Numerical Method for solving Singular Perturbed Problem

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Certificate

I hereby certify that the Project Dissertation titled **" Numerical Method for solving Singular Perturbed Problem"** which is submitted by Nikita Kaushik (2K19/MSCMAT/33) and Kirti Rani (2K19/MSCMAT/14) to the Department of Applied Mathematics, Delhi Technological University in partial fulfilment of the requirement for the award of the degree Master of Science, is a record of the project work carried out by the students under my supervision.

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Abstract

Mathematical equations are used to model a variety of phenomena in natural sciences and engineering. Various parameters are included in these mathematical equations. The solutions of these equations are affected by minor changes in these parameters. This minor alteration is called perturbation and the corresponding parameter is known as the perturbation parameter.
It is difficult to find the exact solutions of these mathematical equations. Therefore, the alternative way is to find their approximate solutions. These solutions are obtained by using the approximation techniques. These Perturbation techniques further pave the way to Perturbation theory.
We begin with perturbation theory in chemical kinetics. With the introduction of Michaelis-Menten mechanism and steady state approximation the concept of singular perturbation theory in chemical kinetics is studied. As we move further, we discuss a weakly coupled system of m-equations and study a highly significant numerical method i.e. q-stage runge Kutta method.
We then discuss a number of iterative methods to solve initial- and/or boundary-value problems in ordinary and partial differential equations. As a series of iterates, these iterative procedures have the solution or a close approximation to it. We present and evaluate an iterative analytic approach based on the Lagrange multiplier technique to estimate the multiscale solution.
Iteration is used to achieve closed-form analytic approximations to nonlinear boundary value problems. In a general setting, variational theory and Liouville–Green transforms are used to obtain the Lagrange multiplier optimally. We have taken singular peturbed problem to test the method and also compare it with the exact solution. Further, two test partial differential equations problems are taken into account and the findings of a detailed comparative study are discussed.

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Chapter 1

Introduction

Mathematics is used in almost every aspet of our day to day life. The progress in applicable mathematics has been brought about by the extension and development of many important approaches and techniques [1]. Differential equations is a broad and significant branch of mathematics in today's world. The topic of calculus has long been a source of both theoretical and functional science, and it remains so today.

Definition 1.0.1 *An equation which involves differentials or differential coefficients is called a differential equation i.e an equation involving dependent and independent variable and the derivative of dependent variable w.r.t independent variable*

Furthermore, differential equations can be divides into two categories which are again divided into different subcategories. Most two important subcategories are ordinary differential equations and partial differential equations.

Definition 1.0.2 *Ordinary differential equations are differential equations that include the ordinary derivatives of one or more dependent variables with respect to a single independent variable.*

Example 1.0.1

$$2\frac{d^2y}{dx^2} + xy^2\frac{dy}{dx} = 0 \tag{1.0.1}$$

Definition 1.0.3 *A partial differential equation is a differential equation that includes partial derivatives of one or more dependent variables with respect to more than one independent variable.*

Example 1.0.2

$$\frac{\delta v}{\delta x} - 2\frac{\delta v}{\delta t} = v^2 \tag{1.0.2}$$

The linear and non-linear differential equations are then separated. When a differential equation is expressed in the form of a polynomial, the derivatives and dependent

variable are in the first degree, and the coefficients of the various terms are either constants or functions of the independent variable, while when a non linear differential equation is expressed in the form of a polynomial, the derivatives and dependent variable are in more than degree one, and the coefficients of the various terms are either constants or functions of the independent variable. .
Now that we've classified differential equations in various ways, let's have a look at where and how they come from. In this way, we can get a sense of the wide range of things on which differential equations and its methods can be applied.
In various branches of science and engineering differential equations are used to solve a number equations we highlight a few of these issues here, which can be easily expanded in filling several pages.

1. Dilemma of evaluating a projectile's, rocket's, satellite's, or planet's motion.
2. In an electric vehicle, deciding the charge or current is a challenge.
3. The analysis of the radioactive substance's rate of decomposition or the population's rate of growth.
4. The problem of identifying curves with specific geometrical properties.

Differential equations are formed by the mathematical formulation of such problems. Ordinary differential equation (ODE) [11] with variable coefficient are used in a wide range of applications. Euler equation, Bessel equation, Legendre equation, and Laguerre equation are examples of these equations. The literature has also looked at many nonlinear ODE which are with variable coefficient, such as Duffing equation, Thomas-Fermi equation, and Van der Pol equation[11]. Linear and nonlinear ordinary differential equations which are with variable coefficients playing an important role in applied mathematics, physics, engineering.

Researchers wanted to develop accurate methods to solve a wide range of linear and nonlinear equations and integral equations without having to make any tangible assumptions or discretize the variables.
The term singular perturbation was coined in the 1940s. The subject and the techniques associated with it have evolved over time in order to find approximate solutions to complex problems.

Differential equations with at least one small parameter are commonly used to express such problems. Mathematical problems that make substantial use of small parameter were probably described first by J.H. Poincare [1].

Mathematical equations are used to model a variety of phenomena in natural sciences and engineering. Various parameters are included in these mathematical equations.

The solutions of these equations are affected by minor changes in these parameters. This minor alteration is called perturbation and the corresponding parameter is known as the perturbation parameter.

It is difficult to find the exact solutions of these mathematical equations. Therefore, the alternative way is to find their approximate solutions. these solutions are obtained by using the approximation techniques. These Perturbation techniques further pave the way to Perturbation theory.
The fundamental concept of perturbation theory is to approach the solution of a problem by looking for the solution of its neighbouring problem. The theory examines the behaviour of solutions locally. This can be done by inserting a small dimensionless quantity. called the perturbation parameter (ϵ), into the problem. The approximate solution of the problem can be then written in formal power series of perturbation parameter. Hence, perturbation theory measures the effect of small disturbances.
The perturbtion is categorized into two classes, namely, regular perturbation and singular perturbation.

This classification depends on the effect of such disturbances when the effect is small, then perturbation is termed as regular and when the effect is large, then perturbation is termed as singular perturbation.
A precise definition of the regularly and singularly perturbed problem is as follows-

Definition 1.0.4 *A problem $f(y(x), \epsilon) = 0$ is said to be Regular Perturbation Problem if it depends on the perturbation parameter ϵ in such a way that its solution $y_\epsilon(x)$ converges uniformly to the solution $y_0(x)$ of the limiting case $(f(y(x), 0))$ over the domain of existence as $\varepsilon \to 0$.*

Definition 1.0.5 *A problem $f(y(x), \epsilon) = 0$ is said to be a Singular Perturbation Problem if it depends on the perturbation parameter ϵ in such a way that its solution $y_\epsilon(x)$ is not uniformly convergent to the solution $y_0(x)$ of the limiting case $(f(y(x), 0))$ as $\varepsilon \to 0$.*

In this paper,singular perturbation is used in chemical kinetics and weakely coupled system which is of convection diffustion equation. The study of the time evolution of chemical reactions is one of the main problems of chemical kinectics .The [2]. The effort to simulate chemical processes using simple reactions leads to time multiscaling.

As a result, the activity of various components in such reaction mixture varies greatly, those product which are appear in the output have a short lifespan and quickly stabilised. One of its example is that the concentrations of active particles such as radicals, ions, and other ions change rapid and noticable in $10^{-6} sec$, whereas changes in a stable substance take several hours. Despite their short lifetime or rapid stabilisation,such type of things gives an important point in the overall process. In most

cases, it is impossible to distinguish between fast and slow processes, necessitating continuously time noticing of the total kinetic process in order to grasp the essential quality of appearence. The scanning of kinetic curves can be done effectively with the help of mathematical models.
To solve equations such as linear, nonlinear, original, and boundary value problem, He invented variational iteration method and homotopy perturbation method. While Inokuti, Sekine, and Mura [12] are credited with inventing the variational iteration process, He was first to fully realise the technique's potential. He also recognised the physical significance of the variational iteration process, as well as its consistency with physical problems, and applied this promising technique to a wide range of linear and nonlinear, ordinary, partial, deterministic, and stochastic differential equations [12]. He also developed homotopy perturbation method by combining two techniques: regular homotopy and perturbation. Taking full advantage of the standard homotopy and perturbation methods, homotopy perturbation method was developed. A perturbed model is a mathematical model with a minimal number of parameters. An unperturbed or reduced model is the equivalent degenerate model. From a statistical standpoint, the model is singularly perturbed if the small parameter influences the highest-order differential coefficient. The solutions to these equations have a multi-scale nature. There are narrow areas where the solution switches quickly and has steep gradients. Using variational iteration method and homotopy perturbation method a broad range of functional equations have been studied. Thus, solution is given in an infinite series in these methods, which typically leads to an accurate solution.
We look at two-point boundary value problems that are singularly perturbed

$$\epsilon y''(t) + p(t)y'(t) + q(t)y(t) = f(t), \quad t \in [a, b] \tag{1.0.3}$$

with boundary conditions

$$y(a) = \gamma \quad y(b) = \alpha \tag{1.0.4}$$

Where γ and α are real constants whereas ϵ is small positive parameter
$(0 < \epsilon \leq 1)$. On $[a, b]$, we conclude $p(t), q(t), , f(t)$ are the continuously differentiable functions. The Fluid mechanics, the chemical reactor theory, the reaction-diffusion processes, and the geophysics are all examples of two-point boundary value problems (1.0.3 ,1.0.4) . A small parameter is multiplied to the highest derivative in these problems. In certain areas, the solution changes instantly, and in others, it changes slowly. In thin transition boundary or interior layers where solution can shift quickly, but the solution behave consistently and change very slowly away from the layers. Singular perturbation problems can be solved using a number of techniques. Our aim is to introduce He's variational iteration method as an alternative to existing methods for solving singularly perturbed two-point boundary value problems [13], and we use four numerical examples to demonstrate the method.
In chapter 1, we are going to apply steady state approximation in Chemical kinetics. The steady state approximation, also known as the stationary state approximation, entails setting the rate of change of a reaction intermediate in a reaction mechanism

to zero in order to simplify the kinetic equations by making the rate of formation equal to the rate of destruction.
The overall chemical change is caused by a reaction mechanism, which is a step-by-step sequence of elementary reactions.
In practise, it suffices if the rates of formation and destruction are nearly equal, implying that the net rate of variation of the intermediate's concentration is small compared to the rates of formation and destruction, and the intermediate's concentration varies slowly.
If the convective coupling matrix B is diagonal, the system is weakly coupled, and the system is coupled only through the lower-order reaction terms.
For a class of singularly perturbed weakly coupled systems, we use an efficient asymptotic-numerical method (q-stage Runge-kutta method, discussed in Chapter 3).
In chapter 4, we have discussed Variational iteration method for solving ordinary differential equations and partial differential equation.

Key words: Steady state approximation, singular perturbation, convection diffusion equations, q-stage runge kutta method

Chapter 2

Singular Perturbation Theory in Chemical Kinetics

2.1 Introduction

The Michaelis-Menten mechanism is given as follows-

$$A + B \rightarrow C \rightarrow A + D \tag{2.1.1}$$

Here,
A is the enzyme, B represents the substrate, D denotes the product and C is the intermediate(Enzyme-Substarte Complex)

Enzymes are highly effective catalyst. As a result, we expect that the conversion of C to enzyme and product is considered to be a faster process than the formation of the enzyme-substarte complex.

Now, using steady state approximation,

$$\text{Rate of formation} = \text{Rate of disappearance}$$

$$k_1[B][A] = k_{-1}[AB] + k_2[AB]$$

The rates of formation and degradation are nearly the same in steady state.
Here, k_1, k_{-1} and k_2 are rate constants. k_1 has different units as compared to k_{-1} and k_2. We need to remove units from the equation, so that we might be able to make a comparison in the processes taking place in the mechanism.

2.2 Equations and Results

The mechanism consists of the following rate equations-

$$\begin{aligned}
\frac{dA}{dt} &= -k_1AB + k_{-1}C + k_2C \\
\frac{dB}{dt} &= -k_1AB + k_{-1}C \\
\frac{dC}{dt} &= k_1AB - k_{-1}C - k_2C \\
\frac{dD}{dt} &= k_2C
\end{aligned}$$

Since, the enzymes is unchanges throughout the reaction, we express the total enzyme concentration as a sum of enzyme-substarte complex.

The follwing two equations are given as a result of mass-conservation-

$$\begin{aligned}
A_0 &= A + C \\
B_0 &= B + C + D
\end{aligned}$$

These conservation relations are consequences of the differential equations-

$$\begin{aligned}
\frac{dA}{dt} + \frac{dC}{dt} &= k_{-1}C - k_1AB + k_2C + k_1AB - k_{-1}C - k_2C \\
\frac{dB}{dt} + \frac{dC}{dt} + \frac{dD}{dt} &= k_{-1}C - k{-}1AB + k_1AB - k_{-1}C - k_2C + k_2C
\end{aligned}$$

Both of these equations give zero.

Therefore, on substitution, we obtain,

$$\frac{dB}{dt} = -k_1(A_0 - C)B + k_{-1}C \tag{2.2.1}$$

$$\frac{dC}{dt} = k_1(A_0 - C)B - (k_{-1} + k_2)C \tag{2.2.2}$$

The SSA is based on the idea that the concentrations of highly reactive intermediates like C eventually reach a rough equilibrium between production and destruction, resulting in a small net rate of change of C.

In order to make this balance evident, we first have to select new measurement scales for each of the variables(B, C and t) that are all of unit magnitude.

Defining new variables b, c and τ by-

$$b = \frac{B}{\tilde{B}}, \tag{2.2.3}$$

$$c = \frac{C}{\tilde{C}}, \tag{2.2.4}$$

$$\tau = \frac{t}{\tilde{t}} \tag{2.2.5}$$

Here, the tilde symbol denotes a measurement scale with the same units as the corresponding variable, implying that the new variables obtained are of unit magnitude.

Consider,

$B(0) = B_0$, therefore, $B = B_0$ and $(B, C) = (B_0, 0)$.

$C(t)$ will be maximum when $\frac{dC}{dt} = 0$. Therefore, we have,

$$\begin{aligned} k_1(A_0 - C)B - (k_{-1} + k_2)C &= 0 \\ k_1 A_0 B - k_1 C B &= k_{-1}C + k_2 C \\ k_1 E_0 S &= k_{-1}C + k_1 C S + k_2 C \\ &= (k_{-1} + k_1 S + k_2)C \end{aligned}$$

Or $C = \frac{k_1 A_0 B}{k_{-1} + k_2 + k_1 B}$

Before much B has been consumed, we expect the maximum in C to be reached early in the reaction.

Therefore, substituting $B = B_0$, we get,

$$\begin{aligned} C &= \frac{k_1 A_0 B_0}{k_1 B_0 + k_{-1} + k_2} \\ &= \frac{A_0 B_0}{B_0 + k_B} \end{aligned}$$

where $k_B = \frac{k_{-1} + k_2}{k_1}$ is the Michaelis constant.
Now,

$$\begin{aligned} B &= bB_0 \\ C &= c\tilde{C} = c\frac{A_0 B_0}{B_0 + k_B} \\ and, t &= \tau\tilde{t} \end{aligned}$$

$$\frac{dB}{dt} = \frac{d(bB_0)}{d(\tau\tilde{t})} = \frac{B_0 db}{\tilde{t} d\tau}$$

We know that,

$$\begin{aligned}
\frac{dB}{dt} &= -k_1(A_0 - C)B + k_{-1}C \\
\implies \frac{B_0}{\tilde{t}}\frac{db}{d\tau} &= -k_1(A_0 - C)B + k_{-1}C \\
&= -k_1 b B_0 \left\{ A_0 - c\frac{A_0 B_0}{B_0 + k_B} \right\} + k_{-1}\frac{cA_0 B_0}{B_0 + k_B} \\
\frac{db}{d\tau} &= \tilde{t}\left[\left(-k_1 A_0 b - \frac{cB_0}{B_0 + k_B} \right) + k_{-1}\frac{cA_0}{B_0 + k_B} \right] \\
&= \tilde{t}\left[-k_1 A_0 b \left(1 - \frac{cB_0}{B_0 + k_B} \right) + k_{-1}\frac{cA_0}{B_0 + k_B} \right]
\end{aligned}$$

Similarly, for C,

$$\begin{aligned}
\frac{dC}{dt} &= k_1(A_0 - C)B - (k_{-1} + k_2)C \\
\frac{d(c\tilde{C})}{d(\tau\tilde{t})} &= k_1(A_0 - C)B - (k_{-1} + k_2)C \\
\frac{\tilde{C}}{\tilde{t}}\frac{dc}{d\tau} &= \left[k_1 b B_0 \left(A_0 - c\frac{A_0 B_0}{B_0 + k_B} \right) \right] - \left[(k_{-1} + k_2)c\frac{A_0 B_0}{B_0 + k_B} \right]
\end{aligned}$$

$$\frac{\tilde{C}}{\tilde{t}}\frac{dc}{d\tau} = A_0\left[k_1 b B_0\left(1 - c\frac{A_0}{B_0+k_B}\right) - \left((k_{-1}+k_2)c\frac{B_0}{B_0+k_B}\right)\right] \quad (2.2.6)$$

$$\frac{dc}{d\tau}\frac{A_0 B_0}{B_0+k_B} = \tilde{t}A_0 B_0\left[k_1 b\left(1 - c\frac{B_0}{B_0+k_B}\right) - \left((k_{-1}+k_2)\frac{c}{B_0+k_B}\right)\right]$$

$$\frac{dc}{d\tau} = (B_0+k_B)\tilde{t}\left[k_1 b\left(1 - c\frac{B_0}{B_0+k_B}\right) - \left(k_B k_1\frac{c}{B_0+k_B}\right)\right]$$

$$= \tilde{t}k_1(B_0+k_B)\left[b\left(1 - c\frac{B_0}{B_0+k_B}\right) - \left(k_B\frac{c}{B_0+k_B}\right)\right]$$

Let, $\alpha = \frac{B_0}{B_0+k_B}$, where α is a parameter between 0 and 1.

Our two rate equations become :-

$$\frac{db}{d\tau} = \tilde{t}\left[-k_1 A_0 b\left(1 - c\frac{B_0}{B_0+k_B}\right) + k_{-1}c\frac{A_0}{B_0+k_B}\right] \quad (2.2.7)$$

$$\frac{dc}{d\tau} = \tilde{t}k_1(B_0+k_B)[b(1-\alpha c) - c(1-\alpha)] \quad (2.2.8)$$

The terms in the square brackets of the second of these equations are now clearly balanced. Each is made up of a product of quantities which are of unit magnitude.

Now, we are left to decide $\tilde{t}$.

The back reaction $C \to A+B$ is not particularly important for many enzymes. Under these circumstances, the first term $-k_1 A_0 b(1-\alpha c)\tilde{t}$ should dominate $\frac{db}{d\tau}$.

This implies that we should use $\tilde{t}^{-1} = k_1 A_0$, to achieve the desired balance between the derivative and its term. In other words, we expect $(k_1 A_0)^{-1}$ to represent a slow time scale that controls the rate at which the substarte is consumed.

If the above hypothesis does not hold true for a specific enzyme, a different choice for $\tilde{t}$ should be made.

In any case, our choice leads to -

$$\begin{aligned} \frac{db}{d\tau} &= -b(1-\alpha c) + c\frac{k_{-1}}{k_{-1}+k_2}(1-\alpha) \\ \frac{dc}{d\tau} &= \frac{B_0+k_B}{A_0} b(1-\alpha c) - c(1-\alpha). \end{aligned}$$

We define,

$$\begin{aligned} \beta &= \frac{k_{-1}}{k_{-1}+k_2}, \\ and \quad \mu &= \frac{A_0}{B_0+k_B} \end{aligned}$$

Then we have

$$\frac{db}{d\tau} = -b(1-\alpha c) + \beta c(1-\alpha), \tag{2.2.9}$$

$$and \quad \mu\frac{dc}{d\tau} = b(1-\alpha c) - c(1-\alpha). \tag{2.2.10}$$

2.3 Conclusion

It is frequently stated that the Steady state approximation holds true when the enzyme concentration (μ) is small. If μ is negligible, then our differential equation (2.2.11) is close to the algebraic equation

$$0 \approx b(1 - \alpha c) - c(1 - \alpha) \tag{2.3.1}$$

i.e. the SSA.

A small value of μ implies that

$$A_0 << B_0 + k_B$$

This implies that the SSA will hold true under this condition. it is to be noted that this is a sufficient, but not a necessary condition. If time scaling is done in some other way, we discover other conditions which lead to the validity of the SSA.

We began with two differential equations (2.2.1) and (2.2.2). We discovered that one of the differential equations degenerates to an algebraic equation.

The equation (2.2.8) requires the values of both b and c to specify the initial condition, while the equation (2.2.9) requires only a value for b, the value of c being computed from equation (2.3.1).

Here, we considered the limit $\mu \to 0$. In practice, this limit can never be reached. Such problems are called singular perturbation problems [5].

Chapter 3

Singular Perturbation Theory in Convection-Diffusion Equations

3.1 Introduction

We are already familiar with differentail equations in which at the highest derivatives there is a small parameter. These type of equations are known as singularly perturbed differential equations. To obtain the solution of singularly perturbed systems, an efficient asymptotic-numerical method is presented in this chapter. A weakly coupled convection dominated system of m-equations is analyzed.

$$\begin{aligned} \mathcal{L}u := \epsilon u'' + Eu' + Su &= f \\ u(0) = 0, u(1) = 0 \ \ in \ \ \Omega &= (0,1) \end{aligned} \tag{3.1.1}$$

where $\epsilon = diag(\epsilon_1, ..., \epsilon_m)$, the permutation parameters $0 < \epsilon_i << 1$ for $1 = 1, ..., m$ and $u = (u_1, ..., u_m) \in (C^2(\Omega) \cap C(\bar{\Omega}))^m$.
The problem (3.1.1) is considered to be the most widespread fundamental sub-problem in science and engineering.

Now, $E = diag(e_1, ..., e_m)$ represents the convective matrix, $S = (s_{ij})_{m \times m}$ is the coupling matrix and the source vector $f = (f_1, ..., f_m)^T$ are assumed to be sufficiently smooth on $\bar{\Omega}$.
Moreover, for each i, j
$e_i > 0, s_{ii} \leq 0$ and $s_{ij} \geq 0 \ \forall i \neq j$.

The ith component of the system (3.1.1) is given by -

$$\epsilon_i u_i'' + e_i(x)u_i' + \sum_{j=1}^{m} s_{ij}(x)u_j = f_i(x) \qquad (3.1.2)$$
$$u_i(0) = 0, u_i(1) = 0$$

in $\Omega = (0,1)$ for $i = 1, 2, ..., m$.

For $x \in \Omega$, it can be transformed to

$$\epsilon_i u_i'' + e_i(x)u_i' + s_{ii}(x)u_i = f(x) - \sum_{j=1, j\neq i}^{m} s_{ij}(x)u_j \qquad (3.1.3)$$

$$u_i(0) = 0, u_i(1) = 0, u_i(1) = 0$$

The system is referred as to be coupled weakly if the convective coupling matrix S is daigonal, therefore, only through the lower-order reaction terms the given system is coupled.
Using the following theorems and Results, it follows that

$$||u_i|| + \sum_{j=1, j\neq i}^{m} Z_{ij}||u_j|| \leq min\left\{\left\|\frac{f_i}{s_{ii}}, \frac{f_i}{a_i}\right\|\right\} \qquad (3.1.4)$$

where the matrix $\boldsymbol{Z} = (Z_{ij})_{m\times m}$ is such that $Z_{ii} = 1$ and $Z_{ij} = -min\left\|\frac{s_{ij}}{s_{ii}}, \frac{s_{ij}}{a_i}\right\|$, for $i \neq j$, where $\|.\|$ denotes the maximum norm over Ω.

3.1.1 Theorems and Results

Now, we make use of the test function $T : x \to T(x) = 1 - x$, and observe that the differential operator $\mathcal{L}$ in $\mathcal{L}u := -\epsilon u'' - bu' + cu = f$ satisfies the hypothesis of lemma(3.1.1) , because $\mathcal{L}T \geq \beta > 0$.

Consequently, it obeys a comparison princuple and in association with lemma(3.1.2), we get,

$$|u(x)| \leq max\left\{|Z_0|, |Z_1|\right\} + (1-x)\left\|\frac{f}{b}\right\|_\infty$$

$$\text{for } x \in [0,1]$$

$$\implies \|v\|_\infty \leq \left\|\frac{\mathcal{L}v}{b}\right\|_\infty \;\forall\; v \in C^2[0,1] \;\; with \;\; v(0) = v(1) = 0.$$

Alternatively, if $c > 0$ on $[0,1]$, then lemma(3.1.2) with $Z = 1$ yields

$$\|v\|_\infty \leq \left\|\frac{\mathcal{L}v}{c}\right\|_\infty \;\forall\; v \in C^2[0,1] \;\; with \;\; v(0) = v(1) = 0 \tag{3.1.5}$$

Theorem 3.1.1 *Set $\beta^* = \epsilon_d(\mu_1 - \mu_0)$. The operator $\mathcal{L}$ satisfies*

$\|v\|_\infty \leq \left\|\frac{\mathcal{L}v}{c}\right\|_\infty \forall v \in \dot{X}^{1,\infty}(0,1) \cap X^{2,\infty}(0,1)$

$\|v\|_\infty \leq \frac{2}{\beta^*}\|\mathcal{L}v\|_1 \forall v \in \dot{X}^{1,\infty}(0,1) \cap X^{2,1}(0,1)$

$\|v'\|_1 \leq \frac{2}{\beta^*} v \|\mathcal{L}v\|_1 \forall v \in \dot{X}^{1,\infty}(0,1) \cap X^{2,1}(0,1)$

and

$|||v|||_\infty \leq \|\mathcal{L}v\|_{-1,\infty} \forall v \in X^{1,\infty}(0,1).$

Lemma 3.1.1 *let $\exists$ a function $T \in C^2(0,1) \cap C[0,1]$ with $T > 0$ on $[0,1]$ and $\mathcal{L}T > 0$ in $(0,1)$. Then the operator $\mathcal{L}$ is inverse monotone.*

Lemma 3.1.2 *Suppose $\exists$ a function $T \in C^2(0,1) \cap C[0,1]$ with $T > 0$ on $[0,1]$ and $\mathcal{L}T > 0$ in $(0,1)$. Then for any function $v \in C^2(0,1) \cap C[0,1]$ with $v(0) = v(1) = 0$.*

$$|v| \leq T \left\| \frac{\mathcal{L}v}{\mathcal{L}T} \right\|_\infty \text{ in } [0,1]$$

we have

$$\epsilon_i u_i'' + e_i u_i' + \sum_{j=1}^{m} s_{ij} u_j = f_i$$

Rewriting above equation as

$$\epsilon_i u_i'' + e_i u_i' = f_i - \sum_{j=1}^{m} s_{ij} u_j$$

Then using (3.1.5), we get,

$$\|u_i\| + \sum_{j=1, j\neq i}^{m} Z_{ij} \|u_j\| \leq min \left\{ \left\| \frac{f_i}{s_{ii}} \right\|, \left\| \frac{f_i}{e_i} \right\| \right\}, i = 1, ..., m$$

where the matrix $\tau = (Z_{ij})_{m\times m}$ is so that $Z_{ii} = 1, Z_{ij} = -min \left\{ \left\| \frac{s_{ij}}{s_{ii}} \right\|, \left\| \frac{s_{ij}}{e_i} \right\| \right\}$ and $\|.\|$ denotes the maximum norm over Ω.
The uniqueness of the solution and maximum norm stability of the operator $\mathcal{L}$ follows from-

Theorem 3.1.2 *Assume that the matrix A has non-negative diagonal entries. Suppose also that all entries of A lie in $C[0,1]$. Assume that $\tau(A,s)$ is inverse monotone. Then for $k = 1, ..., l$ one has*

$$\|v_k\|_\infty \leq \sum_{m=1}^{l} (\tilde{\tau})_{km} min \left\{ \left\| \frac{(\mathcal{L}v)_m}{a_{mm}} \right\|_\infty, \left\| \frac{(\mathcal{L}v)_m}{s_m} \right\|_\infty \right\} \tag{3.1.6}$$

for any function $v = (v_1, ..., v_l)^T \in (C^2(0,1) \cap C[0,1])^l$ with $v(0) = v(1) = 0$.

Under the hypothesis of the above theorem, the boundary value problem has a unique solution u, and $\|u\|_\infty \leq C \|f\|_\infty$ for some constant C.

Theorem 3.1.3 *If $a = (a_1, a_2, ..., a_m)$ is the solution of problem* (3.1.1)*, then*

$$\left\|a_i^{(k)}\right\| = \begin{cases} O\left(\varepsilon_i^{-k}\right) & \text{if } k = 1, 2, \\ O\left(\varepsilon_p^{-k}\right) & \text{if } k \geq 3, \end{cases}$$

for $i = 1, ..., m$, where $\epsilon_p := min_{1 \leq i \leq m} \epsilon_i$.

Proof:- We begin by stating the following theorem,

Theorem 3.1.4 *If $e(x) > 0, s(x)$ and $c(x)$ are sufficiently smooth functions, then the solution $a(x)$ of the problem* (3.1.1) *satisfies $\left|a^{(i)}(x)\right| \leq C\left\{1 + \epsilon^{-i} exp\left(\underline{a}\frac{x}{\epsilon}\right)\right\}$, where C is an arbitrary constant. [3]*

Using the statement of the above theorem, we get the required bounds for $k = 1, 2$. Differentiating (3.1.2) for $k = 3$, we obtain,

$$\begin{aligned} \|a_i'''\| &\leq \frac{\|e_i\|}{\epsilon_i}\|a_i''\| + \frac{\|e_i\|}{\epsilon_i}\|a_i'\| + \frac{\|f_i'\|}{\epsilon_i} + \sum_{j=1}^{m}\left(\frac{\|s_{ij}\|}{\epsilon_i}\|a_j\| + \frac{\|s_{ij}\|}{\epsilon_i}\left\|a_j'\right\|\right) \\ &\leq C\left(\epsilon_i^{-3} + \epsilon_i^{-2} + \epsilon_i^{-1} + \sum_{j=1}^{m}\left(\epsilon_i^{-1} + \epsilon_i^{-1}\epsilon_j^{-1}\right)\right) \\ &\leq C\left(\epsilon_p^{-3} + \epsilon_p^{-2} + \epsilon_p^{-1} + \sum_{j=1}^{m}\left(\epsilon_i^{-3}\right)\right) \\ &\leq C\left(\epsilon_p^{-3}\right), \end{aligned}$$

where $\epsilon_p := min_{1 \leq i \leq m} \epsilon_i$. for $k \geq 3$, on further differentiation we obtain the required result.

3.2 Solution methodology

Breaking the solution a of (3.1.1) into regular(r) and singular(s) components as $a = r + s$. We then expand the regular component r in the perturbation series of the form

$$r = r_0 + \epsilon r_1 + \epsilon^2 r_2 + ... + \epsilon^k r_k + \epsilon^{k+1} R(x, \epsilon), \tag{3.2.1}$$

where for each $i = 0, 1, ..., k, r_i = (r_{i1}, ..., r_{im})^T$ represents a column vector. Moreover, the regular component r_i satisfies the following system of IVPs in Ω :

$$\begin{aligned} Ev_0' + Sv_0 &= f & (3.2.2)\\ r_0(1) &= 0 \\ Ev_i' + Sv_i &= -r_{i-1} & (3.2.3)\\ r_i(1) &= 0 \end{aligned}$$

For $i = 1, 2, ..., k$ and

$$\begin{aligned} \epsilon V'' + RV' + SV &= -v_k'' & (3.2.4)\\ V(0) &= 0, \\ V(1) &= 0. \end{aligned}$$

Let $\bar{\Omega}^N := x_n : x_n = nh; n = 0, 1, .., N$ be the discrete domain with equally spaced notes on $\bar{\Omega}$, where N is a natural number. Here, $h = \frac{1}{N}$ represents uniform mesh diameter. Following that, the result below provides a high-order approximation of the solution's smooth component.

Theorem 3.2.1 *Suppose explicit q-stage Runge-Kutta method is used to approximate the solutions of (3.2.2) and (3.2.3) and if* $(v_i)_n$ *be the approximated equivalent of* v_i *for* $i = 0, 1, ..., k$, *then*

$$\|v(x_n) - (v)_n\| = \mathcal{O}(h_p); p \leq q, \tag{3.2.5}$$

where

$$(v)_n = (v_0)_n + \epsilon(v_1)_n + \epsilon^2(v_2)_n + ... + \epsilon^k(v_k)^n \tag{3.2.6}$$

Proof:- The system of IVPs (3.2.2) can be written as

$$v_0' = F_0(x, v_0), v_0(1) = 0 \tag{3.2.7}$$

where $F_0 : \Omega \times \mathcal{R}^m \to \mathcal{R}^m$ is defined as $F_0(x, v_0) = -A^{-1}Bv_0 + A^{-1}f$. The system (3.2.7) has a unique solution because the matrices A, B, and f are continuous on $\bar{\Omega}$ and $|A| \neq 0$.
Because the degenerate system does not satisfy the condition at $x = 0$, its contribution to the (3.1.1) solution is limited to those values of x that are not $x = 0$. As a result, the solution is the same as the outer solution. Using the explicit q-stage Runge-Kutta method on (3.2.7):

$$(v_0)_{n+1} = (v_0)_n - h\sum_{r=1}^{q} w_r^0 k_r^0, \tag{3.2.8}$$

where

$$k_r^0 = F_0(x_n - \alpha_r^0 h, (v_0)_n - h\sum_{s=1}^{q} \beta_{rs}^0 k_s^0), \beta_{rs}^0 = 0 for s \geq r and \sum_{s=1}^{q} \beta_{rs}^0 = \alpha_r^0 \tag{3.2.9}$$

for $r = 1, ..., q$.

$p(\zeta) = \zeta - 1$ is the corresponding characteristic polynomial. Clearly, $\rho(1) = 0$ and the root condition is satisfied by the polynomial. Furthermore, let the $\sum_{r=1}^{q} w_r^0 = 1$ to meet the q-stage Runge-Kutta method's stability and consistency requirements.

Consequently, Lax-Ritchmayer theorem(A consistent finite difference scheme for a partial differential equation for which the initial-value problem is well posed is convergent if and only if stable) leads to converegence. It is easy to follow from [3] that

$$\|v_0(x_n) - (v_0)_n\| = \mathcal{O}(h^p); p \leq q \tag{3.2.10}$$

3.3 q-Stage Runge Kutta Method

Consider the initial value problem (IVP)

$$y' = f(x, y), \quad y(a) = \eta, f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m.$$

The q-stage Runge Kutta method for the above IVP is defined as

$$y_{n+1} = y_n + h \sum_{i=1}^{q} b_i k_i,$$

where h is the step size and

$$k_i = f\left(x_n + c_i h, y_n + h \sum_{j=1}^{q} a_{ij} k_j\right), \qquad i = 1, 2, \ldots, q.$$

$$c_i = \sum_{j=1}^{q} a_{ij} k_j, \qquad i = 1, 2, \ldots, q.$$

The method is said to be Explicit Runge Kutta method if $a_{ij} = 0$ for all $j \geq i$ and the method is said to be implicit Runge-Kutta method if $a_{ij} \neq 0$ for some $j > i$.

Stability of Explicit Runge-Kutta Method

The stability of a numerical method is observed when a minor change in the initial conditions or data, results into a correspondingly minor change in subsequent approximations. A Runge-Kutta method is said to satisfy root condition if all of the roots of the characteristic polynomial have modulus less than or equal to unity and those of modulus unity are simple.

In other words, we can say that all the root being complex must lie in or on the unit circle, and there must be no multiple roots on the unit circle. The significance of root condition lies in the fact that it is the necessary and sufficient condition for the stability of the RK-methods.

Consistency of Runge-Kutta method

A numerical method is said to be consistent if the successive local truncation error tends to zero as the step size tends to zero.

The necessary and sufficient condition for a general Runge-Kutta method to be consistent is that

$$\sum_{i=1}^{q} b_i = 1.$$

	u_1		u_2	
x	Exact	Approximate	Exact	Approximate
0.00	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.01	-0.009800000	-0.0098999974	1.9700000000	1.9799999984
0.02	-0.019500000	-0.0195999974	1.9500000000	1.9599999984
0.03	-0.029000000	-0.0290999995	1.9300000000	1.9399999984
0.04	-0.038300000	-0.0383999983	1.9200000000	1.9199999984
0.05	-0.047400000	-0.0474999986	1.9000000000	1.8999999984
0.06	-0.056300000	-0.0563999976	1.8700000000	1.8799999984
0.07	-0.065200000	-0.0650999984	1.8600000000	1.8599999984
0.08	-0.073500000	-0.0735999976	1.8300000000	1.8399999984
0.09	-0.081800000	-0.0818999977	1.8100000000	1.8199999984
0.10	-0.091000000	-0.0899999979	1.8000000000	1.7999999984
0.20	-0.162000000	-0.1599999993	1.7000000000	1.5999999984
0.30	-0.220000000	-0.2099999979	1.5000000000	1.3999999984
0.40	-0.230000000	-0.2399999986	1.3000000000	1.1999999985
0.50	-0.240000000	-0.2499999987	1.0000000000	0.9999999985
0.60	-0.230000000	-0.2399999979	0.8000000000	0.7999999985
0.70	-0.220000000	-0.2099999990	0.6000000000	0.5999999986
0.80	-0.150000000	-0.1599999994	0.4000000000	0.3999999987
0.90	-0.090000000	-0.0899999995	0.2000000000	0.1999999988
1.00	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Table 3.1: Numerical computations for above example with $\epsilon_1 = \epsilon_2 = 10^{-20}$, $N = 100$.

3.4 Numerical Results and Examples

Consider the weakly coupled system below

$$\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \begin{pmatrix} u_1'' \\ u_2'' \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} + \begin{pmatrix} -5 & 4 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \tag{3.4.1}$$

with homogeneous boundary conditions. Here, $f_1(x)$ and $f_2(x)$ are those functions of x for which exact solution of the problem comes

$$u_1(x) = \frac{1 - e^{-2x/\varepsilon_1}}{1 - e^{-2/\varepsilon_1}} - \frac{1 - e^{-3x/\varepsilon_2}}{1 - e^{-3/\varepsilon_2}} + x^2 - x; \;\; u_2(x) = \frac{1 - e^{-2x/\varepsilon_1}}{1 - e^{-2/\varepsilon_1}} + \frac{1 - e^{-3x/\varepsilon_2}}{1 - e^{-3/\varepsilon_2}} - 2x. \tag{3.4.2}$$

The following table represents the numerical computations for the above example.

3.5 Conclusion

In this chapter, we introduce the q-stage Runge Kutta method for a class of weakly coupled system that are singularly perturbed. We analyze a weakly coupled system of $m-$ equations. the solution of the system is decomposed into regualr and singular components and we move further in order to find our solution. The q-stage Runge Kutta method is an efficient asymptotic numerical method.

The numerical scheme is found to be robust in terms of the perturbation parameter, and it produces much better results than existing numerical schemes.[4].

This scheme gives more accurate solutions in comparison to other schemes. The method presented here is simple to implement and can easily be extended to even more general situations with a little tweaking. [4].

Chapter 4

Variational Iteration Method

A Chinese mathematician, Ji-huan, firstly introduced the variational iteration approach in 1999 which can be used to solve both partial as well as ordinary differential equations without taking use of prohibitive hypotheses that might alter the physical structure of the solution. For both weakly and strongly nonlinear equations, the variational iteration approach is the most efficient and convenient method .This approach is also very efficient on comparing with other methods like the adomian method, perturbation method, and so on. On solving, if an exact solution exists, this approach provides speedily convergent successive approximations of it ; or else, some type of approximations were used for numerical purposes. The limiting assumptions that are used to treat nonlinear terms are a flaw in current numerical techniques. For nonlinear operators, the VIM has no such unique criteria, like linearization, Adomian polynomials, and so on. Another significant benefit of the VIM approach is its ability to significantly reduce calculation size while retaining high numerical accuracy. Furthermore, this method's capacity allows it to handle a vast range of theoretical and computational applications in our real-world problems.
The VIM is an iterative method which is used for solving nonlinear ordinary and partial differential equations that relies on the use of the lagrange multiplier, restricted variations, and correction functional. The solution is given as series of iterates, and approach does not require the existence of small parameters in the differential equation. The non linearities do not have to be differentiable with respect to the dependent variable and its derivatives for the method to work. As like many perturbation method [13] as like other non linear analytical approaches , that approach need not rely on small parameters, and initially our guess could be easily selected even with some unknown parameters, allowing it to be used in a wide range of non-linear problems that do not need linearization or small perturbations.

4.1 Brief analysis of the method

We will go through the basic concepts, in this section that underpin the variational iteration process. Now,take a look at the following nonlinear equation:

$$\mathcal{L}y(s) \equiv L(y(s)) + N(y(s)) = f(s) \tag{4.1.1}$$

where L stands for linear, here, N for nonlinear, and $f(s)$ for the given analytic inhomogeneous term. By using variational theory, we can also write a correction functional in the following format:

$$y_{n+1}(s) = y_n(s) + \int_0^s \lambda(t)\{Ly_n(t) + N\tilde{y}_n(t) - f(t)\}dt, \;\; n \geq 0 \tag{4.1.2}$$

where λ stands for lagrange's multiplier that could be easily calculated using variational theory , integration by parts and liouville-Green transforms. Also, $y_n(x)$ is nth approximation of $y(x)$and $\tilde{y}_n(x)$ be the restricted variation which means , $\delta\tilde{y}_n = 0$ so, on the first step we will calculate the value of lagrange multiplier λ then select Ji-huan a appropriate intial function which satisfies the boundary conditions, y_0 then successive approximations y_n of the function y can be easily obtained through correction function (4.1.1). Therefore, exact solution of the problem (4.1.2) :

$$y(x) = lim_{n\to\infty} y_n(x)$$

4.2 Variational iteration method for Ordinary differential equations

In 2000 the variational iteration approach is used to solve autonomous ordinary differential equation . Momani et al.(2006) investigate the application of this approach to the Helmholtz equation. In Abdou et al.(2005), burger and coupled burger equation can be solved using this approach [12, 13] . To coupled Schrondinger KdV equations and shallow water equations which are presented by Abdou et al.(2005) is its main application [15]. Numerical methods are oftenly used since most differential equation donot have exact analytic solution ,such approaches are effective while solving nonlinear and linear issues, as it provides analytical solutions and has some advantages over traditional numerical methods. This approach was used to solve a system of ordinary differential equations by Biazar et al (2004) [15]. Applying Variational iteration method to Singularity perturbed problem;

Example 4.2.1 Consider the following two-point boundary problem:

$$\epsilon\frac{d^2y(t)}{dt^2} - 4\frac{dy(t)}{dt} = 0, \quad t \in (0,1); \quad y(0) = U, \;\; y(1) = V, \tag{4.2.1}$$

here ϵ,is small peturbation [10]. Then, with respect to root $(0, 4/\epsilon)$ of applicable characteristic polynomial, exact solution is given as

$$\begin{aligned} y(t) &= \frac{-V + U\exp(4/\epsilon)}{-1+\exp(4/\epsilon)} - \frac{U-V}{-1+\exp(4/\epsilon)}\exp(4t/\epsilon) \\ &= \frac{\exp(4t/\epsilon) - \exp(4/\epsilon)}{1-\exp(4/\epsilon)} \quad \textit{for U=1 and V=0.} \end{aligned} \tag{4.2.2}$$

The correction functional with respect to the equation (4.2.1) can be written as

$$\begin{aligned} y_{n+1}(t) &= y_n(t) + \int_0^t \lambda(s)\left[\epsilon\frac{d^2y_n(s)}{ds^2} - \frac{4dy_n(s)}{ds}\right]ds \\ &= y_n(t) - \epsilon\frac{d\lambda(s)}{ds}y_n(s)|_{s=0}^t - 4\lambda(s)y_n(s)|_{s=0}^t + \epsilon\lambda(s)\frac{dy_n(s)}{ds}|_{s=0}^t \\ &\quad + \int_0^t \left(\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\frac{d\lambda(s)}{ds}\right)y_n(s)ds. \end{aligned} \tag{4.2.3}$$

Proceed variation with respect to y_n and then creating correction functional (4.2.3) stationary, i.e., $\delta y_{n+1} = 0$:

$$\begin{aligned} \delta y_{n+1}(t) &= \left(1 - \epsilon\frac{d\lambda(s)}{ds} - 4\lambda(s)\right)_{s=t}\delta y_n(t) + \epsilon\lambda(s)|_{s=t}\delta y_n'(t) \\ &\quad + \int_0^t \left(\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\frac{d\lambda(s)}{ds}\right)\delta y_n(s)ds \\ &= 0. \end{aligned}$$

Therefore,we obtained the Euler-Lagrange equation

$$\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\frac{d\lambda(s)}{ds} = 0 \tag{4.2.4}$$

and the stationary conditions are

$$\left(1 - \epsilon\frac{d\lambda}{ds} - 4\lambda(s)\right)_{s=t} = 0 \quad \text{and} \quad \lambda(s)|_{s=t} = 0. \tag{4.2.5}$$

Consequently, Euler equation (4.2.4) combining with stationary conditions (4.2.5) gives

$$\lambda(s) = \frac{1}{4}\left(1 - \exp 4\left(\frac{t-s}{\epsilon}\right)\right).$$

Hence,we can write the variational iteration formulas as

$$y_{n+1}(t) = y_n(t) + \int_0^t \frac{1}{4}\left(1 - \exp 4\left(\frac{t-s}{\epsilon}\right)\right)\left(\epsilon\frac{d^2y(t)}{dt^2} - 4\frac{dy(t)}{dt}\right)ds. \tag{4.2.6}$$

We consider the linearly independent solutions of (4.2.1) to obtain iterate series and starts with initial approximation $y_0 = A + B\exp(4t)$, where c and d denotes free constants that can be set with the use of boundary conditions. Using formula (4.2.6), we have

$$\begin{aligned} y_1 &= A + B\exp(t) + B(\epsilon - 4)\frac{1}{4}\int_0^t \left(1 - \exp 4\left(\frac{t-s}{\epsilon}\right)\right)\exp(s)ds \\ &= A + B - B\frac{\epsilon}{4}\left(1 - \exp\left(\frac{4t}{\epsilon}\right)\right). \end{aligned}$$

If we inflict boundary conditions $U = 1$ and $V = 0$, and so, in such type the solution we obtained by variation iteration method at first iteration given as

$$y_1(t) = \frac{\exp(4t/\epsilon) - \exp(4/\epsilon)}{1 - \exp(4/\epsilon)},$$

which is indeed the exact solution (4.2.2).

Example 4.2.2 Consider the following boundary value problem

$$\epsilon\frac{d^2y(t)}{dt^2} + 4y(t) = 0, \quad t \in (0,1); \quad y(0) = U, \;\; y(1) = V, \tag{4.2.7}$$

Here ϵ is small peturbation [6]. And linearly independent solutions $sin(2/\sqrt{\epsilon})t$ and $cos(2/\sqrt{\epsilon})t$ correspond to the large imaginary roots of related characteristic polynomial then exact solution is:

$$\begin{aligned} y(t) &= acos(2/\sqrt{\epsilon})t + \frac{V - Ucos(2/\sqrt{\epsilon})}{sin(2/\sqrt{\epsilon})}.sin(2/\sqrt{\epsilon})t \\ &= cos(2/\sqrt{\epsilon})t + \frac{2 - cos(2/\sqrt{\epsilon})}{sin(2/\sqrt{\epsilon})}.sin(2/\sqrt{\epsilon})t \quad \text{for U=1 and V=2} \end{aligned} \tag{4.2.8}$$

Then correction functional corresponding to 4.2.6 is:

$$\begin{aligned} y_{n+1}(t) &= y_n(t) + \int_0^t \lambda(s)\left[\epsilon\frac{d^2y_n(s)}{ds^2} + 4y_n(s)\right]ds \\ &= y_n(t) - \epsilon\frac{d\lambda(s)}{ds}y_n(s)|_{s=0}^t + \epsilon\lambda(s)\frac{dy_n(s)}{ds}|_{s=0}^t \\ &\quad + \int_0^t \left(\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\lambda\right)y_n(s)ds. \end{aligned} \tag{4.2.9}$$

by imposing the variation and by considering the restricted variation i.e $\delta y_{n+1} = 0$ equation 4.2.9 becomes

$$\begin{aligned} \delta y_{n+1}(t) &= \left(1 - \epsilon\frac{d\lambda(s)}{ds}\right)\delta y_n(s) + \epsilon\lambda(s)|_{s=0}^t\delta y_n'(s) \\ &\quad + \int_0^t \left(\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\lambda\right)\delta y_n(s)ds. \\ &= 0 \end{aligned} \tag{4.2.10}$$

Therefore, we get Euler-Lagrange equation as

$$\epsilon\frac{d^2\lambda(s)}{ds^2} + 4\lambda(s) = 0 \tag{4.2.11}$$

and the stationary conditions obtained are

$$\left(1 - \epsilon\frac{d\lambda}{ds}\right)_{s=t} = 0 \quad \text{and} \quad \lambda(s)|_{s=t} = 0. \tag{4.2.12}$$

Consequently, (4.2.11) together with stationary conditions (4.2.12) yield

$$\lambda(s) = \frac{1}{\sqrt{\epsilon}} sin\left(\frac{2s-2t}{\sqrt{\epsilon}}\right).$$

As a result,we can write variational iteration formulas as

$$y_{n+1}(t) = y_n(t) + \int_0^t \frac{1}{\sqrt{\epsilon}} sin\left(\frac{2s-2t}{\sqrt{\epsilon}}\right)\left[\epsilon\frac{d^2y_n(s)}{ds^2} + 4y_n(s)\right]ds \tag{4.2.13}$$

We consider the linearly independent solutions of (4.2.6) to obtain iterate series. and starts with initial approximation $y_0 = C_1\ cos2t + C_2\ sin2t$, where C_1 and C_2 denotes free constants that can be determined using boundary conditions. Using formula (4.2.13), we have

$$\begin{aligned} y_1 &= C_1 cos2t + C_2 sin2t + \frac{4(1-\epsilon)}{\sqrt{\epsilon}}\int_0^t sin\left(\frac{2s-2t}{\sqrt{\epsilon}}\right)(C_1 cos2t + C_2 sin2t)ds \\ &= cos(2/\sqrt{\epsilon})t + \frac{2 - cos(2/\sqrt{\epsilon})}{sin(2/\sqrt{\epsilon})}.sin(2/\sqrt{\epsilon})t \quad \text{for a=1 and b=2} \end{aligned} \tag{4.2.14}$$

which indeed the exact solution (4.2.8)

4.2.1 Problem Description

Consider the following nonlinear singularly perturbed two-point boundary value problem [10]:

$$\begin{aligned} &\epsilon y'' = h(t, y(t), y'(t)); \quad t \in (0,1) \\ &y(0) = \beta, \quad y(1) = \alpha. \end{aligned} \tag{4.2.15}$$

When dealing with singularly perturbed boundary value problems, it's common to want results that guarantee the solution's survival and provide an estimate position of boundary layers that appear in the proof. Most of these results were obtained using a maximum theory statement. [15]. The solution is approximated in terms of values on interval's boundary within its interval existence. We will create a few such supporting results in the following discussion.

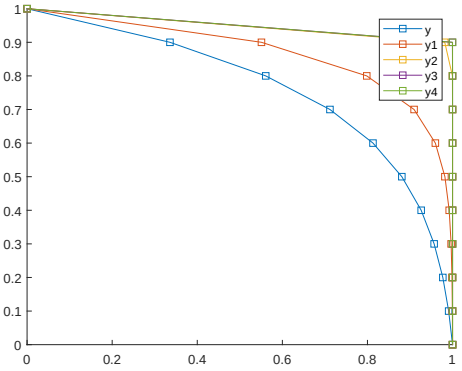

Figure 4.1: Numerical solution of example 4.2.1 for different values of ϵ.

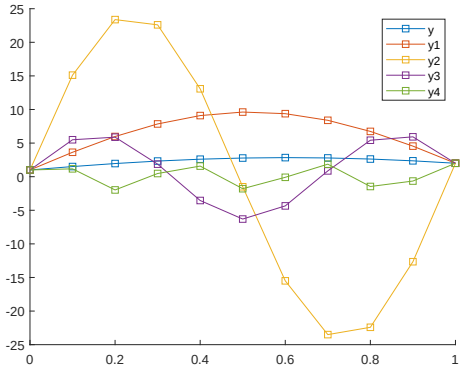

Figure 4.2: Numerical solution of example 4.2.2 for different values of ϵ.

4.2.2 Solution Methodology

We will try to understand using two possible cases separately [10]:

$$h(t,y(t),y'(t)) \;=\; \begin{cases} g(t,y(t),y'(t)) - p(t)y'(t) - q(t)y(t);\ p(t)\neq 0 & \text{and} \\ g(t,y(t),y'(t)), \end{cases} \tag{4.2.16}$$

i.e., The first one arises when the linear element of $h(t,y(t),y'(t))$ can be directly extracted, and the second case is otherwise.

Case-I:$h(t,y(t),y'(t)) = g(t,y(t),y'(t)) - p(t)y'(t) - q(t)y(t);\ p(t)\neq 0.$
So, we can write the correction functional corresponding to (4.2.15) as

$$\begin{aligned} y_{n+1}(t) &= y_n(t) + \int_0^t \lambda(\xi)\left(\epsilon y''(\xi) + p(\xi)y'(\xi) + q(\xi)y(\xi) - \tilde{g}(\xi,y,y')\right)d\xi \\ &= y_n(t) + \epsilon\left\{\lambda(\xi)y'(\xi)|_0^t - \int_0^t \frac{d\lambda(\xi)}{d\xi}y'(\xi)d\xi\right\} \\ &\quad + \left\{\lambda(\xi)p(\xi)y(\xi)|_0^t - \int_0^t\left(\frac{d\lambda(\xi)}{d\xi}p(\xi) + \lambda(\xi)\frac{dp(\xi)}{d\xi}\right)y(\xi)d\xi\right\} \\ &\quad + \int_0^t \lambda(\xi)q(\xi)y(\xi)d\xi - \int_0^t \lambda(\xi)\tilde{g}(\xi,y,y')d\xi \\ &= y_n(t) + \epsilon\left\{\lambda(\xi)y'(\xi)|_0^t - \left(\frac{d\lambda(\xi)}{d\xi}y(\xi)|_0^t - \int_0^t \frac{d^2\lambda(\xi)}{d\xi^2}y(\xi)\right)\right\} \\ &\quad + \left\{\lambda(\xi)p(\xi)y(\xi)|_0^t - \int_0^t\left(\frac{d\lambda(\xi)}{d\xi}p(\xi) + \lambda(\xi)\frac{dp(\xi)}{d\xi}\right)y(\xi)d\xi\right\} \\ &\quad + \int_0^t \lambda(\xi)q(\xi)y(\xi)d\xi - \int_0^t \lambda(\xi)\tilde{g}(\xi,y,y')d\xi \end{aligned} \tag{4.2.17}$$

Here $\lambda(\xi,t) := \lambda(\xi)$ is Lagrange multiplier, it should be determined and $\tilde{g}(\xi,y(\xi),y'(\xi))$ this function is denoting the restricted variation of nonlinear source term (i.e., $\delta\tilde{g}=0$). We will use variational theory to calculate Lagrange's multiplier, take variation w.r.t independent variable y_n (notice that $\delta y_n(0)=0$) and making the correctional functional (4.2.17) stationary, means.,$\delta y_{n+1}=0$:

$$
\begin{aligned}
\delta y_{n+1}(t) &= \delta y_n(t) + \epsilon\left\{\lambda(\xi)\delta y'(\xi)|_{\xi=t} - \left(\frac{d\lambda(\xi)}{d\xi}\delta y(\xi)|_{\xi=t} - \int_0^t \frac{d^2\lambda(\xi)}{d\xi^2}\delta y(\xi)\right)\right\} \\
&\quad + \left\{\lambda(\xi)p(\xi)\delta y(\xi)|_{\xi=t} - \int_0^t \left(\frac{d\lambda(\xi)}{d\xi}p(\xi) + \lambda(\xi)\frac{dp(\xi)}{d\xi}\right)\delta y(\xi)d\xi\right\} \\
&\quad + \int_0^t \lambda(\xi)q(\xi)\delta y(\xi)d\xi - \int_0^t \lambda(\xi)\delta\tilde{g}(\xi, y, y')d\xi \\
&= \left(1 - \epsilon\frac{d\lambda(\xi)}{d\xi} + p(\xi)\lambda(\xi)\right)\delta y_n(\xi)|_{\xi=t} + \epsilon\lambda(\xi)\delta\frac{dy_n(\xi)}{d\xi}|_{\xi=t} \\
&\quad + \int_0^t \left(\epsilon\frac{d^2\lambda(\xi)}{d\xi^2} - p(\xi)\frac{d\lambda(\xi)}{d\xi} + \left(q(\xi) + \frac{dp(\xi)}{d\xi}\right)\lambda(\xi)\right)\delta y_n(\xi)d\xi \\
&= 0. \qquad (4.2.18)
\end{aligned}
$$

Therefore, Euler-Lagrange's equation becomes

$$
\left.\begin{aligned}
\epsilon\tfrac{d^2\lambda(\xi)}{d\xi^2} - p(\xi)\tfrac{d\lambda(\xi)}{d\xi} + \left(q(\xi) + \tfrac{dp(\xi)}{d\xi}\right)\lambda(\xi) = 0 \\
\left(1 - \epsilon\tfrac{d\lambda(\xi)}{d\xi} + p(\xi)\lambda(\xi)\right)_{\xi=t} = 0 \\
\lambda(\xi)_{\xi=t} = 0
\end{aligned}\right\} \qquad (4.2.19)
$$

Here Lagrange multiplier (λ), obtained by the use of Liouville's green transformation. Defining Liouville-Green transformation x, $\phi(x)$ and $w(\xi)$ as follows

$$
\left.\begin{aligned}
x = \phi(\xi) = -\tfrac{1}{\epsilon}\textstyle\int p(\xi)d\xi \\
\psi(\xi) = \phi'(\xi) = -\tfrac{1}{\epsilon}p(\xi) \\
w(x) = \psi(\xi)\lambda(\xi).
\end{aligned}\right\} \qquad (4.2.20)
$$

It follows that

$$
\frac{d\lambda(\xi)}{d\xi} = \frac{\phi'(\xi)}{\psi(\xi)}\frac{dw}{dx} - \frac{\psi'(\xi)}{\psi^2(\xi)}w, \quad \text{and} \qquad (4.2.21a)
$$

$$
\frac{d^2\lambda(\xi)}{d\xi^2} = \frac{\phi'^2(\xi)}{\psi(\xi)}\frac{d^2w}{dx^2} + \left(\frac{\phi''(\xi)}{\psi(\xi)} - 2\frac{\phi'(\xi)\psi'(\xi)}{\psi^2(\xi)}\right)\frac{dw}{dx} - \left(\frac{\psi''(\xi)}{\psi^2(\xi)} - 2\frac{\psi'^2(\xi)}{\psi^3(\xi)}\right)w. \qquad (4.2.21b)
$$

Substitute (4.2.21) into (4.2.19), it gives

$$
\begin{aligned}
&\frac{d^2w}{dx^2} + \left(\frac{\phi''(\xi)}{\psi^2(\xi)} - 2\frac{\phi'(\xi)\psi'(\xi)}{\psi^3(\xi)} - \frac{p(\xi)\phi'(\xi)}{\epsilon\psi^2(\xi)}\right)\frac{dw}{dx} \\
&+ \left(\frac{q(\xi)}{\epsilon\psi^2(\xi)} - \frac{\psi''(\xi)}{\psi^3(\xi)} + 2\frac{\psi'^2(\xi)}{\psi^4(\xi)} + \frac{p(\xi)\psi'(\xi)}{\epsilon\psi^3(\xi)} + \frac{p'(\xi)}{\epsilon\psi^2(\xi)}\right)w = 0.
\end{aligned}
$$

$$
\frac{d^2w}{dx^2} + \frac{dw}{dx} = \epsilon\left(\mathcal{G}(\xi,\epsilon)w(x) - \mathcal{F}(\xi)\frac{dw}{dx}\right) \qquad (4.2.22)
$$

where

$$\mathcal{F}(\xi) = \frac{p'(\xi)}{p^2(\xi)} \text{ and } \mathcal{G}(\xi,\epsilon) = \left(\epsilon\frac{p''(\xi)}{p^3(\xi)} - 2\epsilon\frac{p'^2(\xi)}{p^3(\xi)} - \frac{2p'(\xi)+q(\xi)}{p^2(\xi)}\right).$$

Since $p(\cdot) \in C^2[0,1]$, $q(\cdot) \in C[0,1]$ and $\mathcal{F}(\cdot)$, $\mathcal{G}(\cdot,\epsilon)$ are bounded on $[0,1]$, we have

$$\epsilon\left(\mathcal{G}(\xi,\epsilon)w(x) - \mathcal{F}(\xi)\frac{dw}{dx}\right) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Therefore, (4.2.22) reduces to

$$\frac{d^2w}{dx^2} + \frac{dw}{dx} \approx 0$$

and hence

$$w(x) = C_1 + C_2\exp(-x). \tag{4.2.23}$$

In view of (4.2.20), (4.2.23) yields

$$\lambda(\xi,t) = -\frac{\epsilon}{p(\xi)}\left(C_1 + C_2\exp\left(\frac{1}{\epsilon}\int_t^\xi p(s)ds\right)\right). \tag{4.2.24}$$

where C_1 and C_2 are arbitrary constants . Consequently, boundary conditions yields

$$\lambda(\xi,t) = -\frac{1}{p(\xi))}\left(1 - \exp\left(\frac{1}{\epsilon}\int_t^\xi p(s)ds\right)\right). \tag{4.2.25}$$

As a result, iteration formula for (4.2.15)is given as

$$\begin{aligned} y_{n+1}(t) &= y_n(t) + \int_0^t -\frac{1}{p(\xi))}\left(1 - \exp\left(\frac{1}{\epsilon}\int_t^\xi p(s)ds\right)\right)\times \\ &\quad \left(\epsilon\frac{d^2y(\xi)}{d\xi^2} + p(\xi)\frac{dy(\xi)}{d\xi} + q(\xi)y(\xi) - f\left(\xi, u(\xi), \frac{dy(\xi)}{d\xi}\right)\right)d\xi \end{aligned} \tag{4.2.26}$$

Case-II: $h(t,y(t),y'(t)) = g(t,y(t),y'(t))$; **contain purely nonliner/implicit terms.**

The correction functional with respect to (4.2.15) is given as

$$\begin{aligned} y_{n+1}(t) &= y_n(t) + \int_0^t \lambda(\xi)\left(\epsilon y''(\xi) - \tilde{g}(\xi,y,y')\right)d\xi \\ &= y_n(t) + \epsilon\left\{\lambda(\xi)y'(\xi)|_0^t - \int_0^t \frac{d\lambda(\xi)}{d\xi}y'(\xi)d\xi\right\} - \int_0^t \lambda(\xi)\tilde{g}(\xi,y,y')d\xi \\ &= y_n(t) + \epsilon\left\{\lambda(\xi)y'(\xi)|_0^t - \left(\frac{d\lambda(\xi)}{d\xi}y(\xi)|_0^t - \int_0^t \frac{d^2\lambda(\xi)}{d\xi^2}y(\xi)\right)\right\} \\ &\quad - \int_0^t \lambda(\xi)\tilde{g}(\xi,y,y')d\xi \end{aligned} \tag{4.2.27}$$

Here $\lambda(\xi,t) := \lambda(\xi)$ is the Lagrange multiplier and $\tilde{g}(\xi, y(\xi), y'(\xi))$ denoting the restricted variation of the nonlinear source term. We will use variational theory in order to calculate Lagrange's multiplier, we will take variation w.r.t independent variable y_n (notice that $\delta y_n(0) = 0$) and $\delta y_{n+1} = 0$:

$$\begin{aligned}
\delta y_{n+1}(t) &= \delta y_n(t) + \epsilon \left\{ \lambda(\xi)\delta y'(\xi)|_0^t - \left(\frac{d\lambda(\xi)}{d\xi}\delta y(\xi)|_0^t - \int_0^t \frac{d^2\lambda(\xi)}{d\xi^2}\delta y(\xi) \right) \right\} \\
&\quad - \int_0^t \lambda(\xi)\delta\tilde{g}(\xi, y, y')d\xi \\
&= \left(1 - \epsilon\frac{d\lambda(\xi)}{d\xi}\right)_{\xi=t} \delta y(t) + \epsilon\lambda(\xi)|_{\xi=t}\delta y'(t) + \epsilon\int_0^t \frac{d^2\lambda(\xi)}{d\xi^2}\delta y(\xi)d\xi \\
&= 0. \qquad (4.2.28)
\end{aligned}$$

Therefore, Euler-Lagrange's equation becomes

$$\frac{d^2\lambda(\xi)}{d\xi^2} = 0; \quad \left(1 - \epsilon\frac{d\lambda(\xi)}{d\xi}\right)_{\xi=t} = 0, \quad \lambda(\xi)_{\xi=t} = 0$$

which in turn yields

$$\lambda = \frac{\xi - t}{\epsilon}.$$

As a result, iteration formula reads

$$y_{n+1}(t) = y_n(t) + \int_0^t \left(\frac{\xi - t}{\epsilon}\right) \left(\epsilon\frac{d^2y(\xi)}{d\xi^2} - g\left(\xi, u(\xi), \frac{dy(\xi)}{d\xi}\right)\right) d\xi. \quad (4.2.29)$$

4.3 Variational iteration method for Partial differential equations

For several years, nonlinear differential equations in engineering and applied mathematics have been the focus of intense research. In the analysis of very mechanic systems and other fields of research, partial differential equation systems are used. For instance, consider wave propagation. Current methods for solving these equations interface with sever computation when many systems must be solved. The Variational iteration method (VIM) is used to solve these equations . This is critical when designing an iteration function.
Applying variational iteration method to solve partial differential equations:

Example 4.3.1 Parabolic partial differential equation
Let us consider a problem [8]:

$$y_t = y_{xx} + e^{-x}(cost - sint) \qquad (4.3.1)$$

with initial conditions $y(x,0) = x$ and there are boundary conditions[14]

$$y(0,t) = sint, \quad y(1,t) = \frac{1+sint}{e}$$

Which is readily apparent to provide the exact solution

$$y(x,t) = x + e^{-x}(cost - sint) \tag{4.3.2}$$

The correction functional corresponding to 4.3.1 can be written as:

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t \lambda \left(\frac{\partial y_n(x,\xi)}{\partial \xi} - \frac{\partial^2 y_n(x,\xi)}{\partial x^2} - e^{-x}cost + e^{-x}sint \right) d\xi \tag{4.3.3}$$

where λ is the lagrange multiplier, which can be easily obtained using variational theory:

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \delta \int_0^t \lambda \left(\frac{\partial y_n(x,\xi)}{\partial \xi} - \frac{\partial^2 y_n(x,\xi)}{\partial x^2} - e^{-x}cost + e^{-x}sint \right) d\xi \tag{4.3.4}$$

Applying by parts ,we obtain $\lambda = -1$

$$\delta y_{n+1}(x,t) = \delta y_n(x,t) + \delta \int_0^t (-1) \left(\frac{\partial y_n(x,\xi)}{\partial \xi} - \frac{\partial^2 y_n(x,\xi)}{\partial x^2} - e^{-x}cost + e^{-x}sint \right) d\xi \tag{4.3.5}$$

Now,take arbitrary intial approximation which satisfying the intial condition

$$y_0(x,t) = x$$

substitute initial condition into (4.3.5)

$$y_1(x,t) = x - e^{-x} + e^{-x}sint + e^{-x}cost \tag{4.3.6}$$

similarly,

$$y_2(x,t) = x + 2e^{-x}sint - e^{-x}t \tag{4.3.7}$$

$$y_3(x,t) = x + e^{-x}sint + e^{-x} - e^{-x}cost - \frac{1}{2}e^{-x}t^2 \tag{4.3.8}$$

In the same way rest of the iterations can also be obtained

The outcomes of this analysis were compared to those of an exact solution and results are approximately same. In terms of accuracy and effectiveness, the variational iteration method was found to be effective and a strong mathematical tool for solving differential equations.

t	y_{exact}	y_{vim}
0.1	0.190333	0.190333
0.3	0.367396	0.367397
0.5	0.533801	0.531465
0.7	0.68292	0.674006
0.9	0.808783	0.794705

Table 4.1: Numerical results for $x = 0.1$

Example 4.3.2 Consider a non linear partial differential equation

$$\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial t^2} = u^2 - \left(\frac{\partial u}{\partial t}\right)^2 \tag{4.3.9}$$

with initial condition $u(t, 0) = e^t$ [9]. Determine the value of the nonlinear parabolic equation in equation using the variational iteration method.
The correction functional corresponding to 4.3.9

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda \left(\frac{\partial u_n(t,\xi)}{\partial \xi} - \frac{\partial^2 u_n(t,\xi)}{\partial t^2} - u_n^2(t,\xi) + \left(\frac{\partial u(t,\xi)}{\partial t}\right)^2 \right) d\xi \tag{4.3.10}$$

where λ is the lagrange multiplier, which can be easily obtained using variational theory:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^x \lambda \left(\frac{\partial u_n(t,\xi)}{\partial \xi} - \frac{\partial^2 u_n(t,\xi)}{\partial t^2} - u_n^2(t,\xi) + \left(\frac{\partial u(t,\xi)}{\partial t}\right)^2 \right) d\xi \tag{4.3.11}$$

Applying by parts ,we obtain $\lambda = -1$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^x (-1) \left(\frac{\partial u_n(t,\xi)}{\partial \xi} - \frac{\partial^2 u_n(t,\xi)}{\partial t^2} - u_n^2(t,\xi) + \left(\frac{\partial u(t,\xi)}{\partial t}\right)^2 \right) d\xi \tag{4.3.12}$$

Now,take arbitrary intial approximation which satisfying the intial condition

$$u(t, 0) = e^t$$

substitute initial condition into (4.3.12)

$$\begin{aligned} u_1(x,t) &= e^t(1+x), \\ u_2(x,t) &= e^t\left(1+x+\frac{x^2}{2!}\right), \\ u_3(x,t) &= e^t\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right), \\ &\vdots \\ u_n(x,t) &= e^x\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots+\frac{t^n}{n!}\right), \\ &\vdots \end{aligned}$$

Taking $n \to \infty$ gives us the exact solution to Equation 4.3.9, so we get

$$u_n(x,t) = e^x\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots+\frac{t^n}{n!}\cdots\right) \tag{4.3.13}$$

or

$$u(x,t) = e^{x+t} \tag{4.3.14}$$

The number of iterations used to solve Equation 4.3.9 decides its accuracy. Figures (a) and (b) show an intermediate comparison of the exact and approximate solutions for $n = 2$ and $n = 8$.

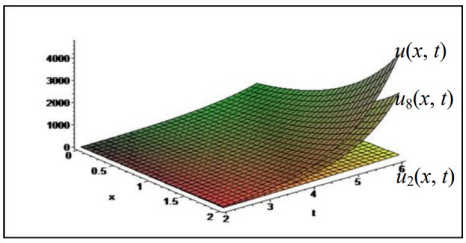

Figure 4.3: (a) Comparison of exact soluion with $0 \leq x \leq 2$

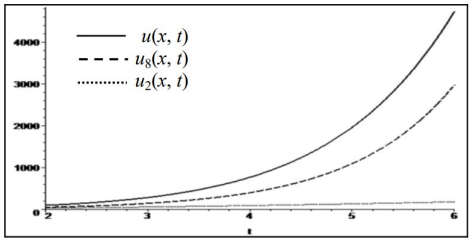

Figure 4.4: (b) Comparison of exact solution with $x = 2$

Comparision of the exact solution with u_2 and u_8 with $2 \leq t \leq 6$:
$(a)\ \ 0 \leq x \leq 2, (b)\ \ x = 2$

Figure 3.1 and 3.2 shows that the curve formed by $u_8(x,t)$ is more closely connected to $u(x,t)$ than the other curves. This means that using further iterations to complete an approximation will get you closer to the exact completion.

4.3.1 Problem Description

Consider the following parabolic partial differential equation,

$$\begin{aligned} &\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} - a(x,t)y = f(x,t) \\ &y(x,0) = \alpha, \quad y(0,t) = y(1,t) = \beta, \quad t, x \in (0,1) \end{aligned} \tag{4.3.15}$$

while dealing with such type of problems , it's common to want results that guarantee the solution's survival . Here we consider a general parabolic partial differential equation and lagrange multiplier can be calculated by the use of variational theory , integration by parts or liouville-green transform and we obtained a iterative formula for the given problem.

4.3.2 Solution Methodology

To solve eqn 4.3.15 using variational iteration method , the correctional function is defines as :

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t \lambda \left(\frac{\partial y_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{y}_n(x,\xi)}{\partial x^2} - a y_n - f(x,\xi) \right) d\xi \tag{4.3.16}$$

where λ is the lagrange multiplier that need to be determined and $\tilde{y}_n$ is the restricted variation i.e $\tilde{y}_n = 0$ and lagrange multiplier can be easily obtained using variational theory and to find the optimal value of λ we have:

$$\begin{aligned}
\delta y_{n+1}(x,t) &= \delta y_n(x,t) + \delta \int_0^t \lambda \left(\frac{\partial y_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{y}_n(x,\xi)}{\partial x^2} - a y_n - f(x,\xi) \right) d\xi \\
&= \delta y_n(x,t) + \delta \int_0^t \lambda \left(\frac{\partial y_n(x,\xi)}{\partial \xi} - a y_n \right) d\xi \\
&= \delta y_n + \left[\lambda \delta y_n |_0^t - \int_0^t \lambda' \delta y_n d\xi \right] - \int_0^t a \delta y_n d\xi \\
&= \delta y_n (1+\lambda) - \int_0^t \delta y_n (\lambda' + a) d\xi \\
&= 0
\end{aligned} \tag{4.3.17}$$

Therefore , the Euler lagrange equation becomes:

$$\left.\begin{aligned} \lambda'(\xi) + a = 0 \\ 1 + \lambda = 0 \end{aligned}\right\} \tag{4.3.18}$$

Therefore, we find the value of λ and write it as:

$$\begin{aligned}
\lambda'(\xi) &= -a \\
\lambda(\xi) &= a(t-\xi) - 1
\end{aligned}$$

As a result , we obtained the iteration formula as:

$$y_{n+1}(x,t) = y_n(x,t) + \int_0^t (a(t-\xi) - 1) \left(\frac{\partial y_n(x,\xi)}{\partial \xi} - \frac{\partial^2 y_n(x,\xi)}{\partial x^2} - a y_n - f(x,\xi) \right) d\xi \tag{4.3.19}$$

Chapter 5

Conclusion

Nonlinear problems are often linearized using the quasi-linearization technique. The linear version of the problem is then solved using either analytic or numerical methods. However, linearizing nonlinear problems reduces the precision of the solution in several ways. We also proposed an iterative analytic approach in this paper to resolve some of the relevant issues. The methodology presented has been shown to be effective. The proposed method provides us with an analytic approximation that is extremely accurate. The results are, in fact, comparable to simple analytic solutions.The approach presented is simple to apply, and it can easily be generalised to even more general situations such as problems with discontinuous source terms and mathematical physics evolution equations with minor modifications. Since this method justifies its efficiency and delivers promising findings with just a few iterations, and without any restrictive assumptions, the numerical results obtained are indistinguishable.
Further, we have taken parabolic partial differential equations and iterative method is successful in solving PDE's too. We clearly observed that the exact solution is obtained by taking infinte iterations and it can be shown that increasing the number of iterations leads to a solution that is similar to the exact solution. Therefore, this approach can be used to get a close approximation to the exact solution.

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