Some Results Related to Fixed Point Theorem in Complex Valued *b*-Metric Space

A dissertation submitted to the Delhi Technological University in partial fulfilment of the requirements for the award of degree of

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I,Komal Verma, Roll No.s 2K19/MSCMAT/19 of Master in Science (Mathematics), hereby declare that the project dissertation titled Some Results Related to Fixed Point Theorem In Complex Valued b-Metric Space which is submitted by me to the Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree of Master of Science in Mathematics, is original and not copied from any source without proper citation.This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship or other similar title or recognition.

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CERTIFICATE

I hereby certify that the Project Dissertation titled Some Results Related to Fixed Point Theorem In Complex Valued b-Metric Space which has been submitted by Komal Verma (2K19/MSCMAT/19) of Department of Applied Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement for the award of the degree of Master of Mathematics, is a record of the project work carried out by the students under my supervision. To the best of my knowledge, this work has not been submitted in part or full for any Degree or Diploma to this University.

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Jamkhongam Touthang
Supervisor

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ABSTRACT

We gave some new results related to common fixed point in complex valued b-metric space for a pair of mappings satisfying more general contraction conditions of complex valued b-metric spaces introduced by Bakhtin [5]. Further, we provided related results for composition of metrics in complex valued b-metric space.

Contents

Chapter 1

Introduction

This chapter introduces the basic concepts of fixed point theorem which include metric spaces, complete metric spaces, contraction mapping, complex valued metric spaces and complex valued b-metric space, and some results which help to understand about the main concept of our dissertation.

1.1 Metric Spaces

A function $d : A \times A \to \mathbb{R}$ to find distance between elements a, b of an arbitrary set A, which could consist of vectors in $\mathbb{R}^n$, functions, sequences, matrices, etc is called a metric or distance. The distance between elements $a, b \in A$ is given by the number $d(a, b)$.

Definition 1.1.1. Define a function $d : A \times A \to \mathbb{R}$ on an arbitrary set A, the pair (A, d) is said to be metric space iff $\forall\ a, b \in A$, $d(a, b)$ fulfil these properties:

(i) $d(a, b) \geq 0$ (Non-negativeness)

(ii) $d(a, b) = 0 \iff a = b$ (Identification)

(iii) $d(a, b) = d(b, a)$ (Symmetry)

(iv) For $a, b, c \in A$, $d(a, b) \leq d(a, c) + d(c, b)$ (Triangular inequality)

2.1.2. Results

2.1.2.1 Euclidean Metric Suppose $A = \mathbb{R}$ and define a function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(a, b) = |a - b|\ \forall\ a, b \in A$ is a metric space (i.e., *Standard Metric Space*). In general, let $A = \mathbb{R}^n$:

$$\mathbb{R}^n = \{(z_1, z_2, ..., z_n) | z_i \in \mathbb{R}, i = 1\ to\ n\}$$

and for $a = (a_1, a_2, .., a_n), b = (b_1, b_2, .., b_n) \in \mathbb{R}^n$, defines the funtion $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$d(a, b) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

is a metric on $\mathbb{R}^n$, called the *Euclidean Metric*

2.1.2.2 Discrete Metric Space A function $d : A \times A \to \mathbb{R}$ is defined on a set A by

$$d(a_1, a_2) = \begin{cases} 0 & if\ a_1 = a_2 \\ 1 & if\ a_1 \neq a_2 \end{cases}$$

(A, d) forms a metric space.

1.2 Complete metric space

We know that any convergent sequence in a metric space (A, d) is Cauchy but a Cauchy sequence in a metric space (A, d) may or may not be a convergent sequence in that metric space.

Definition 1.2.1. A metric space (A, d) is complete iff every Cauchy sequence in A converges to a point in A.
Open intervals (a, b) are not complete.
Standard metric space of real numbers $\mathbb{R}$ is complete. In general, $\mathbb{R}^n$ with standard metric space is complete.
The space $C_b([a, b])$ which is bounded, real valued continuous functions on interval $[a, b]$, is complete.

Some important results of Complete metric spaces

There are so many results which include that In a metric space (A, d), any sequence which is Cauchy is bounded; In a metric space, Any convergent sequence is a Cauchy sequence; a complete subspace of a metric space (A, d) is a closed subset; and many more. Some of the important results are:

Theorem 1.2.2. *A subset (which is closed) of a complete metric space is a complete subspace.*

Theorem 1.2.3. *Consider a metric space (A, d) and a complete and bounded subset $B \subseteq A$. Then B is compact.*

Theorem 1.2.4. *Suppose (A, d) is a complete metric space. If (F_n) is a sequence of non-empty closed subsets of A with the property that $F_{n+1} \subseteq F_n\ \forall\ n \in N$ and $(diam(F_n)) \to\ 0$, then $\cap_{n=1}^{\infty} F_n$ is a singleton.*

1.3 Fixed point theorem

The Banach fixed point theorem is one of the most fundamental and important theorems in metric space which have many useful applications in mathematics as well as in real life.

Theorem 1.3.1. ***(Contraction Mapping Theorem or Banach Fixed point theorem)***. *Suppose a map G from a complete metric space (X, d) to itself is a contraction. Then, G has a unique fixed-point and under the action of iterates of $G : X \to X$, all points converge with exponential speed to it.*

Theorem 1.3.2. ***(Kannan Fixed Point Theorem [?])*** *Suppose that f and g are a self map on a complete metric space (X, d). If $\exists$ a constant $\alpha \in [0, 1/2)$ such that*

$$d(f(a), g(b)) \leq \alpha(d(a, f(a)) + d(b, g(b))) \; \forall \; a, b \in X,$$

then f and g have a unique common fixed point.
In particular, if $\exists$ a constant $\beta \in [0, 1/2)$ such that $d(f(a), f(b)) \leq \beta(d(a, f(a) + d(b, f(b))) \; \forall \; a, b \in X$, then f has a unique fixed point.

Theorem 1.3.3. ***(Gupta and Srivastava [?])*** *Suppose f and g are a self map on a complete metric space (X, d). If $\exists$ a non-negative integers $i, \; i$ and constants $\alpha, \beta \in [0, 1)$ such that*

$$d(f^i(a), g^j(b)) \leq \alpha d(a, f^i(a)) + \beta d(b, g^j(b))$$

$\forall \; a, b \in X$, then f and g have a unique common fixed point.

Theorem 1.3.4. ***(Suzuki [?])*** *Suppose that f is a self map on a metric space (X, d) and a function $\theta_1 : [0, 1] \to (1/2, 1]$ is defined by*

$$\theta_1(s) := \begin{cases} 1 & if \; 0 \leq s \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-s}{s^2} & if \; \frac{\sqrt{5}-1}{2} \leq s \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+s} & if \; \frac{1}{\sqrt{2}} \leq s < 1. \end{cases}$$

Suppose that there exists a constant $s \in [0, 1)$ such that
$\theta_1(s)d(a, f(a)) \leq d(a, b)$ implies $d(f(a), f(b)) \leq sd(a, b) \; \forall \; a, b \in X$.
If X is complete, then f has a unique fixed point z. Moreover $\lim f^n(a) = z \; \forall a \in X$.
Conversely, if every self map f on X satisfying the above condition then f has a fixed point, then X is complete.

1.4 Complex Valued Metric Space

Azam et al. *[1] was the first one to establish the concept of Complex Valued Metric Spaces and then investigate the existence and uniqueness of the fixed point results for mapping reassuring the rational inequalities.*
Suppose the set of complex numbers be $\mathbb{C}$ *and* $w_1, w_2 \in \mathbb{C}$ *and define a partial order* $\precsim$ *on* $\mathbb{C}$ *as follows:*

$$w_1 \precsim w_2 \; if \; and \; only \; if \; Re(w_1) \leq Re(w_2), \; Im(w_1) \leq Im(w_2).$$

$w_1 \precsim w_2$ *if we have one of the following cases:*

$$(i) \; Re(w_1) = Re(w_2), \; Im(w_1) < Im(w_2)$$
$$(ii) \; Re(w_1) < Re(w_2), \; Im(w_1) = Im(w_2)$$
$$(iii) \; Re(w_1) < Re(w_2), \; Im(w_1) < Im(w_2)$$
$$(iv) \; Re(w_1) = Re(w_2), \; Im(w_1) = Im(w_2)$$

$w_1 \precnsim w_2$ *if* $w_1 \neq w_2$ *and one of* (i), (ii) *and* (iii) *is acceptable and* $w_1 \prec w_2$ *if only* (iii) *is served. We also have*

$$0 \precsim w_1 \; but \; w_1 \precnsim w_2 \implies |w_1| < |w_2|$$
$$w_1 \precsim w_2 \prec w_3 \implies w_1 \prec w_3$$

Definition 1.4.1. Let A be a nonempty set. A mapping $d : A \times A \to \mathbb{C}$ is said to be complex valued metric on A if the following conditions are satisfied:

(i) $0 \precsim d(a, b)$ $\forall a, b \in A$ and $d(a, b) = 0$ if and only if $a = b$;

(ii) $d(a, b) = d(b, a) \forall \; a, b \in A$:

(iii) $d(a, b) \precsim d(a, c) \; + \; d(c, b) \; \forall a, b, c \; \in \; A$

Then (A, d) is complex valued metric space.

Example 1.4.2. *Let* $A = \mathbb{C}$. *Then* (A, d) *is a complex valued metric space for the mapping* $d : A \times A \to \mathbb{C}$ *defined by*

$$d(a, b) = |a - b| + i|a - b|, \; \forall \; a, b \; \in \; A.$$

Definition 1.4.3. [4] Suppose that (A, d) is a Complex-valued metric space and $\{a_n\}$ is a sequence in A and $a \in A$. If for every $e \in \mathbb{C}$ with $0 \prec e$, $\exists \; N \in \mathbb{N}$ such that $\forall \; n > N$, $d(a_n, a) \prec e$, then we state that $\{a_n\}$ is convergent, $\{a_n\}$ converges to a and a is the limit of $\{a_n\}$. Symbolically, $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ as $n \to \infty$.

If for every $e \in \mathbb{C}$, with $0 \prec e$, $\exists \; N \in \mathbb{N}$ such that $\forall \; n > N$, $d(a_n, a_{n+m}) \prec e$ where $m \in \mathbb{N}$, then we say $\{a_n\}$ is said to be a Cauchy sequence.

If every Cauchy sequence in A is convergent in A, then (A, d) is said to be a Complete Complex valued metric space.

Example 1.4.4. *Suppose $A = \mathbb{C}$. Then (A, d) is a Complex valued Metric space for the mapping $d : A \times A \to \mathbb{C}$ defined by*

$$d(a, b) = |a - b| + i|a - b|, \; \forall \; a, b \; \in \; A.$$

Lemma 1.4.5. *Suppose (A, d) is a Complex valued-metric space and $\{a_n\}$ a sequence in A. Then $\{a_n\}$ is a convergent sequence with limit $a \iff |d(a_n, a)| \to 0$ as $n \to \infty$.*

Lemma 1.4.6. *Suppose (A, d) is a Complex valued metric space and $\{a_n\}$ a sequence in A. Then $\{a_n\}$ is said to be a Cauchy sequence $\iff |d(a_n, a_{n+m})| \to 0$ as $n \to \infty$.*

1.5 Complex valued *b*-metric space

Definition 1.5.1. [4] Suppose A is a nonempty set and let $t \geq 1$ be a given real number. Define a mapping $d : A \times A \to \mathbb{C}$. Then d is said to be Complex value b-metric on A if these conditions hold:

(i) $0 \precsim d(a, b) \; \forall a, b \in A$ and $d(a, b) = 0$ if and only if $a = b$;

(ii) $d(a, b) = d(b, a) \forall \; a, b \in A$:

(iii) $d(a, b) \precsim t[d(a, c) \; + \; d(c, b)] \; \forall a, b, c \; \in \; A$

Then (A, d) is called a Complex-valued b-metric space. Thus a complex valued metric space is a particular case of complex valued b-metric space.

Definition 1.5.2. [4] Suppose that (A, d) is a Complex-valued b-metric space and $\{a_n\}$ is a sequence in A and $a \in A$. If for every $e \in \mathbb{C}$ with $0 \prec e$, $\exists \; N \in \mathbb{N}$ such that $\forall \; n > N$, $d(a_n, a) \prec e$, then we state that $\{a_n\}$ is convergent, $\{a_n\}$ converges to a and a is the limit of $\{a_n\}$. Symbolically, $\lim_{n \to \infty} a_n = a$ or $a_n \to a$ as $n \to \infty$.

If for every $e \in \mathbb{C}$, with $0 \prec e$, $\exists \; N \in \mathbb{N}$ such that $\forall \; n > N$, $d(a_n, a_{n+m}) \prec e$ where $m \in \mathbb{N}$, then we say $\{a_n\}$ is said to be a Cauchy sequence.

If every Cauchy sequence in A is convergent in A, then (A, d) is said to be a Complete Complex valued b-metric space.

Lemma 1.5.3. *Suppose (A, d) is a Complex valued b-metric space and $\{a_n\}$ a sequence in A. Then $\{a_n\}$ is a convergent sequence with limit $a \iff |d(a_n, a)| \to 0$ as $n \to \infty$.*

Lemma 1.5.4. *Suppose (A, d) is a Complex valued b-metric space and $\{a_n\}$ a sequence in A. Then $\{a_n\}$ is said to be a Cauchy sequence $\iff |d(a_n, a_{n+m})| \to 0$ as $n \to \infty$.*

Lemma 1.5.5. *[6] Let* $\alpha, \beta \in (0,1)$ *and* $a, b \in A$. *If* $X,\ Y$ *satisfy*

$$d(Xa, YXa) \precsim \alpha d(a, Xa) + \beta \frac{d(a, Xa)d(Xa, YXa)}{1 + d(a, Xa)}$$

$$d(XYb, Yb) \precsim \alpha d(Yb, b) + \beta \frac{d(Yb, XYb)d(b, Yb)}{1 + d(Yb, b)}$$

then

$$|d(Xa, YXa)| \leq \alpha |d(a, Xa)| + \beta |d(Xa, YXa)|,$$
$$|d(XYb, Yb)| \leq \alpha |d(Yb, b)| + \beta |d(Yb, XYb)|$$

respectively.

Lemma 1.5.6. *[7] Let* $\{a_n\}$ *be a sequence in* A *and* $h \in (0,1)$. *If* $x_n = |d(a_n, a_{n+1})|$ *satisfies*

$$x_n \leq h x_{n-1},\ \forall\ n\ \in \mathbb{N}$$

then $\{a_n\}$ *is a Cauchy sequence.*

Chapter 2

Main Results

Proposition 2.0.1. *Suppose (A, d) is a Complex valued b-metric space and $a_0 \in A$. Define the mappings $X, Y : A \to A$ and a sequence $\{a_n\}$ by*

$$a_{2n+1} = Xa_{2n}, \quad a_{2n+2} = Ya_{2n+1}, \ \ \forall\, n = 0, 1, 2, ...$$

Suppose $\exists$ a mapping $\lambda : A \times A \to [0, 1)$ such that $\lambda(YXa, \alpha) \leq \lambda(a, \alpha)$ and $\lambda(XYa, \alpha) \leq \lambda(a, \alpha)$ $\forall\, a \in A$ and for fixed element $\alpha \in A$. Then

$$\lambda(a_{2n}, \alpha) \leq \lambda(a_0, \alpha)\ ,\ \lambda(a_{2n+1}, \alpha) \leq \lambda(a_1, \alpha), \ \ \forall\, n = 0, 1, 2, \ldots$$

Proof. Let $a \in A$. Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}\lambda(a_{2n}, \alpha) = \lambda(YXa_{2n-2}, \alpha) \leq \lambda(a_{2n-2}, \alpha) &= \lambda(YXa_{2n-4}, \alpha)\\ &\leq \lambda(a_{2n-4}, \alpha) \leq \ldots \leq \lambda(a_0, \alpha)\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda(a_{2n+1}, \alpha) = \lambda(XYa_{2n-1}, \alpha) \leq \lambda(a_{2n-1}, \alpha) &= \lambda(XYa_{2n-3}, \alpha)\\ &\leq \lambda(a_{2n-3}, \alpha) \leq \ldots \leq \lambda(a_1, \alpha)\end{aligned}$$

□

Example 2.0.2. *Let $A = \{1, 1/2, 1/2^2, 1/2^3, \ldots\}$ and define $d : A \times A \to \mathbb{C}$ as*

$$d(a, b) = |a - b|^2 + i|a - b|^2$$

Clearly, (A, d) is a Complex valued b-metric space with $t = 2$.
Define $X, Y : A \to A$ by

$$X\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}} = Y\left(\frac{1}{2^n}\right), \ \forall\, n = 0, 1, 2, \ldots.$$

Consider the sequence $\{a_n\}$ *where* $a_n = \frac{1}{2^n},\ \forall\ n = 0, 1, 2, \ldots.$
Then $a_0 = 1,\ a_1 = \frac{1}{2} \in A$. *Clearly,* $Xa_{2n} = a_{2n+1}$ *and* $Ya_{2n+1} = a_{2n+2}$.
Consider a mapping $\lambda : A \times A \to [0, 1)$ *defined by* $\lambda(a, \alpha) = \frac{a}{2} + \frac{\alpha}{3},\ \forall\ a \in A$ *so that for fixed* $\alpha = \frac{1}{2} \in A$, $\lambda(a, \alpha) = \frac{a}{2} + \frac{1}{6}$.
Clearly, $\lambda(YXa, \alpha) \leq \lambda(a, \alpha),\ \lambda(XYa, \alpha) \leq \lambda(a, \alpha),\ \forall\ a \in A$ *and for fixed* $\alpha = \frac{1}{2} \in A$.
Further

$$\lambda(a_{2n}, \alpha) = \frac{1}{2^{2n}.2} + \frac{1}{6} \leq \frac{1}{2} + \frac{1}{6} = \lambda(a_0, \alpha)$$

and

$$\lambda(a_{2n+1}, \alpha) = \frac{1}{2^{2n+1}.2} + \frac{1}{6} \leq \frac{1}{2.2} + \frac{1}{6} = \frac{(1/2)}{2} + \frac{1}{6} = \lambda(a_1, \alpha)$$

which verifies proposition 2.1.

Theorem 2.0.3. *Suppose* (A, d) *is a complete Complex valued* b*-metric space with coefficient* $t \geq 1$. *Define self mappings* $X, Y : A \to A$. *Suppose* $\exists$ *mappings* $\lambda, \mu : A \times A \to [0, 1)$ *such that* $\forall\ a, b \in A$ *and for fixed* $\alpha \in A$,

(i) $\lambda(YXa, \alpha) \leq \lambda(a, \alpha), \lambda(XYa, \alpha) \leq \lambda(a, \alpha)$,
$\mu(YXa, \alpha) \leq \mu(a, \alpha), \mu(XYa, \alpha) \leq \mu(a, \alpha)$

$$(ii)\ \ d(Xa, Yb) \precsim \left[\frac{\lambda(a, \alpha) + \lambda(b, \alpha)}{2}\right] d(a, b) + \left[\frac{\mu(a, \alpha) + \mu(b, \alpha)}{2}\right] \left[\frac{d(a, Xa)d(b, Yb)}{1 + d(a, b)}\right]$$

(iii) $2t\lambda(a, \alpha) + 2\mu(a, \alpha) < 1\ \forall a \in A$ *and for fixed* $\alpha\ \in A$

Then, X, Y *have a unique common fixed point in* X.

Proof. Let $a, b \in A$ and α a fixed point in A.

$$d(Xa, YXa) \precsim \left[\frac{\lambda(a, \alpha) + \lambda(Xa, \alpha)}{2}\right] d(a, Xa) + \left[\frac{\mu(a, \alpha) + \mu(Xa, \alpha)}{2}\right] \left[\frac{d(a, Xa)d(Xa, YXa)}{1 + d(a, Xa)}\right]$$

$$|d(Xa, YXa)| \leq \left[\frac{\lambda(a, \alpha) + \lambda(Xa, \alpha)}{2}\right] |d(a, Xa)| + \left[\frac{\mu(a, \alpha) + \mu(Xa, \alpha)}{2}\right] |d(Xa, YXa)| \tag{2.0.1}$$

Similarly,

$$\begin{aligned}
d(XYb, Yb) &\precsim \left[\frac{\lambda(Yb,\alpha)+\lambda(b,\alpha)}{2}\right] d(Yb,b) \\
&\qquad + \left[\frac{\mu(Yb,\alpha)+\mu(b,\alpha)}{2}\right]\left[\frac{d(Yb,XYb)d(b,Yb)}{1+d(Yb,b)}\right] \\
|d(XYb, Yb)| &\leq \left[\frac{\lambda(Yb,\alpha)+\lambda(b,\alpha)}{2}\right] |d(Yb,b)| \\
&\qquad + \left[\frac{\mu(Yb,\alpha)+\mu(b,\alpha)}{2}\right] |d(Yb,XYb)|
\end{aligned} \tag{2.0.2}$$

Let $a_0, a_1 \in A$ be arbitrary and let $\{a_n\}$ be a sequence defined by

$$a_{2n+1} = Xa_{2n}, \quad a_{2n+2} = Ya_{2n+1}, \ \forall\ n = 0,1,2,\ldots$$

as in proposition 2.1. Now, for $k = 0,1,2,\ldots$,

$$\begin{aligned}
|d(a_{2k+1}, a_{2k})| &= |d(XYa_{2k-1}, Ya_{2k-1})| \\
&\leq \left[\frac{\lambda(Ya_{2k-1},\alpha)+\lambda(a_{2k-1},\alpha)}{2}\right] |d(Ya_{2k-1}, a_{2k-1})| \\
&\qquad + \left[\frac{\mu(Ya_{2k-1},\alpha)+\mu(a_{2k-1},\alpha)}{2}\right] |d(Ya_{2k-1}, XYa_{2k-1})| \\
&= \left[\frac{\lambda(a_{2k},\alpha)+\lambda(a_{2k-1},\alpha)}{2}\right] |d(a_{2k}, a_{2k-1})| \\
&\qquad + \left[\frac{\mu(a_{2k},\alpha)+\mu(a_{2k-1},\alpha)}{2}\right] |d(a_{2k}, a_{2k+1})| \\
&\leq \left[\frac{\lambda(a_0,\alpha)+\lambda(a_1,\alpha)}{2}\right] |d(a_{2k-1}, a_{2k})| \\
&\qquad + \left[\frac{\mu(a_0,\alpha)+\mu(a_1,\alpha)}{2}\right] |d(a_{2k}, a_{2k+1})|
\end{aligned}$$

$$\therefore \qquad |d(a_{2k+1}, a_{2k})| \leq \left[\frac{\lambda(a_0,\alpha)+\lambda(a_1,\alpha)}{2-[\mu(a_0,\alpha)+\mu(a_1,\alpha)]}\right] |d(a_{2k-1}, a_{2k})| \tag{2.0.3}$$

Similarly,

$$|d(a_{2k+2}, a_{2k+1})| \leq \frac{\lambda(a_0,\alpha)+\lambda(a_1,\alpha)}{2-[\mu(a_0,\alpha)+\mu(a_1,\alpha)]} |d(a_{2k}, a_{2k+1})| \tag{2.0.4}$$

By condition (iii), $h = \dfrac{\lambda(a_0,\alpha)+\lambda(a_1,\alpha)}{2-[\mu(a_0,\alpha)+\mu(a_1,\alpha)]} < 1.$

This gives

$$|d(a_{2k+1}, a_{2k})| \leq h|d(a_{2k}, a_{2k-1})| \text{ and } |d(a_{2k+2}, a_{2k+1})| \leq h|d(a_{2k+1}, a_{2k})|$$

Therefore

$$|d(a_{2n+1}, a_{2n})| \leq h|d(a_{2n}, a_{2n-1})|, \ \forall\, n \geq 0.$$

From Lemma 1.8, we deduce that $\{a_n\}$ is a Cauchy sequence in (A, d). By completeness property of complex valued b-metric A, $\exists\ a \in A$ such that $a_n \to a$ as $n \to \infty$.
We will show that a is the common fixed point of X and Y. By condition (ii) and Proposition 2.1, we have

$$\begin{aligned}
d(a, Xa) &\precsim t\{d(a, Ya_{2n+1}) + d(Ya_{2n+1}, Xa)\}\\
&= t\{d(a, a_{2n+2}) + d(Xa, Ya_{2n+2})\}\\
&\precsim t\left[d(a, a_{2n+2}) + \left\{\frac{\lambda(a,\alpha) + \lambda(a_{2n+1},\alpha)}{2}\right\} d(a, a_{2n+1})\right.\\
&\qquad \left. + \left\{\frac{\mu(a,\alpha) + \mu(a_{2n+1},\alpha)}{2}\right\}\left\{\frac{d(a, Xa)d(a_{2n+1}, Ya_{2n+1})}{1 + d(a, a_{2n+1})}\right\}\right]\\
&\precsim t\left[d(a, a_{2n+2}) + \left\{\frac{\lambda(a,\alpha) + \lambda(a_1,\alpha)}{2}\right\} d(a, a_{2n+1})\right.\\
&\qquad \left. + \left\{\frac{\mu(a,\alpha) + \mu(a_1,\alpha)}{2}\right\}\left\{\frac{d(a, Xa)d(a_{2n+1}, a_{2n+2})}{1 + d(a, a_{2n+1})}\right\}\right]
\end{aligned}$$

Letting $n \to \infty$, we obtain $d(a, Xa) = 0$ giving $Xa = a$.
Similarly,

$$\begin{aligned}
d(a, Ya) &\precsim t\{d(a, Xa_{2n}) + d(Xa_{2n}, Ya)\}\\
&\precsim t\left[d(a, a_{2n+1}) + \left\{\frac{\lambda(a_{2n},\alpha) + \lambda(a,\alpha)}{2}\right\} d(a_{2n}, a)\right.\\
&\qquad \left. + \left\{\frac{\mu(a_{2n},\alpha) + \mu(a,\alpha)}{2}\right\}\left\{\frac{d(a_{2n}, Xa_{2n})d(a, Ya)}{1 + d(a_{2n}, a)}\right\}\right]\\
&\precsim t\left[d(a, a_{2n+1}) + \left\{\frac{\lambda(a_0,\alpha) + \lambda(a,\alpha)}{2}\right\} d(a_{2n}, a)\right.\\
&\qquad \left. + \left\{\frac{\mu(a_0,\alpha) + \mu(a,\alpha)}{2}\right\}\left\{\frac{d(a_{2n}, a_{2n+1})d(a, Ya)}{1 + d(a_{2n}, a)}\right\}\right]
\end{aligned}$$

Letting $n \to \infty$, we have $d(a, Ya) = 0$ giving $Ya = a$.
The uniqueness can be easily established by using conditions (ii) and (iii). □

Taking $\mu = 0$ in Theorem 2.3, we get the following corollary.

Corollary 2.0.4. *Suppose (A, d) is a complete Complex valued b-metric space with coefficient $t \geq 1$. Define self mapping $X, Y : A \to A$. Suppose $\exists$ a mapping $\lambda : A \times A \to [0, 1)$ such that $\forall\ a, b \in A$ and for fixed $\alpha \in A$,*

(i) $\lambda(YXa,\alpha) \leq \lambda(a,\alpha), \lambda(XYa,\alpha) \leq \lambda(a,\alpha)$

(ii) $d(Xa,Yb) \precsim \dfrac{\lambda(a,\alpha)+\lambda(b,\alpha)}{2} d(a,b)$

(iii) $2t\lambda(a,\alpha) < 1$ $\forall a \in A$ *and for fixed* $\alpha \in A$.

Then, X, Y *have a unique common fixed point in* A. *Taking* $X = Y$ *in Theorem 2.3, we obtain this result.*

Corollary 2.0.5. *Suppose* (A,d) *be a complete Complex valued* b*-metric space with coefficient* $t \geq 1$. *Define self mapping* $Y : A \to A$. *Suppose* $\exists$ *mappings* λ, μ : $A \times A \to [0,1)$ *such that* $\forall$ $a,b \in A$ *and for fixed* $\alpha \in A$,

(i) $\lambda(Y^2 a,\alpha) \leq \lambda(a,\alpha), \quad \mu(Y^2 a,\alpha) \leq \mu(a,\alpha)$

$$(ii)\ \ d(Ya,Yb) \precsim \left[\frac{\lambda(a,\alpha)+\lambda(b,\alpha)}{2}\right] d(a.b) + \left[\frac{\mu(a,\alpha)+\mu(b,\alpha)}{2}\right]\left[\frac{d(a,Ya)d(b,Yb)}{1+d(a,b)}\right]$$

(iii) $2t\lambda(a,\alpha) + 2\mu(a,\alpha) < 1$ $\forall a \in A$ *and for fixed* α $\in A$.

Then Y *has a unique fixed point in* A.

Example 2.0.6. *Let* $A = [0,1]$. *Define* $d : A \times A \to \mathbb{C}$ *as*

$$d(a,b) = |a-b|^2 + i|a-b|^2.$$

Clearly, (A,d) *is a complex valued* b*-metric space with* $t = 2$.
Define X, $Y : A \to A$ *as*

$$X(a) = \frac{a}{11},\ Y(b) = \frac{b}{11},\ \forall\ a,\ b \in A.$$

For fixed $\alpha = 1/7 \in A$, *let* λ, $\mu : A \times A \to [0,1)$ *be defined as*

$$\lambda(a,\alpha) = \frac{a}{11} + \frac{\alpha}{4},\ \mu(a,\alpha) = \frac{a\alpha}{110},\ \forall\ a \in A$$

Clearly,

$$\begin{aligned} &2t\lambda(a,\alpha) + 2\mu(a,\alpha) < 1, \\ &\lambda(YXa,\alpha) \leq \lambda(a,\alpha),\ \lambda(XYa,\alpha) \leq \lambda(a,\alpha), \\ &\mu(YXa,\alpha) \leq \mu(a,\alpha),\ \mu(XYa,\alpha) \leq \mu(a,\alpha). \end{aligned}$$

Also, $\forall\ a, b \in A$*, we have*

$$0 \precsim d(a,b), d(Xa,Yb), \frac{d(a,Xa)d(b,Yb)}{1+d(a,b)}$$

It is enough to show that

$$d(Xa,Yb) \precsim \frac{\lambda(a,\alpha)+\lambda(b,\alpha)}{2} d(a,b)$$

Now

$$\begin{aligned}
d(Xa,Yb) = d(a/11, b/11) &= \left|\frac{a}{11} - \frac{b}{11}\right|^2 + i\left|\frac{a}{11} - \frac{b}{11}\right|^2 \\
&= \frac{1}{121}\{|a-b|^2 + i|a-b|^2\} \\
&\precsim \frac{1}{7.4}\{|a-b|^2 + i|a-b|^2\} \\
&\precsim \left(\frac{1}{7\times 4} + \frac{a}{2\times 11} + \frac{b}{2\times 11}\right) d(a,b) \\
&= \frac{1}{2}\left(\frac{a}{11} + \frac{\alpha}{4} + \frac{b}{11} + \frac{\alpha}{4}\right) d(a,b) \\
&= \left[\frac{\lambda(a,\alpha)+\lambda(b,\alpha)}{2}\right] d(a,b)
\end{aligned}$$

All conditions of Theorem 2.3 are satisfied. Also $a = 0$ *remains fixed point under* X, Y *and it is unique.*

Theorem 2.0.7. *Let* (A, d) *be a complete complex valued* b*-metric space with coefficient* $t \geq 1$*. Define self mapping* $X, Y : A \to A$*. If* $\exists$ *mappings* $\lambda,\ \mu : A \times A \to [0, 1)$ *such that* $\forall\ a, b \in A$ *and for fixed* $\alpha\ \in A$*,*

(i) $\lambda(YXa,\alpha) \leq \lambda(a,\alpha), \lambda(XYa,\alpha) \leq \lambda(a,\alpha)$
$\mu(YXa,\alpha) \leq \mu(a,\alpha), \mu(XYa,\alpha) \leq \mu(a,\alpha)$

(ii) $$\begin{aligned} d(Xa,Yb) \precsim &\left[\frac{\lambda(a,\alpha)+\lambda(b,\alpha)}{2}\right] d(a,b) \\ &+ \left[\frac{\mu(a,\alpha)+\mu(b,\alpha)}{2}\right]\left[\frac{d(a,Xa)d(b,Yb)}{d(a,Yb)+d(b,Xa)+d(a,b)}\right]\end{aligned}$$
$\forall\ a, b \in A$ *such that* $a \neq b$*,* $d(a,Yb) + d(b,Xa) + d(a,b) \neq 0$ *where* $2\lambda(a,\alpha) + 2t\mu(a,\alpha) < 1\ \forall a \in A$ *and for fixed* $\alpha\ \in A$

or

$d(Xa,Yb) = 0$ *if* $d(a,Yb) + d(b,Xa) + d(a,b) = 0.$

Then, X, Y *have a unique common fixed point.*

Proof. Let $a, b \in A$ and $\alpha \in A$ be fixed.
Let $a_0, a_1 \in A$ be arbitrary and $\{a_n\}$ a sequence defined as

$$a_{2n+1} = Xa_{2n}, \qquad a_{2n+2} = Ya_{2n+1}, \ \ \forall\, n = 0, 1, 2, \ldots$$

as defined in Proposition 2.1.

$$\begin{aligned} d(a_{2n+1}, a_{2n}) &= d(Xa_{2n}, Ya_{2n-1}) \\ &\precsim \left[\frac{\lambda(a_{2n}, \alpha) + \lambda(a_{2n-1}, \alpha)}{2}\right] d(a_{2n}, a_{2n-1}) + \left[\frac{\mu(a_{2n}, \alpha) + \mu(a_{2n-1}, \alpha)}{2}\right] \\ &\qquad \times \left[\frac{d(a_{2n}, Xa_{2n}) d(a_{2n-1}, Ya_{2n-1})}{d(a_{2n}, Ya_{2n-1}) + d(a_{2n-1}, Xa_{2n}) + d(a_{2n}, a_{2n-1})}\right] \\ &= \left[\frac{\lambda(a_0, \alpha) + \lambda(a_1, \alpha)}{2}\right] d(a_{2n}, a_{2n-1}) + \left[\frac{\mu(a_0, \alpha) + \mu(a_1, \alpha)}{2}\right] \\ &\qquad \times \left[\frac{d(a_{2n}, a_{2n+1}) d(a_{2n-1}, a_{2n})}{d(a_{2n}, a_{2n}) + d(a_{2n-1}, a_{2n+1}) + d(a_{2n}, a_{2n-1})}\right] \end{aligned}$$

$$\begin{aligned} |d(a_{2n+1}, a_{2n})| \leq& \left[\frac{\lambda(a_0, \alpha) + \lambda(a_1, \alpha)}{2}\right] |d(a_{2n}, a_{2n-1})| \\ &+ \left[\frac{\mu(a_0, \alpha) + \mu(a_1, \alpha)}{2}\right] \frac{|d(a_{2n}, a_{2n+1})||d(a_{2n-1}, a_{2n})|}{|d(a_{2n-1}, a_{2n+1})| + |d(a_{2n}, a_{2n-1})|} \end{aligned} \tag{2.0.5}$$

(using triangular inequality of complex valued b-metric space)

$$\begin{aligned} |d(a_{2n+1}, a_{2n})| \leq& \left[\frac{\lambda(a_0, \alpha) + \lambda(a_1, \alpha)}{2}\right] |d(a_{2n}, a_{2n-1})| \\ &+ t\left[\frac{\mu(a_0, \alpha) + \mu(a_1, \alpha)}{2}\right] |d(a_{2n-1}, a_{2n})| \\ =& \frac{1}{2}\left[(\lambda(a_0, \alpha) + \lambda(a_1, \alpha)) + t(\mu(a_0, \alpha) + \mu(a_1, \alpha))\right] |d(a_{2n-1}, a_{2n})| \\ |d(a_{2n+1}, a_{2n})| \leq& \frac{1}{2}[(\lambda(a_0, \alpha) + \lambda(a_1, \alpha)) + t(\mu(a_0, \alpha) + \mu(a_1, \alpha))]|d(a_{2n-1}, a_{2n})| \end{aligned}$$

By condition (ii), $2\lambda(a, \alpha) + 2t\mu(a, \alpha) < 1$ so that $h = \frac{1}{2}[(\lambda(a_0, \alpha) + \lambda(a_1, \alpha)) + t(\mu(a_0, \alpha) + \mu(a_1, \alpha))] < 1$. Therefore,

$$|d(a_{2n+1}, a_{2n})| \leq h|d(a_{2n}, a_{2n-1})|.$$

By Lemma 1.8, $\{a_n\}$ is a Cauchy sequence. By completeness of A, there exists $a \in A$ such that $a_n \to a$ as $n \to \infty$.
We now show that a is a common fixed point of X and Y.

By condition (ii) and Proposition 2.1, we have

$$\begin{aligned}
d(a, Xa) &\precsim t\{d(a, Ya_{2n+1}) + d(Ya_{2n+1}, Xa)\} \\
&= t\{d(a, a_{2n+2}) + d(Xa, Ya_{2n+1})\} \\
&\precsim t\left[d(a, a_{2n+2}) + \left\{\frac{\lambda(a,\alpha) + \lambda(a_{2n+1},\alpha)}{2}\right\} d(a, a_{2n+1})\right. \\
&\qquad \left. + \left\{\frac{\mu(a,\alpha) + \mu(a_{2n+1},\alpha)}{2}\right\}\left\{\frac{d(a, Xa)d(a_{2n+1}, Ya_{2n+1})}{d(a, Ya_{2n+1}) + d(a_{2n+1}, Xa) + d(a, a_{2n+1})}\right\}\right] \\
&\precsim t\left[d(a, a_{2n+2}) + \left\{\frac{\lambda(a,\alpha) + \lambda(a_1,\alpha)}{2}\right\} d(a, a_{2n+1})\right. \\
&\qquad \left. + \left\{\frac{\mu(a,\alpha) + \mu(a_1,\alpha)}{2}\right\}\left\{\frac{d(a, Xa)d(a_{2n+1}, a_{2n+2})}{d(a, a_{2n+2}) + d(a_{2n+1}, Xa) + d(a, a_{2n+1})}\right\}\right]
\end{aligned}$$

Letting $n \to \infty$,we get $d(a, Xa) = 0$ which yields $Xa = a$.
Similarly,

$$\begin{aligned}
d(a, Ya) &\precsim s[d(a, Xa_{2n}) + d(Xa_{2n}, Ya)] \\
&\precsim s\left[d(a, a_{2n+1}) + \left\{\frac{\lambda(a_{2n},\alpha) + \lambda(a,\alpha)}{2}\right\} d(a_{2n}, a)\right. \\
&\qquad \left. + \left\{\frac{\mu(a_{2n},\alpha) + \mu(a,\alpha)}{2}\right\}\left\{\frac{d(a_{2n}, Xa_{2n})d(a, Ya)}{d(a_{2n}, Ya) + d(a, Xa_{2n}) + d(a_{2n}, a)}\right\}\right] \\
&\precsim s\left[d(a, a_{2n+1}) + \left\{\frac{\lambda(a_0,\alpha) + \lambda(a,\alpha)}{2}\right\} d(a_{2n}, a)\right. \\
&\qquad \left. + \left\{\frac{\mu(a_0,\alpha) + \mu(a,\alpha)}{2}\right\}\left\{\frac{d(a_{2n}, a_{2n+1})d(a, Ya)}{d(a_{2n}, Ya) + d(a, a_{2n+1}) + d(a_{2n}, a)}\right\}\right]
\end{aligned}$$

Letting $n \to \infty$,we get $d(a, Ya) = 0$ which gives $Ya = a$.
The uniqueness follows from condition (ii). □

By taking $X = Y$ in Theorem 2.5, we get this corollary.

Corollary 2.0.8. *Suppose (A, d) is a complete complex valued b-metric space with coefficient $t \geq 1$. Define self mapping $Y : A \to A$. If $\exists$ mappings $\lambda,\ \mu : A \times A \to [0, 1)$ such that $\forall\ a, b \in A$ and for fixed $\alpha\ \in A$,*

(i) $\lambda(Ya, \alpha) \leq \lambda(a, \alpha)$
$\mu(Ya, \alpha) \leq \mu(a, \alpha)$

$$(ii)\quad d(Ya, Yb) \precsim \left[\frac{\lambda(a,\alpha) + \lambda(b,\alpha)}{2}\right] d(a, b) + \left[\frac{\mu(a,\alpha) + \mu(b,\alpha)}{2}\right]\left[\frac{d(a, Ya)d(b, Yb)}{d(a, Yb) + d(b, Ya) + d(a, b)}\right]$$

$\forall\ a, b \in A$ *such that* $a \neq b$, $d(a, Yb) + d(b, Ya) + d(a, b) \neq 0$ *where* $2\lambda(a, \alpha) + 2t\mu(a, \alpha) < 1\ \forall a \in A$ *and for fixed* $\alpha\ \in A$

or

$d(Ya, Yb) = 0$ *if* $d(a, Yb) + d(b, Ya) + d(a, b) = 0$.

Then, Y *have a unique fixed point.*

Taking $t = 1$ *in Theorems 2.3 and 2.5, we obtain these theorems for complex valued metric space which is a particular case of complex valued* b*- metric space.*

Theorem 2.0.9. *Suppose* (A, d) *is a complete complex valued metric space. Define self mappings* $X, Y : A \to A$. *Suppose* $\exists$ *a mapping* $\lambda, \mu : A \times A \to [0, 1)$ *such that* $\forall\ a, b \in A$ *and for fixed* $\alpha\ \in A$,

(i) $\lambda(YXa, \alpha) \leq \lambda(a, \alpha), \lambda(XYa, \alpha) \leq \lambda(a, \alpha)$
$\mu(YXa, \alpha) \leq \mu(a, \alpha), \mu(XYa, \alpha) \leq \mu(a, \alpha)$

$$(ii)\ \ d(Xa, Yb) \precsim \left[\frac{\lambda(a, \alpha) + \lambda(b, \alpha)}{2}\right] d(a, b) + \left[\frac{\mu(a, \alpha) + \mu(b, \alpha)}{2}\right]\left[\frac{d(a, Xa)d(b, Yb)}{1 + d(a, b)}\right]$$

(iii) $2\lambda(a, \alpha) + 2\mu(a, \alpha) < 1\ \forall\ a \in A$ *and for fixed* $\alpha\ \in A$.

Then, X, Y *have a unique common fixed point in* A.

Proof. By Theorem 2.3 and equations (2.3) and (2.4), we get

$$h = \frac{\lambda(a_0, \alpha) + \lambda(a_1, \alpha)}{2 - [\mu(a_0, \alpha) + \mu(a_1, \alpha)]} < 1$$

and

$$|d(a_{2n+1}, a_{2k})| \leq h|d(a_{2n}, a_{2n-1})|,\ \forall\ n \geq 0.$$

From Lemma 1.8, $\{a_n\}$ is a Cauchy sequence in (A, d). By completeness of A, $\exists$ $a \in A$ such that $a_n \to a$ as $n \to \infty$.
By condition (ii) and Proposition 2.1,

$$\begin{aligned}
d(a, Xa) &\precsim d(a, Ya_{2n+1}) + d(Ya_{2n+1}, Xa) \\
&= d(a, a_{2n+2}) + d(Xa, Ya_{2n+2}) \\
&\precsim d(a, a_{2n+2}) + \left[\frac{\lambda(a, \alpha) + \lambda(a_{2n+1}, \alpha)}{2}\right] d(a, a_{2n+1}) \\
&\qquad + \left[\frac{\mu(a, \alpha) + \mu(a_{2n+1}, \alpha)}{2}\right]\left[\frac{d(a, Xa)d(a_{2n+1}, Ya_{2n+1})}{1 + d(a, a_{2n+1})}\right] \\
&\precsim d(a, a_{2n+2}) + \left[\frac{\lambda(a, \alpha) + \lambda(a_1, \alpha)}{2}\right] d(a, a_{2n+1}) \\
&\qquad + \left[\frac{\mu(a, \alpha) + \mu(a_1, \alpha)}{2}\right]\left[\frac{d(a, Xa)d(a_{2n+1}, a_{2n+2})}{1 + d(a, a_{2n+1})}\right]
\end{aligned}$$

Letting $n \to \infty$, we get $d(a, Xa) = 0$ which gives $Xa = a$
Similarly,

$$\begin{aligned} d(a, Ya) &\precsim d(a, Xa_{2n}) + d(Xa_{2n}, Ya) \\ &\precsim d(a, a_{2n+1}) + \left[\frac{\lambda(a_{2n}, \alpha) + \lambda(a, \alpha)}{2}\right] d(a_{2n}, a) \\ &\quad + \left[\frac{\mu(a_{2n}, \alpha) + \mu(a, \alpha)}{2}\right] \left[\frac{d(a_{2n}, Xa_{2n}) d(a, Ya)}{1 + d(a_{2n}, a)}\right] \\ &\precsim d(a, a_{2n+1}) + \left[\frac{\lambda(a_0, \alpha) + \lambda(a, \alpha)}{2}\right] d(a_{2n}, a) \\ &\quad + \left[\frac{\mu(a_0, \alpha) + \mu(a, \alpha)}{2}\right] \left[\frac{d(a_{2n}, a_{2n+1}) d(a, Ya)}{1 + d(a_{2n}, a)}\right] \end{aligned}$$

Letting $n \to \infty$, we obtain $d(a, Ya) = 0$ which gives $Ya = a$.
The uniqueness follows from conditions (ii) and (iii).

□

The result for complex valued metric space is obtained by taking $\mu = 0$.

Corollary 2.0.10. *Suppose (A, d) is a complete complex valued metric space. Define self mappings X, $Y : A \to A$. If $\exists$ a mapping $\lambda : A \times A \to [0, 1)$ such that $\forall\ a, b \in A$ and for fixed $\alpha \in A$,*

(i) $\lambda(YXa, \alpha) \leq \lambda(a, \alpha), \lambda(XYa, \alpha) \leq \lambda(a, \alpha)$,

(ii) $d(Xa, Yb) \precsim \dfrac{\lambda(a, \alpha) + \lambda(b, \alpha)}{2} d(a, b)$,

(iii) $2\lambda(a, \alpha) < 1\ \forall a \in A$ *and for fixed* $\alpha \in A$.

Then, X, Y have a unique common fixed point in A.

Corollary 2.0.11. *Suppose (A, d) is a complete complex valued metric space and Define self mapping $Y : A \to A$. Suppose $\exists$ mappings $\lambda, \mu : A \times A \to [0, 1)$ such that $\forall\ a, b \in A$ and for fixed $\alpha \in A$,*

(i) $\lambda(Y^2 a, \alpha) \leq \lambda(a, \alpha), \quad \mu(Y^2 a, \alpha) \leq \mu(a, \alpha)$,

(ii) $d(Ya, Yb) \precsim \left[\dfrac{\lambda(a, \alpha) + \lambda(b, \alpha)}{2}\right] d(a, b) + \left[\dfrac{\mu(a, \alpha) + \mu(b, \alpha)}{2}\right] \left[\dfrac{d(a, Ya) d(b, Yb)}{1 + d(a, b)}\right]$,

(iii) $2\lambda(a, \alpha) + 2\mu(a, \alpha) < 1\ \forall a \in A$ *and for fixed* $\alpha \in a$.

Then, Y has a unique fixed point in A.

Theorem 2.0.12. *Suppose (A, d) is a complete complex valued metric space and Define self mappings $X, Y : A \to A$. Suppose $\exists$ mappings $\lambda, \mu : A \times A \to [0, 1)$ such that $\forall\ a, b \in A$ and for fixed $\alpha\ \in A$,*

(i) $\lambda(YXa, \alpha) \leq \lambda(a, \alpha), \lambda(XYa, \alpha) \leq \lambda(a, \alpha)$
$\mu(YXa, \alpha) \leq \mu(a, \alpha), \mu(XYa, \alpha) \leq \mu(a, \alpha)$,

$$(ii)\ \ d(Xa, Yb) \precsim \left[\frac{\lambda(a, \alpha) + \lambda(b, \alpha)}{2}\right] d(a, b) + \left[\frac{\mu(a, \alpha) + \mu(b, \alpha)}{2}\right]\left[\frac{d(a, Xa)d(b, Yb)}{d(a, Yb) + d(b, Xa) + d(a, b)}\right]$$

$\forall\ a, b \in A$ *such that* $a \neq b$, $d(a, Yb) + d(b, Xa) + d(a, b) \neq 0$ *where* $2\lambda(a, \alpha) + 2\mu(a, \alpha) < 1\ \forall a \in A$ *and for fixed* $\alpha\ \in A$

or

$d(Xa, Yb) = 0$ *if* $d(a, Yb) + d(b, Xa) + d(a, b) = 0$.

Then, X, Y have a unique common fixed point.

Proof. By equation (2.5) and triangular inequality condition of complex valued metric space, we have

$$\begin{aligned} |d(a_{2n+1}, a_{2n})| &\leq \left[\frac{\lambda(a_0, \alpha) + \lambda(a_1, \alpha)}{2}\right] |d(a_{2n}, a_{2n-1})| \\ &\qquad + \left[\frac{\mu(a_0, \alpha) + \mu(a_1, \alpha)}{2}\right] |d(a_{2n-1}, a_{2n})| \\ &= \frac{1}{2}[(\lambda(a_0, \alpha) + \lambda(a_1, \alpha)) + (\mu(a_0, \alpha) + \mu(a_1, \alpha))]|d(a_{2n-1}, a_{2n})| \\ |d(a_{2n+1}, a_{2n})| &\leq \frac{1}{2}[(\lambda(a_0, \alpha) + \lambda(a_1, \alpha)) + (\mu(a_0, \alpha) + \mu(a_1, \alpha))]|d(a_{2n-1}, a_{2n})| \end{aligned}$$

By condition (ii), $2\lambda(a, \alpha) + 2\mu(a, \alpha) < 1$ so that $h = \frac{1}{2}[(\lambda(a_0, \alpha) + \lambda(a_1, \alpha)) + (\mu(a_0, \alpha) + \mu(a_1, \alpha))] < 1$. Therefore

$$|d(a_{2n+1}, a_{2n})| \leq h|d(a_{2n}, a_{2n-1})|$$

By Lemma 1.8, $\{a_n\}$ is a Cauchy sequence. By completeness of A, $\exists\ a \in A$ such that $a_n \to a$ as $n \to \infty$.
We now show that a is a fixed point of X and Y.

By condition (ii) and Proposition 2.1, we have

$$
\begin{aligned}
d(a,Xa) &\precsim d(a,Ya_{2n+1}) + d(Ya_{2n+1},Xa)\\
&= d(a,a_{2n+2}) + d(Xa,Ya_{2n+1})\\
&\precsim d(a,a_{2n+2}) + \left[\frac{\lambda(a,\alpha)+\lambda(a_{2n+1},\alpha)}{2}\right] d(a,a_{2n+1})\\
&\qquad + \left[\frac{\mu(a,\alpha)+\mu(a_{2n+1},\alpha)}{2}\right]\left[\frac{d(a,Xa)d(a_{2n+1},Ya_{2n+1})}{d(a,Ya_{2n+1})+d(a_{2n+1},Xa)+d(a,a_{2n+1})}\right]\\
&\precsim d(a,a_{2n+2}) + \left[\frac{\lambda(a,\alpha)+\lambda(a_1,\alpha)}{2}\right] d(a,a_{2n+1})\\
&\qquad + \left[\frac{\mu(a,\alpha)+\mu(a_1,\alpha)}{2}\right]\left[\frac{d(a,Xa)d(a_{2n+1},a_{2n+2})}{d(a,a_{2n+2})+d(a_{2n+1},Xa)+d(a,a_{2n+1})}\right]
\end{aligned}
$$

Letting $n \to \infty$,we get $d(a,Xa) = 0$ which gives $Xa = a$.
Similarly,

$$
\begin{aligned}
d(a,Ya) &\precsim d(a,Xa_{2n}) + d(Xa_{2n},Ya)\\
&\precsim d(a,a_{2n+1}) + \left[\frac{\lambda(a_{2n},\alpha)+\lambda(a,\alpha)}{2}\right] d(a_{2n},a)\\
&\qquad + \left[\frac{\mu(a_{2n},\alpha)+\mu(a,\alpha)}{2}\right]\left[\frac{d(a_{2n},Xa_{2n})d(a,Ya)}{d(a_{2n},Ya)+d(a,Xa_{2n})+d(a_{2n},a)}\right]\\
&\precsim d(a,a_{2n+1}) + \left[\frac{\lambda(a_0,\alpha)+\lambda(a,\alpha)}{2}\right] d(a_{2n},a)\\
&\qquad + \left[\frac{\mu(a_0,\alpha)+\mu(a,\alpha)}{2}\right]\left[\frac{d(a_{2n},a_{2n+1})d(a,Ya)}{d(a_{2n},Ya)+d(a,a_{2n+1})+d(a_{2n},a)}\right]
\end{aligned}
$$

Letting $n \to \infty$,we get $d(a,Ya) = 0$ which yields $Ya = a$.
The uniqueness follows from condition (ii). □

By putting $X = Y$, we get this corollary.

Corollary 2.0.13. *Suppose (A,d) is a complete complex valued metric space and define a self mapping $Y : A \to A$. Let $\exists$ mappings $\lambda, \mu : A \times A \to [0,1)$ such that $\forall\ a,b \in A$ and for fixed $\alpha \in A$,*

(i) $\lambda(Ya,\alpha) \le \lambda(a,\alpha), \quad \mu(Ya,\alpha) \le \mu(a,\alpha)$,

(ii) $$d(Ya,Yb) \precsim \left[\frac{\lambda(a,\alpha)+\lambda(b,\alpha)}{2}\right] d(a,b) + \left[\frac{\mu(a,\alpha)+\mu(b,\alpha)}{2}\right]\left[\frac{d(a,Ya)d(b,Yb)}{d(a,Yb)+d(b,Ya)+d(a,b)}\right]$$

$\forall\ a,b \in A$ *such that* $a \neq b$, $d(a,Yb)+d(b,Ya)+d(a,b) \neq 0$ *where*

$$2\lambda(a,\alpha)+2\mu(a,\alpha)<1\ \forall a\in A \textit{ and for fixed } \alpha\ \in A$$
$$\textit{or}$$
$$d(Ya,Yb)=0 \textit{ if } d(a,Yb)+d(b,Ya)+d(a,b)=0.$$

Then, YT has a unique fixed point.

Proposition 2.0.14. *Suppose (A,d) is a complex valued b-metric space with coefficient $t\geq 1$ and Define a mapping $f:\mathbb{C}\to\mathbb{C}$ such that*

(i) $f(a)=0$ *if and only if* $a=0$,

(ii) $b\precsim a$ *then* $f(b)\precsim f(a)$ *and* $f(la+mb)=lf(a)+mf(b)$ *where* $l,\ m$ *are constants.*

Then (A,fod) is complex valued b-metric space.

Proof. For all $a,b,c\in A$,

(i) $0\precsim fod(a,b)$ as $0\precsim d(a,b)$ using condition (ii)
and $fod(a,b)=0\iff f(d(a,b))=0\iff d(a,b)=0\iff a=b$

(ii) $fod(a,b)=fod(b,a)\iff f(d(a,b))=f(d(b,a))\iff d(a,b)=d(b,a)$

(iii) As $d(a,b)\precsim t\{d(a,c)+d(c,b)\}$ and $f(la+mb)=lf(a)+mf(b)$
$fod(a,b)\precsim t\{fod(a,c)+fod(c,b)\}$

Then, (A,fod) is a complex valued b-metric space. □

Example 2.0.15. *Let $d:A\times A\to\mathbb{C}$ defined by $d(a,b)=|a-b|^2+i|a-b|^2$. Clearly, (A,d) is complex valued b-metric space with coefficient $t=2$. Define a function $f:\mathbb{C}\to\mathbb{C}$ as*

$$f(a)=\begin{cases} a & if\ a\in\mathbb{C}-\{0\}\\ 0 & if\ a=0\end{cases}$$

satisfying

(i) $f(a)=0$ *if and only if* $a=0$

(ii) $b\precsim a$ *then* $f(b)\precsim f(a)$ *and* $f(la+mb)=lf(a)+mf(b)$

Then, (A,fod) is a complex valued b-metric space.

Let the mappings $f,g:\mathbb{C}\to\mathbb{C}$ be defined such that

(i) $f(a)=0\ and\ g(a)=0$ *if and only if* $a=0$

(ii) $b\precsim a$ *then* $f(b)\precsim f(a),\ g(b)\precsim g(a)$ *and* $f(la+mb)=lf(a)+mf(b),\ g(la+mb)=lg(a)+mg(b)$

Then, $(A, (f+g)od)$ *is a complex valued* b*-metric space as:*

(i) *Clearly* $0 \precsim ((f+g)od)(a,b)$ *and* $((f+g)od)(a,b) = 0 \iff (f+g)(d(a,b)) = 0 \iff d(a,b) = 0 \iff a = b$

(ii) $((f+g)od)(a,b) = ((f+g)od)(b,a) \iff (f+g)(d(a,b)) = (f+g)(d(b,a)) \iff d(a,b) = d(b,a)$

(iii) *As* $d(a,b) \precsim t\{d(a,c) + d(c,b)\}$ *and* $f(la+mb) = lf(a) + mf(b),\ g(la+mb) = lg(a) + mg(b)$

$$\begin{aligned}(f+g)(d(a,b)) &\precsim (f+g)(t\{d(a,c)+d(c,b)\}) \\ &= t\{(f+g)(d(a,c)) + (f+g)(d(c,b))\}\end{aligned}$$

$$fod(a,b) \precsim t\{fod(a,c) + fod(c,b)\}$$

Similarly, we can prove that $(A, (fog)od)$ *and* $(A, (gof)od)$ *are complex valued* b*-metric spaces where* $fog : \mathbb{C} \to \mathbb{C}$ *and* $gof : \mathbb{C} \to \mathbb{C}$ *satisfy*

(i) $fog(a) = 0$ *and* $gof(a) = 0$ *if and only if* $a = 0$

(ii) $b \precsim a$ *then* $fog(b) \precsim fog(a),\ gof(b) \precsim gof(a)$ *and*

$$fog(la+mb) = l.fog(a) + m.fog(b),$$

$$gof(la+mb) = l.gof(a) + m.gof(b)$$

The mapping d *in condition* (ii) *of Theorem 2.0.3 can be replaced by* fod*,* $((f+g)od)$*,* $((fog)od)$ *and* $((gof)od)$*. For instance, the following relation holds:*

$$\begin{aligned}fod(Xa,Yb) \precsim &\left[\frac{\lambda(a,\alpha)+\lambda(b,\alpha)}{2}\right] fod(a,b) \\ &+ \left[\frac{\mu(a,\alpha)+\mu(b,\alpha)}{2}\right]\left[\frac{fod(a,Xa).fod(b,Yb)}{1+fod(a,b)}\right]\end{aligned}$$

Similar results can be obtained by replacing the mapping d *with the composition of mappings -* fod*,* $((f+g)od)$*,* $((fog)od)$ *and* $((gof)od)$*, in Theorems 2.0.5.*

Theorem 2.0.16. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be defined as*

(1) $f(a) = 0$ *if and only if* $a = 0$

(2) $b \precsim a$ *then* $f(b) \precsim f(a)$ *and* $f(la+mb) = lf(a) + mf(b)$

Suppose (A, fod) *is a complete complex valued* b*-metric space with coefficient* $t \geq 1$ *and Define self mappings* $X, Y : A \to A$*. If* $\exists$ *mappings* $\lambda, \mu : A \times A \to [0,1)$ *such that* $\forall\ a, b \in A$ *and for fixed* $\alpha\ \in A$*,*

(i) $\lambda(YXa,\alpha) \leq \lambda(a,\alpha), \lambda(XYa,\alpha) \leq \lambda(a,\alpha)$
$\mu(YXa,\alpha) \leq \mu(a,\alpha), \mu(XYa,\alpha) \leq \mu(a,\alpha)$

$$(ii)\ fod(Xa,Yb) \precsim \left[\frac{\lambda(a,\alpha)+\lambda(b,\alpha)}{2}\right] fod(a,b) + \left[\frac{\mu(a,\alpha)+\mu(b,\alpha)}{2}\right]\left[\frac{fod(a,Xa)fod(b,Yb)}{1+fod(a,b)}\right]$$

(iii) $2t\lambda(a,\alpha)+2\mu(a,\alpha)<1\ \forall a \in X$ *and for fixed* $\alpha\ \in A$.

Then, X, Y *have a unique common fixed point in* A.

Proof. Let $a, b \in A$. For fixed $\alpha \in A$,

$$fod(Xa,YXa) \precsim \left[\frac{\lambda(a,\alpha)+\lambda(Xa,\alpha)}{2}\right] fod(a,Xa) + \left[\frac{\mu(a,\alpha)+\mu(Xa,\alpha)}{2}\right]\left[\frac{fod(a,Xa)fod(Xa,YXa)}{1+fod(a,Xa)}\right]$$

$$|fod(Xa,YXa)| \leq \left[\frac{\lambda(a,\alpha)+\lambda(Xa,\alpha)}{2}\right] |fod(a,Xa)| + \left[\frac{\mu(a,\alpha)+\mu(Xa,\alpha)}{2}\right] |fod(Xa,YXa)|$$

Similarly,

$$fod(XYb,Yb) \precsim \left[\frac{\lambda(Yb,\alpha)+\lambda(b,\alpha)}{2}\right] fod(Yb,b) + \left[\frac{\mu(Yb,\alpha)+\mu(b,\alpha)}{2}\right]\left[\frac{fod(Yb,XYb)fod(b,Yb)}{1+fod(Yb,b)}\right]$$

$$|fod(XYb,Yb)| \leq \left[\frac{\lambda(Yb,\alpha)+\lambda(b,\alpha)}{2}\right] |fod(Yb,b)| + \left[\frac{\mu(Yb,\alpha)+\mu(b,\alpha)}{2}\right] |fod(Yb,XYb)|$$

Let $a_0, a_1 \in X$ be arbitrary and $\{a_n\}$ a sequence defined by

$$a_{2n+1} = Xa_{2n}, \quad a_{2n+2} = Ya_{2n+1}, \quad \forall n = 0, 1, 2, \ldots$$

as defined in Proposition 2.1.

For all $k = 0, 1, 2, \ldots$,

$$\begin{aligned}
|fod(a_{2k+1}, a_{2k})| &= |fod(XYa_{2k-1}, Ya_{2k-1})| \\
&\leq \left[\frac{\lambda(Ya_{2k-1}, \alpha) + \lambda(a_{2k-1}, \alpha)}{2}\right] |fod(Ya_{2k-1}, a_{2k-1})| \\
&\qquad + \left[\frac{\mu(Ya_{2k-1}, \alpha) + \mu(a_{2k-1}, \alpha)}{2}\right] |fod(Ya_{2k-1}, XYa_{2k-1})| \\
&= \left[\frac{\lambda(a_{2k}, \alpha) + \lambda(a_{2k-1}, \alpha)}{2}\right] |fod(a_{2k}, a_{2k-1})| \\
&\qquad + \left[\frac{\mu(a_{2k}, \alpha) + \mu(a_{2k-1}, \alpha)}{2}\right] |fod(a_{2k}, a_{2k+1})| \\
&\leq \left[\frac{\lambda(a_0, \alpha) + \lambda(a_1, \alpha)}{2}\right] |fod(a_{2k}, a_{2k-1})| \\
&\qquad + \left[\frac{\mu(a_0, \alpha) + \mu(a_1, \alpha)}{2}\right] |fod(a_{2k}, a_{2k+1})|
\end{aligned}$$

$$|fod(a_{2k+1}, a_{2k})| \leq \left[\frac{\lambda(a_0, \alpha) + \lambda(a_1, \alpha)}{2 - (\mu(a_0, \alpha) + \mu(a_1, \alpha))}\right] |fod(a_{2k}, a_{2k-1})| \tag{2.0.6}$$

Similarly, we get

$$|fod(a_{2k+2}, a_{2k+1})| \leq \left[\frac{\lambda(a_0, \alpha) + \lambda(a_1, \alpha)}{2 - (\mu(a_0, \alpha) + \mu(a_1, \alpha))}\right] |fod(a_{2k+1}, a_{2k})| \tag{2.0.7}$$

By condition (iii),

$$h = \frac{\lambda(a_0, \alpha) + \lambda(a_1, \alpha)}{2 - (\mu(a_0, \alpha) + \mu(a_1, \alpha))} < 1.$$

Therefore, equations (2.0.6) and (2.0.7) respectively become

$$|fod(a_{2k+1}, a_{2k})| \leq h|fod(a_{2k}, a_{2k-1})|,$$

$$|fod(a_{2k+2}, a_{2k+1})| \leq h|fod(a_{2k+1}, a_{2k})|$$

Hence, for all $n \geq 0$,

$$|fod(a_{2n+1}, a_{2n})| \leq h|fod(a_{2n}, a_{2n-1})|.$$

From Lemma 1.8, $\{a_n\}$ is a Cauchy sequence in (A, fod). By completeness of A, $\exists\, a \in A$ such that $a_n \to a$ as $n \to \infty$. We now show that a is the common fixed point of X and Y. By condition (ii) and Proposition 2.1, we have

$$\begin{aligned}
fod(a,Xa) &\precsim t\{fod(a,Ya_{2n+1})+fod(Xa_{2n+1},Xa)\}\\
&= t\{fod(a,a_{2n+2})+fod(Xa,Ya_{2n+2})\}\\
&\precsim t\left[fod(a,a_{2n+2})+\left\{\frac{\lambda(a,\alpha)+\lambda(a_{2n+1},\alpha)}{2}\right\}fod(a,a_{2n+1})\right.\\
&\qquad \left.+\left\{\frac{\mu(a,\alpha)+\mu(a_{2n+1},\alpha)}{2}\right\}\left\{\frac{fod(a,Xa)fod(a_{2n+1},Ya_{2n+1})}{1+fod(a,a_{2n+1})}\right\}\right]\\
&\precsim t\left[fod(a,a_{2n+2})+\left\{\frac{\lambda(a,\alpha)+\lambda(a_1,\alpha)}{2}\right\}fod(a,a_{2n+1})\right.\\
&\qquad \left.+\left\{\frac{\mu(a,\alpha)+\mu(a_1,\alpha)}{2}\right\}\left\{\frac{fod(a,Xa)fod(a_{2n+1},a_{2n+2})}{1+fod(a,a_{2n+1})}\right\}\right]
\end{aligned}$$

Letting $n\to\infty$, we get $fod(a,Xa)=0$ which yields $Xa=a$.
Similarly,

$$\begin{aligned}
fod(a,Ya) &\precsim t\{fod(a,Xa_{2n})+fod(Xa_{2n},Ya)\}\\
&\precsim t\left[fod(a,a_{2n+1})+\left\{0\frac{\lambda(a_{2n},\alpha)+\lambda(a,\alpha)}{2}\right\}fod(a_{2n},a)\right.\\
&\qquad \left.+\left\{\frac{\mu(a_{2n},\alpha)+\mu(a,\alpha)}{2}\right\}\left\{\frac{fod(a_{2n},Xa_{2n})fod(a,Ya)}{1+fod(a_{2n},a)}\right\}\right]\\
&\precsim t\left[fod(a,a_{2n+1})+\left\{\frac{\lambda(a_0,\alpha)+\lambda(a,\alpha)}{2}\right\}fod(a_{2n},a)\right.\\
&\qquad \left.+\left\{\frac{\mu(a_0,\alpha)+\mu(a,\alpha)}{2}\right\}\left\{\frac{fod(a_{2n},a_{2n+1})fod(a,Ya)}{1+fod(a_{2n},a)}\right\}\right]
\end{aligned}$$

Letting $n\to\infty$, we have $fod(a,Ya)=0$ which gives $Ya=a$.
The uniqueness follows from conditions (ii) and (iii). $\square$

Chapter 3

Conclusions

In this paper, we explained some new results related to common fixed point for a pair of mappings satisfying more general contraction conditions represented by rational expressions having point dependent control functions of two variables as coefficient in complex valued b*-metric space. Then we gave related results for composition of function and complex valued* b*- metric space.*

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