{\delta}e^{\mu z}+\frac{\gamma(\lambda-1)}{\delta}e^{-\mu z}\right)\\
&>1+\lambda\mu x+\frac{\alpha\lambda}{\delta}+\frac{\beta(\lambda+1)}{\delta}\frac{1}{e^{|\mu|}}+\frac{\gamma(\lambda-1)}{\delta}\frac{1}{e^{|\mu|}}=:S(x),
\end{aligned}$$

for $A,B>0$ and $\delta\ge 0$. For the given range of $|\lambda\mu|$, here we deal with two cases:

Case 1: If $-1\le\lambda\mu\le 0$, then $1+\lambda\mu x>1+\lambda\mu\ge 0$ and we obtain

$$S(x)>\frac{\alpha\lambda}{\delta}+\frac{\beta(\lambda+1)}{\delta}\frac{1}{e^{|\mu|}}+\frac{\gamma(\lambda-1)}{\delta}\frac{1}{e^{|\mu|}}. \tag{2.3.6}$$

Case 2: If $0\le\lambda\mu\le 1$, then $1+\lambda\mu x>1-\lambda\mu\ge 0$ and we obtain the same equation (2.3.6) as in the case 1.

Furthermore, we see that $S(x)\ge 0$ provided $\alpha\lambda\ge(\gamma-\beta-\lambda(\beta+\gamma))/e^{|\mu|}$, which is trivial in view of the conditions stated in the hypothesis. Thus the conditions of Theorem 2.2.2 now holds, hence the result.

(b) Let $\beta=0$ and $q(z)=e^{\mu z}$ in Theorem 2.2.3. Then, $\operatorname{Re}(zQ'(z)/Q(z))=1+(\lambda-1)\mu x$. So, we get Q is starlike (univalent) in $\mathbb{D}$ whenever $|\mu(\lambda-1)|\le 1$. Consider

$$\begin{aligned}
&\operatorname{Re}\left(\frac{\gamma(\lambda-1)}{\delta}+\frac{\alpha\lambda}{\delta}q(z)+\frac{\beta(\lambda+1)}{\delta}q^2(z)+(\lambda-2)\frac{zq'(z)}{q(z)}+\left(1+\frac{zq''(z)}{q'(z)}\right)\right)\\
&=\operatorname{Re}\left(\frac{\gamma(\lambda-1)}{\delta}+\frac{\alpha\lambda}{\delta}e^{\mu z}+(\lambda-2)\mu z+1+\mu z\right)\\
&>1+(\lambda-1)\mu x+\frac{\gamma(\lambda-1)}{\delta}+\frac{\alpha\lambda}{\delta}\frac{1}{e^{|\mu|}}=:S(x),
\end{aligned}$$

whenever $\alpha\lambda/\delta \geq 0$. Now we observe that $S(x) \geq 0$ provided $\gamma(\lambda-1)/\delta \geq 0$ and which completes the proof. □

By taking $\mu = 1$ in Corollary 2.3.16 (a) and (b), we obtain the following result, respectively.

Corollary 2.3.17. Let $p \in \mathcal{P}_0$.

(a) Let $\delta > 0$, $|\lambda| < 1$ and $\alpha\lambda \geq -(A+B)$, where $A = \beta(\lambda+1)/e \geq 0$ and $B = \gamma(\lambda-1)/e \geq 0$. If p satisfies

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec e^{\lambda z}\left(\alpha + \beta e^{z}, +\gamma e^{-z} + \delta z\right),$$

then $p(z) \prec e^{z}$.

(b) Let $|\lambda - 1| \leq 1$, $\alpha\lambda/\delta \geq 0$ and $\gamma(\lambda-1) \geq 0$. If p satisfies

$$p^{\lambda}(z)\left(\alpha + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^{2}(z)}\right) \prec e^{\lambda z}\left(\alpha + \gamma e^{-z} + \delta z e^{-z}\right),$$

then $p(z) \prec e^{z}$.

In Theorem 2.2.2 and Theorem 2.2.3, we take $q(z) = \sqrt{1+z}$ to obtain the following result:

Corollary 2.3.18. Let $p \in \mathcal{P}_0$.

(a) If $-2 \leq \lambda \leq 2$, $\gamma(\lambda-1) \geq 0$, $\beta(\lambda+1) \geq 0$, $-(2\sqrt{2}\gamma + \delta)/4 < \alpha \leq -3\gamma/(2\sqrt{2})$, $\delta > \max(0; \sqrt{2}\gamma)$ and p satisfies

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec (1+z)^{\lambda/2}\left(\alpha + \beta\sqrt{1+z} + \frac{\gamma}{\sqrt{1+z}} + \frac{\delta z}{2(1+z)}\right),$$

then $p(z) \prec \sqrt{1+z}$.

(b) If $-1 \leq \lambda \leq 3$, $\alpha\lambda/\delta \geq 0$, $\beta(\lambda+1)/\delta \geq 0$, $-1/4 < \gamma/\delta \leq 0$ and p satisfies

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p^{2}(z)}\right) \prec (1+z)^{\lambda/2}\left(\alpha + \beta\sqrt{1+z} + \frac{\gamma}{\sqrt{1+z}} + \frac{\delta z}{2(1+z)^{3/2}}\right),$$

then $p(z) \prec \sqrt{1+z}$.

Proof. (a) Now, to achieve the desired result we apply Theorem 2.2.2 with $q(z) = \sqrt{1+z}$. So we have $Q(z) = \delta z(1+z)^{\lambda/2-1}/2$, furthermore which yields the expression

$zQ'(z)/Q(z) = 1 + (\lambda/2-1)(z/(1+z))$. Since $zQ'(z)/Q(z)$ is bounded only if $\lambda \leq 2$, so to find the upper bound we evaluate it on the boundary $z = e^{i\theta}$ $(-\pi \leq \theta < \pi)$, which eventually yields $\mathrm{Re}(zQ'(z)/Q(z)) = (\lambda+2)/4 \geq 0$ for $\lambda \geq -2$. Clearly Q is starlike (univalent) in $\mathbb{D}$. Also, we have

$$\begin{aligned}
&\mathrm{Re}\left(\frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}q(z) + \frac{\gamma(\lambda-1)}{\delta}\frac{1}{q(z)} + (\lambda-1)\frac{zq'(z)}{q(z)} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right)\\
&= \mathrm{Re}\left(1 + \frac{\alpha\lambda}{\delta} + \frac{\beta(\lambda+1)}{\delta}\sqrt{1+z} + \frac{\gamma(\lambda-1)}{\delta}\frac{1}{\sqrt{1+z}} + \left(\frac{\lambda}{2}-1\right)\frac{z}{1+z}\right)\\
&\geq 1 + \frac{\alpha\lambda}{\delta} + \frac{\gamma(\lambda-1)}{\delta}\frac{1}{\sqrt{2}} + \frac{\lambda}{4} - \frac{1}{2} = \lambda\left(\frac{1}{4} + \frac{\alpha}{\delta} + \frac{\gamma}{\sqrt{2}\delta}\right) + \frac{1}{2} - \frac{\gamma}{\sqrt{2}\delta} =: S,
\end{aligned}$$

since $\mathrm{Re}(\sqrt{1+z}) > 0$, $\mathrm{Re}(1/\sqrt{1+z}) > 1/\sqrt{2}$, $\beta(\lambda+1)/\delta$ and $\gamma(\lambda-1)/\delta \geq 0$. Now, to complete the proof, we find the conditions on the parameters such that $S \geq 0$. For this, it suffices to show that

$$\lambda \geq \frac{\frac{\gamma}{\sqrt{2}\delta} - \frac{1}{2}}{\frac{\alpha}{\delta} + \frac{\gamma}{\sqrt{2}\delta} + \frac{1}{4}}, \tag{2.3.7}$$

which is a valid expression when $\alpha > -(2\sqrt{2}\gamma + \delta)/4$ and $\delta > 0$. Also, from the inequality $-2 \leq \lambda \leq 2$, we have $\alpha \leq -3\gamma/(2\sqrt{2})$. So equation (2.3.7) holds true. The derived range of α is meaningful only if $\delta \geq \sqrt{2}\gamma$, which establishes part (a).

(b) The result follows from Theorem 2.2.3 by taking $q(z) = \sqrt{1+z}$. Since $-1 \leq \lambda \leq 3$, on the similar lines of part (a), we have $Q(z) = \delta(1+z)^{(\lambda-3)/2}z/2$ is starlike (univalent) in $z \in \mathbb{D}$. Also, we have

$$\begin{aligned}
&\mathrm{Re}\left(1 + \frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}q(z) + \frac{\beta(\lambda+1)}{\delta}q^2(z) + (\lambda-2)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right)\\
&= \mathrm{Re}\left(1 + \frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}\sqrt{1+z} + \frac{\beta(\lambda+1)}{\delta}(1+z) + \frac{\lambda}{4} + (\lambda-3)\frac{z}{2(1+z)}\right)\\
&\geq 1 + \frac{\gamma(\lambda-1)}{\delta} + \frac{\lambda-3}{4} = \lambda\left(\frac{\gamma}{\delta} + \frac{1}{4}\right) - \frac{\gamma}{\delta} + \frac{1}{4} =: S,
\end{aligned}$$

for the given range of constants in the hypothesis (b). To complete the proof, we now show $S \geq 0$, which holds if

$$\lambda \geq \frac{\frac{\gamma}{\delta} - \frac{1}{4}}{\frac{\gamma}{\delta} + \frac{1}{4}}.$$

The above inequality holds true as $-1/4 < \gamma/\delta \leq 0$ and this completes the proof. □

The following lemmas are required to prove some subsequent results involving

Janowski function.

Lemma 2.3.19. For $-1 \le B < A \le 1$, let $J(z) = (1+Az)/(1+Bz)$ satisfies $\min_{|z|\le 1} \operatorname{Re} J(z) = (1-A)/(1-B)$ and $\min_{|z|\le 1} \operatorname{Re}(1/J(z)) = (1+B)/(1+A)$.

Proof. Consider

$$\operatorname{Re}(J(e^{i\theta})) = \frac{1+(A+B)\cos\theta + AB}{1+A^2+2A\cos\theta} =: \tilde{J}(\theta), \quad -\pi < \theta \le \pi,$$

which attains minimum value at $\theta = \pi$ by the second derivative test and the minimum value is given by $\tilde{J}(\pi) = (1-A)/(1-B)$. Now, consider $J(-z) = (1-Az)/(1-Bz)$. In view of the fact that the images of unit disk under $J(z)$ and $J(-z)$ are same, thus we obtain

$$\min_{|z|\le 1} \operatorname{Re}\left(\frac{1+Bz}{1+Az}\right) = \frac{1+B}{1+A},$$

as in this case, $-A$ takes the role of B and $-B$ takes the role of A. This completes the proof. □

Similarly, we can find the minimum and maximum values of the real part of the functions $(1-Bz)/(1+Bz)$ and $z(A-B)/((1+Az)(1+Bz))$, respectively, as given in the following lemmas:

Lemma 2.3.20. For $-1 \le B < 1$, the minimum of real part of $f(z) = (1-Bz)/(1+Bz)$ is $(1-|B|)/(1+|B|)$.

Lemma 2.3.21. For $-1 < B < A \le 1$. Consider $f(z) = (A-B)z/((1+Az)(1+Bz))$, then $\max_{|z|\le 1} \operatorname{Re} f(z) = (A-B)/(1+A)(1+B)$, whenever $(1+AB)(1-A)(1-B) > 8AB$ and $\min_{|z|\le 1} \operatorname{Re} f(z) = (B-A)/((1-A)(1-B))$, whenever $(1+AB)(1+A)(1+B) > 8AB$.

We employ the above Lemmas in our following results dealing with Janowski function in Theorem 2.2.2 and Theorem 2.2.3.

Corollary 2.3.22. Let $p \in \mathcal{P}_0$, $\alpha/\delta \ge (B-A)/((1+A)(1+B))$, $(1+AB)(1-A)(1-B) > 8AB$, where $-1 < B < A \le 1$, and

(a) If $0 \le \lambda \le 1$, $\gamma\delta \le 0$, $\beta/\delta \ge (A-B)(1-B)/((1-A^2)(1+B))$ and p satisfies

$$p^{\lambda}(z)\left(\alpha + \beta p(z) + \frac{\gamma}{p(z)} + \delta\frac{zp'(z)}{p(z)}\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\lambda}\left(\alpha + \beta\left(\frac{1+Az}{1+Bz}\right) + \gamma\left(\frac{1+Bz}{1+Az}\right) + \delta\frac{(A-B)z}{(1+Az)(1+Bz)}\right),$$

then $p(z) \prec (1+Az)/(1+Bz)$.

(b) If $\alpha\delta \geq 0$, $0 \leq \lambda \leq 2$, $(B-A)/((1+A)(1+B)) \leq \gamma/\delta \leq 2(B-A)/((1+A)(1+B))$ and p satisfies

$$p^{\lambda}(z)\left(\alpha+\frac{\gamma}{p(z)}+\delta\frac{zp'(z)}{p^2(z)}\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\lambda}\left(\alpha+\gamma\left(\frac{1+Bz}{1+Az}\right)+\delta\frac{(A-B)z}{(1+Az)^2}\right),$$

then $p(z) \prec (1+Az)/(1+Bz)$.

Proof. (a) The result is followed by taking $q(z) = (1+Az)/(1+Bz)$ in Theorem 2.2.2. We have $Q(z) = (\delta z(A-B))/((1+Az)^{1-\lambda}(1+Bz)^{1+\lambda})$, and

$$\frac{zQ'(z)}{Q(z)} = 1+(\lambda-1)\frac{Az}{1+Az}-(1+\lambda)\frac{Bz}{1+Bz} =: K(z).$$

According to Remark 1, we let $-1 \leq \lambda \leq 1$, which implies $K(e^{i\theta}) \geq 0$. So, clearly $Q(z)$ is starlike (univalent) in $\mathbb{D}$. Also, we have

$$\begin{aligned}
&\operatorname{Re}\left(\frac{\alpha\lambda}{\delta}+\frac{\beta(\lambda+1)}{\delta}q(z)+\frac{\gamma(\lambda-1)}{\delta}\frac{1}{q(z)}+(\lambda-1)\frac{zq'(z)}{q(z)}+\left(1+\frac{zq''(z)}{q'(z)}\right)\right)\\
&=\operatorname{Re}\left(\frac{\alpha\lambda}{\delta}+\frac{\beta(\lambda+1)}{\delta}\left(\frac{1+Az}{1+Bz}\right)+\frac{\gamma(\lambda-1)}{\delta}\left(\frac{1+Bz}{1+Az}\right)\right.\\
&\qquad\left.+(\lambda-1)\frac{(A-B)z}{(1+Az)(1+Bz)}+\frac{1-Bz}{1+Bz}\right) =: S(z).
\end{aligned}$$

Now, Lemma 2.3.19–Lemma 2.3.21 yield

$$\begin{aligned}
S(z) &\geq \frac{\alpha\lambda}{\delta}+\frac{\beta(\lambda+1)}{\delta}\left(\frac{1-A}{1-B}\right)+\frac{\gamma(\lambda-1)}{\delta}\left(\frac{1+B}{1+A}\right)+(\lambda-1)\left(\frac{A-B}{(1+A)(1+B)}\right)+\left(\frac{1-|B|}{1+|B|}\right)\\
&=\lambda\left(\frac{\alpha}{\delta}+\frac{A-B}{(1+A)(1+B)}\right)+\frac{\beta(\lambda+1)}{\delta}\left(\frac{1-A}{1-B}\right)-\left(\frac{A-B}{(1+A)(1+B)}\right)+\frac{1-|B|}{1+|B|}\\
&\quad+\frac{\gamma(\lambda-1)}{\delta}\left(\frac{1+B}{1+A}\right),
\end{aligned} \tag{2.3.8}$$

for $\beta\delta \geq 0$ and $\gamma\delta \leq 0$. To complete the proof, it suffices to prove the second condition of Theorem 2.2.2. For this, we show equation (2.3.8) greater than or equal to 0, which is possible when $0 \leq \lambda \leq 1$, $\alpha/\delta \geq (B-A)/(1+A)(1+B)$ and $\beta/\delta \geq (A-B)(1-B)/((1-A^2)(1+B))$. Now, the result follows immediately.

(b) The result is obtained by taking $\beta = 0$ and $q(z) = (1+Az)/(1+Bz)$ in Theorem 2.2.3. So, we have

$$Q(z) = \frac{\delta z(A-B)}{(1+Az)^{2-\lambda}(1-Bz)^{\lambda}},$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + (\lambda - 2)\left(\frac{Az}{1+Az}\right) - \frac{\lambda Bz}{1+Bz}, \qquad 0 \le \lambda \le 2, \tag{2.3.9}$$

which is clearly bounded below for the mentioned range of λ. Thus computing equation (2.3.9) on the boundary of $\mathbb{D}$, we get $Q(z)$ is starlike (univalent) for the given range of λ. Also, from condition (2) of Theorem 2.2.3, we have

$$\operatorname{Re} H(z) := \operatorname{Re}\left(\frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}\left(\frac{1+Az}{1+Bz}\right) + (\lambda-2)\left(\frac{A-Bz}{(1+Az)(1+Bz)}\right) + \left(\frac{1-Bz}{1+Bz}\right)\right).$$

Now, by using Lemma 2.3.19–Lemma 2.3.21, we deduce

$$\begin{aligned}\operatorname{Re} H(z) &\ge \frac{\gamma(\lambda-1)}{\delta} + \frac{\alpha\lambda}{\delta}\left(\frac{1-A}{1-B}\right) + \left(\frac{1-|B|}{1+|B|}\right) + (\lambda-2)\left(\frac{A-B}{(1+A)(1+B)}\right) \quad (\alpha\delta \ge 0)\\ &= \lambda\left(\frac{\gamma}{\delta} + \frac{A-B}{(1+A)(1+B)}\right) + \frac{\alpha\lambda}{\delta}\left(\frac{1-A}{1-B}\right) + \frac{1-|B|}{1+|B|} - \left(\frac{\gamma}{\delta} + \frac{2(A-B)}{(1+A)(1+B)}\right). \quad (2.3.10)\end{aligned}$$

To achieve the desired result it is required to show that equation (2.3.10) is greater than or equal to 0, which is possible when

$$\frac{B-A}{(1+A)(1+B)} \le \frac{\gamma}{\delta} \le \frac{2(B-A)}{(1+A)(1+B)}.$$

This completes the proof. □

With $\lambda = 1$ and $q(z) = \phi_{PAR}(z) := 1 + (2/\pi^2)(\log((1+\sqrt{z})/(1-\sqrt{z})))^2$ in Theorem 2.2.2, the result follows as:

Corollary 2.3.23. Let $\beta/\delta \ge \max(0; -\alpha/\delta)$. If $p \in \mathcal{P}_0$ satisfies

$$\begin{aligned}&\alpha p(z) + \beta p(z)^2 + \delta z p'(z)\\ &\prec \left(1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\left(\alpha + \beta\left(1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right) + \frac{4\delta}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)\right).\end{aligned}$$

Proof. As we know that $\phi_{PAR}(z)$ is a convex function, we obtain $Q(z) = \delta z q'(z)$ is starlike (univalent) in $\mathbb{D}$, by Alexander's Theorem, which is stated in Theorem B. Also, from condition (2) of Theorem 2.2.2, we have

$$\operatorname{Re} H(z) := \operatorname{Re}\left(\frac{\alpha}{\delta} + \frac{2\beta}{\delta}\left(1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right) + 1 + \frac{zq''(z)}{q'(z)}\right).$$

Now let us assume $g(\theta) = \frac{1}{2} + \frac{2}{\pi^2}\log(\cot(\theta/4)))^2$, and evaluating the above equation at

$z = e^{i\theta}$ $(-\pi < \theta \le \pi)$, we obtain

$$\begin{aligned} \operatorname{Re} H(e^{i\theta}) &\ge \operatorname{Re}\left(\frac{\alpha}{\delta} + \frac{2\beta}{\delta}\left(1 + \frac{2}{\pi^2}(\log(i\cot(\theta/4)))^2\right)\right) \\ &=: \frac{\alpha}{\delta} + \frac{2\beta}{\delta} g(\theta). \end{aligned} \tag{2.3.11}$$

For $\theta = \pi$, a simple calculation shows that

$$\begin{aligned} g''(\theta) = &\csc^2(\theta/4)\log(\cot(\theta/4))/(4\pi^2) + (\csc^2(\theta/4)\sec(\theta/4)^2)/(4\pi^2) \\ &- (\log(\cot(\theta/4)))\sec^2(\theta/4)/(4\pi^2) > 0. \end{aligned}$$

On substituting $g(\pi) = 1/2$, the minimum value of $g(\theta)$ in equation (2.3.11), we get $\operatorname{Re} H(e^{i\theta}) \ge 0$. Hence, both the conditions of Theorem 2.2.2 get satisfied. The result follows now. □

2.4 Applications to Starlike Functions

On substituting $p(z) = zf'(z)/f(z)$ in Corollaries 2.3.3, 2.3.11, 2.3.5 and 2.3.13 respectively, the result follows as:

***Example* 2.** Let $f \in \mathcal{S}$.

(i)(a) $1 + 2\beta/\delta > 0$, $\gamma\delta \le 0$, $\alpha\delta \ge 0$ and $0 \le \lambda \le 1$.

(b) $\alpha\lambda/\delta \ge 0$, $\beta(\lambda+1)/\delta \ge 0$, $\gamma(\lambda-1)/\delta \ge 0$ and $|\lambda| \le 1$.

If either (a) or (b) holds and f satisfies

$$\begin{aligned} &\left(\frac{zf'(z)}{f(z)}\right)^\lambda \left(\alpha + \gamma\frac{f(z)}{zf'(z)} + (\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) \\ &\prec \left(\frac{1+z}{1-z}\right)^\lambda \left(\alpha + \beta\left(\frac{1+z}{1-z}\right) + \gamma\left(\frac{1-z}{1+z}\right) + \frac{2\delta z}{1-z^2}\right), \end{aligned}$$

then $f \in \mathcal{S}^*$.

(ii)(a) $1 + 2\alpha/\delta > 0$, $\gamma\delta \ge 0$ and $1 \le \lambda \le 2$.

(b) $\alpha\lambda/\delta$, $\gamma(\lambda-1)/\delta \ge 0$ and $-1 \le \lambda \le 2$.

If either (a) or (b) holds and f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^\lambda \left(\alpha + \gamma\frac{f(z)}{zf'(z)} + \delta\left(\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1\right)\right) \prec \left(\frac{1+z}{1-z}\right)^\lambda \left(\alpha + \gamma\frac{1-z}{1+z} + \frac{2\delta z}{(1+z)^2}\right),$$

then $f \in \mathcal{S}^*$.

By taking $\alpha = \gamma = 0$ and $\lambda = 1$ in Example (2)(i)(a), we have the following result of Ravichandran and Kumar [85].

Remark 8. [85] Let $1 + 2\beta/\delta > 0$. If $f \in \mathcal{S}$ and

$$\frac{zf'(z)}{f(z)}\left((\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) \prec \beta\left(\frac{1+z}{1-z}\right)^2 + \frac{2\delta z}{(1-z)^2},$$

then $f \in \mathcal{S}^*$.

By taking $\lambda = \alpha = \gamma = 0$ and $\delta = 1$ in Example (2)(ii)(b), we have the following result of Obradović and Tuneski [79].

Remark 9. If $f \in \mathcal{S}$ and

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{2z}{(1+z)^2},$$

then $f \in \mathcal{S}^*$.

By taking $p(z) = 2f'(z)/(2+z)$, $\alpha = \gamma = 0$ and $\lambda = 1$ in the Corollary 2.3.3, we get the sufficient condition for f to belong to the class $\mathcal{K}$, as given below.

***Example* 3.** Let $1 + 2\beta/\delta > 0$. If $f \in \mathcal{S}$ satisfies

$$\frac{2f'(z)}{(2+z)^2}(2\beta f'(z) - \delta z) + \frac{2\delta z f''(z)}{2+z} \prec \beta\left(\frac{1+z}{1-z}\right)^2 + \frac{2\delta z}{(1-z)^2}.$$

then $f \in \mathcal{K}$.

As an application of Corollary 2.3.17, where we assume $p(z) = zf'(z)/f(z)$, we deduce the following result.

***Example* 4.** Let $f \in \mathcal{S}$.

(i) If $\delta > 0$, $\alpha\lambda \geq -(A+B)$, where $A = \beta(\lambda+1)/e \geq 0$, $B = \gamma(\lambda-1)/e \geq 0$ and $|\lambda| \leq 1$ and f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \gamma\frac{f(z)}{zf'(z)} + (\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) \prec e^{\lambda z}(\gamma e^{-z} + \alpha + \beta e^{z} + \delta z),$$

then $f \in \mathcal{S}^*_e$.

(ii) If $|\lambda-1|\leq 1$, $\alpha\lambda/\delta\geq 0$ and $\gamma(\lambda-1)\geq 0$ and f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha+\beta\frac{zf'(z)}{f(z)}+\gamma\frac{f(z)}{zf'(z)}+\delta\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}-1\right)\right)$$
$$\prec e^{\lambda z}\left(\alpha+\beta e^{z}+\gamma e^{-z}+\delta z e^{-z}\right),$$

then $f\in\mathcal{S}^*_e$.

The above result deals with the sufficient conditions for analytic functions to belong to the class $\mathcal{S}^*_e$. Now, letting $p(z)=zf'(z)/f(z)$ in the Corollaries 2.3.4 and 2.3.6, the couple of results follow, respectively.

***Example* 5.** Let $f\in\mathcal{S}$.

(i) If $1+\beta/\delta>0$, $\gamma\delta\leq 0$, $-1+(2\alpha+\beta)/\delta\geq 0$ and $0\leq\lambda\leq 1$ and f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha+\gamma\frac{f(z)}{zf'(z)}+(\beta-\delta)\frac{zf'(z)}{f(z)}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right)$$
$$\prec\left(\frac{1}{1-z}\right)^{\lambda}\left(\alpha+\beta\frac{1}{1-z}+\gamma(1-z)+\frac{\delta z}{1-z}\right),$$

then $f\in\mathcal{S}^*(1/2)$.

(ii) If $1+\alpha/\delta>0$, $(2\gamma+\alpha)/\delta\geq 1$ and $1\leq\lambda\leq 2$ and f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha+\gamma\frac{f(z)}{zf'(z)}+\delta\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}-1\right)\right)\prec\left(\frac{1}{1-z}\right)^{\lambda}\left(\alpha+(1-z)\gamma+\delta z\right),$$

then $f\in\mathcal{S}^*(1/2)$.

The above result gives the sufficient conditions for analytic functions to be starlike of order 1/2. Let $\lambda=\alpha=\gamma=0$, $\delta=\beta=1$ and $\lambda=\alpha=\delta=1$, $\gamma=0$, respectively in Example (5)(i) and (ii), we obtain the following known result of Marx [64] and Strohh*ä*cker [109].

Remark 10. A convex function is starlike of order 1/2.

Recall, a result by Mocanu [69] for $\rho-$convex functions, furnished below in the remark and is needed for the upcoming result.

Remark 11. Let ρ be an arbitrary real number, and suppose that $f(z)$ is $\rho-$convex. If $\rho\geq 1$, then $f(z)$ is convex.

By taking $\lambda = \alpha = \gamma = 0$, $\beta = 1$ and $\delta = \rho$ in Example 5(i), we deduce the result as follows:

Corollary 2.4.1. Let $1/\rho \geq 1$ and if f satisfies

$$(1-\rho)\frac{zf'(z)}{f(z)} + \rho\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+\rho z}{1-z},$$

then $zf'(z)/f(z) \prec 1/(1-z)$.

The Remark 10, Corollary 2.4.1 and Remark 11 yield the following result.

Corollary 2.4.2. A $\rho-$convex function is starlike of order $1/2$ for $\rho > 0$.

As an application of Corollary 2.3.18(a) and (b), where we take $p(z) = zf'(z)/f(z)$ to obtain the following results, respectively, in Example 6.

***Example* 6.** Let $f \in \mathcal{S}$.

(i) If $-2 \leq \lambda \leq 2$, $\gamma(\lambda - 1) \geq 0$, $\beta(\lambda + 1) \geq 0$, $\delta > \max(0, \sqrt{2}\gamma)$ and $-(2\sqrt{2}\gamma + \delta)/4 < \alpha \leq -3\gamma/(2\sqrt{2})$ and f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \gamma\frac{f(z)}{zf'(z)} + (\beta-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)$$
$$\prec (1+z)^{\lambda/2}\left(\frac{\gamma}{\sqrt{1+z}} + \alpha + \beta\sqrt{1+z} + \delta\left(\frac{z}{2(1+z)}\right)\right),$$

then $f \in \mathcal{SL}^*$.

(ii) If $-1 \leq \lambda \leq 3$, $\alpha\lambda/\delta \geq 0$, $\beta(\lambda+1)/\delta \geq 0$ and $-1/4 < \gamma/\delta \leq 0$ and f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha + \beta\frac{zf'(z)}{f(z)} + \gamma\frac{f(z)}{zf'(z)} + \delta\left(\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1\right)\right)$$
$$\prec (1+z)^{\lambda/2}\left(\alpha + \beta\sqrt{1+z} + \frac{\gamma}{\sqrt{1+z}} + \frac{\delta z}{2(1+z)^{3/2}}\right),$$

$f \in \mathcal{SL}^*$.

The above result provides the sufficient conditions for functions in $\mathcal{S}$ to belong to the class $\mathcal{SL}^*$. Now, by taking $p(z) = zf'(z)/f(z)$ in Corollary 2.3.22(a) and (b), then we deduce the following, which serves as the sufficient conditions for functions in $\mathcal{S}$ to belong to the class $\mathcal{S}^*[A,B]$, involving Janowski function.

***Example* 7.** Let $f \in \mathcal{S}$ and $(1+AB)(1-A)(1-B) > 8AB$ $(-1 < B < A \leq 1)$.

(i) If $0 \le \lambda \le 1$, $\alpha/\delta \ge (B-A)/((1+A)(1+B))$ and $\beta/\delta \ge (A-B)(1-B)/((1-A^2)(1+B))$ and f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha+\gamma\frac{f(z)}{zf'(z)}+(\beta-\delta)\frac{zf'(z)}{f(z)}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right)$$
$$\prec\left(\frac{1+Az}{1+Bz}\right)^{\lambda}\left(\alpha+(\beta-\delta)\left(\frac{1+Az}{1+Bz}\right)+\delta\frac{(A-B)z}{(1+Az)(1+Bz)}\right),$$

then $f \in \mathcal{S}^*[A,B]$.

(ii) If $\alpha\delta \ge 0$, $0 \le \lambda \le 2$ and $(B-A)/((1+A)(1+B)) \le \gamma/\delta \le 2(B-A)/((1+A)(1+B))$ and f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(\alpha+\gamma\frac{f(z)}{zf'(z)}+\delta\left(\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}-1\right)\right)$$
$$\prec\left(\frac{1+Az}{1+Bz}\right)^{\lambda}\left(\alpha+\gamma\left(\frac{1+Bz}{1+Az}\right)+\delta\left(\frac{(A-B)z}{(1+Az)^2}\right)\right),$$

then $f \in \mathcal{S}^*[A,B]$.

From Corollary 2.3.23, we get the following result by taking $p(z) = zf'(z)/f(z)$.

***Example* 8.** Let $f \in \mathcal{S}$ and $\beta/\delta \ge \max(0;-\alpha/\delta)$. If f satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\alpha+(\beta-\delta)\frac{zf'(z)}{f(z)}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right)$$
$$\prec\left(1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)\left(\alpha+\beta\left(1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)+\frac{4\delta}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right),$$

then $f \in \mathcal{S}_P$.

Remark 12. By replacing $p(z) = f'(z)$ in each of the Corollaries of section 2.3, one can find several univalence criterion for analytic functions, as a by-product of Noshiro-Warschawski Theorem, stated in Theorem A.

Concluding Remarks

Our findings generalize some of the previously known results, apart from extending the result obtained by Mocanu [69] for $\rho-$ convex class. In literature, we come across many subclasses of $\mathcal{S}$, among which we established relation between some famous classes such as (a) class of starlike functions of reciprocal order and $\mathcal{N}(\kappa)$; (b) ρ-convex class and $\mathcal{S}^*(1/2)$. Furthermore, obtained argument based results involving *oblique sector function*, a name which is coined by us for a function $\left(\frac{1+cz}{1-z}\right)^{(\eta_1+\eta_2)/2}$, where the parameters take the values given by (1.2.7).

Chapter 3

Certain Exact Differential Subordinations

In this chapter, we introduce the concept of exact differential subordinations in the complex plane, which is analogous to exact differential equations on the real line. Let h be a non vanishing convex univalent function and p be an analytic function in $\mathbb{D}$. We consider the following first order differential subordination

$$\psi_j(p(z), zp'(z)) \prec h(z) \quad (j = 1, 2),$$

where the admissible functions ψ_1 and ψ_2 are given by $\psi_1 := (\beta p(z) + \gamma)^{-\alpha}\left(\frac{(\beta p(z)+\gamma)}{\beta(1-\alpha)} + zp'(z)\right)$ *and* $\psi_2 := \frac{1}{\sqrt{\gamma\beta}}\arctan\left(\sqrt{\frac{\beta}{\gamma}}p^{1-\alpha}(z)\right) + \left(\frac{1-\alpha}{\beta p^{2(1-\alpha)}(z)+\gamma}\right)\frac{zp'(z)}{p^{\alpha}(z)}$. *We find the dominants as well the best dominant (say q) of the solution of the above differential subordination satisfying* $\psi_j(q(z), nzq'(z)) = h(z)$. *We prove that* $\psi_j(q(z), zq'(z)) = h(z)$ *is an exact differential equation and q is a convex univalent function in* $\mathbb{D}$. *Furthermore, we estimate the sharp lower bound of* $\operatorname{Re} p$ *for different choices of h and as an application of our results, derive a univalence criterion for functions in* $\mathcal{A}$.

3.1 Introduction

In the recent times, we come across certain first order differential subordination implications involving Ma-Minda functions (see [1, 18, 71]). This topic has attracted consistent study, leading to numerous types of differential subordinations being introduced. For instance, Briot-Bouquet differential subordination has been defined in the literature analogous to Briot-Bouquet differential equations on real line. This differential subordination has a great deal of applications in univalent function theory. In 2005, Kanas and Kowalczyk [38] studied a special type of differential subordination, namely Briot-Bouquet-Bernoulli differential subordination based on the combination of Bernoulli and Briot-Bouquet differential equations on real line. Motivated by their work, in this chapter, we introduce a new concept of exact differential subordinations and deduce certain results pertaining to it. Furthermore, we derive some special cases of these results by considering various choices of Ma-Minda functions.

Recall the Nevanlinna's criterion, which states that a function $f \in \mathcal{A}$ is starlike if and only if $\operatorname{Re}(zf'(z)/f(z)) > 0$ in $\mathbb{D}$. For functions in $\mathcal{A}$, the convex characterization [5] and the univalence criterion [78, Lemma 1.1, p.10769] yield the characterization for convex univalent functions in $\mathcal{A}$ as follows:

Conclusion 3.1.1. A function $f \in \mathcal{A}$ is convex univalent in $\mathbb{D}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{3.1.1}$$

Geometrically, it is also known that a function $f(z)$ is convex (starlike) if it maps $\mathbb{D}$ onto a convex (starlike) domain. The characterization for convex functions $f \in \mathcal{A}$, given in Conclusion 3.1.1 can be extended to convex functions in $\mathcal{H}[a,1]$ as follows:

Conclusion 3.1.2. Let $F(z) = a + a_1 z + a_2 z^2 + \cdots \in \mathcal{H}[a,1]$. Then F is convex univalent in $\mathbb{D}$ if and only if it satisfies

$$\operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{3.1.2}$$

Proof. Since $F(z) \in \mathcal{H}[a,1]$, the function $f(z) = \frac{F(z)-a}{a_1} \in \mathcal{A}$ $(a_1 \neq 0)$ and we get

$$\operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) = \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right). \tag{3.1.3}$$

Now, if we assume F is convex univalent in $\mathbb{D}$, then geometrically so is the function $f(z)$. Thus the inequality (3.1.2) follows using Conclusion 3.1.1 and equation (3.1.3). Conversely, let inequality (3.1.2) holds, then (3.1.1) holds true using equation (3.1.3). Thus by Conclusion 3.1.1, we get f is convex univalent in $\mathbb{D}$ and geometrically so is $F(z)$. This completes the proof. □

The theory of differential subordinations is extensively studied in [67]. For some more work in this direction, one may refer to [3, 51, 106]. The set of analytic and univalent functions $\tilde{q}$ on $\overline{\mathbb{D}}\backslash E(\tilde{q})$, where $E(\tilde{q}) = \{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta}\tilde{q}(z) = \infty\}$, such that $\tilde{q}'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D}\backslash E(\tilde{q})$ is denoted by $\widetilde{Q}$. The theory of admissible functions paved a way in finding the dominants of the solutions of differential subordinations altogether with a distinctive approach. The admissible function is defined as follows:

Definition 3.1.1. [67] Let Ω be a set in $\mathbb{C}$, $\tilde{q} \in \widetilde{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega,\tilde{q}]$, consists of the functions $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$, which satisfy the admissibility condition

$$\psi(r,s;z) \notin \Omega, \tag{3.1.4}$$

whenever $r = \tilde{q}(\zeta)$, $s = m\zeta\tilde{q}'(\zeta)$, $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}\backslash E(\tilde{q})$ and $m \geq n$.

In particular, if Ω is a simply connected domain, $\Omega \neq \mathbb{C}$ and h conformally maps $\mathbb{D}$ onto Ω, we denote the class $\Psi_n[\Omega,\tilde{q}]$ by $\Psi_n[h,\tilde{q}]$. Recently, in [47, 72, 73], authors have studied the applications of admissibility conditions for various known classes of starlike functions. We consider a special case of the class $\Psi_n[\Omega,\tilde{q}]$ by taking $\Omega = \{w : \operatorname{Re} w > 0\}$, $\tilde{q}(\mathbb{D}) = \Omega$, $\tilde{q}(0) = 1$, $E(\tilde{q}) = \{1\}$ and $\tilde{q} \in \widetilde{Q}$, wherein the above admissibility condition, given in (3.1.4) reduces to

$$\psi(\rho i,\sigma;z) \notin \Omega,$$

where ρ and $\sigma \in \mathbb{R}$, $\sigma \leq -(n/2)|1 - i\rho|^2$, $z \in \mathbb{D}$ and $n \geq 1$. Set $\Psi_n\{1\} := \Psi_n[\Omega,\tilde{q}]$ for the above particular case.

3.2 Concept of Exact Differential Subordinations

Let us recall that a first order differential equation of the type:

$$M(x,y)dx + N(x,y)dy = 0, \tag{3.2.1}$$

where $M(x,y)$ and $N(x,y)$ have continuous partial derivatives in some domain $D \subset \mathbb{R}^2$, is called an exact differential equation if it satisfies $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. The solution of such a differential equation (3.2.1), is given by

$$\int M(x,y)\, dx + \int \widetilde{N}(y)\, dy = c, \tag{3.2.2}$$

where c is the integration constant and $\widetilde{N}(y)$ is equal to that part of the expression $N(x,y)$, which is independent of x. This concept we carry to the complex plane by considering the following general first order differential subordination implication:

Implication A. Let $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be an analytic function and h be univalent in $\mathbb{D}$. For some suitable a_0, if $p \in \mathcal{H}[a_0,n]$ satisfies

$$\psi(p(z), zp'(z); z) \prec h(z),$$

then $p \prec q$, where q be the best (a_0,n)-dominant satisfying the differential equation

$$\psi(q(z), nzq'(z)) = h(z).$$

In view of the Implication A, we furnish the following definition:

Definition 3.2.1. Let ψ, p and q are as stated in the Implication A. A first order differential subordination of the form

$$\psi(p(z), zp'(z); z) \prec h(z)$$

is said to be *exact differential subordination*, if the following conditions hold:
(i) There exists $\widetilde{M}(z,p)$ and $N(z,p)$ satisfying $\frac{\partial \widetilde{M}(z,p)}{\partial p} = \frac{\partial N(z,p)}{\partial z}$ such that

$$\psi(p(z), zp'(z)) = \widetilde{M}(z,p) + N(z,p)\, p'(z).$$

(ii) When $n = 1$, the equation $\psi(q(z), nzq'(z)) = h(z)$ reduces to an exact differential equation of the form:

$$M(z,q)\, dz + N(z,q)\, dq = 0,$$

where $M(z,q) := \widetilde{M}(z,q) - h(z)$ and $\frac{\partial M(z,q)}{\partial q} = \frac{\partial N(z,q)}{\partial z}$.

In this chapter, we mainly focus on the following exact type of differential subordinations:

$$(\beta p(z)+\gamma)^{-\alpha}\left(\frac{(\beta p(z)+\gamma)}{\beta(1-\alpha)}+zp'(z)\right)\prec h(z), \tag{3.2.3}$$

and

$$\frac{1}{\sqrt{\gamma\beta}}\arctan\left(\sqrt{\frac{\beta}{\gamma}}p^{1-\alpha}(z)\right)+\left(\frac{1-\alpha}{\beta p^{2(1-\alpha)}(z)+\gamma}\right)\frac{zp'(z)}{p^{\alpha}(z)}\prec h(z), \tag{3.2.4}$$

where $p\in\mathcal{H}[a_0,n]$ for an appropriate a_0 and h is a non vanishing convex univalent function in $\mathbb{D}$ with $h(0)=a$. We denote the expression on left side of (3.2.3) and (3.2.4) by $\psi_1(p(z),zp'(z))$ and $\psi_2(p(z),zp'(z))$, respectively. The speciality of the above two differential subordinations is that they generalize a result studied by Hallenbeck and Ruscheweyh [31]. Using the concept of admissibility conditions, we obtain the dominant and best dominant q of the solutions of the above differential subordinations, which satisfy

$$\psi_j(q(z),nzq'(z))=h(z)\quad (j=1,2). \tag{3.2.5}$$

Now we show that the above two differential subordinations are exact. Take $\widetilde{M}(z,p):=\frac{(\beta p+\gamma)^{1-\alpha}}{\beta(1-\alpha)}$ and $N(z,p):=\frac{z}{(\beta p+\gamma)^{\alpha}}$ in (3.2.3), then we have $\frac{\partial\widetilde{M}(z,p)}{\partial p}=\frac{\partial N(z,p)}{\partial z}$. Furthermore, for $n=1$, the equation (3.2.5) with $j=1$, reduces to the following exact differential equation in z and q:

$$M(z,q)\,dz+N(z,q)\,dq=0, \tag{3.2.6}$$

as $\frac{\partial M(z,q)}{\partial q}=\frac{\partial N(z,q)}{\partial z}$, where $M(z,q)=\frac{(\beta q+\gamma)^{1-\alpha}}{\beta(1-\alpha)}-h(z)$. From Definition 3.2.1, we get (3.2.3) is an exact differential subordination. Since (3.2.6) is an exact differential equation, we can obtain its solution analogous to (3.2.2) as follows:

$$\int M(z,q)\,dz+\int\widetilde{N}(q)\,dq=c,$$

where $\widetilde{N}(q)$ is equal to that part of the expression $N(z,q)$, which is independent of z. Upon simplification of the above equation, we get

$$q(z)=\frac{\left(\dfrac{\beta(1-\alpha)}{z}\int_0^z h(t)dt\right)^{\frac{1}{1-\alpha}}-\gamma}{\beta}. \tag{3.2.7}$$

Interestingly, the above solution (3.2.7) of the exact differential equation (3.2.6) obtained analogous to (3.2.2) coincides with the best dominant of the associated differential subordinations (3.2.3), which we illustrate in the subsequent section. Now we prove that the subordination (3.2.4) is also exact. For this, let us take

$$\widetilde{M}(z,p) := \frac{1}{\sqrt{\gamma\beta}}\arctan\left(\sqrt{\frac{\beta}{\gamma}}p^{1-\alpha}\right) \text{ and } N(z,p) := \left(\frac{1-\alpha}{\beta p^{2(1-\alpha)}+\gamma}\right)\frac{z}{p^{\alpha}},$$

in subordination (3.2.4). We calculate that $\partial\widetilde{M}(z,p)/\partial p = \partial N(z,p)/\partial z$, which reveals that (3.2.4) satisfies the condition (i) of Definition 3.2.1. The second condition of Definition 3.2.1 also holds, as equation (3.3.14) for $n=1$, reduces to the differential equation (3.2.6), with $M(z,q) := (1/\sqrt{\gamma\beta})\arctan\left(\sqrt{\frac{\beta}{\gamma}}q^{1-\alpha}\right) - h(z)$ and $N(z,q) := \frac{(1-\alpha)z}{q^{\alpha}(\beta q^{2(1-\alpha)}+\gamma)}$, where $\frac{\partial M(z,q)}{\partial q} = \frac{\partial N(z,q)}{\partial z}$, which shows that the differential equation, given in (3.3.14) is exact for $n=1$. This clearly shows that (3.2.4) is an exact differential subordination. Consequently, the equation (3.2.2) yields the solution for equation (3.3.14) whenever $n=1$ as follows:

$$q(z) = \left(\sqrt{\frac{\gamma}{\beta}}\tan\left(\frac{\sqrt{\gamma\beta}}{z}\int_0^z h(t)dt\right)\right)^{\frac{1}{1-\alpha}}, \tag{3.2.8}$$

which coincides with (3.3.10) for $n=1$. Note that the above solution (3.2.8) of the exact differential equation (3.2.6) also coincides with the best dominant of the associated differential subordinations (3.2.4). In what follows, let us presume convex to mean convex univalent.

3.3 Main Results

We employ the famous Hallenbeck and Ruscheweyh [31] result with $\gamma = 1$ to prove our main results.

Lemma 3.3.1. *[31]* Let h be convex in $\mathbb{D}$ having $h(0) = a$. If $P \in \mathcal{H}[a,n]$ satisfies

$$P(z) + zP'(z) \prec h(z),$$

then $P(z) \prec Q(z) \prec h(z)$, where

$$Q(z) = \frac{1}{nz^{1/n}}\int_0^z h(t)t^{(1/n)-1}dt, \tag{3.3.1}$$

is convex and is the best (a,n)−dominant.

We state below that part of the result which is relevant in proving our results.

Lemma 3.3.2. [67] Let $\psi:\mathbb{C}^2\to\mathbb{C}$ and h be univalent in $\mathbb{D}$. Suppose that the differential equation

$$\psi(q(z), nzq'(z)) = h(z)$$

has a solution q, with $q(0)=a$. Furthermore, assume $q\in\widetilde{Q}$ and $\psi\in\Psi_n[h,q]$. If $p\in\mathcal{H}[a,n]$, $\psi(p(z),zp'(z))$ is analytic in $\mathbb{D}$, and p satisfies $\psi(p(z),zp'(z))\prec h(z)$ then $p\prec q$, and q is the best (a,n)-dominant.

The following theorem deals in finding the best dominant of the exact type differential subordination (3.2.3).

Theorem 3.3.3. Suppose h be a non-vanishing convex function with $h(0)=a$. Let $\beta(\neq 0)$, γ be complex numbers and $-1\le\alpha\le 0$. If $a_0=((\beta(1-\alpha)a)^{1/(1-\alpha)}-\gamma)/\beta$ and $p\in\mathcal{H}[a_0,n]$ satisfies

$$\frac{(\beta p(z)+\gamma)^{1-\alpha}}{\beta(1-\alpha)}+\frac{zp'(z)}{(\beta p(z)+\gamma)^{\alpha}}\prec h(z), \tag{3.3.2}$$

then $p(z)\prec q(z)\prec H(z)$, where $H(z):=((\beta(1-\alpha)h(z))^{\frac{1}{1-\alpha}}-\gamma)/\beta$ and

$$q(z)=\frac{\left(\dfrac{\beta(1-\alpha)}{nz^{1/n}}\displaystyle\int_0^z h(t)t^{(1/n)-1}dt\right)^{\frac{1}{1-\alpha}}-\gamma}{\beta}. \tag{3.3.3}$$

Furthermore, $H(z)$ and $q(z)$ are convex in $\mathbb{D}$ and q is the best (a_0,n)- dominant.

Proof. We first prove that q and H are convex in $\mathbb{D}$. Since the function h here satisfies the conditions of h of Lemma 3.3.1, the expression (3.3.3) becomes

$$q(z)=\frac{(\beta(1-\alpha)Q(z))^{1/(1-\alpha)}-\gamma}{\beta}, \tag{3.3.4}$$

where Q is given by (3.3.1). Thus q is well-defined and analytic in $\mathbb{D}$ as $\beta\neq 0$. Let $P(z)=\frac{(\beta p(z)+\gamma)^{1-\alpha}}{\beta(1-\alpha)}$, then $P\in\mathcal{H}[a,n]$ and the subordination (3.3.2) becomes $P(z)+zP'(z)\prec h(z)$. The Lemma 3.3.1 yields $P(z)\prec Q(z)\prec h(z)$, which implies $(\beta p(z)+\gamma)^{1-\alpha}\prec\beta(1-\alpha)Q(z)\prec\beta(1-\alpha)h(z)$. Since $h(z)$ does not vanish in $\mathbb{D}$, we have $q'(z)\neq 0$ and therefore $1+zq''(z)/q'(z)=: s(z)$ is analytic in $\mathbb{D}$. The logarithmic differentiation of (3.3.4) gives

$$1+\frac{zQ''(z)}{Q'(z)}=1+\frac{zq''(z)}{q'(z)}+A(z)=:\tilde{\psi}(s(z);z), \tag{3.3.5}$$

$$A(z):=-\alpha\beta\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{\alpha}{\alpha-1}\left(\frac{zQ'(z)}{Q(z)}\right)$$

and we can write (3.3.5) as $\tilde{\psi}(r) = r + A(z)$. By Lemma 3.3.1, the function Q is convex, thus from Conclusion 3.1.2, we have $\operatorname{Re}\tilde{\psi}(s(z)) > 0$. Now to prove q is convex, by Conclusion 3.1.2, it suffices to show $\operatorname{Re} s(z) > 0$. To accomplish this task, we use [67, Theorem 2.3i, p. 35] and so we need to show that $\tilde{\psi} \in \Psi_n\{1\}$. Furthermore, by using the admissibility condition for the same, it is equivalent to prove that $\operatorname{Re}\tilde{\psi}(\rho i; z) \not> 0$. From equation (3.3.5), we obtain

$$\operatorname{Re}\tilde{\psi}(\rho i) = \operatorname{Re}(\rho i + A(z)) = \operatorname{Re} A(z).$$

Using the fact that $Q(z) \neq 0$ for all $z \in \mathbb{D}$, we get $zQ'(z)/Q(z)$ is analytic in $\mathbb{D}$. Since $P(z) \prec Q(z)$, we have $Q(0) = a (\neq 0)$, which implies

$$\left.\frac{zQ'(z)}{Q(z)}\right|_{z=0} = 0$$

and also we have $Q'(z) \neq 0$ for all z, therefore $zQ'(z)/Q(z)$ lies on either side of the imaginary axis. So clearly we have $\operatorname{Re} A(z) \not> 0$. Thus, q is convex in $\mathbb{D}$. We compute that

$$1 + \frac{zh''(z)}{h'(z)} = 1 + \frac{zH''(z)}{H'(z)} + B(z),$$

where

$$B(z) = -\alpha\beta \frac{zH'(z)}{\beta H(z) + \gamma} = \frac{\alpha}{\alpha - 1} \frac{zh'(z)}{h(z)}. \tag{3.3.6}$$

In the same way as function q is proved to be convex in $\mathbb{D}$, we may show that the function H is convex in $\mathbb{D}$, since $zh'(z)/h(z)$ $(h(0) = a)$ and $zQ'(z)/Q(z)$ behave alike. We now proceed to show $p(z) \prec q(z) \prec H(z)$ and q is the best (a_0, n)- dominant. Let

$$\psi_1(r,s) := \frac{(\beta r + \gamma)^{1-\alpha}}{\beta(1-\alpha)} + \frac{s}{(\beta r + \gamma)^{\alpha}},$$

which corresponds to left hand side of the subordination (3.3.2). First we show $p \prec H$ by proving $\psi_1 \in \Psi_n[h, H]$, or equivalently

$$\psi := \psi_1(H(\zeta), m\zeta H'(\zeta)) = \frac{(\beta H(\zeta) + \gamma)^{1-\alpha}}{\beta(1-\alpha)} + \frac{m\zeta H'(\zeta)}{(\beta H(\zeta) + \gamma)^{\alpha}} \notin h(\mathbb{D}),$$

whenever $|\zeta| = 1$ and $m \geq n$. From hypothesis, replacing H by its expression in terms of h, in ψ, we obtain

$$\left|\arg\left(\frac{\psi - h(\zeta)}{\zeta h'(\zeta)}\right)\right| = |\arg m| < \pi/2.$$

As the function $h(\mathbb{D})$ is convex and $m \geq 1$, $h(\zeta) \in h(\partial\mathbb{D})$ and $\zeta h'(\zeta)$ is the outer normal to $h(\partial\mathbb{D})$ at $h(\zeta)$, we arrive at the conclusion that $\psi \notin h(\mathbb{D})$ and therefore we obtain $p(z) \prec H(z)$. Now, we show $p \prec q$. A simple calculation shows that q, given by (3.3.3) satisfies the following differential equation

$$\frac{(\beta q(z)+\gamma)^{1-\alpha}}{\beta(1-\alpha)} + \frac{nzq'(z)}{(\beta q(z)+\gamma)^{\alpha}} = \psi_1(q(z), nzq'(z)) = h(z). \tag{3.3.7}$$

Now, we apply Lemma 3.3.2 to show q is the best dominant. Without loss of generality, we can suppose h and q are analytic and univalent on $\overline{\mathbb{D}}$ and $q'(\zeta) \neq 0$ for $|\zeta| = 1$, which shows that $q \in \widetilde{Q}$. To complete the proof, we now show $\psi_1 \in \Psi_n[h, q]$. This is equivalent to show that

$$\acute{\psi} := \psi_1(q(\zeta), m\zeta q'(\zeta)) = \frac{(\beta q(\zeta)+\gamma)^{1-\alpha}}{\beta(1-\alpha)} + \frac{m\zeta q'(\zeta)}{(\beta q(\zeta)+\gamma)^{\alpha}} \notin h(\mathbb{D}),$$

whenever $|\zeta| = 1$ and $m \geq n$. Using equations (3.3.3) and (3.3.7), we obtain

$$Q(\zeta) + \frac{m}{n}(h(\zeta) - Q(\zeta)) = \acute{\psi}.$$

Now we have $Q(z) \prec h(z)$ from Lemma 3.3.1. This clearly implies $Q(\zeta) \notin h(\mathbb{D})$ and also we know $m/n \geq 1$, it is easy to observe that $\acute{\psi} \notin h(\mathbb{D})$. Thus the proof follows. □

Let $n = 1$, then the best dominant q, given in (3.3.3) coincides with the solution (3.2.7) obtained by solving its associated exact differential equation. By taking $\beta = 1$ and $\gamma = 0$ in Theorem 3.3.3, the result follows as:

Corollary 3.3.4. Let h be a non vanishing convex function in $\mathbb{D}$ with $h(0) = a$, $-1 \leq \alpha \leq 0$ and $a_0 = ((1-\alpha)a)^{1/(1-\alpha)}$. If $p \in \mathcal{H}[a_0, n]$ satisfies

$$\frac{p^{1-\alpha}(z)}{1-\alpha} + \frac{zp'(z)}{p^{\alpha}(z)} \prec h(z),$$

then $p(z) \prec q(z) \prec H(z)$, where $H(z) := ((1-\alpha)h(z))^{\frac{1}{1-\alpha}}$ and

$$q(z) = \left(\frac{(1-\alpha)}{nz^{1/n}} \int_0^z h(t)t^{(1/n)-1}dt\right)^{\frac{1}{1-\alpha}}.$$

Furthermore, $H(z)$ and $q(z)$ are convex in $\mathbb{D}$ and q is the best (a_0, n)- dominant.

In Corollary 3.3.4, if we choose $\alpha = -1$, h as any convex function with $h(0) = 1$ and

$\operatorname{Re} h(z) > 0$, then we obtain the following result:

Corollary 3.3.5. Let h be a convex function with $h(0) = 1$ and $\operatorname{Re} h(z) > 0$. If $p \in \mathcal{H}[1, n]$ satisfies the following

$$p^2(z) + 2zp(z)p'(z) \prec h(z),$$

then

$$p(z) \prec q(z) = \sqrt{Q(z)}, \text{ where } Q(z) = \frac{1}{nz^{1/n}} \int_0^z h(t)t^{(1/n)-1}dt. \tag{3.3.8}$$

The function q is the best $(1, n)$- dominant and is convex in $\mathbb{D}$.

Remark 13. The Corollary 3.3.5 improves the already known result of Miller and Mocanu [67, Theorem 3.1e, p. 77], by additionally establishing q, given in (3.3.8) is convex in $\mathbb{D}$ apart from being the best dominant. Moreover, the Corollary 3.3.4 generalizes the known result [67, Theorem 3.1e, p. 77] for all α $(-1 \leq \alpha \leq 0)$ and extends the same by proving the convexity of the function q as well.

In the following theorem, we deal with another exact type differential subordination associated with (3.2.4).

Theorem 3.3.6. Let β, $\gamma (\neq 0)$ be complex numbers, $-1 \leq \alpha \leq 0$, h be a non vanishing convex function in $\mathbb{D}$ with $h(0) = a$, $|\sqrt{\gamma\beta}h(z)| < \pi/2$ and $a_0 = (\sqrt{\gamma/\beta}\tan(\sqrt{\gamma\beta}a))^{1/(1-\alpha)}$. If $p \in \mathcal{H}[a_0, n]$ satisfies

$$\frac{1}{\sqrt{\gamma\beta}} \arctan\left(\sqrt{\frac{\beta}{\gamma}} p^{1-\alpha}(z)\right) + \left(\frac{1-\alpha}{\beta p^{2(1-\alpha)}(z) + \gamma}\right) \frac{zp'(z)}{p^{\alpha}(z)} \prec h(z), \tag{3.3.9}$$

then $p(z) \prec q(z) \prec H(z)$, where $H(z) = (\sqrt{\gamma/\beta}\tan(\sqrt{\gamma\beta}h(z)))^{1/(1-\alpha)}$ and

$$q(z) = \left(\sqrt{\frac{\gamma}{\beta}} \tan\left(\frac{\sqrt{\gamma\beta}}{nz^{1/n}} \int_0^z h(t)t^{(1/n)-1}dt\right)\right)^{\frac{1}{1-\alpha}}. \tag{3.3.10}$$

Furthermore, $H(z)$ and $q(z)$ are convex in $\mathbb{D}$ and q is the best $(a_0, n)-$dominant.

Proof. We first show that q and H are convex in $\mathbb{D}$. As the function h here satisfies the condition of h of Lemma 3.3.1, thus q reduces to

$$q(z) = \left(\sqrt{\frac{\gamma}{\beta}} \tan\left(\sqrt{\gamma\beta} Q(z)\right)\right)^{\frac{1}{1-\alpha}}, \tag{3.3.11}$$

where Q is defined in (3.3.1) and q is well defined and analytic in $\mathbb{D}$. Let $P(z) = (1/\sqrt{\gamma\beta})\arctan\left(\sqrt{\beta/\gamma}p^{1-\alpha}(z)\right)$, then the subordination (3.3.9) reduces to $P(z)+zP'(z) \prec h(z)$ and clearly $P \in \mathcal{H}[a,n]$. Now from Lemma 3.3.1, we have $P(z) \prec Q(z) \prec h(z)$, which is equivalent to $\arctan\left(\sqrt{\frac{\beta}{\gamma}}p^{1-\alpha}(z)\right) \prec \sqrt{\gamma\beta}Q(z) \prec \sqrt{\gamma\beta}h(z)$. We have $q'(z) \neq 0$ as $h(z) \neq 0$ for all $z \in \mathbb{D}$ and therefore $1 + zq''(z)/q'(z) =: g(z)$ is analytic in $\mathbb{D}$. The logarithmic differentiation of the function q, given in (3.3.11) yields

$$1+\frac{zQ''(z)}{Q'(z)} = 1+\frac{zq''(z)}{q'(z)} + A(z) =: \tilde{\psi}(g(z);z), \tag{3.3.12}$$

where

$$\begin{aligned} A(z) &= -\left(\alpha + 2(1-\alpha)\beta\frac{q^{2(1-\alpha)}(z)}{\gamma+\beta q^{2(1-\alpha)}(z)}\right)\left(\frac{zq'(z)}{q(z)}\right) \\ &= \frac{\alpha}{\alpha-1}\left(\sum_{k=0}^{\infty}(-1)^k\frac{(2\sqrt{\gamma\beta}Q(z))^{2k}}{(2k+1)!}\right)^{-1}\left(\frac{zQ'(z)}{Q(z)}\right) \\ &\quad -2\sqrt{\gamma\beta}\tan(\sqrt{\gamma\beta}Q(z))zQ'(z), \end{aligned} \tag{3.3.13}$$

and we can write $\tilde{\psi}(r) = r + A(z)$. By Lemma 3.3.1, the function Q is convex, thus from Conclusion 3.1.2, we have $\operatorname{Re}\tilde{\psi}(g(z)) > 0$. In order to prove q is convex, by Conclusion 3.1.2, it suffices to show $\operatorname{Re} g(z) > 0$. Now, to achieve this, we use [67, Theorem 2.3i, p. 35], so then we only need to show $\tilde{\psi} \in \Psi_n\{1\}$. Furthermore, by using the admissibility condition for the same, it is equivalent to show that $\operatorname{Re}\tilde{\psi}(\rho i;z) \not> 0$. From equation (3.3.12), we deduce $\operatorname{Re}\tilde{\psi}(\rho i) = \operatorname{Re}(\rho i + A(z)) = \operatorname{Re} A(z)$. Now, using the fact $Q(z) \neq 0$ for all $z \in \mathbb{D}$, we get A is analytic in $\mathbb{D}$. Since $P(z) \prec Q(z)$, we have $Q(z) \in \mathcal{H}[a,1]$, thus

$$\left(\sum_{k=0}^{\infty}(-1)^k\frac{(2\sqrt{\gamma\beta}Q(z))^{2k}}{(2k+1)!}\right)^{-1}\left(\frac{zQ'(z)}{Q(z)}\right)\Bigg|_{z=0} = 0$$

and $\tan(\sqrt{\gamma\beta}Q(z))zQ'(z)|_{z=0} = 0$, which ensures either $A(z) \equiv 0$ or lies on either side of the imaginary axis. Therefore, from equation (3.3.13), we have $\operatorname{Re} A(z) \not> 0$, which proves q is convex in $\mathbb{D}$. Similarly, we can prove the function H is convex in $\mathbb{D}$ as we compute

$$1+\frac{zh''(z)}{h'(z)} = 1+\frac{zH''(z)}{H'(z)} + A(z),$$

where A is as defined in (3.3.13), with q being replaced by H and Q by h. We now show $p(z) \prec q(z) \prec H(z)$ and q is the best (a_0,n)- dominant. For this we assume the following expression

$$\psi_2(r,s) := \frac{1}{\sqrt{\gamma\beta}} \arctan\left(\sqrt{\frac{\beta}{\gamma}} r^{1-\alpha} \right) + \frac{(1-\alpha)s}{r^{\alpha}(\beta r^{2(1-\alpha)} + \gamma)},$$

which corresponds to left hand side of the subordination (3.3.9). Now first we show $p \prec H$ by proving $\psi_2 \in \Psi_n[h,H]$, or equivalently

$$\psi_2(H(\zeta), m\zeta H'(\zeta)) = \frac{1}{\sqrt{\gamma\beta}} \arctan\left(\sqrt{\frac{\beta}{\gamma}} H^{1-\alpha}(\zeta) \right) + \frac{(1-\alpha)m\zeta H'(\zeta)}{H^{\alpha}(\zeta)(\beta H^{2(1-\alpha)}(\zeta) + \gamma)} =: \psi \notin h(\mathbb{D}),$$

whenever $|\zeta| = 1$ and $m \geq n$. Now, replacing H by its expression in terms of h in ψ, we get

$$\left| \arctan\left(\frac{\psi - h(\zeta)}{\zeta h'(\zeta)} \right) \right| = |\arg m| < \frac{\pi}{2}.$$

Since the function h is convex, $h(\zeta) \in h(\partial\mathbb{D})$ and $\zeta h'(\zeta)$ is the outer normal to $h(\partial\mathbb{D})$ at $h(\zeta)$, thus we obtain $\psi \notin h(\mathbb{D})$, which further implies $p(z) \prec H(z)$. A computation shows that q, given by (3.3.10) satisfies the differential equation

$$\frac{1}{\sqrt{\gamma\beta}} \arctan\left(\sqrt{\frac{\beta}{\gamma}} q^{1-\alpha}(z) \right) + \frac{(1-\alpha)nzq'(z)}{q^{\alpha}(z)(\beta q^{2(1-\alpha)}(z) + \gamma)} =: \psi_2(q(z), nzq'(z)) = h(z). \quad (3.3.14)$$

We apply Lemma 3.3.2 to show q is the best dominant. Without loss of generality, we assume h and q are analytic and univalent on $\overline{\mathbb{D}}$ and $q'(\zeta) \neq 0$ for $|\zeta| = 1$, which shows that $q \in \widetilde{Q}$. In order to complete the proof, it suffices to show $\psi_2 \in \Psi_n[h,q]$, which is equivalent to show that

$$\psi_2(q(\zeta), m\zeta q'(\zeta)) = \frac{1}{\sqrt{\gamma\beta}} \arctan\left(\sqrt{\frac{\beta}{\gamma}} q^{1-\alpha}(\zeta) \right) + \frac{(1-\alpha)m\zeta q'(\zeta)}{q^{\alpha}(\zeta)(\beta q^{2(1-\alpha)}(\zeta) + \gamma)} =: \acute{\psi} \notin h(\mathbb{D}),$$

whenever $|\zeta| = 1$ and $m \geq n$. From equations (3.3.1) and (3.3.14), we have $Q(\zeta) + (m/n)(h(\zeta) - Q(\zeta)) = \acute{\psi}$. As we have $m/n \geq 1$ and from Lemma 3.3.1, we have $Q(z) \prec h(z)$, therefore evidently, we get $\acute{\psi} \notin h(\mathbb{D})$. This completes the proof. □

3.4 Special Cases

For detail information on hypergeometric function, one may refer to [67, p. 5-7]. Here, we recall the definitions of a couple of hypergeometric functions which are needed in this section. Suppose a and c are complex numbers with $c \neq 0, -1, -2, \ldots$.

The function

$$ {}_1F_1(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tz}dt, $$

is called the confluent hypergeometric function and the following

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt, $$

is the Gaussian hypergeometric function. We now obtain the following corollary for different choices of h in Theorem 3.3.3, when β is a real number.

Corollary 3.4.1. Let $-1 \le \alpha \le 0$, $\beta > 0$ and γ be a complex number. If $p \in \mathcal{H}[(\beta(1-\alpha))^{1/(1-\alpha)} - \gamma)/\beta, n]$ and $\psi_1(p(z), zp'(z))$, given in (3.2.3), satisfies the following:

(i) $\psi_1 \prec (1+Az)/(1+Bz)$, $-1 \le B < A \le 1$, with
$\lambda_1 := {}_2F_1(1, 1/n, 1+1/n; B) - \frac{A}{n+1}\, {}_2F_1(1, 1+1/n, 2+1/n; B)$

(ii) $\psi_1 \prec e^{\mu z}$, $(|\mu| \le 1)$, with $\lambda_2 := {}_1F_1(1/n, 1/n+1; -\mu)$

(iii) $\psi_1 \prec \sqrt{1+\kappa z}$, $\kappa \in [0,1]$, with $\lambda_3 := {}_2F_1(-1/2, 1/n, 1+1/n; \kappa)$

(iv) $-\frac{\eta_2 \pi}{2} < \arg \psi_1 < \frac{\eta_1 \pi}{2}$, where $\eta = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$, $\eta' = \frac{\eta_1 + \eta_2}{2}$, $0 < \eta_1, \eta_2 \le 1$, and $c = e^{\eta \pi i}$, with
$\lambda_4 := \sum_{j=0}^{\infty} \left(\frac{\binom{\eta'}{j}(-c)^j {}_2F_1(\eta', 1/n+j, 1+1/n+j; -1)}{1+nj} \right)$,

then for the above parts (i)-(iv), respectively we have $\operatorname{Re} p(z) > \zeta_i(\alpha, \beta, \gamma, \lambda_i)$, where

$$ \zeta_i(\alpha, \beta, \gamma, \lambda_i) = \operatorname{Re}(((\beta(1-\alpha)\lambda_i)^{1/(1-\alpha)} - \gamma)/\beta), $$

with i corresponding to its appropriate integral value from 1 to 4. The result is sharp.

Proof. Let $h(z) = (1+Az)/(1+Bz)$, $e^{\mu z}$, $\sqrt{1+\kappa z}$ and $((1+cz)/(1-z))^{\eta'}$, respectively for the parts (i)-(iv). Then from Theorem 3.3.3, clearly $\operatorname{Re} p(z) > \min_{|z| \le 1} \operatorname{Re} q(z)$, where q is given by (3.3.4). Therefore, it suffices to find the minimum value of real part of Q in each of the parts, where Q is given by (3.3.1).

(i) We have $h(z) = (1+Az)/(1+Bz)$, then from equation (3.3.1), we get

$$ \begin{aligned} Q(z) &= \frac{1}{nz^{1/n}} \int_0^z \left(\frac{1+At}{1+Bt} \right) t^{(1/n)-1} dt \\ &= \frac{1}{n} \int_0^1 (1+Btz)^{-1} t^{(1/n)-1} dt + \frac{Az}{n} \int_0^1 (1+Btz)^{-1} t^{1/n} dt \\ &= {}_2F_1(1, 1/n, 1+1/n; -Bz) + \frac{Az}{n+1}\, {}_2F_1(1, 1+1/n, 2+1/n; -Bz). \end{aligned} $$

Now, we have $\min_{|z|\leq 1} \operatorname{Re} Q(z) = Q(-1) = \lambda_1$, where

$$\lambda_1 := {}_2F_1\left(1, 1/n, 1+1/n; B\right) - \frac{A}{n+1}\, {}_2F_1\left(1, 1+1/n, 2+1/n; B\right).$$

Hence the result.

(ii) We have $h(z) = \exp(\mu z)$, then from equation (3.3.1) we obtain

$$\begin{aligned} Q(z) &= \frac{1}{nz^{1/n}} \int_0^z e^{\mu t} t^{1/n-1} dt \\ &= \tfrac{1}{n} \int_0^1 e^{\mu t z} t^{1/n-1} dt \\ &= {}_1F_1\left(1/n, 1/n+1; \mu z\right). \end{aligned}$$

Clearly, $\min_{|z|\leq 1} \operatorname{Re} Q(z) = Q(-1) = \lambda_2$, where $\lambda_2 := {}_1F_1\left(1/n, 1/n+1; -\mu\right)$. This completes the proof for part (ii).

(iii) Let $h(z) = \sqrt{1+\kappa z}$, then from equation (3.3.1), we have

$$\begin{aligned} Q(z) &= \frac{1}{nz^{1/n}} \int_0^z \sqrt{1+\kappa t}\; t^{(1/n)-1} dt \\ &= \frac{1}{n} \int_0^1 \sqrt{1+\kappa t z}\; t^{(1/n)-1} dt \\ &= {}_2F_1\left(-1/2, 1/n, 1+1/n; -\kappa z\right). \end{aligned}$$

Thus, we have $\min_{|z|\leq 1} \operatorname{Re} Q(z) = Q(-1) = \lambda_3$, where $\lambda_3 := {}_2F_1\left(-1/2, 1/n, 1+1/n; \kappa\right)$ and that completes the proof for part (iii).

(iv) We have $h(z) = \left(\frac{1+cz}{1-z}\right)^{\eta'}$, then from equation (3.3.1), we have

$$\begin{aligned} Q(z) &= \frac{1}{nz^{1/n}} \int_0^z \left(\frac{1+ct}{1-t}\right)^{\eta'} t^{(1/n)-1} dt \\ &= \frac{1}{nz^{1/n}} \int_0^z \left(\sum_{j=0}^{\infty} \tbinom{\eta'}{j} c^j t^{(1/n+j-1)} (1-t)^{-\eta'} \right) dt \\ &= \frac{1}{nz^{1/n}} \sum_{j=0}^{\infty} \left(\tbinom{\eta'}{j} c^j \int_0^1 (tz)^{(1/n+j-1)} (1-tz)^{-\eta'} z dt \right) \\ &= \frac{1}{n} \sum_{j=0}^{\infty} \left(\tbinom{\eta'}{j} (cz)^j \int_0^1 t^{(1/n+j-1)} (1-tz)^{-\eta'} dt \right) \\ &= \sum_{j=0}^{\infty} \left(\frac{\tbinom{\eta'}{j} (cz)^j {}_2F_1(\eta', 1/n+j, 1+1/n+j; z)}{1+nj} \right). \end{aligned}$$

Now, $\min_{|z|\le 1} \operatorname{Re} Q(z) = Q(-1) = \lambda_4$, where

$$\lambda_4 := \sum_{j=0}^{\infty} \left(\frac{\binom{\eta'}{j}(-c)^j {}_2F_1(\eta', 1/n + j, 1 + 1/n + j; -1)}{1 + nj} \right).$$

Hence the result. □

Remark 14. If A and B are replaced with each other in Corollary 3.4.1(i), then we have $\min_{|z|\le 1} \operatorname{Re} Q(z) = Q(1)$. Further by taking $A = 2a - 1$ ($a \in [0,1)$), $B = 1$, $\alpha = -1$, $\beta = 1$ and $\gamma = 0$, the above result reduces to the result of Miller and Mocanu [67, Corollary 3.1e.1, p. 79].

We now obtain the following corollary for different choices of h in Theorem 3.3.6, when β is a real number.

Corollary 3.4.2. Suppose $\gamma(\neq 0)$ be a complex number, $-1 \le \alpha \le 0$ and $\beta > 0$. If $p \in \mathcal{H}[(\sqrt{\gamma/\beta}\tan(\sqrt{\gamma\beta}a))^{1/(1-\alpha)}, n]$ and $\psi_2(p(z), zp'(z))$, given by (3.2.4) satisfies the following:

(i) $\psi_2 \prec (1 + Az)/(1 + Bz)$, where $-1 \le B < A \le 1$ such that $|\sqrt{\gamma\beta}(1 + Az)/(1 + Bz)| < \pi/2$

(ii) $\psi_2 \prec e^{\mu z}$, where $|\mu| \le 1$ such that $|\sqrt{\gamma\beta}e^{\mu z}| < \pi/2$

(iii) $\psi_2 \prec \sqrt{1 + \kappa z}$, where $\kappa \in [0,1]$ such that $|\sqrt{\gamma\beta}\sqrt{1 + \kappa z}| \le \pi/2$

(iv) $-\eta_2\pi/2 < \arg\psi_2 < \eta_1\pi/2$, $\eta = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$, where $0 < \eta_1, \eta_2 \le 1$, and $c = e^{\eta\pi i}$ such that $|\sqrt{\gamma\beta}((1 + cz)/(1 - z))^{\eta'}| < \pi/2$, $(\eta' = (\eta_1 + \eta_2)/2)$,

then respectively, for the above parts (i)-(iv), we have $\operatorname{Re} p(z) > \xi_i(\alpha, \beta, \gamma, \lambda_i) (i = 1,2,3,4)$, where

$$\xi_i(\alpha, \beta, \gamma, \lambda_i) = \operatorname{Re}\left(\sqrt{\gamma/\beta}\tan\left(\sqrt{\gamma\beta}\lambda_i\right)\right)^{\frac{1}{1-\alpha}}$$

and λ_i is same as in Corollary 3.4.1. This result is sharp.

Proof. Here, we assume $h(z) = (1 + Az)/(1 + Bz)$, $e^{\mu z}$, $\sqrt{1 + \kappa z}$ and $((1 + cz)/(1 - z))^{\eta'}$ in Theorem 3.3.6, respectively for the above parts (i)-(iv). Then clearly, $\operatorname{Re} p(z) > \min_{|z|\le 1} \operatorname{Re} q(z)$, where $q(z)$ is defined by (3.3.11). Now, to complete the poof it suffices to have the minimum value of $\operatorname{Re} Q(z)$ in each of the above parts, where Q is given by (3.3.1). Now, $Q(z)$ for the above parts (i)-(iv) is same as calculated in each of the parts of Corollary 3.4.1, respectively. We directly substitute the minimum value of each function Q in equation (3.3.11), respectively for all the parts and the result follows at once. □

3.5 Examples and Applications

Here below, we provide some illustrations of our results.

***Example* 9.** If $p \in \mathcal{H}[1,1]$ satisfies

$$p(z) + zp'(z) \prec \frac{1+z}{1-z}, \tag{3.5.1}$$

then $p(z) \prec q(z) := \frac{-2\log(1-z)}{z} - 1 \prec \frac{1+z}{1-z}$. Moreover, q is the best (1,1)- dominant and is convex in $\mathbb{D}$.

Proof. By taking $h(z) = \frac{1+z}{1-z}$, $n = 1$ and $\alpha = 0$ in Corollary 3.3.4, we obtain $Q(z) = q(z) = -2(\log(1-z))/z - 1$. Now the assertion follows at once from Corollary 3.3.4 as $h(0) = 1$ and $h(z) \neq 0$ for all $z \in \mathbb{D}$. Moreover, the quantities $zQ'(z)/Q(z)$ and $zh'(z)/h(z)$ lie on either side of the imaginary axis as we have

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\left(\frac{2(z-(-1+z)\log(1-z))}{(-1+z)(z+2\log(1-z))}\right) \not> 0$$

and

$$\operatorname{Re}\frac{zh'(z)}{h(z)} = \frac{2z}{1-z^2} \not> 0,$$

for $z \in \mathbb{D}$, as proved in the proof of Theorem 3.3.3. □

Remark 15. The above result can also be stated as if $p \in \mathcal{H}[1,1]$ satisfies the inequality $\operatorname{Re}(p(z) + zp'(z)) > 0$, then

$$\begin{aligned}\operatorname{Re} p(z) &> 2\log 2 - 1 \\ &= {}_2F_1(1,1,2;-1) - \frac{1}{2}\, {}_2F_1(1,2,3;-1),\end{aligned}$$

which is a special case of Corollary 3.4.1(i) with $A = 1$, $B = -1$, $\beta = 1$, $\gamma = 0$ and $\alpha = 0$.

***Example* 10.** Let $a_0 = 2((2/3)^{3/4} - 1)$. If $p \in \mathcal{H}[a_0, 1]$ satisfies the differential subordination

$$\frac{3(p(z)/2+1)^{4/3}}{2} + zp'(z)\left(\frac{p(z)}{2}+1\right)^{1/3} \prec \exp(z),$$

then $p(z) \prec q(z) \prec H(z)$, where $H(z) = 2((2e^z/3)^{3/4} - 1)$ and $q(z) = 2((2(e^z - 1)/(3z))^{3/4} - 1)$. Moreover, q is the best $(a_0, 1)$- dominant and is a convex function.

Proof. Let $h(z) = \exp(z)$, $n = 1$, $\alpha = -1/3$, $\beta = 1/2$ and $\gamma = 1$ in Theorem 3.3.3, then $Q(z) = (e^z - 1)/z$. Now h here with the given constants satisfy the hypothesis of Theorem 3.3.3, the assertion follows from the same. Moreover, in accordance with the proof of Theorem 3.3.3, we observe that

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\left(\frac{1 + e^z(z-1)}{e^z - 1}\right) \not> 0$$

and similarly the quantity $\operatorname{Re} zh'(z)/h(z) = \operatorname{Re} z \not> 0$, therefore both the quantities lie on either side of the imaginary axis. □

***Example* 11.** Let $a_0 = ((0.5)\tan(0.5))^{3/5}$. If $p \in \mathcal{H}[a_0, 1]$ satisfies

$$2\arctan(2p^{5/3}(z)) + \frac{20}{3}\left(\frac{zp^{2/3}(z)p'(z)}{1 + 4p^{10/3}(z)}\right) \prec \frac{2+z}{2-z},$$

then

$$p(z) \prec q(z) := \left(\tan\left(\frac{-1}{2} - \frac{2}{z}\log\left(\frac{2-z}{2}\right)\right)/2\right)^{3/5}. \tag{3.5.2}$$

Moreover, the function q is the best $(a_0, 1)$-dominant and is convex in $\mathbb{D}$.

Proof. Let $h(z) = \frac{2+z}{2-z}$, $n = 1$, $\alpha = -2/3$, $\beta = 1$ and $\gamma = 1/4$ in Theorem 3.3.6, then we get $Q(z) = -1 - ((4/z)\log((2-z)/z))$ and $q(z)$ is given by (3.5.2). As the function h and all the constants satisfy the hypothesis of Theorem 3.3.6, the result follows at once. Moreover, we observe that the quantity in (3.3.13) for the present case also lie on either side of the imaginary axis, which is in accordance with the proof of Theorem 3.3.6. □

Next, we obtain a sufficient condition for univalence of $f \in \mathcal{A}$ in the following result.

Theorem 3.5.1. Let $f \in \mathcal{A}$. If f satisfies

$$\operatorname{Re}(f'(z) + zf''(z)) > 0, \tag{3.5.3}$$

then f is univalent in $\mathbb{D}$ and in fact, $\operatorname{Re} f'(z) > 2\log 2 - 1$.

Proof. Let $p(z) = f'(z)$, then we have $p \in \mathcal{H}[1, 1]$. We observe that equation (3.5.3) is equivalent to equation (3.5.1) when $p(z) = f'(z)$. Since p satisfies the hypothesis of

Example 9, we obtain $\operatorname{Re} f'(z) > 2\log 2 - 1 > 0$, therefore f is univalent by Noshiro-Warschawski's result, which is stated in Theorem A. □

The result by Miller and Mocanu [66] with $n = 1$ is stated in the following lemma.

Lemma 3.5.2. [66] Let $a \in [0,1)$ and $\chi = \chi(a)$ be defined as

$$\chi = (2(1-a)\beta(1) + (2a-1))^{1/2},$$

where $\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt$. If $f \in \mathcal{A}$, then

$$\operatorname{Re} f'(z) > a \Rightarrow \operatorname{Re} \sqrt{\frac{f(z)}{z}} > \chi(a).$$

Corollary 3.5.3. Let $f \in \mathcal{A}$ and $a = 2\log 2 - 1$. If f satisfies the equation (3.5.3), then
(i) $\operatorname{Re} \sqrt{\frac{f(z)}{z}} > \chi(a)$.
(ii) $\operatorname{Re} \dfrac{f(z)}{z} > a$.

Proof. (i) The proof follows directly from Theorem 3.5.1 and Lemma 3.5.2.
(ii) Let $p(z) = f(z)/z$, then we have $p \in \mathcal{H}[1,1]$. Thus Remark 15 yields $\operatorname{Re} f'(z) > 0$ implies $\operatorname{Re}(f(z)/z) > a$. Thus the result follows now using Theorem 3.5.1. □

Concluding Remarks

We introduced a new concept of exact differential subordinations using the idea of exact differential equations on real line. The speciality of the two exact type of differential subordinations considered here, is that the dominants obtained in each of the cases are convex in $\mathbb{D}$. Furthermore, several examples are provided in support of our claims. One of the highlights of this chapter is that we generalized and improved the result obtained by Miller and Mocanu [67, Corollary 3.1e.1, p. 79].

Chapter 4

A Special Type of Ma-Minda Function

In the present chapter, we throw light on the geometrical significance of each of the conditions of Ma-Minda functions. Consequently, we introduce and examine a special type of Ma-Minda functions by differing the orientation of its Ma-Minda counterpart. Further, study a newly defined subclass of starlike functions involving a special type of Ma-Minda function for obtaining inclusion and radius results. In addition, we establish some majorization and Bloch function norms related results.

4.1 Introduction

In the previous chapters, we chiefly dealt with the differential subordinations results for Ma-Minda functions. However, there are problems like radius and inclusion, which are also studied by many authors in the past, that has glorified the theory in various possible ways, some of which are illustrated in [22, 65]. The estimation of coefficient bounds for normalized analytic functions in the classes associated with Ma-Minda functions has always been a trendy area of research, for instance there have been a number of published articles in this direction [55, 68, 87, 88, 107]. Motivated by these works, in this chapter, we examine the conditions imposed on Ma-Minda funtions, consequently introduce a *special type of Ma-Minda functions* and studied it in the direction of radius and inclusion results apart from majorization and Bloch function norm.

We come across the following observations, enlisted below, while examining the geometry of a function defined on $\mathbb{D}$ in general, which are of great use in deriving our results:

1. A function with real coefficients is always symmetric with respect to the real axis, but not conversely. For example, consider the functions:

$$f_1(z) = iz,\ f_2(z) = 1 + iz,\ f_3(z) = \frac{1+iz}{1-z^2},$$

   which are symmetric with respect to the real axis but do not have real coefficients. The converse holds under special conditions, namely if f is symmetric with respect to the real axis with $f(0) = 0$ and $f'(0)$ is some non zero real number, then the function f has real coefficients. To justify our claim, consider the functions f_1, f_2 and f_3, where $f_i'(0)$ is not a real number for all $i = 1,2,3$, therefore f_i do not have real coefficients despite being symmetric to real axis.

2. Let $f(z)$ be an analytic function with real coefficients and $f(0) = 0$. Then f is typically real if and only if its first coefficient is positive. Thus $\phi'(0) > 0$ implies $\phi - 1$ is typically real, where ϕ is a Ma-Minda function.

3. Geometrically, it is evident that the real part of an analytic function attains its maximum or minimum value on the real line if and only if the function is symmetric with respect to real axis and convex in the direction of imaginary axis.

In the past, authors do considered functions other than Ma-Minda. For instance Kargar et al. [41, 42] and Uralegaddi et al. [111] dealt with such functions to define their classes. In view of the same, in this chapter, we classify functions into Ma-Minda, non-Ma-Minda or a special type of Ma-Minda function. We now define Ma-Minda function on the basis of its deep rooted conditions:

Definition 4.1.1. An analytic univalent function ϕ with $\phi'(0) > 0$, satisfying:

A. $\operatorname{Re}\phi(z) > 0 \ (z \in \mathbb{D})$
B. $\phi(\mathbb{D})$ symmetric about the real axis and starlike with respect to $\phi(0) = 1$

is called a *Ma-Minda function* and let $\mathscr{M}$ denotes the class of all such functions. Let $\widetilde{\mathscr{M}_{\mathbf{A}}}$ denotes the class of all *non-Ma-Minda functions of type-**A***, which are obtained by relaxing the condition in **A**.

Note that the functions considered by Kargar et al. [42] and Uralegaddi et al. [111] belong to $\widetilde{\mathscr{M}_{\mathbf{A}}}$, see also [41].

The Ma-Minda function ϕ is considered as univalent and therefore $\phi'(0) \neq 0$. Since $\phi(\mathbb{D})$ is symmetric about the real axis and if $\phi'(0)$ is any non-zero real number, then ϕ has real coefficients. To address distortion theorem, Ma-Minda perhaps restricted $\phi'(0)$ to be positive instead of any non-zero real number. However, it plays no role in establishing the coefficient, radius, inclusion, subordination, and other similar results for the classes $\mathcal{C}(\phi)$ and $\mathcal{S}^*(\phi)$. This very fact, which is under gloom until now, has been brought to daylight in this chapter, by replacing the condition $\phi'(0) > 0$ with $\phi'(0) < 0$. Note that $\phi(z)$ and $\Phi(z) := \phi(-z)$ both map unit disk to the same image but with opposite orientation. Thus $\Phi(z)$ differs from its Ma-Minda counterpart by mere a rotation and is therefore non-typically real, but still, image domain invariant and rest all properties are intact. Such $\Phi(z)$ can be considered as *a special type of Ma-Minda* function. We now premise the above notion in the following definition:

Definition 4.1.2. An analytic univalent function Φ defined on the unit disk $\mathbb{D}$ is said to be a *special type of Ma-Minda* if $\operatorname{Re}\Phi(\mathbb{D}) > 0$, $\Phi(\mathbb{D})$ is symmetric with respect to the real axis, starlike with respect to $\Phi(0) = 1$ and $\Phi'(0) < 0$. Further, it has a power series expansion of the form:

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} C_n z^n = 1 + C_1 z + C_2 z^2 + \cdots \ (C_1 < 0). \tag{4.1.1}$$

Let us denote the class of all such special type of Ma-Minda functions by $\mathscr{M}^{\circ}$.

Recently, Altinkaya et al. [7] considered a special type of Ma-Minda function, given by $g(z) = \alpha(1-z)/(\alpha-z)$, $(\alpha > 1)$ to define and study their class. Note that the classes $\mathcal{S}^*(\Phi)$ and $\mathcal{C}(\Phi)$ can be defined on the similar lines of (1.2.4), by replacing ϕ with Φ. We introduce here a special type of Ma-Minda function, given by

$$\psi(z) := 1 - \log(1+z) = 1 - z + \frac{z^2}{2} - \frac{z^3}{3} + \cdots, \tag{4.1.2}$$

which maps the unit disk onto a parabolic region for its boundary curve τ, see Figure 4.1. Although $\phi(\mathbb{D}) = \Phi(\mathbb{D})$, at times considering Φ is advantageous over its counterpart ϕ, which is evident from the example $1-\log(1+z)$, dealt here. Therefore, the special type of Ma-Minda functions can now be considered in defining Ma-Minda classes for computational convenience and especially in establishing distortion and growth bounds. We now list in Table 1, a few examples of $\phi \in \mathcal{M}$ and its counter part $\Phi \in \mathcal{M}^\circ$:

$\phi(z)$	$\Phi(z)$
$\cos\sqrt{-z}$	$\cos\sqrt{z}$
$\sqrt{1+z}$	$\sqrt{1-z}$
$1-\log(1-z)$	$1-\log(1+z)$

Table 4.1: Examples of Ma-Minda and its counter part special type of Ma-Minda functions.

Distortion and Growth Theorems: We define the functions $d_{\Phi n}(z)$ and $t_{\Phi n}(z)$ $(n = 1,2,3,\cdots)$ in a similar fashion as Ma-Minda [61] defines $k_{\phi n}(z)$ and $h_{\phi n}(z)$, respectively. Thus the structural formula of $d'_{\Phi n}$ and $t_{\Phi n}$ is given by:

$$d'_{\Phi n}(z) = \exp\int_0^z \frac{\Phi(t^n)-1}{t}dt \text{ and } t_{\Phi n}(z) = z\exp\int_0^z \frac{\Phi(t^n)-1}{t}dt. \tag{4.1.3}$$

Here, we set $d_\Phi := d_{\Phi 1}(z)$ and $t_\Phi := t_{\Phi 1}(z)$. Ma and Minda [61] proved the distortion and growth theorems for the functions in the classes $\mathcal{C}(\phi)$ and $\mathcal{S}^*(\phi)$, where $\phi \in \mathcal{M}$. However, these results differ for functions in $\mathcal{M}^\circ$, which we examine in the case of an example ψ, given by (4.1.2) and eventually generalize it. So, let us consider the class $\mathcal{C}(\psi)$, the structural formula, given in (4.1.3) yields:

$$d'_\psi(z) = \exp\sum_{k=1}^{\infty}\frac{(-z)^k}{k^2}.$$

A numerical computation shows that $d'_\psi(1/2) \approx 0.63864$ and $d'_\psi(-1/2) \approx 1.79004$. Let the function f be such that:

$$f'(z) = d'_{\psi 2} = \exp \sum_{k=1}^{\infty} \frac{(-1)^k (z)^{2k}}{2k^2},$$

clearly, $f \in \mathcal{C}(\psi)$. A numerical computation shows that $|f'(1/2)| \approx 0.88874$. Hence

$$d'_{\psi}(r) \leq |f'(z_0)| \leq d'_{\psi}(-r), \text{ for } z_0 = r = \frac{1}{2}.$$

Thus functions in $\mathcal{C}(\psi)$ violate distortion theorem bounds proved in [61], which shows that $\phi'(0) > 0$ is inevitable in obtaining the distortion theorem, given in [61].

Remark 16. Let $\phi \in \mathscr{M}$ and its counter part $\Phi \in \mathscr{M}^{\circ}$ then $\Phi(\mathbb{D}) = \phi(\mathbb{D})$, which implies $\mathcal{C}(\Phi) = \mathcal{C}(\phi)$ and $\mathcal{S}^*(\Phi) = \mathcal{S}^*(\phi)$. Therefore to obtain distortion and growth theorems for functions in $\mathcal{C}(\Phi)$ and $\mathcal{S}^*(\Phi)$, it suffices to replace $\phi(z)$ by $\Phi(-z)$, in the result [61, Corollary 1, p. 159].

Using the above Remark and the fact $d'_{\Phi}(z) = d'_{\phi}(-z)$, we deduce the growth and distortion bounds for functions in $\mathscr{M}^{\circ}$ as follows:

Theorem 4.1.1. Let $|z_0| = r < 1$.

(1) Suppose $f \in \mathcal{C}(\Phi)$. Then
   - (i) Growth Theorem: $d_{\Phi}(r) \leq |f(z_0)| \leq -d_{\Phi}(-r)$.
   - (ii) Distortion Theorem: $d'_{\Phi}(r) \leq |f'(z_0)| \leq d'_{\Phi}(-r)$.

(2) Suppose $f \in \mathcal{S}^*(\Phi)$. Then
   - (i) Growth Theorem: $t_{\Phi}(r) \leq |f(z_0)| \leq -t_{\Phi}(-r)$.
   - (ii) Distortion Theorem: $t'_{\Phi}(r) \leq |f'(z_0)| \leq t'_{\Phi}(-r)$, when we additionally assume $\min_{|z|=r} = |\Phi(z)| = \Phi(r)$ and $\max_{|z|=r} |\Phi(z)| = \Phi(-r)$.

The equality holds for some non zero z_0 if and only if f is a rotation of d_{Φ} and t_{Φ}, respectively for (1) and (2).

We now introduce the following classes involving ψ, a special type of Ma-Minda function:

$$\mathcal{S}^*_l := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 - \log(1+z) \right\} \text{ and } \mathcal{C}_l := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec 1 - \log(1+z) \right\}.$$

Let us now establish the structural formula for functions in $\mathcal{S}^*_l$. By the structural formula (4.1.3), we get a function $f \in \mathcal{S}^*_l$ if and only if there exists an analytic function

q, satisfying $q(z) \prec \psi(z)$ such that

$$f(z) = z\exp\left(\int_0^z \frac{q(t)-1}{t}dt\right). \tag{4.1.4}$$

Now, we provide some examples of functions in the class $\mathcal{S}^*_l$. For this, let us assume

$$\psi_1(z) = 1 - \frac{z}{6}, \psi_2(z) = \frac{4-z}{4+z}, \psi_3(z) = 1 - z\sin\frac{z}{4} \text{ and } \psi_4(z) = \frac{8-2z}{8+z}.$$

A geometrical observation leads to $\psi_i(\mathbb{D}) \subset \psi(\mathbb{D})$ $(i = 1,2,3,4)$. Thus $\psi_i(z) \prec \psi(z)$ for $z \in \mathbb{D}$. Now, the functions $f_i'^{}s$ belonging to the class $\mathcal{S}^*_l$ corresponding to each of the functions $\psi_i's$ are determined by the structural formula (4.1.4) as follows:

$$f_1(z) = z\exp\left(\frac{-z}{6}\right), f_2(z) = \frac{16z}{(4+z)^2}, f_3(z) = z\exp\left(4\left(1+\cos\frac{z}{4}\right)\right) \text{ and } f_4(z) = z - \frac{z^2}{8}.$$

In particular, for $q(z) = \psi(z) = 1 - \log(1+z)$, the corresponding function obtained is as follows:

$$f_0(z) = z\exp\left(\int_0^z \frac{-\log(1+t)}{t}dt\right) = z - z^2 + \frac{3}{4}z^3 - \frac{19}{36}z^4 + \frac{107}{288}z^5 + \cdots, \tag{4.1.5}$$

acts as an extremal function in many cases for $\mathcal{S}^*_l$.

Remark 17. The distortion and growth theorems for $\mathcal{C}_l$ and $\mathcal{S}^*_l$ can be obtained from that of $\mathcal{C}(\Phi)$ and $\mathcal{S}^*(\Phi)$, given in Theorem 4.1.1.

4.2 Radius Problems

Besides majorization, this section chiefly focuses on estimating various radius constants associated with $\mathcal{S}^*_l$. We begin with establishing the following bounds meant for $\mathcal{S}^*_l$:

Theorem 4.2.1. Let $f \in \mathcal{S}^*_l$. Then we have for $|z| = r < 1$,

$$1 - \log(1+r) \leq \operatorname{Re}\frac{zf'(z)}{f(z)} \leq 1 - \log(1-r) \tag{4.2.1}$$

and

$$\left|\operatorname{Im}\frac{zf'(z)}{f(z)}\right| \leq \tan^{-1}\left(\frac{r}{\sqrt{1-r^2}}\right). \tag{4.2.2}$$

The bounds are sharp.

Proof. Since $f \in \mathcal{S}_l^*$, we have $zf'(z)/f(z) \prec 1-\log(1+z)$. Thus by the definition of subordination, we have

$$\frac{zf'(z)}{f(z)} = 1-\log|q(z)| - i\arg(q(z)), \tag{4.2.3}$$

where $q(z) = 1+\omega(z)$, ω is a Schwarz function satisfying $\omega(0) = 0$ and $|\omega(z)| \le |z|$. Let $q(z) = u+iv$, then $(u-1)^2+v^2 < 1$. For $|z| = r$, we have

$$|q(z)-1| \le r. \tag{4.2.4}$$

Squaring both sides of the above equation yields

$$T : (u-1)^2+v^2 \le r^2. \tag{4.2.5}$$

Clearly T represents the equation of the disk with center: $(1,0)$ and radius r, for which the point $(0,0)$ lies outside the disk T. From (4.2.4) and (4.2.5), we have

$$|q(z)| \le 1+r \text{ and } |q(z)| \ge 1-r.$$

Since, $\log x$ is an increasing function on $[1,\infty)$, we have

$$\log(1-r) \le \log|q(z)| \le \log(1+r).$$

Thus we get the desired result (4.2.1) by considering real part in (4.2.3). If $v = au$, representing the equation of tangent to the boundary of disk T, which passes through the origin O, then the tangent and the boundary of the disk have a common point, hence from (4.2.5), we obtain

$$(1+a^2)u^2-2u+1-r^2 = 0.$$

By the definition of tangent, we get

$$1-(1+a^2)(1-r^2) = 0, \text{ further which yields } a = \pm\frac{r}{\sqrt{1-r^2}}.$$

Finally, we deduce

$$\tan^{-1}\left(\frac{-r}{\sqrt{1-r^2}}\right) \le \arg q(z) \le \tan^{-1}\left(\frac{r}{\sqrt{1-r^2}}\right),$$

the desired result (4.2.2) follows. □

Theorem 4.2.2. Let $f \in \mathcal{S}_l^*$. Then the followings hold:

(i) f is starlike of order ν in $|z| < \exp(1-\nu)-1$ whenever $1-\log 2 \le \nu < 1$.

(ii) $f \in \mathcal{M}(\kappa)$ in $|z| < 1-\exp(1-\kappa)$ whenever $\kappa > 1$.

(iii) f is convex of order ν in $|z| < \tilde{r}(\nu) < 1$ whenever $0 \le \nu < 1$, where $\tilde{r}(\nu)$ is the smallest positive root of the equation:

$$(1-r)(1-\log(1+r))(1-\log(1+r)-\nu)-r = 0, \tag{4.2.6}$$

for the given value of ν.

(iv) $f \in k-\mathcal{ST}$ in $|z| < r(k)$ whenever $k > 0$, where $r(k)$ is the smallest positive root of the equation

$$1+r-e(1-r)^k = 0, \tag{4.2.7}$$

for the given value of k. In particular, for $k = 1$, f is parabolic starlike in $|z| < \frac{e-1}{e+1}$.

(v) f is strongly starlike of order η in $|z| < r(\eta)$ whenever $0 < \eta \le \eta_0 \approx 0.514674$, where

$$r(\eta) = \sqrt{2\left(1-\frac{1}{\sqrt{1+\tan^2\left(\tan\frac{\eta\pi}{2}\right)}}\right)}. \tag{4.2.8}$$

Proof. (i) Since $f \in \mathcal{S}_l^*$, we obtain the following from Theorem 4.2.1.

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \ge 1-\log(1+r), \quad |z| = r < 1,$$

which yields the inequality $\operatorname{Re}(zf'(z)/f(z)) > \nu$, whenever $1-\log 2 \le \nu < 1$, which holds true in the open disk of radius $\exp(1-\nu)-1$. For the function f_0, given in (4.1.5) and $z_0 = \exp(1-\nu)-1$, we have $\operatorname{Re}(z_0 f_0'(z)/f_0(z)) = \nu$. Hence this result is sharp.

(ii) From Theorem 4.2.1, we get

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \le 1-\log(1-r), \quad |z| = r < 1,$$

which yields the inequality $\operatorname{Re}(zf'(z)/f(z)) < \kappa$, for $\kappa > 1$, that holds true in the open disk of radius $1-\exp(1-\kappa)$. For the function f_0, given in (4.1.5) and $z_0 = \exp(1-\kappa)-1$, we get

$$\operatorname{Re}\left(\frac{z_0 f_0'(z)}{f_0(z)}\right) = \kappa.$$

Therefore, the result is sharp.

(iii) Let $f \in \mathcal{S}_l^*$. Now, $f \in C(\nu)$, whenever

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \nu.$$

Since we have $zf'(z)/f(z) = 1-\log(1+\omega(z))$, where ω is a Schwarz function, we obtain

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}(1-\log(1+\omega(z))) - \operatorname{Re}\left(\frac{z\omega'(z)}{(1+\omega(z))(1-\log(1+\omega(z)))}\right). \tag{4.2.9}$$

The function ω satisfies the following inequality, given in [74]

$$|\omega'(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2}. \tag{4.2.10}$$

Using the above inequality, (4.2.9) reduces to

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \frac{(1-r)(1-\log(1+r))^2-r}{(1-r)(1-\log(1+r))} =: \chi(r).$$

Let $\chi(r,\nu) := \chi(r)-\nu = (1-r)(1-\log(1+r))(1-\log(1+r)-\nu)-r$. Clearly, $\chi(0,\nu)>0$ and $\chi(1,\nu)<0$ for all $\nu \in [0,1)$. Thus, there must exist $\tilde{r}(\nu)$ such that $\chi(r,\nu) \geq 0$ for all $r \in [0,\tilde{r}(\nu)]$, where $\tilde{r}(\nu)$ is the smallest positive root of the equation (4.2.6). Hence the result.

(iv) Let $f \in \mathcal{S}_l^*$. Now, $f \in k-\mathcal{ST}$ whenever

$$\operatorname{Re}(1-\log(1+\omega(z))) > k|\log(1+\omega(z))| \quad (z \in \mathbb{D}),$$

for some Schwarz function ω. Let $g(\omega(z)) := |\log(1+\omega(z))| = |\log|1+\omega(z)| + i\arg(1+\omega(z)|$, $\omega(z) = Re^{it}$, where $R \leq |z| = r < 1$ and $-\pi < t < \pi$. Now, consider

$$g^2(R,t) = \left(\frac{1}{2}\log(1+R^2+2R\cos t)\right)^2 + \left(\arctan\left(\frac{R\sin t}{1+R\cos t}\right)\right)^2,$$

which upon partially differentiating with respect to t, yields

$$h(R,t) := \frac{R\left(2(R+\cos t)\arctan\dfrac{R\sin t}{1+R\cos t} - \log(1+R^2+2R\cos t)\sin t\right)}{1+R^2+2R\cos t}.$$

Then clearly, the function $h(R,t) \geq 0$ for $t \in [0,\pi]$ and $h(R,t) \leq 0$ for $t \in [-\pi,0]$. Thus

$\max_{-\pi\le t\le\pi} g(R,t) = \max\{g(R,-\pi), g(R,\pi)\}$, which yields

$$|\log(1+\omega(z))| \le |\log(1-R)| \le |\log(1-r)|. \tag{4.2.11}$$

The inequality (4.2.11) and Theorem 4.2.1 reveal that the result follows at once by showing

$$1-\log(1+r) \ge k|\log(1-r)|,$$

whenever $m(r,k) := 1+r-e(1-r)^k \le 0$. Clearly, $m(0,k) < 0$ and $m(1,k) > 0$ for fixed value of k. Thus, there must exist $r(k)$ such that $m(r,k) \le 0$ for all $r \in [0, r(k)]$, where $r(k)$ is the smallest positive root of the equation (4.2.7). Hence the result.

(v) Since $f \in \mathcal{S}_l^*$, we have

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \le |\arg(1-\log(1+z))|.$$

Now, $f \in \mathcal{SS}^*(\eta)$ whenever

$$\left|\arg\left(1-\frac{1}{2}\log((1+x)^2+y^2) - i\arctan\frac{y}{1+x}\right)\right| < \eta\frac{\pi}{2},$$

for $|z| = \sqrt{x^2+y^2} < 1$. Consider the function f_0, given in (4.1.5) and let us assume

$$z_0 = \frac{1}{\sqrt{1+A^2}} - 1 + i\frac{A}{\sqrt{1+A^2}},$$

where $A = \tan(\tan\eta\pi/2)$. We have $|z_0| = \sqrt{2(1-1/(\sqrt{1+A^2})} < 1$, whenever $\eta \le \eta_0$ and

$$\begin{aligned}\left|\arg(1-\log(1+z_0))\right| &= \left|\arg\frac{z_0 f_0'(z_0)}{f_0(z_0)}\right| \\ &= |\arg(1-i\arctan(\tan(\tan\eta\pi/2)))| \\ &= \left|\arctan(-\arctan(\tan(\tan\eta\pi/2)))\right| \\ &= \arctan(\arctan(\tan(\tan\eta\pi/2))) \\ &= \eta\pi/2.\end{aligned}$$

A geometrical observation reveals that $f \in \mathcal{SS}^*(\eta)$ whenever $|z| < r(\eta)$, where $r(\eta)$ is given by (4.2.8) and therefore the result is sharp. □

Our next result involves the concept of majorization. For any two analytic functions

f and g, we say f is *majorized* by g in $\mathbb{D}$, denoted by $f \ll g$, if there exists an analytic function $\mu(z)$ in $\mathbb{D}$, satisfying

$$|\mu(z)| \leq 1 \text{ and } f(z) = \mu(z)g(z).$$

The following majorization result involves the class $\mathcal{S}^*_l$:

Theorem 4.2.3. Let $f \in \mathcal{A}$. Suppose that $f \ll g$ in $\mathbb{D}$, where $g \in \mathcal{S}^*_l$. Then

$$|f'(z)| \leq |g'(z)|, \text{ for } |z| \leq \tilde{r},$$

where $\tilde{r}$ is the smallest positive root of the following equation:

$$(1-r^2)(1-\log(1+r)) - 2r = 0. \tag{4.2.12}$$

Proof. Since $g \in \mathcal{S}^*_l$, we have $zg'(z)/g(z) \prec 1-\log(1+z)$. Then there exists a Schwarz function $\omega(z)$ such that

$$\frac{zg'(z)}{g(z)} = 1-\log(1+\omega(z)). \tag{4.2.13}$$

Let $\omega(z) = Re^{it}$, $R \leq |z| = r < 1$ and $-\pi < t < \pi$. Consider the function

$$h(R,t) := |1-\log(1+Re^{it})|^2 = \left(1-\frac{1}{2}\log(1+R^2+2R\cos t)\right)^2 + \left(\arctan\frac{R\sin t}{1+R\cos t}\right)^2,$$

which upon differentiation with respect to t yields

$$h_t(R,t) := \frac{R\left(2(R+\cos t)\arctan\dfrac{R\sin t}{1+R\cos t} - (-2+\log(1+R^2+2R\cos t))\sin t\right)}{1+R^2+2R\cos t}.$$

Then clearly, the function $h_t(R,t) \geq 0$ for $t \in [0,\pi)$ and $h_t(R,t) \leq 0$ for $t \in [-\pi,0]$. Thus $\min_{-\pi\leq t<\pi} h(R,t) = h(R,0)$, which yields $|1-\log(1+\omega(z))| \geq 1-\log(1+R) \geq 1-\log(1+r)$. Further, condition (4.2.13) yields

$$\left|\frac{g(z)}{g'(z)}\right| = \frac{|z|}{|1-\log(1+\omega(z))|} \leq \frac{r}{1-\log(1+r)}. \tag{4.2.14}$$

By the definition of majorization, we get $f(z) = \mu(z)g(z)$, which upon differentiation, gives

$$f'(z) = \mu(z)g'(z) + g(z)\mu'(z) = g'(z)\left(\mu(z) + \mu'(z)\frac{g(z)}{g'(z)}\right). \tag{4.2.15}$$

The function μ satisfies the inequality (4.2.10), using this for μ and substituting the inequality (4.2.14) in (4.2.15), we obtain

$$|f'(z)| \leq K(r,\zeta)|g'(z)|,$$

where $K(r,\zeta) = \zeta + \frac{r(1-\zeta^2)}{(1-r^2)(1-\log(1+r))}$ for $|\mu(z)| = \zeta$ $(0 \leq \zeta \leq 1)$. To achieve the result, it suffices to show that

$$1 - K(r,\zeta) = \frac{(1-\zeta)((1-r^2)(1-\log(1+r)) - r(1+\zeta))}{(1-r^2)(1-\log(1+r))} \geq 0,$$

equivalent to show

$$\eta(r,\zeta) := (1-r^2)(1-\log(1+r)) - r(1+\zeta) \geq 0.$$

For $\zeta = 1$, $\eta(r,\zeta)$ attains its minimum value, which is given by

$$\eta(r,1) =: \eta(r) = (1-r^2)(1-\log(1+r)) - 2r.$$

Clearly, $\eta(0) = 1 > 0$ and $\eta(1) = -2 < 0$. In view of these inequalities there must exist $\tilde{r}$ such that $\eta(r) \geq 0$ for all $r \in [0,\tilde{r}]$, where $\tilde{r}$ is the smallest positive root of the equation (4.2.12). Hence the proof. □

4.3 Inclusion Relations

In this section, we give inclusion relations between the classes $\mathcal{S}^*_l$ and various other subclasses of starlike functions, namely $\mathcal{S}^*(\nu)$, $\mathcal{SS}^*(\eta)$, $\mathcal{ST}(1,b)$ and $\mathcal{S}^*(q_c)$.

Theorem 4.3.1. The class $\mathcal{S}^*_l$ satisfies the following inclusion properties:

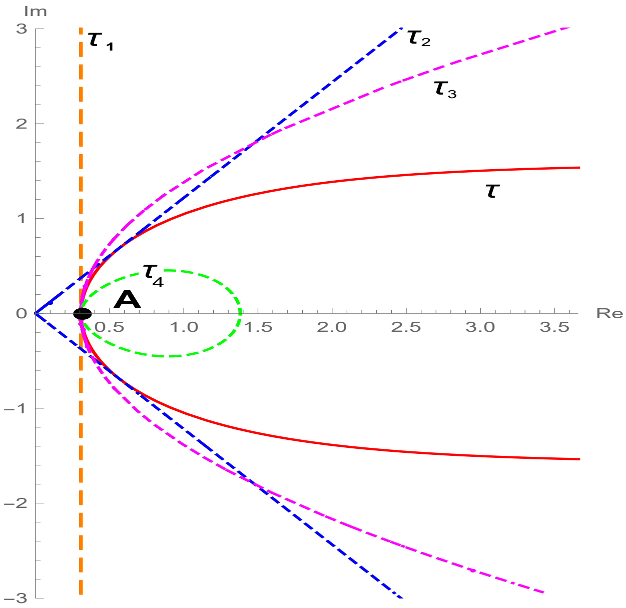
(i) $\mathcal{S}^*_l \subset \mathcal{S}^*(\nu) \subset \mathcal{S}^*$ for $0 \leq \nu \leq 1 - \log 2$.

(ii) $\mathcal{S}^*_l \subset \mathcal{SS}^*(\eta) \subset \mathcal{S}^*$ for $2\tilde{f}(\theta_0)/\pi \leq \eta \leq 1$, where θ_0 is the smallest positive root of the equation $-2 + \log(2(1+\cos\theta)) + \theta\tan\theta/2 = 0$ and $\tilde{f}(\theta) = \arg(1 - \log(1+e^{i\theta}))$, $\theta \in [0,\pi]$.

(iii) $\mathcal{S}^*_l \subset \mathcal{ST}(1,b)$ for $b \leq 1 - 2\log 2$.

(iv) $\mathcal{S}^*(q_c) \subset \mathcal{S}^*_l \subset \mathcal{S}^*$ for $c \leq c_0$, where $c_0 = \log 2(2 - \log 2)$.

The above constants in each part is best possible. The pictorial representation of the result is depicted in the following **Figure 4.1**.



$$\tau : |\exp(1-w)-1| = 1$$

$$\tau_1 : \operatorname{Re} w = 1-\log 2$$

$$\tau_2 : |\arg w| = \frac{7029}{12500}\frac{\pi}{2}$$

$$\tau_3 : \operatorname{Re} w - |w-1| = 1-2\log 2$$

$$\tau_4 : |w^2(z)-1| = \log 2(2-\log 2)$$

$$A = 1-\log 2$$

Figure 4.1: Boundary curves of best dominants and a subordinant of $\psi(z) = 1-\log(1+z)$.

Proof. (i) Since $f \in \mathcal{S}^*_l$, we have $zf'(z)/f(z) \prec 1-\log(1+z)$. Theorem 4.2.1 yields the following:

$$1-\log 2 = \min_{|z|=1} \operatorname{Re}(1-\log(1+z)) < \operatorname{Re}\frac{zf'(z)}{f(z)}.$$

Hence, the result follows.

(ii) Let $f \in \mathcal{S}^*_l$. Then, we have

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &< \max_{|z|=1}|\arg(1-\log(1+z))| \\ &= \max_{-\pi\leq\theta\leq\pi}\left|\arctan\left(\frac{-\theta}{2-\log(2(1+\cos\theta))}\right)\right| \\ &=: \max_{-\pi\leq\theta\leq\pi}|\tilde{f}(e^{i\theta})|.\end{aligned}$$

Due to the symmetricity of the function $\tilde{f}(\theta)$, we consider $\theta \in [0,\pi]$ and $\tilde{f}'(\theta) = 0$ yields $-2+\log(2(1+\cos(\theta_0)))+\theta_0\tan(\theta_0/2) = 0$, where $\theta_0 \approx 1.37502$. A calculation shows that $\tilde{f}''(\theta) < 0$, which implies $\max_{0\leq\theta\leq\pi}\tilde{f}(\theta) = \tilde{f}(\theta_0) \approx 0.88329$. Thus $f \in \mathcal{SS}^*(\eta)$, for the given range of η.

(iii) Let us consider the domain $\Omega_b := \{w \in \mathbb{C} : \operatorname{Re} w > |w-1|+b\}$, whose boundary represents a parabola, for $w = x+iy$, given by:

$$x = \frac{y^2}{2(1-b)} + \frac{1+b}{2},$$

whose vertex is given by: $((1+b)/2, 0)$. In order to prove the result, it suffices to show

$$\begin{aligned} h(\theta) &:= \operatorname{Re}(1-\log(1+z)) - |\log(1+z)| \\ &= 1 - \frac{1}{2}\log(2(1+\cos\theta)) - \sqrt{\frac{1}{4}\log^2(2(1+\cos\theta)) + \frac{\theta^2}{4}} > b, \end{aligned}$$

for $z = e^{i\theta}$. A numerical computation shows that

$$\min_{-\pi\leq\theta\leq\pi} h(\theta) = h(0) = 1 - 2\log 2.$$

Hence the result.

(iv) Since $f \in \mathcal{S}^*(q_c)$, we have $zf'(z)/f(z) \prec \sqrt{1+cz}$ and

$$\sqrt{1-c} = \min_{|z|=1} \sqrt{1+cz} < \operatorname{Re}\frac{zf'(z)}{f(z)} < \max_{|z|=1} \sqrt{1+cz} = \sqrt{1+c}.$$

Similar analysis can be carried out for the imaginary part bounds and therefore by using Theorem 4.2.1, we deduce the result. □

4.4 Further Results

We recall the set $\mathcal{B}$, the space of all Bloch functions. An analytic function f is said to be a Bloch function if it satisfies

$$\chi_{\mathcal{B}}(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty. \tag{4.4.1}$$

Also, $\mathcal{B}$ is a Banach space with the norm $||.||_{\mathcal{B}}$ defined by

$$||f||_{\mathcal{B}} = |f(0)| + \chi_{\mathcal{B}}(f), \; f \in \mathcal{B}. \tag{4.4.2}$$

Now, we give below a result involving Bloch function norm for the functions in the class $\mathcal{S}^*_l$.

Theorem 4.4.1. The set $\mathcal{S}^*_l \subseteq \mathcal{B}$. Furthermore, if $f \in \mathcal{S}^*_l$, then $||f||_{\mathcal{B}} \leq x$, where $x \approx 1.27429$.

Proof. If $f \in \mathcal{S}^*_l$, then $zf'(z)/f(z)g(z) := 1 - \log(1+\omega(z))$. By the structural formula, given in (4.1.4), we have

$$f(z) = z\exp\left(\int_0^z \frac{g(t)-1}{t}dt\right).$$

Upon differentiating f and further considering the modulus, we obtain

$$\begin{aligned}|f'(z)| =&|g(z)|\left|\exp\int_0^z \frac{g(t)-1}{t}dt\right|\\ \leq&|1-\log(1+\omega(z))|\exp\left(\int_0^z \frac{|\log(1+\omega(t))|}{|t|}dt\right). \end{aligned} \tag{4.4.3}$$

Let $t=re^{i\theta_1}$ and $\omega(t)=Re^{i\theta_2}$, where $R\leq r=|t|<1$ and $-\pi\leq\theta_1,\theta_2<\pi$. Now, by using the similar technique used in the proof of part (v) of Theorem 4.2.2 and in Theorem 4.2.3, equation (4.4.3) reduces to

$$|f'(z)|\leq(1-\log(1-R))\exp\left(|\log(1-R)|\int_{-\pi}^{\pi} e^{i\theta_1}d\theta_1\right)\leq 1-\log(1-r).$$

Thus we have $(1-|z|^2)|f'(z)|\leq g(r):=(1-r^2)(1-\log(1-r))$, which upon differentiation, gives $g'(r)=1-r+2r\log(1-r)$. By taking $g'(r)=0$, yields $r_0\approx 0.453105$. Now $g''(r_0)<0$, yields $\max_{0\leq r<1} g(r)=g(r_0)\approx 1.27429<\infty$. Using (4.4.1), we obtain $\mathcal{S}^*_l\subseteq\mathcal{B}$. We can now, estimate the norm $\|f\|_{\mathcal{B}}$ for the functions in the class $\mathcal{S}^*_l$. By using the definition of norm, given in (4.4.2), we have $\|f\|_{\mathcal{B}}\leq f(0)+1.27429$. By the normalization of the function f, the result follows at once. □

The following theorem gives the sufficient condition for the given function g to belong to the class $\mathcal{S}^*_l$.

Theorem 4.4.2. Let $m,n\geq 1$ and $0\leq\lambda\leq 1$. Then, $g(z)=z\exp(\alpha)\in\mathcal{S}^*_l$, where

$$\alpha=\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\lambda\left(\frac{(-z)^{nk}}{n}-\frac{(-z)^{mk}}{m}\right)+\frac{(-z)^{mk}}{m}\right).$$

Proof. For the given α, we have

$$g(z)=z\left(\exp\left(\frac{(\lambda-1)}{m}z^m\left(1-\frac{z^m}{4}+\frac{z^{2m}}{9}-\cdots\right)-\frac{\lambda}{n}z^n\left(1-\frac{z^n}{4}+\frac{z^{2n}}{9}-\cdots\right)\right)\right).$$

Then, we have

$$\begin{aligned}\frac{zg'(z)}{g(z)}&=1+(\lambda-1)z^m\left(1-\frac{z^m}{2}+\frac{z^{2m}}{3}-\cdots\right)-\lambda z^n\left(1-\frac{z^n}{2}+\frac{z^{2n}}{3}-\cdots\right)\\ &=\lambda(1-\log(1+z^n))+(1-\lambda)(1-\log(1+z^m)).\end{aligned}$$

We observe that $1-\log(1+z^t)\prec 1-\log(1+z)=:\psi(z)$ for all $t\geq 1$ and the function ψ is convex in $|z| < 1$. Thus the result follows at once when $0 \le \lambda \le 1$. □

When $m = n$, the above Theorem yields the following result:

Corollary 4.4.3. Let $n \ge 1$ and let $\alpha = \frac{1}{n}\left(\sum_{k=1}^{\infty} \frac{(-z)^{nk}}{k^2}\right)$. Then $g(z) = z\exp(\alpha) \in \mathcal{S}_l^*$.

Theorem 4.4.4. The class $\mathcal{S}_l^*$ is not a vector space.

Proof. For if, the class $\mathcal{S}_l^*$ is a vector space, then this class preserves an additive property, that is, whenever two functions belong to the class $\mathcal{S}_l^*$, then their sum also belongs to the class $\mathcal{S}_l^*$. Let f_1 and $f_2 \in \mathcal{S}_l^*$. Then, using (4.1.3), we obtain

$$f_1(z) = z\exp\left(\int_0^z \frac{-\log(1+\omega_1(t))}{t}dt\right) \text{ and } f_2(z) = z\exp\left(\int_0^z \frac{-\log(1+\omega_2(t))}{t}dt\right), \quad (4.4.4)$$

for some Schwarz functions ω_1 and ω_2. Now, sum of the functions, $f_1 + f_2$ to be in $\mathcal{S}_l^*$, there should exist some Schwarz function $\omega(z)$ such that

$$(f_1 + f_2)(z) = z\exp\left(\int_0^z \frac{-\log(1+\omega(t))}{t}dt\right). \quad (4.4.5)$$

Substituting equation (4.4.4) in (4.4.5), we get

$$\omega(z) = \frac{\exp(-z(A'(z)\exp A(z) + B'(z)\exp B(z)))}{\exp A(z) + \exp B(z)} - 1,$$

where

$$A(z) = \int_0^z \frac{-\log(1+\omega_1(t))}{t}dt \quad \text{and} \quad B(z) = \int_0^z \frac{-\log(1+\omega_2(t))}{t}dt.$$

Then $\omega(0) = 0$ and $|\omega(z)| < 1$ for all A and B. If we choose $\omega_1(z) = z$ and $\omega_2(z) = z^2$, then clearly $f_1(z)$ and $f_2(z)$ are the members of $\mathcal{S}_l^*$. In this case, $\omega(z) \approx 1.03053$ at $z = -\left(\frac{1}{2} + i\frac{2}{3}\right)$, which contradicts the existence of Schwarz function $\omega(z)$ satisfying $|\omega(z)| < 1$. Hence the assertion follows. □

The following theorem is an immediate consequence of the Growth Theorem of $\mathcal{S}^*(\Phi)$.

Theorem 4.4.5. Let $f \in \mathcal{S}_l^*$. Then we have

$$|f(z)| \le |z|\exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\right) = |z|L \quad (z \in \mathbb{D}),$$

where $L \approx 0.822467$.

Proof. In view of Remark 17, we get

$$t_\Psi(r) \leq |f(z)| \leq -t_\Psi(-r).$$

For $|z| = r$, we have

$$\log\left|\frac{f(z)}{z}\right| \leq \int_0^r \frac{\log(1+t)}{t}dt \leq \int_0^1 \frac{\log(1+t)}{t}dt = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

The convergent nature of the series on the right side of the above equality yields the desired result. □

Concluding Remarks

We classified Ma-Minda function based on its conditions and extensively studied all its geometric aspects. As a result, introduced a special type of Ma-Minda function Φ and defined the classes $\mathcal{S}^*(\Phi)$ (and $\mathcal{C}(\Phi)$) on the similar lines of $\mathcal{S}^*(\phi)$ (and $\mathcal{C}(\phi)$). Furthermore, studied a special type of Ma-Minda function, namely $1-\log(1+z)$ and obtained sharp radius and inclusion results for functions in class $\mathcal{S}^*(1-\log(1+z))$. Also, the idea of non-Ma-Minda and a special type of Ma-Minda coined here can be used as a future scope to define new classes and studied in the direction pointed out here.

Chapter 5

Coefficient Estimates of Certain Analytic Functions

In this chapter, we find the sharp bounds of various initial coefficients and certain Hankel determinants for functions in $\mathcal{A}(g,h,\varphi)$, where φ is a Ma-Minda function or a special type of Ma-Minda function. Special cases of our class $\mathcal{A}(g,h,\varphi)$, in particular $\mathcal{SL}^*$ and $\mathcal{S}_l$ are extensively studied for coefficient estimates.

5.1 Introduction

As a consequence of the concept of non-Ma-Minda and special type of Ma-Minda functions introduced in the previous chapter, an avenue opens up for further exploration in parallel to the concept of Ma-Minda function. This chapter, deals with both type of classes, involving *Ma-Minda function* and the *special type of Ma-Minda function* for various coefficient related bound estimates.

We first consider a special case of our class, given by (1.2.3), involving Ma-Minda function as follows:

$$\mathcal{M}_{g,h}(\phi) := \left\{ f \in \mathcal{A} : \frac{(f*g)(z)}{(f*h)(z)} \prec \phi(z),\, \phi \in \mathscr{M} \right\}, \tag{5.1.1}$$

where Taylor series expansion of g, h is given by (1.2.2) and $g_n, h_n > 0$ with $g_n - h_n > 0$. This class is studied by Murugusundaramoorthy et al. [70] and the authors have obtained Fekete-Szegö bound for functions in the class (5.1.1). In the present chapter, we extend the work by establishing the sharp bounds of fourth coefficient $|a_4|$, second Hankel determinant $|a_2a_4 - a_3^2|$ and the quantity $|a_2a_3 - a_4|$ for functions in the class $\mathcal{M}_{g,h}(\phi)$. The importance of this class lies in unification of various subclasses of $\mathcal{S}$, discussed in detail in the upcoming section. Note that as a special case, some of our results reduce to many earlier known results of Lee et al. [55], Mishra et al. [68] and Singh [101], which are pointed out here.

Now, we consider the counterpart $\mathcal{M}_{g,h}(\Phi)$ of the class $\mathcal{M}_{g,h}(\phi)$, by simply replacing ϕ by Φ, a special type of Ma-Minda function. By taking $g(z) = (z(1 + (2\alpha - 1)z))/(1 - z)^3$ and $h(z) = (z(1 + (\alpha - 1)z))/(1 - z)^2$ in $\mathcal{M}_{g,h}(\Phi)$, we obtain the following class:

$$\mathcal{M}_\alpha(\Phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z) + \alpha z^2 f''(z)}{\alpha z f'(z) + (1-\alpha) f(z)} \prec \Phi(z),\ (0 \le \alpha \le 1) \right\}.$$

Furthermore, the power series expansion of g and h, respectively yield

$$g_n = \begin{cases} 2(1+\alpha), & n = 2 \\ 3(1+2\alpha), & n = 3 \\ 4(1+3\alpha), & n = 4 \end{cases} \quad \text{and} \quad h_n = \begin{cases} 1+\alpha, & n = 2 \\ 1+2\alpha, & n = 3 \\ 1+3\alpha, & n = 4. \end{cases} \tag{5.1.2}$$

Set $\mathcal{S}_l(\alpha) := \mathcal{M}_\alpha(\psi)$, where $\psi(z) = 1 - \log(1 + z)$. Now, $\mathcal{S}_l^* := \mathcal{S}_l(0)$ and $\mathcal{C}_l := \mathcal{S}_l$. We obtain here the sharp bounds of initial coefficients such as a_2, a_3, a_4 and a_5, Fekete-Szegö functional and second Hankel determinant for functions in $\mathcal{S}_l(\alpha)$.

5.2 Coefficient Bounds for Functions in $\mathcal{M}_{g,h}(\phi)$ and $\mathcal{M}_{g,h}(\Phi)$

In this section, we establish sharp bounds of various initial coefficients and certain Hankel determinants for functions in $\mathcal{M}_{g,h}(\phi)$ and $\mathcal{M}_{g,h}(\Phi)$. Since the results obtained here involve the general Ma-Minda function ϕ and a general special type of Ma-Minda function Φ, so generalize many earlier well-known results. As a special case of $\mathcal{M}_{g,h}(\phi)$ and $\mathcal{M}_{g,h}(\Phi)$, we study the classes $\mathcal{SL}^*$ and $\mathcal{S}_l(\alpha)$, respectively for certain coefficient estimates.

Evidently the class $\mathcal{M}_{g,h}(\phi)$ unifies various subclasses of $\mathcal{S}$ for different choices of g and h. A few of the same are enlisted in the table below for ready reference:

$\mathcal{M}_{\mathbf{g,h}}(\phi)$	$(\mathbf{f}*\mathbf{g})(\mathbf{z})/(\mathbf{f}*\mathbf{h})(\mathbf{z})$	$\mathbf{g}(\mathbf{z})$	$\mathbf{h}(\mathbf{z})$
$\mathcal{S}^*(\phi)$	$\dfrac{zf'(z)}{f(z)}$	$\dfrac{z}{(1-z)^2}$	$\dfrac{z}{1-z}$
$\mathcal{S}^*_s(\phi)$	$\dfrac{2zf'(z)}{f(z)-f(-z)}$	$\dfrac{z}{(1-z)^2}$	$\dfrac{z}{1-z^2}$
$\mathcal{C}_s(\phi)$	$\dfrac{(2zf'(z))'}{(f(z)-f(-z))'}$	$\dfrac{z(1+z)}{(1-z)^3}$	$\dfrac{z(1+z^2)}{(1-z^2)^2}$
$\mathcal{M}_\alpha(\phi)$	$\dfrac{zf'(z)+\alpha z^2 f''(z)}{\alpha z f'(z)+(1-\alpha)f(z)}$	$\dfrac{z(1+(2\alpha-1)z)}{(1-z)^3}$	$\dfrac{z(1+(\alpha-1)z)}{(1-z)^2}$

Table 5.1: Various subclasses of $\mathcal{S}$ involving general ϕ for different choices of g and h in $\mathcal{M}_{g,h}(\phi)$.

Note that $\mathcal{S}^*_s(\phi)$ and $\mathcal{C}_s(\phi)$ are respectively, the class of starlike functions and convex functions with respect to symmetric points. A function $f\in\mathcal{S}$ is in the class $\mathcal{S}^*_s(\phi)$ if it satisfies

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \phi(z),\ z\in\mathbb{D}$$

and is in the class $\mathcal{C}_s(\phi)$ if it satisfies

$$\frac{(2zf'(z))'}{(f(z)-f(-z))'} \prec \phi(z),\ z\in\mathbb{D}.$$

We recall that the Taylor series expansion of $\phi(z) = 1 + \sum_{i=1}^{\infty} B_i z^i$ ($B_1 > 0$) and $\Phi(z) = 1 + \sum_{i=1}^{\infty} C_i z^i$ ($C_1 < 0$).

Murugusundaramoorthy et al. [70, Theorem 2.1, p. 250] in their work obtained the result of Fekete-Szegö functional bound for functions in the class $\mathcal{M}_{g,h}(\phi)$. Since $\mathcal{M}_{g,h}(\Phi(z)) = \mathcal{M}_{g,h}(\phi(z))$, we state below the parallel result for functions in $\mathcal{M}_{g,h}(\Phi(z))$ by simply replacing each B_i by $(-1)^i C_i$.

Theorem 5.2.1. If $f \in \mathcal{M}_{g,h}(\Phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{C_2}{g_3 - h_3} - \dfrac{\mu C_1^2}{(g_2 - h_2)^2} + \dfrac{(g_2 h_2 - h_2^2) C_1^2}{(g_3 - h_3)(g_2 - h_2)^2}, & \mu \leq \kappa_1; \\ \dfrac{-C_1}{g_3 - h_3}, & \kappa_1 \leq \mu \leq \kappa_2; \\ \dfrac{-C_2}{g_3 - h_3} + \dfrac{\mu C_1^2}{(g_2 - h_2)^2} - \dfrac{(g_2 h_2 - h_2^2) C_1^2}{(g_3 - h_3)(g_2 - h_2)^2}, & \mu \geq \kappa_2, \end{cases}$$

where

$$\kappa_1 = \frac{(g_2 - h_2)^2 (C_2 + C_1) + h_2 (g_2 - h_2) C_1^2}{(g_3 - h_3) C_1^2} \text{ and } \kappa_2 = \frac{(g_2 - h_2)^2 (C_2 - C_1) + h_2 (g_2 - h_2) C_1^2}{(g_3 - h_3) C_1^2}.$$

The result is sharp for f satisfying:

$$\frac{(f * g)(z)}{(f * h)(z)} = \begin{cases} \Phi(z), & \mu < \kappa_1 \text{ or } \mu > \kappa_2; \\ \Phi(z^2), & \kappa_1 < \mu < \kappa_2; \\ \Phi(\Psi(z)), & \mu = \kappa_1; \\ \Phi(-\Psi(z)), & \mu = \kappa_2, \end{cases}$$

for some g and h, where $\Psi(z) = \dfrac{z(z + \eta)}{1 + \eta z}$ $\quad (0 \leq \eta \leq 1)$.

Remark 18. We find that the bound of Fekete-Szegö $|a_3 - \mu a_2^2| \leq B_1/2(g_3 - h_3)$ when $\sigma_1 \leq \mu \leq \sigma_2$, as stated in [70, Theorem 2.1, p. 250], is incorrect and it should be $|a_3 - \mu a_2^2| \leq B_1/(g_3 - h_3)$, which is appropriately corrected in Theorem 5.2.1 for our case.

The next result deals with the sharp second Hankel determinant bound for functions in $\mathcal{M}_{g,h}(\phi)$, which generalizes many earlier known results.

Theorem 5.2.2. Let $f \in \mathcal{M}_{g,h}(\phi)$ and either

$$(g_3 - h_3)^2 \leq L \quad \text{or} \quad L < (g_3 - h_3)^2 \leq 2L \tag{5.2.1}$$

holds, where $L=(g_2-h_2)(g_4-h_4)$. Then

$$|a_2a_4-a_3^2|\le\begin{cases}\dfrac{B_1^2}{(g_3-h_3)^2}, & \text{when } X \text{ holds ;}\\[2ex] \dfrac{|M|}{(g_2-h_2)^4(g_3-h_3)^2(g_4-h_4)}, & \text{when } Y_1 \text{ or } Y_2 \text{ holds ;}\\[2ex] \dfrac{B_1^2(|T|+B_1(g_3-h_3)^2(g_2-h_2)-2B_1(g_2-h_2)^2(g_4-h_4))^2}{4S(g_3-h_3)^2(g_4-h_4)} \\ \quad +\dfrac{B_1^2}{(g_3-h_3)^2}, & \text{when } Z \text{ holds,}\end{cases}$$

where,

$$X:|M|-B_1^2(g_2-h_2)^4(g_4-h_4)\le 0 \text{ and } |T|+B_1(g_3-h_3)^2(g_2-h_2)-2B_1(g_2-h_2)^2(g_4-h_4)\le 0, \tag{5.2.2}$$

$$\begin{aligned}Y_1:&|T|+B_1(g_3-h_3)^2(g_2-h_2)-2B_1(g_2-h_2)^2(g_4-h_4)\ge 0 \text{ and } 2|M|-B_1|T|(g_2-h_2)^2\\&-B_1^2(g_3-h_3)^2(g_2-h_2)^3\ge 0,\\Y_2:&|M|-B_1^2(g_2-h_2)^4(g_4-h_4)\ge 0 \text{ and } |T|+B_1(g_3-h_3)^2(g_2-h_2)\\&-2B_1(g_2-h_2)^2(g_4-h_4)\le 0,\end{aligned}$$

$$\begin{aligned}Z:&|T|+B_1(g_3-h_3)^2(g_2-h_2)-2B_1(g_2-h_2)^2(g_4-h_4)>0 \text{ and } 2|M|-B_1|T|(g_2-h_2)^2\\&-B_1^2(g_3-h_3)^2(g_2-h_2)^3\le 0,\end{aligned} \tag{5.2.3}$$

with $S:=-(|M|-B_1|T|(g_2-h_2)^2-B_1^2(g_3-h_3)^2(g_2-h_2)^3+B_1^2(g_2-h_2)^4(g_4-h_4))$,

$$\begin{aligned}M:=\ &B_1^4\Big(-h_2^2(g_2-h_2)^2(g_4-h_4)+(g_3-h_3)\Big(g_2g_3h_2^2-g_3h_2^3+g_2^2h_2h_3-3g_2h_2^2h_3\\&+2h_2^3h_3+(g_3-h_3)(-g_2h_2^2+h_2^3)\Big)\Big)-B_2^2(g_2-h_2)^4(g_4-h_4)+B_1B_3(g_3-h_3)^2\\&(g_2-h_2)^3+B_1^2B_2\Big((g_3-h_3)(g_2-h_2)^2\Big(g_3h_2+g_2h_3-2h_2h_3-2h_2(g_2-h_2)(g_4-h_4)\Big)\Big)\end{aligned} \tag{5.2.4}$$

and

$$T := 2B_2(g_2-h_2)^2(g_4-h_4)+2B_1^2h_2(g_2-h_2)(g_4-h_4)-B_1^2g_3h_2(g_3-h_3)$$
$$-B_1^2g_2h_3(g_3-h_3)+2B_1^2h_2h_3(g_3-h_3)-2B_2(g_3-h_3)^2(g_2-h_2). \quad (5.2.5)$$

Proof. The series expansion of the functions f, g and h yield

$$\frac{(f*g)(z)}{(f*h)(z)} = 1+a_2(g_2-h_2)z+(a_3(g_3-h_3)+a_2^2h_2(h_2-g_2))z^2+(a_4(g_4-h_4)$$
$$-a_2(a_2^2h_2^2(-g_2+h_2)+a_3(g_3h_2+g_2h_3-2h_2h_3)))z^3+\cdots. \quad (5.2.6)$$

Here, we define a function p in $\mathcal{P}$ as follows:

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+p_1z+p_zz^2+\cdots. \quad (5.2.7)$$

Then, we have $\omega(z)=\frac{p(z)-1}{p(z)+1}$, clearly is a Schwarz function. Since $(f*g)(z)/(f*h)(z) \prec \phi(z)$, we get

$$\frac{(f*g)(z)}{(f*h)(z)}=\phi(\omega(z)). \quad (5.2.8)$$

Now using (5.2.6), (1.2.5) and expression of ω in terms of p in (5.2.8), we get

$$a_2=\frac{B_1p_1}{2(g_2-h_2)},\ a_3=\frac{B_2p_1^2(g_2-h_2)-B_1(p_1^2-2p_2)(g_2-h_2)+B_1^2p_1^2h_2}{4(g_2-h_2)(g_3-h_3)}$$

and

$$a_4=\frac{1}{8(g_2-h_2)(g_3-h_3)(g_4-h_4)}\Big(p_1(-2B_2p_1^2+B_3p_1^2+4B_2p_2)(g_2-h_2)(g_3-h_3)$$
$$+B_1^3p_1^3h_2h_3-B_1^2p_1(p_1^2-2p_2)\big(g_3h_2+(g_2-2h_2)h_3\big)+B_1\Big(p_1^3\big(g_2(g_3+(B_2-1)h_3)$$
$$+h_2\big((B_2-1)g_3+h_3-2B_2h_3\big)\big)-4p_1p_2(g_2-h_2)(g_3-h_3)+4p_3(g_2-h_2)(g_3-h_3)\Big)\Big).$$

Let us assume $p_1 =: p \in [0,2]$ and applying Lemma D, we deduce

$$a_2a_4-a_3^2 := \frac{1}{16(g_2-h_2)^4(g_3-h_3)^2(g_4-h_4)}\Big(p^4M-p^2\gamma(4-p^2)(g_2-h_2)^2B_1T$$
$$-(4-p^2)^2\gamma^2B_1^2(g_2-h_2)^4(g_4-h_4)-p^2\gamma^2(4-p^2)B_1^2(g_3-h_3)^2(g_2-h_2)^3$$
$$+2B_1^2p(4-p^2)(g_3-h_3)^2(g_2-h_2)^3\eta(1-|\gamma|^2)\Big),$$

where T and M are given by (5.2.5) and (5.2.4), respectively. Applying triangular inequality in the above equation with the assumption that $x := |\gamma|$, we obtain

$$\begin{aligned}|a_2a_4-a_3^2| \leq &\frac{1}{16(g_2-h_2)^4(g_3-h_3)^2(g_4-h_4)}\Big(|M|p^4+p^2(4-p^2)x^2B_1^2(g_3-h_3)^2(g_2-h_2)^3)\\ &+2B_1^2(g_3-h_3)^2(g_2-h_2)^3p(4-p^2)(1-x^2)+B_1^2(g_2-h_2)^4(g_4-h_4)(4-p^2)^2x^2\\ &+B_1|T|p^2x(4-p^2)(g_2-h_2)^2\Big) =: G(p,x).\end{aligned}$$

The function $G(p,x)$ is an increasing function of x in the closed interval [0,1], when either of the conditions in (5.2.1) hold. Thus $\max_{0\leq x\leq 1} G(p,x) = G(p,1) =: F(p)$. On solving further, $F(p)$ becomes

$$\begin{aligned}F(p) &= \frac{1}{16(g_2-h_2)^4(g_3-h_3)^2(g_4-h_4)}\bigg(\Big(|M|-B_1(g_2-h_2)^2\Big(B_1(g_2-h_2)^2(g_4-h_4)-|T|\\ &\quad -B_1(g_2-h_2)(g_3-h_3)^2\Big)\Big)p^4+4B_1(g_2-h_2)^2\Big(|T|+B_1(g_2-h_2)(g_3-h_3)^2\\ &\quad -2B_1(g_2-h_2)^2(g_4-h_4)\Big)p^2+16B_1^2(g_2-h_2)^4(g_4-h_4)\bigg)\\ &=: \frac{1}{16(g_2-h_2)^4(g_3-h_3)^2(g_4-h_4)}\Big(Ap^4+Bp^2+C\Big). \qquad (5.2.9)\end{aligned}$$

We recall that

$$\max_{0\leq t\leq 4}(At^2+Bt+C) = \begin{cases} C, & B\leq 0, A\leq -\frac{B}{4};\\ 16A+4B+C, & B\geq 0,\ A\geq -\frac{B}{8} \text{ or } B\leq 0,\ A\geq -\frac{B}{4};\\ \dfrac{4AC-B^2}{4A}, & B>0,\ A\leq -\frac{B}{8}.\end{cases} \qquad (5.2.10)$$

Now, simply (5.2.10) yields the desired result when applied on (5.2.9). □

Remark 19. We notice that the bound evaluated in [55, Theorem 1, p. 3] for the case (3), reproduced below:

$$|a_2a_4-a_3^2| \leq \frac{B_1^2}{12}\left(\frac{3|4B_1B_3-B_1^4-3B_2^2|-4B_1|B_2|+4B_1^2-|B_2|^2}{|4B_1B_3-B_1^4-3B_2^2|-2B_1|B_2|-B_1^2}\right),$$

has a typographical error and the corrected form will be:

$$|a_2a_4-a_3^2| \leq \frac{B_1^2}{12}\left(\frac{3|4B_1B_3-B_1^4-3B_2^2|-4B_1|B_2|-4B_1^2-|B_2|^2}{|4B_1B_3-B_1^4-3B_2^2|-2B_1|B_2|-B_1^2}\right).$$

Remark 20. In view of the first case (and fourth case with $\alpha = 1$) of Table 5.1, Theorem 5.2.2 reduces to the result obtained by Lee et al. [55] which provides the sharp second Hankel determinant bound for functions in Ma-Minda class $\mathcal{S}^*(\phi)$ (and $\mathcal{C}(\phi)$).

Remark 21. The second Hankel determinant bound for functions in the class $\mathcal{M}_{g,h}(\Phi)$ can be obtained from Theorem 5.2.2 by replacing each B_i by $(-1)^i C_i$.

The following couple of corollaries can be obtained from Theorem 5.2.2, in view of second and third cases of Table 5.1, respectively.

Corollary 5.2.3. Let $f \in \mathcal{S}_s^*(\phi)$. Then we have

$$|a_2a_4 - a_3^2| \leq \begin{cases} B_1^2/4, & \text{when X holds;} \\ |M|/256, & \text{when } Y_1 \text{ or } Y_2 \text{ holds ;} \\ \dfrac{B_1^2}{4} - \dfrac{B_1^2(|T|-24B_1)^2}{64(|M|-4B_1|T|+32B_1^2)}, & \text{when Z holds,} \end{cases}$$

where $M = 16B_1^2B_2 - 64B_2^2 + 32B_1B_3$, $T = 16B_2 - 4B_1^2$ and

$$X : |M| - 64B_1^2 \leq 0 \text{ and } |T| - 24B_1 \leq 0.$$
$$Y_1 : |M| - 2B_1|T| - 16B_1^2 \geq 0 \text{ and } |T| - 24B_1 \geq 0.$$
$$Y_2 : |M| - 64B_1^2 \geq 0 \text{ and } |T| - 24B_1 \leq 0.$$
$$Z : |M| - 2B_1|T| - 16B_1^2 \leq 0 \text{ and } |T| - 24B_1 > 0.$$

Corollary 5.2.4. Let $f \in \mathcal{C}_s(\phi)$. Then we have

$$|a_2a_4 - a_3^2| \leq \begin{cases} B_1^2/36, & \text{when X holds;} \\ |M|/147456, & \text{when } Y_1 \text{ or } Y_2 \text{ holds ;} \\ \dfrac{B_1^2}{36} - \dfrac{B_1^2(|T|-368B_1)^2}{2304(|M|-16B_1|T|+1792B_1^2)}, & \text{when Z holds,} \end{cases}$$

where $M = 128(9B_1^2B_2 - 32B_2^2 + 18B_1B_3)$, $T = 8(28B_2 - 9B_1^2)$ and,

$$X : |M| - 4096B_1^2 \leq 0 \text{ and } |T| - 368B_1 \leq 0.$$
$$Y_1 : |M| - 8B_1|T| - 1152B_1^2 \geq 0 \text{ and } |T| - 368B_1 \geq 0.$$
$$Y_2 : |T| - 368B_1 \leq 0 \text{ and } |M| - 4096B_1^2 \geq 0.$$
$$Z : |T| - 368B_1 > 0 \text{ and } |M| - 8B_1|T| - 1152B_1^2 \leq 0.$$

Remark 22. Note that the second Hankel determinant bound for the functions in the classes $\mathcal{S}_s^*(\Phi)$ and $\mathcal{C}_s(\Phi)$ can be obtained similarly to the Corollaries 5.2.3 and 5.2.4, respectively by replacing each B_i by $(-1)^i C_i$.

The class $\mathcal{M}_{g,h}(\phi)$ reduces to various subclasses of $\mathcal{S}$ (see Table 5.1), which are generalized subclasses involving ϕ and these can be further specialized for different choices of ϕ. In view of this, Theorem 5.2.2, besides being sharp, is of major importance in this chapter for it reduces to many previously known bounds of second Hankel determinant for various subclasses of $\mathcal{S}$ associated with specific function in place of ϕ, as illustrated in the following Table. Let $\xi(z) := \frac{1+z}{1-z}$ and $\xi_\nu(z) := \frac{1+(1-2\nu)z}{1-z}$.

$\mathbf{g(z)}$	$\mathbf{h(z)}$	$\phi(\mathbf{z})$	$\mathcal{M}_{\mathbf{g,h}}(\phi)$	$\lvert\mathbf{a_2a_4 - a_3^2}\rvert \leq$	**Reference**
$\dfrac{z(1+z)}{(1-z)^3}$	$\dfrac{z}{(1-z)^2}$	$\xi(z)$	$\mathcal{C}$	1/8	[36]
$\dfrac{z}{(1-z)^2}$	$\dfrac{z}{1-z^2}$	$\xi(z)$	$\mathcal{S}_s^*$	1	[68]
$\dfrac{z(1+z)}{(1-z)^3}$	$\dfrac{z(1+z^2)}{1-z^2}$	$\xi(z)$	$\mathcal{C}_s$	1/9	[68]
$\dfrac{z}{(1-z)^2}$	$\dfrac{z}{1-z}$	$\sqrt{1+z}$	$\mathcal{SL}^*$	1/16	[88]
		$\xi_\nu(z)$	$\mathcal{S}^*(\nu)$	$1-\nu^2$	[110]
		$\left(\dfrac{1+z}{1-z}\right)^\beta$	$\mathcal{SS}^*(\eta)$	η^2	[110]

Table 5.2: Various subclasses of $\mathcal{S}$ for particular values of g, h and ϕ in $\mathcal{M}_{g,h}(\phi)$.

Here we require the function $H(q_1,q_2)$, given in [6, Lemma 3], to establish our results in what follows. Expressing the fourth coefficient a_4 and $a_2a_3 - a_4$ for the function $f \in \mathcal{M}_{g,h}(\phi)$ in terms of the Schwarz function $\omega(z) = 1 + \omega_1 z + \omega_2 z^2 + \cdots$, we obtain the bounds as follows:

$$|a_4| \leq \frac{B_1}{g_4 - h_4} H(q_1, q_2), \tag{5.2.11}$$

where

$$q_1 = \frac{2B_2(g_2-h_2)(g_3-h_3) + B_1^2(g_3h_2 + g_2h_3 - 2h_2h_3)}{B_1(g_2-h_2)(g_3-h_3)} \tag{5.2.12}$$

and

$$q_2 = \frac{B_3(g_2-h_2)(g_3-h_3) + B_1^3 h_2h_3 + B_1B_2(g_3h_2 + g_2h_3 - 2h_2h_3)}{B_1(g_2-h_2)(g_3-h_3)}. \tag{5.2.13}$$

Also

$$|a_2a_3 - a_4| \leq \frac{B_1}{g_4 - h_4} H(q_1, q_2), \tag{5.2.14}$$

where

$$q_1 = \frac{2B_2(g_2-h_2)^2(g_3-h_3) + B_1^2(g_2-h_2)(-g_4+g_3h_2+g_2h_3-2h_2h_3+h_4)}{B_1(g_2-h_2)^2(g_3-h_3)} \tag{5.2.15}$$

and

$$q_2 = \frac{1}{B_1(g_2-h_2)^2(g_3-h_3)}\Big(B_3(g_2-h_2)^2(g_3-h_3) + B_1B_2(g_2-h_2)(-g_4+g_3h_2+g_2h_3-2h_2h_3 + h_4) + B_1^3h_2(-g_4+g_2h_3-h_2h_3+h_4)\Big). \tag{5.2.16}$$

Remark 23. Note that the bound of fourth coefficient and $|a_2a_3 - a_4|$ for functions in $\mathcal{M}_{g,h}(\Phi)$ can be obtained from (5.2.11) and (5.2.14), respectively by replacing each B_i by $(-1)^iC_i$.

5.3 About $\mathcal{SL}^*$, a Special Case of $\mathcal{M}_{g,h}(\phi)$

As a special case of the class $\mathcal{M}_{g,h}(\phi)$, here we consider the subclass $\mathcal{SL}^*$ [105], given by

$$\mathcal{SL}^* := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \quad z \in \mathbb{D} \right\},$$

where $g(z) = z/(1-z)^2$, $h(z) = z/(1-z)$ and $\phi(z) = \sqrt{1+z}$. Evidently, if we choose $\omega = zf'(z)/f(z)$ then the analytic characterization of the class $\mathcal{SL}^*$ can be expressed as $|\omega^2 - 1| < 1$, which indeed is the interior of the right loop of the lemniscate of Bernoulli, with the boundary equation $\gamma_1 : (u^2+v^2)^2 - 2(u^2-v^2) = 0$. In 2009, Sokół [103] obtained the sharp bounds of the initial coefficients a_2, a_3 and a_4 for functions in this class, further it is conjectured that $|a_{n+1}| \leq 1/(2n)$ whenever $n \geq 1$, with the extremal function f satisfying $zf'(z)/f(z) = \sqrt{1+z^n}$. Later Ravichandran and Verma [87] gave the proof for the sharp estimate of the fifth coefficient with the extremal function for functions in $\mathcal{SL}^*$ using the characterization of positive real part functions in terms of certain positive semi-definite Hermitian form. Sokół [104] also dealt the radius problems for the class $\mathcal{SL}^*$. Recently Ali et al. [4] have examined the radius of starlikeness associated with the lemniscate of Bernoulli. Some differential subordination results involving lemniscate of Bernoulli are studied in [3, 50].

For $\phi(z) = \sqrt{1+z}$, we have $B_1 = 1/2$, $B_2 = -1/8$, $B_3 = 1/16$, $B_4 = -5/128$ and $B_5 = 7/256$, where B_i's are the coefficients of $\phi(z)$.

Remark 24. For functions in the class $\mathcal{SL}^*$, Raza and Malik [88] obtained the Fekete-Szegö functional bound, which is a special case of the result obtained in [70, Theorem 2.1, p. 250].

In view of the first case of Table 5.1 for $\phi(z) = \sqrt{1+z}$, Theorem 5.2.2, equations (5.2.11) and (5.2.14), respectively yield the following couple of examples.

***Example* 12.** Let $f \in \mathcal{SL}^*$. Then

$$|a_2a_4 - a_3^2| \leq \frac{1}{16}.$$

Thus Theorem 5.2.2 generalizes the result obtained by Raza and Malik [88].

***Example* 13.** Let $f \in \mathcal{SL}^*$. Then
(i) $|a_4| \leq 1/6$
(ii) $|a_2a_3 - a_4| \leq 1/6$.
The result is Sharp.

Remark 25. The above Examples are the results obtained in [88] for functions in the class $\mathcal{SL}^*$.

5.3.1 Sharp Bounds of Hankel Determinants $H_3(1)$ and $H_2(3)$

In 2013, Raza and Malik [88] obtained the sharp bounds of $|H_2(1)|$ and $|H_2(2)|$ and the upper bound of $|H_3(1)|$ for functions in the class $\mathcal{SL}^*$. Thus the sharp estimate of $|H_3(1)|$ and $|H_2(3)|$ for $\mathcal{SL}^*$ was open until now.

The upper bound of $|H_3(1)|$ was proved to be $\frac{43}{576}$ (see [88]), which has been eventually improved in this section to a sharp estimate of $\frac{1}{36}$. We use the novel idea of incorporating the recently derived formula for the fourth coefficient of Carathéodory functions, in place of the routine triangle inequality to achieve the sharp bounds of the Hankel determinants $H_3(1)$ and $H_2(3)$ for functions in the well-known class $\mathcal{SL}^*$.

Theorem 5.3.1. If $f \in \mathcal{SL}^*$, then we have

$$|H_3(1)| \leq \frac{1}{36}. \tag{5.3.1}$$

The bound is sharp.

Proof. Let $f \in \mathcal{SL}^*$ then from [87, p. 509], we have

$$a_2 = \frac{p_1}{4}, \quad a_3 = \frac{1}{8}p_2 - \frac{3}{64}p_1^2, \quad a_4 = \frac{1}{12}p_3 - \frac{7}{96}p_1p_2 + \frac{13}{768}p_1^3 \tag{5.3.2}$$

and

$$a_5 = -\frac{1}{16}\left(\frac{49}{384}p_1^4 - \frac{17}{24}p_1^2p_2 + \frac{1}{2}p_2^2 + \frac{11}{12}p_1p_3 - p_4\right). \tag{5.3.3}$$

On simplifying the equation (1.3.6), we get

$$H_3(1) = 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5. \tag{5.3.4}$$

Since the class $\mathcal{P}$ is invariant under the rotation, the value of p_1 lies in the interval $[0,2]$. Let $p := p_1$ and substituting the above values of $a_i's$ in (5.3.4), we obtain

$$\begin{aligned} H_3(1) = &\frac{1}{2359296}\Big(689p^6 - 3368p^4p_2 + 3520p^3p_3 + 24064pp_2p_3 + 3008p^2p_2^2 \\ &- 16128p^2p_4 - 13824p_2^3 - 16384p_3^2 + 18432p_2p_4\Big). \end{aligned}$$

Using the equalities (1.3.3)-(1.3.5) and upon simplification, we arrive at

$$H_3(1) = \frac{1}{2359296}\Big(\nu_1(p,\gamma) + \nu_2(p,\gamma)\eta + \nu_3(p,\gamma)\eta^2 + \psi(p,\gamma,\eta)\rho\Big).$$

Where $\rho, \eta, \gamma \in \overline{\mathbb{D}}$,

$$\begin{aligned} \nu_1(p,\gamma) &:= 29p^6 + 944p^2\gamma^2(4-p^2)^2 - 640p^2\gamma^3(4-p^2)^2 - 2304\gamma^3(4-p^2)^2 + 128p^2\gamma^4(4-p^2)^2 \\ &\quad - 116p^4\gamma(4-p^2) + 752p^4\gamma^2(4-p^2) - 3456p^2\gamma^2(4-p^2) - 864p^4\gamma^3(4-p^2), \\ \nu_2(p,\gamma) &:= (4-p^2)(1-|\gamma|^2)\Big(224p^3 + 3456p^3\gamma + (4-p^2)(2432p\gamma - 512p\gamma^2)\Big), \\ \nu_3(p,\gamma) &:= (4-p^2)(1-|\gamma|^2)\Big(4096(4-p^2) - 512|\gamma|^2(4-p^2) + 3456p^2\overline{\gamma}\Big), \\ \psi(p,\gamma,\eta) &:= (4-p^2)(1-|\gamma|^2)(1-|\eta|^2)\Big(-3456p^2 + 4608\gamma(4-p^2)\Big). \end{aligned}$$

Furthermore, by taking $x := |\gamma|$, $y := |\eta|$ and using the fact $|\rho| \le 1$, we have

$$|H_3(1)| \le \frac{1}{2359296}\Big(|\nu_1(p,\gamma)| + |\nu_2(p,\gamma)|y + |\nu_3(p,\gamma)|y^2 + |\psi(p,\gamma,\eta)|\Big) \le S(p,x,y),$$

where

$$S(p,x,y) := \frac{1}{2359296}\Big(s_1(p,x) + s_2(p,x)y + s_3(p,x)y^2 + s_4(p,x)(1-y^2)\Big) \tag{5.3.5}$$

with

$$\begin{aligned}
s_1(p,x) &:= 29p^6 + 944p^2x^2(4-p^2)^2 + 640p^2x^3(4-p^2)^2 + 2304x^3(4-p^2)^2 + 128p^2x^4(4-p^2)^2\\
&\qquad + 116p^4x(4-p^2) + 752p^4x^2(4-p^2) + 3456p^2x^2(4-p^2) + 864p^4x^3(4-p^2),\\
s_2(p,x) &:= (4-p^2)(1-x^2)(224p^3 + (4-p^2)(2432px + 512px^2) + 3456p^3x),\\
s_3(p,x) &:= (4-p^2)(1-x^2)(4096(4-p^2) + 512x^2(4-p^2) + 3456p^2x),\\
s_4(p,x) &:= (4-p^2)(1-x^2)(3456p^2 + 4608x(4-p^2)).
\end{aligned}$$

Now we need to maximize $S(p,x,y)$ in the closed cuboid $T : [0,2]\times[0,1]\times[0,1]$. We establish this by finding the maximum values in interior of the six faces, on the twelve edges and in the interior of T.

I. First we proceed with interior points of T. Let $(p,x,y) \in (0,2)\times(0,1)\times(0,1)$. In an attempt to find the points where maximum value is attained in the interior of T, we partially differentiate equation (5.3.5) with respect to y and on algebraic simplification, we get

$$\begin{aligned}
\frac{\partial S}{\partial y} = &\ \frac{1}{73728}(4-p^2)(1-x^2)(8y(x-1)(4(4-p^2)(x-8) + 27p^2)\\
&+ p(4x(4-p^2)(19+4x) + p^2(7+108x))).
\end{aligned}$$

Now $\frac{\partial S}{\partial y} = 0$ yields

$$y = \frac{p(4x(4-p^2)(19+4x) + p^2(7+108x))}{8(x-1)(4(4-p^2)(8-x) - 27p^2)} =: y_0.$$

For the existence of critical points, y_0 should lie in the interval $(0,1)$, which is possible only when

$$p^3(7+108x) + 4px(4-p^2)(19+4x) + 32(1-x)(8-x)(4-p^2) < 216p^2(1-x) \tag{5.3.6}$$

and $27p^2 > 4(4-p^2)(8-x)$. Now, to find solutions for these equations, we proceed by hit and trial method. If we assume p tends to 2, then equation (5.3.6) holds for all $x < \frac{101}{216}$. In fact, we observe for all $x \in \left(\frac{101}{216}, 1\right)$, there does not exist any $p \in (0,2)$ such that equation (5.3.6) holds. Now, if we assume x tends to 0, then equation (5.3.6) holds for all $p > 1.48946$. In fact, a numerical computation shows that for $p \in (0, 1.48946)$, there does not exist any $x \in (0,1)$ such that equation (5.3.6) holds. Thus $(1.48946, 2) \times \left(0, \frac{101}{216}\right)$ is the domain for the solutions of equation (5.3.6). Now, further calculation shows that $\frac{\partial S}{\partial p}\Big|_{y=y_0} \neq 0$ in $(1.48946, 2) \times \left(0, \frac{101}{216}\right)$. Hence the function S has no critical point in $(0,2) \times (0,1) \times (0,1)$.

II. Here we consider interior of all the six faces of the cuboid T.

On the face $p = 0$, $S(p,x,y)$ reduces to

$$r_1(x,y) := S(0,x,y) = \frac{2(1-x^2)(y^2(x-1)(x-8)+9x)+9x^3}{576}, \tag{5.3.7}$$

where $x, y \in (0,1)$. We note that r_1 has no critical point in $(0,1) \times (0,1)$ since

$$\frac{\partial r_1}{\partial y} = \frac{y(1-x^2)(x-1)(x-8)}{144} \neq 0, \quad x, y \in (0,1). \tag{5.3.8}$$

On the face $p = 2$, $S(p,x,y)$ reduces to

$$S(2,x,y) = \frac{29}{36864}, \quad x, y \in (0,1). \tag{5.3.9}$$

On the face $x = 0$, $S(p,x,y)$ reduces to

$$\begin{aligned} S(p,0,y) &= \frac{128y^2(512-364p^2+59p^4)+224p^3y(4-p^2)+13824p^2-3456p^4+29p^6}{2359296} \\ &=: r_2(p,y), \quad p \in (0,2) \text{ and } y \in (0,1). \end{aligned} \tag{5.3.10}$$

We solve $\frac{\partial r_2}{\partial y} = 0$ and $\frac{\partial r_2}{\partial p} = 0$ to determine the points where maxima occur. On solving $\frac{\partial r_2}{\partial y} = 0$, we get

$$y = y_1 := -\frac{7p^3}{8(128-59p^2)}. \tag{5.3.11}$$

For the given range of y, we should have $y_1 \in (0,1)$, which is possible only if $p > p_0$, $p_0 \approx 1.47292$. A computation shows that $\partial r_2/\partial p = 0$ implies

$$256y^2(-182+59p^2) - 112y(-12p+5p^3) + 87p^4 - 6912p^2 + 13824 = 0. \tag{5.3.12}$$

Substituting the value of y as y_1 from equation (5.3.11) in (5.3.12) and upon simplification, we obtain

$$75497472 - 107347968p^2 + 51265024p^4 - 8426096p^6 + 95167p^8 = 0. \tag{5.3.13}$$

A numerical computation shows that the solution of (5.3.13) in the interval $(0,2)$ is $p \approx 1.39732$. Thus r_2 has no critical point in $(0,2)\times(0,1)$.
On the face $x = 1$, $S(p,x,y)$ reduces to

$$r_3(p,y) := S(p,1,y) = \frac{36864 + 22784p^2 - 7920p^4 + 9p^6}{2359296}, \quad p \in (0,2). \tag{5.3.14}$$

Solving $\frac{\partial r_3}{\partial p} = 0$, we get a critical point at $p =: p_0 \approx 1.2008$. A simple calculation shows that r_3 attains its maximum value ≈ 0.0225817 at p_0.

On the face $y = 0$, $S(p,x,y)$ reduces to

$$S(p,x,0) = \frac{1}{2359296}\Big(29p^6 + (4-p^2)((4-p^2)(944p^2x^2 + 640p^2x^3 - 2304x^3 + 4608x + 128p^2x^4) + 116p^4x + 752p^4x^2 + 864p^4x^3 + 3456p^2x^2\Big) =: r_4(p,x).$$

A computation shows that

$$\frac{\partial r_4}{\partial x} = \frac{1}{2359296}\Big((8192p^2 - 576p^4 + 512p^6)x^3 + (30720p^2 - 4992p^4 - 672p^6)x^2 + (30208p^2 - 9088p^4 + 384p^6)x + 73728 - 36864p^2 + 5072p^4 - 116p^6\Big),$$

$$\frac{\partial r_4}{\partial p} = \frac{1}{2359296}\Big((4096p - 4096p^3 + 768p^5)x^4 + (3840p - 6656p^3 - 1344p^5)x^3 + (30208p - 18176p^3 + 1152p^5)x^2 + (-73728p + 20288p^3 - 696p^5)x + 1344p - 13824p^3 + 174p^5\Big).$$

A numerical calculation shows that there does not exist any solution for the system of equations $\frac{\partial r_4}{\partial x} = 0$ and $\frac{\partial r_4}{\partial p} = 0$ in $(0,2)\times(0,1)$.

On the face $y = 1$, $S(p,x,y)$ reduces to

$$S(p,x,1) = \frac{1}{2359296}\Big(29p^6 + (4-p^2)(116p^4x + 752p^4x^2 + 3456p^2x^2 + 864p^4x^3 + (1-x^2)(224p^3 + 3456p^2x + 3456p^3x) + (4-p^2)((1-x^2)(2432px + 512px^2 + 4096 + 512x^2) + 944p^2x^2 + 640p^2x^3 + 2304x^3 + 128p^2x^4))\Big) =: r_5(p,x).$$

Proceeding on the similar lines as in the previous case for face $y = 0$, again there is no solution for the system of equations $\frac{\partial r_5}{\partial x} = 0$ and $\frac{\partial r_5}{\partial p} = 0$ in $(0,2)\times(0,1)$.

III. Now we calculate the maximum values achieved by $S(p,x,y)$ on the edges of cuboid T.
Considering the equation (5.3.10), we have

$$t_1(p) := S(p,0,0) = (29p^6 - 3456p^4 + 13824p^2)/2359296.$$

It is easy to verify that the function $t_1'(p) = 0$ for $p =: \lambda_0 = 0$ and $p =: \lambda_1 \approx 1.43285$ in the interval $[0,2]$. We observe that λ_0 is the point of minima and the maximum value of $t_1(p)$ is ≈ 0.00596162, attained at λ_1. Hence

$$S(p,0,0) \leq 0.00596162, \quad p \in [0,2].$$

Evaluating the equation (5.3.10) at $y = 1$, we obtain $t_2(p) := S(p,0,1) = (65536 - 32768p^2 + 896p^3 + 4096p^4 - 224p^5 + 29p^6)/2359296$. It is easy to verify that $t_2'(p)$ is a decreasing function in $[0,2]$, hence attains its maximum value at $p = 0$. Thus

$$S(p,0,1) \leq S(0,0,1) \leq \frac{1}{36}, \quad p \in [0,2].$$

In view of the equation (5.3.10) and by straightforward calculation, the maximum value of $S(0,0,y)$ is attained at $y = 1$. This implies

$$S(0,0,y) \leq S(0,0,1) = \frac{1}{36}, \quad y \in [0,1].$$

As equation (5.3.14) is independent of x, we have

$$S(p,1,1) = S(p,1,0) = t_3(p) := (9p^6 - 7920p^4 + 22784p^2 + 36864)/2359296.$$

Now, $t_3'(p) = 45568p - 31680p^3 + 54p^5 = 0$ for $p =: \lambda_2 = 0$ and $p =: \lambda_3 \approx 1.2008$ in the interval $[0,2]$, where λ_2 is a point of minima and $t_3(p)$ attains its maximum value at λ_3. We conclude that

$$S(p,1,1) = S(p,1,0) \leq 0.0225817, \quad p \in [0,2].$$

Substituting $p = 0$ in equation (5.3.14), we obtain $S(0,1,y) = 1/64$. The equation (5.3.9) is independent of all the variables p, x and y. Thus the value of $S(p,x,y)$ on the edges $p = 2$, $x = 1$; $p = 2$, $x = 0$; $p = 2$, $y = 0$ and $p = 2$, $y = 1$, respectively, is given by

$$S(2,1,y) = S(2,0,y) = S(2,x,0) = S(2,x,1) = 29/36864, \quad x,y \in [0,1].$$

Equation (5.3.10), yields $S(0,0,y) = y^2/36$. A simple computation shows that

$$S(0,0,y) \leq \frac{1}{36}, \quad y \in [0,1].$$

Using equation (5.3.7), we get $t_4(x) := S(0,x,1) = (16 - 14x^2 + 9x^3 - 2x^4)/576$. A simple computation shows that the function t_4 is decreasing in $[0,1]$ and hence attains its maximum value at $x = 0$. Thus

$$S(0,x,1) \leq S(0,0,1) = \frac{1}{36}, \quad x \in [0,1].$$

Once again, by using equation (5.3.7), we get $t_5(x) := S(0,x,0) = -x(x^2-2)/64$. Performing a simple calculation, we get $t_5'(x) = 0$ for $x =: x_0 = \sqrt{2}/\sqrt{3}$ and for $0 \leq x < x_0$, t_5 is an increasing function and for $x_0 < x \leq 1$, it's a decreasing function. Thus it attains maximum value at x_0. Hence

$$S(0,x,0) \leq S(0,x_0,0) = 0.0170103, \quad x \in [0,1].$$

In view of the cases I-III, the inequality (5.3.1) holds. Let the function $f : \mathbb{D} \to \mathbb{C}$ be as follows

$$f(z) = z \exp\left(\int_0^z \frac{\sqrt{1+t^3} - 1}{t} dt \right) = z + \frac{z^4}{6} + \cdots . \tag{5.3.15}$$

The sharpness of bound $|H_3(1)|$ is justified by an extremal function f given by (5.3.15), which belongs to $\mathcal{SL}^*$. For this function f, we have $a_2 = a_3 = a_5 = 0$ and $a_4 = 1/6$, which clearly shows that $|H_3(1)| = 1/36$ using equation (5.3.4). This completes the proof. □

We now estimate the bound for the Hankel determinant $H_2(3)$.

Theorem 5.3.2. Let $f \in \mathcal{SL}^*$. Then we have

$$|H_2(3)| \leq \frac{1}{36}. \tag{5.3.16}$$

The result is sharp.

Proof. We proceed here on the similar lines as in the proof of Theorem 5.3.1. Now, substituting the equalities (5.3.2)-(5.3.3) in $H_2(3) = a_3a_5 - a_4^2$. and with the assumption $p_1 =: p \in [0,2]$, we deduce

$$H_2(3) = \frac{1}{1179648}\Big(103p^6 - 712p^4p_2 - 4608p_2^3 + 1984p^2p_2^2 + 5888pp_2p_3 - 160p^3p_3 - 8192p_3^2 - 3456p^2p_4 + 9216p_2p_4\Big). \tag{5.3.17}$$

Using the equalities (1.3.3)-(1.3.5) and simplifying the terms in the expression (5.3.17), we get

$$H_2(3) = \frac{1}{1179648}\Big(\zeta_1(p,\gamma) + \zeta_2(p,\gamma)\eta + \zeta_3(p,\gamma)\eta^2 + \xi(p,\gamma,\eta)\rho\Big),$$

where ρ, η and $\gamma \in \overline{\mathbb{D}}$,

$$\begin{aligned}
\zeta_1(p,\gamma) &:= -5p^6 - 80p^2\gamma^2(4-p^2)^2 + 64p^2\gamma^4(4-p^2)^2 + 160p^2\gamma^3(4-p^2)^2 - 4p^4\gamma(4-p^2) \\
&\quad - 104p^4\gamma^2(4-p^2) + 576p^2\gamma^2(4-p^2) + 144p^4\gamma^3(4-p^2), \\
\zeta_2(p,\gamma) &:= 16p(4-p^2)(1-|\gamma|^2)(-5p^2 - 36p^2\gamma - 16\gamma^2(4-p^2) - 20\gamma(4-p^2)), \\
\zeta_3(p,\gamma) &:= 64(4-p^2)(1-|\gamma|^2)(-32(4-p^2) - 4\gamma^2(4-p^2) - 9p^2\gamma), \\
\xi(p,\gamma,\eta) &:= 576(4-p^2)(1-|\gamma|^2)(1-|\eta|^2)(p^2 + 4\gamma(4-p^2)).
\end{aligned}$$

By taking $x := |\gamma|$, $y := |\eta|$ and using the fact $|\rho| \le 1$, we get

$$|H_2(3)| \le \frac{1}{1179648}\Big(|\zeta_1(p,\gamma)| + |\zeta_2(p,\gamma)|y + |\zeta_3(p,\gamma)|y^2 + |\xi(p,\gamma,\eta)|\Big) \le F(p,x,y),$$

$$F(p,x,y) := \frac{1}{1179648}\Big(q_1(p,x) + q_2(p,x)y + q_3(p,x)y^2 + q_4(p,x)(1-y^2)\Big) \tag{5.3.18}$$

with

$$\begin{aligned}
q_1(p,x) &:= 5p^6 + 80p^2x^2(4-p^2)^2 + 64p^2x^4(4-p^2)^2 + 160p^2x^3(4-p^2)^2 + 4p^4x(4-p^2) \\
&\quad + 104p^4x^2(4-p^2) + 576p^2x^2(4-p^2) + 144p^4x^3(4-p^2), \\
q_2(p,x) &:= 16p(4-p^2)(1-x^2)(5p^2 + 36p^2x + 16x^2(4-p^2) + 20x(4-p^2)), \\
q_3(p,x) &:= 64(4-p^2)(1-x^2)(32(4-p^2) + 4x^2(4-p^2) + 9p^2x), \\
q_4(p,x) &:= 576(4-p^2)(1-x^2)(p^2 + 4x(4-p^2)).
\end{aligned}$$

In order to complete the proof, we need to maximize the function $F(p,x,y)$ in the closed cuboid $T:[0,2]\times[0,1]\times[0,1]$. For this, we find the maximum values of F in T by considering all the twelve edges, interior of the six faces and in the interior of T.

I. We proceed with interior points of T. Let us assume $(p,x,y)\in(0,2)\times(0,1)\times(0,1)$. To determine the points where the maximum value occur in the interior of T, we partially differentiate equation (5.3.18) with respect to y and we obtain

$$\begin{aligned}\frac{\partial F}{\partial y} =&\frac{1}{73728}(4-p^2)(1-x^2)(8y(x-1)(4(4-p^2)(x-8)+9p^2)\\ &+p(4x(4-p^2)(5+4x)+p^2(5+36x))).\end{aligned}$$

Now, $\frac{\partial F}{\partial y}=0$ yields

$$y=\frac{p(4x(4-p^2)(5+4x)+p^2(5+36x))}{8(x-1)(4(4-p^2)(8-x)-9p^2)}=:y_1.$$

Now, y_1 should lie in the interval $(0,1)$ for the existence of the critical points. Thus we have

$$p^3(5+36x)+4px(4-p^2)(5+4x)+32(1-x)(8-x)(4-p^2)<72p^2(1-x) \tag{5.3.19}$$

and $4(4-p^2)(8-x)<9p^2$. We use hit and trial method to obtain solution for these inequalities. If p tends to 2, then equation (5.3.19) holds for all $x<\frac{31}{72}$. In fact, we observe for every $x\in\left[\frac{31}{72},1\right)$, there does not exist any $p\in(0,2)$ such that (5.3.19) holds. Let us now assume x tends to 0, then equation (5.3.19) holds for all $p>1.79154$. In fact, a numerical computation shows that for $p\in(0,1.79154]$, there does not exist any $x\in(0,1)$ such that equation (5.3.19) holds. Thus $(1.79154,2)\times\left(0,\frac{31}{72}\right)$ is the domain for the solutions of equation (5.3.19). Now, furthermore calculation shows that $\frac{\partial F}{\partial p}\Big|_{y=y_1}\neq 0$ in $(1.79154,2)\times\left(0,\frac{31}{72}\right)$. Therefore, we conclude F has no critical point in $(0,2)\times(0,1)\times(0,1)$.

II. Now, we consider the interior of all the six faces of the cuboid T.
On the face $p=0$,

$$k_1(x,y):=F(0,x,y)=\frac{1-x^2}{288}\Big(y^2(x-1)(x-8)+9x\Big),\quad x,y\in(0,1). \tag{5.3.20}$$

A simple calculation shows that $\partial k_1/\partial y=\partial r_1/\partial y$. Thus equation (5.3.8) implies k_1 has no critical point in $(0,1)\times(0,1)$.

On the face $p = 2$,

$$F(2, x, y) = \frac{5}{18432}, \quad x, y \in (0, 1). \tag{5.3.21}$$

On the face $x = 0$, $F(p, x, y)$ reduces to $F(p, 0, y)$, given by

$$k_2(p, y) := \frac{64y^2(512 - 292p^2 + 41p^4) + 80p^3y(4 - p^2) + 2304p^2 - 576p^4 + 5p^6}{1179648}, \tag{5.3.22}$$

$p \in (0, 2)$ and $y \in (0, 1)$. On solving $\frac{\partial k_2}{\partial y} = 0$, we get

$$y = \frac{5p^3}{8(41p^2 - 128)} =: y_1. \tag{5.3.23}$$

For the given range of y, y_1 should lie in the interval $(0, 1)$, which holds only if $p > p_0$, $p_0 \approx 1.7669$. The computation shows that $\frac{\partial k_2}{\partial p} = 0$ implies

$$y^2(5248p^2 - 18688) + 40y(12p - 50p^3) + 2304 - 1152p^2 + 15p^4 = 0. \tag{5.3.24}$$

Let $p > p_0$ and substituting equation (5.3.23) in (5.3.24) and performing lengthy calculation, we deduce

$$1048576 - 1196032p^2 + 449216p^4 - 57582p^6 + 615p^8 = 0. \tag{5.3.25}$$

The numerical computation shows that the solution of (5.3.25) for $p \in (0, 2)$ is $p =: p_0 \approx 1.35957$. Thus k_2 has no critical point in $(0, 2) \times (0, 1)$.
On the face $x = 1$,

$$k_3(p) := F(p, 1, y) = \frac{7168p^2 - 2000p^4 + 57p^6}{1179648}, \quad p \in (0, 2). \tag{5.3.26}$$

To attain maximum value of k_3, we solve $\partial k_3 / \partial p = 0$ and get critical point at $p =: p_0 \approx 1.39838$. Simple calculation shows that k_3 attains its maximum value ≈ 0.00576045 at p_0.

On the face $y = 0$,

$$\begin{aligned} F(p, x, 0) = &\frac{1}{1179648}\Big(5p^6 + (4 - p^2)((4 - p^2)(2304x(1 - x^2) + 80p^2x^2 + 160p^2x^3 + 64p^2x^4) \\ &+ 4p^4x + 576p^2x^2 + 104p^4x^2 + 144p^4x^3 + 576p^2(1 - x^2)\Big) =: k_4(p, x). \end{aligned}$$

A complex computation shows that

$$\frac{\partial k_4}{\partial p} = \frac{1}{589824}\Big(2304p - 1152p^3 + 15p^5 + (-18432p + 4640p^3 - 12p^5)x + (1280p - 448p^3 - 72p^5)x^2 + (20992p - 6016p^3 + 48p^5)x^3 + (1024p - 1024p^3 + 192p^5)x^4\Big)$$

and

$$\frac{\partial k_4}{\partial x} = \frac{1}{294912}\Big((p^2-4)((-256p^2 + 64p^4)x^3 + (6912 - 2208p^2 + 12p^4)x^2 + (-160p^2 - 12p^4)x - 2304 + 576p^2 - p^4)\Big).$$

The numerical computation shows that there does not exist any solution for the system of equations $\frac{\partial k_4}{\partial p} = 0$ and $\frac{\partial k_4}{\partial x} = 0$ in $(0,2)\times(0,1)$.

On the face $y = 1$, $F(p,x,y)$ reduces to $F(p,x,1)$ given by

$$k_5(p,x) := \frac{1}{1179648}\Big(5p^6 + (4-p^2)((4-p^2)(80p^2x^2 + 64p^2x^4 + 160p^2x^3 + (1-x^2)(256px^2 + 320px + 256(8+x^2))) + 4p^4x + 104p^4x^2 + 576p^2x^2 + 144p^4x^3 + (1-x^2)(80p^3 + 576p^3x + 576p^2x))\Big).$$

Proceeding on the similar lines as in the previous case on the face $y = 0$, again, the system of equations $\partial k_5/\partial p = 0$ and $\partial k_5/\partial x = 0$ have no solution in $(0,2)\times(0,1)$.

III. We now consider the maximum values attained by $F(p,x,y)$ on the edges of the cuboid T:
In view of the equation (5.3.22), we have $F(p,0,0) = l_1(p) := 5p^6 - 576p^4 + 2304p^2)/1179648$. It is easy to compute that $l_1'(p) = 0$ for $p =: \lambda_0 = 0$ and $p =: \lambda_1 \approx 1.43351$ in the interval $[0,2]$, where λ_0 is the point of minima and λ_1 is the point of maxima. Hence

$$F(p,0,0) \le 0.00198843, \quad p \in [0,2].$$

Again, considering the equation (5.3.22), we obtain $F(p,0,1) = l_2(p) := (32768 - 16384p^2 + 320p^3 + 2048p^4 - 80p^5 + 5p^6)/1179648$. Now, we note that l_2 is a decreasing function in $[0,2]$ and hence attains its maximum value at $p = 0$. Thus

$$F(p,0,0) \le F(0,0,0) = 1/36, \quad p \in [0,2].$$

Now, we observe that the equation (5.3.26) does not depend on the value of y, hence we get $F(p,1,1) = F(p,1,0) = l_3(p) := (7168p^2 - 2000p^4 + 57p^6)/1179648$. It is easy to verify that the function l_3 has two critical points at $p = 0$ and $p =: \lambda_2 \approx 1.39838$ in the interval $[0,2]$, where the maximum value is attained at λ_2. Thus

$$F(p,0,0) = F(p,1,0) \leq 0.0057645, \quad p \in [0,2].$$

On substituting $p = 0$ in (5.3.26), we get $F(0,1,y) = 0$. In view of equation (5.3.21), which is independent of all the variables p, x and y, the value of $F(p,x,y)$ on the edges $p = 2$, $x = 0$; $p = 2$, $x = 1$; $p = 2$, $y = 0$ and $p = 2$, $y = 1$, respectively, is given by

$$F(2,0,y) = F(2,1,y) = F(2,x,0) = F(2,x,1) = 5/18432, \quad x,y \in [0,1].$$

Evaluating equation (5.3.22) at $p = 0$, we get $l_4(y) := F(0,0,y) = y^2/36$. It is easy to verify that l_4 is an increasing function of y and hence attains maximum value at $y = 1$ in $[0,1]$. Thus

$$F(0,0,y) \leq F(0,0,1) = \frac{1}{36}, \quad y \in [0,1].$$

Using equation (5.3.20), we get $l_5(x) := F(0,x,1) = (8 - 7x^2 - x^4)/288$. Since l_5 is decreasing function in $[0,1]$, it attains maximum value at $x = 0$. Thus

$$F(0,x,1) \leq F(0,0,1) = \frac{1}{36}, \quad x \in [0,1].$$

Substituting $y = 0$ in equation (5.3.20), we obtain the function $F(0,x,0) = l_6(x) := x(1 - x^2)/32$. A simple calculation shows that the function $l_6'(x) = 0$ at $x =: x_0 = \sqrt{3}/3$ and it is increasing in $(0,x_0)$ and decreasing in $(x_0,1)$. Hence it attains the maximum value at $x = x_0$. Thus we conclude

$$F(0,x,0) \leq \frac{\sqrt{3}}{144}, \quad x \in [0,1].$$

Taking into account all the cases I-III, the inequality (5.3.16) holds. For the function given in (5.3.15), which belongs to the class $\mathcal{SL}^*$, $a_3 = a_5 = 0$ and $a_4 = 1/6$. Thus $|H_2(3)| = 1/36$ for this function, which also proves the result is sharp. This completes the proof. □

5.3.2 Further Results

In the following theorem, we obtain the Zalcman coefficient inequality for $n = 3$ for functions in the class $\mathcal{SL}^*$.

Theorem 5.3.3. Let $f \in \mathcal{SL}^*$. Then

$$|a_3^2 - a_5| \le \frac{1}{8}.$$

The estimate is sharp.

Proof. Using equations (5.3.2) and (5.3.3), we get

$$a_3^2 - a_5 = \frac{125}{12288}p_1^4 - \frac{43}{768}p_1^2 p_2 + \frac{3}{64}p_2^2 + \frac{11}{192}p_1 p_3 - \frac{1}{16}p_4. \tag{5.3.27}$$

Applying Lemma E with $a = 125/768$, $b = 43/72$, $c = 11/24$ and $d = 3/4$ in the equation (5.3.27), we obtain

$$|a_3^2 - a_5| \le \frac{1}{8}.$$

Let the function $f : \mathbb{D} \to \mathbb{C}$, be defined as follows:

$$f(z) = z\exp\left(\int_0^z \frac{\sqrt{1+t^4}-1}{t}dt\right) = z + \frac{z^5}{8} + \cdots. \tag{5.3.28}$$

The equality holds for the function given in (5.3.28), which belongs to $\mathcal{SL}^*$ as $a_3 = 0$ and $a_5 = 1/8$, which contributes to the sharpness of the inequality. This completes the proof. □

Now, we derive the necessary and sufficient condition for a function $f \in \mathcal{S}$ to belong to the class $\mathcal{SL}^*$ in the following theorem, involving the convolution concept.

Theorem 5.3.4. A function $f \in \mathcal{S}$ is in the class $\mathcal{SL}^*$ if and only if

$$\frac{1}{z}(f * H_t(z)) \neq 0, \quad (z \in \mathbb{D}) \tag{5.3.29}$$

where

$$H_t(z) = \frac{z}{(1-z)(1-S(t))}\left(\frac{1}{1-z} - S(t)\right)$$

and

$$S(t) = \sqrt{t} + i\left(\pm\sqrt{\sqrt{1+4t}-(t+1)}\right), \quad (0 < t < 2).$$

Proof. Define $p(z) = zf'(z)/f(z)$. As we know $p(0) = 1$, to prove the result, it suffices to show that $f \in \mathcal{SL}^*$ if and only if $p(z) \notin \gamma_1$, where

$$\gamma_1 = \{(u^2+v^2)^2 - 2(u^2-v^2) = 0\}.$$

By taking $u^2 = t$, we can give the parametric representation of the curve γ_1 as follows

$$S(t) = \sqrt{t} + i\left(\pm\sqrt{\sqrt{1+4t}-(t+1)}\right), \quad (0 < t < 2).$$

For $f \in \mathcal{S}$, we have

$$\frac{z}{(1-z)^2} * f(z) = zf'(z) \quad \text{and} \quad \frac{z}{1-z} * f(z) = f(z). \tag{5.3.30}$$

Using the above equations (5.3.29) and (5.3.30), we get

$$\frac{1}{z}(f * H_t(z)) = \frac{f(z)}{z(1-S(t))}\left(\frac{zf'(z)}{f(z)} - S(t)\right) \neq 0,$$

which clearly shows that $zf'(z)/f(z) \neq S(t)$. Hence $1/(z(f * H_t(z))) \neq 0$ if and only if $p(z) \notin \gamma_1$ if and only if $f \in \mathcal{SL}^*$. □

Theorem 5.3.5. The function

$$\Theta(z) = \frac{z}{1-\rho z}, \quad (z \in \mathbb{D})$$

belongs to the class $\mathcal{SL}^*$ if $|\rho| \leq 1/4$.

Proof. By the definition of the class $\mathcal{SL}^*$, it suffices to show that the following inequality holds for the given range of ρ.

$$\left|\left(\frac{1}{1-\rho z}\right)^2 - 1\right| < 1. \tag{5.3.31}$$

The above inequality (5.3.31) holds whenever

$$|2\rho z - \rho^2 z^2| < 1 + |\rho z|^2 - 2\,\mathrm{Re}(\rho z),$$

which in turn holds if

$$2|\rho z| \leq 1 - 2|\rho z|,$$

which holds if $|\rho| \leq \frac{1}{4}$. Hence the function $\Theta(z) \in \mathcal{SL}^*$. □

5.4 About $\mathcal{S}_l(\alpha)$, a Special Case of $\mathcal{M}_{g,h}(\Phi)$

Here to study a special case of the class $\mathcal{M}_{g,h}(\Phi)$, we choose a subclass $\mathcal{S}_l(\alpha)$, given by

$$\mathcal{S}_l(\alpha) = \left\{ f \in \mathcal{A} : \frac{zf'(z) + \alpha z^2 f''(z)}{\alpha z f'(z) + (1-\alpha) f(z)} \prec 1 - \log(1+z), \ (0 \leq \alpha \leq 1) \right\},$$

where $g(z) = z/(1-z)^2$, $h(z) = z/(1-z)$ and $\Phi(z) = \psi(z) := 1 - \log(1+z)$. In the following example, we establish a Fekete-Szegö result for the class $\mathcal{S}_l(\alpha)$, obtained by taking the aforesaid values of $\Phi(z)$, $g(z)$ and $h(z)$ in Theorem 5.2.1.

***Example* 14.** Let $f \in \mathcal{S}_l(\alpha)$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{3}{4(1+2\alpha)} - \dfrac{\mu}{(1+\alpha)^2}, & \mu \leq \dfrac{(1+\alpha)^2}{4(1+2\alpha)} =: \kappa_1; \\ \dfrac{1}{2(1+2\alpha)}, & \dfrac{(1+\alpha)^2}{4(1+2\alpha)} \leq \mu \leq \dfrac{5(1+\alpha)^2}{4(1+2\alpha)}; \\ \dfrac{\mu}{(1+\alpha)^2} - \dfrac{3}{4(1+2\alpha)}, & \mu \geq \dfrac{5(1+\alpha)^2}{4(1+2\alpha)} =: \kappa_2. \end{cases}$$

The result is sharp.

Proof. Since $f \in \mathcal{S}_l(\alpha) = \mathcal{M}_\alpha(\psi(z))$, we have $C_1 = -1$, $C_2 = 1/2$ and $C_3 = -1/3$. The result follows from Theorem 5.2.1 by substituting the values of $g_i's$ and $h_i's$ from (5.1.2). Equality holds whenever f satisfies:

$$\frac{(f*g)(z)}{(f*h)(z)} = \begin{cases} 1 - \log(1+z), & \mu < \kappa_1 \text{ or } \mu > \kappa_2; \\ 1 - \log(1+z^2), & \kappa_1 < \mu < \kappa_2; \\ 1 - \log\left(1 + \frac{z(z+\eta)}{1+\eta z}\right), & \mu = \kappa_1; \\ 1 - \log\left(1 - \frac{z(z+\eta)}{1+\eta z}\right), & \mu = \kappa_2. \end{cases}$$

□

***Example* 15.** Let $f \in \mathcal{S}_l(\alpha)$. Then

$$\text{(i)} \quad |a_3 - a_2^2| \leq \frac{1}{2(1+2\alpha)}, \quad \text{(ii)} \quad |a_3| \leq \frac{3}{4(1+2\alpha)}.$$

These inequalities are sharp.

The proof directly follows from Example 14.

***Example* 16.** Let $f \in \mathcal{S}_l(\alpha)$. Then

$$|a_2a_4 - a_3^2| \leq \begin{cases} \dfrac{1}{4(1+2\alpha)^2}, & 0 \leq \alpha \leq \frac{2+\sqrt{15}}{11}; \\ \dfrac{31\alpha^4 + 136\alpha^3 - 14\alpha^2 - 24\alpha - 3}{2(61\alpha^2 - 20\alpha - 5)(1+\alpha)(1+3\alpha)(1+2\alpha)^2}, & \frac{2+\sqrt{15}}{11} \leq \alpha \leq 1. \end{cases}$$

Proof. For $f \in \mathcal{M}_\alpha(\psi)$, we have $C_1 = -1$, $C_2 = 1/2$ and $C_3 = -1/3$. Furthermore, note that $\mathcal{M}_\alpha(\phi(z)) = \mathcal{M}_\alpha(\phi(-z)) = \mathcal{M}_\alpha(\Phi(z))$. Thus using Theorem 5.2.2 and Remark 21, we obtain

$$M = \frac{1}{12}(1+\alpha)^3(61\alpha^2 - 20\alpha - 5) \text{ and } T = -(1 + 5a + 17\alpha^2 + 13\alpha^3).$$

To get the desired estimate, we now consider the following cases:

(i) For $0 \leq \alpha \leq (2+\sqrt{15})/11$, it is easy to verify that M and T satisfy the inequalities given in (5.2.2), respectively.

(ii) For $(2+\sqrt{15})/11 < \alpha \leq 1$, it is easy to verify that the inequalities (5.2.3) hold true for M and T.

Now, the assertion follows at once from Theorem 5.2.2. □

***Example* 17.** Let $f \in \mathcal{S}_l(\alpha)$, then

$$\text{(i)} \quad |a_4| \leq \frac{19}{36(1+3\alpha)} \quad \text{(ii)} \quad |a_2a_3 - a_4| \leq \frac{1}{3(1+3\alpha)}.$$

The result is sharp.

Proof. For $f \in \mathcal{M}_\alpha(\psi)$, we have $C_1 = -1$, $C_2 = 1/2$, $C_3 = -1/3$.
(i) The equations (5.2.12), (5.2.13) and Remark 23 yield $q_1 = -5/2$ and $q_2 = 19/12$. The result follows from (5.2.11) and extremal functions f, up to rotations can be obtained when f satisfies

$$\frac{zf'(z) + \alpha z^2 f''(z)}{\alpha z f'(z) + (1-\alpha) f(z)} = 1 - \log(1+z).$$

(ii) Upon replacing each B_i by $(-1)^i C_i$ in (5.2.15) and (5.2.16), we get

$$q_1 = -\frac{5\alpha^2+3\alpha+1}{(1+\alpha)(1+2\alpha)} \text{ and } q_2 = \frac{19\alpha^2-12\alpha-4}{6(1+\alpha)(1+2\alpha)}.$$

Here we observe that q_1 and q_2 belong to D_2, which is given in [6, Lemma 3]. Therefore the extremal functions f, up to rotations can be obtained when f satisfies

$$\frac{zf'(z)+\alpha z^2 f''(z)}{\alpha z f'(z)+(1-\alpha)f(z)} = 1-\log(1+z^3).$$

This completes the proof. □

Theorem 5.4.1. Let $f \in \mathcal{S}_l(\alpha)$. Then, we have $|a_5| \leq 107/(288(1+4\alpha))$. The result is sharp.

Proof. The equations (5.1.2), (5.2.6), (5.2.7) and (5.2.8) with $\psi(z)$ in place of $\phi(z)$, yield a_5 in terms of p_1, p_2, p_3 and p_4 as follows

$$\begin{aligned} |a_5| &= \frac{1}{4(1+4\alpha)}\left|\frac{695}{1152}p_1^4 - \frac{27}{16}p_1^2p_2 + \frac{1}{2}p_2^2 + \frac{13}{12}p_1p_3 - \frac{1}{2}p_4\right| \\ &=: \frac{1}{8(1+4\alpha)}\left|P + \frac{1}{6}p_1Q - \frac{1}{24}p_1^2R\right| \\ &\leq \frac{1}{8(1+4\alpha)}\left(|P| + \frac{1}{6}|p_1||Q| + \frac{1}{24}|p_1|^2|R|\right), \end{aligned}$$

where $P = p_1^4 - 3p_1^2p_2 + p_2^2 + 2p_1p_3 - p_4$, $Q = p_3 - 2p_1p_2 + p_1^3$ and $R = p_2 - (23/24)p_1^2$. Since $|P|, |Q| \leq 2$ from (1.3.2) and $|R| \leq 2$ from (1.3.1), we obtain

$$|a_5| \leq \frac{1}{8(1+4\alpha)}\left(2 + \frac{2}{3} + \frac{1}{12}|p_1|^2 - \frac{1}{576}|p_1|^4\right).$$

Let us assume $G(p_1) := |p_1|^2/12 - |p_1|^4/576$. Then, the formula given in (5.2.10) yields the bound when $A = -1/576$, $B = 1/12$ and $C = 0$. By taking $p_1 = 1$, $p_2 = 2$, $p_3 = -2/3$ and $p_4 = -1/64$, we obtain the result is sharp. □

Using Examples 15–Theorem 5.4.1, we can estimate the bound for $H_3(1)$ for functions in the class $\mathcal{S}_l(\alpha)$, which is stated below in the following theorem:

Theorem 5.4.2. Let $f \in \mathcal{S}_l(\alpha)$. Then $|H_3(1)| \leq g(\alpha)$, where

$$g(\alpha) = \frac{949 + 11388\alpha + 52493\alpha^2 + 114974\alpha^3 + 117180\alpha^4 + 42568\alpha^5}{1728(1+4\alpha)(1+3\alpha)^2(1+2a)^4},$$

when $0 \leq \alpha \leq \frac{2+\sqrt{15}}{11}$ and

$$g(\alpha) = \frac{1}{1728(1+\alpha)(1+4\alpha)(1+3\alpha)^2(1+2\alpha)^3(61\alpha^2-20\alpha-5)}\Big(-5069-76035\alpha-385994\alpha^2 - 619570\alpha^3 + 831511\alpha^4 + 3545777\alpha^5 + 3327024\alpha^6 + 1298324\alpha^7\Big),$$

when $\frac{2+\sqrt{15}}{11} \leq \alpha \leq 1$.

Remark 26. By taking $\alpha = 0$ and 1, we get all the above bounds for the classes $\mathcal{S}_l^*$ and C_l, respectively.

Theorem 5.4.3. Let $f \in \mathcal{S}_l^*$. Then

$$|H_3(1)| \leq \frac{1}{9}$$

The result is sharp.

The proof of the above Theorem is on the similar lines of proof of Theorem 5.3.1. Thus we just give an outline of the proof, which is as follows:

For functions in $\mathcal{S}_l^*$, we have

$$H_3(1) = \frac{1}{663552}\Big(157p^6 - 1404p^4p_2 + 5184p^2p_2^2 - 10368p_2^3 - 1056p^3p_3 + 10368pp_2p_3 - 18432p_3^2 - 5184p^2p_4 + 20736p_2p_4\Big).$$

We apply Lemma D on the above equation and further maximizes the obtained function $S(p,x,y)$, by taking $x := |\gamma|$, $y := |\eta|$ in the closed cuboid $T : [0,2]\times[0,1]\times[0,1]$ by considering twelve edges, interior of the six faces and in the interior of T. We calculate that there is no critical point in the interior of T. The maximum value of S, among all the twelve edges and the six faces, is attained on the edges $S(p,0,1)$, $S(0,x,1)$ and $S(0,0,y)$. Let the function $f : \mathbb{D} \to \mathbb{C}$, be defined as follows:

$$f(z) = z\exp\left(\int_0^z \frac{-\log(1+t^3)}{t}\,dt\right) = z - \frac{z^4}{3} + \cdots, \tag{5.4.1}$$

clearly which belongs to $\mathcal{S}_l^*$ and the equality holds for this function as $a_2 = a_3 = a_5 = 0$

and $a_4 = -1/3$.

As we know the function z is univalent in $\mathbb{D}$, we have $z^n \prec z$, which further implies $1+z^n \prec 1+z$ $(n \geq 1)$. Now, there exists a Schwarz function $\omega(z)$ such that $1+z^n = 1+\omega(z)$. Since $|z| < 1$ and $|\omega(z)| < 1$, we can view $1+z$ and $1+\omega(z)$ as a shifted unit disk. Thus the branch of the log function is well defined and we can write:

$$1 - \log(1+z^n) = 1 - \log(1+\omega(z)).$$

Hence $1-\log(1+z^n) \prec 1-\log(1+z)$ for all $n \geq 1$. Let us define a function f_n in the class $\mathcal{A}$ as:

$$f_n(z) = z + a_{2,n}z^2 + a_{3,n}z^3 + \cdots = z + \sum_{m=2}^{\infty} a_{m,n}z^m.$$

We consider the subclass $\mathcal{S}^*_{l,n}$ of $\mathcal{S}^*_l$ consisting of the functions f_n satisfying

$$\frac{zf_n'(z)}{f_n(z)} = 1 - \log(1+z^n) \quad (n \geq 1),$$

which upon simplification yields

$$zf_n'(z) = f_n(z)(1-\log(1+z^n)).$$

Further, we have

$$\sum_{m=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^k \frac{a_{m,n}}{k} z^{nk+m}\right) = \sum_{k=1}^{\infty}(k-1)\, a_{k,n}\, z^k,$$

where $a_{1,n} = 1$. On comparing the coefficients of like power terms on either side of the above equation, we get a special pattern due to which we conjecture the following:

Conjecture 5.4.4. Let $f_n \in \mathcal{S}^*_{l,n}$. Then for $m \geq 1$, we have

$$|a_{m,n}| \leq |a_{m,1}|.$$

Concluding Remarks

The class $\mathcal{M}_{g,h}(\phi)$ unifies various sub-classes of analytic functions, mentioned here. We have obtained generalized sharp bound of second Hankel determinant for functions in the class $\mathcal{M}_{g,h}(\phi)$ and our results reduce to many earlier known bounds. In this chapter, using the novel idea of incorporating the recently derived formula for the fourth coefficient of Carathéodory functions, in place of the routine triangle inequality to achieve the sharp bounds of the Hankel determinants $H_3(1)$ and $H_2(3)$ for the well-known class $\mathcal{SL}^*$, stands out as an important theme. In fact, we improve the bound $|H_3(1)|$ to a sharp estimate of $1/36$ from $43/576$ for functions in $\mathcal{SL}^*$. Following our work, recently some authors such as Kumar and Kamaljeet [48], Kumar et al. [49], Riaz et al. [89] and Wang et al. [112] used the same technique and similar arrangements of the expressions to obtain their results.

Chapter 6

A Novel Subclass of Starlike Functions

In the past, several subclasses of starlike functions are defined involving real part and modulus of certain expressions of functions under study, combined by way of an inequality. In a similar way, here we introduce a new class $\mathscr{S}^*_\alpha$, consisting of normalized analytic univalent function f in the open unit disk $\mathbb{D}$, satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} - \alpha\right| \quad (0 \leq \alpha < 1).$$

Evidently, $\mathscr{S}^*_\alpha \subset \mathcal{S}^*$, the class of starlike functions. We first establish $\mathscr{S}^*_\alpha \subset \mathcal{S}^*(q_\alpha)$, the class of analytic function f satisfying $zf'(z)/f(z) \prec q_\alpha(z)$, where q_α is an extremal function in many cases. We obtain certain inclusion and radius results for both the classes $\mathscr{S}^*_\alpha$ and $\mathcal{S}^*(q_\alpha)$. Furthermore, we estimate the bounds of logarithmic coefficients, inverse coefficients and Fekete-Szegö functional for functions in $\mathcal{S}^*(q_\alpha)$.

6.1 Introduction

In the recent past, the class $\mathcal{S}^*(\phi)$ has been studied extensively for various choices of Ma-Minda function ϕ and enough discussion has been made on the topic in the previous chapters. In this direction, the present chapter is no more an exception, as it deals with a subclass $\mathcal{S}^*(\phi)$ by considering a specific function in place of ϕ, which is the solution of a differential subordination obtained by reformulating a differential inequality. Until recently, it has become a common practice of several authors considering inequalities involving real part and modulus of certain expressions of w to define their classes, where w can be zf'/f, f/z, $1+zf''/f'$ or $f'(z)$. A few such are mentioned below:

Inequalities	Authors
$\operatorname{Re} w > k\|w-1\|\ (0<k\leq 1)$	Sim et al. [100]
$\operatorname{Re} w > \|w-1\|$	Rønning [95]
$\left\|\dfrac{zw'-w^2}{w^2}-1\right\| < b\ (0<b\leq 1)$	Silverman [99]
$\left\|w-\dfrac{1}{w}\right\| < 2$	Raina and Sokół [84]
$\operatorname{Re}(1+zw'/w) \geq \|w-1\|$	Mahzoon et al. [63]

Table 6.1: Certain types of inequalities involving w, considered by various authors in the literature.

Motivated by the above works, in a similar way, here we define our class by means of an inequality with the expressions of w and zw'/w as follows:

$$\operatorname{Re} w > \left|\frac{zw'}{w}-\alpha\right| \quad (0\leq\alpha<1).$$

In the present investigation, we choose $w = zf'/f$ for our study, which yields the following subclass of $\mathcal{S}^*$.

Definition 6.1.1. Let $\mathscr{S}^*_\alpha$ denotes the class of functions $f\in\mathcal{S}$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}-\alpha\right| \quad (0\leq\alpha<1), \tag{6.1.1}$$

clearly $\mathscr{S}^*_\alpha \subseteq \mathcal{S}^*$. Note that the other choices of ω are still open and can be explored in a similar way. The identity function $f(z)=z$ satisfies the inequality (6.1.1) for all $0\leq\alpha<1$, hence $\mathscr{S}^*_\alpha$ is non-empty. For $\alpha=0$, set $\mathscr{S}^* := \mathscr{S}^*_0$. We deduce $\mathscr{S}^*_\alpha$ as a subclass of another fascinating class of starlike functions, in the following section, by showing $f \in \mathscr{S}^*_\alpha$ implies

$$\frac{zf'(z)}{f(z)} \prec q_\alpha(z),$$

where q_α is given by (6.2.2). While studying the class $\mathscr{S}^*_\alpha$, we encountered the function q_α, which was first introduced by MacGregor [62] and is further explored in this chapter in context of our class $\mathscr{S}^*_\alpha$. Note that $q_\alpha(z)$ plays a key role in establishing the MacGregor's result [62], that every convex function of order α is starlike of order m, where

$$m = \begin{cases} \dfrac{2\alpha-1}{2-2^{2(1-\alpha)}} & \alpha \neq 1/2, \\ \dfrac{1}{\log 4}, & \alpha = 1/2. \end{cases}$$

Apart from this, MacGregor [62] also proved $q_\alpha(z)$ is univalent and

$$\min_{|z|\leq 1} \operatorname{Re} q_\alpha(z) = q_\alpha(-1), \tag{6.1.2}$$

one can refer [67] as well for the same. We obtain certain sharp coefficient bounds for functions in $\mathcal{S}^*(q_\alpha)$ and discuss many inclusion and radius results pertaining to the classes $\mathcal{S}^*(q_\alpha)$ and $\mathscr{S}^*_\alpha$.

6.2 Class $\mathscr{S}^*_\alpha$

We begin this section by showing existence of certain analytic functions other than identity function in $\mathscr{S}^*_\alpha$.

***Example* 18.** Let $f_\gamma(z) = z + \gamma z^2$. If $|\gamma| < r_0$, where r_0 is the smallest such $r < 1$ satisfying the equation:

$$\begin{aligned} &\alpha^2(2r-1)^2(r-1)^3 - 33r^2 + 57r^3 - 48r^4 + 16r^5 + 2\alpha(r-1)^2 r(2r-1) \\ &+ (2r-1)(r-1)(\alpha + r - 3\alpha r + 2\alpha r^2)(2-4r) + 9r = 1 \quad (0 \leq \alpha < 1), \end{aligned}$$

then $f_\gamma \in \mathscr{S}^*_\alpha$. Moreover, $r_0 \in (0, 1/4]$.

Proof. We know $f_\gamma \in \mathcal{S}$ whenever $|\gamma| \leq 1/2$, thus $r_0 \in (0, 1/2]$. A simple calculation yields

$$\frac{zf_\gamma'(z)}{f_\gamma(z)} = \frac{1+2\gamma z}{1+\gamma z} \text{ and } 1 + \frac{zf_\gamma''(z)}{f_\gamma'(z)} - \frac{zf_\gamma'(z)}{f_\gamma(z)} = \frac{\gamma z}{(1+\gamma z)(1+2\gamma z)}.$$

Let $\gamma z = re^{i\theta}$, $0 \le r \le 1/2$ and $\theta \in [-\pi, \pi)$. Now, to prove the result, we show the following inequality holds:

$$\operatorname{Re}\left(\frac{1+2re^{i\theta}}{1+re^{i\theta}}\right) > \left|\frac{re^{i\theta}}{(1+re^{i\theta})(1+2re^{i\theta})} - \alpha\right| \quad (0 \le \alpha < 1),$$

which is equivalent to show

$$\begin{aligned} h(\alpha, r, x) := &- \sqrt{\frac{\alpha^2 + r^2 - 6\alpha r^2 + 9\alpha^2 r^2 + 4\alpha^2 r^4 + 2\alpha(-1+3\alpha) r(1+2r^2) x + 4\alpha^2 r^2 (2x^2-1)}{1 + 9r^2 + 4r^4 + 6(r+2r^3)x + 4r^2(2x^2-1)}} \\ &+ \frac{1+3rx+2r^2}{1+r^2+2rx} > 0, \end{aligned}$$

where $x := \cos\theta$. The function h is increasing with respect to $x \in [-1,1]$, thus it suffices to show

$$\begin{aligned} h(\alpha, r, -1) &= \frac{2r-1}{r-1} - \sqrt{\frac{\alpha^2 + r^2 - 6\alpha r^2 + 13\alpha^2 r^2 + 4\alpha^2 r^4 - 2\alpha(-1+3\alpha) r(1+2r^2)}{1 + 13r^2 + 4r^4 - 6(r+2r^3)}} \\ &=: h(\alpha, r) > 0. \end{aligned} \tag{6.2.1}$$

We observe that $h(\alpha, 0) > 0$ and $h(\alpha, 1/2) < 0$. In fact, graphically we observe $h(\alpha, r) \le 0$ for all α if $r \ge 1/4$. Since h is a continuous function of r, there must exist $r_0 \in (0, 1/4]$, which is a smallest positive root of $h(\alpha, r) = 0$ by Intermediate value property. Consequently (6.2.1) holds for $0 < r < r_0$. Therefore, we conclude that $f_\gamma \in \mathscr{S}^*_\alpha$ whenever $|\gamma| < r_0$. □

Theorem 6.2.1. Let $f \in \mathscr{S}^*_\alpha$ for $0 \le \alpha < 1$. Then

$$\frac{zf'(z)}{f(z)} \prec q_\alpha(z) := \begin{cases} \dfrac{(1-2\alpha)z}{(1-z)(1-(1-z)^{1-2\alpha})}, & \alpha \ne 1/2 \\ \dfrac{-z}{(1-z)\log(1-z)}, & \alpha = 1/2. \end{cases} \tag{6.2.2}$$

Proof. Let us define $p(z) = zf'(z)/f(z)$. Then equation (6.1.1) reduces to

$$\begin{aligned} \operatorname{Re} p(z) &> \left|\frac{zp'(z)}{p(z)} - \alpha\right| \\ &\ge \operatorname{Re}\left(\alpha - \frac{zp'(z)}{p(z)}\right), \end{aligned}$$

which yields $\operatorname{Re}\left(p(z) + \frac{zp'(z)}{p(z)}\right) > \alpha$. Using the subordination concept, we write the above inequality as follows:

$$p(z)+\frac{zp'(z)}{p(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (0 \le \alpha < 1). \tag{6.2.3}$$

From [67, Theorem 3.3 d, p. 109], we have

$$q(z)+\frac{zq'(z)}{q(z)} = \frac{1+(1-2\alpha)z}{1-z}, \tag{6.2.4}$$

where q is the best dominant of the subordination (6.2.3). On solving equation (6.2.4), we obtain

$$q(z)=\left(\frac{z}{(1-z)^{2(1-\alpha)}}\right)\left(\int_0^z (1-t)^{-2(1-\alpha)}dt\right)^{-1} \quad (\alpha \neq 1/2)$$

and for $\alpha = 1/2$

$$q(z)=\left(\frac{z}{1-z}\right)\left(\int_0^z \frac{dt}{1-t}\right)^{-1}.$$

Hence the result follows. □

We consider a function $f \in \mathcal{A}$ and further suppose

$$\frac{z}{f'(z)} := z+\sum_{n=2}^{\infty} c_n z^n \text{ and } \frac{z}{f(z)} := 1+\sum_{n=1}^{\infty} b_n z^n, \tag{6.2.5}$$

where $b_n \in \mathbb{R}$ and $c_n \ge 0$.

Theorem 6.2.2. Let $f \in \mathcal{S}$ be of the form (6.2.5), belong to $\mathscr{S}^*_\alpha$. Then

$$\sum_{n=2}^{\infty}(n+\alpha-2)c_n < (1-\alpha).$$

Proof. Here, $f \in \mathscr{S}^*_\alpha$ of the form (6.2.5), gives

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}-\alpha\right| < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right). \tag{6.2.6}$$

A simple computation yields

$$z\left(\frac{z}{f(z)}\right)' = \frac{z}{f(z)}-\left(\frac{z}{f(z)}\right)^2 f'(z)$$

and

$$z\left(\frac{z}{f'(z)}\right)' = \frac{z}{f'(z)}-\left(\frac{z}{f'(z)}\right)^2 f''(z).$$

Now, to prove the result, we substitute the above equations in the inequality (6.2.6) and deduce

$$\left|1-\frac{z\left(\frac{z}{f'(z)}\right)'}{\frac{z}{f'(z)}}+\frac{z\left(\frac{z}{f(z)}\right)'}{\frac{z}{f(z)}}-\alpha\right|<\operatorname{Re}\left(\frac{\frac{z}{f(z)}-z\left(\frac{z}{f(z)}\right)'}{\frac{z}{f(z)}}\right)$$

if and only if

$$\left|1-\frac{1+\sum_{n=2}^{\infty}nc_nz^{n-1}}{1+\sum_{n=2}^{\infty}c_nz^{n-1}}+\frac{\sum_{n=1}^{\infty}nb_nz^n}{1+\sum_{n=1}^{\infty}b_nz^n}-\alpha\right|<\operatorname{Re}\left(1-\frac{\sum_{n=1}^{\infty}nb_nz^n}{1+\sum_{n=1}^{\infty}b_nz^n}\right).$$

Now if $z\in\mathbb{D}$ is real and tends to 1^- through reals, then from the above inequality we deduce

$$\frac{1+\sum_{n=2}^{\infty}nc_n}{1+\sum_{n=2}^{\infty}c_n}<2-\alpha.$$

Therefore, the results follows now. □

By using Theorem 6.2.1, we define another interesting subclass of $\mathcal{S}^*$ as follows:

Definition 6.2.1. Let $\mathcal{S}^*(q_\alpha)$ denote the class of analytic functions $f\in\mathcal{S}$, satisfying

$$\frac{zf'(z)}{f(z)}\prec q_\alpha(z)\quad (z\in\mathbb{D},\ 0\le\alpha<1). \tag{6.2.7}$$

It is clear from Theorem 6.2.1, $\mathscr{S}^*_\alpha\subset\mathcal{S}^*(q_\alpha)$, therefore $\mathcal{S}^*(q_\alpha)$ is non-empty. Since $\mathscr{S}^*_\alpha$ reduces to $\mathcal{S}^*(1/2)$ for $\alpha=0$, it generalizes a subclass of $\mathcal{S}$.

6.3 About $\mathcal{S}^*(q_\alpha)$ and Coefficient Estimates

A function $f\in\mathcal{S}^*(q_\alpha)$ if and only if there exists an analytic function s, satisfying $s(z)\prec q_\alpha(z)$ such that

$$f(z)=z\exp\int_0^z\frac{s(t)-1}{t}dt. \tag{6.3.1}$$

Specifically, for $s(z)=q_\alpha(z)$, the structural formula (6.3.1) yields

$$f_\alpha(z)=\begin{cases}\dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \alpha\neq\dfrac{1}{2};\\[2ex] -\log(1-z), & \alpha=\dfrac{1}{2}.\end{cases} \tag{6.3.2}$$

Then the Taylor series of $f_\alpha(z)$ $(0 \leq \alpha < 1)$ is given as follows

$$f_\alpha(z) = z + (1-\alpha)z^2 + (3-5\alpha+2\alpha^2)\frac{z^3}{3} + (6-13\alpha+9\alpha^2-2\alpha^3)\frac{z^4}{6} + \cdots.$$

Also, it can be expressed as the following

$$f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\prod_{j=2}^{n}(j-2\alpha)}{n!}z^n\right) \quad (0 \leq \alpha < 1),$$

which is an extremal function for many cases of $\mathcal{S}^*(q_\alpha)$. Interestingly, $f_\alpha(z)$ is also the extremal function for several results such as distortion and coefficient bounds for the class $C(\alpha)$ (see [21, 67]).

Remark 27. Using the proof of Theorem 6.2.1, we deduce

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \Rightarrow p(z) \prec q_\alpha(z),$$

which further yields

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \Rightarrow \frac{zf'(z)}{f(z)} \prec q_\alpha(z). \tag{6.3.3}$$

The function $f_\alpha(z)$, given by (6.3.2) is an extremal function for the above differential subordination implication. This result (6.3.3) is initially proved by MacGregor [62] and later in [67, p. 113–115], using the following :

$$\min_{|z|\leq 1}(\operatorname{Re} q_\alpha(z)) = q_\alpha(-1) = \begin{cases} \dfrac{2\alpha-1}{2-2^{2(1-\alpha)}}, & \alpha \neq 1/2 \\ \dfrac{1}{\log 4}, & \alpha = 1/2, \end{cases} \tag{6.3.4}$$

$\min_{|z|\leq r} \operatorname{Re} q_\alpha(z) = q_\alpha(-r)$ and $\max_{|z|\leq r} \operatorname{Re} q_\alpha(z) = q_\alpha(r)$.

It is important to note that an analytic univalent function $q_\alpha(z)$ is a Ma-Minda function, as $q_\alpha(0) = 1$, $\operatorname{Re} q_\alpha(z) > 0$ $(0 \leq \alpha < 1)$ and some further calculation reveals $q'_\alpha(0) > 0$. Also, the Taylor series of $q_\alpha(z)$ is given as follows:

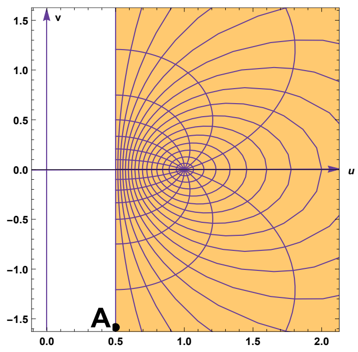
$$q_\alpha(z) = 1 + (1-\alpha)z + (3-4\alpha+\alpha^2)\frac{z^2}{3} + (2-3\alpha+\alpha^2)\frac{z^3}{2} + (45-72\alpha+26\alpha^2+2\alpha^3-\alpha^4)\frac{z^4}{45} + \cdots,$$

which shows the function is symmetric with respect to the real axis as it has real coefficients. For detailed analysis for the geometry of functions defined on $\mathbb{D}$, see Chapter 4.

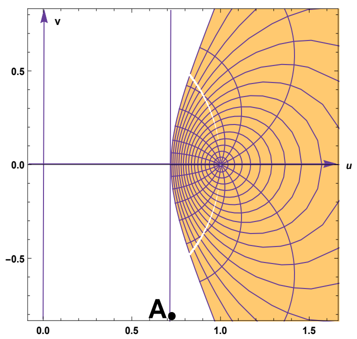
Furthermore, the function $q_\alpha(z)$ is starlike with respect to $q_\alpha(0) = 1$, as we calculate

$$\operatorname{Re}\left(\frac{e^{i\theta} q'_\alpha(e^{i\theta})}{q_\alpha(e^{i\theta}) - 1}\right) > 0 \qquad (-\pi \leq \theta < \pi,\ 0 \leq \alpha < 1),$$

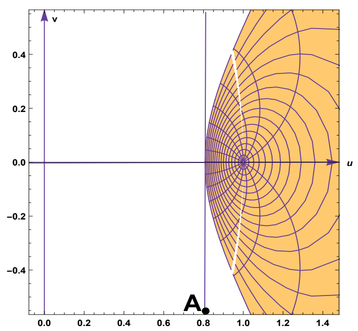
by performing a highly complex computation using Mathematica 11.0, which otherwise is not easy manually. Thus $q_\alpha(z)$ is a Ma-Minda function as it satisfies all the conditions meant for the same. The Figure 6.1 depicts $q_\alpha(\mathbb{D})$ for various $\alpha \in [0,1)$.



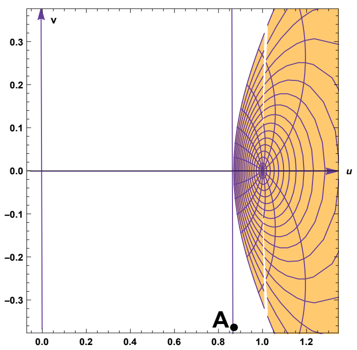
$\alpha = 0$, $\mathbf{A} = 0.5$



$\alpha = 1/2$, $\mathbf{A} = 0.721348$



$\alpha = 2/3$, $\mathbf{A} = 0.807887$



$\alpha = 7/9$, $\mathbf{A} = 0.869128$

Figure 6.1: Image of unit disk $\mathbb{D}$ under the function $q_\alpha(z)$ for various α.

Observation: It is clear that $q_0(z)$ is a convex function but it is not the case for every q_α. In fact, using Mathematica 11.0, the graph of $\operatorname{Re}\left(1 + \frac{z q''_\alpha(z)}{q'_\alpha(z)}\right)\Big|_{z=e^{i\theta}}$ $(-\pi \leq \theta < \pi)$, is not positive for some $\alpha \in (0,1)$. Thus we coin below a problem which is open at present.

Open Problem: Find the range of α in $(0,1)$ for which $q_\alpha(\mathbb{D})$ is convex.

Based on certain subordination results proved in [61], we have $f(z)/z \prec f_\alpha(z)/z$ and the following:

Theorem 6.3.1. Let $f \in \mathcal{S}^*(q_\alpha)$ and $|z_0| = r < 1$. Then

(i) *Distortion Theorem*: $f'_\alpha(-r) \leq |f'(z_0)| \leq f'_\alpha(r)$, where

$$f'_\alpha(r) = (1-r)^{2(\alpha-1)}.$$

(ii) *Growth Theorem*: $-f_\alpha(-r) \le |f(z_0)| \le f_\alpha(r)$, where

$$f_\alpha(r) = \begin{cases} \dfrac{1-(1-r)^{2\alpha-1}}{2\alpha-1}, & \alpha \ne 1/2; \\ -\log(1-r), & \alpha = 1/2. \end{cases}$$

(iii) *Rotation Theorem*: $|\arg(f(z_0)/z_0)| \le \max_{|z|=r} \arg(f_\alpha(z)/z)$.

Equality holds at some $z_0 \ne 0$ if and only if f is a rotation of f_α.

Remark 28. It is noteworthy that these growth, distortion and rotation results also hold in the case of $C(\alpha)$ [25, Theorem 1, p. 139] as well. It is due to the fact that the classes $\mathcal{S}^*(q_\alpha)$ and $C(\alpha)$ have the same extremal function $f_\alpha(z)$.

Now, we proceed with various coefficient estimates for functions in $\mathcal{S}^*(q_\alpha)$. Using Fekete-Szegö bound, given in [70], we obtain sharp bound $|a_2| \le 1-\alpha$ for functions in $\mathcal{S}^*(q_\alpha)$ and the following:

Theorem 6.3.2. Let $f \in \mathcal{S}^*(q_\alpha)$. Then we have

$$|a_3 - ta_2^2| \le \begin{cases} \dfrac{(3-2\alpha)(1-\alpha)}{3} - t(1-\alpha)^2, & t \le \dfrac{3-4\alpha}{6(1-\alpha)} \\ \dfrac{(1-\alpha)}{2}, & \dfrac{3-4\alpha}{6(1-\alpha)} \le t \le \dfrac{9-4\alpha}{6(1-\alpha)} \\ t(1-\alpha)^2 - \dfrac{(3-2\alpha)(1-\alpha)}{3}, & t \ge \dfrac{9-4\alpha}{6(1-\alpha)}. \end{cases}$$

The result is sharp.

We omit the proof as it is a straightforward substitution. By taking $t = 0$ and 1, respectively in the above result, we have

Corollary 6.3.3. Let $f \in \mathcal{S}^*(q_\alpha)$. Then we have

(i) $$|a_3| \le \begin{cases} \dfrac{(2\alpha-3)(\alpha-1)}{3}, & \alpha \le 3/4 \\ \dfrac{1-\alpha}{2}, & \alpha \ge 3/4. \end{cases}$$

The result is sharp and the extremal function is given by $f_\alpha(z)$, given by (6.3.2) for $\alpha \le 3/4$ and $\tilde{f}_\alpha(z) = \sqrt{\frac{(1-z^2)^{2\alpha-1}-1}{2\alpha-1}}$ for $\alpha \ge 3/4$.

(ii) $|a_3 - a_2^2| \le \dfrac{1-\alpha}{2}$.

The result is sharp for $\tilde{f}_\alpha(z)$, as defined in the above part (i) for $\alpha \ne 1/2$ and $f_{1/2}(z) := \sqrt{\log(1-z^2)}$ for $\alpha = 1/2$.

Recall that the inverse of a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ is given by $f^{-1}(f(z)) = z$, $z \in \mathbb{D}$ and $f(f^{-1}(w)) = w$ ($|w| < r_0$, $r_0 \geq 1/4$), for which

$$f^{-1}(w) = w - a_2 w^2 - (2a_2^2 - a_3)w^3 - (5a_3^2 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{6.3.5}$$

Corollary 6.3.4. Let $f \in \mathcal{S}^*(q_\alpha)$ and the inverse be given by $f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n$. Then

(i) $|b_2| \leq 1 - \alpha$,

(ii) $|b_3| \leq \begin{cases} 2(1-\alpha)^2 - \dfrac{(3-2\alpha)(1-\alpha)}{3}, & 0 \leq \alpha \leq 3/8 \\ \dfrac{1-\alpha}{2}, & 3/8 \leq \alpha \leq 1. \end{cases}$

The inequalities are sharp.

Proof. On comparing the coefficients of f^{-1}, given in Taylor series expansion (6.3.5) with that in the hypothesis, we get $|b_2| = |a_2| \leq (1-\alpha)$, for which $f_\alpha(z)$ acts as an extremal function. We find $|b_3| = |2a_2^2 - a_3|$ and now the result follows at once from Theorem 6.3.2, when $t = 2$. □

The logarithmic coefficients β_n for functions $f \in \mathcal{A}$ is given by:

$$\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} 2\beta_n z^n \quad (z \in \mathbb{D}).$$

In the literature, the sharp bound $|\beta_n| \leq 1/n$ ($n \geq 1$) for functions in $\mathcal{S}^*$ is already proved and equality holds for the Koebe function. Here we derive the bounds of $|\beta_n|$ for functions in the class $\mathcal{S}^*(q_\alpha)$. For this, let us define a set

$$E(\alpha) = \{\alpha \in [0,1) : q_\alpha(\mathbb{D}) \text{ is convex }\}.$$

Theorem 6.3.5. Let $f \in \mathcal{S}^*(q_\alpha)$ ($\alpha \in E(\alpha)$). Then, we have

$$|\beta_n| \leq \frac{1-\alpha}{2n}.$$

The inequality is sharp.

Proof. For $f \in \mathcal{S}^*(q_\alpha)$, we have, $zf'(z)/f(z) \prec q_\alpha(z)$, equivalent to

$$z\left(\log \frac{f(z)}{z}\right)' \prec q_\alpha(z) - 1,$$

which further yields

$$\sum_{n=1}^{\infty} 2n\beta_n z^n \prec \sum_{n=1}^{\infty} B_n z^n =: q_\alpha(z) - 1.$$

Applying Rogosinski's result [94], we obtain $2n|\beta_n| \leq |B_1| := 1-\alpha$. Hence the result. □

The following result gives the necessary and sufficient condition for functions in $\mathcal{S}^*(q_\alpha)$ in terms of convolution .

Theorem 6.3.6. A function $f \in \mathcal{S}$ belongs to $\mathcal{S}^*(q_\alpha)$ if and only if it satisfies

$$\frac{1}{z}\left(f(z) * \frac{z - \lambda z^2}{(1-z)^2}\right) \neq 0 \quad (z \in \mathbb{D}), \tag{6.3.6}$$

where

$$\lambda := \lambda(\theta) = \begin{cases} \dfrac{1-2\alpha}{(1-2\alpha) - (e^{-i\theta} - 1)(1-(1-e^{i\theta})^{1-2\alpha})}, & \alpha \neq 1/2 \\ \left(1 - (1-e^{-i\theta}) \log(1-e^{i\theta})\right)^{-1}, & \alpha = 1/2, \end{cases} \qquad (\theta \in [-\pi, \pi)).$$

Proof. Let $\alpha \neq 1/2$. Then from Definition 6.2.1, we have $f \in \mathcal{S}^*(q_\alpha)$ if and only if it satisfies the subordination (6.2.7), or equivalently

$$\frac{zf'(z)}{f(z)} \neq \frac{(1-2\alpha)e^{i\theta}}{(1-e^{i\theta})(1-(1-e^{i\theta})^{1-2\alpha})} \quad (z \in \mathbb{D}, \theta \in [-\pi, \pi)),$$

further which yields

$$f'(z) \neq \left(\frac{(1-2\alpha)e^{i\theta}}{(1-e^{i\theta})(1-(1-e^{i\theta})^{1-2\alpha})}\right)\frac{f(z)}{z}.$$

Also, it can be expressed as

$$\frac{1}{z}\left(zf'(z) - \frac{(1-2\alpha)e^{i\theta}}{(1-e^{i\theta})(1-(1-e^{i\theta})^{1-2\alpha})} f(z)\right) \neq 0. \tag{6.3.7}$$

As we recall

$$zf'(z) = f(z) * \frac{z}{(1-z)^2} \text{ and } f(z) = f(z) * \frac{z}{1-z},$$

(6.3.7) becomes

$$\frac{1}{z}\left(f(z) * \frac{z - \frac{(1-2\alpha)e^{i\theta}}{(1-e^{i\theta})(1-(1-e^{i\theta})^{1-2\alpha})}(z-z^2)}{(1-z)^2}\right) \neq 0,$$

which upon simplification reduces to (6.3.6). In a similar way, the result follows for

$\alpha = 1/2$. □

Now, using the convolution properties, we have

$$\frac{1}{z}\left(f(z) * \frac{z-\lambda z^2}{(1-z)^2}\right) = \frac{1}{z}\left((1-\lambda)zf'(z) + \lambda f(z)\right).$$

Substituting the above equation in (6.3.6) along with the Taylor series expansion of $f \in \mathcal{A}$, we deduce the following result:

Corollary 6.3.7. A function $f \in \mathcal{S}$ belongs to $\mathcal{S}^*(q_\alpha)$ if and only if it satisfies

$$1 \neq \sum_{n=2}^{\infty} \rho(\theta) a_n z^{n-1} := \begin{cases} \displaystyle\sum_{n=2}^{\infty} \frac{(e^{-i\theta}-1)(1-(1-e^{i\theta})^{1-2\alpha})n-(1-2\alpha)}{(1-2\alpha)-(e^{-i\theta}-1)(1-(1-e^{i\theta})^{1-2\alpha})} a_n z^{n-1}, & \alpha \neq 1/2 \\ \displaystyle\sum_{n=2}^{\infty} \frac{(1-e^{-i\theta})\log(1-e^{i\theta})n-1}{1-(1-e^{-i\theta})\log(1-e^{i\theta})} a_n z^{n-1}, & \alpha = 1/2. \end{cases}$$

Consequently, the next result follows.

Corollary 6.3.8. If $f \in \mathcal{S}$ satisfies

$$\sum_{n=2}^{\infty} |\rho(\theta)||a_n| < 1,$$

then $f \in \mathcal{S}^*(q_\alpha)$.

6.4 Inclusion Relations

We establish below inclusion relations of class $\mathscr{S}^*_\alpha$ with some well-known classes of analytic functions.

Theorem 6.4.1. Let $0 \leq \alpha < 1$, then $\mathscr{S}^*_\alpha \subset C(\alpha)$.

Proof. Let $f \in \mathscr{S}^*_\alpha$, then inequality (6.1.1) yields

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > -\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} - \alpha\right),$$

which upon a straightforward calculation yields

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha.$$

This completes the proof. □

Silverman [99] introduced the class

$$\mathcal{G}_b := \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b,\ 0 < b \leq 1 \right\}.$$

In the following result we show the relation between the classes $\mathcal{G}_b$ and $\mathcal{S}_q^*$.

Theorem 6.4.2. $\mathcal{S}_q^* \subset \mathcal{G}_1$.

Proof. Since $\mathcal{S}_q^* := \mathcal{S}_q^*(0)$, we have $\alpha = 0$. Let $f \in \mathcal{S}_q^*$, then inequality (6.1.1) becomes

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right|,$$

which implies

$$\left| \frac{zf'(z)}{f(z)} \right| > \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right|,$$

therefore, we have

$$\left| \left(\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \right) - 1 \right| < 1.$$

Hence the result. □

In the following result, we deal with Mocanu's $\rho-$ convex class [69].

Theorem 6.4.3. Let $f \in \mathcal{S}_q^*$, then f is -1-convex function.

Proof. Let $f \in \mathcal{S}_q^*$, then we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right),$$

if and only if

$$\operatorname{Re}\left(2\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)} \right) \right) > 0.$$

This completes the proof. □

The known inclusion $\mathscr{S}_\alpha^* \subset \mathcal{S}^*(q_\alpha)$ and (6.3.4) directly yield the following result.

Corollary 6.4.4. We have $\mathscr{S}^*_\alpha \subset \mathcal{S}^*(q_\alpha) \subset \mathcal{S}^*(\gamma)$ whenever $0 \leq \gamma \leq \frac{2\alpha-1}{2(1-2^{1-2\alpha})}$ for $\alpha \neq 1/2$, or whenever $0 \leq \gamma \leq 1/\log 4$ for $\alpha = 1/2$.

Let $\mathcal{Q}(\alpha) := \{f \in \mathcal{A} : \operatorname{Re}(f(z)/z) > \alpha\}$ $(0 \leq \alpha < 1)$. Recall $\mathcal{S}^*(1/2) \subset \mathcal{Q}(1/2)$ from [67, p. 57] and the constant $1/2$ is the best possible. The following result deals with inclusion relation associating the class $\mathcal{Q}(\alpha)$.

Theorem 6.4.5. Let $f \in \mathcal{S}^*(q_\alpha)$. Then we have

$$\operatorname{Re}\frac{f(z)}{z} > \gamma(\alpha) := \begin{cases} \dfrac{3-2\alpha-2^{2(1-\alpha)}}{4-2\alpha-3(2^{1-2\alpha})}, & \alpha \neq 1/2 \\ \\ \dfrac{2(1-\log 4)}{2-3\log 4}, & \alpha = 1/2, \end{cases} \tag{6.4.1}$$

which also implies $\mathscr{S}^*_\alpha \subset \mathcal{S}^*(q_\alpha) \subset \mathcal{Q}(\gamma(\alpha))$.

Proof. For brevity, let us denote $\gamma := \gamma(\alpha)$. We observe that $0 < \gamma \leq 1/2$ for $0 \leq \alpha < 1$. Let us define an analytic function p with $p(0) = 1$ as

$$p(z) = \frac{1}{1-\gamma}\left(\frac{f(z)}{z} - \gamma\right),$$

which upon simplification gives

$$\frac{zf'(z)}{f(z)} = \xi(p(z), zp'(z)) := 1 + \frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z)+\gamma},$$

where

$$\xi(r,s) := 1 + \frac{(1-\gamma)s}{(1-\gamma)r+\gamma}.$$

Let $\alpha' := \frac{2\alpha-1}{2(1-2^{1-2\alpha})}$ $(\alpha \neq 1/2)$ and otherwise $\alpha' := \frac{1}{\log 4}$. As $f \in \mathcal{S}^*(q_\alpha)$, then Corollary 6.4.4 yields

$$\xi(p(z), zp'(z)) \subset \{w \in \mathbb{C} : \operatorname{Re} w > \alpha'\} =: \Omega_{\alpha'}.$$

Let $\lambda, \tau \in \mathbb{R}$ be such that $\lambda \leq \frac{-1}{2}(1+\tau^2)$. Then we have

$$\begin{aligned} \operatorname{Re}\{\xi(i\tau,\lambda)\} &= \operatorname{Re}\left(1 + \frac{\lambda(1-\gamma)}{(1-\gamma)i\tau+\gamma}\right) \\ &= 1 + \frac{\lambda\gamma(1-\gamma)}{(1-\gamma)^2\tau^2+\gamma^2} \\ &\leq 1 - \left(\frac{\gamma(1-\gamma)}{2}\right)\left(\frac{(1+\tau^2)}{(1-\gamma)^2\tau^2+\gamma^2}\right) = 1 - \frac{\gamma(1-\gamma)}{2}h(\tau^2), \end{aligned} \tag{6.4.2}$$

where

$$h(\tau^2) := \frac{(1+\tau^2)}{(1-\gamma)^2\tau^2+\gamma^2},$$

is a decreasing function of τ^2 for all γ, so we deduce the following

$$\frac{1}{(1-\gamma)^2} \leq h(\tau^2) \leq \frac{1}{\gamma^2}.$$

Using the above inequality in (6.4.2), we obtain

$$\operatorname{Re}\{\xi(i\tau,\lambda)\} \leq 1 - \frac{\gamma(1-\gamma)}{2(1-\gamma)^2} = \frac{2-3\gamma}{2(1-\gamma)} = \alpha'.$$

This clearly shows that $\operatorname{Re}\{\xi(i\tau,\lambda)\} \notin \Omega_{\alpha'}$. Now, from [67, Theorem 2.3i., p. 35], we conclude $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$. Therefore, we obtain $\mathcal{S}^*(q_\alpha) \subset Q(\gamma(\alpha))$. In addition, the fact $\mathscr{S}^*_\alpha \subset \mathcal{S}^*(q_\alpha)$ completes the proof. □

6.5 Radius Problems

The purpose of this section is to estimate various radius constants associated with $\mathscr{S}^*_\alpha$ and $\mathcal{S}^*(q_\alpha)$.

Theorem 6.5.1. Let $f \in \mathcal{SL}^*$. Then $f \in \mathscr{S}^*_\alpha$ whenever $|z| < \tilde{r}(\alpha) < 1$, where $\tilde{r}(\alpha)$ is the smallest positive root of

$$2(1-r)(\sqrt{1-r}-\alpha)-r = 0 \quad (0 \leq \alpha < 1). \tag{6.5.1}$$

The result is sharp.

Proof. For $f \in \mathcal{SL}^*$, there exists a Schwarz function $\omega(z) = Re^{it}$ $(0 \leq R \leq r = |z| < 1; 0 \leq t < 2\pi)$ such that

$$\frac{zf'(z)}{f(z)} = \sqrt{1+\omega(z)}.$$

A computation shows

$$\min_{|\omega(z)|=R} \operatorname{Re}\left(\sqrt{1+\omega(z)}\right) = \sqrt{1-R} \geq \sqrt{1-r} \tag{6.5.2}$$

and the following from Schwarz Pick inequality

$$\frac{|z||\omega'(z)|}{1-|\omega(z)|} \leq \frac{|z|(1+|\omega(z)|)}{1-|z|^2} \leq \frac{|z|}{1-|z|}. \tag{6.5.3}$$

To prove the result, it suffices to show (6.1.1) holds. Therefore we consider

$$\begin{aligned}\operatorname{Re}\left(\sqrt{1+\omega(z)}\right)-\left|\frac{z\omega'(z)}{2(1+\omega(z))}-\alpha\right| &\geq \operatorname{Re}\left(\sqrt{1+\omega(z)}\right)-\alpha-\frac{|z||\omega'(z)|}{2(1-|\omega(z)|)}\\ &\geq \sqrt{1-r}-\alpha-\left(\frac{r}{2(1-r)}\right), \qquad (6.5.4)\end{aligned}$$

using the inequalities (6.5.2) and (6.5.3) with $|z|=r$. A calculation reveals (6.5.4) further becomes greater than 0 provided $r<\tilde{r}(\alpha)$. For sharpness, let us consider a function

$$\tilde{f}(z)=\frac{4z\exp(2(\sqrt{1+z}-1))}{(1+\sqrt{1+z})^2},$$

belonging to $\mathcal{SL}^*$. Let α be given by (6.5.1), then equality holds in (6.1.1) for $\tilde{f}(z)$ at $z=-\tilde{r}(\alpha)$. □

Theorem 6.5.2. Let $f\in\mathcal{S}_l^*$. Then $f\in\mathscr{S}_\alpha^*$ whenever $|z|<\tilde{r}(\alpha)<1$, where $\tilde{r}(\alpha)$ is the smallest positive root of

$$(1-\log(1+r))(1-r)(1-\log(1+r)-\alpha)-r=0 \quad (0\leq\alpha<1).$$

Proof. For $f\in\mathcal{S}_l^*$, we have

$$\frac{zf'(z)}{f(z)}=1-\log(1+\omega(z)),$$

where $\omega(z)=Re^{it}$ $(0\leq R\leq r=|z|<1; 0\leq t<2\pi)$. To prove the result, it suffices to show

$$\operatorname{Re}(1-\log(1+\omega(z)))-\left|\frac{-z\omega'(z)}{(1+\omega(z))(1-\log(1+\omega(z)))}-\alpha\right|>0.$$

We also have

$$\begin{cases}\min_{|\omega(z)|=R}\operatorname{Re}(1-\log(1+\omega(z)))\\ \text{and } \min_{|\omega(z)|=R}|1-\log(1+\omega(z))|\end{cases} = \quad 1-\log(1+R)\geq 1-\log(1+r),$$

(see Chapter 4). By taking these inequalities and (6.5.3) with $|z|=r$, we obtain

$$\begin{aligned}\left|\frac{-z\omega'(z)}{(1+\omega(z))(1-\log(1+\omega(z)))}-\alpha\right| &\leq \frac{|z||\omega'(z)|}{(1-|\omega(z)|)|1-\log(1+\omega(z))|}+\alpha\\ &\leq \frac{r}{(1-r)(1-\log(1+r))}+\alpha,\end{aligned}$$

which further shows

$$\mathrm{Re}(1-\log(1+\omega(z)))-\left|\frac{-z\omega'(z)}{(1+\omega(z))(1-\log(1+\omega(z)))}-\alpha\right|\geq\frac{(1-r)(1-\log(1+r))^2-r}{(1-r)(1-\log(1+r))}-\alpha,$$

greater than 0 provided $r<\tilde{r}(\alpha)$. □

Theorem 6.5.3. Let $f\in\mathcal{S}^*_e$. We have $f\in\mathscr{S}^*_\alpha$ in $|z|<\tilde{r}(\alpha)$, where $\tilde{r}(\alpha)$ is the smallest positive root of

$$\begin{cases}4(1-r^2)(e^{-r}-\alpha)-(1+r^2)^2=0, & 0\leq\alpha<0.246646\\ e^{-r}-\alpha-r=0, & 0.246646\leq\alpha<1.\end{cases} \tag{6.5.5}$$

Proof. For $f\in\mathcal{S}^*_e$, we have $zf'(z)/f(z)=e^{\omega(z)}$, where $|\omega(z)|=R\leq r=|z|$. A straightforward calculation gives

$$\min_{|\omega(z)|=R}\mathrm{Re}\left(e^{\omega(z)}\right)=e^{-R}\geq e^{-r}.$$

The following is the well-known inequality [20] for the derivative of Schwarz function

$$|\omega'(z)|\leq\begin{cases}1, & |z|\leq\sqrt{2}-1\\ \dfrac{(1+r^2)^2}{4r(1-r^2)}, & |z|\geq\sqrt{2}-1.\end{cases} \tag{6.5.6}$$

Now we show inequality (6.1.1) holds, by proving $\mathrm{Re}(e^{\omega(z)})-|z\omega'(z)-\alpha|>0$. For this, we consider the following with $|z|=r$ and further applying (6.5.6), we obtain

$$\mathrm{Re}(e^{\omega(z)})-|z\omega'(z)-\alpha|\geq\mathrm{Re}(e^{\omega(z)})-\alpha-|z||\omega'(z)|\geq\begin{cases}e^{-r}-\alpha-r, & r\leq\sqrt{2}-1\\ e^{-r}-\alpha-\dfrac{(1+r^2)^2}{4(1-r^2)}, & r\geq\sqrt{2}-1,\end{cases} \tag{6.5.7}$$

greater than 0 provided $r<\tilde{r}(\alpha)$, given in (6.5.5). Note that $\tilde{r}(\alpha)\in(\sqrt{2}-1,1]$ for $\alpha<0.246646$ and $\tilde{r}(\alpha)\in(0,\sqrt{2}-1]$ for $\alpha\geq0.246646$. This proves the result. □

Recent literature studies have detailed some new subclasses of $\mathcal{S}^*$, which are obtained by choosing different ϕ in $\mathcal{S}^*(\phi)$, amongst, noteworthy are, $\mathcal{S}^*_{SG}$ [22], for $\phi(z)=2/(1+e^{-z})$, $\mathcal{S}^*_S$ [17], obtained by taking $\phi(z)=1+\sin z$ and $\mathcal{S}^*_C$, first studied in [97] for $\phi(z)=1+4z/3+2z^2/3$.

Theorem 6.5.4. Let $f\in\mathcal{S}^*_{SG}$. We have $f\in\mathscr{S}^*_\alpha$ in $|z|<\tilde{r}(\alpha)$, where $\tilde{r}(\alpha)$ is the smallest positive root of

$$\begin{cases} 4(1-r^2)(1+e^{-r})(2-\alpha(1+e^r))-(1+r^2)^2(1+e^r)=0, & 0\leq\alpha<0.546407\\ (1+e^{-r})(2-\alpha(1+e^r))-r(1+e^r)=0, & 0.546407\leq\alpha<1. \end{cases}$$

Proof. For $f\in\mathcal{S}^*_{SG}$, we have $zf'(z)/f(z)=2/(1+e^{-\omega(z)})$, for some Schwarz function $\omega(z)=Re^{it}$, $(0\leq R\leq r=|z|<1; 0\leq t<2\pi)$. To prove the result, it suffices to show (6.1.1) holds for f in consideration here. Therefore we consider the following

$$\operatorname{Re}\left(\frac{2}{1+e^{-\omega(z)}}\right)-\left|\frac{z\omega'(z)e^{-\omega(z)}}{1+e^{-\omega(z)}}-\alpha\right|\geq\operatorname{Re}\left(\frac{2}{1+e^{-\omega(z)}}\right)-\alpha-\frac{|z||\omega'(z)|}{|1+e^{\omega(z)}|} \tag{6.5.8}$$

From [22], we have

$$\min_{|\omega(z)|=R}\operatorname{Re}\left(\frac{2}{1+e^{-\omega(z)}}\right)=\frac{2}{1+e^R}\geq\frac{2}{1+e^r}$$

and a computation shows that $\min_{|\omega(z)|=R}|1+e^{\omega(z)}|=1+e^{-R}\geq 1+e^{-r}$. Using these inequalities in right side of the inequality (6.5.8) and further applying (6.5.6) with $|z|=r$, (6.5.8) finally reduces to

$$\operatorname{Re}\left(\frac{2}{1+e^{-\omega(z)}}\right)-\left|\frac{z\omega'(z)e^{-\omega(z)}}{1+e^{-\omega(z)}}-\alpha\right|\geq\begin{cases}\dfrac{2}{1+e^r}-\alpha-\left(\dfrac{r}{1+e^{-r}}\right), & r\leq\sqrt{2}-1\\[2ex] \dfrac{2}{1+e^r}-\alpha-\left(\dfrac{(1+r^2)^2}{4(1-r^2)(1+e^{-r})}\right), & r\geq\sqrt{2}-1,\end{cases}$$

which is greater than 0 provided $r<\tilde{r}(\alpha)$. Note that for $\alpha<0.546407$, $\tilde{r}(\alpha)\in(\sqrt{2}-1,1]$ and for $\alpha\geq 0.546407$, $\tilde{r}(\alpha)\in(0,\sqrt{2}-1]$. This completes the proof. □

Theorem 6.5.5. Let $f\in\mathcal{S}^*_S$.Then $f\in\mathscr{S}^*_\alpha$ in $|z|<\tilde{r}(\alpha)$, where $\tilde{r}(\alpha)$ is the smallest positive root of

$$(1-\sin r)(1-\sin r\cosh r-\alpha)-r\cosh r=0\quad(0\leq\alpha<1).$$

Proof. Let $f\in\mathcal{S}^*_S$. Then for some Schwarz function $\omega(z)=Re^{it}$ $(0\leq R\leq r=|z|<1; 0\leq t<2\pi)$, we have $zf'(z)/f(z)=1+\sin(\omega(z))$. To prove the result, we show (6.1.1) holds for f considered here. For this we consider

$$\operatorname{Re}(1+\sin(\omega(z)))-\left|\frac{z\omega'(z)\cos(\omega(z))}{1+\sin(\omega(z))}-\alpha\right|\geq\operatorname{Re}(1+\sin(\omega(z)))-\alpha-\frac{|z||\omega'(z)||\cos(\omega(z))|}{|1+\sin(\omega(z))|}. \tag{6.5.9}$$

A calculation shows that $\operatorname{Re}(1+\sin(\omega(z))) = 1+\sin(R\cos t)\cosh(R\sin t)$ and

$$\sin(R\cos t) \geq -\sin R \text{ and } \cosh(R\sin t) \leq \cosh(R).$$

Thus we have, $\operatorname{Re}(1+\sin(\omega(z))) \geq 1-\sin R\cosh R \geq 1-\sin r\cosh r$. Also

$$\max_{|\omega(z)|=R} |\cos(\omega(z))| = \cosh R \leq \cosh r \text{ and } \min_{|\omega(z)|=R} |1+\sin(\omega(z)| = 1-\sin R \geq 1-\sin r.$$

Using these inequalities in right side of the inequality (6.5.9) and further applying (6.5.6) with $|z| = r$, (6.5.9) eventually reduces to

$$\operatorname{Re}(1+\sin(\omega(z))) - \left|\frac{z\omega'(z)\cos(\omega(z))}{1+\sin(\omega(z))} - \alpha\right| \geq 1-\sin r\cosh r - \alpha - \left(\frac{r\cosh r}{1-\sin r}\right) > 0,$$

provided $r < \tilde{r}(\alpha) \leq \sqrt{2}-1$ for every $0 \leq \alpha < 1$. □

Theorem 6.5.6. Let $f \in \mathcal{S}^*_C$ and $0 \leq \alpha < 3/4$. Then $f \in \mathscr{S}^*_\alpha$ in $|z| < \tilde{r}(\alpha)$, where $\tilde{r}(\alpha)$ is the smallest positive root of

$$(3-2r^2)(1-r^2-2\alpha) - 6\sqrt{3}r(1+r) = 0 \quad (0 \leq \alpha < 3/4).$$

Proof. For $f \in \mathcal{S}^*_C$, there exists a Schwarz function $\omega(z) = Re^{it}$ $(0 \leq R \leq r = |z| < 1; 0 \leq t < 2\pi)$ such that $zf'(z)/f(z) = 1 + \frac{4\omega(z)+2\omega^2(z)}{3}$. We consider

$$\begin{aligned}
&\operatorname{Re}\left(1+\frac{4\omega(z)+2\omega^2(z)}{3}\right) - \frac{4}{3}\left|\frac{z\omega'(z)(1+\omega(z))}{1+\dfrac{4\omega(z)+2\omega^2(z)}{3}} - \alpha\right| \\
&\quad \geq \operatorname{Re}\left(1+\frac{4\omega(z)+2\omega^2(z)}{3}\right) - \frac{4\alpha}{3} - \frac{4}{3}\left(\frac{|z||\omega'(z)|(1+|\omega(z)|)}{\left|1+\dfrac{4\omega(z)+2\omega^2(z)}{3}\right|}\right). \qquad (6.5.10)
\end{aligned}$$

Let

$$\kappa(R,t) := \operatorname{Re}\left(1+\frac{4\omega(Re^{it})+2\omega^2(Re^{it})}{3}\right).$$

Then a calculation shows that $\kappa_t(R,t) = 0$ at either $R = 0$ or $t = 0,\ \pi,\ \arccos(-1/(2R))$. Evaluating $\kappa(R,t)$ at these values of R and t, we obtain

$$\min_{0<R\leq r} \kappa(R,t) = \kappa(R,\arccos(-1/(2R))) = \frac{2}{3}\left(1-R^2\right) \geq \frac{2}{3}\left(1-r^2\right) \text{ and } \kappa(0,t) = 1.$$

Similarly, we calculate

$$\min_{|\omega(z)|=R\neq 0}\left|1+\frac{4\omega(z)+2\omega^2(z)}{3}\right| = \frac{3-2R^2}{\sqrt{27}} \geq \frac{3-2r^2}{\sqrt{27}}\ (r\neq 0) \text{ and } \left|\left|1+\frac{4\omega(z)+2\omega^2(z)}{3}\right|\right|_{R=0} = 1.$$

Using these inequalities in right side of the inequality (6.5.10) and further applying (6.5.6) with $|z| = r$, (6.5.10) finally reduces to

$$\mathrm{Re}\left(1+\frac{4\omega(z)+2\omega^2(z)}{3}\right)-\frac{4}{3}\left|\frac{z\omega'(z)(1+\omega(z))}{1+\dfrac{4\omega(z)+2\omega^2(z)}{3}}-\alpha\right| \geq \begin{cases} \dfrac{2}{3}(1-r^2)-\dfrac{4\alpha}{3}-\dfrac{4\sqrt{3}r(1+r)}{3-2r^2} & r\neq 0 \\ 1-\dfrac{4\alpha}{3} & r=0, \end{cases}$$

which is greater than 0 provided $r < \tilde{r}(\alpha) \leq \sqrt{2}-1$ and $0 \leq \alpha < 3/4$. □

Remark 29. Interestingly in Theorem 6.5.5–Theorem 6.5.6, the respective root $\tilde{r}(\alpha)$ lies in $(0, \sqrt{2}-1]$.

We now proceed to establish radius constants associated with $\mathcal{S}^*(q_\alpha)$.

Theorem 6.5.7. Let $f \in \mathcal{S}^*(q_\alpha)$ $(0 \leq \alpha < 1)$. Then the followings hold:

(i) f is starlike of order γ in $|z| < \tilde{r}$ whenever the following (a) or (b) holds.

(a) $\frac{2\alpha-1}{2(1-2^{1-2\alpha})} < \gamma < 1$ for $\alpha \neq 1/2$, where $\tilde{r}$ is the smallest such $r < 1$ satisfying the equation

$$(2\alpha-1)r-\gamma(1+r)(1-(1+r)^{1-2\alpha}) = 0.$$

(b) $\frac{1}{\log 4} < \gamma < 1$ for $\alpha = 1/2$, where $\tilde{r}$ is the smallest such $r < 1$ satisfying the equation

$$r-\gamma(1+r)\log(1+r) = 0.$$

(ii) Let $\beta > 1$ then $f \in \mathcal{M}(\beta)$ in $|z| < r_0$ if there exists r_0, the smallest such $r < 1$ satisfying the equation

$$(1-2\alpha)r-\beta(1-r)(1-(1-r)^{1-2\alpha}) = 0\ (\alpha\neq 1/2) \quad \text{or} \quad r+\beta(1-r)(\log(1-r)) = 0\ (\alpha = 1/2),$$

else $r_0 = 1$.

Proof. Let $f \in \mathcal{S}^*(q_\alpha)$. Then f satisfies the subordination (6.2.7).

(i) Let $\alpha \neq 1/2$. Thus we have

$$\mathrm{Re}\frac{zf'(z)}{f(z)} \geq \frac{(2\alpha-1)r}{(1+r)(1-(1+r)^{1-2\alpha})} =: \rho(r,\alpha), \quad |z| = r < 1.$$

We find $r \in [0,1)$ such that $\rho(r,\alpha) > \gamma$ for given α and γ. Let us assume $\tilde{\rho}(r,\alpha,\gamma) := (2\alpha-1)r/((1+r)(1-(1+r)^{1-2\alpha}))-\gamma$. Now, we observe $\tilde{\rho}(0,\alpha,\gamma)=1-\gamma>0$ and $\tilde{\rho}(1,\alpha,\gamma)=\frac{2\alpha-1}{2(1-2^{1-2\alpha})}-\gamma<0$ for every γ in the given range. So, there must exist $\tilde{r}$ such that $\tilde{\rho}(r,\alpha,\gamma)\geq 0$ for all $r\in(0,\tilde{r}]$, where $\tilde{r}$ is as defined in the hypothesis. Similarly we can prove the result for $\alpha=1/2$. This completes the proof for part (i).

(ii) Let $\alpha \neq 1/2$. We have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \leq \frac{(1-2\alpha)r}{(1-r)(1-(1-r)^{1-2\alpha})} =: \mu(r,\alpha), \quad |z|=r<1.$$

Now, to find $r\in[0,1)$ such that $\mu(r,\alpha)-\beta<0$ for given α and β, let us assume

$$\tilde{\mu}(r,\alpha,\beta) := \frac{(1-2\alpha)r}{(1-r)(1-(1-r)^{1-2\alpha})}-\beta.$$

Clearly, $\tilde{\mu}(0,\alpha,\beta)<0$ and suppose $\tilde{r}:=1-\epsilon$, $(\epsilon\approx 0)$, then $\mu(\tilde{r},\alpha)>0$, which implies either $\tilde{\mu}(\tilde{r},\alpha,\beta)<0$ or $\tilde{\mu}(\tilde{r},\alpha,\beta)>0$ depending on the value of β. Thus, there must exist r_0 such that $\tilde{\mu}(r,\alpha\beta)\leq 0$ for all $r\in(0,r_0]$, where r_0 is as defined in the hypothesis. On similar lines, proof follows for $\alpha=1/2$ and that concludes the proof of part (ii). □

Remark 30. In Theorem 6.5.7(i), we exclude the case when $0\leq\gamma\leq\frac{2\alpha-1}{2(1-2^{1-2\alpha})}$ for $\alpha\neq 1/2$ and $0\leq\gamma\leq\frac{1}{\log 4}$ for $\alpha=1/2$ as for these ranges of γ, the result holds in $|z|<1$ and it becomes the inclusion result instead, given in Corollary 6.4.4.

Theorem 6.5.8. Let $f\in\mathcal{S}^*(q_\alpha)$ and $\beta\geq 1$. Then for $|z|<\tilde{r}$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\beta,$$

where $\gamma:=\gamma(\alpha)$, given by (6.4.1) and $\tilde{r}$ is the smallest such $r<1$ satisfying the equation

$$2(1-\gamma)r-\beta(1-r)(1-|1-2\gamma|r)=0.$$

Proof. As $f\in\mathcal{S}^*(q_\alpha)$, thus from (6.4.1), we deduce the following

$$\frac{f(z)}{z}=\frac{1+(1-2\gamma)\omega(z)}{1-\omega(z)},$$

for some Schwarz function $\omega(z)$. Logarithmic differentiation of the above equation yields

$$\frac{zf'(z)}{f(z)}-1=\frac{2(1-\gamma)z\omega'(z)}{(1+(1-2\gamma)\omega(z))(1-\omega(z))}.$$

Using the triangle inequality on the modulus of above equation, we get

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{2(1-\gamma)|z||\omega'(z)|}{(1-|1-2\gamma||\omega(z)|)(1-|\omega(z)|)} \tag{6.5.11}$$

Upon applying the Schwarz-Pick inequality

$$|\omega'(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2} \quad (z \in \mathbb{D})$$

and $|\omega(z)| \leq |z|$ in equation (6.5.11), we deduce

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}-1\right| &\leq \frac{2(1-\gamma)|z|(1+|z|)}{(1-|1-2\gamma||z|)(1-|z|^2)}\\ &= \frac{2(1-\gamma)r}{(1-|1-2\gamma|r)(1-r)} =: u(r,\gamma).\end{aligned}$$

Now to show $u(r,\gamma)-\beta<0$. For this consider, $\tilde{u}(r,\gamma,\beta) := 2(1-\gamma)r-\beta(1-r)(1-|1-2\gamma|r)$. We observe $\tilde{u}(0,\gamma,\beta) = -\beta < 0$ and $\tilde{u}(1,\gamma,\beta) = 2(1-\gamma) > 0$. Thus, there must exist $\tilde{r}$ such that $\tilde{u}(r,\gamma,\beta) \leq 0$ for all $r \in [0,\tilde{r}]$, where $\tilde{r}$ is defined in the hypothesis. Hence the result. □

Remark 31. Since $\mathscr{S}^*_\alpha \subset \mathcal{S}^*(q_\alpha)$, Theorem 6.5.7 and Theorem 6.5.8 hold even for $\mathscr{S}^*_\alpha$, however, they need not be sharp.

Concluding Remarks

A new class $\mathscr{S}^*_\alpha$ involving real part and modulus of certain expression of functions, combined by way of an inequality has been defined here. The class under consideration also implies a subclass of starlike functions $\mathcal{S}^*(q_\alpha)$, where $q_\alpha(z)$ is a well-known function and has certain interesting properties. The coefficient bounds obtained are all sharp. The radius results for $\mathscr{S}^*_\alpha$ is deduced in an interesting manner using the famous Schwarz Pick inequality. For instance, we considered f to be in various classes such as $\mathcal{SL}^*$, $\mathcal{S}^*_e$, $\mathcal{S}^*_{SG}$, $\mathcal{S}^*_S$ and $\mathcal{S}^*_C$ and then obtain the largest $|z|=r<1$ for which $f \in \mathscr{S}^*_\alpha$.

Future Scope

- The idea of exact differential subordinations introduced by us is new and novel. Here, it has been dealt for first order differential subordinations and can be extended for higher differential subordinations.

- The concept of non-Ma-Minda and a special type of Ma-Minda functions, which are introduced here are of great use and pave a way for further explorations in parallel to the concept of Ma-Minda function.

- The classes $\mathcal{S}^*(\Phi)$, $\mathcal{C}(\Phi)$, $\mathcal{S}^*_l$, $\mathcal{C}_l$, $\mathcal{M}_{g,h}(\phi)$ and $\mathcal{M}_\alpha(\Phi)$ studied here are all special cases of $\mathcal{A}(g,h,\varphi)$ and it leaves ample scope open for further studies in specializing the class for different choices of g and h together with altered conditions on φ.

- The estimation of sharp third order Hankel determinant for functions in subclass $\mathcal{SL}^*$ of $\mathcal{S}$ could induce the curiosity to make an attempt for further finding the sharp bound of higher order Hankel determinants such as fourth order for functions in various subclasses of $\mathcal{S}$.

- The sharp bound of third order Hankel determinant can be obtained for various subclasses of starlike functions related to Ma-Minda functions such as $1+\sin z$, $\exp(z)$ and $\frac{1+z}{1-z}$, etc by using the fourth coefficient formula of Carathéodory functions.

  - In fact we are working in this direction and using the same technique, we have achieved the sharp bound of $|H_3(1)|$ for functions in $\mathcal{S}^*$. This problem has been in the trend since many years as various attempts was made by several authors and eventually we established the sharp result as $4/9$. Thus the bound obtained is an improvement over the value $5/9$, recently proved in [118].

- Here, we investigated a new subclass of $\mathcal{S}$ by choosing $\omega = zf'/f$ in the expression $\operatorname{Re}\omega > \left|\frac{z\omega'}{\omega} - \alpha\right|$. As a future task, we can replace ω by f/z, $1+zf''/f'$ or f' to define various other new classes of analytic functions and study in a similar way as done here. Also, we can consider other combinations of ω and its derivatives to define new subclasses of $\mathcal{S}$ and study.

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List of Publications

1. **Shagun Banga** and S. Sivaprasad Kumar; *Applications of differential subordinations to certain classes of starlike functions*, Journal of the Korean Mathematical society **57** (2020), no. 2, 331–357. **SCIE, Impact Factor (0.583)**

2. **Shagun Banga** and S. Sivaprasad Kumar; *The sharp bounds of the second and third Hankel determinants for the class* $\mathcal{SL}^*$, Mathematica Slovaca **70** (2020), no. 4, 849–862. **SCIE, Impact Factor (0.770)**

3. S. Sivaprasad Kumar and **Shagun Banga**; *On certain exact differential subordinations involving convex dominants*, Mediterranean Journal of Mathematics **18** (2021), no. 6, Paper No. 260, 15 pp. **SCIE, Impact Factor (1.400)** [1]

4. S. Sivaprasad Kumar and **Shagun Banga**; *On a special type of Ma-Minda function*, Applied Mathematics-A Journal of Chinese Universities. (**Accepted**) **SCIE, Impact Factor (0.657)**

5. **Shagun Banga** and S. Sivaprasad Kumar; *A novel subclass of starlike functions*. (Communicated)

6. **Shagun Banga** and S. Sivaprasad Kumar; *Sharp bound of third Hankel determinant for a class of starlike and a subclass of q-starlike functions*. (Under Review)

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