INVESTIGATION OF NONLINEAR SYSTEMS USING TAIL EQUIVALENT LINEARIZATION METHOD

Thesis submitted in partial fulfilment of the requirements for award of the degree of

Master of Technology

In

Structural Engineering

Submitted By
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CANDIDATES DECLARATION

I hereby declare that the project work entitled "INVESTIGATION OF NONLINEAR SYSTEMS USING TAIL EQUIVALENT LINEARIZATION METHOD" submitted to Department of Structural Engineering DTU is a record of the original work done by **PRATEEK MISHRA** under the guidance of **Mr. G.P AWADHIYA**, Department of Civil Engineering, DTU and this project work has not performed the basis for the award of any Degree or Diploma/fellowship and similar project, if any.

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CERTIFICATE

This is to certify that the work presented in this project report entitled "INVESTIGATION OF NONLINEAR SYSTEMS USING TAIL EQUIVALENT LINEARIZATION METHOD" has been submitted to the Delhi Technological University, Delhi-110042, in fulfilment for the requirement for the award of the degree of **MASTER OF TECHNOLOGY (STRUCTURAL ENGINEERING)** by the candidate **MR. PRATEEK MISHRA (2K17/STE/14)** under my supervision.

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ABSTRACT

Development of a non-parametric linearization method to be used for analysis of nonlinear random vibration has been done. It operates on a discretized representation of the stochastic inputs and combines the ideas from the first order reliability method (FORM). For nonlinear system a particular response threshold is defined and the equivalent linear system is characterized by matching the "design points" of the nonlinear and linear responses in the space of the standard normal variables which is derived by discretizing the excitation. As a consequence of this definition, the first order approximation of tail probability of nonlinear system is equalized to tail probability of linear system. Thus, a unit-impulse response function of the input excitation is required for the representation of TELS. This system has been organized in order to tackle the inadequacy of conventional equivalent linearization method. Our objectives are investigation and thorough understanding of analysis of stochastic non-linear system by tail equivalent linearization method along with the calculations of certain nonlinear response characteristics. Furthermore study is presented on method of random vibrational analysis especially on equivalent linearization method and also gives brief review on reliability analysis of structure, first order reliability analysis (FORM). This purpose of this study is to look at the effects of different parameters on the system. At the design point, linearization of the limit-state surface is done in order to distinctly define a linear system, which is denoted as TELS.

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CHAPTER 1

INTRODUCTION

1.1 General

Most of the structural and mechanical engineering problems involve parameters that are stochastic and non-linear. In a way, it can be said that these parameters are the mother of all problems in analysis. Moreover, they cannot be overlooked as they are important considerations in determining the reliability of structural and mechanical systems under extreme loading conditions, e.g.

- Inelastic structural response to strong ground motion in case of earthquakes
- Response of high rise structures against turbulent winds
- Response of offshore structures to wave loads taking into effect the material and geometric non-linearity.

A method for non-linear stochastic dynamic analysis is the needed as the presently existing methods are either restricted to special cases or are unsuitable for reliability analysis. Consideration of nonlinearity during the safety evaluation of a structure is necessary as failure of the structure mostly happens in the nonlinear range of behaviour.

Various methods have been developed for the study of nonlinear random vibrations over the last few decades. A typical drawback of the other methods is that they are limited to specialized systems or excitation and are thus not fit for practical application despite being more accurate. The Monte Carlo Simulation method does not suffer any limitations, but it requires very difficult and lengthy calculations.

As we know, the functions of second moment of the response of a linear system represent the characteristics of the system, therefore the solutions need to follow an iterative pattern. Furthermore, a Gaussian distribution is preferred for assumption of the probability distribution. Thus, the probability distribution may not be truly correct, especially in the tail region. Therefore, the results aren't very accurate although the method is quite accurate in estimation of the mean square response.

1.2 Objective and scope of study

The objectives are examination and intensive comprehension of investigation of nonlinear systems by Tail Equivalent Linearization Method just as calculation of certain nonlinear reaction attributes. Appropriate calculation for finding the design point has been displayed.

This study to display intensive examination of nonlinear dynamic analysis utilizing TELM (Tail Equivalent Linearization Method), and impact of different parameters on the system, for example, discrete portrayal of stochastic excitation, characterization of linear system and so on. Aside from TELM for the utilization of white noise process, for contemplating of TELM we need essential thought regarding arbitrary vibration investigation and strategies for basic reliability investigation.

TELM depends on FORM and ELM or random vibration.

TELM is mix of FORM and ELM implies unwavering quality investigation and random vibration investigation. In this examination we give brief survey of both the techniques.

Various uses of this technique in structural building documented has been examined for stationary and non-stationary problems, for both single & multi-degree of freedom systems, hysteretic material models, demonstrating its legitimacy and precision.

1.3 Organization of Report

The following report is composed into twelve sections.

In the primary section a short survey of TELM and significance of this strategy are given. For comprehension TELM, we require a decent learning of arbitrary vibration examination and dependability investigation of structure with the goal that we likewise require a survey of both techniques for examination.

The second section is literature review in which works by past researchers on Tail Equivalent Linearization Method are clarified.

In part three, the various strategies for nonlinear stochastic examination are diagrammed. This incorporates traditional strategies, simulation techniques and linearization methods.

In part four, the characteristics of linear system are clarified.

In part five, we learn about reliability. Different terms utilized in reliability are explained. We learn about normal distribution function. Lastly first order reliability method is examined. Nonlinear system is changed into linear system and we compute the design point and further reliability is determined.

In section six, the different steps to discretize nonlinear stochastic procedure is clarified. This incorporates time-domain discretization and frequency-domain discretization.

In part seven, we perceive how to utilize FORM to tackle stochastic dynamic problems.

In part eight, we consider how to recognize linear system in both time & frequency domains.

In section nine, we learn about the Tail Equivalent Linearization Method. A brief presentation of TELM is given. At that point the different steps in TELM are clarified. Thereafter iterative calculations to discover the structure design point is shown.

In part ten, we learn about the different qualities of the TELM. A numerical example to demonstrate the different attributes of TELM is used by utilizing a SDOF inelastic hysteretic oscillator based on Buoc Wen Model. Before we tackle the abovementioned issue, we should know the various strategies which are utilized to assess dynamic response. We unravel a numerical example given in A. K. Chopra book by linear interpolation and furthermore, Newmark's method.

In part eleven, we learn about the deficiencies and impediments of TELM lastly

In section twelve, we acquire the conclusions from the entire report.

CHAPTER 2

LITERATURE REVIEW

Kazuya Fujimara, Armen Der Kiureghian, displayed TELM which uses the benefits of FORM. TELM is non-parametric linearization technique. The "design points" of nonlinear and linear responses are derived by discretization of the excitation and coordinated to characterize the equivalent linear system. The first order approximation of tail probability of nonlinear system is equalized to tail probability of linear system. The name comes from this property. The impulse response function is obtained by the knowledge of the design point and it uniquely describes the equivalent linear system.

Luca Garre, Armen Der Kiureghian, broadened the past work on the method to the frequency domain. The frequency response function was introduced to characterize a TELS. This approach was very helpful in case of stationary input and response functions, found very frequently in marine structures. Considering linear waves, the TELS has properties, such as the representation of multi-support excitations and invariance from the scale of the excitation. The computational proficiency of TELM is largely enhanced as a result of the last property. Discretizing input excitation to a limited arrangement of standard normal variables is an essential prerequisite of TELM. The strategy was at first created in earthquake engineering. Thus, after this a comparing meaning of TELS was gotten with regards to its unit impulse response functions. Various uses of this strategy for non-stationary as well as stationary problems in the field of structural designing have been examined.

Armen Der Kiureghian and Kazuya Fujimura, another elective methodology for processing fragility curve for nonlinear structures is proposed. This methodology is proposed. The approach utilizes an as of late created technique for the nonlinear dynamic analysis using TELM. The methodology keeps away from repeated time-history examination.

Offering a reasonable option for fragility investigation, the proposed technique has constraints. For instance, right now it is just material to non-degrading frameworks, and just a single part of ground movement was considered. Moreover, response slope calculations are required and in this manner, a dynamic investigation code with this capacity must be utilized. By and by, the proposed strategy offers an option in contrast to a kind of investigation for which couple of other reasonable choices are by and by accessible.

Caughey TK proposed summing up to the instance of nonlinear systems along with the random excitation. The strategy is connected to an assortment of issues and results are contrasted and definite arrangements of the Fokker-Planck condition for those situations where the Fokker-Planck system may be connected. Exchange ways to deal with the issue are examined including the trademark work.

Heonsang Koo, Armen Der Kiureghian, Kazuya Fujimara, The structure point excitation for white noise input into an indirect versatile SDOF oscillator is indistinguishable from the identical representation of free-vibration reaction when discharged from the target threshold. For these cases just a guess to the structure design point is acquired. In the event that essential the guess can be utilized as a beginning stage in a calculation to get the exact structure design point.

Armen Der Kiureghian, The geometry of random vibration issues in the space of the standard random variables acquired by discretizing the input procedures is portrayed. Linear systems exposed to Gaussian excitation, straightforward geometric structures, for example, vectors, planes and ellipsoids, portray the issue of intrigue. For non-Gaussian reactions, non-direct geometric structures portray the issues. Surmised answers for such issues are acquired by the utilization of FORM and SORM.

M.Ababneh, M.Salah, K.Alwidyan, in his paper, an examination between the ideal linear model and Jacobian linearization method is led. The exhibition of these two linearization techniques are delineated utilizing two benchmark nonlinear systems, these are transformed pendulum system; and Duffing chaos system, linearization of nonlinear dynamical systems. Optimal linear model is an online linearization strategy for finding a neighbourhood model that is linear in both the state and control terms.

CHAPTER 3

METHODS OF NONLINEAR STOCHASTIC ANALYSIS

3.1 Introduction

Classical methods: Fokker-Plank equation, Moment Closure, Perturbation methods, stochastic averaging.

Simulation methods: Monte Carlo Simulation (MCS), Markov Chain Monte Carlo (MCMC), Orthogonal Plane Sampling (OPS), Importance Sampling (IS), Latin Hypercube Sampling (LHS), etc.

Linearization methods: Equivalent Linearization Method (ELM), TELM.

The old style strategies are significant and rich approaches, however they are restricted to particular systems. The vast group of simulations has no hypothetical breaking points be that as it may, a few of these techniques lack computational efficiency for high reliability issues. The last class of strategies offers an effective and genuinely precise calculation of the reaction distribution for most structural designs.

3.2 Classical methods

3.2.1 FOKKER-PLANCK EQUATION - It was inferred with regards to statistical mechanics, a partial differential equation depicting development in time of probability density function. Solving this condition gives the definite probabilistic structure of the response consistently.

3.2.2 MOMENT OF CLOSURE - It is a rough strategy for assessing the statistical moments. As a rule, statistical moments are administered by endless coupled conditions; a closure procedure is utilized to acquire an approximate solution as far as a limited arrangement of minutes. The precision of the arrangement relies upon the order of closure.

3.2.3 PERTURBATION - Among the traditional strategies, these are presumably the initial ones to be utilized in nonlinear random excitation. They are genuinely broad techniques to comprehend deterministic and/or potentially nonlinear issues.

These depend on power series expansion keeping "significant" terms only. Differential equations figured for every term. The technique is somewhat direct. In any case, because of the idea of the detailing, the expansion terms quickly turn complex in case of high-order terms. Moreover, these techniques are generally constrained to slightly nonlinear systems.

3.2.4 STOCHASTIC AVERAGING - Basically, the strategy approximates response vector with diffusive Markov vector with the likelihood thickness capacity administered by FP condition. The strategy is intended to ascertain the coefficient function by taking out the impact of intermittent terms using stochastic averaging. This strategy is relevant to a wide range of SDOF systems; however it discovers its constraints when connected to MDOF systems.

3.3 Simulations methods

3.3.1 MONTE CARLO SIMULATION - Because of its effortlessness, it is the most widely and frequently used method. There are no hypothetical restrictions inferable from the idea of the methodology. For systems with very high reliability, where the most distant tail of the distribution is of interest, numerous simulation techniques have been created. The two head classifications are IS and MCMC techniques.

3.3.2 MARKOV CHAIN MONTE CARLO METHOD - A Markov chain having ideal distribution in its equilibrium state is developed for examining from complicated probability densities. Gibbs sampling discovered the underlying foundations in image processing, great references for MCMC techniques.

3.4 Linearization Methods

3.4.1 Equivalent Linearization Method

The ELM is the most famous strategy utilized in nonlinear elements. It is widely popular because it is very simple and has a wide range of applications. The most engaging element of each linearization technique is that, as soon as we acquire the linear system, all the linear theory can be easily connected.

The equivalent needs to be specifically defined in the approximation of nonlinear response in terms of an "equivalent" linear response system (Caughey 1963). The ELS is defined by removing a type of variation between the nonlinear and linear systems. Various techniques are established depending on the type of variation minimised:

- Ordinary ELM – limits the variance of error among nonlinear and linear responses; a distribution is to be assumed, normally Gaussian (for example Atalik and Utku 1976; Wen 1976). Gaussian distribution is utilized on the grounds that it highly simplifies every one of the calculations. This technique functions admirably on the off chance that you are assessing the variation of the nonlinear response. It gives very exact outcomes. Anyway you are keen on tail probabilities (the probability of a response exceeding a given higher threshold), this technique does not functions admirably. This does not function admirably especially due to the Gaussian distribution function. We realize that regardless of whether input is Gaussian, the yield of a nonlinear framework isn't Gaussian, so this Gaussian distribution is fairly limited. Hence the following two strategies attempt to beat this issue.

- Minimize higher moments of error (Naess 1995) – this technique is utilized for a specific type of elastic nonlinear system wherein the restoring force has a polynomial structure. Since higher moments are being looked at, accentuation is set in the tail thus showing more satisfactory results in the tail. Yet, the technique is confined again on account of the polynomial structure.
- Minimizing the difference in mean crossing up rates at a selected threshold (Casciati 1993). By this we can get great outcomes in the tail. Anyway the estimation of the up crossing rate response is very vague.

3.4.2 Tail Equivalent Linearization Method

This technique characterizes the direct framework by equating first order approximation of tail probability of nonlinear response to tail probability of linear response (Fujimara and Der Kiureghian 2007). Since it is managing the tails the precision is improved in the tail area.

CHAPTER 4

CHARACTERISTICS OF A LINEAR SYSTEM

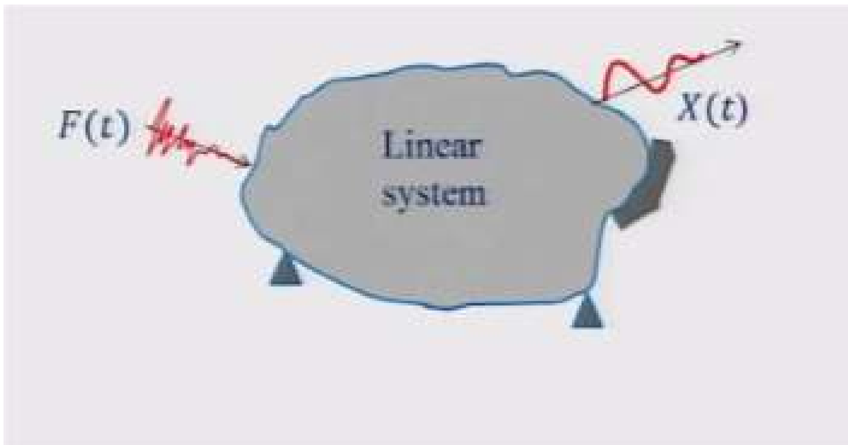


Fig 4.1 Linear System

Think about a linear system exposed to one excitation F(t) and one response X(t).

For one input-output pair (F(t), X(t)), a stable linear system is totally characterized by both of the accompanying:

- h(t) = impulse response function(IRF); i.e. reaction to F(t)=δ(t)
- H(ω) = frequency response function(FRF); i.e. amplitude to steady state response to F(t) = exp(iωt) (complex harmonic function).

In the event that you have both of these functions for a stable linear system, at that point you have totally described the system. You don't have to know the geometry, boundary conditions, and so forth. So for any input you can contribute the comparing output.

CHAPTER 5

RELIABILITY

5.1 Reliability Analysis

5.1.1 Reliability:

Geotechnical structural quality over some time under standard conditions is known as reliability. We can also say that it is the success probability.

5.1.2 Methods of reliability:

i. First Order Reliability Method (FORM)

ii. Second Order Reliability Method (SORM)

iii. Monte Carlo Sampling (MCS)

iv. Numerical Integration (NI)

v. Increased Variance Sampling (IVS)

5.1.3 Mean:

It is the first central moment and can be defined as average value. It also approximates the central tendency of the data.

5.1.4 Variance:

It is the second central moment and it tells us how much the data values spread about the mean.

5.1.5 Coefficient of variation (cov):

It gauges the scattering of data. Higher estimation of cov speaks to the higher dispersion about its mean.

5.1.5 Covariance:

It demonstrates the extent of linear relationship among two random variables.

$$Cov\ (x,y) = E\left((x-mx)\,(y-my)\right) = E\,(xy - mx\,my)$$
$$= E(xy) - E(x)E(y)$$

The vulnerabilities in a variable can be evaluated utilizing a numerical model fulfilling various capacities.

5.1.6 Normal distribution:

It is the most generally known and utilized of all distributions considered. Since it approximates numerous natural phenomena very accurately, it has developed as a standard of reference for most of the problems of probability.

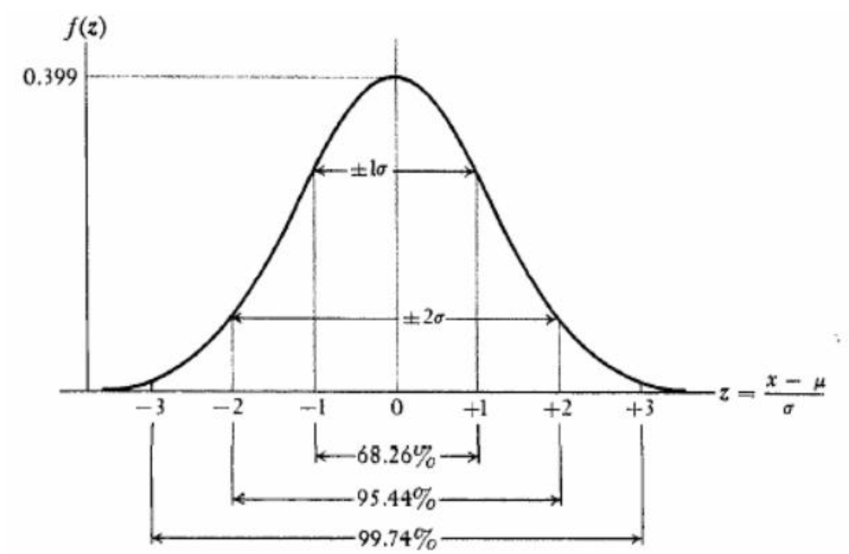


Fig. 5.1 Standard Normal Distribution Curve

5.1.7 Properties of Normal distribution:

- Ranges from – ∞ to + ∞.
- It is impeccably symmetric.
- Mean, median and mode values are always same.

The standard expression for a normal density function is

$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

The value of reliability is obviously $(1 - Pf)$. Let 'R' = resistance and 'S' = load, then the structure will fail on the chance that 'R' < 'S' and failure probability can then be written as

$$Pf = P[R \leq S] = P[(R - S) \leq 0].$$

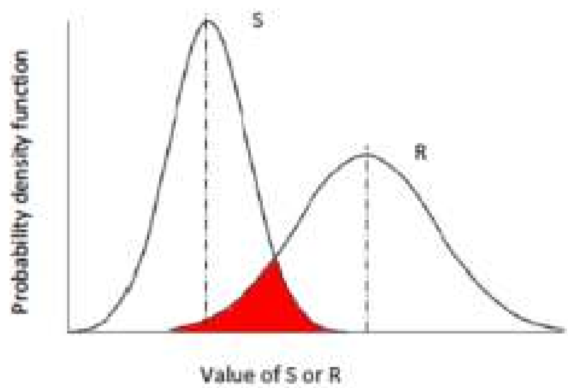


Fig. 5.2 Shaded area is the failure probability

The shaded area as depicted in the above figure is the probability of failure and it is mathematically written as

$$Pf = \int_{-\infty}^{+\infty} GR(r)GS(s)ds$$

Reliability,

$$R = 1 - \int_{-\infty}^{+\infty} GR(r)GS(s)ds$$

Where, GR(r) is cumulative distributive function of 'R' and GS(s) is the cumulative distributive function of 'S'.

A mathematical model that can relate the variables like load and resistance is known as the limit state function. It can be written as

$$Z = (R - S) = f(R, S) = f(X_1, X_2, X_3, \dots\dots, X_n)$$

Where, 'Z' = margin of safety

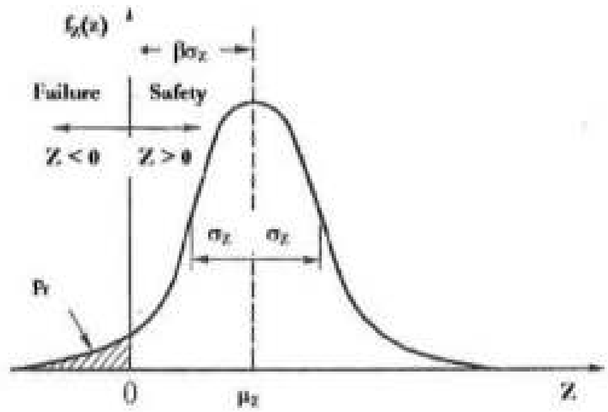


Fig. 5.3 Distribution of safety margin (Melchers 2002**)**

The equation for the reliability index was given by Cornell as

$$\beta = \frac{\mu_z}{\sigma_z} \text{ and}$$

$P_f = \Phi\ (-\beta)$ is the cdf of the given standard normal variable.

5.2 First order reliability method (FORM)

It is an all-around created method for solving reliability problems which is an essential component of structural reliability analysis.

X = vector of random variables

$G(X)$ = limit state function (G(X) <= 0 →failure event)

$P_f = Pr[g(x) <= 0]$ = failure probability

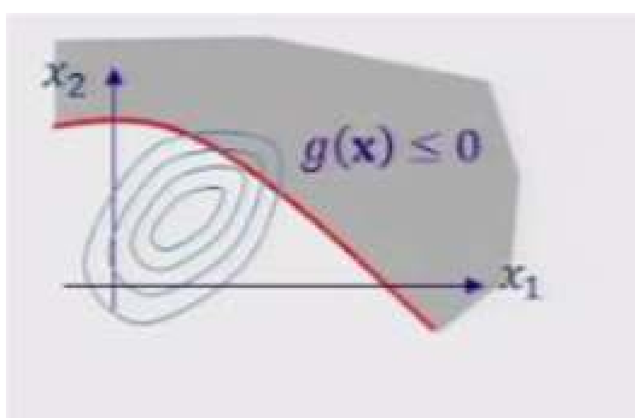


Fig 5.4 Geometry of random variables

The image above depicts the idea regarding geometry in the space of the random variables x_1 and x_2. The shapes are speaking to the form of the likelihood thickness capacity of these arbitrary factors. The limit state surface is shown by the red line where this limit state function takes zero values and the grey area is the failure space. Estimating the probability of these random variables to be lying in this failure region is the main work at hand.

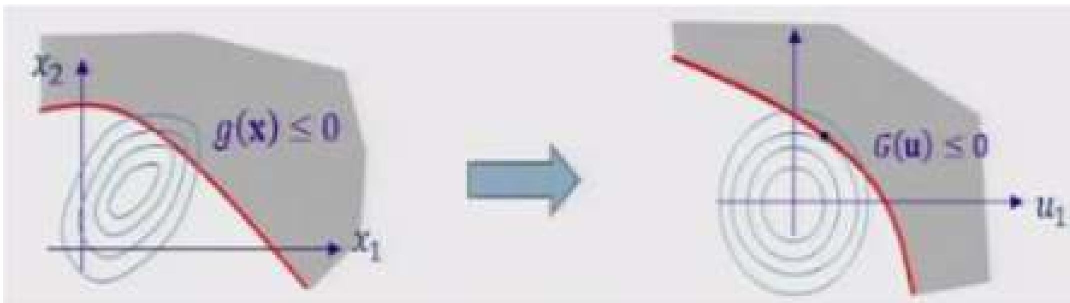


Fig 5.5 Transformation - x to u space

The FORM takes care of this issue by making a transformation. The transformation is made from the 'x' space to the 'u' space (a vector of standard normal variables). There is no estimation required here and this should be possible until the random variables are continuous and possess a strictly increasing joint cumulative distributive function (cdf).

- $u = u(x)$ - Transformed to normal space.
- $G(u) = g\{u(x)\}$ - Limit state function in the transformed space.

The benefit of doing this (x-u space transformation) is that the probability densities have circular and hyper circular contours in the u space in higher dimensions. So it is a standard space and in this space there are basic properties regarding probability calculations.

Next we discover the closest point from the origin, i.e. design point.

$$u^* = \arg min \left[\left| |u| \right| | \, G(u) = 0 \right] = \text{design point}$$

The surface is the linearized at this point.

$$\beta = \left| |u^*| \right| = \text{reliability index}$$

Reliability index is the distance of the design point from the origin.

The first order approximation of the failure probability can be obtained by obtaining the probability of failure that is defined by a particular hyper plane which is half space probability in the normal space. Generally, it is standard normal probability function assessed at minus the distance from the origin.

$$P_f = \Phi(-\beta) \text{ => FORM approximation}$$

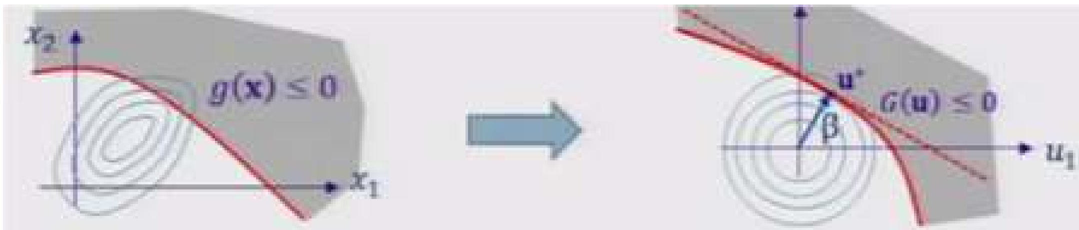


Fig 5.6 Reliability index and Design point representation

Probability density decays exponentially in accordance with the distance from the origin is the only reason why this approximation works so well. Hence the differences between the hyper plane and the actual surface become negligible as we go farther away from the origin.

CHAPTER 6

DISCRETE REPRESENTATION OF STOCHASTIC PROCESS

Discretizing the stochastic process is necessary and representation in terms of random variables only so as to utilize a time-invariant stochastic problem.

6.1 General form of a zero-mean Gaussian process

$F(t) = s(t).u$

$s(t) = [s(t_1) \dots \dots . . s(t_n)]$ = vector of deterministic basis function which upholds the time evaluation of the excitation.

$u = [u_1 \dots . . u_n]$ = vector containing standard normal variables responsible for stochasticity.

This is a way of separating variation in time and stochasticity.

There are different ways of doing this.

6.2 Time domain discretization

$s_i(t) = q(t). h_f(t - t_i)$ where,

$h_f(t - t_i) = impulse\ response\ function\ of\ a\ linear\ filter$

$q(t) =$ modulating function that modulates the process in time

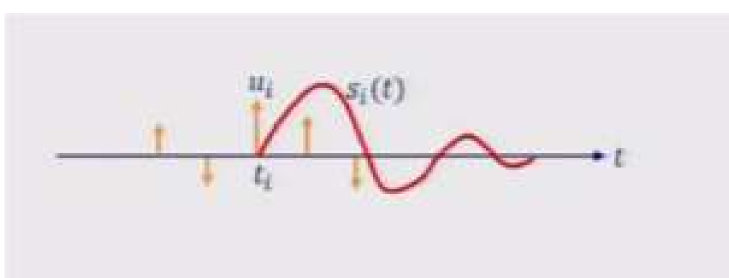


Fig 6.1 Time domain discretization

Discretized time has been shown in the above image. A random impulse and the filter response to that impulse are obtained and they can be added up as the first equation is a summation equation. The final result obtained after summing up these impulses is shown hereafter.

Fig 6.2 Frequency domain discretization

The above process is non-stationary in both time domain & in the frequency domain.

6.3 Frequency-domain discretization

$$F(t) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega, \text{ where}$$

$$F(\omega) = \int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt$$

The above process is well known for decomposing a process into its frequency components.

CHAPTER 7

FORM SOLUTION OF STOCHASTIC DYNAMIC PROBLEMS

7.1 Definitions

- $F(t) = s(t).u$ = discretized stochastic excitation
- $X(t,u)$ = response to discretized stochastic excitation (the response is a function of time but also implicitly the function of the random variable u, there could be huge numbers of them relying upon how you discretize length, etc.)
- $Pr(x < X(t,u))$ = tail probability at the threshold 'x' at time 't' (the tail probability tells about the probability that at any given time 't', the response exceeds a particular threshold response 'x')

7.2 Reliability Formulation

- $G(u,x) = x - X(t,u)$ => this is the limit state function G(), because the random variables are already in the standard normal space
- $Pr(x < X(t,u) = Pr(G(u,x) \leq 0)$ => tail probability is now the probability that the limit state function assumes a value that is less than zero
- $u^* = \arg\min\left(||u||\middle| G(u,x) = 0\right) => \mathit{this\ is\ the\ design\ point}$
- $\beta(x) = ||u^*(x)|| =>$ reliability index

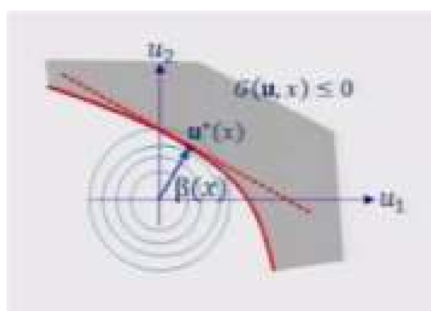


Fig 7.1 Reliability index (β (x))

The only exception here from the one shown before is that here the limit state function contains the threshold too as a parameter.

$$Pr(x < X(t,u)) = \Phi(-\beta(x)) => \text{FORM approximation of tail probability}$$

Since β is not necessarily proportional to 'x', the distribution obtained is also not Gaussian.

7.2.1 Reliability formulation - in case of linear system

Using the superposition principle

- $X(t,u) \;=\; a(t).u$ where, $a_i(t)$ = vector of responses to deterministic functions $s_i(t)$.
- $G(u,x) = x - a(t).u$, we can notice that the limit state function is a linear function of u.
- $u^* \;= \frac{x}{||a(t)||}.\frac{\mathrm{a(t)}}{||a(t)||}$
- $\beta(x) = \frac{x}{||a(t)||}$, we see that the reliability index is proportional to the threshold x.
- $Pr(x < X(t,u)) = \Phi(-\beta(x))$, since the tail probability β is proportional to x, it can be concluded that the response is Gaussian.

CHAPTER 8

IDENTIFICATION OF THE LINEAR SYSTEM

Once the design point is known for a particular input-output pair, linear system can easily be identified. Knowing the design point will also lead to the knowledge of the linear vector 'a' and as soon as we get the vector 'a', the system can be identified in either of the two domains, viz. the time domain and the frequency domain.

8.1 Time Domain Analysis

Duhamel's integral gives the response *X(t)* of a general linear, time-invariant dynamical system.

$$\mathrm{a_i(t)} = \int_0^t h(t-t_i).s(t_i)dt_i \,, i = 1,2,\dots,n$$

where $a_i(t)$ is the response of the system to the deterministic function s_i.

The discretized form of the Duhamel's integral can be written as

$$a_i(t) = \sum_{i=1}^{n} h(t-t_i).s(t_i)\Delta t \,, \forall\, i = 1,2,\dots,n$$

Here, *h(.)* is impulse response function (IRF) of system for particular input-output pair.

Provided that a_i and s_i are known, '*h*' can be calculated at various time steps. Thus we can directly get the IRF of the system by the design point. Therefore we can know the type of linear system being used without knowing the linear system but by knowing the design point.

8.2 Frequency Domain Analysis

$$|H(\omega_i)| = \frac{\sqrt{a_i(t)^2 + \bar{a}_i(t)^2}}{\sigma_i}$$

$$\theta_i = \tan^{-1}[\frac{a_i(t)}{\bar{a}_i(t)}]$$

$$H(\omega_i) = |H(\omega_i)| \exp(i\theta_i)$$

We see that the modulus and the phase angle of the frequency response function (FRF) can be computed if we have a_i, which in turn is easily obtained is the design point is known.

Thus we confirm that with the knowledge of design point, both the IRF and the FRF can be easily obtained.

The FRF can also be computed if we know the IRF of the system by the following equation

$$H(\omega) = \int_0^\infty h(t) \exp(-i\omega t)\, dt$$

CHAPTER 9

THE TAIL EQUIVALENT LINEARIZATION METHOD

9.1 Introduction

TELM was introduced as a developed linearization technique for the purpose of nonlinear stochastic dynamic analysis.

It uses the time-invariant first order reliability method (FORM) to make a fairly accurate estimate of the tail of the response distribution of a nonlinear system when subjected to a stochastic input making.

Discretization of the input process is done and it is represented as a set of normal variables in TELM. A limit-state surface is defined for various response thresholds. A non-parametric and unique linear system is defined by linearizing the limit-state surface at the design point. This linear system is known as the TELS.

9.2 Steps in TELM

- For a given threshold 'x' and time 't' the tail probability problem is formulated in terms of the limit-state function $G(u,t) = x - X(t,u)$
- Design point 'u*' is found out.
- Gradient vector of tangent plane is then found.
- The TELS corresponding to the gradient vector 'a' is then identified by its IRF '*h(t)*' or its FRF '*H(ω)*'. The tangent that hyper plane defines the TELS. Although the calculations are pretty simple, the determination of the design point is not that easy a task.

9.3 Iterative algorithms for solving design point

$$u^*(x) = \arg\min\{||u|| \; G(u,x) = 0\}$$

Repeated calculations of *X(t, u)* and the gradient of the response for the selected values of 'u' is required.

The solution to constraint optimization problem gives us the design point. Distance of limit state surface from the origin has to be minimised.

The nonlinear problem has to be solved repetitively. Generally, this calculation does not require a lot of iterations. Convergence in results is found in not more than 10-20 steps.

Due to the huge amount of variables used in the problem, using the method of finite differences to calculate the response gradient can prove to be a very cumbersome process.

The direct differentiation method is used in order to avoid this problem.

Small increments are assumed between successive thresholds. This gives a fair approximation to the design point of next threshold, x_{i+1}, using the following equations

$$\hat{u}^*(x_{i+1}) = u^*(x_i) + \lambda \frac{u^*(x_i) - u^*(x_{i-1})}{||u^*(x_i) - u^*(x_{i-1})||}$$

$$X(\hat{u}^*) = x_{i+1}$$

The approximate design point '$\hat{u}^*$' is located 'λ' distance away from the previous design point '$u^*(x_i)$' on the line which connecting previous 2 design points, using first equation. Utilization of the second equation is that it finds the point on limit-state surface for that response threshold, i.e. x_{i+1}. The approximate solution '$\hat{u}^*(x_{i+1})$' is then used as a starting point.

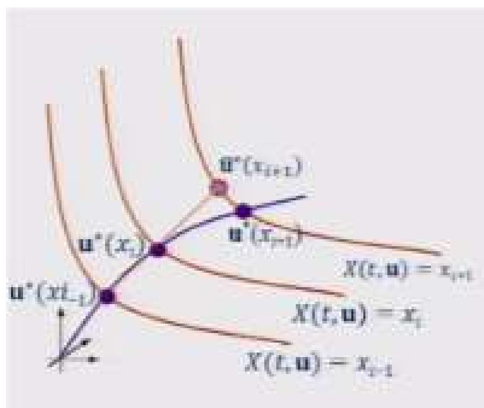


Fig 9.1 Representation of design point

$x_1 = 0$ is selected as the first threshold. A small value is selected for x_2 where the response of the structure is nearly linear. When the first two points have been determined, the search algorithm can be started.

CHAPTER 10

CHARACTERICTICS OF THE TAIL EQUIVALENT LINEARIZATION METHOD

For a given threshold '*x*' and time '*t*' :

The tail probability of the TELS system = First order approximation of the tail probability of the nonlinear system response

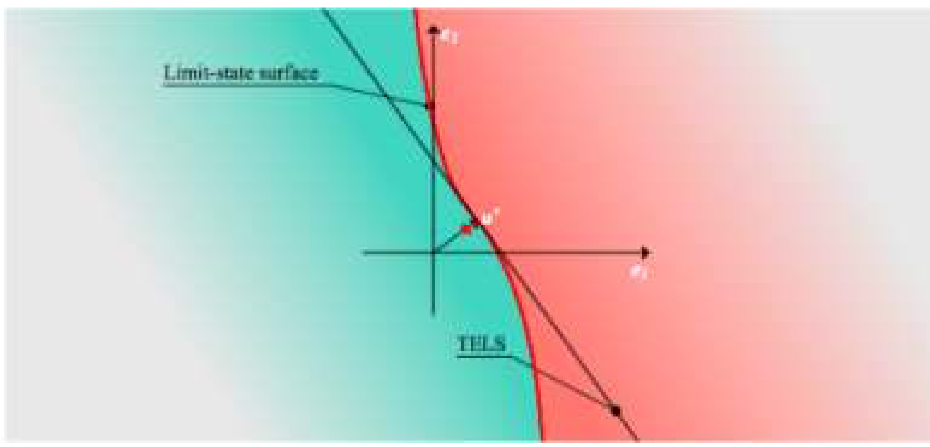


Fig 10.1 TELS of the nonlinear response

- TELM is a non-parametric method unlike the ELM and other linearization methods.
- The IRF/FRF of the linear system shares a one-to-one relationship with the design point of the tail distribution. Specifically speaking, the IRF/FRF can be defined solely by the coordinates of the design point. It is quite a relief that no parameters need to be defined and computed through optimizations for the TELS, making it a non-parametric system in a complete sense.
- Design point excitation $F^*(t) = s(t).u^*$ gives us the most probable realization of the excitation so as to offer ascent to the occasion $\{x \leq X(t,u)\}$. In order to find the design point excitation, we first find the design point and then substitute it back in the equation of the discretized point excitation.

10.1 Numerical Example

Here, an SDOF oscillator with inelastic material behaviour is considered to numerically investigate the properties of TELM. Both the frequency and time domains are utilized to solve the problem. A symmetric Bouc-Wen material model is used in order to describe the force-displacement relationship.

The condition of differentiability of the limit state & response function is important as it confirms that the limit-state surface has a tangent hyper plane at the design point.

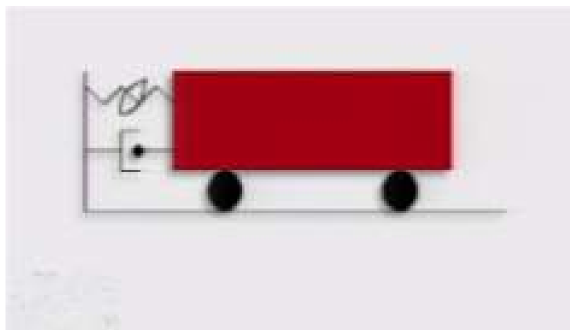


Fig. 10.2 SDOF oscillator with inelastic material behaviour

A hysteretic oscillator is considered to obtain further insight into the nature of the system. This oscillator is described as:

$$m\ddot{X}(t) + c\dot{X}(t) + k[\alpha X(t) + (1-\alpha)Z(t)] = F(t)$$

Where, $m = 3.0x10^5\ (kg)$,

$c = 1.5x10^2\ \left(kN\frac{s}{m}\right)$,

And $k = 2.1x10^4\ (kN/m)$

The degree of hysteresis is controlled by the parameter 'α'.

$\alpha = 0.1$

The excitation process is described by the equation:

$$F(t) = -\ddot{U}_g(t)$$

Where, $\ddot{U}_g(t)$ gives base acceleration modelled as white-noise process.

A finite value of the intensity of the white noise process produces results as shown below since the scale of the excitation has no effect on the TELS.

The term $Z(t)$ follows the Bouc–Wen hysteresis law.

$$Z(t) = -\gamma|\dot{X}||Z(t)|^{n-1}Z(t) - \eta|Z(t)|^{n}\dot{X}(t) + A\dot{X}(t)$$

Where $\gamma = \eta = \frac{1}{2\sigma_o^n}$ in which $\sigma_o^2 = \frac{\pi S m^2}{ck}$ is the mean square response of the linear *(α* = 1*)* oscillator, and the selected parameters are $n = 3$ and $A = 1$.

The values of stiffness, initial displacement, mass damping ratio, natural time period, velocity can be changed and the graphs obtained hereafter will change correspondingly.

Thus example investigating the various properties of TELM is solved on MATLAB.

Values of various parameters that are given in the problem statement have been utilized in the code and the values of some other parameters have been assumed. The predefined functions such as “linsquare, pwelch, linsquare, hilbert” have been put to use which are predefined functions in MATLAB. A graph is plotted between the ground acceleration and time, depicting the variations in the impulse response functions (IRFs) and the frequency response functions (FRFs), thus showing the true nature of TELM. The problem has been solved in both the time and frequency domains.

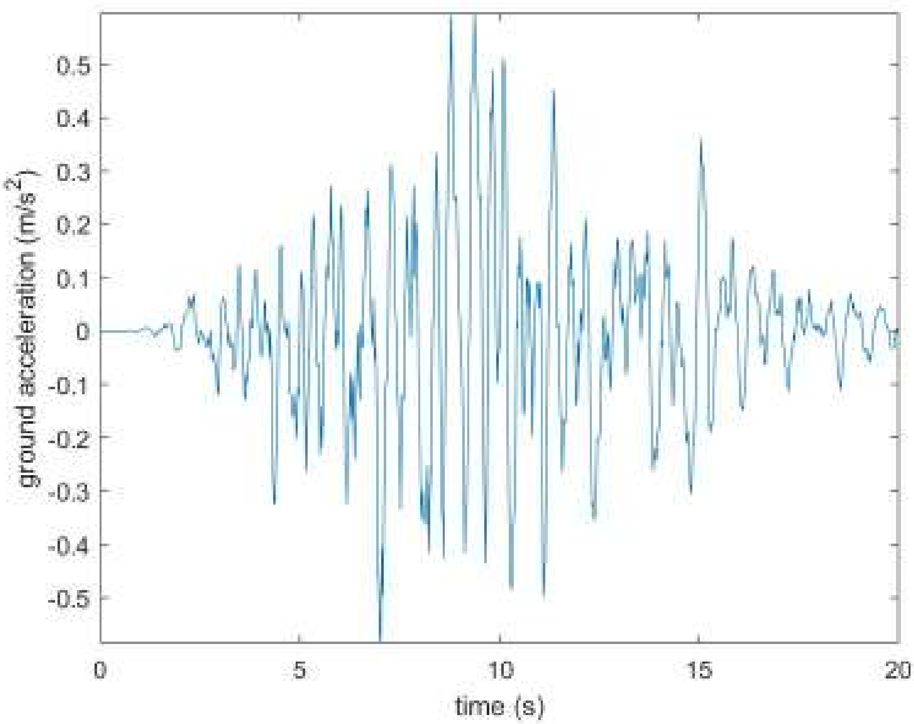


Fig.10.3 ground acceleration v/s time graph

The figure above shows that the ground acceleration $\ddot{U}_g(t)$ first reaches to a peak and then its effect starts diminishing after a while.

- TELS is not dependent on the scaling of the excitation because neither the direction of the design point nor the shape of the limit-state surface varies due to the effect of this scaling, i.e. $h(t, x)$ and $H(\omega, x)$ for the excitation $sF(t)$ are invariant of 's'.

This characteristic of TELS is very useful in getting fragility curves. The conditional probability of an event of interest conditioned on scale of excitation is known as fragility.

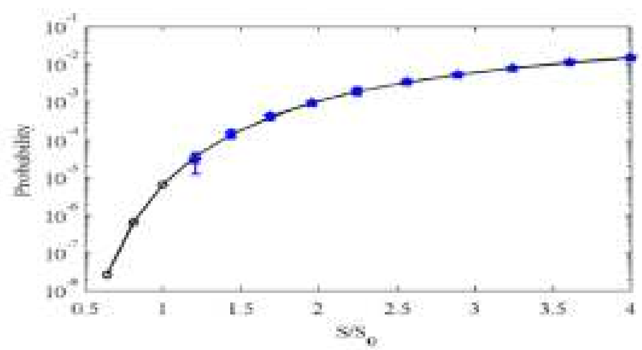


Fig.10.4 Fragility curve for given threshold

Every curve has been developed from one design point.

- For broad-band excitations, there is a very mildly dependency of the TELS on the frequency content. So, a white-noise approximation is used to describe the IRF/FRF.

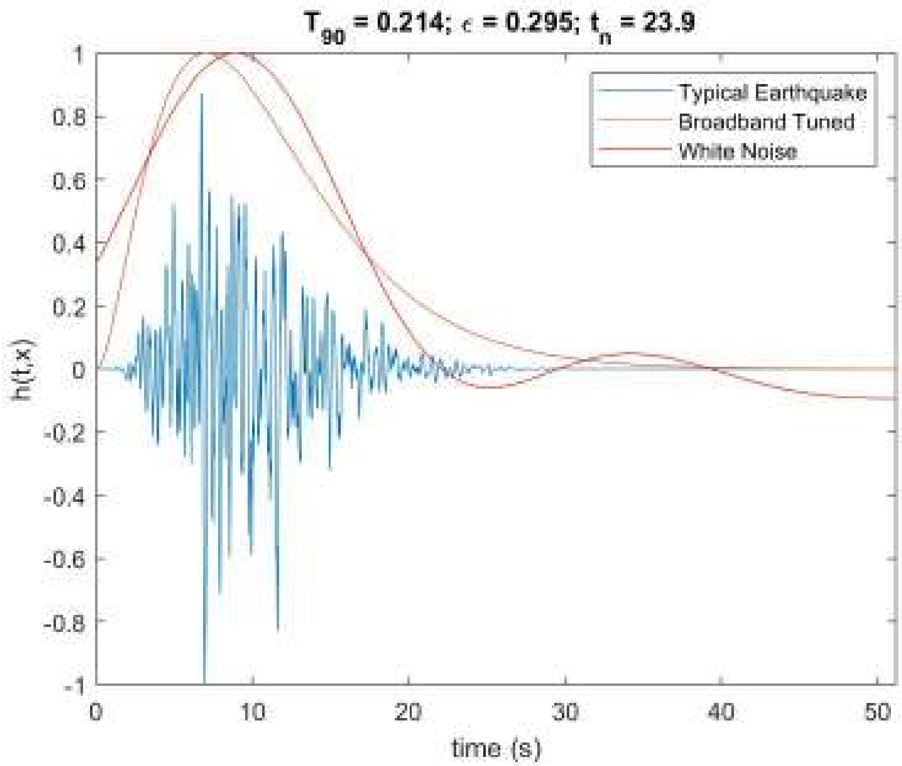


Fig.10.5 IRFs of TELS for the response of hysteretic oscillator

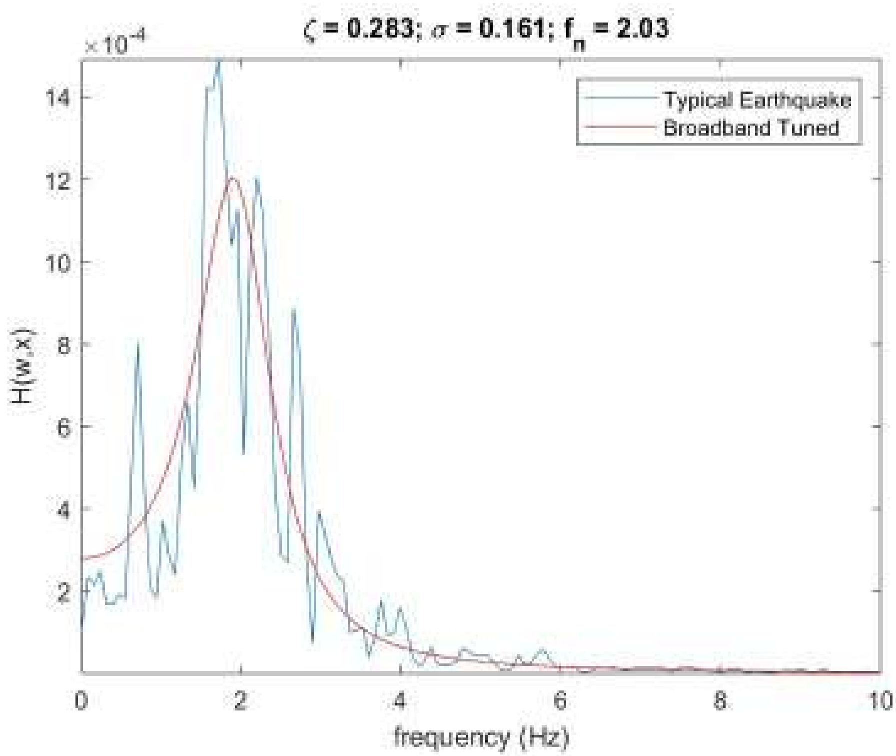


Fig.10.6 FRFs of TELS for the response of hysteretic oscillator

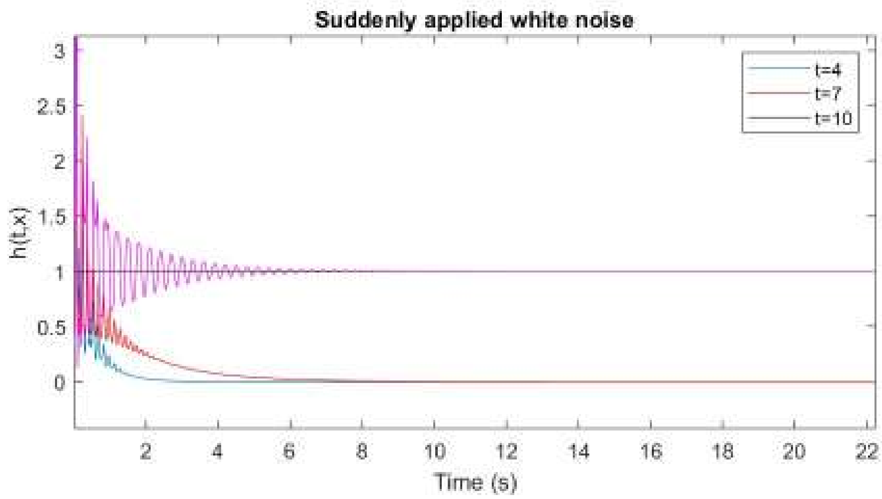


Fig.10.7 Effect of non-stationarity on TELS.

The above figure presents comparisons between the corresponding IRFs for $t_n = 4s, 7s\ and\ 10s,$ which are plotted for the interval (0,5s). It is clearly evident that in the event of the occurrence of a suddenly applied stationary excitation, the IRF has very little dependence on t_n. On one hand, a single IRF is adequate per threshold for a stationary process, and on the other the IRF for each time point needs to be determined repetitively for a non-stationary process. This is the methodology that is followed in the ELM. The ELS needs to be obtained at every time step for a non-stationary excitation.

There is a very strong dependency of the TELS on the selected thresholds:

$$h(t) \quad => \quad h(t,x), \ H(\omega) \quad => \quad H(\omega,x)$$

As opposed to the conventional linear system, where there is only one linear equivalent linear system that has to be found and applied for all thresholds, in the tail equivalent linear system we have a different linear system for each threshold.

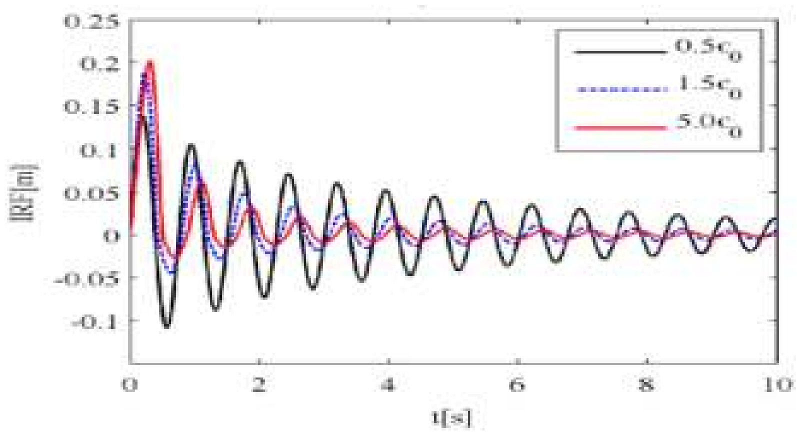


Fig.10.8 Variation of IRF for selected thresholds

We see that as the threshold value 'x' increases, the corresponding dissipation is faster. Though, for the unit response function of linear oscillator these curves are not typical.

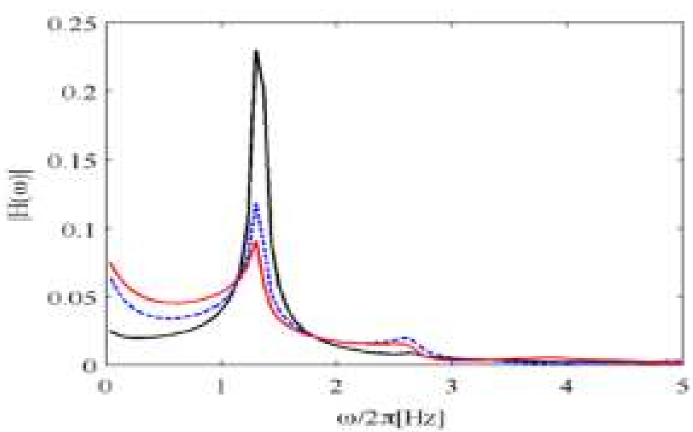
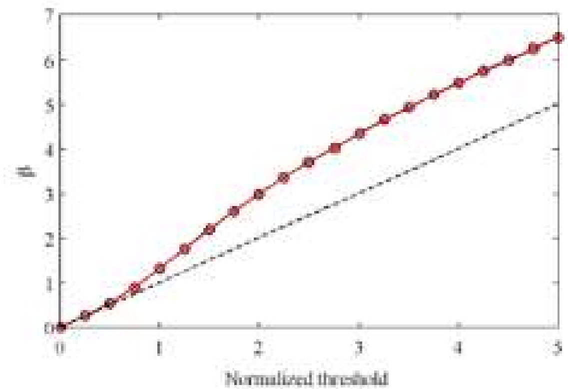


Fig.10.9 Variation of FRF of selected threshold

The above graph makes it clear that upon increasing the threshold value the peak of the FRF drops.

$$Pr[x \leq X(t,u)] \quad = \quad \Phi(-\beta(x))$$

Since the reliability index is not proportional to the threshold 'x', the TELM can present the non-Gaussian distribution.



$Blue => linear(\alpha = 1);\ Red => nonlinear(\alpha = 0.1)$

Fig.10.10 Variation of reliability index with threshold

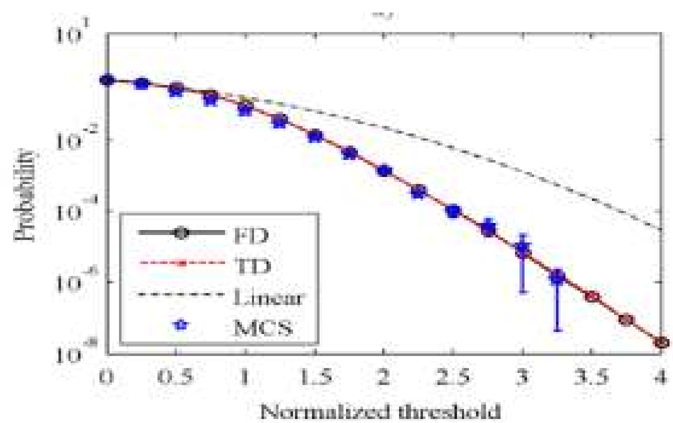


Fig.10.11 Variation of complementary cdf with threshold

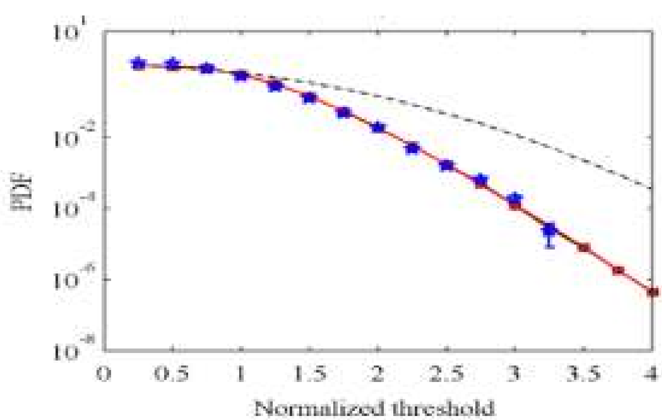


Fig.10.12 Variation of probability density function with threshold

The curve appears to be parabolic in the case that the response is Gaussian. In case of a linear response, the curve is parabolic. The curve in the figure above is not parabolic. The tail of the curve is going down as a straight line.

TELS is invariant of time t for a stationary response. Thus the TELSs determined at one time point are enough to compute every statistical property of the response, viz.

- Point-in-time distribution: $\Pr[x \leq X(t, u)]$
- Mean up crossing rate
- First passage probability

TELM can be extended to MDOF systems with sufficient ease.

CHAPTER 11

LIMITATIONS AND SHORTCOMINGS OF TELS

- All the traditional disadvantages of FORM automatically become the drawbacks of TELM. In particular, the error due to the linearization approximation cannot be measured, implying that precision of TELM cannot be known in beforehand. Also, TELM asks for unmistakably greater analysis than ELM.
- Several repeated computations are required in TELM as we use the direct differentiation method.
- The continuous differentiability of the nonlinear response is a must. Thus the use of smooth or smoothened constitutive laws becomes compulsory otherwise the tangent plane cannot be defined. A pure elasto-plastic oscillator cannot be used without smoothening. Also, the transitions between the different systems have to be smooth.
- TELM is not very accurate for strongly stiffening systems (e.g. Duffin oscillator with a strong cubic term) or when abrupt behaviour in the system behaviour is involved in the non-linearity, i.e. the limit state surface should be well behaving.
- Also, TELM cannot be applied to degrading systems.

CHAPTER 12

CONCLUSIONS

- In general, TELM is another linearization method for non-linear stochastic dynamic excitations.
- TELM is non parametric.
- According to its name, TELM gives us better results for tail probabilities.
- TELM proves to be very convenient for fragility analysis.
- TELM can be applied to stationary as well as non-stationary response.
- It can also be applied to MDoF systems with multi-component excitations.
- Therefore, in TELM also the accuracy of the method depends on the nature of the nonlinearity.
- For the application of TELM, the nonlinear response should have continuous differentiability.
- TELM can also capture the non-Gaussian distributions of nonlinear response.

APPENDIX-I

Matlab code for numerical example solved in chapter 10:

1. Main.m

```
%%TELM for noonlinear random vibration
warning off;
f1=linspace(0,40,2048);
zeta1=0.3;
sigma1=0.3;
fn1=2;
T90=0.3;
eps=0.4;
tn1=20;
f0=0;
Fs=100;
NFFT=2^12;

[y1,t1]=seiTELS(sigma1,fn1,zeta1,f1,T90,eps,tn1);

input.Vs1 =[200 300 2000]; %(m/s)
input.rho1 =[2000 2100 2400]; %(kgr/m3)
input.damp1 =[0.04 0.03 0.01];
input.freq1 =linspace(f0,Fs,NFFT); %frequency range
input.layer_thick1=[10 10]; %(m) !no thickness for bedrock!

%Call function
[f,U,A,B]=HOR_IRF(input);
%Frequency response function
FRF_linear=U(1,:)./U(end,:);
FRF_firstorder=U(2,:)./U(end,:);
FRF_secondorder=U(3,:)./U(end,:);
FRF_thirdorder=U(2,:)./U(end,:)+1;
%%plot
```

```
figure
plot(t1,y1);
xlabel('time in sec')
ylabel('ground-acceleration in (m/s^2)')
axis tight; xlim([0 20]);
set(gcf,'color','w')
guessEnvelop=[0.33,0.43,50];
guessKT=[1,1,5];
[T90,eps,tn1,zeta1,sigma1,fn1]=KTPSD(t1,y1,guessEnvelop,guessKT,'dataplot','yes'
);
%plot
figure;
plot(f,abs(FRF_linear));
hold on;
plot(f,abs(FRF_firstorder),'r');
hold on;
plot(f,abs(FRF_secondorder),'k');
hold on;
plot(f,abs(FRF_thirdorder),'m');
xlim([0 Fs/2])
xlabel('Time in sec')
ylabel('h(t,x)')
title('Suddenly applied white noise');
legend('t=4','t=7','t=10');
```

2. seiTELS.m

```
function[y,t] = seiTELS(sigma1,fn1,zeta1,f1,T90,eps,tn)
% [y,t] = seiTELS(fn1,sigma1,zeta1,f1,T90,eps,tn) generates one time series
% corresponding to the acceleration record from a seismometer.
% The function requires 7 inputs, and gives 2 outputs.
```

Initialisation

```
w1 = 2*pi*f1;
```

```
fs1 = f1(end);

dt1 = 1/fs1;

f01= median(diff(f1));

Nfreq1 = numel(f1);

t1 = 0:dt:dt*(Nfreq1-1);
```

Generation of the spectrum S

```
Fn1 = fn1*2*pi;
s01 = 2*zeta1*sigma1.^2./(pi.*fn1.*(4*zeta1.^2+1));
A1 = fn1.^4+(2*zeta1*fn1*w1).^2;
B1 = (fn1.^2-w1.^2).^2+(2*zeta1*fn1.*w1).^2;
S1 = s0.*A1./B1;
```

Time series generation - Monte Carlo simulation

```
A1 = sqrt(2.*S1.*f01);

B1 =cos(w'*t1 + 2*pi.*repmat(rand(Nfreq1,1),[1,Nfreq1]));

x1 = A1*B1;
```

Envelop function E

```
b1 = -eps.*log(T90)./(1+eps.*(log(T90)-1));

c1 = b1./eps;

a1 = (exp(1)./eps).^b1;

E1 = a1.*(t1./tn1).^b1.*exp(-c1.*t1./tn1);
```

Envelop multiplied with stationary process to get y

```
y = x1.*E1;

end
```

3.HOR_IRF.m

```
function [f,U,A,B] = HOR_IRF(input)
```

```
if length(input.Vs)~=length(input.layer_thick)+1
   disp('There is a problem with the number of velocities Vs assigned to the various
layers')
   disp(' ')
   disp('Solution: Assign velocities for all soil layers and for the bedrock')
end
%frequency vector
f=input.freq;
%circular frequency vector
omega=2*pi*input.freq;
%imaginary 'i'
clear i; i=sqrt(-1);
%complex shear wave velocity
Vsstar=input.Vs.*(1+i*input.damp);
%thickness of the soil layers
h=input.layer_thick;
%number of soil layers + bedrock
layernum=length(input.Vs);
%complex impedance ratio on layer interfaces
az=zeros(layernum-1);
for i1=1:layernum-1
   az(i1)=input.rho(i1)*Vsstar(i1)/(input.rho(i1+1)*Vsstar(i1+1));
end
%Initialization of matrices
kstar=zeros(layernum,length(input.freq));
A=zeros(layernum,length(input.freq));
B=zeros(layernum,length(input.freq));
U=zeros(layernum,length(input.freq));
%Calculate transfer functions
for i1=1:layernum %Loop for the soil layers
   for i2=1:length(input.freq) %Loop for the frequencies
      kstar(i1,i2)=omega(i2)./Vsstar(i1); %complex wave number
kstar=omega/Vsstar
      if i1==1
         A(i1,i2) = 0.5*exp(i*kstar(i1,i2)*input.layer_thick(i1)) + 0.5*exp(-
i*kstar(i1,i2)*input.layer_thick(i1));
         B(i1,i2) = 0.5*exp(i*kstar(i1,i2)*input.layer_thick(i1)) + 0.5*exp(-
i*kstar(i1,i2)*input.layer_thick(i1));
         U(i1,i2) = A(i1,i2)+B(i1,i2);
      else
         A(i1,i2) = 0.5*A(i1-1,i2) * (1+az(i1-1)) * exp(i*kstar(i1-
1,i2)*input.layer_thick(i1-1)) + 0.5*B(i1-1,i2) * (1+az(i1-1)) * exp(-i*kstar(i1-
1,i2)*input.layer_thick(i1-1));
         B(i1,i2) = 0.5*A(i1-1,i2) * (1-az(i1-1)) * exp(i*kstar(i1-
1,i2)*input.layer_thick(i1-1)) + 0.5*B(i1-1,i2) * (1+az(i1-1)) * exp(-i*kstar(i1-
1,i2)*input.layer_thick(i1-1));
         U(i1,i2) = A(i1,i2)+B(i1,i2);
      end
   end
end
```

```
N=length(input.freq);
%Complex conjugates for "perfect" ifft
if round(rem(N,2))==1
   ia=2:1:(N+1)/2;
   ib=N:-1:(N+3)/2;
else
   ia=2:1:N/2;
   ib=N:-1:N/2+2;
end
A(:,ib)=conj(A(:,ia));
B(:,ib)=conj(B(:,ia));
U(:,ib)=conj(U(:,ia));
end
```

4. KTPSD.m

```
function[T90,eps,tn1,zeta1,sigma1,fn1] =
KTPSD(t1,y,guessEnvelop,guessKT,varargin)
whiteNoise = inputParser();

whiteNoise.CaseSensitive = false;

whiteNoise.addOptional('f3DB1',0.05);
whiteNoise.addOptional('tolX1',1e-8);
whiteNoise.addOptional('tolFun1',1e-8);
whiteNoise.addOptional('dataPlot','no');
whiteNoise.parse(varargin{:});
tolX1 = whiteNoise.Results.tolX1 ;
tolFun1 = whiteNoise.Results.tolFun1 ;
f3DB1 = whiteNoise.Results.f3DB1 ;
dataPlot = whiteNoise.Results.dataPlot ;
narginchk(4,8)
```

Get envelop parameters

```
dt1 = median(diff(t));

h11=fdesign.lowpass('N,F3dB1',8,f3DB1,1/dt1);
d11 = design(h11,'butter');
Y1 = filtfilt(d11.sosMatrix,d1.ScaleValues, abs(hilbert(y)));
Y1 = Y1./max(abs(Y1));
options=optimset('Display','off','TolX1',tolX1,'TolFun1',tolFun1);
coeff1 = lsqcurvefit(@(para,t) Envelop(para,t), guessEnvelop, t1, Y1,
[0.01,0.01,0.1], [3,3,100], options);
eps = coeff1(1);
T90 = coeff1(2);
tn1 = coeff1(3);
```

Get stationary perameters for the spectrum

```
E =Envelop(coeff1,t);

X1 = y1./E1;
x(1)=0;
[PSD,freq]=pwelch(x1,[],[],[],1/median(diff(t)));
coeff2 = lsqcurvefit(@(para,t) KT(para,freq), guessKT, freq, PSD, [0.01,0.01,1],
[5,5,100], options);
zeta1 = coeff2(1);
sigma1 = coeff2(2);
fn1 = coeff2(3);
```

dataPLot (optional)

```
if strcmpi(dataPlot,'yes')
spectra = KT(coeff2,freq);
figure
% subplot(211)
plot(t1,y1./max(abs(y1)),t1,Envelop(coeff1,t1),t1,Y1,'r')
legend('Typical Earthquake','Broadband Tuned','White Noise')
title([' T_{90} = ',num2str(coeff1(2),3),'; \epsilon = ',num2str(coeff1(1),3),'; t_{n} =
',num2str(coeff1(3),3)]);
xlabel('time (s)')
ylabel('h(t,x)');
axis tight ;
figure;
% subplot(212)
plot(freq,PSD,freq,spectra,'r')
legend('Measured','Fitted envelop')
legend('Typical Earthquake','Broadband Tuned')
title([' \zeta = ',num2str(coeff2(1),3),'; \sigma = ',num2str(coeff2(2),3),'; f_{n} =
',num2str(coeff2(3),3)]);
xlabel('frequency (Hz)')
ylabel('H(w,x)')
axis tight
xlim([0 10]);
set(gcf,'color','w')
end
function E = Envelop(para,t)
eps0 = para(1);
eta0 = para(2);
tn0 = para(3);
b = -eps0.*log(eta0)./(1+eps0.*(log(eta0)-1));
c = b./eps0;
a = (exp(1)./eps0).^b;
E = a.*(t./tn0).^b.*exp(-c.*t./tn0);
end
function S = KT(para,freq)
zeta01 = para(1);
```

```
sigma01 = para(2);
omega01 = 2*pi.*para(3);
w1 =2*pi*freq;
s0 = 2*zeta01*sigma01.^2./(pi.*omega01.*(4*zeta01.^2+1));
A1 = omega01.^4+(2*zeta01*omega01*w1).^2;
B1 = (omega01.^2-w1.^2).^2+(2*zeta01*omega01.*w1).^2;
S1 = s0.*A1./B1;
end
end
```

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