CERTAIN APPROXIMATION METHODS OF CONVERGENCE FOR LINEAR POSITIVE OPERATORS

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By

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(Enrollment No.: 2K16/Ph.D/AM/04)

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DECLARATION

I declare that the research work reported in this thesis entitled “Certain Approximation Methods of Convergence For Linear Positive Operators" for the award of the degree of Doctor of Philosophy in Mathematics has been carried out by me under the supervision of Prof. Naokant Deo, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

The research work embodied in this thesis, except where otherwise indicated, is my original research. This thesis has not been submitted by me earlier in part or full to any other University or Institute for the award of any degree or diploma. This thesis does not contain other person's data, graphs or other information, unless specifically acknowledged.

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CERTIFICATE

This is to certify that the research work embodied in the thesis entitled **"Certain Approximation Methods of Convergence For Linear Positive Operators"** submitted by **Mr. Ram Pratap** with enrollment number (2K16/Ph.D/AM/04) is the result of his original research carried out in the Department of Applied Mathematics, Delhi Technological University, Delhi, for the award of Doctor of Philosophy under the supervision of Prof. Naokant Deo.

It is further certified that this work is original and has not been submitted in part or fully to any other University or Institute for the award of any degree or diploma.

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

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Date : **(Ram Pratap)**

Place : Delhi, India.

Dedicated to My Parents

&

Teachers

Contents

Abstract

The present thesis deals with the approximation behavior of various linear positive operators, their modifications, and approximation properties like rate of convergence, error estimation, and graphical comparison.

We segregate the thesis in seven chapters.

Chapter 1 is an introduction that contains a history of approximation theory, basic definitions, and converging tools which play an important role in approximation theory.

Chapter 2 is divided into three sections. In the first section, we have considered the Kantorovich form of $\alpha-$Bernstein operators introduced by Chen et al. [36]. We discussed some auxiliary properties and study the direct local approximation theorem, Voronovskaya type asymptotic, and function of bounded variation for $\alpha-$Bernstein-Kantorovich operators.
In the second section, we have considered the q-analogue of α-Bernstein Kantorovich operators. For these proposed operators, we studied some convergence properties by using first and second-order modulus of continuity.
In the last section, we have proposed the Stancu type generalization of the family of Bernstein Kantorovich operators involving parameter $\alpha \in [0,1]$ with Shifted Knots. These operators provide the pliability to approximate on the interval [0,1] and over its subintervals. For the proposed operators, we investigate some basic results of approximation and their rate of convergence in terms of first and second-order modulus of continuity, Lipschitz class, and Lipschitz-type function. We also estimate the global rate of convergence of the operators with the help of the Ditzian-Totik modulus of smoothness. Moreover, the p^{th} order generalization of the operators is established. Some numerical simulation and graphical comparisons are given for a better depiction of theoretical results.

In chapter 3, we propose the Kantorovich type generalized Szász-Mirakyan operators based on Jain and Pethe operators [101]. We study local approximation results in terms of classical modulus of continuity as well as Ditzian-Totik moduli of smoothness. Further, we establish the rate of convergence in a class of absolutely continuous functions having a derivative coinciding a.e. with a function of bounded variation.

In chapter 4, we propose the integral form of Jain and Pethe operators associated with the Baskakov operators and study some basic properties. We estimate the rate of convergence, Voronovskaja type asymptotic estimate formula, and weighted approximation of these operators.

In chapter 5, we consider new operators, which are defined by Gupta and Srivastava [90]. They considered a general sequence of positive linear operators and gave the modified form of their previous operators [142]. As these operators preserve linear functions, we call these operators as genuine Gupta-Srivastava operators. Here we discuss some basic properties, direct results, rate of convergence for a class of functions whose derivatives are of bounded variation, and weighted approximation for our considered operators.

In chapter 6, we propose the Bézier variant of the Gupta-Srivastava operators [90] and discuss some direct convergence results by using Lipschitz type spaces, Ditzian-Totik modulus of smoothness, weighted modulus of continuity, and for functions whose derivatives are of bounded variation. In the end, some graphical representation for comparison with other variants has been presented.

In chapter 7, we consider mixed approximation operators based on the second-kind beta transform by using Szász-Mirakyan operators. For the proposed operators, we establish direct result, Voronovskaya type theorem, quantitative Voronovskaya type theorem, Grüss Voronovskaya type theorem, weighted approximation, and functions of bounded variation.

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List of Symbols and Notations

$\mathbb{N}$	Set of natural numbers.
$\mathbb{N}_0$	the set of natural number including zero,
$\mathbb{R}$	Set of real numbers,
$\mathbb{R}^+$	the set of positive real numbers,
$[a,b]$	a closed interval,
(a,b)	an open interval,
$C[a;b]$	the set of all real-valued and continuous function defined in $[a,b]$
$C^r[a,b]$	the set of all real-valued, r-times continuously differentiable function $(r \in \mathbb{N})$,
$C[0,\infty)$	the set of all continuous functions defined on $[0,\infty)$,
$C_B[0,\infty)$	the set of all bounded functions on $C[0,\infty)$,
$C_B^r[0,\infty)$	the set of all r-times continuously differentiable functions in $C_B[0,\infty)$ $(r \in \mathbb{N})$
$B_2[0,\infty)$	the set of all functions φ defined on $[0,\infty)$ satisfying the condition $\lvert\varphi(x)\rvert \leq M(1+x^2)$, M is a positive constant,
$C_2[0,\infty)$	the subspace of all continuous function in $B_2[0,\infty)$,
$C_2^*[0,\infty)$	the subspace of all function $\varphi \in C_2[0,\infty)$ for which $\lim\limits_{x\to\infty} \frac{\varphi(x)}{1+x^2}$ exists and finite.
$\lVert.\rVert_*$	$\lVert\varphi\rVert_* = \mathop{Sup}\limits_{x>0} \frac{\lvert\varphi\rvert}{1+x^2}$,
$Lip_\xi M$	the set of all $C[a;b]$- functions which holds the Lipschitz condition $\lvert\varphi(t)-\varphi(x)\rvert \leq M\lvert t-x\rvert^\xi$ for all $t,x \in [a,b]$, $0<\xi\leq 1$, $M>0$,
$\prod_n$	the linear space of all real polynomials with the degree at most n,
e_n	denotes the n-th monomials with $e_n : [a,b] \to \mathbb{R}$, $e_n(x) = x^n$, $n \in \mathbb{N}_0$,

Chapter 1

Introduction

In this introductory chapter, we present an extensive literature of the title of doctoral thesis, a part of the field approximation theory. Concretely, we present with essential definitions and certain rudimentary properties in connection with linear positive operators which plays a significant role in approximation theory. This chapter easily explains our interest in the field.

Approximation Theory is a branch of mathematical analysis that allows us to approximate the given real-valued continuous functions to simple functions like algebraic and trigonometric polynomials. For over the decades, the convergence of such sequences is a broad and useful area in approximation theory.

Approximation theory plays an important role in both aspects computational as well as theoretical. In a computational aspect, it deals with computational practicalities, precise error estimation, the order of approximation, and many more. On the other side, it plays an important role in concern with existence and uniqueness questions, and applications in other theoretical parts. Mainly, based on theoretical aspects, we begin our introduction with a short history.

1.1 Historical background

The foundation of approximation theory was laid down by P. L. Chebyshev in 1853 by introducing an interesting problem:

A continuous function φ in $[a,b]$ can be written by a polynomials $p(x) = \sum_{j=0}^{n} d_j x^j$ with

at most degree n ($n \in \mathbb{N}$). In such condition, the maximum error can be minimized for any $x \in [a,b]$ by controlling error $\max_{x\in[a,b]} |\varphi(x) - p(x)|$.
For better understanding, the fundamental development of Chebyshev problem was realized by Karl Weierstrass in 1885 in such a way:
For $\varphi \in C[a,b]$ and $\varepsilon > 0$, then $\exists$ a real polynomial $p(x)$ in such a way

$$|\varphi(x) - p(x)| < \varepsilon,\ x \in [a,b].$$

For $\varphi \in C[a,b]$ and $\varepsilon > 0$, then $\exists$ a real polynomial $p(x)$ in such a way

$$|\varphi(x) - p(x)| < \varepsilon,\ x \in [a,b].$$

The proof provided by Weierstrass was complicated and not easy to understand. In 1905, E. Borel [33] introduced the interpolation process which provides a polynomial $p(x)$ that converges uniformly in $[a,b]$. For a detailed explanation and to enhance the knowledge in approximation theory, we refer some books to readers [14, 51].

1.2 Positive linear operators

Let A be the space of linear functions, then the mapping $L_n : A \to A$ is said to be positive linear operators if it fulfills the properties given below:

(i) $L_n(a\varphi + b\psi; x) = aL_n(\varphi; x) + bL_n(\psi; x)$

(ii) $L_n(\varphi; x) \geq 0,\ \forall\ \varphi \geq 0,$

where $\varphi, \psi \in A$ and $a, b \in \mathbb{R}$.

Proposition 1.2.1. *Let $L_n : A \to A$ be a positive linear operators. Then the following inequalities holds:*

(i) If $\varphi, \psi \in A$ with $\varphi \leq \psi$, then $L_n(\varphi; x) \leq L_n(\psi; x)$.

(ii) $\forall\ \varphi \in A$, we have $|L_n(\varphi; x)| \leq L_n(|\varphi|; x)$.

1.2.1 The Bernstein operators

The simple and easy proof of Weierstrass theorem was provided by S.N. Bernstein [30] in 1912 by proposing polynomials known as Bernstein polynomials or operators:

$$B_n(\varphi;x)=\sum_{k=0}^{n}p_{n,k}(x)\varphi\left(\frac{k}{n}\right),\quad x\in[0,1],\tag{1.1}$$

and $p_{n,k}(x)=\begin{pmatrix} n \\ k \end{pmatrix}x^k(1-x)^{n-k}$. These operators prove Weierstrass theorem in $C[0,1]$ and can be extended to $C[a,b]$ by linear substitution.

1.2.2 The Szász-Mirakyan operators

In 1941 S. Mirakyan and 1950 O. Szász [147] introduced the extension of the Bernstein operators for half non-negative real axis $\mathbb{R}^+$ as:
If $\varphi\in C(\mathbb{R}^+)$, the Szász-Mirakyan operators Q_n are given by

$$Q_n(\varphi;x)=\sum_{k=0}^{\infty}e^{-nx}\frac{(nx)^k}{k!}\varphi\left(\frac{k}{n}\right),\quad x\in\mathbb{R}^+.\tag{1.2}$$

1.2.3 The Baskakov operators

Another extension of Bernstein operators for $\mathbb{R}^+$ given by Baskakov [29] in 1957 are as follows:

$$V_n(\varphi;x)=\sum_{k=0}^{\infty}\begin{pmatrix} n+k-1 \\ k \end{pmatrix}\frac{x^k}{(1+x)^{n+k}}\varphi\left(\frac{k}{n}\right),\quad x\in\mathbb{R}^+.\tag{1.3}$$

1.2.4 The Bernstein-Kantorovich operators

To approximate Lebesgue integral functions, the above-proposed operators are not suitable. For such functions L. V. Kantorovich [110] proposed the integral form of (1.1) as:

$$\hat{P}_n(\varphi;x)=(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\varphi(t)dt.\tag{1.4}$$

1.2.5 The Szász-Kantorovich operators

Totik [149] proposed the integral form of (1.2) which are given as:

$$\hat{Q}_n(\varphi;x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} \varphi(t)dt, \tag{1.5}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

1.2.6 The Baskakov-Kantorovich operators

Introduced by Totik [149], the Kantorovich form of Baskakov operators (1.3) are given by

$$\hat{V}_n(\varphi;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} \varphi(t)dt, \tag{1.6}$$

where

$$v_{n,k}(x) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix}\frac{x^k}{(1+x)^{n+k}}.$$

1.2.7 The Bernstein-Durrmeyer operators

A more generalized integral modification of Bernstein operators obtained by replacing $\varphi\left(\frac{k}{n}\right)$ over integral basis function in operators (1.1). This integral form introduced by Durrmeyer [58] and known as Bernstein-Durrmeyer operators are given by

$$\tilde{T}_n(\varphi;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\,\varphi(t)dt. \tag{1.7}$$

1.2.8 The Szász-Durrmeyer operators

Mazhar & Totik [121] in 1985 studied the integral modification of (1.2) and termed as Szász-Durrmeyer operators. They are represented by

$$\tilde{S}_n(\varphi;x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k}(t)\,\varphi(t)dt. \tag{1.8}$$

1.2.9 The Baskakov-Durrmeyer operators

Sahai & Prasad [140] in 1985 introduced the Durrmeyer form of Baskakov operators, which are given as:

$$\tilde{M}_n(\varphi;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x)\int_0^{\infty} v_{n,k}(t)\,\varphi(t)dt. \tag{1.9}$$

1.3 Qualitative result

The next result provides the most impactful conditions for which the function approximation process by linear positive operators is realizable. This is a qualitative problem and the result is known as Bohman-Korovkin theorem:

If these three conditions $L_n(e_0;x) = 1 + p_n(x)$, $L_n(e_1;x) = x + q_n(x)$ and $L_n(e_2;x) = x^2 + r_n(x)$ are satisfied by the sequence of linear positive operators $L_n : C[a,b] \to C[a,b]$ such that $p_n(x), q_n(x)$ and $r_n(x)$ uniformly converges to 0 in $[a,b]$ then $L_n(\varphi;x) \to \varphi(x)$ uniformly for any function $\varphi \in C[a,b]$. This outcome was presented by three mathematicians autonomously in three succeeding years T. Popoviciu [136] in 1551, H. Bohman [32] in 1952, and P. P. Korovkin [113] in 1953 individually. The contribution of T. Popoviciu was acknowledged after a very long time, therefore, this result remained as Bohman-Korovkin theorem.

This result fails to establish uniform convergence in unbounded interval therefore some limitations are required. Introduce

$$C^*(\mathbb{R}^+) = \left\{\varphi \in C(\mathbb{R}^+) : |\varphi(x)| \leq C_\varphi(1+x^2) \text{ and} \lim_{x\to\infty}\frac{|\varphi(x)|}{1+x^2} = K_\varphi < \infty\right\}.$$

epped with the norm

$$\|\varphi\|_* = \sup_{x\geq 0}\frac{|\varphi(x)|}{1+x^2}$$

and C_φ is a positive constant that depends on function φ.

In view of Gadjiev papers [63, 64], the Korovkin type theorem in unbounded interval holds in the space $C^*(\mathbb{R}^+)$ and has the following form:

"A sequence of positive linear operators L_n which satisfy the conditions

$$\lim_{n\to\infty} \left\| L_n(e_i;x) - x^i \right\|_* = 0, \quad i = 0,1,2,$$

we get that for any function $\varphi \in C^*(\mathbb{R}^+)$

$$\lim_{n\to\infty} \|L_n(\varphi(t;x) - \varphi\|_* = 0,"$$

where $e_m(x) = x^m$, for $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are the test functions.

1.4 Preliminaries

Definition 1.4.1. H. Lebesgue in 1910 introduced the modulus of continuity for $\varphi \in C[a,b]$ and $j \in \mathbb{N}$ as follows:

$$\omega_j(\varphi;\delta) = \sup\left\{ \left| \Delta_k^j \varphi(x) \right| ; |k| \leq \delta, x \in [a,b] \right\}, \delta \geq 0. \tag{1.10}$$

For $j = 1$, it denotes the first order modulus of continuity $\omega(.;\delta)$ and given by

$$\omega(\varphi;\delta) = \max_{|x-t|\leq\delta} |\varphi(x) - \varphi(t)|, x,t \in [a,b]. \tag{1.11}$$

This definition is also shown in Ph.D. thesis of Jackson [100] in 1911.

Proposition 1.4.1. *For $\varphi \in C[a,b]$, $\omega(\varphi;.)$ hold following properties:*

(i) *$\omega(\varphi;.)$ is non-decreasing, non-negative and uniformly continuous function in $\mathbb{R}^+$.*

(ii) *When $\delta^+ \to 0$ then $\omega(\varphi,\delta) = 0$.*

(iii) *For each $s \geq 0$, the following inequality holds:*

$$\omega(\varphi, s\delta) \leq (1+s)\omega(\varphi,\delta), \ \delta \geq 0.$$

(iv) *For each $x,t \in [a,b]$, it follows:*

$$|\varphi(x) - \varphi(t)| \leq \omega(\varphi, |x-t|) \leq \left(1 + \frac{|x-t|}{\delta}\right) \omega(\varphi,\delta), \ \delta \geq 0.$$

Definition 1.4.2. In 1963, Peetre [134] introduced another tool to estimate the smoothness of function for $\varphi \in C_B[a,b]$ termed as Peetre-K-Functional and given by

$$K_2(\varphi;\delta) = \inf_{\psi \in C_B^2[a,b]} \left\{ \|\varphi - \psi\| + \delta \left\| \psi'' \right\| \right\}, \delta > 0 \tag{1.12}$$

where $C_B^2[a,b] = \{\psi \in C_B[a,b] : \psi', \psi'' \in C_B[a,b]\}$. There exists $M > 0$ such that

$$K_2\left(\varphi, \delta^2\right) \leq M\omega_2\left(\varphi, \delta\right) \tag{1.13}$$

conformable from([52]).

Definition 1.4.3. Let $\lambda \in [0,1]$, $\phi(x) : \mathbb{R}^+ \to \mathbb{R}$ is an admissible weight function and $\varphi \in C_B(\mathbb{R}^+)$, conformable ([79], p. 51) or [103], the Ditzian-Totik modulus of smoothness are defined as

$$\omega_{\phi^\lambda}(\varphi,\delta) = \sup_{0 \leq h \leq \delta} \sup_{x \pm \frac{h\phi^\lambda(x)}{2} \in \mathbb{R}^+} \left| \varphi\left(x + \frac{h\phi^\lambda(x)}{2}\right) - \varphi\left(x - \frac{h\phi^\lambda(x)}{2}\right) \right|, \tag{1.14}$$

$$\omega_{\phi^\lambda}^2\left(\varphi,\delta\right) = \sup_{0 < h \leq \delta} \sup_{x \pm h\phi^\lambda(x) \in \mathbb{R}^+} \left| \Delta_{h\phi^\lambda}^2 \varphi\left(x\right) \right|, \delta > 0, \tag{1.15}$$

where

$$\Delta_{h\phi^\lambda}^2 \varphi\left(x\right) = \varphi\left(x + h\phi^\lambda\left(x\right)\right) - 2\varphi\left(x\right) + \varphi\left(x - h\phi^\lambda\left(x\right)\right).$$

and their K-functionals are:

$$K_{\phi^\lambda}(\varphi,\delta) = \inf_{\psi \in W_\lambda} \left\{ \|\varphi - \psi\| - \delta \left\| \phi^\lambda g' \right\| \right\}, \tag{1.16}$$

$$K_{\phi^\lambda}^2\left(\varphi,\delta^2\right) = \inf_{\psi \in D_\lambda^2} \left\{ \|\varphi - \psi\| + \delta^2 \left\| \phi^{2\lambda} \psi'' \right\| \right\}, \tag{1.17}$$

where $W_\lambda = \left\{ \varphi \in C_B(\mathbb{R}^+) : \varphi' \in A.C._{loc}(\mathbb{R}^+), \left\| \phi^\lambda \varphi' \right\| < \infty \right\}$,
$D_\lambda^2 = \left\{ \varphi \in C_B(\mathbb{R}^+) : \varphi' \in \text{A.C.}_{\text{loc}}(\mathbb{R}^+), \left\| \phi^{2\lambda} \varphi'' \right\| < \infty \right\}$. and $A.C._{loc}$ means locally absolutely continuous functions φ on $\mathbb{R}^+$.

In [[55], Theorem 2.1.1], there exists a constant $M > 0$ such that

$$M^{-1}\omega_{\phi^\lambda}(\varphi,\lambda) \leq K_{\phi^\lambda}(\varphi,\delta) \leq M\omega_{\phi^\lambda}(\varphi,\delta). \tag{1.18}$$

and

$$M^{-1}\omega_{\phi^\lambda}^2(\varphi,\delta) \leq K_{\phi^\lambda}^2(\varphi,\delta^2) \leq M\omega_{\phi^\lambda}^2(\varphi,\delta). \tag{1.19}$$

Definition 1.4.4. In [131, 132], the Lipschitz-type spaces are as follows:

$$Lip_M(\xi) = \left\{ \varphi \in C_B(\mathbb{R}^+) : |\varphi(t) - \varphi(x)| \leq M \frac{|t-x|^\xi}{(t+x)^{\xi/2}} \right\}. \tag{1.20}$$

For fixed $a, b > 0$, we have

$$Lip_M^{a,b}(\xi) := \left\{ \varphi \in C_B(\mathbb{R}^+) : |\varphi(t) - \varphi(x)| \leq M \frac{|t-x|^\xi}{(t+ax^2+bx)^{\xi/2}}; x, t \in \mathbb{R}^+ \right\}, \tag{1.21}$$

where $M > 0$ and $\xi \in (0, 1]$.

Definition 1.4.5. For $\varphi \in C_2(\mathbb{R}^+)$, Ispir [97] introduced the weighted modulus of continuity $\Omega(\varphi;\delta)$ as:

$$\Omega(\varphi;\delta) = \sup_{0 \leq \beta < \delta} \frac{|\varphi(x+\beta) - \varphi(x)|}{(1+\beta^2)(1+x^2)}. \tag{1.22}$$

Yüksal and Ispir [154] also considered the alternate form of above definition as follows:

$$\Omega(\varphi;\delta) = \sup_{x \in \mathbb{R}^+, 0 < \beta < \delta} \frac{\varphi(x+\beta) - \varphi(x)}{1 + (x+\beta)^2}. \tag{1.23}$$

Proposition 1.4.2. *It is shown in [[97] ,p.359-360] that for every $\varphi \in C_2(\mathbb{R}^+)$, $\Omega(.;\delta)$ has the properties:*

(i) $\lim_{\delta \to 0} \Omega(\varphi;\delta) = 0$;

(ii) $\Omega(\varphi; d\delta) \leq 2(1+d)(1+\delta^2)\Omega(\varphi;\delta) \quad p > 0.$;

(iii) $|\varphi(t) - \varphi(x)| \leq (1+(t-x)^2)(1+x^2)\Omega(\varphi;|t-x|)$
$\leq 2\left(1 + \frac{|t-x|}{\delta}\right)(1+\delta^2)\Omega(\varphi;\delta)(1+(t-x)^2)(1+x^2).$

Definition 1.4.6. Let DBV$[a,b]$ denotes the class of all absolutely continuous functions φ defined on the interval $[a,b]$, having a derivative φ' equivalent with a function of bounded variation on $[a,b]$. It may be observed that for $\varphi \in DBV[a,b]$, we may write

$$\varphi(x) = \int_0^x \psi(t)dt + \varphi(0),$$

where $\varphi(t)$ is a function of bounded variation on $[a,b]$.

Chapter 2

The family of Bernstein-Kantorovich operators

2.1 α-Bernstein-Kantorovich operators

2.1.1 Introduction

Chen et al. [36] in the year 2018, introduced the generalization of Bernstein operators for fixed α, $0 \leq \alpha \leq 1$, $\varphi \in C[0,1]$ and $n \in \mathbb{N}$ as follows:

$$B_{n,\alpha}(\varphi;x) = \sum_{k=0}^{n} \varphi_k p_{n,k}^{(\alpha)}(x), \;\; x \in [0,1], \tag{2.1}$$

where $\varphi_k = \varphi\left(\frac{k}{n}\right)$. For $n > 2$ the basis function $p_{n,k}^{(\alpha)}(x)$ of degree n is defined by $p_{1,0}^{(\alpha)}(x) = 1-x$, $p_{1,1}^{(\alpha)}(x) = x$, and

$$p_{n,k}^{(\alpha)}(x) = (1-\alpha)(p_{n-2,k}(x) + p_{n-2,k-2}(x)) + \alpha p_{n,k}(x). \tag{2.2}$$

Where $p_{n,k}(x)$ given in (1.1) and they studied another interesting proof of the Weierstrass approximation theorem [152]. They also studied its fundamental properties, the rate of convergence, and the Voronovskaya type asymptotic estimate formula. When $\alpha = 1$, the operators (2.1) reduce to the classical Bernstein operators (1.1).

Many authors have discussed Kantorovich modification of various linear positive operators see [3, 8, 13, 21, 54, 69, 91, 106, 116, 124, 130] and studied several approximation results.

Here, we consider the Kantorovich form of operators (2.1) as follows:

$$\hat{K}_{n,\alpha}(\varphi(t);x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \varphi(t)\,dt, \tag{2.3}$$

Very recently, Acar et al. [3] introduced the genuine Bernstein-Durrmeyer type operators with parameter α. They studied several approximation results and also shown graphically.

This section aims to give some basic properties and study the direct local approximation theorem, Voronovskaya type asymptotic estimate formula, and functions whose derivatives are of bounded variation for our considered operators (2.3).

2.1.2 Preliminaries

Here, we give some auxiliary results which help us to prove main results

Lemma 2.1.1. *[124] For $0 \leq x \leq 1$ and $n \in \mathbb{N}$, we have*

i) $\hat{K}_{n,\alpha}(1;x) = 1$;

ii) $\hat{K}_{n,\alpha}(t;x) = \frac{2nx+1}{2(n+1)}$;

iii) $\hat{K}_{n,\alpha}(t^2;x) = \frac{1}{3(n+1)^2}\left[3\left\{\left(n^2-n-2(1-\alpha)\right)x^2+(4n+2(1-\alpha))x\right\}+1\right]$.

Lemma 2.1.2. *[124] The m^{th} moments are as follows:*

$$\mu_n^m(x) = \hat{K}_{n,\alpha}((t-x)^m;x),$$

then we have

(i) $\mu_n^1(x) = \frac{1-2x}{2(n+1)}$;

(ii) $\mu_n^2(x) = \frac{1}{3(n+1)^2}\left\{3(-n+2\alpha-1)x^2+3(n-2\alpha+1)x+1\right\}$;

(iii) $\mu_n^4(x) = \frac{1}{5(n+1)^4}\left\{5\left(3n^2-4(2+3\alpha)n-(131-132\alpha)\right)x^4 + 10\left(-3n^2+4(2+3\alpha)n+(131-132\alpha)\right)x^3 + 5\left(3n^2-(13+12\alpha)n-2(80-81\alpha)\right)x^2+5(5n+(33-32\alpha))x+1\right\}$.

Lemma 2.1.3. *For adequately large n, we have* *(i)* $\lim_{n\to\infty} n\mu_n^1(x) = \frac{1-2x}{2}$;

(ii) $\lim_{n\to\infty} n\mu_n^2(x) = x(1-x)$;

(iii) $\lim_{n\to\infty} n^2\mu_n^4(x) = 3(x(1-x))^2$.

Lemma 2.1.4. *The following inequalities hold as a consequence of Lemma 2.1.2 for adequately large n:*

(i) $\mu_n^1(x) \le \frac{1}{2(n+1)}$;

(ii) $\mu_n^2(x) \le \frac{1}{4(n+1)}$.

Proof. From Lemma 2.1.2, we have

$$\mu_n^1(x) = \frac{(1-2x)}{2(n+1)}.$$

The maximum value of $(1-2x)$ is 1 on $[0,1]$, therefore

$$\mu_n^1(x) \le \frac{1}{2(n+1)}.$$

Also

$$\mu_n^2(x) \le \frac{1}{(n+1)}x(1-x) \le \frac{1}{4(n+1)}.$$

Maximum value of $x(1-x)$ is $\frac{1}{4}$ in $[0,1]$. □

Lemma 2.1.5. *Let $\varphi \in C[0,1]$ and α be a fixed number then, we have*

$$\left\|\hat{K}_{n,\alpha}(\varphi(t);x)\right\| \le \|\varphi\|.$$

Proof. From the operators (2.3), we obtain

$$\begin{aligned}\left|\hat{K}_{n,\alpha}(\varphi(t);x)\right| &= \left|(n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \varphi(t)dt\right| \\ &\le (n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |\varphi(t)|\,dt \\ &\le \hat{K}_{n,\alpha}(1;x)\,\|\varphi\| = \|\varphi\|.\end{aligned}$$

□

2.1.3 Direct results

Here, we discuss the rate of convergence for the proposed operators (2.3) in terms of the usual Lipschitz class $Lip_{\xi}M$

Theorem 2.1.6. *For every $\varphi \in Lip_{\xi}M$, we have*

$$\left|\hat{K}_{n,\alpha}(\varphi(t);x)-\varphi(x)\right| \leq M\left(\mu_n^2(x)\right)^{\frac{\xi}{2}},$$

where

$$\mu_n^2(x)=\frac{1}{3(n+1)^2}\left\{3\left(-n+2\alpha-1\right)x^2+3\left(n-2\alpha+1\right)x+1\right\},$$

and $\xi > 0$.

Proof. By monotonicity property of the operators $\hat{K}_{n,\alpha}(.;x)$, we get

$$\begin{aligned}\left|\hat{K}_{n,\alpha}(\varphi(t);x)-\varphi(x)\right| &\leq \hat{K}_{n,\alpha}\left(\left|\varphi(t)-\varphi(x)\right|;x\right)\\ &\leq (n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left|\varphi(t)-\varphi(x)\right|dt\\ &\leq (n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}|t-x|^{\xi}dt.\end{aligned}$$

Making use of Hölder's inequality, we acquire

$$\begin{aligned}\left|\hat{K}_{n,\alpha}(\varphi(t);x)-\varphi(x)\right| &\leq M\left[(n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2dt\right]^{\frac{\xi}{2}}\\ &\qquad\times\left[(n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}dt\right]^{\frac{2-\xi}{2}}\\ &= M\left(\mu_n^2(x)\right)^{\frac{\xi}{2}}\end{aligned}$$

□

2.1.4 Function of bounded variation

For our convenience the alternative form of the operators (2.3) may be written as:

$$\hat{K}_{n,\alpha}\left(\varphi(t);x\right)=\int_0^1 U_{n,\alpha}\left(x;t\right)\varphi(t)dt, \tag{2.4}$$

where

$$U_{n,\alpha}(x,t) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\,\chi_{n,k}(t),$$

and $\chi_{n,k}(t)$ indicates the characteristic function in $\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$ w.r.t $[0,1]$.

Lemma 2.1.7. *For $0 \leq x \leq 1$ and for adequately large n, we obtain*

(i) Since $0 \leq y < x$, therefore

$$\beta_n(x,y) = \int_0^y U_{n,\alpha}(x,t)\,dt \leq \frac{C}{4(n+1)(x-y)^2}.$$

(ii) If $x < z \leq 1$, we get

$$1 - \beta_n(x,z) = \int_z^1 U_{n,\alpha}(x,t)\,dt \leq \frac{C}{4(n+1)(z-x)^2}.$$

Theorem 2.1.8. *For each $x \in (0,1)$, $\varphi \in DBV[0,1]$ and adequately large n, we have*

$$\begin{aligned}
\left|\hat{K}_{n,\alpha}(\varphi(t);x) - \varphi(x)\right| &\leq \frac{1}{4(n+1)}\left|\varphi'(x+) + \varphi'(x-)\right| + \frac{1}{4}\sqrt{\frac{1}{(n+1)}}\left|\varphi'(x+) - \varphi'(x-)\right| \\
&+ \frac{x}{\sqrt{n}} V_{x-\frac{x}{\sqrt{n}}}^{x}(\varphi_x') + \frac{C}{4(n+1)x}\sum_{s=1}^{[\sqrt{n}]} \vee_{x-\frac{x}{k}}^{x}(\varphi_x'). \\
&+ \frac{(1-x)}{\sqrt{n}} \overset{x+(1-x)/\sqrt{n}}{\underset{x}{V}} (\varphi')_x + \frac{C}{4(n+1)(1-x)}\sum_{s=1}^{[\sqrt{n}]} \overset{x+(1-x)/k}{\underset{x}{V}} ((\varphi')_x),
\end{aligned}$$

where $\vee_a^b \varphi(x)$ denotes the total variation of φ on $[0,1]$ and φ_x is an auxiliary operator given by

$$\varphi_x(t) = \begin{cases} \varphi(t) - \varphi(x-), & 0 \leq t < x \\ 0, & t = x \\ \varphi(t) - \varphi(x+), & x < t < 1. \end{cases}$$

Proof. Since $\hat{K}_{n,\alpha}(1;x) = 1$, for all $x \in [0,1]$, we have

$$\begin{aligned}
\hat{K}_{n,\alpha}(\varphi(t);x) - \varphi(x) &= \int_0^1 (\varphi(t) - \varphi(x))U_{n,\alpha}(x;t)\,dt \\
&= \int_0^1 U_{n,\alpha}(x;t)\left(\int_x^t \varphi'(u)du\right)dt, \qquad (2.5)
\end{aligned}$$

for $\varphi \in DBV[0,1]$, we can write

$$\varphi'(u) = \frac{1}{2}\left(\varphi'(x+) + \varphi'(x-)\right) + \varphi_x{}'(u) + \frac{1}{2}\left(\varphi'(x+) - \varphi'(x-)\right) sgn(u-x) \\ + \delta_x(u)\left(\varphi'(u) - \frac{1}{2}\left(\varphi'(x+) + \varphi'(x-)\right)\right), \tag{2.6}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x, \\ 0, & u \neq x. \end{cases}$$

It is obvious that

$$\int_0^1 \left(\int_x^t \left(\varphi'(u) - \frac{1}{2}\left(\varphi'(x+) + \varphi'(x-)\right) \right) \delta_x(u)\, du \right) U_{n,\alpha}(x,t)\, dt = 0. \tag{2.7}$$

Using (2.4), we have

$$\int_0^1 \left(\int_x^t \frac{1}{2}\left(\varphi'(x+) + \varphi'(x-)\right) du \right) U_{n,\alpha}(x,t)\, dt \\ = \frac{1}{2}\left(\varphi'(x+) + \varphi'(x-)\right) \hat{K}_{n,\alpha}((t-x);x). \tag{2.8}$$

Moreover,

$$\int_0^1 \left(\int_x^t \frac{1}{2}\left(\varphi'(x+) - \varphi'(x-)\right) sgn(u-x)\, du \right) U_{n,\alpha}(x,t)\, dt \\ = \int_0^1 \frac{1}{2}\left(\varphi'(x+) - \varphi'(x-)\right)(t-x)\, U_{n,\alpha}(x,t)\, dt \\ \leq \frac{1}{2}\left|\varphi'(x+) - \varphi'(x-)\right| \int_0^1 |t-x|\, U_{n,\alpha}(x,t)\, dt \\ \leq \frac{1}{2}\left|\varphi'(x+) - \varphi'(x-)\right| \hat{K}_{n,\alpha}(|t-x|;x) \\ \leq \tfrac{1}{2}\left|\varphi'(x+) - \varphi'(x-)\right| \left(\hat{K}_{n,\alpha}\left((t-x)^2;x\right)\right)^{1/2}. \tag{2.9}$$

Using Lemma 2.1.3 and from equation (2.6) to (2.9), we obtain

$$\begin{aligned}
\hat{K}_{n,\alpha}(\varphi;x)-\varphi(x) &\leq \frac{1}{2}\left(\varphi'(x+)+\varphi'(x-)\right)\hat{K}_{n,\alpha}((t-x);x) \\
&\quad+\frac{1}{2}\left|\varphi'(x+)-\varphi'(x-)\right|\left(\hat{K}_{n,\alpha}\left((t-x)^2;x\right)\right)^{1/2} \\
&\quad+\int_0^1\left(\int_x^t {\varphi_x}'(u)\,du\right)U_{n,\alpha}(x,t)\,dt \\
&\leq \frac{1}{4(n+1)}\left(\varphi'(x+)+\varphi'(x-)\right)+\frac{1}{4}\sqrt{\frac{1}{(n+1)}}\left|\varphi'(x+)-\varphi'(x-)\right| \\
&\quad+\int_0^1\left(\int_x^t {\varphi_x}'(u)\,du\right)U_{n,\alpha}(x,t)\,dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|\hat{K}_{n,\alpha}(\varphi;x)-\varphi(x)\right| &\leq \frac{1}{4(n+1)}\left|\varphi'(x+)+\varphi'(x-)\right|+\frac{1}{4}\sqrt{\frac{1}{(n+1)}}\left|\varphi'(x+)-\varphi'(x-)\right| \\
&\quad+J_{1,n,x}+J_{2,n,x}, \qquad (2.10)
\end{aligned}$$

where

$$J_{1,n,x}=\left|\int_0^x\left(\int_x^t {\varphi_x}'(u)\,du\right)U_{n,\alpha}(x,t)\,dt\right|,$$

and

$$J_{2,n,x}=\left|\int_x^1\left(\int_x^t {\varphi_x}'(u)\,du\right)U_{n,\alpha}(x,t)\,dt\right|.$$

Applying Lemma 2.1.5, by integrating by parts and take $y=x-\frac{x}{\sqrt{n}}$, we acquire

$$\begin{aligned}
J_{1,n,x} &= \left|\int_0^x\left(\int_x^t {\varphi_x}'(u)\,du\right)U_{n,\alpha}(x,t)\,dt\right| \\
&= \left|\int_0^x \beta_n(x,t)\varphi_x'(t)dt\right| \\
&\leq \int_0^y \beta_n(x,t)\left|\varphi_x'(t)\right|dt+\int_y^x \beta_n(x,t)\left|\varphi_x'(t)\right|dt
\end{aligned}$$

Since $\varphi_x'(x)=0$ and $\beta_n(x,t)\le 1$, it implies

$$\int\limits_{x-\frac{x}{\sqrt{n}}}^{x} \beta_n(x,t)\left|\varphi_x'(t)\right|dt = \int\limits_{x-\frac{x}{\sqrt{n}}}^{x} \left|\varphi_x'(t)-\varphi_x'(x)\right|\beta_n(x,t)dt$$
$$\le \int\limits_{x-\frac{x}{\sqrt{n}}}^{x} V_t^x(\varphi_x')dt \le \frac{x}{\sqrt{n}} V_{x-\frac{x}{\sqrt{n}}}^{x}(\varphi_x').$$

Again using Lemma 2.1.5 and put $y = x-\frac{x}{\sqrt{n}}$,

$$\int_0^{x-\frac{x}{\sqrt{n}}} \left|\varphi_x{}'(t)\right|\beta_n(x,t)\,dt \le \frac{C}{4(n+1)}\int_0^{x-\frac{x}{\sqrt{n}}} \frac{\left|\varphi_x{}'(t)\right|}{(x-t)^2}dt$$
$$\le \frac{C}{4(n+1)x}\int_1^{\sqrt{n}} \vee_{x-\frac{x}{u}}^{x}\left(\varphi_x{}'\right)du$$
$$\le \frac{C}{4(n+1)x}\sum_{k=1}^{[\sqrt{n}]} \vee_{x-\frac{x}{k}}^{x}\left(\varphi_x{}'\right).$$

Thus

$$J_{1,n,x} \le \frac{x}{\sqrt{n}} V_{x-\frac{x}{\sqrt{n}}}^{x}(\varphi_x') + \frac{C}{4(n+1)x}\sum_{k=1}^{[\sqrt{n}]} \vee_{x-\frac{x}{k}}^{x}\left(\varphi_x{}'\right). \tag{2.11}$$

By using Lemma 2.1.5 we can write

$$J_{2,n,x} \le J_{2,n,x}^{(1)} + J_{2,n,x}^{(2)},$$

where

$$J_{2,n,x}^{(1)} = \left|\int\limits_{x}^{z}\left(\int\limits_{t}^{x}\varphi_x'(u)du\right)d_t\left(1-\beta_n(x,t)\right)\right|,$$

and

$$J_{2,n,x}^{(2)} = \left|\int\limits_{z}^{1}\left(\int\limits_{t}^{x}\varphi_x'(u)du\right)d_t\left(1-\beta_n(x,t)\right)\right|.$$

Applying integration by parts as well as using Lemma 2.1.5, since $(1-\beta_n(x,t))\le 1$ and putting $z=x+\frac{(1-x)}{\sqrt{n}}$, we have

$$\begin{aligned}
J_{2,n,x}^{(2)} &= \left|\int_{z}^{1}\left(\int_{t}^{x}\varphi_x^{'}(u)du\right)d_t\left(1-\beta_n(x,t)\right)\right| \\
&\le \frac{C}{4(n+1)}\int_{z}^{1}\bigvee_{x}^{t}(\varphi')_x\frac{1}{(t-x)^2}dt \\
&\le \frac{C}{4(n+1)}\int_{x+\frac{(1-x)}{\sqrt{n}}}^{1}\bigvee_{x}^{t}(\varphi')_x\frac{1}{(t-x)^2}dt.
\end{aligned}$$

By substituting $u=(1-x)\big/(t-x)$, we obtain

$$\begin{aligned}
J_{2,n,x}^{(2)} &\le \frac{C}{4(n+1)(1-x)}\int_{1}^{\sqrt{n}}\bigvee_{x}^{x+(1-x)/u}(\varphi')_x du \\
&\le \frac{C}{4(n+1)(1-x)}\sum_{k=1}^{[\sqrt{n}]}\int_{k}^{k+1}\bigvee_{x}^{x+(1-x)/k}(\varphi')_x du \\
&\le \frac{C}{4(n+1)(1-x)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+(1-x)/k}((\varphi')_x).
\end{aligned}$$

Again applying integration by parts as well as using Lemma 2.1.5, for calculating $J_{2,n,x}^{(1)}$, since $(1-\beta_n(x,t))\le 1$ and putting $z=x+\frac{(1-x)}{\sqrt{n}}$, we get

$$\begin{aligned}
J_{2,n,x}^{(1)} &= \left|\int_{x}^{z}\left(\int_{t}^{x}\varphi_x^{'}(u)du\right)d_t\left(1-\beta_n(x,t)\right)\right| \le \int_{x}^{z}\bigvee_{x}^{t}(\varphi')_x dt \\
&\le \frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+(1-x)/\sqrt{n}}(\varphi')_x = J_{2,n,x}^{(1)}+J_{2,n,x}^{(2)}.
\end{aligned}$$

Now

$$J_{2,n,x}\le \frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+(1-x)/\sqrt{n}}(\varphi')_x+\frac{C}{4(n+1)(1-x)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+(1-x)/k}((\varphi')_x) \tag{2.12}$$

From equation (2.10) to (2.12), we get

$$\left|\hat{K}_{n,\alpha}(\varphi;x)-\varphi(x)\right| \leq \frac{1}{4(n+1)}\left|\varphi'(x+)+\varphi'(x-)\right|+\frac{1}{4}\sqrt{\frac{C}{(n+1)}}\left|\varphi'(x+)-\varphi'(x-)\right|$$
$$+\frac{x}{\sqrt{n}}V_{x-\frac{x}{\sqrt{n}}}^{x}(\varphi_x')+\frac{C}{4(n+1)x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}(\varphi_x').$$
$$+\frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+(1-x)/\sqrt{n}}(\varphi')_x+\frac{C}{4(n+1)(1-x)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+(1-x)/k}((\varphi')_x).$$

The Theorem is proved. □

2.1.5 Voronovskaya type theorem

Theorem 2.1.9. *Let $\varphi \in C[0,1]$. If φ is twice differentiable in $x \in [0,1]$ and φ'' is continuous at x, then the following limit holds:*

$$\lim_{n\to\infty} n\left[\hat{K}_{n,\alpha}(\varphi(t);x)-\varphi(x)\right]=\tfrac{(1-2x)}{2}\varphi'(x)+\tfrac{(1-x)x}{2}\varphi''(x)\ .$$

Proof. Using Taylor's formula there exists η lying between x and t such that

$$\varphi(t)=\varphi(x)+(t-x)\varphi'(x)+\frac{(t-x)^2}{2!}\varphi''(x)+\xi(t,x)(t-x)^2,$$

where

$$\xi(t,x)=\frac{\varphi''(\eta)-\varphi''(x)}{2},$$

and φ is a continuous function which vanishes as $t\to x$. Applying the operator $\hat{K}_{n,\alpha}$ to the above equality, we get

$$\hat{K}_{n,\alpha}(\varphi(t)-\varphi(x);x)=\varphi'(x)\hat{K}_{n,\alpha}((t-x);x)+\frac{\varphi''(x)}{2!}\hat{K}_{n,\alpha}\left((t-x)^2;x\right)$$
$$+\hat{K}_{n,\alpha}\left(\xi(t,x)(t-x)^2;x\right).$$

Therefore

$$\lim_{n\to\infty} n\left[\hat{K}_{n,\alpha}(\varphi(t)-\varphi(x);x)\right]=\varphi'(x)\lim_{n\to\infty} n\left[\hat{K}_{n,\alpha}((t-x);x)\right]$$
$$+\frac{\varphi''(x)}{2!}\lim_{n\to\infty}\left[\hat{K}_{n,\alpha}\left((t-x)^2;x\right)\right]$$
$$+\lim_{n\to\infty} n\left[\hat{K}_{n,\alpha}\left(\xi(t,x)(t-x)^2;x\right)\right]. \tag{2.13}$$

In order to estimate the last term of equation (2.13), for each given $\varepsilon > 0$, choose $\delta > 0$ in such a way

$$\xi(t,x) < \varepsilon \text{ for } |t-x| < \delta.$$

Therefore, if we take $|t-x| < \delta$ then

$$\left|\xi(t,x)(t-x)^2\right| < \varepsilon(t-x)^2.$$

Since $\xi(t,x) < M$, while if $|t-x| \geq \delta$, we obtain

$$\left|\xi(t,x)(t-x)^2\right| \leq \frac{M}{\delta^2}(t-x)^4.$$

Therefore, we get

$$\hat{K}_{n,\alpha}\left(\xi(t,x)(t-x)^2;x\right) \leq \varepsilon\mu_n^2 + \frac{M}{\delta^2}\mu_n^4.$$

By direct calculations, we obtain

$$\mu_n^4(x) = O\left(\frac{1}{n^2}\right),$$

and we conclude

$$\lim_{n\to\infty} n\left[\hat{K}_{n,\alpha}\left(\xi(t,x)(t-x)^2;x\right)\right] = 0.$$

Therefore, we achieve the required result. □

2.2 Q-analogue of generalized Bernstein-Kantorovich operators

2.2.1 Introduction

The first time in 1987, Bernstein operators based on q-integers were introduced by Lupas [117] and they are rational functions. Again in 1997, Phillips [135] introduced the q-Bernstein polynomials known as Phillips q-Bernstein operators. In the past decade, linear positive operators based on q-integers is an active area of research. For more articles related to this approach, we refer to readers [16, 35, 86, 155].

For any $n \in \mathbb{N}_0$ the q-integer $[n]_q$ is defined by

$$[n]_q = 1+q+q^2+q^3+...+q^{n-1} = \frac{1-q^n}{1-q} \; (n \in \mathbb{N}), \; [0]_q = 0$$

and the q-factorial $[n]_q!$ by

$$[n]_q! = [1]_q[2]_q...[n]_q, \; [0]_q! = 1$$

. For integer $0 \leq k \leq n$ the $q-$binomial, or the Gaussian coefficient is defined is by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

In [26] (on page no. 11), the Jackson definite integral of the function f is denoted by

$$\int_0^a f(t)d_qt = (1-q)\sum_{n=0}^{\infty} f(aq^n)\,q^n, \;\; a \in \mathbb{R}.$$

Notice that the series on the right hand side is guaranteed to be convergent as soon as the function f is such that for some $M > 0$, $l > -1$, $|f| \leq Mx^l$ in a right neighborhood of $x = 0$.

The Jackson integral in a generic interval $[a,b]$:

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Chai et al. [35] have considered the q-analogue of (2.1) are as follows:

$$B_{n,q}^{(\alpha)}(\varphi;x) = \sum_{k=0}^{n} \varphi_k p_{n,q,k}^{(\alpha)}(x), \tag{2.14}$$

where

$$p_{n,q,k}^{(\alpha)}(x) = \left(\begin{bmatrix} n-2 \\ k \end{bmatrix}_q (1-\alpha)x + \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q (1-\alpha)\,q^{n-k-2}\left(1-q^{n-k-1}x\right) \right.$$
$$\left. + \begin{bmatrix} n \\ k \end{bmatrix}_q \alpha x\left(1-q^{n-k-1}x\right) \right) x^{k-1}(1-x)_q^{n-k-1},$$

$q \in (0,1]$ and $\varphi_k = \varphi\left(\frac{[k]_q}{[n]_q}\right)$. For detailed explanation (see [26]).

Motivated from the above stated work, we consider the q-analogue of the operators (2.3) as follows:

$$K_{n,q}^{(\alpha)}(\varphi;x) = [n+1]_q \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} \varphi(t)d_qt, \tag{2.15}$$

and $p_{n,q,k}^{(\alpha)}(x)$ is given in (2.14). For $\alpha = 1$ and $q = 1$ the operators (2.15) reduce to Bernstein Kantorovich operators.

In this section, we estimate the moments of the proposed operators and discuss the rate of convergence using usual and second-order modulus of smoothness.

2.2.2 Preliminaries

Here, we prove some auxiliary results to show our main results.

Lemma 2.2.1. *From [35], we have*
$B_{n,q}^{(\alpha)}(1;x) = 1$, $B_{n,q}^{(\alpha)}(t;x) = x$ *and*

$$B_{n,q}^{(\alpha)}(t^2;x) = x^2 + \frac{x(1-x)}{[n]_q} + \frac{(1-\alpha)q^{n-1}[2]_q x(1-x)}{[n]_q^2}.$$

Lemma 2.2.2. *The moments of the proposed operators:*

(i) $K_{n,q}^{(\alpha)}(1;x) = 1$;

(ii) $K_{n,q}^{(\alpha)}(t;x) = \frac{2q[n]_q}{[2]_q[n+1]_q}x + \frac{1}{[2]_q[n+1]_q}$;

(iii) $K_{n,q}^{(\alpha)}(t^2;x) = \frac{3q^2[n]_q^2}{[3]_q[n+1]_q^2}x^2 + \frac{3q^2}{[3]_q[n+1]_q^2}\left([n]_q + (1-\alpha)q^{n-1}[2]_q\right)x(1-x)$
$+\frac{3q[n]_q x}{[3]_q[n+1]_q^2} + \frac{1}{[3]_q[n+1]_q^2}.$

Proof. From [129], $\int\limits_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} 1 d_q t = \frac{1}{[n+1]_q}$, $\int\limits_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} t d_q t = \frac{2q[k]_q}{[2]_q[n+1]_q^2} + \frac{1}{[2]_q[n+1]_q^2}$ and

$$\int\limits_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} t^2 d_q t = \frac{3q^2[k]_q^2}{[3]_q[n+1]_q^3} + \frac{3q[k]_q}{[3]_q[n+1]_q^3} + \frac{1}{[3]_q[n+1]_q^3}.$$

It is easy to say that $K_{n,q}^{(\alpha)}(1;x) = 1$.

In view of Lemma 2.2.1 and for $f(t) = t$, we have

$$\begin{aligned} K_{n,q}^{(\alpha)}(t;x) =& [n+1]_q \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \int\limits_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} t d_q t \\ =& [n+1]_q \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \left(\frac{2q[k]_q}{[2]_q[n+1]_q^2} + \frac{1}{[2]_q[n+1]_q^2} \right) \\ =& \frac{[n]_q}{[n+1]_q} \left(\frac{2q}{[2]_q} \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \frac{[k]_q}{[n]_q} + \frac{1}{[2]_q[n]_q} \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \right) \\ =& \frac{2q[n]_q x + 1}{[2]_q[n+1]_q}. \end{aligned}$$

Similarly, for $f(t) = t^2$, we can estimate. So here we skip the proof. □

Lemma 2.2.3. *The central moments for the operators* (2.15) *are as follows:*

(i) $K_{n,q}^{(\alpha)}(t-x;x) = \frac{2q[n]_q}{[2]_q[n+1]_q} x + \frac{1}{[2]_q[n+1]_q}$;

(ii) $K_{n,q}^{(\alpha)}((t-x)^2;x) = \left(\frac{3q^2[n]^2}{[3]_q[n+1]_q^2} - \frac{4q[n]_q}{[2]_q[n+1]_q} + 1 \right) x^2$
$+ \frac{3q^2}{[3]_q[n+1]_q} \left([n]_q + [2]_q(1-\alpha)q^{n-1} \right) x(1-x) + \left(\frac{3q[n]_q}{[3]_q[n+1]_q^2} - \frac{2}{[2]_q[n+1]_q} \right) x$
$+ \frac{1}{[3]_q[n+1]_q^2}$.

Proof. Using linearity property of the operators (2.15) and Lemma 2.2.2, we get the required results. □

Lemma 2.2.4. *Let* $0 < q < 1$ *and* $c \in [0, qd]$, $d > 0$. *Then the inequality*

$$\int\limits_c^d |t-x| \, d_q t \le \left(\int\limits_c^d (t-x)^2 d_q t \right)^{\frac{1}{2}} \left(\int\limits_c^d d_q t \right)^{\frac{1}{2}}.$$

Proof. For the proof of the Lemma (see [129]). □

2.2.3 Direct results

Theorem 2.2.5. *For $0 < q \leq 1$, $q = \{q_n\}$ be a sequence converging to 1 as $n \to \infty$. Then, for all $\varphi \in C[0,1]$ and $\alpha \in [0,1]$, it implies $K_{n,q}^{(\alpha)}(\varphi;x)$ converges to $\varphi(x)$ uniformly on $[0,1]$ for sufficiently large n.*

Proof. From Lemma 2.2.2, $\lim_{n\to\infty} q_n = 1$, we get
$\lim_{n\to\infty} K_{n,q}^{(\alpha)}(1;x) = 1$, $\lim_{n\to\infty} K_{n,q}^{(\alpha)}(t;x) = x$ and $\lim_{n\to\infty} K_{n,q}^{(\alpha)}(t^2;x) = x^2$. Then by Bohaman-Korovokin theorem $\lim_{n\to\infty} K_{n,q}^{(\alpha)}(\varphi(t);x) = \varphi(x)$ converges uniformly on $[0,1]$. □

Theorem 2.2.6. *For $\varphi \in C[0,1]$, $q \in (0,1)$, $\lambda > 0$ and $\alpha \in [0,1]$, we have*

$$\left| K_{n,q}^{(\alpha)}(\varphi;x) - \varphi(x) \right| \leq \lambda \omega_2 \left(\varphi; \sqrt{\mu_{n,2}^q(x) + (\mu_{n,1}^q(x))^2} \right) + \omega \left(\varphi; \mu_{n,1}^q(x) \right),$$

where $\mu_{n,1}^q(x)$ and $\mu_{n,2}^q(x)$ are first and second order central moments of the operators (2.15).

Proof. We define the auxiliary operators

$$\hat{K}_{n,q}^{(\alpha)}(\varphi;x) = K_{n,q}^{(\alpha)}(\varphi;x) - \varphi\left(\frac{2q[n+1]_q x + 1}{[2]_q [n+1]_q} \right) + \varphi(x). \tag{2.16}$$

From (2.16), we have

$$\hat{K}_{n,q}^{(\alpha)}(t-x;x) = 0.$$

Suppose $\psi \in C^2[0,1]$, $x,t \in [0,1]$, by Tylor's expansion

$$\psi(t) = \psi(x) + (t-x)\psi'(x) + \int_x^t (t-u)\psi''(u)du.$$

Applying $\hat{K}_{n,q}^{(\alpha)}(.;x)$ in above equation, we have

$$\hat{K}_{n,q}^{(\alpha)}(\psi;x) = \psi(x) + \hat{K}_{n,q}^{(\alpha)} \left(\int_x^t (t-u)\psi''(u)du; x \right).$$

Therefore,

$$\begin{aligned}
\left|\hat{K}_{n,q}^{(\alpha)}(\psi;x)-\psi(x)\right| \leq& \left|K_{n,q}^{(\alpha)}\left(\int_x^t (t-u)\psi''(u)du;x\right)\right| \\
&+\left|\left(\int_x^{\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}} \left(\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}-x\right)\psi^{''}(u)du;x\right)\right| \\
\leq& K_{n,q}^{(\alpha)}\left(\int_x^t |t-x|\,\psi^{''}(u)du;x\right) \\
&+\left|\left(\int_x^{\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}} \left|\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}-u\right|\left|\psi''(x)\right|du;x\right)\right| \\
\leq& \left[K_{n,q}^{(\alpha)}((t-x)^2;x)+\left(\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}-x\right)^2\right]\left\|\psi''\right\|. \qquad (2.17)
\end{aligned}$$

From (2.16), we have

$$\left|K_{n,q}^{(\alpha)}(\varphi;x)\right| \leq \|\varphi\|\, K_{n,q}^{(\alpha)}(1;x)+2\|\varphi\| = 3\|\varphi\|. \qquad (2.18)$$

From (2.16), (2.17) and (2.18), we have

$$\begin{aligned}
\left|K_{n,q}^{(\alpha)}(\varphi;x)-\varphi(x)\right| \leq& \left|K_{n,q}^{(\alpha)}(\varphi-\psi;x)\right|+|\varphi-\psi| \\
&+\left|\varphi\left(\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}\right)-\varphi(x)\right| \\
\leq& 4\|\varphi-\psi\|+\left(\mu_{n,2}^q(x)+{\mu_{n,1}^q}^2(x)\right)+\left|\varphi\left(\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}\right)-\varphi(x)\right|.
\end{aligned}$$

Now taking infimum on the right hand side of the above inequality over $\psi \in C^2[0,1]$, we get

$$\leq 4K_2\left(\varphi;\mu_{n,2}^q(x)+{\mu_{n,1}^q}^2(x)\right)+\omega\left(\varphi;\mu_{n,1}^q(x)\right)$$

From (1.13), we get

$$\left|K_{n,q}^{(\alpha)}(\varphi;x)-\varphi(x)\right| \leq \lambda\,\omega_2\left(\varphi;\sqrt{\mu_{n,2}^q(x)+{\mu_{n,1}^q}^2(x)}\right)+\omega\left(\varphi;\omega_{n,1}^q(x)\right).$$

Hence the result. $\square$

Theorem 2.2.7. *Let $q_n \in (0,1)$ be a sequence converging to 1 and α is fixed. Then for $\varphi \in C[0,1]$, we have*

$$\left|K_{n,q}^{(\alpha)}(\varphi;x)-\varphi(x)\right| \leq 2\omega(\varphi;\mu_{n,2}^{q}(x)),$$

where $\mu_{n,2}^{q}(x)=\left(K_{n,q}^{(\alpha)}((t-x)^2;x)\right)^{\frac{1}{2}}$.

Proof. For non-decreasing function $\varphi \in C[0,1]$. Using linearity and monotonicity of $K_{n,q}^{(\alpha)}$, we have

$$\begin{aligned}\left|K_{n,q}^{(\alpha)}(\varphi;x)-\varphi(x)\right| \leq& K_{n,q}^{(\alpha)}\left(|\varphi(t)-\varphi(x)|;x\right)\\ \leq& \omega(\varphi;\delta)\left(1+\frac{1}{\delta}K_{n,q}^{(\alpha)}\left(|t-x|;x\right)\right).\end{aligned}$$

Applying Lemma 2.2.4 with $c=\frac{q[k]_q}{[n+1]_q}$ and $d=\frac{[k+1]_q}{[n+1]_q}$, we get

$$\left|K_{n,q}^{(\alpha)}(\varphi;x)-\varphi(x)\right| \leq \omega(\varphi;x)\left\{1+\frac{[n+1]_q}{\delta}\sum_{k=0}^{n}p_{n,q,k}^{(\alpha)}(x)\left(\int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}}(t-x)^2 d_q t\right)^{\frac{1}{2}}\times\left(\int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} d_q t\right)^{\frac{1}{2}}\right\}.$$

Using Hölder's inequality for sums, we have

$$\begin{aligned}\left|K_{n,q}^{(\alpha)}(\varphi;x)-\varphi(x)\right| =& \omega(\varphi;x)\left\{1+\frac{1}{\delta}\left([n+1]_q\sum_{k=0}^{n}p_{n,q,k}^{(\alpha)}(x)\int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}}(t-x)^2 d_q t\right)^{\frac{1}{2}}\right.\\ &\left.\times\left([n+1]_q\sum_{k=0}^{n}p_{n,q,k}^{(\alpha)}(x)\int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} d_q t\right)^{\frac{1}{2}}\right\}\\ =& \omega(\varphi;x)\left\{1+\frac{1}{\delta}\left(K_{n,q}^{(\alpha)}((t-x)^2;x)\right)^{\frac{1}{2}}\right\}.\end{aligned}$$

By choosing $\delta=\mu_{n,2}^{q}(x)$, we get required result. □

2.3 The family of Bernstein-Stancu-Kantorovich operators with shifted knots

2.3.1 Introduction

Recently, Gadjiev and Ghorbanalizadeh [65] proposed another generalization of Bernstein-Stancu operators with shifted knots. These shifted knots provide the pliability to the operators and also allow us to approximate in the interval [0,1] and over its subintervals. These operators are defined as follows

$$B_n^{(r,s)}(\varphi;x) = \left(\frac{n+s_2}{n}\right)^n \sum_{k=0}^{n} b_n^{(r_2,s_2)}(x)\varphi\left(\frac{k+r_1}{n+s_1}\right), \tag{2.19}$$

where

$$b_n^{(r_2,s_2)}(x) = \begin{pmatrix} n \\ k \end{pmatrix} \left(x - \frac{r_2}{n+s_2}\right)^k \left(\frac{n+r_2}{n+s_2} - x\right)^{n-k}$$

and $\frac{r_2}{n+s_2} \leq x \leq \frac{n+r_2}{n+s_2}$, $r_j, s_j \geq 0$ $(j = 1,2)$ with condition $0 \leq r_1 \leq r_2 \leq s_1 \leq s_2$. For $r_2 = s_2 = 0$, we obtain Bernstein-Stancu operators and for $r_1 = r_2 = s_1 = s_2 = 0$, we get classical Bernstein operators. Içöz and Kantorovich [94] considered the Kantorovich-type modification of (2.19) and discussed its rate of convergence using usual modulus of continuity. Recently, Rahman et al. [137] defined the λ-Bernstein-Kantorovich operators with shifted knots and discussed some basic results along with its rate of convergence, graphical comparisons, and error estimation tables. For more articles, we refer to readers $[16, 96, 98, 102, 125, 143, 150, 156, 157]$.

Motivated from the above stated work, we consider the family of Bernstein-Stancu-Kantorovich operators with shifted knots. For $\varphi \in C[0,1]$, the operators are defined as follows:

$$K_{n,\alpha}^{(r,s)}(\varphi;x) = \left(\frac{n+s}{n}\right)^n (n+s+1) \sum_{k=0}^{n} p_{n,k,\alpha}^{(r,s)}(x) \int_{\frac{k+r}{n+s+1}}^{\frac{k+r+1}{n+s+1}} \varphi(t)dt, \tag{2.20}$$

where

$$p_{n,k,\alpha}^{(r,s)}(x)=\left[\begin{pmatrix} n-2 \\ k-1 \end{pmatrix}(1-\alpha)\left(x-\frac{r}{n+s}\right)+\begin{pmatrix} n-2 \\ k-2 \end{pmatrix}(1-\alpha)\left(\frac{n+r}{n+s}-x\right)\right.$$
$$\left.+\begin{pmatrix} n \\ k \end{pmatrix}\alpha\left(x-\frac{r}{n+s}\right)\left(\frac{n+r}{n+s}-x\right)\right]\left(x-\frac{r}{n+s}\right)^{k-1}\left(\frac{n+r}{n+s}-x\right)^{n-k-1},$$

$x\in\left[\frac{r}{n+s},\frac{n+r}{n+s}\right]$ and $r,s\geq 0$ with condition $0\leq r\leq s$. In particular cases:

*Case*1. For $r=s=0$, the operators (2.20) reduce to family of Bernstein Kantorovich operators (2.3).

*Case*2. For $r=s=0$ and $\alpha=1$ the operators (2.20) reduce to classical Bernstein-Kantorovich operators (1.4).

The aim here to define the family of Bernstein-Stancu-Kantorovich operators with shifted knots and estimate its moments and central moments to study local and global rate of convergence and discuss the p^{th} order generalization of operators (2.20). In support of theoretical results, we have also shown numerical and graphical comparisons for the proposed operators.

2.3.2 Preliminareis

Lemma 2.3.1. *The moments of the operators* (2.20) *are given as:*

(i) $K_{n,\alpha}^{(r,s)}(1;x)=1$;

(ii) $K_{n,\alpha}^{(r,s)}(t;x)=\frac{1}{n(n+s+1)}[(n+s)x+1)(n+(1-\alpha)s]$;

(iii) $K_{n,\alpha}^{(r,s)}(t^2;x)=\frac{1}{3n^3(n+s+1)^2}\left[(3n^5+3((3-\alpha)s+(3\alpha-4))\,n^4+3((3-2\alpha)s^2+(10\alpha-12)s+4(1-\alpha))n^3+3((1-\alpha)s^3+(12-11\alpha)s^2+12(1-\alpha)s)n^2+12(1-\alpha)(s-3)s^2n-12(1-\alpha)s^3)x^2+((12-6\alpha)n^4+(21-15\alpha-6(1-\alpha)r)s+6(3-2\alpha)r-3(1-\alpha))n^3+((9-6r)(1-\alpha)s^2-3((1-\alpha)-2(7-6\alpha)r)s-24(1-\alpha)r)n^2+24(1-\alpha)(s-2)srn-24(1-\alpha)rs^2)x+(4-3\alpha-6r)n^3+(3(1-\alpha)(3-r)rs+3r((1-\alpha)-(2-\alpha)r))n^2-12(1-\alpha)s(s-2)rn+12(1-\alpha)r^2s\right]$.

Lemma 2.3.2. *The central moments* $\mu_{n,\alpha,j}^{(r,s)}(x)=K_{n,\alpha}^{(r,s)}((t-x)^j;x)$ *for* $j=1,2$ *of the operators* (2.20) *are given as:*

(i) $\mu_{n,\alpha,1}^{(r,s)}(x)=\frac{1}{n(n+s+1)}[(((1-\alpha)s-1)n+(1-\alpha)s^2)x+(n+(1-\alpha)s)]$

(ii) $\mu_{n,\alpha,2}^{(r,s)}(x) = \frac{1}{3n^3(n+s+1)^2}[(-3((1-\alpha)s+(4-3\alpha))n^4 - 6((1-\alpha)s^2+(7-6\alpha)s)n^3 - 3((1-\alpha)s^3+(14-13\alpha)s^2+4(1-\alpha)s)n^2 - 12(1-\alpha)(s-3)s^2n + 12(1-\alpha)s^3)x^2 + (6(1-\alpha)n^4 + 3((1-\alpha)(3-2r)s + 2(3-2\alpha)r - (3-\alpha))n^3 + (3(1-\alpha)(1-2r)s^2 + (-9(1-\alpha)+6(7-6\alpha)r)s - 24(1-\alpha)r)n^2 + 24(1-\alpha)(s-2)rn - 24(1-\alpha)rs^2)x + (4-3\alpha-6r)n^3 + 3((1-\alpha)(r-3)rs + (r-2)r^2 + (1-\alpha)r)n^2 - 12(1-\alpha)(s-1)r^2n + 12(1-\alpha)r^2s]$.

Lemma 2.3.3. *For $\alpha \in [0,1]$, $x \in \left[\frac{r}{n+s}, \frac{n+r}{n+s}\right]$ and $\varphi \in C[0,1]$, we have*

$$\left|K_{n,\alpha}^{(r,s)}(\varphi;x)\right| \leq \|\varphi\|.$$

Proof. From (2.20) and Lemma 2.3.1, we have

$$\begin{aligned}\left|K_{n,\alpha}^{(r,s)}(\varphi;x)\right| &\leq \left(\frac{n+s}{n}\right)^n (n+s+1)\sum_{k=0}^{n} p_{n,k,\alpha}^{(r,s)}(x)\int_{\frac{k+r}{n+s+1}}^{\frac{k+r+1}{n+s+1}} |\varphi|\,dt \\ &\leq \|\varphi\|\left(\frac{n+s}{n}\right)^n (n+s+1)\sum_{k=0}^{n} p_{n,k,\alpha}^{(r,s)}(x)\int_{\frac{k+r}{n+s+1}}^{\frac{k+r+1}{n+s+1}} 1\,dt \\ &\leq \|\varphi\|.\end{aligned}$$

□

2.3.3 Direct results

Theorem 2.3.4. *For $\varphi \in C[0,1]$ and $\alpha \in [0,1]$, we acquire*

$$\lim_{n\to\infty}\max_{\frac{r}{n+s}\leq x\frac{n+r}{n+s}}\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| = 0.$$

Proof. From Lemma 2.3.1 and for $i = 0,1,2$, we obtain

$$\lim_{n\to\infty}\max_{\frac{r}{n+s}\leq x\leq\frac{n+r}{n+s}}\left|K_{n,\alpha}^{(r,s)}(t^i;x)-x^i\right| = 0. \tag{2.21}$$

Now consider a sequence of operators:

$$K_{n,\alpha}^{(r,s)}(\varphi(t);x) = \begin{cases} K_{n,\alpha}^{(r,s)}(\varphi(t);x)\; if\; \frac{r}{n+s}\leq x\leq\frac{n+r}{n+s}, \\ \varphi(x) \qquad\quad if\; x\in\left[0,\frac{r}{n+s}\right]\bigcup\left[\frac{n+r}{n+s},1\right]. \end{cases}$$

Then,

$$\left|K_{n,\alpha}^{(r,s)}(\varphi(t);x)-\varphi(x)\right| = \max_{\frac{r}{n+s}\le x\le \frac{n+r}{n+s}} \left|K_{n,\alpha}^{(r,s)}(\varphi(t);x)-\varphi(x)\right|. \tag{2.22}$$

Using (2.21) and (2.22), we get

$$\lim_{n\to\infty}\left|K_{n,\alpha}^{(r,s)}(t^i;x)-x^i\right| = 0,\ k=0,1,2.$$

Using well-known Korovkin theorem [24,63,113] and in view of (2.22) follows the result. □

Theorem 2.3.5. *For* $\alpha\in[0,1]$, $x\in\left[\frac{r}{n+s},\frac{n+r}{n+s}\right]$ *and* $\varphi\in C[0,1]$, *we acquire*

$$\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| \le C\omega_2\left(\varphi;\frac{\mu_{n,\alpha,2}^{(r,s)}(x)}{4}\right)+\omega\left(\varphi;\left|\mu_{n,\alpha,1}^{(r,s)}(x)\right|\right),$$

where $\mu_{n,\alpha,1}^{(r,s)}(x)$ *and* $\mu_{n,\alpha,2}^{(r,s)}(x)$ *are given in Lemma 2.3.2.*

Proof. Define an auxiliary operators as follows:

$$\widehat{K_{n,\alpha}^{(r,s)}}(\varphi;x) = K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi\left(x+\mu_{n,\alpha,1}^{(r,s)}(x)\right)+\varphi(x). \tag{2.23}$$

On account of Lemma 2.3.1, we obtain

$$\widehat{K_{n,\alpha}^{(r,s)}}(1;x)=1,\ \widehat{K_{n,\alpha}^{(r,s)}}(t;x)=x.$$

For $\psi\in C^2[0,1]$, by Taylor's expansion

$$\psi(t)=\psi(x)+(t-x)\psi'(x)+\int_x^t (t-v)\psi''(v)dv.$$

Applying $\widehat{K_{n,\alpha}^{(r,s)}}(.;x)$ in above equation, we obtain

$$\begin{aligned}\widehat{K_{n,\alpha}^{(r,s)}}(\psi(t);x)-\psi(x) =& \widehat{K_{n,\alpha}^{(r,s)}}((t-x);x)\psi'(x)+\widehat{K_{n,\alpha}^{(r,s)}}\left(\int_x^t (t-v)\psi''(v)dv;x\right)\\ =& \widehat{K_{n,\alpha}^{(r,s)}}\left(\int_x^t (t-v)\psi''(v)dv;x\right)\\ &-\widehat{K_{n,\alpha}^{(r,s)}}\left(\int_x^{x+\mu_{n,\alpha,1}^{(r,s)}(x)}\left(x+\mu_{n,\alpha,1}^{(r,s)}(x)-v\right)\psi''(v)dv;x\right).\end{aligned}$$

$$\begin{aligned}\left|\widehat{K_{n,\alpha}^{(r,s)}}(\psi(t);x)-\psi(x)\right| \leq & \widehat{K_{n,\alpha}^{(r,s)}}\left(\left|\int_x^t (t-v)\psi''(v)dv\right|;x\right)\\ & +\widehat{K_{n,\alpha}^{(r,s)}}\left(\int_x^{x+\mu_{n,\alpha,1}^{(r,s)}(x)} \left|\left(x+\mu_{n,\alpha,1}^{(r,s)}(x)-v\right)\right| \left|\psi''(v)\right|dv;x\right)\\ \leq & K_{n,\alpha}^{(r,s)}((t-x)^2;x)\left\|\psi''\right\|+\left(\mu_{n,\alpha,1}^{(r,s)}(x)\right)^2\left\|\psi''\right\|\\ \leq & \left(\mu_{n,\alpha,2}^{(r,s)}(x)+\left(\mu_{n,\alpha,1}^{(r,s)}(x)\right)^2\right)\left\|\psi''\right\|. \qquad (2.24)\end{aligned}$$

From (2.23), we have

$$\left|\widehat{K_{n,\alpha}^{(r,s)}}(\varphi(t);x)\right| \leq 3\left\|\varphi\right\|. \qquad (2.25)$$

Since

$$\begin{aligned}\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| = & \left|\widehat{K_{n,\alpha}^{(r,s)}}(\varphi(t);x)-\varphi(x)+\varphi\left(x+\mu_{n,\alpha,1}^{(r,s)}(x)\right)-\varphi(x)\right|\\ \leq & \left|\widehat{K_{n,\alpha}^{(r,s)}}(\varphi-\psi;x)\right|+\left|\widehat{K_{n,\alpha}^{(r,s)}}(\psi;x)-\psi(x)\right|\\ & +\left|\varphi(x)-\psi(x)\right|+\left|\varphi\left(x+\mu_{n,\alpha,1}^{(r,s)}(x)\right)-\varphi(x)\right|.\end{aligned}$$

From (2.24) and (2.25), we have

$$\begin{aligned}\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| \leq & 4\left\|\varphi-\psi\right\|+\left(\mu_{n,\alpha,2}^{(r,s)}(x)+\left(\mu_{n,\alpha,1}^{(r,s)}(x)\right)^2\right)\left\|\psi''\right\|\\ & +\omega\left(\varphi;\left|\mu_{n,\alpha,1}^{(r,s)}(x)\right|\right).\end{aligned}$$

Taking infimum over $\varphi \in C^2[0,1]$, we have

$$\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| \leq 4K_2\left(\varphi;\frac{\mu_{n,\alpha,2}^{(r,s)}(x)}{4}\right)+\omega\left(\varphi;\left|\mu_{n,\alpha,1}^{(r,s)}(x)\right|\right).$$

In view of relation (1.13), we obtain the required result. □

Theorem 2.3.6. *For $\varphi \in Lip_{\xi}M$, $\alpha \in [0,1]$ and for each $x \in \left[\frac{r}{n+s},\frac{n+r}{n+s}\right]$, we have*

$$\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| \leq M\left(\mu_{n,\alpha,2}^{(r,s)}(x)\right)^{\frac{\xi}{2}},$$

where $\mu_{n,\alpha,2}^{(r,s)}(x)$ is given in Lemma 2.3.2.

Proof. By linearity and monotonicity of the operators $K_{n,\alpha}^{(r,s)}$, we have

$$\begin{aligned}\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| = \left|K_{n,\alpha}^{(r,s)}(\varphi(t)-\varphi(x);x)\right| &\leq K_{n,\alpha}^{(r,s)}(|\varphi(t)-\varphi(x)|;x)\\ &\leq M K_{n,\alpha}^{(r,s)}\left(|t-x|^{\xi};x\right).\end{aligned}$$

Owing to Hölder's inequality with $p=\frac{\xi}{2}$, $q=\frac{2-\xi}{2}$, we have

$$\begin{aligned}\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| &\leq M\left(K_{n,\alpha}^{(r,s)}((t-x)^2;x)\right)^{\frac{\xi}{2}}\left(K_{n,\alpha}^{(r,s)}(1;x)\right)^{\frac{2-\xi}{2}}\\ &\leq M\left(\mu_{n,\alpha,2}^{(r,s)}(x)\right)^{\frac{\xi}{2}}.\end{aligned}$$

Hence the proof. □

Theorem 2.3.7. *For $\varphi \in Lip_M^{(a,b)}(\xi)$, we have*

$$\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| \leq M\left(\frac{\mu_{n,\alpha,2}^{(r,s)}(x)}{ax^2+bx}\right)^{\frac{\xi}{2}},$$

where $\mu_{n,\alpha,2}^{(r,s)}(x)$ provided in Lemma 2.1.2.

Proof. First, we show the statement is true for $\xi=1$.
we can write

$$\begin{aligned}&\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right|\\ &\leq K_{n,\alpha}^{(r,s)}(|\varphi(t)-\varphi(x)|;x)\\ &\leq \left(\frac{n+s}{n}\right)^n(n+s+1)\sum_{k=0}^{n}p_{n,k,\alpha}^{(r,s)}(x)\int_{\frac{k+r}{n+s+1}}^{\frac{k+r+1}{n+s+1}}|\varphi(t)-\varphi(x)|\,dt\\ &\leq M\left(\frac{n+s}{n}\right)^n(n+s+1)\sum_{k=0}^{n}p_{n,k,\alpha}^{(r,s)}(x)\int_{\frac{k+r}{n+s+1}}^{\frac{k+r+1}{n+s+1}}\frac{|t-x|}{(ax^2+bx+t)^{\frac{1}{2}}}dt.\end{aligned}$$

Since $(ax^2+bx+t)^{-\frac{1}{2}} \leq (ax^2+bx)^{-\frac{1}{2}}$, we have

$$\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| \leq \frac{M}{(ax^2+bx)^{\frac{1}{2}}}\left(\frac{n+s}{n}\right)^n(n+s+1)\sum_{k=0}^{n}p_{n,k,\alpha}^{(r,s)}(x)\int_{\frac{k+r}{n+s+1}}^{\frac{k+r+1}{n+s+1}}|t-x|\,dt.$$

Applying Cauchy-Schwarz inequality in above, we obtain

$$\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| \leq M\sqrt{\frac{\mu_{n,\alpha,2}^{(r,s)}(x)}{ax^2+bx}}.$$

Hence the statement is true for $\xi = 1$. Next, we prove the same for $\xi \in (0,1)$. Owing to Hölder's inequality with $p=\frac{\xi}{2}$, $q=\frac{2-\xi}{2}$, we acquire

$$\begin{aligned}\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| &\leq \left(\frac{n+s}{n}\right)^n (n+s+1)\sum_{k=0}^{n} p_{n,k,\alpha}^{(r,s)}(x)\int_{\frac{k+r}{n+s+1}}^{\frac{k+r+1}{n+s+1}} |\varphi(t)-\varphi(x)|\,dt \\ &\leq \frac{M}{(ax^2+bx)^{\frac{\xi}{2}}}\left(\frac{n+s}{n}\right)^n (n+s+1)\sum_{k=0}^{n} p_{n,k,\alpha}^{(r,s)}(x)\int_{\frac{k+r}{n+s+1}}^{\frac{k+r+1}{n+s+1}} |t-x|^{\xi}\,dt \\ &\leq \frac{M\left(K_{n,\alpha}^{(r,s)}((t-x)^2;x)\right)^{\frac{\xi}{2}}}{(ax^2+bx)^{\frac{\xi}{2}}} = M\left(\frac{\mu_{n,\alpha,2}^{(r,s)}(x)}{ax^2+bx}\right)^{\frac{\xi}{2}}.\end{aligned}$$

which is clearly true for $0<\xi<1$. Hence the proof. □

Theorem 2.3.8. *Let $\alpha \in [0,1]$, $\varphi \in C[0,1]$ and for concave function ϕ^2 ($\phi \neq 0$), we have*

$$\left|\widehat{K_{n,\alpha}^{(r,s)}}(\varphi;x)-\varphi(x)\right| \leq C\omega_2^{\phi}\left(\varphi;\frac{\left(\mu_{n,\alpha,2}^{(r,s)}(x)+\left(\mu_{n,\alpha,1}^{(r,s)}(x)\right)^2\right)^{\frac{1}{2}}}{2\phi(x)}\right)+\omega_\eta\left(\varphi;\frac{\mu_{n,\alpha,1}^{(r,s)}(x)}{\eta(x)}\right),$$

where $\mu_{n,\alpha,1}^{(r,s)}(x)$ and $\mu_{n,\alpha,2}^{(r,s)}(x)$ are given in Lemma 2.3.2.

Proof. From (2.23), it is clear that $\widehat{K_{n,\alpha}^{(r,s)}}(1;x)=1$, $\widehat{K_{n,\alpha}^{(r,s)}}(t;x)=x$ and $\widehat{K_{n,\alpha}^{(r,s)}}(t-x;x)=0$. Let $y=\lambda x+(1-\lambda)t$, $\lambda \in [0,1]$. Since ϕ^2 ($\phi \neq 0$) is concave function in $[0,1]$ implies that $\phi^2(y) \geq \lambda\phi^2(x)+(1-\lambda)\phi^2(t)$ and hence

$$\frac{|t-y|}{\phi^2(y)} \leq \frac{\lambda\,|x-t|}{\lambda\phi^2(x)+(1-\lambda)\phi^2(t)} \leq \frac{|t-x|}{\phi^2(x)}. \tag{2.26}$$

We can write

$$\begin{aligned}\left|\widehat{K_{n,\alpha}^{(r,s)}}(\varphi;x)-\varphi(x)\right| &\leq \left|\widehat{K_{n,\alpha}^{(r,s)}}(\varphi-\psi;x)\right|+\left|\widehat{K_{n,\alpha}^{(r,s)}}(\psi;x)-\psi(x)\right|+|\varphi(x)-\psi(x)| \\ &\leq 4\,\|\varphi-\psi\|+\left|\widehat{K_{n,\alpha}^{(r,s)}}(\psi;x)-\psi(x)\right|. \end{aligned} \tag{2.27}$$

By Taylor's formula and operating $\widehat{K_{n,\alpha}^{(r,s)}}(.;x)$, we have

$$\begin{aligned}
&\left|\widehat{K_{n,\alpha}^{(r,s)}}(\psi;x)-\psi(x)\right| \\
&\leq K_{n,\alpha}^{(r,s)}\left(\int_x^t |t-y|\left|\psi''(y)\right|dy;x\right)+\left|\int_x^{x+\mu_{n,1,\alpha}^{(r,s)}(x)}\left|x+\mu_{n,\alpha,1}^{(r,s)}(x)-y\right|\left|\psi''(y)\right|dy\right| \\
&\leq \left\|\phi^2\psi''\right\|_{C[0,1]}K_{n,\alpha}^{(r,s)}\left(\int_x^t \frac{|t-y|}{\phi^2(y)}dy;x\right)+\left\|\phi^2\psi''\right\|_{C[0,1]}\left|\int_x^{x+\mu_{n,\alpha,1}^{(r,s)}(x)}\frac{\left|x+\mu_{n,\alpha,1}^{(r,s)}(x)-y\right|}{\phi^2(y)}dy\right| \\
&\leq \phi^{-2}(x)\left\|\phi^2\psi''\right\|_{C[0,1]}\left(\mu_{n,\alpha,2}^{(r,s)}(x)+\left(\mu_{n,\alpha,1}^{(r,s)}(x)\right)^2\right). \qquad (2.28)
\end{aligned}$$

From (2.26), (2.27) and using the definition of K-functional, we obtain

$$\begin{aligned}
\left|\widehat{K_{n,\alpha}^{(r,s)}}(\varphi;x)-\varphi(x)\right| &\leq \phi^{-2}(x)\left\|\phi^2\psi''\right\|_{C[0,1]}\left(\mu_{n,\alpha,2}^{(r,s)}(x)+\left(\mu_{n,\alpha,1}^{(r,s)}(x)\right)^2\right)+4\|\varphi-\psi\|_{C[0,1]} \\
&\leq C\omega_2^{\phi}\left(\varphi;\frac{\left(\mu_{n,\alpha,2}^{(r,s)}(x)+\left(\mu_{n,\alpha,1}^{(r,s)}(x)\right)^2\right)^{\frac{1}{2}}}{2\phi(x)}\right).
\end{aligned}$$

Using (1.14), we have

$$\begin{aligned}
\left|\varphi\left(x+\mu_{n,\alpha,1}^{(r,s)}(x)\right)-\varphi(x)\right| &= \left|\varphi\left(x+\eta(x)\frac{\mu_{n,\alpha,1}^{(r,s)}(x)}{\eta(x)}\right)-\varphi(x)\right| \\
&\leq \omega_\eta\left(\varphi;\frac{\mu_{n,\alpha,1}^{(r,s)}(x)}{\eta(x)}\right)
\end{aligned}$$

Therefore the following inequality satisfy:

$$\begin{aligned}
\left|K_{n,\alpha}^{(r,s)}(\varphi;x)-\varphi(x)\right| &\leq \left|\widehat{K_{n,\alpha}^{(r,s)}}(\varphi;x)-\varphi(x)\right|+\left|\varphi\left(x+\mu_{n,\alpha,1}^{(r,s)}(x)\right)-\varphi(x)\right| \\
&\leq C\omega_2^{\phi}\left(\varphi;\frac{\left(\mu_{n,\alpha,2}^{(r,s)}(x)+\left(\mu_{n,\alpha,1}^{(r,s)}(x)\right)^2\right)^{\frac{1}{2}}}{2\phi(x)}\right)+\omega_\eta\left(\varphi;\frac{\mu_{n,\alpha,1}^{(r,s)}(x)}{\eta(x)}\right).
\end{aligned}$$

□

2.3.4 Generalization of p^{th} order of the operators

In this section, we consider the p^{th} order generalization of the operators (2.20) as follows:

$$K_{n,\alpha,p}^{(r,s)}(\varphi;x) = \left(\frac{n+s}{n}\right)^n (n+s+1) \sum_{k=0}^{n} p_{n,k,\alpha}^{(r,s)}(x) \int_{\frac{k+r}{n+s+1}}^{\frac{k+r+1}{n+s+1}} \sum_{j=0}^{p} \varphi^{(j)}(u) \frac{(x-u)^j}{j!} du, \tag{2.29}$$

where $\varphi \in C^p[0,1]$ and $p \in \mathbb{N}_0$. For the generalized operators (2.29), it can be easily seen that at $p = 0$ reduces to the operators (2.20). For more details, the readers can refer to $[56, 94, 133, 137]$.

Theorem 2.3.9. *Suppose $\alpha \in [0,1]$, $\varphi \in C^p[0,1]$ and $\varphi^{(p)} \in Lip_M(\xi)$, then for all $x \in \left[\frac{r}{n+s}, \frac{n+r}{n+s}\right]$, we have*

$$\left| K_{n,\alpha,p}^{(r,s)}(\varphi;x) - \varphi(x) \right| \leq \frac{M}{(p-1)!} \frac{\xi}{\xi+p} B(\xi,p) . \left\| K_{n,\alpha}^{(r,s)}(\varphi;x) - \varphi(x) \right\|,$$

where B is a Beta function.

Proof. From (2.29), we have

$$\varphi(x) - K_{n,\alpha,p}^{(r,s)}(\varphi;x) = \left(\frac{n+s}{n}\right)^n (n+s+1) \sum_{k=0}^{n} p_{n,k,\alpha}^{(r,s)}(x) \times \int_{\frac{k+r}{n+s+1}}^{\frac{k+r+1}{n+s+1}} \left(\varphi(x) - \sum_{j=0}^{p} \varphi^{(j)}(u) \frac{(x-u)^j}{j!} \right) du. \tag{2.30}$$

By Taylor's theorem (see [112]), we get

$$\varphi(x) - \sum_{j=0}^{p} \varphi^{(j)}(u) \frac{(x-u)^j}{k!} = \frac{(x-u)^p}{(p-1)!} \int_0^1 (1-y)^{p-1} \left(\varphi^{(p)}(u + y(x-u)) - \varphi^{(p)}(u) \right) dy \tag{2.31}$$

Since $\varphi^{(p)} \in Lip_\xi M$, we have

$$\left| \varphi^{(p)}(u + y(x-u)) - \varphi^{(p)}(u) \right| \leq M y^\xi |x-u|^\xi. \tag{2.32}$$

From the definition of Beta function, we get

$$\int_0^1 y^{\xi}(1-y)^{p-1}dy = B(1+\xi,p) = \frac{\xi}{\xi+p}B(\xi,p). \tag{2.33}$$

Using relation (2.32) and (2.33) in (2.31), we get

$$\left|\varphi(x) - \sum_{j=0}^{p}\varphi^{(j)}(u)\frac{(x-u)^j}{k!}\right| \leq M\frac{|x-u|^p}{(p-1)!}\int_0^1 y^{\xi}|x-u|^{\xi}(1-y)^{p-1}dy$$
$$= M\frac{|x-u|^{p+\xi}}{(p-1)!}\int_0^1 y^{\xi}(1-y)^{p-1}dy = M\frac{|x-u|^{p+\xi}}{(p-1)!}\frac{\xi}{\xi+p}B(\xi,p). \tag{2.34}$$

From (2.30) and (2.34), we obtain required result. □

2.3.5 Graphical results

Example 1. Let $\varphi(x) = x^3\cos(3\pi x)$, $\alpha = 0.2$, $r = 4$, $s = 5$ and $n \in \{10, 15, 30\}$. The convergence of the defined operators $K_{n,\alpha}^{(r,s)}$ towards the function $\varphi(x)$ and the absolute error $\varepsilon_{n,\alpha}^{(r,s)}$ of the operators are shown in figure 2.1 and figure 2.2 respectively. The absolute error of the operators are also computed in table 2.1 for some values in $\left[\frac{r}{n+s}, \frac{n+r}{n+s}\right]$.

Example 2. Let $\varphi(x) = x^2\sin\left(2\pi\left(x+\frac{1}{2}\right)^2\right)$ and $n = 50$. The comparison of convergence of proposed operators $K_{n,\alpha}^{(r,s)}$ (cyan, $\alpha = 0.25$, $r = 4$, $s = 5$), family of Bernstein-Kantorovich operators (2.4) (black, $\alpha = 0.25$, $r = 0$, $s = 0$), classical Bernstein-Kantorovich operators (2.3) (red, $\alpha = 1.0$, $r = 0$, $s = 0$) and absolute errors $\varepsilon_{n,\alpha}^{(r,s)}$ of the operators are shown in figure 2.3 and figure 2.4 respectively. The absolute error of the operators are also computed for some values in $\left[\frac{r}{n+s}, \frac{n+r}{n+s}\right]$ which is shown in table 2.2.

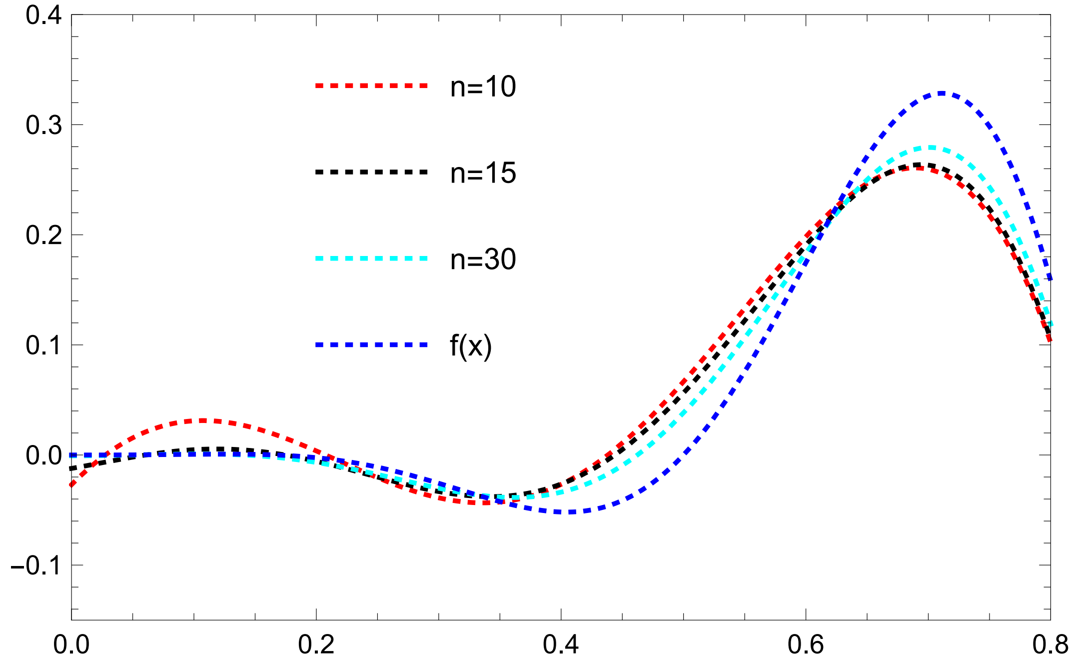

Figure 2.1: Convergence of the operator $K_{n,\alpha}^{(r,s)}(\varphi;x)$ towards function $\varphi(x)$ for ($\alpha = 0.2$, $r = 4$, $s = 5$).

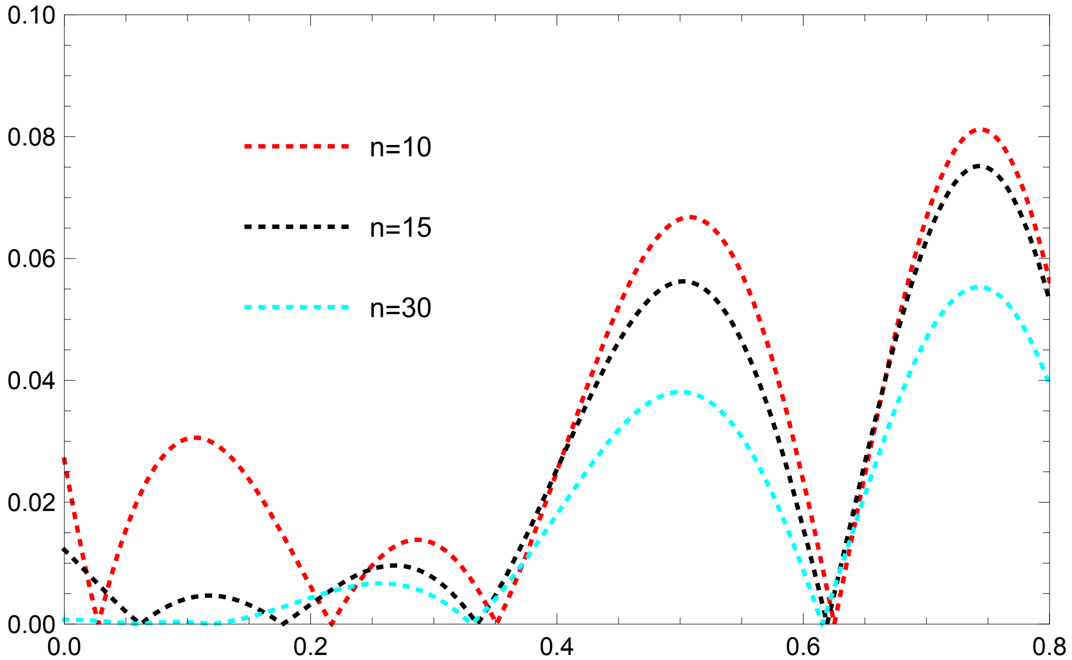

Figure 2.2: The absolute error $\varepsilon_{n,\alpha}^{(r,s)}$ for ($\alpha = 0.2$, $r = 4$, $s = 5$).

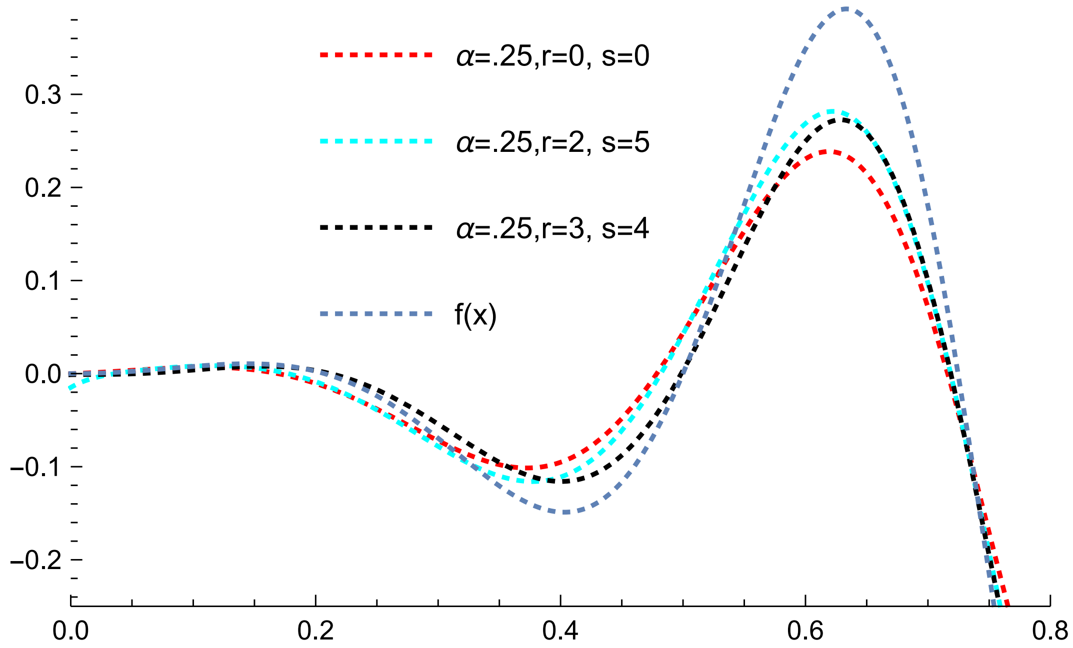

Figure 2.3: x for ($n = 50$)

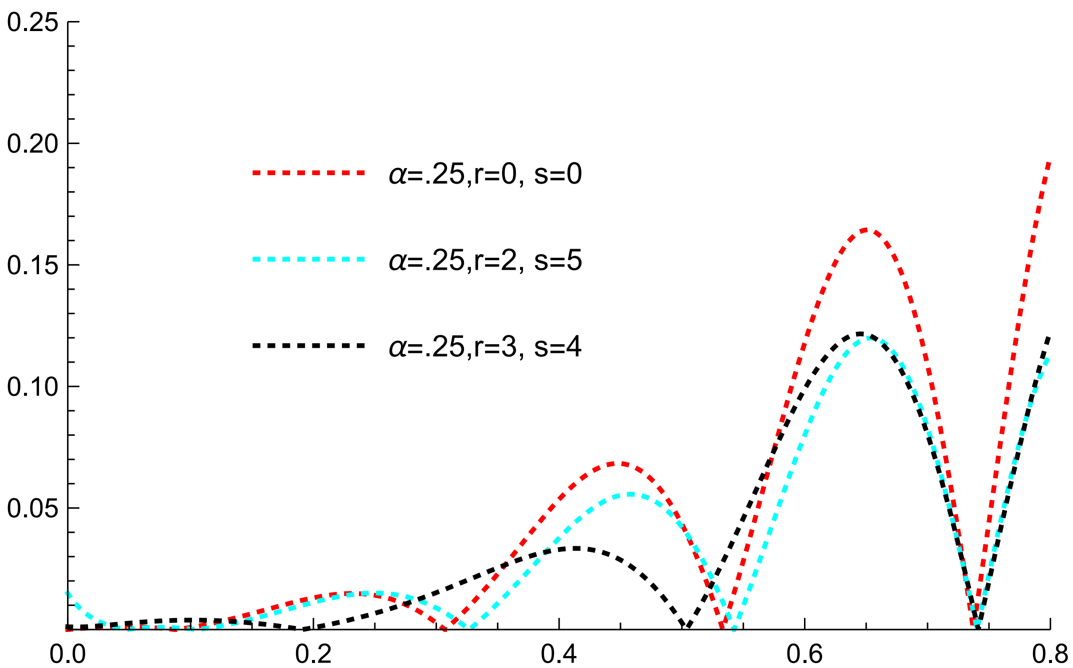

Figure 2.4: The absolute error $\varepsilon_{n,\alpha}^{(r,s)}$ for ($n = 50$).

Table 2.1: Absolute error of the operators $K_{n,\alpha}^{(r,s)}$ with function $\varphi(x) = x^3 \cos(3\pi x)$ for $\alpha = 0.2$, $r = 4$, $s = 5$.

x	n=10	n=15	n=30
0.1	.030426	.004204	.000300
0.2	.006182	.003235	.004314
0.3	.013229	.007541	.004520
0.4	.025076	.025405	.018002
0.5	.066466	.056224	.038098
0.6	.023637	.015840	.008822
0.7	.066703	.062968	.046879
0.8	.056277	.053033	.039276

Table 2.2: Absolute error of the operators with function $\varphi = x^2 \sin\left(2\pi\left(x+\frac{1}{2}\right)^2\right)$ for $n = 50$.

x	$\alpha = 0.25$, $r = 0$, $s = 0$	$\alpha = 0.25$, $r = 2$, $s = 5$	$\alpha = 0.25$, $r = 3$, $s = 4$
0.1	.000904	.000260	.003880
0.2	.013064	.011202	.000805
0.3	.002768	.008868	.015166
0.4	.053540	.037694	..032918
0.5	.043356	.042120	.002801
0.6	.118005	..080048	.098996

Chapter 3

Approximation by Kantorovich form of modified Szász-Mirakyan operators

3.1 Introduction

In 1977, Jain and Pethe [101] generalized the well-known Szász-Mirakyan operators [147] as:

$$S_n^{[\alpha]}(\varphi;x) = (1+n\alpha)^{-\frac{x}{\alpha}} \sum_{k=0}^{\infty} \left(\alpha + \frac{1}{n}\right)^{-k} \frac{x^{(k,-\alpha)}}{k!} \varphi\left(\frac{k}{n}\right)$$
$$= \sum_{k=0}^{\infty} s_{n,k}^{[\alpha]}(x)\, \varphi\left(\frac{k}{n}\right), \tag{3.1}$$

where

$$s_{n,k}^{[\alpha]}(x) = (1+n\alpha)^{-\frac{x}{\alpha}} \left(\alpha + \frac{1}{n}\right)^{-k} \frac{x^{(k,-\alpha)}}{k!},$$

$x^{(k,-\alpha)} = x(x+\alpha)\ldots(x+(k-1)\alpha)$, $x^{(0,-\alpha)} = 1$ and φ is any function of exponential type such that

$$|\varphi(t)| \le Ke^{At} \quad (t \ge 0),$$

for some finite constants $K, A > 0$. Here $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ is such that

$$0 \le \alpha_n \le \frac{1}{n}.$$

Notice that as $n \to \infty$ α tends to zero.

The operators $S_n^{[\alpha]}$ have also been considered by Stancu [145], Mastroianni [120], Della

Vecchia and Kocic [44] and Finta [61, 62].

Abel and Ivan [1] gave the following alternate form of operators (3.1) (by putting $c = \frac{1}{n\alpha}$):

$$S_{n,c}(\varphi;x) = \sum_{k=0}^{\infty} \left(\frac{c}{1+c}\right)^{ncx} \begin{pmatrix} ncx+k-1 \\ k \end{pmatrix} (1+c)^{-k} \varphi\left(\frac{k}{n}\right), \qquad x \geq 0, \tag{3.2}$$

where $c = c_n \geq \beta(n = 0,1,2,......)$ for certain constant $\beta > 0$. Also, for a particular case $\alpha = \frac{1}{n}$, the operators (3.1) reduce to another form, which was considered by Agratini [19] as follows:

$$S_n^{\left[\frac{1}{n}\right]}(\varphi;x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} \varphi\left(\frac{k}{n}\right), \tag{3.3}$$

where

$$(nx)_k = nx(nx+1)\dots(nx+k-1), \qquad k \geq 1,$$

and $(nx)_0 = 1$. These operators (3.3) are special cases of Lupaş operators [118]. The operators (3.3) have also been studied in [60] and [122].

Agratini [20] modified the operators (3.3) into integral form in Kantorovich sense as:

$$T_n(\varphi;x) = n \sum_{k=0}^{\infty} 2^{-nx} \frac{(nx)_k}{2^k k!} \int_{k/n}^{(k+1)/n} \varphi(t)\,dt, \tag{3.4}$$

and studied some approximation properties. Very recently, Dhamija and Deo considered generalized positive linear operators based on Pólya-Eggenberger and inverse Pólya-Eggenberger distribution in [54] and they gave Kantorovich variant of these generalized operators in [50]. Several researchers have given some interesting results on Kantorovich variant of various operators see [11, 27, 28, 34, 37–43, 46, 83, 105, 128]. Motivated by the above works, for any bounded and integrable function φ defined on $\mathbb{R}^+$, we also modify the operators (3.1) in Kantorovich form:

$$L_n^{[\alpha]}(\varphi;x) = n \sum_{k=0}^{\infty} s_{n,k}^{[\alpha]}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \varphi(t)\,dt. \tag{3.5}$$

Special Cases:

1. For $\alpha = 0$ in (3.5), we get Szász Kantorovich operators given by Totik in [149].

2. For $\alpha = \frac{1}{n}$ in (3.5), we obtain another Kantorovich operators considered by Agratini [20].

The focus of this chapter is to study the approximation properties of modified Kantorovich operators (3.5). First, we obtain local approximation formula via modulus of continuity of second-order then we use Ditzian-Totik moduli of smoothness to discuss the rate of convergence of our operators. Finally, we establish the rate of convergence for functions having derivatives of bounded variation. The properties discussed in this article can be found in some recent papers and books like [17, 23, 48, 57, 72, 73, 79, 80, 85, 95, 98, 103, 108, 141].

3.2 Preliminaries

In order to prove the main convergence properties of operators (3.5), we need the following basic results:

Lemma 3.2.1. *[145] For the generalized Szász-Mirakyan operators* (3.1) *hold*

(i) $S_n^{[\alpha]}(1;x) = 1;$

(ii) $S_n^{[\alpha]}(t;x) = x;$

(iii) $S_n^{[\alpha]}\left(t^2;x\right) = x^2 + \left(\alpha + \frac{1}{n}\right)x.$

Proposition 3.2.2. *For the operators* (3.1)*, there hold the following higher order moments:*

(i) $S_n^{[\alpha]}\left(t^3;x\right) = x^3 + 3\left(\alpha + \frac{1}{n}\right)x^2 + \left(2\alpha^2 + \frac{3\alpha}{n} + \frac{1}{n^2}\right)x;$

(ii) $$S_n^{[\alpha]}\left(t^4;x\right) = x^4 + 6\left(\alpha + \frac{1}{n}\right)x^3 + \left(11\alpha^2 + \frac{18\alpha}{n} + \frac{7}{n^2}\right)x^2 + \left(6\alpha^3 + \frac{12\alpha^2}{n} + \frac{7\alpha}{n^2} + \frac{1}{n^3}\right)x.$$

Proof. By definition we can write

$$\begin{aligned}
S_n^{[\alpha]}\left(t^3;x\right) &= (1+\alpha n)^{-\frac{x}{\alpha}} \sum_{k=1}^{\infty} \frac{x(x+\alpha)\dots(x+(k-1)\alpha)n^k}{k!(1+\alpha n)^k} \frac{k^3}{n^3} \\
&= \frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3} \sum_{k=1}^{\infty} \frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\dots\left(\frac{x}{\alpha}+k-1\right)\alpha^k n^k}{(k-1)!(1+\alpha n)^k} (k(k-1)+k) \\
&= \frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3} \sum_{k=2}^{\infty} \frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\dots\left(\frac{x}{\alpha}+k-1\right)\alpha^k n^k}{(k-2)!(1+\alpha n)^k} k \\
&\quad + \frac{1}{n} S_n^{[\alpha]}\left(t^2;x\right) \\
&= \frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3} \sum_{k=3}^{\infty} \frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\dots\left(\frac{x}{\alpha}+k-1\right)\alpha^k n^k}{(k-3)!(1+\alpha n)^k}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=2}^{\infty}\frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\dots\left(\frac{x}{\alpha}+k-1\right)\alpha^k n^k}{(k-2)!(1+\alpha n)^k}\\
&+\frac{1}{n}S_n^{[\alpha]}\left(t^2;x\right)\\
&=\frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{\alpha}+k+2\right)!\alpha^{k+3}n^{k+3}}{k!\left(\frac{x}{\alpha}-1\right)!(1+\alpha n)^{k+3}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\sum_{k=0}^{\infty}\frac{\left(\frac{x}{\alpha}+k+1\right)!}{k!\left(\frac{x}{\alpha}+1\right)!}\left(\frac{\alpha n}{1+\alpha n}\right)^{k+2}\\
&+\frac{1}{n}S_n^{[\alpha]}\left(t^2;x\right)\\
&=\frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{\alpha}+k+2\right)!\alpha^{k+3}n^{k+3}}{k!\left(\frac{x}{\alpha}-1\right)!(1+\alpha n)^{k+3}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2(1+\alpha n)^{-\frac{x}{\alpha}}}{n}\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\alpha^2\left(1-\frac{\alpha n}{1+\alpha n}\right)^{-\left(\frac{x}{\alpha}+2\right)}\\
&+\frac{1}{n}S_n^{[\alpha]}\left(t^2;x\right)\\
&=\frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{\alpha}+k+2\right)!\alpha^{k+3}n^{k+3}}{k!\left(\frac{x}{\alpha}-1\right)!(1+\alpha n)^{k+3}}+\frac{2}{n}\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\alpha^2\\
&+\frac{1}{n}\left[x^2+\left(\alpha+\frac{1}{n}\right)x\right]\\
&=x^3+3\left(\alpha+\frac{1}{n}\right)x^2+\left(2\alpha^2+\frac{3\alpha}{n}+\frac{1}{n^2}\right)x.
\end{aligned}
$$

Similarly, we can prove the expression for $S_n^{[\alpha]}\left(t^4;x\right)$. □

Lemma 3.2.3. *For Kantorovich operators* (3.5)*, we have*

(i) $L_n^{[\alpha]}(1;x)=1$;

(ii) $L_n^{[\alpha]}(t;x)=x+\frac{1}{2n}$;

(iii) $L_n^{[\alpha]}\left(t^2;x\right)=x^2+\left(\alpha+\frac{2}{n}\right)x+\frac{1}{3n^2}$;

(iv) $L_n^{[\alpha]}\left(t^3;x\right)=x^3+3\left(\alpha+\frac{3}{2n}\right)x^2+\left(2\alpha^2+\frac{9\alpha}{2n}+\frac{7}{2n^2}\right)x+\frac{1}{4n^3}$;

(v) $L_n^{[\alpha]}\left(t^4;x\right)=x^4+\left(6\alpha+\frac{8}{n}\right)x^3+\left(11\alpha^2+\frac{24\alpha}{n}+\frac{15}{n^2}\right)x^2$
$\qquad+\left(6\alpha^3+\frac{16\alpha^2}{n}+\frac{15\alpha}{n^2}+\frac{6}{n^3}\right)x+\frac{1}{5n^4}.$

Proof. Taking into account of Lemma 3.2.3 and Proposition 3.2.2, we can easily get the desired result. □

Remark 3.2.4. *By simply applying Lemma 3.2.3, we have*

$$L_n^{[\alpha]}(t-x;x) = x + \frac{1}{2n} - x = \frac{1}{2n},$$

$$\begin{aligned} L_n^{[\alpha]}\left((t-x)^2;x\right) &= L_n^{[\alpha]}\left(t^2;x\right) - 2xL_n^{[\alpha]}(t;x) + x^2 L_n^{[\alpha]}(1;x) \\ &= x^2 + \left(\alpha + \frac{2}{n}\right)x + \frac{1}{3n^2} - 2x\left(x + \frac{1}{2n}\right) + x^2 \\ &= \left(\alpha + \frac{1}{n}\right)x + \frac{1}{3n^2}, \end{aligned}$$

$$\begin{aligned} L_n^{[\alpha]}\left((t-x)^3;x\right) &= L_n^{[\alpha]}\left(t^3;x\right) - 3xL_n^{[\alpha]}\left(t^2;x\right) + 3x^2 L_n^{[\alpha]}(t;x) - x^3 L_n^{[\alpha]}(1;x) \\ &= x^3 + 3\left(\alpha + \frac{3}{2n}\right)x^2 + \left(2\alpha^2 + \frac{9\alpha}{2n} + \frac{7}{2n^2}\right)x + \frac{1}{4n^3} \\ &\quad - 3x\left(x^2 + \left(\alpha + \frac{2}{n}\right)x + \frac{1}{3n^2}\right) + 3x^2\left(x + \frac{1}{2n}\right) - x^3 \\ &= \frac{3x^2}{n} + \left(2\alpha^2 + \frac{9\alpha}{2n} + \frac{5}{2n^2}\right)x + \frac{1}{4n^3}, \end{aligned}$$

and

$$\begin{aligned} L_n^{[\alpha]}\left((t-x)^4;x\right) &= L_n^{[\alpha]}\left(t^4;x\right) - 4xL_n^{[\alpha]}\left(t^3;x\right) + 6x^2 L_n^{[\alpha]}\left(t^2;x\right) \\ &\quad - 4x^3 L_n^{[\alpha]}(t;x) + x^4 L_n^{[\alpha]}(1;x) \\ &= x^4 + \left(6\alpha + \frac{8}{n}\right)x^3 + \left(11\alpha^2 + \frac{24\alpha}{n} + \frac{15}{n^2}\right)x^2 \\ &\quad + \left(6\alpha^3 + \frac{16\alpha^2}{n} + \frac{15\alpha}{n^2} + \frac{6}{n^3}\right)x + \frac{1}{5n^4} \\ &\quad - 4x\left(x^3 + 3\left(\alpha + \frac{3}{2n}\right)x^2 + \left(2\alpha^2 + \frac{9\alpha}{2n} + \frac{7}{2n^2}\right)x + \frac{1}{4n^3}\right) \\ &\quad + 6x^2\left(x^2 + \left(\alpha + \frac{2}{n}\right)x + \frac{1}{3n^2}\right) - 4x^3\left(x + \frac{1}{2n}\right) + x^4 \\ &= \left(3\alpha^2 + \frac{6\alpha}{n} + \frac{3}{n^2}\right)x^2 + \left(6\alpha^3 + \frac{16\alpha^2}{n} + \frac{15\alpha}{n^2} + \frac{5}{n^3}\right)x + \frac{1}{5n^4}. \end{aligned}$$

Lemma 3.2.5. *Let φ be a bounded function defined on $\mathbb{R}^+$ with $\|\varphi\| = \sup_{x\in\mathbb{R}^+} |\varphi(x)|$, then*

$$\left|L_n^{[\alpha]}(\varphi;x)\right| \le \|\varphi\|.$$

Lemma 3.2.6. *For $n \in \mathbb{N}$, we have*

$$L_n^{[\alpha]}\left((t-x)^2;x\right) \leq \frac{C}{n}\delta_n^2(x),$$

where $\delta_n^2(x) = \phi^2(x) + \frac{1}{n}$ and $\phi^2(x) = x$.

Now we can write the operators (3.5) in other form as:

$$L_n^{[\alpha]}(\varphi;x) = \int_0^\infty K_n^{[\alpha]}(x,t)\,\varphi(t)\,dt, \tag{3.6}$$

where

$$K_n^{[\alpha]}(x,t) = n\sum_{k=0}^{\infty} s_{n,k}^{[\alpha]}(x)\,\chi_{n,k}(t),$$

and $\chi_{n,k}(t)$ indicates the characteristic function on $\left[\frac{k}{n},\frac{k+1}{n}\right]$ w.r.t. $\mathbb{R}^+$.

Lemma 3.2.7. *For adequately large n and $x \in \mathbb{R}^+$:*

(i) Since $0 \leq y < x$, therefore

$$\beta_n(x,y) = \int_0^y K_n^{[\alpha]}(x,t)dt \leq \frac{C\delta_n^2(x)}{n(x-y)^2}.$$

(ii) If $x < z < \infty$ then we get

$$1 - \beta_n(x,z) = \int_z^\infty K_n^{[\alpha]}(x,t)dt \leq \frac{C\delta_n^2(x)}{n(z-x)^2}.$$

3.3 Direct results

Theorem 3.3.1. *Let $\varphi \in C(\mathbb{R}^+) \cap E$ and $\alpha(n)$ be a sequence converging to zero for adequately large n,*

$$\lim_{n\to\infty} L_n^{(\alpha)}(\varphi;x) = \varphi(x)$$

uniformly on each compact subset of $\mathbb{R}^+$, where $C(\mathbb{R}^+)$ is the space of all real-valued continuous functions on $\mathbb{R}^+$ and

$$E := \left\{\varphi : x \in \mathbb{R}^+, \frac{\varphi(x)}{1+x^2} \text{is convergent as } x \to \infty\right\}$$

Proof. Taking Lemma 3.2.3 into the account and the fact that $\alpha \to 0$ as $n \to \infty$, it is clear that

$$\lim_{n\to\infty} L_n^{(\alpha)}(e_i;x) = x^i, \ \ i = 0,1,2$$

uniformly on each compact subset of $\mathbb{R}^+$. Hence, applying the well-known Korovkin-type theorem [24] regarding the convergence of a sequence of positive linear operators, we get the desired result. □

Theorem 3.3.2. *Let $\varphi \in C_B(\mathbb{R}^+)$, then for any $x \in \mathbb{R}^+$ it follows*

$$\left| L_n^{[\alpha]}(\varphi;x) - \varphi(x) \right| \leq M\omega_2\left(\varphi, \tfrac{1}{2}\delta_n(x)\right) + \omega(\varphi, \beta_n),$$

where M is an absolute constant and
$\delta_n(x) = \left(L_n^{[\alpha]}\left((t-x)^2;x\right) + \left(L_n^{[\alpha]}(t-x;x)\right)^2 \right)^{\frac{1}{2}}, \quad \beta_n = L_n^{[\alpha]}(t-x;x)$
such that both terms δ_n and β_n tends to zero as $n \to \infty$.

Proof. For $x \in \mathbb{R}^+$, consider the operators

$$\hat{L}_n^{[\alpha]}(\varphi;x) = L_n^{[\alpha]}(\varphi;x) - \varphi\left(x + \frac{1}{2n}\right) + \varphi(x). \tag{3.7}$$

Since constants and linear functions are preserved by the operators $\hat{L}_n^{[\alpha]}$. Therefore,

$$\hat{L}_n^{[\alpha]}(t-x;x) = 0. \tag{3.8}$$

Let $\psi \in C_B^2(\mathbb{R}^+)$ and $x,t \in \mathbb{R}^+$. By Taylor's expansion, we have

$$\psi(t) = \psi(x) + (t-x)\psi'(x) + \int_x^t (t-u)\psi''(u)\,du.$$

Applying $\hat{L}_n^{[\alpha]}$ on both sides of the above Taylor's expansion, we get

$$\begin{aligned}\hat{L}_n^{[\alpha]}(\psi;x) - \psi(x) &= \psi'(x)\cdot\hat{L}_n^{[\alpha]}(t-x;x) + \hat{L}_n^{[\alpha]}\left(\int_x^t (t-u)\psi''(u)\,du;x\right) \\ &= L_n^{[\alpha]}\left(\int_x^t (t-u)\psi''(u)\,du;x\right) - \int_x^{x+\frac{1}{2n}} \left(x + \frac{1}{2n} - u\right)\psi''(u)du.\end{aligned}$$

Observe that

$$\left| \int_x^t (t-u)g''(u) \right| \leq (t-x)^2 \cdot \|\psi''\|.$$

Thus

$$\left|\hat{L}_n^{[\alpha]}(\psi;x)-\psi(x)\right|\leq\left(L_n^{[\alpha]}((t-x)^2;x)+\left(L_n^{[\alpha]}(t-x;x)\right)^2\right)\cdot\|\psi''\|.$$

Making use of definition (3.7) of the operators $\hat{L}_n^{[\alpha]}$ and Lemma 3.2.5, we have

$$\begin{aligned}\left|L_n^{[\alpha]}(\varphi;x)-\varphi(x)\right|&\leq\left|\hat{L}_n^{[\alpha]}(\varphi-\psi;x)\right|+\left|\hat{L}_n^{[\alpha]}(\psi;x)-\psi(x)\right|\\&\quad+|\psi(x)-\varphi(x)|+\left|\varphi\left(x+\frac{1}{2n}\right)-\varphi(x)\right|\\&\leq 4\|\varphi-\psi\|+\delta_n^2(x)\left\|\psi''\right\|+\omega(\varphi,\beta_n),\end{aligned}$$

with $\delta_n^2(x)=L_n^{[\alpha]}\left((t-x)^2;x\right)+\left(L_n^{[\alpha]}(t-x;x)\right)^2$ and $\beta_n=L_n^{[\alpha]}(t-x;x)$.

Now taking infimum on the right-hand side over all $\psi\in C_B^2(\mathbb{R}^+)$ and using the relation (1.13), we get

$$\begin{aligned}\left|L_n^{[\alpha]}(\varphi;x)-\varphi(x)\right|&\leq 4K_2\left(\varphi,\frac{{\delta_n}^2(x)}{4}\right)+\omega(\varphi,\beta_n)\\&\leq M\omega_2\left(\varphi,\tfrac{1}{2}\delta_n(x)\right)+\omega(\varphi,\beta_n).\end{aligned}$$

Hence the proof. $\square$

Theorem 3.3.3. *Let $\varphi\in C_B(\mathbb{R}^+)$, then for any $x\in\mathbb{R}^+$ we have*

$$\left|L_n^{[\alpha]}(\varphi;x)-\varphi(x)\right|\leq C\omega_2^{\varphi^\lambda}\left(\varphi,\frac{\delta_n^{(1-\lambda)}(x)}{\sqrt{n}}\right)+\omega\left(\varphi,\frac{1}{2n}\right),$$

where C is an absolute constant and
$\delta_n(x)=\left(L_n^{[\alpha]}\left((t-x)^2;x\right)+\left(L_n^{[\alpha]}(t-x;x)\right)^2\right)^{\frac{1}{2}}$.

Proof. Consider the operators defined by (3.7)

$$\hat{L}_n^{[\alpha]}(\varphi;x)=L_n^{[\alpha]}(\varphi;x)+\varphi(x)-\varphi\left(x+\frac{1}{2n}\right)\tag{3.9}$$

For above considered operators, we can write $\hat{L}_n^{[\alpha]}(1;x)=1$ and $\hat{L}_n^{[\alpha]}(t;x)=x$.
Therefore, definition (3.9), Lemma 3.2.5 and Lemma 3.2.6 gives

$$\hat{L}_n^{[\alpha]}(t-x;x)=0,\quad \hat{L}_n^{[\alpha]}\left((t-x)^2;x\right)\leq\frac{C}{n}\delta_n^2(x)\tag{3.10}$$

$$\text{and}\qquad\left\|\hat{L}_n^{[\alpha]}(\varphi;x)\right\|\leq 3\|\varphi\|.$$

Again, from ([55], p.141), for $t < u < x$, we have

$$\frac{|t-u|}{\varphi^{2\lambda}(u)} \leq \frac{|t-x|}{\varphi^{2\lambda}(x)} \quad \text{and} \quad \frac{|t-u|}{\delta_n^{2\lambda}(u)} \leq \frac{|t-x|}{\delta_n^{2\lambda}(x)}. \tag{3.11}$$

Now

$$\begin{aligned} \left|\hat{L}_n^{[\alpha]}(\varphi;x)-\varphi(x)\right| &\leq \left|\hat{L}_n^{[\alpha]}(\varphi-\psi;x)\right| + \left|\hat{L}_n^{[\alpha]}(\psi;x)-\psi(x)\right| \\ &\quad + |\varphi(x)-\psi(x)| \\ &\leq 4\|\varphi-\psi\| + \left|\hat{L}_n^{[\alpha]}(\psi;x)-\psi(x)\right|. \end{aligned} \tag{3.12}$$

For $\psi \in D_\lambda^2$ and $t,x \in \mathbb{R}^+$, using Taylor's expansion with integral remainder,

$$\psi(t) = \psi(x) + (t-x)\psi'(x) + \int_x^t (t-u)\psi''(u)\,du.$$

Operating $\hat{L}_n^{[\alpha]}$ and using (3.10) and (3.11), we get

$$\begin{aligned} \left|\hat{L}_n^{[\alpha]}(\psi;x)-\psi(x)\right| &= \left|\hat{L}_n^{[\alpha]}\left(\int_x^t (t-u)\psi''(u)\,du;x\right)\right| \\ &\leq \left|L_n^{[\alpha]}\left(\int_x^t (t-u)\psi''(u)\,du;x\right)\right| + \left|\int_x^{x+\frac{1}{2n}}\left(x+\frac{1}{2n}-u\right)\psi''(u)\,du\right| \\ &\leq \left\|\delta_n^{2\lambda}\psi''\right\| L_n^{[\alpha]}\left(\frac{(t-x)^2}{\delta_n^{2\lambda}(x)};x\right) + \left\|\delta_n^{2\lambda}\psi''\right\| \frac{\left(\frac{1}{2n}\right)^2}{\delta_n^{2\lambda}(x)} \\ &= \delta_n^{-2\lambda}(x)\left\|\delta_n^{2\lambda}\psi''\right\| L_n^{[\alpha]}\left((t-x)^2;x\right) + \left\|\delta_n^{2\lambda}g''\right\| \delta_n^{-2\lambda}(x)\frac{1}{(2n)^2} \\ &\leq C\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\delta_n^{2\lambda}\psi''\right\|\right) + \frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\delta_n^{2\lambda}\psi''\right\| \\ &\leq C\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\delta_n^{2\lambda}\psi''\right\|. \end{aligned} \tag{3.13}$$

From (3.12), (3.13) and then using definition of K-functional (corresponding to Ditzian-Totik) along with the relation (1.19),

$$\begin{aligned} \left|\hat{L}_n^{[\alpha]}(\varphi;x)-\varphi(x)\right| &\leq 4\|\varphi-\psi\| + C\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\varphi^{2\lambda}\psi''\right\| \\ &\leq C\omega_{\varphi^\lambda}^2\left(\varphi, \frac{\delta_n^{(1-\lambda)}(x)}{\sqrt{n}}\right). \end{aligned}$$

Hence

$$\begin{aligned}\left|L_n^{[\alpha]}(\varphi;x)-\varphi(x)\right| &\le \left|\hat{L}_n^{[\alpha]}(\varphi;x)-\varphi(x)\right|+\left|\varphi\left(x+\frac{1}{2n}\right)-\varphi(x)\right|\\ &\le C\omega^2_{\varphi^\lambda}\left(\varphi,\frac{\delta_n^{(1-\lambda)}(x)}{\sqrt{n}}\right)+\omega\left(\varphi,\frac{1}{2n}\right).\end{aligned}$$

Thus the proof is complete. □

3.4 Function of bounded variation

Theorem 3.4.1. *Let $\varphi \in DBV(\mathbb{R}^+)$. Then for adequately large n and for each $x \in \mathbb{R}^+$, we get*

$$\begin{aligned}&\left|L_n^{[\alpha]}(\varphi;x)-\varphi(x)\right|\\ &\le \frac{1}{4n}\left|\varphi'(x+)+\varphi'(x-)\right|+\frac{1}{2}\sqrt{\frac{C}{n}}\left|\varphi'(x+)-\varphi'(x-)\right|\\ &\quad+\frac{C\delta_n^2(x)}{nx^2}\left|\varphi(2x)-\varphi(x)-x\varphi'(x+)\right|\\ &\quad+\frac{x}{\sqrt{n}}\vee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}\left(\varphi_x'\right)+\frac{C\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\vee_{x-\frac{x}{k}}^{x+\frac{x}{k}}\left(\varphi_x'\right)\\ &\quad+M(\gamma,r,x)+\frac{|\varphi(x)|}{nx^2}C\delta_n^2(x)+\sqrt{\frac{C}{n}}\delta_n(x)\varphi'(x+),\end{aligned}\tag{3.14}$$

where $\vee_a^b\varphi(x)$ denotes the total variation of φ on $[a,b]$, φ_x is an auxiliary operator given by

$$\varphi_x(t)=\begin{cases}\varphi(t)-\varphi(x-),\ 0\le t<x\\ 0, \qquad t=x\\ \varphi(t)-\varphi(x+),\ x<t<\infty\end{cases}$$

and

$$M(\gamma,r,x)=M2^{\gamma}\left(\int_0^\infty(t-x)^{2r}K_n^{[\alpha]}(x,t)\,dt\right)^{\frac{\gamma}{2r}}.$$

Proof. Because $L_n^{[\alpha]}(1;x)=1$, therefore for all $x\in\mathbb{R}^+$, we obtain

$$\begin{aligned}L_n^{[\alpha]}(\varphi;x)-\varphi(x)&=\int_0^\infty(\varphi(t)-\varphi(x))K_n^{[\alpha]}(x,t)dt\\ &=\int_0^\infty K_n^{[\alpha]}(x,t)\int_x^t\varphi'(u)\,dudt.\end{aligned}\tag{3.15}$$

For $\varphi \in DBV(\mathbb{R}^+)$, we may write

$$\begin{aligned}\varphi'(u) =& \frac{1}{2}\left(\varphi'(x+)+\varphi'(x-)\right)+{\varphi_x}'(u)+\frac{1}{2}\left(\varphi'(x+)-\varphi'(x-)\right)(u-x)\\ &+\delta_x(u)\left(\varphi'(u)-\frac{1}{2}\left(\varphi'(x+)+\varphi'(x-)\right)\right),\end{aligned} \tag{3.16}$$

where

$$\delta_x(u)=\begin{cases}1, & u=x\\ 0, & u\neq x.\end{cases}$$

From the last term of (3.16) and using the property of δ_x, we acquire

$$\int_0^\infty\left(\int_x^t\left(\varphi'(u)-\frac{1}{2}\left(\varphi'(x+)+\varphi'(x-)\right)\right)\delta_x(u)\,du\right)K_n^{[\alpha]}(x,t)\,dt=0. \tag{3.17}$$

Using (3.6), we obtain

$$\begin{aligned}&\int_0^\infty\left(\int_x^t\frac{1}{2}\left(\varphi'(x+)+\varphi'(x-)\right)du\right)K_n^{[\alpha]}(x,t)\,dt\\ &\quad=\frac{1}{2}\left(\varphi'(x+)+\varphi'(x-)\right)L_n^{[\alpha]}((t-x);x).\end{aligned} \tag{3.18}$$

Moreover,

$$\begin{aligned}&=\int_0^\infty\frac{1}{2}\left(\varphi'(x+)-\varphi'(x-)\right)(t-x)K_n^{[\alpha]}(x,t)\,dt\\ &\leq\frac{1}{2}\left|\varphi'(x+)-\varphi'(x-)\right|\int_0^\infty|t-x|K_n^{[\alpha]}(x,t)\,dt\\ &=\frac{1}{2}\left|\varphi'(x+)-\varphi'(x-)\right|L_n^{[\alpha]}(|t-x|;x)\\ &\leq\frac{1}{2}\left|\varphi'(x+)-\varphi'(x-)\right|\left(L_n^{[\alpha]}\left((t-x)^2;x\right)\right)^{1/2}.\end{aligned} \tag{3.19}$$

Using equations (3.15)-(3.19) with Lemma 3.2.7, we have

$$\begin{aligned}L_n^{[\alpha]}(\varphi;x)-\varphi(x)\leq&\frac{1}{2}\left(\varphi'(x+)+\varphi'(x-)\right)L_n^{[\alpha]}((t-x);x)\\ &+\frac{1}{2}\left|\varphi'(x+)-\varphi'(x-)\right|\left(L_n^{[\alpha]}\left((t-x)^2;x\right)\right)^{1/2}\\ &+\int_0^\infty\left(\int_x^t{\varphi_x}'(u)\,du\right)K_n^{[\alpha]}(x,t)\,dt\\ \leq&\frac{1}{4n}\left(\varphi'(x+)+\varphi'(x-)\right)+\frac{1}{2}\sqrt{\frac{C}{n}}\delta_n(x)\left|\varphi'(x+)-\varphi'(x-)\right|\\ &+\int_0^\infty\left(\int_x^t{\varphi_x}'(u)\,du\right)K_n^{[\alpha]}(x,t)\,dt.\end{aligned}$$

Therefore

$$\left|L_n^{[\alpha]}(\varphi;x)-\varphi(x)\right|\leq\frac{1}{4n}\left|\varphi'(x+)+\varphi'(x-)\right|+\frac{1}{2}\sqrt{\frac{C}{n}}\delta_n(x)\left|\varphi'(x+)-\varphi'(x-)\right|$$
$$+A_{nx}+B_{nx}, \tag{3.20}$$

where

$$A_{nx}=\left|\int_0^x\left(\int_x^t\varphi_x{}'(u)\,du\right)K_n^{[\alpha]}(x,t)\,dt\right|,$$

and

$$B_{nx}=\left|\int_x^\infty\left(\int_x^t\varphi_x{}'(u)\,du\right)K_n^{[\alpha]}(x,t)\,dt\right|.$$

Applying Lemma 3.2.7, integrating by parts and taking $y=x-\frac{x}{\sqrt{n}}$, we obtain

$$\begin{aligned}A_{nx}&=\left|\int_0^x\left(\int_x^t\varphi_x{}'(u)\,du\right)d_t\beta_n(x,t)\right|=\left|\int_0^x\beta_n(x,t)\,\varphi_x{}'(t)\,dt\right|\\&\leq\int_0^y|\beta_n(x,t)|\left|\varphi'{}_x(t)\right|dt+\int_y^x|\beta_n(x,t)|\left|\varphi'{}_x(t)\right|dt\\&=\int_0^{x-\frac{x}{\sqrt{n}}}\beta_n(x,t)\left|\varphi'{}_x(t)\right|dt+\int_{x-\frac{x}{\sqrt{n}}}^x\beta_n(x,t)\left|\varphi'{}_x(t)\right|dt.\end{aligned}$$

Since $\varphi_x{}'(x)=0$ and $\beta_n(x,t)\leq 1$, it follows

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x\left|\varphi_x{}'(t)\right|\beta_n(x,t)\,dt&=\int_{x-\frac{x}{\sqrt{n}}}^x\left|\varphi_x{}'(t)-\varphi_x{}'(x)\right|\beta_n(x,t)\,dt\\&\leq\int_{x-\frac{x}{\sqrt{n}}}^x\vee_t^x\left(\varphi_x{}'\right)dt\leq\frac{x}{\sqrt{n}}\vee_{x-\frac{x}{\sqrt{n}}}^x\left(\varphi_x{}'\right).\end{aligned}$$

Again using Lemma 3.2.7 and substituting $t=x-\frac{x}{u}$,

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}\left|\varphi_x{}'(t)\right|\beta_n(x,t)\,dt&\leq\frac{C\delta_n^2(x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{\left|\varphi_x{}'(t)\right|}{(x-t)^2}dt\\&\leq\frac{C\delta_n^2(x)}{nx}\int_1^{\sqrt{n}}\vee_{x-\frac{x}{u}}^x\left(\varphi_x{}'\right)du\\&\leq\frac{C\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\vee_{x-\frac{x}{k}}^x\left(\varphi_x{}'\right).\end{aligned}$$

Thus, $A_{nx}\leq\frac{x}{\sqrt{n}}\vee_{x-\frac{x}{\sqrt{n}}}^x\left(\varphi_x{}'\right)+\frac{C\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\vee_{x-\frac{x}{k}}^x\left(\varphi_x{}'\right)$.
Now, we can write

$$B_{n,x}\leq\left|\int_x^{2x}\left(\int_x^t\varphi_x{}'(u)\,du\right)K_n^{[\alpha]}(x,t)\,dt\right|+\left|\int_{2x}^\infty\left(\int_x^t\varphi_x{}'(u)\,du\right)K_n^{[\alpha]}(x,t)\,dt\right|$$

Also from part (ii) of Lemma 3.2.7, we have

$$K_n^{[\alpha]}(x,t) = d_t\left(1-\beta_n(x,t)\right) \quad \text{for} \qquad t > x$$

Thus,

$$B_{nx} \le B_{1,n,x} + B_{2,n,x},$$

where

$$B_{1,n,x} = \left| \int_x^{2x} \left(\int_x^t \varphi_x{}'(u)\,du \right) d_t\left(1-\beta_n(x,t)\right) \right|$$

and

$$B_{2,n,x} = \left| \int_{2x}^{\infty} \left(\int_x^t \varphi_x{}'(u)\,du \right) K_n^{[\alpha]}(x,t)\,dt \right|.$$

Making use of integration by parts as well as using Lemma 3.2.7, (3.16), $1-\beta_n(x,t) \le 1$ and putting $t = x + \frac{x}{u}$ successively,

$$\begin{aligned}
B_{1,n,x} &= \left| \int_x^{2x} \varphi_x{}'(u)\,du\left(1-\beta_n(x,2x)\right) - \int_x^{2x} \varphi_x{}'(t)\left(1-\beta_n(x,t)\right)dt \right| \\
&\le \left| \int_x^{2x} \left(\varphi'(u) - \varphi'(x+)\right) du \right| \left|1-\beta_n(x,2x)\right| \\
&\quad + \int_x^{2x} \left|\varphi_x{}'(t)\right| \left|1-\beta_n(x,t)\right| dt \\
&\le \frac{C\delta_n^2(x)}{nx^2} \left|\varphi(2x) - \varphi(x) - x\varphi'(x+)\right| \\
&\quad + \int_x^{x+\frac{x}{\sqrt{n}}} \left|\varphi_x{}'(t)\right| \left|1-\beta_n(x,t)\right| dt + \int_{x+\frac{x}{\sqrt{n}}}^{2x} \left|\varphi_x{}'(t)\right| \left|1-\beta_n(x,t)\right| dt \\
&\le \frac{C\delta_n^2(x)}{nx^2} \left|\varphi(2x) - \varphi(x) - x\varphi'(x+)\right| \\
&\quad + \frac{C\delta_n^2(x)}{n} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{\bigvee_x^t(\varphi_x{}')}{(t-x)^2} dt + \int_x^{x+\frac{x}{\sqrt{n}}} \bigvee_x^t\left(\varphi_x{}'\right) dt \\
&\le \frac{C\delta_n^2(x)}{nx^2} \left|\varphi(2x) - \varphi(x) - x\varphi'(x+)\right| \\
&\quad + \frac{C\delta_n^2(x)}{n} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{\bigvee_x^t(\varphi_x{}')}{(t-x)^2} dt + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} \left(\varphi_x{}'\right) \\
&\le \frac{C\delta_n^2(x)}{nx^2} \left|\varphi(2x) - \varphi(x) - x\varphi'(x+)\right| \\
&\quad + \frac{C\delta_n^2(x)}{nx} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{k}} \left(\varphi_x{}'\right) + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} \left(\varphi_x{}'\right).
\end{aligned}$$

Finally, Remark 3.2.4 implies

$$\begin{aligned}
B_{2,n,x} &= \left| \int_{2x}^{\infty} \left(\int_{x}^{t} \left(\varphi'(u) - \varphi'(x+) \right) du \right) K_n^{[\alpha]}(x,t)\, dt \right| \\
&\le \int_{2x}^{\infty} |\varphi(t) - \varphi(x)| K_n^{[\alpha]}(x,t)\, dt + \int_{2x}^{\infty} |t-x| \varphi'(x+) K_n^{[\alpha]}(x,t)\, dt \\
&\le M \int_{2x}^{\infty} t^{\gamma} K_n^{[\alpha]}(x,t)\, dt + |\varphi(x)| \int_{2x}^{\infty} K_n^{[\alpha]}(x,t)\, dt \\
&\quad + \sqrt{\frac{C}{n}} \delta_n(x) \varphi'(x+).
\end{aligned}$$

As it is obvious that $t \le 2(t-x)$ and $x \le t-x$ when $t \ge 2x$, applying Holder's inequality, we get

$$\begin{aligned}
B_{2,n,x} &\le M 2^{\gamma} \left(\int_{0}^{\infty} (t-x)^{2r} K_n^{[\alpha]}(x,t)\, dt \right)^{\frac{\gamma}{2r}} + \frac{C\delta_n^2(x)|\varphi(x)|}{nx^2} \\
&\quad + \sqrt{\frac{C}{n}} \delta_n(x) \varphi'(x+) \\
&= M(\gamma, r, x) + \frac{C\delta_n^2(x)|\varphi(x)|}{nx^2} + \sqrt{\frac{C}{n}} \delta_n(x) \varphi'(x+).
\end{aligned}$$

Estimates of $B_{1,n,x}$ and $B_{2,n,x}$ results

$$B_{n,x} \le \frac{C\delta_n^2(x)}{nx^2} \left| \varphi(2x) - \varphi(x) - x\varphi'(x+) \right|$$

$$\begin{aligned}
&+ \frac{C\delta_n^2(x)}{nx} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{k}} (\varphi_x') + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} (\varphi_x') \\
&+ M(\gamma, r, x) + \frac{C\delta_n^2(x)|\varphi(x)|}{nx^2} + \sqrt{\frac{C}{n}} \delta_n(x) \varphi'(x+).
\end{aligned}$$

Hence values of $A_{n,x}$ and $B_{n,x}$ in (3.20), we get the required result. □

Chapter 4

Approximation by integral form of Jain and Pethe operators

4.1 Introduction

In 1957, Baskakov [29] introduced the Baskakov operators for $\varphi \in C(\mathbb{R}^+)$ and its Durrmeyer form proposed by Sahai and Prasad [140]. In the literature, Durrmeyer forms of several operators are available, which have been studied by many researchers and have presented interesting results, see [2, 15, 25, 53, 58, 75, 76, 78, 80–82, 140].

In the previous chapter, we studied the Kantorovich form of Jain and Pethe operators (3.1). Now we consider the Durrmeyer form of Jain and Pethe operators associated with Baskakov operators (JPDB operators), defined as:

$$P_{n,\alpha}(\varphi(t);x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}^{[\alpha]}(x)\int_{0}^{\infty}\left(\begin{array}{c} n+k-1 \\ k \end{array}\right)\frac{t^k}{(1+t)^{n+k}}\varphi(t)dt. \tag{4.1}$$

The main objective of this chapter is to discuss some direct results, Voronovskaja type theorem and weighted approximation properties for the JPDB operators (4.1).

4.2 Preliminaries

This section is based on some basic results that will be used later for the proof of the main theorems.

Lemma 4.2.1. *The following are some moments of JPDB operators:*

(i) $P_{n,\alpha}(1;x) = 1$;

(ii) $P_{n,\alpha}(t;x) = \frac{(n-3)!}{(n-2)!}(nx + 1!)$;

(iii) $P_{n,\alpha}(t^2;x) = \frac{(n-4)!}{(n-2)!}((nx)^2 + (\alpha n + 4)(nx) + 2!)$;

(iv) $P_{n,\alpha}(t^3;x) = \frac{(n-5)!}{(n-2)!}((nx)^3 + (3\alpha n + 2)(nx)^2 + (2\alpha^2 n^2 + 9\alpha n + 19)(nx) + 3!)$;

(v) $P_{n,\alpha}(t^4;x) = \frac{(n-6)!}{(n-2)!}((nx)^4 + 2(3\alpha n + 8)(nx)^3 + (11(\alpha n)^2 + 48\alpha n + 72)(nx)^2 + (6(\alpha n)^3 + 32(\alpha n)^2 + 72\alpha n + 96)(nx) + 4!)$.

Proof. Making use of Lemma 3.2.1 and Proposition 3.2.2, we obtain

$$P_{n,\alpha}(\varphi;x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x)\int_0^{\infty}\begin{pmatrix} n+k-1 \\ k \end{pmatrix}\frac{t^k}{(1+t)^{n+k}}\varphi(t)dt.$$

For $\varphi(t) = 1$

$$P_{n,\alpha}(1;x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x)\int_0^{\infty}\begin{pmatrix} n+k-1 \\ k \end{pmatrix}\frac{t^k}{(1+t)^{n+k}}dt = 1.$$

For $\varphi(t) = t$

$$\begin{aligned} P_{n,\alpha}(t;x) &= (n-1)\sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x)\int_0^{\infty}\begin{pmatrix} n+k-1 \\ k \end{pmatrix}\frac{t^{k+1}}{(1+t)^{n+k}}dt \\ &= \sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x)\frac{(k+1)}{(n-2)} \\ &= \frac{n}{n-2}\left\{\sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x)\frac{k}{n} + \frac{1}{n}\sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x)\right\} = \frac{nx+1}{n-2}. \end{aligned}$$

For $\varphi(t) = t^2$

$$\begin{aligned}
P_{n,\alpha}(t^2;x) &= (n-1)\sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x) \int_0^{\infty} \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{t^{k+2}}{(1+t)^{n+k}} dt \\
&= \sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x) \frac{k^2+3k+2}{(n-2)(n-3)} \\
&= \frac{n^2}{(n-2)(n-3)} \left\{ \sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x) \frac{k^2}{n^2} + \frac{3}{n} \sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x) \frac{k}{n} + \frac{2}{n^2} \sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x) \right\} \\
&= \frac{n^2}{(n-2)(n-3)} \left\{ x^2 + \left(\alpha + \frac{4}{n}\right) x + \frac{2}{n^2} \right\}.
\end{aligned}$$

Higher order terms follow in similar manner. □

Lemma 4.2.2. *The central moments for the JPDB operators are denoted by* $\mu_{n,m}^{\alpha}(x) = P_{n,\alpha}((t-x)^m;x)$ *and given as:*

(i) $\mu_{n,1}^{\alpha}(x) = \frac{(n-3)!}{(n-2)!}(1+2x)$*;*

(ii) $\mu_{n,2}^{\alpha}(x) = \frac{(n-4)!}{(n-2)!}((n+6)x^2 + (n(\alpha n+1) - 2(n-1))x + 2)$*;*

(iii) $\mu_{n,4}^{\alpha}(x) = \frac{(n-6)!}{(n-2)!}((3n^2+86n+120)x^4 + 2(3\alpha n^3 + (60\alpha+6)n^2 + 146n + 120)x^3$
$+ (3\alpha^2 n^4 + 4\alpha(10\alpha+3)n^2 + 12(15\alpha+1)n^2 + 252n + 240)x^2$
$+ 2(3\alpha^3 n^4 + 16\alpha^2 n^3 + 36\alpha n^2 + 36n + 60)x + 4!)$*,*

Lemma 4.2.3. *Let* $\lim_{n\to\infty} n\alpha = l$*, for* $l \in \mathbb{R}$*. The central moments of Lemma 4.2.2 have the limiting values:*

(i) $\lim_{n\to\infty} n\,\mu_{n,1}^{\alpha}(x) = 1+2x$*;*

(ii) $\lim_{n\to\infty} n\,\mu_{n,2}^{\alpha}(x) = x(x+l-1)$*;*

(iii) $\lim_{n\to\infty} n^2\,\mu_{n,4}^{\alpha}(x) = 3x^2(x^2+(l+2)x+(l+2)^2)$*.*

Lemma 4.2.4. *Let* φ *be a bounded function defined on* $\mathbb{R}^+$ *with*
$\|\varphi\| = \sup_{x\in\mathbb{R}^+} |\varphi(x)|$*, then*

$$|P_{n,\alpha}(\varphi;x)| \le \|\varphi\|.$$

4.3 Direct results

Theorem 4.3.1. *Let* $C(\mathbb{R}^+)$ *be the space of all real-valued continuous functions in* $\mathbb{R}^+$*. Let*

$$E = \left\{ \varphi : x \in \mathbb{R}^+, \frac{\varphi(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}.$$

For $\varphi \in C(\mathbb{R}^+) \cap E$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be such that the limit

$$\lim_{n\to\infty} P_{n,\alpha}(\varphi;x) = \varphi(x)$$

is uniform on each compact subset of $\mathbb{R}^+$.

Proof. Taking Lemma 4.2.1 into the account and the fact that $\alpha \to 0$ as $n \to \infty$, it is clear that

$$\lim_{n\to\infty} P_{n,\alpha}\left(t^i;x\right) = x^i, \quad i = 0,1,2$$

uniformly on each compact subset of $\mathbb{R}^+$. Hence, making use of the well-known Korovkin-type theorem [24], we get the required result. □

Theorem 4.3.2. *Let $\varphi \in C_B(\mathbb{R}^+)$, then for any $x \in \mathbb{R}^+$ it follows that*

$$|P_{n,\alpha}(\varphi(t);x) - \varphi(x)| \le M\,\omega_2\left(\varphi, \frac{\beta_n(x)}{2}\right) + \omega(\varphi, \mu_{n,1}^{\alpha}(x)),$$

where M is an absolute constant, $\beta_n(x) = \sqrt{\mu_{n,2}^{\alpha}(x) - (\mu_{n,1}^{\alpha}(x))^2}$, and the terms β_n and $\mu_{n,1}^{\alpha}(x)$ tend to zero as $n \to \infty$.

Proof. For $x \in \mathbb{R}^+$, consider the auxiliary operators

$$\hat{P}_{n,\alpha}\left(\varphi(t);x\right) = P_{n,\alpha}\left(\varphi(t);x\right) - \varphi\left(\frac{nx+1}{n-2}\right) + \varphi\left(x\right). \tag{4.2}$$

Since the constants and linear functions are preserved by the operators $\hat{P}_{n,\alpha}$, therefore,

$$\hat{P}_{n,\alpha}\left(t-x;x\right) = 0. \tag{4.3}$$

Let $\psi \in C_B^2(\mathbb{R}^+)$ and $x,t \in \mathbb{R}^+$. By Taylor's expansion, we have

$$\psi\left(t\right) = \psi\left(x\right) + (t-x)\psi'\left(x\right) + \int_x^t (t-u)\,\psi''\left(u\right)du.$$

Applying, $\hat{P}_{n,\alpha}$ to both sides of the expansion, we get

$$\begin{aligned}\hat{P}_{n,\alpha}\left(\psi(t);x\right) - \psi(x) &= \psi'(x)\cdot\hat{P}_{n,\alpha}(t-x;x) + \hat{P}_{n,\alpha}\left(\int_x^t (t-u)\,\psi''\left(u\right)du;x\right)\\ &= P_{n,\alpha}\left(\int_x^t (t-u)\,\psi''\left(u\right)du;x\right) - \int_x^{\frac{nx+1}{n-2}}\left(\frac{nx+1}{n-2} - u\right)\psi''(u)du.\end{aligned}$$

We observe that

$$\left|\int_x^t (t-u)\psi''(u)du\right| \leq (t-x)^2\|\psi''\|,$$

thus

$$\left|\hat{P}_{n,\alpha}\left(\psi(t);x\right)-\psi(x)\right| \leq \beta_n^2(x)\|\psi''\|.$$

Making use of (4.2) and Lemma 4.2.2, we obtain

$$\begin{aligned}|P_{n,\alpha}(\varphi(t);x)-\varphi(x)| &\leq \left|\hat{P}_{n,\alpha}(\varphi-\psi;x)\right| + \left|\hat{P}_{n,\alpha}(\psi(t);x)-\psi(x)\right| \\ &\quad + |\psi(x)-\varphi(x)| + \left|\varphi\left(\frac{nx+1}{n-2}\right)-\varphi(x)\right| \\ &\leq 4\|\varphi-\psi\| + \beta_n^2(x)\left\|\psi''\right\| + \omega\left(\varphi,\mu_{n,1}^{\alpha}(x)\right). \qquad (4.4)\end{aligned}$$

Now taking infimum in (4.4) over $\psi \in C_B^2(\mathbb{R}^+)$, we get

$$\begin{aligned}|P_{n,\alpha}(\varphi(t);x)-\varphi(x)| &\leq 4K_2\left(\varphi,\frac{\beta_n{}^2(x)}{4}\right) + \omega\left(\varphi,\mu_{n,1}^{\alpha}(x)\right) \\ &\leq M\omega_2\left(\varphi,\tfrac{1}{2}\beta_n(x)\right) + \omega\left(\varphi,\mu_{n,1}^{\alpha}(x)\right).\end{aligned}$$

□

Theorem 4.3.3. *Let $\varphi \in Lip_M(\xi)$, $0<\xi\leq 1$ and for every positive x, we have*

$$|P_{n,\alpha}(\varphi(t);x)-\varphi(x)| \leq M_{\varphi}\left(\frac{\mu_{n,2}^{\alpha}(x)}{x}\right)^{\xi/2}.$$

Proof. Making the use of definition (1.20) and $\varphi \in Lip_M(\xi)$, then we obtain

$$\begin{aligned}|P_{n,\alpha}(\varphi(t);x)-\varphi(x)| &\leq (n-1)\sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x)\int_0^{\infty}\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}|\varphi(t)-\varphi(x)|\,dt \\ &\leq (n-1)M_{\varphi}\sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x)\int_0^{\infty}\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}\left(\frac{|t-x|^{\xi}}{(t+x)^{\frac{\xi}{2}}}\right)dt.\end{aligned}$$

With $p=2/(2-\xi)$,$q=2/\xi$, we apply the Hölder inequality for integration and get

$$|P_{n,\alpha}(\varphi(t);x)-\varphi(x)| \leq (n-1)M_{\varphi}\sum_{k=0}^{\infty} s_{n,k}^{(\alpha)}(x)\left((n-1)\int_0^{\infty}\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}dt\right)^{(2-\xi)/2}$$

$$\times \left((n-1) \int_0^\infty \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{t^k}{(1+t)^{n+k}} \frac{(t-x)^2}{(t+x)} dt \right)^{\xi/2}$$

Now, applying the Hölder inequality for summation with p and q, we get

$$\begin{aligned}
|P_{n,\alpha}(\varphi(t);x) - \varphi(x)| \leq & M_\varphi \left\{ (n-1) \sum_{k=0}^\infty s_{n,k}^{(\alpha)}(x) \int_0^\infty \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{t^k}{(1+t)^{n+k}} dt \right\}^{(2-\xi)/2} \\
& \times \left\{ (n-1) \sum_{k=0}^\infty s_{n,k}^{(\alpha)}(x) \int_0^\infty \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{t^k}{(1+t)^{n+k}} \left(\frac{(t-x)^2}{(t+x)} \right) dt \right\}^{\xi/2} \\
\leq & \frac{M_\varphi}{x^{\xi/2}} \left(P_{n,\alpha}\left((t-x)^2;x \right) \right)^{\xi/2}.
\end{aligned}$$

Since $\mu_{n,2}^{\alpha}(x) = P_{n,\alpha}\left((t-x)^2;x\right)$. Hence the result follows. □

The Steklov-mean of second-order for sufficiently small h associated with the function $\varphi \in C_B(\mathbb{R}^+)$ is as follows:

$$\varphi_h(x) = \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \left(2\varphi(x+u+v) - \varphi(x+2u+2v)\right) du dv, \ h > 0 \tag{4.5}$$

The definition of Steklov-mean and its properties can be seen in these references [92,148, 157] for detailed explanation .

Proposition 4.3.4. *Some properties of Steklov-mean are as follows:*

(i) $|\varphi - \varphi_h| \leq \frac{13}{4}\,\omega_2(\varphi;h)$,

(ii) $|\varphi'|_\infty \leq \frac{5}{h}, \omega_2(f;h)$,

(iii) $|\varphi''|_\infty \leq \frac{9}{h^2}\,\omega_2(\varphi;h)$.

Theorem 4.3.5. *Let* $\varphi \in C_B(\mathbb{R}^+)$*, then*

$$|P_{n,\alpha}(\varphi(t);x) - \varphi(x)| \leq \frac{13}{2}\omega_2(\varphi;\mu_{n,2}^{\alpha}(x)) + \omega(\varphi;\mu_{n,2}^{\alpha}(x)).$$

Proof. Let $\varphi \in C_B^2(\mathbb{R}^+)$, using Taylor's expansion

$$\varphi_h(t) = \varphi_h(x) + (t-x)\,\varphi_h^{'}(x) + \frac{(t-x)^2}{2}\,\varphi_h^{''}(\eta),$$

where $t \leq \eta \leq x$. Applying $P_{n,\alpha}(\cdot;x)$ in above and use Cauchy-Schwarz inequality, we get

$$|P_{n,\alpha}(\varphi_h(t);x) - \varphi_h(x)| \leq \sqrt{P_{n,\alpha}((t-x)^2;x)}\,||\varphi_h^{'}||_\infty + \frac{1}{2}P_{n,\alpha}((t-x)^2;x)\,||\varphi_h^{''}||_\infty. \tag{4.6}$$

For $\varphi \in C_B^2(\mathbb{R}^+)$, using Proposition 4.3.4 and (4.6), we get

$$\begin{aligned}|P_{n,\alpha}(\varphi(t);x) - \varphi(x)| &\leq |P_{n,\alpha}(\varphi;x) - P_{n,\alpha}(\varphi_h;x)| + |P_{n,\alpha}(\varphi_h;x) - \varphi_h(x)| + |\varphi_h(x) - \varphi(x)| \\ &\leq 2\|\varphi - \varphi_h\|_\infty + |P_{n,\alpha}(\varphi_h;x) - \varphi_h| \\ &\leq \frac{13}{2}\omega_2(\varphi;\mu_{n,2}^{\alpha}(x)) + \omega(\varphi;\mu_{n,2}^{\alpha}(x)).\end{aligned}$$

Choosing $h = \mu_{n,2}^{\alpha}(x)$, we obtain the desire outcome. □

Theorem 4.3.6. *For any $\varphi \in C_B(\mathbb{R}^+)$, $x > 0$, and $\phi(x) = \sqrt{x}$, we have*

$$|P_{n,\alpha}(\varphi(t);x) - \varphi(x)| \leq M\,\omega_\phi\left(\varphi, \frac{\mu_{n,2}^{\alpha}(x)}{\sqrt{x}}\right).$$

Proof. For any $\psi \in W_\phi(\mathbb{R}^+)$, by Taylor's theorem

$$\psi(t) = \psi(x) + \int_x^t \psi^{'}(u)du,$$

we have

$$|\psi(t) - \psi(x)| \leq \left\|\phi\psi^{'}\right\|_\infty \left|\int_x^t \frac{1}{\phi(u)}du\right| = 2\left\|\phi\psi^{'}\right\|_\infty \left|\sqrt{t} - \sqrt{x}\right|,$$

therefore

$$|\psi(t) - \psi(x)| \leq 2\left\|\phi\psi^{'}\right\|_\infty \frac{|t-x|}{\sqrt{t}+\sqrt{x}} \leq 2\left\|\phi\psi^{'}\right\|_\infty \frac{|t-x|}{\sqrt{x}} = 2\left\|\phi\psi^{'}\right\|_\infty \frac{|t-x|}{\phi(x)}.$$

By the property $|P_{n,\alpha}(\varphi;x)| \leq \|\varphi\|$, using Lemma 4.2.2, and the above equation, we get

$$\begin{aligned}|P_{n,\alpha}((\varphi(t);x) - \varphi(x)| &\leq |P_{n,\alpha}\varphi(t) - \psi(x)| + |P_{n,\alpha}(\psi;x) - \psi(x)| + |\psi(x) - \varphi(x)| \\ &\leq 2\|\varphi - \psi\|_\infty + \frac{\left\|\phi\psi^{'}\right\|_\infty}{\phi(x)}P_{n,\alpha}(|t-x|;x).\end{aligned}$$

With the use of Cauchy-Schwarz inequality, we get

$$|P_{n,\alpha}(\varphi(t);x) - \varphi(x)| \leq 2\|\varphi - \psi\|_\infty + \frac{\left\|\phi\psi^{'}\right\|_\infty}{\phi(x)}\mu_{n,2}^{\alpha}(x).$$

Taking the infimum over $\psi \in W(\mathbb{R}^+)$ yields

$$|P_{n,\alpha}(\varphi(t);x)-\varphi(x)| \leq 2K_\phi\left(\varphi, \frac{\mu_{n,2}^{\alpha}(x)}{\sqrt{x}}\right).$$

The desired outcome is obtained by using the equivalence between the Peetre's K-functional and the Ditzian-Totik modulus of smoothness which is given in (1.18). □

4.4 Voronovskaja type theorem

Theorem 4.4.1. *Let $\varphi \in C_2(\mathbb{R}^+)$ and $\alpha = \alpha_n \to 0$ as $n \to \infty$. If there exists first and second derivatives of the function φ at a fixed point $x \in \mathbb{R}^+$ and $\lim_{n\to\infty} n\alpha_n = l \in \mathbb{R}$, then*

$$\lim_{n\to\infty} n\left[P_{n,\alpha}\left(\varphi(t);x\right)-\varphi(x)\right] = (1+2x)\varphi'(x)+\left(x^2+(l+2)x\right)\varphi''(x),$$

uniformly in $[0,a]$, $a>0$.

Proof. By the Taylor's series expansion, we have

$$\varphi(t) = \varphi(x)+(t-x)\varphi^{'}(x)+\frac{(t-x)^2}{2!}\varphi^{''}(x)+\varepsilon(t,x)(t-x)^2, \tag{4.7}$$

where $\varepsilon(t,x)$ is a bounded function approaches to 0 as $t \to x$.
Applying $P_{n,\alpha}(;x)$ on both side of (4.7), we have

$$P_{n,\alpha}(\varphi(t)-\varphi(x);x) = \varphi^{'}(x)P_{n,\alpha}((t-x);x)+\frac{\varphi^{''}(x)}{2!}P_{n,\alpha}((t-x)^2;x)$$
$$P_{n,\alpha}(\varepsilon(t,x)(t-x)^2;x).$$

Now,

$$\lim_{n\to\infty} nP_{n,\alpha}(\varphi(t)-\varphi(x);x) = \lim_{n\to\infty} n\varphi^{'}(x)P_{n,\alpha}((t-x);x)+\lim_{n\to\infty} nP_{n,\alpha}((t-x)^2;x)\varphi^{''}(x)$$
$$+\lim_{n\to\infty} nP_{n,\alpha}(\varepsilon(t,x)(t-x)^2;x). \tag{4.8}$$

From Theorem 4.3.1, Lemma 4.2.3 and applying cauchy-Schwarz in the last term of (4.8) then, we get

$$\lim_{n\to\infty} nP_{n,\alpha}(\varepsilon(t,x)(t-x)^2;x) = 0. \tag{4.9}$$

Again using Lemma 4.2.3, (4.8) and (4.9) then, we get required result. □

4.5 Weighted approximation

Theorem 4.5.1. *Let $\varphi \in C_2(\mathbb{R}^+)$ and ω_d be its modulus of continuity on the finite interval $[0,d], d > 0$. Then*

$$||P_{n,\alpha}(\varphi)-\varphi||_{C[0,d]} \leq 4M(1+d^2)\kappa_n(d)+2\omega_{d+1}(\varphi,\sqrt{\kappa_n(d)}),$$

where $\kappa_n(d) = \max_{x\in[0,d]}\mu^{\alpha}_{n,2}(x)$ and $||\cdot||_{C[0,d]}$ represents the sup norm in $C[0,d]$.

Proof. In [93], for each $0 \leq x \leq d$, $\delta > 0$ and $t \geq 0$ the inequality holds:

$$|\varphi(t)-\varphi(x)| \leq 4M(1+x^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(\varphi;\delta), \tag{4.10}$$

With the use of Cauchy-Schwarz inequality and applying $P_{n,\alpha}(.;x)$, we have

$$\begin{aligned}|P_{n,\alpha}(\varphi(t);x)-\varphi(x)| &\leq 4M(1+d^2)P_{n,\alpha}((t-x)^2;x)+\left(1+\frac{1}{\delta}P_{n,\alpha}(|t-x|;x)\right)\omega_{d+1}(\varphi,\delta)\\ &\leq 4M(1+d^2)\kappa_n(d)+\omega_{b+1}(\varphi,\delta)\left(1+\frac{1}{\delta}\sqrt{\kappa_n(d)}\right).\end{aligned}$$

For our suitability if we choose $\delta = \sqrt{\kappa_n(d)}$, the result follows. □

Theorem 4.5.2. *For $\varsigma > 0$ and $\varphi \in C_2^*(\mathbb{R}^+)$, we have*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^+}\frac{|P_{n,\alpha)}(\varphi(t);x)-\varphi(x)|}{(1+x^2)^{1+\varsigma}} = 0.$$

Proof. Suppose x_0 is an arbitrary and fixed belongs to the set $\mathbb{R}^+$, we obtain

$$\begin{aligned}&\sup_{x\in\mathbb{R}^+}\frac{|P_{n,\alpha}(\varphi(t);x)-\varphi(x)|}{(1+x^2)^{1+\varsigma}} \leq \sup_{x\leq x_0}\frac{|P_{n,\alpha}(\varphi(t);x)-\varphi(x)|}{(1+x^2)^{1+\varsigma}}+\sup_{x>x_0}\frac{|P_{n,\alpha}(\varphi(t);x)-\varphi(x)|}{(1+x^2)^{1+\varsigma}}\\ &\leq ||P_{n,\alpha}(\varphi(t);x)-\varphi||_{C[0,x_0]}+||\varphi||_2\sup_{x>x_0}\frac{|P_{n,\alpha}(1+t^2;x)|}{(1+x^2)^{1+\varsigma}}+\sup_{x>x_0}\frac{|\varphi(x)|}{(1+x^2)^{1+\varsigma}}.\end{aligned}$$

Since $|\varphi(x)| \leq ||\varphi||_2(1+x^2)$, we obtain

$$\sup_{x>x_0}\frac{|\varphi(x)|}{(1+x^2)^{1+\varsigma}} \leq \frac{||\varphi||_2}{(1+x_0^2)^{\varsigma}}.$$

We choose x_0 is sufficiently large and suppose $\varepsilon > 0$, we get

$$\frac{||\varphi||_2}{(1+x_0^2)^{\varsigma}} < \frac{\varepsilon}{6}. \tag{4.11}$$

Since $\lim\limits_{n\to\infty} \sup\limits_{x>x_0} \dfrac{P_{n,\alpha}(1+t^2;x)}{1+x^2} = 1$, we obtain

$$\sup_{x>x_0} \frac{P_{n,\alpha}(1+t^2;x)}{1+x^2} \leq \frac{(1+x_0^2)^{\varsigma}}{\|\varphi\|_2} \frac{\varepsilon}{3} + 1,$$

for adequately large n, we get

$$\|\varphi\|_2 \sup_{x>x_0} \frac{P_{n,\alpha}(1+t^2;x)}{(1+x^2)^{\varsigma+1}} \leq \frac{\|\varphi\|_2}{(1+x_0^2)^{\varsigma}} \sup_{x>x_0} \frac{P_{n,\alpha}(1+t^2;x)}{(1+x^2)} \leq \frac{\varepsilon}{3} + \frac{\|\varphi\|_2}{(1+x_0^2)^{\varsigma}}. \tag{4.12}$$

Applying Theorem 4.5.2, we can find for adequately large n

$$||P_{n,\alpha}(\varphi(t);x) - \varphi(x)||_{C[0,x_0]} < \frac{\varepsilon}{3}. \tag{4.13}$$

Combining the relations (4.11)-(4.13), we obtain

$$\sup_{x\in\mathbb{R}^+} \frac{|P_{n,\alpha}(\varphi(t);x) - \varphi(x)|}{(1+x^2)^{1+\varsigma}} < \varepsilon.$$

□

Theorem 4.5.3. *Let $\varphi \in C_2^*(\mathbb{R}^+)$ and for adequately large n, we acquire*

$$\sup_{x\in\mathbb{R}^+} \frac{|P_{n,\alpha}(\varphi(t);x) - \varphi(x)|}{(1+x^2)^{\frac{5}{2}}} \leq \tilde{\eta(l)}\Omega\left(\varphi; \frac{1}{\sqrt{n}}\right), \tag{4.14}$$

where $\tilde{\eta(l)} > 0$.

Proof. For $x \in \mathbb{R}^+$, $\delta > 0$ and using the property (iii) of Proposition (1.4.2), we have

$$\begin{aligned}|\varphi(t) - \varphi(x)| &\leq \left(1 + (x + |x-t|)^2\right)\Omega(\varphi; |t-x|)\\ &\leq 2(1+x^2)(1+(t-x)^2)\left(1 + \frac{|t-x|}{\delta}\right)\Omega(\varphi;\delta).\end{aligned}$$

Using $P_{n,\alpha}(\cdot;x)$ in above inequality, we obtain

$$|P_{n,\alpha}(\varphi;x) - \varphi(x)| \leq 2(1+x^2)\Omega(\varphi;\delta) \tag{4.15}$$

$$\times\left(1+P_{n,\alpha}((t-x)^2;x)+P_{n,\alpha}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\right).$$

From Lemma 4.2.3, for adequately large n, we obtain the following inequlities

$$nP_{n,\alpha}((t-x)^2;x)\leq\eta(l)(1+x^2) \textit{ and } n^2P_{n,\alpha}((t-x)^4;x)\leq\eta(l)(1+x^2)^2, \tag{4.16}$$

where $\eta(l)>0$.

Using Cauchy-Schwarz inequality in (4.15), we have

$$P_{n,\alpha}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\leq\frac{1}{\delta}\sqrt{P_{n,\alpha}((t-x)^2;x)}+\frac{1}{\delta}\sqrt{P_{n,\alpha}((t-x)^4;x)P_{n,\alpha}((t-x)^2;x)}. \tag{4.17}$$

Combining the estimates (4.15)-(4.17) and assume

$$\tilde{\eta(l)}=2\left(1+\sqrt{\eta(l)}+2\eta(l)\right) \text{ and } \delta=\frac{1}{\sqrt{n}},$$

the result holds. □

Chapter 5

Approximaton by genuine Gupta-Srivastava operators

5.1 Introduction

In 2003, Srivastava and Gupta [142] proposed a general sequence of positive linear operators and studied the rate of convergence for the function of bounded variation. Two years later Ispir and Yuksel [99] considered the Bézier variant of these operators and established the rate of convergence. The operators defined in [142] reproduced constant function only except for the special case $c = 0$, which provides the well-known Phillips operators. These operators and related versions were studied by several researchers, see for instance [4,47,77,89,99,114,119,127,151,153], where many interesting results, like the rate of convergence for functions having derivatives of bounded variation, Voronovskaja type asymptotic formula, better estimates and simultaneous approximation of these operators were established. It is well-known that if operators preserve the linear function, one may get a better approximation. In this direction very recently Gupta and Srivastava [90] proposed a general family of positive linear operators, which preserve constant, as well as linear functions for all $c \in \mathbb{N} \cup \{0\} \cup \{-1\}$, which may be termed as Gupta-Srivastava operators and for all integers m, are defined as:

$$L_{n,c}(\varphi(t);x) = \{n+(m+1)c\} \sum_{k=1}^{\infty} p_{n+mc,k}(x;c) \int_0^{\infty} p_{n+(m+2)c,k-1}(t;c)\varphi(t)dt + p_{n+mc,0}(x;c)\varphi(0), \tag{5.1}$$

where

$$p_{n,k}(x;c)=\frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x).$$

The special cases of operators (5.1) are as follows:

(i) For $c=0$ and $\phi_{n,0}(x)=e^{-nx}$, we get Phillips operators

$$L_{n,0}(\varphi(t);x)=n\sum_{k=1}^{\infty}p_{n,k}(x;0)\int_{0}^{\infty}p_{n,k-1}(t;0)\varphi(t)dt+p_{n,0}(x;0)\varphi(0),$$

where

$$p_{n,k}(x,0)=\frac{e^{-nx}(nx)^k}{k!}\text{ and }x\in[0,\infty).$$

(ii) For $c\in\mathbb{N}$ and $\phi_{n,c}(x)=(1+cx)^{-\frac{n}{c}}$, we get genuine Baskakov-Durrmeyer type operators. These operators are similar to (5.1), where

$$p_{n,k}(x;c)=\frac{\left(\frac{n}{c}\right)_k}{k!}\frac{(cx)_k}{(1+cx)^{\frac{n}{c}+k}}\text{ and }x\in[0,\infty)$$

with $(n)_i$ denotes the rising factorial given by

$$(n)_i=n(n+1)(n+2)...(n+i-1)\ \&\ (n)_0=1(i\in\mathbb{N}).$$

(iii) For $c=-1$ and $\phi_{n,-1}(x)=(1-x)^{-n}$, we have sequence of Bernstein-Durrmeyer operators

$$L_{n,-1}(\varphi;x)=(n-m-1)\sum_{k=1}^{n-m-1}p_{n-m,k}(x,-1)\int_{0}^{1}p_{n-m-2,k-1}(t,-1)\varphi(t)dt$$
$$+p_{n-m,0}(x,-1)\varphi(0)+p_{n-m,n-m}(x,-1),$$

where

$$p_{n,k}(x;-1)=\binom{n}{k}x^k(1-x)^{n-k}\text{ and }x\in[0,1].$$

Recently Acu and collaborators [6, 7] and Deo-Dhamija in [50] studied and many interesting approximation properties and provide a better degree of approximation over the classical operators, direct approximation theorem using the Ditzian-Totik modulus of smoothness and a Voronovskaja type theorem and many more.

In this chapter, we consider operators (5.1) and study fundamental properties, the rate of convergence in terms of second-order modulus of continuity, Lipschitz type space, Voronovskaya type estimates, convergence estimates for the function having derivatives of bounded variation and weighted approximation.

5.2 Preliminaries

In this section, we prove some basic results which are useful to prove several theorems and results.

Lemma 5.2.1. *Let $\varphi(t) = t^i$, $i = 0,1,2,3,4$ and $c \in \mathbb{N} \cup \{0\} \cup \{-1\}$, then we have*

(i) $L_{n,c}(1;x) = 1$;

(ii) $L_{n,c}(t;x) = x$;

(iii) $L_{n,c}\left(t^2;x\right) = \frac{(n+(m+1)c)}{(n+(m-1)c)}x^2 + \frac{2}{(n+(m-1)c)}x$;

(iv) $L_{n,c}\left(t^3;x\right) = \frac{(n+(m+1)c)(n+(m+2)c)}{(n+(m-1)c)(n+(m-2)c)}x^3 + \frac{6(n+(m+1)c)}{(n+(m-1)c)(n+(m-2)c)}x^2 + \frac{6}{(n+(m-1)c)(n+(m-2)c)}x$;

(v) $L_{n,c}\left(t^4;x\right) = \frac{(n+(m+1)c)(n+(m+2)c)(n+(m+3)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^4 + \frac{12(n+(m+1)c)(n+(m+2)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^3$
$+\frac{36(n+(m+1)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^2 + \frac{24}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x$.

Proof. All the moments of operators (5.1), can be obtained in terms of hyper geometric function of order $r \in \mathbb{N}$ for details see [90]. □

Lemma 5.2.2. *The central moment of the operators (5.1) is given as:*

$$\mu_{n,c,s}(x) = L_{n,c}((t-x)^s;x),$$

for $s = 1,2,4$ then, we have

(i) $\mu_{n,c,1}(x) = 0$;

(ii) $\mu_{n,c,2}(x) = \frac{2x(1+cx)}{(n+(m-1)c)}$;

(iii) $\mu_{n,c,4}(x) = \frac{12c^2(n+(m+7)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^4 + \frac{24c^2(13n+(13m+1)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^3$
$+\frac{12(n+(m+9)c)}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x^2 + \frac{24}{(n+(m-1)c)(n+(m-2)c)(n+(m-3)c)}x$.

Lemma 5.2.3. *For $n \in \mathbb{N}$ then, we have*

$$L_{n,c}\left((t-x)^2;x\right) \leq \frac{2\phi^2(x)}{n},$$

where $\phi^2(x) = x(1+cx)$.

Lemma 5.2.4. *For* $\varphi \in C_B(\mathbb{R}^+)$ *and sufficient large n, then we have*

(i) $\lim_{n\to\infty} n\mu_{n,c,2}(x) = 2x(1+cx)$;

(ii) $\lim_{n\to\infty} n^2\mu_{n,c,4}(x) = 12x^2(c^2x^2+26c^2x+1)$.

Lemma 5.2.5. *For* $\varphi \in C_B(\mathbb{R}^+)$ *then, we have*

$$|L_{n,c}(\varphi;x)| \leq \|\varphi\|.$$

5.3 Direct results

Theorem 5.3.1. *Let* $\varphi \in C_B(\mathbb{R}^+)$ *and* $n \to \infty$. *Then* $\lim_{n\to\infty} L_{n,c}(\varphi(t);x) = \varphi(x)$, *uniformly in each compact subset of* $\mathbb{R}^+$.

Proof. From Lemma 5.2.1, $L_{n,c}(1;x) = 1$, $L_{n,c}(t;x) = x$, $L_{n,c}(t^2;x) = x^2$, as $n \to \infty$. By Bohman-Korovkin theorem, we have

$$L_{n,c}(\varphi(t);x) = \varphi(x), \quad \text{as} \quad n \to \infty,$$

uniformly in each compact subset of $\mathbb{R}^+$. □

Theorem 5.3.2. *For the function* $\varphi \in C_B(\mathbb{R}^+)$ *we have*

$$|L_{n,c}(\varphi(t);x) - f(x)| \leq C\omega_2(\varphi;\delta_n(x)),$$

where $\delta_n^2(x) = \mu_{n,c,2}(x) = \frac{2x(1+cx)}{(n+(m-1)c)}$.

Proof. Let the function $\psi \in C_B^2(\mathbb{R}^+)$, and using Taylor's expression then we have

$$\psi(t) = \psi(x) + \psi'(x)(t-x) + \int_x^t (t-u)\psi''(u)du. \tag{5.2}$$

Applying $L_{n,c}(.;x)$ on both side of the expression (5.2) then, we have

$$L_{n,c}(\psi(t);x) - \psi(x) = \psi'(x)L_{n,c}((t-x);x) + L_{n,c}\left(\int_x^t (t-u)\psi''(u)du;x\right).$$

Using Lemma 5.2.3 then, we have

$$L_{n,c}\left(\psi(t);x\right)-\psi(x)=L_{n,c}\left(\int_{x}^{t}(t-u)\psi''(u)du;x\right) \tag{5.3}$$

and we know that

$$\int_{x}^{t}(t-u)\psi''(u)du\leq(t-x)^{2}\left\|\psi''\right\|. \tag{5.4}$$

From (5.3) & (5.4) and Using Lemma 5.2.3, we have

$$\left|L_{n,c}\left(\psi(t);x\right)-\psi(x)\right|\leq L_{n,c}((t-x)^{2};x)\left\|\psi''\right\|=\frac{2x(1+cx)}{(n+(m-1)c)}\left\|\psi''\right\|. \tag{5.5}$$

From Lemma 5.6.1, we have

$$\left|L_{n,c}\left(\varphi(t);x\right)\right|\leqslant\left\|\varphi\right\|. \tag{5.6}$$

Hence from (5.5) and (5.6), we get

$$\begin{aligned}|L_{n,c}(\varphi(t);x)-\varphi(x)| &\leq |L_{n,c}(\varphi(t)-\psi(t);x)-(\varphi(x)-\psi(x))|+|L_{n,c}(\psi(t);x)-\psi(x)|\\ &\leq 2\left\|\varphi-\psi\right\|+\frac{2x(1+cx)}{(n+(m-1)c)}\left\|\psi''\right\|.\end{aligned}$$

Taking infimum on the right hand side of for all $\psi\in C_B^2(\mathbb{R}^+)$ and using (1.13), we get the desired result. □

Theorem 5.3.3. *If $\varphi\in Lip_{\xi}M$, $x\in\mathbb{R}^+$ and $n>(m-1)c$, we have*

$$|L_{n,c}(\varphi(t);x)-\varphi(x)|\leq M\delta_n(x).$$

Proof. Since $\varphi\in Lip_{\xi}M$, $x\in\mathbb{R}^+$ and applying the Hölder's inequality with $p=2/\xi$ and $q=2/(2-\xi)$, we have

$$\begin{aligned}|L_{n,c}(\varphi(t);x)-\varphi(x)| &\leq L_{n,c}(|\varphi(t)-\varphi(x)|;x)\\ &\leq ML_{n,c}(|t-x|^{\xi};x)\end{aligned}$$

$$\begin{aligned}&\leq M(L_{n,c}((t-x)^{2};x)^{\xi/2}\\ &\leq M\delta_n(x),\end{aligned}$$

which is required result. □

5.4 Voronovskaya type theorem

Here, we present the Voronovskaya type theorem for Gupta-Srivastava operators:

Theorem 5.4.1. *Let $\varphi \in C_B(\mathbb{R}^+)$ and if there exists second derivatives of function φ at a fixed point $x \in \mathbb{R}^+$, then we have*

$$\lim_{n\to\infty} n\,[L_{n,c}(\varphi;x) - \varphi(x)] = x(1+cx)\varphi^{''}(x).$$

Proof. By the Taylor's series expansion, we have

$$\varphi(t) = \varphi(x) + (t-x)\varphi^{'}(x) + \frac{(t-x)^2}{2!}\varphi^{''}(x) + \varepsilon(t,x)(t-x)^2, \tag{5.7}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.
Applying $L_{n,c}(;x)$ on both side of (7.11), then we have

$$L_{n,c}(\varphi(t) - \varphi(x);x) = \varphi^{'}(x)L_{n,c}((t-x);x) + \frac{\varphi^{''}(x)}{2!}L_{n,c}((t-x)^2;x)$$
$$L_{n,c}(\varepsilon(t,x)(t-x)^2;x).$$

From Lemma 5.2.2, we get

$$L_{n,c}((t-x);x) = 0$$

,

$$\lim_{n\to\infty} nL_{n,c}(\varphi(t) - \varphi(x);x) = \lim_{n\to\infty} nL_{n,c}((t-x)^2;x)\varphi^{''}(x)$$
$$+ \lim_{n\to\infty} nL_{n,c}(\varepsilon(t,x)(t-x)^2;x). \tag{5.8}$$

From theorem 5.3.1, Lemma 5.2.4 and applying cauchy-Schwarz in the last term of (5.8) then, we get

$$\lim_{n\to\infty} nL_{n,c}(\varepsilon(t,x)(t-x)^2;x) = 0. \tag{5.9}$$

Using Lemma 5.2.4 and (5.8), (5.9) then, we get required result. □

5.5 Quantitative Voronovskaya type theorem

With the help of weighted modulus of continuity $\Omega(.;\delta)$, here we establish the degree of approximation of the function $\varphi \in C_2^*(\mathbb{R}^+)$ for the proposed operators (5.1). Several researchers have discussed this result. For more details see [66, 68, 75, 109].

Theorem 5.5.1. *Let* $\varphi \in C_2^*(\mathbb{R}^+)$ *such that* $\varphi', \varphi'' \in C_2^*(\mathbb{R}^+)$ *and* $c \in \mathbb{N} \cup \{0\} \cup \{-1\}$ *then for sufficient large n and for each* $x \in \mathbb{R}^+$,

$$\left| n\left(L_{n,c}(\varphi;x) - \varphi(x)\right) - x(1+cx)\varphi''(x) \right| = O(1)\Omega\left(\varphi'', \frac{1}{\sqrt{n}}\right),$$

as $n \to \infty$ *holds true.*

Proof. By Taylor's theorem

$$\varphi(t) = \varphi(x) + \varphi'(x)(t-x) + \frac{\varphi''(x)}{2!}(t-x)^2 + h_2(t,x), \tag{5.10}$$

where

$$h_2(t,x) = \frac{\varphi''(\eta) - \varphi''(x)}{2!}(t-x)^2, \tag{5.11}$$

here η is a number lies between t and x.

From the well-known property of the weighted modulus of continuity, we get

$$\left|\varphi''(\eta) - \varphi''(x)\right| \leq 4(1+x^2)(1+\delta^2)^2 \left(1 + \frac{(t-x)^4}{\delta^4}\right)\Omega\left(\varphi'', \delta\right). \tag{5.12}$$

From (5.11) and (5.12), we get

$$h_2(t,x) \leq 8(1+x^2)\Omega\left(\varphi'', \delta\right)\left(1 + \frac{(t-x)^4}{\delta^4}\right)(t-x)^2, \tag{5.13}$$

for $0 < \delta < 1$.

Applying $L_{n,c}(;x)$ on both side of in (5.10) and using Lemma 5.2.2, we get

$$\left| L_{n,c}(\varphi;x) - \varphi(x) - \frac{\varphi''(x)}{2!}L_{n,c}\left((t-x)^2;x\right) \right| \leq L_{n,c}\left(\left|h_2(t,x)\right|;x\right) \tag{5.14}$$

From [90], the order of the convergence of operators

$$L_{n,c}((t-x)^s;x) = O\left(n^{-\left[\frac{s+1}{2}\right]}\right). \tag{5.15}$$

From (5.13), (5.15) and Lemma 5.2.4, we get

$$\begin{aligned} L_{n,c}\left(|h_2(t,x)|;x\right) \leq& 8(1+x^2)\Omega\left(\varphi'';\delta\right)L_{n,c}\left((t-x)^2+\frac{(t-x)^6}{\delta^4};x\right) \\ =& 8(1+x^2)\Omega\left(\varphi'';\delta\right)\left(L_{n,c}\left((t-x)^2;x\right)+\frac{1}{\delta^4}L_{n,c}\left((t-x)^6;x\right)\right) \\ =& 8(1+x^2)\Omega\left(\varphi'';\delta\right)\left(O\left(\frac{1}{n}\right)+\frac{1}{\delta^4}O\left(\frac{1}{n^3}\right)\right) as\, n\to\infty. \end{aligned}$$

By Choosing $\delta=\frac{1}{\sqrt{n}}$, we get

$$L_{n,c}\left(|h_2(t,x)|;x\right)=O(1)\Omega\left(\varphi'',\frac{1}{\sqrt{n}}\right). \tag{5.16}$$

From (5.14), (5.16) and Lemma 5.2.4, we get

$$\left|n\left(L_{n,c}(\varphi;x)-\varphi(x)\right)-x(1+cx)\varphi''(x)\right|=O(1)\Omega\left(\varphi'',\frac{1}{\sqrt{n}}\right),$$

as $n\to\infty$. □

5.6 Function of bounded variation

The operators (5.1) can be written in the following form:

$$L_{n,c}(\varphi(t);x)=\int_0^\infty U_{n,c}(x;t)\varphi(t)dt,$$

where

$$U_{n,c}(x;t)=\{n+(m+1)c\}\sum_{k=1}^{\infty}p_{n+mc,k}(x;c)p_{n+(m+2)c,k-1}(t;c)+p_{n+mc,0}(x;c)\delta(t),$$

$\delta(t)$ is dirac-delta function.

Lemma 5.6.1. *For fixed $x\in\mathbb{R}^+$ and if n is sufficient large then, we have*

(i) $\chi_{n,c}(x;y)=\int_0^y U_{n,c}(x;t)dt\leq 2\frac{\phi^2(x)}{(x-y)^2},\quad 0\leq y\leq x;$

(ii) $1-\chi_{n,c}(x;y)=\int_z^0 U_{n,c}(x;t)dt\leq 2\frac{\phi^2(x)}{n(z-x)^2},\quad x<z<\infty.$

Theorem 5.6.2. *Let $\varphi \in DBV(0,\infty)$ then for all $x>0$ and sufficiently large n, we have*

$$\begin{aligned}
&\left|L_{n,c}(\varphi;x)-\varphi(x)\right| \\
&\leq \sqrt{\frac{1}{2n}}\phi(x)\left|\varphi'(x+)-\varphi'(x-)\right|+\frac{x}{\sqrt{n}}\vee_{x-\frac{x}{\sqrt{n}}}^{x}\left(\varphi_x'\right) \\
&+\frac{2\phi^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\vee_{x-\frac{x}{k}}^{x}(\varphi_x')+\frac{2\phi^2(x)}{nx^2}\left|\varphi(2x)-\varphi(x)-x\varphi'(x+)\right| \\
&\quad+\frac{2\phi^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\vee_{x}^{x+\frac{x}{k}}\left(\varphi_x'\right)+\frac{x}{\sqrt{n}}v_x^{x+\frac{x}{\sqrt{n}}}(\varphi_x')+M(r,\alpha,x) \\
&\quad+2\frac{|\varphi(x)|}{nx^2}\phi^2(x)+\sqrt{\frac{2}{n}}\phi(x)\varphi'(x+),
\end{aligned}$$

where $\vee_a^b\varphi(x)$ denotes the total variation of φ on $[a,b]$ and φ_x is an auxiliary operator given by

$$\varphi_x(t)=\begin{cases}\varphi(t)-\varphi(x-), & 0\leq t<x\\ 0, & t=x\\ \varphi(t)-\varphi(x+), & x<t<\infty\end{cases}$$

Proof. Using Lemma 5.6.1, the proof of this theorem closely follows the idea which is developed in [119]. Therefore here we skip the proof. □

5.7 Weighted approximation

Theorem 5.7.1. *For each $\varphi \in C_2^*(\mathbb{R}^+)$ then, we have*

$$|L_{n,c}(\varphi(t);x)-\varphi(x)|\leq 4M_\varphi(1+x^2)\delta_n^2(x)+2\omega_{d+1}(\varphi;\delta_n(x)),$$

where $\delta_n^2(x)=L_{n,c}((t-x)^2;x)$ and $\omega_{d+1}(\varphi;\delta_n(x))$ is usual modulus of continuity in $[0,d+1]$.

Proof. From [93], for each $x\in[0,d]$ and $t\geq 0$, the accompanying inequality holds:

$$|\varphi(t)-\varphi(x)|\leq 4M_\varphi(1+x^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{d+1}(\varphi;\delta_n(x)),\delta>0.$$

With the use of Cauchy Schwarz inequality, we obtain

$$\begin{aligned}
|L_{n,c}(\varphi(t);x)-\varphi(x)| &\leq 4M_\varphi(1+x^2)L_{n,c}((t-x)^2;x)+\left(1+\frac{L_{n,c}|t-x|}{\delta}\right)\omega_{d+1}(\varphi;\delta_n(x)) \\
&\leq 4M_\varphi(1+x^2)\delta_n^2(x)+\left(1+\frac{\delta_n(x)}{\delta}\right)\omega_{d+1}(\varphi;\delta_n(x)).
\end{aligned}$$

Now choosing $\delta = \delta_n(x)$, we get required results. □

Theorem 5.7.2. *Let $\varphi \in C_2^*(\mathbb{R}^+)$,then, we have*

$$\lim_{n\to\infty} \|L_{n,c}(\varphi(t);x) - \varphi(x)\|_2 = 0$$

Proof. From [63], it suffices to prove that

$$\lim_{n\to\infty} \|L_{n,c}(t^r;x) - x^r\|_2 = 0, \quad r = 0,1,2. \tag{5.17}$$

From lemma 5.2.1, the condition (5.17) is true for $r = 0,1$.
Now for $r = 2$ then, we have

$$\begin{aligned}\left\|L_{n,c}(t^2;x) - x^2\right\|_2 &= \underset{x\geq 0}{Sup}\left(\frac{1}{1+x^2}\right)\cdot\left|\frac{(n+(m+1)c)}{(n+(m-1)c)}x^2 + \frac{2x}{(n+(m-1)c)} - x^2\right| \\ &\leq \underset{x\geq 0}{Sup}\left(\frac{x^2}{1+x^2}\right)\cdot\left|\frac{2c}{n+(m-1)c}\right| + \underset{x\geq 0}{Sup}\left(\frac{x}{1+x^2}\right)\cdot\left|\frac{2}{n+(m-1)c}\right|\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \left\|L_{n,c}(t^2;x) - x^2\right\|_2 = 0.$$

Hence the proof is completed. □

Theorem 5.7.3. *For each $\varphi \in C_2^*(\mathbb{R}^+)$ and $\varsigma > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^+} \frac{|L_{n,c}(\varphi;x) - \varphi(x)|}{(1+x^2)^{1+\varsigma}} = 0.$$

Proof. Suppose x_0 is an arbitrary and fixed belongs to the set $\mathbb{R}^+$, we obtain

$$\begin{aligned}&\sup_{x\in\mathbb{R}^+} \frac{|L_{n,c}(\varphi;x) - \varphi(x)|}{(1+x^2)^{1+\varsigma}} \leq \sup_{x\leq x_0} \frac{|L_{n,c}(\varphi;x) - \varphi(x)|}{(1+x^2)^{1+\varsigma}} + \sup_{x>x_0} \frac{|L_{n,c}(\varphi;x) - \varphi(x)|}{(1+x^2)^{1+\varsigma}} \\ &\leq ||L_{n,c}(\varphi) - \varphi||_{C[0,x_0]} + ||\varphi||_2 \sup_{x>x_0} \frac{|L_{n,c}(1+t^2;x)|}{(1+x^2)^{1+\varsigma}} + \sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\varsigma}}.\end{aligned} \tag{5.18}$$

Since $|\varphi(x)| \leq ||\varphi||_2(1+x^2)$ implies $\sup_{x>x_0} \frac{|\varphi(x)|}{(1+x^2)^{1+\varsigma}} \leq \frac{||\varphi||_2}{(1+x_0^2)^{\varsigma}}$.
We choose x_0 is sufficiently large and suppose $\varepsilon > 0$, we get

$$\frac{||\varphi||_2}{(1+x_0^2)^{\varsigma}} < \frac{\varepsilon}{6}. \tag{5.19}$$

Since $\lim_{n\to\infty} \sup_{x>x_0} \frac{L_{n,c}(1+t^2;x)}{1+x^2} = 1$. Therefore, we acquire

$$\sup_{x>x_0} \frac{L_{n,c}(1+t^2;x)}{1+x^2} \leq \frac{(1+x_0^2)^\varsigma}{\|\varphi\|_2}\frac{\varepsilon}{3}+1,$$

for adequately large n, we have

$$\|\varphi\|_2 \sup_{x>x_0} \frac{L_{n,c}(1+t^2;x)}{(1+x^2)^{\varsigma+1}} \leq \frac{\|\varphi\|_2}{(1+x_0^2)^\varsigma} \sup_{x>x_0} \frac{L_{n,c}(1+t^2;x)}{(1+x^2)} \leq \frac{\varepsilon}{3}+\frac{||\varphi||_2}{(1+x_0^2)^\varsigma}.$$

Applying Theorem 5.7.1, we can find for adequately large n

$$||L_{n,c}(\varphi;x)-\varphi(x)||_{C[0,x_0]} < \frac{\varepsilon}{3}. \tag{5.20}$$

Combining the relations (5.19)-(5.20), we obtain

$$\sup_{x\in\mathbb{R}^+} \frac{|L_{n,c}(\varphi;x)-\varphi(x)|}{(1+x^2)^{1+\varsigma}} < \varepsilon.$$

□

Chapter 6

Approximation by Bézier variant of Gupta-Srivastava operators with certain parameters

6.1 Introduction

Recently, Gupta and Srivastava [90] proposed a general family of a positive linear operators, which preserve constant functions as well as linear functions for all $c \in \mathbb{N} \cup \{0\} \cup \{-1\}$ are given in (5.1). We have discussed several approximation results of these operators in the previous chapter and termed these operators as Gupta-Srivastava operators.

In the year 1972, Bézier [31] have introduced the curves to design the Renault Car known as Bézier curves. These curves are symmetric in nature and also useful in Computer-Aided Design. Motivated by this idea, many researchers have proposed the Bézier variant of positive linear operators and discussed several approximation results see [18, 67, 71, 99, 127].

Inspired from the above-stated work, we propose here the Bézier variant of the operators (5.1), depending upon parameter $\alpha \geq 1$ as follows:

$$F_{n,m}^{c,\alpha}(\varphi(t);x) = \{n+(m+1)c\} \sum_{k=1}^{\infty} Q_{n+mc}^{(\alpha)}(x;c) \int_{0}^{\infty} p_{n+(m+2)c,k-1}(t;c)\varphi(t)dt + Q_{n+mc,0}^{(\alpha)}(x;c)\varphi(0), \tag{6.1}$$

where $Q^{(\alpha)}_{n+mc,k}(x;c) = \left(J_{n+mc,k}(x,c)\right)^{\alpha} - \left(J_{n+mc,k+1}(x,c)\right)^{\alpha}$, $\alpha \geq 1$ with $J_{n+mc,k}(x,c) = \sum\limits_{j=k}^{\infty} p_{n+mc,k}(x,c)$, where $k < \infty$ and otherwise zero. It is obvious that the operators $F^{c,\alpha}_{n,m}(.;x)$ are the linear positive operators. For $\alpha = 1$ the operators (6.1) immediately reduce to the form (5.1). The special cases of operators (6.1) are given below:

(i) For $c = 0$, $\alpha = 1$ and $\phi_{n,0}(x) = e^{-nx}$, we get Phillips operators

$$L^{0}_{n,m}(\varphi(t);x) = n\sum_{k=1}^{\infty} p_{n+mc,k}(x;0)\int_{0}^{\infty} p_{n+mc,k-1}(t;0)\varphi(t)dt + p_{n,0}(x;0)\varphi(0),$$

where

$$p_{n,k}(x,0) = \frac{e^{-nx}(nx)^{k}}{k!} \text{ and } x \in [0,\infty).$$

(ii) For $c \in \mathbb{N}$, $\alpha = 1$ and $\phi_{n,c}(x) = (1+cx)^{-\frac{n}{c}}$, we get genuine Baskakov-Durrmeyer type operators. These operators are similar to (5.1) except for $c = \{0,-1\}$, called summation integral type of operators, where

$$p_{n,k}(x;c) = \frac{\left(\frac{n}{c}\right)_k}{k!}\frac{(cx)_k}{(1+cx)^{\frac{n}{c}+k}},$$

and $(n)_i$ denotes the rising factorial given by

$$(n)_i = n(n+1)(n+2)...(n+i-1) \;\&\; (n)_0 = 1 (i \in \mathbb{N}).$$

(iii) For $c = -1$, $\alpha = 1$ and $\phi_{n,-1}(x) = (1-x)^{n}$, we have a sequence of Bernstein-Durrmeyer operators

$$L^{-1}_{n,m}(\varphi;x) = (n-m-1)\sum_{k=1}^{n-m-1} p_{n-m,k}(x,-1)\int_{0}^{1} p_{n-m-2,k-1}(t,-1)\varphi(t)dt + p_{n-m,0}(x,-1)\varphi(0) + p_{n-m,n-m}(x,-1), \tag{6.2}$$

where

$$p_{n,k}(x;-1) = \binom{n}{k} x^{k}(1-x)^{n-k}.$$

The purpose of this chapter is to investigate the approximation results by using Lipchitz type space, Ditzian-Totik modulus of smoothness, weighted modulus of continuity, and functions whose derivatives are of bounded variation.

6.2 Preliminaries

In this section, we give some auxiliary results to study our main results.

Lemma 6.2.1. *If n is sufficiently large then the central moment of the operators (5.1) are:*

(i) $\mu_{n,2}^{c}(x) \leq C\frac{x(1+cx)}{n}$;

(ii) $\mu_{n,4}^{c}(x) \leq C\frac{(x(1+cx))^2}{n^2}$;

where $C > 0$ *is constant.*

Lemma 6.2.2. *We know that* $\sum\limits_{j=0}^{\infty} p_{n+mc,j}(x,c) = 1$ *and from (6.1), we have*

$$\begin{aligned}
F_{n,m}^{c,\alpha}(1;x) &= \{n+(m+1)c\}\sum_{k=1}^{\infty} Q_{n+mc}^{(\alpha)}(x;c)\int\limits_{0}^{\infty} p_{n+(m+2)c,k-1}(t;c)dt \\
&\quad + Q_{n+mc,0}^{(\alpha)}(x;c) \\
&= \sum_{k=0}^{\infty} Q_{n+mc}^{(\alpha)}(x;c) = (J_{n+mc,0}(x,c))^{(\alpha)} \\
&= \left(\sum_{k=0}^{\infty} p_{n+mc,j}(x,c)\right)^{(\alpha)} = 1.
\end{aligned}$$

Lemma 6.2.3. *For each* $\varphi \in C_B(\mathbb{R}^+)$ *then, we have*

$$\left|F_{n,m}^{c,\alpha}(\varphi(t);x)\right| \leq \|\varphi\|.$$

Proof. It is easy to prove the above result by using Lemma 6.2.2, therefore we skip the proof. □

Lemma 6.2.4. *For every* $\varphi \in C_B(\mathbb{R}^+)$ *then, we have*

$$\left|F_{n,m}^{c,\alpha}(\varphi(t);x)\right| \leq \alpha L_{n,m}^{c}(\|\varphi\|;x).$$

Proof. For $0 \leq c \leq d \leq 1$, $\alpha \geq 1$, using the inequality

$$|c^{\alpha} - d^{\alpha}| \leq \alpha|c-d|,$$

from the definition of $Q^{(\alpha)}_{n+mc,k}(x;c)$, for all $k \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned} 0 &< \left(J_{n+mc,k}(x,c)\right)^{\alpha} - \left(J_{n+mc,k+1}(x,c)\right)^{\alpha} \\ &\le \alpha \left(J_{n+mc,k}(x,c) - J_{n+mc,k+1}(x,c)\right) \\ &= \alpha p_{n+mc}(x,c). \end{aligned}$$

Hence

$$\left|F^{c,\alpha}_{n,m}(\varphi(t);x)\right| \le \alpha L^{c}_{n,m}(\|\varphi\|;x).$$

□

6.3 Direct results

Now we estimate the rate of convergence of the function $\varphi \in Lip_M(\xi)$ by the operators $F^{c,\alpha}_{n,m}(.;x)$.

Theorem 6.3.1. *For $\varphi \in Lip_M(\xi)$ and $0 < \xi \le 1$. Then for $x \in \mathbb{R}^+$, we obtain*

$$\left|F^{c,\alpha}_{n,m}(\varphi(t);x) - \varphi(x)\right| \le \alpha M \left(\frac{\delta^{c}_{n,m}(x)}{x}\right)^{\xi/2},$$

where $\delta^{c}_{n,m}(x) = \sqrt{\frac{2x(1+cx)}{(n+(m-1)c)}}$.

Proof. Using Lemma 6.2.4, we get

$$\begin{aligned} \left|F^{c,\alpha}_{n,m}(\varphi(t);x) - \varphi(x)\right| &\le F^{c,\alpha}_{n,m}(|\varphi(t) - \varphi(x)|;x) \\ &\le \alpha L^{c}_{n,m}(|\varphi(t) - \varphi(x)|;x) \\ &\le \alpha M L^{c}_{n,m}\left(\frac{|t-x|^{\xi}}{(t+x)^{\xi/2}};x\right) \\ &\le \frac{\alpha M}{x^{\xi/2}} L^{c}_{n,m}\left(|t-x|^{\xi};x\right). \end{aligned} \tag{6.3}$$

Using Hölder's inequality by taking $p = 2\xi$ and $q = 2/(2-\xi)$, we get

$$\begin{aligned} L^{c}_{n,m}\left(|t-x|^{\xi};x\right) &\le \left\{L^{c}_{n,m}\left((t-x)^2;x\right)\right\}^{\frac{\xi}{2}} . \left\{L^{c}_{n,m}\left(1^{\frac{2}{(2-\xi)}};x\right)\right\}^{\frac{(2-\xi)}{2}} \\ &\le \left\{L^{c}_{n,m}\left((t-x)^2;x\right)\right\}^{\frac{\xi}{2}} = \left(\delta^{c}_{n,m}(x)\right)^{\frac{\xi}{2}}. \end{aligned} \tag{6.4}$$

From (6.3) and (6.4), we get

$$\left|F_{n,m}^{c,\alpha}\left(\varphi(t);x\right)-\varphi(x)\right|\leqslant \alpha M\left(\frac{\delta_{n,m}^{c}(x)}{x}\right)^{\frac{\xi}{2}}.$$

Hence the proof. □

Theorem 6.3.2. *For $\varphi \in C_B(\mathbb{R}^+)$ and $0 \leq \lambda \leq 1$, we get*

$$\left|F_{n,m}^{c,\alpha}\left(\varphi(t);x\right)-\varphi(x)\right|\leq C\omega_{\phi^{\lambda}}\left(\varphi;\frac{\phi^{1-\lambda}(x)}{\sqrt{n}}\right),$$

for sufficient large n and C is a positive constant independent from φ and n.

Proof. For $\psi \in W_\lambda$, we get

$$\psi(t)=\psi(x)+\int_{x}^{t}\psi'(u)du. \tag{6.5}$$

Applying $F_{n,m}^{c,\alpha}$ in (6.5) and using Hölder's inequality then, we have

$$\begin{aligned}\left|F_{n,m}^{c,\alpha}\left(\psi(t);x\right)-\psi(x)\right| &\leq F_{n,m}^{c,\alpha}\left(\int_{x}^{t}\left|\psi'\right|du;x\right)\\ &\leq \left\|\phi^{\lambda}\psi'\right\|F_{n,m}^{c,\alpha}\left(\left|\int_{x}^{t}\frac{du}{\phi^{\lambda}(u)}\right|;x\right)\\ &\leq \left\|\phi^{\lambda}\psi'\right\|F_{n,m}^{c,\alpha}\left(|t-x|^{1-\lambda}\left|\int_{x}^{t}\frac{du}{\phi(u)}\right|^{\lambda};x\right).\end{aligned} \tag{6.6}$$

Let us take $A=\left|\int_{x}^{t}\frac{du}{\phi(u)}\right|$ then, we get

$$\begin{aligned}A &\leq \left|\int_{x}^{t}\frac{du}{\sqrt{u}}\right|\left|\left(\frac{1}{\sqrt{1+cx}}+\frac{1}{\sqrt{1+ct}}\right)\right|\\ &\leq 2\left|\sqrt{t}-\sqrt{x}\right|\left(\frac{1}{\sqrt{1+cx}}+\frac{1}{\sqrt{1+ct}}\right)\\ &\leq 2\frac{|t-x|}{\sqrt{x}+\sqrt{t}}\left(\frac{1}{\sqrt{1+cx}}+\frac{1}{\sqrt{1+ct}}\right)\\ &\leq 2\frac{|t-x|}{\sqrt{x}}\left(\frac{1}{\sqrt{1+cx}}+\frac{1}{\sqrt{1+ct}}\right).\end{aligned} \tag{6.7}$$

The inequality $|a+b|^{\lambda} \leq |a|^{\lambda} + |b|^{\lambda}$ holds for $0 \leq \lambda \leq 1$ then from (6.7), we get

$$\left| \int_x^t \frac{du}{\phi(u)} \right|^{\lambda} \leq 2^{\lambda} \frac{|t-x|^{\lambda}}{x^{\lambda/2}} \left(\frac{1}{(1+cx)^{\lambda/2}} + \frac{1}{(1+ct)^{\lambda/2}} \right). \tag{6.8}$$

From (6.6),(6.8) and using Cauchy inequality then, we get

$$\begin{aligned} \left| F_{n,m}^{c,\alpha}(\psi(t);x) - \psi(x) \right| \leq & \frac{2^{\lambda} \left\| \phi^{\lambda} \psi' \right\|}{x^{\lambda/2}} F_{n,m}^{c,\alpha} \left(|t-x| \left(\frac{1}{(1+cx)^{\lambda/2}} + \frac{1}{(1+ct)^{\lambda/2}} \right); x \right) \\ & + \sqrt{F_{n,m}^{c,\alpha}((t-x)^2;x)}.\sqrt{F_{n,m}^{c,\alpha}((1+ct)^{-\lambda};x)}. \end{aligned} \tag{6.9}$$

If n is sufficiently large then we get

$$\left(F_{n,m}^{c,\alpha} \left((t-x)^2; x \right) \right)^{1/2} \leq \sqrt{\frac{2\alpha}{n}} \phi(x), \tag{6.10}$$

where $\phi(x) = \sqrt{x(1+cx)}$.

For each $x \in \mathbb{R}^+$, $F_{n,m}^{c,\alpha}\left((1+ct)^{-\lambda};x\right) \to (1+cx)^{-\lambda}$ as $n \to \infty$. Thus for $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that

$$F_{n,m}^{c,\alpha}\left((1+ct)^{-\lambda};x\right) \leq (1+cx)^{-\lambda} + \varepsilon, \quad \text{for all } n \geq n_0$$

By choosing $\varepsilon = (1+cx)^{-\lambda}$ then, we get

$$F_{n,m}^{c,\alpha}\left((1+ct)^{-\lambda};x\right) \leq 2(1+cx)^{-\lambda}, \quad \text{for all } n \geq n_0. \tag{6.11}$$

From (6.9) to (6.11), we have

$$\begin{aligned} \left| F_{n,m}^{c,\alpha}(\psi(t);x) - \psi(x) \right| & \leq 2^{\lambda} \left\| \phi^{\lambda} \psi' \right\| \sqrt{\frac{2\alpha}{n}} \phi(x) \left(\phi^{-\lambda}(x) + \sqrt{2} x^{-\frac{\lambda}{2}} (1+cx)^{-\frac{\lambda}{2}} \right) \\ & \leq 2^{\lambda+\frac{1}{2}} (1+\sqrt{2}) \left\| \phi^{\lambda} \psi' \right\| \sqrt{\frac{\alpha}{n}} \phi^{1-\lambda}(x). \end{aligned} \tag{6.12}$$

We may write

$$\begin{aligned} \left| F_{n,m}^{c,\alpha}(\varphi(t);x) - \varphi(x) \right| & \leq \left| F_{n,m}^{c,\alpha}(\varphi(t) - \psi(t);x) \right| \\ & \quad + \left| F_{n,m}^{c,\alpha}(\psi(t);x) - \psi(x) \right| + |\psi(x) - \varphi(x)| \\ & \leq 2\|\varphi - \psi\| + \left| F_{n,m}^{c,\alpha}(\psi(t);x) - \psi(x) \right|. \end{aligned} \tag{6.13}$$

From (6.12) to (6.13) and for sufficiently large n, we get

$$\begin{aligned}\left|F_{n,m}^{c,\alpha}(\varphi(t);x)-\varphi(x)\right| &\leq 2\|\varphi-\psi\|+2^{\lambda+\frac{1}{2}}(1+\sqrt{2})\sqrt{\frac{\alpha}{n}}\phi^{1-\lambda}(x)\left\|\phi^{\lambda}\psi'\right\| \\ &\leq C_1\left\{\|\varphi-\psi\|+\frac{\phi^{1-\lambda}(x)}{\sqrt{n}}\left\|\phi^{\lambda}\psi'\right\|\right\} \\ &\leq CK_{\phi^{\lambda}}\left(\varphi,\frac{\phi^{1-\lambda}(x)}{\sqrt{n}}\right), \end{aligned} \tag{6.14}$$

where $C_1 = max(2, 2^{\lambda+\frac{1}{2}}(1+\sqrt{2})\sqrt{\alpha})$ and $C = 2C_1$. From (1.18) and (6.14), we get the required result. □

6.4 Weighted approximation

Theorem 6.4.1. *Let $\varphi \in C_2(\mathbb{R}^+)$, $\alpha > 0$, for fixed m and sufficiently large n then, we have*

$$\sup_{x\in[0,\infty)}\frac{\left|F_{n,m}^{c,\alpha}(\varphi;x)-\varphi(x)\right|}{(1+x)^{5/2}} \leqslant C\Omega\left(\varphi;\frac{1}{\sqrt{n}}\right),$$

where C is positive constant depends on n and φ.

Proof. From the property (iii) of Proposition 1.4.2, we have

$$\begin{aligned}|\varphi(t)-\varphi(x)| &\leq \left(1+(x+|t-x|)^2\right)\Omega(\varphi;|t-x|) \\ &\leq 2(1+x^2)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(\varphi;\delta). \end{aligned} \tag{6.15}$$

Applying $F_{n,m}^{c,\alpha}(.;x)$ on both side of (6.15), we get

$$\begin{aligned}\left|F_{n,m}^{c,\alpha}(\varphi;x)-\varphi(x)\right| \leq \Big[&1+F_{n,m}^{c,\alpha}((t-x)^2;x) \\ &+F_{n,m}^{c,\alpha}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right)\Big]. \end{aligned} \tag{6.16}$$

From Remark 6.2.1, and apply Cauchy-Schwarz inequality in (6.16), we get

$$\begin{aligned}F_{n,m}^{c,\alpha}\left((1+(t-x)^2)\frac{|t-x|}{\delta};x\right) \leq &\frac{1}{\delta}\left(\alpha\mu_{n,2}^{c}(x)\right)^{1/2} \\ &+\frac{1}{\delta}\left(\alpha\mu_{n,4}^{c}(x)\right)^{1/2}\left(\alpha\mu_{n,2}^{c}(x)\right)^{1/2}. \end{aligned} \tag{6.17}$$

Combining the estimate from (6.15) to (6.17) and taking $C = 2(1+\sqrt{\alpha C}+2C)$ and $\delta = \frac{1}{\sqrt{n}}$

then we get the required result. □

6.5 Function of bounded variation

The operators (6.1) can be rewritten in the following form:

$$F_{n,m}^{c,\alpha}(\varphi(t);x)=\int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)\varphi(t)dt, \tag{6.18}$$

where

$$M_{n,m,c}^{(\alpha)}(x,t)=\{n+(m+1)c\}\sum_{k=1}^{\infty}Q_{n+mc,k}^{(\alpha)}(x;c)p_{n+(m+2)c,k}(t,c)+Q_{n+mc,0}^{(\alpha)}(x;c)\delta(t),$$

where $\delta(t)$ is Dirac delta function.

Lemma 6.5.1. *For a fixed $x\in\mathbb{R}^+$ and n is sufficient large then, we have*

(i) $\zeta_{n,c}^{(\alpha)}(x;y)=\int_0^y M_{n,m,c}^{(\alpha)}(x;t)dt\leq\frac{2\alpha x(1+cx)}{n(x-y)^2},\ \ 0\leq y\leq x;$

(ii) $1-\zeta_{n,c}^{(\alpha)}(x;z)=\int_z^\infty M_{n,m,c}^{(\alpha)}(x;t)dt\leq\frac{2\alpha x(1+cx)}{n(z-x)^2},\ \ x\leq z\leq\infty.$

Proof. From (6.18), and using Lemma 6.2.1 then, we have

$$\begin{aligned}\zeta_{n,c}^{(\alpha)}(x;y)&\leq\int_0^y M_{n,m,c}^{(\alpha)}(x;t)\left(\frac{x-t}{x-y}\right)^2dt\\&\leq\frac{\alpha}{(x-y)^2}L_{n,c}\left((e_1-x)^2;x\right)\\&\leq\frac{2\alpha x(1+cx)}{n(x-y)^2}.\end{aligned}$$

We can prove the second part of Lemma in same way. □

Theorem 6.5.2. *Let $\varphi \in DBV(\mathbb{R}^+)$, $x>0$ and for adequately large n large, we have*

$$\begin{aligned}\left|F_{n,m}^{c,\alpha}(\varphi;x)-\varphi(x)\right| \leq & \frac{1}{\alpha+1}\left|\varphi'(x+)+\alpha\varphi'(x-)\right|\sqrt{\frac{2\alpha x(1+cx)}{n}} \\ & +\frac{\alpha}{\alpha+1}\left|\varphi'(x+)-\varphi'(x-)\right|\sqrt{\frac{2\alpha x(1+cx)}{n}} \\ & +\frac{2\alpha(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}\varphi_x'+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}\varphi_x' \\ & +\frac{2\alpha(1+cx)}{nx}\left|\varphi(2x)-\varphi(x)-x\varphi(x+)\right| \\ & +\frac{2\alpha x(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}\varphi_x'+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(\varphi_x') \\ & +M(\gamma,r,x)+\frac{2\alpha(1+cx)}{nx}\left|\varphi(x)\right|+\sqrt{\frac{2\alpha x(1+cx)}{n}}\left|\varphi(x+)\right|.\end{aligned}$$

where $\bigvee_a^b \varphi(x)$ denotes the total variation of φ on $[a,b]$, φ_x is an auxiliary operator given by

$$\varphi_x(t)=\begin{cases}\varphi(t)-\varphi(x-), & 0\leq t<x \\ 0, & t=x \\ \varphi(t)-\varphi(x+), & x<t<\infty\end{cases}. \tag{6.19}$$

Proof. From Remark 6.2.2, $F_{n,m}^{c,\alpha}(1;x)=1$ and using the alternative form of the operators (6.18) for each $x\in\mathbb{R}^+$ then, we have

$$\begin{aligned}F_{n,m}^{c,\alpha}(\varphi(t);x)-\varphi(x) &= \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)\big(\varphi(t)-\varphi(x)\big)dt \\ &= \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)\left(\int_x^t \varphi'(u)du\right)dt \end{aligned} \tag{6.20}$$

For each $\varphi\in DBV(\mathbb{R}^+)$ and from (6.19), we can write

$$\begin{aligned}\varphi'(u)= & \varphi_x'(u)+\frac{1}{\alpha+1}(\varphi'(x+)+\alpha\varphi'(x-)) \\ & +\frac{1}{2}\left(\varphi'(x+)+\alpha\varphi'(x-)\right)\left(\operatorname{sgn}(u-x)+\frac{\alpha-1}{\alpha+1}\right) \\ & \times\delta_x(u)\left[\varphi'(u)-\left(\varphi'(x+)+\varphi'(x-)\right)\right],\end{aligned} \tag{6.21}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x \\ 0, & u \neq x. \end{cases}$$

From (6.20) and (6.21), we have

$$\begin{aligned} F_{n,m}^{c,\alpha}(\varphi(t);x) - \varphi(x) = \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t) \int_x^t \Bigg(& \varphi_x'(u) + \frac{1}{\alpha+1}(\varphi'(x+) + \alpha\varphi\prime(x-)) \\ & + \frac{1}{2}(\varphi'(x+) + \alpha\varphi'(x-)) \left(sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) \\ & \times \delta_x(u)[\varphi'(u) - \frac{1}{2}(\varphi'(x+) + \varphi'(x-)] \Bigg) dudt. \end{aligned} \tag{6.22}$$

It is easy to say that

$$\int_0^\infty M_{n,m,c}^{(\alpha)}(x,t) \int_x^t [\varphi'(u) - \frac{1}{2}(\varphi'(x+) + \varphi'(x-)]\delta_x(u)dudt = 0. \tag{6.23}$$

Now

$$\begin{aligned} B_1 &= \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t) \int_x^t \frac{1}{\alpha+1}(\varphi'(x+) + \alpha\varphi'(x-))dudt. \\ &= \frac{1}{\alpha+1}(\varphi'(x+) + \alpha\varphi'(x-)) \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)(t-x)dt \\ &= \frac{1}{\alpha+1}(\varphi'(x+) + \alpha\varphi'(x-)) F_{n,c}^{(\alpha)}((t-x);x), \end{aligned} \tag{6.24}$$

and

$$\begin{aligned} B_2 &= \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t) \int_x^t \frac{1}{2}(\varphi'(x+) + \alpha\varphi'(x-)) \left(sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) dudt \\ &= \frac{1}{2}(\varphi'(x+) + \alpha\varphi'(x-)) \Bigg(-\int_0^x M_{n,m,c}^{(\alpha)}(x,t) \int_x^t \left(sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) dudt \\ &\quad + \int_x^\infty M_{n,m,c}^{(\alpha)}(x,t) \int_x^t \left(sgn(u-x) + \frac{\alpha-1}{\alpha+1} \right) \Bigg) \\ &\leq \frac{\alpha}{\alpha+1}(\varphi'(x+) + \alpha\varphi'(x-)) \int_0^\infty M_{n,m,c}^{(\alpha)}(x,t)\,|t-x|\,dt \\ &\leq \frac{\alpha}{\alpha+1}(\varphi'(x+) + \alpha\varphi'(x-)) \left(F_{n,c}^{(\alpha)}\left((e_1-x)^2;x \right) \right)^{\frac{1}{2}}, \end{aligned} \tag{6.25}$$

By using Lemma 6.2.1 and Lemma 6.2.4, from (6.22) – (6.25) then, we have

$$F_{n,m}^{c,\alpha}(\varphi;x)-\varphi(x)\leq\left|A_n^{(\alpha)}(\varphi';x)+B_n^{(\alpha)}(\varphi';x)\right|$$
$$+\frac{2\alpha}{\alpha+1}\left|\varphi'(x+)+\alpha\varphi'(x-)\right|\frac{x(1+cx)}{n}$$
$$+\frac{\alpha}{\alpha+1}\left|\varphi'(x+)-\varphi'(x-)\right|\sqrt{\frac{2\alpha x(1+cx)}{n}},\tag{6.26}$$

where

$$A_n^{(\alpha)}(\varphi';x)=\int_0^x\left(\int_x^t\varphi_x'(u)du\right)M_{n,m,c}^{(\alpha)}(x,t)dt,$$

and

$$B_n^{(\alpha)}(\varphi';x)=\int_x^\infty\left(\int_x^t\varphi_x'(u)du\right)M_{n,m,c}^{(\alpha)}(x,t)dt.$$

To estimate $A_n^{(\alpha)}(\varphi';x)$, using integration by parts and applying Lemma 6.5.1 with $y=x-\frac{x}{\sqrt{n}}$, we obtain

$$A_n^{(\alpha)}(\varphi';x)=\left|\int_0^x\left(\int_x^t\varphi_x'(u)du\right)d_t\zeta_{n,c}^{(\alpha)}(x;t)\right|$$
$$=\left|\int_0^x\zeta_{n,c}^{(\alpha)}(x;t)\varphi_x'(t)dt\right|$$

$$\leq\int_0^y\left|\varphi_x'(t)\right|\left|\zeta_{n,c}^{(\alpha)}(x;t)\right|dt+\int_0^y\left|\varphi_x'(t)\right|\left|\zeta_{n,c}^{(\alpha)}(x;t)\right|dt$$
$$\leq\frac{2\alpha x(1+cx)}{n}\int_0^y\bigvee_t^x\varphi_x'(x-t)^2dt+\int_y^x\bigvee_t^x\varphi_x'dt$$
$$\leq\frac{2\alpha x(1+cx)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x\varphi_x'(x-t)^2dt+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x\varphi_x'.\tag{6.27}$$

Substituting $u=\frac{x}{x-t}$, we get

$$\begin{aligned}\frac{2\alpha x(1+cx)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x \varphi_x'(x-t)^2dt &= \frac{2\alpha x(1+cx)}{nx}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^x \varphi_x' du\\ &\leq \frac{2\alpha(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_{x-\frac{x}{k}}^x \varphi_x' du\\ &\leq \frac{2\alpha(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x \varphi_x'. \end{aligned}\tag{6.28}$$

From (6.27) and (6.28), we get

$$A_n^{(\alpha)}(\varphi';x)=\frac{2\alpha(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x \varphi_x' + \frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x \varphi_x'. \tag{6.29}$$

We can write

$$\begin{aligned}B_n^{(\alpha)}(\varphi';x)\leq &\left|\int_x^{2x}\left(\int_x^t \varphi_x'(u)du\right)d_t(1-\zeta_{n,c}^{(\alpha)}(x;t))\right|\\ &+\left|\int_{2x}^{\infty}\left(\int_x^t \varphi_x'(u)du\right)d_tM_{n,m,c}^{(\alpha)}(x,t)\right|.\end{aligned}$$

From the second part of the Lemma 6.5.1, we get

$$M_{n,m,c}^{(\alpha)}(x,t)=d_t((1-\zeta_{n,c}^{(\alpha)}(x;t)) \quad for\ \ t>x.$$

Hence

$$B_n^{(\alpha)}(f';x)=B_{n,1}^{(\alpha)}(\varphi';x)+B_{n,2}^{(\alpha)}(\varphi';x),$$

where

$$B_{n,1}^{(\alpha)}(\varphi';x)=\left|\int_x^{2x}\left(\int_x^t \varphi_x'(u)du\right)d_t(1-\zeta_{n,c}^{(\alpha)}(x;t))\right|,$$

and

$$B_{n,2}^{(\alpha)}(f';x)=\left|\int_{2x}^{\infty}\left(\int_x^t \varphi_x'(u)du\right)d_tM_{n,m,c}^{(\alpha)}(x,t)\right|.$$

Using integration by parts, applying Lemma 6.5.1, $1-\zeta_{n,c}^{(\alpha)}(x;t)\leq 1$ and taking $t=x+\frac{x}{u}$

successively,

$$\begin{aligned}
B_{n,1}^{(\alpha)}(\varphi';x) =& \left| \int_x^{2x} \varphi_x'(u)du(1-\zeta_{n,c}^{(\alpha)}(x;2x)) - \int_x^{2x} f_x'(t)(1-\zeta_{n,c}^{(\alpha)}(x;t))dt \right| \\
\leq & \left| \int_x^{2x} (\varphi'(u)-\varphi'(x+))du \right| \left| 1-\zeta_{n,c}^{(\alpha)}(x;2x) \right| + \left| \int_x^{2x} \varphi_x'(t)(1-\zeta_{n,c}^{(\alpha)}(x;t))dt \right| \\
\leq & \frac{2\alpha(1+cx)}{nx} \left| \varphi(2x)-\varphi(x)-x\varphi(x+) \right| \\
& + \frac{2\alpha x(1+cx)}{n} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{\bigvee_x^t \varphi_x'}{(t-x)^2} dt + \int_x^{x+\frac{x}{\sqrt{n}}} \bigvee_x^t \varphi_x' dt \\
\leq & \frac{2\alpha(1+cx)}{nx} \left| \varphi(2x)-\varphi(x)-x\varphi(x+) \right| \\
& + \frac{2\alpha x(1+cx)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{k}} \varphi_x' + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} (\varphi_x'). \qquad (6.30)
\end{aligned}$$

Using Lemma 6.2.1 then, we have

$$\begin{aligned}
B_{n,2}^{(\alpha)}(\varphi';x) =& \left| \int_{2x}^{\infty} \left(\int_x^t (\varphi'(u)-\varphi'(x+))du \right) M_{n,m,c}^{(\alpha)}(x,t)dt \right| \\
\leq & \int_{2x}^{\infty} |\varphi(t)-\varphi(x)| M_{n,m,c}^{(\alpha)}(x,t)dt + \int_{2x}^{\infty} |t-x| \, |\varphi(x+)| M_{n,m,c}^{(\alpha)}(x,t)dt \\
\leq & \left| \int_{2x}^{\infty} \varphi(t) M_{n,m,c}^{(\alpha)}(x,t)dt \right| + |\varphi(x)| \left| \int_{2x}^{\infty} M_{n,m,c}^{(\alpha)}(x,t)dt \right| \\
& + |\varphi(x+)| \left(\int_{2x}^{\infty} (e_1-x)^2 M_{n,m,c}^{(\alpha)}(x,t)dt \right)^{\frac{1}{2}} \\
\leq & M \int_{2x}^{\infty} t^{\gamma} M_{n,m,c}^{(\alpha)}(x,t)dt + |\varphi(x)| \left| \int_{2x}^{\infty} M_{n,m,c}^{(\alpha)}(x,t)dt \right| \\
& + \sqrt{\frac{2\alpha x(1+cx)}{n}} \, |\varphi(x+)|. \qquad (6.31)
\end{aligned}$$

For $t \geq 2x$, we get $t \leq 2(t-x)$ and $x \leq t-x$, applying Hölder's inequality, we have

$$\begin{aligned}
B_{n,2}^{(\alpha)}(\varphi';x) \leq & M2^{\gamma}\left(\int_{2x}^{\infty}(e_1-x)^{2r}M_{n,m,c}^{(\alpha)}(x,t)dt\right)^{\frac{\gamma}{2r}} \\
& + \frac{2\alpha(1+cx)}{nx}|\varphi(x)| + \sqrt{\frac{2\alpha x(1+cx)}{n}}|\varphi(x+)| \\
= & M(\gamma,c,r,x) + \frac{2\alpha(1+cx)}{nx}|\varphi(x)| + \sqrt{\frac{2\alpha x(1+cx)}{n}}|\varphi(x+)|. \qquad (6.32)
\end{aligned}$$

From (6.30) and (6.32), we get

$$\begin{aligned}
B_{n}^{(\alpha)}(\varphi';x) = & \frac{2\alpha(1+cx)}{nx}|\varphi(2x)-\varphi(x)-x\varphi(x+)| \\
& + \frac{2\alpha x(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}\varphi_x' + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(\varphi_x') \\
& + M(\gamma,c,r,x) + \frac{2\alpha(1+cx)}{nx}|\varphi(x)| + \sqrt{\frac{2\alpha x(1+cx)}{n}}|\varphi(x+)|. \qquad (6.33)
\end{aligned}$$

From (6.26), (6.29) and (6.33), we get our desired result. □

6.6 Graphical results

In operators (6.1) by taking $m=-1$, we obtain Bézier variant of Srivastava-Gupta operators which were propoesd by Ispir and Yüksel [99] and its modification by Neer et al. [127]. For $m=1$ in (6.1), we obtain another Bézier form of Srivastava-Gupta operators considered by Kajla [108]. The proposed operators (6.1) have generalized form with different values of m. Here, we show graphical comparison between operators (6.1) for $m=20$ with discussed operators [99,108,127] for the function $\varphi(x)=x^3-2x^2+x$.

From the graphs we observe here that we have better approximation for the Bézier variant of Gupta-Srivastava operators (6.1), discussed in the present chapter than the other variants of [142], therefore it is justified to study this form of operators.

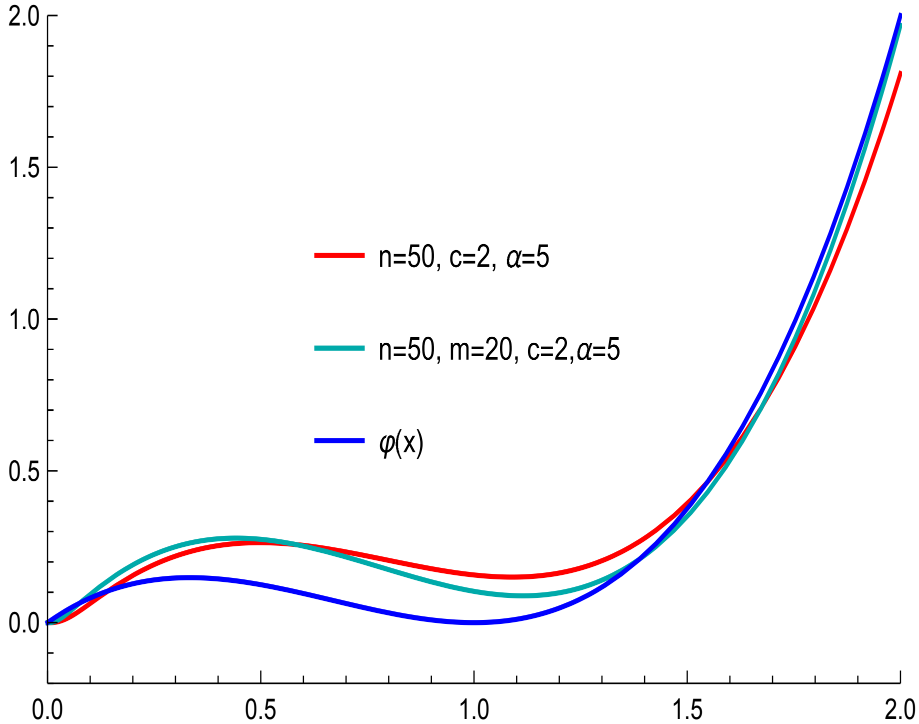

Figure 6.1: Comparison between Bézier variant of Srivastava-Gupta operators [99] (red) with operators (6.1)(cyan) along with function $\varphi(x)$ (blue).

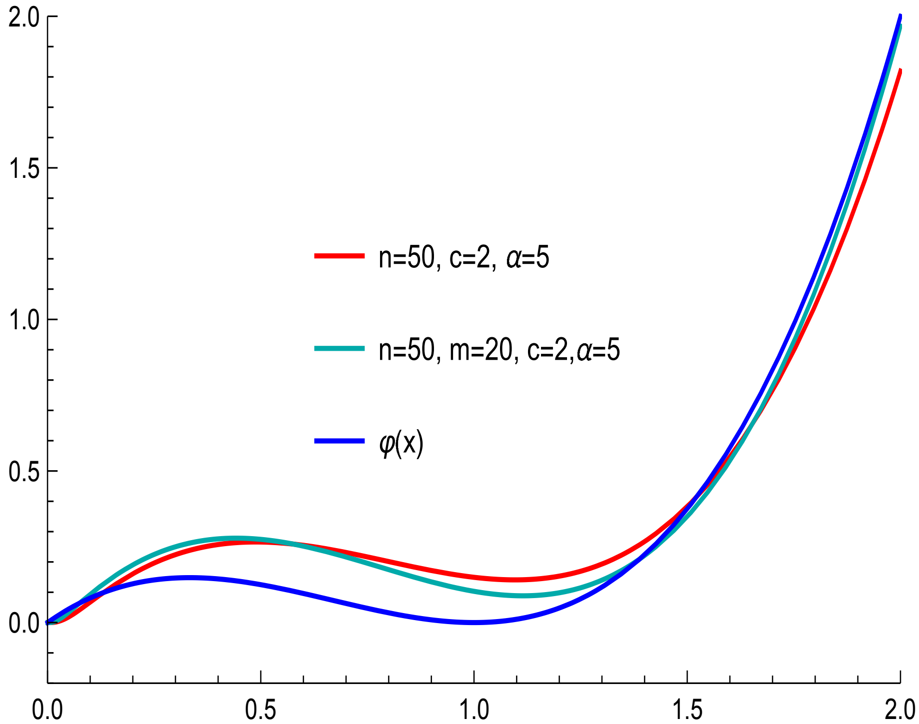

Figure 6.2: Comparison between Bézier variant of Srivastava-Gupta Operators [108] (red) with operators (6.1) (cyan) along with function $\varphi(x)$ (blue).

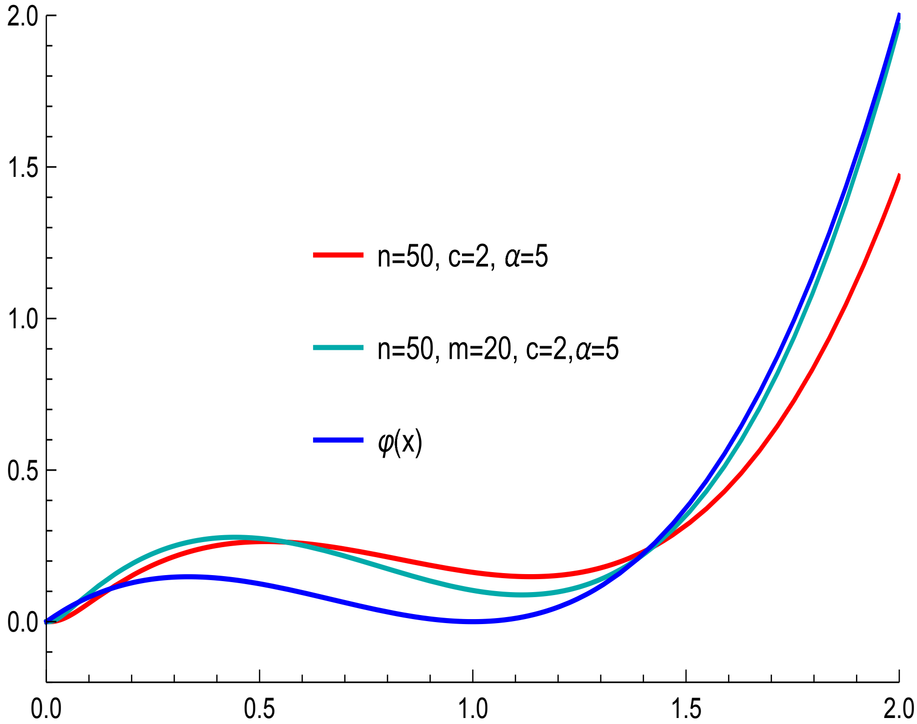

Figure 6.3: Comparison between Bézier variant of modified Srivastava-Gupta operators [127] (red) with operators (6.1) (cyan) along with function $\varphi(x)$ (blue).

Chapter 7

Approximation by mixed positive linear operators based on Second-Kind Beta transform

7.1 Introduction

We know that the probability density function of second-kind beta distribution with positive parameters $c>0$ and $d>0$ is given by

$$b_{c,d}(t)=\frac{t^{c-1}}{\beta(c,d)(1+t)^{c+d}},\quad t>0, \tag{7.1}$$

where $\beta(c,d)$ is the Euler's beta function.

In 1970, Stancu [144] modified of Baskakov operators [29] using a novel idea of inverse Pólya Eggenberger distribution [59]. Gupta et al. [87] also introduced the Durrmeyer modification of Baskakov operators and established some well-known qualitative and quantitative approximation results. Many researchers studied the different types of linear positive operators by using Pólya Eggenberger distribution and inverse Pólya Eggenberger [59] distribution sees [9, 45, 48, 50, 84, 107, 115, 139].

Stancu [146] defined the second-kind beta transform with the help of distribution (7.1). In similar way for $\alpha>0$ and $x\in\mathbb{R}^{+}$, we consider operators $M_n^{[\alpha]}$ as:

$$M_n^{[\alpha]}(\varphi;x)=\frac{1}{\beta\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}\int_0^{\infty}\frac{t^{\frac{x}{\alpha}-1}}{(1+t)^{\frac{1+x}{\alpha}}}M_n(\varphi;x)\,dt, \tag{7.2}$$

where

$$M_n(\varphi;x)=np\sum_{k=0}^{\infty}e^{-nt}\frac{(nt)^k}{k!}\int_0^{\infty}e^{-npt}\frac{(npt)^{kp}}{(kp)!}\varphi(t)dt.$$

Several researchers introduced and studied different types of linear positive operators using second-kind Euler's beta function see [9, 10, 72, 122, 123, 138, 146].

In this chapter, we establish some approximation properties for our proposed operators like direct results, asymptotic estimates, weighted approximation, and also rate of convergence for the class of functions whose derivatives are of bounded variation.

7.2 Preliminaries

This section deals with some basic results to discuss our main results.

Lemma 7.2.1. *For the operators* (7.2), $p>0$ *and* $\alpha=\alpha(n)$ *be a sequence converging to* 0 *then, we have*

(i) $M_n^{[\alpha]}(1;x)=1;$

(ii) $M_n^{[\alpha]}(t;x)=\frac{x}{1-\alpha}+\frac{1}{np};$

(iii) $M_n^{[\alpha]}(t^2;x)=\frac{x(x+\alpha)}{(1-\alpha)(1-2\alpha)}+\left(\frac{p+3}{np}\right)\frac{x}{(1-\alpha)}+\frac{2}{n^2p^2};$

(iv) $M_n^{[\alpha]}(t^3;x)=\frac{x(x+\alpha)(x+2\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)}+\left(\frac{3p+6}{np}\right)\frac{x(x+\alpha)}{(1-\alpha)(1-2\alpha)}$
$+\left(\frac{p^2+6p+11}{n^2p^2}\right)\frac{x}{(1-\alpha)}+\frac{6}{n^3p^3};$

(v) $M_n^{[\alpha]}(t^4;x)=\Big[\frac{x(x+\alpha)(x+2\alpha)(x+3\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)}+\left(\frac{6}{n}+\frac{10}{np}\right)\frac{x(x+\alpha)(x+2\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)}$
$+\left(\frac{7}{n^2}+\frac{30}{n^2p}+\frac{35}{n^2p^2}\right)\frac{x(x+\alpha)}{(1-\alpha)(1-2\alpha)}+\left(\frac{1}{n^3}+\frac{10}{n^3p}+\frac{35}{n^3p^2}+\frac{50}{n^3p^3}\right)\frac{x}{1-\alpha}+\frac{24}{n^4p^4}\Big];$

(vi) $M_n^{[\alpha]}(t^5;x)=\Big[\frac{x(x+\alpha)(x+2\alpha)(x+3\alpha)(x+4\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)(1-5\alpha)}+\left(\frac{10}{n}+\frac{15}{np}\right)\frac{x(x+\alpha)(x+2\alpha)(x+3\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)}$
$+\left(\frac{25}{n^2}+\frac{90}{n^2p}+\frac{85}{n^2p^2}\right)\frac{x(x+\alpha)(x+2\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)}+\left(\frac{15}{n^3}+\frac{105}{n^3p}+\frac{225}{n^3p^2}+\frac{225}{n^3p^3}\right)\frac{x(x+\alpha)}{(1-\alpha)(1-2\alpha)}$
$+\left(\frac{1}{n^4}+\frac{15}{n^4p}+\frac{85}{n^4p^2}+\frac{225}{n^4p^3}+\frac{274}{n^4p^4}\right)\frac{x}{(1-\alpha)}+\frac{120}{n^5p^5}\Big];$

(vii) $M_n^{[\alpha]}(t^6;x)=\Big[\frac{x(x+\alpha)(x+2\alpha)(x+3\alpha)(x+4\alpha)(x+5\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)(1-5\alpha)(1-6\alpha)}$
$+\left(\frac{15}{n}+\frac{21}{np}\right)\frac{x(x+\alpha)(x+2\alpha)(x+3\alpha)(x+4\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)(1-5\alpha)}$
$+\left(\frac{65}{n^2}+\frac{210}{n^2p}+\frac{175}{n^2p^2}\right)\frac{x(x+\alpha)(x+2\alpha)(x+3\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)}$
$+\left(\frac{90}{n^3}+\frac{525}{n^3p}+\frac{1050}{n^3p^2}+\frac{735}{n^3p^3}\right)\frac{x(x+\alpha)(x+2\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)}$

$$+\left(\frac{31}{n^4}+\frac{315}{n^4p}+\frac{1225}{n^4p^2}+\frac{2205}{n^4p^3}+\frac{1625}{n^4p^4}\right)\frac{x(x+\alpha)}{(1-\alpha)(1-2\alpha)}$$
$$+\left(\frac{1}{n^5}+\frac{21}{n^5p}+\frac{175}{n^5p^2}+\frac{735}{n^5p^3}+\frac{1624}{n^5p^4}+\frac{1764}{n^5p^5}\right)\frac{x}{(1-\alpha)}+\frac{720}{n^6p^6}\Bigg].$$

Proof. We estimate the moments of the operators (7.2) by direct computation. So here, we skip the proof. □

Lemma 7.2.2. *The central moments of the operators (7.2) are as follows:*

(i) $M_n^{[\alpha]}(t-x;x)=\frac{\alpha x}{1-\alpha}+\frac{1}{np}$;

(ii) $M_n^{[\alpha]}((t-x)^2;x)=\frac{(2\alpha^2+\alpha)}{(1-\alpha)(1-2\alpha)}x^2+\frac{(p+3)(1-2\alpha)+np\alpha-2(1-\alpha)(1-2\alpha)}{np(1-\alpha)(1-2\alpha)}x+\frac{2}{n^2p^2}$;

(iii)
$$M_n^{[\alpha]}((t-x)^4;x)=\frac{1}{(1-\alpha)(1-2\alpha)(1-3\alpha)(1-4\alpha)n^3p^3}\big[\alpha^2(24\alpha^2+46\alpha+3)n^3p^3x^4$$
$$+2\alpha(-48\alpha^3+3(1+p)+4\alpha^2(-29+9(n-2)p))+\alpha(20+3(n+2)p))n^2p^2x^3$$
$$+\Big(288\alpha^4+3(1+p)^2+\alpha(31+6(n+1)p+(6n-13)p^2)$$
$$+8\alpha^3(57-36(n-2)p+2(2n^2-9n+6)p^2)+\alpha^2(-304+24(2n-11)p$$
$$+(3n^2+12n-20)p^2\big)npx^2+\big(-24+240\alpha-840\alpha^2+1200\alpha^3-576\alpha^4$$
$$+6\alpha^3n^3p^3+4(\alpha^2-4\alpha^3)(3p+5)n^2p^2+(12\alpha^3-7\alpha^2+\alpha)(7p^2+30p+35)np$$
$$+(1-9\alpha+26\alpha^2-24\alpha^3)(p^3+10p^2+35p+50)\big)x\big]+\frac{24}{n^4p^4}.$$

Proof. We estimate the central moments by direct computation, using the linearity property of the operators (7.2) and Lemma 7.2.1. Similarly, we can obtain higher-order central moments. □

Lemma 7.2.3. *If for large n the sequence* $\alpha=\alpha(n)$ *approaches to* 0 *and* $n\alpha(n)$ *to* $l\in\mathbb{R}$, *we have*

(i) $\lim_{n\to\infty} nM_n^{[\alpha]}((t-x);x)=lx+\frac{1}{p}$;

(ii) $\lim_{n\to\infty} nM_n^{[\alpha]}((t-x)^2;x)=lx^2+\left(l+1+\frac{1}{p}\right)x$;

(iii) $\lim_{n\to\infty} n^2M_n^{[\alpha]}((t-x)^4;x)=3x^2\left(l(1+x)+1+\frac{1}{p}\right)^2$;

(iv)
$$\lim_{n\to\infty} n^3M_n^{[\alpha]}((t-x)^6;x)=15l^3x^6+45l^2\left(1+\frac{1}{p}+l\right)x^5+45l\left(1+\frac{1}{p}+l\right)^2x^4$$
$$+(15(1+15p+3p^2+p^3)-3lp(1+2p-p^2)+l^2p^2(3+3p+lp))x^3.$$

Proof. These results are obtained by straightforward computation. Therefore, the details are omitted. □

Lemma 7.2.4. *For* $\varphi\in C_B(\mathbb{R}^+)$, *we have*

$$\left|M_n^{[\alpha]}(\varphi;x)\right|\leq\|\varphi\|.$$

7.3 Direct results

Theorem 7.3.1. *Let $\varphi \in C_2^*(\mathbb{R}^+)$ and for adequately large n, $\alpha = \alpha(n)$ approaches to 0. Then*

$$\lim_{n\to\infty} M_n^{[\alpha]}(\varphi(t);x) = \varphi(x),$$

converges in every compact subset of $\mathbb{R}^+$ uniformly.

Proof. From Lemma 7.2.1, we have

(i) $\lim\limits_{n\to\infty} M_n^{[\alpha]}(1;x) = 1;$

(ii) $\lim\limits_{n\to\infty} M_n^{[\alpha]}(t;x) = x;$

(iii) $\lim\limits_{n\to\infty} M_n^{[\alpha]}(t^2;x) = x^2,$

Then according to Bohman-Korovkin theorem, we can say that $\lim\limits_{n\to\infty} M_n^{[\alpha]}(\varphi(t);x) = \varphi(x)$ converges in each compact subset of $\mathbb{R}^+$ uniformly. □

Theorem 7.3.2. *Let $\varphi \in C_2^*(\mathbb{R}^+)$ and $\alpha = \alpha(n)$ converges to zero as $n \to \infty$ and*

$$\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}.$$

Then for each $x \geq 0$, we have

$$M_n^{[\alpha]}(\varphi_h(t) - \varphi_h(x)) \leq 5\omega(\varphi, \delta_n(x)) + \frac{9}{2}\omega_2(\varphi, \delta_n(x)),$$

where $\delta_n^2(x)$ is second order central moments of considered operators.

Proof. From (4.5), using the Steklov-mean φ_h, we have

$$\varphi_h(t) = \varphi_h(x) + \varphi_h'(x)(t-x) + \frac{\varphi_h''(x)}{2}(t-x)^2 + r(t,x)(t-x)^2, \tag{7.3}$$

where $r(t,x)$ is bounded function and converging to 0 as $t \to x$.
Applying $M_n^{[\alpha]}(.;x)$ in (7.3), we get

$$\begin{aligned} M_n^{[\alpha]}(\varphi_h(t) - \varphi_h(x)) = \varphi_h'(x) M_n^{[\alpha]}((t-x);x) + \frac{\varphi_h''(x)}{2} M_n^{[\alpha]}\left((t-x)^2;x\right) \\ + M_n^{[\alpha]}\left(r(t,x)(t-x)^2;x\right). \end{aligned} \tag{7.4}$$

Applying Cauchy-Schwarz inequality in the last term of (7.4), we have

$$M_n^{[\alpha]}\left(r(t,x)(t-x)^2;x\right) \leq \sqrt{M_n^{[\alpha]}\left(r^2(t,x);x\right)}\sqrt{M_n^{[\alpha]}\left((t-x)^4;x\right)}. \tag{7.5}$$

In view of Theorem 7.3.1, we get

$$\lim_{n\to\infty} M_n^{[\alpha]}(r^2(t,x);x) = r^2(x,x) = 0. \tag{7.6}$$

From Lemma 7.2.4

$$\left|M_n^{[\alpha]}(\varphi(t);x)\right| \leq \|\varphi\|. \tag{7.7}$$

From (7.4)-(7.7), we get

$$\left|M_n^{[\alpha]}(\varphi_h(t)-\varphi_h(x))\right| \leq \|\varphi_h'\|\sqrt{M_n^{[\alpha]}((t-x)^2;x)} + \frac{1}{2}\|\varphi_h''\| M_n^{[\alpha]}((t-x)^2;x)$$

Applying (ii) and (iii) property of Proposition 4.3.4 for sufficiently large n, we have

$$\left|M_n^{[\alpha]}\varphi_h(t)-\varphi_h(x)\right| \leq \frac{5}{h}\omega(\varphi,h)\,\delta_n(x) + \frac{9}{2h^2}\omega_2(\varphi,h)\,\delta_n^2(x).$$

By taking $h=\delta_n(x)$ and substituting in above estimate, we get the required result. □

Theorem 7.3.3. *Let $\varphi \in C_B^1(\mathbb{R}^+)$ and $\alpha = \alpha(n)$ converges to 0 and $n\alpha(n) = l \in \mathbb{R}$ for large n. Then for each $x \geq 0$, we have*

$$\left|M_n^{[\alpha]}(\varphi(t);x)-\varphi(x)\right| \leq |\beta_n(x)|\,\varphi(x) + 2\delta_n(x)\omega\left(\varphi',\delta_n(x)\right),$$

where $\beta_n(x) = M_n^{[\alpha]}((t-x);x)$.

Proof. For $x,t \in \mathbb{R}^+$, the Taylor's expansion is given as

$$\varphi(t)-\varphi(x) = \varphi'(x)(t-x) + \int_x^t \left(\varphi'(u)-\varphi'(x)\right)du. \tag{7.8}$$

Applying $M_n^{[\alpha]}(.;x)$ in (7.8), we get

$$M_n^{[\alpha]}(\varphi(t)-\varphi(x);x) = \varphi'(x)M_n^{[\alpha]}((t-x);x) + M_n^{[\alpha]}\left(\int_x^t(\varphi'(u)-\varphi'(x))du;x\right). \tag{7.9}$$

Using (iv) property of Proposition 1.4.1, we get

$$|\varphi(u)-\varphi(x)| \leq \omega(\varphi,\delta)\left(\frac{|u-x|}{\delta}+1\right),\ \ \delta>0,$$

we obtain

$$\left|\int_x^t (\varphi'(u)-\varphi'(x))du\right| \leq \omega\left(\varphi',\delta\right)\left(\frac{(t-x)^2}{\delta}+|t-x|\right). \tag{7.10}$$

Therefore, from (7.9) and (7.10), we get

$$\left|M_n^{[\alpha]}\left(\varphi(t)-\varphi(x);x\right)\right| \leq \left|\varphi'(x)\right|\left|M_n^{[\alpha]}((t-x);x)\right| + \omega\left(\varphi',\delta\right)\left(\frac{1}{\delta}M_n^{[\alpha]}((t-x)^2;x)\right.$$
$$\left. + M_n^{[\alpha]}\left(|t-x|;x\right)\right).$$

In above expression, applying Cauchy-Schwarz inequality, we get

$$\left|M_n^{[\alpha]}\left(\varphi(t)-\varphi(x);x\right)\right| \leq \left|\varphi'(x)\right|\left|M_n^{[\alpha]}\left((t-x);x\right)\right|$$
$$+\omega\left(\varphi',\delta\right)\left\{\frac{1}{\delta}\sqrt{M_n^{[\alpha]}\left((t-x)^2;x\right)}+1\right\}\sqrt{M_n^{[\alpha]}\left((t-x)^2;x\right)}.$$

When we choose $\delta = \delta_n(x)$ then we get the required result.

7.4 Voronovskaya type theorem

Theorem 7.4.1. *Let $\varphi \in C_B(\mathbb{R}^+)$ and if there exists second derivatives of function φ at a fixed point $x \in \mathbb{R}^+$. The sequence $\alpha = \alpha(n)$ converges to 0 for large n, $p > 0$ and $\lim_{n\to\infty} n\alpha(n) = l$, we have*

$$\lim_{n\to\infty} n\left[\left(M_n^{[\alpha]}(\varphi;x)-\varphi(x)\right)\right] = \left(lx+\frac{1}{p}\right)\varphi'(x)+\frac{x}{2}\left(lx+\left(l+\frac{1}{p}+1\right)\right)\varphi''(x).$$

Proof. By the Taylor's series expansion, we have

$$\varphi(t) = \varphi(x)+(t-x)\varphi^{'}(x)+\frac{(t-x)^2}{2!}\varphi^{''}(x)+\varepsilon(t,x)(t-x)^2, \tag{7.11}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.
Applying $M_n^{[\alpha]}, ;x$ on both side of (7.11), then we have

$$M_n^{[\alpha]}(\varphi(t)-\varphi(x);x) = \varphi^{'}(x)M_n^{[\alpha]}((t-x);x)+\frac{\varphi^{''}(x)}{2!}M_n^{[\alpha]}((t-x)^2;x)$$
$$M_n^{[\alpha]}(\varepsilon(t,x)(t-x)^2;x).$$

Now

$$\lim_{n\to\infty} nM_n^{[\alpha]}(\varphi(t)-\varphi(x);x) = \lim_{n\to\infty} nM_n^{[\alpha]}((t-x);x)\varphi'(x) + \lim_{n\to\infty} nM_n^{[\alpha]}((t-x)^2;x)\varphi''(x)$$
$$+\lim_{n\to\infty} nM_n^{[\alpha]}(\varepsilon(t,x)(t-x)^2;x). \tag{7.12}$$

From Theorem 7.3.1, Lemma 7.2.3 and applying cauchy-Schwarz in the last term of (7.12) then, we get

$$\lim_{n\to\infty} nM_n^{[\alpha]}(\varepsilon(t,x)(t-x)^2;x) = 0. \tag{7.13}$$

Using Lemma 7.2.3 and (7.12), (7.13) then, we get required result. □

7.5 Quantitative Voronovskaya type theorem

With the help of weighted modulus of continuity $\Omega(.;\delta)$, here we establish the degree of approximation of the function $\varphi \in C_2^*(\mathbb{R}^+)$ for the proposed operators (7.2).

Theorem 7.5.1. *Let $\varphi \in C_2^*(\mathbb{R}^+)$ such that $\varphi'(x), \varphi''(x) \in C_2^*(\mathbb{R}^+)$, $\alpha = \alpha(n)$ be a sequence converging to 0 for sufficient large n and p is any fixed positive real number, then*

$$\Bigg| n\left(M_n^{[\alpha]}(\varphi;x)-\varphi(x)\right) - \varphi'(x)\left(\frac{n\alpha x}{1-\alpha}+\frac{1}{p}\right) - \frac{\varphi''(x)}{2!}n\left(\frac{2\alpha^2+\alpha}{(1-\alpha)(1-2\alpha)}x^2\right.$$
$$\left. + \frac{(p+3)(1-2\alpha)+np\alpha-2(1-\alpha)(1-2\alpha)}{np(1-\alpha)(1-2\alpha)}x + \frac{2}{n^2p^2}\right)\Bigg| = O(1)\Omega\left(\varphi''; \frac{1}{\sqrt{n}}\right)$$

as $n \to \infty$.

Proof. By the Taylor's expansion

$$\varphi(t) = \varphi(x) + \varphi'(x)(t-x) + \frac{\varphi''(x)}{2!}(t-x)^2 + \frac{\varphi''(\xi)-\varphi''(x)}{2!}(t-x)^2. \tag{7.14}$$

Let

$$h_2(\varphi,x) = \frac{\varphi''(\xi)-\varphi''(x)}{2!}(t-x)^2. \tag{7.15}$$

Using the well-known property of weighted modulus of continuity, we have

$$\left|\varphi''(\xi)-\varphi''(x)\right| \le 4\left(1+x^2\right)\left(1+\delta^2\right)^2\left(1+\frac{(t-x)^4}{\delta^4}\right)\Omega(\varphi'',\delta). \tag{7.16}$$

From (7.15) and (7.16)

$$h_2(\varphi,x) \leq 2\left(1+x^2\right)\left(1+\delta^2\right)^2\Omega(\varphi'';\delta)\left(1+\frac{(t-x)^4}{\delta^4}\right)(t-x)^2 \tag{7.17}$$

Applying $M_n^{[\alpha]}(.;x)$ on both side of (7.14), we get

$$\left|\left(M_n^{[\alpha]}(\varphi;x)-\varphi(x)\right)-\varphi'(x)M_n^{[\alpha]}((t-x);x)-\frac{\varphi''(x)}{2!}M_n^{[\alpha]}((t-x)^2;x)\right| \\ \leq M_n^{[\alpha]}\left(|h_2(\varphi,x)|;x\right). \tag{7.18}$$

From (7.17) and (7.18), we have

$$\begin{aligned} M_n^{[\alpha]}\left(|h_2(\varphi,x)|;x\right) &\leq 8(1+x^2)\Omega(\varphi'';\delta)M_n^{[\alpha]}\left(\left((t-x)^2+\frac{(t-x)^6}{\delta^4}\right);x\right) \\ &\leq 8(1+x^2)\Omega(\varphi'';\delta)\left(M_n^{[\alpha]}\left((t-x)^2;x\right)+\frac{1}{\delta^4}M_n^{[\alpha]}\left((t-x)^6;x\right)\right) \\ &\leq 8(1+x^2)\Omega(\varphi'';\delta)\left(O\left(\frac{1}{n}\right)+\frac{1}{\delta^4}O\left(\frac{1}{n^3}\right)\right), \end{aligned}$$

as $n \to \infty$.

By choosing $\delta = \frac{1}{\sqrt{n}}$, we get

$$nM_n^{(\alpha)}\left(|h_2(\varphi,x)|;x\right) \leq O(1)\Omega\left(\varphi'',\frac{1}{\sqrt{n}}\right). \tag{7.19}$$

From (7.18) and (7.19) and using Lemma 7.2.3, we obtain the required result. □

7.6 Grüss Voronovskaya type theorem

Gr*ü*ss [74] had taken into account an inequality, known as Grüss inequality. The main significance of this inequality is to find the difference between the integral of a product of two functions and the product of integrals of the two functions. To know more about the Grüss inequality and its application see [5, 66, 68].

Theorem 7.6.1. *Let $\varphi,\psi \in C_2^*(\mathbb{R}^+)$ such that $\varphi',\psi',\varphi'',\psi'' \in C_2^*(\mathbb{R}^+)$ then, for each $x \in \mathbb{R}^+$,*

$$\lim_{n\to\infty} n\left\{M_n^{[\alpha]}(\varphi.\psi;x)-M_n^{[\alpha]}(\varphi;x)M_n^{[\alpha]}(\psi;x)\right\} = x\left(lx+\left(l+1+\frac{1}{p}\right)\right)\varphi'(x)\psi'(x).$$

Proof. For each $x \in \mathbb{R}^+$,

$$n\left\{M_n^{[\alpha]}(\varphi.\psi;x)-M_n^{[\alpha]}(\varphi;x)M_n^{[\alpha]}(\psi;x)\right\}$$

$$\begin{aligned}
&= n\Bigg[\left\{M_n^{[\alpha]}(\varphi.\psi;x)-\varphi(x)\psi(x)-(\varphi\psi)'M_n^{[\alpha]}((t-x);x)-\frac{(\varphi\psi)''}{2!}M_n^{(\alpha)}((t-x)^2;x)\right\}\\
&\quad -g(x)\left\{M_n^{[\alpha]}(\varphi;x)-\varphi(x)-\varphi' M_n^{[\alpha]}((t-x);x)-\frac{\varphi''}{2!}M_n^{[\alpha]}((t-x)^2;x)\right\}\\
&\quad -M_n^{[\alpha]}(\varphi;x)\left\{M_n^{[\alpha]}(\psi;x)-\psi(x)-\psi' M_n^{[\alpha]}((t-x);x)-\frac{\psi''}{2!}M_n^{[\alpha]}((t-x)^2;x)\right\}\\
&\quad +\frac{1}{2!}M_n^{[\alpha]}((t-x)^2;x)\left\{\varphi(x)\psi''(x)+2\varphi'(x)\psi'(x)-\psi''(x)M_n^{[\alpha]}(\varphi;x)\right\}\\
&\quad +M_n^{[\alpha]}((t-x);x)\left\{\varphi(x)\psi'(x)-\psi'(x)M_n^{[\alpha]}(\varphi;x)\right\}\Bigg].
\end{aligned}$$

Applying Theorem 7.3.1 for each $x\in\mathbb{R}^+$, $M_n^{[\alpha]}(\varphi;x)\to\varphi(x)$ as $n\to\infty$ and for $\varphi''\in C_2^*(\mathbb{R}^+)$, $x\in\mathbb{R}^+$ by Theorem 7.3.2 and using Lemma 7.2.3, we get the desired result. □

7.7 Weighted approximation

Theorem 7.7.1. *For every $\varphi\in C_2^*(\mathbb{R}^+)$, d is fixed positive real number then, we have*

$$\left|M_n^{[\alpha]}(\varphi(t);x)-\varphi(x)\right|\le 4M_\varphi(1+x^2)\delta_n^2(x)+2\omega_{d+1}(\varphi;\delta_n(x)),$$

where $\delta_n^2(x)=M_n^{[\alpha]}\left((t-x)^2;x\right)$.

Proof. From [48], for each $x\in[0,d]$ and $t\in\mathbb{R}^+$ then, we have

$$|\varphi(t)-\varphi(x)|\le 4M_\varphi(1+x^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{d+1}(\varphi;\delta_n(x)).$$

Applying $M_n^{[\alpha]}(.;x)$ then using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\left|M_n^{[\alpha]}(\varphi(t);x)-\varphi(x)\right| &\le 4M_\varphi(1+x^2)M_n^{[\alpha]}\left((t-x)^2;x\right)\\
&\quad +\left(1+\frac{\left(M_n^{[\alpha]}\left((t-x)^2;x\right)\right)^{\frac{1}{2}}}{\delta}\right)\omega_{d+1}(\varphi;\delta_n(x))\\
&\le 4M_\varphi(1+x^2)\delta_n^2(x)+\left(1+\frac{\delta_n(x)}{\delta}\right)\omega_{d+1}(\varphi;\delta_n(x)).
\end{aligned}$$

By choosing $\delta=\delta_n(x)$, the required result follows. □

Theorem 7.7.2. *For $\varphi\in C_2^*(\mathbb{R}^+)$, $\alpha=\alpha(n)$ be a sequence converging to 0 for adequately* *large n and* $\lim_{n\to\infty} n\alpha(n) = l \in \mathbb{R}$*,* $\exists\, n_0 \in \mathbb{N}$ *and* $\widetilde{C}(l,p)$*, the positive constant depends on p and l then, we have*

$$\sup_{x\in\mathbb{R}^+} \frac{\left|M_n^{[\alpha]}(\varphi(t);x) - \varphi(x)\right|}{(1+x^2)^{5/2}} \leq \widetilde{C}(l,p)\Omega(\varphi;n^{-1/2}),$$

for $n > n_0$.

Proof. From (iii) property of Proposition 1.4.2 and $x,t \in \mathbb{R}^+$, we have

$$|\varphi(t) - \varphi(x)| \leq 4(1+x^2)(1+(t-x)^2)\left(1+\frac{|t-x|}{\delta}\right)\Omega(\varphi;\delta) \tag{7.20}$$

Applying $M_n^{[\alpha]}(;x)$ in (7.20), we get

$$\begin{aligned}\left|M_n^{[\alpha]}(\varphi(t);x) - \varphi(x)\right| &\leq 4(1+x^2)\Omega(\varphi;\delta)\{1+M_n^{[\alpha]}((t-x)^2;x) \\ &\quad + M_n^{[\alpha]}\left(\frac{(1+(t-x)^2)\,|t-x|}{\delta};x\right)\bigg\}\end{aligned} \tag{7.21}$$

From Lemma 7.2.3, for sufficient large n, we have

$$\begin{aligned} M_n^{[\alpha]}((t-x)^2;x) &\leq \frac{C(l,p)(1+x^2)}{n} \\ \textit{and}\ \ M_n^{[\alpha]}((t-x)^4;x) &\leq \frac{C(l,p)(1+x^2)^2}{n^2}, \end{aligned} \tag{7.22}$$

where $C(l,p) > 0$ depends on l and p.
Applying Cauchy-Schwarz inequality in (7.21), we have

$$\begin{aligned} M_n^{[\alpha]}\left(\frac{\left(1+(t-x)^2\right)|t-x|}{\delta};x\right) &\leq \frac{1}{\delta}\left(M_n^{[\alpha]}((t-x)^2;x)\right)^{\frac{1}{2}} \\ &\quad + \frac{1}{\delta}\left(M_n^{[\alpha]}((t-x)^4;x)\right)^{\frac{1}{2}}\left(M_n^{[\alpha]}((t-x)^2;x)\right)^{\frac{1}{2}}. \end{aligned} \tag{7.23}$$

Combing $(7.21)-(7.23)$ and taking $\delta = \frac{1}{\sqrt{n}}$, we have

$$\widetilde{C}(l,p) = 2(1+\sqrt{C(l,p)}+2C(l,p)).$$

We obtain the required result. $\square$

7.8 Function of bounded variation

For our convenience, the operators (7.2) can be written in the following form

$$M_n^{[\alpha]}(\varphi;x)=\int_0^\infty U_{n,p}^{(\alpha)}(t,x)\varphi(t)dt \tag{7.24}$$

where

$$U_{n,p}^{(\alpha)}(t;x)=\sum_{k=0}^{\infty}v_{n,k}^{(\alpha)}s_{n,k}^{p}(t),$$

$v_{n,k}^{(\alpha)}=\frac{t^{\frac{x}{\alpha}-1}e^{-nt}(nt)^{-k}}{\beta\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)k!(1+t)^{\frac{1+x}{\alpha}}}$ *and* $s_{n,k}^{p}(t)=\frac{npe^{-npt}(npt)^{kp}}{\Gamma kp+1}$.

Lemma 7.8.1. *For each $x\in\mathbb{R}^+$ and for large n, we get*

(i) $\beta_{n,p}(x,t)=\int\limits_0^t U_{n,p}^{(\alpha)}(x,u)\,du\leq\frac{C(l,p)\left(1+x^2\right)}{n(x-t)^2},\quad 0\leq t<x.$

(ii) $1-\beta_{n,p}(x,t)=\int\limits_t^\infty U_{n,p}^{(\alpha)}(x,u)\,du\leq\frac{C(l,p)\left(1+x^2\right)}{n(t-x)^2}\quad x\leq t<\infty,$

where $C(l,p)>0$ depends on p and l.

Theorem 7.8.2. *Let $\varphi\in DBV(\mathbb{R}^+)$, let $\alpha=\alpha(n)$ be a sequence converging to 0 for adequately large n, and let $\lim\limits_{n\to\infty}n\alpha(n)=l$. Then, for $x\in\mathbb{R}^+$ and large n, we acquire*

$$\begin{aligned}\left|M_n^{[\alpha]}(\varphi;x)-\varphi(x)\right|&\leq\left(\frac{\alpha x}{1-\alpha}+\frac{1}{np}\right)\left|\frac{\varphi'(x+)+\varphi'(x-)}{2}\right|\\&+\sqrt{\frac{C(l,p)(1+x^2)}{n}}\left|\frac{\varphi'(x+)-\varphi'(x-)}{2}\right|\\&+\frac{C(l,p)\left(1+x^2\right)}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^{x}\varphi_x{}'\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}\varphi_x{}'\right)\\&+\left(4M+\frac{M+|\varphi(x)|}{x^2}\right)\frac{C(l,p)\left(1+x^2\right)}{n}\\&+\left|\varphi'(x+)\right|\sqrt{\frac{C(l,p)(1+x^2)}{n}}\\&+\frac{C(l,p)\left(1+x^2\right)}{nx^2}\left|\varphi(2x)-\varphi(x)-x\varphi'(x+)\right|\\&+\frac{C(l,p)\left(1+x^2\right)}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{k}}\varphi_x{}'\right),\end{aligned}$$

where $\vee_a^b\varphi(x)$ denotes the total variation of φ on $[a,b]$ and φ_x is an auxiliary operator given *by*

$$\varphi_x(t) = \begin{cases} \varphi(t) - \varphi(x-), & 0 \leq t < x \\ 0, & t = x \\ \varphi(t) - \varphi(x+), & x < t < \infty \end{cases}$$

Proof. The proof of this theorem closely follows the idea which is developed in [87]. Therefore here we skip the proof. □

Chapter 8

Conclusion and Future Scope

8.1 Conclusion

In the year 2017, Chen et. al [36] generalized the Bernstein operators with shape parameter $\alpha \in [0,1]$. For these operators they established several approximation results. We have considered the Kantorovich variant of these operators and studied rate of convergence for functions whose derivatives are of bounded variation and Voronovskaya type theorem. We have also introduced the q-analogue and shifted knots of our considered Kantorovich operators. These operators play an important role to show better convergence with existing operators. We have also studied several approximation properties for these operators.

The Kantorovich and integral form of Jain and Pethe operators [101] have been introduced by us in chapter 3 and chapter 4 respectively. These operators can approximate integrable functions. We have also established local and global approximation properties by using second-order modulus of continuity, Ditzian-Totik modulus of smoothness, Lipchitz type space, Voron-ovskaya type theorem, weighted modulus of continuity and also rate of convergence for functions whose derivatives are of bounded variation.
In 2003, Srivastava and Gupta proposed a sequence of linear positive operators and studied the rate of convergence for functions whose derivatives are of bounded variation. Several mathematicians proposed the modifications of these operators and established approximation results. In 2018, Gupta and Srivastava introduced another modification of their previous operators. For these operators, we have discussed several approximation results. We have also proposed the new generalization of these operators known as the Bézier variant of Gupta-Srivastava operators. For these operators, we have studied

local and global approximation properties, weighted approximation and functions whose derivatives are of bounded variation.
The hybrid types of operators are also playing an important role in approximation theory. These operators proposed by several researchers [70, 88] and established many local and global approximation properties . We have also introduced hybrid type operators based on second kind beta transform and discussed some approximation properties like direct results, Voronovskaya, Grüss Voronovskaya type theorem.

8.2 Future Scope

In the future, we have more opportunities to introduce new generalizations of our considered operators with certain parameters and establish some local and global approximation results. These operators also can be considered in different spaces like (set of complex numbers, L_p space, etc.) to make them more realizable and discuss many qualitative and quantitative properties.
In the year 2003, J. P. King [111] modified linear positive operators which preserves x^2. In several papers, King type modifications have been introduced by many researchers (see [49, 126]) and discussed some approximation properties. This approach can also be applied for our considered operators to extend the study in approximation theory.

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List of Publications

1. Dhamija M., **Pratap R.** and Deo N., Approximation by Kantorovich form of modified Szász-Mirakyan operators, Applied Mathematics and Computation, 317 (2018), 109-120, Elsevier.

2. **Pratap R.** and Deo N., Approximation by genuine Gupta-Srivastava operators, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 113 (2019), 2495-2505, Springer.

3. **Pratap R.** and Deo N., Rate of convergence of Gupta-Srivastava operators based on certain parameters, Journal of Classical Analysis, 14(2) (2019), 137-153, Ele-Math.

4. Deo N. and **Pratap R.**, α-Bernstein-Kantorovich operators, Afrika Matematika, 31 (2020), 609-618, Springer.

5. **Pratap R.** and Deo N., Q-analogue of generalized Bernstein-Kantorovich operators, Proceedings in Mathematical Analysis-I-Approximation Theory, (2020), 67-75, Springer.

6. Deo N. and **Pratap R.**, Approximation by integral form of Jain and Pethe operators, Proceedings in National Academy of Sciences, India Section A, Physical Sciences, (2020), https://doi.org/10.1007/s40010-020-00691-z, Springer.

7. Deo N. and **Pratap R.**, Approximation by mixed positive linear operators based on second-kind beta transform, Asian-European Journal of Mathematics, (2020) World Scientific (accepted).

8. **Pratap R.** and Deo N. , A family of Bernstein-Stancu-Kantorovich operators with shifted knots, Rendiconti del Circolo Matematico di Palermo Series 2, (2020), Springer (accepted).