Project Report

On

Finite Difference Methods for Partial Differential Equations

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Himanshu Pandey (2K19/MSCMAT/09)

Under the Guidance

Of

Dr Aditya Kaushik

Delhi Technological University

(Formerly Delhi College of Engineering)

Certificate

This is to certify that the project report entitled " Finite Difference Methods for Partial Differential Equations" submitted by me to the Delhi Technological University (Formerly Delhi College of Engineering) for partial fulfillment of requirements for the degree of master of science in Mathematics, as well as the review work, were completed by him under my supervision and guidance. It complied with all of the requirements for the submission of his M.Sc. research project report. The contents of this project submitted by him are, in my view, deserving of consideration for an M.Sc. degree, and this thesis has not been submitted to any other institute or university for the award of any degree.

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Preface

We looked at how finite-difference methods (FDM) are used to solve differential equations by approximating derivatives with finite differences such as forward difference, backward difference, and central difference. Both the variable space and the time interval are broken down into a finite number of steps, and the solution's value is determined at these discrete points is approximated by solving algebraic equations containing finite differences and values from nearby points and also we discussed All the three types of equation Heat, Wave and Laplace equation using difference method and also we discuss its convergence stability.

The whole report is divided into these Chapters

- **Chapter 1: Introduction :** This chapters provide intro part of finite difference method in this chapter we will see how discretize the Partial Differential Equation into Finite differences using Forward , backward and Central difference method.

- **Chapter 2: Parabolic Equation:** In this Chapter we will see how a Heat Equation solved by Finite Difference Scheme and also we will discuss which method have higher stability and easy to solve.

- **Chapter 3: Elliptic Equation:** In this chapter we will see the solving technique of Laplace and Poisson Equation.

- **Chapter 4: Hyperbolic Equation:** In this chapter we are discussing two form of Finite difference Scheme to solving Wave Equation.

Abstract

Finite difference methods are well-known computational methods for solving differential equations that include approximating the derivatives using various difference schemes. During the last five decades, theoretical findings related to the precision, stability, and convergence of finite difference schemes (FDS) for differential equations have been discovered. In this paper, we look at various finite difference schemes for numerically solving partial differential equations in the context of heat transfer, wave equations, and the Laplace equation. The use of finite difference method (FDM) grids to solve partial difference equations (PDEs) is presented here. The solution of standard PDEs such as parabolic, hyperbolic, and elliptical form is the subject of this paper. The implicit, explicit, and Crank-Nicolson schemes, as well as the Richardson Method, Du-Fort Method, and Frankel Method, are all used to solve PDEs. Depending on the form of differential equations, equilibrium, and convergence, there are many different forms and methods of FDS. The values at discrete grid points are obtained from numerical solutions of differential equations using finite difference. To obtain the numerical approximations to the time-dependent differential equations needed for computer simulations, explicit and implicit approaches are used. The numerical example results of the explicit and implicit schemes for the heat equation under unique initial and boundary conditions are implemented in this chapter. It also presents a common elliptic partial differential equation to determine the temperature at the inside nodes.

Keywords: Finite Difference Methods, Bender-Schmidt Method, CN Method Matrix Stability, Implicit and Explicit Schemes.

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Chapter 1

Introduction:

Finite difference methods are well-known computational methods for solving differential equations that include approximating the derivatives using various difference schemes. During the last five decades, theoretical findings related to the precision, stability, and convergence of finite difference schemes (FDS) for differential equations have been discovered

PDEs are crucial in the research of a wide range of applied sciences and engineering discipline.

- Fluid dynamics,
- Heat transfer,
- Elasticity,
- Electromagnetic theory,
- Optics, plasma physics,
- Quantum mechanics, etc.

Our goal is to find a discrete approximation that satisfies a given relationship between several of its derivatives on some given region of space and time, as well as some boundary conditions along the edges of its domain, for a PDE such as (Heat, Laplace, and Wave equation). We rarely find an analytic formula for the solution to this problem because it is usually a very difficult problem. then there is a powerful tool Finite Difference Method which replaces the derivative form in the differential equation into finite difference approximation.

1.1 Classification of Second-Order Quasi-Linear PDEs:-

Many physical phenomena like heat flow in a metal rod, waves in the string can be well described by a second order PDE of type

$$R(x,y)\frac{\partial^2 u}{\partial x^2} + S(x,y)\frac{\partial^2 u}{\partial x \partial y} + T(x,y)\frac{\partial^2 u}{\partial y^2} + L\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad (1.1)$$

Where *R*, *S*, *T* are continuous functions of the variables *y*&*x* only, while *L* is a continuous function of $x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ the equation is linear in highest derivatives (second derivative), therefore it is quasi-linear. The PDE (1.1) is classified according to the value of $S^2 - 4RT$ as follows.

1. Elliptic if $S^2 - 4RT < 0$
2. Parabolic if $S^2 - 4RT = 0$
3. Hyperbolic if $S^2 - 4RT > 0$

It is analogous to general second-degree equation $ax^2 + bxy + cy^2 + fx + gy + h = 0$ in coordinate geometry, where it represents an ellipse, parabola and hyperbola if the value of $b^2 - 4ac$ is negative, zero and positive in that order.
The functions *R*, *S*, *T* are continuous functions of the variables y & *x*, hence $S^2 - 4RT$ is also a function of the variables y *and* x. Therefore, the domain of PDE (1.1) is important in describing the classification of PDE; consider the following

Example of a PDE

$$x\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + (x + 2y)\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial u}{\partial y} + u = 0$$

For this PDE we have

$$S^2 - 4RT = (2x)^2 - 4x(x + 2y) = -8xy$$

This PDE is parabolic on x and y axis, elliptic in 1st and 3rd quadrant and hyperbolic in 2nd and 4th quadrant.

Standard P.D.E.:

Here are the three types of Partial Differential equation

- Parabolic Equation: -

Consider the flow of heat in a rod (heat-conducting material) subject to some outer heat source along its boundary. The solution is then governed by the heat equation is given by

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$$

Where the constant $c = \frac{K}{\rho\omega}$ is known as the coefficient of heat conduction and which may be vary upon the medium.
K = Conductivity
ρ = density of the medium
ω = specific heat of the medium
This all equation and constant represents the flow of heat along x axis in a given medium the solution of which gives the temperature at u at a distance x from one end of a uniform rod after a time t.
It actually contains both initial as well as boundary condition. It contains the first order of the derivative of the time, so only one initial condition is required due to presence of second derivative of x variables.

Similarly, two-dimensional heat conduction equation is as follows

$$\frac{\partial u}{\partial t} = c\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c\nabla^2 u$$

Where $\nabla^2 = \nabla\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a two dimension Laplace operator.

- Hyperbolic Equation:-
  The solutions of hyperbolic equations are "wave-like" So the vibration of a tight string between two points are well described by the hyperbolic equation known as wave equation in one dimension

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$$

Here $c = \frac{T}{m}$
Where
T is tension in the string
m is the mass per unit length is a positive constant

All the above two constants depends upon the nature of the strings.
The strings are homogenous, i.e. uniform and elastic. The solution of this equation $u(x,t)$ is tells about the displacement of the string at any point x at any time t from initial position.

Similar expression for 2-dimensional wave equation as follows

$$\frac{\partial^2 u}{\partial t^2} = c\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c\nabla^2 u$$

- Elliptic Equation: -
  This type of equation describes a class of partial differential equations that explain phenomena that do not vary from one moment to the next, such as when heat or fluid flows through a medium with no accumulations. Elliptic partial differential equations have a wide range of applications in mathematics, ranging from harmonic analysis to geometry to lie theory, as well as a variety of other fields.. Two important cases of elliptic equations are Laplace and Poisson equation. These equations also represent heat and wave equation in steady state.

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace Equation)}$$

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad \text{(Poisson Equation)}$$

These equations also describe other physically important phenomena in different branches of science, like electromagnet theory etc. These equations are also known as potential equation as the variable potential and electromagnet potential in various relevant fields of science.

1.2 Initial and Boundary condition

We all know that ODE solutions aren't always special (integration constants appear in many places). PDEs, on the other hand, face the same problem. A collection of boundary or initial conditions is commonly used to characterize PDEs. And we already know that the state of a border is determined by the presence or absence of a boundary. Similarly, an initial condition is similar to a boundary condition, except that it is applied in the time direction. Not all boundary conditions allow for solutions, but usually the physics suggests what makes sense. Therefore, in case of one-dimensional heat conduction equation only one initial condition $[u(x,0) = f(x)]$ is required. It is quite obvious as initially (t = 0), the composition of system is well known. Similarly, two initial conditions $(u(x,0) = f(x)$ and $u_t(x,0) = g(x)]$ are given in case of wave equation as the time derivative is up to second order.

1.3 Finite difference for Partial Derivative

The finite difference method is simple & most powerful tool in a numerical method to solve differential equations with boundary conditions.

In this method, we use finite difference approximations in place of derivative terms involve in the differential equation and then convert into algebraic equation and then we solve this algebraic equation which gives the solution of a given differential equation. This system of linear equations can be solved by any direct or iterative procedure or commonly we use Gauss Seidal method. Most important feature of this method is we can solve it by computer programming very easily. Here, we will discuss the finite difference approximations of partial derivatives using Taylor series

$$u(x+h,y) = u(x,y) + h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

$$u(x-h,y) = u(x,y) - h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

$$u(x+2h,y) = u(x,y) + 2h\frac{\partial u}{\partial x} + \frac{(2h)^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{(2h)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

$$u(x-2h,y) = u(x,y) - 2h\frac{\partial u}{\partial x} + \frac{(2h)^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{(2h)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots$$

$$u(x,y+k) = u(x,y) + k\frac{\partial u}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 u}{\partial y^2} + \frac{k^3}{3!}\frac{\partial^3 u}{\partial y^3} + \cdots$$

$$u(x,y-k) = u(x,y) - k\frac{\partial u}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 u}{\partial y^2} - \frac{k^3}{3!}\frac{\partial^3 u}{\partial y^3} + \cdots$$

$$u(x,y+2k) = u(x,y) + 2k\frac{\partial u}{\partial y} + \frac{(2k)^2}{2!}\frac{\partial^2 u}{\partial y^2} + \frac{(2k)^3}{3!}\frac{\partial^3 u}{\partial y^3} + \cdots$$

$$u(x,y-2k) = u(x,y) - 2k\frac{\partial u}{\partial y} + \frac{(2k)^2}{2!}\frac{\partial^2 u}{\partial y^2} - \frac{(2k)^3}{3!}\frac{\partial^3 u}{\partial y^3} + \cdots$$

$$u(x+h,y+k) = u(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)u + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 u + \cdots$$

We can obtain all these following approximation for the first and second order partial derivatives by using Taylor expansion.

First Order Partial Derivatives:

- Forward Differences

$$\frac{\partial u(x,y)}{\partial x} = \frac{u(x+h,y) - u(x,y)}{h} + O(h)$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{u(x,y+k) - u(x,y)}{k} + O(k)$$

- Backward Difference

$$\frac{\partial u(x,y)}{\partial x} = \frac{u(x,y) - u(x-h,y)}{h} + O(h)$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{u(x,y) - u(x,y-k)}{k} + O(k)$$

- Central Difference

$$\frac{\partial u(x,y)}{\partial x} = \frac{u(x+h,y) - u(x-h,y)}{2h} + O(h^2)$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{u(x,y+k) - u(x,y-k)}{2k} + O(k^2)$$

Second Order Partial Derivative:

- Forward Difference:

$$\frac{\partial^2 u(x,y)}{\partial x^2} = \frac{u(x+2h,y) - 2u(x+h,y) + u(x,y)}{h^2} + O(h)$$

$$\frac{\partial^2 u(x,y)}{\partial y^2} = \frac{u(x,y+2k) - 2u(x,y+k) + u(x,y)}{k^2} + O(k)$$

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{u(x+h,y+k) - u(x,y+k) - u(x+h,y) + u(x,y)}{h^2} + O(h,k)$$

- Backward Difference:

$$\frac{\partial^2 u(x,y)}{\partial x^2} = \frac{u(x,y) - 2u(x-h,y) + u(x-2h,y)}{h^2} + O(h)$$

$$\frac{\partial^2 u(x,y)}{\partial y^2} = \frac{u(x,y) - 2u(x,y-k) + u(x,y-2k)}{k^2} + O(k)$$

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{u(x,y) - u(x,y-k) - u(x-h,y) + u(x-h,y-h)}{h^2} + O(h,k)$$

- Central Difference:-

$$\frac{\partial^2 u(x,y)}{\partial x^2} = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + O(h^2)$$

$$\frac{\partial^2 u(x,y)}{\partial y^2} = \frac{u(x,y+k) - 2u(x,y) + u(x,y-k)}{k^2} + O(k)$$

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{u(x+h,y+k) - u(x+h,y-k) - u(x-h,y+k) + u(x-h,y-h)}{h^2} + O(hk)$$

To replace various derivative terms by their finite difference approximation, we discretize the derivative terms at any point(x_i, y_j). Let the variables x and y be equidistant with spacing h and k respectively. Let us assume that the node points are $u(x_i, y_j) = u_{i,j}$. So we can obtain following approximation for first and second order derivative terms at a point (x_i, y_j).by using the above expression.

First Order Partial Derivative:

- Forward Differences:

$$\frac{\partial u(x_i, y_j)}{\partial x} = \frac{u(x_i + h, y_j) - u(x_i, y_j)}{h} + O(h)$$

$$= \frac{u(x_{i+1}, y_j) - u(x_i, y_j)}{h} + O(h)$$

$$= \frac{u_{i+1,j} - u_{i,j}}{h} + O(h)$$

$$\frac{\partial u(x_i, y_j)}{\partial y} = \frac{u(x_i, y_j + k) - u(x_i, y_j)}{k} + O(k)$$

$$= \frac{u(x_i, y_{j+1}) - u(x_i, y_j)}{k} + O(k)$$

$$= \frac{u_{i,j+1} - u_{i,j}}{k} + O(k)$$

- Backward Differences:

$$\frac{\partial u(x_i, y_j)}{\partial x} = \frac{u(x_i, y_j) - u(x_i - h, y_j)}{h} + O(h)$$

$$= \frac{u(x_i, y_j) - u(x_{i-1}, y_j)}{h} + O(h)$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} + O(h)$$

$$\frac{\partial u(x_i, y_j)}{\partial y} = \frac{u(x_i, y_j) - u(x_i, y_j - k)}{k} + O(k)$$

$$= \frac{u(x_i, y_j) - u(x_i, y_{j-1})}{k} + O(k)$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k)$$

- Central Differences:

$$\frac{\partial u(x_i, y_j)}{\partial x} = \frac{u(x_i + h, y_j) - u(x_i, -hy_j)}{2h} + O(h^2)$$

$$= \frac{u(x_{i+1}, y_j) - u(x_{i-1}, y_j)}{2h} + O(h^2)$$

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2)$$

$$\frac{\partial u(x_i, y_j)}{\partial y} = \frac{u(x_i, y_j + k) - u(x_i, y_j - k)}{2k} + O(k^2)$$

$$= \frac{u(x_i, y_{j+1}) - u(x_i, y_{j-1})}{2k} + O(k^2)$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2)$$

Second Order Derivative:

- Forward Difference:

$$\frac{\partial^2 u(x_i, y_j)}{\partial x^2} = \frac{u(x_i + 2h, y_j) - 2u(x_i + h, y_j) + u(x_i, y_j)}{h^2} + O(h)$$

$$= \frac{u(x_{i+2}, y_j) - 2u(x_{i+1}, y_j) + u(x_i, y_j)}{h^2} + O(h)$$

$$= \frac{u_{i+2,j} - 2u_{i+1,j} + u_{i,j}}{h^2} + O(h)$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial y^2} = \frac{u(x_i, y_j + 2k) - 2u(x_i, y_j + k) + u(x_i, y_j)}{k^2} + O(k)$$

$$= \frac{u(x_i, y_{j+2}) - 2u(x_i, y_{j+1}) + u(x_i, y_j)}{k^2} + O(k)$$

$$= \frac{u_{i,j+2} - 2u_{i,j+1} + u_{i,j}}{k^2} + O(k)$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial x \partial y}$$
$$= \frac{u(x_i + h, y_j + k) - u(x_i + h, y_j) + u(x_i, y_j + k) + u(x_i, y_j)}{hk} + O(h, k)$$

$$= \frac{u(x_{i+1}, y_{j+1}) - u(x_{i+1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_j)}{hk} + O(h, k)$$

$$= \frac{u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1} + u_{i,j}}{hk} + O(h, k)$$

- Backward Differences:

$$\frac{\partial^2 u(x_i, y_j)}{\partial x^2} = \frac{u(x_i, y_j) - 2u(x_i - h, y_j) + u(x_i - 2h, y_j)}{h^2} + O(h)$$

$$= \frac{u(x_i, y_j) - 2u(x_{i-1}, y_j) + u(x_{i-2}, y_j)}{h^2} + O(h)$$

$$= \frac{u_{i,j} - 2u_{i-1,j} + u_{i-2,j}}{h^2} + O(h)$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial y^2} = \frac{u(x_i, y_j) - 2u(x_i, y_j - k) + u(x_i, y_j - 2k)}{k^2} + O(k)$$

$$= \frac{u(x_i, y_j) - 2u(x_i, y_{j-1}) + u(x_i, y_{j-2})}{k^2} + O(k)$$

$$= \frac{u_{i,j} - 2u_{i,j-1} + u_{i,j-2}}{k^2} + O(k)$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial x \partial y}$$

$$= \frac{u(x_i, y_j) - u(x_i - h, y_j) - u(x_i, y_j - k) + u(x_i - h, y_j - k)}{hk} + O(h, k)$$

$$= \frac{u(x_i, y_j) - u(x_{i-1}, y_j) - u(x_i, y_{j-1}) \mp u(x_{i-1}, y_{j-1})}{hk} + O(h, k)$$

$$= \frac{u_{i,j} - u_{i-1,j} - u_{i,j-1} + u_{i-1,j-1}}{hk} + O(h, k)$$

- Central Difference:

$$\frac{\partial^2 u(x_i, y_j)}{\partial x^2} = \frac{u(x_i + h, y_j) - 2u(x_i, y_j) + u(x_i - h, y_j)}{h^2} + O(h^2)$$

$$= \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + O(h^2)$$

$$= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial y^2} = \frac{u(x_i, y_j + k) - 2u(x_i, y_j) + u(x_i, y_j - k)}{k^2} + O(k^2)$$

$$= \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} + O(k^2)$$

$$= \frac{u_{i,j+1} - 2u_{i,j} + u_{i,-1j}}{k^2} + O(k^2)$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial x \partial y}$$
$$= \frac{u(x_i + h, y_j + k) - u(x_i - h, y_j + k) + u(x_i + h, y_j - k) + u(x_i - h, y_j - k)}{4hk}$$
$$+ O(hk)$$

$$= \frac{u(x_{i+1}, y_{j+1}) - u(x_{i-1}, y_{j+1}) - u(x_{i+1}, y_{j-1}) + u(x_{i-1}, y_{j-1})}{4hk} + O(hk)$$

$$= \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4hk} + O(hk)$$

Now, we are going to solve the Heat Conduction, Laplace, Poisson and Wave equation (with initial and boundary condition) representing the important physical system using this discretization method.

Note: The Central difference operator are far better than the forward and backward because we have to neglect terms of order h^2 in central difference where as in forward and backward we neglect order of h

Chapter 2

2.1 Parabolic Equation (Heat Equation)

The heat conduction equation in 1-dimension is a parabolic equation of following form

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

The following five finite difference schemes for 1-dimension heat conduction (2.1) will be discussed.

a) Bender-Schmidt Explicit Scheme
b) Crank-Nicolson (CN) Method
c) General Implicit Scheme
d) Richardson Method
e) Du-Fort and Frankel Method

2.1.1 Bender-Schmidt Explicit Scheme

In this method let us assume that the value of $u(x,t)$ at point (x_i, t_j) is $u_{i,j}$ i.e. $u(x_i, t_j) = u_{i,j}$. Also assume that grid size for the variable t is $\Delta t = k$ and for the variable x is $\Delta x = h$

On Discretizing Eq. (1.2) at point (x_i, t_j) we get

$$\frac{\partial u}{\partial t}\Big|_{(x_i,t_j)} = c\frac{\partial^2 u}{\partial x^2}\Big|_{(x_i,t_j)} \tag{2.2}$$

Here we use the forward difference formula of first order derivative term (for the time variable) and central difference formula of second order derivative term for (space variable) are given by

$$\frac{\partial u(x_i, t_j)}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k)$$

$$\frac{\partial^2 u(x_i, t_j)}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

Using these formulas in Heat Equation and neglecting error terms$O(h)\&O(k)$, we have

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c\, \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

Or

$$u_{i,j+1} - u_{i,j} = \frac{ck}{h^2}(\, u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

Let $r = \frac{ck}{h^2}$ then the Bender-Schmidt explicit Scheme for the solution of Eq. (2.2) is as follows

$$u_{i,j+1} = ru_{i-1,j} + (1 - 2r)u_{i,j} + r\, u_{i+1,j} \tag{2.3}$$

This (1.3) scheme is known as explicit scheme.

Here forward difference formula is used to approximate the time variable term $\frac{\partial u}{\partial t}$. *So, the error term in time derivative approximation is of linear order* $O(k)$ *and in space variable the error term is* $O(h^2)$. *So the total error of this method will be* $O(k) + O(h^2)$*Next method we discuss more accurate and unconditionally stable scheme i.e. Crank Nicolson Method in which both the derivative terms replace by central difference formula*

2.1.2 Crank-Nicolson (CN) Scheme

Let us Discretizing Eq. (2.2) at point $(x_i, t_{j+1/2})$ we get

$$\frac{\partial u}{\partial t}\Big|_{(x_i, t_{j+1/2})} = c\, \frac{\partial^2 u}{\partial x^2}\Big|_{(x_i, t_{j+1/2})} \tag{2.4}$$

Assume the node points in time variable t with step size $k/2$ are $(t_j - k/2, t_j, t_j + k/2$ or $t_{j-1/2}, t_j, t_{j+1/2})$. Using central difference approximation for $\frac{\partial u}{\partial t}$ with these points, we have

$$\frac{\partial u(x_i,t_{j+1/2})}{\partial t} = \frac{u_{i,j+1}-u_{i,j}}{2\left(\frac{k}{2}\right)} = \frac{u_{i,j+1}-u_{i,j}}{k} \tag{2.5a}$$

As the value at the point $(x_i, t_{j+1/2})$are not known therefore the central difference formula $\frac{\partial^2 u(x_i,t_{j+1/2})}{\partial x^2} = \frac{u_{i+1,j+1/2}-2u_{i,j+1/2}+u_{i-1,j+1/2}}{h^2}$ cannot be used for the derivative term $\frac{\partial^2 u}{\partial t^2}|_{(x_i,t_{j+1/2})}$ in equation (1.4) So we will approximate the term $\frac{\partial^2 u}{\partial x^2}|_{(x_i,t_{j+1/2})}$ with average values of $\frac{\partial^2 u}{\partial x^2}$ at (x_i, t_j)& (x_i, t_{j+1}) i.e.

$$\frac{\partial^2 u(x_i, t_{j+1/2})}{\partial x^2} = \frac{1}{2}\left(\frac{\partial^2 u(x_i, t_j)}{\partial x^2} + \frac{\partial^2 u(x_i, t_{j+1})}{\partial x^2}\right)$$

$$= \frac{1}{2}\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}\right) \tag{2.5b}$$

Using Eq. (2.5a) & (2.5b) in eq. (1.4), we get

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c\ \frac{1}{2}\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right)$$

Let $r = \frac{ck}{h^2}$, then the Crank-Nicolson (CN) implicit scheme for the solution of Eq. 1.2 is as follows

$$-ru_{i-1,j+1} + 2(1+r)u_{i,j+1} - ru_{i-1,j+1} = ru_{i-1,j} + 2(1-r)u_{i,j} + ru_{i+1,j} \tag{2.6}$$

It is easy to see that this scheme is implicit scheme, as we can't obtain the solution $u_{i,j+1}$ directly from the scheme. First a set of equation is obtained, and then we solve this set of values of $u_{i,j+1}$ in a row.

2.1.3 General Explicit Scheme

In CN-scheme, equal weightage is given to the both j and j+1 levels, consider Eq. (1.5b)

$$\frac{\partial^2 u(x_i, t_{j+1/2})}{\partial x^2} = \frac{1}{2}\left(\frac{\partial^2 u(x_i, t_j)}{\partial x^2} + \frac{\partial^2 u(x_i, t_{j+1})}{\partial x^2}\right)$$

Let us construct an implicit scheme by giving different weight to different levels, say θ and $1-\theta$ to levels j & $j+1$ respectively then

$$\frac{\partial^2 u(x_i, t_{j+1/2})}{\partial x^2} = \theta \frac{\partial^2 u(x_i, t_j)}{\partial x^2} + (1-\theta)\frac{\partial^2 u(x_i, t_{j+1})}{\partial x^2}$$

It is worth to note down that CN method can be obtained for $\theta = 1/2$. Using Central difference formulas, we have

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c\left(\theta \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + (1-\theta)\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right)$$

Assume $r = \frac{ck}{h^2}$ the general implicit scheme for the solution of Eq. (2.2) is as follows

$$u_{i,j+1} - u_{i,j} = r(\theta(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + (1-\theta)(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1})$$

Or

$$-r(1-\theta)u_{i-1,j+1} + \big(1+2r(1-\theta)\big)u_{i,j+1} - r(1-\theta)u_{i-1,j+1} = r\theta u_{i-1,j} + (1-2r\theta)u_{i,j} + r\theta u_{i+1,j} \tag{2.7}$$

This scheme (1.7) is known as general implicit scheme.

So far, we have learned the Bender-Schmidt explicit scheme, Crank Nicolson and general implicit schemes for the solutions of 1-dimensional heat conduction Eq. (1.2) it is worth to mentioning here that these schemes are two-level-schemes as these involve only two levels(j & $j+1$) of time variable.

Note: Now we have to discuss three level schemes for the solution of Eq. (2.2) known as Richardson scheme and Du-Fort & Frankel Scheme.

2.1.4 Richardson Scheme

Consider Eq. (1.2) at point (x_i, t_j) we get

$$\frac{\partial u}{\partial t}|_{(x_i,t_j)} = c\frac{\partial^2 u}{\partial x^2}|_{(x_i,t_j)}$$

Using central difference formulas for time and space variable

$$\frac{\partial u(x_i,t_j)}{\partial t} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2)$$

$$\frac{\partial^2 u\big(x_i,t_j\big)}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

After putting in the above equation

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = c\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

Or

$$u_{i,j+1} = u_{i,j-1} + 2\frac{ck}{h^2}\big(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\big)$$

$$u_{i,j+1} = u_{i,j-1} + 2r\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \tag{2.8}$$

This expression is known as Richardson scheme.

Note that this scheme in unstable for all values of $r = \frac{ck}{h^2}$. *So we will construct an unconditionally stable Du-Fort and Frankel scheme, which is stable for all values of* r.

2.1.5 Du-Fort and Frankel Scheme

Richardson scheme (2.8) for the Eq. (2.2) is as follows

$$u_{i,j+1} = u_{i,j-1} + 2r\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

Replace the term $u_{i,j} = \frac{1}{2}\left(u_{i,j-1} + u_{i,j+1}\right)$ with average value at $j-1$ & $j+1$ levels as follows

$$u_{i,j+1} = u_{i,j-1} + 2r\left(u_{i+1,j} - 2(u_{i,j-1} + u_{i,j+1}) + u_{i-1,j}\right)$$

Rearranging the terms, Du-Fort and Frankel scheme is given by

$$u_{i,j+1} = \frac{1-2r}{1+2r}u_{i,j-1} + \frac{2r}{1+2r}\left(u_{i+1,j} + u_{i-1,j}\right)$$

So far we have discussed five finite difference schemes for the solution of 1-dimensional heat conduction Eq. (2.2)

We will use all these schemes to compute the values of $u_{i,j} = u(x_i, t_j)$ at various node points. We will construct the following table to collect all the values at one place.

i \ j	0	1	2	....	n
0	$u_{0,0}$	$u_{1,0}$	$u_{2,0}$	$\cdots$	$u_{n,0}$
1	$u_{0,1}$	$u_{1,1}$	$u_{1,2}$	$\cdots$	$u_{n,1}$
2	$u_{0,2}$	$u_{1,2}$	$u_{2,2}$	$\cdots$	$u_{n,2}$
.	.	.	.	...	.
.	.	.	.	$\cdots$	.

Examples (2.1)

Solve the 1-dimensional heat conduction equation

$$2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1 \; with$$

Initial condition $u(x, 0) = x(4 - x)$ *and*

Boundary condition $u(0, t) = 0$ *and* $u(4, t) = 0$

Use Explicit Scheme to find the value $u(x, t)$ up to $t = 5$ second with $\Delta x = 1$ & $\Delta t = 1$

Since $\Delta x = 1$ for $0 \leq x \leq 1$ hence our node points are as follows

$$x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4,$$

Let$u_{i,j} = u(x_i, t_j)$. The initial condition $u(x, 0) = x(4 - x)$ at $t_0 = 0$ provides

$u_{0,0} = u(x_0, t_0) = u(0,0) = 0(4 - 0) = 0$

$u_{1,0} = u(x_1, t_0) = u(1,0) = 1(4 - 1) = 3$

$u_{2,0} = u(x_2, t_0) = u(2,0) = 2(4 - 2) = 4$

$u_{3,0} = u(x_3, t_0) = u(3,0) = 3(4 - 3) = 3$

$u_{4,0} = u(x_4, t_0) = u(4,0) = 4(4 - 4) = 0$ (2.10)

$i(x)$ \ $j(x)$	0	1	2	3	4
0	0	3	4	3	0
1	0				0
2	0				0
3	0				0
4	0				0
5					0

After utilizing the initial and boundary conditions, we will now use the explicit method to find the value of $u(x, t)$ at node points. With $\Delta x = 1$ & $\Delta t = 1$ and $c = 1/2$, our r is given by

$$r = \frac{c\Delta t}{\Delta x^2} = \frac{ck}{h^2} = \frac{\frac{1}{2}(1)}{(1)^2} = \frac{1}{2}$$

Bender-Schmidt Scheme (1.3) is given by

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + r\, u_{i+1,j}$$

For $r = 0.5$, we have

$$u_{i,j+1} = 0.5u_{i-1,j} + (0)u_{i,j} + 0.5\, u_{i+1,j}$$

$$u_{i,j+1} = 1/2\big(u_{i-1,j} + +0.5\, u_{i+1,j}\big)$$

$i(x)$ / $j(x)$	0	1	2	3	4
0(0)	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

Examples (2.2)

Use BSD Explicit scheme to calculate temperature distribution in uniform insulated rod of length 1m with diffusivity constant of the material of the rod is given 1. Both ends of the rod are kept at zero temperature, and initial temperature distribution in the rod is given by the function $u(x,0) = Sin(\pi x)$. Take $\Delta x = 1/4$ and $\Delta t = 1/16$ solve up to $t = 1/8$.

Sol.

The diffusivity constant of the material of the rod is 1, i.e. $c = 1$. So the temperature distribution is given by following heat conduction equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1$$

Both the ends of the rod are kept at zero temperature, therefore boundary condition are given by

$$u(0,t) = u(1,t) = 0$$

Also, initial temperature distribution gives following initial condition

$$u(x,0) = Sin(\pi x)$$

The mathematical model distribution for the given physical problem is complete

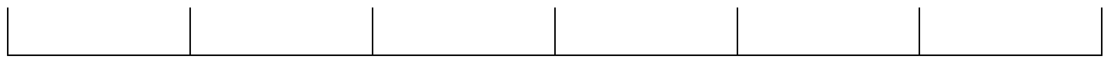
Now we will compute the temperature distribution at various nodes using explicit scheme

With$\Delta x = 1/4, t = 0$, the initial condition $u(x,0) = Sin(\pi x)$ gives

$$u_{1,0} = u(x_1, t_0) = u(1/4, 0\,) = Sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = 0.707107$$

$$u_{2,0} = u(x_2, t_0) = u(1/4, 0\,) = Sin\left(\frac{\pi}{2}\right) = 1$$

$$u_{3,0} = u(x_3, t_0) = u(3/4, 0\,) = Sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} = 0.707107 \qquad (2.12)$$



$x = 0$ $\quad x = \frac{1}{4}$ $\quad x = \frac{1}{2}$ $\quad x = \frac{3}{4}$ $\quad x = 1$

$u = 0$ $\quad u = 0.707107$ $\quad u = 1$ $\quad u = 0.707107$ $\quad u = 0$

The boundary condition $u(0,t) = u(1,t) = 0$ provide

$$u_{0,j} = u\left(x_0, t_j\right) = u(0, t\ ) = 0$$

$$u_{4,j} = u\left(x_4, t_j\right) = u(1, t\ ) = 0 \qquad for\ \forall j = 0,1,2,3, \ldots. \qquad (2.13)$$

In the table form (2.12) and (2.13) are as follows

$i(x)$ \ $j(t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0				0
2(1/8)	0				0

With $\Delta x = 1/4$, $\Delta t = 1/16$ and $c = 1$, the value of constant r is given by

$$r = \frac{c\Delta t}{\Delta x^2} = \frac{ck}{h^2} = \frac{1(1/16)}{(1/4)^2} = 1$$

Explicit scheme (2.3) for $r = 1$ is as follows

$$u_{i,j+1} = u_{i-1,j} - u_{i,j} + u_{i+1,j}$$

Using $j = 0$ in the above formula, we have

$$u_{i,1} = u_{i-1,0} - u_{i,0} + u_{i+1,0}$$

Computing the different values of $u_{i,1}$ for $i = 1,2, and\ 3$ we get second row of the following table

$j(t)$ \ $i(x)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.292803	0.414214	0.292803	0
2(1/8)	0				0

Using $j = 1$ in the above formula, we have

$$u_{i,2} = u_{i-1,1} - u_{i,1} + u_{i+1,1}$$

Computing the different values of $u_{i,1}$ for $i = 1,2, and\ 3$ these values are given in third row of the following table.

$j(t)$ \ $i(x)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.292803	0.414214	0.292803	0
2(1/8)	0	0.121411	0.171392	0.121411	0

Examples (2.3)

Using Crank-Nicolson to figure out temperature allocation in uniform insulated rod of length 1m with diffusivity constant of the material on the rod is given 1. Both ends of the rod are kept at zero temperature, and initial temperature distribution in the rod is given by the function $u(x,0) = Sin(\pi x)$. Take $\Delta x = 1/4$ and $\Delta t = 1/16$ solve up to $t = 1/8$.

Sol.

Crank-Nicolson formula from (1.6) is as follows

$$-ru_{i-1,j+1} + 2(1+r)u_{i,j+1} - ru_{i-1,j+1} = ru_{i-1,j} + 2(1-r)u_{i,j} + ru_{i+1,j}$$

With $r = 1, j = 0$ the CN formula gives following equation for $i = 1,2,3$ ....

$$-u_{2,1} - u_{0,1} + 4u_{1,1} = u_{2,0} + u_{0,0}$$

$$-u_{3,1} - u_{1,1} + 4u_{2,1} = u_{3,1} + u_{1,0}$$

$$-u_{4,1} - u_{2,1} + 4u_{3,1} = u_{4,0} + u_{2,0} \quad (2.14)$$

Using the value from eq. (2.12) & (2.13) in the system (2.14) we get

$$-u_{2,1} + 4u_{1,1} = 1 + 0 = 1$$

$$-u_{3,1} - u_{1,1} + 4u_{2,1} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$-u_{2,1} + 4u_{3,1} = 0 + 1 = 1 \quad (2.15)$$

On solving the system (1.15) we have

$$u_{1,1} = u_{3,1} = 0.386729 \; u_{2,1} = 0.546916 \quad (2.16)$$

$i(x)$ \ $j(t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.386729	0.546916	0.386729	0
2(1/8)	0				0

values of $u_{i,j}$ for $t = \frac{1}{8}$ (or $j = 2$)

with $r = 1$ & $j = 1, i = 1,2,3$, the CN formula (1.6) provides the following linear system

$$-u_{2,2} - u_{0,2} + 4u_{1,2} = u_{2,1} + u_{0,1}$$

$$-u_{3,2} - u_{1,2} + 4u_{2,2} = u_{3,2} + u_{1,1}$$

$$-u_{4,2} - u_{2,2} + 4u_{3,2} = u_{4,1} + u_{2,1} \quad (2.17)$$

Using the known values in the above system, and solving the resulting linear system $u_{1,2} = u_{3,2} = 0.211509$ $u_{2,1} = 0.29912$

$i(x)$ \ $j(t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.386729	0.546916	0.386729	0
2(1/8)	0	0.211509	0.29912	0.211509	0

Examples (2.4)

Use Richardson scheme and Du-Fort & Frankel to figure out temperature distribution in uniform insulated rod of length 1m with "c" constant of the material of the rod is given 1. Both ends of the rod are kept at zero temperature, and initial temperature distribution in the rod is given by the function $u(x,0) = Sin(\pi x)$. Take $\Delta x = 1/4$ and $\Delta t = 1/16$ solve up to $t = 1/8$.

Ans.

The Richardson Scheme (1.8)

$$u_{i,j+1} = u_{i,j-1} + 2r\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

is a three level scheme. To start with Richardson scheme, we must have the values

for at least $j = 0$ & $j = 1$. Let us use for $t = 1/16$ or $j = 1$ from CN- Scheme of Example 2.3

$j(t)$ \ $i(x)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.386729	0.546916	0.386729	0
2(1/8)	0				0

The values of $u_{i,j}$ for $t = 1/8 (or\ j = 2)$ with the aid of Richardson scheme (2.8) for $i = 1,2,3$ are given by

$i(x)$ \ $j(t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.386729	0.546916	0.386729	0
2(1/8)	0	0.254025	0.359246	0.254025	0

Similarly, we can compute values for $j = 3,4$

$i(x)$ \ $j(t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.386729	0.546916	0.386729	0
2(1/8)	0	0.254025	0.359246	0.254025	0
3(3/16)	0	0.149613	0.211586	0.149613	0
4(1/4)	0	0.89121	0.126036	0.089121	0

Du-Fort and Frankal Scheme

This scheme is also there level scheme, again using value of $t = 1/16$ (or $j = 1$) from CN-Scheme

$i(x)$ \ $j(t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.386729	0.546916	0.386729	0
2(1/8)	0				0

On applying Du-Fort and Frankel Scheme

This scheme is also three level schemes, to calculate the values $u_{i,j+1}$ for $t = 1/8$ (or $j = 2$) we have

$$u_{i,j+1} = \frac{1-2r}{1+2r}u_{i,j-1} + \frac{2r}{1+2r}\left(u_{i+1,j} + u_{i-1,j}\right)$$

$$u_{i,2} = \frac{1-2r}{1+2r}u_{i,0} + \frac{2r}{1+2r}\left(u_{i+1,1} + u_{i-1,1}\right)$$

The values for $i = 1,2,3$ are given by third row of the following table

$i(x)$ \ $j(t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.386729	0.546916	0.386729	0
2(1/8)	0	0.128910	0.182306	0.128910	0

Similarly for $j = 3$ and 4 are as follows

$i(x)$ \ $j(t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.386729	0.546916	0.386729	0
2(1/8)	0	0.128910	0.182306	0.128910	0
3(3/16)	0	-0.00737	-0.01042	-0.00737	0
4(1/4)	0	-0.04992	-0.07059	-0.04992	0

2.2 Consistency, Convergence and Stability of Explicit and Crank Nicolson Method

In this segment, we discussed the three important consequences of FDM

1. Consistency
2. Convergence and Order
3. Stability

Let any linear PDE of the following form

$$Lu = 0 \tag{2.18}$$

For Example Heat Equation

$$\frac{\partial u}{\partial t} - c\frac{\partial^2 u}{\partial x^2} = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial^2}{\partial x^2}\right)u = Lu = 0$$

Where linear operator $L \equiv \frac{\partial}{\partial t} - c\frac{\partial^2}{\partial x^2}$

In any FDS, we use finite difference discretization instead of derivative terms of the differential. The discretized equation

$$L_D u = 0 \tag{2.19}$$

is finite difference equation including parameters Δt & Δx. During calculation of solution from the Finite difference equation, various other like rounding error, also occur. Let u be exact solution of Eq. (2.18) and u_D be exact solution of the discretized Eq. (2.19). Again consider that u_C is the final solution obtained from discretized Eq. (2.19). The solution u_c contains following type of errors

a) Truncation Error:
In any FDS, we estimated partial derivatives of given PDE with finite difference approximations (like forward, central differences, etc.). These finite differences are Taylor series approximations of partial derivatives; therefore these finite differences have truncation error. So, the finite difference scheme has truncation error which is also known as **discretization error**. The difference $u - u_D$ is called as "discretization error."
A finite difference scheme is consistent with a PDE if the discretization error vanishes as the grid spacing goes to zero independently.

b) Stability Error:
Once a FDS is created, and then it will be using to obtain the solution of given linear PDE. A finite difference scheme is stable if it gives the bounded solution for a stable PDE. The difference $u_c - u_D$ is called as "stability error."

Since u is the exact solution of differential Eq. (2.18) and u_c is the final solution obtained from the scheme (2.19), so the total error is given by

$$\text{Total error: } u - u_c = \underbrace{(u - u_D)}_{discretization\ err.} + \underbrace{(u_D - u_c)}_{Stability\ err.}$$

The total error is the sum of discretization error and stability error.

2.2.1 Consistency

Here we will discuss our method is consistent with a PDE if the variation $u - u_D$ between the solutions of both the equations (i.e. Truncation error) vanishes as the size of the grid spacing goes to zero independently. If T is truncation error then we can write Eq. (2.18) as follows

$$Lu = L_D(u) + \text{T}$$

If the finite difference approximation $L_D(u)$ is tending towards Lu (truncation error vanishes) this will be happened when the grid spacing between independent variable are going to zero; then the finite difference scheme is supposed to be consistent. In any finite difference scheme, we using finite difference approximations in place of partial derivatives of given partial differential equation. It is obvious that finite differences Taylor series approximations of partial derivatives. Hence, these finite differences have truncation error. For example, consider following Taylor series expansion

$$u(x, t + \Delta t) = u(x, t) + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 u}{\partial t^2} + \cdots$$

On rearranging the terms we get,

$$\frac{\partial u}{\partial t} = \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} - \frac{\Delta t^2}{2!} \frac{\partial^2 u}{\partial t^2} + \cdots$$

On neglecting Δt and its higher powers, we get following forward difference approximation of the partial derivative $\frac{\partial u}{\partial t}$

$$\frac{\partial u}{\partial t} = \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} + O(\Delta t)$$

A finite difference method is well-suited with a PDE if the truncation error vanishes if the grid spacing between different independent variables tends to zero.

For example, consider Bender–Schmidt explicit scheme (2.3) for 1-dimensional heat conduction equation $\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}$$

The truncation error (T) of finite difference approximation tends to zero with $\Delta t, \Delta x \to 0$. Therefore, the scheme is consistent with Heat conduction equation in one space variable.

2.2.2 Consistency of Explicit Scheme

Let us assume the grid spacing for variables $t\ \&\ x$ are $k = \Delta t\ \&\ h = \Delta x$ respectively Consider the following Taylor series expansion

$$u(x, t+k) = u(x,t) + k\frac{\partial u}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 u}{\partial t^2} + \cdots (2.20)$$

$$u(x+h, t) = u(x,t) + h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots (2.21)$$

$$u(x-h, t) = u(x,t) - h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots (2.22)$$

Using Eq. (1.20) we have

$$\frac{u(x,t+k) - u(x,t)}{k} = \frac{\partial u}{\partial t} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3} + \cdots (2.23)$$

Addition of Eq. (1.21) and (1.22) gives the following expression for $\frac{\partial^2 u}{\partial x^2}$

$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} = \frac{\partial^2 u}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial t^4} + \frac{h^4}{360}\frac{\partial^6 u}{\partial t^6} + \cdots (2.24)$$

From Eq. (2.23) and (2.24) we obtain

$$\frac{u(x,t+k) - u(x,t)}{k} - c\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + T = \frac{\partial u}{\partial t} - c\frac{\partial^2 u}{\partial x^2}$$

(2.25)

Where T is Truncation error. It is given by

$$T = c\left(\frac{h^2}{12}\frac{\partial^4 u}{\partial t^4} + \frac{h^4}{360}\frac{\partial^6 u}{\partial t^6} + \cdots\right) - \left(\frac{k}{2}\frac{\partial^2 u}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3} + \cdots\right)$$

This scheme (2.25) is Bender-Schmidt Explicit Scheme. It is easy to see that truncation error $T \to 0\ as\ k, h \to 0$. So the explicit scheme is compatible with Heat conduction Equation

We can minimize the truncation error by selecting the suitable value of r. using the expression $\frac{\partial}{\partial t} = c\frac{\partial^2}{\partial x^2}$ from heat conduction equation $\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$, we have

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^4 u}{\partial x^4}\ , \qquad \frac{\partial^3 u}{\partial t^3} = c^3\frac{\partial^6 u}{\partial x^6}\cdots$$

Consider the Truncation error T in the following form

$$T = \left(c\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}\right) + \left(c\frac{h^4}{360}\frac{\partial^6 u}{\partial x^6} - \frac{k^2}{6}\frac{\partial^3 u}{\partial t^3}\right) + \cdots$$

$$= \left(c\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} - \frac{k}{2}c^2\frac{\partial^4 u}{\partial x^4}\right) + \left(c\frac{h^4}{360}\frac{\partial^6 u}{\partial x^6} - \frac{k^2}{6}c^3\frac{\partial^6 u}{\partial x^6}\right) + \cdots$$

Let $r = \frac{ck}{h^2} = \frac{1}{6}$, then the first terms vanishes, and truncation error reduces to the following expression

$$T = -\frac{k^2c^3}{15}\frac{\partial^6 u}{\partial x^6} + \cdots$$

So, the truncation error is of the highest order for the value $r = \frac{ck}{h^2} = \frac{1}{6}$

2.2.3 Convergence and Order

The FDS is said to be convergent to the exact solution if the discretization error $(u - u_D)$ tends to zero as step sizes of various independent variables approach zero. For example, the explicit scheme of one-dimensional heat conduction equation is convergent to the exact solution as $u \to u_D$ for $\Delta t, \Delta x \to 0$.

The order is the rate at which the finite difference scheme tends to the exact solutions as the grid sizes go to zero. In other words, the minimum degree of error terms present in the finite difference approximation of the derivative terms of the differential equation is the order. For example, in Bender–Schmidt Explicit Scheme (16.3), we approximate the time derivative term with forward difference

$$\frac{\partial u(x_i, t_j)}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k)$$

While the derivative term containing spatial variable is approximated with central differences formula

$$\frac{\partial^2 u(x_i, t_j)}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

Hence the explicit scheme has truncation error of order $O(k) + O(h^2)$

2.2.4 Stability

The stability error $(u - u_D)$ tends to zero with $\Delta t, \Delta x \to 0$ then the scheme is said to be a stable scheme. The exact solutions of a given PDE must be examined to discuss the stability of any finite difference scheme. If the PDE is itself unstable, then the solution obtained by finite difference scheme is also unstable. But the numerical solution must be bounded for a stable PDE. Stability analysis is performed only for linear PDEs. So, nonlinear PDEs must be linearized locally for stability analysis. There are several methods for stability analysis of a finite difference scheme. We only discuss Matrix Method for Explicit and CN scheme.

2.2.5 Matrix Method Stability for Explicit Method

The explicit scheme is given by

$$u_{i,j+1} = ru_{i-1,j} + (1 - 2r)u_{i,j} + r\, u_{i+1,j} \quad i = 1,2,3 \ldots .(n-1) \qquad (1.26)$$

The values $u_{i,j}$ computed by this method have certain error, let these approximation values be $u^*_{i,j}$

$$u_{i,j+1}^* = ru_{i-1,j}^* + (1-2r)u_{i,j}^* + r\, u_{i+1,j}^* \quad i = 1,2,3 \dots .(n-1) \tag{1.27}$$

Consider the error $e_{i,j} = u_{i,j} - u_{i,j}^*$ on subtracting eq. (1.27) from eq. (1.26)

$$e_{i,j+1} = re_{i-1,j} + (1-2r)e_{i,j} + r\, e_{i+1,j} \quad i = 1,2,3 \dots .(n-1) \tag{1.28}$$

Note that for linear finite difference scheme; the error equation is same as that of this scheme itself. Let the boundary condition be prescribed at ends points, so the error in these values is zero i.e.

$$e_{0,j} = e_{n,j} = 0 \qquad \forall\, j = 1,2,3, \dots n$$

The equation (1.28) can be written in matrix form as follows

$$\begin{bmatrix} e_{1,j+1} \\ e_{2,j+1} \\ e_{3,j+1} \\ \cdot \\ \cdot \\ e_{n-1,j+1} \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 & 0 & \cdot & 0 & 0 \\ r & 1-2r & r & 0 & \cdot & 0 & 0 \\ 0 & r & 1-2r & r & \cdot & 0 & 0 \\ . & . & . & 1-2r & \cdot & 0 & 0 \\ . & . & . & . & \cdot & . & . \\ 0 & 0 & 0 & 0 & \cdot & r & 1-2r \end{bmatrix} \begin{bmatrix} e_{1,j} \\ e_{2,j} \\ e_{3,j} \\ \cdot \\ \cdot \\ e_{n,j} \end{bmatrix}$$

In compact form, Let

$$E_{j+1} = AE_j$$

Where

$$E_{j+1} = \begin{bmatrix} e_{1,j+1} \\ e_{2,j+1} \\ e_{3,j+1} \\ \cdot \\ \cdot \\ e_{n-1,j+1} \end{bmatrix} A = \begin{bmatrix} 1-2r & r & 0 & 0 & \cdot & 0 & 0 \\ r & 1-2r & r & 0 & \cdot & 0 & 0 \\ 0 & r & 1-2r & r & \cdot & 0 & 0 \\ . & . & . & 1-2r & \cdot & 0 & 0 \\ . & . & . & . & \cdot & . & . \\ 0 & 0 & 0 & 0 & \cdot & r & 1-2r \end{bmatrix}, E_j \begin{bmatrix} e_{1,j} \\ e_{2,j} \\ e_{3,j} \\ \cdot \\ \cdot \\ e_{n-1,j} \end{bmatrix}$$

Let error vector at initial time $t = t_0$ be E_0 so $E_1 = AE_0$. similarly $E_2 = AE_1$. After m numbers of steps, we have

$$E_m = A^m E_0. \tag{1.29}$$

The initial error vector E_0 is a finite quantity. The scheme is stable if $A^m \to 0$ as $m \to \infty$. Let we study the behavior of matrix A containing the elements depend upon the value$r = \frac{ck^2}{h}$. Let the eigenvalues $\lambda_1, \lambda_2, \lambda_3 \dots . \lambda_{n-1}$ of matrix A be distinct with corresponding eigenvectors $X_1, X_2, X_3 \dots . X_{n-1}$

$$AX_i = \lambda_i X_i \qquad for\ i = 1,2,3 \dots n-1$$

The error vector E_0 can be written as linear combination of eigenvector as follows

$$E_0 = c_1X_1 + c_2X_2 + c_3X_3 \dots + c_{n-1}X_{n-1}$$

Pre-Multiply with matrix A, we get

$$AE_0 = A(c_1X_1 + c_2X_2 + \cdots + c_{n-1}X_{n-1})$$
$$= Ac_1X_1 + Ac_2X_2 + \cdots + Ac_{n-1}X_{n-1})$$

Here $AX_i = X_i\lambda_i$

$$= c_1\lambda_1X_1 + c_2\lambda_2X_2 + \cdots + c_{n-1}\lambda_{n-1}X_{n-1}$$

Again Pre-Multiply by A

$$A^2E_0 = c_1\lambda_1^2X_1 + c_2\lambda_2^2X_2 + \cdots + c_{n-1}\lambda_{n-1}^2X_{n-1}$$

Repeating up to m times successively we obtain

$$A^mE_0 = c_1\lambda_1^mX_1 + c_2\lambda_2^mX_2 + \cdots + c_{n-1}\lambda_{n-1}^mX_{n-1}$$

The scheme is stable if the error term $E_m = A^mE_0$ is finite as limit $m \to \infty$ So we must have

$$|\lambda_i| \le 1; \forall\ 1 \le i \le n \quad or \quad |\lambda_{max}| \le 1$$

The maximum absolute eigenvalue must be less than or equal to unity. The quantity $r = ck/h^2$ is always positive

Using Brauer Theorem for matrix A, we have

$|\lambda - (1-2r)| \le r$ and $|\lambda - (1-2r)| \le 2r$

On simplifying we get

$1-3r \le \lambda \le 1-r$ and $1-4r \le \lambda \le 1$

For $|\lambda| \le 1$, We must have

$$-1 \le 1-3r \le 1 \ , -1 \le 1-r \le 1 \ , -1 \le 1-4r \le 1$$
$$0 \le r \le \frac{2}{3}, \quad 0 \le r \le 2, \quad 0 \le r \le \frac{1}{2}$$

All these condition satisfy for $0 \le r \le \frac{1}{2}$ So explicit scheme is stable scheme for $0 \le r \le \frac{1}{2}$

2.2.6 Matrix Method Stability of CN Scheme

The CN scheme is given by

$$-ru_{i-1,j+1} + 2(1+r)u_{i,j+1} - ru_{i-1,j+1} = ru_{i-1,j} + 2(1-r)u_{i,j} + ru_{i+1,j}$$

Proceeding in a similar manner as in explicit scheme, we have

$$AE_{j+1} = BE_j \tag{1.30}$$

Where

$$A = \begin{bmatrix} 2(1+r) & -r & 0 & 0 & \cdot & 0 & 0 \\ -r & 2(1+r) & -r & 0 & \cdot & 0 & 0 \\ 0 & -r & 2(1+r) & -r & \cdot & 0 & 0 \\ . & . & . & 2(1+r) & \cdot & 0 & 0 \\ . & . & . & . & \cdot & . & . \\ 0 & 0 & 0 & 0 & \cdot & -r & 2(1+r) \end{bmatrix}, E_j \begin{bmatrix} e_{1,j+1} \\ e_{2,j+1} \\ e_{3,j+1} \\ . \\ . \\ e_{n-1,j+1} \end{bmatrix}$$

$$B = \begin{bmatrix} 2(1-r) & r & 0 & 0 & \cdot & 0 & 0 \\ r & 2(1-r) & r & 0 & \cdot & 0 & 0 \\ 0 & r & 2(1-r) & r & \cdot & 0 & 0 \\ . & . & . & 2(1-r) & \cdot & 0 & 0 \\ . & . & . & . & \cdot & . & . \\ 0 & 0 & 0 & 0 & \cdot & r & 2(1-r) \end{bmatrix}, E_j \begin{bmatrix} e_{1,j} \\ e_{2,j} \\ e_{3,j} \\ . \\ . \\ e_{n-1,j} \end{bmatrix}$$

From (1.30) we have

$$E_{j+1} = A^{-1}BE_j$$

The CN scheme is stable if the modulus of eigenvalues of $A^{-1}B$ is less than or equal to unity. For simplification, consider the following matrix

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdot & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdot & 0 & 0 \\ 0 & . & 2 & r & \cdot & 0 & 0 \\ . & . & . & 2 & \cdot & 0 & 0 \\ . & . & . & . & \cdot & . & . \\ 0 & 0 & 0 & 0 & \cdot & -1 & 2 \end{bmatrix}$$

Let us assume that the eigenvalue of matrix C is μ. Using again Brauer Theorem for matrix C, we have

$$-2 \leq |\mu - 2| \leq 2 \rightarrow 0 \leq \mu \leq 4$$

The matrix $A^{-1}B = (2I + rC)^{-1}(2I - rC)$. let the eigenvalue of matrix $A^{-1}B$ be λ then

$$\lambda = \frac{2 - r\mu}{2 + r\mu}$$

Since r and μ both are non-negative quantities, so we have $|\lambda| \leq 1$. Hence, the CN scheme is stable for each r. i.e. unconditionally stable scheme.

Summarization:

Method	Level	Stability	Difficulty	Truncation Error
Explicit	2	Stable for $r \leq 1/2$	Easy ($u_{i,j}$ can be obtained directly)	$O(\Delta t) + O(\Delta x^2)$
Crank-Nicolson	2	Unconditionally Stable	Difficult (System of equation in $u_{i,j}$ has to solve)	$O(\Delta t^2) + O(\Delta x^2)$
Richardson	3	Unstable	Easy ($u_{i,j}$ can be obtained directly)	$O(\Delta t^2) + O(\Delta x^2)$
Du-Fort & Frankel	3	Unconditionally Stable	Easy ($u_{i,j}$ can be obtained directly)	$O(\Delta t^2) + O(\Delta x^2)$

Chapter 3

3.1 Elliptic Equation (Laplace and Poisson Equation)

All time-dependent problems are known as "transient" problems. As time increases, all transient problems tend to steady state i.e. the problems are independent upon time (mathematically $\frac{\partial u}{\partial t} = 0$). some physical process come to the steady state in a very short span of time. Therefore we are interested in final stage for this process. Parabolic and hyperbolic equation in two and three dimension tends to elliptic equations in their steady state condition. For example, heat conduction and wave equation reduce to the following Laplace Equation an d Poisson equation.

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} \qquad \textit{Lapalce Equation}$$

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = f(x, y) \quad \textit{Poisson Equation}$$

These equations are also known as potential equations as the variable u represents the gravitational potential, velocity potential, and electromagnetic potential in various relevant fields of science. Finite difference approximation will be used for the solutions of various types of these elliptic equations.

In this section, Laplace and Poisson equations for the following cases are discussed with the help of finite difference approximations to obtain solutions at the pivotal points.

1. Dirichlet Condition
2. Symmetric problem

3.1.1 Laplace Equation

Consider the Laplace Equation

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} \qquad (1.30)$$

Let x_i & y_i be equally spaced points with spacing h & k , respectively and $u(x_i, y_i) = u_{i,j}$ The central differences for derivative terms are given by

$$\frac{\partial^2 u(x_i, y_j)}{\partial x^2} = \frac{u(x_i + h, y_j) - 2u(x_i, y_j) + u(x_i - h, y_j)}{h^2} + O(h^2)$$

$$= \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + O(h^2)$$

$$= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial y^2} = \frac{u(x_i, y_j + k) - 2u(x_i, y_j) + u(x_i, y_j - k)}{k^2} + O(k^2)$$

$$= \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} + O(k^2)$$

$$= \frac{u_{i,j+1} - 2u_{i,j} + u_{i,-1j}}{k^2} + O(k^2)$$

Note that both $h \, and \, k$ are spacing values of spatial variables, therefore we can consider the square meshes for our calculations. Let $h = k$ using central difference approximation in Laplace equation at point (x_i, y_j), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,-1j}}{k^2} + O(h^2) = 0$$

On neglecting the terms $O(h^2)$ we have

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,-1j}}{h^2} = 0$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,-1j} - 4u_{i,j} = 0$$

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,-1j}\right) \qquad (1.31)$$

This formula is known as standard 5-points formula. It becomes clearer the following figure that the value at any point is the average of four other points, situated at the lower, upper left and right sides.

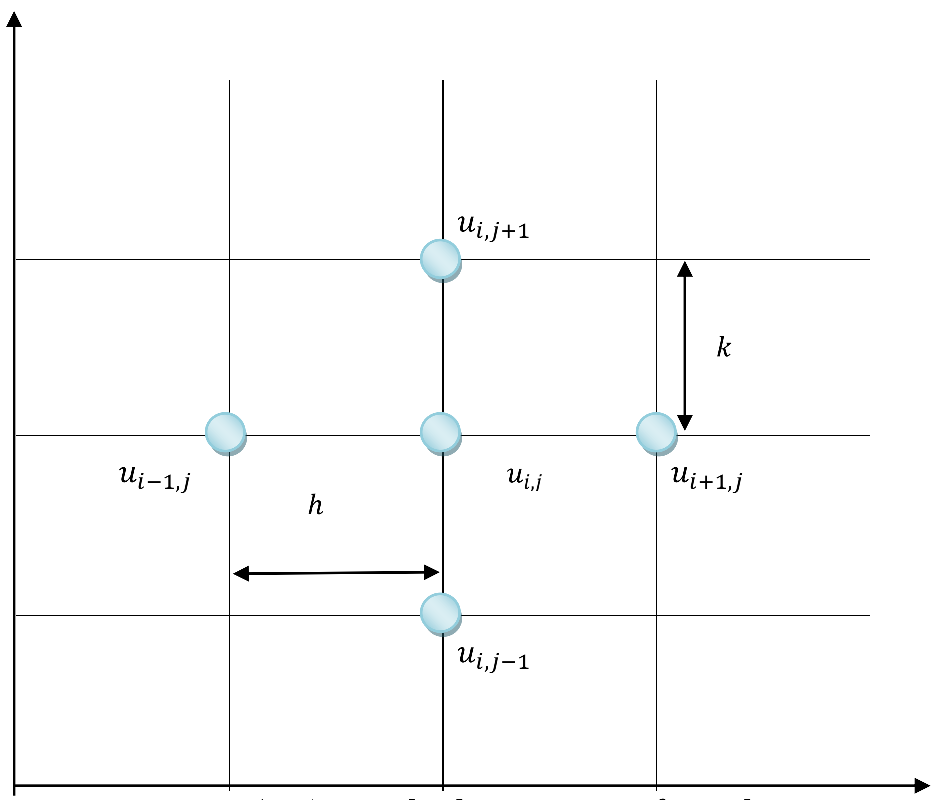


Fig. (1.1) Standard Five Points formula

Similarly we can obtained following diagonal 5-points formula by considering diagonal points for finite differences

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j+1} + u_{i-1,j-1} + u_{i-1,j+1} + u_{i,+1j-1}\right) \qquad (1.32)$$

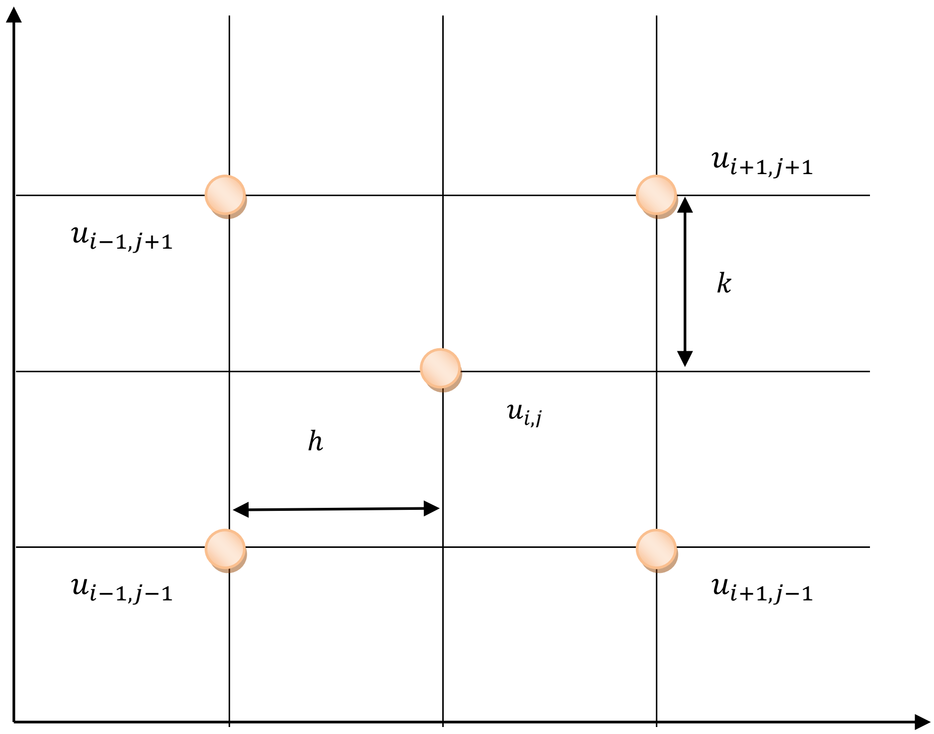
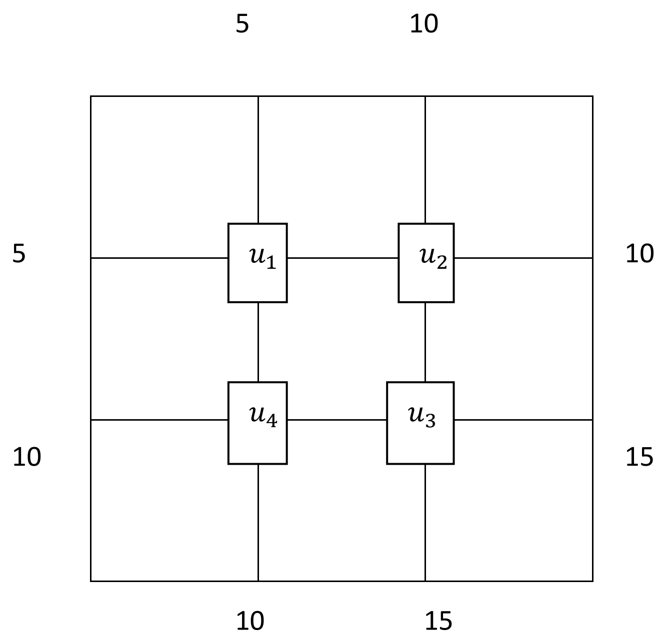


Fig. (1.2) Diagonal Five Points formula

It is easy to see from the Fig. 1.2, that the value of the function at any point is an average of the values at the diagonal points of the square meshes. But the value obtained from diagonal 5-points formula is less accurate, in general, as the points under discussion are at more distance in this case as compared to the standard 5-points formula. So, we will use the diagonal formula only when the standard formula is not applicable to the grid, or to ease the computation.

Examples (3.1) Dirichlet conditions only

Solve the Laplace Equation $\nabla^2 u = u_{xx} + u_{yy} = 0$ for the square mesh with the boundary values (Dirichlet condition) as shown in the following figure. Use Gauss-Seidal method till two consecutive iterations have same value up to three decimal points. Take initial approximation $u_1^0 = 0, u_2^0 = 0, u_3^0 = 0, u_4^0 = 0$



Ans.

Dirichlet condition (function value $u(x,t)$) are given at both the boundaries for the variables x and y. It is easy to apply standard 5-Points formula (1.31)

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,-1j}\right)$$

Using the function values $u(x,t)$ at the nodal points, we get following four equations

$$u_1 = \frac{1}{4}(5 + 5 + u_2 + u_4)$$

$$u_2 = \frac{1}{4}(10 + 10 + u_1 + u_3)$$

$$u_3 = \frac{1}{4}(15 + 15 + u_2 + u_4)$$

$$u_4 = \frac{1}{4}(10 + 10 + u_1 + u_3)$$

On solving this system of simultaneous linear Equation by Gauss-Seidal method with initial approximation $[0,0,0,0]^T$, the following iteration $[u_1, u_2, u_3, u_4]$ are obtained

u_1	u_2	u_3	u_4
2.500000	5.625000	8.906250	7.851562
5.869141	8.693848	11.636353	9.376373
7.017555	9.663477	12.259962	9.819380
7.370714	9.907669	12.431763	9.950619
7.464572	9.974084	12.481175	9.986437
7.490130	9.992826	12.494816	9.996237
7.497266	9.998020	12.498564	9.998958
7.499245	9.999453	12.499602	9.999712
7.499791	9.999848	12.499890	9.999920
7.499942	9.999958	12.499969	9.999978

The last two iterations are equal up to three decimal digits. So the solution is given by values at 10th iteration.

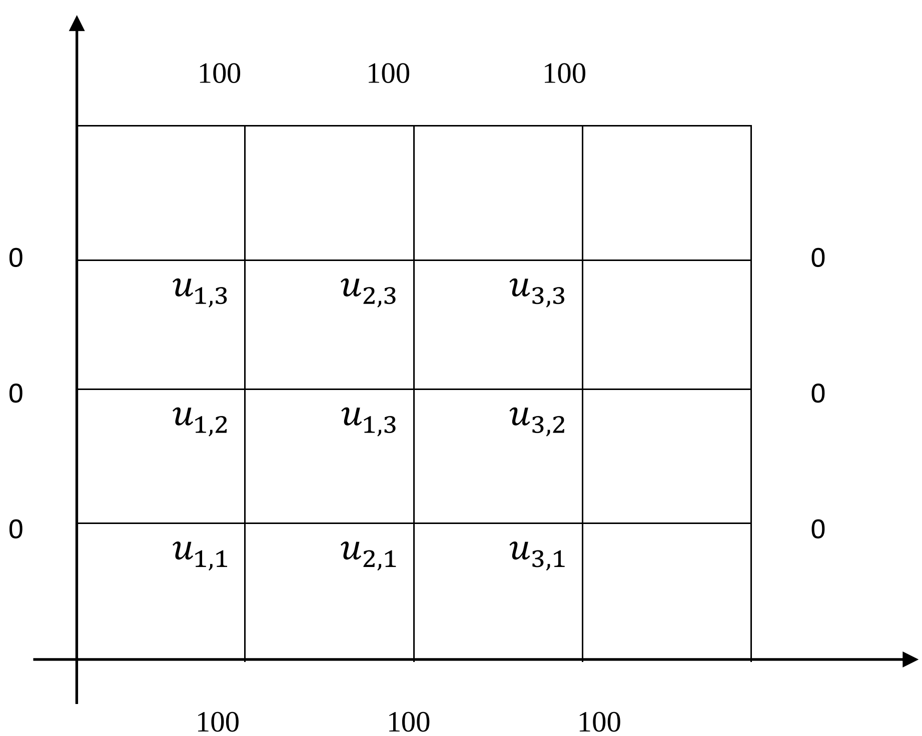
So exact answer are

$$u_1 = 7.5 \quad u_2 = 10 \quad u_3 = 12.5 \quad u_4 = 10$$

Examples (3.2) Symmetric Problem

To the find the steady state temperature distribution in a thin rectangular plate, whose edges $x = 0$ & $x = 2$, are kept at 0°C (in ice), and edges $y = 0$ & $y = 2$, are kept at temperature 100°C (in boiling water). Find the values of temperature at the nodal points of the rectangular region with mess length 0.5. Compute ten iterations of Gauss–Seidel method for temperature distribution on the grid.

Ans. The temperature at the edges $x = 0$ & $x = 2$, are kept at 0°C (in ice), and edges $y = 0$ & $y = 2$, are kept at temperature 100°C. The grid spacing $h = 0.5$ is shown in figure.



The steady state temperature distribution in a thin rectangular plate is defined by the Laplace Equation. The boundary condition and Laplace equation $(u_{xx} + u_{yy})$ both are symmetrical around the lines $x = 1$ & $y = 1$. therefore the values about these lines are equal i.e.

$u_{11} = u_{31}, u_{12} = u_{32}, u_{13} = u_{33}$ Symmetry about $x = 1$

$u_{11} = u_{13}, u_{21} = u_{23}, u_{3,1} = u_{33}$ Symmetry about $y = 1$

So the values $u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2}$ need to be computed only. Using standard 5-Points formula and symmetries (from above equation), we have following simplified equation for $i, j = 1,2$

$$At\ (1,1) \qquad 4u_{1,1} - u_{2,1} - u_{1,2} = 100$$

$$At\ (2,1) \qquad 4u_{2,1} - u_{1,1} - u_{2,2} = 100$$

$$At\ (1,2) \qquad 4u_{1,2} - u_{2,2} - u_{1,1} = 0$$

$$At\ (2,2) \qquad 2u_{2,2} - u_{1,2} - u_{2,1} = 0$$

The ten iteration of Gauss Seidal method for $u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2}$ are as follows. The zero vectors are used as initial approximation.

$u_{1,1}$	$u_{2,1}$	$u_{1,2}$	$u_{2,2}$
25.000000	37.500000	12.500000	25.000000
37.500000	50.000000	25.000000	37.500000
43.750000	56.250000	31.250000	43.750000

46.875000	59.375000	34.375000	46.875000
48.437500	60.937500	35.937500	48.437500
49.218750	61.718750	36.718750	49.218750
49.609375	62.109375	37.109375	49.609375
49.804688	62.304688	37.304688	49.804688
49.902344	62.402344	37.402344	49.902344
49.951172	62.451172	37.451172	49.951172

The final iteration is approximately equals to

$$u_{1,1} = 50 \; u_{2,1} = 62.5 \; u_{1,2} = 37.5 \; u_{2,2} = 50$$

3.1.2 Poisson Equation

Consider the Poisson Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = f(x, y) \qquad (1.30)$$

Let x_i & y_i be equally spaced points with spacing h & k , respectively and $(x_i, y_i) = u_{i,j}$. The central differences for derivative terms are given by

$$\frac{\partial^2 u(x_i, y_j)}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

$$\frac{\partial^2 u(x_i, y_j)}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,-1j}}{k^2} + O(k^2)$$

The Poisson equation at any point (x_i, y_j),

$$\frac{\partial^2 u}{\partial x^2}|_{(x_i,y_j)} + \frac{\partial^2 y}{\partial y^2}|_{(x_i,y_j)} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,-1j}}{k^2}$$
$$= f(x_i, y_j)$$

Let $h = k$ then

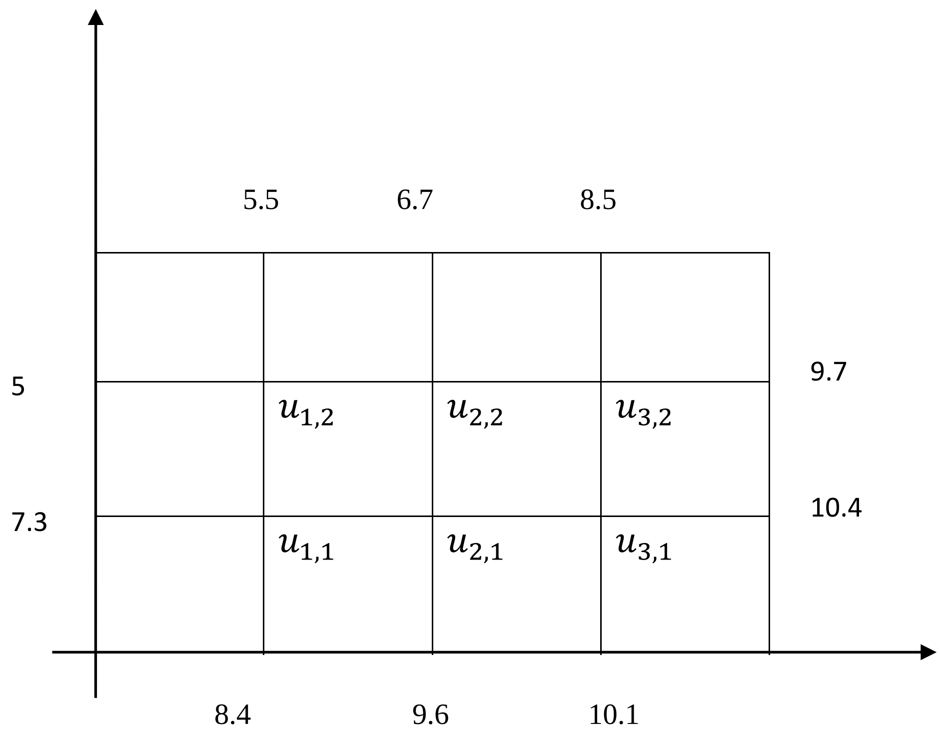
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} + O(h^2) = 0$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,-1j} - 4u_{i,j} = h^2 f(x_i, y_j) \qquad (1.31)$$

Now we will discuss few examples of Poisson Equation.

Examples (3.3) Dirichlet conditions only

Solve the Poisson equation $\nabla^2 u = u_{xx} + u_{yy} = x + y$ for square mesh whose edges $x = 0$ & $x = 0.8$, and edges $y = 0$ & $y = 0.6$ are kept at the temperature shown in the following figure. Find the values of $u(x, y)$ at the nodal points of the rectangular region with mess length 0.2. Use Gauss–Seidel iterative method to compute values at nodal points until the difference between successive values at each point is less than 0.005.



Ans. The edges of the rectangular region are $x = 0, x = 0.8$ and $y = 0, y = 0.6$ with the mess length 0.2. So value of x & y are as follows

x	y
$x_0 = 0$	$y_0 = 0$
$x_1 = 0.2$	$y_1 = 0.2$
$x_2 = 0.4$	$y_2 = 0.4$
$x_3 = 0.6$	$y_3 = 0.6$
$x_4 = 0.8$	

(1.32)

Let the value of $u(x_i, y_j) = u_{ij}$ we have

$$u_{10} = 8.4, u_{20} = 9.6, u_{30} = 10.1, u_{01} = 7.3, u_{02} = 5$$

$$u_{13} = 5.5, u_{23} = 6.7, u_{33} = 8.5, u_{41} = 10.4, u_{42} = 9.7$$

On replacing the derivative terms with central differences in the Poisson equation $u_{xx} + u_{yy} = x + y$ at the point (x_i, y_j) we have

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,-1j} - 4u_{i,j} = h^2 f(x_i, y_j) \qquad (1.33)$$

For $i = 1,2,3 \; and \; j = 1,2$, we obtain

$$At\ (1,1) \qquad u_{21} + u_{01} + u_{12} + u_{10} - 4u_{11} = (0.2)^2(x_1 + y_1)$$

$$At\ (2,1) \qquad u_{31} + u_{11} + u_{22} + u_{20} - 4u_{21} = (0.2)^2(x_2 + y_1)$$

$$At\ (3,1) \qquad u_{41} + u_{21} + u_{32} + u_{30} - 4u_{31} = (0.2)^2(x_3 + y_1)$$

$$At\ (1,2) \qquad u_{22} + u_{02} + u_{13} + u_{11} - 4u_{12} = (0.2)^2(x_1 + y_2)$$

$$At\ (2,2) \qquad u_{32} + u_{12} + u_{23} + u_{21} - 4u_{22} = (0.2)^2(x_2 + y_2)$$

$$At\ (3,2) \qquad u_{42} + u_{22} + u_{33} + u_{31} - 4u_{32} = (0.2)^2(x_3 + y_2)$$

Using the values of x_2, x_2, x_2 from Eq. (1.32) and (1.33)

$$u_{21} + u_{12} - 4u_{11} = -15.684$$

$$u_{31} + u_{11} + u_{22} - 4u_{21} = -9.576$$

$$u_{21} + u_{32} - 4u_{31} = -20.468$$

$$u_{22} + u_{11} - 4u_{12} = -10.476$$

$$u_{32} + u_{12} + u_{21} - 4u_{22} = -6.668$$

$$u_{22} + u_{31} - 4u_{32} = -18.16$$

On solving the above system of simultaneous linear equation by Gauss Seidal with initial approximation [0 0 0 0 0 0] the following iteration $[u_{11}\ u_{21}\ u_{31}\ u_{12}\ u_{22}\ u_{13}]$ are

Iteration 1
3.921000 3.374250 5.960563 3.599250 3.410375 6.882734
Iteration 2
5.664375 6.152828 8.375891 4.887688 6.147813 8.170926
Iteration 3

6.681129 7.695209 9.083534 5.826235 7.090093 8.583406
Iteration 4
7.301361 8.262747 9.328539 6.216864 7.432755 8.730324
Iteration 5
7.540903 8.469549 9.416968 6.362414 7.557571 8.783635
Iteration 6
7.628991 8.544883 9.449129 6.415641 7.603040 8.803042
Iteration 7
7.661131 8.572325 9.460842 6.435042 7.619602 8.810111
Iteration 8
7.672842 8.582321 9.465109 6.442111 7.625636 8.812686
Iteration 9
7.677108 8.585963 9.466662 6.444686 7.627833 8.813623
Iteration 10
7.678662 8.587290 9.467228 6.445624 7.628634 8.813966

The final solution is as follows
$u_{11} = 7.678662$ $u_{21} = 8.587290$ $u_{31} = 9.467228$ $u_{31} = 9.467228$
$u_{12} = 6.445624$ $u_{22} = 7.628634$ $u_{13} = 8.813966$ $u_{13} = 8.813966$

Examples (3.4) Symmetric Problem

Solve the Poisson equation $\nabla^2 u = u_{xx} + u_{yy} = x^2 + y^2$ for square mesh whose edges $x = 0$ & $x = 2$, and edges $y = 0$ & $y = 2$ are kept at the temperature 100^0C (in boiling water). Find the values of $u(x, y)$at the nodal points of the rectangular region with mess length 0.5. Use Gauss–Seidel iterative method to compute values at nodal points until the difference between successive values at each point is less than 0.005. Use symmetry.

Ans. The edges of the rectangular region are $x = 0, x = 2$ and $y = 0, y = 2$ with the mess length 0.5. So value of x & y are as follows

x	y
$x_0 = 0$	$y_0 = 0$
$x_1 = 0.5$	$y_1 = 0.5$
$x_2 = 1$	$y_2 = 1$
$x_3 = 1.5$	$y_3 = 1.5$
$x_4 = 2$	$y_4 = 2$

(1.34)

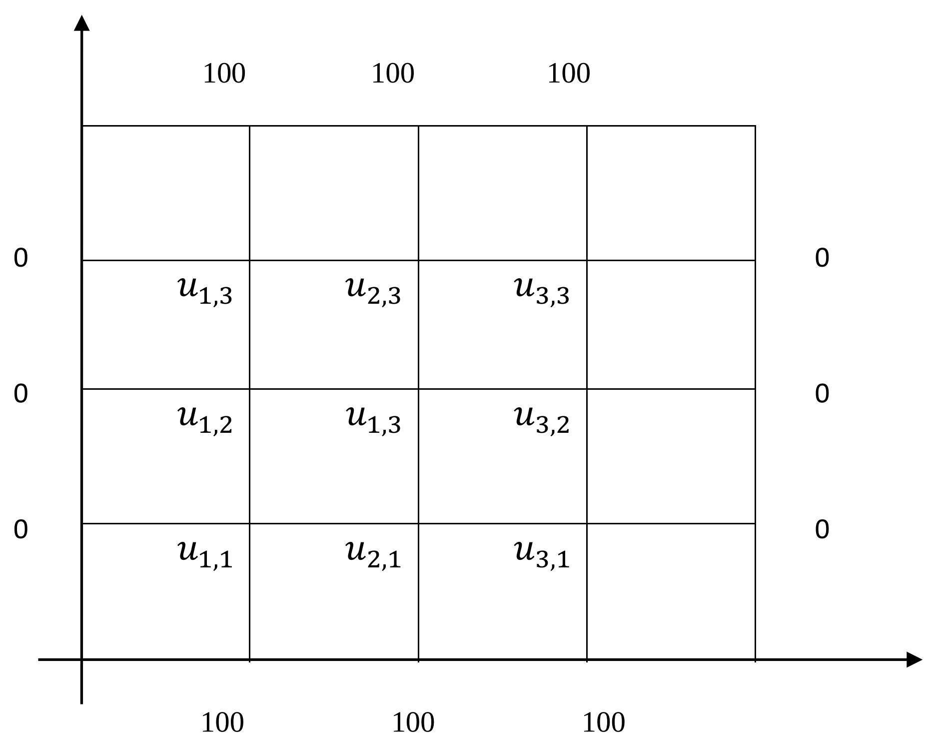
Let the value of $u(x_i, y_j) = u_{ij}$ we have

$u_{01} = 0,\ u_{02} = 0,\ u_{03} = 0, \quad x = 0$

$u_{41} = 0,\ u_{42} = 0,\ u_{43} = 0, \quad x = 2$

$u_{10} = 100,\ u_{20} = 100,\ u_{30} = 100, \quad y = 0$

$u_{14} = 100,\ u_{24} = 100,\ u_{34} = 100 \quad y = 2$ (1.35)



Let us solve the Poisson Equation $u_{xx} + u_{yy} = x^2 + y^2$ with symmetry consideration. Since the boundary condition and the Poisson Equation are symmetrical around lines$x = 1$ & $y = 1$; so we can assume the values about lines are equal i.e.

$$u_{11} = u_{31}\,, u_{12} = u_{32}\,, u_{13} = u_{33} \quad \text{Symmetry about } x = 1$$

$$u_{11} = u_{13}\,, u_{21} = u_{23}, u_{3,1} = u_{33} \quad \text{Symmetry about } y = 1 \tag{1.36}$$

So the values $u_{1,1}\,, u_{2,1}\,, u_{1,2}\,, u_{2,2}$need to be computed only

The Poisson Equation $u_{xx} + u_{yy} = x^2 + y^2$ is given by

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,-1j} - 4u_{i,j} = h^2 f\left(x_i, y_j\right)$$

We need to find the values of $u_{1,1}\,, u_{2,1}\,, u_{1,2}\,, u_{2,2}$ So

$$At\ (1,1) \qquad u_{21} + u_{01} + u_{12} + u_{10} - 4u_{11} = (0.5)^2(x_1^2 + y_1^2)$$

$$At\ (2,1) \qquad u_{31} + u_{11} + u_{22} + u_{20} - 4u_{21} = (0.5)^2(x_2^2 + y_1^2)$$

$$At\ (1,2) \qquad u_{22} + u_{02} + u_{13} + u_{11} - 4u_{12} = (0.5)^2(x_1^2 + y_2^2)$$

$$At\ (2,2) \qquad u_{32} + u_{12} + u_{23} + u_{21} - 4u_{22} = (0.5)^2(x_2^2 + y_2^2)$$

(1.37)

Using symmetries from Eq. (1.34) the values of $x_i\,, y_j$ & $u_{i,j}$ from equation (1.35) & (1.36) becomes

$$u_{21} + u_{12} - 4u_{11} = -100 + (0.5)^2(0.5) = -99.875$$

$$2u_{11} + u_{22} - 4u_{21} = -100 + (0.5)^2(0.5) = -99.6875$$

$$u_{22} + 2u_{11} - 4u_{12} = (0.5)^2(1.25) = 0.3125$$

$$2u_{12} + 2u_{21} - 4u_{22} = (0.5)^2(2) = 0.5$$

The 16 iterations of Gauss Seidal method for $u_{1,1}\,, u_{2,1}\,, u_{1,2}\,, u_{2,2}$ are as follows. The zero vectors are used as initial approximation.

Iteration 1

24.968750 37.406250 12.406250 24.781250

Iteration 2

37.421875 49.828125 24.828125 37.203125

Iteration 3

43.632812 56.039062 31.039062 43.414062

Iteration 4

46.738281 59.144531 34.144531 46.519531

Iteration 5

48.291016 60.697266 35.697266 48.072266

Iteration 6

49.067383 61.473633 36.473633 48.848633

Iteration 7

49.455566 61.861816 36.861816 49.236816

Iteration 8

49.649658 62.055908 37.055908 49.430908

Iteration 9

49.746704 62.152954 37.152954 49.527954

Iteration 10

49.795227 62.201477 37.201477 49.576477

Iteration 11

49.819489 62.225739 37.225739 49.600739

Iteration 12

49.831619 62.237869 37.237869 49.612869

Iteration 13

49.837685 62.243935 37.243935 49.618935

Iteration 14

49.840717 62.246967 37.246967 49.621967

Iteration 15

49.842232 62.248482 37.248482 49.623482

Iteration 16
49.842991 62.249241 37.249241 49.624241

The final solution is given by

$$u_{11} = u_{13} = u_{31} = u_{33} = 49.842991$$

$$u_{21} = u_{23} = 62.249241$$

$$u_{12} = u_{32} = 37.249241$$

$$u_{22} = 49.624241$$

Chapter 4

4.1 Hyperbolic Equation (Wave Equation)

Vibrations in a tightly stretched string between two points are well described by the following hyperbolic equation known as 1-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2} \qquad (1.38)$$

Here $c = \frac{T}{m}$ (where T is tension in the string and m is the mass per unit length) is a positive constant, and it depends on the nature of the strings. The strings are homogenous, i.e. uniform and elastic. The dependent variable $u(x,t)$ is the displacement of the string at any point x and at any time t from equilibrium position.

We will discuss explicit and implicit scheme for the solution of wave Eq. (1.38).

4.1.1 Explicit scheme

Let the value of $u(x,t)$ at point (x_i, y_j) be u_{ij} i.e. $u(x_i, y_j) = u_{ij}$. Also assume that step sizes for the variable x and t are $\Delta x = h$ & $\Delta y = k$, respectively

Let us discretized the Eq. (1.38) at point (x_i, y_j)

$$\frac{\partial^2 u}{\partial t^2}|_{(x_i,y_j)} = c\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial t^2}|_{(x_i,y_j)} \qquad (1.39)$$

Using central difference formulas in this equation, and neglecting the error terms in discretization, we have

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = c\ \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$u_{i,j+1} = \frac{ck^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + 2u_{i,j} - u_{i,j-1}$$

Let $r = \frac{ck^2}{h^2}$ then we have

$$u_{i,j+1} = r\left(u_{i+1,j} + u_{i-1,j}\right) + 2(1-r)u_{i,j} - u_{i,j-1} \qquad (1.40)$$

The explicit scheme (1.40) is stable for $r \le 1$; Let us discuss the unconditionally stable implicit scheme

4.1.2 Implicit scheme

In this scheme, the central difference formula for time derivative and average of central difference at $j-1$ & $j+1$ levels for space derivative are used in Eq. (1.39). After neglecting the error terms in discretization, we have

$$\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2} = c\,\frac{1}{2}\left(\frac{u_{i+1,j-1}-2u_{i,j-1}+u_{i-1,j-1}}{h^2} + \frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}\right) \qquad (1.41)$$

Let $r = \frac{ck^2}{h^2}$ then we have

$$-ru_{i-1,j+1} + 2(1+r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j-1} - 2(1+r)u_{i,j-1} + ru_{i+1,j-1} + 4u_{i,j}$$

This scheme is known as Implicit scheme.

Examples (4.1)

Solve the wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2}$$

With

Initial condition $u(x,0) = \begin{cases} 0.1(x) & 0 \le x \le 1/2 \\ 0.1(1-x) & 1/2 \le x \le 1 \end{cases}$

and $\frac{\partial u}{\partial t}\Big|_{t=0} = 0$

and boundary condition $u(0,t) = u(1,t) = 0$

Take that step size for t is 0.1 ($k = 0.1$) and step size for x is 0.25 ($h = 0.25$) Use explicit scheme to compute the solution up to time $t = 0.3$. Use central difference formula for derivative terms in the initial condition

Ans. It is given that step size for t is 0.1 ($k = 0.1$) and step size for x is 0.25 ($h = 0.25$). So we have $x = 0,0.25,0.5,0.75,1$ and $t = 0,0.1,0.2,0.3$

Use the initial condition

Initial condition $u(x,0) = \begin{cases} 0.1(x) & 0 \leq x \leq 1/2 \\ 0.1(1-x) & 1/2 \leq x \leq 1 \end{cases}$

And boundary condition

$$u(0,t) = u(1,t) = 0$$

We have following table for values of $u_{ij} = u(x_i, y_j)$

$i(x)$ \ $j(t)$	0(0)	1(0.25)	2(0.5)	3(0.75)	4(1)
0(0)	0	0.025	0.05	0.025	0
1(0.1)	0				0
2(0.2)	0				0
3(0.3)	0				0

The explicit scheme is used for wave Equation

$$u_{i,j+1} = r\left(u_{i+1,j} + u_{i-1,j}\right) + 2(1-r)u_{i,j} - u_{i,j-1}$$

Using $r = \frac{ck^2}{h^2} = 4\frac{(0.1)^2}{(0.25)^2} = 0.64$, we have

$$u_{i,j+1} = 0.64\left(u_{i+1,j} + u_{i-1,j}\right) + 0.72u_{i,j} - u_{i,j-1} \qquad (1.42)$$

Equation (1.42) can be used to compute the various nodal values for different j.

$j = 1$

At time $t = 0$ the initial condition is $\frac{\partial u}{\partial t}|_{t=0}$=0 the central difference formula provides the following equation

$$\frac{\partial u}{\partial t}|_{t=0} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}|_{j=0} = \frac{u_{i,1} - u_{i,-1}}{2k}|_{j=0}$$

$$\rightarrow u_{i,1} = u_{i,-1} \qquad (1.43)$$

The scheme (1.42) for $j = 0$ is given by

$$u_{i,1} = 0.64\left(u_{i+1,0} + u_{i-1,0}\right) + 0.72u_{i,0} - u_{i,-1}$$

Using Eq. (1.43)

$$u_{i,1} = 0.32\left(u_{i+1,0} + u_{i-1,0}\right) + 0.36u_{i,0}$$

For $i = 1,2,3$, we can easily obtain the following values

$u_{1,1} = 0.32\left(u_{2,0} + u_{0,0}\right) + 0.36u_{1,0} = 0.32(0.05 + 0) + 0.36(0.025) = 0.025$

$u_{1,1} = 0.32\left(u_{3,0} + u_{1,0}\right) + 0.36u_{2,0} = 0.32(0.025 + 0.025) + 0.36(0.5) = 0.034$

$u_{1,1} = 0.32\left(u_{4,0} + u_{2,0}\right) + 0.36u_{3,0} = 0.32(0 + 0.05) + 0.36(0.025) = 0.025$

The following table shows this value in the second row

$i(x)$ \ $j(t)$	0(0)	1(0.25)	2(0.5)	3(0.75)	4(1)
0(0)	0	0.025	0.05	0.025	0
1(0.1)	0	0.025	0.034	0.025	0
2(0.2)	0				0
3(0.3)	0				0

$j = 2$

The explicit scheme (1.42) for $j = 1$ gives the following

$$u_{i,2} = 0.64\left(u_{i+1,1} + u_{i-1,1}\right) + 0.72u_{i,1} - u_{i,0}$$

For $i = 1,2,3$, we can easily obtain the following value

$u_{1,2} = 0.64\left(u_{2,1} + u_{0,1}\right) + 0.72u_{1,1} - u_{1,0} = 0.64(0.034 + 0) + 0.72(0.025) - 0.025 = 0.01476$

$u_{2,2} = 0.64\left(u_{3,0} + u_{1,1}\right) + 0.72u_{2,1} - u_{2,0} = 0.64(0.025 + 0.025) + 0.72(0.5) - 0.05 = 0.00648$

$u_{3,2} = 0.64\left(u_{4,1} + u_{2,1}\right) + 0.72u_{3,1} - u_{3,0} = 0.64(0 + 0.034) + 0.72(0.025) - 0.025 = 0.01476$

$i(x)$ \ $j(t)$	0(0)	1(0.25)	2(0.5)	3(0.75)	4(1)
0(0)	0	0.025	0.05	0.025	0
1(0.1)	0	0.025	0.034	0.025	0
2(0.2)	0	0.01476	0.00648	0.01476	0
3(0.3)	0				0

$j = 3$

The explicit scheme (1.42) for $j = 2$ gives the following

$$u_{i,3} = 0.64\left(u_{i+1,2} + u_{i-1,2}\right) + 0.72u_{i,2} - u_{i,1}$$

For $i = 1,2,3$, we can easily obtain the following value

$u_{1,3} =$

$0.64\left(u_{2,2} + u_{0,1}\right) + 0.72u_{1,2} - u_{1,1} = 0.64(0.00648 + 0) + 0.72(0.01476) - 0.025 = -0.0102256$

$u_{2,3} =$

$0.64\left(u_{3,2} + u_{1,2}\right) + 0.72u_{2,2} - u_{2,1} = 0.64(0.01476 + 0.01476) + 0.72(0.00648) - 0.034 = -0.0104416$

$u_{3,3} =$

$0.64(u_{4,2} + u_{2,2}) + 0.72u_{3,2} - u_{3,1} = 0.64(0.00648 + 0) + 0.72(0.01476) - 0.025 = -0.0102256$

$i(x)$ \ $j(t)$	0(0)	1(0.25)	2(0.5)	3(0.75)	4(1)
0(0)	0	0.025	0.05	0.025	0
1(0.1)	0	0.025	0.034	0.025	0
2(0.2)	0	0.01476	0.00648	0.01476	0
3(0.3)	0	-0.0102256	-0.0104416	-0.0102256	0

Conclusion

The research has revealed that the size of the mesh is significant to arrive at an accurate solution when using finite difference method, the smaller the size of the mesh the closer is the numerical result to the exact solution. The value of the solution at these discrete points is approximated by solving algebraic equations containing finite differences and values from neighboring points, and both the spatial domain and time interval are divided into a finite number of steps. The values at discrete grid points are obtained from numerical solutions of differential equations using finite difference. To obtain the numerical approximations to the time-dependent differential equations needed for computer simulations, explicit and implicit approaches are used. The numerical example results of the explicit and implicit schemes for the heat equation under unique initial and boundary conditions are implemented in this chapter. It also presents a common elliptic partial differential equation to determine the temperature at the inside nodes.

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