Construction of a Family of Positive But Not Completely Positive Map For the Detection of Entangled States

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by

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CERTIFICATE

I hereby certify that the dissertation entitled **Construction of a Family of Positive But Not Completely Positive Map For the Detection of Bound Entangled States** which is submitted by **Richa Rohira and Shreya Sanduja**, Department of Mathematics, Delhi Technological University, Delhi in partial fulfillment of the requirement of the award of the degree of Master of Science, is a record of dissertation work carried out by the students under my supervision. To the best of my knowledge, this work has not been submitted in part of full for any Degree or Diploma to this University or elsewhere.

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ABSTRACT

We have constructed a family of map. The map is shown to be positive when imposing certain condition on the parameters. Then we show that the constructed map can never be completely positive. After tuning the parameters, we found that the map still remain positive but it is not completely positive. We then use the positive but not completely positive map in the detection of entanglement of a composite bipartite quantum system such as Bound Entangled States (BES) and Negative Partial Transpose Entangled States (NPTES).

Contents

Chapter 1

Introduction

1.1 Introduction to Entanglement

From the past times, everyone has raised the issue of scarceness from quantum mechanics and that is why the quantum correlation has been the topic of debates and research.[1] It plays a central role in many information processing protocols such as quantum cryptography [4], quantum super dense coding [5] and quantum teleportation [6, 7]. The potential offered by quantum entanglement to computing, security and communication makes it a topic of vital interest to researchers all across the globe. The breach of Bell-type inequalities tells the fundamentally different nature of quantum correlations in comparison to classical correlations. Discussing such correlations, Schrodinger first used the term entanglement and assumed it to be the predictable trait of quantum mechanics. Bohm then inspected entanglement in an easier way, that of a pair of spins in the singlet state, which are still the review of the organizations of quantum mechanics and quantum information. Using this progress, Bell successfully updated the study of quantum entanglement by finding it. Now it is known as Bell's inequality that must be followed by systems that are correlated. The potential offered by the systematic use of such entangled systems as resources for quantum information, communication, and quantum computing has led to many engaging procedures. [3, 2]
The use of entangled resources is to find engaging and leading results in quantum information and communication.These long-range correlations with no classical analogs thus distinguish the quantum world from classical world. Besides, quantum correlations not only put focus on the complex nature of entanglement but also give physical help to quantum computing and quantum communication.
Moreover, a separable state is defined as any state $\rho = H_A \otimes H_B$ that is written as,

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i, 0 \leq p_i \leq 1, \sum_i p_i = 1 \tag{1.1}$$

If (1.1) does not hold, then the state is known as **Entangled State**.

One of the major problem in QIT is the detection of entanglement in a quantum mechanical system. A pure two-qubit entangled system always violated Bell-CHSH inequality and thus detected by Bell-CHSH operator [8, 9]. On the other hand, the Bell-CHSH inequality fails to detect the several mixed bipartite entangled state. This loophole can be fixed using Peres-Horodecki (PH) positive partial transpose (PPT) criteria, which is required for the detection of entanglement in $2 \otimes 2$ and $2 \otimes 3$ systems [10, 11]. If the dimensions are higher then all states with negative partial transpose (NPT) are entangled but the states with PPT may or may not be entangled [12]. The entangled states which are expressed by a density matrix, are positive under partial transposition are known as bound entangled states. Thus, the separability problem can also be framed as studying whether states with PPT are entangled or not.

The separability problem can be tackled to certain extent by witness operator [13, 14]. Witness operators are hermitian operators with at least one negative eigenvalues. The witness operators are more powerful than Bell inequalities in the sense that it can detect multipartite entanglement in different cuts, if some prior information about the state under investigation is provided. They not only detect multipartite entanglement in different cuts but also detect genuine entanglement and classify entanglement in multipartite system. They are observables and thus experimentally realizable also.

A map $\Lambda : M_{d_1}(C) \rightarrow M_{d_2}(C)$ is known as a positive map if $\Lambda(A) \in M_{d_2}(C)$ wiil be positive, for any positive $A \in M_{d_1}(C)$. Unfortunately, the structure of positive map is not completely understood and still it is under extensive research [15, 16, 17]. Indecomposable maps plays an important role among all positive linear maps due to the fact that it can detect positive partial transpose entangled states. Let Λ be a positive map and let $I_d : M_d(C) \rightarrow M_d(C)$ be a identity map. TNow, we can state that Λ is completely positive if for all d, the extended map $I_d \otimes \Lambda$ is positive.

The separability problem can be reformulated in terms of positive maps [11] as follows: Let us suppose that H_{d_1} and H_{d_2} represent two Hilbert spaces with dimensions d_1 and d_2 respectively. A bipartite quantum state described by the density operator $\sigma \in H_{d_1} \otimes H_{d_2}$ is separable iff $(I_{d_1} \otimes \Lambda)\sigma$ is positive for any positive map Λ. Thus there is a deep relation between the theory of detection of entanglement and operator theory. This linkage has been established by Choi-Jamiolkowski isomorphism [18, 19].

1.2 Positive and Completely Positive Map

A hermitian matrix A is known as a positive semi definite if A has all positive eigen values. Consequently, a mapping $\Lambda : M_{d_1}(C) \rightarrow M_{d_2}(C)$ is known to be positive iff the mapping maps the positive semi definite matrices of $M_{d_1}(C)$ to positive semi definite elements of $M_{d_1}(C)$.

Assume that T_{d_1} represents the transpose mapping on $M_{d_1}(C)$, then if the mapping, $\Lambda \otimes T_{d_1}$ is positiven then the map, Λ is known to be co positive.

For the identity mapping, $I_k : M_k(C) \rightarrow M_k(C)$, if the mapping, $I_k \otimes \Lambda : M_k(C) \otimes M_{d_1}(C) \rightarrow M_k(C) \otimes M_{d_2}(C)$ is positive, then the map, Λ is said to be k-positive.

The map, Λ is known as a **completely positive mapping**, iff it is k-positive for $\forall k \in \mathrm{N}$.

1.3 Some Results using Positive Definite Matrix

Lemma-1: If $a \geq b$, matrix $[a^{i+j-1} - ba^{i+j-2}]$ of order, $(r+1) \times (r+1)$ is positive semi definite (psd) where $1 \leq i, j \leq r+1, r = 0, 1, 2...$

Proof: It is known that the matrix a^{i+j-2} is psd for all $a \in \Re$
The rank of the matrix a^{i+j-2} is one and it has a non-negative trace.
Also, the matrix $[a-b]$ is psd as $a \geq b$. So, the Schur product given by,

$$a^{i+j-2} \circ [a-b] = [a^{i+j-1} - ba^{i+j-2}]$$

is also psd.

Theorem 1: Let Φ be a positive unital linear mapping defined on $M(n) \rightarrow M(l)$. Consider a Hermitian element of $M(n)$, A such that the spectrum of A is in $[m, M]$. Then

$$[\Phi(A^{i+j-1}) - m\Phi(A^{i+j-2})]_{(r+1)\times(r+1)} \geq 0 \tag{1.2}$$

and

$$[M\Phi(A^{i+j-2}) - \Phi(A^{i+j-1})]_{(r+1)\times(r+1)} \geq 0 \tag{1.3}$$

Proof: Using the eigenvalues of the matrices, we have

$$[\Phi(A^{i+j-1}) - m\Phi(A^{i+j-2})] = \textstyle\sum_{k=1}^{n}[(\lambda_k^{i+j-1} - m\lambda_k^{i+j-2})\Phi(P_k)]$$

$$[\Phi(A^{i+j-1}) - m\Phi(A^{i+j-2})] = \sum_{k=1}^{n}[(\lambda_k^{i+j-1} - m\lambda_k^{i+j-2}) \otimes \Phi(P_k)] \tag{1.4}$$

Here $\otimes$ is the matrix tensor product. By equation (3), and using lemma 1, we can say that $[\Phi(A^{i+j-1}) - m\Phi(A^{i+j-2})]$ is result of the addition of tensor products of two psd matrices and hence it is psd.

On similar note, from equation (2), we can say that

$$[(M\lambda_k^{i+j-2} - \lambda_k^{i+j-1})]$$

is also psd.

Theorem 2: Consider the elements Φ, A, m and M as defined in the first theorem. Let the matrix, A, be positive definite. Then

$$[\Phi(A^{i+j-2}) - m\Phi(A^{i+j-3})]_{(r+1)\times(r+1)} \geq 0 \tag{1.5}$$

and

$$[M\Phi(A^{i+j-3}) - \Phi(A^{i+j-2})]_{(r+1)\times(r+1)} \geq 0 \tag{1.6}$$

Lemma-2: Let $x \geq y \geq z$ and consider a psd matrix $[y^{i+j-2}(x-y)(y-z)]$ of order, $(r+1) \times (r+1)$ and $1 \leq i, j \leq r+1, r = 0, 1, 2...$

Proof: The expression, $[y^{i+j-2}(x-y)(y-z)]$ can be written in the form of a Schur product as,

$$[y^{(i+j-2)}(x-y)(y-z)] = [y^{i+j-2}] \circ [(x-y)(y-z)]$$

The RHS of the expression is clearly psd.

Theorem 3: Consider the elements Φ, A, m and M as defined in the first theorem. Let the matrix, A, be a positive definite matrix. Then,

$$[\Phi(A^{i+j-2})(A - mI)(MI - A)]_{(r+1)\times(r+1)} \geq 0 \quad (1.7)$$

Proof: Using the lemma 2, we have

$$[(\lambda_k^{i+j-2}] \circ [(\lambda_k - m)(M - \lambda_k)] = [(\lambda_k^{i+j-2}(\lambda_k - m)(M - \lambda_k)]$$

This is psd. Now we can use same steps as in Theorem 1.

Theorem 4: Let Φ be a positive unital linear mapping defined on $M(n) \to M(l)$. Consider a Hermitian element of $M(n)$, A such that A has unique eigenvalues $\lambda_1 < \lambda_2 < ...\lambda_k$. Then,

$$[\Phi(A^{i+j-2})(A - \lambda_{j-1}I)(A - \lambda_j I]_{(r+1)\times(r+1)} \geq 0 \quad (1.8)$$

Proof: Since, all the $\lambda_k(k = 1, 2, ..., n)$ does not lie inside the interval $(\lambda_{j-1}, \lambda_j), j = 2, 3, ..., k$.
Now, we have

$$[(\lambda_k^{i+j-2}] \circ [(\lambda_k - \lambda_{j-1})(\lambda_k - \lambda_j] = [(\lambda_k^{i+j-2}(\lambda_k - \lambda_{j-1})(\lambda_k - \lambda_j)]$$

This is psd. Now we can use same steps as in Theorem 1.

1.4 Preliminaries

Definition 1: Tensor Product
The tensor product of two matrices, $X = [x_{ij}]_{1\leq i,j\leq m,n}$ and $Y = [y_{kl}]_{1\leq k,l\leq p,q}$ is given by;

$$X \otimes Y = \begin{pmatrix} x_{11}Y & x_{12}Y & \cdots & x_{1n}Y \\ x_{21}Y & x_{22}Y & \cdots & x_{2n}Y \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1}Y & x_{m2}Y & \cdots & x_{mn}Y \end{pmatrix}_{mq\times np}$$

Definition 2: Partial Transposition
Let A be a partitioned block matrix of the form

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

The partial transposition of the block matrix A is given by;

$$A^{\Gamma} = \left(\begin{array}{cc} X^T & Y^T \\ Z^T & W^T \end{array} \right)$$

Definition 3: Contraction
A matrix V is said to be a contraction if

$$\|V\| \leq 1 \tag{1.9}$$

where $\|.\|$ denote the operator norm.

Let A be a partitioned block matrix of the form

$$B = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \tag{1.10}$$

Definition 4: Choi Matrix
Consider $\phi : M_{d_1}(C) \to M_{d_2}(C)$ to be a linear mapping, then the Choi matrix corresponding to ϕ is given by;

$$C_{\phi} := \sum_{i,j=0}^{1} |i\rangle \langle j| \otimes \phi \left(|i\rangle \langle j| \right)$$

Definition 5: Choi-Jamiolkowski Isomorphism
Consider $\Sigma : B\left(M_{d_1}\left(C\right), M_{d_2}\left(C\right)\right) \to M_{d_1}\left(C\right) \otimes M_{d_2}\left(C\right)$ to be a mapping, given by; $\phi \to C_{\phi}$. The mapping is linear and bijective, and is known as Choi-Jamiolkowski Isomorphism.

Definition 6: Separable States
A pure state $|\psi\rangle \in H$ is called separable state if we can find states $\left|\phi^A\right\rangle \in H_A$ and $\left|\phi^B\right\rangle \in H_B$, such that

$$|\psi\rangle = \left|\phi^A\right\rangle \otimes \left|\phi^B\right\rangle$$

If this condition does not hold, then the state $|\psi\rangle$ is known as **Entangled State**.

Definition 7: Density Operator

An operator ρ that satisfies the following conditions is known as a density operator:
(i) $Tr(\rho)$ is equal to one.
(ii) ρ is a positive operator.

Definition 8: Bound Entangled State
Bound entangled states are the states which are entangled and have positive partial transposition.

Result-1 [21]: The block matrix (1.10) is positive semidefinite if and only if X and Z are positive semidefinite and there exists a contraction V such that

$$Y = X^{\frac{1}{2}} V Z^{\frac{1}{2}} \tag{1.11}$$

Result-2: Descarte's Rule of Sign The number of most number of positive roots of an function $f(x)$ is equal to the number of changes of sign when the exponents of the polynomials are arranged in decreasing order.
The number of negative roots of an equation can be found out by the determining the number of changes in sign of the function $f(-x)$.

Result- 3 [20]: Let $\phi : M_m(C) \rightarrow M_n(C)$ be a linear map. Then the following statements are equivalent:
(i) ϕ is completely positive.
(ii) C_ϕ is positive semidefinite, where C_ϕ denote the Choi matrix of ϕ.

Result-4 [22]:
For positive definite blocks X and Z, the matrix A given in (1.10) is positive semidefinite iff $X \geq Y Z^{-1} Y^{\dagger}$.

Result-5 : PPT Criterion
A Quantum state, ρ is said to be entangled if the partial transpose of the state, ρ^{Γ}, is negative definite. Otherwise, the state is said to be separable.

Motivation of the Thesis:
The motivation of this work is as follows: The construction and studying the structure of new positive map may give useful insight in the understanding of positive map, which gives us the first motivation of this work. Secondly, we find that the problem of constructing the positive but not completely positive map and its relation in the detection of entanglement may take one step further in the development of not only operator theory but also quantum information theory.

Chapter 2

Construction of a family of map

In this chapter, we will construct a map and derive the condition for which the map is positive. Further, we will probe that whether the constructed map is completely positive. Moreover, we will provide the explicit matrix form of the map, which is positive but not completely positive.

2.1 Defining a Map

Let us take a positive integer $n \ (n \geq 2)$ and then we define a general family of map $\Phi : M_n(C) \rightarrow M_n(C) \otimes M_n(C)$ as

$$\Phi_{\alpha,\beta}(A) = \alpha((A + A^T) \otimes I_n) + \beta(|\psi_+\rangle\langle\psi_+|)^\Gamma \tag{2.1}$$

where A denote $n \times n$ matrix, $\alpha, \beta \in R$, Γ represent the partial transposition, I_n denote the identity matrix of order n and $|\psi_+\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |ii\rangle$.
To discuss our result, we will fix $n = 2$ and re-define the map $\Phi : M_2(C) \rightarrow M_2(C) \otimes M_2(C)$ as

$$\Phi_{\alpha,\beta}(A) = \alpha((A + A^T) \otimes I_2) + \beta(|\psi_+\rangle\langle\psi_+|)^\Gamma \tag{2.2}$$

where I_2 denote the identity matrix of order 2 and $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
For any $a, d \geq 0$ and $b, c \in$ R, we can take the input matrix $A \in M_2(R)$ of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.3}$$

In matrix notation, the output of the map $\Phi_{\alpha,\beta}$ can be expressed as

$$\Phi_{\alpha,\beta} = \begin{pmatrix} 2a\alpha + \frac{\beta}{2} & 0 & \alpha(b+c) & 0 \\ 0 & 2a\alpha & \frac{\beta}{2} & \alpha(b+c) \\ \alpha(b+c) & \frac{\beta}{2} & 2d\alpha & 0 \\ 0 & \alpha(b+c) & 0 & 2d\alpha + \frac{\beta}{2} \end{pmatrix} \tag{2.4}$$

2.2 Conditions for which a map ϕ will be positive

We will derive here the conditions for which Φ represent a positive map. The map Φ will be positive if the matrix represented by $\Phi_{\alpha,\beta}$ given in (2.4) is a positive semi-definite matrix. To accomplish this task, we re-express $\Phi_{\alpha,\beta}$ in a block matrix form as

$$\Phi_{\alpha,\beta} = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \tag{2.5}$$

where

$$X = \begin{pmatrix} 2a\alpha + \frac{\beta}{2} & 0 \\ 0 & 2a\alpha \end{pmatrix}, Y = \begin{pmatrix} \alpha(b+c) & 0 \\ \frac{\beta}{2} & \alpha(b+c) \end{pmatrix},$$
$$Z = \begin{pmatrix} 2d\alpha & 0 \\ 0 & 2d\alpha + \frac{\beta}{2} \end{pmatrix} \tag{2.6}$$

Applying Result-2 on $\Phi_{\alpha,\beta}$, we can state that the matrix $\Phi_{\alpha,\beta}$ will be positive semidefinite if the below mentioned conditions hold:

$$(i) X \geq 0 \Rightarrow 2a\alpha \geq 0 \text{ and } 4a\alpha + \beta \geq 0 \tag{2.7}$$

$$(ii) Z \geq 0 \Rightarrow 2d\alpha \geq 0 \text{ and } 4d\alpha + \beta \geq 0 \tag{2.8}$$

$$(iii) \qquad \|V\| = \|X^{\frac{1}{2}} Y Z^{\frac{1}{2}}\| \leq 1$$
$$\Rightarrow \| \begin{pmatrix} \frac{\alpha(b+c)}{\sqrt{d\alpha(4a\alpha+\beta)}} & 0 \\ \frac{\beta}{4\alpha\sqrt{ad}} & \frac{\alpha(b+c)}{\sqrt{a\alpha(4d\alpha+\beta)}} \end{pmatrix} \| \leq 1 \tag{2.9}$$

where $\|V\|$ denote the operator norm of V.
Conditions (i) and (ii) given by (2.7) and (2.8) are collectively given by

$$2\alpha(a+d) + \beta \geq 0, \;\; \alpha \geq 0 \tag{2.10}$$

Now our task is to take into account condition (iii) in which we need to calculate the operator norm of the matrix V. Operator norm of the matrix V is defined as the maximum eigenvalue of $V^T V$. The eigenvalue of $V^T V$ can be calculated from the characteristic equation of $V^T V$. The characteristic equation of $V^T V$ is given by

$$\lambda^2 - k_1 \lambda + \frac{k_2}{4} = 0 \tag{2.11}$$

where $k_1 = \frac{\alpha(b+c)^2}{(4a\alpha+\beta)d} + \frac{\beta}{4a\alpha} + \frac{\beta^2}{16ad\alpha^2} + \frac{d}{a}$ and $k_2 = \frac{(b+c)^2(\beta+4d\alpha)}{ad(4a\alpha+\beta)}$.
Since $a \geq 0$ and $d \geq 0$ from the earlier assumptions and using equations (2.7) and (2.8), we can infer that $k_1 \geq 0$ and $k_2 \geq 0$. Thus, it is clear from Descarte's rule of

sign that the two roots of the characteristic equation given by (2.11) will be positive. If λ_1 and λ_2 denote two positive eigenvalues of V^TV then they are given by

$$\lambda_1 = \frac{1}{2}(k_1 + \sqrt{k_1^2 - k_2})$$
$$\lambda_2 = \frac{1}{2}(k_1 - \sqrt{k_1^2 - k_2}) \tag{2.12}$$

Since both the eigenvalues are positive so $\|V\| = max\{\lambda_1, \lambda_2\} = \lambda_1$. The condition (iii) says that $\|V\| \leq 1$ which implies

$$4(1 + \sqrt{k_1^2 - k_2} - k_2) \geq 1 \tag{2.13}$$

The map $\Phi_{\alpha,\beta}$ is positive if equations (2.10) and (2.13) holds simultaneously. In particular, the map $\Phi_{\alpha,\beta}$ will be positive for $\alpha \geq 0$ and $\beta = 0$.

2.3 Is the map ϕ completely positive?

In this section, we will investigate the fact that whether the map ϕ is completely positive. To do this, we begin with the construction of Choi matrix corresponding to the positive operator $\Phi_{\alpha,\beta}$. The Choi matrix $C_{\Phi_{\alpha,\beta}}$ is defined as [18]

$$C_{\Phi_{\alpha,\beta}} = \sum_{i,j=0}^{1} |i\rangle\langle j| \otimes \Phi_{\alpha,\beta}(|i\rangle\langle j|) \tag{2.14}$$

where $|i\rangle$ represent the basis state in two-dimensional Hilbert space.
The Choi matrix $C_{\Phi_{\alpha,\beta}}$ can be re-expressed in terms of matrix as

$$C_{\Phi_{\alpha,\beta}} = \begin{pmatrix} 2\alpha + \frac{\beta}{2} & 0 & 0 & 0 & \frac{\beta}{2} & 0 & \alpha & 0 \\ 0 & 2\alpha & \frac{\beta}{2} & 0 & 0 & 0 & \frac{\beta}{2} & \alpha \\ 0 & \frac{\beta}{2} & 0 & 0 & \alpha & \frac{\beta}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta}{2} & 0 & \alpha & 0 & \frac{\beta}{2} \\ \frac{\beta}{2} & 0 & \alpha & 0 & \frac{\beta}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta}{2} & \alpha & 0 & 0 & \frac{\beta}{2} & 0 \\ \alpha & \frac{\beta}{2} & 0 & 0 & 0 & \frac{\beta}{2} & 2\alpha & 0 \\ 0 & \alpha & 0 & \frac{\beta}{2} & 0 & 0 & 0 & 2\alpha + \frac{\beta}{2} \end{pmatrix}, \tag{2.15}$$

To show the completely positivity of a positive map $\Phi_{\alpha,\beta}$, we need to the show that the choi matrix $C_{\Phi_{\alpha,\beta}}$ corresponding to the positive map $\Phi_{\alpha,\beta}$ is positive semidefinite. We first express the choi matrix in block form as

$$C_{\Phi_{\alpha,\beta}} = \begin{pmatrix} P & Q \\ Q^* & R \end{pmatrix} \tag{2.16}$$

where

$$P = \begin{pmatrix} 2\alpha + \frac{\beta}{2} & 0 & 0 & 0 \\ 0 & 2\alpha & \frac{\beta}{2} & 0 \\ 0 & \frac{\beta}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta}{2} \end{pmatrix}, Q = \begin{pmatrix} \frac{\beta}{2} & 0 & \alpha & 0 \\ 0 & 0 & \frac{\beta}{2} & \alpha \\ \alpha & \frac{\beta}{2} & 0 & 0 \\ 0 & \alpha & 0 & \frac{\beta}{2} \end{pmatrix},$$

$$R = \begin{pmatrix} \frac{\beta}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta}{2} & 0 \\ 0 & \frac{\beta}{2} & 2\alpha & 0 \\ 0 & 0 & 0 & 2\alpha + \frac{\beta}{2} \end{pmatrix} \tag{2.17}$$

Following Result-3, we can show that the choi matrix $C_{\Phi_{\alpha,\beta}}$ is positive semidefinite if and only if the below mentioned conditions are satisfied:

$$(i)\ \ P \geq 0 \ \text{ holds when } \ \beta = 0 \ \text{ and } \ \alpha \geq 0 \tag{2.18}$$

$$(ii)\ \ R \geq 0 \ \text{ holds when } \ \beta = 0 \ \text{ and } \ \alpha \geq 0 \tag{2.19}$$

$$\begin{aligned} &(iii)\ \ P - QR^{-1}Q^* \geq 0 \ \text{ holds for either} \\ &(\alpha = 0 \ \text{ and } \ \beta \neq 0) \ \text{ or } \ (\alpha > 0 \ \text{ and } \ 4\alpha + \beta < 0) \\ &\text{ or } \ (\alpha > 0 \ , \ 3\alpha + 2\beta \geq 0 \ \text{ and } \ \beta \neq 0) \end{aligned} \tag{2.20}$$

It can be easily observe that the conditions (i), (ii) and (iii) does not hold simultaneously. Thus the map $\Phi_{\alpha,\beta}$ is not completely positive.

2.4 Conditions for which a map ϕ will be positive but not completely positive

In the previous sections, we have derived the condition for which the map Φ will be positive and later we proved that the positive map Φ cannot be completely positive. In this section, we will derive the common interval of α for which the map Φ will be positive but not completely positive simultaneously.
Without any loss of generality, let us consider the 2×2 positive matrix $A_1 \in M_2(\mathrm{R})$ as

$$A_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{1}{9} & 2 \end{pmatrix} \tag{2.21}$$

Further, taking $\beta = -\gamma(\gamma > 0)$, the output of the mapping can be represented by the matrix as

$$\Phi_{\alpha,-\gamma}(A_1) = \begin{pmatrix} \frac{\alpha}{2} - \frac{\gamma}{2} & 0 & \frac{4\alpha}{9} & 0 \\ 0 & \frac{\alpha}{2} & -\frac{\gamma}{2} & \frac{4\alpha}{9} \\ \frac{4\alpha}{9} & -\frac{\gamma}{2} & 4\alpha & 0 \\ 0 & \frac{4\alpha}{9} & 0 & 4\alpha - \frac{\gamma}{2} \end{pmatrix} \tag{2.22}$$

It can be easily shown that the map $\Phi_{\alpha,-\gamma}$ always produces a positive matrix at the output if $\gamma > 0$ and $\alpha \geq \frac{9\gamma}{2\sqrt{146}}$. Thus $\Phi_{\alpha,-\gamma}$ represent a positive map if $\gamma > 0$ and $\alpha \geq \frac{9\gamma}{2\sqrt{146}}$. Furthermore, the Choi matrix corresponding to the positive map $\Phi_{\alpha,-\gamma}$ is given by

$$C_{\Phi_{\alpha,-\gamma}} = \begin{pmatrix} m_1 & 0 & 0 & 0 & -\frac{\gamma}{2} & 0 & \alpha & 0 \\ 0 & 2\alpha & -\frac{\gamma}{2} & 0 & 0 & 0 & -\frac{\gamma}{2} & \alpha \\ 0 & -\frac{\gamma}{2} & 0 & 0 & \alpha & -\frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\gamma}{2} & 0 & \alpha & 0 & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & 0 & \alpha & 0 & -\frac{\gamma}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\gamma}{2} & \alpha & 0 & 0 & -\frac{\gamma}{2} & 0 \\ \alpha & -\frac{\gamma}{2} & 0 & 0 & 0 & -\frac{\gamma}{2} & 2\alpha & 0 \\ 0 & \alpha & 0 & -\frac{\gamma}{2} & 0 & 0 & 0 & m_1 \end{pmatrix} \tag{2.23}$$

where $m_1 = 2\alpha - \frac{\gamma}{2}$.
The eigenvalues of $C_{\Phi_{\alpha,-\gamma}}$ are given by

$$\begin{aligned} \mu_1 &= \frac{-\gamma + \sqrt{4\alpha^2 + \gamma^2}}{2}, \;\; \mu_2 = \frac{-\gamma - \sqrt{4\alpha^2 + \gamma^2}}{2} \\ \mu_3 &= \frac{4\alpha - \gamma + \sqrt{4\alpha^2 + \gamma^2}}{2}, \\ \mu_4 &= \frac{4\alpha - \gamma - \sqrt{4\alpha^2 + \gamma^2}}{2}, \\ \mu_5 &= \alpha + \sqrt{\frac{4\alpha^2 + \gamma^2 + \sqrt{16\alpha^2 + 4\alpha^2\gamma^2 + \gamma^4}}{2}} \\ \mu_6 &= \alpha + \sqrt{\frac{4\alpha^2 + \gamma^2 - \sqrt{16\alpha^2 + 4\alpha^2\gamma^2 + \gamma^4}}{2}} \\ \mu_7 &= \alpha - \sqrt{\frac{4\alpha^2 + \gamma^2 + \sqrt{16\alpha^2 + 4\alpha^2\gamma^2 + \gamma^4}}{2}} \\ \mu_8 &= \alpha - \sqrt{\frac{4\alpha^2 + \gamma^2 - \sqrt{16\alpha^2 + 4\alpha^2\gamma^2 + \gamma^4}}{2}} \end{aligned} \tag{2.24}$$

It can be observed that the Choi matrix $C_{\Phi_{\alpha,-\gamma}}$ has at least one negative eigenvalues for any α and γ. Therefore, $\Phi_{\alpha,-\gamma}$ is not completely positive map for any α and γ. Thus for $\gamma > 0$ and $\alpha \geq \frac{9\gamma}{2\sqrt{146}}$, the map $\Phi_{\alpha,-\gamma}$ is positive but not completely positive.

Chapter 3

Detection of entangled States

In this chapter, we will use the constructed map which is positive but not completely positive to detect negative partial transpose entangled states and bound entangled states. We will construct the Choi matrix from the positive map that can be considered as a witness operator. A witness operator W is a hermitian operator, which satisfies the following properties:

$$\begin{aligned}&(i)\ \ Tr(W\rho_s) \geq 0, \ \ \text{for all separable state} \ \ \rho_s\\&(ii)\ Tr(W\rho_e) < 0, \ \ \text{for at least one entangled state} \ \ \rho_e\end{aligned} \tag{3.1}$$

3.1 Detection of Bound Entangled State

In this section, our task is to detect Bound Entangled States with the help of a positive but not completely positive map.

Recalling 2.22 and considering $\gamma = 2$. We then choose a value of α in the interval $\alpha \geq \frac{9}{\sqrt{146}}$. Taking $\alpha = \frac{3}{4}$, the matrix given in (2.22) reduces to

$$\Phi_{\frac{3}{4},-2}(A_1) = \begin{pmatrix} \frac{11}{8} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{3}{8} & -1 & \frac{1}{3} \\ \frac{1}{3} & -1 & 3 & 0 \\ 0 & \frac{1}{3} & 0 & 4 \end{pmatrix} \tag{3.2}$$

In particular, the map $\Phi_{\frac{3}{4},-2}$ represent a positive map. Using this positive map, we can construct the Choi matrix which is given below:

$$C_{\Phi_{\frac{3}{4},-2}} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & -1 & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 & 0 & -1 & \frac{3}{4} \\ 0 & -1 & 0 & 0 & \frac{3}{4} & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \frac{3}{4} & 0 & -1 \\ -1 & 0 & \frac{3}{4} & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \frac{3}{4} & 0 & 0 & -1 & 0 \\ \frac{3}{4} & -1 & 0 & 0 & 0 & -1 & \frac{3}{2} & 0 \\ 0 & \frac{3}{4} & 0 & -1 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad (3.3)$$

The Choi matrix $C_{\Phi_{\frac{3}{4},-2}}$ has at least one negative eigenvalues and thus it does not represent a positive semidefinite matrix. Hence $\Phi_{\frac{3}{4},-2}$ is a positive but not completely positive map.

Next our task is to show that $C_{\Phi_{\frac{3}{4},-2}}$ act as witness operator and for this it is sufficient to show that there exist at least one entangled states described by the density operator ρ_e for which $Tr(C_{\Phi_{\frac{3}{4},-2}}\rho_e) < 0$. Then we can say that the entangled state will be detected by $C_{\Phi_{\frac{3}{4},-2}}$.

Let us consider a quantum state described by the density operator ρ_b which is given by

$$\rho_b = \frac{1}{1+7b} \begin{pmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+b}{2} & 0 & 0 & \frac{\sqrt{1-b^2}}{2} \\ b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & \frac{\sqrt{1-b^2}}{2} & 0 & 0 & \frac{1+b}{2} \end{pmatrix} \qquad (3.4)$$

where the state parameter satisfies $0 \leq b \leq 1$. The state ρ_b is shown to be a bound entangled state by range criterion [12].

We are now in a position to show the utility of the operator $C_{\Phi_{\frac{3}{4},-2}}$ in the detection of entanglement. To accomplish this task, we calculate $Tr(C_{\Phi_{\frac{3}{4},-2}}\rho_b)$, which is given by

$$Tr(C_{\Phi_{\frac{3}{4},-2}}\rho_b) = \frac{b-1}{4(1+7b)} < 0 \qquad (3.5)$$

Thus the bound entangled state ρ_b detected by the witness operator $C_{\Phi_{\frac{3}{4},-2}}$.

3.2 Detection of Negative Partial Transpose Entangled State

Let us consider a quantum state described by the density operator ρ_{NPT} which is given by

$$\rho_{NPT} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.6}$$

It can be easily shown that the state (3.6) represent a negative partial transpose entangled state. Now, our task is to construct a witness operator which can detect it. To accomplish this task, let us start with the positive input matrix which is given by

$$A_2 = \begin{pmatrix} 3 & \frac{1}{3} \\ \frac{1}{9} & 2 \end{pmatrix} \tag{3.7}$$

Considering $\beta = -\gamma(\gamma > 0)$ and applying the map on A_2, we get the output matrix in the form

$$\Phi_{\alpha,-\gamma}(A_2) = \begin{pmatrix} 6\alpha - \frac{\gamma}{2} & 0 & \frac{4\alpha}{9} & 0 \\ 0 & 6\alpha & -\frac{\gamma}{2} & \frac{4\alpha}{9} \\ \frac{4\alpha}{9} & -\frac{\gamma}{2} & 4\alpha & 0 \\ 0 & \frac{4\alpha}{9} & 0 & 4\alpha - \frac{\gamma}{2} \end{pmatrix} \tag{3.8}$$

The map $\Phi_{\alpha,-\gamma}$ will be positive map if $\gamma > 0$ and $\alpha \geq \frac{9\gamma}{90-2\sqrt{27}}$. In the next step, we fix $\gamma = 1$ and then choose a value of α from the interval $\alpha \geq \frac{9}{90-2\sqrt{27}}$. Taking $\alpha = \frac{1}{8}$, the matrix given in (3.8) reduces to

$$\Phi_{\frac{1}{8},-1}(A_2) = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{18} & 0 \\ 0 & \frac{3}{4} & \frac{-1}{2} & \frac{1}{18} \\ \frac{1}{18} & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{18} & 0 & 0 \end{pmatrix} \tag{3.9}$$

Therefore, the particular form of the map $\Phi_{\frac{1}{8},-1}$ represent a positive map. Using this positive map, we can construct the Choi matrix as

$$C_{\Phi_{\frac{1}{8},-1}} = \begin{pmatrix} \frac{-1}{4} & 0 & 0 & 0 & \frac{-1}{2} & 0 & \frac{1}{8} & 0 \\ 0 & \frac{1}{4} & \frac{-1}{2} & 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{8} \\ 0 & \frac{-1}{2} & 0 & 0 & \frac{1}{8} & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & 0 & \frac{1}{8} & 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 & \frac{1}{8} & 0 & \frac{-1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & \frac{1}{8} & 0 & 0 & \frac{-1}{2} & 0 \\ \frac{1}{8} & \frac{-1}{2} & 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{8} & 0 & \frac{-1}{2} & 0 & 0 & 0 & \frac{-1}{4} \end{pmatrix}, \tag{3.10}$$

The Choi matrix $C_{\Phi_{\frac{1}{8},-1}}$ has at least one negative eigenvalues and thus it does not represent a positive semidefinite matrix. Hence $\Phi_{\frac{1}{8},-1}$ is a positive but not completely positive map.
We will now show that $C_{\Phi_{\frac{3}{4},-2}}$ act as witness operator and it detect the state (3.6). To detect the state described by the density operator ρ_{NPT}, we calculate $Tr(C_{\Phi_{\frac{3}{4},-2}}\rho_{NPT})$, which is given by

$$Tr(C_{\Phi_{\frac{1}{8},-1}}\rho_{NPT}) = \frac{-1}{6} < 0 \tag{3.11}$$

Thus the negative partial transpose entangled state ρ_{NPT} detected by the witness operator $C_{\Phi_{\frac{1}{8},-1}}$.

Chapter 4

Conclusion

- To summarize, we have constructed a map which is applied on $n \times n$ matrix and as a result, we obtain $n^2 \times n^2$ matrix at the output. The mapping constructed here is general and work for higher order matrices also. But to simplify the discussion, we have taken $n = 2$ and then showed that the map is positive under certain conditions.
- Further, we have shown that the constructed map can never be completely positive and also obtained the conditions for which the map is positive but not completely positive.
- Lastly, we have discussed that the Choi matrix constructed from the positive map can act as a witness operator and take part in the detection of bound entangled state and negative partial transpose entangled state.

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