Information Theoretic Measures For Assessing Financial Markets

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by

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under the supervision of

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July 2021 **Enroll. No.: 2K17/PhD/AM/10**

DECLARATION

I hereby declare that the research work which is being presented in the thesis entitled "**Information Theoretic Measures For Assessing Financial Markets**" submitted by me to the Delhi Technological University, Delhi for the award of the degree of *Doctor of Philosophy in Mathematics* has been carried out by me under the supervision and guidance of Dr. H.C. Taneja, Professor in the Department of Applied Mathematics, Delhi Technological University, Delhi, India.

The research work embodied in this thesis is my original research and to the best of my knowledge has not been submitted earlier in part or full or in any other form to any other University or Institute for the award of any degree or diploma.

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CERTIFICATE

This is to certify that the thesis entitled **"Information Theoretic Measures For Assessing Financial Markets"** submitted by **Luckshay Batra** in the Department of Applied Mathematics Delhi Technological University, Delhi, India for the award of degree of *Doctor of Philosophy in Mathematics* is an original contribution with existing knowledge and faithful record of research work carried out by him under my guidance and supervision. I have gone through the work reported in this thesis, and in my opinion, it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy.

To the best of my knowledge, the work reported in this thesis is original and has not been submitted partially or fully to any other university or institution in any form for the award of any degree or diploma.

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July 2021 **(LUCKSHAY BATRA)**

Dedicated to my beloved parents

Shri Bhim Sen Batra

&

Smt. Madhu Batra

Contents

Preface

The development in the field of information theory can be traced back to 1948 with the publication of C.E. Shannon's landmark paper [76] entitled *"A Mathematical Theory of Communication"* in the *Bell System Technical Journal*. This theory set in motion a revolution in communication system engineering. The application of various information theoretic measures, for example, Shannon entropy measure [76] and its various generalizations like one-parametric: Renyi entropy [68], Arimoto entropy [2], Tsallis entropy [89]; and two-parametric: Varma entropy [93], Sharma and Mittal entropy [77] and Sharma and Taneja entropy [85] have been widely applied in finance [19, 28, 48, 84] to measure the diversity and regularity of price fluctuations across a broad spectrum of markets (i.e., stock, currency, future, commodity, and cryptocurrency).

This thesis aims to examine how entropy can be useful to analyze financial markets through an analytical approach, an emergent field of interest with a lot to learn. Many researchers have tried to explore the behavior of various information theoretic measures on the volatility modeling [9, 14, 73], portfolio-selection problem [10, 95], and time series analysis [18, 44]. In this thesis, we have studied four aspects which have the application of information theoretic tools as a common thread.

Firstly, we have used entropy to measure the uncertainty in the financial markets since the standard deviation/variance can discern only linear relationships. We find it worth to investigate deeper into the working of these entropic measures such as Renyi, Tsallis, Approximate, and Sample entropies in the situations of financial turmoil. As the second aspect, Kullback measure of relative information [40] has been used to examine and contrast different forecasting models such as ARIMA, Holt-Winters, and LSTM in terms of performance to predict the financial data series.

As the generalized entropy measures are useful due to flexibility provided by additional parameters, so thirdly, we have studied a rich class of one and two parametric information measures for portfolio selection problems such as portfolio diversification and portfolio optimization to quantify risk and measure risk-adjusted performance in the capital market. Further, as the fourth aspect, we have applied the information theoretic measures approach to the Black-Scholes option pricing model with stochastic volatility and Quanto-option pricing model, as it is a useful and interesting concept not explored so much. The reported work is organized as follows:

Chapter 1 provides a summary of the related literature as well as an introduction to the mathematical concepts used. It gives an outline of Information theory, entropy measures, and its generalizations. Mutual information and its variants are also discussed. Also, a brief survey of stochastic differential equation and Option pricing theory have been carried out. The motivation for the problems undertaken along with a plan of work has been discussed at the end.

In Chapter 2, we have modeled the implied volatility as a linear combination of historical volatility and entropy, and found that the model was heavily dependent on the entropy. Also, we have considered seven different estimators of Shannon entropy; Tsallis entropy and Renyi entropy for various values of their parameters to characterize the volatility in the stock market, where we have done in-depth empirical analysis among generalized information theoretic measures. We have also observed some equivalence between information theoretic measures and statistical measures normally employed to capture the randomness in financial time series.

In Chapter 3, we have applied forecasting models such as Autoregressive Integrated Moving Average (ARIMA), Holt-Winter and Long Short Term Memory (LSTM) network to forecast the behavior of some well-known stock market indices and have compared the accuracy of these forecasting models by using the Kullback measure of relative information. We conclude that the ARIMA forecasting model outperforms the other two for one specific index, and the Holt-Winters model works better for prediction of the other two indices.

In Chapter 4, we present an alternative method for improving the accuracy of portfolio risk assessment by using a rich class of information theoretic measures. We analyze the effectiveness of Markowitz's Mean-Variance model with the models which replace expected portfolio variance with measures of information (uncertainty of the portfolio allocations to the different assets). The empirical analysis is carried out on the historical data of Indian financial stock indices by application of portfolio optimization problem with information measures as the objective function and constraints derived from the return and the risk. Our findings indicate that some generalized information measures with parameters can be used as an adequate supplement to traditional portfolio optimization model such as the mean-variance model.

In Chapter 5, we have used the Kullback measure of relative information to obtain risk-neutral measures of the stock option price and volatility. Based on theoretical analysis, when the underlying financial asset is calculated using a stochastic volatility model, we have obtained a second-order parabolic partial differential equation, the generalized Black-Scholes equation. Also, to investigate the analytical solution of this generalized Black-Scholes equation, we have used the Laplace transform homotopy perturbation method.

In Chapter 6, we have extended the information theoretic approach as studied in Chapter 5 to the Quanto option pricing model. Numerical results for the assumed financial parameters demonstrate that the method is effective, and this approach will help to study the financial behavior of the Quanto option pricing problems.

In the end, we conclude the work reported in this thesis and also give further scope of the study.

List of Figures

List of Tables

Chapter 1

Introduction And Literature Survey

1.1 An outline of Information Theory

Information Theory is a branch of applied mathematics dealing with problems such as information processing, storage, retrieval, and decision-making. Basically, it deals with all the theoretical problems which come across in the transmission of information over the communication channels. Information Theory has found applications in various fields like electrical engineering, financial mathematics, statistical modeling and image processing, etc. Although Nyquist [62] and Hartley [29] made the first attempt in this direction by characterizing the entropy measure for equally probable events, yet the theoretical foundation for all the developments in the area of digital communication dates backs to the work of Shannon [76] and others in the mid of the 20th century. All this resulted in the advancement of information theory as a mathematical discipline. The theory basically considers the following three fundamental questions:

- **Compression:** How much data can be compressed so that an identical copy of the uncompressed data can be recovered by another person?
- **Lossy data compression:** How much can data be compressed so that another person can recover an approximate copy of the uncompressed data?
- **Channel capacity:** How quickly reliable communication is possible from the source to destination through a noisy medium?

Being an electrical engineer Shannon's goal was to get maximum line capacity with minimum distortion. He was more concerned with the technical problems of high-fidelity message transmission than with the semantic nature of the message or its pragmatic effect on the listener. The second half of the twentieth century was characterized by the enormous development of systems in which the transmitted information (analog signal) is coded in a digital form. In spite of the fact that Shannon presented entropy [76] as a measure of uncertainty in the communication theory, the information measures have kept on finding diverse applications in a variety of disciplines including mathematics, physics, biological sciences, pattern recognition, etc., refer to [39, 72, 90]. It has broad applications in finance too especially in portfolio selection, refers to Fernholz [25], who studied entropy as diversification in financial markets. Nawrocki and Harding [60] proposed state-value weighing of the entropy and tested using a portfolio selection heuristic algorithm. The results showed that weighing the entropy value will increases the investment performance of the entropy risk measures.

Tang and Song [86] have given a more detailed description of the model assumptions and the mathematics derivation process of the formula and have analyzed the sensitivity of the value of an option. According to their empirical analysis, the Black-Scholes model features strong usefulness, but also has some limitations and thus further research was required. Finkelstein and Friedberg [24] have used entropy as a measure of industrial concentration and market competitiveness. Dedu and Toma [19] designed few techniques for modeling financial data and solving decision-making problems based on risk theory and information theory. They proposed a new risk measure given by the convex combination of two measures: Value-at-Risk (VaR) and Shannon Entropy and resulting in different consequences of the optimization problem, depending upon the proportion of the two measures used.

Sheraz et al. [78] presented the entropic approach so as to assess the volatile stock index by using entropy measures: Tsallis entropy [89], Shannon entropy [76], Renyi entropy [68] and Approximate entropy [66]. They concluded that the entropic approach for the volatility of the stock market was another approach which investigates the new horizons for future research. Tunaru [92] reviewed some important aspects of the application of entropy-related concepts to option pricing. Li et al. [44] constructed financial networks based on the data from the Chinese financial market and examined the different types of network entropies. Bose and Hamacher [14] proposed two alternate information theoretical approaches (Super-information, and Super-mutual-information) to assess volatility in financial markets. Motivated by the aforementioned work by researchers, in this thesis, we study the ability of information theoretic measures to capture the irregularities in a financial time series, analysis of portfolio selection problems and also have explored the Black-Scholes model [12, 55], a pricing model for financial instruments, applying information theoretic measures.

1.2 Entropy Measure and Generalizations

1.2.1 Entropy Measures

Shannon Entropy

Shannon introduced a measure of information or entropy for a general finite complete probability distribution and gave a characterization theorem for the entropy measure introduced. Entropy as defined by Shannon [76] and added upon by other physicists is closely related to thermodynamical entropy. In fact, Shannon borrowed the idea of entropy from the second law of thermodynamics, according to this law the universe is winding down from an organized state to chaos, moving from predictability to uncertainty. How much information a message contains is measured by the extent it combats entropy. Consider the random variable $X = \{X_1, X_2, ...X_n\}$ with probability distribution $P = \{p_1, p_2, ..., p_n\}$, then the Shannon's entropy is defined as

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \ , \ \ 0 \leq p_i \leq 1, \ \ \sum_{i=1}^{n} p_i = 1. \tag{1.2.1}$$

Here it is assumed that $0 \log 0 = 0$; the base of logarithm is being considered as e, unless stated otherwise. Shannon entropy provides the measure of average uncertainty associated with the outcome of a random experiment or a measure of information conveyed through the knowledge of the probabilities associated with the events. It satisfies the following important properties which are usually considered desirable for a measure of uncertainty defined in terms of probability distributions:

1. **Non-negativity:** $H(P)$ is always non-negative, that is,

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \geq 0;$$

   The result is obvious since $-p_i \log p_i \geq 0$ for all i. It becomes zero if one $p_i = 1$ and rest are zeros.

2. **Maxima:** $H(p_1, p_2, \ldots\ldots, p_n) \leq \log n$, with equality when $p_i = \frac{1}{n}$ for all i.

3. **Continuity:** $H(p_1, p_2, \ldots\ldots, p_n)$ is a continuous function of p_i's, that is, a slight change in the probabilities p_i's results in the slight change in the uncertainty measure also.

4. **Symmetry:** $H(p_1, p_2, \ldots\ldots, p_n)$ is a symmetric function of p_i's, that is, it is invariant with respect to the order of the outcomes.

5. **Grouping (or, Branching) Property:**

$$H\{p_1, p_2, p_3, \cdots, p_n\} = H\{p_1 + \cdots + p_r,\ p_{r+1} + \cdots + p_n\} + (p_1 + \cdots + p_r) \times$$

$$H\left(\frac{p_1}{\sum_{i=1}^{r} p_i}, \cdots, \frac{p_r}{\sum_{i=1}^{r} p_i}\right) + (p_{r+1} + \cdots + p_n) H\left(\frac{p_{r+1}}{\sum_{i=r+1}^{n} p_i}, \cdots, \frac{p_n}{\sum_{i=r+1}^{n} p_i}\right).$$

   for $r = 1, 2, \cdots n-1$.

6. **Additivity**: If $P = (p_1, p_2, \ldots\ldots, p_n)$ and $Q = (q_1, q_2, \ldots\ldots, q_n)$ are two independent probability distributions, then

$$H(P \bullet Q) = H(P) + H(Q),$$

   where $P \bullet Q$ is the joint probability distribution, that is, for two independent distributions entropy of the joint distribution is the sum of the entropies of the two marginal distributions.

Analogous to (1.2.1), for a continuous random variable with probability density function $f(x)$, the measure of uncertainty is defined as

$$H(X) = -E[\log f(X)] = -\int_0^{\infty} f(x) \log f(x) dx\,. \tag{1.2.2}$$

In general, this measure is termed as *differential entropy*, for more details refer to McEliece [52]. Unlike the uncertainty measure (1.2.1) defined for discrete random variable which is always positive, the differential entropy can take a negative value as well.

Pal and Pal entropy

Pal and Pal [64] proposed another entropy measure called exponential entropy given by

$$Ep(X) = \sum_i p_i(e^{(1-p_i)} - 1), \;\; 0 \le p_i \le 1, \;\; \sum_{i=1}^{n} p_i = 1. \tag{1.2.3}$$

This entropy measure has an advantage over Shannon entropy, like in the case of uniform distribution of probabilities of a random variable $X = \{x_1, x_2, ..., x_n\}$, $Ep(X)$ has a finite upper bound $e-1$ as $n \to \infty$.

Sine entropy

Sine entropy [98] of a random variable X is defined by

$$Si(X) = \sum_i sin(\pi \; p_i), \;\; 0 \le p_i \le 1, \;\; \sum_{i=1}^{n} p_i = 1. \tag{1.2.4}$$

This entropy keeps invariant under arbitrary translation, and also can measure the uncertainty of some set which Shannon entropy cannot.

1.2.2 One-Parametric Entropy Measures

We have seen that Shannon entropy satisfies a number of useful properties like non-negativity, continuity, symmetry, additivity, grouping, etc. Though Shannon's entropy is at the focus in information theory, yet the idea of information is so rich enabling no single definition that will have the capacity to measure information legitimately. Hence, many researchers presented the parametric entropies as a mathematical generalization of Shannon's entropy. These entropies have some additional parameters and tend to Shannon entropy when these parameters approach their limiting values.

Renyi entropy

Alfred Renyi [68] proposed a generalized form of Shannon entropy, known as Renyi entropy, depending on a parameter r. The Renyi entropy is defined as

$$R_r(X) = \frac{log \sum_{i=1}^{n} p_i^r}{1-r}, \quad 0 < r < \infty, \quad r \neq 1. \tag{1.2.5}$$

Properties of Renyi entropy are similar to the Shannon entropy i.e., it is additive and has a maximum for $p_i = \frac{1}{n}$, but it contains additional parameter r and thus forms a parametric family of entropy measures that give weights to extremely rare and regular events completely different. This entropy became widely useful for its applications in finance [15, 36, 38].

Arimoto entropy

Arimoto entropy [2] with parameter a, a generalization of Shannon entropy, is given by

$$A_a(X) = \frac{a}{a-1}\left[1-\left(\sum p_i^a\right)^{\frac{1}{a}}\right], \quad a > 0,\ a \neq 1. \tag{1.2.6}$$

$A_a(X)$ tends to Shannon entropy when $a \to 1$.

Tsallis entropy

Tsallis [89] proposed a generalized non-additive entropy with a parameter q. In the case for $q < 1$, rare events are emphasized and for $q > 1$ frequent events prevail. Tsallis entropy is defined as

$$T_q(X) = \frac{1-\sum_{i=1}^{n} p_i^q}{q-1}, \quad 0 < q < \infty, \quad q \neq 1. \tag{1.2.7}$$

There are diverse applications of Tsallis entropy in finance [59, 75]. Tsallis and Renyi both proposed generalized entropies that for $q \to 1$ and $r \to 1$ reduce to the Shannon entropy. These generalized entropy measures have been used to analyze highly volatile financial sectors [6].

1.2.3 Two-Parametric Entropy Measures

Varma entropy

Varma entropy [93] of order α and type β is a generalized form of Renyi entropy. This two-parametric entropy, denoted by $H_{\alpha,\beta}(X)$, is given by

$$H_{\alpha,\beta}(X) = \frac{1}{\beta-\alpha} log\left(\sum_i p_i^{\alpha+\beta-1}\right),\ 0 \leq \beta - 1 < \alpha < \beta,\ \alpha+\beta \neq 2. \qquad (1.2.8)$$

It reduces to Renyi entropy when $\beta = 1$. So Renyi entropy becomes a particular case of Varma entropy.

Sharma and Mittal entropy

Sharma and Mittal [77] introduced a two parametric entropy measure given by

$$SM_{\alpha,\beta}(X) = \frac{1}{1-\beta}\left[\left(\sum_i p_i^{\alpha}\right)^{\frac{1-\beta}{1-\alpha}} - 1\right], \qquad (1.2.9)$$

for $\alpha \neq \beta,\ \alpha,\beta > 0$ and $\alpha \neq 1 \neq \beta$. Sharma-Mittal entropy $SM_{\alpha,\beta}(X)$ reduces to Renyi entropy and Tsallis entropy when $\beta \to 1$ and $\beta \to \alpha$ respectively.

Sharma and Taneja entropy

Sharma-Taneja entropy [85] is given by

$$ST_{\alpha,\beta}(X) = \frac{1}{2^{1-\alpha} - 2^{1-\beta}} \sum_i (p_i^{\alpha} - p_i^{\beta}), \quad (\alpha,\beta) \neq (1,1). \qquad (1.2.10)$$

This entropy reduces to Shannon entropy when $(\alpha,\beta) \to (1,1)$ and by imposing $\beta = 1$, this entropy reduces to Tsallis entropy.

Along with these entropies, some other entropies which have been used in this thesis are discussed below.

Approximate and Sample entropies

In 1991, Pincus [66] introduced Approximate entropy that measures the intricacy and regularity of a time series which is relatively short and noisy. This entropy calculates the statistical likelihood that the trend will remain the same on a larger scale. Series that are highly entropic, approximate random systems that usually contain very less repetitive trends than those with low entropy values. Approximate entropy was introduced to measure the regularity of non-linear systems [11,23,47,67]. The description of Approximate entropy (ApEn) is as follows:

Given a finite time series T=$\{t_1,t_2,...,t_N\}$, define two vectors of m-dimension, $x_i = t_i,t_{i+1},...,t_{i+m-1}$ and $y_j = t_j,t_{j+1},...,t_{j+m-1}$ for $1 \leq i,j \leq N-m+1$, whose distance, d is expressed as

$$d = \max_{k=0,1,...,m-1} |y_{j+k} - x_{i+k}|. \tag{1.2.11}$$

If d is smaller than a specified tolerance r then sequence vectors x_i and y_j are called *similar*. For each of the $N-m+1$ elements, the *relative frequency* of finding similar vectors is

$$f_i(r) = \frac{n_i}{N-m+1}, \tag{1.2.12}$$

where n_i is calculated as the total number of vectors y_j similar to x_i. Next, one calculates the *average regularity* as

$$\xi^m(r) = \frac{\sum_i \log f_i(r)}{N-m+1}, \tag{1.2.13}$$

from which the approximate entropy can be evaluated by comparing the average regularity of adjacent dimensions m and $m+1$ as

$$ApEn(m,r,N) = \xi^m(r) - \xi^{m+1}(r), \tag{1.2.14}$$

where m and r are variable parameters.

Lower ApEn value stipulates that the time series is deterministic and a higher value stipulates randomness. ApEn depicts a family of statistics, with distinct regularity measures over a range of tolerances r and dimensions m.

Next a modification of ApEn, the *Sample entropy* is an alternative method given by Richman and Mooran [69] to eliminate ApEn's biasness towards regularity. Like ApEn, the algorithm remains the same except that when calculating the relative frequency, it excludes self-matching vectors. Sample entropy has been utilized substantially for analyzing financial markets [96].

1.2.4 Joint and Conditional Entropies

The joint entropy is the uncertainty measure allied with a joint probability distribution of two discrete random variables, say $X = \{x_1, x_2, x_3, ..., x_s\}$ and $Y = \{y_1, y_2, y_3, ..., y_t\}$, associated with some experiment with all possible outcomes, st is defined as

$$H(X,Y) = -\sum_{i=1}^{s}\sum_{j=1}^{t} p_{ij} log\ p_{ij}, \tag{1.2.15}$$

where $p_{ij} = p(x_i, y_j) = P(X = x_i, Y = y_j)$ is the joint probability density function of random variables X and Y.

The conditional entropy, also called the equivocation of the random variable $Y = \{y_1, y_2, y_3, ..., y_t\}$ given $X = \{x_1, x_2, x_3, ..., x_s\}$ is defined as

$$H(Y|X) = -\sum_{i=1}^{s}\sum_{j=1}^{t} p_{ij} log\ p_{j|i}, \tag{1.2.16}$$

where $p_{j|i} = p(y_j|x_i) = P(Y = y_j|X = x_i)$ is the conditional probability density function of the random variable Y given X.

The joint and conditional entropy measures follow some properties as given below

- $H(Y|X) \leq H(Y)$.
- $H(X,Y) \leq H(X) + H(Y)$.
- $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$.

1.3 Mutual Information and its Variants

In this section we discuss the concept of mutual information and its variants like global correlation coefficient and variation of information.

1.3.1 Mutual Information

This information measure gives the reduction in uncertainty of one random variable Y due to the knowledge of another random variable X. The mutual information, also called the *transinformation* of Y relative to X, is defined as

$$\begin{aligned} I(Y|X) =& H(Y) - H(Y|X) \\ =& H(Y) + H(X) - H(X,Y) \\ =& H(X) - H(X|Y), \end{aligned}$$

where $H(Y|X)$ and $H(X|Y)$ are the conditional entropies and $H(X,Y)$ is the joint entropy of two random variables X and Y.
If $p_i = P(X = x_i)$ and $p_j = P(Y = y_j)$ are marginal probability distributions of X and Y respectively; and $p_{ij} = p(x_i, y_j) = P(X = x_i, Y = y_j)$ is the joint probability density function of X and Y, then it can be shown that

$$I(Y|X) = \sum_{i=1}^{s} \sum_{j=1}^{t} p_{ij} \, log \, \frac{p_{ij}}{p_i p_j}. \tag{1.3.1}$$

This information measure follows some properties, as given below

- Non-negative: $I(Y|X) \geq 0$,
- Symmetric: $I(Y|X) = I(X|Y)$.

Next, we summarize some of the extensions of mutual information measure as follows:

- The *relative mutual information* measure is directly comparable to the coefficient of determination (R^2), because both estimates the proportion of variability demonstrated by an independent variable in the dependent variable [21]. This information measure is defined as

$$RMI_{XY} = \frac{I(X,Y)}{H(Y)}. \tag{1.3.2}$$

- For the two discrete random variables say, X and Y, the *normalized version of mutual information* is defined as

$$NMI_{XY} = \frac{I(X,Y)}{\sqrt{H(X)H(Y)}}. \tag{1.3.3}$$

In addition to capturing strong linear relationship, this non-linear approach based on mutual information also captures the non-linearity found in the volatile market data which was not otherwise captured by the approach based on correlation coefficient.

1.3.2 Global Correlation Coefficient

This standardized measure of mutual information can be used as a predictability measure based on the distribution of empirical probability, and is defined as

$$\lambda_{XY} = \sqrt{1 - e^{-2I(X,Y)}}. \tag{1.3.4}$$

It ranges from 0 to 1 and records the linear as well as the nonlinear dependence between two discrete random variables, say X and Y, and is therefore easily comparable with the linear correlation coefficient [81],

$$\lambda_{XY} = \begin{cases} 0, & \text{if } X \text{ contains no information on } Y \\ 1, & \text{if there is a perfect relationship between the vectors } X \text{ and } Y. \end{cases}$$

1.3.3 Variation of Information

This information measure is a true estimate of distance between two discrete random variables, say X and Y and, as such it obeys the triangle inequality [53]. It is closely related with mutual information measure and is defined as

$$\begin{aligned} VI_{XY} =& H(X) + H(Y) - 2I(X,Y) \\ =& H(X,Y) - I(X,Y) \\ =& H(X|Y) + H(Y|X). \end{aligned}$$

Alternatively, variation of information can be given as

$$VI_{XY} = \begin{cases} 0, & \text{when } X \text{ is equal to } Y \\ < H(X,Y), & \text{when } X \text{ and } Y \text{ are dependent} \\ H(X,Y), & \text{when } X \text{ and } Y \text{ are independent,} \end{cases}$$

where $H(X)$ is the entropy of the discrete random variable $X = \{x_1, x_2, x_3, ..., x_n\}$ and $I(X,Y)$ is mutual information between two discrete random variables $X = \{x_1, x_2, x_3, ..., x_n\}$ and $Y = \{y_1, y_2, y_3, ..., y_n\}$ and $H(X,Y)$ is the joint entropy of X and

Y.

1.4 Specific Stochastic Processes

1.4.1 Brownian Motion

In 1827, Brown identified the peculiar motion of a small particle fully immersed in a liquid or gas. During 19th century, Brownian motion was applied to model the price movements of stocks and commodities.
A stochastic process $\{W(t),\ t \geq 0\}$ is said to be a Brownian motion if it satisfies the following properties:

- $W(0) = 0$.
- For $t > 0$, the sample path of $W(t)$ is continuous.
- $\forall n$, and $0 \leq t_0 < t_1 < t_2 < ... < t_n$, increments $W(t_i) - W(t_{i-1}),\ i = 1, 2, ..., n$ are independent and stationary.
- For $0 \leq s < t < \infty$, $W(t) - W(s)$ is normallly distributed random variable with mean 0 and variance $t - s$.

A Brownian motion is also called a *Wiener process*. The sample path of Weiner process is continuous, and Wiener process can also be visualized as a scaling limit of a *symmetric random walk*. These paths are also essentially nowhere differential.
The Wiener process is not wide sense stationary and this is because for $s < t$, the covariance function $Cov(W(t), W(s))$ is not a function of $(t - s)$. In fact

$$\begin{aligned}
Cov(W(t), W(s)) &= E[W(t) * W(s)] - E[W(t)] * E[W(s)] \\
&= E[W(t) * W(s)] \\
&= E[(W(t) - W(s) + W(s)) * W(s)] \\
&= E[(W(t) - W(s)) * W(s)] + E[W(s)^2] \\
&= E[W(s)^2] = Var(W(s)) = s.
\end{aligned}$$

Hence $Cov(W(t), W(s)) = min(s, t)$. Given, $W(t)$ the future $W(t + h)$ for any $h > 0$ only depends on the increment $W(t + h) - W(t)$ and this is independent of the past. Thus $\{W(t),\ t \geq 0\}$ is a *Markov process*.

Properties of Brownian motion

Let $\{W(t),\ t > 0\}$ be a Weiner process. Then

- $\{-W(t),\ t \geq 0\}$ is a Wiener process (*Symmetric*).
- $\{\frac{1}{\sqrt{c}}W(ct),\ t \geq 0\}$ is a Wiener process for each fixed $c > 0$ (*Scaling*).
- taking $\dot{W}(0) = 0$, $\dot{W}(t) = t * W(\frac{1}{t})$, $t > 0$, the process $\{\dot{W}(t),\ t \geq 0\}$ is a Wiener process (*Time Inversion property*).

A general Brownian motion need not have $W(0) = 0$ and $\sigma^2 = 1$. Therefore, we define a general Brownian motion with drift μ and variance σ^2 as follows

A stochastic process $\{X(t),\ t \geq 0\}$ is said to be a Brownian motion with drift μ and volatility σ if

$$X(t) = \mu t + \sigma W(t),$$

where $W(t)$ is a Brownian motion, $\mu \in (-\infty, \infty)$ and $\sigma > 0$ are constants. This is a generalization of standard Brownian motion. In this process, the mean function $E[X(t)] = \mu t$ and Covariance function $Cov(W(s), W(t)) = \sigma^2 min(s,t)$, for $s,t \geq 0$.

Brownian Bridge

A standard Brownian motion bridge $\{X(t),\ t \in [0,1]\}$ is defined as

$$X(t) = W(t) - tW(1),$$

where $W(t)$ is a standard Brownian motion. Clearly $X(0) = X(1) = 0$ and since for $0 < t < 1$, $W(t) \sim N(0,t)$, and $X(t) \sim N(0, t(1-t))$. The covariance function is $s(1-t)$ for $0 \leq s \leq t \leq 1$. Therefore, the Brownian bridge is a Gaussian process but is not a Brownian motion. For a fixed $T > 0$, the general Brownian bridge $\{X(t),\ t \in [0,T]\}$ can be defined as

$$X(t) = W(t) - \frac{t}{T}W(T), \quad 0 \leq t \leq T.$$

Also, the covariance function is given by

$$Cov(X(s),X(t)) = min\{s,t\} + \frac{st}{T} - \frac{t}{T}min\{s,T\} - \frac{s}{T}min\{t,T\}.$$

Note, $Cov(X(t),X(s)) < Cov(W(t),W(s))$, and most uncertainty is in the middle of the bridge, with zero uncertainty at the nodes. Also, the increments in a brownian bridge are not independent.

1.4.2 Geometric Brownian Motion

A stochastic process $X(t),\ t \geq 0$ is said to be a geometric Brownian motion if

$$X(t) = X(0)e^{W(t)},$$

where $W(t)$ is standard Brownian motion. Note that Brownian motion has independent increment, hence given $X(t)$, the future $X(t+h)$ only depends on the future increments of the Brownian motion, i.e. $X(t+h) = X(t)e^{W(t+h)-W(t)}$. Thus future is independent of the past and therefore the Markov property is satisfied. Hence, $X(t),\ t \geq 0$ is a Markov process. Because, a geometric Brownian motion is non-negative, it provides for a more realistic model of stock prices.

How does geometric Brownian motion relate to stock prices? One possibility is to think of modeling the rate of the stock price as a Brownian motiion. Suppose that the stock price $S(t)$ at time t is given by

$$S(t) = S(0)e^{H(t)},$$

where $S(0)$ is an initial price and $H(t) = \mu t + \sigma W(t)$ is a Brownian motion with drift. In this case, $H(t)$ represents a continuously compound rate of return of the stock price over the period of time $[0,t]$. Here, $H(t)$ refers to the logarithmic growth of the stock price, satisfies

$$H(t) = log(S(t)) - log(S(0)),$$

and therefore $log(S(t))$ has a normal distribution with mean $\mu t + log(S(0))$ and variance $\sigma^2 t$. Here, logarithm is to the base e.

1.4.3 Levy Process

A stochastic process $\{X(t),\ t \geq 0\}$ is said to be a Levy process if it satisfies the following properties:

- $X(0) = 0$,
- $\forall n$, and for $0 \leq t_0 < t_1 < t_2 < ... < t_n$, increments $X(t_i) - X(t_{i-1}),\ i = 1,2,...,n$ are independent and stationary,
- for $a > 0$, $P(|X(t) - X(s)| > a) \to 0$, when $t \to s$.

Note, for $X(t) = bt$ where b is constant, then $\{X(t),\ t \geq 0\}$ is a Levy process. Also, Brownain motion $\{W(t),\ t \geq 0\}$ is a Levy process in $\mathbb{R}$ that has continuous paths and has the Gaussian distribution with mean 0 and variance Δt for its increments $W(t + \Delta t) - W(t)$. The most general continuous Levy process in $\mathbb{R}$ has the form

$$X(t) = bt + cW(t),\ t \geq 0,\ b,c \in \mathbb{R}.$$

In the continuous time modeling, the dynamics of stock price is described by a stochastic differential equation (SDE). By Oksendal [63],“ *... equation we obtain by allowing randomness in the coefficients of a differential equation is called a stochastic differential equation*”. It is explained as below.
Consider a stochastic process X_t, and a partition $0 = h_0 < h_1 < ... < h_n = t$ of the time interval $[0,t]$. Then, the difference equation

$$X_{h_{i+1}} - X_{h_i} = \mu\left(h_i, X_{h_i}\right)\left(h_{i+1} - h_i\right) + \sigma\left(h_i, X_{h_i}\right)\left(W_{h_{i+1}} - W_{h_i}\right); \quad i = 0,1,\ldots,n-1. \tag{1.4.1}$$

As the increments $h_{i+1} - h_i \to 0$, the above eq.(1.3.1) is transformed to

$$dX_t = \mu\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dW_t. \tag{1.4.2}$$

This is the general form of a SDE where the first term on the right side is deterministic with drift μ and the second term is a random term with diffusion σ. W_t represents the standard Brownian motion (BM). It must be noted that as BM is no where differentiable so dW_t just represents the increment $W_{t+dt} - W_t$ on the interval $[t, t+dt]$, and $(dW_t)^2 = (W_{t+dt} - W_t)^2 = dt$ is the quadratic variation of W_t.

Also,

$$dW_t dt = 0, \quad (dt)^2 = 0, \tag{1.4.3}$$

and Eq.(1.4.2) can be expressed as an integral

$$X_t = X_0 + \int_0^t \mu(h, X_h)\, dh + \int_0^t \sigma(h, X_h)\, dW_h. \tag{1.4.4}$$

Here, $\int_0^t \sigma(h, X_h)\, dW_h$ is the integral with respect to the standard Brownian motion W_t and is known as the *stochastic integral*.

1.5 Option Pricing Theory

1.5.1 Volatility

Volatility as a concept is intuitive and simple, measures dispersion about a central tendency. It is a measure of how far an asset's current price deviates from its average past price. More this deviation, more is the volatility. Volatility can indicate the conviction or endurance underneath a stock price movement. Volatility is of mainly two types, *Historical* and *Implied*.

- *Historical Volatility* is the annualized volatility σ, the standard deviation of the annual logarithmic returns of an instrument, defined as $\sigma_T = s\sqrt{\Delta T}$ where ΔT stands for trading days per year (usually 252), s is the standard deviation of stock price returns over time. Then, σ gives us an estimate of the historical volatility of the stock. The closing stock price S_i is usually observed daily over a period of time to calculate the historical volatility of a stock price, and then we let $u_i = log\left(\frac{S_i}{S_{i-1}}\right)$ for $i = 0, 1, 2, ..., n$ and the standard deviation s of $u_i{}'s$ is given as $s = \sqrt{\frac{1}{n-1}\sum(u_i - \bar{u})^2}$ where $\bar{u}$ is the mean of u_i. The volatility is given by $\sigma\sqrt{T}$, and since s is an estimate of σ we can write volatility as $\hat{\sigma} = s\sqrt{T}$.

- On the other hand, *Implied volatility* is the volatility resulting from the market observed option prices. It is used to monitor the opinion of the market or of a trader about the volatility of the security. Just like market as a whole, implied volatility is subject to variability. We will come to implied volatility calculation in Subsection 1.5.3.

1.5.2 Options

Option is a financial security (contract) whose value is derived from (or depends on) an underlying security (stock, stock index, foreign currency etc.). It is a contract giving the option buyer (or holder) a right (without any obligation) to buy or sell a specified underlying asset in future by a fixed date and at a fixed price which is decided when the contract is initiated. This fixed price is known as the "strike price or exercise price" and the fixed date is known as the "maturity or expiration date".

Since an option gives its holder a right without any obligation, the holder needs to pay some amount in the beginning of the contract to the option seller (or writer) to get this right. This amount is called the *premium or price of the option*.

Any option can be categorized as a "call option" or a "put option" depending upon, whether the holder gets the right to buy or the right to sell respectively without any obligation. An option can be either exercised only at maturity (European option) or at any time up to maturity (American option). These options are also called the vanilla options. Vanilla options are the path-independent options because these options are independent of the historical prices of the underlying assets.

Options are the most traded financial securities and can be used to manage risk. For example, if an investor buys a risky asset (say a stock), its future is not known today. Its price can fall in the future and the investor may have to bear the loss. Instead of buying a stock if a person buys a call option on that stock, he gets the right to wait and buy this asset in future (without any obligation) if its price increases the strike price. If the price does not reach the agreed price, he can exit the contract without exercising his right. In this case, his loss will be the premium amount only which was paid at the beginning of the contract.

Here, we consider the European call options only and its payoff (or the value of option at maturity T) is given as:

$$h(X_T) = (X_T - K)^+ = \max(X_T - K, 0), \tag{1.5.1}$$

where X_T is stock price at maturity and K is the strike price.

The European call option is said to be "In The Money" (ITM) if $X_T > K$, "At The

Money" (ATM) if $X_T = K$ and "Out The Money" (OTM) if $X_T < K$. The buyer of a call option exercises his right only if he gets some benefit, which is possible when $X_T > K$, otherwise he'll not exercise his right. This results in the payoff as given in Eq.(1.5.1).

The most interesting question is 'To get this right without obligation, how much one should pay for an option contract?'. Every option pricing model tries to find out the fair price of an option.

Before 1970s, there was not any standard way to price the options. Most of the previous work on the valuation of options include Sprenkle [82], Ayrus [3], Boness [13], Samuelson [70], Baumol et al. [8] and Chen [16], but none were widely accepted until the significant breakthrough by Fischer Black, Myron Scholes and Robert Merton during 1970s, called the Black-Scholes Model.

1.5.3 Black-Scholes Model

In 1973, there came a revolution in the field of options trading when Fischer Black and Myron Scholes [12] and Robert C. Merton [55] published their work on 'Pricing options'. Black and Scholes [12] gave an option pricing formula for the European options which was further developed by Merton [55] in the same year. Black-Scholes model became a milestone in the field of modern finance and for their contribution Myron Scholes and Robert C. Merton received the "Nobel prize in Economics" in 1997. Due to his sudden demise in 1995, Fischer Black could not share this prize.

The Black-Scholes model assumes that the market is complete, i.e., a market with equilibrium price for every asset in every possible state of the world, stock pays no dividend and follows a Geometric Brownian Motion. Volatility and risk free rate of interest are constant through out the option period. The model gives the formula for the pricing of European call options. This model is explained as follows:

Let X_t be the price of the underlying asset at time t, whose dynamics is governed by the geometric Brownian motion given as

$$dX_t = \mu X_t dt + \sigma X_t dW_t^x, \tag{1.5.2}$$

where W_t^x is the standard Brownian motion, $\mu \in (-\infty, \infty)$ is the growth term (drift) and $\sigma > 0$ is the volatility. Let B_t be the bond price at time t and r be the risk free interest rate, then the dynamics of B_t is given as

$$dB_t = rB_t dt. \tag{1.5.3}$$

Let the price of the European call option at time t be $C(x,t)$, where $x = X_t$. This price satisfies the partial differential equation (PDE)

$$\frac{\partial C}{\partial t} + rx\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} - rC = 0, \tag{1.5.4}$$

with the terminal condition

$$h(X_T) = (X_T - K)^+. \tag{1.5.5}$$

This PDE is called the Black-Scholes equation. For its derivation one may refer to Luenberger [46]. Eq.(1.5.4) can also be written as

$$\mathscr{L}_{BS} C = 0, \tag{1.5.6}$$

where

$$\mathscr{L}_{BS} = \frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} - r, \tag{1.5.7}$$

is Black-Scholes operator for the European options. The solution of this equation gives the Black-Scholes formula for pricing options, mentioned next.

The implied volatility is the value of σ when substituted into the Black-Scholes-Merton formula [12] for the European call option price: $C = S_0 N(d_1) - Ke^{-rT} N(d_2)$ where, N(.) is a cumulative distribution function for a standardized normal random variable, i.e., $0 < N(x) < 1$, and r and σ are kept as constant, given by $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp^{-\frac{1}{2}y^2} dy$ where, $d_1 = \frac{\log(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{\log(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ gives the value of C. But its not possible to invert the above equation and express σ as an explicit function of the other parameters. Hence, some numerical methods like Bisection Method or Newton Raphson Method is used to find this volatility. Just like market as a whole, implied volatility is subject to variability. When the demand of a stock is high, the price is likely to rise, and so does implied volatility, leading to a higher option premium. A long-dated option leads to high implied volatility while short-dated option often leads to low implied volatility. It is directly related to market opinion, which in turn affects the pricing of the options. Implied volatilities are forward looking while historical volatilities are backward looking.

Black-Scholes Formula

The European call option price, at time t, is obtained in the closed form as

$$C(X_t, t, T, r, \sigma, K) = X_t N(w_1) - Ke^{-r(T-t)} N(w_2), \tag{1.5.8}$$

here

$$\begin{aligned} w_1 &= \frac{\log\left(\frac{X_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ w_2 &= w_1 - \sigma\sqrt{T-t}, \end{aligned} \tag{1.5.9}$$

with the strike price K, maturity T and the current time t. $N(.)$ represents the cumulative distribution function (CDF) of the standard normal random variable.

The European call and put option prices of an underlying asset X_t, with same strike price K and expiration time T are related with "Put-Call parity formula" given as

$$C_t - P_t = X_t - Ke^{-r(T-t)}, \tag{1.5.10}$$

here, C_t and P_t represents the prices of call and put option respectively at time t with risk free interest rate r. Black-Scholes formula had a major impact on the market as the investors began to feel more comfortable in trading options.

Evolution of dynamic volatility modeling

Volatility is not constant and has many empirical characteristics like volatility smile and skew, mean-reversion, leverage, volatility clustering, variation on different time scales etc., refer to [26, 29, 58]. There has been a logical progress in the field of volatility modeling by the development of new models as an improvement of the existing ones. Time dependent volatility models, where $\sigma = f(t)$, were able to capture the variation of option prices with maturity dates [55]. The local volatility models, where $\sigma = f(t, X_t)$ [17, 20, 22], also addressed volatility smiles and leverage effect (correlation of stock price and volatility), whereas stochastic volatility models, where $\sigma = f(Y_t)$, were able to assimilate all the features of local volatility models and some other empirical volatility characteristics like mean-reversion and volatility clustering (autocorrelation of volatility). Our work is mainly concentrated on the stochastic volatility models and their extensions, so we start with the general representation of the stochastic volatility models.

The general representation of the single factor stochastic volatility model is given below. The dynamics of stock price process X_t is given by

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t^x, \tag{1.5.11}$$

where, $\sigma = f(Y_t)$ and Y_t is the stochastic volatility factor whose dynamics is given by

$$dY_t = a(Y_t)\,dt + b(Y_t)\,dW_t^y, \tag{1.5.12}$$

here, $a(Y_t)$ and $b(Y_t)$ are some functions of Y_t. The standard Brownain motions W_t^x and W_t^y have the correlation structure $E[dW_t^x dW_t^y] = \rho dt$ where, ρ represents a correlation between them.

Some of the most notable single factor stochastic volatility models are given in Ball and Roma [5], Heston [33], Hull and White [37], Scott [74] and Stein and Stein [83]. Hull and White [37] considered the volatility dynamics (1.5.12) to follow process (1.5.2), Scott [74] and Stein and Stein [83] considered the volatility dynamics to follow Ornstein-Uhlenbeck (OU) process, whereas, Ball and Roma [5] and Heston [33] considered volatility dynamics to follow Cox-Ingersoll-Ross (CIR) process.

1.6 Motivation and Plan of Work

Considering the importance and various applications of information theoretic measures and their generalizations in different fields of science and engineering, and in view of the above literature review, we are motivated to apply information theoretic measures to analyze the financial markets [21]. In this thesis, we have studied four aspects in this context which have the application of information theoretical tools as a common thread.

Firstly, we have used entropy to measure the uncertainty in the financial markets since the standard deviation/variance can discern only linear relationships. We find it worth to investigate deeper into the working of these entropic measures such as Renyi, Tsallis, Approximate, and Sample entropies in the situations of financial turmoil. As the second aspect, Kullback measure of relative information

[40] has been used to examine and contrast different forecasting models such as ARIMA, Holt-Winters, and LSTM in terms of performance to predict the financial data series.

As the generalized entropy measures are useful due to flexibility provided by additional parameters, so thirdly, we have studied a rich class of one and two parametric information measures for portfolio selection problems such as portfolio diversification and portfolio optimization to quantify risk and measure risk-adjusted performance in the capital market. Further, as the fourth aspect, we have applied the information theoretic measures approach to the Black-Scholes option pricing model with stochastic volatility and the Quanto-option pricing model, as it is a useful and interesting concept not explored so much.
This thesis includes six chapters, including the current chapter on introduction and literature survey. The reported work is organized as follows:

In Chapter 2 titled, "*Analysing Financial Markets Using Entropy Measures*", we have applied the concept of entropy measure to capture the disorder or uncertainty in highly volatile financial markets without putting any additional constraint on the probability distribution function. We have modeled implied volatility as a linear combination of historical volatility and entropy and found that the implied volatility was heavily dependent on the entropy. We use the closing price data[1] from Jan 2000 to Dec 2017 of top 10 Indian companies: Tata Consultancy Services (TCS), Hindustan Unilever Limited (HUL), India Tobacco Company (ITC), Oil and Natural Gas Corporation Limited (ONGC), Infosys, Reliance, Maruti Suzuki, ICICI Bank, HDFC Bank, State Bank of India (SBI); top four public sector banks: Bank of Baroda, Punjab National Bank (PNB), IDBI Bank, State Bank of India (SBI); and top four private sector banks: Yes Bank, Kotak Bank, HDFC Bank, Axis Bank in terms of their market cap in the year 2018.
Next, we have considered seven different estimators of Shannon entropy; Tsallis entropy and Renyi entropy for various values of their parameters; and also Approximate entropy and Sample entropy to characterize the volatility in the stock market. We have done in-depth empirical analysis among aforementioned information theoretic measures and can affirm that Sample entropy measures more the regularity of time series rather than its complexity; and in comparison it with

[1]Source - Official website of National Stock Exchange of India.

Approximate entropy, a similar class, Sample entropy is more consistent measure and provides improved analysis of the stock market regularity. For this analysis, we have taken the data from the official website of National Stock Exchange of India, we consider the NIFTY stock closing prices into account - NIFTY 50, NIFTY 100, NIFTY 200, NIFTY 500 & NIFTY stock in sectoral - NIFTY Auto, NIFTY Bank, NIFTY IT, NIFTY Pharma for the period of 2016-2018.

Also, we have taken into account the information theoretic concepts such as entropy, conditional entropy, and mutual information and their dynamic extensions for studying the association among the randomness of different financial time series. We have also checked some equivalence between information theoretic measures and statistical measures normally employed to capture the randomness in financial time series. We observe that mutual information and its dynamical extensions provide a good approach to study the association between several international stock indices. The mutual information and conditional entropy have shown a decent efficiency compared to the measures of statistical dependence. To analyze, we have obtained daily closing price data for eighteen years of international stock market indices namely, IBOVESPA(Brazil), Hang Seng(Hong Kong), SSE Composite(China), Dow Jones Industrial Average(United States), CAC 40(France), DAX(France), NASDAQ(United States), Nifty 50(India) and, BSE Sensex(India) from https://in.finance.yahoo.com/, the free financial data online platform.

The part of the work reported in this chapter has been published in two research papers entitled, "*Evaluating volatile stock markets using information theoretic measures*", **Physica A: Statistical Mechanics and its Applications, 537, 122711 (2020)** and "*Entropy as a measure of implied volatility in options market*", **AIP Conference Proceedings 2183 (1), 110005 (2019)**.

In Chapter 3 titled, "*Comparative Performance of Forecasting Models using Relative Information*", we have applied forecasting models such as ARIMA, Holt-Winter, and Long Short Term Memory network to forecast the behavior of some well-known stock market indices, namely NIFTY 50, Dow Jones Industrial Average and S&P BSE SENSEX monthly closing price data; and have compared the accuracy of these forecasting models by using Kullback measure of relative information. We conclude that the ARIMA forecasting model outperforms over the other two for NIFTY 50 index and Holt-Winters model works better for prediction for other two indices. The part of the work presented in this chapter is communicated entitled, "*Comparison of forecasting models using information theoretic approach in financial market prediction*" and also presented in **3rd International Conference of Mathematical Sciences (ICMS 2019)** held at Maltepe University, Istanbul, Turkey, Sept. 2019.

In Chapter 4 titled, "*Information Measures Based Portfolio Optimization*", we have provided a rich class of information theoretical measures designed to enhance the accuracy of portfolio risk assessments. We analyze the effectiveness of Markowitz's mean-variance model with the models which replace expected portfolio variance with measures of information (uncertainty of the portfolio allocations to the different assets). The empirical analysis is carried out on the historical data of Indian financial stock indices by applying portfolio optimization problem with information measures as the objective function and constraints derived from the return and the risk.
Our findings indicate that the information measures with parameters can be used as an adequate supplement to traditional portfolio optimization model such as the mean-variance model. However, we should take care of the fact that generalized entropy measures are highly sensitive to the values of their parameters. So the market analysts need to adjust these parametric values as per their risk and return capacities. The part of the work presented in this chapter has been published in a research paper entitled, "*Portfolio optimization based on generalized information theoretic measures*", **Communications in Statistics - Theory and Methods, (2020)** and also presented in the **International Conference on Recent Trends in Mathematics and Its Applications to Graphs, Networks and Petri Nets (ICRTMA-GPN-2020)** held at School of Computational and Integrative Sciences, JNU, July 2020.

In Chapter 5 titled, "*Black-Scholes Model With Stochastic Volatility Using Relative Information*", we have derived the risk-neutral measures of the stock options price and volatility by incorporating constrained minimization of the Kullback measure of relative information. We obtain a second-order parabolic partial differential equation, the generalized Black-Scholes equation based on the theoretical analysis when the underlying financial asset is estimated using a stochastic volatility model. Also, to investigate the analytical solution of this generalized Black-Scholes equation, we have used Laplace transform homotopy perturbation method. The work presented in this chapter has been published in a research paper entitled, "*On Black-Scholes option pricing model with stochastic volatility: an information theoretic approach*", **Stochastic Analysis and Applications, 39(2), 327-338 (2021)**.

In Chapter 6 titled, "*Quanto-Option Black-Scholes Model Using Relative Information*", we extend the information theoretic approach to Quanto option pricing model. Numerical results for the assumed parameters demonstrate that the method is effective and this approach will help to study the financial behavior of the Quanto option pricing problems. The work reported in the present chapter is communicated entitled, "*Information theoretic approach to quanto option pricing model*".

Lastly, we have presented the conclusion of the work reported in this thesis and further scope of work, followed by a bibliography and the list of publications.

Chapter 2

Analysing Financial Markets Using Entropy Measures

2.1 Introduction

There has been a growing debate on the stock market volatility over the past few years [9]. Shiller [93] asserted that the measured volatility of the stock market was erratic mostly with the predictions of models based on current values. Schwert [73] brought a new insight into this, who asked "Why does stock market volatility change over time?".

The basic characteristic of the entropy that it plays a vital role in retrieving the ubiquitous functionalities of a system from its microscopic feature makes it applicable to measure the randomness in diversified systems. Entropy quantifies the price return variations, the ability to retrieve information, is possibly its most striking property. Entropy studies the behaviour of trends in financial sector, for instance, sessions with several regular trends will tend to be less entropic than those which are having relatively fewer occurrences. The concept of entropy plays an alternative way to look at the stock market volatility. This measure can be ap-

The part of the results reported in this chapter have been published in the paper entitled **Evaluating volatile stock markets using information theoretic measures** in *Physica A: Statistical Mechanics and its Applications*, 2020, 537; and also have been published in the paper entitled **Entropy as a measure of implied volatility in options market** in *AIP Conference Proceedings 2183*, 110005 (2019).

plied to describe the non-linear dynamics of volatility.

This chapter considers three aspects, first we liken techniques, one based on the standard deviation, a statistic, and another based on the investor's expectation on the future movements of the underlying asset price (implied) and the last, centred on the concept of Shannon entropy. The empirical analysis is carried out so as to find some relationship between the three different aspects. It is also worth investigating deeper into the working of these entropic measures in the situations of financial turmoil.

Next, we extend the information theoretic measure approach to estimate the comparison between highly volatile stock indices and sectors. We have considered seven different estimators of Shannon entropy; Tsallis entropy and Renyi entropy for various values of their parameters; and also Approximate entropy and Sample entropy.

Some researchers have considered information measures theoretic approach to study this statistical dependence because of their potential to identify a stochastic relationship as a whole (including linearity and non-linearity both) refer to [6, 7, 18, 28, 101], and so making it a general approach. For example network statistics based on mutual information can successfully replace statistics based on coefficient of correlation [80]. The correlation structure of time series data for financial securities incorporates important statistics for many real world applications such as portfolio diversification, risk measure and asset pricing modeling [49, 91]. Traditionally, non-linear correlations with higher order moments have been studied [21]. Therefore, we apply these information measures for studying the association among the randomness of different financial time series. We have also checked some equivalence between information theoretic measures and statistical measures normally employed to capture the randomness in financial time series. We have proposed a way of integrating non-linear dynamics and dependencies in the study of financial markets using several information entropic measures and their dynamical extensions in the mutual information like normalized mutual information measure, relative mutual information rate and variation of information. We have shown that this approach leads to better results than other studies using a correlation-based approach on the basis of nine international stock indices which are traded continuously during eighteen years i.e. (2001-2018).

The chapter is organized as follows. Section 2.2 models the implied volatility as a linear combination of historical volatility and entropy. Section 2.3 explores the information theoretic approach to analyse the stock indices and sectors which are highly volatile. Section 2.4 presents equivalence between various statistical measures with the information theoretic measures. Section 2.5 concludes the chapter.

2.2 Volatility and Entropy Model and its Analysis

This section examines whether entropy is a good measure for volatility in stock returns and also compares historical volatility with implied volatility. Our concern is to model Implied Volatility (IV) as a function of Historical Volatility (HV) and Entropy (S). As per Gulko [28], the entropy based implied volatility subsumes all the information about historical volatility and entropy. So, we propose a model as:

$$IV = aHV + bS + c \tag{2.2.1}$$

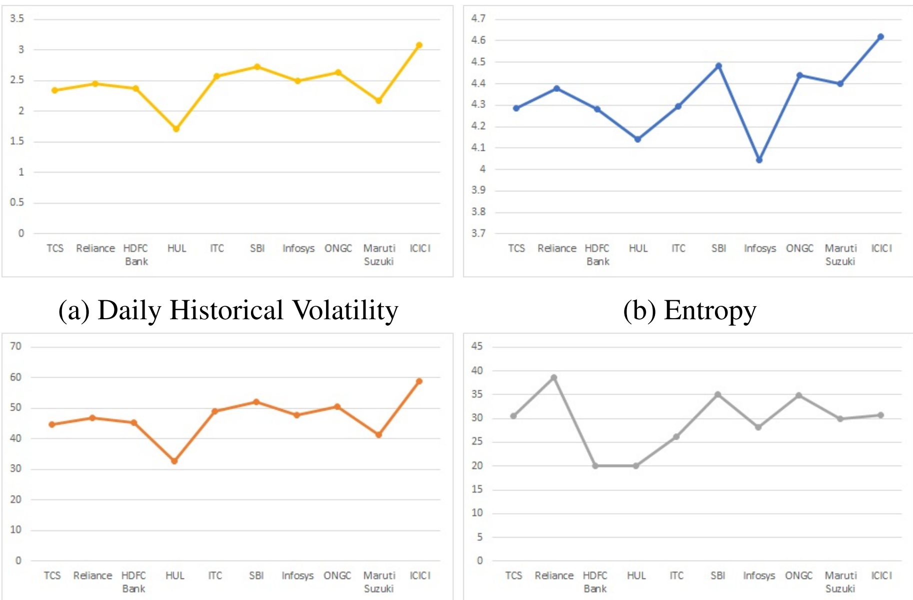
where a,b and c are constants, and to estimate these, we use the closing price data[2] from Jan 2000 to Dec 2017 of top 10 Indian companies: Tata Consultancy Services (TCS), Hindustan Unilever Limited (HUL), India Tobacco Company (ITC), Oil and Natural Gas Corporation Limited (ONGC), Infosys, Reliance, Maruti Suzuki, ICICI Bank, HDFC Bank, State Bank of India (SBI); four public sector banks: Bank of Baroda, Punjab National Bank (PNB), IDBI Bank, State Bank of India (SBI); and four private sector banks: Yes Bank, Kotak Bank, HDFC Bank, Axis Bank in terms of their market cap in the year 2018 which has 4478 observations to make our analysis meaningful. The aforementioned data is given in Tables 2.1, 2.3, and 2.4 and the main reason of choosing this period was that the period had several scenarios of markets in it, the crash due to the economic crisis as well as the sharp rise due to stimulus package given by the Indian government for the revival.
By using multiple regression, the estimated values for the constants come out to be $a = -0.05$, $b = 41.46$, $c = -147.93$. Here we observe that variation in S effects IV much more as compared to the variation in HV. So entropy can be seen as a suitable alternate to implied volatility in financial markets.

The data has daily closing prices of the index which was used to calculate the

[2] Source - Official website of National Stock Exchange of India.

daily returns. With the help of these daily returns, we calculate the annualized historical volatility. We compute a time series of historical volatility by taking a calculation window of 1 year and the time step is 1 day. As we constructed a time series for historical volatility, we did the same for entropic measure i.e., Shannon entropy so as to compare them and visualize them correctly. The entropy was calculated using the natural logarithm and the unit for entropy here is nats.

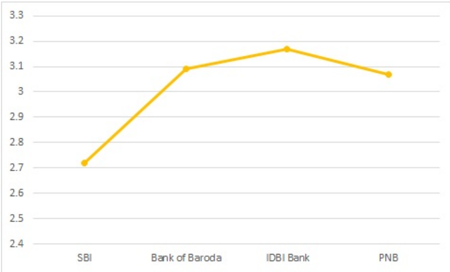
After analyzing the time series, we find that entropy moves in a similar way as of volatility, see Figure 2.1 (for Indian companies), Figure 2.2 (for Indian public sector banks) and Figure 2.3 (for Indian private sector banks), and it has been seen that in periods of crash or jumps (highly volatile), volatility and entropy show similar signals but in period of stability, entropy doesn't give much information about the uncertainty. We have built correlation matrices for sector-wise analysis to check the relation among entropy, historical volatility and implied volatility, see Tables 2.2, 2.3, and 2.4. We can also say that the entropy is mean reverting similar to volatility and hence it can be used in mean reversion trading strategies.



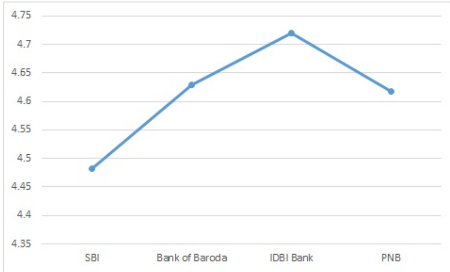
(a) Daily Historical Volatility (b) Entropy

(c) Annualized Historical Volatility (d) Implied Volatility

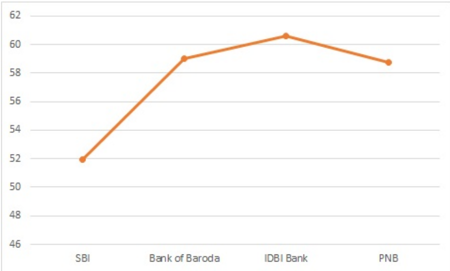
Figure 2.1: Comparative analysis of the entropy and volatility for Indian Companies



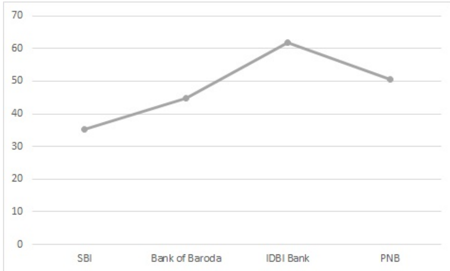
(a) Daily Historical Volatility



(b) Entropy

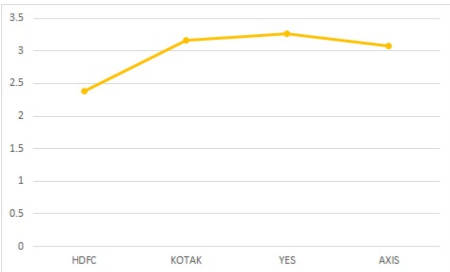


(c) Annualized Historical Volatility

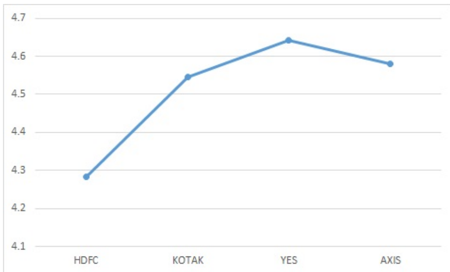


(d) Implied Volatility

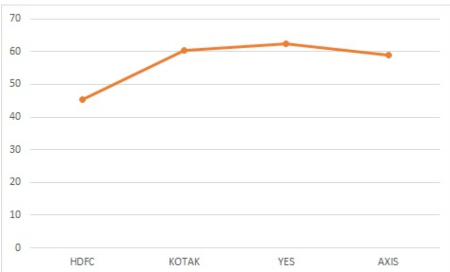
Figure 2.2: Comparative analysis of the entropy and volatility for Public Sector Banks



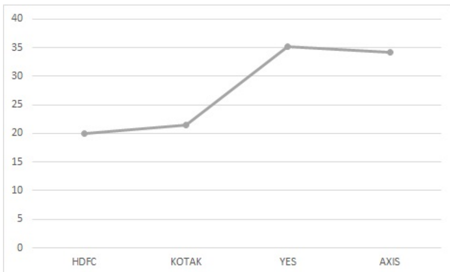
(a) Daily Historical Volatility



(b) Entropy



(c) Annualized Historical Volatility



(d) Implied Volatility

Figure 2.3: Comparative analysis of the entropy and volatility for Private Sector Banks

	TCS	**RELIANCE**	**HDFC BANK**	**HUL**	**ITC**	**SBI**	**INFOSYS**	**ONGC**	**MARUTI SUZUKI**	**ICICI**
Daily Historical Volatility	2.34	2.45	2.38	1.71	2.57	2.72	2.49	2.64	2.17	3.08
Annualized Historical Volatility	44.73931994	46.73316927	45.44156973	32.64344619	49.11858614	51.9632587	47.6258941	50.43246184	41.36674589	58.8840549
Implied Volatility	30.5938721	38.57421875	20.01953125	20.11108398	26.18408203	35.1257324	28.0456543	34.79003906	30.01403809	30.76171875
Entropy	4.285956147	4.379801726	4.280798401	4.142849766	4.295392422	4.483919655	4.042877496	4.43936491	4.401005022	4.619018274

Table 2.1: Entropy and Volatility results for selected Indian Companies

	Entropy	Annualized Historical Volatility	Implied Volatility
Entropy	1		
Annualized Historical Volatility	0.65639891	1	
Implied Volatility	0.547485879	0.5137703	1

Table 2.2: Correlation Matrix for Indian Companies

Entropy and Volatility results

	SBI	**Bank of Baroda**	**IDBI Bank**	**PNB**
Daily Historical Volatility	2.72	3.09	3.17	3.07
Annualized Historical Volatility	51.9632587	59.01722391	60.61496826	58.71762192
Implied Volatility	35.1257324	44.79980469	61.76757813	50.65917969
Entropy	4.482365622	4.628906387	4.720556759	4.616763381

Correlation Matrix

	Entropy	Annualized Historical Volatility	Implied Volatility
Entropy	1		
Annualized Historical Volatility	0.962957291	1	
Implied Volatility	0.951469855	0.877646967	1

Table 2.3: Public Sector Banks

Entropy and Volatility results

	HDFC	**KOTAK**	**YES**	**AXIS**
Daily Historical Volatility	2.38	3.16	3.26	3.08
Annualized Historical Volatility	45.44156973	60.40155195	62.29837139	58.92868316
Implied Volatility	20.01953125	21.45385742	35.15625	34.11865234
Entropy	4.282262489	4.546316805	4.641434414	4.581236584

Correlation Matrix

	Entropy	Annualized Historical Volatility	Implied Volatility
Entropy	1		
Annualized Historical Volatility	0.983360564	1	
Implied Volatility	0.771749481	0.643492335	1

Table 2.4: Private Sector Banks

2.3 Volatile Markets Analysis

Here we compute different entropy measures from the data obtained to analyze the underlying financial markets. From the official website of National Stock Exchange of India, we consider the NIFTY stock closing prices into account - NIFTY 50, NIFTY 100, NIFTY 200, NIFTY 500 & NIFTY stock in sectoral - NIFTY Auto, NIFTY Bank, NIFTY IT, NIFTY Pharma for the period of 2016-2018. For the Shannon entropy measure, we first use different estimators to assess the consistency of the results. In Tables 2.5, 2.8, 2.9 and 2.10 we consider the following estimators to calculate the Shannon entropy [30] with the flattening constants a play the role of pseudo-counts.

- Maximum Likelihood (ML).
- Miller-Madow (MM) : biased corrected Maximum Likelihood.
- Jeffreys : Shannon entropy Bayesian estimate with $a = \frac{1}{2}$.
- Laplace : Shannon entropy Bayesian estimate with $a = 1$.
- Schurmann-Grassberger (SG) : Shannon entropy Bayesian estimate with a = 1/(length of underlying time series).
- Chao Shen (CS) : Horvitz-Thompson estimator applied to the problem of entropy estimation, with additional coverage correction as proposed by Good.
- Shrink entropy : employs James-Stein-type shrinkage at the level of cell frequencies.

The main objective of using various measures of entropy is to liken their performance and deviation between results. Tables 2.5-2.10 show the results obtained.

	NIFTY minute wise	NIFTY day wise
Shannon		
ML	10.73183	4.762456
MM	11.81106	4.971733
Jeffreys	10.7358	4.783969
Laplace	10.73631	4.794722
SG	10.73183	4.763124
CS	28.95428	5.318577
Shrink	10.73668	4.820282

Table 2.5: Entropy results for NIFTY index from Jan 2018 to June 2018

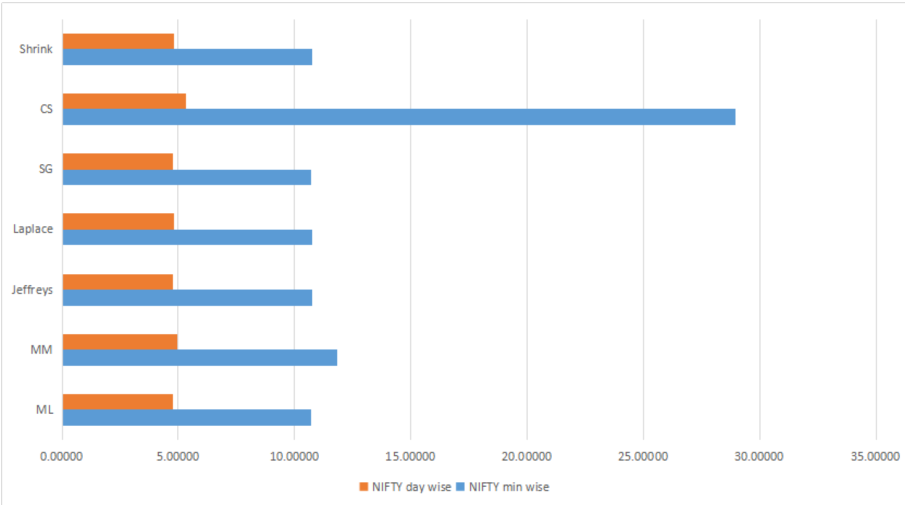


Figure 2.4: Shannon estimators results for NIFTY minute wise and day wise data

		NIFTY 50	NIFTY 100	NIFTY 200	NIFTY 500
Tolerance	**Dimension**				
0.1 σ	1	2.453312	2.480094	2.481278	2.478506
	2	0.9984871	0.9454796	0.9397409	0.9286534
0.15 σ	1	2.245176	2.296259	2.278354	2.27682
	2	1.33853	1.296562	1.293055	1.295919
0.2 σ	1	2.075483	2.087393	2.084737	2.086328
	2	1.460449	1.460357	1.41661	1.427835
0.25 σ	1	1.905598	1.912649	1.919878	1.905752
	2	1.471024	1.469146	1.470134	1.493677
0.3 σ	1	1.760218	1.76128	1.761205	1.754683
	2	1.464866	1.448892	1.464682	1.446751

Table 2.6: Approximate Entropy results for selected NIFTY stock indices

	NIFTY 50	NIFTY 100	NIFTY 200	NIFTY 500
Tolerance				
0.1 σ	2.93492	2.797733	2.805012	2.575432
0.2 σ	2.033375	2.0367	1.984498	1.984444
0.3 σ	1.632652	1.600092	1.601844	1.568093

Table 2.7: Sample Entropy results for selected NIFTY stock indices

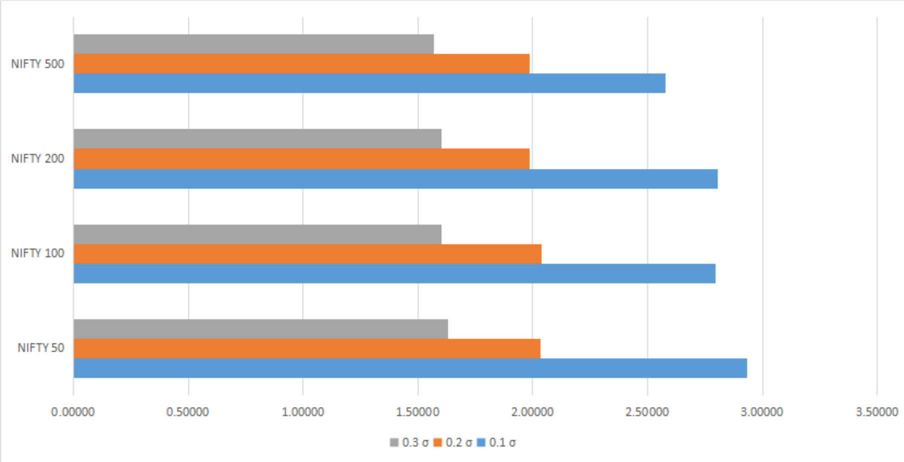
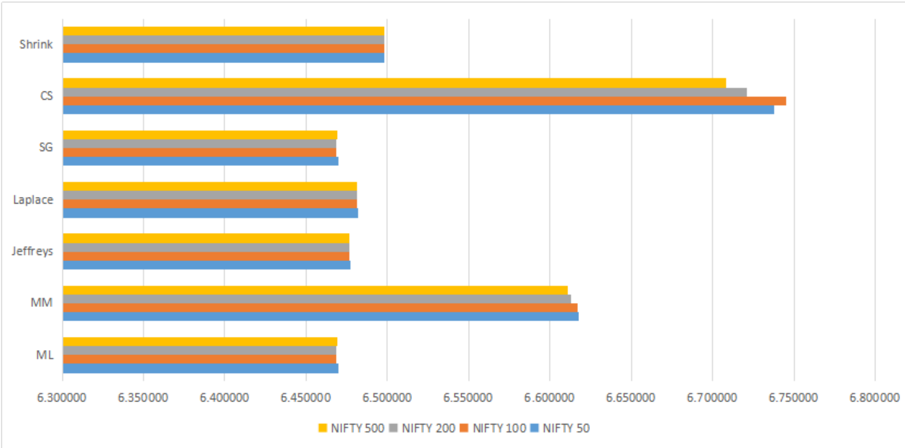


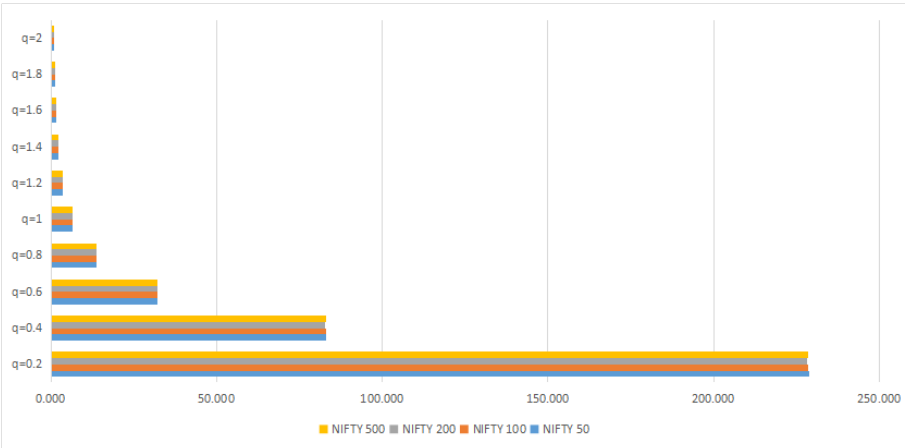
Figure 2.5: Sample entropy with different tolerance for NIFTY stock indices

	NIFTY 50	NIFTY 100	NIFTY 200	NIFTY 500
Shannon				
ML	6.469935	6.468499	6.468767	6.469134
MM	**6.617608**	**6.6172**	**6.613147**	**6.611208**
Jeffreys	6.477448	6.476464	6.476518	6.476734
Laplace	6.482168	6.481448	6.481405	6.481536
SG	6.46997	6.468536	6.468803	6.469169
CS	6.73791	6.745052	6.720968	6.708602
Shrink	6.498282	6.498282	6.498282	6.498282
Tsallis				
q=0.2	228.894	228.3279	228.1869	228.4059
q=0.4	83.05012	82.89198	82.81183	82.85537
q=0.6	32.20349	32.15995	32.1261	32.13327
q=0.8	13.74028	13.72848	13.7159	13.71653
q=1	**6.636976**	**6.633829**	**6.629494**	**6.629312**
q=1.2	3.682577	3.681754	3.680336	3.680194
q=1.4	2.328711	2.3285	2.328054	2.327993
q=1.6	1.637382	1.637329	1.637193	1.637171
q=1.8	1.244449	1.244436	1.244396	1.244389
q=2	0.998895	0.9988919	0.9988804	0.9988783
Renyi				
r=0.2	6.519452	6.516373	6.515606	6.516798
r=0.25	6.526381	6.523292	6.522326	6.523436
r=0.5	6.561816	6.558688	6.556683	6.557372
r=1	**6.636976**	**6.633829**	**6.629494**	**6.629312**
r=2	6.80787	6.805136	6.794805	6.792868
r=4	7.255002	7.254384	7.227422	7.22835
r=16	1.882505	1.839721	1.857602	1.93357
Approximate	1.460449	1.460357	1.41661	1.427835
Sample	2.033375	2.0367	1.984498	1.984444

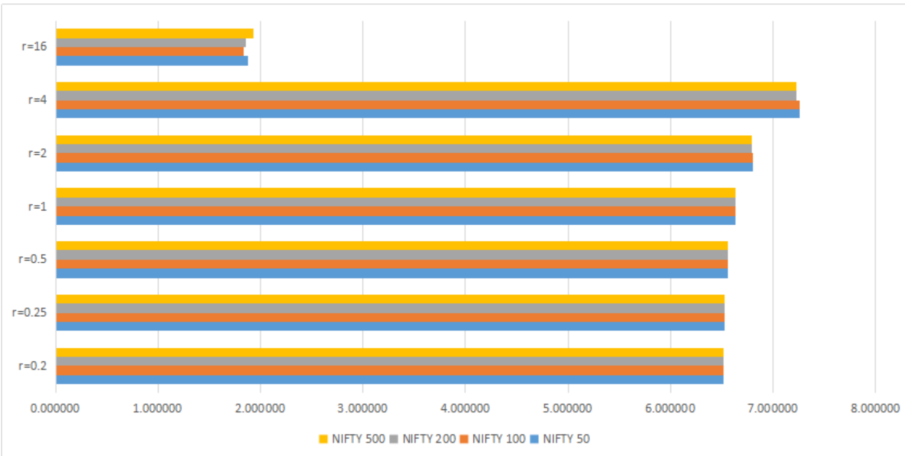
Table 2.8: Entropy results for selected NIFTY stock indices



(a) Shannon entropies



(b) Tsallis entropies

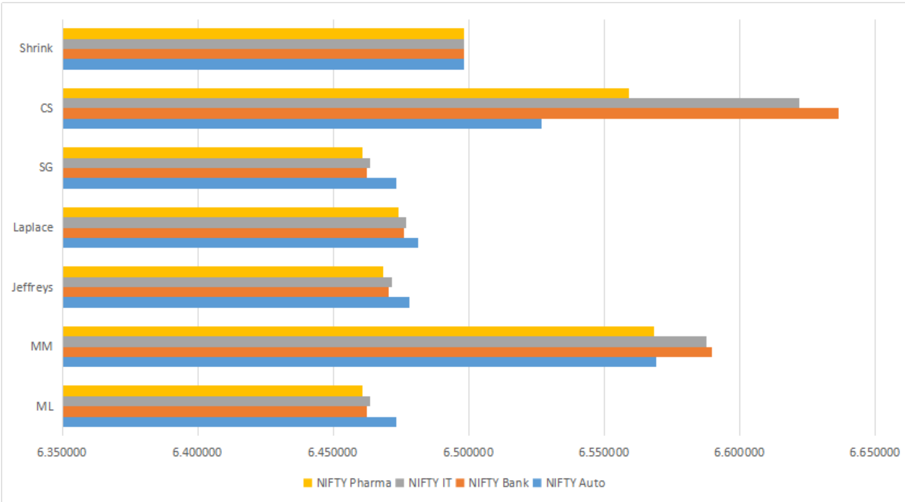


(c) Renyi entropies

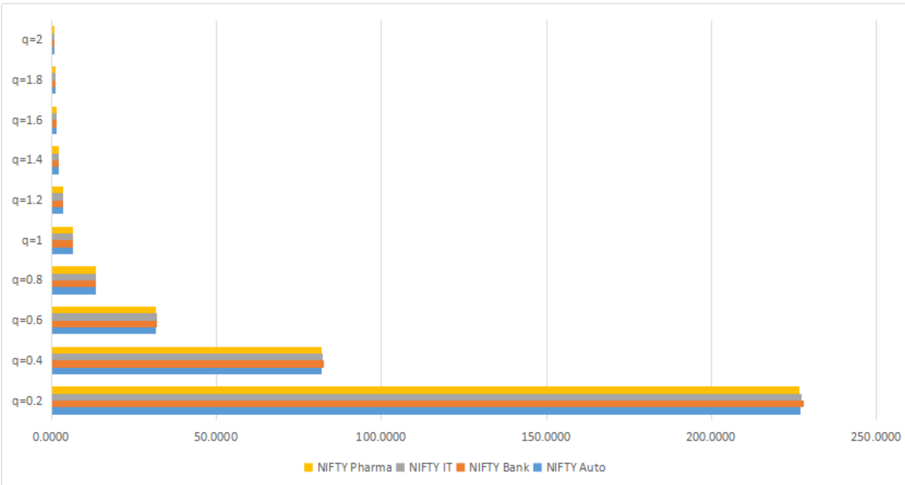
Figure 2.6: Shannon, Tsallis and Renyi entropy results for selected NIFTY stock indices

	NIFTY Auto	**NIFTY Bank**	**NIFTY IT**	**NIFTY Pharma**
Shannon				
ML	6.473271	6.462192	6.463629	6.460898
MM	**6.569173**	**6.589525**	**6.587767**	**6.568279**
Jeffreys	6.478032	6.470574	6.471589	6.468561
Laplace	6.481401	6.476173	6.476912	6.473905
SG	6.473292	6.462229	6.463665	6.46093
CS	6.526887	6.636346	6.621999	6.558987
Shrink	6.498282	6.498282	6.498282	6.498282
Tsallis				
q=0.2	227.0767	227.9727	227.3463	226.5249
q=0.4	81.94662	82.4432	82.25018	81.79209
q=0.6	31.72982	31.9347	31.8761	31.68594
q=0.8	13.56319	13.63774	13.62019	13.55058
q=1	**6.575724**	**6.600955**	**6.595767**	**6.572068**
q=1.2	3.662466	3.670595	3.66908	3.6614
q=1.4	2.322359	2.324883	2.324446	2.322047
q=1.6	1.635437	1.636198	1.636073	1.635346
q=1.8	1.243869	1.244093	1.244058	1.243843
q=2	0.998726	0.9987902	0.9987804	0.9987183
Renyi				
r=0.2	6.509543	6.514438	6.511018	6.506518
r=0.25	6.513536	6.519668	6.516147	6.510503
r=0.5	6.533784	6.546168	6.542127	6.530647
r=1	**6.575724**	**6.600955**	**6.595767**	**6.572068**
r=2	6.665574	6.717341	6.709269	6.659568
r=4	6.865558	6.952645	6.936996	6.850211
r=16	1.876934	1.820595	1.833627	2.052759
Approximate	1.457835	1.456819	1.451147	1.47301
Sample	1.91334	2.048367	2.002481	2.074121

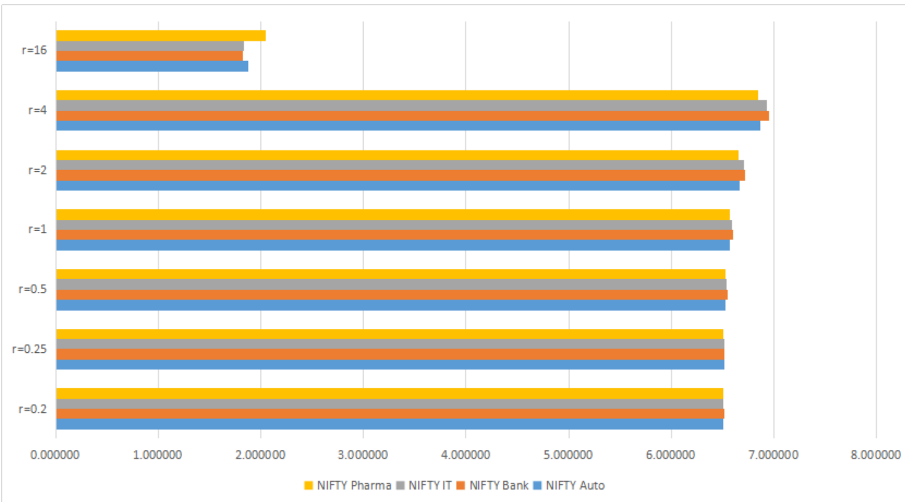
Table 2.9: Entropy results for selected NIFTY Sectoral indices



(a) Shannon entropies



(b) Tsallis entropies

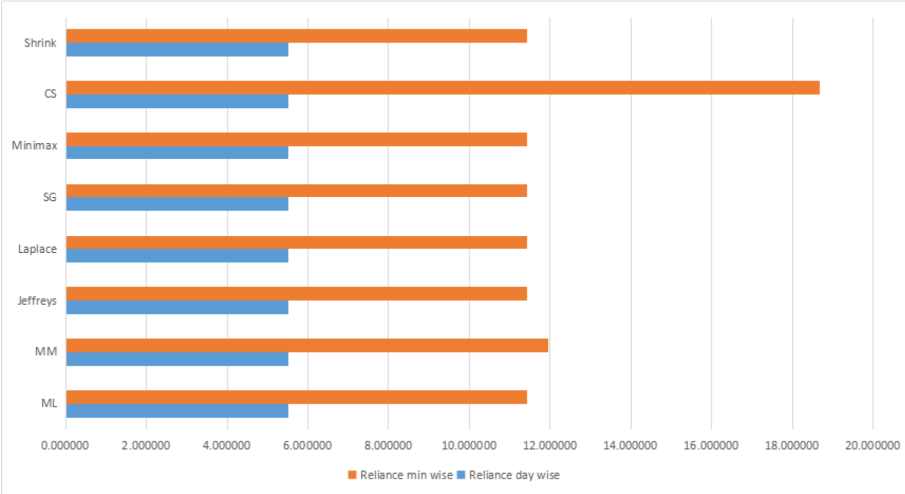


(c) Renyi entropies

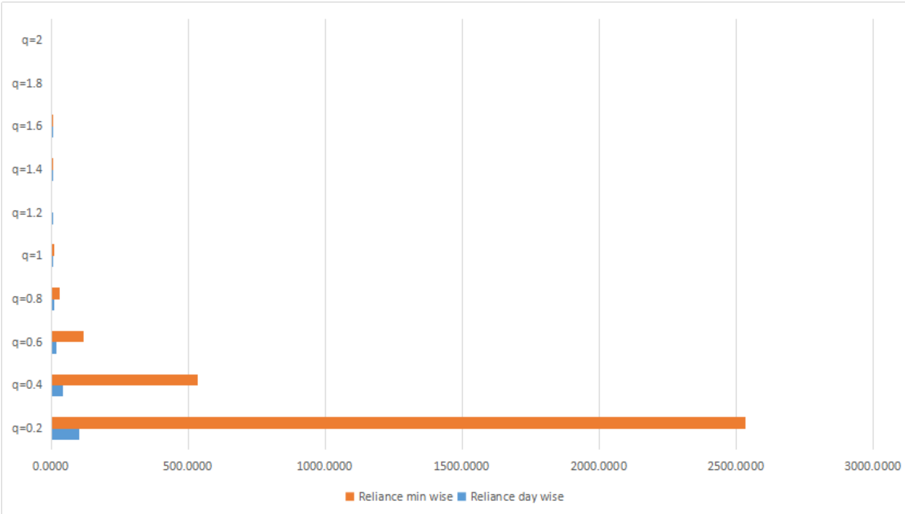
Figure 2.7: Shannon, Tsallis and Renyi entropy results for selected NIFTY sectoral indices

	Reliance day wise	**Reliance minute wise**
Shannon		
ML	5.508918	11.43187
MM	**5.518777**	11.95902
Jeffreys	5.50915	11.43516
Laplace	5.509324	11.43599
SG	5.508921	11.43187
Minimax	5.509131	11.43195
CS	5.508918	18.66754
Shrink	5.513429	11.43694
Tsallis		
q=0.2	101.4863	2535.621
q=0.4	43.87956	533.0535
q=0.6	20.19893	119.9994
q=0.8	10.07189	30.90599
q=1	**5.518873**	**10.00761**
q=1.2	3.342548	4.348634
q=1.4	2.225495	2.461381
q=1.6	1.606096	1.664045
q=1.8	1.234976	1.249856
q=2	0.9960279	1
Renyi		
r=0.2	5.511277	9.519429
r=0.25	5.51175	9.543102
r=0.5	5.514118	9.672091
r=1	**5.518873**	**10.00761**
r=2	5.528459	17.47514
r=4	5.54793	9.823324
r=16	1.87016	1.900184

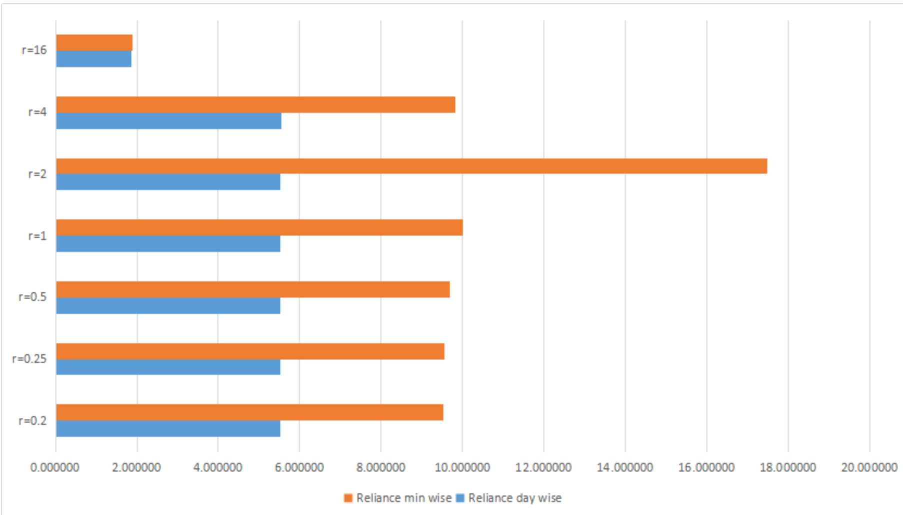
Table 2.10: Entropy results for selected Reliance stock from Jan 2017 to Dec 2017



(a) Shannon entropies



(b) Tsallis entropies



(c) Renyi entropies

Figure 2.8: Minute wise and Day wise analysis of Shannon, Tsallis and Renyi entropy results for selected Indian Reliance stock data

From the results availablle in Tables 2.5 and 2.10, we observed that entropy values of minute-wise returns increased more than twice the day-wise returns, see Figs. 2.4 and 2.8. From Table 2.10, it is also observed that as Tsallis paramter q increases from 0.2 to 2.0 with step size 0.2, difference between day-wise entropy value and minute-wise entropy value is approximately zero. Similar behaviour is exhibited by Renyi entropy when the parameter r varies between 0.2 and 16. In terms of minute-wise anlysis, values of Tsallis entropy with parameter $q \to 1$, and Renyi entropy with parameter $r \to 1$, ML estimator of Shannon entropy is the most appropriate estimator that satisfies the defination, i.e. for $q \to 1$ Tsallis entropy and $r \to 1$ Renyi entropy, both approach Shannon entropy. On the other hand, day-wise analysis indicates that the values of Tsallis entropy with $q \to 1$ and Renyi entropy with $r \to 1$, MM estimator of Shannon entropy is the most appropriate among other seven estimators which satisfies its entropy value upto 3-decimal places, see Table 2.10.

The behavior of the volatility of stock indices and sectors relies on the values of Tsallis and Renyi entropies with parameters q and r respectively, concludes that the financial series is nonlinear. By analysing the NIFTY index sector-wise, NIFTY Pharma index is the most volatile index, see Table 2.9; and from Tables 2.6, 2.7 and 2.8, NIFTY 50 and NIFTY 100 are the most volatile or we can say that these indices are most uncertain as per indices-wise analysis for the time period 2016 to 2018. To be more informative with our empirical results, it is desirable to use Sample and Approximate entropic values, since Tsallis and Renyi entropies behave similarly, see Figs. 2.6 and 2.7.

2.4 Equivalence between Statistical Measures and Information Theoretic Measures

As the entropy of a stock, refer to [21], can be estimated as

$$H(X) = I(X,Y) + H(X|Y) \tag{2.4.1}$$

where Y represents the stock index and X denotes the stock, and I(X,Y) represents mutual information.

Further, for Y as independent and X as dependent variable, the total variance of the stock (TSS=σ_Y^2) can be subdivided into two parts: the regression sum of squares (ESS=$\beta^2\sum(X_i$-$\bar{X})^2$), i.e. the systematic risk, and the residual sum of squares (RSS=$\sum(Y_i$-$\bar{Y})^2$-$\beta^2\sum(X_i$-$\bar{X})^2$), i.e. the specific risk; the coefficient β tests the sensitivity of stock's rate of return to the risk premium. Thus, the equation is $TSS = ESS + RSS$.

In this section we intend to show the equivalence between statistical measures and information theoretic measures for presenting a decent solution to the problem of dependence in the financial series data, refer to Table 2.11. To this purpose we have considered fourteen time series which are made of the daily closing price of the international stock indices and their selective stocks which were traded continuously between 01/01/2001 and 31/12/2018 (about 4433 observations in each time series). In order to apply entropic measures in operation and

Information Theoretical Measures	**Statistical Measures**
$H(Y)$ Entropy	(TSS) Total sum of squares
$H(Y\|X)$ Conditional entropy	(RSS) Residual sum of squares
I(X,Y) Mutual information	cov(X,Y) Covariance or (ESS) Explained sum of squares
$NMI_{X,Y}$ Normalized mutual information	R_{XY} Correlation
RMI_{XY} Relative mutual information	R^2_{XY} Coefficient of determination

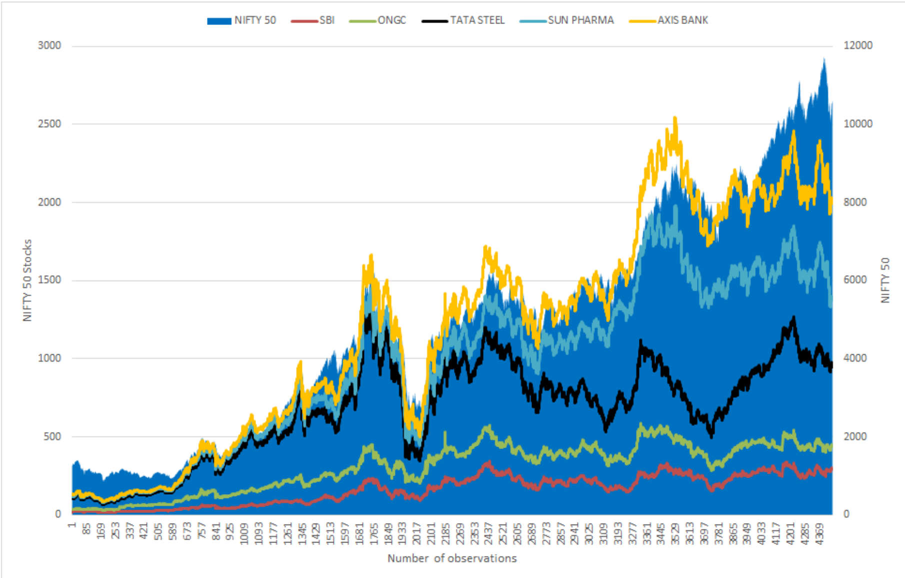
Table 2.11: Equivalences between Statistical measures and Information theoretic measures

to find out their properties, we have obtained daily closing price data of eighteen years of international stock market indices namely, IBOVESPA(Brazil), Hang Seng(Hong Kong), SSE Composite(China), Dow Jones Industrial Average(United States), CAC 40(France), DAX(France), NASDAQ(United States), Nifty 50(India) and BSE Sensex(India) from https://in.finance.yahoo.com/, the free financial data online platform. We have studied a comparison of two Indian market indices namely, NIFTY 50 and S&P BSE Sensex with their common stocks. During the complete selection process, daily closing price data filtered to select stocks listed in their respective indices and as a result due to inadequate data, five stocks were selected from the study. In Fig.2.9 we have presented daily closing price movement of the index NIFTY 50 of National Stock Exchange of India and S&P Bombay Stock Exchange Sensitive Index with their respective stocks. Fig.2.10 presents the allocation of the aforementioned Indian stock indices with their selected stocks. Fig.2.11 gives the details of the distribution of the aforementioned International stock indices. Figs.2.12, 2.13 and 2.14 give the visual comaprison of information and statistical measures.

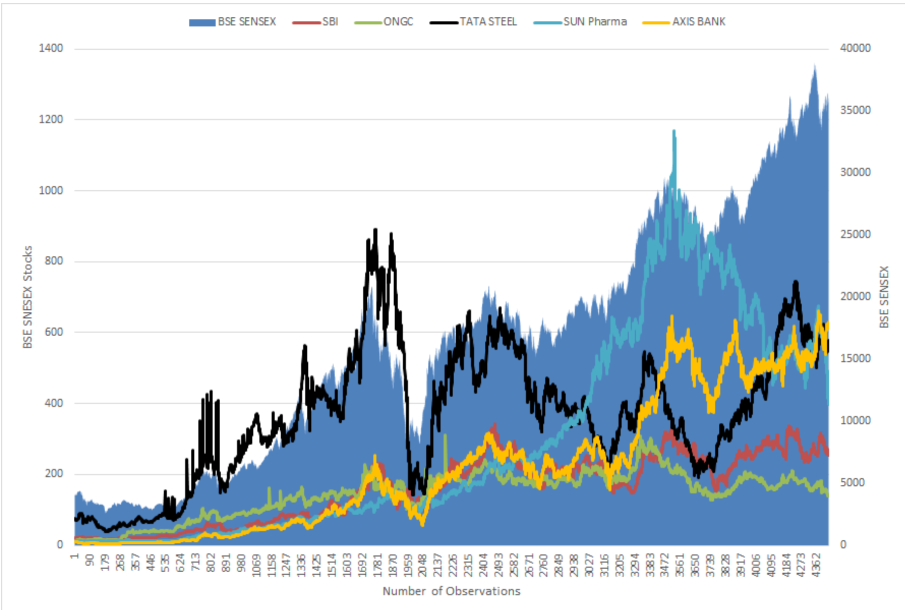
From the data given in Tables 2.12 to 2.15, we have presented four correlation matrix plots of above-mentioned international indices with their stocks, with the distribution of each stock/indices shown on the diagonal; on the lower diagonal, the bi-variate scatter plots with the fitted higher order degree polynomial are shown; on the upper diagonal, the correlation value plus the significance level (p-values: 0, 0.001, 0.01, 0.05, 0.1) as stars("***", "**", "*", ".", " ") is shown, refer to Fig.2.15. Here also, we have discussed the choice of bin counts for the nature of calculation of statistical and information theoretic measures, see Tables 2.16 to 2.21. In general, the count of bins should be selected on the basis of availability of the number of observations of time series data. Also from Figs. 2.12 and 2.13, if we increase bin counts, entropy and TSS measures become inversely symmetric.

Our main emphasis is on determining the dependence level between each stock and their respective stock index, i.e. to select a stock which is more dependent on its relative index and this has been estimated by the information theoretic measures. And, to compare two different indices with their stocks and measure their level of dependence, concept of mutual information is used as an analogue to covariance and normalised mutual information is calculated akin to the Pearson's coefficient of correlation. From Tables 2.16 and 2.17, we observed that for 342 and 341 total number of bins count for the indices NIFTY 50 and BSE sensex respectively, we come to a conclusion that BSE Sensex is more uncertain stock index as compared to NIFTY 50 and also Axis Bank is the underlying stock which is more dependent among other stocks. In other words, by observing the Axis Bank stock, more information of NIFTY 50 and BSE sensex can be obtained.

Also, by observing ONGC stock, remaining uncertainty about the aforementioned indices is more than other stocks. We have also come to conclusion that the measure of Global correlation coefficient can be useful if lower count of bins are considered otherwise this measure gives similar output to all stock variables. Thus in this case, the large count of data bins would not be a reasonable choice for comparative analysis of information theoretic measures with statistical measures, as even in very small change in the prices no depth-patterns are to be found. From Table 2.18, as per maximum count of bins (i.e.403), the index BSE sensex can be used as the appropriate variable for the dependency of the index NIFTY 50. From Table 2.19, as per maximum count of bins (i.e.1132), the international index DAX can be used as the appropriate variable for the dependency of DOW Jones in comparison of other indices, like CAC 40 and NASDAQ.



(a) NIFTY 50



(b) BSE SENSEX

Figure 2.9: Nature of indices with their stocks

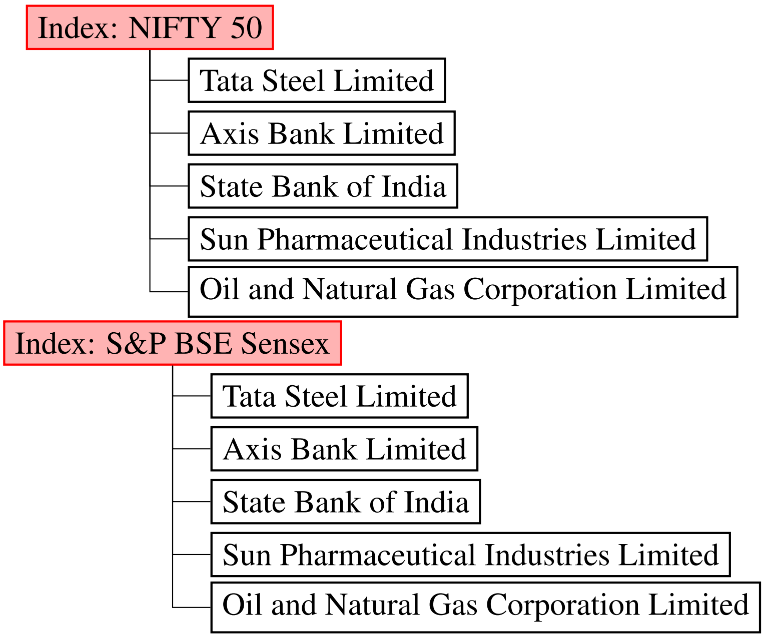


Figure 2.10: Distribution of Indian market indices with their stocks

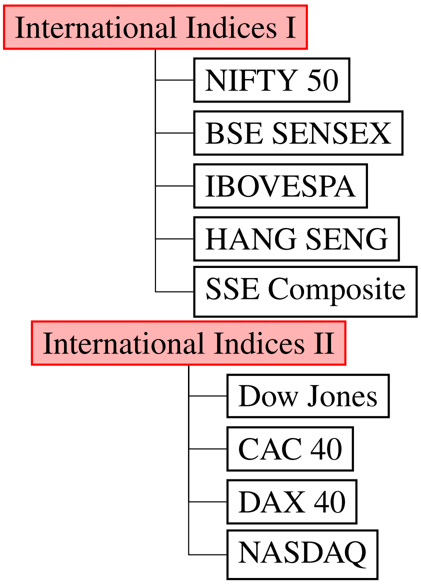


Figure 2.11: Distribution of International market indices

	NIFTY 50	SBI	ONGC	TATA STEEL	SUN PHARMA	AXIS BANK
NIFTY 50	1					
SBI	0.912775	1				
ONGC	0.751958	0.877591	1			
TATA STEEL	0.62706	0.70924	0.729808397	1		
SUN PHARMA	0.849815	0.741449	0.632811246	0.247712201	1	
AXIS BANK	0.968125	0.889382	0.684453591	0.496628753	0.902948749	1

Table 2.12: Correlation between NIFTY 50 index and its stocks

	BSE SENSEX	SBI	ONGC	TATA STEEL	SUN Pharma	AXIS BANK
BSE SENSEX	1					
SBI	0.906307159	1				
ONGC	0.770824431	0.852104665	1			
TATA STEEL	0.616349493	0.690282939	0.71104882	1		
SUN Pharma	0.847512935	0.747856596	0.635405704	0.23767116	1	
AXIS BANK	0.960704283	0.883724153	0.67863293	0.480740583	0.901383062	1

Table 2.13: Correlation between BSE SENSEX index and its stocks

	NIFTY 50	BSE SENSEX	IBOVESPA	HANG SENG	SSE COMPOSITE
NIFTY 50	1				
BSE SENSEX	0.993128661	1			
IBOVESPA	0.846516617	0.854876742	1		
HANG SENG	0.902019599	0.914449061	0.916779164	1	
SSE COMPOSITE	0.629972979	0.621347381	0.665743543	0.72005024	1

Table 2.14: Correlation between International Indices I

	DOW JONES	CAC 40	DAX	NASDAQ
DOW JONES	1			
CAC 40	0.518270674	1		
DAX	0.949384611	0.57404269	1	
NASDAQ	0.9903752	0.464697828	0.948926251	1

Table 2.15: Correlation between International Indices II

	Entropy	Conditional Entropy	RSS	Mutual Information	Covariance	TSS/ESS	Normalized Mutual Informaion	Correlation	Relative Mutual Information	R^2	Global Correlation Coefficient	Beta	Joint Entropy	Variation of Information
Number of Bins: 13														
NIFTY 50	2.3654					624968								
SBI	2.458	1.193993	525211.4	1.171394	14421.8	99756.55	0.4858	0.39952	0.49522298	0.15962	0.950757881	0.27691	3.652042	1.575007
ONGC	2.2503	1.271086	611529.3	1.094301	9615.42	13438.65	0.47432	0.14664	0.462630851	0.0215	0.942298616	0.18463	3.521353	1.5579
TATA STEEL	2.3183	1.558285	271045.3	0.807102	39897.1	353922.7	0.34466	0.75253	0.341213552	0.56631	0.894958652	0.76606	3.876556	1.751986
SUN PHARMA	2.012	1.247659	311598.8	1.117728	70651.4	313369.2	0.51235	0.70811	0.472534938	0.50142	0.945016745	1.35658	3.259693	1.463545
AXIS BANK	2.2408	0.950231	311899.7	1.415156	50735.6	313068.3	0.61468	0.70777	0.59827673	0.50093	0.970054409	0.97417	3.191061	1.332631
Number of Bins: 114														
NIFTY 50	4.404851					155446								
SBI	4.513777	2.363595	109584.07	2.041256	602.407	45861.93	0.45778527	0.5431707	0.452227924	0.29503441	0.99153162	0.43791	6.877372	2.199117096
ONGC	4.270089	2.54428	154842.6142	1.860571	90.2212	603.3858	0.42900482	0.06230283	0.435721832	0.00388164	0.987822702	0.06559	6.814369	2.225712919
TATA STEEL	4.423323	2.637154	117870.67	1.767696	543.832	37575.33	0.40046784	0.4916563	0.399630775	0.24172592	0.985318563	0.39533	7.060477	2.300604486
SUN PHARMA	4.011162	2.17026	141939.42	2.234591	756.637	13506.58	0.53161513	0.2947698	0.557093181	0.08688923	0.994255157	0.55003	6.181422	1.986663283
AXIS BANK	4.220391	2.135119	152160.018	2.269732	324.841	3285.982	0.5264203	0.1453928	0.537801355	0.02113907	0.994646103	0.23614	6.35551	2.02133075
Number of Bins: 234														
NIFTY 50	5.094405					82294								
SBI	5.197257	2.411424	66285.66	2.682981	133.919	16008.34	0.5214153	0.4410512	0.526652475	0.19452616	0.997660783	0.37916	7.608681	2.219391809
ONGC	4.955445	2.661887	81787.8593	2.432518	29.4292	506.1407	0.48413669	0.07842446	0.477488146	0.0061504	0.996136762	0.08332	7.617332	2.277018665
TATA STEEL	5.120726	1.939751	68218.6	2.474352	117.957	14075.4	0.48445002	0.4135673	0.4856999	0.17103791	0.996447392	0.33397	7.060477	2.14152399
SUN PHARMA	4.641876	2.379532	78821.269	2.714873	163.412	3472.731	0.55828507	0.2054242	0.532912676	0.0421991	0.997805487	0.46267	7.021408	2.075219266
AXIS BANK	4.915216	2.338495	80765.575	2.75591	76.2446	1528.425	0.55074047	0.1362818	0.540967984	0.01857273	0.997978582	0.21587	7.253711	2.120801971
Number of Bins: 342														
NIFTY 50	5.453974					59342								
SBI	5.562694	2.301391	48867	3.152583	63.9736	10474.9	0.572358	0.42014	0.578034109	0.17652	0.999086161	0.36761	7.864085	2.170599
ONGC	5.32691	2.57858	58991	2.875393	13.9912	350.645	0.533461	0.07687	0.527210617	0.00591	0.998408594	0.0804	7.90549	2.242788
TATA STEEL	5.485732	2.485931	49847	2.968042	56.5161	9495.09	0.542621	0.40001	0.544198047	0.16001	0.998677948	0.32476	7.971663	2.236878
SUN PHARMA	5.005167	2.36412	56861	3.089854	78.3197	2480.87	0.591388	0.20447	0.566532587	0.04181	0.998963947	0.45005	7.369287	2.068679
AXIS BANK	5.275659	2.291045	58604	3.162929	31.0763	738.054	0.58965	0.11152	0.579931074	0.01244	0.999104884	0.17858	7.566704	2.098517
Number of Bins: 494														
NIFTY 50	5.80256					42468								
SBI	5.906887	2.14323	35651.97	3.65933	31.1724	6816.03	0.62504664	0.400622	0.619502286	0.16049799	0.99966842	0.36187	8.050117	2.095420483
ONGC	5.672488	2.42322	42191.98	3.37934	7.23935	276.02	0.5890271	0.08061933	0.595742115	0.00649948	0.999419451	0.08404	8.095708	2.171720056
TATA STEEL	5.838767	2.292056	35612.2	3.510505	27.998	6855.81	0.603113714	0.4017892	0.60124081	0.16143456	0.999553439	0.32502	8.130823	2.149492498
SUN PHARMA	5.323432	2.318918	41213.97	3.483641	35.5314	1254.03	0.626798167	0.1718393	0.654397577	0.02952875	0.999528784	0.41248	7.64235	2.039291298
AXIS BANK	5.616465	2.514358	41923.79	3.510505	15.8073	544.213	0.614933634	0.1132018	0.625038169	0.01281465	0.999553439	0.1835	8.130823	2.149492498
Number of Bins: 1482														
NIFTY 50	6.777287					17490								
SBI	6.880173	1.438093	15580.71	5.339193	3.61783	1909.3	0.781894219	0.3304012	0.776025981	0.10916495	0.999988481	0.30635	8.318266	1.725999131
ONGC	6.665101	1.679381	17431.45	5.097907	0.70358	58.5507	0.758508763	0.05785902	0.764865679	0.00334767	0.999981337	0.05958	8.344482	1.801825463
TATA STEEL	6.819702	1.524932	15687.37	5.252355	3.19311	1802.63	0.772579952	0.3210395	0.770173682	0.10306636	0.999986296	0.27038	8.344634	1.758487703
SUN PHARMA	6.270829	1.853948	17199.72	4.923339	3.85145	290.284	0.755212766	0.1288299	0.785117725	0.01659714	0.999973538	0.32613	8.124777	1.78925627
AXIS BANK	6.585871	1.63262	17377.21	5.144667	1.50709	112.793	0.77005673	0.08030552	0.78116729	0.00644898	0.999983003	0.12762	8.218491	1.753232443

Table 2.16: NIFTY 50 and its stocks

	Entropy	Conditional Entropy	RSS	Mutual Information	Covariance	TSS/ESS	Normalized Mutual Information	Correlation	Relative Mutual Information	R^2	Global Correlation Coefficient	Beta	Joint Entropy	Variation of Information
Number of Bins: 13														
BSE SENSEX	2.366105					634618								
SBI	2.454473	1.211355	507692.9	1.15475	16513	126925.1	0.479172444	0.4472162	0.488038358	0.20000233	0.94904642	0.3122445	3.665828	1.584638129
ONGC	2.248063	1.261312	617196.6	1.104793	10970.17	17421.43	0.479026642	0.1656859	0.466924756	0.027451817	0.94353268	0.207435	3.509375	1.550671467
TATA STEEL	2.326745	1.565863	254483.2	0.8002426	40779.83	380134.8	0.34105959	0.7739495	0.338210942	0.598997829	0.89342119	0.7711064	3.892608	1.758512269
SUN PHARMA	2.008173	1.277061	332931.9	1.089044	69300.58	301686.1	0.499606245	0.6894797	0.460268669	0.475382257	0.94166986	1.310406	3.285234	1.48195479
AXIS BANK	2.227992	1.029244	329834.5	1.336861	50682.83	304783.5	0.582253937	0.6930101	0.565004934	0.480262999	0.96488603	0.9583624	3.257236	1.385775956
Number of Bins: 31														
BSE SENSEX	3.152247					427756								
SBI	3.280626	1.687102	299283.3	1.465146	5517.767	128472.7	0.455609134	0.5480339	0.464794161	0.300341156	0.97294324	0.38698	4.967728	1.871518635
ONGC	3.042491	1.755437	399161.8	1.396811	4018.9	28594.25	0.451037733	0.2585481	0.443115974	0.06684712	0.96891736	0.2818593	4.797928	1.844211756
TATA STEEL	3.153858	2.010777	306297.5	1.141471	6719.067	121458.5	0.362020942	0.5328634	0.362113438	0.283943403	0.9476372	0.4712313	5.164635	2.00578264
SUN PHARMA	2.799282	1.575705	197588.5	1.576542	18222.33	230167.5	0.530728121	0.7335402	0.500132762	0.538081225	0.97840678	1.277995	4.374987	1.672855343
AXIS BANK	3.046814	1.429468	203804.3	1.72278	12906.5	223951.7	0.555900072	0.7235676	0.546524432	0.523550072	0.98392739	0.9051773	4.476282	1.659367952
Number of Bins: 143														
BSE SENSEX	4.622784					127290								
SBI	4.724562	2.429193	101753.6	2.193592	347.0634	25536.36	0.469378591	0.4479013	0.47451752	0.200615575	0.9937627	0.387171	7.153755	2.227142339
ONGC	4.491146	2.599029	125777	2.023755	101.7817	1513.032	0.444147833	0.1090252	0.437778404	0.011886494	0.99122863	0.1135439	7.090175	2.250870943
TATA STEEL	4.652212	2.648838	98543.81	1.973947	343.3099	28746.19	0.425651275	0.4752181	0.427003944	0.225832243	0.99030536	0.3829837	7.30105	2.308051776
SUN PHARMA	4.197779	2.233289	118472.4	2.389496	472.8239	8817.573	0.542431275	0.2631949	0.516895447	0.069271555	0.9957889	0.5274648	6.431068	2.010366136
AXIS BANK	4.432353	2.248833	124754.5	2.373951	207.8028	2535.488	0.524448418	0.1411346	0.513532754	0.019918975	0.99565563	0.2318171	6.681186	2.075387916
Number of Bins: 341														
BSE SENSEX	5.452604					61410								
SBI	5.547628	2.311644	51572.05	3.14096	63.57059	9837.95	0.5710929	0.4002513	0.576047701	0.160201103	0.99906466	0.3519622	7.859272	2.17216758
ONGC	5.319979	2.561535	61220.0765	2.89107	11.29412	189.924	0.5367866	0.06097936	0.530218222	0.003718482	0.99845775	0.06253053	7.881514	2.23393017
TATA STEEL	5.490891	2.477449	52035.551	2.975155	55.82941	9374.45	0.54373364	0.3907089	0.545639295	0.152653445	0.99869663	0.3091028	7.96834	2.23454358
SUN PHARMA	4.98351	2.347892	58636.813	3.104712	86.90882	2773.19	0.59559587	0.2125054	0.569399868	0.045158545	0.9989943	0.4811757	7.331402	2.05589153
AXIS BANK	5.260563	2.289714	60892.7931	3.162891	26.05882	517.207	0.59056286	0.09177251	0.580069816	0.008422194	0.99910482	0.1442762	7.550277	2.09460879
Number of Bins: 403														
BSE SENSEX	5.616512					51504								
SBI	5.70792	2.252051	43313.94	3.364461	46.60945	8190.06	0.59421446	0.3987706	0.599030323	0.159017991	0.99940191	0.363797	7.959971	2.14371407
ONGC	5.47974	2.491126	51260.0962	3.125386	9.124378	243.904	0.56336562	0.06881591	0.556463869	0.004735629	0.99903505	0.07121777	7.970866	2.2012451
TATA STEEL	5.651517	2.40696	42810.679	3.209552	42.28607	8693.32	0.56967682	0.4108397	0.571449327	0.168789259	0.99918461	0.330052	8.058477	2.20202747
SUN PHARMA	5.146876	2.326516	49158.159	3.289996	60.63433	2345.84	0.61191369	0.2134169	0.585772095	0.045546773	0.99930583	0.4732642	7.473392	2.04533518
AXIS BANK	5.412613	2.284297	51126.8659	3.332216	17.46517	377.134	0.60436083	0.08557116	0.593289216	0.007322423	0.99936206	0.1363195	7.69691	2.08918501

Table 2.17: BSE SENSEX and its stocks

	Entropy	Mutual Information	Correlation	Joint Entropy	Conditional Entropy	Relative Mutual Information
Number of Bins: 31						
NIFTY 50	3.15029					
BSE SNESEX	3.15202	2.11485	0.98778	4.187461	1.03544	0.670950162
IBOVESPA	3.2263	1.51731	0.07871	4.859279	1.632984	0.470293324
HANG SENG	3.201	1.37211	-0.1299	4.979185	1.778181	0.428649261
SSE COMPOSITE	2.9525	1.11249	0.44629	4.9903	2.037799	0.376795808
Number of Bins: 143						
NIFTY 50	4.61974					
BSE SNESEX	4.62246	2.69859	0.95322	6.543614	1.921151	0.583799589
IBOVESPA	4.71728	2.1575	0.10364	7.179524	2.462246	0.457360368
HANG SENG	4.70464	2.05491	-0.1061	7.26947	2.564834	0.436783845
SSE COMPOSITE	4.44981	1.89907	0.38445	7.170491	2.720678	0.426774114
Number of Bins: 341						
NIFTY 50	5.45048					
BSE SNESEX	5.45222	3.37328	0.91252	7.529427	2.077205	0.618697294
IBOVESPA	5.56159	3.06538	0.08978	7.946684	2.385099	0.551170575
HANG SENG	5.54328	2.98039	-0.0806	8.013371	2.47009	0.537658113
SSE COMPOSITE	5.28904	2.78918	0.34756	7.950344	2.661302	0.527350322
Number of Bins: 403						
NIFTY 50	5.60935					
BSE SNESEX	5.61617	3.55583	0.9062	7.669686	2.053519	0.633141963
IBOVESPA	5.72184	3.29735	0.08618	8.033831	2.311995	0.576275517
HANG SENG	5.70409	3.20949	-0.0809	8.103951	2.399858	0.562664739
SSE COMPOSITE	5.44616	3.0275	0.33483	8.028012	2.581848	0.555896407

Table 2.18: International market indices I

	Entropy	Mutual Information	Correlation	Joint Entropy	Conditional Entropy	Relative Mutual Information
Number of bins: 16						
DOW JONES	2.397346					
CAC 40	2.593225	0.745483	0.24985	4.245088	1.651863	0.287473397
DAX	2.61923	1.233584	0.53358	3.782992	1.163762	0.470972003
NASDAQ	2.359363	1.589967	0.76201	3.166742	0.807379	0.673896726
Number of bins: 283						
DOW JONES	5.133843					
CAC 40	5.414879	2.424536	0.25323	8.124186	2.709307	0.447754419
DAX	5.430414	2.631433	0.35362	7.932825	2.502411	0.484573184
NASDAQ	5.143222	2.87793	0.58449	7.399136	2.255914	0.559557802
Number of bins: 556						
DOW JONES	5.790399					
CAC 40	6.069288	3.525484	0.226	8.334203	2.264915	0.580872748
DAX	6.088816	3.61385	0.32142	8.265364	2.176548	0.593522616
NASDAQ	5.799075	3.589182	0.55368	8.000291	2.201216	0.61892319
Number of bins: 1132						
DOW JONES	6.418159					
CAC 40	6.698651	4.722922	0.19908	8.393888	1.695237	0.70505569
DAX	6.71238	4.75384	0.28448	8.376699	1.664319	0.708219737
NASDAQ	6.422752	4.543293	0.51441	8.297619	1.874867	0.707374814

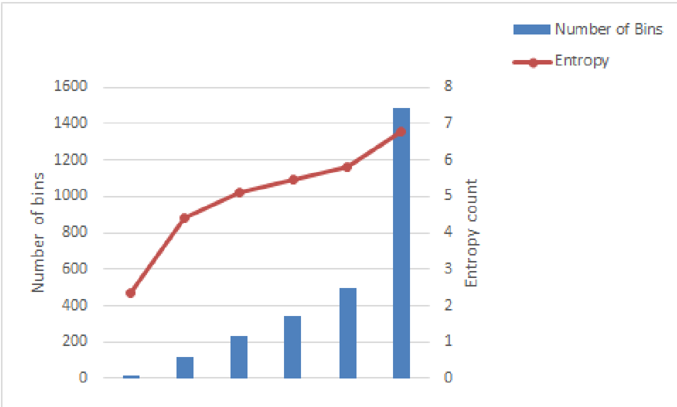
Table 2.19: International market indices II

	BSE SENSEX				
Number of Bins	13	31	143	341	403
Entropy	2.366105	3.152247	4.622784	5.452604	5.616512
TSS	634618	427756	127290	61410	51504

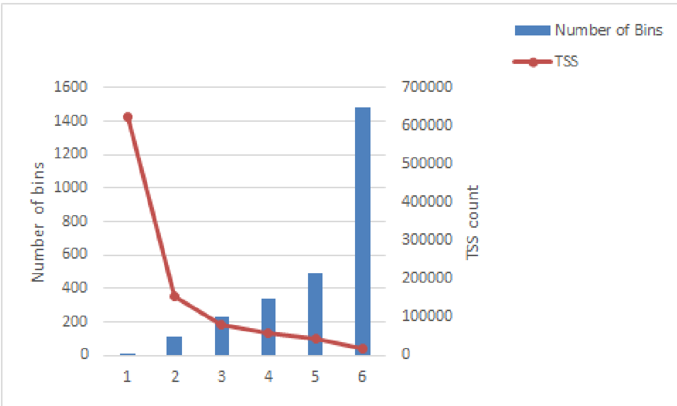
Table 2.20: Comparision of Shannon entropy and Total square of sum of BSE Sensex stocks

	NIFTY 50					
Number of Bins	13	114	234	342	494	1482
Entropy	2.36539	4.404851	5.094405	5.453974	5.80256	6.777287
TSS	624968	155446	82294	59342	42468	17490

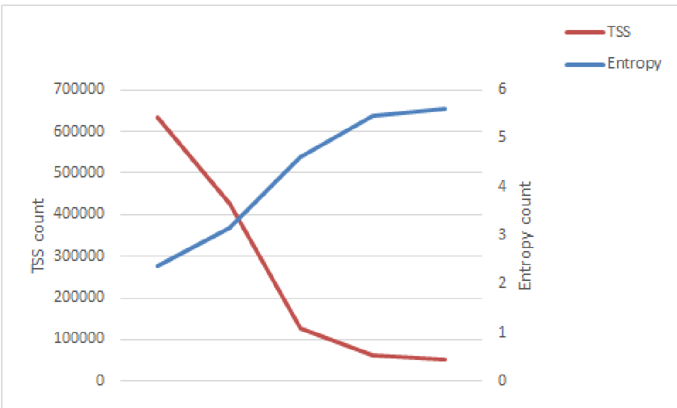
Table 2.21: Comparision of Shannon entropy and Total square of sum of NIFTY 50 stocks



(a) Entropy with bins

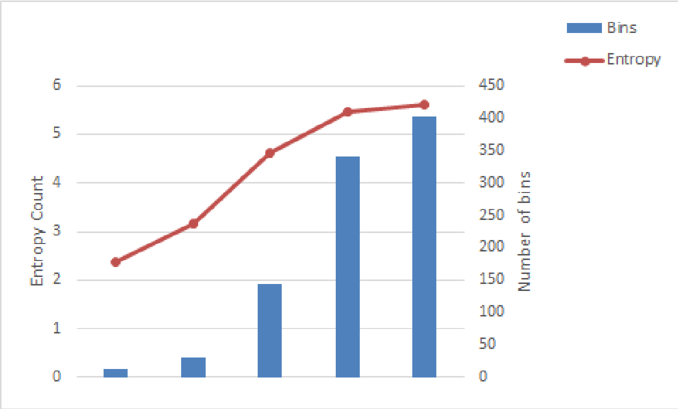


(b) TSS with bins

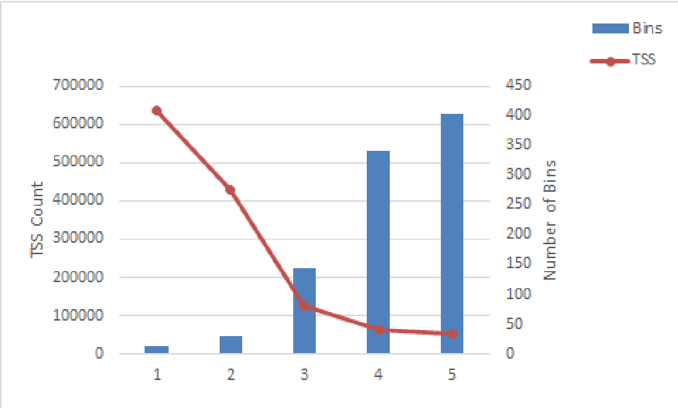


(c) Entropy and TSS

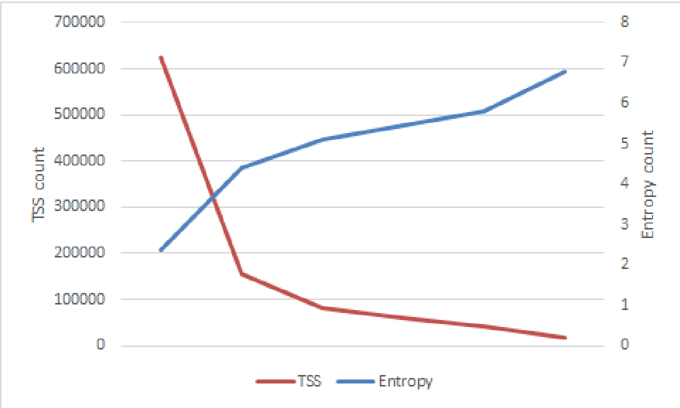
Figure 2.12: Comparision of Shannon entropy and Total square of sum of BSE sensex stocks



(a) Entropy with bins

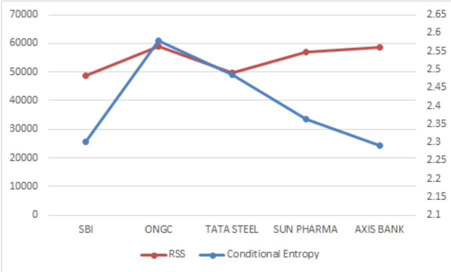


(b) TSS with bins

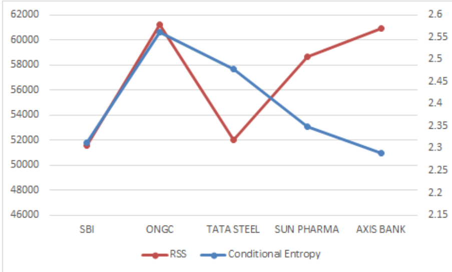


(c) Entropy and TSS

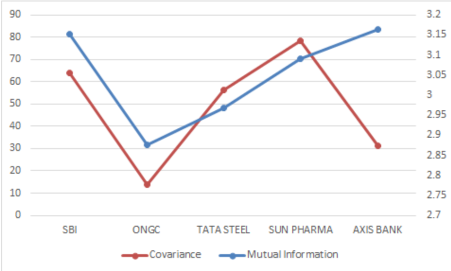
Figure 2.13: Comparision of Shannon entropy and Total square of sum of NIFTY 50 stocks



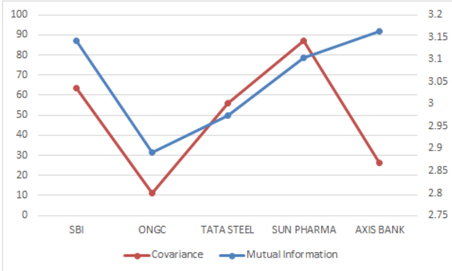
(a) Conditional entropy and RSS with Nifty stocks



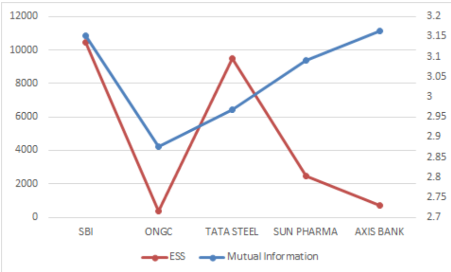
(b) Conditional entropy and RSS with BSE sensex stocks



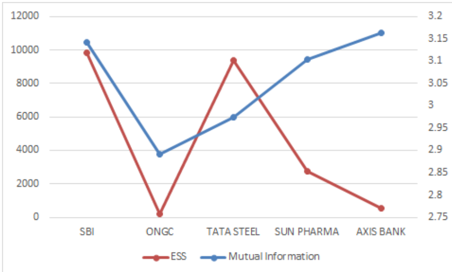
(c) Mutual information and covariance with Nifty stocks



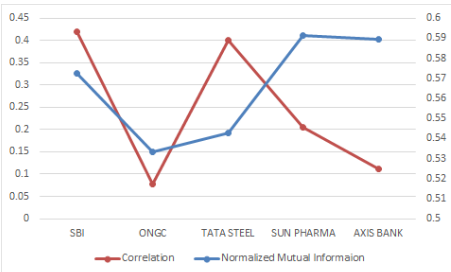
(d) Mutual information and covariance with BSE sensex stocks



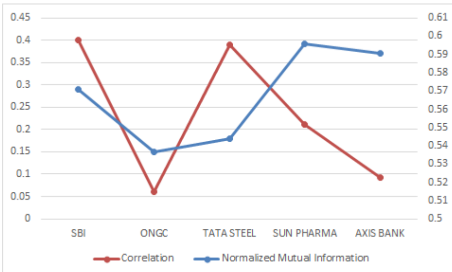
(e) Mutual information and ESS with Nifty stocks



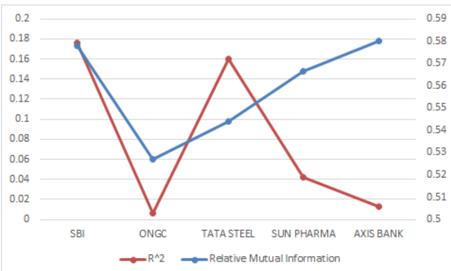
(f) Mutual information and ESS with BSE sensex stocks



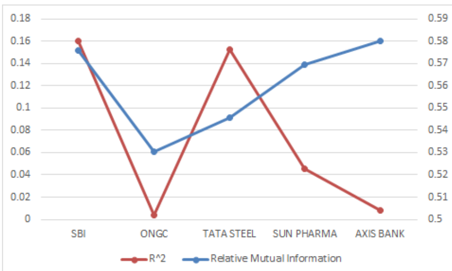
(g) Correlation and Normalized mutual information with Nifty stocks



(h) Correlation and Normalized mutual information with BSE sensex stocks

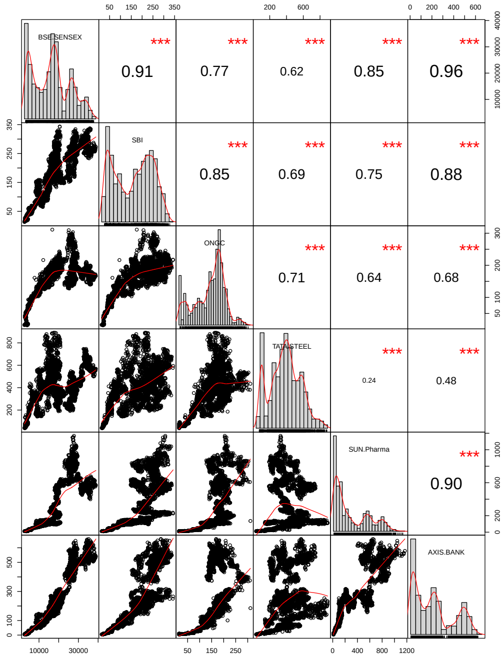


(i) Relative mutual information and R^2 with Nifty stocks

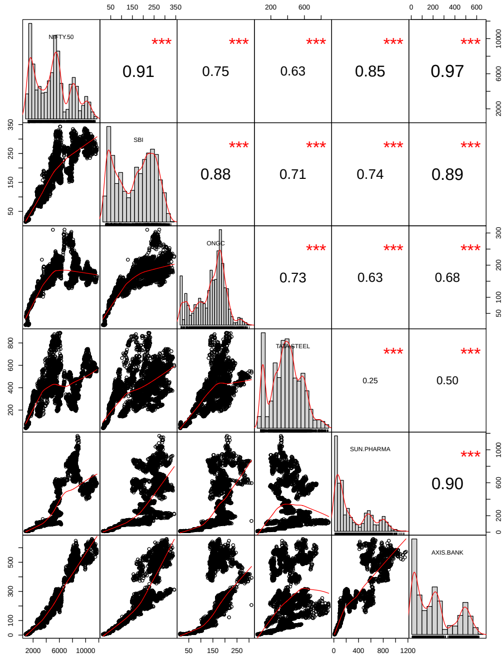


(j) Relative mutual information and R^2 with BSE sensex stocks

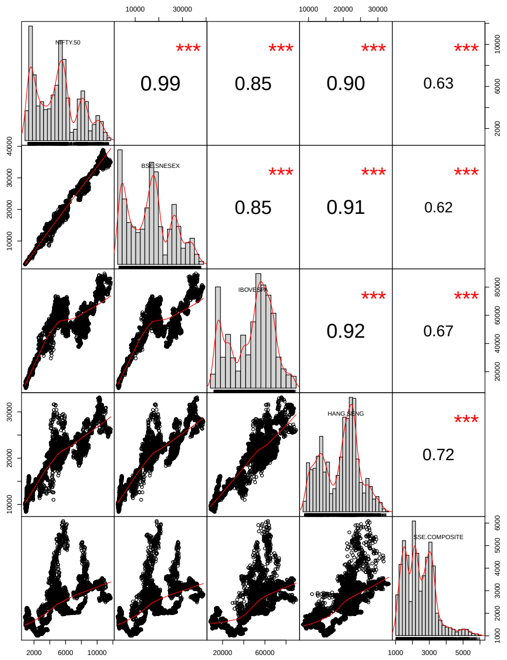
Figure 2.14: Comparision of information and statistical meaures of NIFTY 50 and BSE sensex stocks



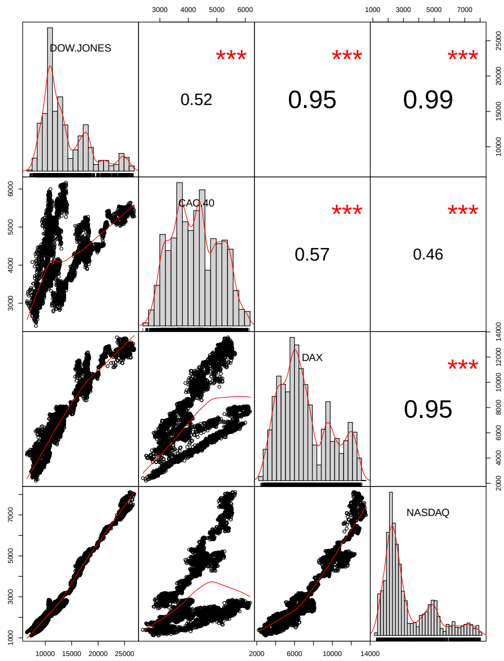
(a) Correlation matrix of index BSE Sensex and its stocks SBI, ONGC, Tata Steel, Sun Pharma, Axis Bank



(b) Correlation matrix of index Nifty 50 and its stocks SBI, ONGC, Tata Steel, Sun Pharma, Axis Bank



(c) Correlation matrix of indices: Nifty 50, BSE Sensex, IBOVESPA, Hang Seng, SSE Composite



(d) Correlation matrix of indices: Dow Jones, CAC 40, DAX, NASDAQ

Figure 2.15: Correlation matrix plot

We find that the information measures like mutual information is an ideal way to expand the financial network of stock market indices based on the statistical measures, while the measure of variation of information presents a different side of the markets but produces network with troubling characteristics. The proposed methodology is sensitive to the choice of data bin counts in which the daily closing prices of market data are separated and requires large sample size. Also, the comparison of entropy measures with statistical measures of dependent stock indices, like Shannon entropy with variance of a stock, conditional entropy with residual sum of squares, coefficient of determination and correlations with the relative mutual information rate and normalized mutual information measure, repectively indicates the further existence of non-linear relationships that are not identified by statistical correlation measure and hence presents a different side of the markets.

The coefficient of correlation is a long-standing estimator of the degree of statistical dependence; however mutual information has advantages over it. Also, from the resulted values of Tables 2.17 and 2.18, the Shannon entropy, mutual information and the conditional entropy perform well in accordance with the systematic risk and the specific risk derived through the linear market model.

2.5 Conclusion

The main idea of this chapter is that entropy can be used as an ure for volatility. The decisive advantage of this approach resides in its ability to capture disorder or uncertainty in the system without putting any constraint on the probability distribution function. We have modelled implied volatility as a linear combination of historical volatility and entropy, and found that the model was heavily dependent on the entropy.

We have also used the information theoretic approach to analyse the stock indices and sectors which are highly volatile. We have used entropy measures: Shannon, Tsallis and Renyi entropy and the Approximate & Sample entropy as an alternate way to characterize the volatility in stock market, where we have done indepth empirical analysis among aforementioned information theoretic measures and can affirm that Sample entropy measures more the regularity of time series rather than its complexity; and in comparison it with Approximate entropy, a similar class, Sample entropy is more consistent measure and provide improved analysis of the stock market regularity. We have also observed that generalized information theoretic measures i.e., Tsallis and Renyi are not perfect for analysing the time series of stock market data, like minute wise time series data; also we have analyzed that which specific estimator of Shannon entropy is appropriate for the given data type set.

We have also used the information theoretic concepts such as entropy, conditional entropy and mutual information and their extensions to analyse the statistical dependence of the financial market. This approach studies the level of similarities between regression analysis and the information theoretic measures, i.e. Shannon entropy measures total sum of squares of the dependent variable, relative mutual information measures coefficient of correlation, conditional entropy measures residual sum of squares and many more. The ability of this approach is to check the non-linear dependence of market data without any specific probability distribution. We have used different international stock indices to address our problem. The mutual information and conditional entropy have shown a decent efficiency compared to the measures of statistical dependence. The results are able to infer that the information entropic approach observes the dependencies and is a more general measure to study the financial market, in addition these measures can provide potential sources of information to financial market investors.

Chapter 3

Comparative Performance Of Forecasting Models Using Relative Information

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3.1 Introduction

Various forecasting models have been evolved in literature over the years, of which the Holt-Winters [94] and Autoregressive Integrated Moving Average (ARIMA) [100] are two traditional techniques that aren't just widely accepted but also exceptionally good predictors of the financial time series. Recently, Artificial Neural Networks have been studied extensively and used in the financial market prediction, refer to [71, 87]. Flexible non-linear modeling capacity of neural networks is the key advantage of deep learning. With these networks, there's no need to define a specific model form, but instead it is based on the traits depicted from the given time series. Deep learning methods in time series forecasting are capable of assessing data structure and patterns such as complexity and non-linearity. Long Short Term Memory (LSTM) network, in particular has been

The work presented in this chapter is communicated entitled **Comparison of forecasting models using information theoretic approach in financial market prediction** and part of the work has been presented in 3rd International Conference of Mathematical Sciences (ICMS 2019) held at Maltepe University, Istanbul, Turkey, Sept., 2019.

used in the prediction of time series in the financial sector, refer to [27, 34, 71]. Next, Holt-Winters doesn't require stationarity the way an ARIMA model does, it is specifically designed for seasonal data. ARIMA requires the calculation of a set of data based parameters, whereas LSTM does not require parameters of this nature.

In this chapter we intend to examine different forecasting models in terms of their performance on a time series which is considered complex to predict. We have compared the performance of three models such as ARIMA, Holt-Winters and LSTM by applying Kullback measure of relative information [40], an information theoretic measure. The study shows that traditional model ARIMA performs well on NIFTY 50 stock index monthly data and the performance of Holt-Winters is better for prediction of Dow Jones Industrial Average and S&P BSE SENSEX index data, in comparison with LSTM, an application of deep learning. Infact, in terms of specific indices under consideration their performance come in the order of ARIMA, Holt-Winters and LSTM, Holt-Winters being the best. This is also exhibited by their empirical outputs.

The chapter is organised as follows. Section 3.2 outlines the time series forecasting models. Section 3.3 discusses the information theroetic measure employed. Section 3.4 analyses NIFTY 50, Dow Jones Industrial Average and S&P BSE SENSEX indices monthly closing price data using ARIMA, Holt-Winters and LSTM models. Their comparision in forecasting abilities have been measured using Kullback measure of relative information. Finally, the chapter concludes with Section 3.5.

3.2 Time Series Forecasting Models

3.2.1 ARIMA

There are two pattern-based forms of the model exhibited by the time series. Basically, when the data series shows strong seasonal trends, one can use seasonal-ARIMA model; and a non-seasonal ARIMA model can be used to depict the series having weak seasonal trends, its general form is denoted as ARIMA(p, d, q), where p denotes the number of observations from past time values used to obtain future forecast values; d denotes the times at which the given time series is differenced to obtain stationarity; and q represents the moving average of the model's past forecasting errors.

In an extension to the non-seasonal model, seasonal-ARIMA model incorporates both seasonal and non-seasonal factors in a multiplicative way, and its general form can be expressed as ARIMA(p, d, q)(P, D, Q)[s], where (p, d, q) shows the non-seasonal component and (P, D, Q)[s] represents the seasonal component of the model and s is the number of seasonal periods, P is the order of *seasonal autoregressive*, D is the *seasonal differencing* and Q denotes the *seasonal moving average* term.

We have used monthly closing prices of the NIFTY 50 index, Dow Jones Industrial Average and S&P BSE SENSEX from 1 January 2001 to 31 December 2017 to predict price movements during the year 2018, see Fig.3.1. For this we applied *auto.arima*() function in R, that searches for feasible model for NIFTY 50 and S&P BSE SENSEX within the specified order constraints and returns best fitted class of ARIMA: ARIMA(0,1,0)(1,0,0)[12] with drift and ARIMA(2,1,2)(2,0,0)[12] with drift for Dow Jones Industrial Average index. Table 3.4 gives the statistics indicating the suitability of fitting of ARIMA model in different stock market indices and the resulting forecast is shown in Figs.3.8(a), 3.9(a), 3.10(a).

3.2.2 Holt-Winters

This method is used on the data series showing trends such as increasing or decreasing with existence of seasonality. There are generally two variants of Winter's smoothing method, depending on whether the seasonal pattern, is modeled in an additive or multiplicative process. Winters generalized Holt's linear method to develop such a technique, termed as called Holt Winters [35]. In addition to the Holt's linear method equations, a seasonal equation is added:

$$L_t = \alpha(y_t - S_{t-s}) + (1-\alpha)(L_{t-1} + T_{t-1}) \quad for \;\; 0 \leq \alpha \leq 1,$$
$$T_t = \beta(L_t - L_{t-1}) + (1-\beta)T_{t-1} \quad for \;\; 0 \leq \beta \leq 1,$$
$$S_t = \gamma(y_t - L_t) + (1-\gamma)S_{t-s} \quad for \;\; 0 \leq \gamma \leq 1$$

then we have a forecast equation:

$$F_{t+k} = L_t + kT_t + S_{t+k-s}$$

where, y_t is the time series, L_t is series level, T_t is trend, F_{t+k} is future forecast for k-periods, s is the seasonal cycle length and α, β and γ are probability values. An alternative Holt Winter's multiplicative model multiplies the trended forecast by the seasonality.

To smoothen the time series, we have used exponential smoothing in R via the *HoltWinters*() function to find the smoothing parameters. For our data we have used monthly prices of the NIFTY 50, Dow Jones Industrial Average and S&P BSE SENSEX indices from 1 January 2001 to 31 December 2017 to predict price movement during the year 2018. The output obtained, "Holt–Winter's exponential smoothing with trend and additive seasonal component" and Table 3.6 gives the smoothing parameters; the resulting forecast is shown in Figs.3.8(b),3.9(b),3.10(b).

3.2.3 Long Short-Term Memory

Long Short-Term Memory (LSTM) [34], a special form of recurrent networks which offers less risks when compared to the other techniques. In regular recurrent neural network, small weights are repeatedly multiplied through several time steps and the gradients tends to zero, a condition is known as *vanishing gradient problem*. Here, the main purpose of LSTM is to perform better and tackle the vanishing gradient problem that recurrent networks would suffer when dealing with large data sets. There are three gates in LSTM layer: forget gate, input gate and output gate. For x_t as input vector at time step *t*, W weight matrix and *b* as biasness, these are given as:

Forget Gate: $F_t = \sigma(W_F[h_{t-1}, x_t] + b_F)$
decides which data information is to be removed from the cell state;
Input Gate:
tanh layer: $T_t = \tanh(W_T[h_{t-1}, x_t] + b_T)$

Sigmoid layer: $S_t = \sigma(W_S[h_{t-1}, x_t] + b_S)$

Cell state: $C_t = F_t C_{t-1} + S_t T_t$, old state C_{t-1} is upadted as C_t;
Output Gate: $O_t = \sigma(W_o[h_{t-1}, x_t] + b_o)$
here the sigmoid layer filters outgoing cell state and sets output values to the range $[-1, 1]$, C_t passed through $tanh$ function. At last, Hidden state h_t to be

passed on to the next cell: $h_t = O_t \tanh(C_t)$.

The monthly closing price data set has been divided into two subsets, training and test. For training, 95% of the data set, i.e. 204 obsevations were used and the remaining 5%, i.e. 11 observations were used to test the model accuracy by predicting the price movement of next 11 months of the year 2018 and compare with real market data. We have used package Keras in R, a comprehensive library that runs on top of Tensorflow for classification and prediction of our data.

Actual Price	Predicted Price	Actual Price	Predicted Price	Actual Price	Predicted Price
11027.7	10624.06	11027.7	10636.64	11027.7	10310.877
10492.85	10721.48	10492.85	10652.06	10492.85	10419.642
10113.7	10821.31	10113.7	11161.59	10113.7	9872.923
10739.35	10929.37	10739.35	11425.52	10739.35	10002.227
10736.15	11030.77	10736.15	11917.63	10736.15	10161.443
10714.3	11150.79	10714.3	11946.06	10714.3	9605.24
11356.5	11244.81	11356.5	12152.35	11356.5	9705.577
11680.5	11396.4	11680.5	11780.87	11680.5	9388.377
10930.45	11494.21	10930.45	11849.78	10930.45	9258.09
10386.6	11592.14	10386.6	12226.14	10386.6	8963.906
10876.75	11718.32	10876.75	11827.45	10876.75	8645.627
10862.55	11828.83	10862.55	11816.78		
ARIMA		Holt-winters		LSTM	

Table 3.1: Predicted analysis of the monthly data of NIFTY 50 index for the year of 2018 by forecasting models

Actual Price	Predicted Price	Actual Price	Predicted Price	Actual Price	Predicted Price
26149.4	23733.6	26149.4	24688.4	26149.4	24568.69
25029.2	24438.1	25029.2	24891.9	25029.2	23577.33
24103.1	24897.8	24103.1	25102.8	24103.1	22602.46
24163.2	25549.3	24163.2	25167	24163.2	22162.85
24415.8	26063.5	24415.8	25255.8	24415.8	22111.56
24271.4	26467.5	24271.4	25431.6	24271.4	21548.65
25415.2	27084.8	25415.2	25596.6	25415.2	21289.01
25964.8	26877.4	25964.8	25640.2	25964.8	21148.49
26458.3	27012.8	26458.3	25769.4	26458.3	20901.22
25115.8	27532.6	25115.8	25946.4	25115.8	21012.33
25538.5	27615	25538.5	26155.3	25538.5	20102.75
23327.5	27136.9	23327.5	26298.1		
ARIMA		Holt-winters		LSTM	

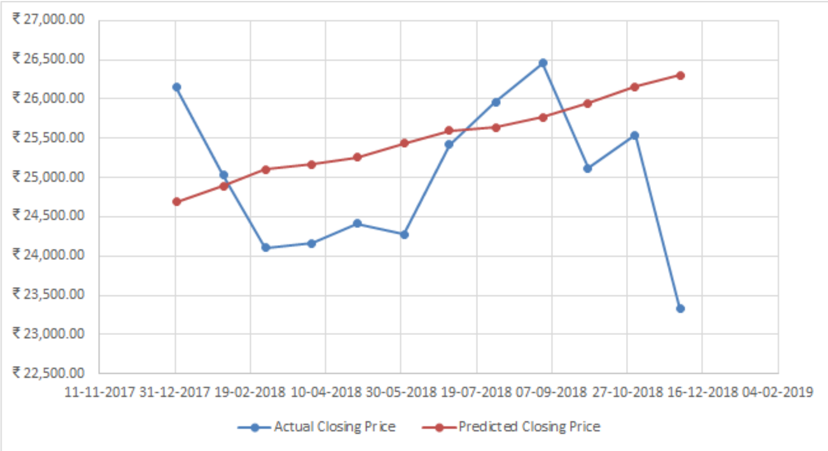
Table 3.2: Predicted nnalysis of the monthly data of Dow Jones Industrial Average index for the year of 2018 by forecasting models

(a) NIFTY 50

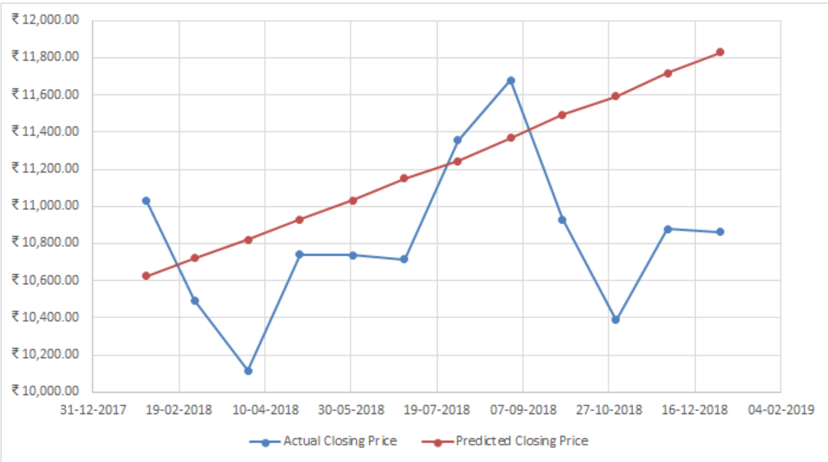
(b) DOW JONES

(c) SENSEX

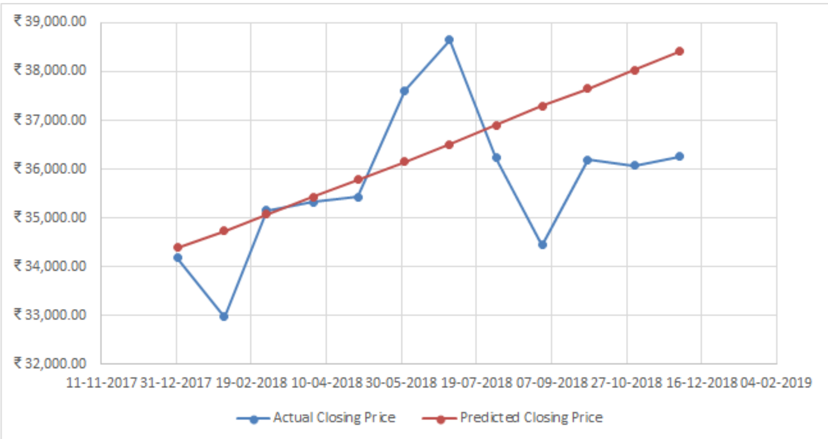
Figure 3.1: Monthly closing price of the selected indices from Jan 2001 to Dec 2018.



(a) DOW JONES

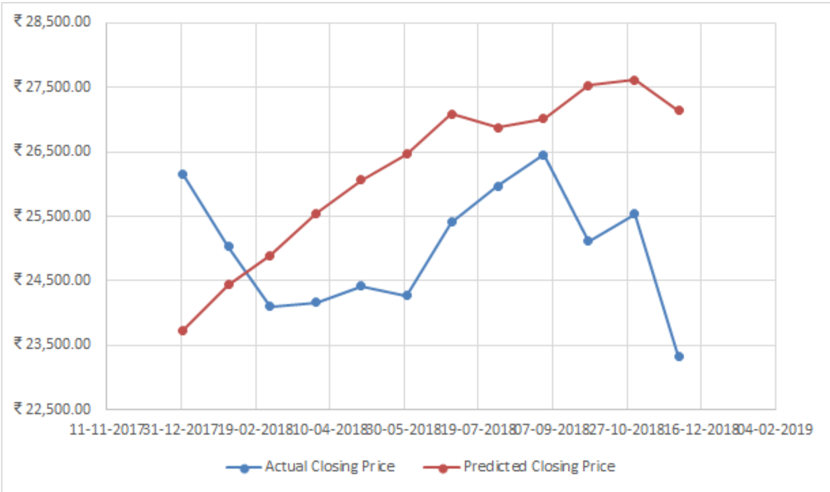


(b) NIFTY 50

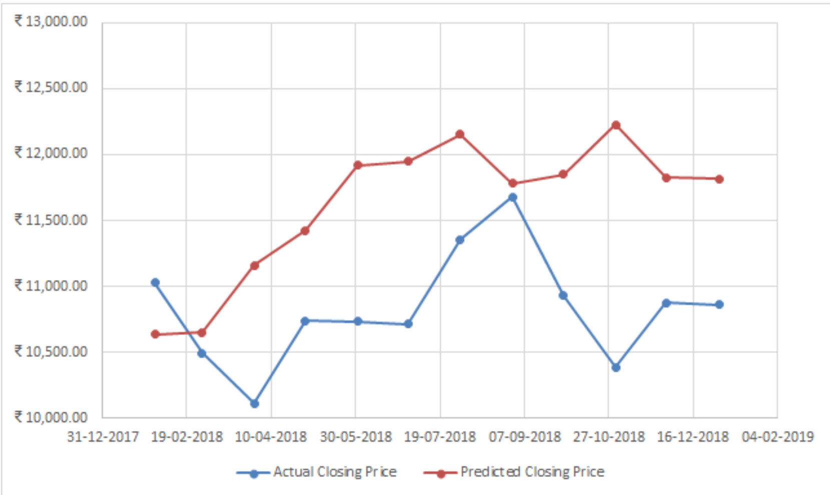


(c) SENSEX

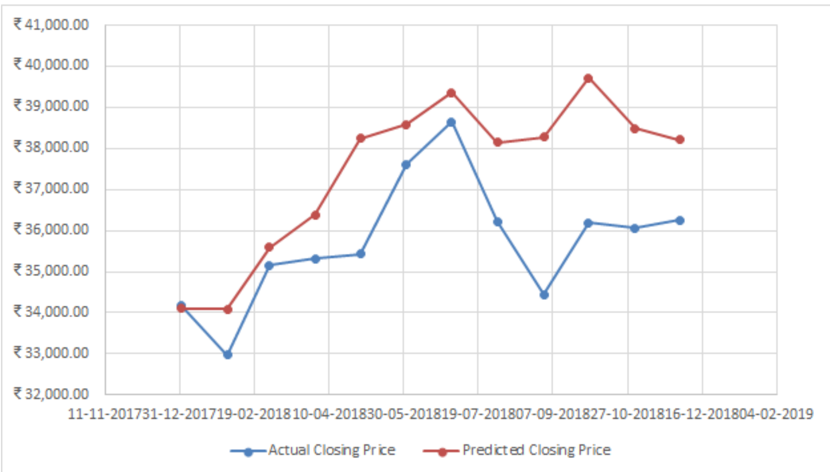
Figure 3.2: Prediction analysis with ARIMA forecasting model



(a) DOW JONES

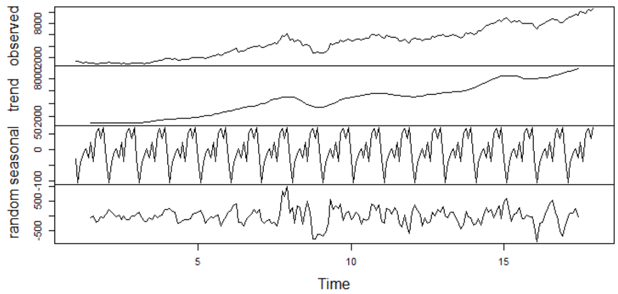


(b) NIFTY 50

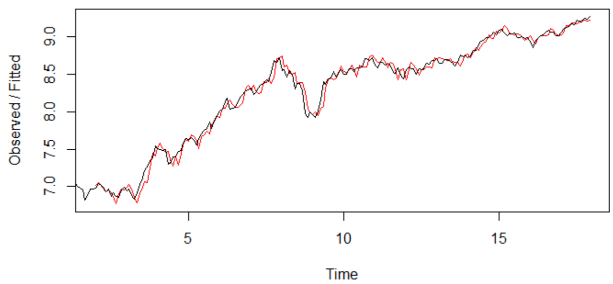


(c) SENSEX

Figure 3.3: Prediction analysis with Holt Winters forecasting model

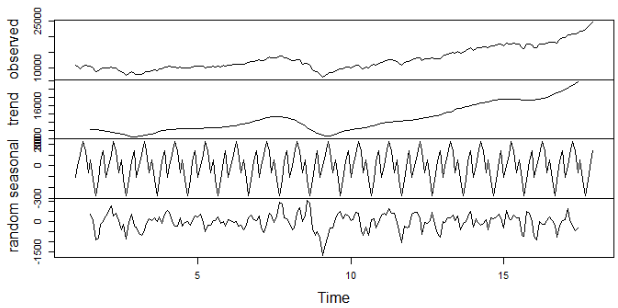


(a) Decomposition of additive time series

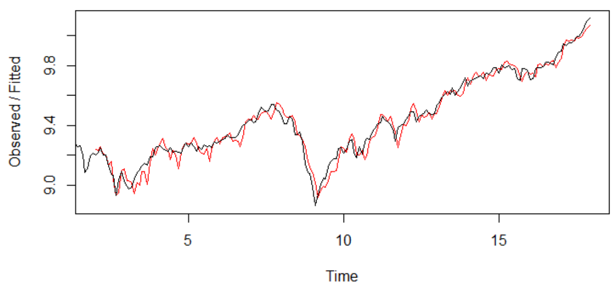


(b) Holt-Winters Filtering

Figure 3.4: Decomposition and Filteration on NIFTY 50 index

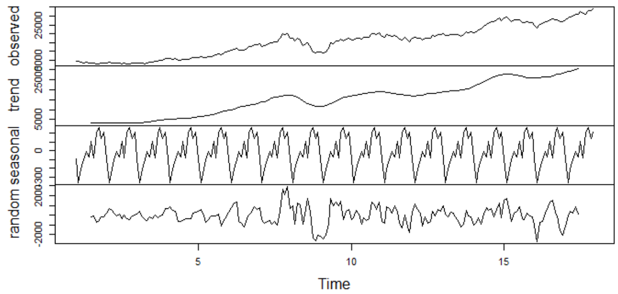


(a) Decomposition of Additive time series

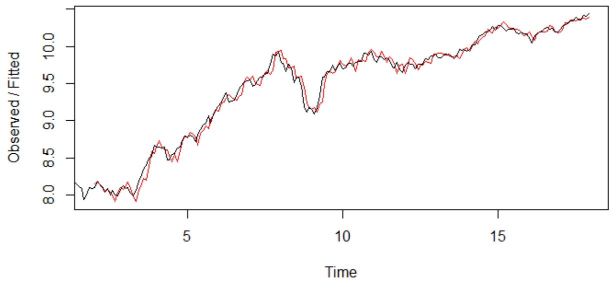


(b) Holt-Winters ailtering

Figure 3.5: Decomposition and Filteration on Dow Jones Industrial Average index



(a) Decomposition of additive time series

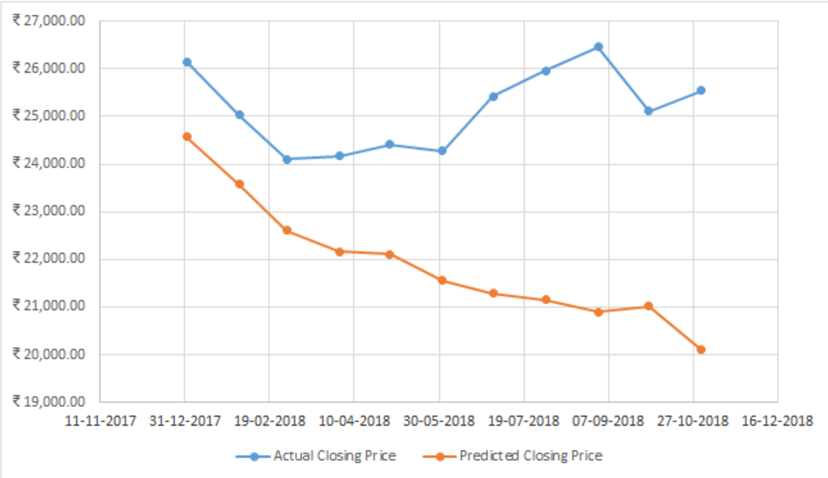


(b) Holt-Winters Filtering

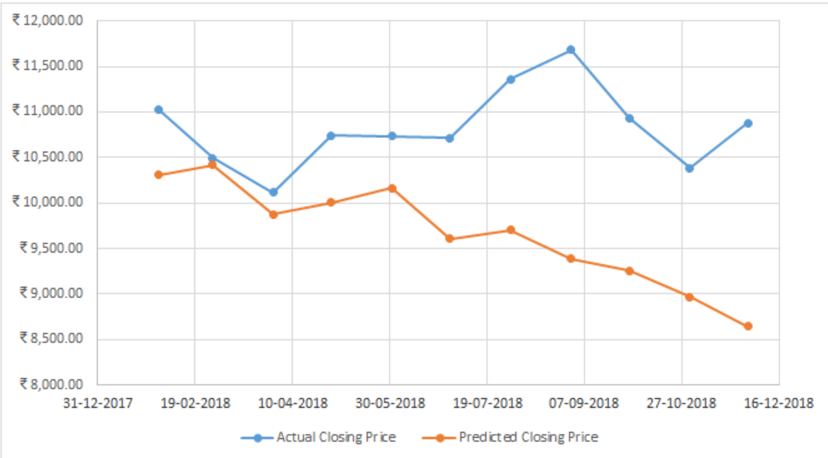
Figure 3.6: Decomposition and Filteration on S&P BSE SENSEX index

Actual Price	Predicted Price	Actual Price	Predicted Price	Actual Price	Predicted Price
34184.04	34392.01	34184.04	34107.59	34184.03	33396.89
32968.68	34730.18	32968.68	34093.09	32968.67	33460.67
35160.36	35075.85	35160.36	35587.22	35160.35	31531.26
35322.38	35434.89	35322.38	36390.45	35322.37	31978.03
35423.48	35782.49	35423.48	38247.55	35423.48	32762.48
37606.58	36157.48	37606.58	38578.14	37606.57	31169.15
38645.07	36507.05	38645.07	39351.79	38645.07	31393.34
36227.14	36898.5	36227.14	38140.62	36227.14	30165.94
34442.05	37288.81	34442.05	38275.77	34442.05	29868.04
36194.3	37644.26	36194.3	39706.52	36194.30	28990.86
36068.33	38035.94	36068.33	38490.08	36068.32	27903.5
36256.69	38416.15	36256.69	38208.35		
ARIMA		Holt-winters		LSTM	

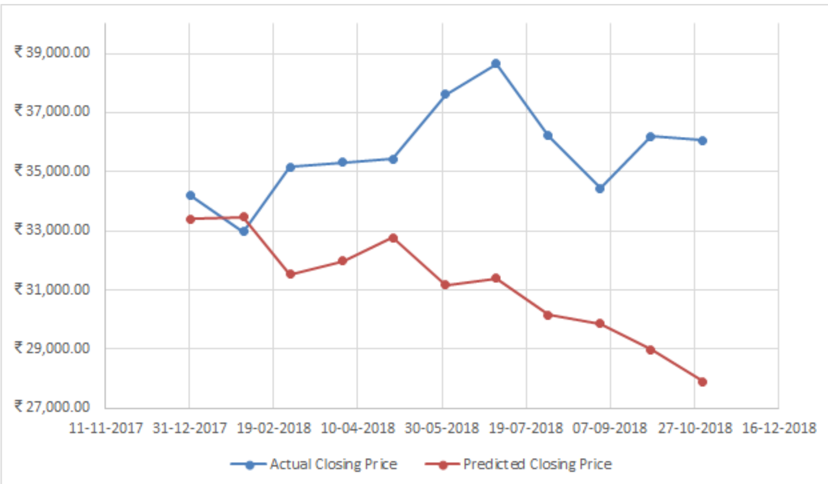
Table 3.3: Predicted analysis of the monthly data of S&P BSE SENSEX index for the year of 2018 by forecasting models



(a) DOW JONES



(b) NIFTY 50



(c) SENSEX

Figure 3.7: Prediction analysis with LSTM forecasting model

3.3 Kullback Measure of Relative Information

Kullback measure of relative information [40] in case of two discrete probability distributions $P = \{p_1, p_2, p_3, ..., p_n\}$ & $Q = \{q_1, q_2, q_3, ..., q_n\}$ defined on the same probability space is given by

$$H(P||Q) = \sum_{i=1}^{n} p_i log\left(\frac{p_i}{q_i}\right). \tag{3.3.1}$$

Here P can be considered as the actual probability distribution and Q as the predicted one.
The measure (3.3.1) can be seen as a measure of deviation of the distribution $\{q_i\}$ from that of actual distribution $\{p_i\}$. Infact, $H(P||Q) > 0$ and $H(P||Q) = 0$ iff $p_i = q_i \ \forall \ i$ and also $H(P||Q) \neq H(Q||P)$. We have used this relative information measure to obtain the comparative perfomance of the forecasting models ARIMA, Holt-Winters and LSTM employed to forecast the aforementioned indices monthly price data.

3.4 Empirical Results

The analysis, interpretation and modeling of the financial data is done by using Microsoft Office Excel 2013 (15.0.4420.1017, 2012 Microsoft Corporation©) and statistical software R (RStudio© 1.2.5019, 2009-2019 RStudio, Inc.), an important tool for time series forecasting. Fig.3.1 shows the monthly price data from 1 January 2001 to 31 December 2018 of three popular stock indices, namely NIFTY 50, Dow Jones Industrial Average (DJI) and S&P BSE SENSEX (BSESN).

First, NIFTY 50, a broad-based index of India's National Stock Exchange (NSE), consists of 50 large and liquid NSE-listed companies reflects the overall condition of the economy of India. We acquired the historical monthly data of NIFTY 50 index from https://www.quandl.com/data/NSE. Next, Dow Jones Industrial Average (DJI), an index comprises of 30 major companies trading on the NYSE and the NASDAQ; and S&P BSE SENSEX (BSESN), the benchmark index of the Bombay Stock Exchange (BSE) in India, consisting 30 of the most traded stocks on the BSE and we acquired the historical data of DJI and BSESN indices from https://finance.yahoo.com.

Index	σ^2=$\frac{\textbf{residual sum of squares}}{\textbf{n}}$	**log likelihood**($log(l)$)	**AIC**=$-2log(l)+2k$	**AICc**=$-2log(l)+2k+\frac{2k(k+1)}{n-k-1}$	**BIC**=$klog(n)-2log(l)$
NIFTY 50	0.00459	259.41	-512.83	-512.71	-502.89
DJI	0.00157	370.46	-724.91	-724.17	-698.41
BSESN	0.004365	264.53	-523.07	-522.95	-513.13

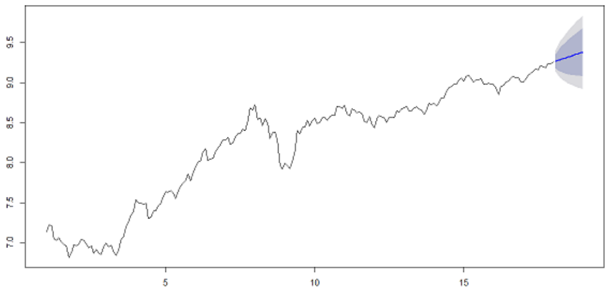
Table 3.4: ARIMA with Akaike information criterion (AIC) and Bayesian information criterion (BIC) with *n* as sample size and *k* as model parameters

	NIFTY 50		**Dow Jones Industrial Average**		**S&P BSE SENSEX**	
Forecasting Models	Shannon entropy	Kullback's information	Shannon entropy	Kullback's information	Shannon entropy	Kullback's information
ARIMA	2.205262	0.190598	2.28213	0.153591	2.238645	0.168842
Holt-winters	2.205262	0.527308	2.28213	0.141208	2.238645	0.140076
LSTM	2.090948	0.338762	1.94397	0.184999	2.079004	0.555111

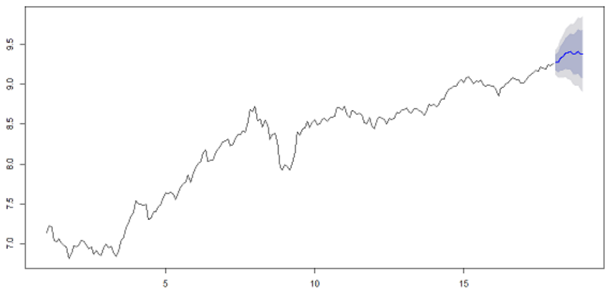
Table 3.5: Entropic analysis on forecasting models

Index	α	β	γ
NIFTY 50	0.7881153	0.01451547	1
DJI	0.7681814	0.01662513	1
BSESN	0.7714529	0.01668882	1

Table 3.6: Resulting smoothing parameters

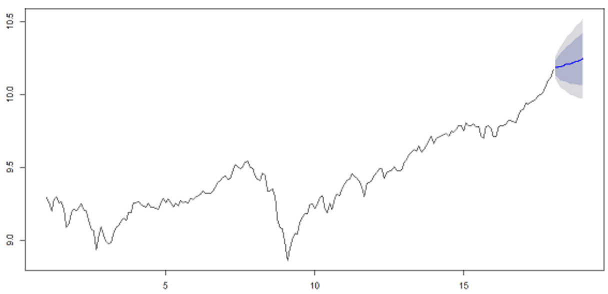


(a) ARIMA(0,1,0)(1,0,0)(12) with drift Forecast

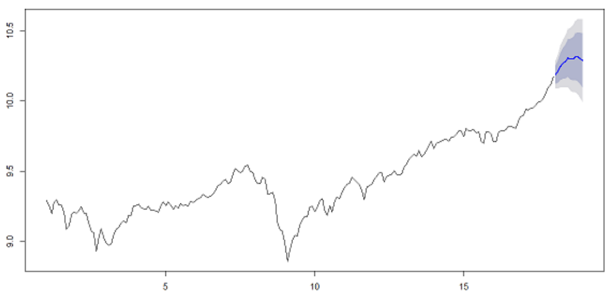


(b) Holt-Winters Forecast

Figure 3.8: NIFTY50 prediction analysis with ARIMA and Holt Winters forecasting model

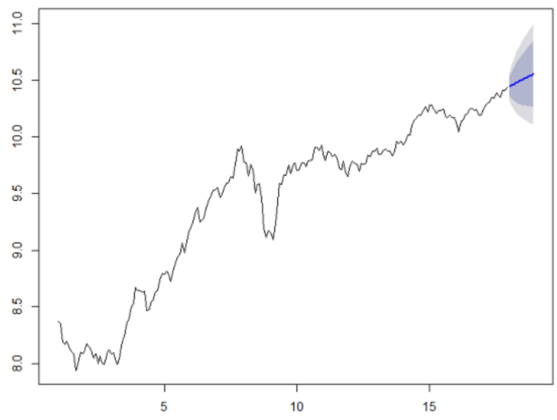


(a) ARIMA(2,1,2)(2,0,0)(12) with drift Forecast

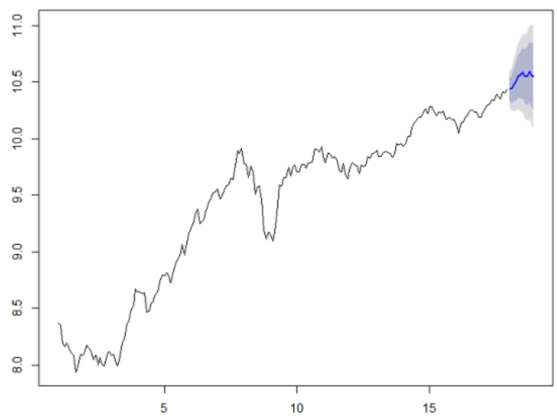


(b) Holt-Winters Forecast

Figure 3.9: Dow Jones prediction analysis with ARIMA and Holt Winters forecasting model



(a) ARIMA(0,1,0)(1,0,0)(12) with drift Forecast



(b) Holt-Winters Forecast

Figure 3.10: BSE SENSEX prediction analysis with ARIMA and Holt Winters forecasting model

The total period is subdivided into three subperiods, at the beginning of fiscal year 2003-04, the economy of India was in a boom stage driven largely by foreign investments until it was halted by the year 2008 global financial crisis. Recovery had been impeded by provisional shocks in some exogenous factors in the fiscal year 2016-17, while since the beginning of the period 2017-18, the market has picked momentum primarily from the introduction of GST and several other policies by the Indian government.

Figs. 3.8,3.9 and 3.10 display the forecast analysis of the aforementioned index by using ARIMA and Holt-Winters forecasting models. Tables 3.1,3.2 and 3.3 contrast NIFTY 50, DJI and BSESN actual market price and predicted price for the year 2018 by using ARIMA, Holt-Winters and LSTM network, respectively. Next, Holt-Winters decomposition annd filteration analysis are shown in Figs. 3.4,3.5 and 3.6 and further Figs. 3.2,3.3 and 3.7 depict the actual (blue line) vs. forecast values (orange line) through the aforementioned time series forecasting models.

Table 3.5 gives the calculated values of Kullback measure of relative information for the forecasts given by Seasonal-ARIMA, Seasonal Additive Holt-Winters and LSTM network. Since, the calculated value of the Kullback relative information is the least in case of ARIMA among all the three so the forecastings made by traditional ARIMA model are the best one in comparison to the other two in case of the NIFTY 50 monthly closing prices. Thus based on the results from Table 3.5, we conclude that ARIMA outperforms LSTM and Holt-Winters models in case of the NIFTY 50 monthly closing prices; and Holt-Winters is better in comparison to ARIMA for DJI and BSESN indices.

3.5 Conclusion

This chapter forecasts three stock indices, namely NIFTY 50, Dow Jones Industrial Average(DJI) and S&P BSE SENSEX(BSESN) monthly data by using ARIMA, Holt-Winter and LSTM network. Their performance have been compared by using Kullback measure of relative information. In comparison to other forecasting models, ARIMA model has accuracy of stock price prediction for the selected NIFTY 50 stock index; and Holt-Winters works better for predicting the DJI and BSESN indices monthly data. LSTM, a neural network method is not as much accurate to predict the stock market monthly data.

Chapter 4

Information Measures Based Portfolio Optimization

4.1 Introduction

The Markowitz Mean-Variance (MV) [51] is the most commonly used formulation in portfolio selection. The drawback of this model is that it is weakly concentrated for asset allocation which results in low diversity. In the present chapter, we have explored the effectiveness of an information theoretic measures in optimizing a portfolio to quantify risk and measure risk-adjusted performance in the capital market. In the conventional portfolio problem for a specified expected return, the expected portfolio variance is minimized [51]. This approach of minimizing the portfolio variance has been replaced by maximization of Shannon entropy, refers to [10], since both these approaches lead to equalization of allocation of assets in the portfolio. Instead of maximization of Shannon entropy, we have explored this approach further by maximizing the generalized measure of entropies. The flexibility given by the additional parameters helps us to achieve better alloca-

The part of the work presented in this chapter has been published in the paper entitled **Portfolio optimization based on generalized information theoretic measures** in *Communications in Statistics - Theory and Methods, 2020* and some work has been communicated with the title **Comparative Study Of Information Measures In Portfolio Optimization Problems** and also presented in the International Conference on Recent Trends in Mathematics and Its Applications to Graphs, Networks and Petri Nets (ICRTMA-GPN-2020) held at School of Computational and Integrative Sciences, JNU, July 2020.

tion. Also, we have extended this optimization problem by including additional non-linear inequality constraint on portfolio variance.

In this chapter, we have considered the problem of portfolio diversification by maximizing the entropy of the asset allocation. Further, since Shannon entropy can be considered as an alternative measure to the volatility (variance), so we employ this concept for portfolio optimization by replacing the minimization of the variance of allocation with the minimization of entropy. This aspect has been studied using various measure of entropy. By using the stock data of Indian indices NIFTY 50 and BSE S&P SENSEX, portfolio optimization problems are formulated with MV, Shannon entropy [76], Pal and Pal entropy [64] and Sine entropy [98]; and one parameter Renyi entropy [68], Arimoto entropy [2], Tsallis entropy [89]; and two parameters Varma entropy [93], Sharma and Mittal entropy [77] and Sharma and Taneja entropy [85]. Information theoretic measures portfolios are generated by placing the specific measure in the objective function. Based on the empirical findings from the performance measures such as diversity index [95] and the award-risk ratio [79], the selection of a well-diversified entropy portfolio model with the given level of return and minimum risk is carried out.

In Section 4.2, we discuss the Mean-Variance model by Markowitz. Section 4.3 considers the maximization of various information theoretic measures based portfolio models and their analysis. In Section 4.4, we have considered minimization of information theoretic measures based portfolio models and their qualitative and quantitative analysis of the financial data and optimization of portfolio models is carried out; some concluding comments are made in Section 4.5.

4.2 Markowitz Mean-Variance Model

The problem of portfolio selection is faced by a financial investor with a given initial wealth say W; the problem is of allocation of the wealth W among available scope of investment. In other words, the problem of portfolio selection is to find the optimal weights of portfolio $w = \{w_1, w_2, ..., w_n\}$, where each $w_i \geq 0$ is the share of W invested in the asset i. A portfolio is said to be *completely diversified* if $w_i > 0$, for all i, and it is said to be *concentrated or a single asset portfolio* if $w_i = 1$ for some i.

The landmark paper of Markowitz [51] entitled, as *"Portfolio selection* in the

Journal of Finance" marked the beginning of Portfolio Theory and transformed the portfolio selection. If m_i denotes the portfolio sample mean for the i^{th} asset, then $\mu_0 = \sum_{i=1}^{n} m_i w_i$ is the expected return and $w' \sum w$ is the expected portfolio variance. In the mean-variance framework proposed by Markowitz, the volatility i.e. the expected portfolio variance is used as a measure of risk and the sample mean as a measure of return and MV model is designed as

$$\min_{w} \; w'\Sigma w \quad \text{subject to} \begin{cases} m_1 w_1 + m_2 w_2 + ... + m_n w_n = \mu_0 \\ w_1 + w_2 + ... + w_n = 1, \; w_i \geq 0 \text{ for all } i. \end{cases} \tag{4.2.1}$$

"*An investor who knew future returns with certainty would invest in only one security, namely the one with the highest future return. If several securities had the same, highest, future return then the investor would be indifferent between any of these, or any combination of these. In no case would the investor actually prefer a diversified portfolio. But diversification is a common and reasonable investment practice. Why? To reduce uncertainty!* ", refer to Markowitz [51].

The sensitivity of MV model is that it is weakly concentrated for asset allocation and thus results in low diversity. Entropy has been considered as an alternative measure of the volatility in the financial market. It measures the randomness and it has a characteristic of measuring non-linear variability which may results in well diversification of optimal portfolio weights.

4.3 Portfolio Diversification By Maximizing Entropy Measures

In this section, we have considered the problem of portfolio diversification by maximizing the entropy of the asset allocation. The various portfolio models are defined as follows:

Shannon Entropy Portfolio Model

$$\max_{w} \; -\sum_{k=1}^{n} w_k log \; w_k \quad \text{subject to} \begin{cases} \sum_{k=1}^{n} m_k w_k = \mu_0 \\ \sum_{k=1}^{n} w_k = 1. \end{cases} \tag{4.3.1}$$

$$\max_{w} \; -\sum_{k=1}^{n} w_k log\ w_k \quad \text{subject to} \begin{cases} \sum_{k=1}^{n} m_k w_k = \mu_0 \\ \sqrt{w'\Sigma w} \leq \sigma_0 \\ \sum_{k=1}^{n} w_k = 1. \end{cases} \tag{4.3.2}$$

Tsallis Entropy Portfolio Model

For $0 < t < \infty$, $t \neq 1$

$$\max_{w} \; \frac{1}{1-t}\left(\sum_{k=1}^{n} w_k^t - 1\right) \quad \text{subject to} \begin{cases} \sum_{k=1}^{n} m_k w_k = \mu_0 \\ \sum_{k=1}^{n} w_k = 1. \end{cases} \tag{4.3.3}$$

$$\max_{w} \; \frac{1}{1-t}\left(\sum_{k=1}^{n} w_k^t - 1\right) \quad \text{subject to} \begin{cases} \sum_{k=1}^{n} m_k w_k = \mu_0 \\ \sqrt{w'\Sigma w} \leq \sigma_0 \\ \sum_{k=1}^{n} w_k = 1. \end{cases} \tag{4.3.4}$$

Renyi Entropy Portfolio Model

For $0 < r < \infty$, $r \neq 1$

$$\max_{w} \; \frac{1}{1-r} log\left(\sum_{k=1}^{n} w_k^r\right) \quad \text{subject to} \begin{cases} \sum_{k=1}^{n} m_k w_k = \mu_0 \\ \sum_{k=1}^{n} w_k = 1. \end{cases} \tag{4.3.5}$$

$$\max_{w} \; \frac{1}{1-r} log\left(\sum_{k=1}^{n} w_k^r\right) \quad \text{subject to} \begin{cases} \sum_{k=1}^{n} m_k w_k = \mu_0 \\ \sqrt{w'\Sigma w} \leq \sigma_0 \\ \sum_{k=1}^{n} w_k = 1. \end{cases} \tag{4.3.6}$$

Two Parametric Entropy Portfolio Model

For $\beta - 1 < \alpha < \beta$, $\beta \geq 1$ and $\alpha + \beta \neq 2$

$$\max_{w} \; \frac{1}{\beta - \alpha} log\left(\sum_{k=1}^{n} w_k^{\alpha+\beta-1}\right) \quad \text{subject to} \begin{cases} \sum_{k=1}^{n} m_k w_k = \mu_0 \\ \sum_{k=1}^{n} w_k = 1. \end{cases} \tag{4.3.7}$$

$$\max_{w} \frac{1}{\beta-\alpha} log\left(\sum_{k=1}^{n} w_k^{\alpha+\beta-1}\right) \quad \text{subject to} \begin{cases} \sum_{k=1}^{n} m_k w_k = \mu_0 \\ \sqrt{w'\Sigma w} \leq \sigma_0 \\ \sum_{k=1}^{n} w_k = 1. \end{cases} \tag{4.3.8}$$

Empirical Results

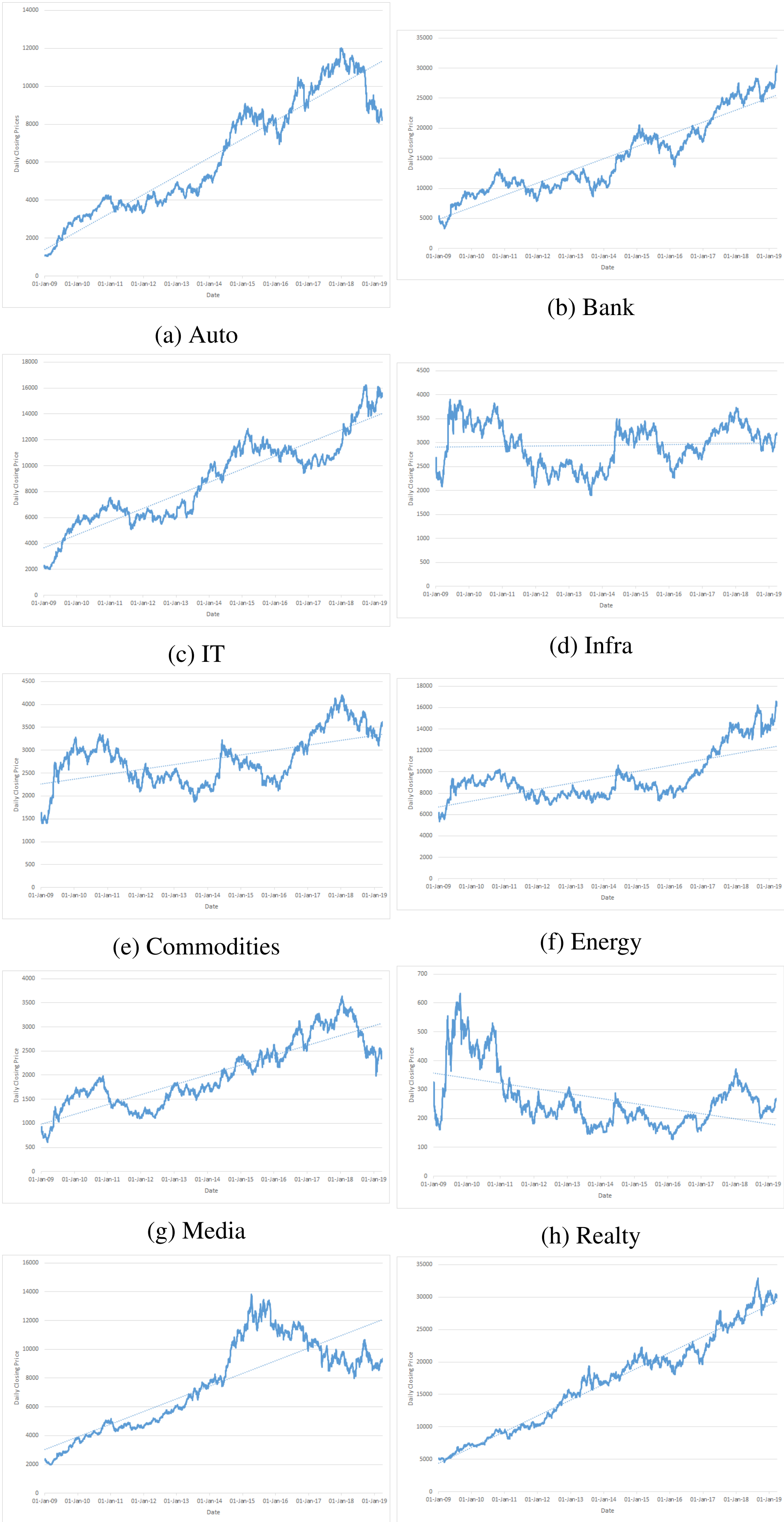
We apply the above designed portfolio models on a financial market data that consists of 10 sectors associated with the NIFTY index, a broad-based index of India's National Stock Exchange (NSE). To analyze the implementation of the aforementioned portfolio models, Ten years (1st January 2009-31th March 2019) daily closing price data on the 10 sectors of a NIFTY index, viz. Auto, Bank, IT, Infra, Commodities, Energy, Media, Realty, Pharma and FMCG were tested into the models. Fig.4.1 indicates daily closing prices with the trend of these data obtained from the National Stock Exchange of India's official website. We performed an optimization algorithm program in *MATLAB version R2015a (8.5.0.197613)* for different values of parameters in different entropy measures.

Next, summary statistics of various NIFTY sectoral indices portfolios are reported in Table 4.1. In the model presented an investor can pick a parameter r in Renyi entropy model, parameter t in Tsallis entropy model and parameters α and β in Varma entropy model according to their risk estimation. Table 4.2 is obtained by setting the Tsallis parameter t level at 0.005, 0.05, 0.5, 2,...; Table 4.3 is obtained by setting the Renyi parameter l̊evel at 0.005, 0.05, 0.5, 2,... Table 4.4 is obtained by setting the Varma entropy parameters α and β level at {0.5,1.5,2.5,3.5,4.5,5.5,6.5,7.5,8.5,9.5,49.5,99.5} and {1,2,3,4,5,6,7,8,9,10,50,100} repectively. These tables present portfolio weights and their performance measures.

We calculate Award Risk ratio (ARR), analogous to the Sharpe ratio [79], the popular and widely accepted performance measure for a portfolio selection model, defined as

$$\text{ARR} = \frac{R_p}{\sigma_p}, \tag{4.3.9}$$

where R_p= portfolio return and σ_p= portfolio risk. If $ARR < 1$, then it is considered as sub-optimal; If $1 \leq ARR \leq 2$, then it is acceptable as good by investors; if $ARR > 2$, then it is rated as very good and if $ARR \geq 3$, then it is treated excellent.



(a) Auto (b) Bank

(c) IT (d) Infra

(e) Commodities (f) Energy

(g) Media (h) Realty

(i) Pharma (j) FMCG

Figure 4.1: Daily closing prices with trendline of the 10 NIFTY sectors from 1 Jan 2009 to 31 Mar 2019.

Next, to measure the diversification in portfolios, we use the diversity index [95],

$$\text{Diversity} = 1 - \sum_{i=1}^{n} w_i^2 = \begin{cases} 0, & \text{no diversification} \\ 1, & \text{ultimate diversification.} \end{cases} \tag{4.3.10}$$

We observe that Tsallis entropy with parameter $t = 0.05$ offers highest portfolio return, and with parameters $t = 4$ and $t = 5$ offers the most diversification among its other parameters, see Table 4.2; Renyi entropy with parameter $\alpha = 0.005$ offers highest portfolio return and with parameter $r = 2$ offers the most diversification among its other parameters, see Table 4.3; two-parameter Varma entropy with parameters $\alpha = 3.5$ and $\beta = 4$ and $\alpha = 49.5$ and $\beta = 50$ offers highest portfolio return and with parameter $\alpha = 99.5$ and $\beta = 100$ offers the most diversification among its other parameters, see Table 4.4. Markowitz MV model offers minimum variance.

We further calculate the ARR of the aforementioned models for different values of desired portfolio returns and variances. ARRs at different desired returns to compare the performances of models is discussed in Tables 4.2-4.4, and the relation between return, diversity, variance, and ARR are displayed in Figs. 4.2-4.4. Next, from Figs.4.5-4.9 we can easily see that without volatility as constraint, the max entropy approach yields a frontier that is lower than mean variance's frontier, and after adding volatility as constraint, the max entropy approach has the same frontier as MV approach.

As a whole, two-parameter entropy model with $\alpha = 3.5$ and $\beta = 4$ and $\alpha = 49.5$ and $\beta = 50$ offers higher portfolio return than all other portfolio models; MV model without volatility as a constraint offers lower variance than all other portfolio models; for ultimate diversity, two-parameter entropy model with $\alpha = 99.5$ and $\beta = 100$ is the most diversified among all other portfolio models with different parameters considered in this study.

	Media	**Pharma**	**Realty**	**Auto**	**Energy**	**FMCG**	**Bank**	**IT**	**Commodities**	**Infrastructure**
Mean	0.000514	0.000607	0.000219	0.000886	0.000486	0.000762	0.000825	0.000854	0.000415	0.000172
Variance	0.000237	0.000133	0.000572	0.000170	0.000174	0.000121	0.000246	0.000190	0.000171	0.000197
Skewness	-0.1623	0.1320	0.0965	0.5957	0.5126	0.1026	0.6804	0.0762	0.5950	0.8658
Kurtosis	12.3852	9.5701	9.8230	10.9396	14.8451	6.5995	13.2914	12.0545	14.6825	17.7267
Correlation	1.000000									
	0.412567	1.000000								
	0.579855	0.401222	1.000000							
	0.561284	0.477209	0.625032	1.000000						
	0.492587	0.444364	0.615379	0.618941	1.000000					
	0.368791	0.412357	0.402087	0.496641	0.439960	1.000000				
	0.538867	0.422522	0.698282	0.687977	0.669249	0.456311	1.000000			
	0.307784	0.339815	0.333865	0.398103	0.383417	0.326309	0.397811	1.000000		
	0.608835	0.505966	0.739428	0.742035	0.865344	0.496412	0.763330	0.413518	1.000000	
	0.590190	0.472709	0.758973	0.717259	0.748015	0.468312	0.781527	0.401915	0.850221	1.000000

Table 4.1: Mean, Variance and Correlation matrix of 10 Portfolios Daily Returns

	Portfolio Models													
	Minimum Variance	**Maximum Shannon entropy**	**Maximum Tsallis entropy with t=0.005**	**Maximum Tsallis entropy with t=0.05**	**Maximum Tsallis entropy with t=0.5**	**Maximum Tsallis entropy with t=2**	**Maximum Tsallis entropy with t=3**	**Maximum Tsallis entropy with t=4,5**	**Maximum Tsallis entropy with t=6**	**Maximum Tsallis entropy with t=7**	**Maximum Tsallis entropy with t=8**	**Maximum Tsallis entropy with t=9,…**	**Maximum Shannon entropy (with Volatility)**	**Maximum Tsallis entropy with t=0.005, 0.05, 0.5, 2,… (with Volatility)**
Portfolios														
Media	0.0456	0.0548	0.1418	0	0	0.0854	0	0.0998	0.0984	0.0948	0.089	0.0779	0.0456	0.0456
Pharma	0.2912	0.0779	0.1674	0.2777	0.4539	0.1079	0	0.1001	0.1009	0.1028	0.106	0.106	0.2912	0.2912
Realty	0.0001	0.0177	0	0	0	0.0132	0	0.0987	0.0902	0.0688	0.0346	0	0.0001	0.0001
Auto	0.026	0.2268	0.1312	0.1389	0	0.1762	0.6263	0.1011	0.1086	0.1274	0.1575	0.1915	0.026	0.026
Energy	0.0904	0.0492	0.1199	0	0	0.0786	0	0.0997	0.0976	0.0923	0.0838	0.0693	0.0904	0.0904
FMCG	0.3635	0.1413	0.1564	0.2238	0.5461	0.146	0	0.1007	0.1052	0.1165	0.1346	0.1536	0.3635	0.3635
Bank	0.0011	0.1796	0.1447	0.1901	0	0.1613	0	0.1009	0.1069	0.122	0.1462	0.1728	0.0011	0.011
IT	0.178	0.2003	0.1386	0.1695	0	0.1683	0	0.101	0.1077	0.1246	0.1515	0.1815	0.178	0.178
Commodities	0.0018	0.0375	0	0	0	0.0611	0	0.0994	0.0956	0.086	0.0707	0.0474	0.0018	0.0018
Infrastructure	0.0023	0.0148	0	0	0	0.0019	0.3737	0.0986	0.0889	0.0647	0.0261	0	0.0023	0.0023
Return	0.00069	0.00074	0.0007	0.00076	0.00069	0.00072	0.00061	0.00057	0.00059	0.00062	0.00068	0.00074	0.00069	0.00069
Variance	0.0089	0.0103	0.0098	0.0095	0.0095	0.0101	0.0125	0.0112	0.011	0.0107	0.0103	0.01	0.0089	0.0089
Diversity	0.74044904	0.84278915	0.85567154	0.7886348	0.49574958	0.86427039	0.46809662	0.89999234	0.89954436	0.89539813	0.8796936	0.85257904	0.74044904	0.74032925
ARR	0.07752809	0.07184466	0.071428571	0.08	0.072631579	0.071287129	0.0488	0.050892857	0.053636364	0.057943925	0.066019417	0.074	0.07752809	0.07752809

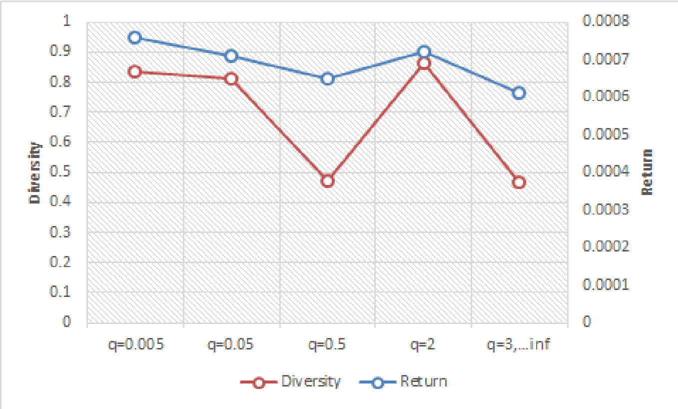
Table 4.2: Tsallis entropy with different parameters Portfolios and their performances

	Portfolio Models								
	Minimum Variance	**Maximum Shannon entropy**	**Maximum Renyi entropy with r=0.005**	**Maximum Renyi entropy with r=0.05**	**Maximum Renyi entropy with r=0.5**	**Maximum Renyi entropy with r=2**	**Maximum Renyi entropy with r=3,...**	**Maximum Shannon entropy (with Volatility)**	**Maximum Renyi entropy with r=0.005, 0.05, 0.5, 2,... (with Volatility)**
Portfolios									
Media	0.0456	0.0548	0.076	0.1561	0	0.0854	0	0.0456	0.0456
Pharma	0.2912	0.0779	0.1138	0.2741	0	0.1079	0	0.2912	0.2912
Realty	0.0001	0.0177	0	0	0	0.0132	0	0.0001	0.0001
Auto	0.026	0.2268	0.2041	0.0928	0	0.1762	0.6263	0.026	0.026
Energy	0.0904	0.0492	0.0607	0	0.3868	0.0786	0	0.0904	0.0904
FMCG	0.3635	0.1413	0.1658	0.2023	0.6132	0.146	0	0.3635	0.3635
Bank	0.0011	0.1796	0.1854	0.1504	0	0.1613	0	0.0011	0.011
IT	0.178	0.2003	0.1942	0.1243	0	0.1683	0	0.178	0.178
Commodities	0.0018	0.0375	0	0	0	0.0611	0	0.0018	0.0018
Infrastructure	0.0023	0.0148	0	0	0	0.0019	0.3737	0.0023	0.0023
Return	0.00069	0.00074	0.00076	0.00071	0.00065	0.00072	0.00061	0.00069	0.00069
Variance	0.0089	0.0103	0.0099	0.0096	0.0101	0.0101	0.0125	0.0089	0.0089
Diversity	0.74044904	0.84278915	0.83635582	0.8128942	0.47437152	0.86427039	0.46809662	0.74044904	0.74032925
ARR	0.07752809	0.07184466	0.076767677	0.073958333	0.064356436	0.071287129	0.0488	0.07752809	0.07752809

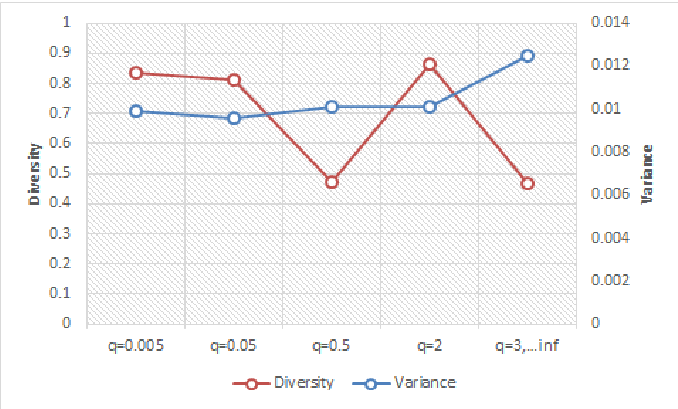
Table 4.3: Renyi entropy with different parameters Portfolios and their performances

	Portfolio Models															
	Minimum Variance	Maximum Shannon entropy	Maximum Shannon entropy (with Volatility)	Maximum Varma entropy with α=0.5 and β=1	Maximum Varma entropy with α=1.5 and β=2	Maximum Varma entropy with α=2.5 and β=3	Maximum Varma entropy with α=3.5 and β=4	Maximum Varma entropy with α=4.5 and β=5	Maximum Varma entropy with α=5.5 and β=6	Maximum Varma entropy with α=6.5 and β=7	Maximum Varma entropy with α=7.5 and β=8	Maximum Varma entropy with α=8.5 and β=9	Maximum Varma entropy with α=9.5 and β=10	Maximum Varma entropy with α=49.5 and β=50	Maximum Varma entropy with α=99.5 and β=100	Maximum Varma entropy with irrespective of values of α and β (with volatility)
Portfolios																
Media	0.0456	0.0548	0.0456	0	0.073	0.0951	0.0356	0.0539	0.0495	0.0466	0.0444	0.0429	0.0416	0.0072	0.1	0.0456
Pharma	0.2912	0.0779	0.2912	0	0.1124	0.1328	0.1607	0.1656	0.1712	0.1749	0.1776	0.1796	0.1811	0.1966	0.1	0.2912
Realty	0.0001	0.0177	0.0001	0	0	0	0	0	0	0	0	0	0	0	0.1	0.0001
Auto	0.026	0.2268	0.026	0	0.2041	0.1835	0.2071	0.1995	0.1983	0.1975	0.1969	0.1965	0.1962	0.1994	0.1	0.026
Energy	0.0904	0.0492	0.0904	0.3868	0.0587	0.0685	0	0	0	0	0	0	0	0	0.1	0.0904
FMCG	0.3635	0.1413	0.3635	0.6132	0.1667	0.1657	0.1924	0.189	0.19	0.1907	0.1911	0.1914	0.1917	0.1986	0.1	0.3635
Bank	0.0011	0.1796	0.0011	0	0.1861	0.1753	0.2005	0.1948	0.1946	0.1944	0.1943	0.1942	0.1942	0.199	0.1	0.011
IT	0.178	0.2003	0.178	0	0.1946	0.1792	0.2037	0.1971	0.1964	0.1959	0.1956	0.1954	0.1952	0.1992	0.1	0.178
Commodities	0.0018	0.0375	0.0018	0	0.0045	0	0	0	0	0	0	0	0	0	0.1	0.0018
Infrastructure	0.0023	0.0148	0.0023	0	0	0	0	0	0	0	0	0	0	0	0.1	0.0023
Return	0.00069	0.00074	0.00069	0.00065	0.00076	0.00074	0.00078	0.00077	0.00077	0.00077	0.00077	0.00077	0.00077	0.00078	0.00057	0.00069
Variance	0.0089	0.0103	0.0089	0.0101	0.0099	0.0099	0.0098	0.0098	0.0098	0.0098	0.0098	0.0098	0.0098	0.0097	0.0112	0.0089
Diversity	0.74044904	0.84278915	0.74044904	0.47437152	0.83662323	0.84465643	0.81130604	0.81735473	0.8163753	0.81569752	0.81518621	0.81476242	0.81441222	0.80281264	0.9	0.74032925
ARR	0.07752809	0.07184466	0.07752809	0.064356436	0.076767677	0.074747475	0.079591837	0.078571429	0.078571429	0.078571429	0.078571429	0.078571429	0.078571429	0.080412371	0.050892857	0.07752809

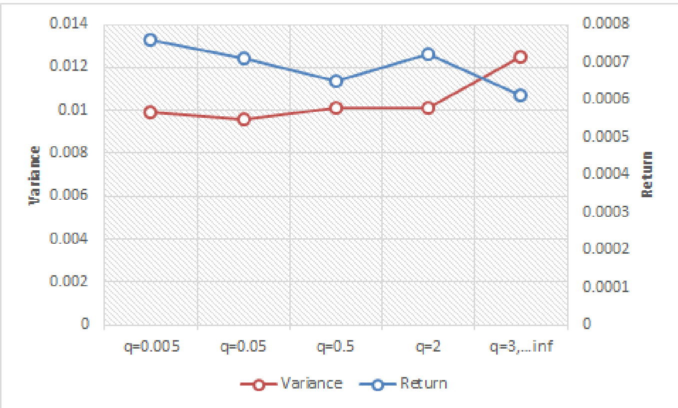
Table 4.4: Two parameter Varma entropy with different parameters Portfolios and their performances



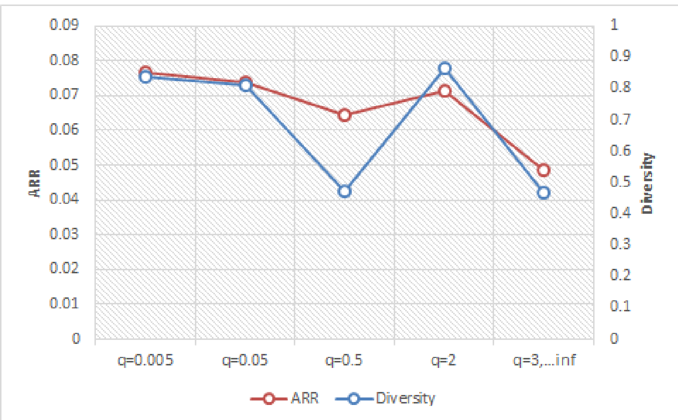
(a) Return and Diversity



(b) Variance and Diversity

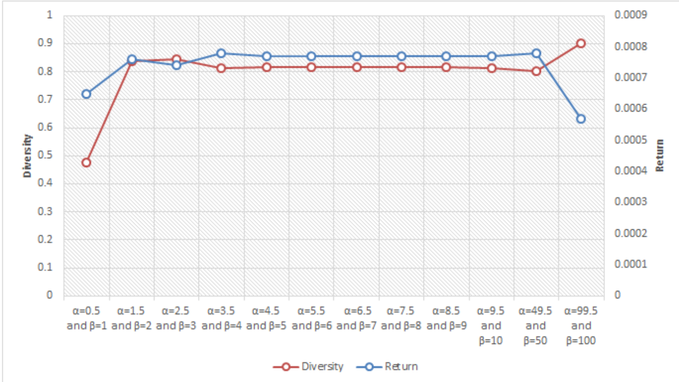


(c) Return and Variance

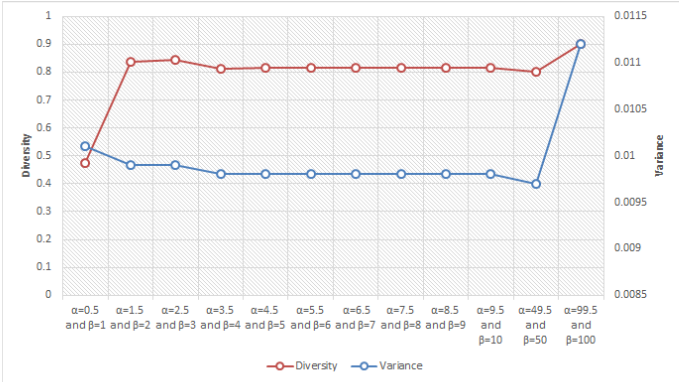


(d) ARR And Diversity

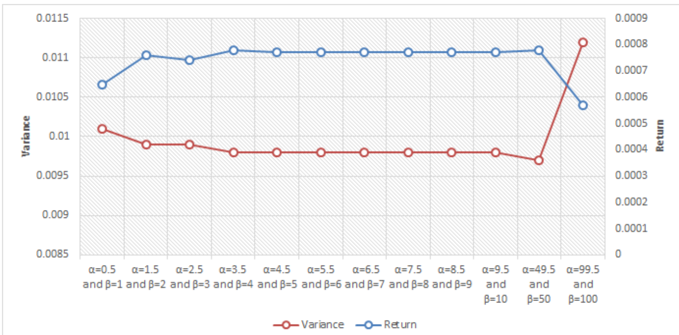
Figure 4.2: Renyi entropy performance



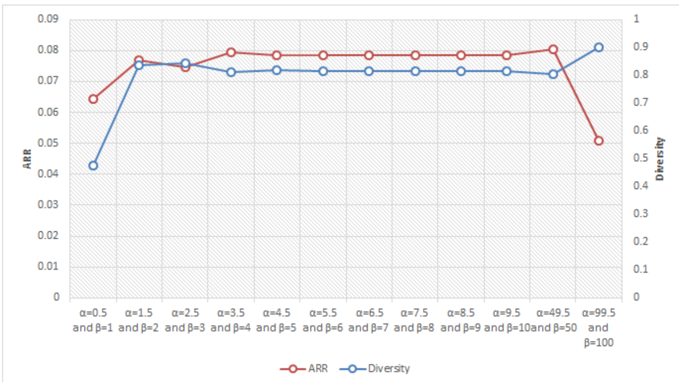
(a) Return and Diversity



(b) Variance and Diversity

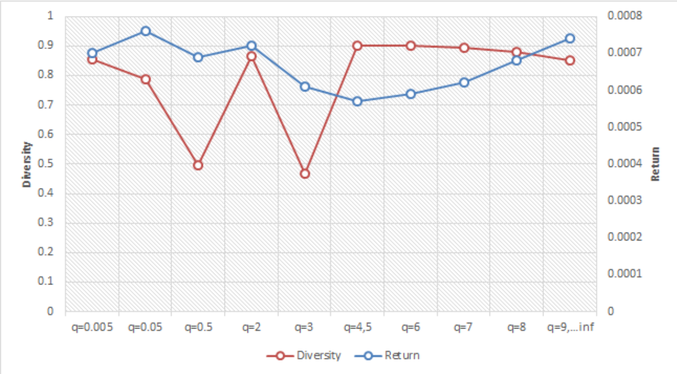


(c) Return and Variance

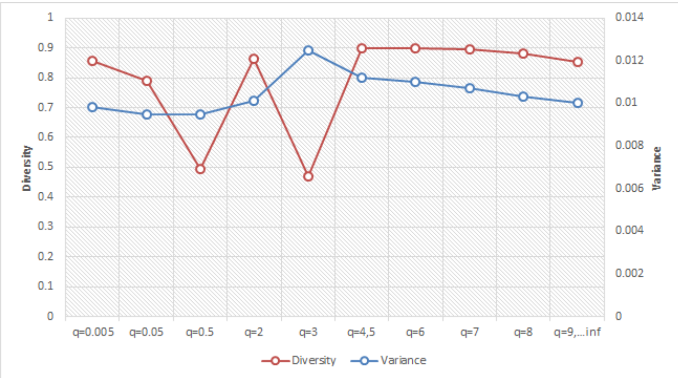


(d) ARR And Diversity

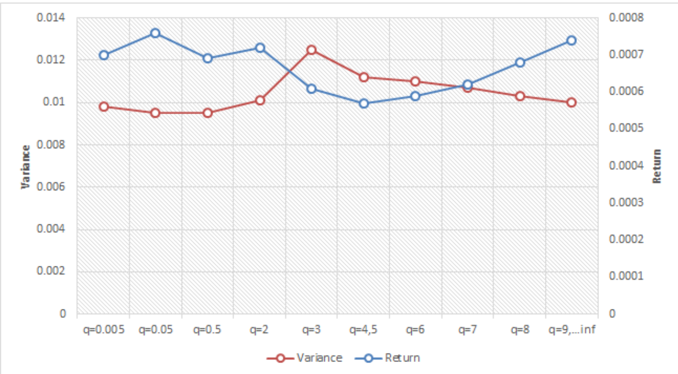
Figure 4.3: Two parameter Varma entropy performance



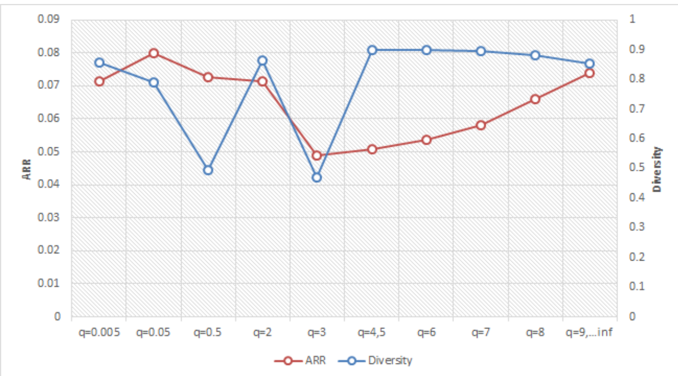
(a) Return and Diversity



(b) Variance and Diversity

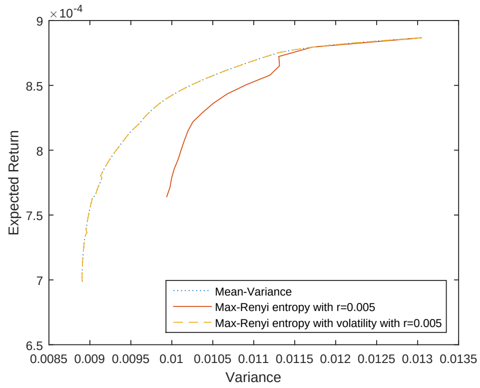


(c) Return and Variance

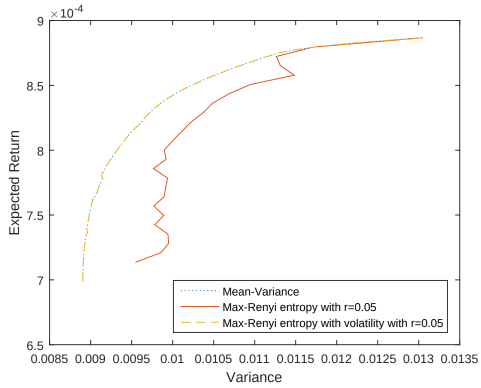


(d) ARR And Diversity

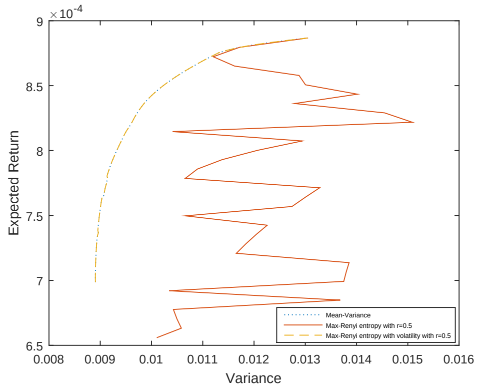
Figure 4.4: Tsallis entropy performance



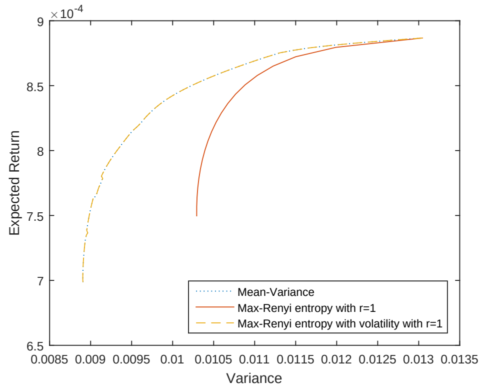
(a) Renyi entropy with parameter r=0.005



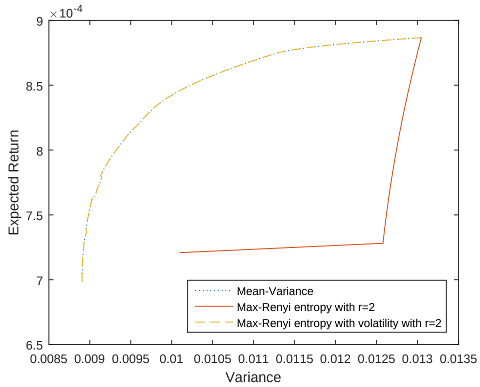
(b) Renyi entropy with parameter r=0.05



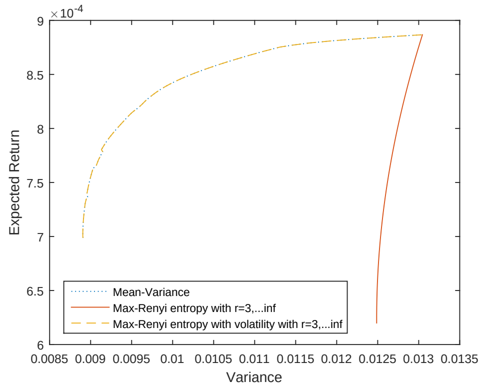
(c) Renyi entropy with parameter r=0.5



(d) Renyi entropy with parameter r=1

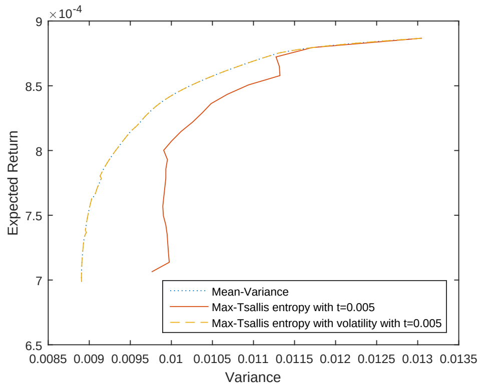


(e) Renyi entropy with parameter r=2

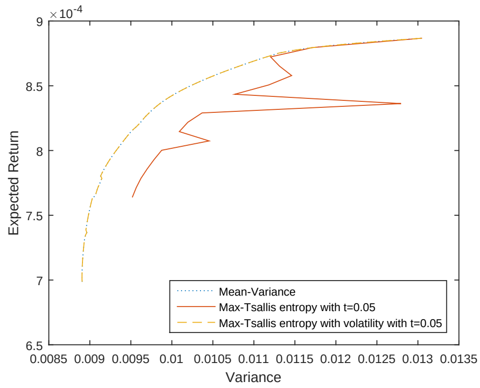


(f) Renyi entropy with parameter r=3,4,5...

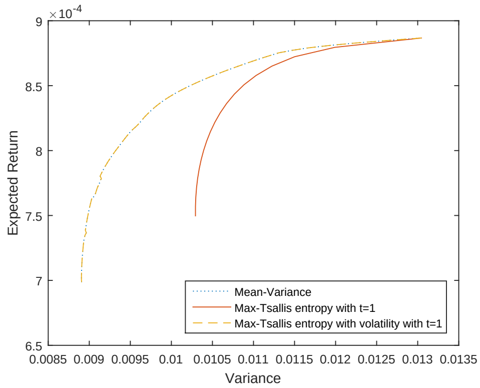
Figure 4.5: Maximum Renyi entropy with various parameter



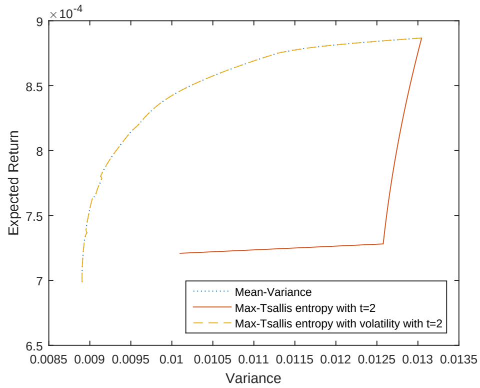
(a) Tsallis entropy with parameter t=0.005



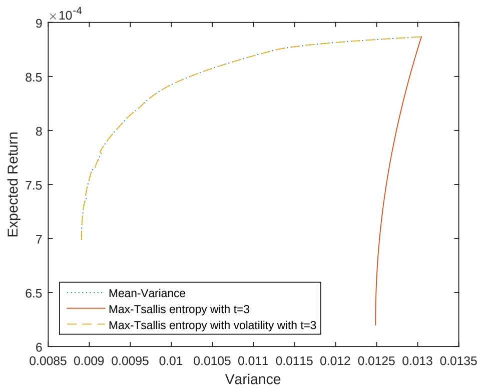
(b) Tsallis entropy with parameter t=0.05



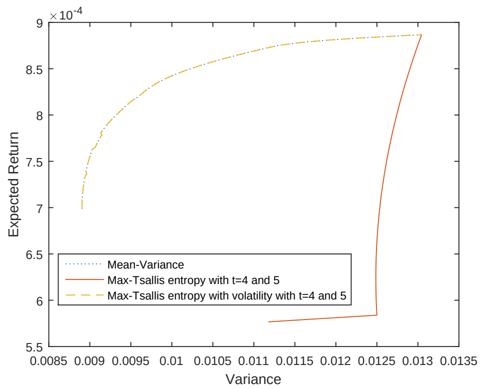
(c) Tsallis entropy with parameter t=1



(d) Tsallis entropy with parameter t=2

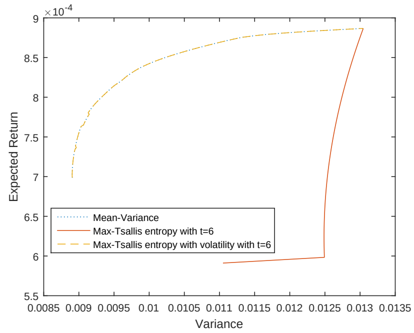


(e) Tsallis entropy with parameter t=3

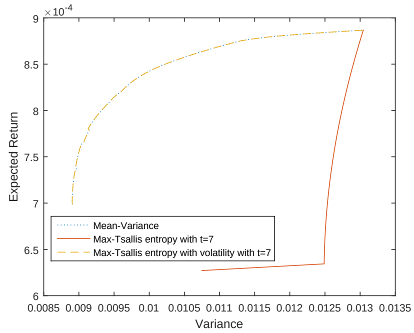


(f) Tsallis entropy with parameter t=4 and 5

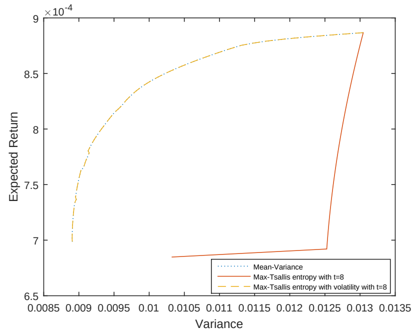
Figure 4.6: Maximum Tsallis entropy with various parameter



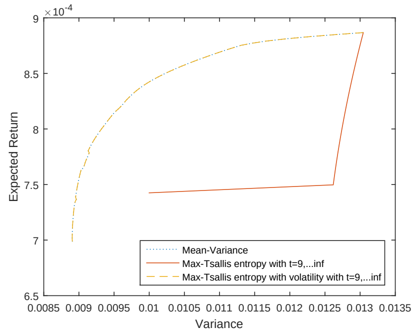
(a) Tsallis entropy with parameter t=6



(b) Tsallis entropy with parameter t=7

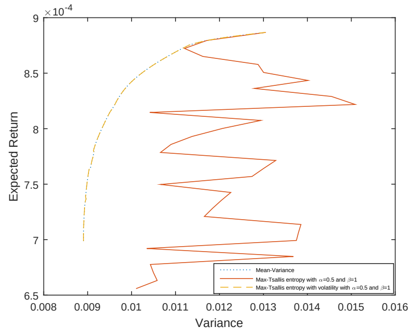


(c) Tsallis entropy with parameter t=8

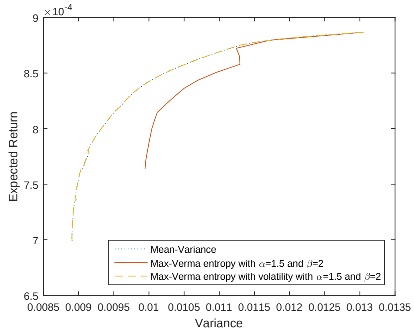


(d) Tsallis entropy with parameter t=9,10,11...

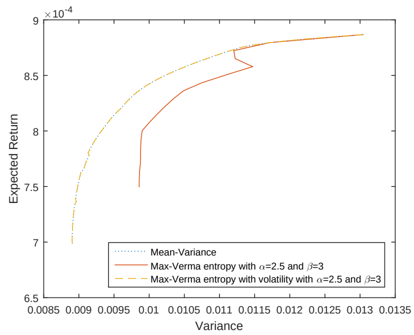
Figure 4.7: Maximum Tsallis entropy with various parameter



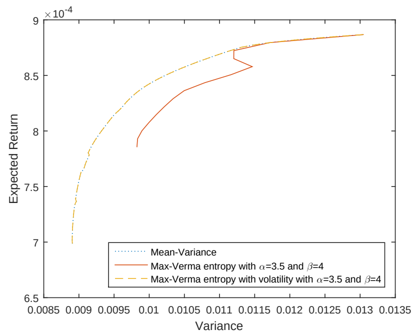
(a) Varma entropy with parameter α=0.5 and β=1



(b) Varma entropy with parameter α=1.5 and β=2

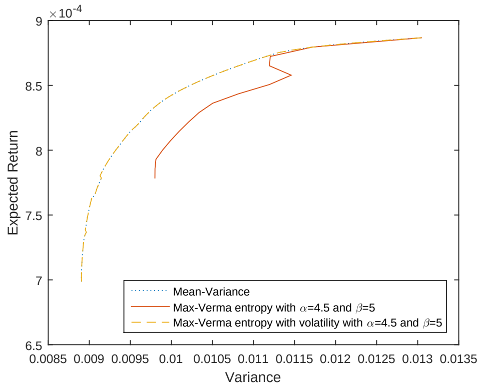


(c) Varma entropy with parameter α=2.5 and β=3

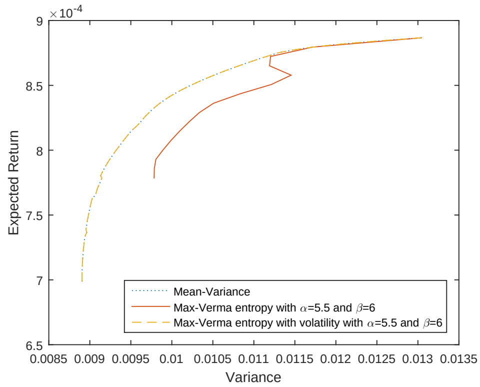


(d) Varma entropy with parameter α=3.5 and β=4

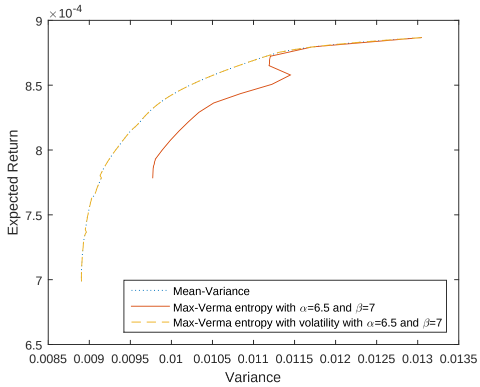
Figure 4.8: Maximum Varma entropy with various parameter



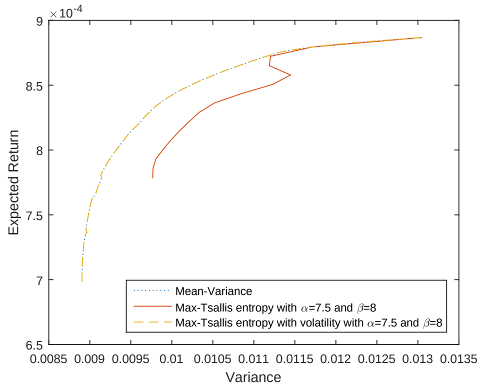
(a) Varma entropy with parameter α=4.5 and β=5



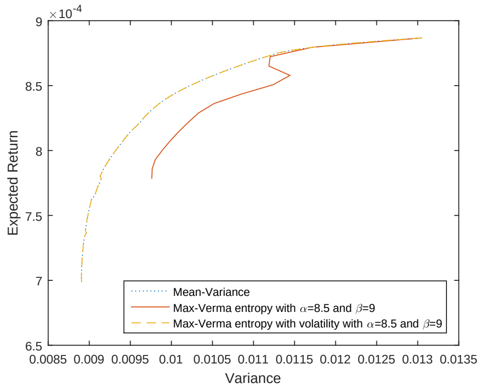
(b) Varma entropy with parameter α=5.5 and β=6



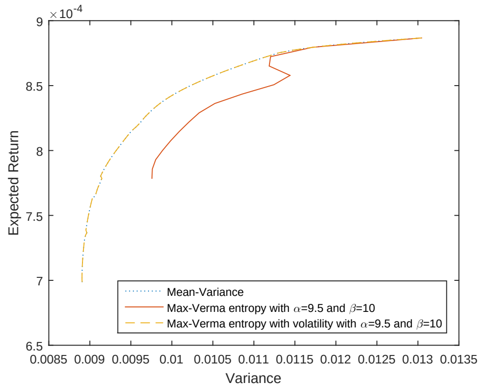
(c) Varma entropy with parameter α=6.5 and β=7



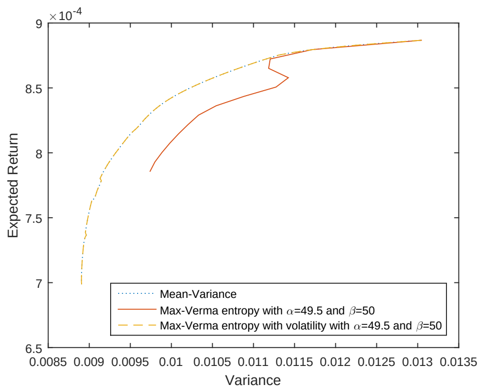
(d) Varma entropy with parameter α=7.5 and β=8



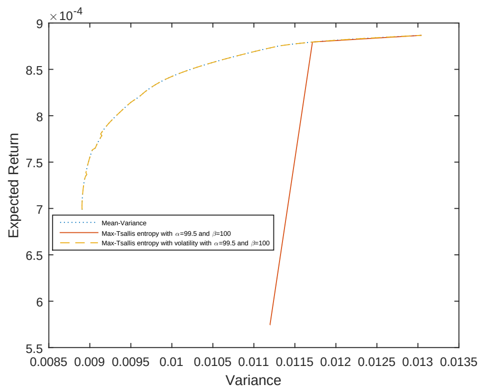
(e) Varma entropy with parameter α=8.5 and β=9



(f) Varma entropy with parameter α=9.5 and β=10



(g) Varma entropy with parameter α=49.5 and β=50



(h) Varma entropy with parameter α=99.5 and β=100

Figure 4.9: Maximum Varma entropy with various parameter

4.4 Portfolio Optimization By Minimizing Entropy Measures

It has been observed that the measures of information provide a measure of randomness and this can be considered as an alternate to conventional measure of volatility in the stock market [6, 99]. In this section, we use this characteristic of information measures in a portfolio optimization problem to achieve diversification. We have replaced the expected variance of the relative asset allocation by the expected entropy of the relative allocation of the assets. We measure portfolio risk as the uncertainty measure say, $H(w)$ of $w = \{w_1, w_2, ..., w_n\}$ and instead of minimizing $H(w)$ we will maximize $-H(w)$. The various portfolio models have been designed on the basis of information theoretic measures on the following generalized information measures, as introduced in Chapter 1. The models are as given below.

4.4.1 Entropy Models

Shannon entropy portfolio model

$$\max_{w} \; (w_1 log\ w_1 + w_2 log\ w_2 + ... + w_n log\ w_n)$$

$$\text{subject to} \begin{cases} m_1 w_1 + m_2 w_2 + ... + m_n w_n = \mu_0 \\ w_1 + w_2 + ... + w_n = 1, \; w_i \geq 0 \text{ for all } i. \end{cases}$$

Pal and Pal entropy portfolio model

$$\max_{w} \; (w_1(1 - e^{(1-w_1)}) + w_2(1 - e^{(1-w_2)}) + ... + w_n(1 - e^{(1-w_n)}))$$

$$\text{subject to} \begin{cases} m_1 w_1 + m_2 w_2 + ... + m_n w_n = \mu_0 \\ w_1 + w_2 + ... + w_n = 1, \; w_i \geq 0 \text{ for all } i. \end{cases}$$

Sine entropy portfolio model

$$\max_{w} \; -(sin(\pi\, w_1) + sin(\pi\, w_2) + \ldots + sin(\pi\, w_n))$$

$$\text{subject to} \begin{cases} m_1 w_1 + m_2 w_2 + \ldots + m_n w_n = \mu_0 \\ w_1 + w_2 + \ldots + w_n = 1,\; w_i \geq 0 \text{ for all } i. \end{cases}$$

4.4.2 One-Parametric Entropy Portfolio Models

Renyi entropy portfolio model

For $0 < r < \infty$, $r \neq 1$

$$\max_{w} \; \frac{1}{r-1} log(w_1^r + w_2^r + \ldots + w_n^r)$$

$$\text{subject to} \begin{cases} m_1 w_1 + m_2 w_2 + \ldots + m_n w_n = \mu_0 \\ w_1 + w_2 + \ldots + w_n = 1,\; w_i \geq 0 \text{ for all } i. \end{cases}$$

Arimoto entropy portfolio model

For $\alpha > 0,\; \alpha \neq 1$

$$\max_{w} \; \frac{\alpha}{1-\alpha}\left(1 - \left(w_1^\alpha + w_2^\alpha + \ldots + w_n^\alpha\right)^{\frac{1}{\alpha}}\right)$$

$$\text{subject to} \begin{cases} m_1 w_1 + m_2 w_2 + \ldots + m_n w_n = \mu_0 \\ w_1 + w_2 + \ldots + w_n = 1,\; w_i \geq 0 \text{ for all } i. \end{cases}$$

Tsallis entropy portfolio model

For $0 < t < \infty$, $t \neq 1$

$$\max_{w} \; \frac{1}{1-t}\left(1 - (w_1^t + w_2^t + \ldots + w_n^t)\right)$$

$$\text{subject to} \begin{cases} m_1 w_1 + m_2 w_2 + \ldots + m_n w_n = \mu_0 \\ w_1 + w_2 + \ldots + w_n = 1,\; w_i \geq 0 \text{ for all } i. \end{cases}$$

4.4.3 Two-Parametric Entropy Portfolio Models

Varma entropy portfolio model

For $\beta - 1 < \alpha < \beta$, $\beta \geq 1$ and $\alpha + \beta \neq 2$

$$\max_{w} \frac{1}{\alpha - \beta} log\left(w_1^{\alpha+\beta-1} + w_2^{\alpha+\beta-1} + \ldots + w_n^{\alpha+\beta-1} \right)$$
$$\text{subject to} \begin{cases} m_1 w_1 + m_2 w_2 + \ldots + m_n w_n = \mu_0 \\ w_1 + w_2 + \ldots + w_n = 1, \ w_i \geq 0 \text{ for all } i. \end{cases}$$

Sharma and Mittal entropy portfolio model

For $\alpha \neq \beta$, where $\alpha, \beta > 0$ and $\alpha \neq \beta \neq 1$

$$\max_{w} \frac{1}{1 - \beta} \left(1 - (w_1^{\alpha} + w_2^{\alpha} + \ldots w_n^{\alpha})^{\frac{1-\beta}{1-\alpha}} \right)$$
$$\text{subject to} \begin{cases} m_1 w_1 + m_2 w_2 + \ldots + m_n w_n = \mu_0 \\ w_1 + w_2 + \ldots + w_n = 1, \ w_i \geq 0 \text{ for all } i. \end{cases}$$

Sharma and Taneja entropy portfolio model

For $\alpha \neq \beta \neq 1$

$$\max_{w} \frac{1}{2^{1-\beta} - 2^{1-\alpha}} \left((w_1^{\alpha} + w_2^{\alpha} + \ldots + w_n^{\alpha}) - (w_1^{\beta} + w_2^{\beta} + \ldots + w_n^{\beta}) \right)$$
$$\text{subject to} \begin{cases} m_1 w_1 + m_2 w_2 + \ldots + m_n w_n = \mu_0 \\ w_1 + w_2 + \ldots + w_n = 1, \ w_i \geq 0 \text{ for all } i. \end{cases}$$

4.4.4 Single-Index Portfolio Entropy

Philippatos and Wilson [65] considered a model to maximize the expected portfolio return as well as to minimize the portfolio entropy. The entropy of a single-index portfolio is defined as

$$H(S_i, I) = H(I) + \sum_{i=1}^{n} w_i H(S_i|I), \tag{4.4.1}$$

where $\{S_i, i = 1,2,...,n\}$ is the set of stocks and I is a financial market index and w_i is the fraction of funds invested in stock i. Given the required individual, joint, and conditional subjective probabilities, we have used this entropy of single-index portfolio to calculate the joint uncertainty of the assets S_i correlated with a market index I.

Empirical Results

For the optimal portfolio modeling, we consider a financial data that consists of 5 stocks viz. SBI, ONGC, TATA STEEL, SUN PHARMA, and AXIS BANK listed with NIFTY 50 as well as with BSE SENSEX, two flagship indices of Indian financial market. In order to analyze the working of proposed portfolio models, we have considered eighteen years (01/01/2001 to 31/12/2018) daily closing price data.

The distribution of daily closing price data of the aforementioned Indian stock indices with their selected stocks, for pictorial representation, refer to Fig. 2.9. The given data is obtained from https://in.finance.yahoo.com/, the free financial data online platform. We have also presented two correlation matrix plots as well as the conditional entropy to check the statistical behavior between the above-mentioned international indices and their stocks, refer to Fig.4.10. Fig.4.11 can be referred for visual comparison of generalized entropy measures by using single-index portfolio entropy.

Further, we implement a portfolio program in MATLAB version *R2015a (8.5.0.197613)* for different values of parameters of generalized information entropic measures. Table 4.5 is obtained by considering Shannon, Pal and Pal and Sine entropy measures, and compare their performance with the traditional mean-variance model. From the data given in Tables 4.6-4.10, we set the parameter level from 0.005 to 160 of one parametric entropy measures, namely Renyi, Arimoto and Tsallis to compare the efficiency of portfolio risk and expected return between mean-variance model and one parameter entropy measure models; similarly to compare the performance of two parametric entropy models, namely Varma, Sharma and Mittal, and Sharma and Taneja, Tables 4.11-4.15 are obtained by setting the two parameters level at {0.505, 1.5, 9.5, 39.5, 79.5, 159.5, 319.5} and {1.005, 2, 10, 40, 80, 160, 320} respectively.

		Portfolio models							Portfolio models			
		Minimum Variance	**Maximum Shannon entropy**	**Maximum Pal and Pal Exponential entropy**	**Maximum Sine entropy**				**Minimum Variance**	**Maximum Shannon entropy**	**Maximum Pal and Pal Exponential entropy**	**Maximum Sine entropy**
BSE Sensex							**NIFTY 50**					
SBI	**Weights**	0.2666	0.2291	0.2277	0.2258		SBI	**Weights**	0.183	0.1501	0.1524	0.1638
ONGC		0.1685	0.2242	0.2236	0.2226		ONGC		0.1626	0.1395	0.1377	0.1379
TATA Steel		0.0376	0.1408	0.1398	0.1383		TATA Steel		0.077	0.1258	0.1174	0.0903
SUN Pharma		0.3458	0.2158	0.2163	0.2168		SUN Pharma		0.3872	0.1923	0.2053	0.2328
Axis Bank		0.1815	0.1901	0.1926	0.1964		Axis Bank		0.1903	0.3923	0.3873	0.3752
	Return	0.0011	0.0012	0.0012	0.0012			**Return**	0.001	1.10E-03	0.0011	0.0011
	Variance	0.0123	0.0142	0.0141	0.0141			**Variance**	0.0133	0.015	0.0149	0.0146
	Diversity	0.74659854	0.79471526	0.79473126	0.79476151			**Diversity**	0.7480053	0.7513055	0.75188081	0.75102818
	ARR	0.089430894	0.084507042	0.085106383	0.085106383			**ARR**	0.075188	7.33E-02	0.073825503	0.075342466

Table 4.5: Shannon, Pal and Pal, and Sine entropy portfolios and their performances

		Portfolio models							Portfolio models			
		Minimum Variance	**Maximum Renyi entropy**	**Maximum Arimoto entropy**	**Maximum Tsallis entropy**				**Minimum Variance**	**Maximum Renyi entropy**	**Maximum Arimoto entropy**	**Maximum Tsallis entropy**
BSE Sensex							**NIFTY 50**					
SBI		0.2666	0.2313	0.2	0.2313		SBI		0.183	0.1489	0.2	0.1489
ONGC		0.1685	0.2251	0.2	0.2251		ONGC		0.1626	0.1421	0.2	0.1421
TATA STEEL	**Weights**	0.0376	0.1422	0.2	0.1422		TATA STEEL	**Weights**	0.077	0.1336	0.2	0.1336
SUN Pharma		0.3458	0.2148	0.2	0.2148		SUN Pharma		0.3872	0.1777	0.2	0.1777
AXIS BANK		0.1815	0.1865	0.2	0.1865		AXIS BANK		0.1903	0.3977	0.2	0.3977
	Return	0.0011	0.0012	0.0012	0.0012			**Return**	0.001	0.0011	0.0009	0.0011
	Variance	0.0123	0.0142	0.0161	0.0142			**Variance**	0.0133	0.0152	0.0144	0.0152
	Diversity	0.74659854	0.79468817	0.8	0.7946882			**Diversity**	0.74800531	0.7500448	0.8	0.7500448
	ARR	0.08943089	0.08450704	0.074534161	0.084507			**ARR**	0.07518797	0.0723684	0.0625	0.0723684

Table 4.6: One parameter $\alpha = 0.005$ and 0.01 entropy portfolios and their performances

		Portfolio models			
		Minimum Variance	**Maximum Renyi entropy**	**Maximum Arimoto entropy**	**Maximum Tsallis entropy**
BSE Sensex					
SBI		0.2666	0.2272	0.2272	0.2272
ONGC		0.1685	0.2234	0.2234	0.2234
TATA STEEL	**Weights**	0.0376	0.1394	0.1394	0.1394
SUN Pharma		0.3458	0.2165	0.2165	0.2165
AXIS BANK		0.1815	0.1935	0.1935	0.1935
	Return	0.0011	0.0012	0.0012	0.0012
	Variance	0.0123	0.0141	0.0141	0.0141
	Diversity	0.74659854	0.79472574	0.79472574	0.7947257
	ARR	0.08943089	0.085106383	0.085106383	0.0851064

		Portfolio models			
		Minimum Variance	**Maximum Renyi entropy**	**Maximum Arimoto entropy**	**Maximum Tsallis entropy**
NIFTY 50					
SBI		0.183	0.154	0.154	0.154
ONGC		0.1626	0.1366	0.1366	0.1366
TATA STEEL	**Weights**	0.077	0.1119	0.1119	0.1119
SUN Pharma		0.3872	0.2135	0.2135	0.2135
AXIS BANK		0.1903	0.3841	0.3841	0.3841
	Return	0.001	0.0011	0.0011	0.0011
	Variance	0.0133	0.0148	0.0148	0.0148
	Diversity	0.74800531	0.7519878	0.7519878	0.7519878
	ARR	0.07518797	0.0743243	0.0743243	0.0743243

Table 4.7: One parameter $\alpha = 2$ entropy portfolios and their performances

		Portfolio models							Portfolio models			
		Minimum Variance	**Maximum Renyi entropy**	**Maximum Arimoto entropy**	**Maximum Tsallis entropy**				**Minimum Variance**	**Maximum Renyi entropy**	**Maximum Arimoto entropy**	**Maximum Tsallis entropy**
BSE Sensex							**NIFTY 50**					
SBI		0.2666	0.2205	0.2205	0.2272		SBI		0.183	0.253	0.253	0.2521
ONGC		0.1685	0.2194	0.2194	0.2234		ONGC		0.1626	0.1092	0.1092	0.1103
TATA STEEL	**Weights**	0.0376	0.1335	0.1335	0.1394		TATA STEEL	**Weights**	0.077	0	0	0
SUN Pharma		0.3458	0.2174	0.2174	0.2165		SUN Pharma		0.3872	0.2982	0.2982	0.2978
AXIS BANK		0.1815	0.2091	0.2091	0.1935		AXIS BANK		0.1903	0.3396	0.3396	0.3398
	Return	0.0011	0.0012	0.0012	0.0012			**Return**	0.001	0.0011	0.0011	0.0011
	Variance	0.0123	0.014	0.014	0.0141			**Variance**	0.0133	0.0143	0.0143	0.0143
	Diversity	0.74659854	0.79443557	0.79443557	0.7947257			**Diversity**	0.74800531	0.719815	0.719815	0.7201306
	ARR	0.08943089	0.085714286	0.085714286	0.0851064			**ARR**	0.07518797	0.0769231	0.0769231	0.0769231

Table 4.8: One parameter $\alpha = 10$ entropy portfolios and their performances

		Portfolio models						Portfolio models			
		Minimum Variance	**Maximum Renyi entropy**	**Maximum Arimoto entropy**	**Maximum Tsallis entropy**			**Minimum Variance**	**Maximum Renyi entropy**	**Maximum Arimoto entropy**	**Maximum Tsallis entropy**
BSE Sensex						**NIFTY 50**					
SBI		0.2666	0.2181	0.2181	0.2272	SBI		0.183	0.3075	0.3075	0.154
ONGC		0.1685	0.2179	0.2179	0.2234	ONGC		0.1626	0.0439	0.0439	0.1366
TATA STEEL	**Weights**	0.0376	0.1312	0.1312	0.1394	TATA STEEL	**Weights**	0.077	0	0	0.1119
SUN Pharma		0.3458	0.2174	0.2174	0.2165	SUN Pharma		0.3872	0.3194	0.3194	0.2135
AXIS BANK		0.1815	0.2154	0.2154	0.1935	AXIS BANK		0.1903	0.3291	0.3291	0.3841
	Return	0.0011	0.0012	0.0012	0.0012		**Return**	0.001	0.0011	0.0011	0.0011
	Variance	0.0123	0.0139	0.0139	0.0141		**Variance**	0.0133	0.0144	0.0144	0.0148
	Diversity	0.74659854	0.79407862	0.79407862	0.7947257		**Diversity**	0.74800531	0.6931934	0.6931934	0.7519878
	ARR	0.08943089	0.086330935	0.086330935	0.0851064		**ARR**	0.07518797	0.0763889	0.0763889	0.0743243

Table 4.9: One parameter $\alpha = 40$ entropy portfolios and their performances

		Portfolio models							Portfolio models			
		Minimum Variance	**Maximum Renyi entropy**	**Maximum Arimoto entropy**	**Maximum Tsallis entropy**				**Minimum Variance**	**Maximum Renyi entropy**	**Maximum Arimoto entropy**	**Maximum Tsallis entropy**
BSE Sensex							**NIFTY 50**					
SBI	**Weights**	0.2666	0.2176	0.2176	0.2272		SBI	**Weights**	0.183	0.3213	0.3213	0.154
ONGC		0.1685	0.2175	0.2175	0.2234		ONGC		0.1626	0.0278	0.0278	0.1366
TATA STEEL		0.0376	0.1306	0.1306	0.1394		TATA STEEL		0.077	0	0	0.1119
SUN Pharma		0.3458	0.2174	0.2174	0.2165		SUN Pharma		0.3872	0.3243	0.3243	0.2135
AXIS BANK		0.1815	0.2169	0.2169	0.1935		AXIS BANK		0.1903	0.3267	0.3267	0.3841
	Return	0.0011	0.0012	0.0012	0.0012			**Return**	0.001	0.0011	0.0011	0.0011
	Variance	0.0123	0.0139	0.0139	0.0141			**Variance**	0.0133	0.0145	0.0145	0.0148
	Diversity	0.74659854	0.79397926	0.79397926	0.7947257			**Diversity**	0.74800531	0.6840901	0.6840901	0.7519878
	ARR	0.08943089	0.086330935	0.086330935	0.0851064			**ARR**	0.07518797	0.0758621	0.0758621	0.0743243

Table 4.10: One parameter $\alpha = 160$ entropy portfolios and their performances

		Portfolio Models							Portfolio Models			
		Minimum Variance	**Maximum Varma entropy**	**Maximum Sharma and Mittal entropy**	**Maximum Sharma and Taneja entropy**				**Minimum Variance**	**Maximum Varma entropy**	**Maximum Sharma and Mittal entropy**	**Maximum Sharma and Taneja entropy**
BSE Sensex							**NIFTY 50**					
SBI	**Weights**	0.2666	0.2301	0.2301	0.2301		SBI	**Weights**	0.183	0.1493	0.1493	0.1493
ONGC		0.1685	0.2247	0.2247	0.2247		ONGC		0.1626	0.1409	0.1409	0.1409
TATA STEEL		0.0376	0.1415	0.1415	0.1415		TATA STEEL		0.077	0.1302	0.1302	0.1302
SUN Pharma		0.3458	0.2154	0.2154	0.2154		SUN Pharma		0.3872	0.1843	0.1842	0.1843
AXIS BANK		0.1815	0.1883	0.1883	0.1883		AXIS BANK		0.1903	0.3953	0.3953	0.3953
	Return	0.0011	0.0012	0.0012	0.0012			**Return**	0.001	0.0011	0.0011	0.0011
	Variance	0.0123	0.0142	0.0142	0.0142			**Variance**	0.0133	0.0151	0.0151	0.0151
	Diversity	0.74659854	0.7946876	0.7946876	0.7946876			**Diversity**	0.74800531	0.75067608	0.75071293	0.75067608
	ARR	0.08943089	0.084507042	0.084507042	0.084507042			**ARR**	0.07518797	0.07284768	0.072847682	0.07284768

Table 4.11: Two parameter $\alpha = 0.505$ and $\beta = 1.005$ entropy portfolios and their performances

		Portfolio Models							Portfolio Models			
		Minimum Variance	**Maximum Varma entropy**	**Maximum Sharma and Mittal entropy**	**Maximum Sharma and Taneja entropy**				**Minimum Variance**	**Maximum Varma entropy**	**Maximum Sharma and Mittal entropy**	**Maximum Sharma and Taneja entropy**
BSE Sensex							**NIFTY 50**					
SBI		0.2666	0.75	0.2281	0.2228		SBI		0.183	0	0.1515	0.205
ONGC		0.1685	0	0.2238	0.2208		ONGC		0.1626	0	0.1379	0.1828
TATA STEEL	**Weights**	0.0376	0.25	0.1401	0.1356		TATA STEEL	**Weights**	0.077	0.5	0.1199	0
SUN Pharma		0.3458	0	0.2162	0.2171		SUN Pharma		0.3872	0	0.2022	0.2561
AXIS BANK		0.1815	0	0.1918	0.2037		AXIS BANK		0.1903	0.5	0.3886	0.3561
	Return	0.0011	0.0012	0.0012	0.0012			**Return**	0.001	0.0011	0.0011	0.0011
	Variance	0.0123	0.0231	0.0141	0.014			**Variance**	0.0133	0.0202	0.0149	0.0144
	Diversity	0.74659854	0.375	0.79472626	0.79459406			**Diversity**	0.74800531	0.5	0.75176053	0.73216474
	ARR	0.08943089	0.051948052	0.085106383	0.085714286			**ARR**	0.07518797	0.05445545	0.073825503	0.07638889

Table 4.12: Two parameter $\alpha = 1.5$ and $\beta = 2$ entropy portfolios and their performances

		Portfolio Models							Portfolio Models			
		Minimum Variance	**Maximum Varma entropy**	**Maximum Sharma and Mittal entropy**	**Maximum Sharma and Taneja entropy**				**Minimum Variance**	**Maximum Varma entropy**	**Maximum Sharma and Mittal entropy**	**Maximum Sharma and Taneja entropy**
BSE Sensex							**NIFTY 50**					
SBI		0.2666	0.75	0.2272	0.75		SBI		0.183	0	0.2508	0
ONGC		0.1685	0	0.2234	0		ONGC		0.1626	0	0.1132	0
TATA STEEL	**Weights**	0.0376	0.25	0.1394	0.25		TATA STEEL	**Weights**	0.077	0.5	0	0.5
SUN Pharma		0.3458	0	0.2165	0		SUN Pharma		0.3872	0	0.2954	0
AXIS BANK		0.1815	0	0.1935	0		AXIS BANK		0.1903	0.5	0.3406	0.5
	Return	0.0011	0.0012	0.0012	0.0012			**Return**	0.001	0.0011	0.0011	0.0011
	Variance	0.0123	0.0231	0.0141	0.0231			**Variance**	0.0133	0.0202	0.0143	0.0202
	Diversity	0.74659854	0.375	0.79472574	0.375			**Diversity**	0.74800531	0.5	0.7210156	0.5
	ARR	0.08943089	0.051948052	0.085106383	0.051948052			**ARR**	0.07518797	0.05445545	0.076923077	0.05445545

Table 4.13: Two parameter $\alpha = 9.5$ and $\beta = 10$ entropy portfolios and their performances

		Portfolio Models							Portfolio Models			
		Minimum Variance	**Maximum Varma entropy**	**Maximum Sharma and Mittal entropy**	**Maximum Sharma and Taneja entropy**				**Minimum Variance**	**Maximum Varma entropy**	**Maximum Sharma and Mittal entropy**	**Maximum Sharma and Taneja entropy**
BSE Sensex							**NIFTY 50**					
SBI	**Weights**	0.2666	0.75	0.2272	0.2272		SBI	**Weights**	0.183	0	0.154	0.2192
ONGC		0.1685	0	0.2234	0.2234		ONGC		0.1626	0	0.1366	0.2266
TATA STEEL		0.0376	0.25	0.1394	0.1394		TATA STEEL		0.077	0.5	0.1119	0.2369
SUN Pharma		0.3458	0	0.2165	0.2165		SUN Pharma		0.3872	0	0.2135	0.1944
AXIS BANK		0.1815	0	0.1935	0.1935		AXIS BANK		0.1903	0.5	0.3841	0.1229
	Return	0.0011	0.0012	0.0012	0.0012			**Return**	0.001	0.0011	0.0011	0.0009
	Variance	0.0123	0.0231	0.0141	0.0141			**Variance**	0.0133	0.0202	0.0148	0.015
	Diversity	0.7465985	0.375	0.79472574	0.79472574			**Diversity**	0.748005	0.5	0.7519878	0.791586
	ARR	0.0894309	0.0519481	0.08510638	0.08510638			**ARR**	0.075188	0.0544554	0.0743243	0.06

Table 4.14: Two parameter $\alpha = 39.5$ and $\beta = 40$ entropy portfolios and their performances

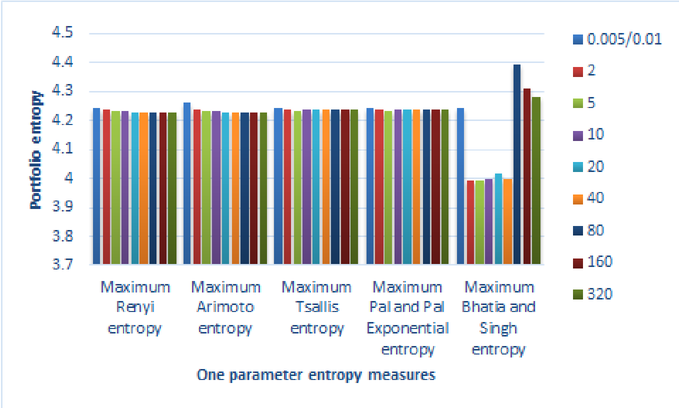
		Portfolio Models							Portfolio Models			
		Minimum Variance	**Maximum Varma entropy**	**Maximum Sharma and Mittal entropy**	**Maximum Sharma and Taneja entropy**				**Minimum Variance**	**Maximum Varma entropy**	**Maximum Sharma and Mittal entropy**	**Maximum Sharma and Taneja entropy**
BSE Sensex							**NIFTY 50**					
SBI	**Weights**	0.2666	0.75	0.2272	0.2272		SBI	**Weights**	0.183	0	0.154	0.154
ONGC		0.1685	0	0.2234	0.2234		ONGC		0.1626	0	0.1366	0.1366
TATA STEEL		0.0376	0.25	0.1394	0.1394		TATA STEEL		0.077	0.5	0.1119	0.1119
SUN Pharma		0.3458	0	0.2165	0.2165		SUN Pharma		0.3872	0	0.2135	0.2135
AXIS BANK		0.1815	0	0.1935	0.1935		AXIS BANK		0.1903	0.5	0.3841	0.3841
	Return	0.0011	0.0012	0.0012	0.0012			**Return**	0.001	0.0011	0.0011	0.0011
	Variance	0.0123	0.0231	0.0141	0.0141			**Variance**	0.0133	0.0202	0.0148	0.0148
	Diversity	0.7465985	0.375	0.79472574	0.79472574			**Diversity**	0.748005	0.5	0.7519878	0.751988
	ARR	0.0894309	0.0519481	0.08510638	0.08510638			**ARR**	0.075188	0.0544554	0.0743243	0.074324

Table 4.15: Two parameter $\alpha = 79.5, 159.5, 319.5, \ldots$ and $\beta = 80, 160, 320, \ldots$ entropy portfolios and their performances

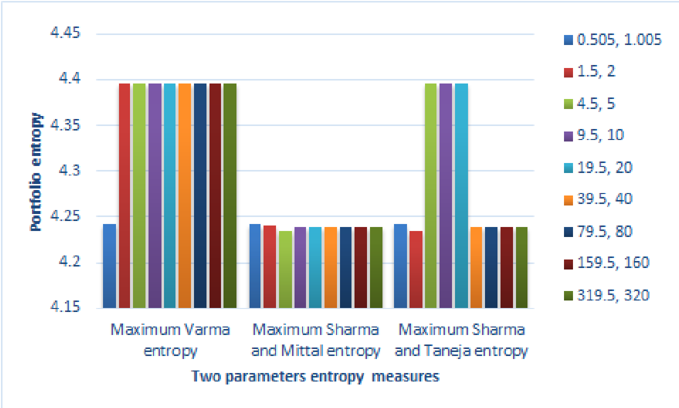
(a) BSE SENSEX

(b) NIFTY 50

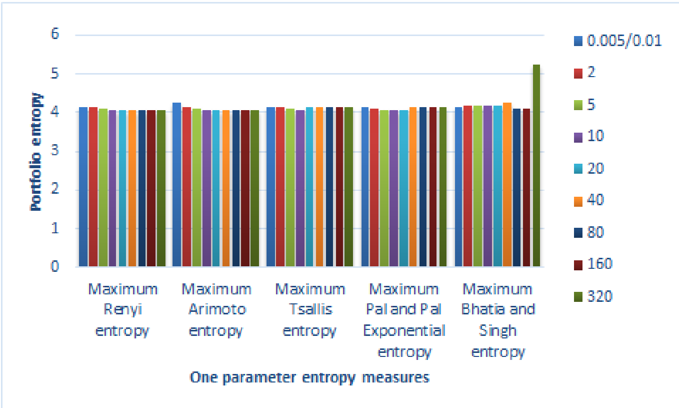
Figure 4.10: Correlation matrix plot of stock Indices and their stocks



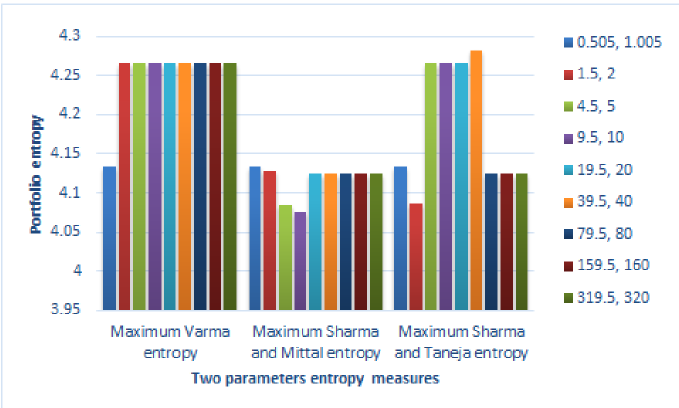
(a) BSE SENSEX on one parametric entropy measures



(b) BSE SENSEX on two parametric entropy measures

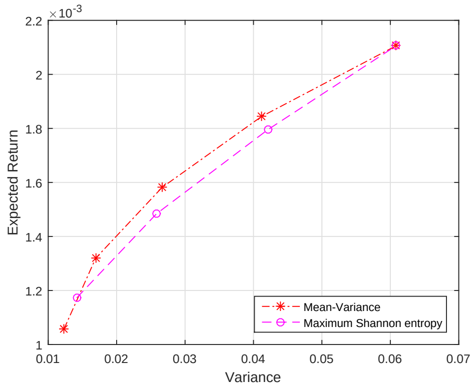


(c) NIFTY 50 on one parametric entropy measures

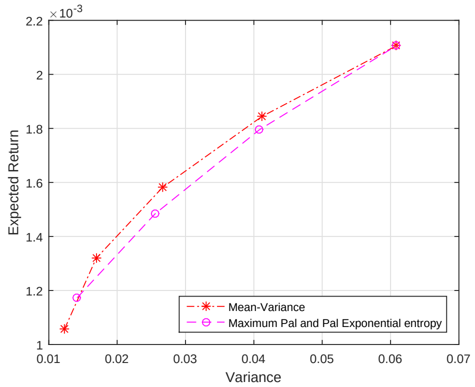


(d) NIFTY 50 on two parametric entropy measures

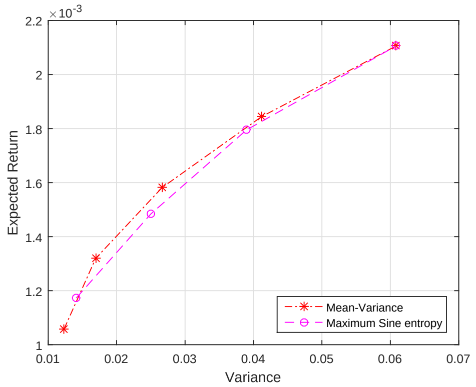
Figure 4.11: Index portfolio entropy



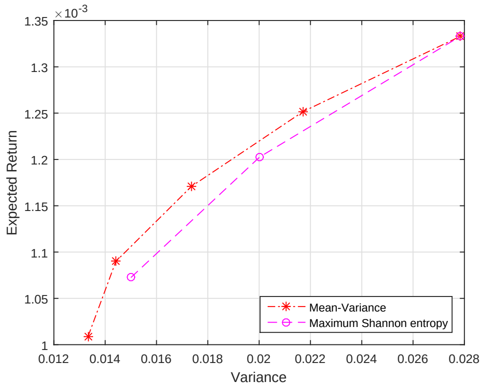
(a) Shannon entropy on BSE SENSEX



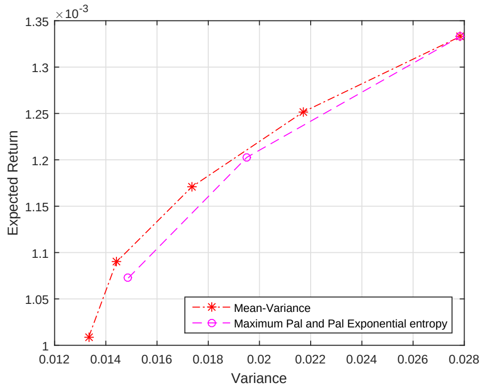
(b) Pal and Pal entropy on BSE SENSEX



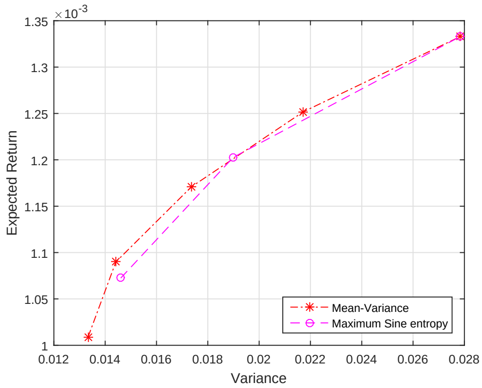
(c) Sine entropy on BSE SENSEX



(d) Shannon entropy on NIFTY 50

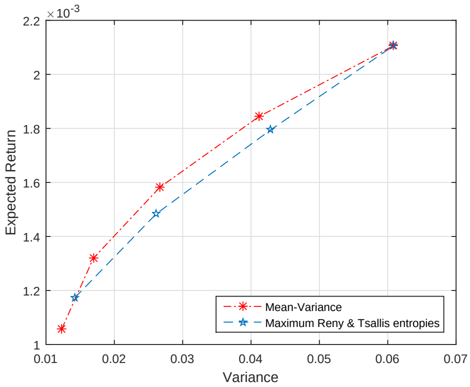


(e) Pal and Pal entropy on NIFTY 50

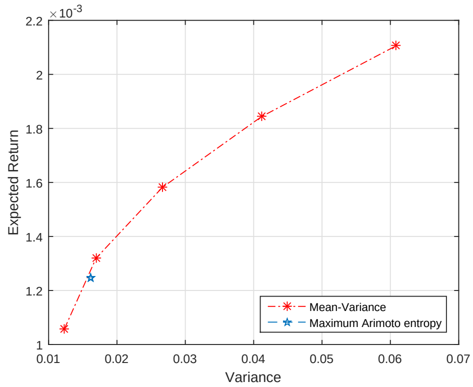


(f) Sine entropy on NIFTY 50

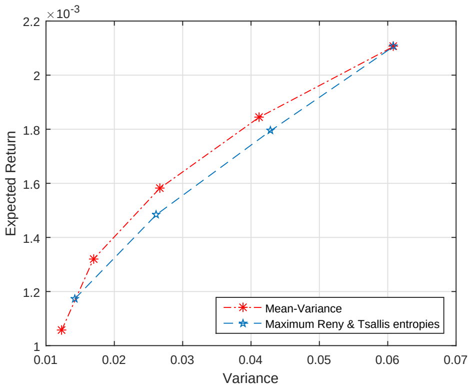
Figure 4.12: Performance of Shannon, Pal and Pal, and Sine entropy measures



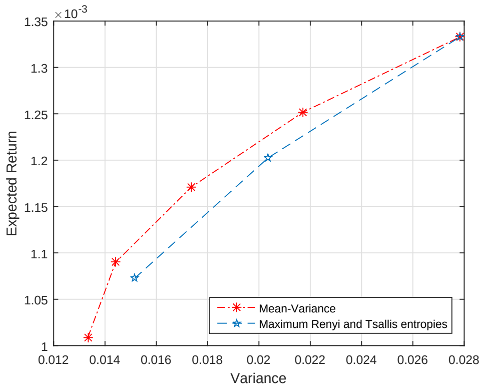
(a) Renyi entropy on BSE SENSEX



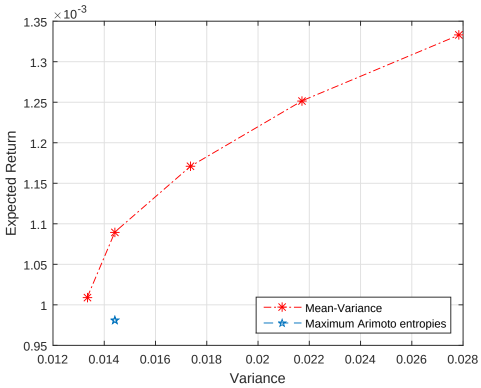
(b) Arimoto entropy on BSE SENSEX



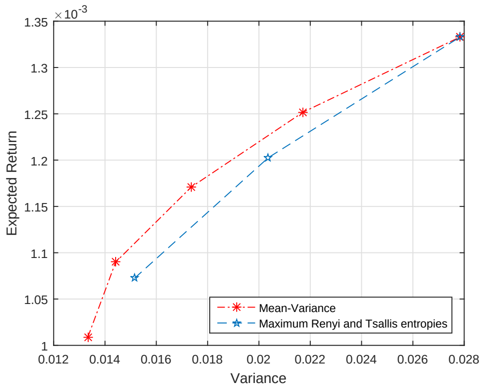
(c) Tsallis entropy on BSE SENSEX



(d) Renyi entropy on NIFTY 50

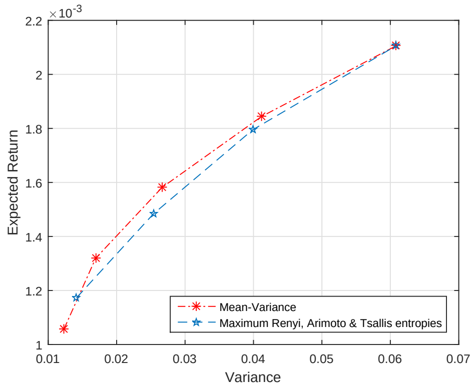


(e) Arimoto entropy on NIFTY 50

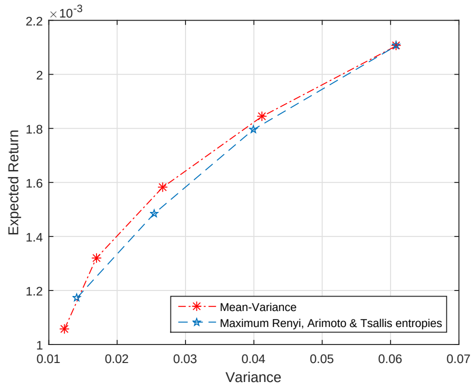


(f) Tsallis entropy on NIFTY 50

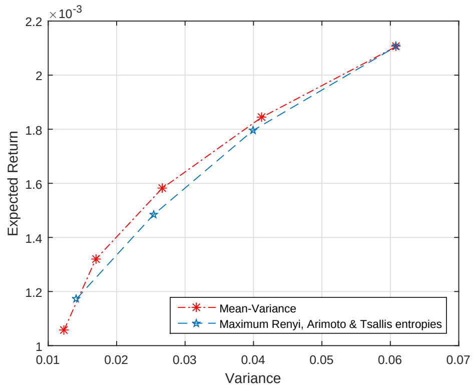
Figure 4.13: Performance of one parametric entropy measures with parameter: 0.005 and 0.01



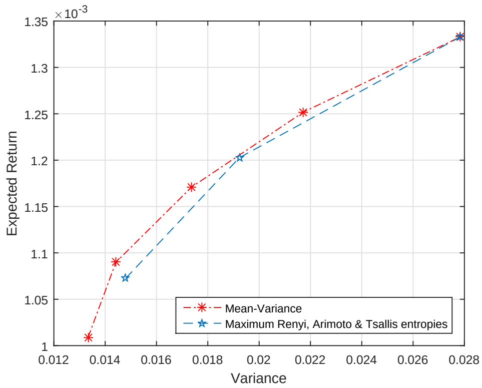
(a) Renyi entropy on BSE SENSEX



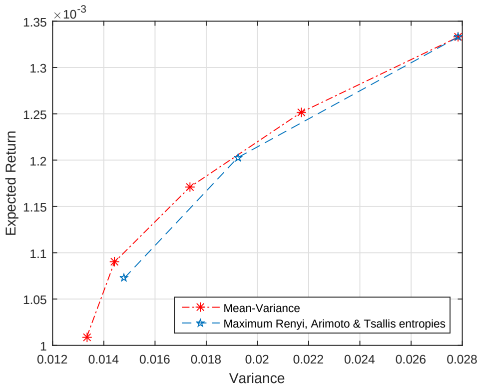
(b) Arimoto entropy on BSE SENSEX



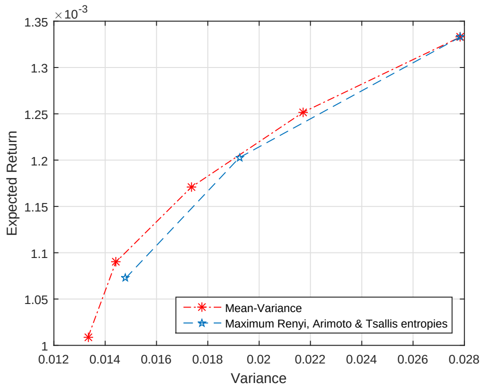
(c) Tsallis entropy on BSE SENSEX



(d) Renyi entropy on NIFTY 50

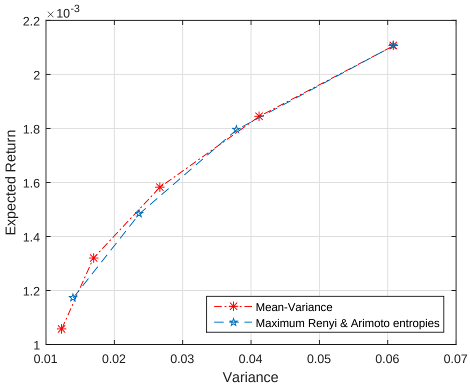


(e) Arimoto entropy on NIFTY 50

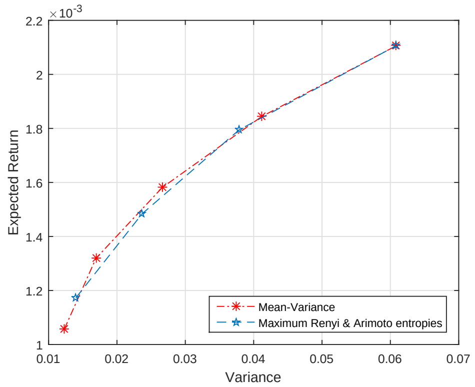


(f) Tsallis entropy on NIFTY 50

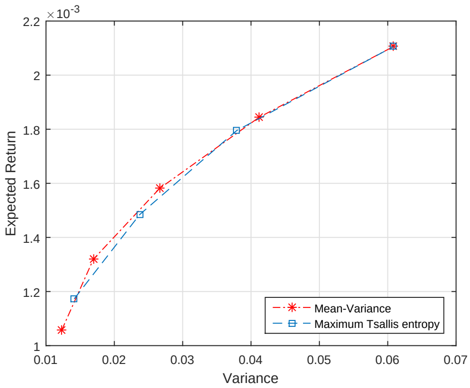
Figure 4.14: Performance of one parametric entropy measures with parameter: 2



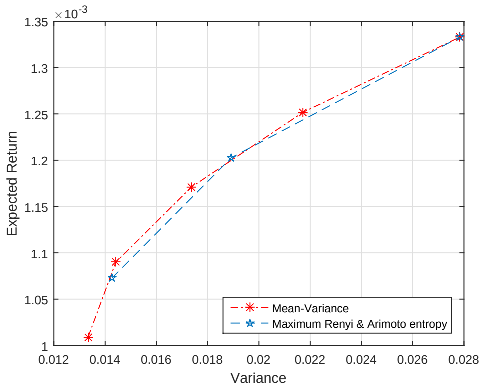
(a) Renyi entropy on BSE SENSEX



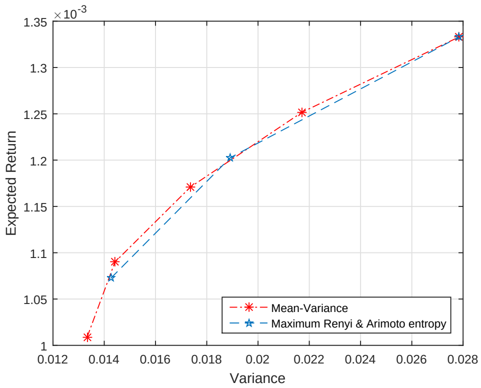
(b) Arimoto entropy on BSE SENSEX



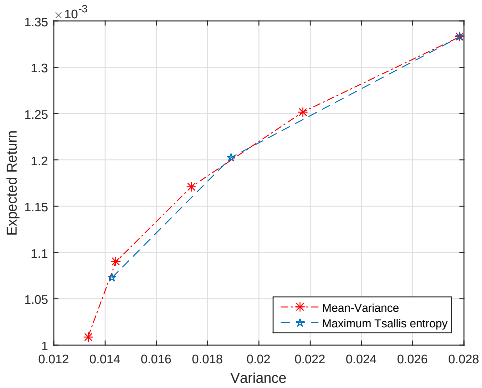
(c) Tsallis entropy on BSE SENSEX



(d) Renyi entropy on NIFTY 50

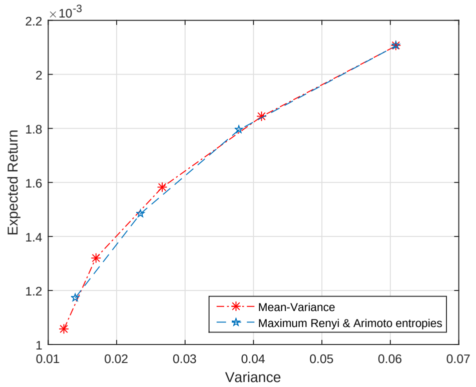


(e) Arimoto entropy on NIFTY 50

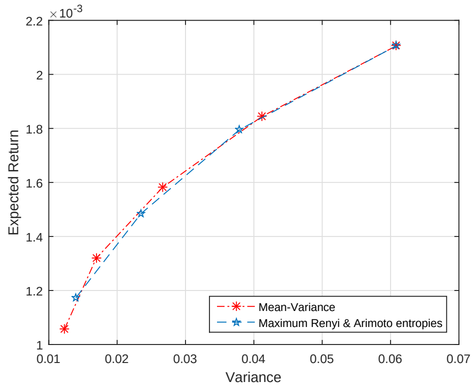


(f) Tsallis entropy on NIFTY 50

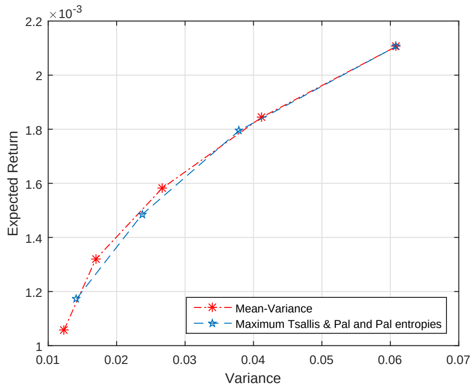
Figure 4.15: Performance of one parametric entropy measures with parameter: 10



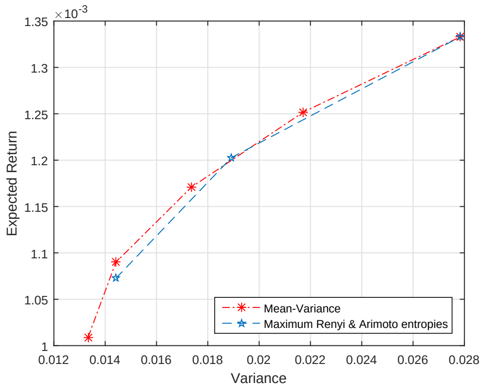
(a) Renyi entropy on BSE SENSEX



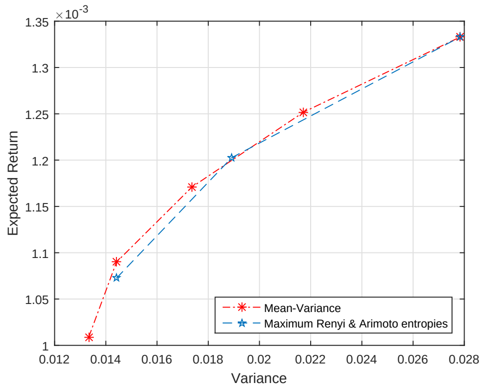
(b) Arimoto entropy on BSE SENSEX



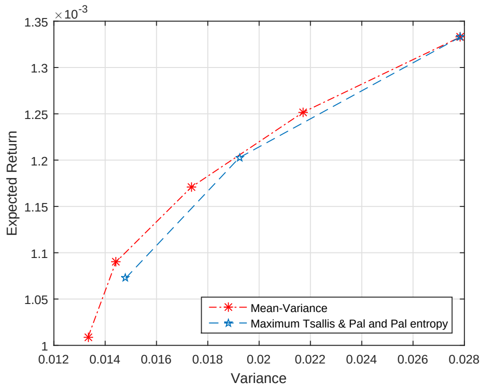
(c) Tsallis entropy on BSE SENSEX



(d) Renyi entropy on NIFTY 50

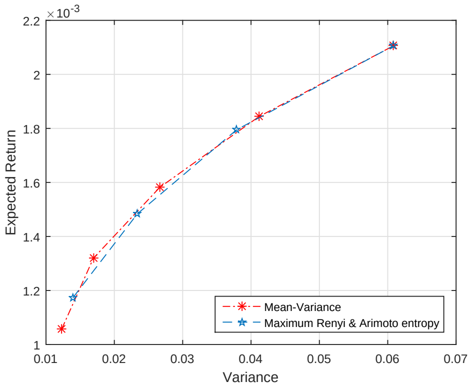


(e) Arimoto entropy on NIFTY 50

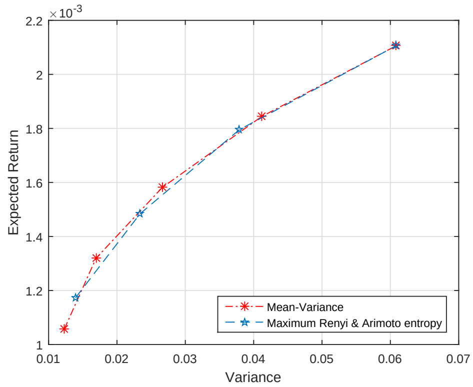


(f) Tsallis entropy on NIFTY 50

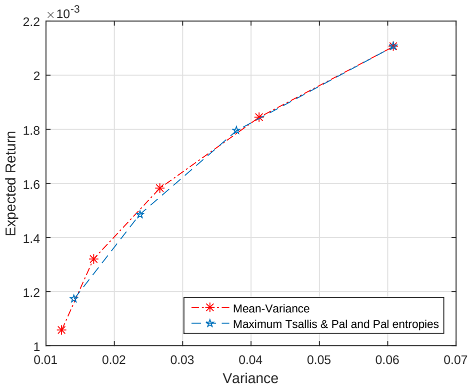
Figure 4.16: Performance of one parametric entropy measures with parameter: 40



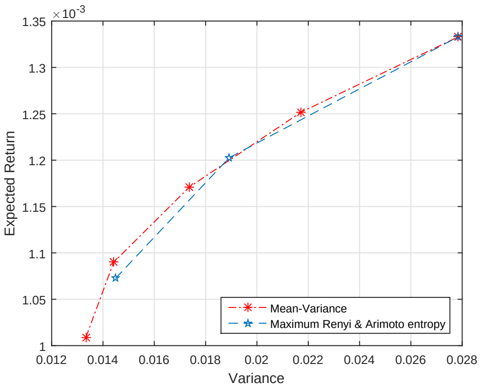
(a) Renyi entropy on BSE SENSEX



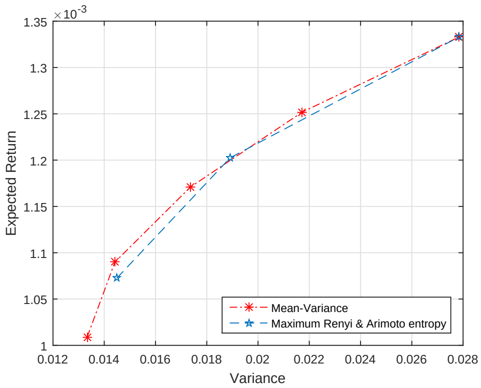
(b) Arimoto entropy on BSE SENSEX



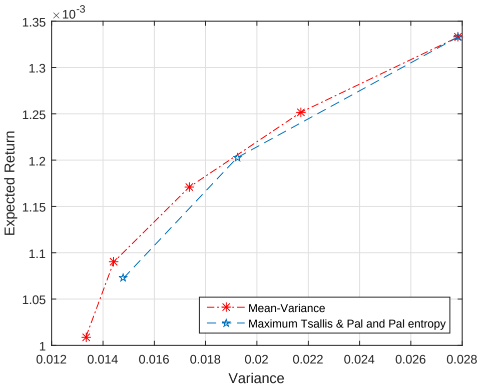
(c) Tsallis entropy on BSE SENSEX



(d) Renyi entropy on NIFTY 50

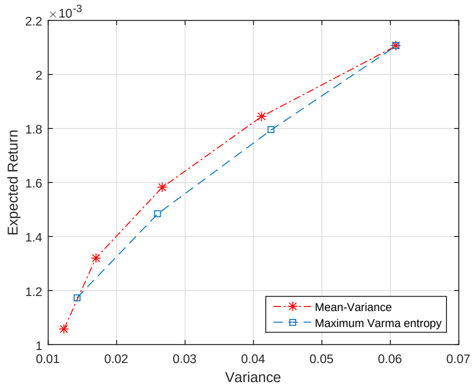


(e) Arimoto entropy on NIFTY 50

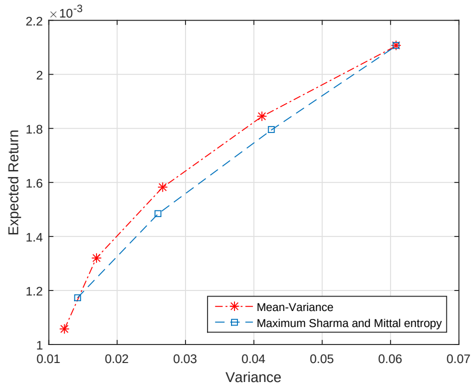


(f) Tsallis entropy on NIFTY 50

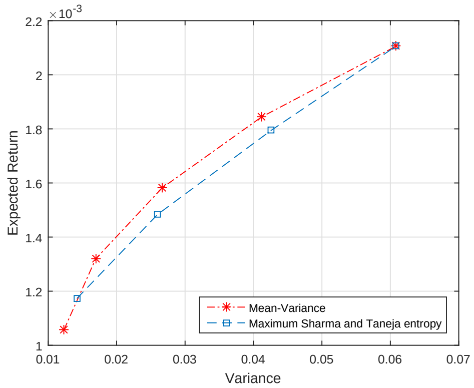
Figure 4.17: Performance of one parametric entropy measures with parameter: 160



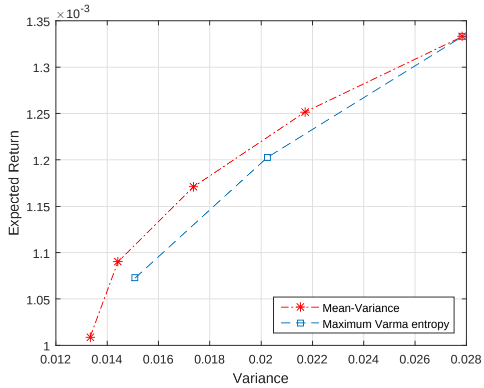
(a) Varma entropy on BSE SENSEX



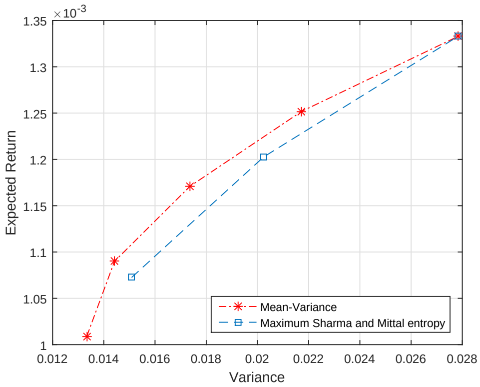
(b) Sharma and Mittal entropy on BSE SENSEX



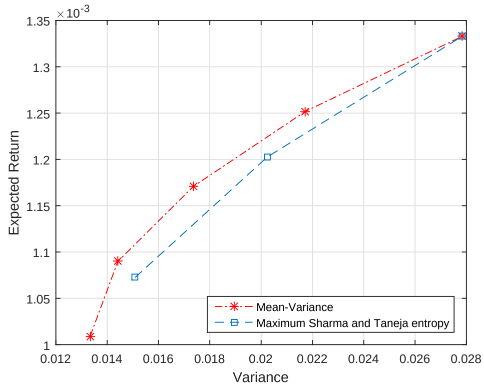
(c) Sharma and Taneja entropy on BSE SENSEX



(d) Varma entropy on NIFTY 50

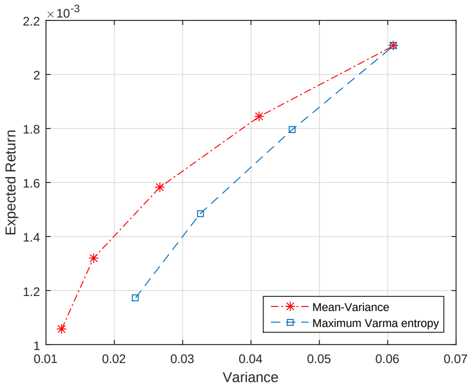


(e) Sharma and Mittal entropy on NIFTY 50

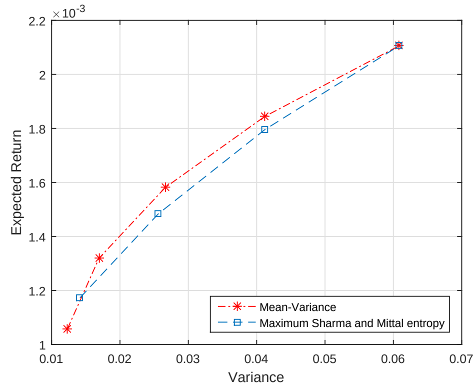


(f) Sharma and Taneja entropy on NIFTY 50

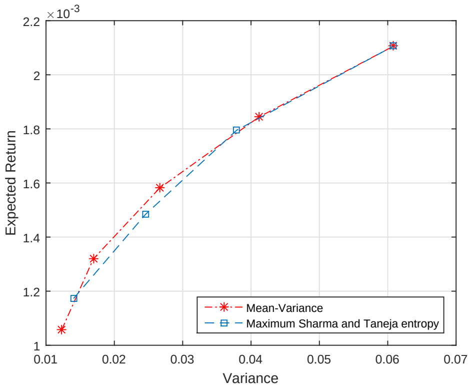
Figure 4.18: Performance of two parametric entropy measures with two parameters: 0.505 & 1.005



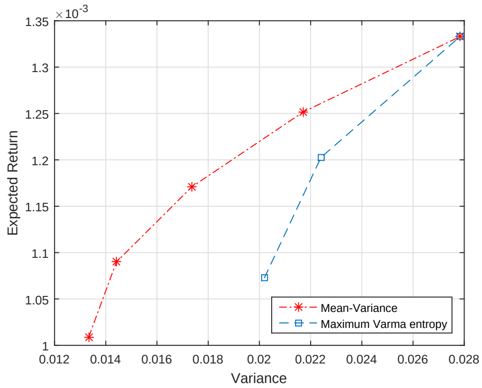
(a) Varma entropy on BSE SENSEX



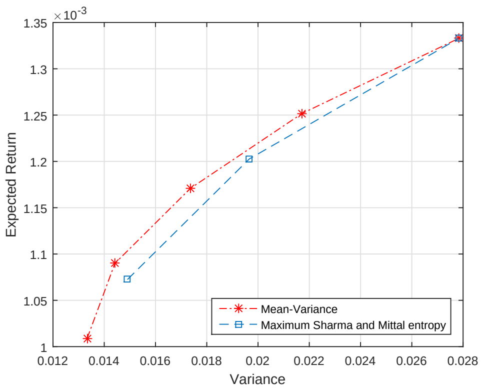
(b) Sharma and Mittal entropy on BSE SENSEX



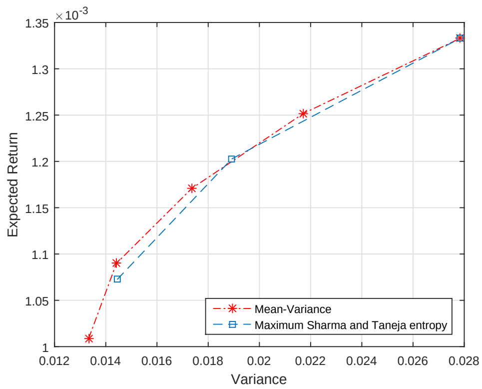
(c) Sharma and Taneja entropy on BSE SENSEX



(d) Varma entropy on NIFTY 50

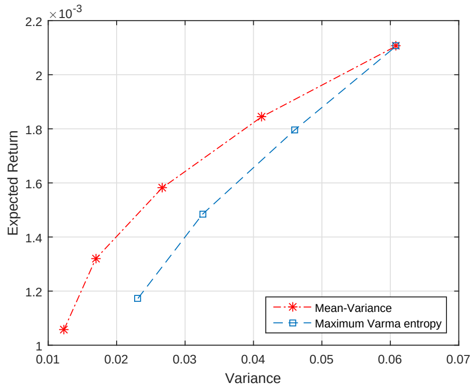


(e) Sharma and Mittal entropy on NIFTY 50

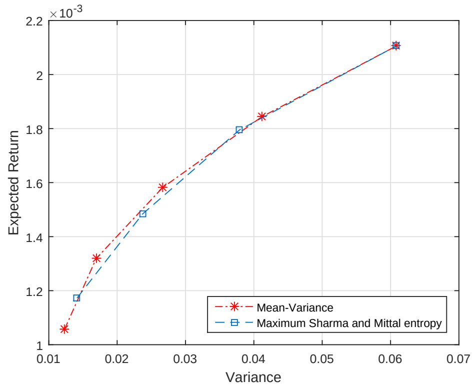


(f) Sharma and Taneja entropy on NIFTY 50

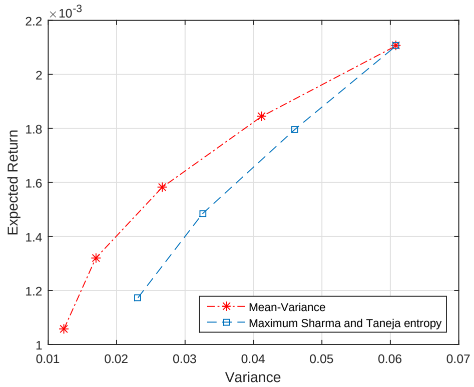
Figure 4.19: Performance of two parametric entropy measures with two parameters: 1.5 & 2



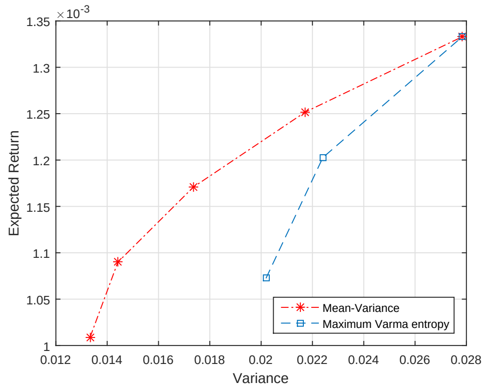
(a) Varma entropy on BSE SENSEX



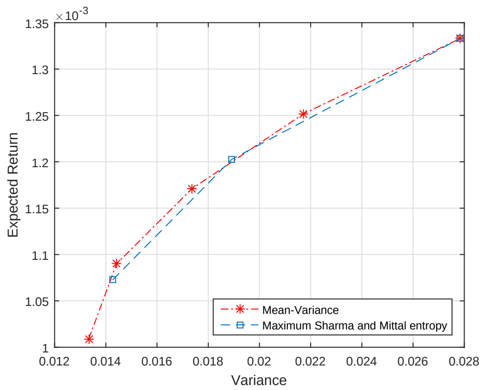
(b) Sharma and Mittal entropy on BSE SENSEX



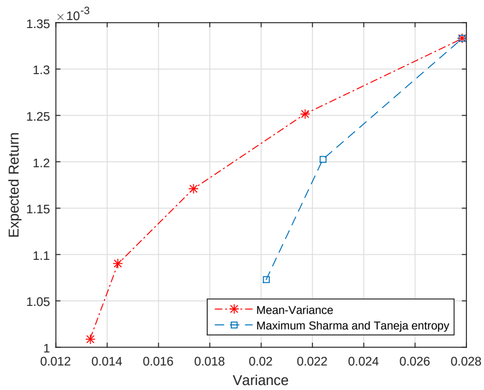
(c) Sharma and Taneja entropy on BSE SENSEX



(d) Varma entropy on NIFTY 50

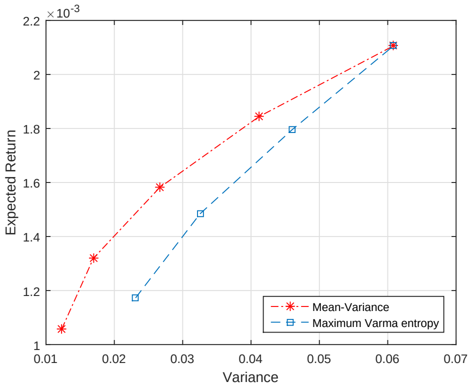


(e) Sharma and Mittal entropy on NIFTY 50

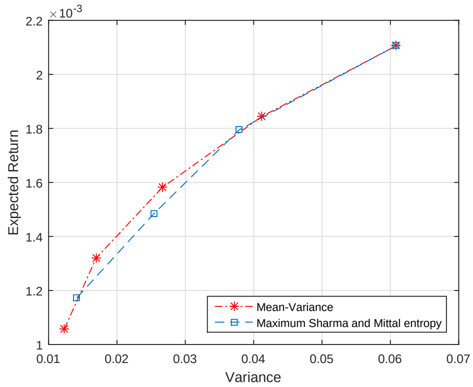


(f) Sharma and Taneja entropy on NIFTY 50

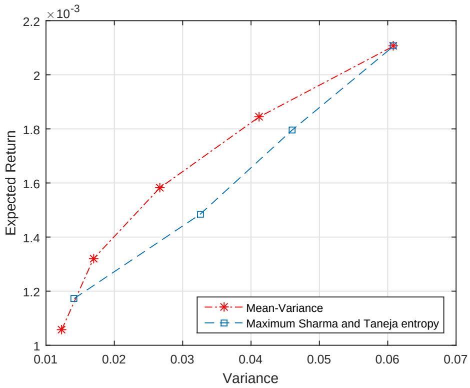
Figure 4.20: Performance of two parametric entropy measures with two parameters: 9.5 & 10



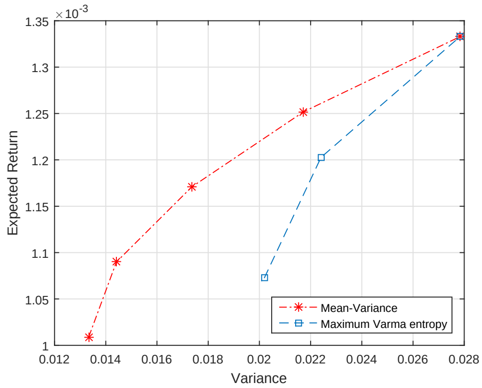
(a) Varma entropy on BSE SENSEX



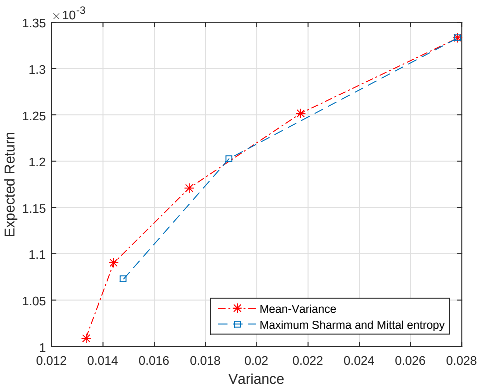
(b) Sharma and Mittal entropy on BSE SENSEX



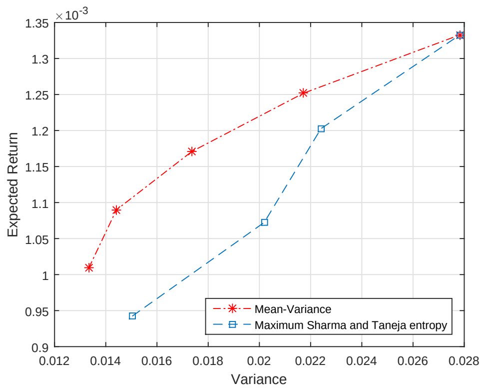
(c) Sharma and Taneja entropy on BSE SENSEX



(d) Varma entropy on NIFTY 50

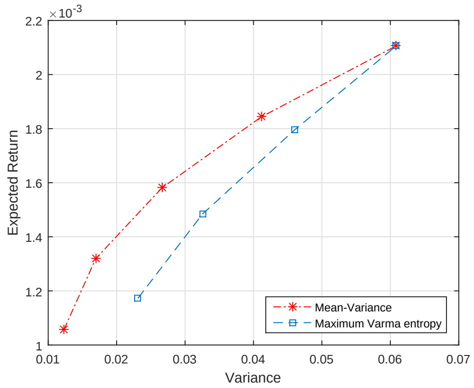


(e) Sharma and Mittal entropy on NIFTY 50

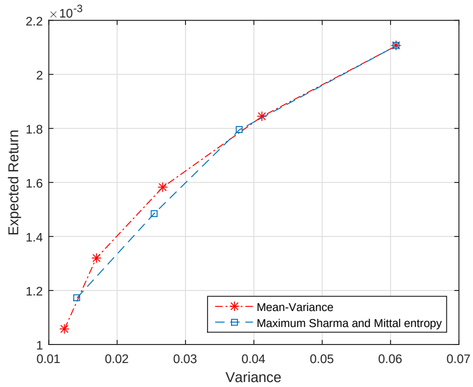


(f) Sharma and Taneja entropy on NIFTY 50

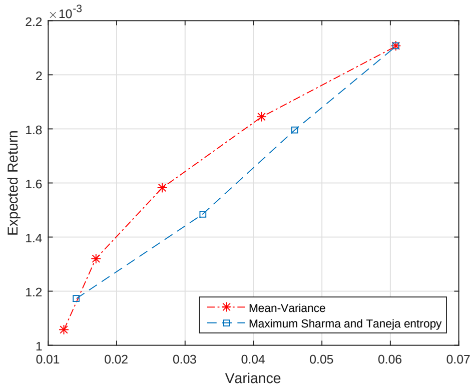
Figure 4.21: Performance of two parametric entropy measures with two parameters: 39.5 & 40



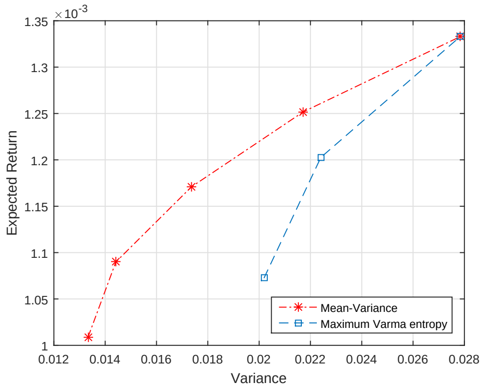
(a) Varma entropy on BSE SENSEX



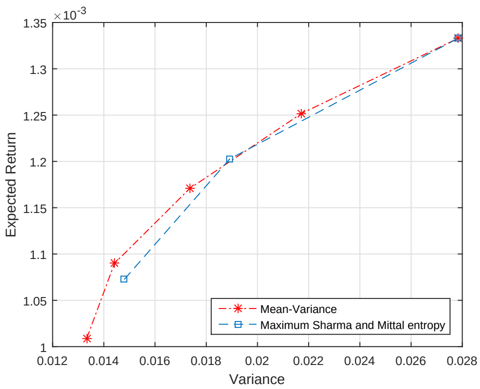
(b) Sharma and Mittal entropy on BSE SENSEX



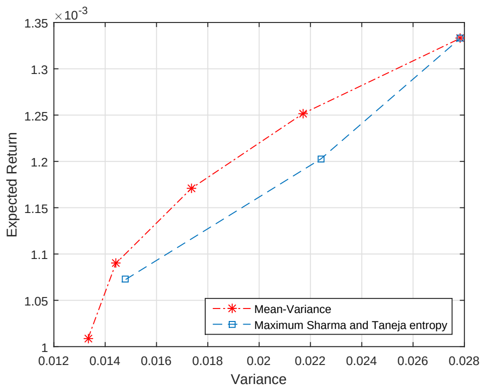
(c) Sharma and Taneja entropy on BSE SENSEX



(d) Varma entropy on NIFTY 50



(e) Sharma and Mittal entropy on NIFTY 50



(f) Sharma and Taneja entropy on NIFTY 50

Figure 4.22: Performance of two parametric entropy measures with two parameters: 79.5 and 159.5 and 319.5 & 80 and 160 and 320

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We observe that Shannon entropy measure offers more diversity with less ARR as compared to others non-generalized entropy portfolio models and MV approach, see Table 4.5. Also, the Arimoto entropy measure offers the most diversification with less ARR as compared to the others one parameter entropy portfolio models and MV approach. As per two parameters entropy measures when one parameter is nearly $1/2$ and other is 1, Varma, Sharma and Mittal, and Sharma and Taneja entropy measures offer high value of diversifications with less ARR as compared to MV method for both the Indian market indices. Next from Figs. 4.12-4.22, we can easily see that the minimum information theoretic measure approach yields a portfolio frontier that is either lower than or equal to the frontier of mean-variance.

As a whole, the minimum entropy model offers higher portfolio returns than the traditional MV approach; classical MV model offers lower variance than all other entropy portfolio models; for perfect diversity, one parameter Arimoto entropy model with $\alpha = 0.005$ and 0.01 is the most diversified among all other portfolio models with different parameters.

4.5 Conclusion

We have considered a class of information theoretical measures based portfolios designed to assess the accuracy of risk assessment. We have analyzed the effectiveness of Markowitz's Mean Variance model in comparison to the model in which the expected portfolio variance has been replaced by measure of information, i.e., the uncertainty of the portfolio allocation to the different assets. According to the study, using some stocks of indices NIFTY50 and BSE SENSEX, the portfolio models with parametric information theoretic measures, with a possible parameter combination, yield well-diversification than the classical MV portfolio model. For the given level of portfolio return while minimizing the risk level, the information theoretic measure approach gives a better idea of risk assessments in portfolio selection problems. However, one should take care of the fact that generalized entropy measures are highly sensitive to the values of their parameters and so the market analysts need to adjust these parametric values as per their risk and return capacities.

Chapter 5

Black-Scholes Model With Stochastic Volatility Using Relative Information

5.1 Introduction

Black and Scholes [12] and Merton [56] reformed the option pricing theory by demonstrating how to hedge continuously the exposure on the option's short position. Merton's significant contribution in the work of Black & Scholes was the arbitrage-based proof of the option pricing formula; for his dynamic programming approach, he began time financial modeling for optimal portfolio and market consumption policies.

Black-Scholes model [12] assumes constant value for volatility over the option duration, but in a real scenario, volatility fluctuates with the level of demand and supply of an asset. Mariani *et al.*citenm derived the generalized form of Black-Scholes model by using Stochastic volatility and transaction costs. An information-theoretic measure approach to derive the Black-Scholes equation has been given by Abedi *et al.* [1], in which they have used the method of minimizing Kullback relative entropy measure. Entropy modeling complements the stochastic process modeling, the advantage of which is that it can unify the different models

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that have been developed across various fields of engineering and science. Many authors have worked on the information theoretic approach to study the aspect of volatility in the financial market [6, 7, 28, 48].

In this chapter, we apply the entropy stock dynamics to model the price of European options and for this, we have derived the risk-neutral probability density function of the asset option price and volatility within the information-theoretic dynamic formalism. In addition, we specify how the probability function changes over a period of time modelled as a Kolmogorov-backward equation. We arrive at the Black-Scholes option pricing model [50] when the volatility is a traded asset, i.e. we derive a generalized Black-Scholes differential equation using the information-theoretic approach. The equation is generalized in the sense that the market volatility which is taken as deterministic (constant) in the Black-Scholes formula, has been taken here stochastic in nature. Further, we propose the generalized Black-Scholes model based on the Liouville-Caputo-type fractional derivative [41], as time-fractional derivative approach is better than the standard Black Scholes model [54]. We use the Laplace Homotopy perturbation method [88] to find the approximate analytical solution of the time fractional Black-Scholes equation in the form of a convergent series.

The chapter is organised as follows: A brief overview of the stochastic volatility model is given in Section 5.2. In Section 5.3, we briefly highlight the dynamics of the stock options price and stock volatility and derive the risk-neutral probability function by integrating the risk-neutral information measure. In Section 5.4, the generalized second-order partial differential equation is derived by taking derivative of expected payoff with respect to time at maturity. The approximate analytical solution of the Black Scholes equation is carried out in Section 5.5. Section 5.6 gives numerical demonstration and discussion and Section 5.7 concludes the chapter.

5.2 Stochastic Volatility Model

The fundamental assumption in modeling the asset using a random walk motion is that the volatility is deterministic over time which is an impractical assumption. Any model where the volatility is unpredicted or random is considered as a stochastic volatility model. The given market is incomplete and free of arbitration and we do not take into account that jumps can happen, as this will result in non-linear partial differential equations which leads to complexity. But one may consider jumps and can switch to work on alternate approaches of stochastic volatility model [45] (i.e. jump-diffusion model processes or more general Levy processes). Change in price S and volatility σ of a stock can be defined by considering the stochastic volatility model as

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma(t) dX_1(t), \tag{5.2.1}$$

and

$$\frac{d\sigma(t)}{\sigma(t)} = \alpha dt + \beta dX_2(t), \tag{5.2.2}$$

where, $\mu, \alpha \in (-\infty, \infty)$ are the constants and the two Brownian motions $X_1(t)$ and $X_2(t)$ are correlated with correlation coefficient ρ, that is

$$E(dX_1(t)dX_2(t)) = \rho dt, \tag{5.2.3}$$

refer to [37]. Here β is constant coefficient of Brownian motion $X_2(t)$ and σ, the volatility index works as a perfect proxy for the stochastic volatility.

5.3 Entropy Probabilistic Dynamics of Options

In finance, the simplest model of prices and volatility of stock is lognormal distribution. We consider the dynamics of stock price S and volatility σ as:

$$f(S) = log(S) \tag{5.3.1}$$

$$f(\sigma) = log(\sigma) \tag{5.3.2}$$

and scale invariance in stochastic processes is defined as

$$P(f(\lambda k)) = \lambda^{\theta} P(f(k)) \text{ where } \theta = \begin{cases} 0, & \text{white noise} \\ -1, & \text{pink noise} \\ -2, & \text{brownian noise} \end{cases} \tag{5.3.3}$$

P(f) stands for expected power of the frequency *f*; λ a real constant, is scaling factor and k is any arbitrary variable. We consider the case of white noise in the stock market, therefore take $\theta = 0$. The reduced form of (5.3.3) is given as

$$P(f(\lambda k)) = P(f(k)) \tag{5.3.4}$$

It can be shown that the probability density of price and volatility functions are invariant under the scaling transformation. The probability density distribution would be a scalar function under the scaling transformation of the stock price and volatility [1].

Next, by using Taylor's expansion upto second order accuracy we consider change in price fuction (5.3.1) as

$$log\frac{\dot{S}(t)}{S(t)} \approx \frac{dS(t)}{S(t)} - \frac{1}{2}\left(\frac{dS(t)}{S(t)}\right)^2 \tag{5.3.5}$$

and change in current volatility function (5.3.2) as

$$log\frac{\dot{\sigma}(t)}{\sigma(t)} \approx \frac{d\sigma(t)}{\sigma(t)} - \frac{1}{2}\left(\frac{d\sigma(t)}{\sigma(t)}\right)^2 \tag{5.3.6}$$

where, $\dot{S}(t) = S(t) + dS(t)$ and $\dot{\sigma}(t) = \sigma(t) + d\sigma(t)$ stand for current stock price and stock volatility, repectively at time $t + dt$. Further, changes in price S and volatlity σ of a stock can be defined by considering the stochastic volatility model as given in Section 5.2. By squaring (5.2.1) and (5.2.2), for $dX^2 \to dt$ as $dt \to 0$ we get

$$\left(\frac{dS(t)}{S(t)}\right)^2 = \sigma^2 dt \tag{5.3.7}$$

$$\left(\frac{d\sigma(t)}{\sigma(t)}\right)^2 = \beta^2 dt \tag{5.3.8}$$

Now using (5.2.1), (5.3.7) in (5.3.5) and (5.2.2), (5.3.8) in (5.3.6) respectively, we obtain the information pertaining to change in prices and change in volatility:

$$log\frac{\dot{S}(t)}{S(t)} \approx \mu dt + \sigma(t)dX_1(t) - \frac{1}{2}\sigma^2 dt \tag{5.3.9}$$

$$log\frac{\dot{\sigma}(t)}{\sigma(t)} \approx \alpha dt + \beta dX_2(t) - \frac{1}{2}\beta^2 dt \tag{5.3.10}$$

This information is applicable to dynamic variables S and σ and is implemented in the form of constraints, where higher order terms of *dt* tends to zero.

Next, assigning the normal distribution of the log price $P_S(log\dot{S}|logS)$ and log stock volatility $P_\sigma(log\dot{\sigma}|log\sigma)$ require utilizing the Kullback measure of relative information [40] which can be defined separately as

$$K_S(P||Q) = \int P_S(log\dot{S}|logS)\ log\frac{P_S(log\dot{S}|logS)}{Q_S(log\dot{S}|logS)}d\ log\dot{S} \tag{5.3.11}$$

and

$$K_\sigma(P||Q) = \int P_\sigma(log\dot{\sigma}|log\sigma)\ log\frac{P_\sigma(log\dot{\sigma}|log\sigma)}{Q_\sigma(log\dot{\sigma}|log\sigma)}d\ log\dot{\sigma} \tag{5.3.12}$$

where $Q_S(log\dot{S}|logS)$ and $Q_\sigma(log\dot{\sigma}|log\sigma)$ are the prior probability distributions corresponding to stock price and stock volatility, respectively.

Instead of minimizing Kullback measure of relative entropy $K(P||Q)$, we will maximize $-K(P||Q)$. Subject to normalization factor N and (5.3.9) as the prior information constraint we obtain the transition probability of the stock price S as

$$P_S(log\dot{S}|logS) = \frac{e^{-\frac{\zeta}{2}\left(log\frac{\dot{S}}{S}\right)^2+\eta\left(log\frac{\dot{S}}{S}\right)}}{N(\zeta,\eta,logS)}; \tag{5.3.13}$$

This can be rewritten as

$$P_S(log\dot{S}|logS) = \frac{e^{-\frac{1}{2\sigma^2 dt}\left(log\dot{S}-(logS+\mu dt+\sigma(t)dX_1(t)-\frac{1}{2}\sigma^2 dt)\right)^2}}{N(\zeta,\eta,logS)} \tag{5.3.14}$$

where $\zeta = (\sigma^2 dt)^{-1}$ and $\eta = \frac{1}{2}\left(\frac{2\mu}{\sigma^2}-1\right)$ are Lagrange multipliers corresponding to the constraint (5.3.9) and normalization factor $N = \int e^{-\frac{\zeta}{2}\left(log\frac{\dot{S}}{S}-\frac{\eta}{\zeta}\right)^2} d\ log\dot{S}$.
Similarly, for the case of stock volatility, we consider (5.3.10), the prior information, as constraint and obtain transition probability function of the stock volatility σ as

$$P_\sigma(log\dot{\sigma}|log\sigma) = \frac{e^{-\frac{1}{2\sigma^2 dt}\left(log\dot{\sigma}-(log\sigma+\alpha dt+\beta(t)dX_2(t)-\frac{1}{2}\sigma^2 dt)\right)^2}}{F(\lambda,\xi,log\sigma)} \tag{5.3.15}$$

where $\lambda = (\sigma^2 dt)^{-1}$ and $\xi = \frac{1}{2}\left(\frac{2\alpha}{\beta^2}-1\right)$ are Lagrange multipliers corresponding to the constraint (5.3.10) and normalization factor $F = \int e^{-\frac{\lambda}{2}\left(log\frac{\dot{\sigma}}{\sigma}-\frac{\xi}{\lambda}\right)^2} d\ log\dot{\sigma}$.

Next by making the transformation from log price back to price of a stock option as

$$P_S(\dot{S}|S) = \frac{1}{\dot{S}} P(log\dot{S}|logS) \tag{5.3.16}$$

and from log stock volatility back to volatility of a stock as

$$P_\sigma(\dot{\sigma}|\sigma) = \frac{1}{\dot{\sigma}} P(log\dot{\sigma}|log\sigma), \tag{5.3.17}$$

the stock price probability is given by

$$P_S(\dot{S}|S) = \frac{e^{-\frac{1}{2\sigma^2 dt}\left(log\dot{S}-(logS+\mu dt+\sigma(t)dX_1(t)-\frac{1}{2}\sigma^2 dt)\right)^2}}{N(\zeta,\eta,logS)\ \dot{S}} \tag{5.3.18}$$

This is the log-normal distribution for very small change in time interval dt in option price S. Similarly probability of the volatility function

$$P_\sigma(\dot{\sigma}|\sigma) = \frac{e^{-\frac{1}{2\sigma^2 dt}\left(log\dot{\sigma}-(log\sigma+\alpha dt+\beta(t)dX_2(t)-\frac{1}{2}\beta^2 dt)\right)^2}}{F(\lambda,\xi,log\sigma)\ \dot{\sigma}} \tag{5.3.19}$$

is the log-normal distribution for change in stock volatility σ.

Next, by using risk-neutral measure, we simply estimate the option's value at maturity by its expected payoff. With r as a risk free rate and the risk-neutrality constraint given as

$$\mu(t) = \alpha = r, \tag{5.3.20}$$

the risk neutral probability function of stock price by imposing the constraint (5.3.20) is as follows

$$P(log\dot{S}|logS) = \frac{e^{-\frac{1}{2\sigma^2 dt}\left(log\dot{S}-(logS+rdt+\sigma(t)dX_1(t)-\frac{1}{2}\sigma^2 dt)\right)^2}}{N(\zeta,\eta,logS)} \tag{5.3.21}$$

and the risk neutral probability density of stock volatility is

$$P(log\dot{\sigma}|log\sigma) = \frac{e^{-\frac{1}{2\sigma^2 dt}\left(log\dot{\sigma}-(log\sigma+rdt+\beta(t)dX_2(t)-\frac{1}{2}\sigma^2 dt)\right)^2}}{F(\lambda,\xi,log\sigma)}. \tag{5.3.22}$$

By using the risk-neutral probability to value the European call and put option, with E as the strike price and S_0 as the current stock price, the required expected payoff at maturity is given by

$$V(S,\sigma,T) = max(S-E,0) \tag{5.3.23}$$

where, the expected payoff for the call option at maturity denoted by V_C is given as

$$V_C(S,\sigma,T) = \int_E^\infty (S-E)\ P(S,T|S_0)dS \tag{5.3.24}$$

and the expected payoff at the maturity for a put option denoted by V_P is given as

$$V_P(S,\sigma,T) = \int_0^E (S-E)\ P(S,T|S_0)dS \tag{5.3.25}$$

Here, we look into the aspect that how the probability distribution in certain state changes over time which comes out to be Kolmogorov-backward equation. We use entropic instant which incorporates both option price and volatility, defined as

$$p(log\dot{S},log\dot{\sigma},\dot{t}) = \int \left[p(logS,log\sigma,t)\ P(log\dot{S}|logS,log\dot{\sigma}|log\sigma)\right]\ dlogS(t)\ dlog\sigma(t) \tag{5.3.26}$$

where $p(logS,log\sigma,t)$ and $p(log\dot{S},log\dot{\sigma},\dot{t})$ are distributions representing all information available at an instant time t and the next instant time $\dot{t} = t+dt$, respectively. We have a generalized Kolmogorov-backward equation for the probability density $p(logS,log\sigma,t)$, a differential form of (5.3.26):

$$\begin{aligned}\frac{\partial}{\partial t}p(logS,log\sigma,t) = &-\left(\mu-\frac{\sigma^2}{2}\right)\frac{\partial}{\partial logS}p(logS,t)-\left(\alpha-\frac{\beta^2}{2}\right)\frac{\partial}{\partial log\sigma}p(log\sigma,t)\\ &-\frac{1}{2}\left[\sigma^2\frac{\partial^2}{\partial(logS)^2}p(logS,t)+\beta^2\frac{\partial^2}{\partial(log\sigma)^2}p(log\sigma,t)\right]\\ &-\rho\sigma\beta\frac{\partial^2}{\partial logS\ \partial log\sigma}p(logS,log\sigma,t),\end{aligned} \tag{5.3.27}$$

to be used for the derivation of the generalized Black-Scholes equation in the next section.

5.4 Derivation of Black-Scholes Equation with Stochastic Volatility

In order to derive the generalized Black-Scholes equation with stochastic volatility, we begin with the payoff equation as

$$V(logS, log\sigma, E, t) = \int (S-E)\ P(log\dot{S}|logS, log\dot{\sigma}|log\sigma)\ dlogS(\dot{t})\ dlog\sigma(\dot{t}). \quad (5.4.1)$$

The time derivate of value of an option is given as

$$\frac{\partial}{\partial t}V(logS, log\sigma, E, t) = \int (S-E)\ \frac{\partial}{\partial t}P(log\dot{S}|logS, log\dot{\sigma}|log\sigma)\ dlogS(\dot{t})\ dlog\sigma(\dot{t}) \quad (5.4.2)$$

Next by using (5.3.27), we get

$$\begin{aligned}\frac{\partial}{\partial t}P\big(log\dot{S}|logS, log\dot{\sigma}|log\sigma\big) &= -\left(r-\frac{\sigma^2}{2}\right)\frac{\partial}{\partial logS}P(log\dot{S}|logS)\\ &-\left(r-\frac{\beta^2}{2}\right)\frac{\partial}{\partial log\sigma}P(log\dot{\sigma}|log\sigma)\\ &-\frac{1}{2}\left[\sigma^2\frac{\partial^2}{\partial(logS)^2}P(log\dot{S}|logS)+\beta^2\frac{\partial^2}{\partial(log\sigma)^2}P(log\dot{\sigma}|log\sigma)\right]\\ &-\rho\sigma\beta\frac{\partial^2}{\partial logS\ \partial log\sigma}P(log\dot{S}|logS, log\dot{\sigma}|log\sigma, t).\end{aligned} \quad (5.4.3)$$

Using (5.4.3) in (5.4.2), we get

$$\begin{aligned}\frac{\partial V}{\partial t} = \int (S-E)\Bigg\{&\left[-\left(r-\frac{\sigma^2}{2}\right)\frac{\partial}{\partial logS}P(log\dot{S}|logS)\right]\\ &+\left[-\left(r-\frac{\beta^2}{2}\right)\frac{\partial}{\partial log\sigma}P(log\dot{\sigma}|log\sigma)\right]\\ &-\frac{1}{2}\left[\sigma^2\frac{\partial^2}{\partial(logS)^2}P(log\dot{S}|logS)+\beta^2\frac{\partial^2}{\partial(log\sigma)^2}P(log\dot{\sigma}|log\sigma)\right]\\ &-\rho\sigma\beta\frac{\partial^2}{\partial logS\ \partial log\sigma}P(log\dot{S}|logS, log\dot{\sigma}|log\sigma)\Bigg\}\ dlogS(\dot{t})\ dlog\sigma(\dot{t}).\end{aligned}$$

This implies

$$\begin{aligned}\frac{\partial V}{\partial t} = &-\left(r-\frac{\sigma^2}{2}\right)\frac{\partial V}{\partial logS}-\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial(logS)^2}-\left(r-\frac{\beta^2}{2}\right)\frac{\partial V}{\partial log\sigma}\\ &-\frac{\beta^2}{2}\frac{\partial^2 V}{\partial(log\sigma)^2}-\frac{\partial^2 V}{\partial logS\partial log\sigma}\rho\sigma\beta\end{aligned}$$

$$\frac{\partial V}{\partial t}+\frac{\sigma^2S^2}{2}\frac{\partial^2 V}{\partial S^2}+\frac{\beta^2\sigma^2}{2}\frac{\partial^2 V}{\partial \sigma^2}+rS\frac{\partial V}{\partial S}+r\sigma\frac{\partial V}{\partial \sigma}+\frac{\partial^2 V}{\partial S\partial \sigma}\rho\sigma^2\beta S=0 \tag{5.4.4}$$

Now substitute $B=Ve^{-r(T-t)}$, which stands for both put and call options, in (5.4.4); we have the required partial differential equation

$$\frac{\partial B}{\partial t}+\frac{\sigma^2S^2}{2}\frac{\partial^2 B}{\partial S^2}+\frac{\beta^2\sigma^2}{2}\frac{\partial^2 B}{\partial \sigma^2}+rS\frac{\partial B}{\partial S}+r\sigma\frac{\partial B}{\partial \sigma}+\frac{\partial^2 B}{\partial S\partial \sigma}\rho\sigma^2\beta S-rB=0 \tag{5.4.5}$$

For $\rho\in(-1,1)$, the above equation is parabolic second-order partial differential equation. Next, we need to impose appropriate boundary conditions to this problem to solve the equation for both the call and put options.

5.5 Approximate Analytical Solution

In this secton, we find the approximate analytical solution of (5.4.5) by Laplace transform homotopy perturbation method, where fractional derivatives are described in the sense of the Liouville-Caputo fractional derivative. Consider (5.4.5), for $S,\sigma\in[0,\infty)$ and $t\in[0,T]$

$$\frac{\partial B}{\partial t}+\frac{\sigma^2S^2}{2}\frac{\partial^2 B}{\partial S^2}+\frac{\beta^2\sigma^2}{2}\frac{\partial^2 B}{\partial \sigma^2}+rS\frac{\partial B}{\partial S}+r\sigma\frac{\partial B}{\partial \sigma}+\frac{\partial^2 B}{\partial S\partial \sigma}\rho\sigma^2\beta S-rB=0 \tag{5.5.1}$$

with terminal condition:

$$B(S,\sigma,T)=max\{\varepsilon_1S+\varepsilon_2\sigma-E,0\} \tag{5.5.2}$$

and boundary conditions:

$$B(S,\sigma,t)=\begin{cases}0, & as\ (S,\sigma)\to(0,0)\\ \varepsilon_1S+\varepsilon_2\sigma-Ee^{-r(T-t)}, & as\ (S,\sigma)\to(\infty,\infty).\end{cases} \tag{5.5.3}$$

where ε_1 and ε_2 are the constant coefficients for the asset prices S and volatility σ, respectively. To analyse (5.5.1) we use the transformations

$$\begin{aligned} x &= log_e(S) - rt + \frac{\sigma^2 t}{2}, & (5.5.4) \\ y &= log_e(\sigma) - rt + \frac{\beta^2 t}{2} & (5.5.5) \end{aligned}$$

This gives $x, y \in (-\infty, \infty)$ and $t \in [0, T]$ for $S, \sigma \in [0, \infty)$, (5.5.1) transforms to

$$\frac{\partial B}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 B}{\partial x^2} + \frac{\beta^2}{2}\frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 B}{\partial x \partial y}\rho\sigma\beta - rB = 0 \qquad (5.5.6)$$

with terminal condition:

$$B(x,y,T) = max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}} + \varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}} - E, 0\} \qquad (5.5.7)$$

and boundary conditions:

$$B(x,y,t) = \begin{cases} 0, & as\ (x,y) \to (-\infty, -\infty) \\ \varepsilon_1 e^{x+rt-\frac{\sigma^2 t}{2}} + \varepsilon_2 e^{y+rt-\frac{\beta^2 t}{2}} - Ee^{-r(T-t)}, & as\ (x,y) \to (\infty, \infty). \end{cases} \qquad (5.5.8)$$

Using $V(x,y,t) = e^{-r(T-t)}B(x,y,t)$ for $x, y \in (-\infty, \infty)$ and $t \in [0, T]$ in (5.5.6), we obtain

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x \partial y}\rho\sigma\beta = 0 \qquad (5.5.9)$$

with terminal condition:

$$V(x,y,T) = max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}} + \varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}} - E, 0\} \qquad (5.5.10)$$

and boundary conditions:

$$V(x,y,t) = \begin{cases} 0, & as\ (x,y) \to (-\infty, -\infty) \\ \varepsilon_1 e^{x+rT-\frac{\sigma^2 t}{2}} + \varepsilon_2 e^{y+rT-\frac{\beta^2 t}{2}} - E, & as\ (x,y) \to (\infty, \infty). \end{cases} \qquad (5.5.11)$$

Further, we transform the final boundary value problem to an initial boundary value problem by changing the forward time variable from t to $\tau = T - t$ in (5.5.9), and for $x, y \in (-\infty, \infty)$ and $\tau \in [0, T]$ we get

$$-\frac{\partial V}{\partial \tau} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x \partial y}\rho\sigma\beta = 0 \tag{5.5.12}$$

with initial condition:

$$V(x,y,0) = max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}} + \varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}} - E, 0\} \tag{5.5.13}$$

and boundary conditions:

$$V(x,y,\tau) = \begin{cases} 0, & as\ (x,y) \to (-\infty,-\infty) \\ \varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}+\frac{\sigma^2\tau}{2}} + \varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}+\frac{\beta^2\tau}{2}} - E, & as\ (x,y) \to (\infty,\infty). \end{cases} \tag{5.5.14}$$

Replacing $\frac{\partial V}{\partial \tau}$ by $\mathscr{L}_\tau^\lambda V$, the Liouville-Caputo [41] type time-fractional derivative of order $\lambda \in (0,1]$, we have the subsequent fractional Black Scholes equation for $x, y \in (-\infty, \infty)$ and $\tau \in [0, T]$, given by

$$\mathscr{L}_\tau^\lambda V = \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x \partial y}\rho\sigma\beta \tag{5.5.15}$$

with initial condition:

$$V(x,y,0) = max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}} + \varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}} - E, 0\} \tag{5.5.16}$$

and boundary conditions:

$$V(x,y,\tau) = \begin{cases} 0, & as\ (x,y) \to (-\infty,-\infty) \\ \varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}+\frac{\sigma^2\tau}{2}} + \varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}+\frac{\beta^2\tau}{2}} - E, & as\ (x,y) \to (\infty,\infty). \end{cases} \tag{5.5.17}$$

Next we find the solution of (5.5.15) subject to conditions (5.5.16)-(5.5.17) by using the Laplace transform of the Liouville-Caputo fractional derivative [57, 88], given as

$$L[V(x,y,\tau)] = \frac{1}{s}max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}} + \varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}} - E, 0\} + \frac{1}{s^\lambda}L\left[\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x \partial y}\rho\sigma\beta\right]. \tag{5.5.18}$$

By using the inverse Laplace transformation in (5.5.18), we obtain

$$\begin{aligned} V(x,y,\tau) =& L^{-1}\left[\frac{1}{s}max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}}+\varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}}-E,0\}+\right. \\ & \left.\frac{1}{s^\lambda}L\left[\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}+\frac{\beta^2}{2}\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial x \partial y}\rho\sigma\beta\right]\right] \end{aligned} \tag{5.5.19}$$

Further, by using Homotopy perturbation method [31, 32], we construct a real valued homotopy $V(x,y,\tau,p)$ defined on $(-\infty,\infty)\times(-\infty,\infty)\times[0,T]\times[0,1]$ which satisfies the following

$$(1-p)(V(x,y,\tau,p)-V_0(x,y,\tau))+p(V(x,y,\tau,p)-V(x,y,\tau))=0 \tag{5.5.20}$$

where $p\in[0,1]$ is an impending parameter and V_0 is the initial approximation [4], defined as $V_0(x,y,\tau)=V(x,y,0)+\tau^\lambda e^{(x+y)}$. Substituting in (5.5.20), we have

$$\begin{aligned} V(x,y,\tau,p) =& max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}}+\varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}}-E,0\}+(1-p)\tau^\lambda e^{(x+y)}+ \\ & p\left(L^{-1}\left[\frac{1}{s^\lambda}L\left[\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}+\frac{\beta^2}{2}\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial x \partial y}\rho\sigma\beta\right]\right]\right). \end{aligned} \tag{5.5.21}$$

The solution of the time-fractional Black-Scholes equation (5.5.15) with initial and boundary conditions (5.5.16) and (5.5.17) can be expressed as

$$\begin{aligned} V(x,y,\tau,p) =& \sum_{k=0}^{\infty} p^k \theta_k(x,y,\tau) \\ =& max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}}+\varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}}-E,0\}+(1-p)\tau^\lambda e^{(x+y)}+ \\ & p\left(L^{-1}\left[\frac{1}{s^\lambda}L\left[\frac{\sigma^2}{2}\sum_{k=0}^{\infty}p^k\frac{\partial^2 \theta_k}{\partial x^2}+\frac{\beta^2}{2}\sum_{k=0}^{\infty}p^k\frac{\partial^2 \theta_k}{\partial y^2}+\sum_{k=0}^{\infty}p^k\frac{\partial^2 \theta_k}{\partial x \partial y}\rho\sigma\beta\right]\right]\right). \end{aligned} \tag{5.5.22}$$

By comparing correponding powers of p

$$\theta_k(x,y,\tau)=\begin{cases} max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}}+\varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}}-E,0\}+\tau^\lambda e^{(x+y)}, & k=0 \\ -\tau^\lambda e^{(x+y)}+L^{-1}\left[\frac{1}{s^\lambda}L\left[\frac{\sigma^2}{2}\frac{\partial^2 \theta_0}{\partial x^2}+\frac{\beta^2}{2}\frac{\partial^2 \theta_0}{\partial y^2}+\frac{\partial^2 \theta_0}{\partial x \partial y}\rho\sigma\beta\right]\right], & k=1 \\ -\tau^\lambda e^{(x+y)}+L^{-1}\left[\frac{1}{s^\lambda}L\left[\frac{\sigma^2}{2}\frac{\partial^2 \theta_{k-1}}{\partial x^2}+\frac{\beta^2}{2}\frac{\partial^2 \theta_{k-1}}{\partial y^2}+\frac{\partial^2 \theta_{k-1}}{\partial x \partial y}\rho\sigma\beta\right]\right], & k\geq 2. \end{cases} \tag{5.5.23}$$

For $k \geq 1$, the general form of (5.5.23) can be written as

$$\theta_k(x,y,\tau) = \frac{\tau^{k\lambda}}{\Gamma(k\lambda+1)}\left(\left(\frac{\sigma^2}{2}\right)^k max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}},0\} + \left(\frac{\beta^2}{2}\right)^k max\{\varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}},0\}\right) + \frac{\tau^{(k+1)\lambda}\Gamma(\lambda+1)}{\Gamma((k+1)\lambda+1)} e^{(x+y)}\left(\frac{\sigma^2}{2}+\frac{\beta^2}{2}+\rho\sigma\beta\right)^k - \frac{\tau^{k\lambda}\Gamma(\lambda+1)}{\Gamma(k\lambda+1)} e^{(x+y)}\left(\frac{\sigma^2}{2}+\frac{\beta^2}{2}+\rho\sigma\beta\right)^{k-1}. \tag{5.5.24}$$

The solution of (5.5.15) is given as

$$V(x,y,\tau,p) = max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}} + \varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}} - E, 0\} + \tau^{\lambda} e^{(x+y)} + \sum_{k=0}^{\infty} p^{(k+1)}\left(\frac{\tau^{(k+1)\lambda}}{\Gamma((k+1)\lambda+1)}\left(\left(\frac{\sigma^2}{2}\right)^{k+1} max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}},0\} + \left(\frac{\beta^2}{2}\right)^{k+1} max\{\varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}},0\}\right) + \frac{\tau^{(k+2)\lambda}\Gamma(\lambda+1)}{\Gamma((k+2)\lambda+1)} e^{(x+y)}\left(\frac{\sigma^2}{2}+\frac{\beta^2}{2}+\rho\sigma\beta\right)^{k+1} - \frac{\tau^{(k+1)\lambda}\Gamma(\lambda+1)}{\Gamma((k+1)\lambda+1)} e^{(x+y)}\left(\frac{\sigma^2}{2}+\frac{\beta^2}{2}+\rho\sigma\beta\right)^{k}\right). \tag{5.5.25}$$

For $p \to 1$, the above equation becomes

$$V(x,y,\tau) = V(x,y,\tau,1) = max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}} + \varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}} - E, 0\} + \tau^{\lambda} e^{(x+y)} + \sum_{k=0}^{\infty}\left(\frac{\tau^{(k+1)\lambda}}{\Gamma((k+1)\lambda+1)}\left(\left(\frac{\sigma^2}{2}\right)^{k+1} max\{\varepsilon_1 e^{x+rT-\frac{\sigma^2 T}{2}},0\} + \left(\frac{\beta^2}{2}\right)^{k+1} max\{\varepsilon_2 e^{y+rT-\frac{\beta^2 T}{2}},0\}\right) + \frac{\tau^{(k+2)\lambda}\Gamma(\lambda+1)}{\Gamma((k+2)\lambda+1)} e^{(x+y)}\left(\frac{\sigma^2}{2}+\frac{\beta^2}{2}+\rho\sigma\beta\right)^{k+1} - \frac{\tau^{(k+1)\lambda}\Gamma(\lambda+1)}{\Gamma((k+1)\lambda+1)} e^{(x+y)}\left(\frac{\sigma^2}{2}+\frac{\beta^2}{2}+\rho\sigma\beta\right)^{k}\right). \tag{5.5.26}$$

This is the required approximate analytical solution of time-fractional Black-Scholes equation when the underlying financial asset is estimated using a stochastic volatility model.

5.6 Numerical Demonstration and Discussion

The performance of explicit solution (5.5.26) on the option price $V(x,y,t)$ on the basis of assumed parameters as given in Table5.1, is given in Fig.5.1 to Fig.5.3. For $\tau = 0$: $-2 \leq x \leq 5$ and $-5 \leq y \leq 5$, the option value V is zero but V increases exponentially when $y \geq 5$. Also for $\tau = 0.5$, and 1: $-2 \leq x \leq 4$ and $-5 \leq y \leq 5$, the option value V is zero but when $x \geq 4$ and $y \geq 5$, V increases exponentially, refer to Fig.5.1. According to Fig.5.2, for $0 \leq \tau \leq 1$ and $0 \leq y \leq 2$, the option value V

Parameters	Symbol	Value
Strike price of the option	E	100
Risk-free interest rate	r	5%
Maturity time of the option (years)	T	1
Volatility of the underlying asset S	σ	0.6
Correlation	ρ	0.8
Brownian motion $X_2(t)$ coefficient	β	0.4
Fractional-time derivative order	λ	0.9
Weight of asset price S	ε_1	2
Weight of volatility σ	ε_2	1

Table 5.1: Selection of parameters

is zero when $x = 0,2,4,8$, and V increases exponentially when $y \geq 3$ but also as x tends from 0 to 8 and for $\tau = 0$, the option value V converges to zero. Next, for $0 \leq \tau \leq 1$ and $0 \leq x \leq 3$, the option value V is zero when $y = 0,2,4,8$, and V increases exponentially when $x \geq 3$ but also as y tends from 0 to 8 and for $x = 0$, the option value V converges to zero, see Fig.5.3.

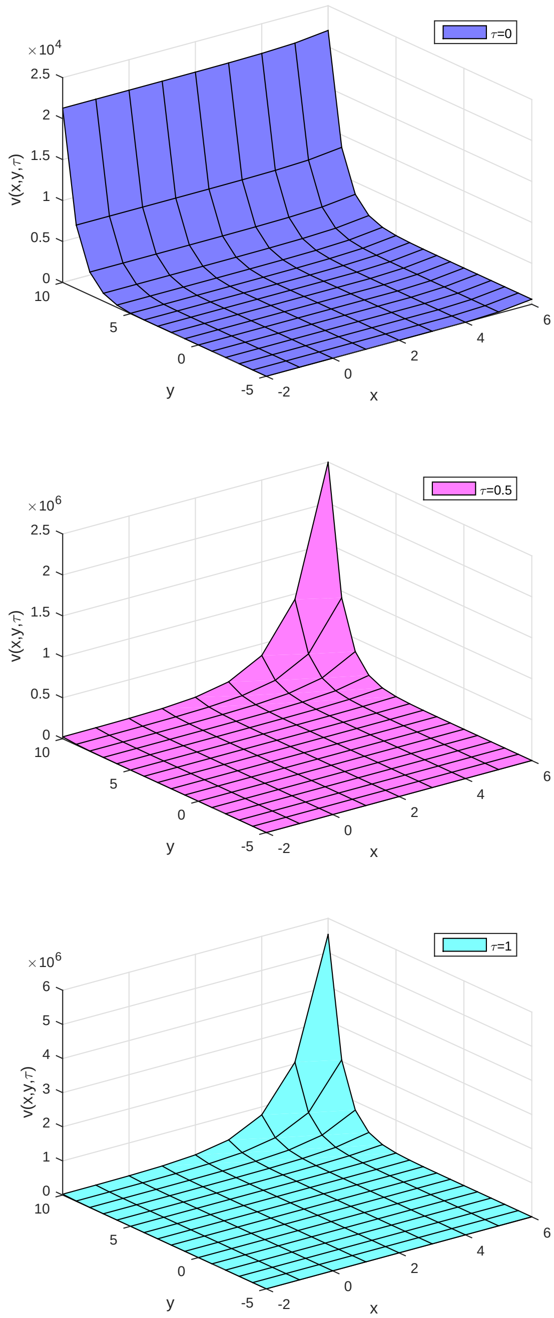

Figure 5.1: Performance of explicit solution on the option price $V(x, y, \tau)$ at $\tau = 0, 0.5, 1$

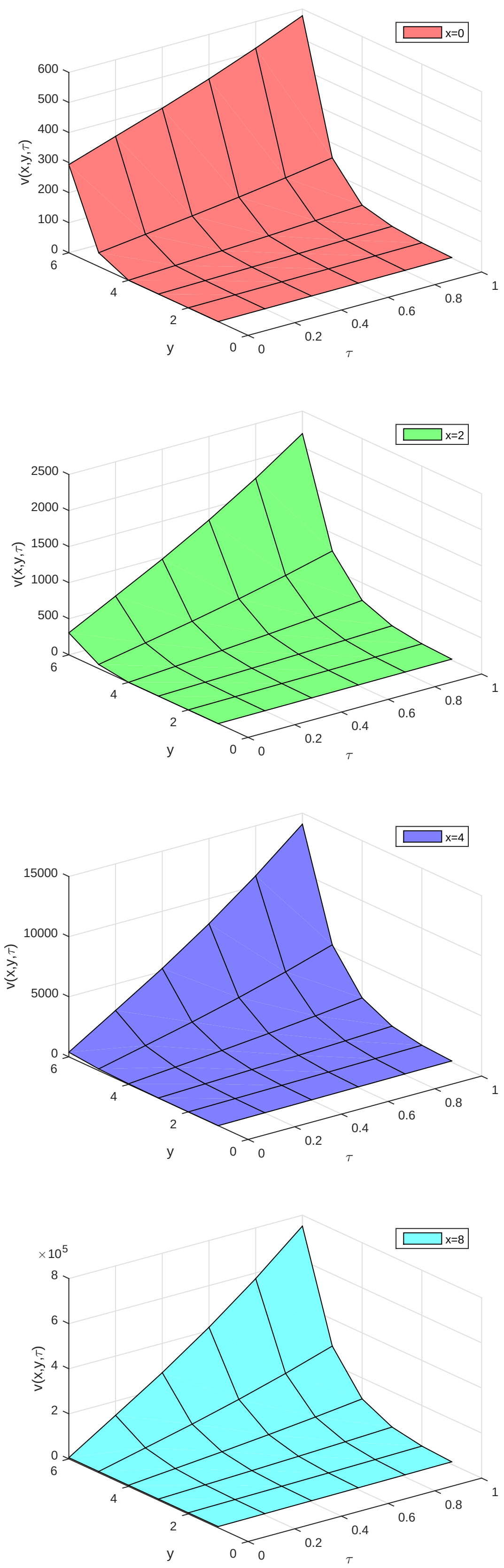

Figure 5.2: Performance of explicit solution on the option price $V(x,y,\tau)$ at $x = 0,2,4,8$

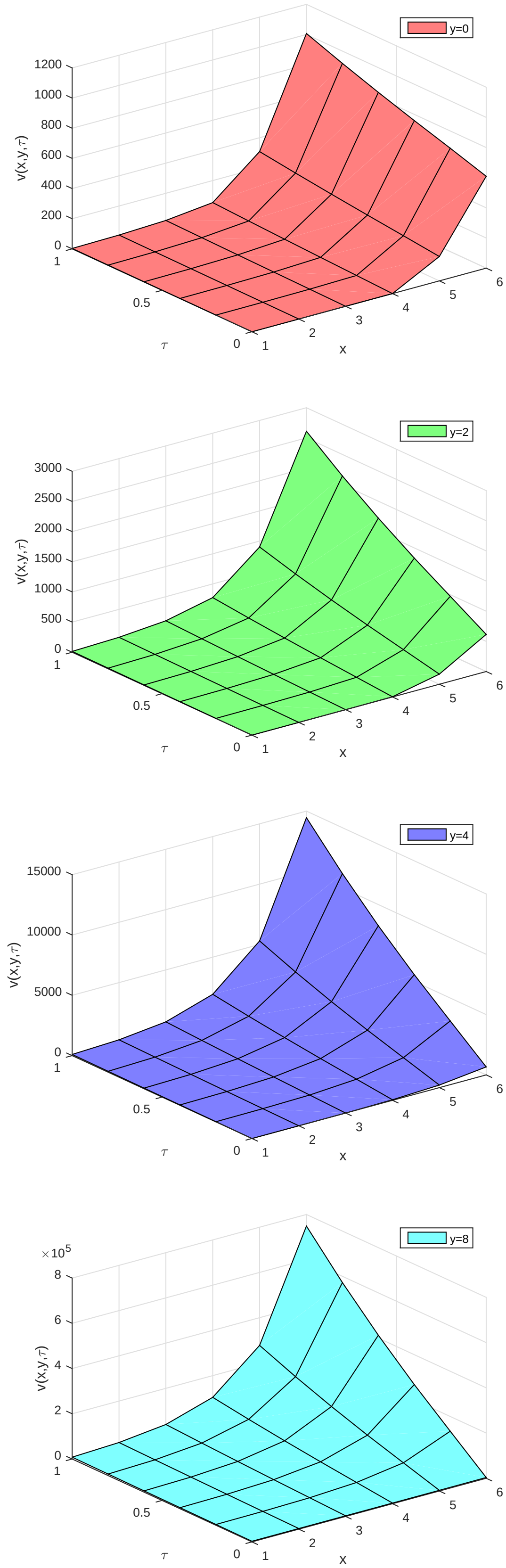

Figure 5.3: Performance of explicit solution on the option price $V(x,y,\tau)$ at $y = 0,2,4,8$

5.7 Conclusion

We have derived the generalized Black-Scholes differential equation which give price of European option when a stochastic volatility model may approximate the underlying asset. This problem is relevant and driven when high-frequency data in the prescribed financial market is considered. Also by integrating the risk-neutral information measure, we have derived the risk-neutral probability density function of stock price and volatility, as the solution of minimizing Kullback relative entropy. We have applied Laplace transform homotopy perturbation method for the approximate analytical solution of the desired Black-Scholes equation where time derivative is assumed as a Liouville-Caputo time fractional derivative. This approach will help to study the financial behavior of the generalized problem.

Chapter 6

Quanto-Option Black-Scholes Model Using Relative Information

6.1 Introduction

In the preceding chapter, we have studied the Black-Scholes option pricing model with stochastic volatility using information theoretic approach. In the present chapter, we have extended that study when the number of underlying assets is two and as a result obtain a two-dimensional parabolic partial differential equation, refer to [97].

Over the past 20 years, the main focus of research has centered on an effective computational approach to the multi-asset option pricing model [42, 97]. The quanto option pricing model is a sort of multi-asset option model, a cross-currency derivative approach in which the payoff of the asset is converted into another currency at a pre-specified rate at the option's maturity. It is an important model for pricing financial derivatives and the numerical solutions of this two-dimensional Black-Scholes equation is significant from theoretical as well as practical aspects.

Many researchers have worked on the information theoretic approach to ex-

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amine the aspect of volatility in the financial sector [6, 7, 28]. In the preceding chapter by using the minimization of the Kullback measure of relative information, we have worked on information-theoretic measure approach to derive the Black-Scholes option pricing model with stochastic volatility. In this chapter, by using the information theoretic approach we obtain a two-dimensional parabolic partial differential equation, a Quanto-option Black-Scholes equation, where the number of underlying assets is two, refer to [97]. Also, we apply the entropy dynamics of multi-asset to model the European options price and for this, we have derived the risk-neutral probability density functions of the multi-asset option price within the information-theoretic dynamic formalism. In addition, modeled as a Kolmogorov-backward equation, we specify how the probability function changes over a period of time.

The chapter is organized as follows: In Section 6.2, we briefly outline the dynamics of the two asset option prices and derive the risk-neutral probability function by integrating the risk-neutral information measure. In Section 6.3, the Quanto option Black-Scholes equation is derived. The approximate analytical solution of the derived Black-Scholes equation is carried out in Section 6.4 by using the Laplace Homotopy perturbation method [88]. Section 6.5 illustrates the performance of the model numerically and Section 6.6 concludes the chapter.

6.2 Entropy Probabilistic Dynamics of Quanto-Option

In finance, the simplest model of the asset price is lognormal distribution. We consider S_1 as the foreign asset price, and S_2 as the foreign currency exchange rate against the domestic asset price with dynamics of S_1 and S_2 as

$$f(S_1) = log(S_1) \tag{6.2.1}$$

$$f(S_2) = log(S_2). \tag{6.2.2}$$

Here, we consider the case of white noise in the financial sector, and therefore we take the scale invariance as

$$P(f(\lambda x)) = P(f(x)) \tag{6.2.3}$$

where *P(f)* stands for expected power of the function *f*; λ is a scaling factor and x is an arbitrary variable. Also, the probability densities of (6.2.1) and (6.2.2) are invariant under the scaling transformation.

Next, we consider a change in S_1 by using Taylor's expansion up to second-order accuracy as

$$log\frac{\dot{S_1}(t)}{S_1(t)} \approx \frac{dS_1(t)}{S_1(t)} - \frac{1}{2}\left(\frac{dS_1(t)}{S_1(t)}\right)^2 \tag{6.2.4}$$

and similarly, change in S_2 as

$$log\frac{\dot{S_2}(t)}{S_2(t)} \approx \frac{dS_2(t)}{S_2(t)} - \frac{1}{2}\left(\frac{dS_2(t)}{S_2(t)}\right)^2 \tag{6.2.5}$$

where, $\dot{S_1}(t) = S_1(t) + dS_1(t)$ and $\dot{S_2}(t) = S_2(t) + dS_2(t)$ stand for the stock prices S_1 and S_2, repectively at time $t+dt$. Further, changes in two underlying asset prices S_1 and S_2 can be defined by considering that each asset price follows a geometric Brownian motion as

$$\frac{dS_1(t)}{S_1(t)} = \mu_1 dt + \sigma_1 dX_1(t) \tag{6.2.6}$$

$$\frac{dS_2(t)}{S_2(t)} = \mu_2 dt + \sigma_2 dX_2(t) \tag{6.2.7}$$

where $\mu_1, \mu_2 \in (-\infty, \infty)$ are growth constants and σ_1 is the volatility of S_1 and σ_2 is the volatility of S_2, and the two Brownian motions $X_1(t)$ and $X_2(t)$ satisfy

$$E(dX_1) = E(dX_2) = 0, \text{ and } E(dX_1^2) = E(dX_2^2) = dt,$$

and are correlated with correlation coefficient ρ, that is

$$E(dX_1(t)dX_2(t)) = \rho dt. \tag{6.2.8}$$

By squaring (6.2.6) and (6.2.7), and for $i = 1, 2$, $dX_i^2 \to dt$ as $dt \to 0$, we get

$$\left(\frac{dS_1(t)}{S_1(t)}\right)^2 = \sigma_1^2 dt \tag{6.2.9}$$

$$\left(\frac{dS_2(t)}{S_2(t)}\right)^2 = \sigma_2^2 dt. \tag{6.2.10}$$

Using (6.2.6), (6.2.9) in (6.2.4), and (6.2.7), (6.2.10) in (6.2.5) respectively, we obtain the information pertaining to change in prices S_1 and S_2:

$$log\frac{\dot{S_1}(t)}{S_1(t)} \approx \mu_1 dt + \sigma_1 dX_1(t) - \frac{1}{2}\sigma_1^2 dt \tag{6.2.11}$$

$$log\frac{\dot{S_2}(t)}{S_2(t)} \approx \mu_2 dt + \sigma_2 dX_2(t) - \frac{1}{2}\sigma_2^2 dt \tag{6.2.12}$$

This information is applicable to dynamic variables S_1 and S_2 and is implemented in the form of constraints, where higher-order terms of *dt* tend to zero.

Next, assigning the normal distribution of the log price S_1 as $P_{S_1}(log\dot{S_1}|logS_1)$ and log price S_2 as $P_{S_2}(log\dot{S_2}|logS_2)$ require minimizing the Kullback measure of relative information [40], which are defined as:

$$K_{S_1}(P||Q) = \int P_{S_1}(log\dot{S_1}|logS_1)\ log\frac{P_{S_1}(log\dot{S_1}|logS_1)}{Q_{S_1}(log\dot{S_1}|logS_1)} d\ log\dot{S_1} \tag{6.2.13}$$

$$K_{S_2}(P||Q) = \int P_{S_2}(log\dot{S_2}|logS_2)\ log\frac{P_{S_2}(log\dot{S_2}|logS_2)}{Q_{S_2}(log\dot{S_2}|logS_2)} d\ log\dot{S_2} \tag{6.2.14}$$

where $Q_{S_1}(log\dot{S_1}|logS_1)$ and $Q_{S_2}(log\dot{S_2}|logS_2)$ are the prior probability distributions corresponding to underlying assets S_1 and S_2, respectively. However, instead of minimizing the Kullback measure of relative entropy $K(P||Q)$, we will maximize $-K(P||Q)$.

Subject to normalization factor N as defined below, and (6.2.11) as the prior information constraint, we obtain the transition probability of the asset price S_1 as

$$P_{S_1}(log\dot{S_1}|logS_1) = \frac{e^{-\frac{\zeta}{2}\left(log\frac{\dot{S_1}}{S_1}\right)^2 + \eta\left(log\frac{\dot{S_1}}{S_1}\right)}}{N(\zeta,\eta,logS_1)}, \tag{6.2.15}$$

or

$$P_{S_1}(log\dot{S_1}|logS_1) = \frac{e^{-\frac{1}{2\sigma_1^2 dt}\left(log\dot{S_1} - (logS_1 + \mu_1 dt + \sigma_1 dX_1(t) - \frac{1}{2}\sigma_1^2 dt)\right)^2}}{N(\zeta,\eta,logS_1)} \tag{6.2.16}$$

where normalization factor $N(\zeta,\eta,logS_1) = \int e^{-\frac{\zeta}{2}\left(log\frac{\dot{S_1}}{S_1} - \frac{\eta}{\zeta}\right)^2} d\ log\dot{S_1}$, and $\zeta = (\sigma_1^2 dt)^{-1}$, $\eta = \frac{1}{2}\left(\frac{2\mu_1}{\sigma_1^2} - 1\right)$ are Lagrange multipliers corresponding to the constraint (6.2.11). Similarly, for the case of asset price S_2 with (6.2.12) as the prior information constraint, we obtain the transition probability function of the asset price S_2 as

$$P_{S_2}(log\dot{S_2}|logS_2) = \frac{e^{-\frac{1}{2\sigma_2^2 dt}\left(log\dot{S_2}-(logS_2+\mu_2 dt+\sigma_2 dX_2(t)-\frac{1}{2}\sigma_2^2 dt)\right)^2}}{N(\lambda,\xi,logS_2)} \tag{6.2.17}$$

where normalization factor $N(\lambda,\xi,logS_2) = \int e^{-\frac{\lambda}{2}\left(log\frac{\dot{S_2}}{S_2}-\frac{\xi}{\lambda}\right)^2} d\ log\dot{S_2}$, and $\lambda = (\sigma_2^2 dt)^{-1}$, $\xi = \frac{1}{2}\left(\frac{2\mu_2}{\sigma_2^2}-1\right)$ are Lagrange multipliers corresponding to the constraint (6.2.12).

Next by making the transformation from log price back to asset price S_1 as

$$P_{S_1}(\dot{S_1}|S_1) = \frac{1}{\dot{S_1}}P(log\dot{S_1}|logS_1) \tag{6.2.18}$$

and from log price back to asset price S_2 as

$$P_{S_2}(\dot{S_2}|S_2) = \frac{1}{\dot{S_2}}P(log\dot{S_2}|logS_2), \tag{6.2.19}$$

the probability of asset price S_1 is given by

$$P_{S_1}(\dot{S_1}|S_1) = \frac{e^{-\frac{1}{2\sigma_1^2 dt}\left(log\dot{S_1}-(logS_1+\mu_1 dt+\sigma_1 dX_1(t)-\frac{1}{2}\sigma_1^2 dt)\right)^2}}{N(\zeta,\eta,logS_1)\ \dot{S_1}} \tag{6.2.20}$$

and for S_2 is given by

$$P_{S_2}(\dot{S_2}|S_2) = \frac{e^{-\frac{1}{2\sigma_2^2 dt}\left(log\dot{S_2}-(logS_2+\mu_2 dt+\sigma_2 dX_2(t)-\frac{1}{2}\sigma_2^2 dt)\right)^2}}{N(\lambda,\xi,logS_2)\ \dot{S_2}} \tag{6.2.21}$$

Equ. (6.2.20)-(6.2.21) are the log-normal distributions for a very small change in time interval dt in option price S_1 and S_2 respectively.

Next, we simply estimate the option's value at maturity by its expected payoff by using the risk-neutral measure. With r_1 as a domestic risk-free rate and r_2 as the foreign risk-free rate, the risk-neutrality constraints are given as

$$\mu_1 = r_1 - d_1, \tag{6.2.22}$$

$$\mu_2 = r_2 - d_2 \tag{6.2.23}$$

where, $d_1 = r_1 - r_2 + d + \sigma_1\sigma_2\rho$ and $d_2 = r_2$, with d as the dividend constant.

The risk-neutral probability function of stock price S_1 by imposing the constraint (6.2.22) is

$$P(log\dot{S}_1|logS_1) = \frac{e^{-\frac{1}{2\sigma_1^2 dt}\left(log\dot{S}_1-(logS_1+(r_1-d_1)dt+\sigma_1 dX_1(t)-\frac{1}{2}\sigma_1^2 dt)\right)^2}}{N(\zeta,\eta,logS_1)} \tag{6.2.24}$$

and the risk-neutral probability density of asset price S_2 by imposing the constraint (6.2.23) is

$$P(log\dot{S}_2|logS_2) = \frac{e^{-\frac{1}{2\sigma_2^2 dt}\left(log\dot{S}_2-(logS_2+(r_2-d_2)dt+\sigma_2 dX_2(t)-\frac{1}{2}\sigma_2^2 dt)\right)^2}}{N(\lambda,\xi,logS_2)}. \tag{6.2.25}$$

By using the risk-neutral probability to value the European call and put option, with E as the strike price, the expected payoff for the call option at maturity, denoted by V_C, is given as

$$V_C(S_1,S_2,T) = max\{\alpha S_1+\beta S_2-E,0\} \tag{6.2.26}$$

and the expected payoff at the maturity for a put option, denoted by V_P, is given as

$$V_P(S_1,S_2,T) = max\{E-(\alpha S_1+\beta S_2),0\}; \tag{6.2.27}$$

where α and β are the constant coefficients for the asset prices S_1 and S_2, respectively.

Next, we look into the aspect how does the probability distribution change over time which comes out to be the Kolmogorov-backward equation. We use entropic instant which incorporates both the asset prices, defined as

$$p(log\dot{S}_1,log\dot{S}_2,\dot{t}) = \int \left[p(logS_1,logS_2,t)\ P(log\dot{S}_1|logS_1,log\dot{S}_2|logS_2)\right] dlogS_1(t)\ dlogS_2(t) \tag{6.2.28}$$

where $p(logS_1,logS_2,t)$ and $p(log\dot{S}_1,log\dot{S}_2,\dot{t})$ are distributions representing the information available at an instant time t and the next instant time $\dot{t}=t+dt$, respectively. We have a generalized Kolmogorov-backward equation for the probability density $p(logS_1,logS_2,t)$, a differential form of (6.2.28) given as

$$\frac{\partial}{\partial t}p(logS_1,logS_2,t) = -\left(\mu_1-\frac{\sigma_1^2}{2}\right)\frac{\partial}{\partial logS_1}p(logS_1,t)-\left(\mu_2-\frac{\sigma_2^2}{2}\right)\frac{\partial}{\partial logS_2}p(logS_2,t)$$
$$-\frac{1}{2}\left[\sigma_1^2\frac{\partial^2}{\partial(logS_1)^2}p(logS_1,t)+\sigma_2^2\frac{\partial^2}{\partial(logS_2)^2}p(logS_2,t)\right]$$
$$-\rho\sigma_1\sigma_2\frac{\partial^2}{\partial logS_1\ \partial logS_2}p(logS_1,logS_2,t). \tag{6.2.29}$$

This will be used for the derivation of the Quanto option Black-Scholes equation in the next section.

6.3 Derivation of Quanto-Option Black-Scholes Equation

In order to derive the Black-Scholes equation of the quanto option, we begin with the payoff equation as

$$V(logS_1,logS_2,E,t) = \int(\alpha S_1+\beta S_2-E)\ P(log\dot{S}_1|logS_1,log\dot{S}_2|logS_2)\ dlogS_1(\dot{t})\ dlogS_2(\dot{t}). \tag{6.3.1}$$

and the derivative of the value of an option with respect to time variable is given as

$$\tfrac{\partial}{\partial t}V(logS_1,logS_2,E,t) = \int(\alpha S_1+\beta S_2-E)\ \tfrac{\partial}{\partial t}P(log\dot{S}_1|logS_1,log\dot{S}_2|logS_2)\ dlogS_1(\dot{t})\ dlogS_2(\dot{t}) \tag{6.3.2}$$

Next, by using (6.2.29) in (6.3.2), we get

$$\frac{\partial V}{\partial t} = \int(\alpha S_1+\beta S_2-E)\left\{\left[-\left((r_1-d_1)-\frac{\sigma_1^2}{2}\right)\frac{\partial}{\partial logS_1}P(log\dot{S}_1|logS_1)\right]\right.$$
$$+\left[-\left((r_2-d_2)-\frac{\sigma_2^2}{2}\right)\frac{\partial}{\partial logS_2}P(log\dot{S}_2|logS_2)\right]$$
$$-\frac{1}{2}\left[\sigma_1^2\frac{\partial^2}{\partial(logS_1)^2}P(log\dot{S}_1|logS_1)+\sigma_2^2\frac{\partial^2}{\partial(logS_2)^2}P(log\dot{S}_2|logS_2)\right]$$
$$\left.-\rho\sigma_1\sigma_2\frac{\partial^2}{\partial logS_1\ \partial logS_2}P(log\dot{S}_1|logS_1,log\dot{S}_2|logS_2)\right\}\ dlogS_1(\dot{t})\ dlogS_2(\dot{t}).$$

This implies

$$\frac{\partial V}{\partial t} = -\left((r_1-d_1)-\frac{\sigma_1^2}{2}\right)\frac{\partial V}{\partial logS_1} - \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial (logS_1)^2} - \left((r_2-d_2)-\frac{\sigma_2^2}{2}\right)\frac{\partial V}{\partial logS_2}$$
$$- \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial (logS_2)^2} - \frac{\partial^2 V}{\partial logS_1 \partial logS_2}\rho\sigma_1\sigma_2,$$

or

$$\frac{\partial V}{\partial t} + \frac{\sigma_1^2 S_1^2}{2}\frac{\partial^2 V}{\partial S_1^2} + \frac{\sigma_2^2 S_2^2}{2}\frac{\partial^2 V}{\partial S_2^2} + (r_1-d_1)S_1\frac{\partial V}{\partial S_1} + (r_2-d_2)S_2\frac{\partial V}{\partial S_2} + \frac{\partial^2 V}{\partial S_1 \partial S_2}\rho\sigma_1\sigma_2 S_1 S_2 = 0 \tag{6.3.3}$$

Now substitute $B = Ve^{-r_1T+r_1t}$, which stands for both put and call options, in (6.3.3); we have the required partial differential equation

$$\frac{\partial B}{\partial t} + \frac{\sigma_1^2 S_1^2}{2}\frac{\partial^2 B}{\partial S_1^2} + \frac{\sigma_2^2 S_2^2}{2}\frac{\partial^2 B}{\partial S_2^2} + (r_1-d_1)S_1\frac{\partial B}{\partial S_1} + (r_2-d_2)S_2\frac{\partial B}{\partial S_2} + \frac{\partial^2 B}{\partial S_1 \partial S_2}\rho\sigma_1\sigma_2 S_1 S_2 - r_1 B = 0. \tag{6.3.4}$$

For $\rho \in (-1,1)$, the above equation is a two-dimensional Black-Scholes equation of the quanto option, second-order partial differential equation and we need to impose appropriate boundary conditions to this problem to solve the equation for both the call and put options.

6.4 Approximate Analytical Solution

Here we find the approximate analytical solution of (6.3.4) by the Laplace transform homotopy perturbation method, where fractional derivatives are described in the sense of the Liouville-Caputo fractional derivative. Consider (6.3.4), for $S_1, S_2 \in [0,\infty)$ and $t \in [0,T]$

$$\frac{\partial B}{\partial t} + \frac{\sigma_1^2 S_1^2}{2}\frac{\partial^2 B}{\partial S_1^2} + \frac{\sigma_2^2 S_2^2}{2}\frac{\partial^2 B}{\partial S_2^2} + (r_1-d_1)S_1\frac{\partial B}{\partial S_1} + (r_2-d_2)S_2\frac{\partial B}{\partial S_2} + \frac{\partial^2 B}{\partial S_1 \partial S_2}\rho\sigma_1\sigma_2 S_1 S_2 - r_1 B = 0 \tag{6.4.1}$$

with the terminal condition:

$$B(S_1,S_2,T) = max\{\alpha S_1 + \beta S_2 - E, 0\} \tag{6.4.2}$$

and boundary conditions:

$$B(S_1,S_2,t)=\begin{cases}0, & as\ (S_1,S_2)\to(0,0)\\ \alpha S_1+\beta S_2-Ee^{-r_1(T-t)}, & as\ (S_1,S_2)\to(\infty,\infty).\end{cases} \tag{6.4.3}$$

To analyse (6.4.1) we use the transformations

$$x = log_e(S_1)-(r_1-d_1)t+\frac{\sigma_1^2 t}{2}, \tag{6.4.4}$$

$$y = log_e(S_2)-(r_2-d_2)t+\frac{\sigma_2^2 t}{2} \tag{6.4.5}$$

This gives $x,y\in(-\infty,\infty)$ and $t\in[0,T]$ for $S_1,S_2\in[0,\infty)$, (6.4.1) transforms to

$$\frac{\partial B}{\partial t}+\frac{\sigma_1^2}{2}\frac{\partial^2 B}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 B}{\partial y^2}+\frac{\partial^2 B}{\partial x\partial y}\rho\sigma_1\sigma_2-r_1B=0 \tag{6.4.6}$$

with the terminal condition:

$$B(x,y,T)=max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}}+\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}}-E,0\} \tag{6.4.7}$$

and boundary conditions:

$$B(x,y,t)=\begin{cases}0, & as\ (x,y)\to(-\infty,-\infty)\\ \alpha e^{x+(r_1-d_1)t-\frac{\sigma_1^2 t}{2}}+\beta e^{y+(r_2-d_2)t-\frac{\sigma_2^2 t}{2}}-Ee^{-r_1(T-t)}, & as\ (x,y)\to(\infty,\infty).\end{cases} \tag{6.4.8}$$

Using $V(x,y,t)=e^{-r_1(T-t)}B(x,y,t)$ for $x,y\in(-\infty,\infty)$ and $t\in[0,T]$ in (6.4.6), we obtain

$$\frac{\partial V}{\partial t}+\frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial x\partial y}\rho\sigma_1\sigma_2=0 \tag{6.4.9}$$

with terminal condition:

$$V(x,y,T)=max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}}+\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}}-E,0\} \tag{6.4.10}$$

and boundary conditions:

$$V(x,y,t)=\begin{cases}0, & as\ (x,y)\to(-\infty,-\infty)\\ \alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 t}{2}}+\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 t}{2}}-E, & as\ (x,y)\to(\infty,\infty).\end{cases} \tag{6.4.11}$$

Further, we transform the final boundary value problem to an initial boundary value problem by changing the forward time variable from t to $\tau = T - t$ in (6.4.9), and for $x, y \in (-\infty, \infty)$ and $\tau \in [0, T]$ we get

$$-\frac{\partial V}{\partial \tau} + \frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x \partial y}\rho\sigma_1\sigma_2 = 0 \tag{6.4.12}$$

with an initial condition:

$$V(x,y,0) = max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}} + \beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}} - E, 0\} \tag{6.4.13}$$

and boundary conditions:

$$V(x,y,\tau) = \begin{cases} 0, \ \ as\ (x,y) \to (-\infty,-\infty) \\ \alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}+\frac{\sigma_1^2 \tau}{2}} + \beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}+\frac{\sigma_2^2 \tau}{2}} - E, \ \ as\ (x,y) \to (\infty,\infty). \end{cases} \tag{6.4.14}$$

Replacing $\frac{\partial V}{\partial \tau}$ by $\mathscr{L}_\tau^\lambda V$, the Liouville-Caputo [41] type time-fractional derivative of order $\lambda \in (0,1]$, we have the subsequent fractional Black Scholes equation for $x, y \in (-\infty, \infty)$ and $\tau \in [0, T]$, given by

$$\mathscr{L}_\tau^\lambda V = \frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x \partial y}\rho\sigma_1\sigma_2 \tag{6.4.15}$$

with an initial condition:

$$V(x,y,0) = max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}} + \beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}} - E, 0\} \tag{6.4.16}$$

and boundary conditions:

$$V(x,y,\tau) = \begin{cases} 0, \ \ as\ (x,y) \to (-\infty,-\infty) \\ \alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}+\frac{\sigma_1^2 \tau}{2}} + \beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}+\frac{\sigma_2^2 \tau}{2}} - E, \ \ as\ (x,y) \to (\infty,\infty). \end{cases} \tag{6.4.17}$$

Next, we find the solution of (6.4.15) subject to conditions (6.4.16)-(6.4.17) by using the Laplace transform of the Liouville-Caputo fractional derivative [57, 88], given as

$$L[V(x,y,\tau)] = \frac{1}{s}max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}} + \beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}} - E, 0\} + \frac{1}{s^\lambda}L\left[\frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial x^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x \partial y}\rho\sigma_1\sigma_2\right]. \tag{6.4.18}$$

By using the inverse Laplace transformation in (6.4.18), we obtain

$$\begin{aligned} V(x,y,\tau) =& L^{-1}\left[\frac{1}{s}max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}}+\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}}-E,0\}+\right. \\ &\left.\frac{1}{s^{\lambda}}L\left[\frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial x\partial y}\rho\sigma_1\sigma_2\right]\right] \end{aligned} \tag{6.4.19}$$

Further, by using Homotopy perturbation method [31, 32], we construct a real valued homotopy $V(x,y,\tau,p)$ defined on $(-\infty,\infty)\times(-\infty,\infty)\times[0,T]\times[0,1]$ and satisfying the following

$$(1-p)(V(x,y,\tau,p)-V_0(x,y,\tau))+p(V(x,y,\tau,p)-V(x,y,\tau))=0 \tag{6.4.20}$$

where $p\in[0,1]$ is an impending parameter and V_0 is the initial approximation, defined as $V_0(x,y,\tau)=V(x,y,0)+\tau^{\lambda}e^{(x+y)}$, refer to [4]. Substitute this in (6.4.20), we have

$$\begin{aligned} V(x,y,\tau,p) =& max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}}+\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}}-E,0\}+(1-p)\tau^{\lambda}e^{(x+y)}+ \\ & p\left(L^{-1}\left[\frac{1}{s^{\lambda}}L\left[\frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial x\partial y}\rho\sigma_1\sigma_2\right]\right]\right). \end{aligned} \tag{6.4.21}$$

The solution of the time-fractional Black-Scholes equation (6.4.15) with initial and boundary conditions (6.4.16) and (6.4.17) can be expressed as

$$\begin{aligned} V(x,y,\tau,p) =& \sum_{k=0}^{\infty}p^k\theta_k(x,y,\tau) \\ =& max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}}+\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}}-E,0\}+(1-p)\tau^{\lambda}e^{(x+y)}+ \\ & p\left(L^{-1}\left[\frac{1}{s^{\lambda}}L\left[\frac{\sigma_1^2}{2}\sum_{k=0}^{\infty}p^k\frac{\partial^2\theta_k}{\partial x^2}+\frac{\sigma_2^2}{2}\sum_{k=0}^{\infty}p^k\frac{\partial^2\theta_k}{\partial y^2}+\sum_{k=0}^{\infty}p^k\frac{\partial^2\theta_k}{\partial x\partial y}\rho\sigma_1\sigma_2\right]\right]\right). \end{aligned} \tag{6.4.22}$$

By comparing correponding powers of p

$$\theta_k(x,y,\tau)=\begin{cases} max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}}+\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}}-E,0\}+\tau^{\lambda}e^{(x+y)}, & k=0 \\ -\tau^{\lambda}e^{(x+y)}+L^{-1}\left[\frac{1}{s^{\lambda}}L\left[\frac{\sigma_1^2}{2}\frac{\partial^2\theta_0}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2\theta_0}{\partial y^2}+\frac{\partial^2\theta_0}{\partial x\partial y}\rho\sigma_1\sigma_2\right]\right], & k=1 \\ -\tau^{\lambda}e^{(x+y)}+L^{-1}\left[\frac{1}{s^{\lambda}}L\left[\frac{\sigma_1^2}{2}\frac{\partial^2\theta_{k-1}}{\partial x^2}+\frac{\sigma_2^2}{2}\frac{\partial^2\theta_{k-1}}{\partial y^2}+\frac{\partial^2\theta_{k-1}}{\partial x\partial y}\rho\sigma_1\sigma_2\right]\right], & k\geq 2. \end{cases} \tag{6.4.23}$$

For $k \geq 1$, the general form of (6.4.23) can be written as

$$\begin{aligned}\theta_k(x,y,\tau) =& \frac{\tau^{k\lambda}}{\Gamma(k\lambda+1)}\left(\left(\frac{\sigma_1^2}{2}\right)^k max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}},0\}\right.\\ &+\left.\left(\frac{\sigma_2^2}{2}\right)^k max\{\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}},0\}\right)\\ &+\frac{\tau^{(k+1)\lambda}\Gamma(\lambda+1)}{\Gamma((k+1)\lambda+1)}e^{(x+y)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^k\\ &-\frac{\tau^{k\lambda}\Gamma(\lambda+1)}{\Gamma(k\lambda+1)}e^{(x+y)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^{k-1}.\end{aligned} \tag{6.4.24}$$

The solution of (6.4.15) is given as

$$\begin{aligned}&V(x,y,\tau,p) = max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}}+\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}}-E,0\}+\tau^{\lambda}e^{(x+y)}+\\ &\sum_{k=0}^{\infty}p^{(k+1)}\left(\frac{\tau^{(k+1)\lambda}}{\Gamma((k+1)\lambda+1)}\left(\left(\frac{\sigma_1^2}{2}\right)^{k+1}max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}},0\}+\left(\frac{\sigma_2^2}{2}\right)^{k+1}\right.\right.\\ &\left.max\{\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}},0\}\right)+\frac{\tau^{(k+2)\lambda}\Gamma(\lambda+1)}{\Gamma((k+2)\lambda+1)}e^{(x+y)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^{k+1}\\ &\left.-\frac{\tau^{(k+1)\lambda}\Gamma(\lambda+1)}{\Gamma((k+1)\lambda+1)}e^{(x+y)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^{k}\right).\end{aligned} \tag{6.4.25}$$

For $p \to 1$, the above equation becomes

$$\begin{aligned}&V(x,y,\tau) = V(x,y,\tau,1) = max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}}+\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}}-E,0\}+\\ &\tau^{\lambda}e^{(x+y)}+\sum_{k=0}^{\infty}\left(\frac{\tau^{(k+1)\lambda}}{\Gamma((k+1)\lambda+1)}\left(\left(\frac{\sigma_1^2}{2}\right)^{k+1}max\{\alpha e^{x+(r_1-d_1)T-\frac{\sigma_1^2 T}{2}},0\}+\left(\frac{\sigma_2^2}{2}\right)^{k+1}\right.\right.\\ &\left.max\{\beta e^{y+(r_2-d_2)T-\frac{\sigma_2^2 T}{2}},0\}\right)+\frac{\tau^{(k+2)\lambda}\Gamma(\lambda+1)}{\Gamma((k+2)\lambda+1)}e^{(x+y)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^{k+1}\\ &\left.-\frac{\tau^{(k+1)\lambda}\Gamma(\lambda+1)}{\Gamma((k+1)\lambda+1)}e^{(x+y)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^{k}\right).\end{aligned} \tag{6.4.26}$$

This is the required approximate analytical solution of the time-fractional two-dimensional Black-Scholes quanto option pricing model.

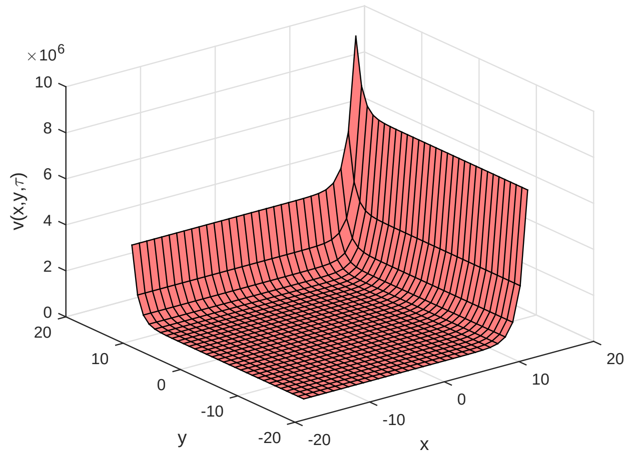
6.5 Numerical Demonstration and Discussion

In this section, we demonstrate the performance of the solution (6.4.26) on the basis of the parameters as given in Table 6.1; $V(x,y,\tau)$ is the price of the quanto option say, in INR, S_1 is the price of foreign risk asset say, in USD, S_2 is the exchange rate of the foreign currency against the domestic one. We observed

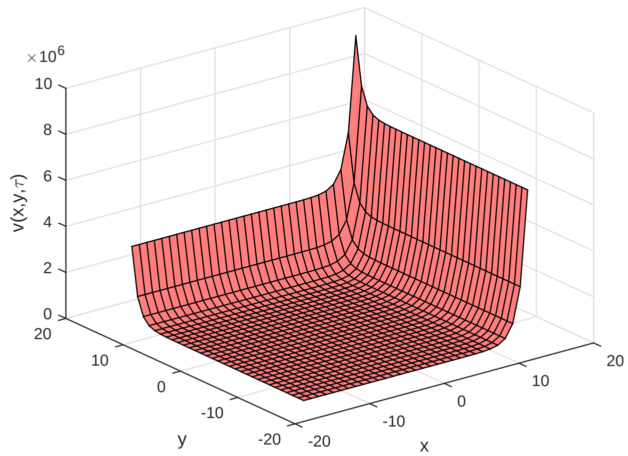
Quanto Options		
Parameters	Symbol	Value
Strike price of the option price S_1	E	100
Domestic risk-free interest rate	r_1	8%
Foreign risk-free interest rate	r_2	4%
Maturity time of the option (years)	T	1
Volatility of the underlying asset S_1	σ_1	0.2
Volatility of the underlying asset S_2	σ_2	0.1
Correlation of asset to domestic currency	ρ	0.2
Dividend	d	3%
Fractional-time derivative order	λ	0.9
Weight of asset S_1	α	2
Weight of asset S_2	β	1

Table 6.1: Selection of parameters

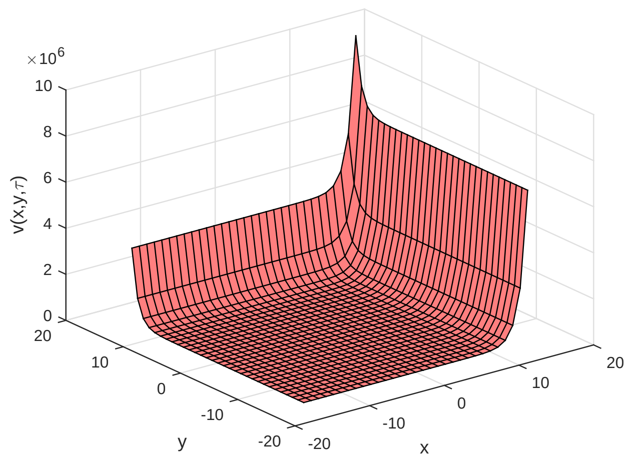
that as x and y increases, option value V increases exponentially at $\tau = 0, 0.5, 1$ refer to Fig.6.1. For $0 \leq \tau \leq 1$ and $0 \leq y \leq 2$, the option value V is zero when $-1 \leq x \leq 4$, and V increases exponentially when $x \geq 4$. But when y increases from 8 to 10, option value V becomes non-zero and rises exponentially with a higher increasing rate. Also, when y increases from 16 to 100, and τ becomes more dominating for V as compared to x, i.e. as τ increases V increases, see Fig.6.2. For $0 \leq \tau \leq 1$ and $0 \leq x \leq 2$, the option value V is zero when $-1 \leq y \leq 4$, and V increases exponentially when $y \geq 4$. But when x increases from 8 to 10, option value V becomes non-zero and rises exponentially with a further higher increasing rate. Also, when x increases from 16 to 100, and τ becomes more dominating for V as compared to y, i.e. as τ increases V increases, see Fig.6.3.



(a) time τ=0

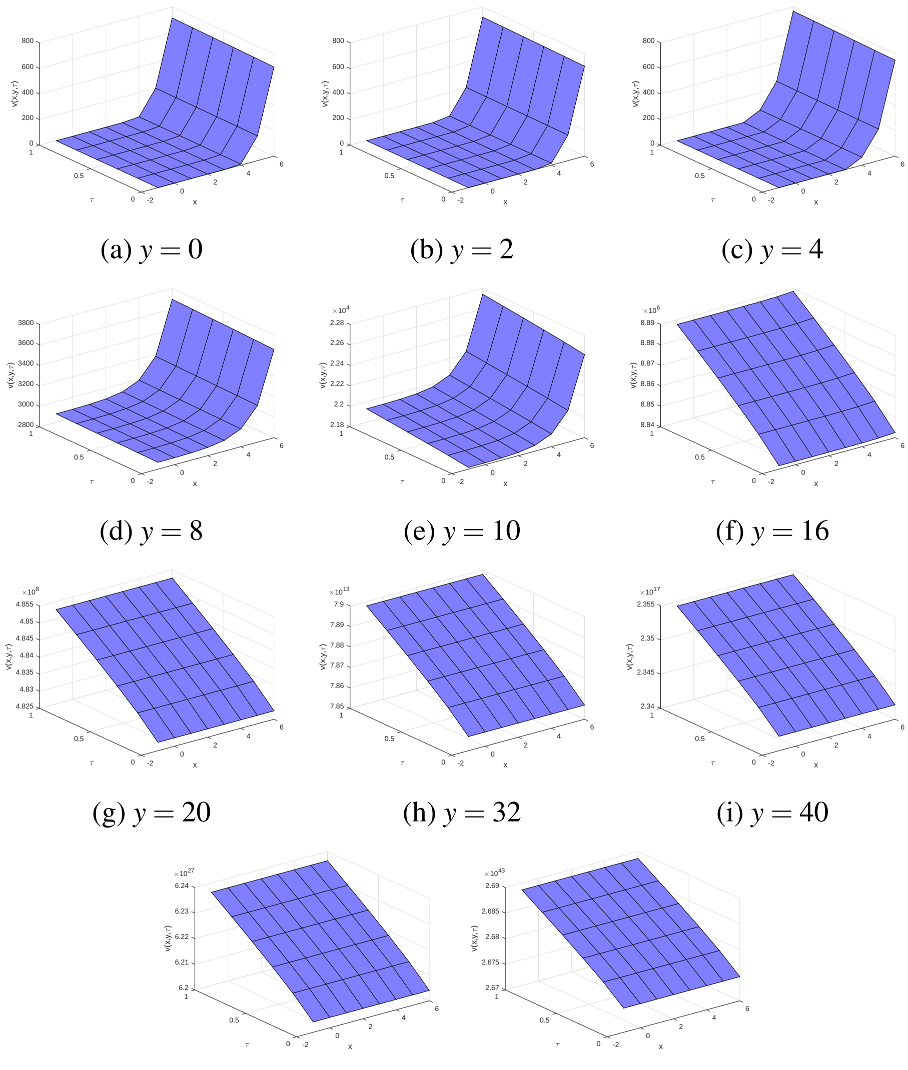


(b) time τ=0.5



(c) time τ=1

Figure 6.1: Performance of explicit solution on the option price $V(x,y,\tau)$ at $\tau = 0, 0.5, 1$



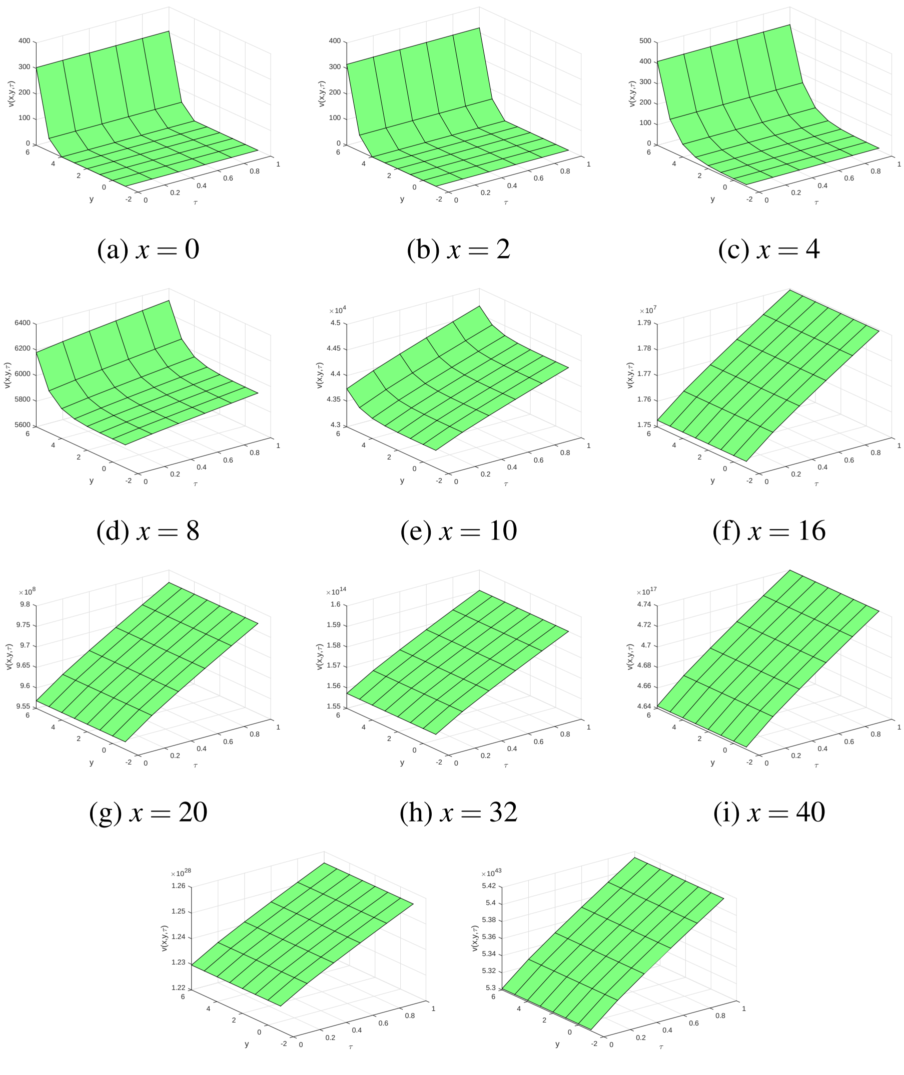
(a) $y = 0$ (b) $y = 2$ (c) $y = 4$

(d) $y = 8$ (e) $y = 10$ (f) $y = 16$

(g) $y = 20$ (h) $y = 32$ (i) $y = 40$

(j) $y = 64$ (k) $y = 100$

Figure 6.2: Performance of explicit solution on the option price $V(x, y, \tau)$ for different values of y.



(a) $x = 0$ (b) $x = 2$ (c) $x = 4$

(d) $x = 8$ (e) $x = 10$ (f) $x = 16$

(g) $x = 20$ (h) $x = 32$ (i) $x = 40$

(j) $x = 64$ (k) $x = 100$

Figure 6.3: Performance of explicit solution on the option price $V(x,y,\tau)$ for different values of x.

6.6 Conclusion

We have derived the Black-Scholes differential equation of the quanto option which gives the price of a European option when underlying financial assets price follows a geometric Brownian motion. Also by integrating the risk-neutral information measure, we have derived the risk-neutral probability density functions of multi-assets price, as the solution of minimizing Kullback relative entropy. We have applied Laplace transform homotopy perturbation method for the approximate analytical solution of the desired Black-Scholes equation where time derivative is assumed as a Liouville-Caputo time-fractional derivative. Numerical results for the assumed parameters demonstrate that the method is effective and this approach will help to study the financial behavior of the quanto option pricing problems.

Summary and Future Scope Of The Work

Here we summarize the work which has been presented in this thesis and also provide some scope for further investigations that can be performed based on the results reported.

Summary of the Work Reported

Entropy can be used as an alternative for volatility in the financial sector. The decisive advantage of this approach resides in its ability to capture disorder or uncertainty in the financial system without putting any constraint on the probability distribution function. In Chapter 2, we have modeled implied volatility as a linear combination of historical volatility and entropy and have found that the model was heavily dependent on entropy. Entropy studies the behavior of trends in the financial sector. For instance, sessions with several regular trends tend to be less entropic than those which are having relatively fewer occurrences. We have considered seven different estimators of Shannon entropy; Tsallis entropy and Renyi entropy for various values of their parameters; and also Approximate entropy and Sample entropy to characterize the volatility in stock market. We have done in-depth empirical analysis among generalized information theoretic measures and find that Sample entropy measures more the regularity of time series rather than its complexity; and in comparison with Approximate entropy, a similar class, Sample entropy is more consistent measure and provides improved analysis of the stock market regularity.

Also, we have taken into account the information theoretic concepts such as entropy, conditional entropy and mutual information and their dynamic extensions for studying the association among the randomness of different financial time series. We have also checked some equivalence between information theoretic measures and statistical measures normally employed to capture the randomness in financial time series.

In Chapter 3, we have examined different forecasting models in terms of their performance on a time series which is considered complex to predict. We have compared the performance of three models such as ARIMA, Holt-Winters and LSTM by applying Kullback measure of relative information, and find that traditional model ARIMA performs well on NIFTY 50 stock index monthly data and the performance of Holt-Winters is better for prediction of Dow Jones Industrial Average and S&P BSE SENSEX index data, in comparison with LSTM, an application of deep learning.

In Chapter 4, we have considered the problem of portfolio diversification by maximizing the entropy of the asset allocation. Further, since Shannon entropy can be considered as an alternative measure to the volatility (variance), so we have employed this concept for portfolio optimization by replacing the minimization of the variance of allocation with the minimization of entropy. This aspect has been studied using various measures of entropy. Information theoretic measures portfolios are generated by placing the specific measure in the objective function. Based on the empirical findings from the performance measures such as diversity index and the award-risk ratio, the selection of a well-diversified entropy portfolio model with the given level of return and minimum risk is carried out.

In Chapter 5, we have extended the concept of information theory to option price modeling. We have derived the risk-neutral measures of the stock options price and volatility by incorporating a minimization of the Kullback measure of relative information under a specified constraint. We have obtained a second-order parabolic partial differential equation, the generalized Black-Scholes equation based on the theoretical analysis when the underlying financial asset is estimated using a stochastic volatility model. To investigate the approximate analytical solution of this generalized Black-Scholes equation we have used the Laplace transform homotopy perturbation method.

In Chapter 6, by using the information theoretic approach we have obtained a two-dimensional parabolic partial differential equation, a Quanto-option Black-

Scholes equation, where the number of underlying assets is two. Also, we have applied the entropy dynamics of multi-asset to model the European options price and for this, we have derived the risk-neutral probability density functions of the multi-asset option price within the information-theoretic dynamic formalism. In addition, modeled as a Kolmogorov-backward equation, we specify how the probability function changes over a period of time. The numerical results for the assumed parameters demonstrate that the method is effective and this approach will help to study the financial behavior of the quanto option pricing problems.

Further Scope of the Work

While working on this thesis many ideas have emanated which can be useful for further study. The concept of stochastic entropy and extropy [61] in financial markets is still untouched. Stochastic entropy can be used as a substitute for stochastic volatility and this can change the entire approach of research in complex financial structures. The literature on extropy measure [43] and its relation with the Shannon entropy is available. The applications of extropy measures and its generalizations can be explored to discuss the trend and randomness in financial market.

Also, we can go beyond the Markowitz mean-variance framework and can account for higher moments too. By minimizing entropy measures, one can aim to get distribution as far away from the Gaussian as possible, and this is often best realized by introducing more and more constraints in the portfolio optimization problems.

The concept of information measures like Shannon entropy and one parametric entropy, such as Renyi and Tsallis entropies have been studied by many researchers, but still, it can be explored further for various two parametric entropy measures. The non-linear fluctuations in a variety of markets (i.e. stock, future, currency, commodity and cryptocurrency) require powerful tools like entropy for extracting information from the market that is otherwise not visible by standard statistical methods. Further, we can extend the information theoretic approach used in this thesis to a more generalized Black-Scholes European and American option pricing and other path-dependent option models.

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List of Publications

1. **Luckshay Batra** and H.C. Taneja. *Approximate-Analytical Solution to the Information Measure's Based Quanto Option Pricing Model*, *Chaos, Solitons and Fractals*, (2021); Elsevier, SCI, Impact Factor (5.944).

2. **Luckshay Batra** and H.C. Taneja. *On Black-Scholes option pricing model with stochastic volatility: an information theoretic approach*, *Stochastic Analysis and Applications*, 39(2), 327-338, (2021); Taylor & Francis, American Mathematical Society, SCIE, Impact Factor (1.530).

3. **Luckshay Batra** and H.C. Taneja. *Portfolio optimization based on generalised information theoretic measures*, *Communications in Statistics - Theory and Methods*, (2020); Taylor & Francis, American Mathematical Society, SCIE, Impact Factor (0.893).

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5. H.C. Taneja, **Luckshay Batra**, and Pulkit Gaur. *Entropy as a measure of implied volatility in options market*, *AIP Conference Proceedings*, 2183 (1), 110005, (2019).

6. **Luckshay Batra** and H.C. Taneja. *Comparison of forecasting models using information theoretic approach in financial market prediction*, **Communicated**.

7. **Luckshay Batra** and H.C. Taneja. *Information theoretic measures based analysis of financial markets network*, **Communicated**.

8. **Luckshay Batra** and H.C. Taneja. *Comparative study of information measures in portfolio optimization problems*, **Communicated**.

Papers Presented in Conferences

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1. **Luckshay Batra** and H.C.Taneja. *Comparative Study of Information Measures in Portfolio Optimization Problems.* Presented at IEEE SSIT Norbert Wiener in the 21st Century-Virtual Conference at Information Theory and Cybernetics session held at Anna University, Chennai, India, July 25, 2021.

2. **Luckshay Batra** and H.C.Taneja. *Optimizing financial markets using generalised information theoretical measures.* Presented at five days International Conference on Recent Trends in Mathematics and Its Applications to Graphs, Networks and Petri Nets (ICRTMA-GPN-2020) at applied mathematics session held at School of Computational and Integrative Sciences, JNU Delhi, India, July 20-24, 2020.

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4. H.C.Taneja, **Luckshay Batra**, and Pulkit Gaur. *Entropy as a measure of implied volatility in options market.* Presented at 3rd International Conference of Mathematical Sciences (ICMS 2019) at applied statistics session held at Maltepe University, Istanbul, Turkey, Sept. 04-08, 2019.
