A Case Study of Vehicle Toll Plaza Service Time Sequences for Designing Poisson Queue Birth Death Model

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IN

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Submitted by

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2K19/MSCMAT/04

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DEPT. OF APPLIED MATHEMATICS

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CANDIDATE'S DECLARATION

I, Monish, Roll No. 2K19/MSCMAT/04, student of MSc.(Mathematics), hereby declare that the project Dissertation titled "A Case Study of Vehicle Toll Plaza Service Time Sequences for Designing Poisson Queue Birth Death Model" which is submitted by me to the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree of Master of Science, is original and not copied from any source without proper citation. This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship or other similar title or recognition.

Place Delhi
Date May 24, 2021

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CERTIFICATE

I hereby certify that the Project Dissertation titled ”A Case Study of Vehicle Toll Plaza Service Time Sequences for Designing Poisson Queue Birth Death Model” which is submitted by Monish, Roll No. $2K19/MSCMAT/04$ (Dept. of Applied Mathematics), Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree of Master of Science, is a record of the project work carried out by the students under my supervision. To the best of my knowledge this work has not been submitted in part or full for any Degree or Diploma to this University or elsewhere.

Place Delhi

Date May 24, 2021

Prof. L.N Das

SUPERVISOR

ACKNOWLEDGEMENT

I take this opportunity to express my deep sense of gratitude and sincere thanks to all who helped me to complete the work successfully. My first and foremost thanks goes to God Almighty who showered in immense blessings on my effort.

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Monish

Abstract

In this study, we have a designed a queue model from the observation of arrival, services and departure of vehicles at a toll plaza. Using the system process and appropriate algorithm, we have minimized the waiting time and by the way improve the service facility. Based on the arrival frequency of the vehicle numbers the extra service counter are likely to be opened in the toll liners and as a consequence the customer vehicle has to wait less time in a queue. Ultimately, the vehicle can also save fuel oil as compared to waiting vehicle queue lines. In this process, we reformulate the standard Poisson queue birth death theorem that relating vehicle's arrival for services and completion of services theorems in briefly. An application of vehicle queue in the traffic route is also illustrated with an example.

Contents

Chapter 1

INTRODUCTION

Queuing theory is a Branch of Mathematics and sub-part of operation research that studies act of waiting in lines. Q.T is the mathematical study of queuing or waiting in lines with relation to time, customer people in the service locations or item are in the information queues. Queues forms when there are limited resources for providing the service.

Waiting lines, often known as queues, are a common occurrence in our daily lives. The main characteristics of a queuing phenomena are that units arrive at a specific spot called the service centre at regular or irregular intervals of time. for example Trucks arriving at a loading station, clients entering a retail shop, people coming at a movie theatre, ships arriving at a port, letters arriving at a typist's desk, and so on. All of these units are referred to as client admissions or arrivals.

What is Queuing Theory?

Queuing theory is a statistical analysis of line congestion and delays. Queuing theory looks at the arrival mechanism, operation process, number of servers, number of device spaces, and number of customers—which may be people, data-packet, autos, vehicle and so on.Queuing theory, a subset of operations science, can help users make more educated decisions on how to design reliable and cost-effective workflow systems. Providing faster customer service, optimising traffic flow, effectively shipping goods from a warehouse, and developing telecommunications systems, from data networks to contact centres, are all examples of real-world applications of queueing theory.

How Queuing Theory Works? When resources are limited, queues form. In reality, queues make economic sense; without them, overcapacity would be costly. Queuing theory aids in the design of balanced structures that support consumers easily and efficiently while not being prohibitively expensive to operate. The entities queuing for associate degree operation are attenuated by all queuing processes.

At its most fundamental level, queueing theory comprises analysing arrivals at a facility, such as a bank or a toll plaza, and determining the facility's service requirements, such as bankers or customers visitors. The origins of queuing theory can be traced back to the first millennium, when Agner Krarup, a engineer and man of science, published a study of the Kobenhavn central. His contribution to the telephone unit theory of economic networks.

When clients arrive to the service centre in such a manner that either the customer or the facilities must wait with duration or service time, waiting time then we have a queuing problem.The queuing problem are minimize the delay time or waiting time during a service process with due priority.

The reliability of a tollgate operation is determined by the length of the queue line and the time spent on waiting line in the system.The queue is caused by a discrepancy in vehicle clearance compare to overall toll arrival per unit of time.The waiting time is a combination of the length of the line and the time of entry into the queue line.In a peak hour, the length of time it takes to enter the queue grows exponentially with the function of time.

In view of the increasing number of vehicle and to save the fuel, we have taken a problem because of the increasing number of vehicle at toll plaza ,the passengers faces more trouble.The length of queue line made at the toll plaza can be reduced with the help of application of queuing theory suggested rules, the has to wait the minimum waiting time and with this help, we can save the fuel of the vehicle.

Chapter 2

Concept, Terminologies and Notations used in queuing Model

2.1 Components of a Queuing-system

The basic elements of a queuing system are as follows:

1. Arrival Procedure - The sequence in which clients arrive for service is handled by this section of the queuing system. Three factors can be used to describe an input source.

   - Size of the queue - The size of the input source is considered to be finite if the total number of possible customers requesting service is small. On the other hand, if the number of potential clients in need of service is great enough, the input source is termed endless.
   - Pattern of arrivals - some Customers may enter in system at predetermined (regular or otherwise) times, or they may enter at random. Queuing problems are classified as deterministic models when the arrival timings are known with certainty. The arrival pattern is established by the mean arrival rate or the inter-arrival time if the period between consecutive arrivals (inter-arrival times)

is unknown.The probability distribution associated with this random process characterises them. The most common stochastic queuing models assume a Poisson distribution for arrival rates and/or an exponential distribution for inter-arrival intervals.

- Customer behaviour - It's also important to understand how a consumer reacts when he or she first enters in system.Regardless of how long the queue gets, a customer might choose to wait. (patient customers may choose not to enter the line if it grows too lengthy for them) (impatient customer). Patient customers are examples of machines arriving at a plant's maintenance shop. Customers who are impatient

2. Queue Discipline - When a wait has formed, consumers are picked for service according to this rule. The first−come, first−served (FCFS) or first−in, first−out (FIFO) rule is the most frequent queue discipline, in which clients are served in the order in which they arrive.Another queue discipline is the "last in, first out" (LIFO) rule, which states that the system's most recent arrival is served first. Most cargo handling circumstances follow this discipline, with the last thing loaded being withdrawn first.

3. Service Mechanism - Service time and service facilities are two aspects of the service mechanism. The time between the start of service and the end of service is referred to as service time. If there are an unlimited number of servers, there will be no waiting because all customers will be served instantly upon arrival. Customers are served in a set order if the number of servers is limited.

4. Capacity of the system - Customer generation might come from a finite or infinite source. The number of consumers who can be served is limited by a finite resource. The maximum queue size, in other words, has a limit. A queue with forced baulking, in which a consumer is forced to baulk if he arrives at a time when the line size has reached its maximum capacity, can also be considered.

2.2 Practical concept

The two important parts of queuing theory upon which the whole concept is based are

- Customer
- Service provider

2.2.1 Arrival time of a customer

Average-rate / Expected-rate or Mean arrival rate (λ) - It is defined as the rate at which the customers are coming to get the services.
Average arrival-rate - number of customer coming to get services with in given period of time.
To know the average arrival rate we must know the arrival rate or interval time between the arrival of two customers.

2.2.2 Serving to the customers

Average-rate / Expected-rate or Mean service-rate (μ) - It is defined as the Rate at which the customers are served by the service facility(provider).
Average service-rate - number of customer that are served with in given period.

2.2.3 Utilization rate (ρ)

$= \frac{\lambda}{\mu}$ = Average arrival rate / Average service rate
utilization means how much of the capacity is utilized.

2.3 Possibilities of the queue are -

- If $\lambda > \mu$, Then

1. Infinite queue.
2. Busy service provider.
3. Failure of service system.

- If $\lambda = \mu$, Then

1. Customers comes and served.
2. No queue will be there.
3. Service will be busy.

- If $\lambda < \mu$, Then

1. No queue will be there.
2. There is idle time for server.

2.3.1 Formula Derived for probability of exactly $'n'$ customers in the system

1. If there are no customer (n = 0), then service provider are free and equal to idle rate
   p(0) = 1 - R = $1 - \frac{\lambda}{\mu}$
   where R is the utilization rate

2. For customer is 1 (n = 0) then its probability
   p(1) = R.p(o) = R(1 - R) = $\frac{\lambda}{\mu}(1 - \frac{\lambda}{\mu})$

3. For customer is 2 (n = 2), then its probability
   p(2) = R.p(1) = R.R.p(0) = $\frac{\lambda}{\mu}\frac{\lambda}{\mu}(1 - \frac{\lambda}{\mu})$

4. For n customers then its probability
   p(n) = $R^n.\ p(0) = (\frac{\lambda}{\mu})^n(1 - \frac{\lambda}{\mu})$

- The average number of customer in the given system (L_s)

  L_s = utilization rate / idle rate

  $= \frac{\frac{\lambda}{\mu}}{(1-\frac{\lambda}{\mu})} = \frac{\lambda}{\mu-\lambda}$

- The average number of customer in the given queue $L_q = L_s$ - utilization rate $= \frac{\lambda}{\mu-\lambda}$ - $\frac{\lambda}{\mu}$

  $= \frac{\lambda^2}{\mu(\mu-\lambda)}$

- The average waiting time for a customer in given queue -

  w_q = Avage number of customer in the given queue / Arrival rate

  $\frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$

- The average waiting time for a customer in given system w_s = Average number of customer in the system / Average rate

  $w_s = \frac{L_s}{\lambda} = \frac{1}{\mu-\lambda}$

2.3.2 Limitation of Queuing Model

- Customers usually have a certain amount of time to wait.

- It's possible that the arrival rate is state-dependent. When a customer arrives and sees a long line, he or she may decide not to join it and leave without receiving service.

- It's possible that the arriving procedure won't be static. There may be peak and null periods when the λ is higher or lower than the average arrival-rate.

- Customers may not be limitless in number, and queuing(discipline) may not be strictly first come, first served.

- Services may not be provided on a constant basis. It's possible that the service facility will break down, and that the service will be delivered in batches rather than individually.

- It's possible that the queuing mechanism hasn't reached a stable state. Instead, it could be in a transitory state. It's typical when a line is just getting started and the time remaining isn't long enough.

Chapter 3

Design of a queuing model from the case study

3.1 Problem Description

In this report, we discuss a real daily life problem of toll plaza, we take data from National highways Authority of india (NHAI) of one month.In which we calculate average queue length (Including customer currently being served), traffic intensity, waiting time of the system and expected length a non empty queue. The data received from NHAI of 24 hour than we convert into the number of vehicle an hour and also convert into in single counter (Because we have assumed that the data is four different counters) at the toll plaza. we have supposed the service rate depend upon the arrival rate per day. According to problem we have assumed that are 4 counters and we have considered only 1 counter because we have converted the average of 4 counter then waiting time and queue length will be reduced and the main thing are vehicle fuel also saved.

Observation-

1. We have plotted the graph between number of days and number of vehicle.According to this picture, 1000 vehicles have come almost every day.

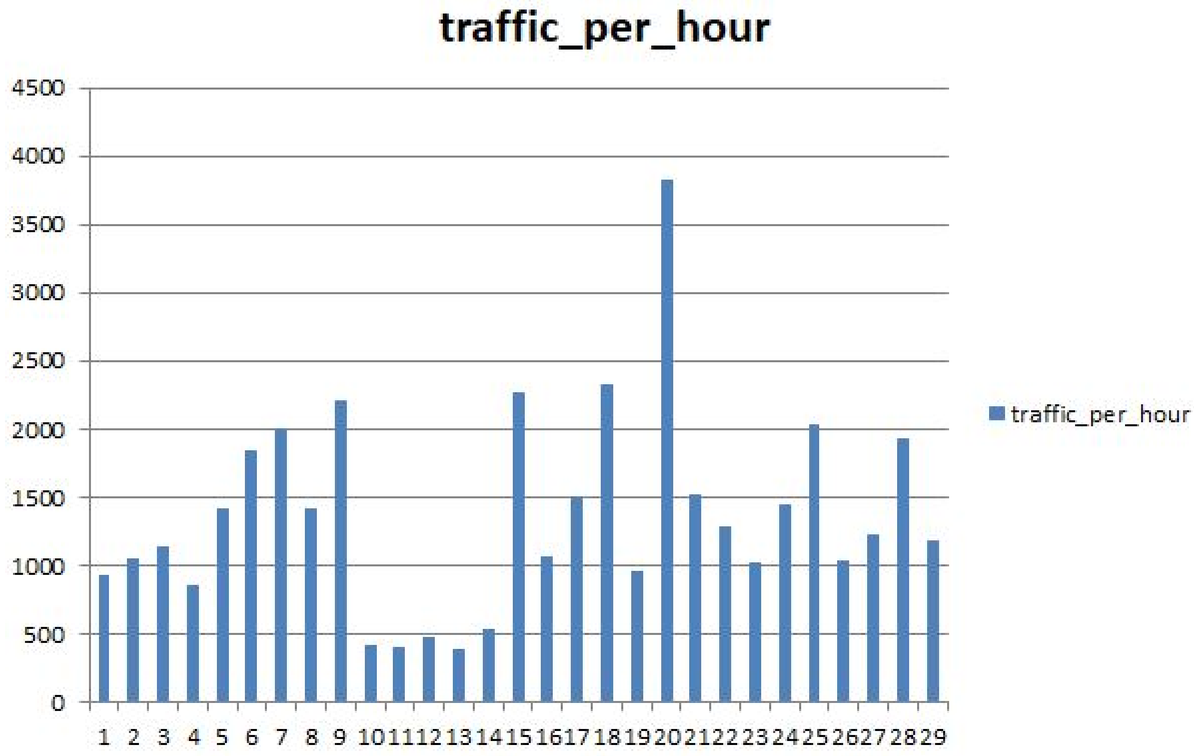

figure 1.1

2

This picture shows traffic intensity and traffic intensity it is ratio of λ and μ. The graph of traffic-intensity represent the utilization rate means the service provider 97% (Average) busy but 3% are free (idle rate).

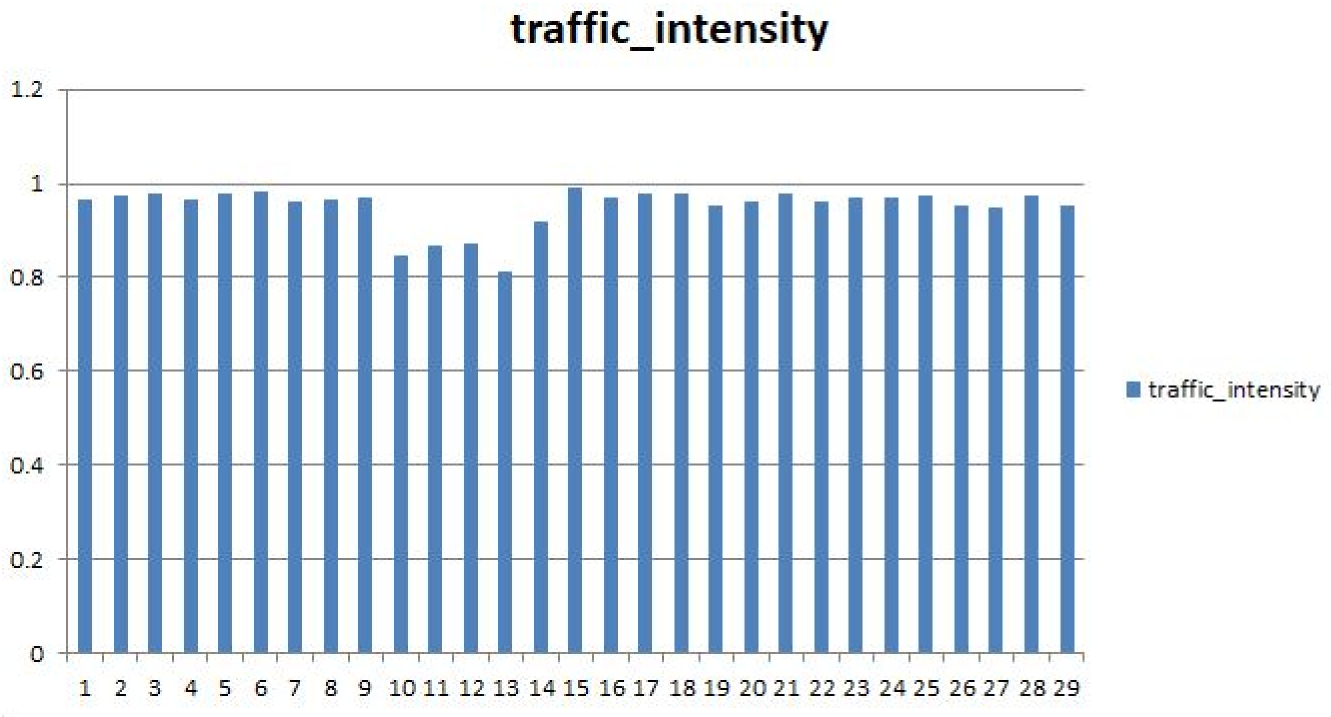

figure 1.2

3

This graph represent the length of the system and graph plotes between number of days and number of vehicles. According to the graph $1-9$ days and $16-19$ days the arrival number of vehicle are 20. due to service provider are busy approximate 97% given by utilization intensity. If the service provider working fast then length of would be reduced.

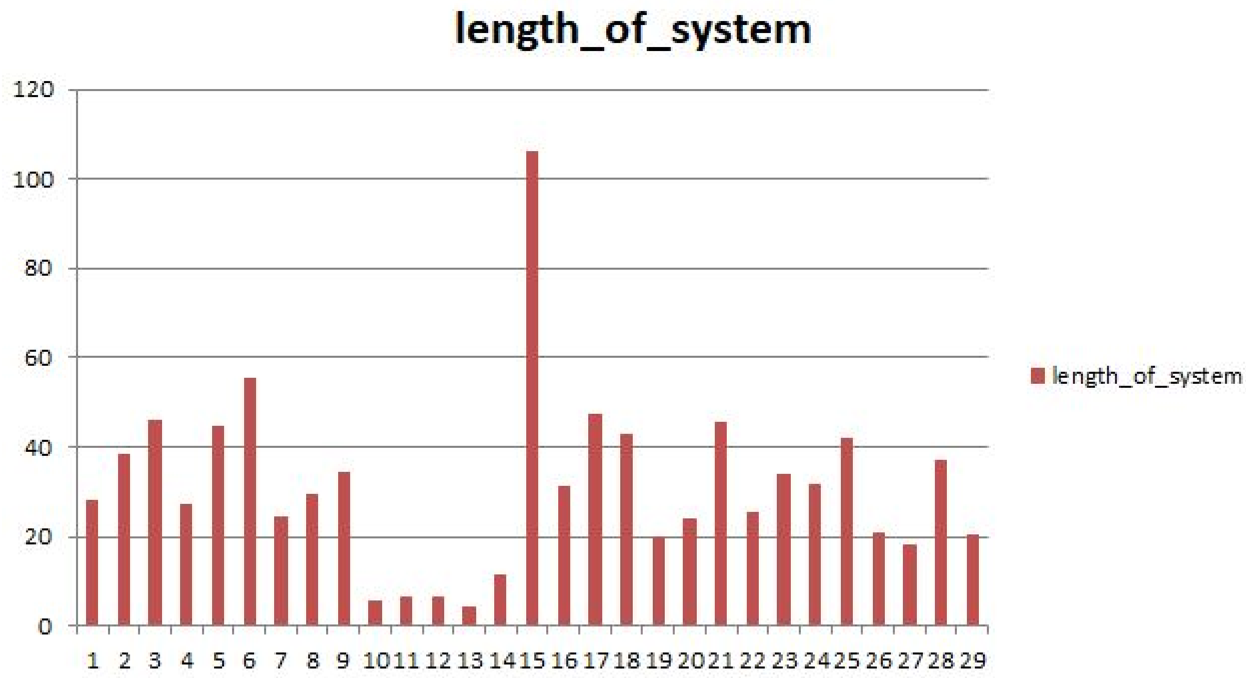

figure 1.3

4

This graph represent the waiting-time of the given system (we have include the waiting time + Service time) and we have plot the graph between the number of days and the number of hours. we convert the the data into the minute (multiply by 60).Then the average arrival rate is $4 - 8$ minute per vehicle.

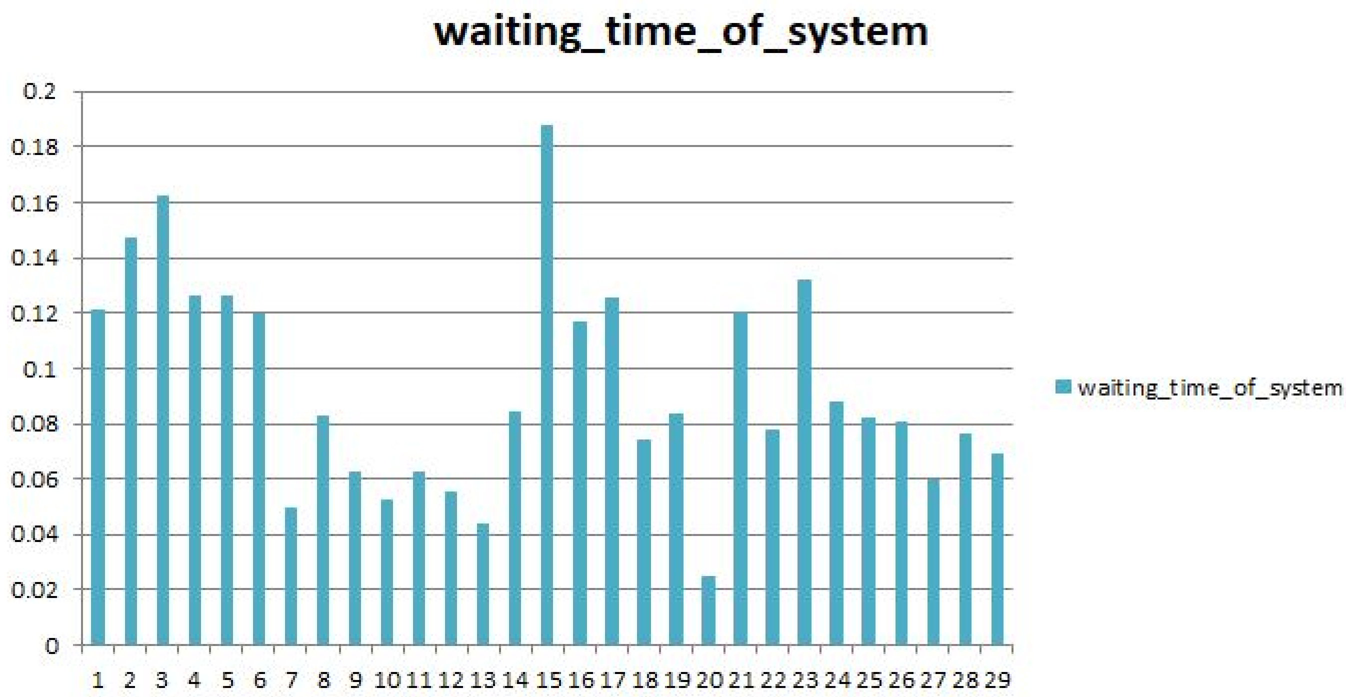

figure 1.4

5

The fifth graph represent the expected graph length of a non-empty queue and the graph plotted between the number of days and the number of vehicles arrived at the toll plaza. In this situation the given length of the system and expected length of system a non-empty queue are approximately same.

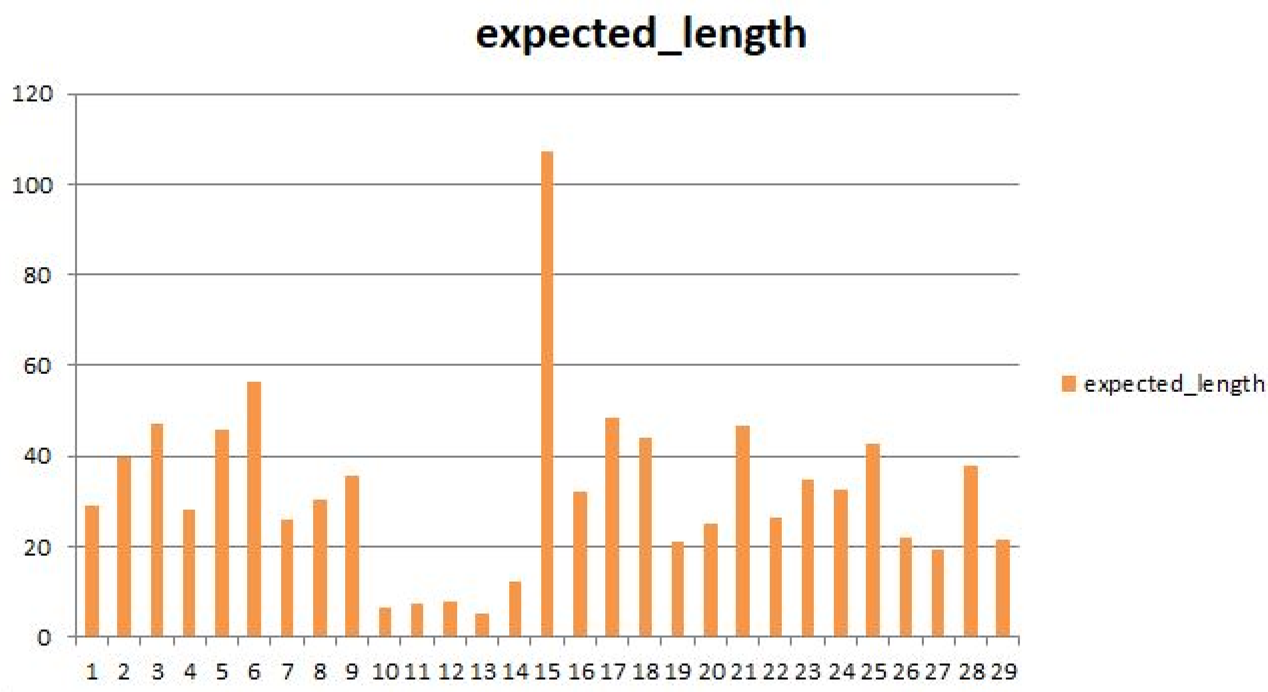
expected_length
120
100
80
60
40
20
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29
expected_length

figure 1.5

traffic_per_day	traffic_per_hour	traffic on counter	name	Service_rate	traffic_intensity
22246	926.9166667	231.7291667	Thakurtola (E	240	0.965538194
25077	1044.875	261.21875	Bankapur	268	0.974696828
27249	1135.375	283.84375	Hirebagewad	290	0.978771552
20742	864.25	216.0625	Durg Bypass (	224	0.964564732
33898	1412.416667	353.1041667	Hattargi	361	0.978127886
44415	1850.625	462.65625	Kognoli	471	0.982285032
47978	1999.083333	499.7708333	Tundla	520	0.961097756
33886	1411.916667	352.9791667	Brahamarakc	365	0.96706621
52993	2208.041667	552.0104167	Cable Stayed	568	0.971849325
10168	423.6666667	105.9166667	Bassi	125	0.847333333
9900	412.5	103.125	Aroli	119	0.866596639
11617	484.0416667	121.0104167	Dhaneshwar	139	0.870578537
9515	396.4583333	99.11458333	Fatehpur	122	0.812414617
12970	540.4166667	135.1041667	Simliya	147	0.919075964
54304	2262.666667	565.6666667	Omallur	571	0.990659661
25484	1061.833333	265.4583333	Nathakkarai	274	0.968826034
36196	1508.166667	377.0416667	Veeracholapu	385	0.979329004
55726	2321.916667	580.4791667	Palayam (Dha	594	0.977237654
23232	968	242	Rasampalaya	254	0.952755906
91814	3825.583333	956.3958333	Nemili (Sripe	996	0.96023678
36353	1514.708333	378.6770833	Joya	387	0.978493755
31026	1292.75	323.1875	Khalghat -MF	336	0.96186756
24618	1025.75	256.4375	Boothakudi	264	0.971354167
34617	1442.375	360.59375	Chittampatti	372	0.969338038
48847	2035.291667	508.8229167	Banglore - Ne	521	0.976627479
25023	1042.625	260.65625	Chalageri	273	0.954784799
29310	1221.25	305.3125	Choundha	322	0.948175466
46360	1931.666667	482.9166667	Brijghat	496	0.973622312
28557	1189.875	297.46875	Usaka (Cham	312	0.953425481

mhu-lamda	length_of_system	waiting_time_of_system	expected_length
8.270833333	28.01763224	0.120906801	29.01763224
6.78125	38.52073733	0.147465438	39.52073733
6.15625	46.10659898	0.162436548	47.10659898
7.9375	27.22047244	0.125984252	28.22047244
7.895833333	44.72031662	0.126649077	45.72031662
8.34375	55.4494382	0.119850187	56.4494382
20.22916667	24.70545829	0.049433574	25.70545829
12.02083333	29.36395147	0.083188908	30.36395147
15.98958333	34.52312704	0.062540717	35.52312704
19.08333333	5.550218341	0.052401747	6.550218341
15.875	6.496062992	0.062992126	7.496062992
17.98958333	6.726693688	0.055587724	7.726693688
22.88541667	4.330905781	0.043695949	5.330905781
11.89583333	11.35726795	0.084063047	12.35726795
5.333333333	106.0625	0.1875	107.0625
8.541666667	31.07804878	0.117073171	32.07804878
7.958333333	47.37696335	0.12565445	48.37696335
13.52083333	42.93220339	0.073959938	43.93220339
12	20.16666667	0.083333333	21.16666667
39.60416667	24.14886902	0.025249868	25.14886902
8.322916667	45.49812265	0.120150188	46.49812265
12.8125	25.22439024	0.07804878	26.22439024
7.5625	33.90909091	0.132231405	34.90909091
11.40625	31.61369863	0.087671233	32.61369863
12.17708333	41.78528657	0.082121471	42.78528657
12.34375	21.1164557	0.081012658	22.1164557
16.6875	18.29588015	0.059925094	19.29588015
13.08333333	36.91082803	0.076433121	37.91082803
14.53125	20.47096774	0.068817204	21.47096774

Chapter 4

Reviewing of Queue Model

4.1 Arrival Distribution theorem (Poisson process / Pure birth process)

Concept - The model in which only arrivals are counted and no departures takes place are called pure birth models. The term birth refers to the arrival of a new calling unit in the system and the death refers to the departure of a served unit.

Theorem- If the arrival are completely random, then the probability distribution of number of arrivals is in fixed time interval follows a poisson distribution.

proof-

In order to derive the arrival distribution in queues we make the three assumptions axioms.

1. There are n units in the system at time t and the probability that exactly one arrival (birth) will occur during small time interval Δt be given by $\lambda(\Delta t) + O(\Delta t)$
   Where λ is the arrival rate independent of t and $O(\Delta t)$ include the terms of higher order of Δt.

2. Secondly, assume that the time Δt is so small that the probability of more than one arrival in time Δt is $O(\Delta t)^2$ i.e almost zero.

3. The number of arrival in non-overlapping intervals are statistically independent.

The probability of n arrivals in a time interval of length t denoted by $P_n(t)$

Case 1 ;

When $n > 0$

There are n units in the system at time t and no arrival takes place during time interval Δt, Hence there will be n units at time $(t + \Delta t)$ also.

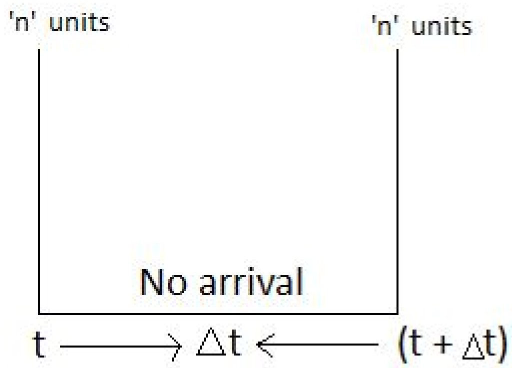

figure 1.1

Probability of two combined events = Probability of n units at time t $*$ probability of no arrival during Δt

$$= P_n(t)(1 - \lambda \Delta t) \tag{4.1}$$

Secondly, there are $(n-1)$ units in the system at time t and one arrival takes places during Δt. Hence there will remain n units in the system at time $(t + \Delta t)$

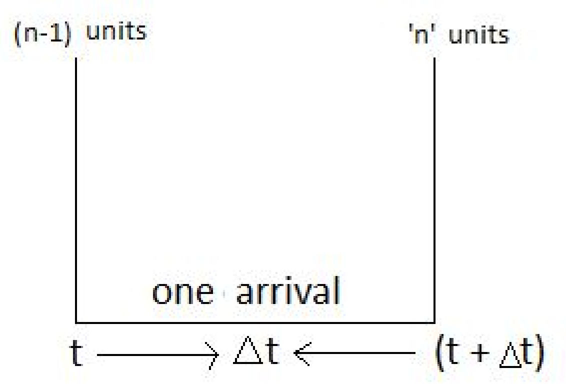

figure 1.2

Probability = prob. of $(n-1)$ units at time t * prob. of one arrival in time Δt

$$= P_{n-1}(t) * \lambda(\Delta t) \tag{4.2}$$

Adding equations (4.1) and (4.2), we get the probability of n arrival at time $(t+\Delta t)$ is :

$$P_n(t+\Delta t) = P_n(1-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t \tag{4.3}$$

Case -2 ;

when $n = 0$

$P_0(t+\Delta t)$ = prob. (no unit at time t) $*$ prob.(no arrival in time Δt)

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) \tag{4.4}$$

Rewriting the Eq. (4.3) and (4.4) after taking terms P_0 to the left hand side , we get

$P_n(t+\Delta t) - P_n(t) = P_n(t)(-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t \qquad (n > 0)$

$P_n(t+\Delta t) - P_0(t) = P_0(t)(-\lambda\Delta t) \qquad (n = 0)$

Dividing the both side of above equation by Δt and than taking limit as $(\Delta t \longrightarrow 0)$

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = [-\lambda P_n(t) + \lambda P_{n-1}(t)]$$

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \tag{4.5}$$

The left hand side becomes by definition of first derivative

and

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_o(t)}{\Delta t} = \frac{-\lambda P_0(t)\Delta t}{\Delta t}$$
$$P_0'(t) = -\lambda P_0(t) \tag{4.6}$$

by definition of first derivative we can write
$lim_{\Delta t \to 0} \frac{P_0(t+\Delta t)-P_o(t)}{\Delta t} = P_0'(t)$
Rewriting it again Eq. 4.5 and 4.6

$$P_0'(t) = \lambda P_0(t) \qquad n = 0 \tag{4.7}$$

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \qquad n > 0 \tag{4.8}$$

This is known as the system of differential difference equation.

now to solve Eq. 4.7 and 4.8

$$\frac{P_0'(t)}{P_0(t)} = -\lambda$$

we can write it as;

$$\frac{d}{dt}[log P_0(t)] = -\lambda \tag{4.9}$$

Integrating on both side of Eq. 4.9

$$log P_0(t) = -\lambda t + A \tag{4.10}$$

where A is constant of integration, using boundary condition to obtain constant of integration

$P_n(0)$ $= 0$ for $n = 0$,

and $= 0$ for $n > 0$

putting $t = 0$ in Eq. 4.10

$log 1$ = -$\lambda(0)$ $+A \implies A = 0$ then using Eq.4.10

$log P_0(t) = -\lambda t$

$$P_0(t) = e^{-\lambda t} \tag{4.11}$$

putting $n = 1$ in Eq. 4.8

$$P_1'(t) = -\lambda P_1(t) + \lambda P_0(t)$$

$$P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t} \tag{4.12}$$

As this the linear differential equation of first order it can be easily solved by multiplying both of eq. 4.12 by integration factor.

$$\begin{aligned}
P_1'(t) + \lambda P_1(t) &= \lambda e^{-\lambda t} \\
e^{\lambda t}[P_1'(t) + \lambda P_1(t)] &= \lambda \\
e^{\lambda t}P_1'(t) + \lambda e^{\lambda t} P_1(t) &= \lambda \\
\frac{d}{dt}[e^{\lambda t} P_1(t)] &= \lambda
\end{aligned} \tag{4.13}$$

integrating both side of eq. 4.13, we get

$$e^{\lambda t} P_1(t) = \lambda t + B \tag{4.14}$$

where B is constant of integration , in order to determine B put $t = 0$ in eq.4.14, we get

$$e^0 P_1(0) = \lambda(0) + B \implies B = 0$$

so eq. 4.14

$$e^{\lambda t} P_1(t) = \lambda t$$

we can write it as

$$P_1(t) = \frac{\lambda t e^{-\lambda t}}{1!} \tag{4.15}$$

putting $n = 2$ in eq. 4.8 and using eq. 4.15

$$\begin{aligned}
P_2'(t) + \lambda P_2(t) &= \lambda P_1(t) \\
P_2'(t) + \lambda P_2(t) &= \frac{\lambda.(\lambda t)e^{-\lambda t}}{1!} \\
e^{\lambda t}P_2'(t) + \lambda e^{\lambda t} P_2(t) &= \frac{\lambda.(\lambda t)}{1!}
\end{aligned}$$

integrating w.r.t 't'

$$e^{\lambda t}P_2(t) = \frac{\lambda^2 t^2}{2!} + C$$
$$e^{\lambda t}P_2(t) = \frac{(\lambda t)^2}{2!} + C$$

put $t = 0,\ P_2(0) = 0$ to obtain C

$$P_2(t) = \frac{(\lambda t)^2}{2!}e^{-\lambda t} \qquad for\ \ n = 2 \tag{4.16}$$

similarly

$$P_3(t) = \frac{(\lambda t)^3}{3!}e^{-\lambda t} \qquad for\ \ n = 3$$

for $n = m$

$$P_m(t) = \frac{(\lambda t)^m}{m!}e^{-\lambda t} \tag{4.17}$$

It can be proved the result Eq. 4.17 is also true for $n = m+1$, then by induction hypothesis result Eq. 4.17 will be true for general values of n.

put $n = m + 1$ in eq. 4.8 and using eq. 4.17 also

$$P'_{m+1}(t) + \lambda P_{m+1}(t) = \frac{\lambda(\lambda t)^m}{m!}e^{-\lambda t}$$
$$\frac{d}{dt}[e^{\lambda t}P_{m+1}(t)] = \frac{\lambda(\lambda t)^m}{m!}$$

Integrating both side of above equation

$$e^{\lambda t}P_{m+1}(t) = \frac{(\lambda t)^{m+1}}{(m+1)^m} + D$$

where D is constant of integration , in order to determine D put $t = 0,\ P_{m+1}(0) = 0$ and $D = 0$, we get

$$P_{m+1}(t) = \frac{(\lambda t)^{m+1}e^{-\lambda t}}{(m+1)m!}$$

Hence in general

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

This is the poisson distribution.

4.2 Pure Death Process(Distribution of Departure)

Theorem -

In this process assume that there are N customers in the system at time $t = 0$, also assume that no arrival (birth) can occur in the system. Departure occur at a rate μ per unit time i.e output rate is μ.

we will derive the distribution of departures from the system on the basis of the following three axioms:

1. Probability (one departure during Δt) $= \mu\Delta t + O(\Delta t)^2$
   $= \mu\Delta t$ $\qquad\qquad$ $O(\Delta t)^2$ is negligible
2. Probability (more than one departure during Δt) $= O(\Delta t)^2 = 0$
3. The number of departures in non-overlapping intervals are statistically independent and identically distributed random variable i.e the process $N(t)$ has independent increment.

First obtain the differential difference equation in three mutually exclusive ways.

Case 1 When $0 < n < N$ (same as pure birth process)

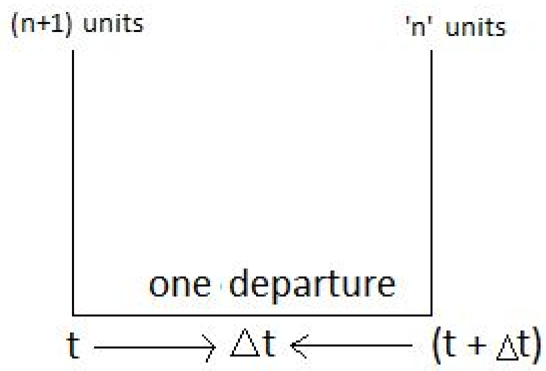

figure 1.3

$$Probability(t + \Delta t) = P_n(t)(1 - \mu\Delta t) + P_{n-1}(t)\mu\Delta t \tag{4.18}$$

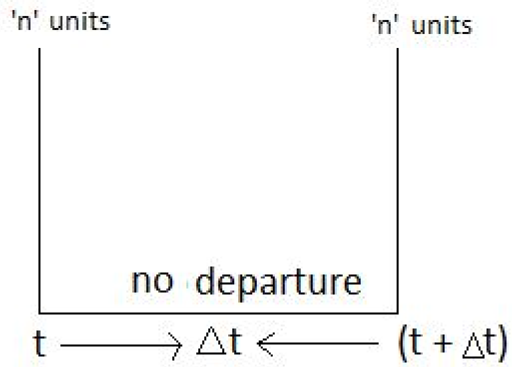

figure 1.4

Case 2, when $n = N$ since there are exactly N units in the system $P_{n+1}(t) = 0$

$$P_N(t + \Delta t) = P_N(t)(1 - \mu\Delta t) \tag{4.19}$$

Case 3, when $n = 0$

$$P_0(t + \Delta t) = P_0(t) + P_1(t)\mu\Delta t \tag{4.20}$$

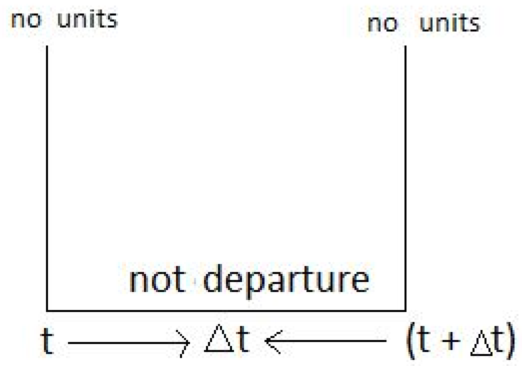

figure 1.5

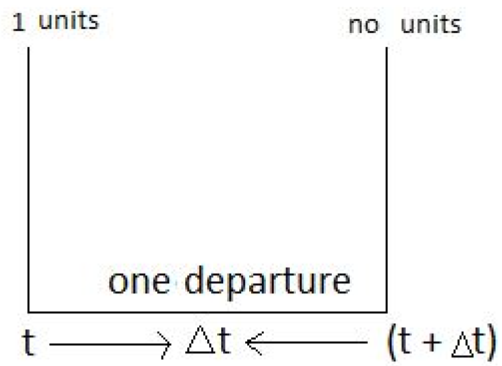

figure 1.6

Since there is no unit in the system at the time t, the question of any departure during Δt does not arrive.

therefore, probability of no departure is unity in this case.

Now using Eqs. 4.18, 4.19 and 4.20; rearranging them and dividing them by Δt and taking limit $\Delta t \longrightarrow 0$

Now eq. 4.19

$$P_N(t + \Delta t) = P_N(t)(1 - \mu\Delta t)$$
$$\lim_{\Delta t \to o} \frac{P_N(t + \Delta t) - P_N(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{-P_N(t)\mu\Delta t}{\Delta t}$$

$$P'_N(t) = -\mu P_N(t) \tag{4.21}$$

Similarly for Eq. 4.18

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lim_{\Delta t \to 0} \left[\frac{-P_n(t)\mu\Delta t}{\Delta t} + \frac{P_{n+1}(t)\mu\Delta t}{\Delta t} \right]$$

$$P'_n(t) = -\mu P_n(t) + \mu P_{n+1}(t) \tag{4.22}$$

Now Eq. 4.20

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{P_1(t)\mu\Delta t}{\Delta t}$$

$$P'_0(t) = \mu P_1(t) \tag{4.23}$$

Now to solve Eqs. 4.21, 4.22 and 4.23, we use iteration method

Step 1 for eq. 4.21

$$P'_N(t) = -\mu P_N(t)$$

$$\frac{P'_N(t)}{P_N(t)} = -\mu$$

$$\frac{d}{dt}[log P_N(t)] = -\mu$$

integrating on both side

$$log P_N(t) = -\mu t + A \tag{4.24}$$

To determine A, use the boundary condition $P_N(t) = 0$ we get $A = 0$

$$log P_N(t) = -\mu t$$

$$P_N(t) = e^{-\mu t} \tag{4.25}$$

Step-2

now eq. 4.22, put $n = N - 1$ and the value $P_N(t)$ from eq. 4.25

$$P'_{N-1}(t) = -\mu P_{N-1}(t) + \mu P_N(t)$$

$$P'_{N-1}(t) = -\mu P_{N-1}(t) + \mu e^{-\mu t}$$

$$P'_{N-1}(t) + \mu P_{N-1}(t) = \mu e^{-\mu t}$$

$$e^{\mu t} P'_{N-1}(t) + e^{\mu t} \mu P_{N-1}(t) = \mu$$

$$P_{N-1}(t) e^{\mu t} = \int \mu e^{\mu t} e^{-\mu t} + B \qquad (I.F = e^{\mu t})$$

$$P_{N-1}(t) = \mu t e^{-\mu t} + B e^{-\mu t} \tag{4.26}$$

Now determine B, put $t = 0$, $P_{N-1}(t) = 0$ in eq. 4.26 we get $B = 0$

$$P_{N-1}(t) = \frac{\mu t e^{-\mu t}}{1!}$$

Step-3

put $n = (N-2)$ in eq. 4.22

$$P_{N-2}(t) = \frac{e^{-\mu t}(\mu t)^2}{2!}$$

putting $n = N-3, N-4, N-5, ...N-i$ and using induction process

$$P_{N-3}(t) = \frac{e^{-\mu t}(\mu t)^3}{3!}$$

....

$$P_{N-i}(t) = \frac{e^{-\mu t}(\mu t)^i}{i!}$$

on letting $n = N - i$

$$P_n(t) = \frac{e^{-\mu t}(\mu t)^{N-n}}{(N-n)!} \tag{4.27}$$

where $n = 1, 2, 3, 4, ...N$

Step-4 To find the $P_0(t)$ use the following procedure

$$1 = \sum_{n=0}^{N} P_n(t) = P_0(t) + \sum_{n=1}^{N} P_n(t)$$

$$P_0(t) = 1 - \sum_{n=1}^{N} P_n(t) = 1 - \sum_{n=1}^{N} \frac{e^{-\mu t}(\mu t)^{N-n}}{(N-n)!} \tag{4.28}$$

now Eq. 4.27 and 4.28

$$P_n(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!}; \qquad for \quad n = 1, 2, 3...N$$

$$= 1 - \sum_{n=1}^{N} \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!} \qquad for \quad n = 0$$

Thus the number of departures in time $'t'$ follows the Truncated poisson distribution.

4.3 Birth and Death Model / Model (M/M/1) :($\infty/FCFS$)

Step-1 To obtain the system of steady- state equation.

The Probability that there will be n units ($n > 0$) in the system at time $t + \Delta t$ may be expressed as the sum of three independent compound probabilities, by using the fundamental properties of probability poisson arrivals and of exponential arrivals times.

1. The product of three probabilities in fig 1.7

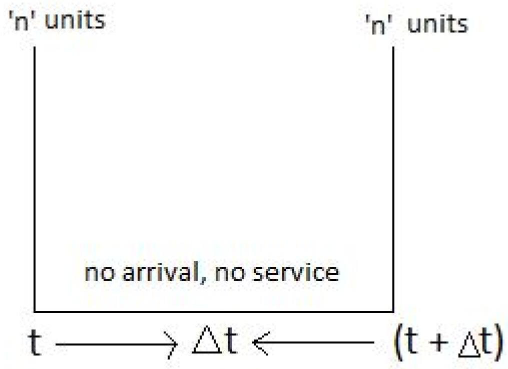

figure 1.7

(a) If there are n units in the system at time t, the probability denoted by $= P_n(t)$

(b) If there is no arrival in time $\Delta t = P_0(\Delta t) = 1 - \lambda\Delta t$

(c) If there is no service in time $\Delta t = \phi_{\Delta t}(0) = 1 - \mu\Delta t$

$$P_n(t)(1 - \lambda\Delta t)(1 - \mu\Delta t) = P_n(t)[1 - (\lambda + \mu)\Delta t] + O_1(\Delta t) \tag{4.29}$$

2. The product of three probabilities in fig(1.8)

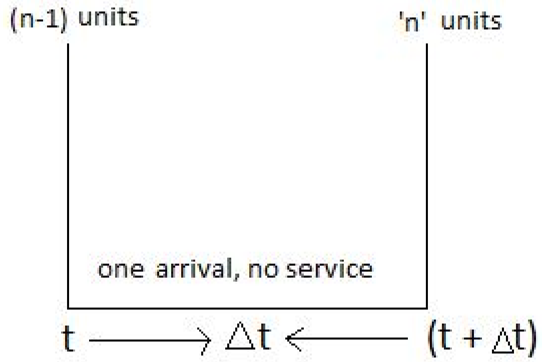

figure 1.8

(a)If there are $n-1$ units in the system at time t, the probability denoted by $= P_{n-1}(t)$

(b)If there is one arrival in time $\Delta t = P_1(\Delta t) = \lambda \Delta t$

(c)If there is no service in time $\Delta t = \phi_{\Delta t}(0) = 1 - \mu \Delta t$

$$P_{n-1}(t)(\lambda \Delta t)(1 - \mu \Delta t) = \lambda P_{n-1}(t)\Delta t + O_2(\Delta t) \tag{4.30}$$

3. The product of three probabilities in fig(1.9)

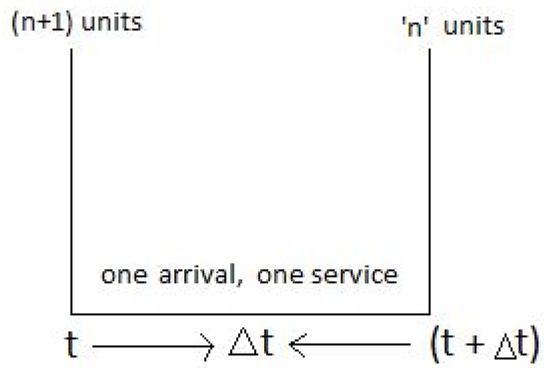

figure 1.9

(a)If there are $n+1$ units in the system at time t, the probability denoted by $= P_{n+1}(t)$

(b)If there is no arrival in time $\Delta t = P_0(\Delta t) = 1 - \lambda \Delta t$

(c)If there is one service in time $\Delta t = \phi_{\Delta t}(1) = \mu \Delta t$

$$P_{n+1}(t)(1-\lambda\Delta t)\mu\Delta t \cong P_{n+1}(t)\mu\Delta t + O_3(\Delta t) \tag{4.31}$$

Now adding above three independent component probabilities, we obtain the probability of n units in the system at time $t+\Delta t$ i.e (eqs. 4.29, 4.30 and 4.31), we get

$$P_n(t+\Delta t) = P_n(t)[1-(\lambda+\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta t + O(\Delta t) \tag{4.32}$$

we consider $[O_1(\Delta t) + O_2(\Delta t) + O_3(\Delta t) = O_(\Delta t)]$

Now Eq. 4.32 can be written as

$$P_n(t+\Delta t) - P_n(t) = -(\Lambda+\mu)\Delta t P_n(t) + \lambda P_{n-1}(t)\Delta t + \mu P_{n+1}(t)\Delta t + O(\Delta t)$$

Now taking limit as $\Delta t \to 0$ on both side

$$\lim_{\Delta t\to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lim_{\Delta t\to 0}\left[-(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{O(\Delta t)}{\Delta t}\right]$$

for higher order terms $\lim_{\Delta t\to 0} \frac{O(\Delta t)}{\Delta t} = 0$

$$\frac{dP_n(t)}{dt} = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) \tag{4.33}$$

In a similar fashion, the probability that there will be no unit (i.e n = 0) in the system at the time $(t+\Delta t)$ will be the sum of the following independent probability.

- Prob.[that there will be no unit in the system at time t, and no arrival in time Δt]
  $= P_0(t)(1-\lambda\Delta t)$ ....(A)
- Probability that there is one unit in the system at the time t, one unit serviced in Δt and no arrival in Δt
  $= P_1(t)\mu(\Delta t)(1-\lambda\Delta t) \cong P_1(t)\mu\Delta t + O(\Delta t)$ ...(B)

now combine the Eqs. A and B of the two probabilities, we get

$$P_0(t+\Delta t) = P_0(t)[1-\lambda\Delta t] + P_1(t)\mu\Delta t + O(\Delta t) \tag{4.34}$$

dividing both side by Δt

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) + \frac{O(\Delta t)}{\Delta t}$$

taking limit $\Delta t \to 0$ both side of above equation

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \qquad for \;\; n = 0 \tag{4.35}$$

since for only steady state probability are considered there as

$$\lim_{t\to\infty} \frac{d[P_n(t)]}{dt} = 0 \qquad for \;\; n \geq 0$$

and

$$\lim_{t\to\infty} P_n(t) = P_n$$

which is independent of t

The eqs. 4.33 and 4.34 can be written as

$$-(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1} = 0 \qquad if \;\; n > 0 \tag{4.36}$$

$$\lambda P_0 + \mu P_1 = 0 \qquad if \;\; n = 0 \tag{4.37}$$

The equation 4.36 and 4.37 constitute the system of steady state difference equation for the above model.

step-2

To solve the system of difference equation

$$-(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1} = 0 \qquad if \;\; n > 0 \tag{4.38}$$

$$\lambda P_0 + \mu P_1 = 0 \qquad if \;\; n = 0 \tag{4.39}$$

Since $P_0 = P_0$

$$P_1 = \frac{\lambda P_0}{\mu} \qquad from \;\; eq.(4.39)$$

now put $n = 1$ in eq.4.38

$-(\lambda+\mu)P_1 + \lambda P_0 + \mu P_2 = 0$

$\frac{-\lambda^2 P_0}{\mu} + \lambda P_0 - \lambda P_0 + \mu P_2 = 0$

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

Simliarly

$$P_2 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \frac{\lambda}{\mu} P_2 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$.$$

$$.$$

$$.$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \qquad\qquad for\ \ n \geq 0$$

Now using

$$\sum_{n=0}^{\infty} = 1$$

$$P_0 + P_1 + P_2 + P_4 + ... = 1$$

$$P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + ... = 1$$

$$P_0 \left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + ...\right] = 1$$

$$P_0 \left[\frac{1}{1 - \frac{\lambda}{\mu}}\right] = 1$$

Since $\frac{\lambda}{\mu} < 1$, so sum of infinite G.P is valid.

$$P_0 = \left(1 - \frac{\lambda}{\mu}\right) \tag{4.40}$$

Substitute above value, we get

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) \tag{4.41}$$

The Eqs. 4.40 and 4.41 gives the probability distribution of the queue length.

4.4 Conclusion

This survey of the vehicle data suggests that the toll plaza's authorities should increase the number of counters.By doing this, we can save the wastage of working hours in our nation which can result in country's growth. We can also reduce the fuel wastage which ultimately will save money and our environment.

4.5 References

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