INCOMPLETE PREFERENCE RELATIONS IN MCDM

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by

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DECLARATION

I declare that the research work reported in this thesis entitled "**Incomplete Preference Relations in MCDM** " for the award of the degree of *Doctor of Philosophy in Mathematics* has been carried out by me under the supervision of *Prof. Anjana Gupta*, Department of Applied Mathematics, Delhi Technological University, Delhi, India.

The research work personified in this thesis, except where otherwise indicated, is my original research. This thesis has not been submitted by me earlier in part or full to any other University or Institute for the award of any degree or diploma. This thesis does not contain other person's data, graphs or other information, unless specifically acknowledged.

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CERTIFICATE

This is to certify that the thesis entitled **"Incomplete Preference Relations in MCDM "** submitted by **Ms. Mamata Sahu** in the Department of Applied Mathematics, Delhi Technological University, Delhi, India for the award of degree of *Doctor of Philosophy in Mathematics*, is a record of bonafide research work carried out by her under my supervision.

I have read this thesis and that, in my opinion, it is fully adequate in scope and quality as a thesis for the degree of Doctor of Philosophy.

To the best of my knowledge the work reported in this thesis is original and has not been submitted to any other Institution or University in any form for the award of any degree or diploma.

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Date : **(MAMATA SAHU)**

Place : Delhi, India.

Dedicated to

My Parents

&

My Son

Contents

Preface

Multi-citeria decision making is concerned with structuring and solving decision and planning problems involving multiple criteria. In a decision scenario, a decision maker is generally required to provide his/her assessments of choices. To communicate the decision maker's preference data, preference relations are exceptionally helpful in various fields of decision-making problem, for example, legislative issues, social brain science, designing, administration, business, and financial aspects, and so on. Sometimes, it has witnessed that in a situation a decision maker might not have a decent comprehension of a specific query, thus he/she can not make an instaneous contrast between each two objects. Consequently it necessities to allow the decision maker to avoid some questionable comparison adaptably. Therefore sometimes, due to lack of time and busy schedule of the decision maker, incomplete preference relations are obtained. In this case, incomplete preference relations are obtained, and the whole process may slow down. In this work we have developed methods for complementing types of incomplete preference relations. Applications of different preference relations in MCDM, are also discussed.

Introductory chapter presents a brief review of uncertainty theory that is Fuzzy set theory and some extension of the fuzzy set theory, to fuzzy relation. Also, this chapter discusses type of preference relations for addressing multi-criteria decision making(MCDM). Thus, the present chapter creates a background, gives the motive of thesis work.

Chapter 2 define $(\widetilde{\alpha},\widetilde{\beta})$-cuts and the resolution form of the interval-valued intuitionistic fuzzy (IVIF) relations to develop a procedure for constructing a hierarchical clustering for IVIF max-min similarity relations. The advantage of the proposed scheme is illustrated in determining the criteria weights in MCDM problems involving IVIF numbers. A complete procedure is drawn to generate criteria weights starting from the criteria-alternative matrix of the MCDM problem with entries provided by a decision maker as interval-valued intuitionistic fuzzy numbers. The chapter is based on a research paper "Hierarchical clustering of interval-valued intuitionistic fuzzy relations and its application to elicit criteria weights in MCDM problems", published in *Opsearch*, springer, **54**, 388–416, (2017).

Chapter 3 propose a characterization of the consistency property using newly defined transitivity property for intuitionistic multiplicative preference relations (IMPR) together with complementing missing elements for incomplete IMPR. Using new transitivity property of IMPR, we have developed two different methods to find the missing element of IMPRs. Acceptably consistent with complete IMPRs is also checked. The another goal of this chapter is to achieve the consistent intuitionistic multiplicative preference relation using graphical approach. We have proposed two different characterization of the consistency for intuitionistic multiplicative preference relation(IMPR). In the first approach, we design an algorithm to achieve the consistency of IMPR by using the cycles of various length in a directed graph. The second approach proves isomorphism between the set of IMPRs and the set of asymmetric multiplicative preference relations. That result is explored to use the methodologies developed for asymmetric multiplicative preference relations to the case of IMPRs and achieve the consistency of asymmetric multiplicative preference relation using directed graph. Also, the above said method is applied for incomplete IMPR, here consistency play an important role. The illustrations are provided to exemplify the designed methods. The chapter is based on a research paper "New transitivity property of intuitionistic fuzzy multiplicative preference relation and its application in missing value estimation", published in *Annals of Fuzzy Mathematics and Informatics*, **16** (1), 71–86 (2018) and "Two different approaches for consistency of intuitionistic multiplicative preference relation using directed graph", is communicated in *Asia-pacific journal of operational research*.

Chapter 4 study the consistency property, and especially the acceptably consistent property, for incomplete interval-valued intuitionistic multiplicative preference relations. We propose a technique to evaluate missing elements which first estimates the initial values for all missing entries in an incomplete interval-valued intuitionistic multiplicative preference relation and then improves them by a local optimization method. A method is developed to estimate the importance of the experts to achieve resultant consistent decision matrix in group decision situations. The proposed method is illustrated using two examples involving group decision scenario. The chapter is based on a research paper "Acceptably consistent incomplete interval-valued intuitionistic multiplicative preference relations", is published in *Soft Computing*, Springer, **22**, 7463–7477 (2018).

In **Chapter 5**, a new definition of additive consistency property of hesitant fuzzy preference relation (HFPR) is given that preserves the property of hesitancy and is used to construct the complete HFPR from incomplete one. The significance of consistency measure for HFPR make sure that the DMs are neither arbitrary nor unreasonable. We develop a method to check consistency level of incomplete HFPR. A numerical example is illustrated to show the applicability of the designed methodology. Group decision-making problem with incomplete HFPR is also considered. The chapter is based on a research paper "Incomplete Hesitant Fuzzy Preference Relation", is published in Journal of *Statistics & Management Systems*, Taylor & Francis, **21**, (8) 1459–1479 (2018).

Chapter 6 developed a method to complete incomplete hesitant multiplicative preference relations (HMPRs). A new definition of multiplicative transitive property of HMPR has given that preserve the hesitancy property and is used to construct the complete HMPR from incomplete one. An optimization model is developed to minimize the error. Also a linear programming model is developed to directly calculate the missing elements of incomplete HMPR. The satisfaction degree and the acceptably consistent of complete HMPR is also checked. The whole procedure is explained with a suitable example. The chapter is based on research paper "A Method to Complement Incomplete Hesitant Multiplicative Preference Relation", published in *International Journal of Research and Analytical Reviews*, **5**, (2)1421–1429 (2018) and "Incomplete Hesitant Multiplicative Preference Relation", **revised version submitted** in Opsearch, Springer.

After chapter 6 we present the summary of the research work carried out in this thesis. Further, the future research plan has been discussed in brief.

Finally, the bibliography and list of publications have been given at the end of the thesis.

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Chapter 1

Introduction

In the present-day world, decision making is turning into a necessary action despite being invaded by various updated technology advancements assisted decision tools. Sometimes, technology is unsuccessful in delivering a decision without thinking about human subjective ability. Human equips with a good insight are expected to capitalize effective decision making to reach a very agreeable decision. One of the most promising decision-making tools that were conceptualized in the early seventies is multi-criteria decision-making theory. The theory of decision making formed a basis for more systematic and rational decision making especially in the situation where multiple criteria need to be accounted. This decision theory does not take so much time to fully recognized with the four terms consolidated to be known as multi-criteria decision making (MCDM). The theory was additionally developed in line with the development of uncertainty and chaos theory. This chapter presents a brief review of uncertainty theory like Fuzzy set theory and some extension of the fuzzy set theory, to the fuzzy relation. Also, this chapter discusses preference relations for addressing MCDM. Another aim of this chapter is to provide a platform to motivate the work carried out in this thesis.

Decision Making is the expression of choosing between two or more courses of 'action.' On the other hand, it must always be remembered that there may not always be a 'correct' decision among the existing choices. There may have been a better choice that had not been considered, or the right information may not have been accessible at the time. Multiple-criteria evaluation problems consist of a finite number of alternatives, explicitly known at the beginning of the solution process. Whereas, in multiple criteria design problems (multiple objective mathematical programming problems) the number of alternatives is either infinite or not countable (when some variables are continuous) or typically very large if countable (when all variables are discrete). The alternatives are not explicitly known in multiple criteria design problems and an alternative (solution) can be found by solving a mathematical model. These two types of problems are considered as subclasses of Multi-Criteria Decision Making problems.

1.1 Different type of fuzzy sets and fuzzy relations

Among all the fields from psychologists, economists, to computer scientists, the fuzzy decision is an important branch of fuzzy theory. Liu and Liao [1] gave a bibliometric analysis of fuzzy decision-related research to find out some underlying patterns and dynamics in the direction of the fuzzy decision. The real world decision-making problem has been broadly perceived that most decisions take place in an environment in which the goals and constraints, are not known precisely, because of their complexity, and hence, the problem cannot be precisely characterized or exactly represented in a crisp value [2].To deal with the kind of qualitative, uncertain data or even not well-organized choice issues, Zadeh [3] suggested employing the fuzzy set theory as a modeling tool for complex systems that can be controlled by humans but are hard to define exactly.

Research on the theory of fuzzy sets has been developed relentlessly since the initiation of the approach by Zadeh [3] and has a significant application in numerous fields like building, medicinal science, sociology, diagram hypothesis and so forth. The fuzzy set theory characterizes the membership of a fuzzy number in $[0,1]$, that describes the level of belonging-ness of the element to the fuzzy set. Be that as it may, indeed, it may not generally be valid that the level of non-membership of a fuzzy set element is equal to one minus the membership degree because there may be some hesitation degree. This framework is not sufficient to model situations where one believes that $x \in A$ with some membership degree although not convinced on the value to be given to it and/or when some additional information is offered on the negation of the positive declaration.

A standout amongst the essential speculations of fuzzy sets (FSs, in shortened form) is interval-valued fuzzy set (IVFS) presented by Zadeh [4] (also see Turksen [5]). The significant thought behind IVFS is that the enrollment degree can barely be exact; an interval can better clarify the vulnerability in it. Similarly, as a helpful development of Zadeh's fuzzy sets [3] is intuitionistic fuzzy sets (IFS) were initial introduced and further developed in an exceedingly series of papers by Atanassov ([6], [7], [8], [9]). The idea of ambiguous sets was independently proposed by Gau and Buehrer [10] and was later Bustince, and Burillo [11] resolved to be identical to IFS. More precisely, an IFS is described by $A_I = \{\langle x, \mu_{A_I}(x), \nu_{A_I}(x)\rangle \mid x \in X\}$, where the functions μ_{A_I}, ν_{A_I} are the membership and non-membership degree of IFS A_I, respectively. Both the membership and non-membership belonging to [0,1], such that $\mu_{A_I}(x) + \nu_{A_I}(x) \leq 1$. Thus, in IFS theory, the non-membership degree $\nu_{A_I}(x)$ is an independent degree with the only condition on it being less than or equal to $1 - \mu_{A_I}(x)$. An IFS can be seen as the first significant departure away from the FS.

Initially the very name intuitionistic fuzzy set was somewhat debatable (see, Dubois et al. [12]), but of late the same nomenclature has found acceptance with the range of research papers using it. For more clarity, sometimes it is also referred to by Atanassov's I-fuzzy set. Yu and Liao [13] make a scientometric review on IFS studies to reveal the most cited papers based on the 1318 references retrieved from SCIE and SSCI databases via Web of science.

The principle thought behind IVFS is that the membership degree can barely be exact; an interval can better clarify the uncertainty in it. Later Deschrijver and kerre [14] demonstrated the two documentations IVFS and IFS are isomorphic.

In every day life, IFS turns out to be a versatile tool. Many researchers has been associated with this theories and applications on IFS to an extensive variety of domains, for instance, multiple criteria decision making ([15], [16], [17], [18]) and pattern recognition [19]. Despite its duties concerning application to various scientific domain, IFS itself has additionally been experiencing extraordinary theoretical advancements in the previous a few decades. A smaller and contemporary study of the basic outcomes in IFS has given by Atanassov [20]. These basic outcomes incorporate , for instance, geometric interpretations of IFS (e.g., [21]- [24]), operators on and relations in IFS (e.g., [25]- [33]) and so on.

A theory of the possibility of the IFS is given in the light of ordinary IVFSs. Making further a stride towards displaying more subjective dubiousness, Atanassov and Gargov [34] presented *interval-valued intuitionistic fuzzy set* (IVIFS). IVIFS is a natural augmentation of IVFS, where both the membership and additionally the non-membership degrees require not to be indicated with accuracy yet rather permitted to lie in the interval $[0,1]$. Mathematically, the IVIFS is defined as

Definition 1.1.1. [34] An IVIFS A_{IVI} in X is described by

$$A_{IVI} = \{\langle x, \widetilde{\mu}_{A_{IVI}}(x), \widetilde{\nu}_{A_{IVI}}(x)\rangle \mid x \in X\},$$

where $\widetilde{\mu}_{A_{IVI}}(x) = [\underline{\mu}_{A_{IVI}}(x), \overline{\mu}_{A_{IVI}}(x)]$, and $\underline{\mu}_{A_{IVI}}(x)$ and $\overline{\mu}_{A_{IVI}}(x)$ are respectively the lower and the upper values of the membership degree, while $\widetilde{\nu}_{A_{IVI}}(x) = [\underline{\nu}_{A_{IVI}}(x), \overline{\nu}_{A_{IVI}}(x)]$, and $\underline{\nu}_{A_{IVI}}(x)$ and $\overline{\nu}_{A_{IVI}}(x)$ are respectively the lower and the upper values of the non-membership degree, such that the following hold.

$$0 \leq \underline{\mu}_{A_{IVI}}(x) \leq \overline{\mu}_{A_{IVI}}(x) \leq 1, \quad 0 \leq \underline{\nu}_{A_{IVI}}(x) \leq \overline{\nu}_{A_{IVI}}(x) \leq 1,$$
$$0 \leq \overline{\mu}_{A_{IVI}}(x) + \overline{\nu}_{A_{IVI}}(x) \leq 1, \quad \forall\, x \in X.$$

Xu and Cai [35] illustrated several real-life applications where the subjective granularity is enhanced by adding an additional layer of flexibility in expressing information using IVIFS. The rich structure supported by strong theory and several good aggregation operators, defined in the literature over the years, makes an IVIF theory an intuitive and computationally efficient for handling uncertain and ambiguous information. The interval-valued membership and non-membership values offer more flexibility regarding describing favoritism (or membership value) and rejection (non-membership).

Also, cluster analysis is an important tool for clustering a data set into groups of similar characteristics. Clustering or the cluster analysis refers to a procedure of grouping an arrangement of objects in such a way that objects classified in one group show more similarity to each other than to the objects in other groups. Clustering analysis as one of the extensively grasped fundamental tools in dealing with information data, that has been connected to the domains of example, pattern recognition [36], data mining [37], and other real-world problems concerning social, medial, biological systems ([38], [39]) and so on. In a genuine world, information utilized for clustering might be dubious and fuzzy, to manage different type of fuzzy details. On the application side of FS and its different variations, Bellmann et al. [40] and Ruspini [41] introduced thoughts of information clustering utilizing FS theory. Fuzzy clustering is found to be useful in variety of areas ([36], [42], [43]). The fuzzy clustering techniques can extensively be ordered into two classifications. One of them is the fuzzy c-means (FCM) algorithm, and its varieties are the natural methodologies in this classification ([44] – [46]). In the cluster analysis, the FCM clustering algorithm is one of the widely acknowledged and adopted technique. The FCM clustering allows one information to have a place with at least two groups all the while with relating membership degrees of belongingness. These techniques require the information to be available in shape of highlight vectors so to compute their distances from the prototypes.

The other important class of fuzzy clustering is based on fuzzy relations. In classical set theory, a relation from a set X to a set Y is formally defined as a subset of the cartesian product $X \times Y$, consequently, a fuzzy relation R from a set X to a set Y is defined as a fuzzy set in the cartesian product $X \times Y$. Zadeh [47] characterized a *fuzzy relation* R_f between two sets X and Y as a fuzzy subset of $X \times Y$ by taking a membership function $\mu_{R_f} : X \times Y \to [0,1]$, with $\mu_{R_f}(x,y)$ equals a membership degree of $(x,y) \in R_f$. At that point, Zadeh [47] moreover characterized a similarity relation that has been effectively connected to various areas of matching, data classification and pattern recognition ([48] – [50]). This class of fuzzy clustering was initially produced for acquiring an agglomerative ("bottom-up") hierarchical clustering. Hence, this approach is to set up various leveled structures for the performance evaluation of vague, humanistic confounded frameworks [51].

Sometimes, the criteria information is uncertain or insufficient, additionally the ambiguity of human thinking, or conflicting with decision-maker discrimination, higher-order fuzziness is needed. In such scenario, the available strategies of fuzzy clustering based on fuzzy relation might not be sufficient to manage these conditions. Despite the fact that, decision makers used fuzzy relation-based techniques to implement pairwise comparisons for the similarity among execution criteria. Sometimes, it is difficult to utilize the membership relation values $\mu_{R_f}(x,y)$ between two criteria x and y to manage higher-order uncertainty. Thus, Guh et al. [52] first generalized the fuzzy relations to interval-valued fuzzy relations(IVFR). Guh et al. [52] proposed a methodology to create hierarchical structure using interval-valued fuzzy similarity relation. Taking inspiration from IFS, Bustince and Burillo [53] characterized intuitionistic fuzzy relation(IFR). A generalization of the notion of IFR is an interval-valued intuitionistic fuzzy relation (IVIFR).

1.2 Preference relation

Decision-making procedures are progressively being utilized in different fields for estimation, determination and prioritization purposes, that is, making preference decisions about a set of different choices. Furthermore, it is also obvious that in many cases of decision problems, the comparison of different alternative activities as indicated by their attractive quality, is not possible using one criterion or a single decision maker(DM). For sure, in the larger part of decision-making problems, strategies have been built up to combine opinions about alternatives identified with various perspectives. These methods depend on pair comparisons, in the sense that processes are connected somewhat of believability of preference of one alternative over another.

In a decision-making process, a DM needs to provide his/her evaluations over alternatives. The pair wise comparison between each pair of objects is known as preference relations, and it is applied in different domains of decision problems to communicate decision maker's preference information data for example, legislative issues, social brain science, designing, administration, business, and financial aspects, and many more.

Amid the previous years, the utilization of preference relations is accepting expanding consideration, and various research is done on this issue, and different kinds of preference relations have been created including fuzzy preference relation (FPR)([54]– [56]), multiplicative preference relation(MPR) ([57], [58]), interval-valued fuzzy preference relation (IV-FPR)([59], [60]), interval-valued multiplicative preference relation (IV-MPR) [61], intuitionistic fuzzy preference relation (IFPR)([16], [20], [62]), intuitionistic multiplicative preference relation (IMPR) ([63] – [65]), interval-valued intuitionistic fuzzy preference relation (IVI-FPR) [66], triangular fuzzy preference relation [67], triangular fuzzy multiplicative preference relation([68] – [70]), linguistic preference relation([71] – [74]), hesitant fuzzy preference relation (HFPR) [75], hesitant multiplicative preference relation (HMPR) [76] and so on.

1.2.1 Fuzzy preference relation

A preference relation of fuzzy nature, considering fuzziness of the preference data given by a DM in most decision-making problems. One of the most important classical preference relations is fuzzy preference relation (FPR)(or additive preference relation)([54]-[56]) where the decision maker provides the preference data using $0-1$ scale. A FPR P

on a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ is represented by a complimentary matrix $P = [p_{ij}]_{n\times n}$ satisfying $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, for all $i, j = 1, 2, \cdots, n$ with $0 \leq p_{ij} \leq 1$, where p_{ij} is the preferred data of the objects x_i over the objects x_j. Particularly, $p_{ij} = 0.5$ shows lack of interest amongst x_i and x_j; $p_{ij} > 0.5$ demonstrates that x_i is preferred to x_j, the bigger p_{ij}, the more prominent the preference level of the option x_i over x_j, $p_{ij} = 1$ shows that x_i is absolutely preferred to x_j; $p_{ij} < 0.5$ demonstrates that x_j is preferred x_i, the smaller p_{ij}, the more noteworthy the preference level of the option x_j over x_i, $p_{ij} = 0$ demonstrates that x_j is absolutely preferred to x_i. Orlovsky [54] study and analysed some important properties of FPR in natural way which allow to introduce a fuzzy set of non-dominated options.

In the decision making association, consistency is a fundamental issue in decision-making problem. Generally according to Herrera-Viedma et al. [77] the question of consistency includes two sub-problem (i) when can an expert be said to be consistent and, (ii) when can a group of experts be viewed as good is called simply consensus measure. Our work of all over the thesis focuses on the consistency measure of preference relation. In 1984, Tanino [78] defined the additive consistent FPR (also see [77], [79]) which satisfy the additive transitive property:

$$p_{ij} + p_{jk} = p_{ik} + 0.5 \ \forall\, i, j, k = 1, 2, \cdots, n \tag{1.2.1}$$

and this equation 1.2.1 is equivalent to

$$p_{ij} = 0.5(w_i - w_j + 1) \ \forall\, i, j, k = 1, 2, \cdots, n \tag{1.2.2}$$

where w_i be the priority vector satisfying $\sum_{i=1}^{n} w_i = 1$, $w_i > 0$, $i = 1, 2, \cdots, n$ which is given by Xu ([79], [80]).

A FPR $P = (p_{ij})_{n\times n}$ is said to be multiplicative consistent FPR if it satisfies the multiplicative transitivity property ([77] – [79]),

$$p_{ij}p_{jk}p_{ki} = p_{ik}p_{kj}p_{ji} \tag{1.2.3}$$

In the literature, much amount of work has been done by the researcher in this direction. Xu and Da [81] originated a technique for improving the consistency of an FPR. Using the additive transitivity property of FPRs, Herrera-Viedma et al. [77] proposed a strategy for the consistency property and using $n-1$ preference data also they constructed consistent fuzzy preference relations. For the inconsistency of a reciprocal preference relation, Ma et al. [82] developed a method to improve inconsistency by obtaining weak transitivity based on additive transitivity property. To calculate the consistency level of FPR, Herrera-Viedma et al. [83] developed the consistency index which is based on the additive transitive property. Based on the multiplicative consistency property Xia et al. [84] investigated the consistency of reciprocal preference relations. Also, they developed an algorithm to improve the consistency level of reciprocal preference relation. Khalili-Damghani et al. [85] use the FPRs to help decision-makers expression concerning the membership values of the fuzzy goals and propose a Goal Programming (GP) approach for portfolio selection that embraces conflicting fuzzy goals with imprecise priorities. Zhou and Xu [86] originated the asymmetric fuzzy preference relation (AFPR). Furthermore, also they derived some properties of AFPR.

1.2.2 Multiplicative preference relation

To manage a complex and unbalanced system, Saaty [57] proposed the multiplicative preference relation(MPR) and using the proportion scale $1/9-9$, also he measure the intensity of the pairwise comparisons of objects. An MPR R over the set X is defined as a reciprocal matrix $R=(r_{ij})_{n\times n}$ is a subset of $X\times X$, satisfying $r_{ij}r_{ji}=1$, where, $r_{ij}\in[1/9,9]$, with $r_{ii}=1$, $\forall\, i,j=1,2,\cdots,n$, where r_{ij} represent the preferred degree of the object x_i to the object x_j.

As a rule the degree of preference r_{ij} is estimated utilizing the proportion scale [87], and specifically as Saaty [57] appeared the $1/9-9$ ratio scale. The preferred degree $r_{ij}=1$ suggests lack of interest among x_i and x_j; $r_{ij}>1$ demonstrates that the alternative x_i is prefer to the alternative x_j, particularly, $r_{ij}=9$ shows that alternative x_i is absolutely preferred to alternative x_j; $r_{ij}<1$ implies that x_j is preferred to x_i, and particularly, $r_{ij}=1/9$ demonstrates that x_j is absolutely preferred to x_i.

The primary contrast between the FPRs and MPRs as measured by using $0-1$ and $1/9-9$ ratio scale respectively. Chiclana et al. [88] proved that MPR and FPR are isomorphic.

The consistency of preferences is related to rationality, which is associated with the transitivity property. To model transitivity, many features are suggested which are inappropriate for preference relations. Also, Saaty [57] introduced the multiplicative transitivity of MPR.

Definition 1.2.1. [57] Let $R=(r_{ij})_{n\times n}$ be a MPR is said to be consistent MPR, if it is satisfied the multiplicative transitivity property

$$r_{ij} = r_{ik}r_{kj}, \; \forall\, i,j = 1,2,\cdots,n \tag{1.2.4}$$

and such MPR is given by

$$r_{ij} = \frac{w_i}{w_j}, \; \forall\, i,j = 1,2,\cdots,n \tag{1.2.5}$$

Be that as it may, in practical an MPR is inconsistent. Saaty [57] recommended a consistency index (CI) and the consistency ratio CR of multiplicative preference relation R to measure the level of inconsistency as:

$$CI = \frac{\lambda_{\max} - n}{n-1}, \quad CR = \frac{CI}{RI_n},$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix $[r_{ij}]$ and RI_n is the random index (depending on the order n of a preference relation matrix)([57]), and quoted in Table 1.1.

Table 1.1: Random index of MPR

n	1	2	3	4	5	6	7	8	9	10
RI_n	0	0	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49

Definition 1.2.2. An MPR R is said to be acceptably consistent if $CR \leq 0.1$, otherwise, R is announced not acceptably consistent (or simply inconsistent).

To manage the unacceptable consistency (i.e Consistency Ratio is greater than 0.1) in MPRs, there might be a methodology, to get new MPRs as shown by decision-maker new judgments. This strategy will continue until the acceptably consistent of MPRs are achieved. For large number of options, this methodology is dependable and exact but not feasible. To avoid this limitation in an MPR, Xu and Wei [89] proposed an algorithm to upgrade the consistency.

In decision-making problem, bunches of strategies with MPRs have been made. Jensen [90] proposes the eigenvector technique. The least square method and the eigenvalue method are given by Saaty and Vargas [91]. In 1985, Cogger and Yu [92] discuss the gradient eigenvector method.

To chose a best option, Herrera et al. [87] characterized a decision model which satisfy the multiplicative property dependent on the fuzzy majority. To calculate the priority weights from an MPR Crawford and Williams [93] developed a logarithmic least square technique. Furthermore, the application of MPRs has been profoundly considered in analytic hierarchy process (AHP) ([57], [61], [90], [94]- [96]).

1.2.3 Interval valued preference relation

In real-world decision-making problems, the expert may have obscure data about the assessment of one option over the other one. Also, because of vulnerability and complexity, decision-maker can not appraise their preference with correct numerical esteem. These circumstances lead by Xu ([59], [60]) to characterize an IV-FPR in which the preference degrees are described using intervals bounds in $[0,1]$. Using IV-FPR, Xu [79] characterize the ideas of compatibility degree and compatibility index and also he extended the detailed analysis of the compatibility in group decision-making (GDM). For overseeing non-homogenous data in GDM problem, Herrera et al. [97] proposed an aggregation procedure composed of fuzzy binary preference relations, IV-FPR, and fuzzy linguistic relations.

On the off chance that a decision maker used an IV-FPR to communicate his/her preference data over criteria. Then, from the IV-FPR the priority weights are obtained and can be utilized to rank the options. Obtaining the weights of criteria and the positioning of options assume two vital parts in a MCDM process. Calculating priority weights from an IV-FPR is a fascinating and essential research point.

To derive the priority weights from various IV-FPR, some practical and straightforward linear programming models are established. Meng et al. [98] brought up inadequacies in the current thoughts of multiplicative consistency for IV-FPR and specific consistency idea for an IV-FPR.

Similarly, the extension of MPRs as interval-valued multiplicative preference relation (IV-MPR) defined by Saaty and Vargas [61] allowing experts to use the interval-values in $(1/9)-9$ scale to record their preferences. Additionally, from IV-MPRs, a Monte Carlo simulation approach is developed by Saaty and Vargas [61] to calculate the priority weight intervals. Arbel [99] converted the detailed prioritization method of IV-MPRs as a linear programming model. To get the weights from inconsistent IV-MPRs, Islam et al. [100] set up a Lexico-graphic goal programming approach. A linear programming model is set up by Xu [101] to derive the priority vector of alternatives by combining interval multiplicative preference with interval attribute values. In IV-MPR scenario, Wang et al. [102] developed another methodology based on linear programming method to obtain consistent interval weights. In 2006, Yager and Xu [103] introduced weighted geometric operator IV-MPR and furthermore, they applied its decision making association. Liu [104] presented the idea of consistency for an IV-MPR and also he derived a formula that approaching to rank interval weights for decision making. Recently, Meng and Tan [105] noticed the limitations in the previously existing consistency concepts for IV-MPR and introduced a new consistency concept for IV-MPR with several desirable properties.

1.2.4 Intuitionistic Preference relation

In some genuine circumstances, a decision maker will most likely be unable to precisely communicate his/her preferences information data for choices because of that (i) the DM may not have an exact or adequate level of information of the problem; (ii) the DM can not segregate expressly how much one option are superior to others [83], in such a situation, the DM may give his/her preferences for options in contrast to a specific degree, yet it is possible that he/she is not so certain about it [106]. Along these lines, it is exceptionally reasonable to represent the preference values of the DM by using intuitionistic fuzzy values instead of correct numerical values ([16], [107], [108]). In such cases, motivated by the idea of an IFS ([20]), an expert prefers to express the imprecise cognition in the sense of positive (preference), negative (non-preference), and hesitant (indeterminacy) degrees leading to an intuitionistic preference relation (IPR). IPR are of two type i.e. IFPR and IMPR.

1.2.5 Intuitinistic fuzzy preference relation

The extension of FPR is IFPR proposed by Szmidt and Kacprzyk [16] and characterized the ideas of intuitionistic fuzzy core and consensus winner. In 2007 Xu [62] introduce the concept of IFPR. An IFPR $\tilde{P}$ on the set is given by a matrix $\tilde{P} = (\tilde{p}_{ij})_{n\times n} \subset X \times X$ with $\tilde{p}_{ij} = (\mu(x_i,x_j), \nu(x_i,x_j))$ for all $i, j \in N$ is called intuitionistic fuzzy number(IFN). For convenience, $\tilde{p}_{ij} = (\mu(x_i,x_j), \nu(x_i,x_j)) = (\mu_{ij}, \nu_{ij})$, where μ_{ij} represents the preference value to which x_i is preferred to x_j and the certainty degree ν_{ij} to which x_i is not preferred to x_j. Additionally, both the membership and non-membership degree should satisfy the condition i.e. $0 \leq \mu_{ij} + \nu_{ij} \leq 1$, $\mu_{ji} = \nu_{ij}$, $\nu_{ji} = \mu_{ij}$, $\mu_{ii} = \nu_{ii} = 0.5$ for all $i, j = 1, 2, \cdots, n$.

Lio and Xu [109] formulated a fractional programming problem to calculate priority weights of IFPR using multiplicative consistency. Xu et al. [110] introduced a new consistency index and used it to define an acceptable consistency for an IFPR. They followed a goal programming approach to improve upon the consensus among several experts and consistency in their IFPRs simultaneously which are not acceptably consistent.

1.2.6 Intuitionistic multiplicative preference relation

On the similar lines, an intuitionistic multiplicative number on an unbalanced scale of $(1/9)-9$ give rise to an IMPR $\tilde{R}$ proposed by Xia et al. [63]. Each entry $\tilde{r}_{ij}$ in the matrix $\tilde{R}$ representing IMPR is a two-tuple, the first component in which is used for depicting the certainty degree of preference of an alternative x_i over an alternative x_j, and the second component is used to indicate the certainty degree of not preferring x_i to x_j. The surplus, one minus the sum of the first and the second components is an uncertainty degree (or an indeterminacy degree or hesitancy degree) of an expert in distinguishing between two extremes. Mathematically, Xia et al. [63] represents the IMPR as:

Definition 1.2.3. [63] An IMPR is $\tilde{R}=[\tilde{r}_{ij}(x_i,x_j)]_{n\times n}$, where $\tilde{r}_{ij}(x_i,x_j)=(\mu(x_i,x_j),\nu(x_i,x_j))$, $i,j\in N$, is an intuitionistic multiplicative number (IMN), and $\mu(x_i,x_j)$ indicates certainty degree to which x_i is preferred to x_j and in case of $\nu(x_i,x_j)$, x_i is not preferred to x_j, and they satisfy the following characteristics:

$$
\begin{aligned}
&1/9\leq\mu(x_i,x_j), \quad \nu(x_i,x_j)\leq 9,\\
&\mu(x_i,x_j)=\nu(x_j,x_i), \quad \nu(x_i,x_j)=\mu(x_j,x_i),\\
&\mu(x_i,x_i)=\nu(x_i,x_i)=1,\ 0<\mu(x_i,x_j)\,\nu(x_i,x_j)\leq 1,\ \forall\, i,j\in N.
\end{aligned}
$$

If $\mu_{ij}(x_i,x_j).\nu_{ij}(x_i,x_j)=1,\ \forall\, i,j\in N$, then the IMPR $\tilde{R}$ is equivalent to a MPR. For convenience and without ambiguity, we shall be denoting $\tilde{r}_{ij}(x_i,x_j)=(\mu(x_i,x_j),\nu(x_i,x_j))$ by $\tilde{r}_{ij}=(\mu_{ij},\nu_{ij})$.

A much amount of research has been done in IMPR scenario. Xu [64] built up a strategy to determine the priority weights of alternatives from an IMPR. Jiang et al. [111] characterized compatibility measure for IMPR and created two models for GDM problems. Xia and Xu [65] were designed some intuitionistic multiplicative aggregation operators and its application in GDM problems. Recently, Zhang and Pedrycz [112] check the consistency of IMPR and generate weights by this relation. Ren [113] extend the intuitionistic multiplicative information into AHP to enhance the ability of AHP in tackling various decision-making problems and verified that the intuitionistic multiplicative weighted geometric aggregation (IMWGA) operator is of excellent characteristics in remaining the consistency of the IMPRs. To adjust the inconsistent IMPRs into an acceptably consistent one, they proposed an iterative process and also, they provide an adjustment process to restore and improves the consistency of inconsistent IMPR. In IMPRs framework, Zhang et al. [114] developed a methodology based on goal programming models to manage unity and consensus of IMPRs. Also, they further proposed the consistency and consensus-based techniques for solving GDM problems. Zhang and Guo [115] proposed two techniques based on complete and incomplete IMPRs in GDM issue. They further discussed the relationship between an IMPR and a normalized intuitionistic multiplicative weight vector. Zhang and Guo [115] also developed a new method primarily based on linear programming approach to verify and improve the consistency of an IMPR.

The more general extension of intuitionistic preference relation is called interval-valued intuitionistic preference relation (short form as IVI-PR). The IVI-PR are two type such as interval-valued intuitionistic fuzzy preference relation (IVI-FPR) and interval-valued intuitionistic multiplicative preference relation (IVI-MPR).

1.2.7 Interval-valued intuitionistic fuzzy preference relation

Like IFPR, IVI-FPR have been broadly connected to the decision making problem. Xu and Chen [116] first defined the IVI-FPR. An IVI-FPR $\tilde{P}_{IVI}$ on the set X is denoted by an interval-valued intuitionistic fuzzy judgement matrix $\tilde{P}_{IVI} = [\tilde{p}_{ij}]_{n\times n}$, where $\tilde{p}_{ij} = (\tilde{\mu}_{ij}, \tilde{v}_{ij})$ is an IVIFV. $\tilde{\mu}_{ij} = [\underline{\mu}_{ij}, \bar{\mu}_{ij}]$ represents the degree to which is an alternative x_i is preferred to the alternative x_j, and $\tilde{v}_{ij} = [\underline{v}_{ij}, \bar{v}_{ij}]$ represents the degree to which is an alternative x_i is non-preferred to the alternative x_j satisfy the conditions: $\tilde{\mu}_{ij} = [\underline{\mu}_{ij}, \bar{\mu}_{ij}] \subset [0,1]$, $\tilde{v}_{ij} = [\underline{v}_{ij}, \bar{v}_{ij}] \subset [0,1]$, $\tilde{\mu}_{ii} = \tilde{v}_{ii} = [0.5, 0.5]$, $0 \leq \bar{\mu}_{ij} + \bar{v}_{ij} \leq 1$, $\tilde{\mu}_{ij} = \tilde{v}_{ji}$, $\tilde{v}_{ij} = \tilde{\mu}_{ji}$ for all $i, j = 1, 2, \cdots, n$.

We know that, in the decision making framework, consistency of the preference relation is a basic issue. The additive consistent IFPR is characterized by Wang [117] using both the membership and non-membership degrees of intuitionistic fuzzy numbers. Gong et al. [118] initiated the characterization of multiplicative consistent IFPR by proposing a change between an IFPR and IV-FPR. In 2011, Xu et al. [119] elucidated multiplicative consistent IFPR, analogous to the additive consistent IFPR [117]. Using some operational laws of interval-valued intuitionistic fuzzy numbers (IVIFNs), Xu and Chen [116] proposed a consistent interval-valued intuitionistic judgment matrix.

Recently, Wan et al. [120] defined consistency and acceptable consistency of an IVI-FPR by separating it into two IFPRs. A new three-phase method is proposed for GDM problems with IVI-FPRs by eliciting the full importance degrees of experts, extracting the most optimistic and pessimistic consistent IFPRs from an IVI-FPR, and further minimizing the deviations and the hesitancy degrees via linear goal programming model. Wan et al. [121] designed a novel likelihood comparison algorithm to rank interval-valued intuitionistic fuzzy values. They also defined the additive consistency of an IVI-FPR according to the additive consistency of IFPR. They proposed a linear programming model by maximizing the group consensus to derive the weights of experts. Subsequently, a collective IVI-FPR is applied to calculate the priority weights. Using the likelihood comparison algorithm, the order of alternatives is generated by ranking their priority weights. Before these papers, Wan et al. [122] put forward a new iterative algorithm to repair the consistency of an IVI-FPR with unacceptable consistency. They established an optimization model by minimizing the deviations between each IVI-FPR and a collective one. A TOPSIS based approach is designed to rank interval-valued Atanassov intuitionistic fuzzy (IVAIF) priority weights in multi-criteria group decision making (MCGDM) problems.

1.2.8 Hesitant fuzzy preference relation

One of the most important extensions of a FS is a hesitant fuzzy set(HFS) proposed by Torra [123] whose participation capacities are spoken to by an arrangement of a few conceivable numerical qualities. The inspiration for presenting HFSs is that it is very hard to decide the membership of an element into a set because there is a set of possible values in this situation. In day to day life, the hesitant fuzzy set is another important tool that expresses the human hesitancy. For representation if number of decision-makers are reluctant about some conceivable incentive as 0.2, 0.5, and 0.7 to judge the participation of a component x to a set A, all things considered, the membership of x to the set A can be spoken by a hesitant fuzzy element (HFE) denoted by 'h_e'$= \{0.2, 0.5, 0.7\}$ which is altogether different from FSs, IVFSs, IFSs, IVIFSs. In the area of HFSs, majorly work has been done on the aggregation of HFEs [124], distance and similarity measures for HFSs ([125], [126]) and some decision making methods based on HFSs [127]. The relationship between the different type of fuzzy sets with HFSs can be found in [128].

Due to absence of learning and shortage of time, the DMs regularly gave their preferences data with a few numerical possible values. Unmistakably, the previously mentioned preference relations do not address such circumstances. Later research on hesitant preference relations has additionally gotten. For example, Zhu and Xu [75] proposed the idea of a HFPR. Each component of HFPR is an HFE, which is an arrangement of functional numerical preference value that denotes the hesitant degree to which an alternative is preferred over another option. HFPRs gave an exact description of the decision maker's hesitation in giving their preference.

Likewise, Zhu and Xu [75] developed a regression method to change HFPRs to FPRs with the highest consistency level. Also, Liao et al. [129] explored the multiplicative consistency of an HFPR and created a few calculations to enhance the gathering accord.

1.2.9 Hesitant Multiplicative Preference relation

As the HFS and the HFPRs are created in light of the $0-1$ scale and symmetrically passed on around 0.5. In various genuine problem, it may be possible that, the consideration of preference data around some value may not dependable and distributed symmetrically. The best example in relation of this concept is while getting the ventures of similar resources, the separation between the evaluations communicating bad information in an organization ought to be not exactly the gap between the degree expressing good information [63]. To address such a condition, Saaty's $1-9$ scale [58] is considered as a beneficial tool which demonstrates the MPR that has been applied in various disciplines [130].

To express the preferences concerning two options, Xia and Xu [130] proposed a methodology to use the unbalanced $1-9$ ratio scale and there by presented the idea of HMPR. Some of the experts in the decision organization in the above illustrations, do not prefer to use the preference value lies between 0 and 1, however, would want to utilize Saaty's proportion scale to give how much the option x_i is better than x_j $(i \neq j)$. In the decision-making association, some experts prefer to give their information as 1/3, some give as 2, and others give as 5. These experts can not able to reach a consensus. In the above cases, according to Xia and Xu [130], hesitant multiplicative element (HME) $\{1/3, 2, 5\}$ as mentioned, is considered when the degrees to which the choice x_i is superior to the choice x_j for $i \neq j$. And furthermore, the HME is the basic unit of the hesitant multiplicative set(HMS) [130], thus, HMPR can be developed. Many research has been done in this scenario ([76] and many more).

1.3 Incomplete preference relations

The above-cited works assume that an expert is always able to provide information on each entry in a preference relation matrix. It may not be the case forever. Here and there, DM might not have a decent comprehension of a specific query, thus he/she can not make an instaneous contrast between each two objects; in this manner it is once in a while important to enable the decision-maker to avoid some questionable comparisons flexibly. In this case, incomplete preference relations are obtained, and the whole process may slow down. At each level of the decision-making process, decision maker need $\frac{n(n-1)}{2}$ judgments to present a complete preference relation, and when n is large, it becomes an onerous task. Therefore sometimes, due to lack of time and busy schedule of the decision maker, incomplete preference relations are obtained. In this section, we briefly present the literature on incomplete FPR and its various variants.

Xu [131] first studied an incomplete FPR. Fedrizzi and Giove [132] characterized a methodology to complete the incomplete FPR. Alonso et al. [133] defined four type of incomplete preference relations: incomplete FPR, incomplete MPR, incomplete IV-PR, incomplete linguistic preference relations. A web consensus support system is developed by Alonso et al. [134] to manage MCGDM problems with various types of incomplete preference relations. Chen et al. [135] computed the missing preference values in an incomplete FPR based on additive consistency. They then constructed the modified consistency matrices which satisfy the additive consistency and order consistency simultaneously. Ure*ñ*a et al. [56] approaches a method to manage incomplete preference relations that estimate the missing information in decision making are desirable. To get a priority vector for GDM problems, Xu et al. [136] proposes a chi-square method (CSM) where DMs' assessment on alternatives is furnished with missing values in an incomplete reciprocal preference relations .

Khalid et al. [137] proposed an upper bound condition to handle an incomplete IV-FPR and utilizing this condition, missing preferences are estimated such that they are expressible which results from the consistent, complete relation. Meng, Tan, and Chen [98] formulated goal-programming models to determine missing values in an incomplete IV-FPR that have the highest consistent level concerning known values.

On the other hand, in the area of incomplete MPR, Harker [96] devised different methods for dropping the complex elicited procedure with incomplete MPR. Nishizawa [138] developed a method to find the various lengths in a directed graph for appraising the elements in an intransitive incomplete MPR.

The process of checking consistency makes sure that the preferences of experts have no self-contradiction. The notion of consistency is the most important tool used to calculate the unknown values in an IPR(Liao et al. [139]). Xu and Cai [140] introduced additive consistent, multiplicative consistent, and acceptably consistent incomplete preference relations. Xu [141] extended an incomplete MPR into a complete MPR using multiplicative transitivity property and utilized a simple iterative procedure to improve consistency. Liu et al. [142] initiated the notion of an incomplete IV-MPR and presented an algorithm to obtain a priority vector from the consistency property for the same. Meng and Tan [105] defined a new notion of consistency for IV-MPR and applied it to formulate a $0-1$ mixed goal programming model to determine the missing values in an incomplete IV-MPR.

Jiang et al. [143] proposed a notion of incomplete IMPR and considered its application to MCGDM problems. They also talked about consistent and acceptably consistent incomplete IMPRs. Two approaches i.e. 'estimating step' and 'adjusting step' were proposed to find all the unknown elements of an incomplete IMPR.

Zhang et al. [144] work on incomplete HFPRs and they proposed an algorithm to estimate the missing data by utilizing the known preference element with the lowest number of judgments and further extended it to more known judgments in an incomplete HFPR.

1.4 Motivation

In reality, multi-criteria decision-making problem has been extensively seen that most choices happen in a situation in which the objectives and requirements, are not known precisely, because of their complexity, and henceforth, the problem cannot be accurately described using the crisp value. To manage the sort of subjective, unverifiable information or on the other hand even not practical decision issues, the uncertainty theory called fuzzy set theory came to existence. The present chapter is introductory which represents the details of fuzzy set theory and extension of this theory to the fuzzy relation. The concept of preference relation has been recently developed to allow the DMs to provide different preference values over two alternatives.

Chapter 2 characterized the $max-min$ similarity relation of IVIFR and further the applicability of this relation construct hierarchical clustering in the area of performance evaluation. To express the application of the proposed hierarchical clustering, we have developed a procedure to calculate the local weights and global weights from the criteria-alternative matrix in MCDM problem.

In chapter 3, we have developed a consistency property of IMPRs. Using this consistency property, we have extended the work in incomplete IMPR scenario. Also, in this chapter, we have checked the consistency of IMPR using the cyclic graph.

To extend the IMPR, we have developed the IVI-MPR in chapter 4. First of all, we study the consistency property, and especially the acceptably consistent property, for incomplete IVI-MPR. To complement the incomplete IVI-MPR, we have developed a local optimization method. A method is designed to estimate the weight vector of DMs to achieve the resultant consistent decision matrix in GDM problem.

In chapter 5, we work on HFPR. This chapter introduced an additive consistency property of HFPR to construct the complete HFPR from incomplete one. We develop a method to check the consistency level of incomplete HFPR. Group decision-making problem with incomplete HFPR is also considered.

The aim of chapter 6 is to characterise two different approaches to calculate the missing element of incomplete hesitant multiplicative preference relations (HMPRs). In the first approach, a new definition of multiplicative transitive property of HMPR has given that preserve the hesitancy property and is used to construct the complete HMPR from incomplete one which involves two-step involving "estimating step" and "adjusting step". Initial values of missing element are calculated using estimating step and an optimization model is developed in the adjusting step to minimize the error. A linear programming model is developed in the second approach to complement HMPR from incomplete one. The satisfaction degree and the acceptably consistent of complete HMPR is also checked. The whole procedure is explained with suitable example.

The summary and the future scope is given after chapter 6.

Chapter 2

Hierarchical Clustering of Interval-Valued Intuitionistic Fuzzy Relations and Its Application to Elicit Criteria Weights in MCDM Problems

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In this chapter[1]*, we apply the* $(\widetilde{\alpha}, \widetilde{\beta})$*-cuts and the resolution form of the interval-valued intuitionistic fuzzy (IVIF) relations to develop a procedure for constructing a hierarchical clustering for IVIF max-min similarity relations. The advantage of the proposed scheme is illustrated in determining the criteria weights in multi-criteria decision making (MCDM) problems involving IVIF numbers. The problem of finding the criteria weights is of critical interest in the domain of MCDM problems. A complete procedure is drawn to generate criteria weights starting from the criteria-alternative matrix of the MCDM problem with entries provided by a decision maker as IVIF numbers.*

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[1]The content of this chapter is based on research paper "Hierarchical clustering of interval-valued intuitionistic fuzzy relations and its application to elicit criteria weights in MCDM problems", *OPSEARCH*, springer **54**, 388–416 (2017).

2.1 Introduction

The concepts of fuzzy set, IVFS, IFS, IVIFS has been broadly discussed in the first chapter.

Using fuzzy set theory, Bellmann et al. [40] and Ruspini [41] gave the ideas of data clustering. Fuzzy clustering is the partitioning of a collection of data into fuzzy subsets or clusters based on similarities between the data in some way. Fuzzy clustering methods are divided into two categories. One of them is fuzzy c-means (FCM) algorithm which is based on distance defined objective function. The other categories of fuzzy clustering based on fuzzy relation which we have discussed in chapter 1. Zadeh [47] defined the similarity relation of fuzzy relation. Hierarchical clustering is applied based on similarity relation.

Guh et al. [52] extended a fuzzy relation (FR) to *interval-valued fuzzy relation* (IVFR) and defined an interval-valued degree of similarity between any two elements $x \in X$ and $y \in Y$ as $[\underline{\mu}_{R_f}(x,y), \overline{\mu}_{R_f}(x,y)]$, with $0 \leq \underline{\mu}_{R_f}(x,y) \leq \overline{\mu}_{R_f}(x,y) \leq 1$.

Mathematically, Let X and Y be two non-empty sets, and $Int([0,1]) = \{(x_1,x_2) \mid x_1,x_2 \in [0,1], x_1 \leq x_2\}$.

Definition 2.1.1. [52] (Interval valued fuzzy relation (IVFR)) An IVFR $\widetilde{R}_f$ on X and Y is a mapping $\widetilde{R}_f : X \times Y \to Int([0,1]$, defined as a fuzzy subset of $X \times Y$, with an interval-valued membership grade $\tilde{\mu}_{\widetilde{R}_f}(x,y) = [\underline{\mu}_{\widetilde{R}_f}(x,y),\ \overline{\mu}_{\widetilde{R}_f}(x,y)],\ \forall\ (x,y) \in X \times Y$.

Definition 2.1.2. [52] (Interval-valued fuzzy proximity relation) An IVFR $\widetilde{R}_f$ on X which is reflexive (that is, $\tilde{\mu}_{\widetilde{R}_f}(x,x) = [1,1]$, $\forall\, x \in X$) and symmetric (that is, $\tilde{\mu}_{\widetilde{R}_f}(x,y) = \tilde{\mu}_{\widetilde{R}_f}(y,x)$, $\forall\, x,y \in X$) is called an interval-valued fuzzy proximity relation.

Definition 2.1.3. [52] (Transitive IVFR) An IVFR $\widetilde{R}_f$ on X is called (max-min) transitive if for all $x,y \in X$, the following hold:

$$\underline{\mu}_{\widetilde{R}_f}(x,y) \geq \max_{z \in X}\{\min\{\underline{\mu}_{\widetilde{R}_f}(x,z), \underline{\mu}_{\widetilde{R}_f}(z,y)\}\} \quad \text{and}$$
$$\overline{\mu}_{\widetilde{R}_f}(x,y) \geq \max_{z \in X}\{\min\{\overline{\mu}_{\widetilde{R}_f}(x,z), \overline{\mu}_{\widetilde{R}_f}(z,y)\}\}.$$

Definition 2.1.4. [52] (Interval-valued fuzzy (max-min) similarity relation) An interval-valued fuzzy proximity relation $\widetilde{R}_f$ on X with transitivity property (in the sense of Definition 2.1.3) is called an interval-valued fuzzy (max-min) similarity relation.

Guh [52] developed some results for IVFR and applied them for performance evaluation by establishing a hierarchical clustering. According to Guh [52], the $\tilde{\alpha}$-cut in the resolution form of the interval-valued similarity relation, is interval-valued with $\widetilde{\alpha} = (\underline{\alpha}, \overline{\alpha})$, $0 \leq \underline{\alpha} \leq \overline{\alpha} \leq 1$. In case of m number of $\widetilde{\alpha}$-cuts for interval-valued similarity relation, say $(\underline{\alpha}_1, \overline{\alpha}_1), \ldots, (\underline{\alpha}_m, \overline{\alpha}_m)$, the following conditions must be satisfied:

$$0 \leq \underline{\alpha}_1 \leq \ldots \leq \underline{\alpha}_m \leq 1, \quad 0 \leq \overline{\alpha}_1 \leq \ldots \leq \overline{\alpha}_m \leq 1.$$

But the same may fails to hold; for instance, if the $\tilde{\alpha}$-cuts are of the form $(0.4, 0.9)$, $(0.5, 0.6)$, $(1, 1)$ then, although $0.4 < 0.5 < 1$ but $0.9 \nleq 0.6 < 1$. On the other hand, Basnet [145] defined the (α, β)-cut for IFS A_I, for $0 \leq \alpha, \beta \leq 1, \ \alpha + \beta \leq 1$, by $A_{I(\alpha,\beta)} = \{x \mid \mu_{A_I}(x) \geq \alpha \text{ and } \nu_{A_I}(x) \leq \beta\}$.

In this chapter, we extend the work of Guh et al. [52] to IVIF scenario. We introduce the definition of $(\widetilde{\alpha}, \widetilde{\beta})$-cuts for an *interval-valued intuitionistic fuzzy relation* (IVIFR) and present decomposed resolution form of IVIFR. The same have been applied to create a hierarchical structure for IVIFR. An MCDM with IVIF numbers is considered, and an entropy measure [146] together with hierarchical clustering is applied to compute the global weights of the criteria involved in the problem. MATLAB codes are developed for all the proposed algorithms and methodologies. All codes are tested on a large number of matrices set-up in the domain backdrops of interval-valued fuzzy (IVF) numbers, intuitionistic fuzzy (IF) numbers, and interval-valued intuitionistic fuzzy (IVIF) numbers.

The now onwards flow of the chapter is as follows. In Section 2.2, we modify the definition of $\widetilde{\alpha}$-cut for *interval-valued fuzzy relation* (IVFR) described in [52]. In Sections 2.3, the hierarchical clustering procedure is developed to include *intuitionistic fuzzy relation* (IFR). Section 2.4 presents some operations on IVIFR along with its resolution form. Section 2.5 describes the hierarchical clustering procedure for IVIFR. Section 2.6 proposed a procedure to compute the global weights of the criteria in an MCDM problem set-up in IVIF environment. Section 2.7 summarizes the work of this chapter.

2.2 $\widetilde{\alpha}$-cut for IVFR

In this section, we revised the definition of $\widetilde{\alpha}$-cut for *interval-valued fuzzy relation* (IVFR) described in [52]. Guh et al. [52] showed that an interval-valued similarity relation $\widetilde{R}_f$ can be decomposed into its resolution form using $\widetilde{\alpha}$-cuts, as follows:

$$\widetilde{R}_f = \bigcup_{\widetilde{\alpha}} (\underline{\mu}_{\alpha}, \overline{\mu}_{\alpha})\, R_{\widetilde{\alpha}} = (\underline{\mu}_{\alpha_1}, \overline{\mu}_{\alpha_1}) R_{\widetilde{\alpha}_1} + (\underline{\mu}_{\alpha_2}, \overline{\mu}_{\alpha_2})\, R_{\widetilde{\alpha}_2} + \ldots + (\underline{\mu}_{\alpha_m}, \overline{\mu}_{\alpha_m})\, R_{\widetilde{\alpha}_m},$$

such that

$$0 \le \underline{\mu}_{\alpha_1} \le \underline{\mu}_{\alpha_2} \le \ldots \le \underline{\mu}_{\alpha_m} \le 1, \quad 0 \le \overline{\mu}_{\alpha_1} \le \overline{\mu}_{\alpha_2} \le \ldots \le \overline{\mu}_{\alpha_m} \le 1, \tag{2.2.1}$$

where $R_{\widetilde{\alpha}_i}$, $i = 1, \ldots, m$, are the similarity relations, and $(\underline{\mu}_{\alpha_i}, \overline{\mu}_{\alpha_i})$, $i = 1, \ldots, m$, are the corresponding $\widetilde{\alpha}$-cut. As discussed above, the following examples further justify our observation that equation 2.2.1 may not always hold.

Example 2.2.1. Consider the following interval-valued proximity relation matrix which is not (min-max) transitive.

$$\widetilde{R}_f^{(0)} = \begin{pmatrix} 1 & (0.1,0.4) & (0.2,0.3) & (0.6,0.9) \\ (0.1,0.4) & 1 & (0.7,0.8) & (0.4,0.7) \\ (0.2,0.3) & (0.7,0.8) & 1 & (0.3,0.5) \\ (0.6,0.9) & (0.4,0.7) & (0.3,0.5) & 1 \end{pmatrix}.$$

Using the conventional procedure, we first convert the above relation into a max-min transitive one. To accomplish this, we take $\tilde{R}_f^{(1)} = \tilde{R}_f^{(0)} \circ \tilde{R}_f^{(0)}$, where $\circ$ stands for the max-min composition, we get

$$\widetilde{R}_f^{(1)} = \begin{pmatrix} 1 & (0.4,0.7) & (0.3,0.5) & (0.6,0.9) \\ (0.4,0.7) & 1 & (0.7,0.8) & (0.4,0.7) \\ (0.3,0.5) & (0.7,0.8) & 1 & (0.4,0.7) \\ (0.6,0.9) & (0.4,0.7) & (0.4,0.7) & 1 \end{pmatrix}.$$

Note that $\tilde{R}_f^{(1)} \neq \tilde{R}_f^{(0)}$. Again obtain $\tilde{R}_f^{(2)} = \tilde{R}_f^{(1)} \circ \tilde{R}_f^{(1)}$. Then,

$$\widetilde{R}_f^{(2)} = \begin{pmatrix} 1 & (0.4,0.7) & (0.4,0.7) & (0.6,0.9) \\ (0.4,0.7) & 1 & (0.7,0.8) & (0.4,0.7) \\ (0.4,0.7) & (0.7,0.8) & 1 & (0.4,0.7) \\ (0.6,0.9) & (0.4,0.7) & (0.4,0.7) & 1 \end{pmatrix}.$$

Since $\widetilde{R}_f^{(2)} \neq \widetilde{R}_f^{(1)}$, we repeat the above procedure to get next relation from $\tilde{R}_f^{(2)}$ as follows:

$$\widetilde{R}_f^{(3)} = \begin{pmatrix} 1 & (0.4,0.7) & (0.4,0.7) & (0.6,0.9) \\ (0.4,0.7) & 1 & (0.7,0.8) & (0.4,0.7) \\ (0.4,0.7) & (0.7,0.8) & 1 & (0.4,0.7) \\ (0.6,0.9) & (0.4,0.7) & (0.4,0.7) & 1 \end{pmatrix}.$$

Now, $\widetilde{R}_f^{(3)} = \widetilde{R}_f^{(2)}$, hence $\widetilde{R}_f^{(2)}$ is an equivalence IVFR.

The $\widetilde{\alpha}$-cuts in $\tilde{R}_f^{(2)}$ are $(0.4,0.7)$, $(0.6,0.9)$, $(0.7,0.8)$, $(1,1)$ with $0.4 < 0.6 < 0.7 < 1$ but $0.7 < 0.9 \nleq 0.8 < 1$, i. e. condition given by equation 2.2.1 does not hold.

Example 2.2.2. Consider the interval-valued fuzzy max-min similarity relation $\widetilde{R}_f^{(0)}$

$$\widetilde{R}_f^{(0)} = \begin{pmatrix} 1 & (0.9,1) & (0.6,0.9) & (0.4,0.9) & (0.4,0.8) \\ (0.9,1) & 1 & (0.6,0.9) & (0.4,0.9) & (0.4,0.8) \\ (0.6,0.9) & (0.6,0.9) & 1 & (0.4,0.9) & (0.4,0.8) \\ (0.4,0.9) & (0.4,0.9) & (0.4,0.9) & 1 & (0.5,0.8) \\ (0.4,0.8) & (0.4,0.8) & (0.4,0.8) & (0.5,0.8) & 1 \end{pmatrix}.$$

It can easily be verified that $\tilde{R}_f^{(0)}$ is reflexive, symmetric and transitive IVFR. The $\widetilde{\alpha}$-cuts are $(0.4,0.8)$, $(0.4,0.9)$, $(0.5,0.8)$, $(0.6,0.9)$, $(0.9,1)$, $(1,1)$. Again the noteworthy point is that although the lower values of cuts satisfy the inequality in (2.2.1), the upper values of cut fail to meet (2.2.1) as $0.8 < 0.9 \nleq 0.8 < 0.9 < 1$.

In this chapter, we attempt to improve this shortcoming and develop a new definition of $\widetilde{\alpha}$-cut.

Definition 2.2.1. The $\widetilde{\alpha}$-cut of an IVFR $\tilde{R}_f$ is defined by $(\max(\underline{\mu}_i),\ \min(\overline{\mu}_i))$, among all $\tilde{\alpha}$-cut $(\underline{\mu}_\alpha, \overline{\mu}_\alpha)$ of the interval-valued similarity relation derived following the approach of Guh et al. [52].

For clarity of the above definition, we propose an algorithm for determining the $\widetilde{\alpha}$-cuts of IVFR.

Algorithm 2.2.1. 1. First determine all $\widetilde{\alpha}$-cuts $(\underline{\mu}_i, \overline{\mu}_i)$ of IVFR by the conventional approach of Guh et al. [52]. Let this collection be denoted by set S.

2. Among all the elements of the set S, choose the one with maximum value of the lower membership and fixed it. Then among all available pairs in S with this lower

membership value, choose the one having minimum upper membership value. This way, we extract the first element in $\widetilde{\alpha}$-cut of IVFR. Note that the first $\widetilde{\alpha}$-cut is a member in S. Reduce the set S by deleting this element from it.

3. Continue the above procedure among all the remaining elements of the reduced S, till finally all elements are searched.

This way we are able to extract $\widetilde{\alpha}$-cut of IVFR from the set S.

Recall the $\widetilde{\alpha}$-cuts of Example 2.2.2. Then, using Definition 2.2.1, the extracted $\widetilde{\alpha}$-cuts of IVFR are given $(1,1)$, $(0.9,1)$, $(0.6,0.9)$, $(0.5,0.8), (0.4,0.8)$. Moreover, the resolution form of the similarity relation is given by

$$
\begin{aligned}
\widetilde{R}_f^{(1)} = 1 &\begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + (0.9,1) \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + (0.6,0.9) \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&+ (0.5,0.8) \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} + (0.4,0.8) \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.
\end{aligned}
$$

A MATLAB code is developed to construct the $\widetilde{\alpha}$-cuts of the IVFR of order m. The code is tested on randomly generated IVFR matrices. One of the illustration is shown below.

Example 2.2.3. Let $\widetilde{B}_f^{(0)}$ be a randomly generated interval-valued fuzzy proximity relation matrix of order 8.

$$
\widetilde{B}_f^{(0)} = \begin{pmatrix}
(1,1) & \binom{0.814724,}{0.936168} & \binom{0.905792,}{0.921919} & \binom{0.126987,}{0.743374} & \binom{0.913376,}{0.916133} & \binom{0.632359,}{0.734167} & \binom{0.09754,}{0.139208} & \binom{0.278498,}{0.348579} \\
\binom{0.814724,}{0.936168} & (1,1) & \binom{0.546882,}{0.920005} & \binom{0.957507,}{0.987032} & \binom{0.964889,}{0.976022} & \binom{0.157613,}{0.958068} & \binom{0.970593,}{0.971606} & \binom{0.957167,}{0.97596} \\
\binom{0.905792,}{0.921919} & \binom{0.546882,}{0.920005} & (1,1) & \binom{0.485376,}{0.681735} & \binom{0.80028,}{0.953169} & \binom{0.141886,}{0.824258} & \binom{0.421761,}{0.529818} & \binom{0.915736,}{0.957005} \\
\binom{0.126987,}{0.743374} & \binom{0.957507,}{0.987032} & \binom{0.485376,}{0.681735} & (1,1) & \binom{0.792207,}{0.884797} & \binom{0.959492,}{0.985673} & \binom{0.655741,}{0.899946} & \binom{0.035712,}{0.763447} \\
\binom{0.913376,}{0.916133} & \binom{0.964889,}{0.976022} & \binom{0.80028,}{0.953169} & \binom{0.792207,}{0.884797} & (1,1) & \binom{0.849129,}{0.890773} & \binom{0.933993,}{0.978858} & \binom{0.678735,}{0.889195} \\
\binom{0.632359,}{0.734167} & \binom{0.157613,}{0.958068} & \binom{0.141886,}{0.824258} & \binom{0.959492,}{0.985673} & \binom{0.849129,}{0.890773} & (1,1) & \binom{0.75774,}{0.797134} & \binom{0.743132,}{0.773699} \\
\binom{0.09754,}{0.139208} & \binom{0.970593,}{0.971606} & \binom{0.421761,}{0.529818} & \binom{0.655741,}{0.899946} & \binom{0.933993,}{0.978858} & \binom{0.75774,}{0.797134} & (1,1) & \binom{0.392227,}{0.695119} \\
\binom{0.278498,}{0.348579} & \binom{0.957167,}{0.97596} & \binom{0.915736,}{0.957005} & \binom{0.035712,}{0.763447} & \binom{0.678735,}{0.889195} & \binom{0.743132,}{0.773699} & \binom{0.392227,}{0.695119} & (1,1)
\end{pmatrix}.
$$

Using the max-min composition, an interval-valued max-min similarity relation is constructed and the same is given as follows:

$$\widetilde{B}_f^{(3)} = \begin{pmatrix} (1,1) & \binom{0.9134,}{0.9362} & \binom{0.9134,}{0.9362} & \binom{0.9134,}{0.9362} & \binom{0.9134,}{0.9362} & \binom{0.9134,}{0.9362} & \binom{0.9134,}{0.9362} & \binom{0.9134,}{0.9362} \\ \binom{0.9134,}{0.9362} & (1,1) & \binom{0.9157,}{0.9570} & \binom{0.9575,}{0.9870} & \binom{0.9649,}{0.9760} & \binom{0.9575,}{0.9857} & \binom{0.9706,}{0.9760} & \binom{0.9572,}{0.9760} \\ \binom{0.9134,}{0.9362} & \binom{0.9157,}{0.9570} & (1,1) & \binom{0.9157,}{0.9570} & \binom{0.9157,}{0.9570} & \binom{0.9157,}{0.9570} & \binom{0.9157,}{0.9570} & \binom{0.9157,}{0.9570} \\ \binom{0.9134,}{0.9362} & \binom{0.9575,}{0.9870} & \binom{0.9157,}{0.9570} & (1,1) & \binom{0.9575,}{0.9760} & \binom{0.9595,}{0.9857} & \binom{0.9575,}{0.9760} & \binom{0.9572,}{0.9760} \\ \binom{0.9134,}{0.9362} & \binom{0.9649,}{0.9760} & \binom{0.9157,}{0.9570} & \binom{0.9575,}{0.9760} & (1,1) & \binom{0.9575,}{0.9760} & \binom{0.9649,}{0.9789} & \binom{0.9572,}{0.9760} \\ \binom{0.9134,}{0.9362} & \binom{0.9575,}{0.9857} & \binom{0.9157,}{0.9570} & \binom{0.9595,}{0.9857} & \binom{0.9575,}{0.9760} & (1,1) & \binom{0.9575,}{0.9760} & \binom{0.9572,}{0.9760} \\ \binom{0.9134,}{0.9362} & \binom{0.9706,}{0.9760} & \binom{0.9157,}{0.9570} & \binom{0.9575,}{0.9760} & \binom{0.9649,}{0.9789} & \binom{0.9575,}{0.9760} & (1,1) & \binom{0.9572,}{0.9760} \\ \binom{0.9134,}{0.9362} & \binom{0.9572,}{0.9760} & \binom{0.9157,}{0.9570} & \binom{0.9572,}{0.9760} & \binom{0.9572,}{0.9760} & \binom{0.9572,}{0.9760} & \binom{0.9572,}{0.9760} & (1,1) \end{pmatrix}.$$

Using Definition 2.2.1, the $\widetilde{\alpha}$-cuts of $\widetilde{B}_f^{(3)}$ are

$$(0.9134, 0.9362),\ (0.9157, 0.9570),\ (0.9572, 0.9760),\ (0.9575, 0.9760),$$
$$(0.9595, 0.9857),\ (0.9649, 0.9760),\ (0.9706, 0.9760),\ (1,1).$$

And the resolution form of $\widetilde{B}_f^{(3)}$

$$= (0.9134, 0.9362)\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} + (0.9157, 0.9570)\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$+ (0.9572, 0.9760)\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} + (0.9575, 0.9760)\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+ (0.9595, 0.9857)\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + (0.9649, 0.9760)\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+(0.9706, 0.9760)\begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + (1,1)\begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consequently, the hierarchical structure of the IVFR is depicted in Fig. 2.1

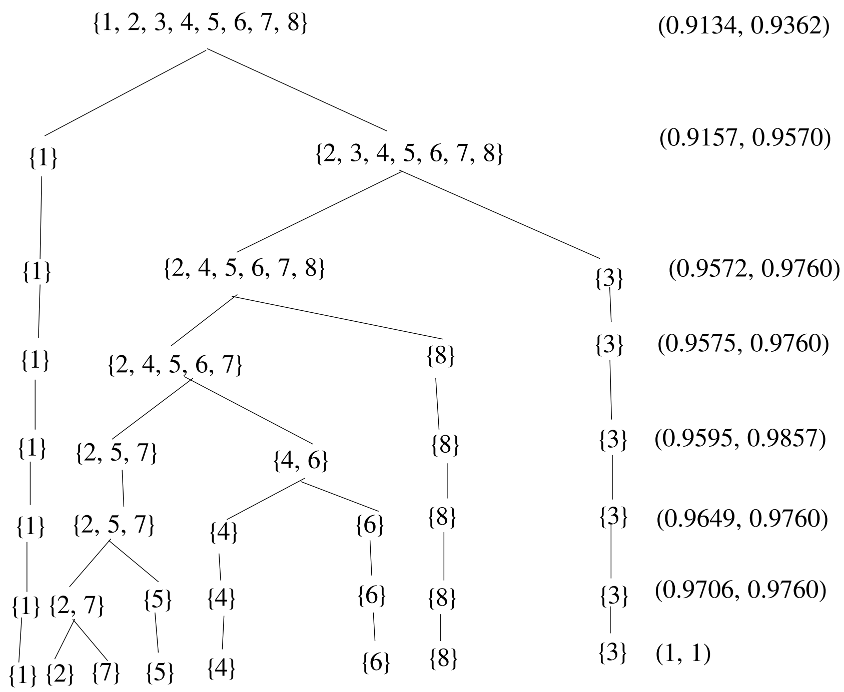

Figure 2.1: Hierarchical clustering for IVFR in Example 2.2.3

2.3 (α, β)-cut for intuitionistic fuzzy relation (IFR)

Intuitionistic Fuzzy Relations (IFRs) has already been studied by many researchers. Commonly, IFRs are intuitionistic fuzzy sets in a cartesian product of universe [53]. In this section we developed an algorithm to construct the (α, β)-cuts for an intuitionistic fuzzy relation (IFR). Taking motivation from IFS, Bustince and Burillo [53] defined IFR as follows.

Definition 2.3.1. (Intuitionistic fuzzy relation (IFR)) [53] An IFR R_I on the two sets X and Y is an intuitionistic fuzzy subset of $X \times Y$ defined by $R_I = \{\langle (x,y), \mu_{R_I}(x,y), \nu_{R_I}(x,y)\rangle \mid x \in X,\ y \in Y\}$, where $\mu_{R_I},\ \nu_{R_I} : X \times Y \to [0,1]$ such that $0 \le \mu_{R_I}(x,y) + \nu_{R_I}(x,y) \le 1,\ \forall (x,y) \in X \times Y$.

Banerjee and Basnet [147] presented some properties of IFR.

Definition 2.3.2 (Intuitionistic fuzzy proximity relation). An IFR R_I on X is called intuitionistic fuzzy proximity relation if it satisfies the following properties:

1. Reflexive: $\mu_{R_I}(x,x) = 1,\ \nu_{R_I}(x,x) = 0, \forall\, x \in X$.

2. Symmetry: $\mu_{R_I}(x,y) = \mu_{R_I}(y,x),\ \nu_{R_I}(x,y) = \nu_{R_I}(y,x),\ \forall\, x, y \in X$.

Definition 2.3.3 (Intuitionistic fuzzy (max-min) similarity relation). An intuitionistic fuzzy proximity relation R_I on X is called an intuitionistic fuzzy (max-min) similarity relation if it satisfy the following transitive relation:

$$\begin{aligned}\mu_{R_I}(x,y) &\geq \max_{z\in X}\{\min\{\mu_{R_I}(x,z), \mu_{R_I}(z,y)\}\},\\ \nu_{R_I}(x,y) &\leq \min_{z\in X}\{\max\{\nu_{R_I}(x,z), \nu_{R_I}(z,y)\}\}.\end{aligned}$$

Definition 2.3.4. The (α, β)-cut of an IFS A_I is defined in [145] as follows:

$$A_{I(\alpha,\beta)} = \{x \in X \mid \mu_{A_I}(x) \geq \alpha,\ \nu_{A_I}(x) \leq \beta\}.$$

We propose an algorithm for determining the (α, β)-cut for intuitionistic fuzzy (max-min) similarity relation.

Algorithm 2.3.1.

1. From all pair of (μ_i, ν_i) in the matrix representation of IFR, first choose the maximum value of the membership μ_i, say μ_i^*.

2. Then among all possible tuples (μ_i^*, ν_i), choose the one with maximum non- membership value. This gives the first (α, β)-cut of IFR.

3. Cross-off this selected cut value from matrix of IFR. And, continue this procedure to remaining elements (μ_i, ν_i) of the matrix of IFR.

Remark 2.3.1. Instead of starting with membership function, one can also start with a non-membership function. For this, among all entries of the matrix representation of R_I, choose the one with minimum non-membership value, and then for this value choose the tuple with minimum membership value also. Continue this procedure with all elements of the IFR till each of them is searched out.

A MATLAB code is designed for finding (α, β)-cuts of IFR R_I of order m. We present below one instance of many examples generated through the code.

Example 2.3.1. Consider a randomly generated IFR proximity relation matrix of order 9 as follows:

$$B_I^{(0)} = \left(\begin{array}{ccccccccc}
(1,0) & \binom{0.814724,}{0.128735} & \binom{0.905792,}{0.029873} & \binom{0.126987,}{0.829556} & \binom{0.913376,}{0.002984} & \binom{0.632359,}{0.1613} & \binom{0.09754,}{0.344341} & \binom{0.278498,}{0.552322} & \binom{0.546882,}{0.36032} \\
\binom{0.814724,}{0.128735} & (1,0) & \binom{0.957507,}{0.007941} & \binom{0.964889,}{0.017196} & \binom{0.157613,}{0.375356} & \binom{0.970593,}{0.019006} & \binom{0.957167,}{0.030384} & \binom{0.485376,}{0.38838} & \binom{0.80028,}{0.055128} \\
\binom{0.905792,}{0.029873} & \binom{0.957507,}{0.007941} & (1,0) & \binom{0.141886,}{0.583262} & \binom{0.421761,}{0.378803} & \binom{0.915736,}{0.013702} & \binom{0.792207,}{0.024727} & \binom{0.959492,}{0.020188} & \binom{0.655741,}{0.330401} \\
\binom{0.126987,}{0.829556} & \binom{0.964889,}{0.017196} & \binom{0.141886,}{0.583262} & (1,0) & \binom{0.035712,}{0.32823} & \binom{0.849129,}{0.0883} & \binom{0.933993,}{0.014773} & \binom{0.678735,}{0.241356} & \binom{0.75774,}{0.061799} \\
\binom{0.913376,}{0.002984} & \binom{,0.157613,}{0.375356} & \binom{0.421761,}{0.378803} & \binom{0.035712,}{0.32823} & (1,0) & \binom{0.743132,}{0.129964} & \binom{0.392227,}{0.42488} & \binom{0.655478,}{0.306936} & \binom{0.171187,}{0..795074} \\
\binom{0.632359,}{0.1613} & \binom{,0.970593,}{0.019006} & \binom{0.915736,}{0.013702} & \binom{0.849129,}{0.0883} & \binom{0.743132,}{0.129964} & (1,0) & \binom{0.706046,}{0.160856} & \binom{0.031833,}{0.134212} & \binom{0.276923,}{0.107951} \\
\binom{0.09754,}{0.344341} & \binom{0.957167,}{0.030384} & \binom{0.792207,}{0.024727} & \binom{0.933993,}{0.014773} & \binom{0.392227,}{0.42488} & \binom{0.706046,}{0.160856} & (1,0) & \binom{0.046171,}{0.245619} & \binom{0.097132,}{0.759057} \\
\binom{0.278498,}{0.552322} & \binom{0.485376,}{0.38838} & \binom{0.959492,}{0.020188} & \binom{0.678735,}{0.241356} & \binom{0.655478,}{0.306936} & \binom{0.031833,}{0.134212} & \binom{0.046171,}{0.245619} & (1,0) & \binom{0.823458,}{0.044892} \\
\binom{0.546882,}{0.36032} & \binom{0.80028,}{0.055128} & \binom{0.655741,}{0.330401} & \binom{0.75774,}{0.061799} & \binom{0.171187,}{0.795074} & \binom{0.276923,}{0.107951} & \binom{0.097132,}{0.759057} & \binom{0.823458,}{0.044892} & (1,0)
\end{array}\right).$$

Applying the max-min composition given in definition 2.3.3, the similarity relation matrix is generated.

$$B_I^{(3)} = \left(\begin{array}{ccccccccc}
(1,0) & \binom{0.9058,}{0.0299} & \binom{0.9058,}{0.0299} & \binom{0.9058,}{0.0299} & \binom{0.9134,}{0.0030} & \binom{0.9058,}{0.0299} & \binom{0.9058,}{0.0299} & \binom{0.9058,}{0.0299} & \binom{0.8235,}{0.0449} \\
\binom{0.9058,}{0.0299} & (1,0) & \binom{0.9575,}{0.0079} & \binom{0.9649,}{0.0172} & \binom{0.9058,}{0.0299} & \binom{0.9706,}{0.0137} & \binom{0.9572,}{0.0172} & \binom{0.9575,}{0.0202} & \binom{0.8235,}{0.0449} \\
\binom{0.9058,}{0.0299} & \binom{0.9575,}{0.0079} & (1,0) & \binom{0.9575,}{0.0172} & \binom{0.9058,}{0.0299} & \binom{0.9575,}{0.0137} & \binom{0.9572,}{0.0172} & \binom{0.9595,}{0.0202} & \binom{0.8235,}{0.0449} \\
\binom{0.9058,}{0.0299} & \binom{0.9649,}{0.0172} & \binom{0.9575,}{0.0172} & (1,0) & \binom{0.9058,}{0.0299} & \binom{0.9649,}{0.0172} & \binom{0.9572,}{0.0148} & \binom{0.9575,}{0.0202} & \binom{0.8235,}{0.0449} \\
\binom{0.9134,}{0.0030} & \binom{0.9058,}{0.0299} & \binom{0.9058,}{0.0299} & \binom{0.9058,}{0.0299} & (1,0) & \binom{0.9058,}{0.0299} & \binom{0.9058,}{0.0299} & \binom{0.9058,}{0.0299} & \binom{0.8235,}{0.0449} \\
\binom{0.9058,}{0.0299} & \binom{,0.9706,}{0.0137} & \binom{0.9575,}{0.0137} & \binom{0.9649,}{0.0172} & \binom{0.9058,}{0.0299} & (1,0) & \binom{0.9572,}{0.0172} & \binom{0.9575,}{0.0202} & \binom{0.8235,}{0.0449} \\
\binom{0.9058,}{0.0299} & \binom{0.9572,}{0.0172} & \binom{0.9572,}{0.0172} & \binom{0.9572,}{0.0148} & \binom{0.9058,}{0.0299} & \binom{0.9572,}{0.0172} & (1,0) & \binom{0.9572,}{0.0202} & \binom{0.8235,}{0.0449} \\
\binom{0.9058,}{0.0299} & \binom{0.9575,}{0.0202} & \binom{0.9595,}{0.0202} & \binom{0.9575,}{0.0202} & \binom{0.9058,}{0.0299} & \binom{0.9575,}{0.0202} & \binom{0.9572,}{0.0202} & (1,0) & \binom{0.8235,}{0.0449} \\
\binom{0.8235,}{0.0449} & \binom{0.8235,}{0.0449} & \binom{0.8235,}{0.0449} & \binom{0.8235,}{0.0449} & \binom{0.8235,}{0.0449} & \binom{0.8235,}{0.0449} & \binom{0.8235,}{0.0449} & \binom{0.8235,}{0.0449} & (1,0)
\end{array}\right).$$

The (α,β)-cuts of $B_I^{(3)}$ are

$$(1,0),\ (0.9706,0.0137),\ (0.9649,0.0172),\ (0.9595,0.0202),\ (0.9575,0.0202),$$
$$(0.9572,0.0202),\ (0.9134,0.0030),\ (0.9058,0.0299),\ (0.8235,0.0449).$$

The resolution form resulting in the hierarchy tree is explained in Fig 2.2.

Resolution form $B_I^{(3)}$

$$= (0.8235, 0.0449)\begin{pmatrix} 1 &&&&&&&& \\ 1 & 1 &&&&&&& \\ 1 & 1 & 1 &&&&&& \\ 1 & 1 & 1 & 1 &&&&& \\ 1 & 1 & 1 & 1 & 1 &&&& \\ 1 & 1 & 1 & 1 & 1 & 1 &&& \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 && \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} + (0.9058, 0.0299)\begin{pmatrix} 1 &&&&&&&& \\ 1 & 1 &&&&&&& \\ 1 & 1 & 1 &&&&&& \\ 1 & 1 & 1 & 1 &&&&& \\ 1 & 1 & 1 & 1 & 1 &&&& \\ 1 & 1 & 1 & 1 & 1 & 1 &&& \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 && \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+(0.9134, 0.0030)\begin{pmatrix} 1 &&&&&&&& \\ 0 & 1 &&&&&&& \\ 0 & 1 & 1 &&&&&& \\ 0 & 1 & 1 & 1 &&&&& \\ 1 & 0 & 0 & 0 & 1 &&&& \\ 0 & 1 & 1 & 1 & 0 & 1 &&& \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 && \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + (0.9572, 0.0202)\begin{pmatrix} 1 &&&&&&&& \\ 0 & 1 &&&&&&& \\ 0 & 1 & 1 &&&&&& \\ 0 & 1 & 1 & 1 &&&&& \\ 0 & 0 & 0 & 0 & 1 &&&& \\ 0 & 1 & 1 & 1 & 0 & 1 &&& \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 && \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+(0.9575, 0.0202)\begin{pmatrix} 1 &&&&&&&& \\ 0 & 1 &&&&&&& \\ 0 & 1 & 1 &&&&&& \\ 0 & 1 & 1 & 1 &&&&& \\ 0 & 0 & 0 & 0 & 1 &&&& \\ 0 & 1 & 1 & 1 & 0 & 1 &&& \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 && \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + (0.9595, 0.0202)\begin{pmatrix} 1 &&&&&&&& \\ 0 & 1 &&&&&&& \\ 0 & 0 & 1 &&&&&& \\ 0 & 1 & 0 & 1 &&&&& \\ 0 & 0 & 0 & 0 & 1 &&&& \\ 0 & 1 & 0 & 1 & 0 & 1 &&& \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 && \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+(0.9649, 0.0172)\begin{pmatrix} 1 &&&&&&&& \\ 0 & 1 &&&&&&& \\ 0 & 0 & 1 &&&&&& \\ 0 & 1 & 0 & 1 &&&&& \\ 0 & 0 & 0 & 0 & 1 &&&& \\ 0 & 1 & 0 & 1 & 0 & 1 &&& \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 && \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + (0.9706, 0.0137)\begin{pmatrix} 1 &&&&&&&& \\ 0 & 1 &&&&&&& \\ 0 & 0 & 1 &&&&&& \\ 0 & 0 & 0 & 1 &&&&& \\ 0 & 0 & 0 & 0 & 1 &&&& \\ 0 & 1 & 0 & 0 & 0 & 1 &&& \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 && \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+(1, 0)\begin{pmatrix} 1 &&&&&&&& \\ 0 & 1 &&&&&&& \\ 0 & 0 & 1 &&&&&& \\ 0 & 0 & 0 & 1 &&&&& \\ 0 & 0 & 0 & 0 & 1 &&&& \\ 0 & 0 & 0 & 0 & 0 & 1 &&& \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 && \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.4 Interval-valued Intuitionistic Fuzzy Relation (IVIFR) and Operations

In this section we focus on extending the methodologies detailed in previous sections on IVIFR.

Definition 2.4.1 (Interval-valued intuitionistic fuzzy relation (IVIFR)). [34] An IVIFR $\widetilde{R}_I$ between two sets X and Y is an interval-valued intuitionistic fuzzy subset of $X \times Y$ asso-

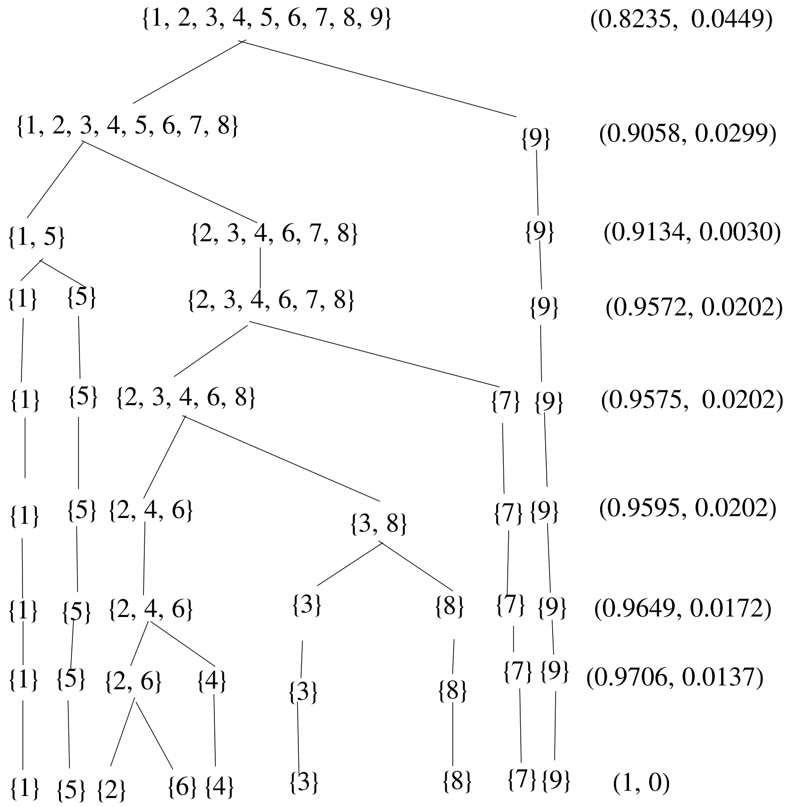

Figure 2.2: Hierarchical clustering for IFR in Example 2.3.1

ciated with an interval-valued membership $\widetilde{\mu}_{\widetilde{R}_I}(x,y) = [\underline{\mu}_{\widetilde{R}_I}(x,y),\ \overline{\mu}_{\widetilde{R}_I}(x,y)]$ and interval-valued non-membership
$\widetilde{\nu}_{\widetilde{R}_I}(x,y) = [\underline{\nu}_{\widetilde{R}_I}(x,y),\ \overline{\nu}_{\widetilde{R}_I}(x,y)]$, for (x,y) in $X \times Y$. The values $\underline{\mu}_{\widetilde{R}_I}(x,y)$ and $\overline{\mu}_{\widetilde{R}_I}(x,y)$ denote the left and right endpoints of $\widetilde{\mu}_{\widetilde{R}_I}(x,y)$ with $0 \leq \underline{\mu}_{\widetilde{R}_I}(x,y) \leq \overline{\mu}_{\widetilde{R}_I}(x,y) \leq 1$, and $\underline{\nu}_{\widetilde{R}_I}(x,y)$ and $\overline{\nu}_{\widetilde{R}_I}(x,y)$ denote the left and right end points of $\widetilde{\nu}_{\widetilde{R}_I}(x,y)$ with $0 \leq \underline{\nu}_{\widetilde{R}_I}(x,y) \leq \overline{\nu}_{\widetilde{R}_I}(x,y) \leq 1$. An IVIFR $\widetilde{R}_I$ on $X = \{x_1, \ldots, x_m\}$ can be represented as follows:

$$\widetilde{R}_I = \begin{bmatrix} ([1,1],[0,0]) & (\tilde{\mu}_{\tilde{R}_I}(x_1,x_2), \tilde{\nu}_{\tilde{R}_I}(x_1,x_2)) & \cdots & (\tilde{\mu}_{\tilde{R}_I}(x_1,x_m), \tilde{\nu}_{\tilde{R}_I}(x_1,x_m)) \\ (\tilde{\mu}_{\tilde{R}_I}(x_2,x_1), \tilde{\nu}_{\tilde{R}_I}(x_2,x_1)) & ([1,1],[0,0]) & \cdots & (\tilde{\mu}_{\tilde{R}_I}(x_2,x_m), \tilde{\nu}_{\tilde{R}_I}(x_2,x_m)) \\ \vdots & \vdots & \cdots & \vdots \\ (\tilde{\mu}_{\tilde{R}_I}(x_m,x_1), \tilde{\nu}_{\tilde{R}_I}(x_m,x_1)) & (\tilde{\mu}_{\tilde{R}_I}(x_m,x_2), \tilde{\nu}_{\tilde{R}_I}(x_m,x_2)) & \cdots & ([1,1],[0,0]) \end{bmatrix}.$$

We define few operations and definitions on IVIFR.

Definition 2.4.2 (Max and min operations for IVIFR).

Let $(\widetilde{\mu}_{\tilde{R}_I}(x_1,x_2), \widetilde{\nu}_{\tilde{R}_I}(x_1,x_2)) = ([\underline{\mu}_{\tilde{R}_I}(x_1,x_2),\ \overline{\mu}_{\tilde{R}_I}(x_1,x_2)],\ [\underline{\nu}_{\tilde{R}_I}(x_1,x_2), \overline{\nu}_{\tilde{R}_I}(x_1,x_2)])$,

and $(\widetilde{\mu}_{\tilde{R}_I}(x_3,x_4), \widetilde{\nu}_{\tilde{R}_I}(x_3,x_4)) = ([\underline{\mu}_{\tilde{R}_I}(x_3,x_4),\ \overline{\mu}_{\tilde{R}_I}(x_3,x_4)],\ [\underline{\nu}_{\tilde{R}_I}(x_3,x_4), \overline{\nu}_{\tilde{R}_I}(x_3,x_4)])$

be two interval-valued intuitionistic fuzzy values of IVIFR $\widetilde{R}_I$ in X. Define

$$\min\{(\tilde{\mu}_{\tilde{R}_I}(x_1,x_2),\ \tilde{\nu}_{\tilde{R}_I}(x_1,x_2),\ (\tilde{\mu}_{\tilde{R}_I}(x_3,x_4), \tilde{\nu}_{\tilde{R}_I}(x_3,x_4))\} = ([\min\{\underline{\mu}_{\tilde{R}_I}(x_1,x_2), \underline{\mu}_{\tilde{R}_I}(x_3,x_4)\},$$
$$\min\{\overline{\mu}_{\tilde{R}_I}(x_1,x_2), \overline{\mu}_{\tilde{R}_I}(x_3,x_4)\}], [\max\{\underline{\nu}_{\tilde{R}_I}(x_1,x_2), \underline{\nu}_{\tilde{R}_I}(x_3,x_4)\}, \max\{\overline{\nu}_{\tilde{R}_I}(x_1,x_2), \overline{\nu}_{\tilde{R}_I}(x_3,x_4)\}]).$$

$$\max\{(\widetilde{\mu}_{\tilde{R}_I}(x_1,x_2),\ \widetilde{\nu}_{\tilde{R}_I}(x_1,x_2)),\ (\widetilde{\mu}_{\tilde{R}_I}(x_3,x_4), \widetilde{\nu}_{\tilde{R}_I}(x_3,x_4))\} = ([\max\{\underline{\mu}_{\tilde{R}_I}(x_1,x_2),\ \underline{\mu}_{\tilde{R}_I}(x_3,x_4)\},$$
$$\max\{\overline{\mu}_{\tilde{R}_I}(x_1,x_2),\ \overline{\mu}_{\tilde{R}_I}(x_3,x_4)\}], [\min\{\underline{\nu}_{\tilde{R}_I}(x_1,x_2),\ \underline{\nu}_{\tilde{R}_I}(x_3,x_4)\}, \min\{\overline{\nu}_{\tilde{R}_I}(x_1,x_2),\ \overline{\nu}_{\tilde{R}_I}(x_3,x_4)\}]).$$

Similar to the IVFR and IFR, we define the similarity relation for IVIFR.

Definition 2.4.3 (Interval-valued intuitionistic fuzzy proximity relation). An IVIFR $\widetilde{R}_I$ in X is called interval-valued intuitionistic fuzzy proximity relation if it satisfies the following two properties:

1. Reflexive: $\widetilde{\mu}_{\widetilde{R}_I}(x,x) = [1,1],\ \widetilde{\nu}_{\widetilde{R}_I}(x,x) = [0,0],\ \forall\, x \in X$.

2. Symmetry: $\widetilde{\mu}_{\widetilde{R}_I}(x,y) = \widetilde{\mu}_{\widetilde{R}_I}(y,x),\ \widetilde{\nu}_{\widetilde{R}_I}(x,y) = \widetilde{\nu}_{\widetilde{R}_I}(y,x),\ \forall\, x,y \in X$.

For example, the following $\widetilde{R}_I$ is an IVIF proximity relation on $X = \{x_1,x_2,x_3,x_4\}$.

$$\widetilde{R}_I = \begin{bmatrix} ([1,1],[0,0]) & ([0.2,0.4],[0.4,0.6]) & ([0.5,0.7],[0.1,0.3]) & ([0.3,0.6],[0.2,0.4]) \\ ([0.2,0.4],[0.4,0.6]) & ([1,1],[0,0]) & ([0.5,0.8],[0.1,0.2]) & ([0.7,0.8],[0.1,0.2]) \\ ([0.5,0.7],[0.1,0.3]) & ([0.5,0.8],[0.1,0.2]) & ([1,1],[0,0]) & ([0.2,0.3],[0.4,0.6]) \\ ([0.3,0.6],[0.2,0.4]) & ([0.7,0.8],[0.1,0.2]) & ([0.2,0.3],[0.4,0.6]) & ([1,1],[0,0]) \end{bmatrix}.$$

Definition 2.4.4 (Interval-valued intuitionistic max-min similarity relation). An IVIF proximity relation $\widetilde{R}_I$ on X is called an interval-valued intuitionistic (max-min) similarity relation if for any $x,y,z \in X$, and any two interval-valued intuitionistic fuzzy values,

$$(\widetilde{\mu}_{\widetilde{R}_I}(x,y),\ \widetilde{\nu}_{\widetilde{R}_I}(x,y)) = \left([\underline{\mu}_{\widetilde{R}_I}(x,y),\ \overline{\mu}_{\widetilde{R}_I}(x,y)],\ [\underline{\nu}_{\widetilde{R}_I}(x,y),\ \overline{\nu}_{\widetilde{R}_I}(x,y)]\right) \text{ and}$$
$$(\widetilde{\mu}_{\widetilde{R}_I}(y,z),\ \widetilde{\nu}_{\widetilde{R}_I}(y,z)) = \left([\underline{\mu}_{\widetilde{R}_I}(y,z),\ \overline{\mu}_{\widetilde{R}_I}(y,z)],\ [\underline{\nu}_{\widetilde{R}_I}(y,z),\ \overline{\nu}_{\widetilde{R}_I}(y,z)]\right),$$

the following transitivity property is satisfied.

$$
\begin{aligned}
\underline{\mu}_{\widetilde{R}_I}(x,z) &\geq \max_{z\in X}\{\min\{\underline{\mu}_{\widetilde{R}_I}(x,y),\underline{\mu}_{\widetilde{R}_I}(y,z)\}\},\\
\overline{\mu}_{\widetilde{R}_I}(x,z) &\geq \max_{z\in X}\{\min\{\overline{\mu}_{\widetilde{R}_I}(x,y),\overline{\mu}_{\widetilde{R}_I}(y,z)\}\},
\end{aligned}
$$

and

$$
\begin{aligned}
\underline{\nu}_{\widetilde{R}_I}(x,z) &\leq \min_{z\in X}\{\max\{\underline{\nu}_{\widetilde{R}_I}(x,y),\underline{\nu}_{\widetilde{R}_I}(y,z)\}\},\\
\overline{\nu}_{\widetilde{R}_I}(x,z) &\leq \min_{z\in X}\{\max\{\overline{\nu}_{\widetilde{R}_I}(x,y),\overline{\nu}_{\widetilde{R}_I}(y,z)\}\}.
\end{aligned}
$$

We now define the $(\tilde{\alpha},\tilde{\beta})$-cuts for IVIFS.

Definition 2.4.5. The $(\tilde{\alpha},\tilde{\beta})$-cut for an IVIFS is described by $A_{IVI(\tilde{\alpha},\tilde{\beta})} = \{x \mid [\underline{\mu},\overline{\mu}] \geq \tilde{\alpha}$ and $[\underline{\nu},\overline{\nu}] \leq \tilde{\beta}\}$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are also interval-valued type.

Note that $(\tilde{\alpha},\tilde{\beta})$-cut is an order pair of membership and non-membership values with $(\tilde{\alpha},\tilde{\beta}) = (\tilde{\mu}_\alpha,\tilde{\nu}_\beta) = ([\underline{\mu}_\alpha,\overline{\mu}_\alpha],\ [\underline{\nu}_\beta,\overline{\nu}_\beta])$, where $0 \leq \underline{\mu}_\alpha \leq \overline{\mu}_\alpha \leq 1$ and $0 \leq \underline{\nu}_\beta \leq \overline{\nu}_\beta \leq 1,\ 0 \leq \overline{\mu}_\alpha + \overline{\nu}_\beta \leq 1$.

To explain Definition 2.4.5, we present an algorithm listing the procedure of constructing all $(\tilde{\alpha},\tilde{\beta})$-cuts in IVIFR $\widetilde{R}_I$.

Algorithm 2.4.1.

1. Among all pairs of elements of the form $([\underline{\mu}_i,\overline{\mu}_i],[\underline{\nu}_i,\overline{\nu}_i])$ in the matrix representation of $\widetilde{R}_I$, choose the maximum value among the lower values of the membership, i. e. $\max_i\{\underline{\mu}_i\}$, and fixed it.

2. Thereafter, among all possible elements in $\widetilde{R}_I$ having this fixed lower membership value (as in preceding step), choose the one having minimum upper value of the membership.

3. Among all elements in $\widetilde{R}_I$ with the fixed interval membership degree (as obtained in the previous two steps), choose those having maximum lower value for the non-membership (fixed this one), and thereafter pick the corresponding maximum upper non-membership value. The respective pair of membership and non-membership degrees is the $(\tilde{\alpha},\tilde{\beta})$-cut.

4. Crossed-off this element from $\widetilde{R}_I$, and continue the above listed procedure with the remaining elements of $\widetilde{R}_I$, till all elements are searched through.

Definition 2.4.6 (Resolution form for interval-valued intuitionistic similarity relation). An IVIF (max-min) similarity relation $\widetilde{R}_I$ on X can be decomposed into a resolution form by using $(\widetilde{\alpha},\widetilde{\beta})$-cuts with corresponding $R_{(\widetilde{\alpha},\widetilde{\beta})}$ being reflexive, symmetric and transitive.

$$\widetilde{R}_I = \bigcup_{(\widetilde{\alpha},\widetilde{\beta})} (\widetilde{\alpha},\widetilde{\beta})R_{(\widetilde{\alpha},\widetilde{\beta})} = \sum_{i=1}^{m}([\underline{\mu}_{\alpha_i},\ \overline{\mu}_{\alpha_i}],\ [\underline{\nu}_{\beta_i},\ \overline{\nu}_{\beta_i}])R_{(\widetilde{\alpha}_i,\widetilde{\beta}_i)},$$
$$\underline{\mu}_i \geq \underline{\mu}_{\alpha_i},\ \ \overline{\mu}_i \geq \overline{\mu}_{\alpha_i},\ \ \underline{\nu}_i \leq \underline{\nu}_{\beta_i},\ \ \overline{\nu}_i \leq \overline{\nu}_{\beta_i},\ \ \overline{\mu}_{\alpha_i} + \overline{\nu}_{\beta_i} \leq 1.$$

With this background, we are ready to explain the clustering scheme for IVIFR.

2.5 Hierarchical Clustering for IVIFR

The subjective information is commonly represented by a matrix which in general is reflexive and symmetry but need not be transitive. However, transitivity is an essential property to build hierarchal clustering [52]. An n-step procedure is employed to get an interval-valued max-min similarity relation matrix using an interval-valued intuitionistic max-min composition, described as follows.

Definition 2.5.1 (Interval-valued intuitionistic max-min composition). Given an initial interval-valued intuitionistic relation matrix $\widetilde{R}_I^{(0)} = (\widetilde{\mu}_{r_{ij}^{(0)}}, \widetilde{\nu}_{r_{ij}^{(0)}})_{m\times m} = ([\underline{\mu}_{r_{ij}^{(0)}}, \overline{\mu}_{r_{ij}^{(0)}}], [\underline{\nu}_{r_{ij}^{(0)}}, \overline{\nu}_{r_{ij}^{(0)}}])$. Then $\widetilde{R}_I^{(n)} = (\widetilde{\mu}_{r_{ij}^{(n)}}, \widetilde{\nu}_{r_{ij}^{(n)}})_{m\times m} = ([\underline{\mu}_{r_{ij}^{(n)}}, \overline{\mu}_{r_{ij}^{(n)}}],\ [\underline{\nu}_{r_{ij}^{(n)}}, \overline{\nu}_{r_{ij}^{(n)}}])$ with

$$\underline{\mu}_{r_{ij}^{(n)}} = \max_k\{\min\{\underline{\mu}_{r_{ik}^{(n-1)}},\ \underline{\mu}_{r_{kj}^{(n-1)}}\}\}, \quad \overline{\mu}_{r_{ij}^{(n)}} = \max_k\{\min\{\overline{\mu}_{r_{ik}^{(n-1)}},\ \overline{\mu}_{r_{kj}^{(n-1)}}\}\},$$

and

$$\underline{\nu}_{r_{ij}^{(n)}} = \min_k\{\max\{\underline{\nu}_{r_{ik}^{(n-1)}},\ \underline{\nu}_{r_{kj}^{(n-1)}}\}\}, \quad \overline{\nu}_{r_{ij}^{(n)}} = \min_k\{\max\{\overline{\nu}_{r_{ik}^{(n-1)}},\ \overline{\nu}_{r_{kj}^{(n-1)}}\}\},$$

$n = 1, 2, \ldots,$ is called an interval-valued intuitionistic relation matrix through an n-step max-min composition.

Theorem 2.5.1. Let $\widehat{R}_I^{(0)} = (\mu_{\widehat{r}_{ij}^{(0)}}, \nu_{\widehat{r}_{ij}^{(0)}})_{m\times m}$ and $\widetilde{R}_I^{(0)} = (\widetilde{\mu}_{r_{ij}^{(0)}}, \widetilde{\nu}_{r_{ij}^{(0)}}) = ([\underline{\mu}_{r_{ij}^{(0)}}, \overline{\mu}_{r_{ij}^{(0)}}], [\underline{\nu}_{r_{ij}^{(0)}}, \overline{\nu}_{r_{ij}^{(0)}}])$ be respectively IFR and IVIFR with $\underline{\mu}_{r_{ij}^{(0)}} \leq \mu_{\widehat{r}_{ij}^{(0)}} \leq \overline{\mu}_{r_{ij}^{(0)}}$ and $\underline{\nu}_{r_{ij}^{(0)}} \leq \nu_{\widehat{r}_{ij}^{(0)}} \leq \overline{\nu}_{r_{ij}^{(0)}}$, $\forall\, i$, j. If $\widehat{R}_I^{(n)} = (\mu_{\widehat{r}_{ij}^{(n)}}, \nu_{\widehat{r}_{ij}^{(n)}})$ and $\widetilde{R}_I^{(n)} = (\widetilde{\mu}_{r_{ij}^{(n)}}, \widetilde{\nu}_{r_{ij}^{(n)}}) = ([\underline{\mu}_{r_{ij}^{(n)}}, \overline{\mu}_{r_{ij}^{(n)}}],\ [\underline{\nu}_{r_{ij}^{(n)}}, \overline{\nu}_{r_{ij}^{(n)}}])_{m\times m}$ are respectively IFR and IVIFR, through n-step max-min compositions, then we have $\underline{\mu}_{r_{ij}^{(n)}} \leq \mu_{\widehat{r}_{ij}^{(n)}} \leq \overline{\mu}_{r_{ij}^{(n)}}$ and $\underline{\nu}_{r_{ij}^{(n)}} \leq \nu_{\widehat{r}_{ij}^{(n)}} \leq \overline{\nu}_{r_{ij}^{(n)}}$, for $n = 1, 2, \ldots$.

Proof. Developing $\underline{\mu}_{r_{ij}^{(1)}}$, $\overline{\mu}_{r_{ij}^{(1)}}$, $\underline{\nu}_{r_{ij}^{(1)}}$, $\overline{\nu}_{r_{ij}^{(1)}}$, $\mu_{\widehat{r}_{ij}^{(1)}}$, and $\nu_{\widehat{r}_{ij}^{(1)}}$, we have,

$$\begin{aligned}
\underline{\mu}_{r_{ij}^{(1)}} &= \max_k\{\min(\underline{\mu}_{r_{ik}^{(0)}}, \underline{\mu}_{r_{kj}^{(0)}})\}, & \overline{\mu}_{r_{ij}^{(1)}} &= \max_k\{\min(\overline{\mu}_{r_{ik}^{(0)}}, \overline{\mu}_{r_{kj}^{(0)}})\} \\
\underline{\nu}_{r_{ij}^{(1)}} &= \min_k\{\max(\underline{\nu}_{r_{ik}^{(0)}}, \underline{\nu}_{r_{kj}^{(0)}})\}, & \overline{\nu}_{r_{ij}^{(1)}} &= \min_k\{\max(\overline{\nu}_{r_{ik}^{(0)}}, \overline{\nu}_{r_{kj}^{(0)}})\} \\
\mu_{\widehat{r}_{ij}^{(1)}} &= \max_k\{\min(\mu_{\widehat{r}_{ik}^{(0)}}, \mu_{\widehat{r}_{kj}^{(0)}})\}, & \nu_{\widehat{r}_{ij}^{(1)}} &= \min_k\{\max(\nu_{\widehat{r}_{ik}^{(0)}}, \nu_{\widehat{r}_{kj}^{(0)}})\}.
\end{aligned}$$

Since $\underline{\mu}_{r_{ij}^{(0)}} \leq \mu_{\widehat{r}_{ij}^{(0)}} \leq \overline{\mu}_{r_{ij}^{(0)}}$, and $\underline{\nu}_{r_{ij}^{(0)}} \leq \nu_{\widehat{r}_{ij}^{(0)}} \leq \overline{\nu}_{r_{ij}^{(0)}}$, $\forall\, i, j$, it is easy to show that $\underline{\mu}_{r_{ij}^{(1)}} \leq \mu_{\widehat{r}_{ij}^{(1)}} \leq \overline{\mu}_{r_{ij}^{(1)}}$, and $\underline{\nu}_{r_{ij}^{(1)}} \leq \nu_{\widehat{r}_{ij}^{(1)}} \leq \overline{\nu}_{r_{ij}^{(1)}}$.
Following induction on n, we can easily obtain the requisite result; completeing the proof. $\square$

Remark 2.5.1. Since $\underline{\mu}_{r_{ij}^{(n)}} \leq \mu_{\widehat{r}_{ij}^{(n)}} \leq \overline{\mu}_{r_{ij}^{(n)}}$, and $\underline{\nu}_{r_{ij}^{(n)}} \leq \nu_{\widehat{r}_{ij}^{(n)}} \leq \overline{\nu}_{r_{ij}^{(n)}}$, $n = 1, 2, 3, \ldots$, we have $\underline{\mu}_{\widetilde{R}_I^{(n)}} \leq \mu_{\widehat{R}_I^{(n)}} \leq \overline{\mu}_{\widetilde{R}_I^{(n)}}$ and $\underline{\nu}_{\widetilde{R}_I^{(n)}} \leq \nu_{\widehat{R}_I^{(n)}} \leq \overline{\nu}_{\widetilde{R}_I^{(n)}}$. It is implied that if a matrix representation of IFR $\widehat{R}_I^{(0)} = (\mu_{\widehat{R}_I^{(0)}},\ \nu_{\widehat{R}_I^{(0)}})$ is taken so as to satisfy $\underline{\mu}_{\widetilde{R}_I^{(0)}} \leq \mu_{\widehat{R}_I^{(0)}} \leq \overline{\mu}_{\widetilde{R}_I^{(0)}}$, $\quad \underline{\nu}_{\widetilde{R}_I^{(0)}} \leq \nu_{\widehat{R}_I^{(0)}} \leq \overline{\nu}_{\widehat{R}_I^{(0)}}$, then we always have $\underline{\mu}_{\widetilde{R}_I^{(n)}} \leq \mu_{\widehat{R}_I^{(n)}} \leq \overline{\mu}_{\widetilde{R}_I^{(n)}}$, and $\underline{\nu}_{\widetilde{R}_I^{(n)}} \leq \nu_{\widehat{R}_I^{(n)}} \leq \overline{\nu}_{\widetilde{R}_I^{(n)}}$, $n = 1, 2, \ldots$.

The following algorithm provides the hierarchical clustering for IVIFR.

Algorithm 2.5.1. Let $\widetilde{R}_I^{(0)} = (\widetilde{\mu}_{\tilde{R}_I^{(0)}}, \widetilde{\nu}_{\tilde{R}_I^{(0)}}) = \big([\underline{\mu}_{\tilde{R}_I^{(0)}},\ \overline{\mu}_{\tilde{R}_I^{(0)}}],\ \ [\underline{\nu}_{\tilde{R}_I^{(0)}},\ \overline{\nu}_{\tilde{R}_I^{(0)}}]\big)$ be an IVIF proximity relation.

1. Initialize $k = 0$.

2. Set $\widetilde{R}_I^{(k+1)} = \widetilde{R}_I^{(k)} \circ \widetilde{R}_I^{(k)}$.

3. If $\widetilde{R}_I^{(k+1)} \neq \tilde{R}_I^{(k)}$, then set $k = k+1$, and go to 2 above.

4. Else an IVIF (max-min) similarity relation $\widetilde{R}_I^{(k)}$ is obtained, and proceed to next step.

5. Decompose $\tilde{R}_I^{(k)}$ into resolution form using $(\widetilde{\alpha}, \widetilde{\beta})$-cuts with its corresponding $R_{(\widetilde{\alpha}, \widetilde{\beta})}$ satisfying the reflexive, symmetric and transitivity properties.

6. Using resolution form, construct the hierarchical partition tree.

A MATLAB code is prepared to implement the above algorithm in which the matrix corresponding to an IVIFR is randomly generated matrix of order m. For illustration purpose, we present the following example.

Example 2.5.1. Consider the IVIF proximity relation:

$$\widetilde{R}_I^{(0)} = \left(\begin{array}{ccccccc}
([1,1],[0,0]) & & & & & & \\
\begin{pmatrix}[0.8147,0.8213],\\ {[0.0956,0.1368]}\end{pmatrix} & ([1,1],[0,0]) & & & & & \\
\begin{pmatrix}[0.9058,0.9857],\\ {[0.0101,0.0113]}\end{pmatrix} & \begin{pmatrix}[0.2785,0.5615],\\ {[0.0801,0.3111]}\end{pmatrix} & ([1,1],[0,0]) & & & & \\
\begin{pmatrix}[0.1269,0.9423],\\ {[0.0103,0.0107]}\end{pmatrix} & \begin{pmatrix}[0.5468,0.8438],\\ {[0.0990,0.1178]}\end{pmatrix} & \begin{pmatrix}[0.9706,0.9787],\\ {[0.0032,0.0034]}\end{pmatrix} & ([1,1],[0,0]) & & & \\
\begin{pmatrix}[0.9134,0.9722],\\ {[0.0074,0.0136]}\end{pmatrix} & \begin{pmatrix}[0.9575,0.9647],\\ {[0.0024,0.0097]}\end{pmatrix} & \begin{pmatrix}[0.9571,0.9591],\\ {[0.0017,0.0048]}\end{pmatrix} & \begin{pmatrix}[0.1418,0.7381],\\ {[0.0549,0.0891]}\end{pmatrix} & ([1,1],[0,0]) & & \\
\begin{pmatrix}[0.6323,0.9109],\\ {[0.0055,0.0396]}\end{pmatrix} & \begin{pmatrix}[0.9648,0.9896],\\ {[0.0057,0.0070]}\end{pmatrix} & \begin{pmatrix}[0.4854,0.5353],\\ {[0.045523,0.231559]}\end{pmatrix} & \begin{pmatrix}[0.4217,0.6051],\\ {[0.1094,0.2311]}\end{pmatrix} & \begin{pmatrix}[0.7922,0.7994],\\ {[0.1252,0.1507]}\end{pmatrix} & ([1,1],[0,0]) & \\
\begin{pmatrix}[0.0975,0.7682],\\ {[0.0223,0.1498]}\end{pmatrix} & \begin{pmatrix}[0.1576,0.1844],\\ {[0.1301,0.5343]}\end{pmatrix} & \begin{pmatrix}[0.8002,0.9647],\\ {[0.0085,0.0338]}\end{pmatrix} & \begin{pmatrix}[0.9157,0.9958],\\ {[0.0003,0.0009]}\end{pmatrix} & \begin{pmatrix}[0.9595,0.9772],\\ {[0.0033,0.0058]}\end{pmatrix} & \begin{pmatrix}[0.6557,0.7871],\\ {[0.0592,0.1077]}\end{pmatrix} & ([1,1],[0,0])
\end{array} \right)$$

Applying the n-step procedure described in Definition 2.5.1, an IVIF symmetric relation $\widetilde{R}_I^{(3)}$ is obtained.

$$\widetilde{R}_I^{(3)} = \left(\begin{array}{ccccccc}
([1,1],[0,0]) & & & & & & \\
\begin{pmatrix}[0.9134,0.9648],\\ {[0.0057,0.0108]}\end{pmatrix} & ([1,1],[0,0]) & & & & & \\
\begin{pmatrix}[0.9134,0.9858],\\ {[0.0057,0.0108]}\end{pmatrix} & \begin{pmatrix}[0.9572,0.9648],\\ {[0.0025,0.0097]}\end{pmatrix} & ([1,1],[0,0]) & & & & \\
\begin{pmatrix}[0.9134,0.9787],\\ {[0.0057,0.0108]}\end{pmatrix} & \begin{pmatrix}[0.9572,0.9648],\\ {[0.0032,0.0097]}\end{pmatrix} & \begin{pmatrix}[0.9706,0.9787],\\ {[0.0032,0.0035]}\end{pmatrix} & ([1,1],[0,0]) & & & \\
\begin{pmatrix}[0.9134,0.9773],\\ {[0.0057,0.0108]}\end{pmatrix} & \begin{pmatrix}[0.9575,0.9648],\\ {[0.0025,0.0097]}\end{pmatrix} & \begin{pmatrix}[0.9572,0.9773],\\ {[0.0017,0.0049]}\end{pmatrix} & \begin{pmatrix}[0.9572,0.9773],\\ {[0.0032,0.0049]}\end{pmatrix} & ([1,1],[0,0]) & & \\
\begin{pmatrix}[0.9134,0.9648],\\ {[0.0055,0.0108]}\end{pmatrix} & \begin{pmatrix}[0.9649,0.9897],\\ {[0.0057,0.0070]}\end{pmatrix} & \begin{pmatrix}[0.9572,0.9648],\\ {[0.0057,0.0097]}\end{pmatrix} & \begin{pmatrix}[0.9572,0.9648],\\ {[0.0057,0.0097]}\end{pmatrix} & \begin{pmatrix}[0.9575,0.9648],\\ {[0.0057,0.0097]}\end{pmatrix} & ([1,1],[0,0]) & \\
\begin{pmatrix}[0.9134,0.9787],\\ {[0.0057,0.0108]}\end{pmatrix} & \begin{pmatrix}[0.9575,0.9648],\\ {[0.0032,0.0097]}\end{pmatrix} & \begin{pmatrix}[0.9572,0.9787],\\ {[0.0032,0.0035]}\end{pmatrix} & \begin{pmatrix}[0.9572,0.9958],\\ {[0.0003,0.0009]}\end{pmatrix} & \begin{pmatrix}[0.9595,0.9773],\\ {[0.0032,0.0049]}\end{pmatrix} & \begin{pmatrix}[0.9575,0.9648],\\ {[0.0057,0.0097]}\end{pmatrix} & ([1,1],[0,0])
\end{array} \right)$$

$\widetilde{R}_I^{(3)}$ is expressed in a resolution form as follows:

$$\widetilde{R}_I^{(3)} = ([0.9134,0.9648],[0.0057,0.0108]) \begin{pmatrix}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & & & \\
1 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 1 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

$$+([0.9572,0.9648],[0.0057,0.0097])\begin{pmatrix}1&&&&&&\\0&1&&&&&\\0&1&1&&&&\\0&1&1&1&&&\\0&1&1&1&1&&\\0&1&1&1&1&1&\\0&1&1&1&1&1&1\end{pmatrix}$$

$$+([0.9575,0.9648],[0.0057,0.0097])\begin{pmatrix}1&&&&&&\\0&1&&&&&\\0&0&1&&&&\\0&0&1&1&&&\\0&1&0&0&1&&\\0&1&0&0&1&1&\\0&1&0&0&1&1&1\end{pmatrix}$$

$$+([0.9595,0.9773],[0.0032,0.0049])\begin{pmatrix}1&&&&&&\\0&1&&&&&\\0&0&1&&&&\\0&0&1&1&&&\\0&0&0&0&1&&\\0&1&0&0&0&1&\\0&0&0&0&1&0&1\end{pmatrix}$$

$$+([0.9649,0.9897],[0.0057,0.0070])\begin{pmatrix}1&&&&&&\\0&1&&&&&\\0&0&1&&&&\\0&0&1&1&&&\\0&0&0&0&1&&\\0&1&0&0&0&1&\\0&0&0&0&0&0&1\end{pmatrix}$$

$$+([0.9706,0.9787],[0.0032,0.0035])\begin{pmatrix}1&&&&&&\\0&1&&&&&\\0&0&1&&&&\\0&0&1&1&&&\\0&0&0&0&1&&\\0&0&0&0&0&1&\\0&0&0&0&0&0&1\end{pmatrix}$$

$$+([1,1],[0,0])\begin{pmatrix}1&&&&&&\\0&1&&&&&\\0&0&1&&&&\\0&0&0&1&&&\\0&0&0&0&1&&\\0&0&0&0&0&1&\\0&0&0&0&0&0&1\end{pmatrix}.$$

The same is summarized by a clustering tree in Fig 2.3.

We now present an application of the hierarchical structures created for IVIFR in multi-criteria decision making (MCDM) problems involving such relations.

2.6 Application to MCDM Problem: Determining the Criteria Weights

An MCDM problem involves comparing n available alternatives $A_1,\dots,A_n$ of one type on m relevant criteria $C_1,,\dots,C_m$ resulting in an $m\times n$ criteria-alternative matrix. The entries of this matrix are provided by an expert/decision maker. A decision-maker has to

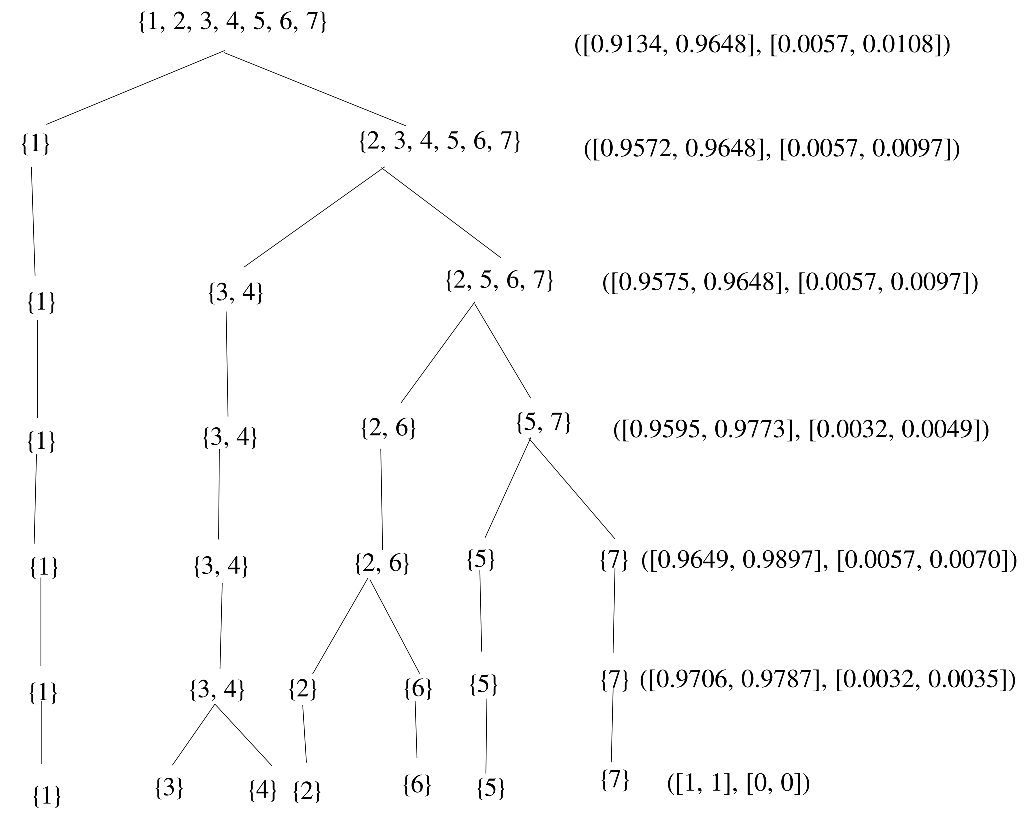

Figure 2.3: Hierarchical clustering for IVIFR for Example 2.5.1

quantify how he/she rates and alternative A_i on criterion C_j. These opinion entries together with the criteria weights are then used to aggregate the entire information so as to rank the alternatives on some preference scale. Various methods are available in literature to rank the alternatives in an MCDM problem, like TOPSIS, VIKOR, PROMETHEE, ELECTREE, to name a few. One can refer to several good texts, see ([53], [145], [147] – [156]) and many more references cited therein, for detailed account of MCDM methodologies and their applications, especially when the MCDM problems are set in intuitionistic fuzzy framework. However, in this work, we only wish to exhibit an application of hierarchical clustering for IVIFR, developed in previous section, to one of the important sub-problem of determining the criteria weights within any MCDM problem. In this section we propose an algorithm to determine the global weights of the criteria in an MCDM problem represented by a criteria-alternative matrix of order $m \times n$ having entries as IVIF numbers.

Algorithm 2.6.1. **Step 1** Seek criteria-alternatives matrix from decision-maker with entries provided are IVIF numbers.

Step 2 Use the following concepts of covariance and correlation coefficient of IVIFS, introduced by Bustince and Burillo [157], to compute the correlation between criteria in the problem.

Definition 2.6.1 (Correlation coefficient for IVIFN). [157] Let $C_p = \{\langle x_j, [\underline{\mu}_{C_p}(x_j),\ \overline{\mu}_{C_p}(x_j)],$ $[\underline{\nu}_{C_p}(x_j),\ \overline{\nu}_{C_p}(x_j)]\}$ and $C_q = \{\langle x_i, [\underline{\mu}_{C_q}(x_j),\ \overline{\mu}_{C_q}(x_j)],\ [\underline{\nu}_{C_q}(x_j),\ \overline{\nu}_{C_q}(x_j)]\}$ be two IVIF numbers. Define the covariance $\mathrm{COV}(C_p, C_q)$ and the correlation $R(C_p, C_q)$ between them, respectively, as follows:

$$\mathrm{COV}(C_p, C_q) = \frac{1}{2}\sum_{j=1}^{n}[\underline{\mu}_{C_p}(x_j)\,\underline{\mu}_{C_q}(x_j) + \overline{\mu}_{C_p}(x_j)\,\overline{\mu}_{C_q}(x_j) + \underline{\nu}_{C_p}(x_j)\,\underline{\nu}_{C_q}(x_j) + \overline{\nu}_{C_p}(x_j)\,\overline{\nu}_{C_q}(x_j)].$$

$$R(C_p, C_q) = \frac{\mathrm{COV}(C_p, C_q)}{\sqrt{\mathrm{COV}(C_p, C_p)\cdot \mathrm{COV}(C_q, C_q)}} \tag{2.6.1}$$

Note that $R(C_p, C_q)$ is an $m \times m$ reflexive and symmetric matrix with crisp entries.

Step 3 Convert the crisp correlation values into IVIFV resulting in a column vector of correlation IVIF values, denoted say by ϑ. We have briefly described a procedure by Yue and Jia [158] to achieve this step in the Appendix-1.

Step 4 Using the max-min composition for membership values and min-max composition for non-membership values, determine the product $\vartheta \times \vartheta$. Take a relation $\widetilde{R}_I \subset \vartheta \times \vartheta$ which is reflexive and symmetric.

Step 5 Construct the IVIF (max-min) similarity matrix from $\widetilde{R}_I$ by invoking Steps 1–4 of Algorithm 4.

Step 6 If the IVIF similarity matrix is order 7 or less, then compute the weights of the criteria using notion of entropy.

Definition 2.6.2. [146] For criterion C_i, the entropy is defined as follows:

$$E(C_i) = \frac{1}{m}\sum_{j=1}^{n}\frac{\left(2 - |\underline{\mu}_{ij} - \underline{\nu}_{ij}| - |\overline{\mu}_{ij} - \overline{\nu}_{ij}| + \underline{\pi}_{ij} + \overline{\pi}_{ij}\right)}{\left(2 + |\underline{\mu}_{ij} - \underline{\nu}_{ij}| + |\overline{\mu}_{ij} - \overline{\nu}_{ij}| + \underline{\pi}_{ij} + \overline{\pi}_{ij}\right)}. \tag{2.6.2}$$

The local weights are computed by

$$w_i = \frac{1 - E(C_i)}{m - \sum\limits_{i=1}^{m} E(C_i)}. \tag{2.6.3}$$

However, for large IVIF similarity matrices, apply Algorithm 4 to construct the hierarchical clustering. Suppose we have k number of disjoint clusters, each having ≤ 7 criteria.

Step 7 Choose a pivot element (criterion) as the one having minimum score value defined by

$$\text{score}_i = \frac{\sum_{j=1}^{n} \overline{\mu}_{ij} - \sum_{j=1}^{n} \overline{\nu}_{ij} + \sum_{j=1}^{n} \underline{\mu}_{ij} - \sum_{j=1}^{n} \underline{\nu}_{ij}}{2}. \tag{2.6.4}$$

Assign this pivot element (criterion) to every cluster.

Step 8 Compute the local weights of the criteria within each cluster using (2.6.2) and (2.6.3).

Step 9 Compute the global weights of the criteria as described by Jalao et al. [159]. For each cluster l, $l = 1, \ldots, k$, divide the local weight of the criterion i in cluster l by the local weight of the pivot element to get $W(i,l)$. The global weight $W_G(i)$ of criterion i is calculated by

$$W_G(i) = \frac{W(i,l)}{\sum_{l=1}^{k} \sum_{i \in S_l} W(i,l) - k + 1}, \quad i = 1, \ldots, m. \tag{2.6.5}$$

The flowchart figure 2.4 summarizes the above procedure.

Example 2.6.1. Suppose an Institute has to determine the most appropriate air conditioning system for installation in its library. A contractor offers 8 feasible alternatives A_j, $j = 1, \ldots, 8$. A decision is to be arrived at based on the following 10 criteria

C_1: size of room,
C_2: correct location of air conditioning system,
C_3: seasonal energy efficiency ratio (SEER),
C_4: coefficient of performance (COP),
C_5: installation and maintenance cost,
C_6: refrigerants and global warming potential (GWP),
C_7: specified cooling power (SCP),
C_8: sound pressure level,
C_9: tonnage of air conditioning system,
C_{10}: miscellaneous.
An Institute expert provides an IVIF criteria-alternatives matrix described in Table 2.1 and the correlation matrix between criteria is calculated using equation 2.6.1 in Table 2.2.

Following the method explained in Appendix-1, construct an IVIF numbers column

Table 2.1: The criteria-alternative matrix

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C_1	$([0.5,0.6],[02,0.3])$	$([0.03,0.2],[0,0.32])$	$([0.7,0.75],[0.10,0.5])$	$([0,0.02],[0.8,0.9])$	$([0.15,0.2],[0.6,0.65])$	$([0,0.3],[0.2,0.5])$	$([0,0.1],[0.08,0.1])$	$([0.5,0.55],[0.3,0.35])$
C_2	$([0,0],[1,1])$	$([0.7,0.8],[0.2,0.2])$	$([0.10,0.15],[0.05,0.06])$	$([0.8,1],[0,0])$	$([0.05,0.07],[0.1,0.2])$	$([0.7,0.75],[0.1,0.15])$	$([0.1,0.2],[0,0.3])$	$([0.2,0.4],[0.2,0.2])$
C_3	$([0.3,0.4],[0.4,0.6])$	$([0.1,0.4],[0.3,0.5])$	$([0.2,0.3],[0.2,0.3])$	$([1,1],[0,0])$	$([0,0.07],[0.2,0.2])$	$([0.1,0.4],[0.3,0.5])$	$([0.15,0.3],[0.2,0.3])$	$([0,0],[1,1])$
C_4	$([0.01,0.3],[0.25,0.30])$	$([0.2,0.5],[0.1,0.2])$	$([0.4,0.5],[0.2,0.3])$	$([0.2,0.3],[0.4,0.5])$	$([0.3,0.5],[0.4,0.5])$	$([0.6,0.6],[0.1,0.2])$	$([0.1,0.4],[0.3,0.5])$	$([1,1],[0,0])$
C_5	$([0.3,0.6],[0.1,0.2])$	$([0.1,0.2],[0.2,0.3])$	$([0.4,0.6],[0.02,0.03])$	$([0.13,0.34],[0,0.3])$	$([0,0.9],[0,0.01])$	$([0.1,0.4],[0.3,0.5])$	$([0.05,0.5],[0.3,0.4])$	$([0.25,0.34],[0.4,0.5])$
C_6	$([0.1,0.2],[0.2,0.7])$	$([0.03,0.04],[0.04,0.06])$	$([0.2,0.3],[0.4,0.6])$	$([0,0],[1,1])$	$([0.4,0.5],[0.3,0.2])$	$([0.8,1],[0,0])$	$([0.1,0.2],[0.6,0.7])$	$([0.01,0.3],[0.25,0.38])$
C_7	$([0.1,0.5],[0.4,0.4])$	$([0.7,0.7],[0.1,0.2])$	$([0.2,0.3],[0.2,0.3])$	$([0.5,0.6],[0.1,0.2])$	$([0.2,0.2],[0.2,0.2])$	$([0.2,0.56],[0.2,0.2])$	$([0.1,0.4],[0.3,0.5])$	$([0.1,0.2],[0.06,0.2])$
C_8	$([0.9,1.0],[0,0])$	$([0.4,0.5],[0.2,0.3])$	$([0,0.08],[0.1,0.2])$	$([0.3,0.5],[0.4,0.5])$	$([0.9,1],[0,0])$	$([0.15,0.2],[0.7,0.75])$	$([0.2,0.3],[0,0.3])$	$([0.05,0.05],[0.03,0.03])$
C_9	$([0.33,0.34],[0.2,0.3])$	$([0.5,0.6],[0.05,0.09])$	$([0,0],[1,1])$	$([0.1,0.2],[0.06,0.5])$	$([0.4,0.5],[0.2,0.3])$	$([0,0.1],[0.75,0.8])$	$([0.1,0.5],[0.02,0.03])$	$([0.05,0.07],[0.1,0.2])$
C_{10}	$([0,0.1],[0.1,0.2])$	$([0.3,0.5],[0.1,0.4])$	$([0.18,0.25],[0,0.2])$	$([0.04,0.44],[0.23,0.3])$	$([0.1,0.4],[0.1,0.4])$	$([0,1],[0,0])$	$([0.1,0.2],[0.06,0.1])$	$([0.1,0.3],[0.03,0.2])$

vector ϑ of order 10 for criteria.

$$\vartheta = \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \end{array} \begin{bmatrix} ([0.0345,0.7238],[0.0122,0.2296]) \\ ([0.1481,0.6187],[0.0371,0.1962]) \\ ([0.0622,0.7934],[0.0133,0.1311]) \\ ([0.0101,0.7929],[0.0985,0.0985]) \\ ([0.0661,0.834],[0.05,0.04996]) \\ ([0.0032,0.8068],[0.0566,0.1333]) \\ ([0.0636,0.9364],[0,0]) \\ ([0.0096,0.7532],[0.0303,0.2069]) \\ ([0.0034,0.8248],[0.0172,01545]) \\ ([0.0183,0.893],[0.0443,0.0443]) \end{bmatrix}$$

Table 2.2: Correlation matrix of criteria

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
C_1	1	0.34143	0.4916	0.7519	0.6753	0.6545	0.5340	0.5456	0.5238	0.5916
C_2	0.34143	1	0.6818	0.6196	0.4701	0.4649	0.8425	0.3626	0.4063	0.7271
C_3	0.4916	0.6818	1	0.43789	0.7069	0.4174	0.7336	0.4799	0.4896	0.5392
C_4	0.7519	0.6196	0.43789	1	0.68274	0.7151	0.71781	0.50639	0.51029	0.7957
C_5	0.6753	0.4701	0.7069	0.6827	1	0.5608	0.6899	0.7058	0.5396	0.6407
C_6	0.6545	0.4649	0.4174	0.7151	0.5608	1	0.5822	0.4546	0.5021	0.712
C_7	0.5340	0.8425	0.7336	0.7178	0.6899	0.5822	1	0.6394	0.6318	0.7758
C_8	0.5456	0.3626	0.4799	0.5064	0.7058	0.4546	0.6394	1	0.7034	0.5102
C_9	0.5238	0.4063	0.4896	0.5103	0.5396	0.5021	0.6318	0.7034	1	0.4751
C_{10}	0.5916	0.7271	0.5392	0.7957	0.6407	0.712	0.7758	0.5102	0.4751	1

Construct $\vartheta \times \vartheta$ using max-min (membership) and min-max (non-membership) operations. Take a relation $\widetilde{R}_I = \vartheta \times \vartheta$ with diagonal entries in $\widetilde{R}_I$ replaced by $([1,1],[0,0])$ to make it reflexive.

Apply the hierarchical clustering (Algorithm 2.5.1), we obtain five clusters $S_1 = \{C_1, C_2, C_3, C_5, C_7, C_{10}\}$, $S_2 = \{C_4\}$, $S_3 = \{C_8\}$, $S_4 = \{C_9\}$, $S_5 = \{C_6\}$. The pivot element, from (2.6.4), turns out to be C_2; insert it in each S_l to get $S_l^{'}$, $l = 1, \ldots, 5$.

$$
S_1^{'} = \begin{array}{c|c|c|c|c|c|c|}
 & C_1 & C_2 & C_3 & C_5 & C_7 & C_{10} \\
\hline
C_1 & ([1,1],[0,0]) & \begin{pmatrix}[0.034,0.62], \\ {[0.037,0.23]}\end{pmatrix} & \begin{pmatrix}[0.034,0.724], \\ {[0.013,0.23]}\end{pmatrix} & \begin{pmatrix}[0.034,0.724], \\ {[0.05,0.23]}\end{pmatrix} & \begin{pmatrix}[0.034,0.724], \\ {[0.012,0.23]}\end{pmatrix} & \begin{pmatrix}[0.018,0.724], \\ {[0.044,0.23]}\end{pmatrix} \\
\hline
C_2 & \begin{pmatrix}[0.034,0.62], \\ {[0.037,0.23]}\end{pmatrix} & ([1,1],[0,0]) & \begin{pmatrix}[0.0622,0.62], \\ {[0.037,0.196]}\end{pmatrix} & \begin{pmatrix}[0.066,0.62], \\ {[0.05,0.196]}\end{pmatrix} & \begin{pmatrix}[0.063,0.62], \\ {[0.037,0.196]}\end{pmatrix} & \begin{pmatrix}[0.018,0.62], \\ {[0.05,0.196]}\end{pmatrix} \\
\hline
C_3 & \begin{pmatrix}[0.034,0.724], \\ {[0.013,0.23]}\end{pmatrix} & \begin{pmatrix}[0.0622,0.62], \\ {[0.037,0.196]}\end{pmatrix} & ([1,1],[0,0]) & \begin{pmatrix}[0.062,0.793], \\ {[0.05,0.131]}\end{pmatrix} & \begin{pmatrix}[0.062,0.793], \\ {[0.13,0.131]}\end{pmatrix} & \begin{pmatrix}[0.018,0.793], \\ {[0.05,0.131]}\end{pmatrix} \\
\hline
C_5 & \begin{pmatrix}[0.034,0.724], \\ {[0.05,0.23]}\end{pmatrix} & \begin{pmatrix}[0.066,0.62], \\ {[0.05,0.196]}\end{pmatrix} & \begin{pmatrix}[0.062,0.793], \\ {[0.05,0.131]}\end{pmatrix} & ([1,1],[0,0]) & \begin{pmatrix}[0.063,0.834], \\ {[0.05,0.05]}\end{pmatrix} & \begin{pmatrix}[0.018,0.534], \\ {[0.05,0.05]}\end{pmatrix} \\
\hline
C_7 & \begin{pmatrix}[0.034,0.724], \\ {[0.012,0.23]}\end{pmatrix} & \begin{pmatrix}[0.063,0.62], \\ {[0.037,0.196]}\end{pmatrix} & \begin{pmatrix}[0.062,0.793], \\ {[0.013,0.131]}\end{pmatrix} & \begin{pmatrix}[0.063,0.834], \\ {[0.05,0.05]}\end{pmatrix} & ([1,1],[0,0]) & \begin{pmatrix}[0.018,0.893], \\ {[0.044,0.044]}\end{pmatrix} \\
\hline
C_{10} & \begin{pmatrix}[0.018,0.724], \\ {[0.044,0.23]}\end{pmatrix} & \begin{pmatrix}[0.018,0.627], \\ {[0.05,0.196]}\end{pmatrix} & \begin{pmatrix}[0.018,0.793], \\ {[0.05,0.131]}\end{pmatrix} & \begin{pmatrix}[0.018,0.834], \\ {[0.05,0.05]}\end{pmatrix} & \begin{pmatrix}[0.018,0.893], \\ {[0.044,0.044]}\end{pmatrix} & ([1,1],[0,0]) \\
\hline
\end{array}
$$

$$S_2^{'} = \begin{array}{c|c|c|} & C_2 & C_4 \\ \hline C_2 & ([1,1],[0,0]) & ([0.0101,0.62],[0.098,0.196]) \\ \hline C_4 & ([0.0101,0.6],[0.098,0.196]) & ([1,1],[0,0]) \\ \hline \end{array}$$

$$S_3^{'} = \begin{array}{c|c|c|} & C_2 & C_8 \\ \hline C_2 & ([1,1],[0,0]) & ([0.0096,0.62],[0.037,0.196]) \\ \hline C_8 & ([0.0096,0.62],[0.037,0.196]) & ([1,1],[0,0]) \\ \hline \end{array}$$

$$S_4^{'} = \begin{array}{c|c|c|} & C_2 & C_9 \\ \hline C_2 & ([1,1],[0,0]) & ([0.0034,0.62],[0.037,0.196]) \\ \hline C_9 & ([0.0034,0.62],[0.037,0.196]) & ([1,1],[0,0]) \\ \hline \end{array}$$

$$S_5^{'} = \begin{array}{c|c|c|} & C_2 & C_6 \\ \hline C_2 & ([1,1],[0,0]) & ([0.0033,0.62],[0.06,0.196]) \\ \hline C_6 & ([0.0033,0.62],[0.06,0.196]) & ([1,1],[0,0]) \\ \hline \end{array}$$

The local and global weights are given in Table 2.3.

Once the global weight of the criteria are known, the entire information in the criteria-alternatives, with entries as IVIF numbers, along with the global weights can be aggregated to rank various alternatives in an MCDM problem. This can be achieved by variety of methods, like TOPSIS [160], VIKOR [161], ELECTRE [162], AHP with preference score [163], to name a few.

2.7 Conclusion

The chapter extends the work of Guh et al. [52] to laid down the procedure for constructing hierarchical clustering for IVIFRs. MATLAB codes are developed for each algorithm presented in this work and the same are implemented on large number of matrices representation of IVIFRs. MATLAB code are given in Appendix-2. The chapter includes instances of few such implementations for illustrative and explanatory purpose. To demonstarte the application of the proposed clustering scheme, an important problem of determining the criteria weights in an MCDM problem, with DM providing entries of criteria-alternative matrix as IVIF values, is addressed. A detailed procedure is presented

Clusters	Criteria	Local weight	Global weight
S_1'	C_1	0.1534	0.097
	C_2	0.1426	0.091
	C_3	0.1706	0.109
	C_5	0.1756	0.112
	C_7	0.1789	0.114
	C_{10}	0.1784	0.113
S_2'	C_2	0.5	0.091
	C_4	0.5	0.091
S_3'	C_2	0.5	0.091
	C_8	0.5	0.091
S_4'	C_2	0.5	0.091
	C_9	0.5	0.091
S_5'	C_2	0.5	0.091
	C_6	0.5	0.091

Table 2.3: Global weights of criteria

with supportive illustration to compute the global weights of the criteria.

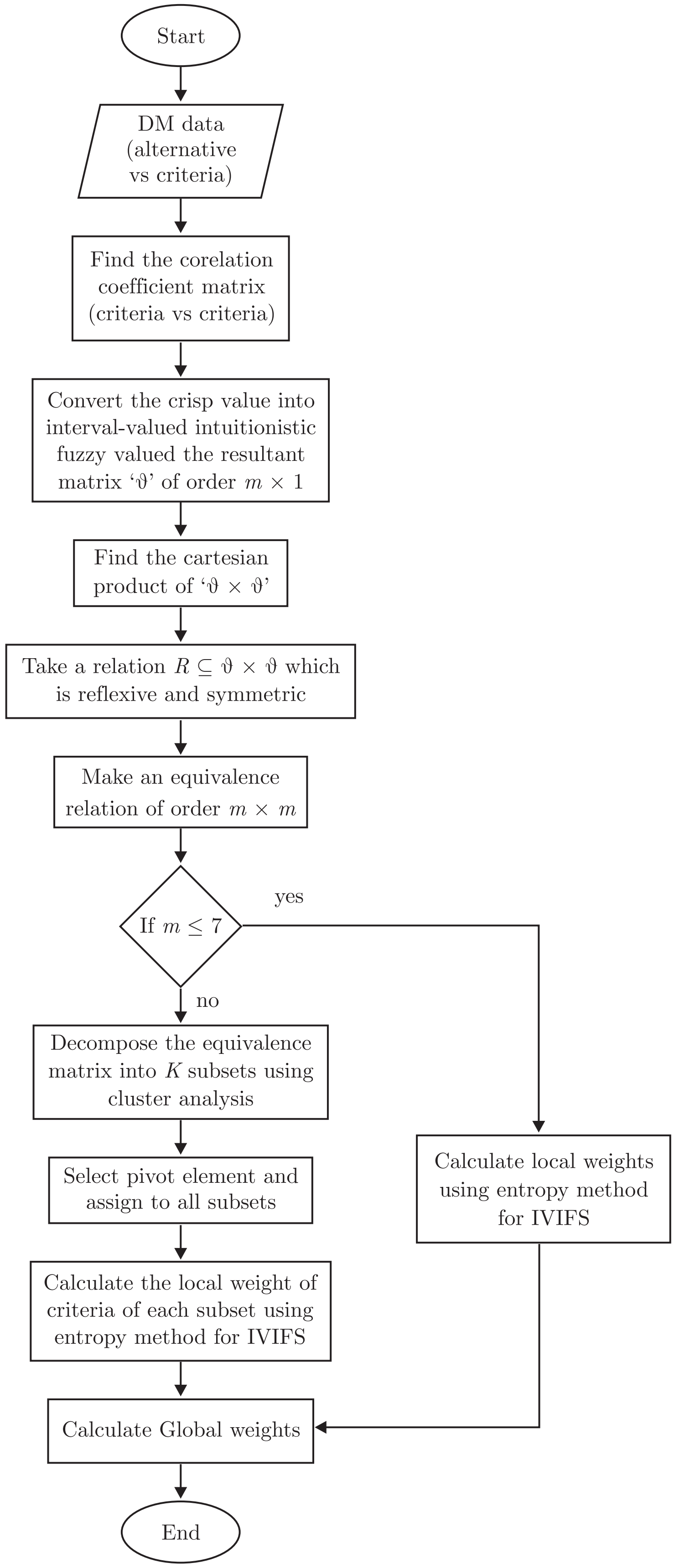

Figure 2.4: Flow chart describing procedure (Algorithm 2.6.1) for eliciting criteria weights

Chapter 3

Consistency of intuitionistic multiplicative preference relation

---

This chapter[1], [2], is dedicated for the consistency of intuitionistic multiplicative preference relation(IMPR). We propose two approach algebraic and graphical. In first approach we propose a characterization of the consistency property using newly defined transitivity property for intuitionistic multiplicative preference relations (IMPR) together with complementing missing elements for incomplete IMPR. Using new transitivity property of IMPR, we have developed two different methods to find the missing element of IMPRs. The first method is two-step procedure method containing estimating step followed by adjusting step. In estimating step, the missing elements of incomplete IMPRs are calculated by using a new transitive property. Sometimes the initial value may not satisfy the conditions of IMPRs. An optimization model is developed in the second step to adjust the initial values that are solved by MATLAB optimization tool. The second proposed method is goal programming model based on new transitivity property to calculate the missing elements directly. Acceptably Consistent with complete IMPRs is also checked. The second approach is to achieve the consistent intuitionistic multiplicative preference relation graphically. We have proposed two different characterization of the consistency for IMPR. In the first method, we propose an algorithm to achieve the consistency of IMPR by using the cycles of various length in a directed graph. The second method proves isomorphism

[1]The result of this chapter is based on a research paper "New transitivity property of intuitionistic multiplicative preference relation and its application in missing value estimation" *Annals of Fuzzy Mathematics and Informatics* **16** (1) 71–86 (2018).

[2]"Two different Approaches for consistency of Intuitionistic Multiplicative Preference Relation using Directed Graph submitted" in *Asia-pacific journal of operational research*.

between the set of IMPRs and the set of asymmetric multiplicative preference relations. That result is explored to use the methodologies developed for asymmetric multiplicative preference relations to the case of IMPRs and achieve the consistency of asymmetric multiplicative preference relation using directed graph. Also, the above said method is applied for incomplete IMPR, here consistency play an important role. The examples are provided to illustrate all the methods mentioning all cases.

3.1 Introduction

Decision making is one of the most important tasks for individuals and organizations and is an interdisciplinary research area attracting researchers from almost all fields from psychologists, economists, to computer scientists.

Consistency plays a significant role in decision-making process. A good amount of researchers has given their attention in decision-making to the use of consistency of preference relations under uncertain environments, which is discussed in chapter 1. For good understanding one may refer to ([77], [89], [164] – [168]).

In 2015 Jiang et al. [143] discussed the consistency and acceptable consistency property of an IMPR. Also, Jiang et al. [143] worked on IMPRs in incomplete scenario, in which the IMPRs split into two MPRs, and the missing elements were calculated by using the consistency of MPRs. In light of it, two strategies were developed to estimate all missing elements of incomplete IMPRs.

At times, it has witnessed that in a situation, decision maker might not have decent comprehension of a specific query, and thus he/she can not make an instaneous contrast between each two objects; subsequently, it necessities to allow the decision maker to avoid some questionable comparisons adaptably. For this situation, incomplete preference relations are obtained, and result of that the whole process may slow down. To introduce a complete preference relation, a decision maker need $\frac{n(n-1)}{2}$ judgments at each level of the decision-making process. When the order of complete preference relation matrix i.e. n is large, it becomes an onerous task. Therefore sometimes, due to lack of time and busy schedule of the decision maker, incomplete preference relations are obtained. Our work focus on both complete and incomplete IMPR.

In this chapter, we have characterized a new consistency property of IMPR. In the view of which, two novel techniques are provided, where one is traditional "two-step procedure methods" and other is goal programming methods for finding the unknown element of incomplete IMPRs. The "two-step procedure methods" is consists of two sub-steps such as: (i) "Estimating step": Initial values are evaluated for the missing elements of the incomplete IMPRs without splitting into two MPRs; (ii)"Adjusting step": An optimization model is developed to adjust the initial values derived from the estimating step which is solved by MATLAB optimization tool. In goal programming model missing elements are estimated from incomplete IMPR without splitting into two MPRs. Both novel methods give the equivalent result. Then the acceptably consistent of IMPR has been checked. Techniques are illustrated with suitable examples.

The another goal of this chapter is to improve the consistency of IMPR using graphical approach. Again we have developed two approaches. In the first approach, we have developed an ordered-pair binary matrix from the IMPR and which is split into two binary arrays. Getting motivation from Nishizawa [138] algorithm, we have checked the consistency of IMPR.

The second approach is to prove the mathematical equivalence between the set of IMPR and the set of asymmetric multiplicative preference relations. This result can thus be exploited to use methodologies developed for MPRs to the case of IMPR and, ultimately, to extend the use of IMPR in decision making and to overcome the computation complexity mentioned above. In other words, this result will allow taking advantage of mature and well-defined methodologies developed for MPRs while controlling the flexibility of IMPR to model vagueness/uncertainty.

Indeed, an issue that can be addressed using the mentioned equivalence is the presence of incomplete IMPR in decision-making process. Using directed graph [138] we check the consistency of asymmetric multiplicative preference relation. The individual comparisons of the two approaches and improvement in consistency of IMPR are also given.

Chapter is organised as follows. In section 3.2, some basic concepts are defined briefly, and new transitivity property of IMPR is defined. In section 3.3, we have developed an algorithm to find the missing elements by using newly defined transitivity property of IMPR and proposed an optimization model for adjusting the initial values. Also a goal programming model is intended to complete the incomplete IMPRs. Two numerical examples illustrate the developed procedures. In this section we have given a comparative analysis of our work with the work of [143] and [169]. Section 3.4 discuss the consistency of IMPR graphically. This section divides three subsection. The subsection 3.4.1 demonstrates the first method that improves the consistency of IMPRs using directed graph. Also this section include incomplete IMPRs also. In the second method, the set of IMPR and the set of asymmetric multiplicative preference relations has proved mathematically isomorphic which is discussed in subsection 3.4.2. Numerical example is also given in this section for both complete or incomplete intuitionistic scenario. In subection 3.4.3, we have given comparison between the two approaches and improve the consistency of IMPR.

3.2 Preliminaries

Let $X = \{x_1, \cdots, x_n\}$ be a discrete set of alternatives/criteria in a decision making problem and set $N = \{1, \cdots, n\}$ be the set of indices. A decision maker needs to provide his/her preferences over the alternatives/criteria using the pairwise comparison method. The preference values are provided by the decision maker from the ratio scale $[1/9, 9]$ introduced by Saaty [58] to estimate and differentiate the intensity of preferences. Based on the above ratio scale, some basic concepts of IMPRs are defined.

In 2013 Xu [64] gave the concept of the consistent property of IMPR by denoting $\tilde{R} = [\tilde{r}_{ij}]_{n\times n} = [(\mu_{ij},\ \nu_{ij})]_{n\times n}$ using multiplicative transitivity property.

Definition 3.2.1. An IMPR $\widetilde{R}$ is called multiplicative transitive if

$$\mu_{ij} = \mu_{ik}\,\mu_{kj}\,, \nu_{ij} = \nu_{ik}\,\nu_{kj}\,, \forall\, i \leq k \leq j,\ \ i,k,j \in N. \tag{3.2.1}$$

It is to take note of that equation 3.2.1, for the condition $i \leq k \leq j$, there is some limitation. While for all $i, j, k \in N$, the transitivity property of MPR is unconstrained. The IMPR is consistent if the transitivity properties given in equation 3.2.1 is satisfied. But, sometimes it may be the case that the transitivity and consistency properties does not hold. For example:

$$\begin{pmatrix} (1,1) & (1/2,1) & (1,1/2) \\ (1,1/2) & (1,1) & (2,1/2) \\ (1/2,1) & (1/2,2) & (1,1) \end{pmatrix}$$

is a consistent IMPR given in [64]. Jiang et al. [143] relax the condition $i \leq k \leq j$, it is follow that $a_{23} = (\mu_{21}\mu_{13}, \nu_{21}\nu_{13}) = (1, 1/4)$. But $a_{23} = (2, 1/2) \neq (1, 1/4)$. This equation is not satisfy because when 'k' comes from the row of lower triangular matrix. To over come this type of transitivity limitation, Jiang et al. [143] proposed a more general consistency property of an IMPR by splitting into two MPRs by using the formula

$$b_{ij}^{(1)} = \begin{cases} \mu_{ij} & i < j \\ 1 & i = j \\ 1/\nu_{ij} & i > j \end{cases} \text{ and } b_{ij}^{(2)} = \begin{cases} \nu_{ij} & i < j \\ 1 & i = j \\ 1/\mu_{ij} & i > j\,. \end{cases} \tag{3.2.2}$$

where the MPRs $B^{(1)} = (b_{ij}^{(1)})_{n\times n}$ and $B^{(2)} = (b_{ij}^{(2)})_{n\times n}$ are the preferred and non-preferred information matrix respectively given by the DM with respect to the alternative x_i over x_j. In view of the above idea, Jiang et al. [143] characterized the consistent IMPR.

Definition 3.2.2. [143] IMPR $\tilde{R} = [\tilde{r}_{ij}]_{n\times n}$ is said to be consistent if both MPRs $B^{(1)}$ and $B^{(2)}$ given by equation 3.2.2 are consistent such that

$$b_{ij}^{(1)} = b_{ik}^{(1)} b_{kj}^{(1)},\ b_{ij}^{(2)} = b_{ik}^{(2)} b_{kj}^{(2)}\ \forall\, i,j,k \in N$$

In this work, instead of splitting IMPRs into MPRs we have defined an new consistency property of IMPR.

Definition 3.2.3. An IMPRs $\tilde{R} = [\tilde{r}_{ij}]_{n\times n}$ is called consistent if it satisfy the transitivity property, where $\tilde{r}_{ij}$ is

$$(\mu_{ij}, \nu_{ij}) = \begin{cases} (\mu_{ik}, \nu_{ik}) \otimes (\mu_{kj}, \nu_{kj}) & \text{if } i \leq k, k \leq j \\ (\frac{1}{\mu_{ki}}, \frac{1}{\nu_{ki}}) \otimes (\mu_{kj}, \nu_{kj}) & \text{if } i \geq k, k \leq j \\ (\mu_{ik}, \nu_{ik}) \otimes (\frac{1}{\mu_{jk}}, \frac{1}{\nu_{jk}}) & \text{if } i < k, k > j \end{cases} \tag{3.2.3}$$

In this work for convenience, we have used the multiplication of two IMNs as the multiplication of two order pairs. Let $a = (\mu, \nu)$, $a_1 = (\mu_1, \nu_1)$ and $a_2 = (\mu_2, \nu_2)$ are intuitionistic multiplicative numbers (IMNs) and $\lambda > 0$, then
$a_1 \otimes a_2 = (\mu_1, \nu_1) \otimes (\mu_2, \nu_2) = (\mu_1\mu_2, \nu_1\nu_2)$
$a^\lambda = \left(\frac{2\mu^\lambda}{(2+\mu)^\lambda - \mu^\lambda}, \frac{(1+2\nu)^\lambda - 1}{2}\right)$ is given by Xia [63] for $\lambda > 0$.

The idea of IMPRs is stretched out to the circumstances where the preference data are given by decision maker is incomplete. Jiang et al. [143] propose to extend the above situation to incomplete IMPR where some elements are missing in the preference relation matrix.

Definition 3.2.4. [143] An IMPR $\tilde{r}_{ij} = (\mu_{ij}, \nu_{ij})_{n\times n}$ is called an incomplete IMPR if some elements in it are missing, and all available elements satisfy the characteristics of IMPR stated in Definition 1.2.3.

3.3 Complementing an incomplete IMPRs

In this section, to evaluate the missing elements from an incomplete IMPRs, we define the consistent property of incomplete IMPR.

Definition 3.3.1. An incomplete IMPR is said to be consistent if all the known element satisfy equation 3.2.3.

In an incomplete MPR $R = (r_{ij})_{n\times n}$, the element r_{ij} and r_{kl} are called adjoining if

$(i, j) \cap (k, l)$ is non empty set, e.g if $r_{i_0 j_0} = r_{i_0 k} \times r_{k j_0}$, where $r_{i_0 k}$ and $r_{k j_0}$ are adjoining known elements and $r_{i_0 j_0}$ be an unknown elements, then $r_{i_0 j_0}$ can be found directly and the corresponding incomplete MPR is called acceptable. MPR R is called an unacceptable incomplete MPR, if there does not exist adjoining known element such that unknown factor can be calculated [66]. In that cases, therefore, it is necessary to return the unacceptable incomplete MPR to the decision maker for revaluation until an acceptable incomplete MPR can be obtained. Wang and Xu [170] showed that if their exit at least $n-1$ judgment provided by the decision maker then an incomplete MPRs are acceptable. In the other sense, there exists at least one known element(except diagonal elements) in each line/column of MPR matrix R given by the decision maker. Later according to Cai and Deng [171], Xu [66] and Alonso et al. [133] prove that in incomplete MPR which is acceptable, there exists at least a set of an $n-1$ number of non-leading diagonal known elements, where each of the criteria is compared at least once, which includes the case when a complete row or column of preference values is known. In this work, we have applied the same above-said applications in the incomplete IMPRs scenario.

Taking inspiration from the work of Jiang et al. [143], we propose a two-step procedure method to estimate the missing values in an incomplete IMPR without splitting IMPRs into two MPRs. The idea is first to evaluate their values using the simple connecting path approach and subsequently improve upon them using an optimization problem.

3.3.1 Estimating step

Harker ([95], [96]) developed a geometric mean method dependent on the connective paths to calculate the missing data from an incomplete MPR. The general structure of a connecting path of length $\ell+1$, denoted by $cp_{(\ell+1)}$, has the following form: $cp_{(\ell+1)}$: $*_{ij} = r_{ik_1} r_{k_1 k_2} \ldots r_{k_\ell j}$, where $r_{ik_1}, r_{k_1 k_2}, \ldots r_{k_\ell j}$ are the known values in the connecting path from i to j, where $i, j, k_1, \ldots, k_\ell \in N$, $0 \leq \ell \leq n-2$, and $*_{ij}$ denotes the missing element to be calculated. The connecting path of length two is an elementary connecting path $cp_{(2)} : *_{ij} = r_{ik_1} r_{k_1 j}$ for $k_1 \in N$, and $k_1 \neq i, j$. Harker ([95], [96]) also contended that the unknown element $*_{ij}$ of MPR can be estimated by using the geometric mean of all elementary connecting paths related to it with no vertex repeats more than once in the path. Consequently, $*_{ij} = \left(\prod_{\xi=1}^{n_\ell} cp_{(\xi)}\right)^{1/n_\ell}$, where n_ℓ is the possible number of connecting fully known paths (that is, no missing entries along the path) from i to j. A major limitation of this technique in some genuine problem is that the number of connecting paths of various length among i and j might be extremely large and computationally unmanageable. For instance, Deschrijver and Kerre [106] presented an example of a matrix of size 10 only with the number of connecting paths exceeding 109,000. Jiang et al. [143] improved this method for incomplete IMPR by taking elementary connecting path instead of all connecting paths of all sizes. In that case the matrix of size 10, the number of all elementary connecting path would not surpass 8, which is much less than 109,000. Here, we have extended this technique in incomplete IMPRs framework. Based on the new consistency property of IMPRs, the initial value of the missing element of incomplete IMPRs can be calculated by using a geometric mean method which is denoted by $\tilde{r}'_{ij}$, where

$$\tilde{r}'_{ij} = (\mu^*_{ij}, \nu^*_{ij}) = \begin{cases} \left(\prod_{k\in T_{ij}}\{(\mu_{ik},\nu_{ik})\otimes(\mu_{kj},\nu_{kj})\}\right)^{1/t_{ij}} & \text{if } i\leq k, k\leq j; \\ \left(\prod_{k\in T_{ij}}\{(\frac{1}{\mu_{ki}},\frac{1}{\nu_{ki}})\otimes(\mu_{kj},\nu_{kj})\}\right)^{1/t_{ij}} & \text{if } i\geq k, k\leq j; \\ \left(\prod_{k\in T_{ij}}\{(\mu_{ik},\nu_{ik})\otimes(\frac{1}{\mu_{jk}},\frac{1}{\nu_{jk}})\}\right)^{1/t_{ij}} & \text{if } i< k, k> j \end{cases} \quad (3.3.1)$$

where $T_{ij} = \{k|(\mu_{ik},\nu_{ik}),(\mu_{kj},\nu_{kj}) \in \Omega\}$, Ω is the set of known element and t_{ij} is the number of element present in the set T_{ij} which indicates that there may exist different pairs of adjoining known elements to find out the unknown elements. The initial values are denoted by $\left(\mu^{*(0)}_{ij}, \nu^{*(0)}_{ij}\right)$.

Remark 3.3.1. It is to note that in equation 3.3.1, $\mu_{kj} \times \frac{1}{\mu_{ki}} \neq \mu_{kj} \times \mu_{ik}$, and it should follow in other expression also.

3.3.2 Adjusting step

The IMPR is consistent if equation 3.2.3 is satisfied. Sometimes initial values of the missing element may not satisfy the conditions of IMPRs. To overcome this difficulty we have developed a local optimization model (Model 3.1) by minimizing the error.

(Model 3.1)

$$\text{Min} \sum_{i,k=1}^{n}\sum_{j=i+1}^{n}\left(\varepsilon^k_{ij_{i\leq k,k\leq j}} + \varepsilon^k_{ij_{i\geq k,k\leq j}} + \varepsilon^k_{ij_{i<k,k>j}}\right)$$

Subject to

$$\varepsilon^k_{ij_{i\leq k,k\leq j}} = |\, (\mu_{ij},\nu_{ij}) - (\mu_{ik}\mu_{kj}, \nu_{ik}\nu_{kj})\,|;$$

$$\varepsilon^k_{ij_{i\geq k,k\leq j}} = \left|(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{kj}}{\mu_{ki}}, \frac{\nu_{kj}}{\nu_{ki}}\right)\right|;$$

$$\varepsilon^k_{ij_{i<k,k>j}} = \left|(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{ik}}{\mu_{jk}}, \frac{\nu_{ik}}{\nu_{jk}}\right)\right|;$$

$$\mu_{ij}\nu_{ij} \leq 1;\ 1/9 \leq \mu_{ij};\ \nu_{ij} \leq 9;$$

$$\mu^{(0)}_{ij} = \mu^{*(0)}_{ij};\ \nu^{(0)}_{ij} = \nu^{*(0)}_{ij};\ i\neq j\neq k;\ i,j,k \in N.$$

where $\mu_{ij}^{*(0)}$ and $\nu_{ij}^{*(0)}$ are the initial value obtain from estimating step. For proper understanding, we have represented an algorithm that illustrates the above methods.

Algorithm 3.3.1. 1. Consider an incomplete IMPRs of $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ in which some elements are missing and the known elements satisfy the transitivity property of equation 3.2.3.

2. The initial value $\tilde{r}_{ij}^{'}$ of the missing element of incomplete IMPRs is calculated using equation 3.3.1.

3. Initial values obtained in step 2 are adjusted by the Model 3.1, and the adjusting values are denoted by $\tilde{r}_{ij}^{''}$.

4. The complete IMPRs $\tilde{R}_c = (\tilde{r}_{c,_{ij}})_{n\times n}$, is obtained where

$$\tilde{r}_{c,_{ij}} = \begin{cases} \tilde{r}_{ij}^{''} & \tilde{r}_{ij} \notin \Omega \\ \tilde{r}_{ij} & \tilde{r}_{ij} \in \Omega. \end{cases} \tag{3.3.2}$$

In the next subsection, we have developed a goal programming model to estimate the missing values.

3.3.3 Goal programming model to estimate the missing values

In 2015 Meng and Chen [169] construct a linear programming model to evaluate the missing value with incomplete MPRs, which is based on consistency index. To cope with incomplete IMPR, this subsection developed a deviation model to evaluate the missing value which is based on new transitivity property which was discussed in section 3.2. Let $\tilde{R} = (\mu_{ij}, \nu_{ij})$ be an incomplete IMPR. We know that $\tilde{R}$ is consistent if and only if equation 3.2.3 holds for the strictly upper triangular elements. To minimize the errors, Approximate the equation 3.2.3. Define

$$\begin{aligned} (\varepsilon_{ij})_{i\le k,k\le j} &= \delta_{ij}\left|(\mu_{ij},\nu_{ij}) - \left(\mu_{ik}\mu_{kj}, \nu_{ik}\nu_{kj}\right)\right| \\ (\varepsilon_{ij})_{i\ge k,k\le j} &= \delta_{ij}\left|(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{kj}}{\mu_{ki}}, \frac{\nu_{kj}}{\nu_{ki}}\right)\right| \\ (\varepsilon_{ij})_{i< k,k> j} &= \delta_{ij}\left|(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{ik}}{\mu_{jk}}, \frac{\nu_{ik}}{\nu_{jk}}\right)\right| \\ &\text{for } i,j = 1,2,\cdots n,\ i<j \end{aligned} \tag{3.3.3}$$

$$\text{where,}\ \delta_{ij} = \begin{cases} 1 & k \in T_{ij} \\ 0 & \text{otherwise} \end{cases}$$

where $T_{ij} = \{k|(\mu_{ik}, \nu_{ik}), (\mu_{kj}, \nu_{kj}) \in \Omega\}$, T_{ij} is the set of known element. Using above equation 3.3.3, we construct the following goal programming model to estimate the missing value

$$
\begin{aligned}
&\mathrm{Min}(\varepsilon_{ij})_{i\le k,k\le j} = \delta_{ij}|(\mu_{ij},\nu_{ij}) - (\mu_{ik}\mu_{kj},\nu_{ik}\nu_{kj})| \\
&\mathrm{Min}(\varepsilon_{ij})_{i\ge k,k\le j} = \delta_{ij}\left|(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{kj}}{\mu_{ki}},\frac{\nu_{kj}}{\nu_{ki}}\right)\right| \\
&\mathrm{Min}(\varepsilon_{ij})_{i<k,k>j} = \delta_{ij}\left|(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{ik}}{\mu_{jk}},\frac{\nu_{ik}}{\nu_{jk}}\right)\right| \\
&\text{subject to,} \\
&\frac{1}{9} \le \mu_{ij};\ \nu_{ij} \le 9;\ (\mu_{ij},\nu_{ij}) \in U; \\
&\text{where } U = \{(\mu_{ij},\nu_{ij})|(\mu_{ij},\nu_{ij}) \text{ is a missing value} \\
&\text{for } i,j = 1,2,\cdots n,\ i<j\}
\end{aligned}
$$

The solution of the minimization problem can be obtained by solving the goal programming Model 3.2.

(Model 3.2)

$$
\begin{aligned}
\mathrm{Min}\, D = \sum_{i,k=1}^{n}\sum_{j=i+1}^{n} &\left(d_{ij,k}^{(+)}\right)_{i\le k,k\le j} + \left(d_{ij,k}^{(-)}\right)_{i\le k,k\le j} + \left(d_{ij,k}^{(+)}\right)_{i\ge k,k\le j} \\
&+ \left(d_{ij,k}^{(-)}\right)_{i\ge k,k\le j} + \left(d_{ij,k}^{(+)}\right)_{i<k,k>j} + \left(d_{ij,k}^{(-)}\right)_{i<k,k>j}
\end{aligned}
$$

subject to,

$$
\begin{aligned}
&\delta_{ij}\{(\mu_{ij},\nu_{ij}) - (\mu_{ik}\mu_{kj},\nu_{ik}\nu_{kj})\} - \left(d_{ij,k}^{(+)}\right)_{i\le k,k\le j} + \left(d_{ij,k}^{(-)}\right)_{i\le k,k\le j} = 0; \\
&\delta_{ij}\left\{(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{kj}}{\mu_{ki}},\frac{\nu_{kj}}{\nu_{ki}}\right)\right\} - \left(d_{ij,k}^{(+)}\right)_{i\ge k,k\le j} + \left(d_{ij,k}^{(-)}\right)_{i\ge k,k\le j} = 0; \\
&\delta_{ij}\left\{(\mu_{ij},\nu_{ij}) - \left(\frac{\mu_{ik}}{\mu_{jk}},\frac{\nu_{ik}}{\nu_{jk}}\right)\right\} - \left(d_{ij,k}^{(+)}\right)_{i<k,k>j} + \left(d_{ij,k}^{(-)}\right)_{i<k,k>j} = 0; \\
&\mu_{ij}\nu_{ij} \le 1;\ 1/9 \le \mu_{ij};\ \nu_{ij} \le 9; \\
&\left(d_{ij,k}^{(+)}\right)_{i\le k,k\le j}, \left(d_{ij,k}^{(-)}\right)_{i\le k,k\le j}, \left(d_{ij,k}^{(+)}\right)_{i\ge k,k\le j}, \left(d_{ij,k}^{(-)}\right)_{i\ge k,k\le j}, \left(d_{ij,k}^{(+)}\right)_{i<k,k>j}, \left(d_{ij,k}^{(-)}\right)_{i<k,k>j} \ge 0;
\end{aligned}
$$

where,

$$
\begin{aligned}
&\left(d_{ij,k}^{(+)}\right)_{i\le k,k\le j} = \left[(\mu_{ij},\nu_{ij}) - (\mu_{ik}\mu_{kj},\nu_{ik}\nu_{kj})\right] \vee 0; \quad \left(d_{ij,k}^{(-)}\right)_{i\le k,k\le j} = \left[(\mu_{ik}\mu_{kj},\nu_{ik}\nu_{kj}) - (\mu_{ij},\nu_{ij})\right] \vee 0; \\
&\left(d_{ij,k}^{(+)}\right)_{i\ge k,k\le j} = \left[(\mu_{ij},\nu_{ij}) - (\frac{\mu_{kj}}{\mu_{ki}},\frac{\nu_{kj}}{\nu_{ki}})\right] \vee 0; \qquad \left(d_{ij,k}^{(-)}\right)_{i\ge k,k\le j} = \left[(\frac{\mu_{kj}}{\mu_{ki}},\frac{\nu_{kj}}{\nu_{ki}}) - (\mu_{ij},\nu_{ij})\right] \vee 0; \\
&\left(d_{ij,k}^{(+)}\right)_{i<k,k>j} = \left[(\mu_{ij},\nu_{ij}) - (\frac{\mu_{ik}}{\mu_{jk}},\frac{\nu_{ik}}{\nu_{jk}})\right] \vee 0; \qquad \left(d_{ij,k}^{(-)}\right)_{i<k,k>j} = \left[(\frac{\mu_{ik}}{\mu_{jk}},\frac{\nu_{ik}}{\nu_{jk}}) - (\mu_{ij},\nu_{ij})\right] \vee 0;
\end{aligned}
$$

For the sake of convenience, here we use,

$\left(d_{ij,k}^{(+)}\right)_{i\le k,k\le j}=\left(d_{\mu_{ij,k}}^{(+)},d_{\nu_{ij,k}}^{(+)}\right)_{i\le k,k\le j},\ \left(d_{ij,k}^{(-)}\right)_{i\le k,k\le j}=\left(d_{\mu_{ij,k}}^{(-)},d_{\nu_{ij,k}}^{(-)}\right)_{i\le k,k\le j},$

$\left(d_{ij,k}^{(+)}\right)_{i\ge k,k\le j}=\left(d_{\mu_{ij,k}}^{(+)},d_{\nu_{ij,k}}^{(+)}\right)_{i\ge k,k\le j},\ \left(d_{ij,k}^{(-)}\right)_{i\ge k,k\le j}=\left(d_{\mu_{ij,k}}^{(-)},d_{\nu_{ij,k}}^{(-)}\right)_{i\ge k,k\le j},$

$\left(d_{ij,k}^{(+)}\right)_{i<k,k>j}=\left(d_{\mu_{ij,k}}^{(+)},d_{\nu_{ij,k}}^{(+)}\right)_{i<k,k>j},\ \left(d_{ij,k}^{(-)}\right)_{i<k,k>j}=\left(d_{\mu_{ij,k}}^{(-)},d_{\nu_{ij,k}}^{(-)}\right)_{i<k,k>j}$

where,

$\left(d_{\mu_{ij,k}}^{(+)}\right)_{i\le k,k\le j}=\left[\log\mu_{ij}-(\log\mu_{ik}+\log\mu_{kj})\right]\vee 0;\ \left(d_{\mu_{ij,k}}^{(-)}\right)_{i\le k,k\le j}=\left[(\log\mu_{ik}+\log\mu_{kj})-\log\mu_{ij}\right]\vee 0;$

$\left(d_{\nu_{ij,k}}^{(+)}\right)_{i\le k,k\le j}=\left[\log\nu_{ij}-(\log\nu_{ik}+\log\nu_{kj})\right]\vee 0;\ \left(d_{\nu_{ij,k}}^{(-)}\right)_{i\le k,k\le j}=\left[(\log\nu_{ik}+\log\nu_{kj})-\log\nu_{ij}\right]\vee 0;$

$\left(d_{\mu_{ij,k}}^{(+)}\right)_{i\ge k,k\le j}=\left[\log\mu_{ij}-(\log\mu_{kj}-\log\mu_{ki})\right]\vee 0;\ \left(d_{\mu_{ij,k}}^{(-)}\right)_{i\ge k,k\le j}=\left[(\log\mu_{kj}-\log\mu_{ki})-\log\mu_{ij}\right]\vee 0;$

$\left(d_{\nu_{ij,k}}^{(+)}\right)_{i\ge k,k\le j}=\left[\log\nu_{ij}-(\log\nu_{kj}-\log\nu_{ki})\right]\vee 0;\ \left(d_{\nu_{ij,k}}^{(-)}\right)_{i\ge k,k\le j}=\left[(\log\nu_{kj}-\log\nu_{ki})-\log\nu_{ij}\right]\vee 0;$

$\left(d_{\mu_{ij,k}}^{(+)}\right)_{i<k,k>j}=\left[\log\mu_{ij}-(\log\mu_{ik}-\log\mu_{jk})\right]\vee 0;\ \left(d_{\mu_{ij,k}}^{(-)}\right)_{i<k,k>j}=\left[(\log\mu_{ik}-\log\mu_{jk})-\log\mu_{ij}\right]\vee 0;$

$\left(d_{\nu_{ij,k}}^{(+)}\right)_{i<k,k>j}=\left[\log\nu_{ij}-(\log\nu_{ik}-\log\nu_{jk})\right]\vee 0;\ \left(d_{\nu_{ij,k}}^{(-)}\right)_{i<k,k>j}=\left[(\log\nu_{ik}-\log\nu_{jk})-\log\nu_{ij}\right]\vee 0$

To illustrate the above procedure we have presented two examples.

Example 3.3.1. Let us consider a decision making problem with five sets of alternatives $x_i,\ i=1,2,\cdots,5$. The decision maker judges these five alternatives by pairwise comparison and provides his/her judgement as $\tilde{r}_{12}=(\mu_{12},\nu_{12})=(5,1/7)$, $\tilde{r}_{14}=(\mu_{14},\nu_{14})=(3,1/7)$, $\tilde{r}_{23}=(\mu_{23},\nu_{23})=(9/5,3/7)$, $\tilde{r}_{25}=(\mu_{25},\nu_{25})=(1/5,3)$, $\tilde{r}_{35}=(\mu_{35},\nu_{35})=(1/9,7)$, $\tilde{r}_{45}=(\mu_{45},\nu_{45})=(1/7,3)$. The matrix representation of the above information is given by

$$\tilde{R}=\begin{bmatrix}(1,1)&(5,1/7)&(*,*)&(3,1/7)&(*,*)\\(1/7,5)&(1,1)&(9/5,3/7)&(*,*)&(1/5,3)\\(*,*)&(3/7,9/5)&(1,1)&(*,*)&(1/9,7)\\(1/7,3)&(*,*)&(*,*)&(1,1)&(1/7,3)\\(*,*)&(3,1/5)&(7,1/9)&(3,1/7)&(1,1)\end{bmatrix}_{5\times 5}$$

The initial value of missing element are calculated using equation 3.3.1, are given in table 3.1.

Some initial values does not satisfies the property of IMPRs e.g $\boldsymbol{\mu_{13}\times\nu_{13}\nleq 1}$. To adjust

Table 3.1: Calculation of Missing element(initial value)

Missing element	Adjoining element	Calculated value
$\left(\mu_{13}^{*(0)}, \nu_{13}^{*(0)}\right)$	(μ_{12}, ν_{12}), (μ_{23}, ν_{23})	$(9, 3/49)$
$\left(\mu_{15}^{*(0)}, \nu_{15}^{*(0)}\right)$	(μ_{12}, ν_{12}), (μ_{25}, ν_{25}) (μ_{14}, ν_{14}), (μ_{45}, ν_{45})	$(1.44877, 0.0297)$
$\left(\mu_{24}^{*(0)}, \nu_{24}^{*(0)}\right)$	(μ_{21}, ν_{21}), (μ_{14}, ν_{14}) (μ_{25}, ν_{25}), (μ_{54}, ν_{54})	$(2.38454, 0.366025)$
$\left(\mu_{34}^{*(0)}, \nu_{34}^{*(0)}\right)$	(μ_{35}, ν_{35}), (μ_{54}, ν_{54})	$(0.777, 2.333)$

Table 3.2: Consistency ratio

Two-step procedure Method	CR(C)	0.0114
	CR(D)	0.0094
Goal programming model (Model 3.2)	CR(C')	0.0144
	CR(D')	0.0094

these values we use an optimization Model 3.1.

$$
\begin{aligned}
&\text{Min } \{|\{(9-\mu_{13})^2 + (3/49-\nu_{13})^2\}^{0.5}| + |\{(1-\mu_{15})^2 + (3/7-\nu_{15})^2\}^{0.5}| \\
&+|\{(3/7-\mu_{15})^2 + (3/7-\nu_{15})^2\}^{0.5}| + |\{(3/5-\mu_{24})^2 + (1-\nu_{24})^2\}^{0.5}| \\
&+|\{(7/5-\mu_{24})^2 + (1-\nu_{24})^2\}^{0.5}| + |\{(0.777-\mu_{34})^2 + (2.333-\nu_{34})^2\}^{0.5}|\}; \\
&\text{Subject to} \\
&\mu_{13} \times \nu_{13} \le 1; \mu_{15} \times \nu_{15} \le 1; \mu_{24} \times \nu_{24} \le 1; \\
&\mu_{34} \times \nu_{34} \le 1; 1/9 \le \mu_{13}; \nu_{13} \le 9; \\
&1/9 \le \mu_{15}; \nu_{15} \le 9; 1/9 \le \mu_{24}; \nu_{24} \le 9; \\
&1/9 \le \mu_{34}; \nu_{34} \le 9; {\mu_{13}}^{*(0)} = 9, {\nu_{13}}^{*(0)} = 3/49; \\
&{\mu_{15}}^{*(0)} = 1.44877, {\nu_{15}}^{*(0)} = 0.0297; \\
&{\mu_{24}}^{*(0)} = 2.3845, {\nu_{24}}^{*(0)} = 0.366; \\
&{\mu_{34}}^{*(0)} = 0.777, {\nu_{13}}^{*(0)} = 2.333;
\end{aligned}
$$

After solving the above optimization model, the adjusting values are given $\mu_{13} = 9, \nu_{13} = 0.111, \mu_{15} = 0.74, \nu_{15} = 0.429, \mu_{24} = 0.912, \nu_{24} = 1, \mu_{34} = 0.441, \nu_{34} = 2.268$. This model is solved by MATLAB optimization tool box. The complete IMPR $\tilde{R}_c$ is given below

$$\begin{bmatrix} (1,1) & (5,1/7) & (9,0.111) & (3,1/7) & (0.74,0.429) \\ (1/7,5) & (1,1) & (9/5,3/7) & (0.912,1) & (1/5,3) \\ (0.111,9) & (3/7,9/5) & (1,1) & (0.441,2.268) & (1/9,7) \\ (1/7,3) & (1,0.912) & (2.268,0.441) & (1,1) & (1/7,3) \\ (0.429,0.74) & (3,1/5) & (7,1/9) & (3,1/7) & (1,1) \end{bmatrix}_{5\times 5}$$

To check the consistency degree, the complete IMPR is split into two MPRs and their corresponding CR values are given in row-1 of table 3.2.

$$C=\begin{pmatrix} 1 & 5 & 9 & 3 & 0.74 \\ \frac{1}{5} & 1 & \frac{9}{5} & 0.912 & \frac{1}{5} \\ \frac{1}{9} & \frac{5}{9} & 1 & 0.441 & \frac{1}{9} \\ \frac{1}{3} & \frac{1}{0.912} & \frac{1}{0.441} & 1 & \frac{1}{7} \\ \frac{1}{0.74} & 5 & 9 & 7 & 1 \end{pmatrix}_{5\times 5} \quad D=\begin{pmatrix} 1 & \frac{1}{7} & 0.111 & \frac{1}{7} & 0.429 \\ 7 & 1 & \frac{3}{7} & 1 & 3 \\ \frac{1}{0.111} & \frac{7}{3} & 1 & 2.268 & 7 \\ 7 & 1 & \frac{1}{2.268} & 1 & 3 \\ \frac{1}{0.429} & \frac{1}{3} & \frac{1}{7} & \frac{1}{3} & 1 \end{pmatrix}_{5\times 5}$$

The example 3.3.1 is also solved by the goal programming model(Model 3.2) and the missing elements are $\mu_{13}=9, \nu_{13}=0.111, \mu_{15}=1, \nu_{15}=0.429, \mu_{24}=1, \nu_{24}=1, \mu_{34}=0.428, \nu_{34}=2.333$. This Model 3.2 is solved using Lingo software. The consistency ratio of both MPRs obtain from two different methods such as two-step procedure method is given in row-1 of table 3.2 and goal programming model method are given in row-2 of table 3.2. Therefore, the complete IMPR $\tilde{R_c}$ is acceptably consistent. In table 3.2 C', D' are two MPRs are obtained by splitting the complete IMPRs where missing element are found from the Model 3.2.

Example 3.3.2. Let us consider a decision making problem with seven sets of alternatives $x_i,\ i=1,2,\cdots,7$. The decision maker judge these seven alternatives by pairwise comparison and provides his/her judgement as follows: $\tilde{r}_{12}=(\mu_{12},\nu_{12})=(3/5,1/4)$, $\tilde{r}_{16}=(\mu_{16},\nu_{16})=(1/5,1/2)$, $\tilde{r}_{23}=(\mu_{23},\nu_{23})=(1/2,8/5)$, $\tilde{r}_{26}=(\mu_{26},\nu_{26})=(1/3,2)$, $\tilde{r}_{34}=(\mu_{34},\nu_{34})=(2/9,15/4)$, $\tilde{r}_{36}=(\mu_{36},\nu_{36})=(2/3,5/4)$, $\tilde{r}_{45}=(\mu_{45},\nu_{45})=(7,1/7)$, $\tilde{r}_{46}=(\mu_{46},\nu_{46})=(3,1/3)$, $\tilde{r}_{56}=(\mu_{56},\nu_{56})=(3/7,7/3)$, $\tilde{r}_{67}=(\mu_{67},\nu_{67})=(1/7,3)$. The matrix representation of the above information is given by $\tilde{R}_1$.

$$\tilde{R}_1 = \begin{bmatrix} (1,1) & (3/5,1/4) & (*,*) & (*,*) & (*,*) & (1/5,1/2) & (*,*) \\ (1/4,3/5) & (1,1) & (1/2,8/5) & (*,*) & (*,*) & (1/3,2) & (*,*) \\ (*,*) & (8/5,1/2) & (1,1) & (2/9,15/4) & (*,*) & (2/3,5/4) & (*,*) \\ (*,*) & (*,*) & (15/4,2/9) & (1,1) & (7,1/7) & (3,1/3) & (*,*) \\ (*,*) & (*,*) & (*,*) & (1/7,7) & (1,1) & (3/7,7/3) & (*,*) \\ (1/2,1/5) & (2,1/3) & (5/4,2/3) & (1/3,3) & (7/3,3/7) & (1,1) & (1/7,3) \\ (*,*) & (*,*) & (*,*) & (*,*) & (*,*) & (3,1/7) & (1,1) \end{bmatrix}_{7\times 7}$$

The initial value of missing element are calculated by using equation 3.3.1 which is given in table 3.3.

Some initial values does not satisfies the property of IMPRs e.g $\mu_{14}, \mu_{17}, \mu_{27}, \mu_{37}, \mu_{57} \ngeq \frac{1}{9}$.
To adjust these value we have solved optimization Model 3.1, that minimize the error.

$$\begin{aligned}
\text{Min } \{ & 2|\{(3/10-\mu_{13})^2+(2/5-\nu_{13})^2\}^{0.5}| + |\{(1/15-\mu_{14})^2+(3/2-\nu_{14})^2\}^{0.5}| \\
& +|\{(7/15-\mu_{15})^2+(3/14-\nu_{15})^2\}^{0.5}| + |\{(1/35-\mu_{17})^2+(3/2-\nu_{17})^2\}^{0.5}| \\
& +2|\{(1/9-\mu_{24})^2+(6-\nu_{24})^2\}^{0.5}| + |\{(7/9-\mu_{25})^2+(6/7-\nu_{25})^2\}^{0.5}| \\
& +|\{(1/21-\mu_{27})^2+(6-\nu_{27})^2\}^{0.5}| + 2|\{(14/9-\mu_{35})^2+(15/28-\nu_{35})^2\}^{0.5}| \\
& +|\{(2/21-\mu_{37})^2+(15/4-\nu_{37})^2\}^{0.5}| + |\{(3/7-\mu_{47})^2+(1-\nu_{47})^2\}^{0.5}| \\
& +|\{(3/49-\mu_{57})^2+(7-\nu_{57})^2\}^{0.5}|\};
\end{aligned}$$

subject to

$\mu_{13}\times\nu_{13}\le 1; \mu_{14}\times\nu_{14}\le 1; \mu_{15}\times\nu_{15}\le 1; \mu_{17}\times\nu_{17}\le 1; \mu_{24}\times\nu_{24}\le 1; \mu_{25}\times\nu_{25}\le 1;$

$\mu_{27}\times\nu_{27}\le 1; \mu_{35}\times\nu_{35}\le 1; \mu_{37}\times\nu_{37}\le 1; \mu_{47}\times\nu_{47}\le 1; \mu_{57}\times\nu_{57}\le 1;$

$1/9\le\mu_{13}; \nu_{13}\le 9; 1/9\le\mu_{14}; \nu_{14}\le 9; 1/9\le\mu_{15}; \nu_{15}\le 9; 1/9\le\mu_{17}; \nu_{17}\le 9;$

$1/9\le\mu_{24}; \nu_{24}\le 9; 1/9\le\mu_{25}; \nu_{25}\le 9; 1/9\le\mu_{27}; \nu_{27}\le 9; 1/9\le\mu_{35}; \nu_{35}\le 9;$

$1/9\le\mu_{37}; \nu_{37}\le 9; 1/9\le\mu_{47}; \nu_{47}\le 9; 1/9\le\mu_{57}; \nu_{57}\le 9;$

${\mu_{13}}^{*(0)} = 0.5237; {\nu_{13}}^{*(0)} = 0.0745; {\mu_{14}}^{*(0)} = 1/15, {\nu_{14}}^{*(0)} = 3/2;$

${\mu_{15}}^{*(0)} = 7/15, {\nu_{15}}^{*(0)} = 3/14; {\mu_{17}}^{*(0)} = 1/35, {\nu_{17}}^{*(0)} = 3/2;$

${\mu_{24}}^{*(0)} = 0.17, {\nu_{24}}^{*(0)} = 3.77; {\mu_{25}}^{*(0)} = 7/9, {\nu_{25}}^{*(0)} = 6/7;$

${\mu_{27}}^{*(0)} = 1/21, {\nu_{27}}^{*(0)} = 6; {\mu_{35}}^{*(0)} = 5.69, {\nu_{35}}^{*(0)} = 0.127;$

${\mu_{37}}^{*(0)} = 2/21, {\nu_{37}}^{*(0)} = 15/4; {\mu_{47}}^{*(0)} = 3/7, {\nu_{47}}^{*(0)} = 1;$

${\mu_{57}}^{*(0)} = 3/49, {\nu_{57}}^{*(0)} = 7;$

Table 3.3: Calculation of Missing element(initial value)

Missing element	Adjoining element	Calculated value
$\left(\mu_{13}^{*(0)}, \nu_{13}^{*(0)}\right)$	(μ_{12},ν_{12}), (μ_{23},ν_{23})	$(3/10, 2/5)$
$\left(\mu_{14}^{*(0)}, \nu_{14}^{*(0)}\right)$	(μ_{16},ν_{16}), (μ_{64},ν_{64})	$(1/15, 3/2)$
$\left(\mu_{15}^{*(0)}, \nu_{15}^{*(0)}\right)$	(μ_{16},ν_{16}), (μ_{65},ν_{65})	$(7/15, 3/14)$
$\left(\mu_{17}^{*(0)}, \nu_{17}^{*(0)}\right)$	(μ_{16},ν_{16}), (μ_{67},ν_{67})	$(1/35, 3/2)$
$\left(\mu_{24}^{*(0)}, \nu_{24}^{*(0)}\right)$	(μ_{23},ν_{23}), (μ_{34},ν_{34}) (μ_{26},ν_{26}), (μ_{46},ν_{46})	$(0.17, 3.77)$
$\left(\mu_{25}^{*(0)}, \nu_{25}^{*(0)}\right)$	(μ_{26},ν_{26}), (μ_{56},ν_{56})	$(7/9, 6/7)$
$\left(\mu_{27}^{*(0)}, \nu_{27}^{*(0)}\right)$	(μ_{26},ν_{26}), (μ_{67},ν_{67})	$(1/21, 6)$
$\left(\mu_{35}^{*(0)}, \nu_{35}^{*(0)}\right)$	(μ_{34},ν_{34}), (μ_{45},ν_{45}) (μ_{36},ν_{36}), (μ_{56},ν_{56})	$(5.69, 0.13)$
$\left(\mu_{37}^{*(0)}, \nu_{37}^{*(0)}\right)$	(μ_{36},ν_{36}), (μ_{67},ν_{67})	$(2/21, 15/4)$
$\left(\mu_{47}^{*(0)}, \nu_{47}^{*(0)}\right)$	(μ_{46},ν_{46}), (μ_{67},ν_{67})	$(3/7, 1)$
$\left(\mu_{57}^{*(0)}, \nu_{57}^{*(0)}\right)$	(μ_{56},ν_{56}), (μ_{67},ν_{67})	$(3/49, 7)$

The adjusting value are given by $(\mu_{13},\nu_{13})=(0.3,0.4)$, $(\mu_{14},\nu_{14})=(0.111,1.5)$, $(\mu_{15},\nu_{15})=(0.467,0.214)$, $(\mu_{17},\nu_{17})=(0.111,1.5)$, $(\mu_{24},\nu_{24})=(0.111,6)$, $(\mu_{25},\nu_{25})=(0.778,0.857)$, $(\mu_{27},\nu_{27})=(0.111,6)$, $(\mu_{35},\nu_{35})=(1.556,0.536)$, $(\mu_{37},\nu_{37})=(0.111,3.748)$, $(\mu_{47},\nu_{47})=(0.429,1)$, $(\mu_{57},\nu_{57})=(0.111,7)$. Example 3.3.2 solved by Model 3.2 also and we obtained the same result. The complete IMPR $\tilde{R}_{c_1}$ is given below.

$$\begin{bmatrix}
(1,1) & (\frac{3}{5},\frac{1}{4}) & (0.3,0.4) & (0.111,1.5) & (0.467,0.214) & (\frac{1}{5},\frac{1}{2}) & (0.111,1.5)\\
(\frac{1}{4},\frac{3}{5}) & (1,1) & (\frac{1}{2},\frac{8}{5}) & (0.111,6) & (0.778,0.857) & (\frac{1}{3},2) & (0.111,6)\\
(0.4,0.3) & (\frac{8}{5},\frac{1}{2}) & (1,1) & (\frac{2}{9},\frac{15}{4}) & (1.556,0.536) & (\frac{2}{3},\frac{5}{4}) & (0.111,3.748)\\
(1.5,0.111) & (6,0.111) & (\frac{15}{4},\frac{2}{9}) & (1,1) & (7,\frac{1}{7}) & (3,\frac{1}{3}) & (0.429,1)\\
(0.214,0.467) & (0.857,0.778) & (0.536,1.556) & (\frac{1}{7},7) & (1,1) & (\frac{3}{7},\frac{7}{3}) & (0.111,7)\\
(\frac{1}{2},\frac{1}{5}) & (2,\frac{1}{3}) & (\frac{5}{4},\frac{2}{3}) & (\frac{1}{3},3) & (\frac{7}{3},\frac{3}{7}) & (1,1) & (\frac{1}{7},3)\\
(1.5,0.111) & (6,0.111) & (3.748,0.111) & (1,0.429) & (7,0.111) & (3,\frac{1}{7}) & (1,1)
\end{bmatrix}$$

To check the consistency degree of IMPR, $\tilde{R}_{c_1}$ split into two MPRs C_1 and D_1, where,

$$C_1=\begin{pmatrix}
1 & \frac{3}{5} & 0.3 & 0.111 & 0.47 & \frac{1}{5} & 0.111\\
\frac{5}{3} & 1 & \frac{1}{2} & 0.111 & 0.778 & \frac{1}{3} & 0.111\\
\frac{1}{0.3} & 2 & 1 & \frac{2}{9} & 1.556 & \frac{2}{3} & 0.111\\
\frac{1}{0.111} & \frac{1}{0.111} & \frac{9}{2} & 1 & 7 & 3 & 0.429\\
\frac{1}{0.467} & \frac{1}{0.778} & \frac{1}{1.556} & \frac{1}{7} & 1 & \frac{3}{7} & 0.111\\
5 & 3 & \frac{3}{2} & \frac{1}{3} & \frac{7}{3} & 1 & \frac{1}{7}\\
\frac{1}{0.111} & \frac{1}{0.111} & \frac{1}{0.111} & \frac{1}{0.429} & \frac{1}{0.111} & 7 & 1
\end{pmatrix}_{7\times 7}$$

$$D_1=\begin{pmatrix}
1 & \frac{1}{4} & 0.4 & 1.5 & 0.214 & \frac{1}{2} & 1.5\\
4 & 1 & \frac{8}{5} & 6 & 0.857 & 2 & 6\\
\frac{1}{0.4} & \frac{5}{8} & 1 & \frac{15}{4} & 0.536 & \frac{5}{4} & 3.748\\
\frac{1}{1.5} & \frac{1}{6} & \frac{4}{15} & 1 & \frac{1}{7} & \frac{1}{3} & 1\\
\frac{1}{0.214} & \frac{1}{0.857} & \frac{1}{0.536} & 7 & 1 & \frac{7}{3} & 7\\
2 & \frac{1}{2} & \frac{4}{5} & 3 & \frac{3}{7} & 1 & 3\\
\frac{1}{1.5} & \frac{1}{6} & \frac{1}{3.748} & 1 & \frac{1}{7} & \frac{1}{3} & 1
\end{pmatrix}_{7\times 7}$$

$CR(C_1)=0.0230$ and $CR(D_1)=0$ both are acceptable threshold value. According to Satty [58] both C_1 and D_1 are acceptably consistent. Therefore $\tilde{R}_{c_1}$ is also acceptably consistent.

3.3.4 A Comparative analysis with existing methods

In this subsection, we compare our proposed method with Jiang et al. [143] for IMPR and Meng and Chen [169] for MPRs.

In 2015, Jiang et al. [143] discussed the consistency property, especially the acceptable consistency of an IMPR by splitting into two MPRs. Based on it, Jiang et al. developed two approaches to complement all missing elements of incomplete IMPRs. According to Jiang et al. [143], the incomplete IMPR split into two MPRs, and the calculation of missing factor involves two steps, i.e. "estimating step" and " adjusting step". In the estimating step, initial values of missing entries are estimated using geometric mean method. Some times the initial values does not satisfies the required condition. To improve these initial values, two unique methodologies are produced: one is local optimization models, which is efficient and other is an iterative method that can work the whole optimization process reasonably.

Using Jiang et al. [143] methods, example 3.3.1 is solved where incomplete IMPRs is split into two MPRs as C and D. Using geometric mean method missing element is calculated and adjusting values are computed by using local optimization model (LOP1) of [143]. Since the consistency ratio (CR) of C and D (using [143]) are 0.0197 and 0.0112. Hence the complete IMPR is acceptably consistent. Consistency ratio of C and D obtained from our methods (both two-step procedure method and Goal programming model) are less than(see Table 3.2) from Jiang et al. [143] methods. Similarly, in the example, 3.3.2 the complete IMPR is also acceptably consistent. Both the model gives the equivalent result.

To measure the multiplicative consistency of an MPR, Meng and Chen [169] proposed the notion of multiplicative geometric consistent index (MGCI). The consistency of an MPR is considered to be unacceptable if the MGCI of an MPR is less than the average value tabulated in Table 1 of [169]. The authors continued their study to include the case of incomplete MPR. They formulated multi-objective programming model to estimate the missing values. Using the goal programming approach and a suitable transformation, the proposed model was converted into an equivalent linear program. The missing values in an MPR were then obtained using the inverse transformation of the optimal solution of the linear program.

In example 3.3.1, the incomplete IMPR is split into two incomplete MPRs using equation 3.2.2. The missing elements of two incomplete MPRs are obtained using Meng and

Chen's linear programming model (LP)(see [169]). Consistency ratio of two MPRs obtained from Meng and Chen's model are 0.0199 and 0.0097 respectively which are less than from both two-step procedure method and Goal programming model Model 3.2.

3.4 Consistency of IMPR using Graphical approach

Now we will discuss the another goal of this chapter. Nishizawa [172] in 1995 proposed an algorithm for the consistency of MPRs by using the cycle of a directed graph. Later Nishizawa [138] proposed two algorithms to find the cycles of various odd and even length using incomplete directed graph. For finding cycles, vertex matrix of order $n \times n$ is needed, where n is the number of vertices, corresponding to the directed graph whose (i, j) element i.e $V(i, j)$ is determined. If one points i is connected to another point j by an arrow, say "$i \to j$", then $V(i, j) = 1$ otherwise 0. Pairwise comparison data is represented by θ or $1/\theta$,where θ is a parameter whose value is greater than 1 in the binary AHP [57]. Nishizawa proposed two algorithms [138] for even and odd length cycle in the incomplete directed graph. From the result Nishizawa [138], one can easily judge the consistency of comparison matrix. If no cycles have found in the directed graph, then the comparison matrix is consistent. The comparison matrix is inconsistent if at least one cycle is found in the directed graph. In case of inconsistency, they find the minimum covering sets [172] among the cycles, and then they eliminate the path of cycles such as the comparison matrix is consistent.

In this chapter, we have discussed two approaches for checking the consistency of IMPRs using graphical approach. Here we are extending the above method of Nishizawa [138] in intuitionistic multiplicative preference relation scenario.

Here we have developed two different methods to check the consistency of IMPR and incomplete IMPRs which are discussed in subsection 3.4.1 and 3.4.2 respectively.

3.4.1 Consistency for IMPR using binary array

The absence of consistency in decision making with preference relations is a big challenge to bring about conflicting conclusions. Numerous strategies on consistency measure and improvement of preference relations with various structures have been exhibited progressively. This subsection checks the consistency of both IMPR and incomplete IMPR.

Let $\tilde{R}$ be the intuitionistic multiplicative preference relation,

$$\tilde{R} = \begin{pmatrix} (\mu_{11}, \nu_{11}) & (\mu_{12}, \nu_{12}) & \cdots & (\mu_{1i}, \nu_{1i}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & \cdots & (\mu_{2i}, \nu_{2i}) & \cdots & (\mu_{2n}, \nu_{2n}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (\mu_{i1}, \nu_{i1}) & (\mu_{i2}, \nu_{i2}) & \cdots & (\mu_{ii}, \nu_{ii}) & \cdots & (\mu_{in}, \nu_{in}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (\mu_{n1}, \nu_{n1}) & (\mu_{n2}, \nu_{n2}) & \cdots & (\mu_{ni}, \nu_{ni}) & \cdots & (\mu_{nn}, \nu_{nn}) \end{pmatrix}$$

We define a ordered pair vertices matrix $V = \{v_{ij}\} = \{(a,b)\}$ as follows, where v_{ij} is an ordered pair (a,b) such that

$$\begin{cases} \text{For } i < j \begin{cases} if\ \mu_{ij} > 1 & \text{then } a = 1, \text{ otherwise } a = 0 \\ if\ \nu_{ij} > 1 & \text{then } b = 1, \text{ otherwise } b = 0 \end{cases} \\ \text{For } i > j \begin{cases} if\ \frac{1}{\nu_{ij}} > 1 & \text{then } a = 1, \text{ otherwise } a = 0 \\ if\ \frac{1}{\mu_{ij}} > 1 & \text{then } b = 1, \text{ otherwise } b = 0 \end{cases} \\ \text{For } i = j \text{ both } a = 0 \text{ and } b = 0. \end{cases} \tag{3.4.1}$$

To illustrate, consider the following IMPR

$$\begin{pmatrix} (1,1) & (1/2, 2/3) & (1/5, 5) \\ (2/3, 1/2) & (1,1) & (4/5, 3/4) \\ (5, 1/5) & (3/4, 4/5) & (1,1) \end{pmatrix}$$

Using the binary order paired vertex matrix V is given below

$$V = \begin{pmatrix} (0,0) & (0,0) & (0,1) \\ (1,1) & (0,0) & (0,0) \\ (1,0) & (1,1) & (0,0) \end{pmatrix}$$

Here, the binary order paired vertex matrix is split into two vertex matrix i.e lower vertex matrix V_L containing the lower element of order pair and upper vertex matrix V_U containing the upper element of order pair.

$$V_L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad V_U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Using the concept of order pair, we have defined a new definition of consistent IMPR.

Definition 3.4.1. The IMPR is consistent if both the corresponding lower and upper vertex matrices are consistent.

In this section, we have applied the graphical approach of Nishizawa [138] to check the consistency of both complete IMPRs and incomplete IMPRs scenario. We have developed an algorithm to illustrate the above method.

Algorithm 3.4.1. Step1: Let us consider an IMPR.
Step2: Order pair vertex matrix is obtained using equation 3.4.1. In case of incomplete IMPRs, missing elements are treated as zero in the vertex matrix.
Step3: The ordered pair vertex matrix is split into lower vertex matrix and upper vertex matrix.
Step4: Then apply the approach of Nishizawa [138] for finding the cycle of even and odd length on both the vertex matrix.
Step5: If any cycle found then the vertex matrix is inconsistent, otherwise consistent.
Step6: IMPR is consistent if both the vertex lower and upper matrices are consistent, otherwise matrix is inconsistent.
Step7: If the IMPR matrix is inconsistent, then remove the minimum number of path that cover the cycles.

The above-said method is illustrated by an example.

Example 3.4.1. Let us consider a decision making problem with four sets of alternatives $x_i, i = 1,2,3,4$. The decision maker judge these four alternatives by pairwise comparison and provides his/her judgement as follows: $\tilde{r}_{12} = (1/2,1/4)$, $\tilde{r}_{13} = (2,1/8)$, $\tilde{r}_{14} = (2/3,1/4)$, $\tilde{r}_{23} = (5,1/7)$, $\tilde{r}_{24} = (7/5,2/3)$, $\tilde{r}_{34} = (6,1/7)$. The matrix representation is given by

$$\tilde{R}_1 = \begin{pmatrix} (1,1) & (\frac{1}{2},\frac{1}{4}) & (2,\frac{1}{8}) & (\frac{2}{3},\frac{1}{4}) \\ (\frac{1}{4},\frac{1}{2}) & (1,1) & (5,\frac{1}{7}) & (\frac{7}{5},\frac{2}{3}) \\ (\frac{1}{8},2) & (\frac{1}{7},5) & (1,1) & (6,\frac{1}{7}) \\ (\frac{1}{4},\frac{2}{3}) & (\frac{2}{3},\frac{7}{5}) & (\frac{1}{7},6) & (1,1) \end{pmatrix}$$

By using equation 3.4.1, the binary order pair vertex matrix is

$$V = \begin{pmatrix} (0,0) & (0,0) & (1,0) & (0,0) \\ (1,1) & (0,0) & (1,0) & (1,0) \\ (0,1) & (0,1) & (0,0) & (1,0) \\ (1,1) & (0,1) & (0,1) & (0,0) \end{pmatrix}$$

The lower and upper vertex matrices are

$$V_L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; V_U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

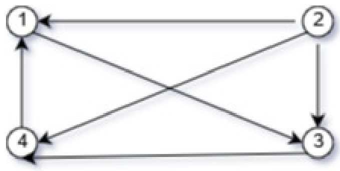

Figure 3.1: Directed graph of V_L of example 3.4.1

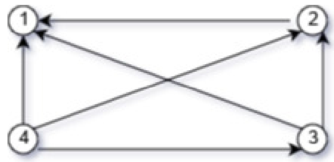

Figure 3.2: Directed graph of V_U of example 3.4.1

There is only one cycle of length 3 is present in the lower vertex matrix. Directly we can found the cycle from figure 3.1. In order to find the cycle of length 3 we use the algorithm of odd length, step (3.2) to step (3.5) ([138]), where $m = 2$. The form of the cycle of length 3 is $(i_0 - k - j_0)$. To find the elements i_0, k, j_0, we need V_L and V_L^2. For this V_L^2 as follows:

$$V_L^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

At the starting we have $i_0 = 1,\ k = 3,\ j_0 = 4$, satisfying the step (3.2) of algorithm of odd length cycle [138], i.e $V_L(1,3) = 1$, $V_L^2(3,1) = 1$, $V_L(3,4) = 1$, $V_L^2(4,3) = 1$, and $V_L(4,1) = 1$. Then the cycle of length 3 is $(1-3-4)$. There is no other cycle available in V_L. Similarly there is no cycle of any length in figure 3.2. The lower vertex matrix V_L is inconsistent and the upper vertex matrix V_U is consistent. Therefore the complete IMPR is inconsistent.

To propose the reason of inconsistency, an algorithm for extinguishing cycles grounded on minimum covering sets is applied ([172]). The cycle-arc incidence matrix is given in Table 3.4.

From table 3.4, we get the several pairs of edges which cover the cycle. we have to choose the pair that eliminate the cycle. In this example if we will change the pair of vertex from any one of them from $(1,4)$ to $(4,1)$ or $(1,3)$ to $(3,1)$ or $(3,4)$ to $(4,3)$ in the original IMPR $\tilde{R}_1$, then the both the lower and upper vertex matrix are consistent. Then IMPR $\tilde{R}_1$ is also consistent.

Table 3.4: Cycle-arc incidence matrix of V_L

Cycle	(1,3)	(1,4)	(3,4)
$(1-3-4)$	1	1	1

Example 3.4.2. Let us examine decision making problem with seven sets of alternatives $x_i,\ i = 1,2,\cdots,7$. The matrix representation of the decision maker judgement is a comparison matrix which is given by

$$\tilde{R}_2 = \begin{pmatrix} (1,1) & (\frac{5}{3},\frac{1}{4}) & (7,\frac{1}{9}) & (3,\frac{1}{7}) & (\frac{5}{3},\frac{1}{7}) & (1,\frac{3}{5}) & (\frac{1}{4},\frac{5}{3}) \\ (\frac{1}{4},\frac{5}{3}) & (1,1) & (\frac{5}{3},\frac{1}{4}) & (\frac{5}{3},\frac{1}{4}) & (\frac{3}{5},1) & (\frac{1}{3},1) & (\frac{1}{4},3) \\ (\frac{1}{9},7) & (\frac{1}{4},\frac{5}{3}) & (1,1) & (1,\frac{3}{5}) & (\frac{1}{4},\frac{5}{3}) & (\frac{1}{3},3) & (\frac{1}{9},7) \\ (\frac{1}{7},3) & (\frac{1}{4},\frac{5}{3}) & (\frac{3}{5},1) & (1,1) & (\frac{3}{5},\frac{3}{5}) & (\frac{1}{3},3) & (\frac{1}{9},7) \\ (\frac{1}{7},\frac{5}{3}) & (1,\frac{3}{5}) & (\frac{5}{3},\frac{1}{4}) & (\frac{3}{5},\frac{3}{5}) & (1,1) & (\frac{1}{3},3) & (\frac{1}{7},3) \\ (\frac{3}{5},1) & (1,\frac{1}{3}) & (3,\frac{1}{3}) & (3,\frac{1}{3}) & (3,\frac{1}{3}) & (1,1) & (\frac{1}{7},3) \\ (\frac{5}{3},\frac{1}{4}) & (3,\frac{1}{4}) & (7,\frac{1}{9}) & (7,\frac{1}{9}) & (3,\frac{1}{7}) & (3,\frac{1}{7}) & (1,1) \end{pmatrix}$$

Using 3.4.1, the binary order pair vertex matrix is given by

$$V = \begin{pmatrix} (0,0) & (1,0) & (1,0) & (1,0) & (1,0) & (0,0) & (0,1) \\ (0,1) & (0,0) & (1,0) & (1,0) & (0,0) & (0,0) & (0,1) \\ (0,1) & (0,1) & (0,0) & (0,0) & (0,1) & (0,1) & (0,1) \\ (0,1) & (0,1) & (1,1) & (0,0) & (0,0) & (0,1) & (0,1) \\ (0,1) & (1,0) & (1,0) & (1,1) & (0,0) & (0,1) & (0,1) \\ (0,1) & (1,0) & (1,0) & (1,0) & (1,0) & (0,0) & (0,1) \\ (1,0) & (1,0) & (1,0) & (1,0) & (1,0) & (1,0) & (0,0) \end{pmatrix}$$

The above vertex matrix split into two binary array.

$$V_L = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad V_U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By applying algorithm of odd and even length cycles [138], there is no cycle found in

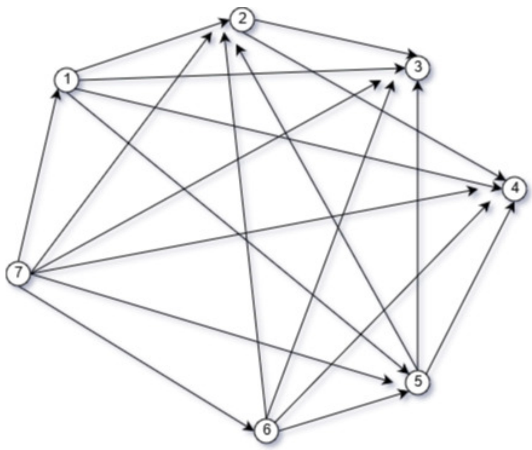

Figure 3.3: Directed graph of V_L of example 3.4.2

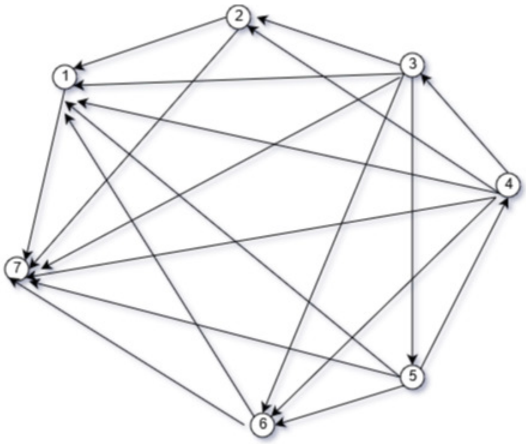

Figure 3.4: Directed graph of V_U of example 3.4.2

Table 3.5: Cycle-arc incidence matrix of V_U

Cycle	(3,4)	(3,5)	(4,5)
$(3-5-4)$	1	1	1

V_L, and one cycle of length three, i.e., $(3-5-4)$ is located in V_U. In order to find the cycle of length 3 in V_U we use step (3.2) to step (3.5) of algorithm (odd length [138]), where $m = 2$. The form of the cycle of length 3 is $(i_0 - k - j_0)$. To locate these element we need V_U and V_U^2. For this V_U^2 as follows:

$$V_U^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 & 1 & 4 \\ 3 & 1 & 0 & 0 & 1 & 1 & 4 \\ 2 & 1 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

At the starting we have $i_0 = 3,\ k = 5,\ j_0 = 4$, satisfying $V_U(3,5) = 1,\ V_U^2(5,3) = 1$, $V_U(5,4) = 1$, $V_U^2(4,5) = 1$, and $V_U(4,3) = 1$. Then the cycle of length 3 is $(3-5-4)$. Similarly there is no cycle of any length in figure 3.4. The lower vertex matrix V_L is consistent and the upper vertex matrix V_U is inconsistent. Therefore the complete IMPR is inconsistent.

To propose the reason of inconsistency, we can find the minimum covering path which cover the cycle [172]. The cycle-arc incidence matrix is given in Table 3.5.

From table 3.5, we get the several pairs of edges which cover the cycle. we have to chose the pair eliminate the cycle. In this example if we will change the pair $(3,4)$ to $(4,3)$ in the original IMPR $\tilde{R}_2$, then both the lower and upper vertex matrix are consistent. Then IMPR $\tilde{R}_2$ is consistent.

The above said method is also applied in incomplete IMPRs scenario which is given below.

For an incomplete preference relation, it is very much essential for known elements to satisfy consistency. We are applying the same graphical approach in incomplete IMPRs scenario. For the incomplete case, we use the measure of inconsistency by the number of the cycle in the graph corresponding to known elements of the IMPR matrix.

Example 3.4.3. We have another illustrations of decision making problem with six sets of alternatives x_i, $i = 1,2,3,4,5,6$. The decision maker judge these six alternatives by pairwise comparison and provides his/her judgement as follows: $\tilde{r}_{12} = (1/5,2)$, $\tilde{r}_{13} = (5,1/6)$, $\tilde{r}_{15} = (1/9,8)$, $\tilde{r}_{16} = (2,1/6)$, $\tilde{r}_{24} = (1/4,1/5)$, $\tilde{r}_{25} = (1/2,1/3)$, $\tilde{r}_{26} = (3,1/5)$, $\tilde{r}_{35} = (6,1/7)$, $\tilde{r}_{45} = (1/5,4)$. The matrix representation is given by

$$\tilde{R}_3 = \begin{pmatrix} (1,1) & (\frac{1}{5},2) & (5,\frac{1}{6}) & (*,*) & (\frac{1}{9},8) & (2,\frac{1}{6}) \\ (2,\frac{1}{5}) & (1,1) & (*,*) & (\frac{1}{4},\frac{1}{5}) & (\frac{1}{2},\frac{1}{3}) & (3,\frac{1}{5}) \\ (\frac{1}{6},5) & (*,*) & (1,1) & (*,*) & (6,\frac{1}{7}) & (*,*) \\ (*,*) & (\frac{1}{5},\frac{1}{4}) & (*,*) & (1,1) & (\frac{1}{5},4) & (*,*) \\ (8,\frac{1}{9}) & (\frac{1}{3},\frac{1}{2}) & (\frac{1}{7},6) & (4,\frac{1}{5}) & (1,1) & (*,*) \\ (\frac{1}{6},2) & (\frac{1}{5},3) & (*,*) & (*,*) & (*,*) & (1,1) \end{pmatrix}$$

The order pair vertex matrix is obtained by using equation 3.4.1

$$V = \begin{pmatrix} (0,0) & (0,1) & (1,0) & (0,0) & (0,1) & (1,0) \\ (1,0) & (0,0) & (0,0) & (0,0) & (0,0) & (1,0) \\ (0,1) & (0,0) & (0,0) & (0,0) & (1,0) & (0,0) \\ (0,0) & (1,1) & (0,0) & (0,0) & (0,1) & (0,0) \\ (1,0) & (1,1) & (0,1) & (1,0) & (0,0) & (0,0) \\ (0,1) & (0,1) & (0,0) & (0,0) & (0,0) & (0,0) \end{pmatrix}$$

The lower and upper vertex matrices are given below

$$V_L = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; V_U = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

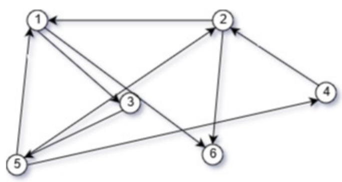

Figure 3.5: Directed graph of V_L of example 3.4.3

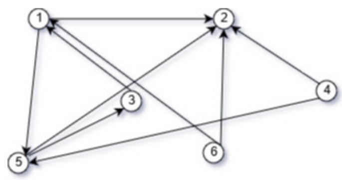

Figure 3.6: Directed graph of V_U of example 3.4.3

In order to find the cycle of length 3 in the lower vertex matrix V_L, we need V_L and V_L^2. For this V_L^2 as follows:

$$V_L^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

At the starting we have $i_0 = 1,\ k = 3,\ j_0 = 5$, satisfying the step (3.2) of algorithm for finding odd length cycle [138] i.e $V_L(1,3) = 1$, $V_L^2(3,1) = 1$, $V_L(3,5) = 1$, $V_L^2(5,3) = 1$, and $V_L(5,1) = 1$. Then the cycle of length 3 is $(1-3-5)$.

Similarly using algorithm [138], there is also one cycle of length 3 in V_U. Here also, for finding the cycle of length 3 in V_U, we need V_U and V_U^2. For this V_U^2 as follows:

$$V_U^2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Also we have $i_0 = 1,\ k = 5,\ j_0 = 3$, satisfying $V_U(1,5) = 1$, $V_U^2(5,1) = 1$, $V_U(5,3) = 1$,

Table 3.6: Cycle-arc incidence matrix of V_L and V_U

Cycle	(1,5)	(1,3)	(3,5)
$(1-3-5)$	1	1	1
$(1-5-3)$	1	1	1

$V_U^2(3,5)=1$, and $V_U(3,1)=1$. Then the cycle of length 3 is $(1-5-3)$.

Both the lower and upper vertex matrix are V_L and V_U is inconsistent. Therefore the incomplete IMPR is inconsistent. Like previous example the cycle-arc incidence matrix is given in Table 3.6. From the Table 3.6, we can find the minimum covering path from the cycle to suggest the cause of inconsistency.

From table 3.6, we get the several pairs of the edges which cover the cycle. We have to chose the few to eliminate the cycle. In this example, if we will change the pair $(3,5)$ to $(5,3)$ in the original IMPR $\tilde{R}_3$, then the both the lower and upper vertex matrix are consistent. Then incomplete IMPR $\tilde{R}_3$ is also consistent.

Example 3.4.4. Let us take an example of incomplete IMPRs of order 10×10 given as below

$$\tilde{R}_4=\begin{bmatrix}
(1,1) & * & (\frac{1}{2},\frac{1}{5}) & (\frac{1}{3},2) & (3,\frac{1}{6}) & * & * & (\frac{8}{3},\frac{1}{8}) & (\frac{1}{7},4) & (5,\frac{1}{5}) \\
* & (1,1) & (\frac{2}{3},\frac{6}{10}) & (\frac{1}{8},\frac{1}{2}) & (\frac{1}{4},1) & * & (\frac{1}{5},5) & * & (\frac{1}{3},3) & * \\
(\frac{1}{5},\frac{1}{2}) & (\frac{6}{10},\frac{2}{3}) & (1,1) & (\frac{3}{5},\frac{2}{3}) & * & (2,\frac{1}{2}) & (4,\frac{1}{8}) & (7,\frac{1}{8}) & (\frac{1}{8},5) & * \\
(2,\frac{1}{3}) & (\frac{1}{2},\frac{1}{8}) & (\frac{2}{3},\frac{3}{5}) & (1,1) & * & (3,\frac{1}{7}) & (5,\frac{1}{6}) & * & (2,\frac{1}{2}) & * \\
(\frac{1}{6},3) & (1,\frac{1}{4}) & * & * & (1,1) & (\frac{1}{8},7) & (8,\frac{1}{9}) & (\frac{1}{6},3) & * & (4,\frac{1}{5}) \\
* & * & (\frac{1}{2},2) & (\frac{1}{7},3) & (7,\frac{1}{8}) & (1,1) & * & * & (\frac{1}{5},4) & (2,\frac{1}{3}) \\
* & (5,\frac{1}{5}) & (\frac{1}{8},4) & (\frac{1}{6},5) & (\frac{1}{9},8) & * & (1,1) & (\frac{1}{9},6) & (\frac{1}{7},2) & (4,\frac{1}{8}) \\
(\frac{1}{8},\frac{8}{3}) & * & (\frac{1}{8},7) & * & (3,\frac{1}{6}) & * & (6,\frac{1}{9}) & (1,1) & * & (6,\frac{1}{6}) \\
(4,\frac{1}{7}) & (3,\frac{1}{3}) & (5,\frac{1}{8}) & (\frac{1}{2},2) & * & (4,\frac{1}{5}) & (2,\frac{1}{7}) & * & (1,1) & (\frac{1}{3},2) \\
(\frac{1}{5},5) & * & * & * & (\frac{1}{5},4) & (\frac{1}{3},2) & (\frac{1}{8},4) & (\frac{1}{6},6) & (2,\frac{1}{3}) & (1,1)
\end{bmatrix}$$

The binary order pair vertex matrix is given by

$$V = \begin{bmatrix} (0,0) & (0,0) & (0,0) & (0,1) & (1,0) & (0,0) & (0,0) & (1,0) & (0,1) & (1,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,1) & (0,0) & (0,1) & (0,0) \\ (1,1) & (1,1) & (0,0) & (0,0) & (0,0) & (1,0) & (1,0) & (1,0) & (0,1) & (0,0) \\ (1,0) & (1,1) & (1,1) & (0,0) & (0,0) & (1,0) & (1,0) & (0,0) & (1,0) & (0,0) \\ (0,1) & (1,0) & (0,0) & (0,0) & (0,0) & (0,1) & (1,0) & (0,1) & (0,0) & (1,0) \\ (0,0) & (0,0) & (0,1) & (0,1) & (1,0) & (0,0) & (0,0) & (0,0) & (0,1) & (1,0) \\ (0,0) & (1,0) & (0,1) & (0,1) & (0,1) & (0,0) & (0,0) & (0,1) & (0,1) & (1,0) \\ (0,1) & (0,0) & (0,1) & (0,0) & (1,0) & (0,0) & (1,0) & (0,0) & (0,0) & (1,0) \\ (1,0) & (1,0) & (1,0) & (0,1) & (0,0) & (1,0) & (1,0) & (0,0) & (0,0) & (0,1) \\ (0,1) & (0,0) & (0,0) & (0,0) & (0,1) & (0,1) & (0,1) & (0,1) & (1,0) & (0,0) \end{bmatrix}$$

The lower and upper vertex matrices are given below:

$$V_L = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, V_U = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

There are three cycles of length 3 is found in the lower vertex matrix i.e. $(1-10-9)$,

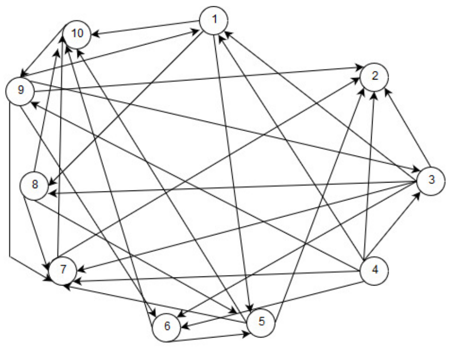

Figure 3.7: Directed graph of V_L of example 3.4.4

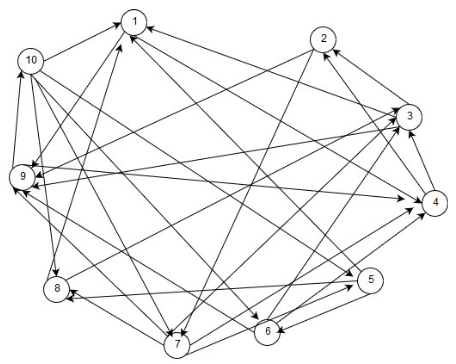

Figure 3.8: Directed graph of V_U of example 3.4.4

$(6-10-9)$ and $(7-10-9)$. In order to find the cycle of length 3 in V_L we use step (3.2) to step (3.5)(algorithm of odd length cycle [138]), where $m=2$. The form of the cycle of length 3 is (i_0-k-j_0). To find the element of the cycle, we need V_L^2 as follows:

$$V_L^2=\begin{bmatrix} 0&1&0&0&1&0&2&0&1&2\\ 0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&3&0&1&1&0&4\\ 2&3&1&0&2&2&2&2&0&3\\ 0&1&0&0&0&0&0&0&1&1\\ 0&1&0&0&0&0&1&0&1&1\\ 0&0&0&0&0&0&0&0&1&0\\ 0&2&0&0&0&0&1&0&1&2\\ 1&2&0&0&2&1&1&2&0&3\\ 1&1&1&0&0&1&1&0&0&0 \end{bmatrix}$$

For the cycle $(1-10-9)$, we have $i_0=1$, $k=10$, $j_0=9$, satisfying $V_L(1,10)=1$, $V_L^2(10,1)>0$, $V_L(10,9)>0$, $V_L^2(9,10)>0$, and $V_L(9,1)=1$. Then the cycle of length 3 is $(1-10-9)$. Similarly for the cycle $(6-10-9)$, we have $i_0=6$, $k=10$, $j_0=9$, such that $V_L(6,10)>0$, $V_L^2(10,6)>0$, $V_L(10,9)>0$, $V_L^2(9,10)>0$, and $V_L(9,6)=1$. For the cycle $(7-10-9)$, we have $i_0=7$, $k=10$, $j_0=9$, such that $V(7,10)>0$, $V_L^2(10,7)>0$, $V_L(10,9)>0$, $V_L^2(9,10)>0$, and $V_L(9,7)=1$.

Next we try to find out the cycle of length 4. The form of the cycle of length 4 is $(i_0-i_1-j_0-j_1)$. By using the algorithm of even length cycle [138], there are five cycle of length 4 that is $(5-10-9-6)$, $(1-10-9-3)$, $(3-6-10-9)$, $(3-7-10-9)$,and $(3-8-10-9)$. To find cycle of length 4, we need V_L^2. For the cycle $(5-10-9-6)$, we have $i_0=5$, $j_0=9$, satisfying step (2.2) of algorithm of even length cycle [138], since $V_L^2(5,9)>0$ and $V_L^2(9,5)>0$, and we have $i_1=10$ and $j_1=6$ satisfying step (2.3) of even length cycle algorithm [138] for $\alpha=1$. Since $V_L(5,10)=1$, $V_L^2(10,6)>0$, $V_L^2(6,10)>0$ and $V_L(9,6)=1$. After that it is confirm $V_L(10,9)=1$ and $V_L(6,5)=1$. Then we have cycle of length 4 is $(5-10-9-6)$. Similarly cycle of length 4 is $(1-10-9-3)$, $(3-6-10-9)$, $(3-7-10-9)$,and $(3-8-10-9)$. To find the cycle of length 5, using the algorithm for odd cycles, we need V_L, V_L^2 and V_L^3. For this V_L^3 as follows:

$$V_L^3 = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 & 4 & 0 & 4 & 5 \\ 1 & 5 & 0 & 0 & 6 & 1 & 5 & 3 & 3 & 10 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 3 & 0 & 0 & 4 & 0 & 4 & 1 & 3 & 7 \\ 1 & 2 & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 3 \end{bmatrix}$$

The form of the cycle of length 5 is $(i_0 - j_1 - k - i_1 - j_0)$. We have $i_0 = 3$, $k = 7$, $j_0 = 9$, satisfying $V_L^2(3,7) > 0$, $V_L^3(7,3) > 0$, $V_L^2(7,9) > 0$, $V_L^3(9,7) > 0$ and $V_L(9,3) = 1$. For $\alpha = 1$, we have to find out i_1 and j_1 where $i_1 = 10$ and $j_1 = 8$ with $V_L^3(3,10) > 0$, $V_L^2(10,3) > 0$, $V_L(7,10) = 1$, $V_L^2(9,8) > 0$, $V_L^3(8,9) > 0$ and $V_L(8,7) = 1$. After that it is confirm $V_L(10,9) = 1$ and $V_L(3,8) = 1$. Then we have cycle of length 5 is $(3-8-7-10-9)$. Similarly cycle of length 5 are obtained i.e $(3-6-5-10-9)$, $(1-8-10-9-3)$, $(1-5-10-9-3)$,and $(5-7-10-9-6)$.

Next we will try to find the cycle of length 6. The form of the cycle of length 6 is $(i_0 - i_1 - i_2 - j_0 - j_1 - j_2)$. We have $i_0 = 1$ $j_0 = 10$ with $V_L^3(1,10) > 0$, $V_L^3(10,1) > 0$. For $\alpha = 1$ we have to find the vertices $i_1 = 5$ and $j_1 = 9$ satisfying step (2.3) of algorithm for even length cycles [138] satisfying $V_L(1,5) = 1$, $V_L^3(5,9) > 0$, $V_L^3(9,5) > 0$, $V_L(10,9) = 1$. Similarly, for $\alpha = 2$ we have $i_2 = 7$ and $j_2 = 3$ satisfying step (2.4) (algorithm of even length cycles [138]) i.e. $V_L(5,7) = 1$, $V_L^3(7,3) > 0$, $V_L^3(3,7) > 0$, $V_L(9,3) = 1$. After that it is confirm $V_L(7,10) = 1$ and $V_L(3,1) = 1$. Then we have cycle of length 6 is $(1-5-7-10-9-3)$. Other cycle of length 6 are given by $(1-8-7-10-9-3)$, $(3-6-5-7-10-9)$, $(3-8-5-7-10-9)$,and $(1-8-5-10-9-3)$. Similarly, one cycle of length 7 is found that is $(1-8-5-7-10-9-3)$. There is no cycle of length 8, 9 and 10 is detected. For easy understanding, all the cycles of V_L are presented in Table 3.7.

Similarly the possible cycles of the upper vertex matrix V_U are given in table 3.8.

Since both the lower and upper matrix V_L and V_U are inconsistent, therefore IMPR $\tilde{R}_4$ is also inconsistent. In this example, from table 3.7 and 3.8, the path $2-7$, $3-4$, $9-10$ which cover all the cycle. If we will change the pair $(3,4)$ to $(4,3)$, $(2,7)$ to $(7,2)$, $(9,10)$ to $(10,9)$, in the original IMPR $\tilde{R}_4$, then the lower vertex matrix V_L and the upper vertex

Table 3.7: All possible cycle of V_L

Length	*Cycles*
3	(1 10 9), (6 10 9), (7 10 9)
4	(5 10 9 6), (1 10 9 3), (3 6 10 9), (3 7 10 9), (3 8 10 9)
5	(3 8 7 10 9), (3 6 5 10 9), (1 8 10 9 3),(1 5 10 9 3) , (5 7 10 9 6)
6	(1 5 7 10 9 3), (1 8 7 10 9 3), (3 6 5 7 10 9), (3 8 5 7 10 9), (1 8 5 10 9 3)
7	(1 8 5 7 10 9 3)
$8 \sim 10$	nothing

matrix V_U are both consistent. Then IMPR $\tilde{R}_4$ is also consistent.

3.4.2 Isomorphism between Intuitionistic multiplicative preference relations(IMPR) and asymmetric multiplicative preference relation

This section discusses the equivalence between the set of IMPR and the set of asymmetric multiplicative preference relation that leads to derive an asymmetric multiplicative preference relation from a given IMPR. From the asymmetric multiplicative preference relation, we obtained the vertex matrix and check for the consistency by using directed graph in IMPR scenario.

Consider an intuitionistic multiplicative preference relation

$$\tilde{R} = \begin{pmatrix} (\mu_{11}, \nu_{11}) & (\mu_{12}, \nu_{12}) & \cdots & (\mu_{1i}, \nu_{1i}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & \cdots & (\mu_{2i}, \nu_{2i}) & \cdots & (\mu_{2n}, \nu_{2n}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (\mu_{i1}, \nu_{i1}) & (\mu_{i2}, \nu_{i2}) & \cdots & (\mu_{ii}, \nu_{ii}) & \cdots & (\mu_{in}, \nu_{in}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (\mu_{n1}, \nu_{n1}) & (\mu_{n2}, \nu_{n2}) & \cdots & (\mu_{ni}, \nu_{ni}) & \cdots & (\mu_{nn}, \nu_{nn}) \end{pmatrix}$$

The above relation can completely characterised using just its upper triangular part, be-

Table 3.8: All possible cycle of V_U

Length	*Cycles*
3	(1 4 3), (2 7 4), (2 7 3), (3 9 4), (1 9 10), (6 9 10),
4	(2 7 4 3), (2 7 8 3), (1 9 4 3), (2 7 9 4), (3 9 10 6), (3 9 10 8), (1 9 10 8), (3 9 10 7), (1 9 10 5), (5 6 9 10), (2 9 4 3)
5	(2 7 5 6 3), (2 7 5 6 4), (1 4 2 7 5),(1 4 2 7 3) , (1 4 2 7 8), (2 7 5 8 3), (2 7 3 9 4), (1 4 2 9 10), (3 9 10 7 8), (1 9 10 8 3), (1 9 10 7 3), (3 9 10 7 4), (1 9 10 6 3),(3 9 10 6 4), (1 9 10 5 8), (2 9 10 8 3), (2 9 10 7 3), (2 9 10 6 3), (2 9 10 6 4), (2 9 10 7 4),
6	(2 7 5 6 4 3), (1 4 3 2 7 5), (1 4 2 7 5 8), (1 4 3 2 7 8), (1 4 2 7 8 3), (1 9 4 2 7 3), (2 9 10 7 4 3), (1 9 4 2 7 8), (2 7 8 3 9 4), (1 9 4 2 7 5), (2 7 5 6 9 4), (2 7 9 10 8 3), (1 4 2 9 10 5), (2 9 10 5 6 4), (1 4 3 9 10 8), (1 9 10 7 4 3), (1 9 10 6 4 3), (3 9 10 5 6 4)
7	(1 4 2 7 5 6 3), (1 4 2 7 5 8 3), (1 9 4 2 7 8 3), (2 7 5 6 3 9 4), (1 9 4 2 7 5 8), (1 4 2 9 10 6 3), (2 7 5 8 3 9 4), (1 9 4 3 2 7 5), (2 7 5 6 9 4 3), (1 4 3 9 10 5 8), (1 9 10 5 6 4 3), (1 4 3 2 9 10 8), (2 9 10 5 6 4 3), (1 4 2 9 10 5 8)
8	(1 9 4 2 7 5 6 3), (1 9 4 2 7 5 8 3), (1 9 4 3 2 7 5 8), (1 9 4 3 2 7 8 10) (1 4 3 2 9 10 5 8) , (1 4 3 2 7 9 10 8)
9	(1 4 2 7 8 3 9 10 5), (1 9 10 5 6 4 2 7 8)
10	(1 9 10 5 6 4 3 2 7 8)

cause the intuitionistic multiplicative element (μ_{ij}, ν_{ij}) is the mirror image of (μ_{ji}, ν_{ji}).

$$U\tilde{R} = \begin{pmatrix} (\mu_{11},\nu_{11}) & (\mu_{12},\nu_{12}) & \cdots & (\mu_{1i},\nu_{1i}) & \cdots & (\mu_{1n},\nu_{1n}) \\ & (\mu_{22},\nu_{22}) & \cdots & (\mu_{2i},\nu_{2i}) & \cdots & (\mu_{2n},\nu_{2n}) \\ & & \ddots & \vdots & \ddots & \vdots \\ & & & (\mu_{ii},\nu_{ii}) & \cdots & (\mu_{in},\nu_{in}) \\ & & & & \ddots & \vdots \\ & & & & & (\mu_{nn},\nu_{nn}) \end{pmatrix}$$

and this can be represented equivalently as the following multiplicative preference relation

$$R=\begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1i} & \cdots & \mu_{1n} \\ \nu_{12} & \mu_{22} & \cdots & \mu_{2i} & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \nu_{1i} & \nu_{2i} & \cdots & \mu_{ii} & \cdots & \mu_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \nu_{1n} & \nu_{2n} & \cdots & \nu_{in}) & \cdots & \mu_{nn} \end{pmatrix}$$

Since $\mu_{ij}=\nu_{ji}$ and $\nu_{ij}=\mu_{ji}$ then the above multiplicative preference relation becomes

$$R=\begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1i} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2i} & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mu_{i1} & \mu_{i2} & \cdots & \mu_{ii} & \cdots & \mu_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{ni}) & \cdots & \mu_{nn} \end{pmatrix}$$

Let $\widetilde{Q}$ denote the set of IMPRs, where
$\widetilde{Q}=\{\tilde{R}=(\tilde{r}_{ij})|\forall i,j:\tilde{r}_{ij}=(\mu_{ij},\nu_{ij}),\mu_{ij},\nu_{ij}\in[1/9,9],\mu_{ii}=\nu_{ii}=1,\mu_{ij}=\nu_{ji},\mu_{ji}=\nu_{ij},0\leq\mu_{ij}\nu_{ij}\leq 1\}$
and $\Re$ be the set of multiplicative preference relations $\Re=\{R=(r_{ij})|\forall i,j:r_{ij}\in[1/9,9]\}$
Define a mapping $f:[1/9,9]\times[1/9,9]\rightarrow[1/9,9]$ by the function $f(x_1,x_2)=x_1$. We can define the following mapping, $F:\widetilde{Q}\rightarrow\Re$ between the set of IMPRs $\widetilde{Q}$ and the set of multiplicative preference relations, $\Re$

$$\{f(\tilde{r}_{ij})\}=\{\mu_{ij}\}\ i.e\ R=F(\tilde{R})$$

The following properties can be proved

Proposition 3.4.1. Function F is well defined, i.e. For given $\tilde{R}\in\widetilde{Q}\Rightarrow f(\tilde{R})\in\Re$.

Proof. Let $\tilde{R}=(\tilde{r}_{ij})\in\widetilde{Q}$.
Here $\tilde{r}_{ij}=(\mu_{ij},\nu_{ij})\Rightarrow f(\tilde{r}_{ij})=f(\mu_{ij},\nu_{ij})=\mu_{ij}\in R$ □

Proposition 3.4.2. Function F is one-one.

Proof. Let $\tilde{R}_1=(\tilde{r}^1_{ij})$ and $\tilde{R}_2=(\tilde{r}^2_{ij})$ are IMPR such that $F(\tilde{R}_1)=F(\tilde{R}_2)$. Then we have that

$$f(\tilde{r}^1_{ij})=f(\tilde{r}^2_{ij})\ \forall\ i,j\Leftrightarrow\mu^1_{ij}=\mu^2_{ij}\ \forall\ i,j.$$

Because of the conditions of $\mu_{ij}^1 = \nu_{ji}^1$ and $\mu_{ij}^2 = \nu_{ji}^2$, then it is obvious that $\nu_{ij}^1 = \nu_{ij}^2$, $\forall\, i, j$ Therefore, we have that

$$(\mu_{ij}^1, \nu_{ij}^1) = (\mu_{ij}^2, \nu_{ij}^2) \Leftrightarrow \tilde{R}_1 = \tilde{R}_2 \;\forall\, i, j$$

$\square$

For the function to be onto, the following conditions to be verified:

$$\forall R \in \mathfrak{R}\; \exists\, \tilde{R} \in \widetilde{Q} : F(\tilde{R}) = R.$$

By the definition of F and $\widetilde{Q}$, we have that $R = (r_{ij}) = (\mu_{ij})$ satisfies:

$$0 \leq r_{ij} r_{ji} = \mu_{ij} \mu_{ji} \leq 1.$$

Thus R is asymmetric multiplicative preference relation that proves the range of the function F is the subset of multiplicative preference relations which are asymmetric.

Theorem 3.4.1. The set of intuitionistic multiplicative preference relations is isomorphic to set of asymmetric multiplicative preference relations.

Proof. We know that when $\tilde{R} \in \widetilde{Q}$ has hesitancy degree always zero, then we have that:

$$\mu_{ij} \nu_{ij} = 1,\; \forall\, i, j \tag{3.4.2}$$

In this case, $F(\tilde{R}) = R$ is also reciprocal, i.e. $r_{ij} r_{ji} = 1 \;\forall\, i, j$. The proof of this is quite simple as we have the following:

$$\forall\, i, j : r_{ij} = f(\tilde{r}_{ij}) = \mu_{ij} \wedge r_{ji} = f(\tilde{r}_{ji}) = \mu_{ji}.$$

Since $\tilde{R} \in \widetilde{Q}$ then we have that $\mu_{ji} = \nu_{ij} \forall i, j$ and by using equation 3.4.2 it is $r_{ij} r_{ji} = \mu_{ij} \mu_{ji} = \mu_{ij} \nu_{ij} = 1 \;\forall\, i, j$. $\square$

Example 3.4.5. Isomorphic asymmetric multiplicative preference relation of IMPR of example 3.4.2 is

$$I_{\tilde{R}_2} = \begin{pmatrix} 1 & \frac{5}{3} & 7 & 3 & \frac{5}{3} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{5}{3} & \frac{5}{3} & \frac{3}{5} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{9} & \frac{1}{4} & 1 & 1 & \frac{1}{4} & \frac{1}{3} & \frac{1}{9} \\ \frac{1}{7} & \frac{1}{4} & \frac{3}{5} & 1 & \frac{3}{5} & \frac{1}{3} & \frac{1}{9} \\ \frac{1}{7} & 1 & \frac{5}{3} & \frac{3}{5} & 1 & \frac{1}{3} & \frac{1}{7} \\ \frac{3}{5} & 1 & 3 & 3 & 3 & 1 & \frac{1}{7} \\ \frac{5}{3} & 3 & 7 & 7 & 3 & 3 & 1 \end{pmatrix}$$

The vertex matrix of above asymmetric MPR is

$$V = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

By using the algorithm of odd and even length cycle [138], there is no cycle is present in the vertex matrix V. Hence $\tilde{R}_2$ is consistent. Also we can directly see from the figure 3.9.

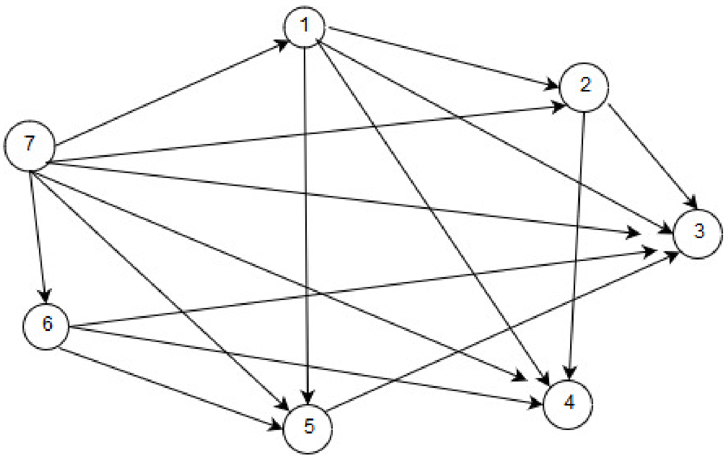

Figure 3.9: Directed graph of example 3.4.5

Also same isomorphism approach is applied in incomplete IMPRs scenario.

Example 3.4.6. Isomorphic asymmetric MPR of of example 3.4.3 is

$$I_{\tilde{R}_3} = \begin{pmatrix} 1 & \frac{1}{5} & 5 & * & \frac{1}{9} & 2 \\ 2 & 1 & * & \frac{1}{4} & \frac{1}{2} & 3 \\ \frac{1}{6} & * & 1 & * & 6 & * \\ * & \frac{1}{5} & * & 1 & \frac{1}{5} & * \\ 8 & \frac{1}{3} & \frac{1}{7} & 4 & 1 & * \\ \frac{1}{6} & \frac{1}{5} & * & * & * & 1 \end{pmatrix}$$

Vertex matrix of the asymmetric MPR, $I_{\tilde{R}_3}$ is

$$V = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using the algorithm [138], the is one cycle of length 3 i.e. $(1-3-5)$ is present. Therefore, $\tilde{R}_3$ is inconsistent. In this example, if we will change the pair $(3,5)$ to $(5,3)$ in the

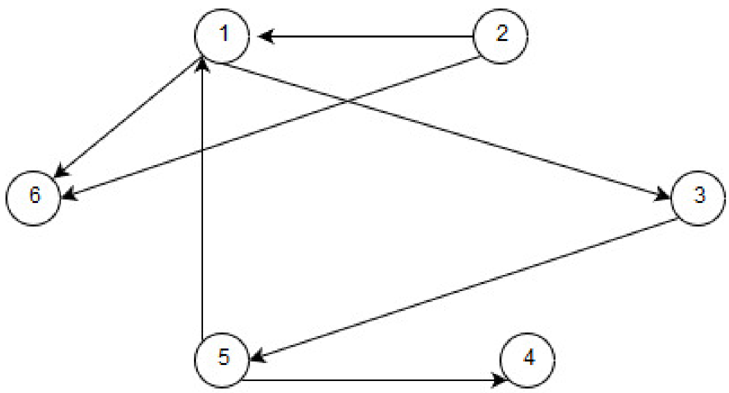

Figure 3.10: Directed graph of 3.4.6

original incomplete IMPR $\tilde{R}_3$, then it is consistent.

Example 3.4.7. Isomorphic asymmetric MPR of example 3.4.4 is

$$I_{\tilde{R}_4} = \begin{pmatrix} 1 & * & \frac{1}{2} & \frac{1}{3} & 3 & * & * & \frac{8}{3} & \frac{1}{7} & 5 \\ * & 1 & \frac{2}{3} & \frac{1}{8} & \frac{1}{4} & * & \frac{1}{5} & * & \frac{1}{3} & * \\ \frac{1}{5} & 0.6 & 1 & \frac{3}{5} & * & 2 & 4 & 7 & \frac{1}{8} & * \\ 2 & \frac{1}{2} & \frac{2}{3} & 1 & * & 3 & 5 & * & 2 & * \\ \frac{1}{6} & 1 & * & * & 1 & \frac{1}{8} & 8 & \frac{1}{6} & * & 4 \\ * & * & \frac{1}{2} & \frac{1}{7} & 7 & 1 & * & * & \frac{1}{5} & 2 \\ * & 5 & \frac{1}{8} & \frac{1}{6} & \frac{1}{9} & * & 1 & \frac{1}{9} & \frac{1}{7} & 4 \\ \frac{1}{8} & * & \frac{1}{8} & * & 3 & * & 6 & 1 & * & 6 \\ 4 & 3 & 5 & \frac{1}{2} & * & 4 & 2 & * & 1 & \frac{1}{3} \\ \frac{1}{5} & * & * & * & \frac{1}{5} & \frac{1}{3} & \frac{1}{8} & \frac{1}{6} & 2 & 1 \end{pmatrix}$$

and the vertex matrix of $I_{\tilde{R}_4}$ is

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

By using the algorithm of odd and even length cycle [138], the corresponding cycles are given in table 3.9,

Therefore, $\tilde{R}_4$ is inconsistent. In table 3.9 it is conclude the path $9-10$ that cover all the cycle. If we will change the pair $(9,10)$ to $(10,9)$, in the original IMPR $\tilde{R}_4$, then $\tilde{R}_4$ is consistent.

3.4.3 Comparison between two different Method using graphical approach:

In example 3.4.2, $\tilde{R}_2$ is inconsistent according to first method has no cycle found in the lower vertex matrix and one cycle of length 3 i.e $(3-5-4)$ found in the upper vertex matrix. But it is consistent according to second method. To suggest the cause of incon-

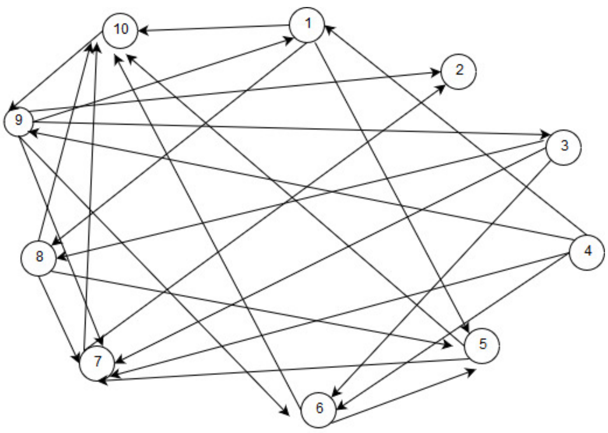

Figure 3.11: Directed graph of Example 3.4.7

Table 3.9: All possible cycle of V

Length	*Cycles*
3	(1 10 9), (6 10 9), (7 10 9)
4	(5 10 9 6), (1 8 10 9), (1 5 10 9), (3 7 10 9), (3 8 10 9), (3 6 10 9)
5	(3 8 7 10 9), (3 6 5 10 9), (1 8 7 10 9),(1 8 5 10 9), (5 7 10 9 6), (3 8 5 10 9), (1 5 7 10 9)
6	(1 8 5 7 10 9), (5 7 10 9 3 6), (3 6 5 7 10 9), (3 8 5 7 10 9), (5 7 10 9 3 8)
$7-10$	nothing

sistency in first method, we will change the path $(3,4)$ to $(4,3)$ in the original IMPR $\tilde{R}_2$, then both the lower and upper vertex are consistent. Therefore $\tilde{R}_2$ is consistent. From the two approach we have conclude that second method is better than first method.

In example 3.4.3, $\tilde{R}_3$ is inconsistent according to first method has one cycle found i.e $(1-3-5)$ in the lower vertex matrix and one cycle of length 3 i.e. $(1-5-3)$ is found in the upper vertex matrix. Also according to second method $\tilde{R}_3$ is inconsistent where one cycle is found in isometric matrix i.e. $(1-3-5)$. In both the methods the path $3-5$ which cover the cycle. We have to choose the couple to eliminate the cycle. In this case if we will change the pair $(3,5)$ to $(5,3)$ in the original IMPR $\tilde{R}_3$, then both the lower and upper vertex are consistent. Then IMPR $\tilde{R}_3$ is also consistent. Therefore method 2 is better than method 1 because less no of cycle is found in method 2.

In example 3.4.4, According to first method, 19 cycles are found in the lower vertex matrix V_L and 78 cycles are found in V_U that are given in table 3.7 and 3.8. Similarly according to second method, 21 cycles are found that are given in table 3.9. From the first method, we conclude that if we change the path $9-10$, $4-3$, and $2-7$ then the IMPR $\tilde{R}_4$ is consistent. But according to second method, if we will change the pair $(9,10)$ to $(10,9)$, in the original IMPR $\tilde{R}_4$, then $\tilde{R}_4$ is consistent. Therefore, second method is better than first method.

3.5 Conclusion

In this chapter, we have introduced a new transitivity property of IMPR. Based on this, we have presented two approaches for completing incomplete IMPRs. In the first approach, missing element can be calculated by using the new transitivity property, and an optimization model has been developed to adjust the initials values. Also goal programming model is developed to calculate the unknown element based on new transitivity property. Also, we have compared our method with Jiang et al. [143] and Meng and Chen [169] method.

In the second approach, we have proposed two different methods to get the characterization of the consistency for IMPR. In the first method, we propose an algorithm to check the consistency of IMPR by using the cycles of various length in a directed graph and same procedure is applied for incomplete IMPRs also. The second method proves isomorphism between the set of IMPRs and the set of asymmetric multiplicative preference relations. Also, consistency property is then checked for asymmetric preference relation using directed graph that is used to get the consistency of IMPR.

Chapter 4

Acceptably Consistent Incomplete Interval-Valued Intuitionistic Multiplicative Preference Relations

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In this chapter[1]*, We study the consistency property, and especially the acceptably consistent property, for incomplete interval-valued intuitionistic multiplicative preference relations. We propose a technique which first estimates the initial values for all missing entries in an incomplete interval-valued intuitionistic multiplicative preference relation and then improves them by a local optimization method. Two examples are presented in order to illustrate applications of the proposed method in group-decision making problems.*

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[1]The content of this chapter is based on research paper "Acceptably Consistent Incomplete Interval-Valued Intuitionistic Multiplicative Preference Relations", *Soft Computing*, Springer, **22**, 7463– 7477 (2018).

4.1 Introduction

Multi-criteria group decision making (MCGDM) problems aim to rank a finite number of alternatives based on specific criteria relevant to the characteristics of alternatives in context. The issues require deliberations from experts/decision makers and stakeholders to compare these alternatives on the pre-identified criteria. Different techniques for aggregating the elicited information thus received dates back to the classical works of [173] (ELECTRE), Saaty ([57], [58]) (AHP), Brans and Vincke [174] (PROMETHEE), Hwang and Yoon [175] (TOPSIS), to name a few. One can refer to the good texts (Figueira et al. [176]; Triantaphyllou [177]) for a detailed description of these methods along with numerous real-life case studies. The process of comparing different alternatives naturally embed the preference of an alternative over the other alternatives by an expert and hence preference relations are an integral part of such problems. Because of varying evaluation scales across the problems, the preference relations have been studied in more than one way, for instance, see ([60], [62], [66], [131], [134], [142], [143], [167], [178]) and many more.

In any of the framework mentioned above, although the scale and the domain of the preference relations may differ, yet an expert has to populate $n(n-1)/2$ judgments for presenting a complete preference relation matrix of order n. It is indeed an enormous task especially when n is reasonably large. Rezaei [179] proposed the best-worst multi-criteria decision-making method which reduces the number of pairwise comparisons of n alternatives to $2n-3$. The technique requires identifying a priori the best among all and the worst among all criteria.

The above-cited works assume that an expert is always able to provide information on each entry in a preference relation matrix. It may not be the case forever. An expert may not have expertise on a particular criterion or considers two alternatives incomparable on satisfied criterion(a). An extreme example is when an expert does not provide any information about an alternative on a particular criterion. This situation is an ignorance in the literature. As a result, an expert declines to make a comparison between two choices leading to an incomplete preference relation, that is, a preference relation matrix with missing entries. The critical issue to address is how to apply the MCGDM methods in incomplete preference relations? The majority of the studies in the literature suggested carrying out a completion process before the aggregation of information in an MCGDM problem.

To the best of our literature survey, there does not exist any study in the literature guiding how to populate the missing entries in an interval-valued intuitionistic multiplicative preference relation (IVI-MPR). This research gap incentivizes us to fixate on this topic in the present study.

The present chapter aims to analyze the class of IVI-MPR and incomplete IVI-MPR preference relations. The proposed work demonstrates an approach to determine the missing elements in an incomplete IVI-MPR. The procedure comprising of the estimating step (based on the elementary connecting paths of length two) and the adjusting step (requiring to solve an optimization problem iteratively) is formulated to determine all missing elements in an incomplete IVI-MPR.

The remainder of the chapter unfolds as follows. Section 4.2 reviews the concepts of IVI-MPR, acceptably consistent IVI-MPR and incomplete IVI-MPR. In Section 4.3, a two-step procedure is devised to compute the missing entries of IVI-MPR. The algorithm uses the geometric mean to calculate the initial estimates of all missing entries in the first step called 'estimating step.' After that, an optimization model is presented to improve these initial estimates in the second step called the 'adjusting step.' Section 4.4 presents an application of incomplete IVI-MPR to MCGDM problems together with illustrative examples. Section 4.5 makes a comparative analysis of the proposed methodology with the two existing works relevant to the context. The paper concludes in Section 4.6.

4.2 Preliminaries

Jiang et al. (2014) introduced the concept of interval-valued intuitionistic multiplicative set (IVI-MS) as follows.

$$\tilde{D} = \{\langle x, \tilde{\mu}_D(x), \tilde{\nu}_D(x)\rangle \mid x \in X\},$$

where $\tilde{\mu}(x) \subseteq [1/9, 9]$ and $\tilde{\nu}(x) \subseteq [1/9, 9]$ are intervals satisfying

$$0 < \sup_{x\in X} \tilde{\mu}(x). \sup_{x\in X} \tilde{\nu}(x) \leq 1.$$

The pair $(\tilde{\mu}_D(x), \tilde{\nu}_D(x))$ is called interval-valued intuitionistic multiplicative number (IVI-MN).

Based on the result of Jiang et al. [180], IVI-MPR is introduced where the preference degree and the non-preference degree of x_i over x_j are now interval-valued. Let us denote

$\widetilde{r}_{ij}(x_i,x_j) = ([\underline{\mu}_{ij}(x_i,x_j), \overline{\mu}_{ij}(x_i,x_j)], [\underline{\nu}_{ij}(x_i,x_j), \overline{\nu}_{ij}(x_i,x_j)])$ by $\widetilde{r}_{ij} = ([\underline{\mu}_{ij}, \overline{\mu}_{ij}], [\underline{\nu}_{ij}, \overline{\nu}_{ij}])$.

Definition 4.2.1. **(IVI-MPR)** An IVI-MPR is defined as $\widetilde{R}_{IVI} = [\widetilde{r}_{ij}]_{n\times n}$, where

$$\widetilde{r}_{ij} = ([\underline{\mu}_{ij}, \overline{\mu}_{ij}]\ [\underline{\nu}_{ij}, \overline{\nu}_{ij}]),\ i, j \in N,$$

and $[\underline{\mu}_{ij}, \overline{\mu}_{ij}]$ indicates the interval of certainty degree to which x_i is preferred to x_j, while $[\underline{\nu}_{ij}, \overline{\nu}_{ij}]$ is the interval of certainty degree to which x_i is not preferred to x_j, and they satisfy the following characteristics:

$$\begin{aligned}
&1/9 \le \underline{\mu}_{ij} \le \overline{\mu}_{ij} \le 9,\ \ 1/9 \le \underline{\nu}_{ij} \le \overline{\nu}_{ij} \le 9,\\
&0 \le \overline{\mu}_{ij}\overline{\nu}_{ij} \le 1,\\
&[\underline{\mu}_{ij}, \overline{\mu}_{ij}] = [\underline{\nu}_{ji}, \overline{\nu}_{ji}], [\underline{\nu}_{ij}, \overline{\nu}_{ij}] = [\underline{\mu}_{ji}, \overline{\mu}_{ji}],\\
&[\underline{\mu}_{ii}, \overline{\mu}_{ii}] = [\underline{\nu}_{ii}, \overline{\nu}_{ii}] = [1,1], \quad \forall\, i, j \in N.
\end{aligned}$$

If we define consistency for IVI-MPR by simply extending the relation in (3.2.1) as follows:

$$\begin{aligned}
([\underline{\mu}_{ij}, \overline{\mu}_{ij}], [\underline{\nu}_{ij}, \overline{\nu}_{ij}]) = ([\underline{\mu}_{ik}, \overline{\mu}_{ik}][\underline{\mu}_{kj}, \overline{\mu}_{kj}], [\underline{\nu}_{ik}, \overline{\nu}_{ik}][\underline{\nu}_{kj}, \overline{\nu}_{kj}]),\\
i \le k \le j, i, j, k \in N, \qquad (4.2.1)
\end{aligned}$$

where,

$$\begin{aligned}
&\underline{\mu}_{ij} = \min\{\underline{\mu}_{ik}\underline{\mu}_{kj}, \underline{\mu}_{ik}\overline{\mu}_{kj}, \overline{\mu}_{ik}\underline{\mu}_{kj}, \overline{\mu}_{ik}\overline{\mu}_{kj}\},\ \overline{\mu}_{ij} = \max\{\underline{\mu}_{ik}\underline{\mu}_{kj}, \underline{\mu}_{ik}\overline{\mu}_{kj}, \overline{\mu}_{ik}\underline{\mu}_{kj}, \overline{\mu}_{ik}\overline{\mu}_{kj}\},\\
&\underline{\nu}_{ij} = \min\{\underline{\nu}_{ik}\underline{\nu}_{kj}, \underline{\nu}_{ik}\overline{\nu}_{kj}, \overline{\nu}_{ik}\underline{\nu}_{kj}, \overline{\nu}_{ik}\overline{\nu}_{kj}\},\ \overline{\nu}_{ij} = \max\{\underline{\nu}_{ik}\underline{\nu}_{kj}, \underline{\nu}_{ik}\overline{\nu}_{kj}, \overline{\nu}_{ik}\underline{\nu}_{kj}, \overline{\nu}_{ik}\overline{\nu}_{kj}\},
\end{aligned}$$

then, an IVI-MPR $\widetilde{R}_{IVI}$ may not be consistent according to (4.2.1).

Example 4.2.1. Consider an IVI-MPR described by

$$\widetilde{R}_{IVI} = \begin{bmatrix} ([1,1],[1,1]) & ([\frac{1}{6},\frac{1}{5}],[3,4]) & ([\frac{1}{3},\frac{1}{3}],[3,3]) \\ ([3,4],[\frac{1}{6},\frac{1}{5}]) & ([1,1],[1,1]) & ([\frac{1}{4},\frac{1}{3}],[\frac{1}{5},\frac{3}{2}]) \\ ([3,3],[\frac{1}{3},\frac{1}{3}]) & ([\frac{1}{5},\frac{3}{2}],[\frac{1}{4},\frac{1}{3}]) & ([1,1],[1,1]) \end{bmatrix}.$$

We have

$$\begin{aligned}
\widetilde{r}_{12}\widetilde{r}_{23} &= ([\underline{\mu}_{12}, \overline{\mu}_{12}][\underline{\mu}_{23}, \overline{\mu}_{23}], [\underline{\nu}_{12}, \overline{\nu}_{12}][\underline{\nu}_{23}, \overline{\nu}_{23}])\\
&= ([0.04167, 0.067], [0.6, 6]) \ne \widetilde{r}_{13},
\end{aligned}$$

hence, the above $\widetilde{R}_{IVI}$ is not consistent according to equation 4.2.1.

To overcome the limitation, imposed by the restriction on indices in the transitivity property, we discuss a more general consistent property of an IVI-MPR by splitting an IVI-MPR into matrices.

An interval-valued intuitionistic multiplicative number (IVI-MN) $\widetilde{r}_{ij} = ([\underline{\mu}_{ij}, \overline{\mu}_{ij}], [\underline{\nu}_{ij}, \overline{\nu}_{ij}])$ contains two parts, $[\underline{\mu}_{ij}, \overline{\mu}_{ij}]$ is the preferred information and $[\underline{\nu}_{ij}, \overline{\nu}_{ij}]$ is the non-preferred information. In this sense, an IVI-MPR $\widetilde{R}_{IVI} = [\widetilde{r}_{ij}]_{n\times n}$ can be splitted into two matrices, $A^{(1)} = [a_{ij}^{(1)}]_{n\times n}$ and $A^{(2)} = [a_{ij}^{(2)}]_{n\times n}$ expressing the preferred information and the non-preferred information in $\widetilde{R}_{IVI}$, respectively. The concrete forms are listed as follows.

$$a_{ij}^{(1)} = \begin{cases} [\underline{\mu}_{ij}, \overline{\mu}_{ij}] & i<j \\ [1,1] & i=j \\ [\frac{1}{\overline{\nu}_{ij}}, \frac{1}{\underline{\nu}_{ij}}] & i>j \end{cases} \text{ and } a_{ij}^{(2)} = \begin{cases} [\underline{\nu}_{ij}, \overline{\nu}_{ij}] & i<j \\ [1,1] & i=j \\ [\frac{1}{\overline{\mu}_{ij}}, \frac{1}{\underline{\mu}_{ij}}] & i>j \end{cases} \tag{4.2.2}$$

For instance, going back to example in (4.2.1) of IVI-MPR $\widetilde{R}_{IVI}$, we have

$$A^{(1)} = \begin{bmatrix} [1,1] & [\frac{1}{6}, \frac{1}{5}] & [\frac{1}{3}, \frac{1}{3}] \\ [5,6] & [1,1] & [\frac{1}{4}, \frac{1}{3}] \\ [3,3] & [3,4] & [1,1] \end{bmatrix}, A^{(2)} = \begin{bmatrix} [1,1] & [3,4] & [3,3] \\ [\frac{1}{4}, \frac{1}{3}] & [1,1] & [\frac{1}{5}, \frac{3}{2}] \\ [\frac{1}{3}, \frac{1}{3}] & [\frac{2}{3}, 5] & [1,1] \end{bmatrix}.$$

Definition 4.2.2. [104] An interval-valued multiplicative reciprocal matrix $S = [\tilde{s}_{ij}]_{n\times n} = ([s_{ij}^{-}, s_{ij}^{+}])_{n\times n}$, where

$$S = \begin{bmatrix} [1,1] & [s_{12}^{-}, s_{12}^{+}] & \cdots & [s_{1n}^{-}, s_{1n}^{+}] \\ [s_{21}^{-}, s_{21}^{+}] & [1,1] & \cdots & [s_{2n}^{-}, s_{2n}^{+}] \\ \vdots & \vdots & \cdots & \vdots \\ [s_{n1}^{-}, s_{n2}^{+}] & [s_{n2}^{-}, s_{n2}^{+}] & \cdots & [1,1] \end{bmatrix},$$

with $0 \le s_{ij}^{-} \le s_{ij}^{+}$, $s_{ij}^{-} s_{ji}^{+} = 1$, $s_{ij}^{+} s_{ji}^{-} = 1$, is said to be consistent if the two multiplicative reciprocal matrices, $S^{(1)} = [s_{ij}^{(1)}]_{n\times n}$ and $S^{(2)} = [s_{ij}^{(2)}]_{n\times n}$, obtained by splitting S as follows, are consistent, where

$$s_{ij}^{(1)} = \begin{cases} s_{ij}^{+} & i<j \\ 1 & i=j \\ s_{ij}^{-} & i>j \end{cases} \text{ and } s_{ij}^{(2)} = \begin{cases} s_{ij}^{-} & i<j \\ 1 & i=j \\ s_{ij}^{+} & i>j\,. \end{cases} \tag{4.2.3}$$

Here, by consistency of matrices $S^{(1)}$ and $S^{(2)}$, we mean, that they both satisfy the transitivity property $s_{ij}^{(1)} = s_{ik}^{(1)}\, s_{kj}^{(1)}$ and $s_{ij}^{(2)} = s_{ik}^{(2)}\, s_{kj}^{(2)}$, $\forall\, i,j,k \in N$.

Motivated by Definition 4.2.2, we propose to further split the matrix $A^{(1)}$ into $C^{(q)}$, $q = 1,2$, and matrix $A^{(2)}$ into $C^{(q)}$, $q = 3,4$, in spirit of (4.2.3). The elements in $C^{(q)}$ satisfy the relation $c_{ij}^{(q)}\, c_{ji}^{(q)} = 1$, $\forall\, i,j \in N$, and $q = 1,2,3,4$. Thus, $[c_{ij}^{(q)}]_{n\times n}$, $q = 1,2,3,4$, are four MPRs obtained from an IVI-MPR $\widetilde{R}_{IVI}$.

Definition 4.2.3. An IVI-MPR $\widetilde{R}_{IVI} = [\widetilde{r}_{ij}]_{n\times n} = ([\underline{\mu}_{ij}, \overline{\mu}_{ij}],\, [\underline{\nu}_{ij}, \overline{\nu}_{ij}])_{n\times n}$, is said to be consistent if the four MPRs $C^{(q)}$, $q = 1,2,3,4$, obtained using (4.2.2) and (4.2.3), are consistent, that means

$$c_{ij}^{(q)} = c_{ik}^{(q)}\, c_{kj}^{(q)}, \quad \forall\, i,\, j,\, k \in N,\; q = 1,2,3,4. \tag{4.2.4}$$

Definition 4.2.4. An IVI-MPR $\widetilde{R} = [\widetilde{r}_{ij}]_{n\times n} = (\,[\underline{\mu}_{ij}, \overline{\mu}_{ij}],\, [\underline{\nu}_{ij}, \overline{\nu}_{ij}]\,)_{n\times n}$ is called acceptably consistent if the four MPRs $C^{(q)}$, $q = 1,2,3,4$, obtained using (4.2.2) and (4.2.3), are acceptably consistent; otherwise $\widetilde{R}_{IVI}$ is called not acceptably consistent or inconsistent.

Consider an IVI-MPR given by

$$\tilde{R}_{IVI} = \begin{bmatrix} ([1,1],[1,1]) & ([3,4],[\frac{1}{5},\frac{1}{4}]) & ([2,3],[\frac{1}{9},\frac{1}{8}]) \\ \\ ([\frac{1}{5},\frac{1}{4}],[3,4]) & ([1,1],[1,1]) & ([\frac{2}{3},\frac{3}{2}],[\frac{1}{4},\frac{1}{2}]) \\ \\ ([\frac{1}{9},\frac{1}{8}],[2,3]) & ([\frac{1}{4},\frac{1}{2}],[\frac{2}{3},\frac{3}{2}]) & ([1,1],[1,1]) \end{bmatrix}.$$

This IVI-MPR is split into the following four MPRs

$$C^{(1)} = \begin{bmatrix} 1 & 4 & 3 \\ \frac{1}{4} & 1 & \frac{3}{2} \\ \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix},\; C^{(2)} = \begin{bmatrix} 1 & 3 & 2 \\ \frac{1}{3} & 1 & \frac{2}{3} \\ \frac{1}{2} & \frac{3}{2} & 1 \end{bmatrix}$$

$$C^{(3)} = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{8} \\ 4 & 1 & \frac{1}{2} \\ 8 & 2 & 1 \end{bmatrix}, \ C^{(4)} = \begin{bmatrix} 1 & \frac{1}{5} & \frac{1}{9} \\ 5 & 1 & \frac{1}{4} \\ 9 & 4 & 1 \end{bmatrix}.$$

Observe $C^{(1)}$ and $C^{(4)}$ are not consistent whereas $C^{(2)}$ and $C^{(3)}$ are consistent (in spirit of Definition 1.2.1). Moreover, $CR < 0.1$ for all four matrices. Hence $\tilde{R}_{IVI}$ is not consistent but acceptably consistent.

Now, if we replace $\underline{\nu}_{12} = \frac{1}{5}$ by $\underline{\nu}_{12} = \frac{1}{7}$ in $\tilde{R}_{IVI}$, then the CR of the MPR $C^{(4)}$ is 0.1392; and then $\tilde{R}_{IVI}$ is not acceptably consistent.

We next propose to extend the above situation to incomplete IVI-MPR where some elements are missing in the preference relation matrix.

Definition 4.2.5. An IVI-MPR $\widetilde{R}_{IVI} = [\widetilde{r}_{ij}]_{n\times n} = ([\underline{\mu}_{ij}, \overline{\mu}_{ij}], [\underline{\nu}_{ij}, \overline{\nu}_{ij}])_{n\times n}$ is called an incomplete IVI-MPR if some elements in it are missing, and all available elements satisfy the characteristics of IVI-MPR stated in Definition 4.2.1.

Definition 4.2.6. An incomplete IVI-MPR $\widetilde{R}_{IVI} = [\widetilde{r}_{ij}]_{n\times n} = ([\underline{\mu}_{ij}, \overline{\mu}_{ij}], [\underline{\nu}_{ij}, \overline{\nu}_{ij}])_{n\times n}$ is called consistent if all known elements of the associated four MPRs $C^{(q)}$, $q = 1,2,3,4$, satisfy condition in (4.2.4).

Definition 4.2.7. An incomplete IVI-MPR $\widetilde{R}_{IVI} = [\widetilde{r}_{ij}]_{n\times n} = ([\underline{\mu}_{ij}, \overline{\mu}_{ij}], [\underline{\nu}_{ij}, \overline{\nu}_{ij}])_{n\times n}$ is called acceptably consistent if the associated four MPRs $C^{(q)}$, $q = 1,2,3,4$, are acceptably consistent; otherwise, $\widetilde{R}_{IVI}$ is called not acceptably consistent or inconsistent.

4.3 Complementing the Incomplete IVI-MPR

In this section, taking inspiration from the work of Jiang et al. [143], we propose a two-step procedure to calculate all missing entries in an IVI-MPR. The idea is first to estimate their values using the simple connecting path approach and subsequently improve upon them using an optimization problem. In the latter step, an optimization problem is constructed so to achieve transitivity and hence acceptably consistent resultant complete IVI-MPR. The objective function of the optimization problem is designed to minimize the overall absolute error occurring in the transitivity equation of the logarithmic values of the (missing) entries in an IVI-MPR. The detailed description of the procedure is laid down in the following subsection.

4.3.1 Estimating Step

Let Ω be the set of all known elements in IVI-MPR $\widetilde{R}_{IVI} = [\widetilde{r}_{ij}]_{n\times n}$; $\Gamma = \{(i,j) \in N \times N \,|\, \widetilde{r}_{ij}$ is missing$\}$; the associated four MPRs be $C^{(q)} = [c_{ij}^{(q)}]_{n\times n}$, $q = 1,2,3,4$; $\Omega^{(q)}$ be the set of all known elements in $C^{(q)}$, $q = 1,2,3,4$; $\Gamma^{(q)} = \{(i,j) \in N\times N \,|\, c_{ij}^{(q)}$ is missing$\}$, $q = 1,2,3,4$. Then, $\Omega = \bigcup_{q=1}^{4}\Omega^{(q)}$ and $\Gamma = \bigcup_{q=1}^{4}\Gamma^{(q)}$.

The missing value $\widetilde{*}_{ij} = ([c_{ij}^{(*2)}, c_{ij}^{(*1)}], [c_{ij}^{(*4)}, c_{ij}^{(*3)}])$ can be estimated by

$$c_{ij}^{(*q)} = \Big(\prod_{k_1\in MC_{ij}^{(q)}} \big(c_{ik_1}^{(q)} c_{k_1 j}^{(q)}\big)\Big)^{\frac{1}{mc_{ij}^{(q)}}},\ q = 1,2,3,4, \tag{4.3.1}$$

where $MC_{ij}^{(q)} = \{k_1 \,|\, c_{ik_1}^{(q)}, c_{k_1 j}^{(q)} \in \Omega^{(q)}\}$, and $mc_{ij}^{(q)}$ is the cardinality of the set $MC_{ij}^{(q)}$, $q = 1,2,3,4$.

If the length of the path $cp_{(\ell+1)}$ (as discussed in section 3.3) increases, then the value of $c_{ij}^{(*q)} \equiv \widetilde{*}_{ij}$ decreases. It means that the value of $*_{ij}$ is largely determined by the paths of smaller lengths. The elementary connecting paths of length 2 are indeed the smallest one and hence significant contributors to estimating the value of $\widetilde{*}_{ij}$. This idea is used in formulating (4.3.1). We shall be denoting the initial values, obtained from (4.3.1), by $\widetilde{*}_{ij}^{((0))}$, $i,j\in N$.

4.3.2 Adjusting Step: Optimization Model

If $\widetilde{R}_{IVI}$ is an incomplete IVI-MPR, then at least one of the associated four MPRs $C^{(q)}$ is an incomplete MPR. In other words, an incomplete IVI-MPR $\widetilde{R}_{IVI}$ can be complemented by complementing $C^{(q)}$. This task is proposed to be accomplished by solving an optimization problem.

If $\widetilde{R}_{IVI}$ is consistent then by Definition 4.2.3, (4.2.4) holds, and hence we have,

$$\begin{aligned} \log c_{ij}^{(q)} &= \log c_{ik}^{(q)} + \log c_{kj}^{(q)}, \\ & i,j,k \in N,\ i \neq j \neq k,\ q = 1,2,3,4. \end{aligned} \tag{4.3.2}$$

Using this, we propose to solve the scalar-optimization problem **Model 4.1** which aims at minimizing the total absolute error in (4.3.2). The formulation of Model 4.1 is in the spirit to ensure that the optimal values lie in the interval range of IVI-MPR. Also, the optimal values of the missing entries will bring the complete IVI-MPR more closer (if not exactly) to acceptably consistent.

(Model 4.1) $\min \quad \sum_{q=1}^{4} \sum_{(i,j)\in\Gamma^{(q)}} \sum_{k\in\Omega^{(q)}} \varepsilon_{ijk}^{c^{(q)}}$

subject to

$$
\begin{array}{rll}
\varepsilon_{ijk}^{c^{(q)}} - |\log c_{ij}^{(q)} - (\log c_{ik}^{(q)} + \log c_{kj}^{(q)})| & = 0 & q = 1,2,3,4, \\
c_{ij}^{(q)} & \geq (1/9) & q = 1,2 \\
c_{ij}^{(q)} & \leq 9 & q = 3,4 \\
c_{ij}^{(q)}\, c_{ij}^{(q+2)} & \leq 1 & q = 1,2,\ i,\ j,k \in N,\ i \neq j \neq k \\
c_{ij}^{(q+1)} & \leq c_{ij}^{(q)} & q = 1,3,\ i < j,\ i,\ j \in N \\
c_{ij}^{(q+1)} & \geq c_{ij}^{(q)} & q = 1,3,\ i > j,\ i,\ j \in N.
\end{array}
$$

Starting from the initial values $c_{ij}^{(*q)((0))}$ for $c_{ij}^{(*q)}$, $(i,j) \in \Gamma^{(q)}$, $q = 1,2,3,4$, Model 4.1 is iteratively solved to generate all unknown elements in the incomplete MPRs $C^{(q)}$, $q = 1,2,3,4$. Consequently the complete IVI-MPR $\widetilde{R}_{IVI}$ can be obtained.

The following two examples illustrate the above methodology for finding the missing elements in an IVI-MPR.

Example 4.3.1. Consider a simple case involving an incomplete IVI-MPR with at most one missing entry in $C^{(q)}$, $q = 1,2,3,4$.

$$
\widetilde{R}_{IVI} = \begin{bmatrix}
([1,1],[1,1]) & ([1/6,1/5],[*,*]) & ([*,1/3],[3,3]) \\
([*,*],[1/6,1/5]) & ([1,1],[1,1]) & ([1/4,*],[1/5,3/2]) \\
([3,3],[*,1/3]) & ([1/5,3/2],[1/4,*]) & ([1,1],[1,1])
\end{bmatrix}
$$

where $*$ denotes the missing element. Using (4.2.2) and (4.2.3), the associated four MPRs are as follows:

$$
C^{(1)} = \begin{bmatrix}
1 & 1/5 & 1/3 \\
5 & 1 & c_{23}^{(*1)} \\
3 & 1/c_{23}^{(*1)} & 1
\end{bmatrix},
C^{(2)} = \begin{bmatrix}
1 & 1/6 & c_{13}^{(*2)} \\
6 & 1 & 1/4 \\
1/c_{13}^{(*2)} & 4 & 1
\end{bmatrix},
$$

$$
C^{(3)} = \begin{bmatrix}
1 & c_{12}^{(*3)} & 3 \\
1/c_{12}^{(*3)} & 1 & 3/2 \\
1/3 & 2/3 & 1
\end{bmatrix},
C^{(4)} = \begin{bmatrix}
1 & c_{12}^{(*4)} & 3 \\
1/c_{12}^{(*4)} & 1 & 1/5 \\
1/3 & 5 & 1
\end{bmatrix}.
$$

The initial values of the missing elements are calculated using (4.3.1) as follows:

$$
c_{23}^{(*1)((0))} = c_{21}^{(1)} c_{13}^{(1)} = 1.67,\ c_{13}^{(*2)((0))} = c_{12}^{(2)} c_{23}^{(2)} = 0.041,
$$
$$
c_{12}^{(*3)((0))} = c_{13}^{(3)} c_{32}^{(3)} = 2,\ c_{12}^{(*4)((0))} = c_{13}^{(4)} c_{32}^{(4)} = 15.
$$

Sometimes the initial values do not lie in the range of IVI-MPRs; like the values $c_{13}^{(*2)((0))}$ and $c_{12}^{(*4)((0))}$ in above example. To adjust these initial values, we solve problem (Model 4.1) using LINGO software, and after 34 iterations we obtain

$$c_{23}^{(*1)} = 0.667, \quad c_{13}^{(*2)} = 0.111, \quad c_{12}^{(*3)} = 5, \quad c_{12}^{(*4)} = 5.$$

Therefore, the complete IVI-MPR is

$$\widetilde{R}_{IVI} = \begin{bmatrix} ([1,1],[1,1]) & ([1/6,1/5],[5,5]) & ([0.111,1/3],[3,3]) \\ ([5,5],[1/6,1/5]) & ([1,1],[1,1]) & ([1/4,0.667],[1/5,3/2]) \\ ([3,3],[0.111,1/3]) & ([1/5,3/2],[1/4,0.667]) & ([1,1],[1,1]) \end{bmatrix}.$$

Since the CR for $C^{(q)}, q = 2,4$, are greater than 0.1, the complete IVI-MPR $\widetilde{R}$ is not acceptably consistent.

Example 4.3.2. Consider an incomplete IVI-MPR with more than one missing element in the associated MPRs.

$$\widetilde{R}_{IVI} = \begin{bmatrix} ([1,1],[1,1]) & ([*,*],[*,*]) & ([0.2,0.4],[0.3,0.5]) & ([0.2,0.4],[0.4,*]) \\ ([*,*],[*,*]) & ([1,1],[1,1]) & ([0.2,0.3],[*,0.7]) & ([*,0.4],[0.4,0.6]) \\ ([0.3,0.5],[0.2,0.4]) & ([*,0.7],[0.2,0.3]) & ([1,1],[1,1]) & ([0.2,0.2],[0.5,0.7]) \\ ([0.4,*],[0.2,0.4]) & ([0.4,0.6],[*,0.4]) & ([0.5,0.7],[0.2,0.2]) & ([1,1],[1,1]) \end{bmatrix}.$$

The associated four MPRs are as follows:

$$C^{(1)} = \begin{bmatrix} 1 & c_{12}^{(*1)} & 0.4 & 0.4 \\ 1/c_{12}^{(*1)} & 1 & 0.3 & 0.4 \\ 1/0.4 & 1/0.3 & 1 & 0.2 \\ 1/0.4 & 1/0.4 & 1/0.2 & 1 \end{bmatrix}, C^{(2)} = \begin{bmatrix} 1 & c_{12}^{(*2)} & 0.2 & 0.2 \\ 1/c_{12}^{(*2)} & 1 & 0.2 & c_{24}^{(*2)} \\ 1/0.2 & 1/0.2 & 1 & 0.2 \\ 1/0.2 & 1/c_{24}^{(*2)} & 1/0.2 & 1 \end{bmatrix},$$

$$C^{(3)} = \begin{bmatrix} 1 & c_{12}^{(*3)} & 0.5 & c_{14}^{(*3)} \\ 1/c_{12}^{(*3)} & 1 & 0.7 & 0.6 \\ 1/0.5 & 1/0.7 & 1 & 0.7 \\ 1/c_{14}^{(*3)} & 1/0.6 & 1/0.7 & 1 \end{bmatrix}, C^{(4)} = \begin{bmatrix} 1 & c_{12}^{(*4)} & 0.3 & 0.4 \\ 1/c_{12}^{(*4)} & 1 & c_{23}^{(*4)} & 0.4 \\ 1/0.3 & 1/c_{23}^{(*4)} & 1 & 0.5 \\ 1/0.4 & 1/0.4 & 1/0.5 & 1 \end{bmatrix}.$$

The following initial values of missing elements are calculated using (4.3.1)

$$c_{12}^{(*1)((0))} = [(c_{13}^{(1)}c_{32}^{(1)})\,(c_{14}^{(1)}c_{42}^{(1)})]^{1/2} = 1.1547,\ c_{12}^{(*2)((0))} = c_{13}^{(2)}c_{32}^{(2)} = 1,$$
$$c_{24}^{(*2)((0))} = c_{23}^{(2)}c_{34}^{(2)} = 0.04,\ c_{12}^{(*3)((0))} = c_{13}^{(3)}c_{32}^{(3)} = 0.7143,\ c_{14}^{(*3)((0))} = c_{13}^{(3)}c_{34}^{(3)} = 0.35,$$
$$c_{12}^{(*4)((0))} = c_{14}^{(4)}c_{42}^{(4)} = 1\,, c_{23}^{(*4)((0))} = c_{24}^{(4)}c_{43}^{(4)} = 0.8.$$

In this case $c_{24}^{(*2)((0))} = 0.04$ does not satisfies the condition of IVI-MPRs. To adjust the initial value solving problem (Model 4.1) in LINGO, after 42 iteration yield, $c_{12}^{(*1)} = 1$, $c_{12}^{(*2)} = 1$, $c_{24}^{(*2)} = 0.111$, $c_{12}^{(*3)} = 0.714$, $c_{14}^{(*3)} = 0.4$, $c_{12}^{(*4)} = 0.714$, $c_{23}^{(*4)} = 0.7$.

It is observed that $C^{(q)}$, $q=2,3,4$, are acceptably consistent while CR of $C^{(1)}=0.155>0.1$, hence it is not acceptably consistent.

4.4 Acceptably Consistent Multi-Criteria Group Decision Making

For a more informed recommendation, it is essential to have a collective opinion of experts and participating stakeholders. Though GDM mostly increases the complexity of decision making yet, it is unavoidable in many strategic decisions. In this section, we aim to extend the results of the earlier section to GDM.

Consider m decision makers/experts and their weight vector $w=(w_1,\ldots,w_m)^t$, with $w_i\geq 0$, and $\sum_{i=1}^m w_i=1$. Set, $M=\{1,\ldots,m\}$. Suppose each decision maker $e_g\,(g\in M)$ provides a preference relation matrix in the form of IVI-MPR

$$\widetilde{R}_{IVI,\,g}=[\widetilde{r}_{ij,g}]_{n\times n}=\left([\underline{\mu}_{ij,g},\overline{\mu}_{ij,g}],\,[\underline{\nu}_{ij,g},\overline{\nu}_{ij,g}]\right)$$

. Before proceeding ahead, we must verify whether each preference matrix $\widetilde{R}_{IVI,g}$ is consistent. This can be accomplished by invoking the following result from Elsner et al. [181](see, Theorem 1) on acceptably consistent aggregation of MPRs.

Proposition 4.4.1. Let m experts provide $n\times n$ acceptably consistent MPRs, $T_1=[t_{ij}^{(1)}],\ldots,$ $T_m=[t_{ij}^{(m)}]$. Let the importance (or weight) of the g-th expert be γ_g, $\gamma_g\in(0,1)$, $g=1,\ldots,m$, $\sum_{g=1}^m\gamma_g=1$. Then, the weighted geometric mean aggregated MPR, $\overline{T}=[\overline{t_{ij}}]_{n\times n}=T_1^{\gamma_1}\,T_2^{\gamma_2}\cdots T_m^{\gamma_m}$, where $\overline{t_{ij}}=\prod\limits_{g=1}^{m}\big(t_{ij}^{(g)}\big)^{\gamma_g}$, $\forall\,(i,j)\in N\times N$, is an acceptably consistent MPR.

Proof. Since T_g, $g=1,\ldots,m$, are acceptably consistent MPRs, by Definition 1.2.2, we have

$$\lambda_{T_g,\max}<0.1(n-1)\,RI_n+n,\;\; g=1,\ldots,m,$$

where $\lambda_{T_g,\max}$ denotes the maximum eigenvalue of the MPR matrix T_g. The above along with $\sum_{g=1}^m\gamma_g=1$, yields

$$\prod_{g=1}^{m}(\lambda_{T_g,\max})^{\gamma_g}<0.1(n-1)\,RI_n+n. \tag{4.4.1}$$

Invoking the following result from (Elsner et al. [181]),

$$\rho(\overline{T}) = \rho(T_1^{\gamma_1} T_2^{\gamma_2} \cdots T_m^{\gamma_m}) \leq \rho(T_1)^{\gamma_1} \rho(T_2)^{\gamma_2} \cdots \rho(T_m)^{\gamma_m},$$

where $\rho(T_g)$ is the spectral radius of the MPR matrix T_g, along with (4.4.1), we get,

$$\lambda_{\overline{T},\max} < 0.1\,(n-1)\,RI_n + n,$$

yielding acceptably consistent MPR $\overline{T}$. □

Remark 4.4.1. It may so happen that each MPR is not acceptably consistent but the weighted geometric mean aggregated MPR is acceptably consistent.

Let m decision makers provide $n \times n$ MPRs out of which k MPRs are acceptably consistent and remaining $m-k$ are not acceptably consistent. Suppose $T_1, \ldots, T_k$ are acceptably consistent while $T_{k+1}, \ldots, T_m$ are not acceptably consistent. By above Proposition, for any scalars $\gamma_1, \ldots, \gamma_k$, all in $(0,1)$ with their sum being 1, the weighted geometric mean aggregated MPR $\overline{T} = T_1^{\gamma_1} T_2^{\gamma_2} \cdots T_k^{\gamma_k}$ is acceptably consistent.

Petra et al. [182] provided the following sufficient condition for the weighted geometric mean aggregated MPR, constituted from the MPRs $\overline{T},\ T_{k+1}, \ldots, T_m$, to be acceptably consistent, and that is,

$$\begin{aligned}(\lambda_{\overline{T},\max})^{\delta} (\lambda_{T_{k+1},\max})^{\delta_1} \cdots (\lambda_{T_m,\max})^{\delta_{m-k}} \\ < (n-1)\beta\, RI_n + n\ ,\end{aligned}$$

where $\lambda_{T,\max}$ is the maximum eigenvalue of the MPR matrix T, and $\delta, \delta_1, \ldots, \delta_{m-k}$, are scalars all in $(0,1)$, with $\delta + \sum_{g=1}^{m-k} \delta_g = 1$. Also, β is the threshold of acceptably consistent[2] aggregated matrix.

Hence, we can identify scalars $\delta,\ \delta_1, \ldots, \delta_{m-k}$ in $(0,1)$ with their sum being 1, in such a way that the weighted geometric mean aggregated MPR of the given m MPRs is acceptably consistent although some of the individual MPRs are not acceptably consistent.

[2] Saaty [183] showed that $\beta = 0.05$ and $\beta = 0.08$ are quite reasonable for $n = 3$ and $n = 4$, respectively, for acceptably consistent aggregated matrix computed by geometric mean. Note, $\beta = 0.1$ in Definition 1.2.2.

Consider the following 4×4 MPRs

$$T_1 = \begin{pmatrix} 1 & 1 & 2 & 6 \\ 1 & 1 & 2 & 3 \\ 1/2 & 1/2 & 1 & 5 \\ 1/6 & 1/3 & 1/5 & 1 \end{pmatrix}, \; T_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1/2 & 1 & 3 & 2 \\ 1/3 & 1/3 & 1 & 2 \\ 1/4 & 1/2 & 1/2 & 1 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 1 & 3 & 5 & 6 \\ 1/3 & 1 & 6 & 9 \\ 1/5 & 1/6 & 1 & 7 \\ 1/6 & 1/9 & 1/7 & 1 \end{pmatrix}, \; T_4 = \begin{pmatrix} 1 & 4 & 3 & 6 \\ 1/4 & 1 & 7 & 8 \\ 1/3 & 1/7 & 1 & 6 \\ 1/6 & 1/8 & 1/6 & 1 \end{pmatrix},$$

$$T_5 = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 1/2 & 1 & 1 & 3 \\ 1/2 & 1 & 1 & 2 \\ 1/3 & 1/3 & 1/2 & 1 \end{pmatrix}$$

The maximum eigenvalues are respectively 4.14, 4.1179, 4.6169, 4.7867, 4.0458. Then, three MPRs T_1, T_2 and T_5 are acceptably consistent (with $n = 4$ and $\beta = 0.08$) while the remaining two MPRs T_3 and T_4 are not so. Now, aggregating the acceptably consistent MPRs with $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$. The aggregated matrix $\overline{T}$ is as follows:

$$\overline{T} = \begin{pmatrix} 1 & 1.5874 & 2.2894 & 4.1601 \\ & 1 & 1.8171 & 2.6207 \\ & & 1 & 2.7144 \\ & & & 1 \end{pmatrix}$$

The MPR $\overline{T}$ has maximum eigenvalue $\lambda_{\overline{T},\max} = 4.0386$, and hence it is acceptably consistent. Next, we choose scalars δ, δ_1, δ_2 so to satisfy the following conditions:

$$\begin{aligned} (4.0386)^{\delta}(4.6169)^{\delta_1}(4.7867)^{\delta_2} &< (n-1)\,\beta\, RI_4 \\ &= 4.2136 \\ \delta + \delta_1 + \delta_2 &= 1 \\ \delta, \delta_1, \delta_2 &\geq 0. \end{aligned}$$

This system has infinitely many solutions. One such solution is $\delta = 0.73,\ \delta_1 = 0.13,\ \delta_2 = 0.14$.

Remark 4.4.2. For $w_g \geq 0,\ g \in M,\ \sum_{g=1}^{m} w_g = 1$, it follows from Petra et al. [182] that a sufficient condition for the weighted geometric mean aggregated MPR (matrix) $C^{(q)}_{agg} = \left(C^{(q)}_{\widetilde{R}_1}\right)^{w_1} \cdots \left(C^{(q)}_{\widetilde{R}_m}\right)^{w_m}$, to be acceptably consistent is that

$$\prod_{g=1}^{m} \left(\lambda_{C^{(q)}_{\widetilde{R}_g},\max}\right)^{w_g} < (n-1)\beta\, RI_n + n, \tag{4.4.2}$$

where $\lambda_{C^{(q)}_{\widetilde{R}_g},\max}$ is the maximum eigenvalue of the MPR matrix $C^{(q)}_{\widetilde{R}_g}$, and β is the threshold of acceptably consistent aggregated matrix.

Returning back to our main discussion, we propose an algorithm which identifies acceptably consistent IVI-MPR in a MCGDM scenario.

Algorithm 4.4.1. 1. Elicit g, $g \in M$, incomplete IVI-MPR $\widetilde{R}_{IVI,t} = [\widetilde{r}_{ij,g}]_{n\times n} = ([\underline{\mu}_{ij,g}, \overline{\mu}_{ij,g}],\ [\underline{\nu}_{ij,g}, \overline{\nu}_{ij,g}]),\ \forall\, i,j \in N$, and represent the missing elements by $*$.

2. Using (4.2.2) and (4.2.3), construct the incomplete MPRs $C^{(*q)}_g$, $q = 1,2,3,4,\ g \in M$.

3. The initial values of missing elements in $C^{(*q)}_g$, $q = 1,2,3,4,\ g \in M$, are calculated using (4.3.1).

4. The initial values adjusted using the local optimization model (Model 4.1), and obtain the complete MPRs $C^{(q)}_g$, $q = 1,2,3,4,\ g \in M$.

5. Check the consistency of $C^{(q)}_g$, $q = 1,2,3,4,\ g \in M$. If the consistency ratio of $C^{(q)}_g < \beta,\ \forall\, q = 1,2,3,4,\ g \in M$, then use geometric mean [3] to aggregate the decision matrices are corresponding to these MPRs.

   Else, compute the weighted geometric mean matrix and test whether it is acceptably consistent using (4.4.2).

The following example illustrates the algorithm.

[3] Aczél and Saaty [184] proved that to synthesize the group judgments, the geometric mean must be used in order to preserve the reciprocal property, that is, $r_{ij}r_{ji} = 1,\ \forall\, i,j \in N$, must hold in the resultant aggregated matrix. Note that the weighted geometric mean preserves reciprocality on aggregation resulting in an aggregated MPR, while the same gets distorted if instead arithmetic mean is used for aggregating two or more MPRs.

Example 4.4.1. Suppose three experts e_g , $g=1,2,3$, with the weight vector $w=(1/3,1/3,1/3)^t$, provide their individual judgement on four alternatives in terms of incomplete IVI-MPRs given by $\widetilde{R}_{IVI,g}$, $g=1,2,3$, where the missing elements are marked distinctly.

$$\widetilde{R}_{IVI,1}=\begin{bmatrix} ([1,1],[1,1]) & ([*,*],[*,*]) & ([0.4,0.6],[0.2,0.4]) & ([0.2,0.3],[0.5,0.6]) \\ ([*,*],[*,*]) & ([1,1],[1,1]) & ([0.6,0.7],[*,0.3]) & ([0.4,0.7],[0.2,0.2]) \\ ([0.2,0.4],[0.4,0.6]) & ([*,0.3],[0.6,0.7]) & ([1,1],[1,1]) & ([0.5,*],[0.2,0.3]) \\ ([0.5,0.6],[0.2,0.3]) & ([0.2,0.2],[0.4,0.7]) & ([0.2,0.3],[0.5,*]) & ([1,1],[1,1]) \end{bmatrix}$$

$$\widetilde{R}_{IVI,2}=\begin{bmatrix} ([1,1],[1,1]) & ([0.3,0.6],[0.3,0.4]) & ([*,*],[*,*]) & ([0.3,0.4],[0.2,0.2]) \\ ([0.3,0.4],[0.3,0.6]) & ([1,1],[1,1]) & ([0.5,0.5],[*,*]) & ([0.2,0.5],[0.3,0.4]) \\ ([*,*],[*,*]) & ([*,*],[0.5,0.5]) & ([1,1],[1,1]) & ([*,*],[0.2,0.5]) \\ ([1,2],[0.3,0.4]) & ([0.3,0.4],[0.2,0.5]) & ([0.2,0.5],[*,*]) & ([1,1],[1,1]) \end{bmatrix}$$

$$\widetilde{R}_{IVI,3}=\begin{bmatrix} ([1,1],[1,1]) & ([0.5,0.6],[0.3,0.4]) & ([0.3,*],[0.2,0.3]) & ([*,0.4],[0.2,0.5]) \\ ([0.3,0.4],[0.5,0.6]) & ([1,1],[1,1]) & ([0.6,0.7],[0.2,0.2]) & ([*,*],[*,*]) \\ ([0.2,0.3],[0.3,*]) & ([0.2,0.2],[0.6,0.7]) & ([1,1],[1,1]) & ([0.3,0.5],[0.2,0.4]) \\ ([0.2,0.5],[*,0.4]) & ([*,*],[*,*]) & ([0.2,0.4],[0.3,0.5]) & ([1,1],[1,1]) \end{bmatrix}.$$

For the first expert preference relation matrix $\widetilde{R}_{IVI,\,1}$, the four incomplete MPRs are obtained using (4.2.2) and (4.2.3) as follows:

$$C^{(1)}_{\widetilde{R}_{IVI,\,1}}=\begin{bmatrix} 1 & c_{12}^{(*1)} & 0.6 & 0.3 \\ 1/c_{12}^{(*1)} & 1 & 0.7 & 0.7 \\ 1/0.6 & 1/0.7 & 1 & c_{34}^{(*1)} \\ 1/0.3 & 1/0.7 & 1/c_{34}^{(*1)} & 1 \end{bmatrix},C^{(2)}_{\widetilde{R}_{IVI,\,1}}=\begin{bmatrix} 1 & c_{12}^{(*2)} & 0.4 & 0.2 \\ 1/c_{12}^{(*2)} & 1 & 0.6 & 0.4 \\ 1/0.4 & 1/0.6 & 1 & 0.5 \\ 1/0.2 & 1/0.4 & 1/0.5 & 1 \end{bmatrix},$$

$$C^{(3)}_{\widetilde{R}_{IVI,\,1}}=\begin{bmatrix} 1 & c_{12}^{(*3)} & 0.4 & 0.6 \\ 1/c_{12}^{(*3)} & 1 & 0.3 & 0.2 \\ 1/0.4 & 1/0.3 & 1 & 0.3 \\ 1/0.6 & 1/0.2 & 1/0.3 & 1 \end{bmatrix},C^{(4)}_{\widetilde{R}_{IVI,\,1}}=\begin{bmatrix} 1 & c_{12}^{(*4)} & 0.2 & 0.5 \\ 1/c_{12}^{(*4)} & 1 & c_{23}^{(*4)} & 0.2 \\ 1/0.2 & 1/c_{23}^{(*4)} & 1 & 0.2 \\ 1/0.5 & 1/0.2 & 1/0.2 & 1 \end{bmatrix}.$$

The initial values are described as follows:

$$\begin{aligned}
c_{12}^{(*1)((0))} &= ((c_{13}^{(1)}c_{32}^{(1)})(c_{14}^{(1)}c_{42}^{(1)}))^{1/2}=0.6060, \\
c_{34}^{(*1)((0))} &= ((c_{31}^{(1)}c_{14}^{(1)})(c_{32}^{(1)}c_{24}^{(1)}))^{1/2}=0.7071, \\
c_{12}^{(*2)((0))} &= ((c_{13}^{(2)}c_{32}^{(2)})(c_{14}^{(2)}c_{42}^{(2)}))^{1/2}=0.577, \\
c_{12}^{(*3)((0))} &= ((c_{13}^{(3)}c_{32}^{(3)})(c_{14}^{(3)}c_{42}^{(3)}))^{1/2}=2, \\
c_{12}^{(*4)((0))} &= c_{14}^{(4)}c_{42}^{(4)}=2.5,\ \ c_{23}^{(*4)((0))}=c_{24}^{(4)}c_{43}^{(4)}=1.
\end{aligned}$$

Solving problem (Model 4.1), using these initial values, we obtain,

$$c_{12}^{(1)} = 0.5,\ c_{12}^{(2)} = 0.5,\ c_{12}^{(3)} = 2,\ c_{12}^{(4)} = 2,$$
$$c_{23}^{(4)} = 0.3,\ c_{34}^{(1)} = 1.$$

Similarly, we work out the missing elements values for the MPRs of the other two experts. For an expert e_2, we obtain,

$$c_{13}^{(1)} = 0.3,\ c_{34}^{(1)} = 1,\ c_{13}^{(2)} = 0.15,\ c_{34}^{(2)} = 0.4,$$
$$c_{13}^{(3)} = 0.4,\ c_{23}^{(3)} = 1.5,\ c_{13}^{(4)} = 0.4,\ c_{23}^{(4)} = 1.5,$$

while for an expert e_3, we have,

$$c_{13}^{(1)} = 0.8,\ c_{24}^{(1)} = 0.67,\ c_{14}^{(2)} = 0.111,\ c_{24}^{(2)} = 0.18,$$
$$c_{24}^{(3)} = 1.25,\ c_{24}^{(4)} = 0.66.$$

It is noted that among the twelve MPRs, $C^{(q)}_{\widetilde{R}_{IVI,1}}$, $q = 1,2,3$, $C^{(q)}_{\widetilde{R}_{IVI,2}}$, $q = 1,3,4$, and $C^{(q)}_{\widetilde{R}_{IVI,3}}$, $q = 1,2$, are acceptably consistent, while $C^{(4)}_{\widetilde{R}_{IVI,1}}$, $C^{(2)}_{\widetilde{R}_{IVI,2}}$, $C^{(3)}_{\widetilde{R}_{IVI,3}}$, and $C^{(4)}_{\widetilde{R}_{IVI,3}}$, are not acceptably consistent.

For $w_1 = w_2 = w_3 = 1/3$, construct the weighted geometric mean judgement matrix of the three matrices $C^{(1)}_{\widetilde{R}_{IVI,1}}$, $C^{(1)}_{\widetilde{R}_{IVI,2}}$, and $C^{(1)}_{\widetilde{R}_{IVI,3}}$,

$$C_{agg}^{(1)} = \begin{pmatrix} 1 & 0.567 & 0.567 & 0.367 \\ 1.778 & 1 & 0.633 & 0.623 \\ 2.083 & 1.619 & 1 & 0.833 \\ 2.778 & 1.640 & 1.333 & 1 \end{pmatrix}.$$

Then, $\lambda_{C_{agg}^{(1)},\max} = 4.103$ which is less than $(n-1)\beta RI_n + n = 4.2136$, for $n = 4$, $\beta = 0.08$. Hence, $C_{agg}^{(1)}$ is acceptably consistent.

In Table 4.1, we summarize the acceptably consistent behavior of the weighted geometric mean matrices for different weights of three experts. For notational convenience the maximum eigenvalue of the weighted geometric mean comparison matrices $C_{agg}^{(q)}$, $q = 1,2,3,4$, is denoted by $\lambda_{\max}$ in Table 4.1.

From Table 4.1, we find that though the individual preference matrices are not acceptably consistent yet their geometric mean aggregated preference matrix turns out to be acceptably consistent for some specific choices of weight vectors w indicating the significance of importance attached to the experts. Also, from Table 4.1, note that (4.4.2) is only a sufficient condition for the aggregated matrix (and hence of MPR) to be acceptably consistent as, for some specific weight vector w, we do have (4.4.2) failing to hold yet the weighted geometric mean $C_{agg}^{(q)}$ is acceptably consistent.

Table 4.1: Consistency of the weighted geometric mean aggregated preference relation matrix

Aggregated matrix	$(w_1, w_2, w_3)^t$	$\prod_{t=1}^{3}\left(\lambda_{C^{(q)}_{\widetilde{R}_{IVI,t}}}, \max\right)^{w_t}$	Is (4.4.2) satisfied?	$\lambda_{\max}$	Is $C_{agg}^{(q)}$, $q = 1,2,3,4$, acceptably consistent?
$C_{agg}^{(1)}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^t$	4.037	yes	4.1033	yes
	$(\frac{9}{20}, \frac{1}{10}, \frac{9}{20})^t$	4.0473	yes	4.0918	yes
	$(\frac{1}{5}, \frac{3}{5}, \frac{1}{5})^t$	5.7133	no	4.0888	yes
	$(\frac{1}{20}, \frac{9}{10}, \frac{1}{20})^t$	4.0145	yes	4.0366	yes
	$(\frac{1}{100}, \frac{98}{100}, \frac{1}{100})^t$	4.0112	yes	4.0160	yes
$C_{agg}^{(2)}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^t$	4.1156	yes	4.0475	yes
	$(\frac{9}{20}, \frac{1}{10}, \frac{9}{20})^t$	4.0400	yes	4.1232	yes
	$(\frac{1}{5}, \frac{3}{5}, \frac{1}{5})^t$	4.2039	yes	4.2650	yes
	$(\frac{1}{20}, \frac{9}{10}, \frac{1}{20})^t$	4.3054	no	4.3242	no
	$(\frac{1}{100}, \frac{98}{100}, \frac{1}{100})^t$	4.3329	no	4.3369	no
$C_{agg}^{(3)}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^t$	4.3418	no	4.1302	yes
	$(\frac{9}{20}, \frac{1}{10}, \frac{9}{20})^t$	4.4533	no	4.2868	no
	$(\frac{1}{5}, \frac{3}{5}, \frac{1}{5})^t$	4.2177	no	4.0333	yes
	$(\frac{1}{20}, \frac{9}{10}, \frac{1}{20})^t$	4.0824	yes	4.0203	yes
	$(\frac{1}{100}, \frac{98}{100}, \frac{1}{100})^t$	4.0471	yes	4.0338	yes
$C_{agg}^{(4)}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^t$	4.5057	no	4.3099	no
	$(\frac{9}{20}, \frac{1}{10}, \frac{9}{20})^t$	4.6591	no	4.9598	no
	$(\frac{1}{5}, \frac{3}{5}, \frac{1}{5})^t$	4.3365	no	4.6849	no
	$(\frac{1}{20}, \frac{9}{10}, \frac{1}{20})^t$	4.1538	yes	4.1048	yes
	$(\frac{1}{100}, \frac{98}{100}, \frac{1}{100})^t$	4.1064	yes	4.1350	yes

4.5 A comparative analysis with existing methods

Research studies on FPR, IFPR, and IVI-FPR exist in the literature. For instance, one may refer to (Xu and Cai [140], Chen et al. [135], Meng, Tan, and Chen [98]). In this chapter, we primarily focused on IVI-MPR rather than IVI-FPR. To the best of our understanding, there does not exist any study in the literature guiding how to populate the missing entries in an IVI-MPR.

In this section, we compare our proposed method with those of Meng and Chen [169] for MPR and Jiang et al. [143] for IMPR and show that our method generalizes their works.

Meng and Chen [169] proposed the notion of the multiplicative geometric consistent index (MGCI) to measure the multiplicative consistency of an MPR. The consistency of an MPR is considered to be unacceptable if the MGCI of an MPR is less than the average value tabulated in Table 1 in their paper. The authors continued their study to include the case of incomplete MPR. They formulated a multi-objective programming model to estimate the missing values. Using the goal programming approach and a suitable transformation, the proposed model was converted into an equivalent linear program. The missing values in an MPR were then obtained using the inverse transformation at the optimal solution of the linear program.

Observe that if we set $q = 1$, $C^{(q)} = C^{(q+\theta)}$, $\theta = 1,2,3$, in Model 4.1 of sub-section 4.3.2, and use only the first three constraints of Model 4.1, then the deviation model (OP) of Meng and Chen [169] follows as a special case of Model 4.1. Our proposed approach thus encompasses the case of incomplete MPR. However, for the wholeness of discussion, we additionally provide the linear goal programming deviation model for IVI-MPR.

Meng and Chen [169] developed a deviation model for an MPR. In the same spirit, we formulate an equivalent deviation model for the Model 4.1 proposed in Section 4.3.

(Model 4.2) $\min\ D = \sum_{q=1}^{4} \sum_{i,j=1,i<j,\,k\in\Omega(q)}^{n} (d_{ij,\,k}^{(q)+} + d_{ij,\,k}^{(q)-})$

subject to

$$\delta_{ij}\left(\log a_{ij}^{(q)} - (\log a_{ik}^{(q)} + \log a_{kj}^{(q)})\right) - d_{ij,\,k}^{(q)+} + d_{ij,\,k}^{(q)-} = 0,$$
$$i<j,\ \forall\, i,\, j\in N,\ k\in\Omega(q),\ q=1,2,3,4$$
$$1/9 \le a_{ij}^{(q)}, q=1,2, \quad a_{ij}^{(q)} \in \Gamma(q)$$
$$a_{ij}^{(q)} \le 9,\ q=3,4, \quad a_{ij}^{(q)} \in \Gamma(q)$$
$$a_{ij}^{(q)} a_{ij}^{(q+2)} \le 1, \quad q=1,2,\ i,j,k\in N,\ i\neq j\neq k$$
$$a_{ij}^{(q+1)} \le a_{ij}^{(q)}, \quad q=1,3,\ i<j,\ i,j\in N$$
$$a_{ij}^{(q+1)} \ge a_{ij}^{(q)}, \quad q=1,3,\ i>j,\ i,j\in N$$
$$d_{ij,\,k}^{(q)+},\ d_{ij,\,k}^{(q)-} \ge 0, \quad i,j=1,\dots,n,\ i<j,$$

where $d_{ij,\,k}^{(q)+} = \left(\log a_{ij}^{(q)} - (\log a_{ik}^{(q)} + \log a_{kj}^{(q)})\right) \vee 0,\ k\in\Omega(q)$,
and $d_{ij,\,k}^{(q)-} = \left((\log a_{ik}^{(q)} + \log a_{kj}^{(q)}) - \log a_{ij}^{(q)}\right) \vee 0,\ k\in\Omega(q)$, and

$$\delta_{ij} = \begin{cases} 1 & k\in\Omega(q) \\ 0 & \text{otherwise} \end{cases}$$

It is observed that both model i.e. Model 4.1 and the above deviation model (Model 4.2) are equivalent and yield the same optimal outputs. Moreover, the model formulated by Meng and Chen [169] is a particular case of the above model and hence of Model 4.1.

The method proposed by us in the current work and the one suggested by Meng and Chen [169] are based fundamentally on the principle of transitivity for MPR. However, unlike the approach by Meng and Chen [169] for MPR, which requires solving a goal programming model, our proposed method for IVI-MPR is a two stage iterative scheme involving initial estimation of missing values and after that improving upon them by solving optimization model (Model 4.1). Though it may so happen that the stage one itself is sufficient to produce optimal values of the missing entries making the second stage redundant.

Consider the following incomplete MPR

$$A = \begin{bmatrix} 1 & 0.6 & \text{x} & 0.4 \\ 1/0.6 & 1 & 0.7 & \text{y} \\ 1/\text{x} & 1/0.7 & 1 & 0.5 \\ 1/0.4 & 1/\text{y} & 1/0.5 & 1 \end{bmatrix}$$

The results on solving the proposed Model 4.1 and the (OP) model of Meng and Chen

[169] are same. Moreover, in this matrix, if 0.6 and 0.4 are changed to 0.2 and 4, respectively, then also the two approaches yield same results; see, Table 4.2.

Table 4.2: Missing entries x and y values and the CR values of the original MPR A (in columns 2-4) and the MPR A with two changes (in columns 5-7)

Optimization Model	x	y	CR	x	y	CR
Meng and Chen (2015) model (OP)	0.579	0.483	0.0097	1.0583	2.6458	0.4169
Our proposed (Model 4.1) model	0.579	0.483	0.0097	1.0583	2.6458	0.4169

In section 4.3, if $C^{(1)} = C^{(2)}$ and $C^{(3)} = C^{(4)}$, then the IVI-MPR gets converted into IMPR studied by Jiang et al. [143]. In fact, our proposed methodology is very much inspired by the work of Jiang et al. [143].

We can conclude that our methodology is an alternative procedure to compute missing entries. The proposed method is indeed not inferior to the one existing for incomplete M-PR in the literature despite IVI-MPR being a generalization of MPR and its other variants.

4.6 Conclusion

The contribution of this chapter is twofold. Firstly, we introduce the notion of acceptably consistent for IVI-MPR, and secondly, we propose a two-step method to populate the missing entries in an incomplete IVI-MPR. The first step of the recommended two-step approach identifies the initial values of all missing entries in an incomplete IVI-MPR and the subsequent second step improves the initial values by solving a linear programming problem. Although certain research is available in the literature on MPRs in the intuitionistic framework yet the proposed integration of IVI-MNs in MPR, especially in the incomplete preference relations, can be considered as a novel contribution of this chapter.

The primary limitation of the proposed procedure for finding the missing entries lies in its estimating step where we have assumed that, for a missing entry $\widetilde{*}_{ij}$, we can always find an elementary connecting path in the given preference relation matrix such that all other entries along this path are completely known. This assumption may not hold. Another concern in the present study is that the weights or different levels of expertise of experts are assumed to be precisely known. A suitable choice of a weight vector ensures acceptably consistent group aggregated preference relation matrix. However, we have not explored a mechanism to guide on the selection of weight vector for experts to enable them to converge to an acceptably consistent aggregated preference matrix. The only sufficient condition available to guide on this choice of weight vector is the inequality in equation 4.4.2. Besides the weights of experts, for aggregation, the weights/importance of the evaluation criteria can be devised using their importance ratings from experts.

Chapter 5

Incomplete Hesitant Fuzzy Preference Relation

Preference relations are obtained in the process of decision making by comparing different alternatives/criteria by the decision maker. In this chapter[1] *we work on hesitant fuzzy preference relation (HFPR). A decision maker might give his/her perception by utilizing* $0-1$ *scale for HFPR to overcome hesitancy and uncertainty, lying in the definition framework of the decision-making problem. The hesitant preference relation is found to be incomplete due to the restraint of the expert's efficient proficiency, experience and shortage of time. Zhang et al. [144] proposed the notion of incomplete HFPR (I-HFPR) and additive consistency property of I-HFPR. The additive consistency property of HFPR defined by Zhang et al. [144] does not satisfy the property of hesitancy given by Zhu and Xu [75]. The study of consistency of preference relation is an important feature to keep away from the puzzling solution. This chapter aims to develop a method with I-HFPRs. A new definition of additive consistency property of HFPR is given that preserves the property of hesitancy and is used to construct the complete HFPR from incomplete one. The significance of consistency measure for HFPR make sure that the DMs are neither arbitrary nor unreasonable. We develop a method to check the consistency level of I-HFPR. Group decision-making problem with I-HFPR is also considered.*

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[1]The work presented in this chapter comprises the results of a research paper entitled "Incomplete Hesitant Fuzzy Preference Relation", *Journal of Statistic and Management Systems*, Taylor & Francis **21**, (8) 1459–1479 (2018).

5.1 Introduction

From the extension of fuzzy set, Torra [123] propose the concept of the Hesitant fuzzy set(HFS) whose membership functions are expressed by a collection of several feasible values. The hesitant fuzzy set is a new important tool that expresses the human hesitancy in daily life. For instance, sometimes, in a decision making problems, decision makers are uncertain about some feasible values such as 0.3, 0.4, and 0.5 to determine the membership of a particular element belonging to a set. Henceforth, the membership of an element in a set is represented by 'h_e'$= \{0.3, 0.4, 0.5\}$ which is known as hesitant fuzzy element (HFE) distinct from the conventional fuzzy sets [3] , IVFS [4], IFS([6], [185], [186]), IVIFS [34] and type-2 fuzzy sets ([187], [188]).

Preference relation is one of the most powerful technique in the decision making frame work. The choice of preferences over the given criteria by making a comparison among each pair of alternative is suggested by decision makers. Different type of preference relations has been proposed. For ready reference one may refer to([54], [58], [60]- [64], [67], [78], [140], [189], [190] – [192]) and not limited to these only.

The preference relation as mentioned above does not explore all promising estimation of preferences of the decision maker because all these preference relations do not think about undecided fuzzy data while making a pairwise comparison in real life situation. However, in a decision organization, to get a more sensible choice outcome, decision maker is approved to give the preferences utilizing 0-1 proportion scale by comparing each pair of alternatives. The fact may conform that, the decision maker while making a decision may not be sure about an accurate value but has uncertainty between several feasible values. These conceivable feasible values can be considered as an HFE and HFPR. It is noteworthy that the extension of FPR is HFPR, where the preferences values are expressed utilizing $0-1$ ratio scale, which is symmetrically conveyed around 0.5. Zhu and Xu [75] give the definition of HFPRs, and they gave some properties on HFPRs. Decision maker requires $\frac{n(n-1)}{2}$ judgements in its entire lower or upper triangular part to represent a complete preference relation of order $n \times n$. In numerous real world circumstances, such like medical diagnosis, personal examination and so forth, by a result of time pressure, lack of sufficient knowledge, sometimes it is challenging to obtain a complete HFPR. In a situation of HFPR, where some elements are missing results in the establishment of an I-HFPR. Therefore, there is a need to proposed a technique to acquire complete HFPR from existing incomplete one.

Consistency plays an important role in a situation where experts have to work in uncertain or vague situations. Zhang et al. [144] defined the concept of I-HFPR and its additive consistency. But Zhang's additive consistency property does not satisfy the property of hesitancy i.e $h_{e_{ij}}^{\sigma(s)} + h_{e_{ji}}^{\sigma(l_{h_{e_{ij}}}-s+1)} = 1$ given by [75]. This chapter aims to propose a new definition of additive consistency property of HFPR that satisfies the said property of hesitancy. We have developed a method to evaluate the missing element of I-HFPR by using additive consistency. Consistency level for HFPR ensures that the DMs are neither arbitrary nor unreasonable. We have also devised a method to check the consistency level of complete HFPR. The above procedure is illustrated by a numerical example. A new algorithm is then given to handle group decision-making problems.

Remainder of this chapter is modified as follows. In section 5.2, some basic concepts of HFS and HFPR are defined briefly. Estimating procedure of missing element from I-HFPR is given in section 5.3. A methodology is proposed to check the consistency level of HFPR is given in section 5.4 and the developed procedure is illustrated via a numerical example. An algorithm is given in section 5.5 to handle the group decision-making problem and is illustrated by an example. Comparison of our work is also given and finally last section constitute the concluding remarks.

5.2 Preliminaries

The concept of hesitant fuzzy set and hesitant fuzzy preference relations are reminded very briefly in this section. The concept of HFS is first characterized by Torra [123] where the membership of an element are the collection of feasible values.

Definition 5.2.1. [123] A HFS on X is defined by $H = \{(x, h(x)) | x \in X\}$, where $h(x)$ is a set of possible membership degree of the element $x \in X$ to the set H that lies between 0 and 1. For convenience $h(x) = h_e$ is called the hesitant fuzzy element (HFE).

For a HFE $h_e = \{h_e^{\sigma(s)} | s = 1, 2, \cdots, l_{h_e}\}$ where l_{h_e} represents the number of elements in the HFE h_e and $\sigma(s)$ denotes the position of HFE. It is important to note that all the possible values of the HFE i.e. h_e are assumed to be arranged in an increasing order and $h_e^{\sigma(s)}$ is the s^{th} smallest value in h_e.

If the length of the HFE is not the same, then the one which is smaller, can be extended to maintain the uniformity by adding the numerical values as given by the following definition.

Definition 5.2.2. [193] Let h_e^+ and h_e^- are the maximum and minimum values respectively in HFE $h_e = \{h_e^{\sigma(s)} | s = 1, 2, \cdots, l_{h_e}\}$ respectively and ζ be an optimized parameter that lies between 0 and 1 which can be decided by the decision makers indicated by their own risk preferences, then the linear combination $\zeta h_e^+ + (1-\zeta) h_e^-$ is called added element that is denoted by $\bar{h}_e$.

Particularly the added element h_e^+ and h_e^- respectively extracted from the conditions $\zeta = 1$ and $\zeta = 0$ that corresponds to optimism and pessimism rules proposed by Xu et al. [125]. Zhu and Xu [75] proposed the idea of HFPRs based on HFEs [123] and FPRs [78] given as follows:

Definition 5.2.3. [75] A HFPR H_R over X is defined by $H_R = (h_{e_{ij}})_{n \times n}$ which is subset of $X \times X$, where $h_{e_{ij}} = \{h_{e_{ij}}^{\sigma(s)} | s = 1, 2, \cdots, l_{h_{e_{ij}}}\}$ is a HFE indicating all the possible preference value to which an alternative x_i is prefer to alternative x_j satisfying

$$h_{e_{ij}}^{\sigma(s)} + h_{e_{ji}}^{\sigma(l_{h_{e_{ij}}} - s + 1)} = 1,\ h_{e_{ii}} = 0.5,\ l_{h_{e_{ij}}} = l_{h_{e_{ji}}},\ \forall i, j \in N. \tag{5.2.1}$$

where $h_{e_{ij}}^{\sigma(s)}$ is the s^{th} smallest element in $h_{e_{ij}}$.

For example, the HFPR, H_R is shown below.

$$\begin{pmatrix} \{0.5\} & \{0.5, 0.6, 0.7\} & \{0.1, 0.4\} \\ \{0.3, 0.4, 0.5\} & \{0.5\} & \{0.2, 0.3, 0.4, 0.5\} \\ \{0.6, 0.9\} & \{0.5, 0.6, 0.7, 0.8\} & \{0.5\} \end{pmatrix}$$

In the above HFPR matrix, length of the preference degree of alternative x_i over x_j are not same. According to definition 5.2.2, the optimised parameter ζ is used to add some elements to an HFPR and finally obtained a normalised HFPR defined as follows

Definition 5.2.4. [193] Let the maximum and minimum elements at $(i, j)^{th}$ position of HFPR, $H_R = (h_{e_{ij}})_{n \times n}$ is represented by $h_{e_{ij}}^+$ and $h_{e_{ij}}^-$ where $i, j = 1, 2, \cdots, n$, respectively. Based on definition 5.2.2 if we add some elements $\bar{h}_{e_{ij}} = \zeta h_{e_{ij}}^+ + (1-\zeta) h_{e_{ij}}^-$ to $h_{e_{ij}}$ if $i \leq j$ and add some elements $\bar{h}_{e_{ji}}$=$(1-\zeta) h_{e_{ji}}^+ + \zeta h_{e_{ji}}^-$ to $h_{e_{ji}}$ if $i \leq j$, where, ζ is an optimized parameter, then the normalized HFPR $\bar{H}_R$=$\{\bar{h}_{e_{ij}}^{\sigma(s)} | s = 1, 2, \cdots, l_{\bar{h}_{e_{ij}}}\}$ satisfying

$$l_{\bar{h}_{e_{ij}}} = \max\{l_{h_{e_{ij}}} | i, j = 1, 2, \cdots, n, i \neq j\},\ \bar{h}_{e_{ij}}^{\sigma(s)} + \bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}} - s + 1)} = 1,\ \bar{h}_{e_{ii}} = \{0.5\}. \tag{5.2.2}$$

where $\bar{h}_{e_{ij}}^{\sigma(s)}$ and $\bar{h}_{e_{ji}}^{\sigma(s)}$ is the s^{th} smallest element in $\bar{h}_{e_{ij}}$ and $\bar{h}_{e_{ji}}$ respectively.

Due to limitation of DMs proficient knowledge, experience or time pressure, the provided preference value in HFPR are incomplete. It might be difficult at times, to get the complete preference relation corresponding to the higher order preference relation. In such a case, the expert is unable to communicate DM opinion over other pairs of alternatives. Then the incomplete HFPR (I-HFPR) is obtained in which some elements are missing. Zhang et al. [144] defined the I-HFPR.

Definition 5.2.5. [144](I-HFPR) The HFPR $H_R = (h_{e_{ij}})_{n\times n}$ where $h_{e_{ij}} = \{h_{e_{ij}}^{\sigma(s)} | s = 1, 2, \cdots, l_{h_{e_{ij}}}\}$ is called an I-HFPR, if some elements in it are missing that is denoted by "$*$" and the other elements are given by DMs should satisfy the condition of definition 5.2.3.

According to definition 5.2.4, an optimized parameter ζ is used for the uniformity of length of I-HFPR. Thus a normalized I-HFPR is obtained which is defined as follows.

Definition 5.2.6. [144](Normalized I-HFPR) I-HFPR is called normalized I-HFPR if the known elements satisfying the equation 5.2.2.

The study of consistency of the HFPR is an important feature to avoid the false solution. Zhang et al. [144] defined the additive consistency i.e

$$\bar{h}_{e_{ij}}^{\sigma(s)} = \bar{h}_{e_{ik}}^{\sigma(s)} - \bar{h}_{e_{jk}}^{\sigma(s)} + 0.5 \ \forall i, j, k = 1, 2, \cdots n, \ s = 1, 2, \cdots l_{\bar{h}_{e_{ij}}} \text{ and } i \leq j \leq k \tag{5.2.3}$$

(where $l_{\bar{h}_{e_{ij}}}$ is the length of the normalized HFE)
The above said additive consistency does not satisfy the property of hesitancy given by Zhu and Xu [75] as discussed below.

$$\bar{h}_{e_{ij}}^{\sigma(s)} + \bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} = \bar{h}_{e_{ik}}^{\sigma(s)} - \bar{h}_{e_{jk}}^{\sigma(s)} + 0.5 + \bar{h}_{e_{jk}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} - \bar{h}_{e_{ik}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} + 0.5 \neq 1.$$

According to Zhang et al. [144], if $i = k$, the additive consistency property becomes

$$h_{e_{ij}}^{\sigma(s)} + h_{e_{ji}}^{\sigma(s)} = h_{e_{ii}}^{\sigma(s)} + 0.5 = 1$$

It again contradicts the property of hesitancy. It is the special case if $l_{\bar{h}_{e_{ij}}} - s + 1 = s$. Therefore we have defined a new formula of the additive consistency of HFPR that satisfies the hesitancy property too.

Definition 5.2.7. Let $H_R = (h_{e_{ij}})_{n\times n}$ be HFPR and $\bar{H}_R = (\bar{h}_{e_{ij}})_{n\times n}$ be its normalized HFPR on X with an optimization operator ζ, satisfying the following condition

$$\bar{h}_{e_{ij}}^{\sigma(s)} = \bar{h}_{e_{ik}}^{\sigma(s)} - \bar{h}_{e_{jk}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} + 0.5, \ \ s = 1, 2, \cdots, l_{\bar{h}_{e_{ij}}} \text{ and } i \leq j \leq k, \ i, j, k \in N. \tag{5.2.4}$$

where $\bar{h}_{e_{ij}}^{\sigma(s)}$, $\bar{h}_{e_{ik}}^{\sigma(s)}$ and $\bar{h}_{e_{jk}}^{\sigma(s)}$ are the s^{th} smallest elements in $\bar{h}_{e_{ij}}$, $\bar{h}_{e_{ik}}$ and $\bar{h}_{e_{jk}}$ respectively then H_R is said to be an additive consistent HFPR.

It is noteworthy that the above additive consistent property satisfy the hesitancy property as follows:

$$\bar{h}_{e_{ij}}^{\sigma(s)} + \bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} = \bar{h}_{e_{ik}}^{\sigma(s)} - h_{e_{jk}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} + 0.5 + \bar{h}_{e_{jk}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} - \bar{h}_{e_{ik}}^{\sigma(s)} + 0.5$$
$$= 1.$$

It is necessary for a HFPR to be additive consistent such that $i \leq j \leq k$, $\forall\ i, j, k \in N$, otherwise the HFPR is reduced to FPR with crisp value. Indeed assume that $\bar{h}_{e_{ij}}^{\sigma(s)} = \bar{h}_{e_{ik}}^{\sigma(s)} - \bar{h}_{e_{jk}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} + 0.5$ and $\bar{h}_{e_{ij}}^{\sigma(s+1)} = \bar{h}_{e_{ik}}^{\sigma(s+1)} - \bar{h}_{e_{jk}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s)} + 0.5$ for all $i, j, k = 1, 2, \cdots, n$ and $s = 1, 2, \cdots l_{\bar{h}_{e_{ij}}} - 1$. For $i = k$, the above equations should also hold, and we have

$$\bar{h}_{e_{ij}}^{\sigma(s)} + \bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} = 1 \text{ and } \bar{h}_{e_{ij}}^{\sigma(s+1)} + \bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s)} = 1$$

By the definition 5.2.4 of the normalized HFE, we must have

$$\bar{h}_{e_{ij}}^{\sigma(s)} \leq \bar{h}_{e_{ij}}^{\sigma(s+1)},\ \bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} \leq \bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s)}$$

Thus,

$$1 = \bar{h}_{e_{ij}}^{\sigma(s)} + \bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} \leq \bar{h}_{e_{ij}}^{\sigma(s+1)} + \bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s)} = 1 \tag{5.2.5}$$

Equation 5.2.5 hold if and only if $\bar{h}_{e_{ij}}^{\sigma(s)} = \bar{h}_{e_{ij}}^{\sigma(s+1)}$, $\bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} = \bar{h}_{e_{ji}}^{\sigma(l_{\bar{h}_{e_{ij}}}-s)}$, which means $H_R = (h_{e_{ij}})_{n\times n}$ may be a crisp fuzzy preference relation if the condition $i \leq j \leq k$ violates. The additive consistency property of HFPR will take part in an important role to construct complete HFPRs based on an I-HFPRs. We define an additive consistent I-HFPR.

Definition 5.2.8. The I-HFPR, $H_R = (h_{e_{ij}})_{n\times n}$ is called additive consistent if all the known elements satisfy the conditions of equation 5.2.4.

5.3 Constructing complete HFPR from an I-HFPR

In I-HFPR, missing element is due to the inability of a DM to measure the degree of preference information of one criteria over other. It has been witnessed that an expert utilize an I-HFPR to judge his/her preference value over the criteria, such that the weights in the precedence are extracted from the I-HFPR which can be further used as the weights of the criteria. Here, we develop a practical technique to derive the weights of the criteria from an I-HFPR. Let $H_R = (h_{e_{ij}})_{n\times n} = \{h_{e_{ij}}^{\sigma(s)} | s = 1, 2, \cdots, l_{h_{e_{ij}}}\}$ be an I-HFPR and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of H_R, where w_i indicates the preference value of

the alternative A_i to all alternative A_j with $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, $i \in N$.
The difference between the value $h_{e_{ij}}^{\sigma(s)} - h_{e_{ji}}^{\sigma(l_{h_{e_{ij}}}-s+1)}$ depicts the strength of preference value of the alternative A_i over A_j. Thus we have

$$h_{e_{ij}}^{\sigma(s)} - h_{e_{ji}}^{\sigma(l_{h_{e_{ij}}}-s+1)} = w_i^{\sigma(s)} - w_j^{\sigma(l_{h_{e_{ij}}}-s+1)} \tag{5.3.1}$$

where $h_{e_{ij}}^{\sigma(s)} = w_i^{\sigma(s)}$, preference value of alternative A_i over A_j at s^{th} position and $h_{e_{ji}}^{\sigma(l_{h_{e_{ij}}}-s+1)} = w_j^{\sigma(l_{h_{e_{ij}}}-s+1)}$, preference value of alternative A_j over A_i at $(l_{h_{e_{ij}}} - s + 1)^{th}$ position

In such cases consistency property must satisfy:

$$h_{e_{ij}}^{\sigma(s)} = h_{e_{ik}}^{\sigma(s)} - h_{e_{jk}}^{\sigma(l_{h_{e_{ij}}}-s+1)} + 0.5,\ \forall\, h_{e_{ij}}, h_{e_{ik}}, h_{e_{jk}} \in \Omega \tag{5.3.2}$$

where Ω denotes the set of known elements. Also from equation 5.3.1 and 5.3.2 we have

$$h_{e_{ij}}^{\sigma(s)} = 0.5(w_i^{\sigma(s)} - w_j^{\sigma(l_{h_{e_{ij}}}-s+1)} + 1),\ \forall\, h_{e_{ij}} \in \Omega \tag{5.3.3}$$

For an I-HFPR, $H_R = (h_{e_{ij}})_{n\times n}$, replace the unknown element '$*$' in H_R by $0.5(w_i^{\sigma(s)} - w_j^{\sigma(l_{h_{e_{ij}}}-s+1)} + 1)$. Then the new I-HFPR becomes $H_R^{'} = (h_{e_{ij}}^{'})_{n\times n}$ where

$$h_{e_{ij}}^{'} = \begin{cases} h_{e_{ij}} & \text{if } h_{e_{ij}} \in \Omega; \\ 0.5(w_i^{\sigma(s)} - w_j^{\sigma(l_{h_{e_{ij}}}-s+1)} + 1) & \text{otherwise.} \end{cases} \tag{5.3.4}$$

From 5.3.4, it is clear that in the HFPR $H_R^{'}$, value not only contain the preference information provided by the DM and also improve the consistency of the I-HFPR, $H_R = (h_{e_{ij}})_{n\times n}$. Following algorithm demonstrates the method to estimate the missing element of an I-HFPR.

Algorithm 5.3.1.

1. Consider an I-HFPR, H_R.

2. Normalized the I-HFPR by using definition 5.2.4 is denoted by $\bar{H}_R$.

3. Replace the unknown element '$*$' in normalized I-HFPR, $\bar{H}_R$ by $0.5(w_i^{\sigma(s)} - w_j^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} + 1)$ and the resultant matrix is $\bar{H}_R^{'}$.

4. Calculate the preference degree $P_i^{\sigma(s)}(w)$, $s = 1, 2, \cdots, l_{\bar{h}_{e_{ij}}}$ of the alternative A_i over

all the other alternatives by using the following formula.

$$P_i^{\sigma(s)}(w) = \sum_{j=1}^{n} \bar{h}_{e_{ij}}^{\prime\sigma(s)},\ i \neq j,\ \ s = \{1,2,\cdots,l_{\bar{h}'_{e_{ij}}}\} \tag{5.3.5}$$

5. Evaluate the collective preference degree $p(w)$ of all alternatives.

$$p(w) = \sum_{i=1}^{n} \sum_{s=1}^{l_{\bar{h}'_{e_{ij}}}} {p_i}^{\sigma(s)}(w),\ s = \{1,2,\cdots,l_{\bar{h}'_{e_{ij}}}\} \tag{5.3.6}$$

6. Since the weight ${w_i}^{\sigma(s)}$ indicates the degree of importance of the alternative A_i. Therefore, weight ${w_i}^{\sigma(s)}$ can be estimated by the ratio of $P_i^{\sigma(s)}(w)$ to $p(w)$ in the collective preference value of all alternatives.

$${w_i}^{\sigma(s)} = \frac{P_i^{\sigma(s)}(w)}{p(w)},\quad \sum_{i=1}^{n} {w_i}^{\sigma(s)} = 1,\ {w_i}^{\sigma(s)} \geq 0,\ \forall s \tag{5.3.7}$$

7. Complete HFPR matrix is obtain by replacing the value of '$*$' by $0.5({w_i}^{\sigma(s)} - {w_j}^{\sigma(l_{\bar{h}_{e_{ij}}}-s+1)} + 1)$.

The scruting of the consistency of HFPR turns into a prominent characteristic to keep away from a critical solution. Following section discuss the consistency level of the HF-PR.

5.4 Consistency level of HFPR:

In the previous section the complete HFPR obtained using the additive consistency. Herrera-viedma et al. [189] developed a method to measure the consistency level of FPR based on error analysis. Motivated by this method Zhu [194], used the concept of error analysis to manage with the selection procedure and the FPR has obtained from the HFPR with highest consistency level. For a given HFPR, represented by a matrix $H = (h_{e_{ij}})_{n\times n} \subset X \times X$, where $X = \{x_1, x_1, \cdots x_n\}$ is the set of alternatives, Zhu [194] first defined some necessary operations. Using equation 5.2.4, all preference degree of all pairs of alternative (x_i, x_j) represented an HFE $h_{e_{ij}}$ $(i \neq j)$ can be estimated using an intermediate alternative x_k $(k \neq i, j)$ which we have defined as follows:

$$h_{e_{ij}^k}^{\sigma(s)} = h_{e_{ik}}^{\sigma(s)} - h_{e_{jk}}^{\sigma(l_{h_{e_{ij}}}-s+1)} + 0.5 \tag{5.4.1}$$

To estimate $h_{e_{ij}^k}$, the alternatives in HFPR, H_R should generally be classified into several sets [194] given as follows:

$$A = \{(i,j)|i,j \in \{1,2,\cdots,n\} \wedge (i \neq j)\}$$
$$OV^A = \{(i,j) \in A|h_{e_{ij}} \text{ is an estimated HFE}\}$$
$$KV^A = (OV^A)^c$$
$$M_{ij}^k = \{k \neq i,j|(i,k),(k,j) \in KV^A\}$$

where A is the set of all pairs of alternatives, OV^A is a set of pairs of alternatives whose preference degree are represented by the HFE $h_{e_{ij}}$ and it is said to be estimated HFE. KV^A is the set of complement $(OV)^A$. M_{ij}^k is the set of the intermediate alternatives x_k, $(k \neq i,j)$ that is used to find $h_{e_{ij}^k}$ by using equation 5.4.1. Consequently by using equation 5.4.1 we may get several HFEs, $h_{e_{ij}^k}$ $(k = 1,2,\cdots,n, k \neq i,j)$ indicating all possible estimated preference degree of the pair of alternatives (i,j).
Motivated from work of Zhu [194] we discuss the following method to check the consistency level for HFPR.
Calculate the average estimated preference degree defined as follows

$$h_{e_{ij}}^{EST} = \frac{\sum \cup_{k \in M_{ij}^k} h_{e_{ij}^k}}{\sum_{k \in M_{ij}^k} (l_{h_{e_{ij}}})} \tag{5.4.2}$$

where $l_{h_{e_{ij}}}$ is the number of all possible preference degree in $h_{e_{ij}^k}$.
Find all possible error between an estimated HFE $h_{e_{ij}}$ and its average estimated preference degree $h_{e_{ij}}^{EST}$ is defined as

$$\varepsilon h_{e_{ij}} = \beta(\cup_{\varepsilon_{ij} \in (h_{e_{ij}} \tilde{-} h_{e_{ij}}^{EST})} |\varepsilon_{ij}|) \tag{5.4.3}$$

where the coefficient $0 < \beta < 1$ is used to make sure each error in $|\varepsilon_{ij}|$ belongs to the interval $[0,1]$. The consistency level of HFPR for ij^{th} position is given as

$$cl_{ij} = 1 - min(\varepsilon h_{e_{ij}}) \tag{5.4.4}$$

The consistency level of each alternative x_i is given as :

$$cl_i = \frac{\sum_{\substack{j=1 \\ i \neq j}}^{n} (cl_{ij} + cl_{ji})}{2(n-1)} \tag{5.4.5}$$

The consistency level of HFPR is given by

$$cl_H = \frac{\sum_{i=1}^{n} cl_i}{n} \tag{5.4.6}$$

Following example illustrates the previous section.

Example 5.4.1. Consider a decision making problem with five sets of alternatives x_i $(i = 1,2,\cdots,5)$. The decision maker judge these five alternatives by pairwise comparison and provides his/her judgement as follows: $h_{e_{12}} = \{0.1,0.6,0.8,0.9\}$, $h_{e_{13}} = \{0.2,0.7\}$, $h_{e_{15}} = \{0.6,0.8\}$, $h_{e_{23}} = \{0.3,0.4,0.6\}$, $h_{e_{25}} = \{0.4,0.5,0.7,1\}$, $h_{e_{45}} = \{0.5,0.6,0.9\}$. The matrix representation of the above information is given by

$$H_R = \begin{bmatrix} \{0.5\} & \{0.1,0.6,0.8,0.9\} & \{0.2,0.7\} & * & \{0.6,0.8\} \\ \{0.1,0.2,0.4,0.9\} & \{0.5\} & \{0.3,0.4,0.6\} & * & \{0.4,0.5,0.7,1\} \\ \{0.3,0.8\} & \{0.4,0.6,0.7\} & \{0.5\} & * & * \\ * & * & * & \{0.5\} & \{0.5,0.6,0.9\} \\ \{0.2,0.4\} & \{0,0.3,0.5,0.6\} & * & \{0.1,0.4,0.5\} & \{0.5\} \end{bmatrix}_{5\times 5}$$

Here '*' represents missing information. Since the length of the HFPR are not same. By using the pessimism or optimism rule to normalized the incomplete HFPR. The normalized I-HFPR is given as follows:

$$\bar{H}_R = \begin{bmatrix} \{0.5\} & \{0.1,0.6,0.8,0.9\} & \{0.2,0.7,0.7,0.7\} & * & \{0.6,0.8,0.8,0.8\} \\ \{0.1,0.2,0.4,0.9\} & \{0.5\} & \{0.3,0.4,0.6,0.6\} & * & \{0.4,0.5,0.7,1\} \\ \{0.3,0.3,0.3,0.8\} & \{0.4,0.4,0.6,0.7\} & \{0.5\} & * & * \\ * & * & * & \{0.5\} & \{0.5,0.6,0.9,0.9\} \\ \{0.2,0.2,0.2,0.4\} & \{0,0.3,0.5,0.6\} & * & \{0.1,0.1,0.4,0.5\} & \{0.5\} \end{bmatrix}$$

Replacing the unknown elements '$*$' of ij^{th} position to $0.5({w_i}^{\sigma(s)} - {w_j}^{\sigma(l_{\bar{h}_{e_{ij}}} - s+1)} + 1)$ in normalized I-HFPR. Let this is denoted by $\bar{H}_R'$. Then calculate the preference degree $P_i^{\sigma(s)}(w)$ of alternative A_i over all the other alternatives using equation 5.3.5 and collective preference $P(w)$ of all alternatives is calculated by using the equation 5.3.6 given as follows:

$P(w) = \sum_{i=1}^{4} \sum_{s=1}^{l_{\bar{h}'_{e_{ij}}}} P_i^{\sigma(s)}(w) = 40$

Weights of I-HFPR can be calculated by solving the system of linear equation using equation 5.3.7 which is given in table 5.1.

The missing elements are obtained using equation 5.3.3 as follows

$h_{14} = \{0.49,0.5,0.51,0.51\}$, $h_{41} = \{0.49,0.49,0.5,0.51\}$, $h_{24} = \{0.48,0.49,0.50,0.51\}$,

$h_{42} = \{0.49,0.5,0.51,0.52\}$, $h_{34} = \{0.49,0.49,0.5,0.51\}$, $h_{43} = \{0.49,0.5,0.51,0.51\}$,

Table 5.1: Weights of the missing element

w_1	w_2	w_3	w_4	w_5
${w_1}^{\sigma(1)} = 0.0347$	${w_2}^{\sigma(1)} = 0.0321$	${w_3}^{\sigma(1)} = 0.0421$	${w_4}^{\sigma(1)} = 0.0492$	${w_5}^{\sigma(1)} = 0.0195$
${w_1}^{\sigma(2)} = 0.0651$	${w_2}^{\sigma(2)} = 0.0397$	${w_3}^{\sigma(2)} = 0.0423$	${w_4}^{\sigma(2)} = 0.0523$	${w_5}^{\sigma(2)} = 0.0272$
${w_1}^{\sigma(3)} = 0.0702$	${w_2}^{\sigma(3)} = 0.0550$	${w_3}^{\sigma(3)} = 0.0477$	${w_4}^{\sigma(3)} = 0.0604$	${w_5}^{\sigma(3)} = 0.0400$
${w_1}^{\sigma(4)} = 0.0728$	${w_2}^{\sigma(4)} = 0.0753$	${w_3}^{\sigma(4)} = 0.0632$	${w_4}^{\sigma(4)} = 0.0609$	${w_5}^{\sigma(4)} = 0.0501$

$h_{35} = \{0.5, 0.5, 0.51, 0.52\}$, $h_{53} = \{0.48, 0.49, 0.5, 0.5\}$.

The complete HFPR matrix is given by

$$\begin{bmatrix} \{0.5\} & \{0.1,0.6,0.8,0.9\} & \{0.2,0.7,0.7,0.7\} & \{0.49,0.5,0.51,0.51\} & \{0.6,0.8,0.8,0.8\} \\ \{0.1,0.2,0.4,0.9\} & \{0.5\} & \{0.3,0.4,0.6,0.6\} & \{0.48,0.49,0.5,0.51\} & \{0.4,0.5,0.7,1\} \\ \{0.3,0.3,0.3,0.8\} & \{0.4,0.4,0.6,0.7\} & \{0.5\} & \{0.49,0.49,0.5,0.51\} & \{.5,0.5,0.51,0.52\} \\ \{0.49,0.49,0.5,0.51\} & \{0.49,0.5,0.51,0.52\} & \{0.49,0.5,0.51,0.51\} & \{0.5\} & \{0.5,0.6,0.9,0.9\} \\ \{0.2,0.2,0.2,0.4\} & \{0,0.3,0.5,0.6\} & \{0.48,0.49,0.5,0.5\} & \{0.1,0.1,0.4,0.5\} & \{0.5\} \end{bmatrix}$$

For checking the consistency level of the complete HFPR, we find all the possible pairs of preference degree represented by an HFE $h_{e_{ij}}$ are calculated using an intermediate alternative x_k $(k \neq i, j)$ by using equation 5.4.1. As an illustration, the estimated HFE of $h_{e_{12}}$ is $h_{12}^3 = \{0.1, 0.6, 0.8, 0.9\}$, $h_{12}^4 = \{0.48, 0.5, 0.52, 0.53\}$, $h_{12}^5 = \{0.1, 0.6, 0.8, 0.9\}$.
The average estimated HFE is calculated using equation 5.4.2 $h_{12}^{EST} = 0.569$.
Calculate the error between estimated HFE and and the its average estimated preference degree using equation 5.4.3, taking $\beta = 2/3$ we have $\varepsilon h_{12} = \{0.313, 0.021, 0.154, 0.2204\}$ and $\min(\varepsilon h_{12}) = 0.021$.
Similarly minimum error is calculated for other element as given below
$\min(\varepsilon h_{13}) = 0.062$, $\min(\varepsilon h_{14}) = 0.036$, $\min(\varepsilon h_{15}) = 0.057$, $\min(\varepsilon h_{21}) = 0.021$, $\min(\varepsilon h_{23}) = 0.0413$, $\min(\varepsilon h_{24}) = 0.0313$, $\min(\varepsilon h_{25}) = 0.0553$, $\min(\varepsilon h_{31}) = 0.0613$, $\min(\varepsilon h_{32}) = 0.0413$, $\min(\varepsilon h_{34}) = 0.0533$, $\min(\varepsilon h_{35}) = 0.122$, $\min(\varepsilon h_{41}) = 0.036$, $\min(\varepsilon h_{42}) = 0.044$, $\min(\varepsilon h_{43}) = 0.0533$, $\min(\varepsilon h_{45}) = 0.0246$, $\min(\varepsilon h_{51}) = 0.058$, $\min(\varepsilon h_{52}) = 0.055$, $\min(\varepsilon h_{53}) = 0.1137$, $\min(\varepsilon h_{54}) = 0.0249$.
Consistency level for a particular alternatives x_i is calculated by using equation 5.4.5, given as $cl_1 = 0.956$, $cl_2 = 0.961$, $cl_3 = 0.931$, $cl_4 = 0.962$, $cl_5 = 0.936$.

Consistency level of the complete HFPR is given by

$$cl_{\bar{H}_R} = \frac{cl_1 + cl_2 + cl_3 + cl_4 + cl_5}{5} = 0.9492$$

The resultant complete HFPR from H_R is consistent with consistency level 94.92%.

5.5 A method for Group decision-making problem with I-HFPR

The present section concern with the issue of group decision making with I-HFPR. A group decision making problem is considered with 'm' decision maker $D = D_1, D_2, \cdots, D_m$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)^T$ the weights vector, representing weights of corresponding DMs. Suppose 'm' DMs provides their judgement over 'n' number of decision alternatives $X = \{x_1, x_2, \cdots, x_n\}$ in form of I-HFPR $H_R^t = (h_{e_{ij}}^t)_{n\times n}, t = 1, 2, \cdots, m$. If the weights of the DMs already known we can use same weights. In this work, we have used criteria to give more weight to DM with less missing values. Following algorithm illustrates the procedure of missing element for group decision problem with I-HFPR.

Algorithm 5.5.1. Let $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)^T$ be the collective weight vector of the I-HFPR $H_R^t = (h_{e_{ij}}^t)_{n\times n}, (t = 1, 2, \cdots m)$.

1. Consider a GDM problem with 'm' I-HFPR provided by 'm' number of DMs.

2. Normalized the I-HFPRs by using definition 5.2.4 denoted by $\bar{H}_R^t,\ t = 1, 2, \cdots, m$.

3. Replace the unknown element '$*$' in $\bar{H}_R^t$ by $0.5(\gamma_i^{\sigma(s)} - \gamma_j^{\sigma(l_{\bar{h}_{e_{ij}}^t} - s + 1)} + 1)$, where $l_{\bar{h}_{e_{ij}}^t}$ is the number of elements present in HFE and construct a new I-HFPR $\bar{H}_R'^t = (\bar{h}_{e_{ij}}'^t)_{n\times n}, t = 1, 2, \cdots, m$, where

$$\bar{h}_{e_{ij}}'^t = \begin{cases} \bar{h}_{e_{ij}}^t & \text{if } \bar{h}_{e_{ij}}^t \neq *; \\ 0.5(\gamma_i^{\sigma(s)} - \gamma_j^{\sigma(l_{\bar{h}_{e_{ij}}^t} - s + 1)} + 1) & \text{if } \bar{h}_{e_{ij}}^t = *. \end{cases} \tag{5.5.1}$$

4. Aggregate the individual decision judgement of HFPR matrix into a collective HF-PR matrix using the additive weighted averaging operator is given as follows.

$$\bar{h}_{e_{ij},Agg}' = \sum_{t=1}^{m} \alpha_t \bar{h}_{e_{ij}}'^t,\ i, j \in N \tag{5.5.2}$$

   where α_t are weight vector of DM.

5. Calculate the missing element using algorithm 5.3.1.

6. Complete aggregated HFPR is obtain.

7. The consistency level of aggregated HFPRs is calculated as discussed in section 5.4.

To illustrates the above algorithm, here we have taken an example.

Example 5.5.1. Let us consider an GDM problem with four alternatives $X = \{x_1, x_2, x_3, x_4\}$ and three decision makers $d_t = \{d_1, d_2, d_3\}$. Let $\alpha = (0.5, 0.2, 0.3)$ be the weights vector of the DMs. These DMs provide their preference information over four alternatives and three I-HFPR matrix H_1, H_2, H_3 are given as follows:

$$H_1 = \begin{bmatrix} \{0.5\} & \{0.2,0.3,0.4\} & * & \{0.6,0.7,0.8\} \\ \{0.6,0.7,0.8\} & \{0.5\} & \{0.1,0.2,0.4\} & * \\ * & \{0.6,0.8,0.9\} & \{0.5\} & * \\ \{0.2,0.3,0.4\} & * & * & \{0.5\} \end{bmatrix}$$

$$H_2 = \begin{bmatrix} \{0.5\} & \{0.5,0.6,0.7\} & * & * \\ \{0.3,0.4,0.5\} & \{0.5\} & * & * \\ * & * & \{0.5\} & * \\ * & * & * & \{0.5\} \end{bmatrix}$$

$$H_3 = \begin{bmatrix} \{0.5\} & * & \{0,0.2,0.4\} & * \\ * & \{0.5\} & * & \{0.5,0.6,0.7\} \\ \{0.6,0.8,1\} & * & \{0.5\} & * \\ * & \{0.3,0.4,0.5\} & * & \{0.5\} \end{bmatrix}_{4\times 4}$$

The known elements of incomplete HFPR should satisfy the condition ${h_{ij}}^{\sigma(s)} = {h_{ik}}^{\sigma(s)} - {h_{jk}}^{\sigma(l_{h_{ij}}-s+1)} + 0.5$. Let $\alpha = (0.5, 0.2, 0.3)^T$ be the weight vector of DMs. It is here to note that the more weights are assigned to decision maker to the less number of missing element. The aggregated matrix $\bar{H}'_{agg}$ is given by

$$\begin{bmatrix}
\{0.5\}, & \begin{Bmatrix} 0.35+0.15(w_1^{\sigma(1)}-w_2^{\sigma(3)}), \\ 0.42+0.15(w_1^{\sigma(2)}-w_2^{\sigma(2)}), \\ 0.49+0.15(w_1^{\sigma(3)}-w_2^{\sigma(1)}) \end{Bmatrix} & \begin{Bmatrix} 0.35(w_1^{\sigma(1)}-w_3^{\sigma(3)}+1), \\ 0.41+0.35(w_1^{\sigma(2)}-w_3^{\sigma(2)}), \\ 0.47+0.35(w_1^{\sigma(3)}-w_3^{\sigma(1)}) \end{Bmatrix} & \begin{Bmatrix} 0.55+0.25(w_1^{\sigma(1)}-w_4^{\sigma(3)}), \\ 0.6+0.25(w_1^{\sigma(2)}-w_4^{\sigma(2)}), \\ 0.65+0.25(w_1^{\sigma(3)}-w_4^{\sigma(1)}) \end{Bmatrix} \\
\begin{Bmatrix} 0.51+0.15(w_2^{\sigma(1)}-w_1^{\sigma(3)}), \\ 0.58+0.15(w_2^{\sigma(2)}-w_1^{\sigma(2)}), \\ 0.65+0.15(w_2^{\sigma(3)}-w_1^{\sigma(1)}) \end{Bmatrix} & \{0.5\} & \begin{Bmatrix} 0.3+0.25(w_2^{\sigma(1)}-w_3^{\sigma(3)}), \\ 0.35+0.25(w_2^{\sigma(2)}-w_3^{\sigma(2)}), \\ 0.45+0.25(w_2^{\sigma(3)}-w_3^{\sigma(1)}) \end{Bmatrix} & \begin{Bmatrix} 0.5+0.35(w_2^{\sigma(1)}-w_4^{\sigma(3)}), \\ 0.53+0.35(w_2^{\sigma(2)}-w_4^{\sigma(2)}), \\ 0.56+0.35(w_2^{\sigma(3)}-w_4^{\sigma(1)}) \end{Bmatrix} \\
\begin{Bmatrix} 0.53+0.35(w_3^{\sigma(1)}-w_1^{\sigma(3)}), \\ 0.59+0.35(w_3^{\sigma(2)}-w_1^{\sigma(2)}), \\ 0.65+0.35(w_3^{\sigma(3)}-w_1^{\sigma(1)}) \end{Bmatrix} & \begin{Bmatrix} 0.55+0.25(w_3^{\sigma(1)}-w_2^{\sigma(3)}), \\ 0.65+0.25(w_3^{\sigma(2)}-w_2^{\sigma(2)}), \\ 0.7+0.25(w_3^{\sigma(3)}-w_2^{\sigma(1)}) \end{Bmatrix} & \{0.5\} & \begin{Bmatrix} 0.5(w_3^{\sigma(1)}-w_4^{\sigma(3)}+1), \\ 0.5(w_3^{\sigma(2)}-w_4^{\sigma(2)}+1), \\ 0.5(w_3^{\sigma(3)}-w_4^{\sigma(1)}+1) \end{Bmatrix} \\
\begin{Bmatrix} 0.35+0.25(w_4^{\sigma(1)}-w_1^{\sigma(3)}), \\ 0.4+0.25(w_4^{\sigma(2)}-w_1^{\sigma(2)}), \\ 0.45+0.25(w_4^{\sigma(3)}-w_1^{\sigma(1)}) \end{Bmatrix} & \begin{Bmatrix} 0.44+0.35(w_4^{\sigma(1)}-w_2^{\sigma(3)}), \\ 0.47+0.35(w_4^{\sigma(2)}-w_2^{\sigma(2)}), \\ 0.5+0.35(w_4^{\sigma(3)}-w_2^{\sigma(1)}) \end{Bmatrix} & \begin{Bmatrix} 0.5(w_4^{\sigma(1)}-w_3^{\sigma(3)}+1), \\ 0.5(w_4^{\sigma(2)}-w_3^{\sigma(2)}+1), \\ 0.5(w_4^{\sigma(3)}-w_3^{\sigma(1)}+1) \end{Bmatrix} & \{0.5\}
\end{bmatrix}$$

The preference degree of the alternative x_i over other alternatives is calculated by using equation 5.3.5 and the collective preference of all alternatives $p(w)$ of $\bar{H}'_{agg}$ has valued 18. Weights are calculated by solving the system of linear equations by using equation 5.3.7

Table 5.2: Weights of the missing element

w_1	w_2	w_3	w_4
$w_1{}^{\sigma(1)} = 0.0684$	$w_2{}^{\sigma(1)} = 0.0720$	$w_3{}^{\sigma(1)} = 0.0879$	$w_4{}^{\sigma(1)} = 0.0700$
$w_1{}^{\sigma(2)} = 0.0791$	$w_2{}^{\sigma(2)} = 0.0810$	$w_3{}^{\sigma(2)} = 0.0979$	$w_4{}^{\sigma(2)} = 0.0753$
$w_1{}^{\sigma(3)} = 0.0899$	$w_2{}^{\sigma(3)} = 0.0929$	$w_3{}^{\sigma(3)} = 0.1049$	$w_4{}^{\sigma(3)} = 0.0807$

and are shown in table 5.2. We obtained complete aggregated HFPR matrix with consistency level 95.25% is given as follows:

$$\bar{H}'_{agg} = \begin{bmatrix} \{0.5\} & \{0.35, 0.42, 0.49\} & \{0.34, 0.4, 0.47\} & \{0.55, 0.601, 0.65\} \\ \{0.51, 0.58, 0.65\} & \{0.5\} & \{0.29, 0.34, 0.45\} & \{0.5, 0.53, 0.57\} \\ \{0.53, 0.6, 0.66\} & \{0.55, 0.66, 0.71\} & \{0.5\} & \{0.5, 0.51, 0.52\} \\ \{0.35, 0.399, 0.45\} & \{0.43, 0.47, 0.5\} & \{0.48, 0.49, 0.45\} & \{0.5\} \end{bmatrix}_{4\times 4}$$

5.5.1 Comparison with comparative work

Zhu and Xu [75] introduce HFPR as given in definition 5.2.3. Xu, Chen et al. [195] introduce a new HFPR as given below.

Definition 5.5.1. A hesitant fuzzy preference relation H_R over $X = \{x_1, x_2, \cdots x_n\}$ is defined by $H_R = (h_{e_{ij}})_{n\times n}$ which is subset of $X \times X$, where $h_{e_{ij}} = \{h_{e_{ij}}^{\sigma(s)} | s = 1, 2, \cdots, l_{h_{e_{ij}}}\}$ is a HFE indicating all the possible preference value to which an alternative x_i is prefer to alternative x_j satisfying

$$h_{e_{ij}}^{\sigma(s)} + h_{e_{ji}}^{\sigma(s)} = 1,\ h_{e_{ii}} = 0.5,\ l_{h_{e_{ij}}} = l_{h_{e_{ji}}},\ \forall i, j \in N.$$

where $h_{e_{ij}}^{\sigma(s)}$ is the s^{th} largest element in $h_{e_{ij}}$.

It is to note that the above equation does not satisfy the hesitancy property of Zhu and Xu [75] as given in equation 5.2.1. Xu, Chen et al. [195] developed a goal programming model by using the additive consistency of FPR to derive the priority weights from an I-HFPRs. By using the goal programming model [195], we have solved our I-HFPR of example 5.4.1 and derived the priority weights. According to Xu Chen et al. [195],

$$h_{e_{ij}}^{\sigma(s)} \text{ or } \cdots \text{ or } h_{e_{ij}}^{\sigma(l_{h_{e_{ij}}} - s + 1)} = 0.5 + \frac{n-1}{2}(w_i - w_j),\ \forall i, j \in N.$$

where w_i and w_j are the priority weights. Then we can get the complete HFPR.

Using the model $(M-8)$ given by Xu, Chen et al. [195], we solved example 5.4.1 and the weights are given by $w_1 = 0.32, w_2 = 0.12, w_3 = 0.22, w_4 = 0.17, w_5 = 0.17$. This model is solved by by Lingo software. The missing elements are $h_{14} = 0.5 + 2(w_1 - w_4) =$

0.8, $h_{24} = 0.5 + 2(w_2 - w_4) = 0.4$, $h_{34} = 0.5 + 2(w_3 - w_4) = 0.6$, $h_{35} = 0.5 + 2(w_3 - w_5) = 0.6$. By using section, 5.4 the consistency level of the complete HFPR is 93.33%.

Remark 5.5.1. (i) According to Xu, Chen, et al. [195] the priority weights has a single value for each hesitant fuzzy preference information. However, our method has different priority weights for each position of the hesitant fuzzy preference information that leads to the more generalized method.
(ii) According to Xu, Chen, et al. [195], some restriction is there for an I-HFPR that the necessary condition of acceptable I-HFPR is that at least one element is present in each row or column, i.e., needs at least $(n-1)$ judgment. Whereas in our method there is no such restriction. If all data are missing then also we can find out the priority weights of I-HFPRs. This point is strong in the case of group decision-making problem.
(iii) Consistency level is also improved in our method.

5.6 Conclusion:

In decision-making problem, to make sure that the DMs are neither arbitrary nor unreasonable, the study of the consistency plays an important role in the situation where the DM has to work in case of vague situations. Zhang et al. [144] defined the additive consistency property of HFPR which not satisfy the property of hesitancy given by Zhu and Xu [75]. In this chapter, we have characterized a new definition of additive consistency property of HFPR which are the common tool to assemble and present preference data given by the DM in decision-making problem. By using additive consistency, we have developed a method that helps to find the missing elements of an incomplete HFPR. Using error analysis, the consistency level of HFPR is also calculated. Also, we have developed an algorithm for group decision making with I-HFPRs and presented two example to illustrates the above methods with highest consistency level.

Chapter 6

Incomplete Hesitant Multiplicative Preference relation

This chapter[1], [2], *discussed another important tool in the process of decision-making that is hesitant multiplicative preference relation (HMPR) for hesitancy and uncertainty in the scale of* $1/9-9$*. Limitation of time, experience and lack of the experts' professional knowledge lead to form an incomplete HMPR. This chapter aims to develop a method to complete incomplete HMPRs. The study of the consistency of HMPRs are very essential feature to keep away from the confusing solution. A new definition of multiplicative transitive property of HMPR has also given that preserve the hesitancy property and is used to construct the complete HMPR from incomplete one. An optimization model is developed to minimize the error. Also, a linear programming model is developed to estimate the unknown element of incomplete HMPR. The satisfaction degree and the acceptably consistent of complete HMPR is also checked. The whole procedure is explained with suitable examples.*

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[1]This chapter is based on a research paper entitled "A method to complement incomplete hesitant multiplicative preference relation" published in *International Journal of Research and Analytical Reviews* **5**, (2)1421–1429 (2018).

[2]"Incomplete hesitant multiplicative preference relation" revised version submitted in *OPSEARCH*, Spinger.

6.1 Introduction

In the framework of decision making problem, a DM is normally requested to give his/her judgement by contrasting the relationship of each pair of object. To communicate the DMs preference value of alternative/criteria, preference relations is utilized as an essential tool. Some of the researchers named the preference relation by pairwise comparison matrices. Different type of preference relation has been developed discussed in chapter 1.

In the above existing preference relations, DM do not provide the hesitancy information by making a comparison among each pair of alternative. Once in a while, to get a more sensible choice outcome, the decision maker is approved to give the preference data about an set of alternatives/criteria. The fact may conform that, the DM while making a decision may not be sure about an single value but has uncertainty between several feasible values. These conceivable feasible values can be considered as hesitant preference relation. The classification of hesitant preference relation is of two type. One of them is HFPR which was briefly discussed in chapter 5.

The preference value in HFPR are expressed using the $0-1$ proportion scale which is symmetrically conveyed around 0.5. Sometimes, the degree of the preference data are unsymmetrically conveyed around some value. Specially the separation between the evaluations communicating good information ought to be more critical than the ones between the evaluations communicating bad information [63]. Saaty [58] developed $1/9-9$ scale to address such problem expressing the MPR which has been used in various field. Some of the DM do not prefer to utilize the preference between 0 and 1. However, he/she might prefer to utilize Saaty's ratio scale to give the data that the alternatives x_i is better than x_j. For instance, in a decision-organization, some experts prefer to give their information as $1/3$, some give as 4, and some give as 5. The representation of the set of possible values as $\{1/3,4,5\}$ which is called hesitant multiplicative element (HME) [130]. Xia and Xu [130] used the $1-9$ scale and characterized the idea of HMPR. In this work, we concentrate only on HMPR.

A complete preference relation of order $n \times n$ need $\frac{n(n-1)}{2}$ judgements in its whole lower or upper triangular part. Because of time pressure, and absence of adequate information, it is complicated to obtain a complete HMPR in many practical situations, such as medical diagnosis, personal examination. The situation in which some elements are missing results an incomplete HMPR. Therefore, it is necessary to develop a method to get complete

HMPR from incomplete one. In this chapter, we have defined the multiplicative transitive property of HMPRs that preserve the hesitancy property of HMPRs. By using this property, we have developed an algorithm to construct a complete HMPRs from incomplete one and developed an optimization model to minimize the error. Zhu et al. [196] developed a method to measure the satisfaction degree of the HMPRs. We have checked the satisfaction degree of new complete HMPRs. An algorithm is illustrated via a suitable example with the satisfaction degree more than 80%. Also, we have checked the acceptably consistent of HMPR.

The rest of the chapter is organized as follows. In section 6.2, some basic concepts of HMPRs are defined briefly. Multiplicative transitive property for HMPRs defined that is used to construct a complete HMPRs from the incomplete one discussed in section 6.3. In section 6.4, also, we have developed a methodology based on linear programming model to calculate the missing data from an incomplete HMPR. Satisfaction degree and acceptably consistent of complete HMPR is checked. An example is given to illustrates the whole procedure. Some concluding remarks are given in the last section.

6.2 Preliminaries

In this section, we discuss some basic concept of hesitant multiplicative set and HMPRs.

Xia and Xu [130] give the definition of hesitant multiplicative set in which elements are the set of possible values.

Definition 6.2.1. ([130]) A hesitant multiplicative set on X is defined by $M = \{(x, b_M(x))|x \in X\}$, where $b_M(x)$ is a set of some values that lies between 1/9 to 9. For the shake of convenience $b_M(x) = b_e$ is called HME.

Given a HME $b_e = \{b_e^{\sigma(s)}|s = 1, 2, \cdots, l_{b_e}\}$ where l_{b_e} denotes the number of elements in the HME b_e, $\sigma(s)$ denotes the position of HME.

Let $b_e^{\sigma(s)}$, $b_{e_1}^{\sigma(s)}$,$b_{e_2}^{\sigma(s)}$, are the s^{th} position value of HME b_e, b_{e_1}, and b_{e_2} respectively. Additionally suppose that $l_{b_{e_1}} = l_{b_{e_2}} = l$, or else we can extend the smaller one by adding the numerical values given as the following definition.

Definition 6.2.2. ([197]) Let b_e^+ and b_e^- are the maximum and minimum elements in HME $b_e = \{b_e^{\sigma(s)}|s = 1, 2, \cdots, l_{b_e}\}$ respectively and δ be an optimized parameter that lies between 0 and 1. This optimized parameter is chosen by the DMs preferences as indicated by their own risk, then $(b_e^+)^{\delta} \times (b_e^-)^{1-\delta}$ is an added element that is denoted by $\bar{b}_e$.

Particularly the added element b_e^+ and b_e^- respectively extracted from the conditions

$\delta = 1$ and $\delta = 0$ that corresponds to the optimism and pessimism rules are taken by the decision makers.

Xia and Xu [130] developed the concept of HMPRs by HMEs and MPRs given as follows.

Definition 6.2.3. ([130]) A HMPR B_R over X is defined by $B_R = (b_{e_{ij}})_{n\times n}$ which is subset of $X \times X$, where $b_{e_{ij}} = \{b_{e_{ij}}^{\sigma(s)} | s = 1, 2, \cdots, l_{b_{e_{ij}}}\}$ is a HME indicating all the possible preference value to which an alternative x_i is prefer to alternative x_j satisfying

$$b_{e_{ij}}^{\sigma(s)} \times b_{e_{ji}}^{\sigma(l_{b_{e_{ji}}} - s+1)} = 1,\ b_{e_{ii}} = \{1\},\ l_{b_{e_{ij}}} = l_{b_{e_{ji}}},\ \forall i, j \in N. \tag{6.2.1}$$

where $b_{e_{ij}}^{\sigma(s)}$ is the s^{th} position element in $b_{e_{ij}}$. Xia and Xu [130] mentioned that HME are assumed in increasing order. But it may be the case that some decision makers gives their preference in decreasing order that also satisfy the hesitancy property (equation 6.2.1). Also Zhang and Guo [197] proved following theorem

Theorem 6.2.1. Let $B = (b_{e_{ij}})_{n\times n}$, where $b_{e_{ij}} = \{b_{e_{ij}}^{\sigma(s)} | s = 1, 2, \cdots, l_{b_{e_{ij}}}\}$ be an HMPR. It can be transfered into HFPR $H = (h_{e_{ij}})_{n\times n}$, $h_{e_{ij}} = \{h_{e_{ij}}^{\sigma(s)} |\ s = 1, 2, \cdots, l_{h_{e_{ij}}}\}$ by using the transformation $h_{e_{ij}}^{\sigma(s)} = \frac{b_{e_{ij}}^{\sigma(s)}}{1+b_{e_{ij}}^{\sigma(s)}}$, then transfer the HFPR H into the HMPR $B^{'} = (b^{'}_{e_{ij}})_{n\times n}$ by using the transformation $b^{'}_{e_{ij}} = \frac{h_{e_{ij}}^{\sigma(s)}}{1-h_{e_{ij}}^{\sigma(s)}}$, where $b^{'}_{e_{ij}} = \{b_{e_{ij}}^{'\,\sigma(s)} |\ s = 1, 2, \cdots, l_{b^{'}_{e_{ij}}}\}$, then $B = B'$.

In our case, we are taking the HME into increase/decrease that also satisfy the theorem 6.2.1. We are not considering the case that HMEs are arranged in increasing order always.

For example let the HMPR, B_R as shown below

$$B_R = \begin{bmatrix} \{1\} & \{\frac{1}{6}, \frac{1}{5}, \frac{1}{4}\} & \{3\} \\ \{4, 5, 6\} & \{1\} & \{4, 2\} \\ \{\frac{1}{3}\} & \{\frac{1}{2}, \frac{1}{4}\} & \{1\} \end{bmatrix}$$

In the above HMPR, length of the preference degree of alternative x_i over x_j are not same. According to definition 6.2.2, optimized parameter δ is used to add few elements to an HMPR. Thus a normalized HMPR is obtained which is defined as follows.

Definition 6.2.4. ([76]) Let $b^{+}_{e_{ij}}$ and $b^{-}_{e_{ij}}$ be the maximum and minimum elements of HMPR, $B_R = (b_{e_{ij}})_{n\times n}$ be respectively, for $i, j = 1, 2, \cdots, n$, and let δ be an optimized parameter. According to definition 6.2.2 if we add some elements $\bar{b}_{e_{ij}}$=$(b^{+}_{e_{ij}})^{\delta} \times (b^{-}_{e_{ij}})^{(1-\delta)}$ to $b_{e_{ij}}$ if $i \leq j$ and add some elements $\bar{b}_{e_{ji}}$=$(b^{+}_{e_{ji}})^{(1-\delta)} \times (b^{-}_{e_{ji}})^{\delta}$ to $b_{e_{ji}}$ if $i \leq j$, then the normalized HMPR $\bar{B}_R$=$\{\bar{b}_{e_{ij}}^{\sigma(s)}|s=1,2,\cdots,l_{\bar{b}_{e_{ij}}}\}$ satisfying

$$l_{\bar{b}_{e_{ij}}}=\max\{l_{b_{e_{ij}}}|i,j=1,2,\cdots,n,i\neq j\},\ \bar{b}_{e_{ij}}^{\sigma(s)}\times\bar{b}_{e_{ji}}^{\sigma(l_{\bar{b}_{e_{ji}}}-s+1)}=1,\ \bar{b}_{e_{ii}}=\{1\}. \quad (6.2.2)$$

where $\bar{b}_{e_{ij}}^{\sigma(s)}$ and $\bar{b}_{e_{ji}}^{\sigma(s)}$ is the s^{th} position element in $\bar{b}_{e_{ij}}$ and $\bar{b}_{e_{ji}}$ respectively.

The normalized HMPR $\bar{B}_R$ $(let\ \delta=1)$ of the HMPR B_R from the above example as shown in below

$$\bar{B}_R=\begin{bmatrix}\{1\} & \{\frac{1}{6},\frac{1}{5},\frac{1}{4}\} & \{3,3,3\}\\ \{4,5,6\} & \{1\} & \{4,4,2\}\\ \{\frac{1}{3},\frac{1}{3},\frac{1}{3}\} & \{\frac{1}{2},\frac{1}{4},\frac{1}{4}\} & \{1\}\end{bmatrix}$$

Sometimes it is complicated to obtain a complete preference relation especially for the preference relation with higher order. It may be the case that an expert is unable to express his/her opinion over other pairs of alternatives, due to the limitation of decision makers' proficient knowledge, shortage of time, the provided preference value in HMPR becomes incomplete. Then the incomplete HMPR is obtained in which some of the elements are missing. We have defined the incomplete HMPR (I-HMPR).

Definition 6.2.5. (I-HMPR) The HMPR $B_R=(b_{e_{ij}})_{n\times n}$ where $b_{e_{ij}}=\{b_{e_{ij}}^{\sigma(s)}|s=1,2,\cdots,l_{b_{e_{ij}}}\}$ is called an I-HMPR, if some elements in it are missing that is denoted by "$*$" and the other elements are given by decision makers should satisfy the condition of definition 6.2.3.

Following is an example of incomplete HMPR,

$$\begin{bmatrix}\{1\} & \{\frac{1}{6},\frac{1}{5}\} & *\\ \{5,6\} & \{1\} & \{2,4\}\\ * & \{\frac{1}{4},\frac{1}{2}\} & \{1\}\end{bmatrix}$$

where "$*$" is the missing element. Based on definition 6.2.4, an optimized parameter δ is used to add some elements to an I-HMPR and a normalized I-HMPR is obtained. The definition of normalized I-HMPR is given as follows.

Definition 6.2.6. (normalized I-HMPR) The incomplete HMPR is called normalized I-HMPR, if the known elements are satisfying equation 6.2.2.

Zhang and Wu [76] mentioned that if $B=(b_{e_{ij}})_{n\times n},\ \ b_{e_{ij}}=\{b_{e_{ij}}^{\sigma(s)}|s=1,2,\cdots,l_{b_{e_{ij}}}\}$ be a HMPR and $\bar{B}=(\bar{b}_{e_{ij}})_{n\times n}$, $\bar{b}_{e_{ij}}=\{\bar{b}_{e_{ij}}^{\sigma(s)}|s=1,2,\cdots,l,\ (l=max\{l_{b_{e_{ij}}}|i,j=1,2,\cdots n,i\neq j\})$ be its normalized HMPR with the optimized parameter δ, then l number of MPR $P^{(s)}=(p_{ij}^{(s)})_{n\times n},(s=1,2,\cdots,l)$ are obtained where,

$$p_{ij} = \begin{cases} \bar{b}_{ij}^{\sigma(s)} & i < j \\ 1 & i = j \\ \bar{b}^{\sigma(l-s+1)} & i > j \end{cases} \tag{6.2.3}$$

In 2014 Zhang and Wu [76] defined the consistent HMPR.

Definition 6.2.7. ([76]) Let $B = (b_{e_{ij}})_{n\times n}$ be a HMPR and $\bar{B} = (\bar{b}_{e_{ij}})_{n\times n}$ be its normalized HMPR with optimized operator δ. If HMPR is said to be consistent if all the MPRs obtained from $\bar{B}$ using 6.2.3 are consistent.

Definition 6.2.8. ([76]) Let $B = (b_{e_{ij}})_{n\times n}$ be a HMPR and $\bar{B} = (\bar{b}_{e_{ij}})_{n\times n}$ be its normalized HMPR with optimized operator δ. If HMPR is acceptably consistent if all the MPRs obtained from $\bar{B}$ are acceptably consistent.

6.3 Constructing complete HMPR from an incomplete HMPR

In this section, before discussing about constructing complete HMPR from an incomplete HMPR, Especially, for a HMPR $B_R = (b_{e_{ij}})_{n\times n}$, we have defined the consistency property.

Definition 6.3.1. Let $B_R = (b_{e_{ij}})_{n\times n}$ and $\bar{B}_R = (\bar{b}_{e_{ij}})$ be HMPR and its normalized HMPR on X with an optimization operator δ, satisfying the following multiplicative transitive property

$$\bar{b}_{e_{ij}}^{\sigma(s)} = \bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \ \forall\, i,j,k \in N, \text{ and } i \neq k \neq j,\ s = 1,2,\cdots,l_{\bar{b}_{e_{ij}}}. \tag{6.3.1}$$

where $\bar{b}_{e_{ij}}^{\sigma(s)}$, $\bar{b}_{e_{ik}}^{\sigma(s)}$ and $\bar{b}_{e_{kj}}^{\sigma(s)}$ are the s^{th} position elements in $\bar{b}_{e_{ij}}$, $\bar{b}_{e_{ik}}$ and $\bar{b}_{e_{kj}}$ respectively then the HMPR, B_R is said to be a multiplicative consistent HMPR with optimization parameter δ. The multiplicative transitive property of HMPRs satisfy the property of hesitancy i.e $\bar{b}_{e_{ij}}^{\sigma(s)} \times \bar{b}_{e_{ji}}^{\sigma(l_{\bar{b}_{e_{ji}}}-s+1)} = 1$.

For example,

$$\begin{bmatrix} \{1\} & \{\frac{1}{4},\frac{1}{3},\frac{1}{2}\} & \{\frac{1}{2},\frac{1}{2},\frac{2}{3}\} \\ \{2,3,4\} & \{1\} & \{\frac{8}{3},\frac{3}{2},1\} \\ \{\frac{3}{2},2,2\} & \{1,\frac{2}{3},\frac{3}{8}\} & \{1\} \end{bmatrix}$$

The above HMPR is consistent. If we replace the value of $b_{13}^{\sigma(3)}$ to $\frac{2}{5}$ in place of $\frac{2}{3}$ in the above HMPR matrix, then it becomes inconsistent.

The multiplicative consistency property of HMPR will take part as an important role to construct complete HMPRs from I-HMPRs. We define a multiplicative consistent I-HMPR.

Definition 6.3.2. The I-HMPR, $B_R = (b_{e_{ij}})_{n\times n}$ is called multiplicative consistent if all the known elements satisfy the conditions of equation 6.3.1.

Zhang and Wu [76] proposed the definition of acceptable incomplete HMPR.

Definition 6.3.3. [76]I-HMPR is said to be acceptable I-HMPR if at least one known element present in each row or each column of HMPR.

Before going to the next section, we have defined an algorithm for constructing a complete HMPR from an incomplete one.

Algorithm 6.3.1.

1. Consider an incomplete HMPR, B_R.

2. Normalized the incomplete HMPR using definition 6.2.4 is denoted by $\bar{B}_R = \bar{b}_{e_{ij}}$.

$$\bar{b}_{e_{ij}} = \begin{cases} * & \text{if } \bar{b}_{e_{ij}} \notin \Omega; \\ \bar{b}_{e_{ij}} & \text{if } \bar{b}_{e_{ij} \in \Omega}. \end{cases} \tag{6.3.2}$$

   Where Ω is the set of known element and '$*$' is the value of the missing element.

3. Initially the missing element can be calculated using the formula

$$\bar{b}_{e_{ij}}^{\sigma(s)(*(0))} = \left\{ \prod_{k\in T_{ij}} \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \right) \right\}^{1/t_{ij}} \tag{6.3.3}$$

   where t_{ij} is the number of the elements of the set $T_{ij} = \{k|\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)}, \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \in \Omega\}$ which indicates that there exist different pairs of adjoining known elements to calculate the missing element $\bar{b}_{e_{ij}}$. It is noted that the subsequent missing elements are evaluated by using the previously calculated missing values.

4. The initial values are obtained.

Sometimes initial values obtained may not lie in between $1/9$ to 9 or does not satisfy the required condition of hesitancy as well as consistency. For solving this type of difficulties, we have developed an optimization model that minimizes the error. MATLAB optimization toolbox is used to solve the optimization model.

The multiplicative transitive property given by equation 6.3.1 of HMPR can be rewritten as

$$\log \bar{b}_{e_{ij}}^{\sigma(s)} = \log \bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} + \log \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \; \forall\, i,j,k \in N, \text{ and } i \neq k \neq j,\; s = 1,2,\cdots,l_{\bar{b}_{e_{ij}}}. \tag{6.3.4}$$

In case of inconsistency, equation 6.3.4 is utilize to minimize the error. To do this we have constructed the following optimization model.

(Model 6.1) Min $\sum_{(i,j)\notin\Omega} \varepsilon_{\bar{b}_{e_{ij}^k}}$

subject to

$$\begin{aligned}
\varepsilon_{\bar{b}_{e_{ij}^k}} - |\log \bar{b}_{e_{ij}}^{\sigma(s)} - \left(\log \bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} + \log \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)}\right)| &= 0,\; i \neq k \neq j,\; s = 1,2\cdots l_{\bar{b}_{e_{ij}}} \\
1/9 - \bar{b}_{e_{ij}}^{\sigma(s)} &\leq 0, s = 1,2\cdots l_{\bar{b}_{e_{ij}}} \\
\bar{b}_{e_{ij}}^{\sigma(s)} - 9 &\leq 0, s = 1,2\cdots l_{\bar{b}_{e_{ij}}} \\
\bar{b}_{e_{ij}}^{\sigma(s)(0)} &= \bar{b}_{e_{ij}}^{\sigma(s)(*(0))}, s = 1,2\cdots l_{\bar{b}_{e_{ij}}} \\
\bar{b}_{e_{ij}}^{\sigma(s)} \times \bar{b}_{e_{ji}}^{\sigma(l_{\bar{b}_{e_{ij}}}-s+1)} &= 1
\end{aligned}$$

where $\bar{b}_{e_{ij}}^{\sigma(s)(*(0))}$ are the intial value obtained from the algorithm 6.3.1.

In the next section, we have developed a linear programming model to calculate the missing element of I-HMPR.

6.4 Linear programming model to complete incomplete HMPR

Many researcher ([63], [67], [84], [131], [198]) noted preference relation may incomplete for numerous reason. To manage the incomplete hesitant multiplicative preference relation, here we have constructed a linear programming model to compute the missing element. The suggested model addresses the situations with ignorance. Let $\bar{B}_R = (\bar{b}_{e_{ij}})_{n\times n}$ ba a normalized I-HMPR is called consistent then equation 6.3.3 is satisfied.

From definition 6.2.3, we know that $\bar{B}_R$ is consistent if and only if the strictly upper triangular elements of I-HMPR satisfy the equation 6.3.3, namely

$$\bar{b}_{e_{ij}}^{\sigma(s)} = \left\{ \prod_{k=1}^{n} \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \right) \right\}^{\frac{1}{n}} \forall\, i,j \in N, \text{ and } i \neq k \neq j,\ i<j,$$
$$s = 1,2,\cdots, l_{\bar{b}_{e_{ij}}}. \qquad (6.4.1)$$

Sometimes, for making consistency of I-HMPR, the missing values does not satisfy the equation 6.4.1. For higher and better approximation, we have minimized the error ε_{ij}, where

$$\varepsilon_{ij} = \left| \bar{b}_{e_{ij}}^{\sigma(s)} - \left\{ \prod_{k=1}^{n} \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \right) \right\}^{\frac{1}{n}} \right| \forall\, i,j \in N, \text{ and } i \neq k \neq j,\ i<j,$$
$$s = 1,2,\cdots, l_{\bar{b}_{e_{ij}}}. \qquad (6.4.2)$$

ε_{ij} is a constant when all the values of equation 6.4.2 are known. Therefore, equations having missing values are only considered. To do this, Let Ω is the set of all known element of HMPRs and

$$\delta_{ij} = \begin{cases} 1 & k \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Define

$$\varepsilon_{ij} = \delta_{ij} \left| \bar{b}_{e_{ij}}^{\sigma(s)} - \left\{ \prod_{k=1}^{n} \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \right) \right\}^{\frac{1}{n}} \right| \forall\, i,j \in N, \text{ and } i \neq k \neq j,\ i<j,$$
$$s = 1,2,\cdots, l_{\bar{b}_{e_{ij}}}. \qquad (6.4.3)$$

Thus, the following multi-objective programming model(Model 6.2) is construct to calculate the missing values:

(Model 6.2)

$$\min \varepsilon_{ij} = \delta_{ij} \left| \bar{b}_{e_{ij}}^{\sigma(s)} - \left\{ \prod_{k=1}^{n} \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}} - s + 1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}} - s + 1)} \right)^{\frac{1}{n}} \right\} \right| \forall\, i, j \in N,$$

$$\text{and } i \neq k \neq j, i < j,\ s = 1, 2, \cdots, l_{\bar{b}_{e_{ij}}}$$

$$\text{Subject to } 1/9 \leq \bar{b}_{e_{ij}}^{\sigma(s)} \leq 9,\ \bar{b}_{e_{ij}}^{\sigma(s)} \in \Gamma,$$

Where $\Gamma = \{\bar{b}_{e_{ij}}^{\sigma(s)} | \bar{b}_{e_{ij}}^{\sigma(s)} \text{is a missing value}, i, j \in N,\ i < j,\ s = 1, 2, \cdots, l_{\bar{b}_{e_{ij}}}\}$

The above mentioned minimization problem (Model 6.2) can be solved by using the following goal programming model

(Model 6.3) $$\sum_{i,j=1, i<j}^{n} \sum_{k \in \Omega, s=1,2,\cdots,l_{\bar{b}_{e_{ij}}}} \left(d_{ij,k}^{\sigma(s)+} + d_{ij,k}^{\sigma(s)-} \right)$$

Subject to

$$\begin{cases} \delta_{ij} \left(\bar{b}_{e_{ij}}^{\sigma(s)} - \left\{ \prod_{k=1}^{n} \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}} - s + 1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}} - s + 1)} \right) \right\}^{\frac{1}{n}} \right) & \\ - d_{ij,k}^{\sigma(s)+} + d_{ij,k}^{\sigma(s)-} = 0; & i, j \in N,\ i < j,\ s = 1, 2, \cdots, l_{\bar{b}_{e_{ij}}} \\ 1/9 \leq \bar{b}_{e_{ij}}^{\sigma(s)} \leq 9 & \bar{b}_{e_{ij}}^{\sigma(s)} \in \Gamma \\ d_{ij,k}^{\sigma(s)+}, d_{ij,k}^{\sigma(s)-} \geq 0 & i, j \in N,\ i < j,\ s = 1, 2, \cdots, l_{\bar{b}_{e_{ij}}} \end{cases}$$

where $d_{ij,k}^{\sigma(s)+} = \left(\bar{b}_{e_{ij}}^{\sigma(s)} - \left\{ \prod_{k=1}^{n} \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}} - s + 1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}} - s + 1)} \right) \right\}^{\frac{1}{n}} \right) \vee 0$ and
$d_{ij,k}^{\sigma(s)-} = \left(\left\{ \prod_{k=1}^{n} \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}} - s + 1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}} - s + 1)} \right) \right\}^{\frac{1}{n}} - \bar{b}_{e_{ij}}^{\sigma(s)} \right) \vee 0$
$\forall\, i, j \in N,\ i < j,\ s = 1, 2, \cdots, l_{\bar{b}_{e_{ij}}}$.

The missing values of I-HMPR are obtained using the model (Model 6.3). Since the above goal programming model (Model 6.3) is nonlinear, to make linear take $z_{ij} = \ln \bar{b}_{e_{ij}}^{\sigma(s)}$ for all $i, j \in N,\ i < j,\ s = 1, 2, \cdots, l_{\bar{b}_{e_{ij}}}$.

Then equation 6.4.1 can be transformed into

$$\begin{aligned} z_{ij} = \ln \bar{b}_{e_{ij}}^{\sigma(s)} &= \ln \left\{ \prod_{k=1}^{n} \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \right) \right\}^{\frac{1}{n}} \\ &= \frac{1}{n} \ln \left\{ \prod_{k=1}^{n} \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \right) \right\} \\ &= \frac{1}{n} \sum_{k=1}^{n} \ln \left(\bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} \times \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \right) \\ &= \frac{1}{n} \sum_{k=1}^{n} \left(\ln \bar{b}_{e_{ik}}^{\sigma(l_{\bar{b}_{e_{ik}}}-s+1)} + \ln \bar{b}_{e_{kj}}^{\sigma(l_{\bar{b}_{e_{kj}}}-s+1)} \right) \\ &= \frac{1}{n} \sum_{k=1}^{n} (z_{ik} + z_{kj}) \end{aligned} \tag{6.4.4}$$

Then we developed a linear programming model (Model 6.4)

(Model 6.4)

$$\sum_{i,j=1,i<j}^{n} \sum_{k\in\Omega,s=1,2,\cdots,l_{\bar{b}_{e_{ij}}}} \left(d_{ij,k}^{\sigma(s)+} + d_{ij,k}^{\sigma(s)-} \right)$$

Subject to

$$\begin{cases} \delta_{ij} \left(z_{ij} - \frac{1}{n} \sum_{k=1}^{n} (z_{ik} + z_{kj}) \right) - d_{ij,k}^{\sigma(s)+} + d_{ij,k}^{\sigma(s)-} = 0, & i,j \in N,\ i<j\ s = 1,2,\cdots,l_{\bar{b}_{e_{ij}}} \\ 1/9 \le z_{ij} \le 9 & z_{ij} \in \Gamma' \\ d_{ij,k}^{\sigma(s)+}, d_{ij,k}^{\sigma(s)-} \ge 0 & i,j \in N,\ i<j\ s = 1,2,\cdots,l_{\bar{b}_{e_{ij}}} \end{cases}$$

where $\Gamma' = \{z_{ij} | z_{ij} \text{is a missing value}, i,j \in N,\ i<j\}$, $d_{ij,k}^{\sigma(s)+} = \left(z_{ij} - \frac{1}{n} \sum_{k=1}^{n} (z_{ik} + z_{kj}) \right) \vee 0$ and $d_{ij,k}^{\sigma(s)-} = \left(\frac{1}{n} \sum_{k=1}^{n} (z_{ik} + z_{kj}) - z_{ij} \right) \vee 0\ \forall i,j \in N,\ i<j\ s = 1,2,\cdots,l_{\bar{b}_{e_{ij}}}$.

To illustrate the above two model, i.e., optimization model (Model 6.1) and linear programming model (Model 6.4), we have taken an example, which is discussed below.

Example 6.4.1. Consider a decision making problem with five sets of alternatives x_i with $i = 1,2,3,4,5$. The matrix representation of decision-maker judgement is a pairwise comparison matrix denoted by B_R given as follows:

$$\begin{bmatrix} \{1\} & \{\frac{7}{4},\frac{5}{6},\frac{1}{3}\} & \{3,5,7\} & * & \{\frac{3}{5},\frac{4}{5}\} \\ \{3,\frac{6}{5},\frac{4}{7}\} & \{1\} & \{4,6,9\} & * & \{\frac{16}{35},\frac{24}{25},\frac{18}{10}\} \\ \{\frac{1}{7},\frac{1}{5},\frac{1}{3}\} & \{\frac{1}{9},\frac{1}{6},\frac{1}{4}\} & \{1\} & * & * \\ * & * & * & \{1\} & \{\frac{1}{4},\frac{1}{4},\frac{1}{4}\} \\ \{\frac{5}{4},\frac{5}{3}\} & \{\frac{10}{18},\frac{25}{24},\frac{35}{16}\} & * & \{4,4,4\} & \{1\} \end{bmatrix}$$

It is to note that above given preference relation is incomplete. The corresponding normalized I-HMPR $\bar{B}_R$ is given by

$$\begin{bmatrix} \{1\} & \{\frac{7}{4},\frac{5}{6},\frac{1}{3}\} & \{3,5,7\} & * & \{\frac{3}{5},\frac{4}{5},\frac{4}{5}\} \\ \{3,\frac{6}{5},\frac{4}{7}\} & \{1\} & \{4,6,9\} & * & \{\frac{16}{35},\frac{24}{25},\frac{18}{10}\} \\ \{\frac{1}{7},\frac{1}{5},\frac{1}{3}\} & \{\frac{1}{9},\frac{1}{6},\frac{1}{4}\} & \{1\} & * & * \\ * & * & * & \{1\} & \{\frac{1}{4},\frac{1}{4},\frac{1}{4}\} \\ \{\frac{5}{4},\frac{5}{4},\frac{5}{3}\} & \{\frac{10}{18},\frac{25}{24},\frac{35}{16}\} & * & \{4,4,4\} & \{1\} \end{bmatrix}$$

The missing element are calculated using equation 6.3.3

$\bar{b}_{e_{14}}^{\sigma(1)} = (\bar{b}_{e_{15}}^{\sigma(3)} \times \bar{b}_{e_{54}}^{\sigma(3)}) = 16/5$, $\bar{b}_{e_{14}}^{\sigma(2)} = (\bar{b}_{e_{15}}^{\sigma(2)} \times \bar{b}_{e_{54}}^{\sigma(2)}) = 16/5$, $\bar{b}_{e_{14}}^{\sigma(3)} = (\bar{b}_{e_{15}}^{\sigma(1)} \times \bar{b}_{e_{54}}^{\sigma(1)}) = 12/5$.

Therefore, $\bar{b}_{e_{14}} = \{\bar{b}_{e_{14}}^{\sigma(1)}, \bar{b}_{e_{14}}^{\sigma(2)}, \bar{b}_{e_{14}}^{\sigma(3)}\} = \{\frac{16}{5}, \frac{16}{5}, \frac{12}{5}\}$

$$\begin{aligned} \bar{b}_{e_{24}}^{\sigma(1)} &= \{(\bar{b}_{e_{21}}^{\sigma(3)} \times \bar{b}_{e_{14}}^{\sigma(3)}) \times (\bar{b}_{e_{25}}^{\sigma(3)} \times \bar{b}_{e_{54}}^{\sigma(3)})\}^{1/2} = 3.142 \\ \bar{b}_{e_{24}}^{\sigma(2)} &= \{(\bar{b}_{e_{21}}^{\sigma(2)} \times \bar{b}_{e_{14}}^{\sigma(2)}) \times (\bar{b}_{e_{25}}^{\sigma(2)} \times \bar{b}_{e_{54}}^{\sigma(2)})\}^{1/2} = 3.84 \\ \bar{b}_{e_{24}}^{\sigma(3)} &= \{(\bar{b}_{e_{21}}^{\sigma(1)} \times \bar{b}_{e_{14}}^{\sigma(1)}) \times (\bar{b}_{e_{25}}^{\sigma(1)} \times \bar{b}_{e_{54}}^{\sigma(1)})\}^{1/2} = 4.189 \end{aligned}$$

Therefore, $\bar{b}_{e_{24}} = \{\bar{b}_{e_{24}}^{\sigma(1)}, \bar{b}_{e_{24}}^{\sigma(2)}, \bar{b}_{e_{24}}^{\sigma(3)}\} = \{3.142, 3.84, 4.189\}$

$$\begin{aligned}
\bar{b}_{e_{34}}^{\sigma(1)} &= \{(\bar{b}_{e_{31}}^{\sigma(3)} \times \bar{b}_{e_{14}}^{\sigma(3)}) \times (\bar{b}_{e_{32}}^{\sigma(3)} \times \bar{b}_{e_{24}}^{\sigma(3)})\}^{1/2} = 0.9153 \\
\bar{b}_{e_{34}}^{\sigma(2)} &= \{(\bar{b}_{e_{31}}^{\sigma(2)} \times \bar{b}_{e_{14}}^{\sigma(2)}) \times (\bar{b}_{e_{32}}^{\sigma(2)} \times \bar{b}_{e_{24}}^{\sigma(2)})\}^{1/2} = 0.64 \\
\bar{b}_{e_{34}}^{\sigma(3)} &= \{(\bar{b}_{e_{31}}^{\sigma(1)} \times \bar{b}_{e_{14}}^{\sigma(1)}) \times (\bar{b}_{e_{32}}^{\sigma(1)} \times \bar{b}_{e_{24}}^{\sigma(1)})\}^{1/2} = 0.399
\end{aligned}$$

Therefore, $\bar{b}_{e_{34}} = \{\bar{b}_{e_{34}}^{\sigma(1)}, \bar{b}_{e_{34}}^{\sigma(2)}, \bar{b}_{e_{34}}^{\sigma(3)}\} = \{0.9153, 0.64, 0.399\}$.

$$\bar{b}_{e_{35}}^{\sigma(1)} = \{(\bar{b}_{e_{31}}^{\sigma(3)} \times \bar{b}_{e_{15}}^{\sigma(3)}) \times (\bar{b}_{e_{32}}^{\sigma(3)} \times \bar{b}_{e_{25}}^{\sigma(3)}) \times (\bar{b}_{e_{34}}^{\sigma(3)} \times \bar{b}_{e_{45}}^{\sigma(3)})\}^{1/3} = 0.2287$$

$$\bar{b}_{e_{35}}^{\sigma(2)} = \{(\bar{b}_{e_{31}}^{\sigma(2)} \times \bar{b}_{e_{15}}^{\sigma(2)}) \times (\bar{b}_{e_{32}}^{\sigma(2)} \times \bar{b}_{e_{25}}^{\sigma(2)}) \times (\bar{b}_{e_{34}}^{\sigma(2)} \times \bar{b}_{e_{45}}^{\sigma(2)})\}^{1/3} = 0.16$$

$$\bar{b}_{e_{35}}^{\sigma(3)} = \{(\bar{b}_{e_{31}}^{\sigma(1)} \times \bar{b}_{e_{15}}^{\sigma(1)}) \times (\bar{b}_{e_{32}}^{\sigma(1)} \times \bar{b}_{e_{25}}^{\sigma(1)}) \times (\bar{b}_{e_{34}}^{\sigma(1)} \times \bar{b}_{e_{45}}^{\sigma(1)})\}^{1/3} = 0.09986$$

Therefore, $\bar{b}_{e_{35}} = \{\bar{b}_{e_{35}}^{\sigma(1)}, \bar{b}_{e_{35}}^{\sigma(2)}, \bar{b}_{e_{35}}^{\sigma(3)}\} = \{0.2287, 0.16, 0.09986\}$.

It is to note that the initial value of $\bar{b}_{e_{35}}$ not lies in between $1/9-9$ ratio scale, therefore an error occurs. Thus the error is minimized using minimize the error, the above optimization model (Model 6.1) solved by MATLAB optimization toolbox. The complete normalized HMPR $\bar{B}_R$ is given below:

$$\begin{bmatrix}
\{1\} & \{\frac{7}{4}, \frac{5}{6}, \frac{1}{3}\} & \{3,5,7\} & \{\frac{16}{5}, \frac{16}{5}, \frac{12}{5}\} & \{\frac{3}{5}, \frac{4}{5}, \frac{4}{5}\} \\
\{3, \frac{6}{5}, \frac{4}{7}\} & \{1\} & \{4,6,9\} & \{3.142, 3.84, 4.189\} & \{\frac{16}{35}, \frac{24}{25}, \frac{18}{10}\} \\
\{\frac{1}{7}, \frac{1}{5}, \frac{1}{3}\} & \{\frac{1}{9}, \frac{1}{6}, \frac{1}{4}\} & \{1\} & \{0.915, 0.64, 0.399\} & \{0.229, 0.16, 0.111\} \\
\{\frac{5}{16}, \frac{5}{16}, \frac{5}{12}\} & \{\frac{1}{4.189}, \frac{1}{3.84}, \frac{1}{3.142}\} & \{\frac{1}{0.399}, \frac{1}{0.64}, \frac{1}{0.915}\} & \{1\} & \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\} \\
\{\frac{5}{4}, \frac{5}{4}, \frac{5}{3}\} & \{\frac{10}{18}, \frac{25}{24}, \frac{35}{16}\} & \{\frac{1}{0.111}, \frac{1}{0.16}, \frac{1}{0.229}\} & \{4,4,4\} & \{1\}
\end{bmatrix}$$

Similarly, using linear programming model (Model 6.4), the missing elements of normalized I-HMPR $\bar{B}_R$ are obtained i.e., $\bar{b}_{14} = \{\frac{16}{5}, \frac{16}{5}, \frac{12}{5}\}$, $\bar{b}_{24} = \{3.142, 3.84, 4.189\}$, $\bar{b}_{34} = \{0.915, 0.64, 0.399\}$, $\bar{b}_{35} = \{0.229, 0.16, 0.111\}$. The above linear programming model (Model 6.4) is solved by LINGO software. Once we have complete HMPR, our next goal is to check the consistency of the obtained relation. Therefore to validate the result, in the next subsection, we have found the satisfaction index and acceptably consistent of HMPR.

6.4.1 Checking the Satisfaction index and Acceptably consistent for HMPR

Zhu and Xu [196] gave a method to calculate the maximum satisfaction degree of HM-PRs that can deal with hesitant judgement and produce a solution of priorities of the objectives of the decision makers. To accomplish the above task, Zhu and Xu [196] developed the following model

$$\begin{aligned}
&\text{Max}\,\lambda\\
&\text{Subject to}\\
&t\lambda + \left(w_i - w_j(b_{e_{ij}}^{\sigma(1)}\ or\ \cdots\ or\ b_{e_{ij}}^{|\sigma(l_{b_{e_{ij}}})|}) \right) \quad \leq t,\\
&t\lambda - \left(w_i - w_j(b_{e_{ij}}^{\sigma(1)} or \cdots or b_{e_{ij}}^{|\sigma(l_{b_{e_{ij}}})|}) \right) \qquad \leq t,\\
&i, j \in N, i < j\\
&\textstyle\sum_{i=1}^{n} w_i = 1,\ w_i \geq 0,\ i \in N
\end{aligned}$$

where λ is a parameter which is used to measure the satisfaction degree of the decision maker to the solution and $w_i,\ w_i \geq 0,\ i \in N$ are the priority weights of the objectives and t is the deviation parameter such that if t is large enough, then $0 \leq \lambda \leq 1$; if t is small or the hesitant judgement given by the decision makers are very inconsistent and then $\lambda < 0$, though the value of priority weights remain same.

The satisfaction degree of the complete normalized HMPR (for $t = 1$) of example 6.4.1 is given by

$Max\,\lambda$;

$subject\ to$:

$\lambda + w_1 - \frac{7}{4}w_2 \leq 1;\ \lambda + w_1 - \frac{5}{6}w_2 \leq 1;\ \lambda + w_1 - \frac{1}{3}w_2 \leq 1;\ \lambda + w_1 - 3w_3 \leq 1;$

$\lambda + w_1 - 5w_3 \leq 1;\ \lambda + w_1 - 7w_3 \leq 1;\ \lambda + w_1 - \frac{16}{5}w_4 \leq 1;\ \lambda + w_1 - \frac{16}{5}w_4 \leq 1;$

$\lambda + w_1 - \frac{12}{5}w_4 \leq 1;\ \lambda + w_1 - \frac{6}{10}w_5 \leq 1;\ \lambda + w_1 - \frac{8}{10}w_5 \leq 1;\ \lambda + w_1 - \frac{8}{10}w_5 \leq 1;$

$\lambda+w_2-4w_3\leq 1;\ \lambda+w_2-6w_3\leq 1;\ \lambda+w_2-9w_3\leq 1;\ \lambda+w_2-3.142w_4\leq 1;$

$\lambda+w_2-3.84w_4\leq 1;\ \lambda+w_2-4.189w_4\leq 1;\ \lambda+w_2-\frac{16}{35}w_5\leq 1;\ \lambda+w_2-\frac{24}{25}w_5\leq 1;$

$\lambda+w_2-\frac{18}{10}w_5\leq 1;\ \lambda+w_3-0.915w_4\leq 1;\ \lambda+w_3-0.64w_4\leq 1;\ \lambda+w_3-0.399w_4\leq 1;$

$\lambda+w_3-0.229w_5\leq 1;\ \lambda+w_3-0.16w_5\leq 1;\ \lambda+w_3-0.111w_5\leq 1;\ \lambda+w_4-\frac{1}{4}w_5\leq 1;$

$\lambda+w_4-\frac{1}{4}w_5\leq 1;\ \lambda+w_4-\frac{1}{4}w_5\leq 1;\ \lambda-w_1+\frac{7}{4}w_2\leq 1;\ \lambda-w_1+\frac{5}{6}w_2\leq 1;$

$\lambda-w_1+\frac{1}{3}w_2\leq 1;\ \lambda-w_1+3w_3\leq 1;\ \lambda-w_1+5w_3\leq 1;\ \lambda-w_1+7w_3\leq 1;$

$\lambda-w_1+\frac{16}{5}w_4\leq 1;\ \lambda-w_1+\frac{16}{5}w_4\leq 1;\ \lambda-w_1+\frac{12}{5}w_4\leq 1;\ \lambda-w_1+\frac{6}{10}w_5\leq 1;$

$\lambda-w_1+\frac{8}{10}w_5\leq 1;\ \lambda-w_1+\frac{8}{10}w_5\leq 1;\ \lambda-w_2+4w_3\leq 1;\ \lambda-w_2+6w_3\leq 1;$

$\lambda-w_2+9w_3\leq 1;\ \lambda-w_2+3.142w_4\leq 1;\ \lambda-w_2+3.84w_4\leq 1;\ \lambda-w_2+4.189w_4\leq 1;$

$\lambda-w_2+\frac{16}{35}w_5\leq 1;\ \lambda-w_2+\frac{24}{25}w_5\leq 1;\ \lambda-w_2+\frac{18}{10}w_5\leq 1;\ \lambda-w_3+0.915w_4\leq 1;$

$\lambda-w_3+0.64w_4\leq 1;\ \lambda-w_3+0.399w_4\leq 1;\ \lambda-w_3+0.229w_5\leq 1;\ \lambda-w_3+0.16w_5\leq 1;$

$\lambda-w_3+0.111w_5\leq 1;\ \lambda-w_4+\frac{1}{4}w_5\leq 1;\ \lambda-w_4+\frac{1}{4}w_5\leq 1;\ \lambda-w_4+\frac{1}{4}w_5\leq 1;$

$w_1+w_2+w_3+w_4+w_5=1;\ w_1,w_2,w_3,w_4,w_5\geq 0$

The satisfaction degree of complete HMPR is 0.8024 i.e 80.24% and the weights vector are $(0.2903, 0.2787, 0.0529, 0.11365, 0.2645)$. The above linear programming problem is solved by LINGO software. For different value of deviation parameter t, has different satisfaction degree but same weights vector that is given in table 6.1.

Table 6.1: Priorities obtained for different t.

	$t=1$	$t=0.5$	$t=0.3$
λ	0.8024	0.6052	0.3420
w_1	0.2903	0.2903	0.2903
w_2	0.2787	0.2787	0.2787
w_3	0.0529	0.0529	0.0529
w_4	0.11365	0.11365	0.11365
w_5	0.2645	0.2645	0.2645

6.4.2 Acceptably consistent for HMPR

In the previous section, we have calculated satisfaction degree of complete HMPR. Zhang and Wu [76] give a method to check the acceptably consistent of HMPR by splitting HMPR to MPRs. Complete HMPR $\bar{B}_R$ obtained from example 6.4.1 is splitted with three MPRs given as $P^{(i)}$, $i=1,2,3$.

$$P^{(1)} = \begin{pmatrix} 1 & \frac{7}{4} & 3 & \frac{16}{5} & \frac{3}{5} \\ \frac{4}{7} & 1 & 4 & 3.142 & \frac{16}{35} \\ \frac{1}{3} & \frac{1}{4} & 1 & 0.915 & 0.229 \\ \frac{5}{16} & \frac{1}{3.142} & \frac{1}{0.915} & 1 & \frac{1}{4} \\ \frac{5}{3} & \frac{35}{16} & \frac{1}{0.229} & 4 & 1 \end{pmatrix}_{5\times5} \quad P^{(2)} = \begin{pmatrix} 1 & \frac{5}{6} & 5 & \frac{16}{5} & \frac{4}{5} \\ \frac{6}{5} & 1 & 6 & 3.84 & \frac{24}{25} \\ \frac{1}{5} & \frac{1}{6} & 1 & 0.64 & 0.16 \\ \frac{5}{16} & \frac{1}{3.84} & \frac{1}{0.64} & 1 & \frac{1}{4} \\ \frac{5}{4} & \frac{25}{24} & \frac{1}{0.16} & 4 & 1 \end{pmatrix}_{5\times5}$$

$$P^{(3)} = \begin{pmatrix} 1 & \frac{1}{3} & 7 & \frac{12}{5} & \frac{4}{5} \\ 3 & 1 & 9 & 4.189 & \frac{18}{10} \\ \frac{1}{7} & \frac{1}{9} & 1 & 0.399 & 0.111 \\ \frac{5}{12} & \frac{1}{4.189} & \frac{1}{0.399} & 1 & \frac{1}{4} \\ \frac{5}{4} & \frac{10}{18} & \frac{1}{0.111} & 4 & 1 \end{pmatrix}_{5\times5}$$

Consistency ratio(CR) of MPRs $P^{(1)}$, $P^{(2)}$, $P^{(3)}$ is 0.0182, 0, 0.0169 respectively which are less than 0.1. Therefore the complete HMPR $\bar{B}_R$ is acceptably consistent.

6.5 Conclusion

In this chapter, we have defined a new multiplicative transitive property of HMPR. We have developed an algorithm to construct a complete HMPRs from an incomplete one. It may be the case that the initial values obtained from algorithm 6.3.1 not lies in the ratio scale $1/9-9$. We have developed an optimization model to minimize the error. Also, we have developed a methodology which is based on linear programming model to complete the incomplete HMPR. Satisfaction degree and acceptably consistent of complete HMPR is also checked. We have presented an example to illustrates the above method.

Summary and future scope of the work

In the real world multi-criteria decision-making problem has been broadly perceived that most decisions take place in an environment in which the goals and constraints, are not known precisely, because of their complexity, and hence, the problem cannot be accurately characterized or correctly represented in a crisp value [2]. Zadeh [3] suggested fuzzy set theory to deal with the kind of qualitative, uncertain data or even not well-organized choice issues, as a modeling tool for complex systems that can be controlled by humans but are hard to define precisely. In this thesis, we have discussed the different type of fuzzy relation, a clustering method based on different type of fuzzy relation.

We have revised the $\bar{\alpha}$-cut of interval-valued fuzzy relation given by Guh et al. [52]. Two algorithms are developed to chose the $\bar{\alpha}$-cut for interval-valued fuzzy relation and (α,β)-cut for intuitionistic fuzzy relation. We have extends the work of Guh et al. [52] to set out the system for constructing hierarchical clustering for interval-valued intuitionistic fuzzy relation. Determining the criteria weights in an MCDM problem, with decision maker providing entries of a criteria-alternative matrix as IVIF value, is tended to demonstrate the utilization of the proposed clustering scheme. A detailed procedure is given a strong outline to calculate the global weights of the criteria. MATLAB codes are developed for each algorithm presented in this work (in chapter 2) and the same are implemented on the large number of matrices representation of IVFR, IFR, IVIFRs. MATLAB codes are given in Appendix 2.

It would be interesting to see if the hierarchical clustering approach can be extended to more than one IVIFR and hence can be applied to MCGDM problems for computing the importance of many experts along with the criteria weight vector.

The applicability of decision-making procedures are constantly increasing in various domains of assessment, selection and prioritization purposes, that is, making preference decisions about a set of different choices. Furthermore, it is also apparent that in many cases, distinct alternatives cannot be compared in view of their desirability in decision situations employing a single expert or one criterion. Obviously, in widely existing decision-making situations, techniques to combine opinions with distinct points of view about alternatives have been established. These techniques involve pairwise comparisons as the methods are linked to some degree of credibility of preference of one alternative over another. In literature, a number of research work has been done on the use of preference relations in several fields. In decision-making problem, the facts may confirm that the decision maker might not have an excellent comprehension on a specific inquiry, and sometimes, he/she can not make a direct comparison between two options or criteria. In that cases, the decision maker might need to communicate their information with incomplete data. It is to note that, a DM require $\frac{n(n-1)}{2}$ judgments for comparison of a complete preference relation. However, due to lack of time and busy schedule of the decision maker, he/she may give less than results the incomplete preference relation.

Among the existing preference relation, IMPR is one of the most useful tool to express the decision makers preference data which is the generalization of MPR. In this work, we have presented the transitivity property of IMPR. In light of this, we have introduced two methodologies for completing incomplete IMPRs. In the first approach, the missing element can be ascertained by utilizing the new transitivity property, and an optimization model has been produced to adjust the initial values. Also, goal programming model is developed in light of new transitivity property to calculate the missing values. In the second approaches, we have developed a method to check the consistency of IMPR by using the cycles of various length in a directed graph and the same procedure apply for incomplete IMPRs also.

Also, we have generalised the IVI-MPR and proposed a two-advance technique to populate the missing entries in an incomplete IVI-MPR. Although certain research papers are available in the literature on MPRs in the intuitionistic framework yet the proposed integration of IVI-MNs in MPR, especially in the incomplete preference relations, can be considered as a novel contribution. The primary goal of the proposed procedure for finding the missing element lies in its estimating step. We can always find an elementary connecting path in the given preference relation matrix such that all other elements along this path are completely known. Sometimes, this assumption may not hold. Another concern in the present study is that the weights or different levels of experts are supposed to be precisely known. A suitable choice of a weight vector ensures acceptably consistent group aggregated preference relation matrix. These grey patches of this study can be quickly filled in near future research taking inspiration from the approaches of very recent works of Wan, Xu, Wang, and their co-researchers (2016-2017). One can also investigate Choquet integral approach for aggregating information in preference relation matrices than the presently used traditional geometric mean operator. Moreover, one can attempt to define group consensus in IVI framework and design a procedure to accomplish the same.

Also in this thesis, we have worked on incomplete HFPR and incomplete HMPR. We have build up different methods that help to find the missing elements from the incomplete one. Comparison of our methods with comparative work are also discussed. The present work can be extended to intuitionistic hesitant multiplicative preference relation, hesitant linguistic preference relation their application in decision-making problem.

APPENDIX

APPENDIX-1

Converting crisp value to an IVIF value (for detail reasoning, plz. refer to [158])

For each column i of the criteria correlation matrix, find the minimum and the maximum values among all values in $(0.5, 1]$, yielding an interval $[\xi_i^l,\ \xi_i^u]$, called the satisfactory interval. Next, for the same i, find the minimum and the maximum values among all values in $[0, 0.5)$ to generate the dissatisfactory interval $[\delta_i^l,\ \delta_i^u]$. If no entry in i-th column of the criteria correlation matrix lies in $(0.5, 1]$ or $[0, 0.5)$, then take the satisfactory interval or the dissatisfactory interval equals $[0.5, 0.5]$.

For instance for the first column corresponding to criterion C_1 in Table 2.2 of chapter 2, the satisfactory interval is $[0.5238, 1]$ and the dissatisfactory interval is $[0.3414, 0.4916]$.

Use the linear transformation $y = 2x - 1$ in the interval $[\xi_i^l,\ \xi_i^u]$ to transform it to interval $[\rho_i^l,\ \rho_i^u]$. Similarly the interval $[\delta_i^l, \delta_i^u]$ be transformed to $[\upsilon_i^l, \upsilon_i^u]$, using the linear transformation $y = 1 - 2x$.

Let $\tau_i = \rho_i^l + \rho_i^u + \upsilon_i^l + \upsilon_i^u$. Apply the transformation,

$$\underline{\mu}_i = \frac{\rho_i^l}{\tau_i},\ \ \overline{\mu}_i = \frac{\rho_i^u}{\tau_i},\ \ \underline{\nu}_i = \frac{\upsilon_i^l}{\tau_i},\ \ \overline{\nu}_i = \frac{\upsilon^u}{\tau_i},$$

to get an IVIF number $([\underline{\mu}_i,\ \overline{\nu}_i],\ [\underline{\nu}_i,\ \overline{\nu}_i]),\ \ i = 1, \ldots, m.$

For instance, from the first column in the criteria correlation matrix in Table 2.2 of chapter 2, $\underline{\mu}_1 = 0.0345,\ \overline{\mu}_1 = 0.7238,\ \underline{\nu}_1 = 0.0121,\ \overline{\nu}_1 = 0.2295.$

APPENDIX-2

1. **MATLAB code for finding $\tilde{\alpha}$-cut of interval valued fuzzy relation(IVFR)**

```
close all;
clear all;
clc;
%making initial matrix for decision maker %
```

```
x = input('Enter the value as to how do you want to take the initial
    data from decision makers : \n1)Through excel sheet \n2)Create
    data through matlab\n ');
if(x==2)
 m = input('Enter number of criterias you want : ');
   membershipmat = zeros(m,m,2);
    for i = 1:m
      for j = 1:m
        if(i==j)
          membershipmat(i,j,1) = 1;
         elseif(i<j)
           membershipmat(i,j,1) = 0+[1*rand];
          else
            membershipmat(i,j,1) = membershipmat(j,i,1);
         end
     end
 end
for i = 1:m
    for j=1:m
        if(i==j)
          membershipmat(i,j,2) = 1;
           elseif(i<j)
            membershipmat(i,j,2) = membershipmat(i,j,1)+
                                  [(1-membershipmat(i,j,1))*rand];
           else
                 membershipmat(i,j,2) = membershipmat(j,i,2);
         end
      end
 end
  a = input('Enter the name of excel file in which you need
      to save the data : ','s');
    xlswrite(a,membershipmat(:,:,1),1);
    xlswrite(a,membershipmat(:,:,2),2);
    else
    a = input('Enter the name of excel file which you need to
```

```
        import : ','s');
    membershipmat(:,:,1) = xlsread(a,'Sheet1');
    membershipmat(:,:,2) = xlsread(a,'Sheet2');
    siz = size(membershipmat(:,:,1));
    m = siz(1,1);
    n = siz(1,2);
end
numberstep = 0;
for i = 1:m
  for j = 1:m
    for k=1:m
       a1(k) = min(membershipmat(i,k,1),membershipmat(k,j,1));
       a2(k) = min(membershipmat(i,k,2),membershipmat(k,j,2));
   end
   midmat1(i,j,1) = max(a1);
   midmat1(i,j,2) = max(a2);
   a1 = [];
   a2 = [];
  end
end
finalmat = midmat1;
midmat1 = membershipmat;
while(isequal(midmat1,finalmat)==0)
  for i = 1:m
     for j = 1:m
        for k=1:m
           a1(k) = min(membershipmat(i,k,1),finalmat(k,j,1));
           a2(k) = min(membershipmat(i,k,2),finalmat(k,j,2));
       end
       midmat1(i,j,1) = max(a1);
       midmat1(i,j,2) = max(a2);
       a1 = [];
       a2 = [];
   end
end
```

```
  midmat2 = finalmat;
  finalmat = midmat1;
  midmat1 = midmat2;
  numberstep = numberstep+1;
end
numberstep
finalmat                    % equivalence matrix

%-------- alpha cut --------%
finalmatt = finalmat;
resmatt = zeros(m);
k=1;
sresmatt=0;
while(sresmatt ~= m^2)
  atemp = find(finalmatt(:,:,1) == max(max(finalmatt(:,:,1))));
  [co1,co2] = ind2sub(m,atemp);
  [s1,s2]=size(co1);
  alphal(k) = max(max(finalmatt(:,:,1)));
  resmatt(atemp) = 1;
  resmat(:,:,k) = resmatt;
  for l = 1:s1
    btemp(l) = finalmatt(co2(l),co1(l),2);
    finalmatt(co2(l),co1(l),1) = nan;
    finalmatt(co2(l),co1(l),2) = nan;
 end
  alphau(k) = min(min(btemp));
  variable = 1;
  atemp = [];
  btemp = [];
  sresmatt=0;
  for i = 1:m
    for j=1:m
       sresmatt = sresmatt + resmatt(i,j);
    end
 end
```

```
   k=k+1;
end
sno = [1:(k-1)];
for i = 1:k-1
   tempresmat(:,:,k-i) = resmat(:,:,i);
end
resmat = tempresmat;
resmat(:,:,1)
colum = 1;
i = 1;
while(i ~= k-1)
   xtemp = find(tempresmat(:,colum,i) == 1);
   ytemp = find(tempresmat(:,colum,i+1) == 0);
   ftemp = intersect(xtemp,ytemp);
   if(isempty(ftemp))
       colum = colum+1;
       if(colum == k-1)   %new part included
          colum = 1;
       end
   else
       resmat(:,:,i+1)
       position = ftemp
       i = i+1;
   end
   xtemp = [];
   ytemp = [];
   ftemp = [];
end %

alphabeta = [flipud(alphal') ,flipud(alphau')]
```

2. **MATLAB code for finding (α,β)-cut of intuitionistic fuzzy relation(IFR)**

```
close all;
clear all;
```

```
clc;
x = input('Enter the value as to how do you want to
    take the initial data from decision makers : \n1)
    Through excel sheet \n2)Create data through
    matlab\n ');
if(x==2)
  m = input('Enter number of criterias you want : ');
  membershipmat = zeros(m,m,2);
   for i = 1:m
     for j = 1:m
        if(i==j)
           membershipmat(i,j,1) = 1;
         elseif(i<j)
           membershipmat(i,j,1) = 0+[1*rand];
         else
           membershipmat(i,j,1) = membershipmat(j,i,1);
       end
    end
 end
  for i = 1:m
     for j=1:m
        if(i==j)
         membershipmat(i,j,2) = 0;
         elseif(i<j)
           membershipmat(i,j,2) = [(1-membershipmat(i,j,1))*rand];
         else
           membershipmat(i,j,2) = membershipmat(j,i,2);
       end
    end
end
  a = input('Enter the name of excel file in which you
      need to save the data : ','s');
    xlswrite(a,membershipmat(:,:,1),1);
    xlswrite(a,membershipmat(:,:,2),2);
    else
```

```
    a = input('Enter the name of excel file which you need to
        import : ','s');
    membershipmat(:,:,1) = xlsread(a,'Sheet1');
    membershipmat(:,:,2) = xlsread(a,'Sheet2');
    siz = size(membershipmat(:,:,1));
    m = siz(1,1);
    n = siz(1,2);
end
numberstep = 0;
for i = 1:m
 for j = 1:m
  for k=1:m
     a1(k) = min(membershipmat(i,k,1),membershipmat(k,j,1));
     a2(k) = max(membershipmat(i,k,2),membershipmat(k,j,2));
  end
    midmat1(i,j,1) = max(a1);
    midmat1(i,j,2) = min(a2);
    a1 = [];
    a2 = [];
    end
end
finalmat = midmat1;
midmat1 = membershipmat;
while(isequal(midmat1,finalmat)==0)
  for i = 1:m
    for j = 1:m
      for k=1:m
        a1(k) = min(membershipmat(i,k,1),finalmat(k,j,1));
        a2(k) = max(membershipmat(i,k,2),finalmat(k,j,2));
      end
        midmat1(i,j,1) = max(a1);
        midmat1(i,j,2) = min(a2);
        a1 = [];
        a2 = [];
   end
```

```
end
    midmat2 = finalmat;
    finalmat = midmat1;
    midmat1 = midmat2;
    numberstep = numberstep+1;
end
numberstep
finalmat
%finalmat = midmat1;
%finalmat                    % equivalence matrix
%-------- alpha cut --------%
finalmatt = finalmat;
resmatt = zeros(m);
k=1;
sresmatt=0;
while(sresmatt ~= m^2)
    atemp = find(finalmatt(:,:,1) == max(max(finalmatt(:,:,1))));
    [co1,co2] = ind2sub(m,atemp);
    [s1,s2]=size(co1);
    alphal(k) = max(max(finalmatt(:,:,1)));
    resmatt(atemp) = 1;
    resmat(:,:,k) = resmatt;
    for l = 1:s1
        btemp(l) = finalmatt(co2(l),co1(l),2);
        finalmatt(co2(l),co1(l),1) = nan;
        finalmatt(co2(l),co1(l),2) = nan;
    end
    alphau(k) = min(min(btemp));
    variable = 1;
    atemp = [];
    btemp = [];
    sresmatt=0;
    for i = 1:m
        for j=1:m
            sresmatt = sresmatt + resmatt(i,j);
```

```
        end
    end
    k=k+1;
end
sno = [1:(k-1)];
for i = 1:k-1
   drespmat(:,:,k-i) = resmat(:,:,i);
end
resmat = tempresmat;
resmat(:,:,1)
colum = 1;
i = 1;
while(i ~= k-1)
   xtemp = find(tempresmat(:,colum,i) == 1);
   ytemp = find(tempresmat(:,colum,i+1) == 0);
   ftemp = intersect(xtemp,ytemp);
   if(isempty(ftemp))
       colum = colum+1;
       if(colum == k-1)   %new part included
          colum = 1;
       end
   else
       resmat(:,:,i+1)
       position = ftemp
       i = i+1;
   end
   xtemp = [];
   ytemp = [];
   ftemp = [];
end
alphabeta = [(flipud(alphal')),(flipud(alphau'))]
```

3. **MATLAB code for finding $(\tilde{\alpha}, \tilde{\beta})$-cut of interval-valued intuitionistic fuzzy relation (IVIFR) and calculate the local and global weight**

```
close all;
```

```
clear all;
clc;
x = input('Enter the value as to how do you want to take the initial
     data from decision makers : \n1)Through excel sheet \n2)Create
     data through matlab\n ');
if(x==2)
    m = input('Enter number of criterias you want : ');
    membershipmat = zeros(m,m,4);
    for i = 1:m
        for j = 1:m
            if(i==j)
               membershipmat(i,j,1) = 1;
            elseif(i<j)
        membershipmat(i,j,1) = 0+[1*rand];
            else
                membershipmat(i,j,1) = membershipmat(j,i,1);
          end
        end
    end
    for i = 1:m
        for j=1:m
            if(i==j)
               membershipmat(i,j,2) = 1;
            elseif(i<j)
            membershipmat(i,j,2) = membershipmat(i,j,1)+
                                 [(1-membershipmat(i,j,1))*rand];
            else
                membershipmat(i,j,2) = membershipmat(j,i,2);
            end
        end
    end
for i = 1:m
  for j = 1:m
     if(i==j)
        membershipmat(i,j,4) = 0;
```

```
          elseif(i<j)
           membershipmat(i,j,4) = 0+[(1-membershipmat(i,j,2))*rand];
            else
           membershipmat(i,j,4) = membershipmat(j,i,4);
       end
   end
end
for i = 1:m
   for j = 1:m
     if(i==j)
       membershipmat(i,j,3) = 0;
        elseif(i<j)
        membershipmat(i,j,3) = 0+[(membershipmat(i,j,4)-0)*rand];
        else
        membershipmat(i,j,3) = membershipmat(j,i,3);
      end
   end
end
 a = input('Enter the name of excel file in which you
         need to save the data : ','s');
     xlswrite(a,membershipmat(:,:,1),1);
     xlswrite(a,membershipmat(:,:,2),2);
     xlswrite(a,membershipmat(:,:,3),3);
     xlswrite(a,membershipmat(:,:,4),4);
     else
     a = input('Enter the name of excel file which you need to
         import : ','s');
     membershipmat(:,:,1) = xlsread(a,'Sheet1');
     membershipmat(:,:,2) = xlsread(a,'Sheet2');
     membershipmat(:,:,3) = xlsread(a,'Sheet3');
     membershipmat(:,:,4) = xlsread(a,'Sheet4');
     siz = size(membershipmat(:,:,1));
     m = siz(1,1);
     m = siz(1,2);
     end
```

```
numberstep = 0;
for i = 1:m
    for j = 1:m
        for k=1:m
            a1(k) = min(membershipmat(i,k,1),membershipmat(k,j,1));
            a2(k) = min(membershipmat(i,k,2),membershipmat(k,j,2));
            a3(k) = max(membershipmat(i,k,3),membershipmat(k,j,3));
            a4(k) = max(membershipmat(i,k,4),membershipmat(k,j,4));
             end
        midmat1(i,j,1) = max(a1);
        midmat1(i,j,2) = max(a2);
        midmat1(i,j,3) = min(a3);
        midmat1(i,j,4) = min(a4);
        a1 = [];
        a2 = [];
        a3 = [];
        a4 = [];
    end
end
finalmat = midmat1;
midmat1 = membershipmat;
while(isequal(midmat1,finalmat)==0)
    for i = 1:m
        for j = 1:m
            for k=1:m
                a1(k) = min(membershipmat(i,k,1),finalmat(k,j,1));
                a2(k) = min(membershipmat(i,k,2),finalmat(k,j,2));
                a3(k) = max(membershipmat(i,k,3),finalmat(k,j,3));
                a4(k) = max(membershipmat(i,k,4),finalmat(k,j,4));
            end
            midmat1(i,j,1) = max(a1);
            midmat1(i,j,2) = max(a2);
            midmat1(i,j,3) = min(a3);
            midmat1(i,j,4) = min(a4);
            a1 = [];
```

```
            a2 = [];
            a3 = [];
            a4 = [];
        end
    end
    midmat2 = finalmat;
    finalmat = midmat1;
    midmat1 = midmat2;
    numberstep = numberstep+1;
end
numberstep
finalmat
%finalmat = midmat1;
%finalmat                    % equivalence matrix
%-------- alpha cut --------%
finalmatt = finalmat;
resmatt = zeros(m);
k=1;
sresmatt=0;
while(sresmatt ~= m^2)
    atemp = find(finalmatt(:,:,1) == max(max(finalmatt(:,:,1))));
    [co1,co2] = ind2sub(m,atemp);
    [s1,s2]=size(co1);
    alphal(k) = max(max(finalmatt(:,:,1)));
    resmatt(atemp) = 1;
    resmat(:,:,k) = resmatt;
    for l = 1:s1
        btemp(l) = finalmatt(co2(l),co1(l),2);
        ctemp(l) = finalmatt(co2(l),co1(l),3);
        dtemp(l) = finalmatt(co2(l),co1(l),4);
        finalmatt(co2(l),co1(l),1) = nan;
        finalmatt(co2(l),co1(l),2) = nan;
        finalmatt(co2(l),co1(l),3) = nan;
        finalmatt(co2(l),co1(l),4) = nan;
    end
```

```
    alphau(k) = min(min(btemp));
    betal(k) = max(max(ctemp));
    betau(k) = max(max(dtemp));
    variable = 1;
    atemp = [];
    btemp = [];
    ctemp = [];
    dtemp = [];
    sresmatt=0;
    for i = 1:m
        for j=1:m
            sresmatt = sresmatt + resmatt(i,j);
        end
    end
    k=k+1;
end
resmat
alphabeta = [flipud(alphal') ,flipud(alphau'),
            flipud(betal'),flipud(betau')]
% finding pivotal element for equivalence matrix
for i = 1:m
    sum1(i) = 0;
    sum2(i) = 0;
    sum3(i) = 0;
    sum4(i) = 0;
    for j = 1:m
        sum1(i) =  sum1(i) + finalmat(i,j,1);
        sum2(i) =  sum2(i) + finalmat(i,j,2);
        sum3(i) =  sum3(i) + finalmat(i,j,3);
        sum4(i) =  sum4(i) + finalmat(i,j,4);
    end
    pel(i) = [sum1(i)+sum2(i)-sum3(i)-sum4(i)]/2;
end
[val,loc] = min(pel);
pivotcri = loc
```

```
%finding local weights
k =  input('Enter number of subets that are formed : ');
localweight = nan(k,k);
for m = 1:k
    criterianum = input('Enter the criteria number which
                  we need to take in subset matrix(if
                  entering first time start with largest
                  set) : ');
 pivotcriloc(m) = find(criterianum == pivotcri);
 subsetmat =  finalmat(criterianum,criterianum,:)
 siz2 = size(subsetmat);
 entropsum = 0;
 for i = 1:siz2(1,1)
  entropm(i) = 0;
  for j = 1:siz2(1,1)
    entropm(i) = entropm(i) + [2-abs(subsetmat(i,j,1)
    -subsetmat(i,j,3))-abs(subsetmat(i,j,2)-subsetmat(i,j,4))
    +(1-subsetmat(i,j,1)-subsetmat(i,j,3))+(1-subsetmat(i,j,2)
    -subsetmat(i,j,4))]/[2+abs(subsetmat(i,j,1)-subsetmat(i,j,3))
    +abs(subsetmat(i,j,2)-subsetmat(i,j,4))+(1-subsetmat(i,j,1)
    -subsetmat(i,j,3))+(1-subsetmat(i,j,2)-subsetmat(i,j,4))];
     end
       entropym(i) =  entropm(i)/siz2(1,1);
       entropsum = entropsum + entropym(i);
    end
for i = 1:siz2(1,1)
 localweightt(i) = [1 - entropym(i)]/[siz2(1,1)-entropsum];
       localweight(i,m) = localweightt(i);
    end
end
zeroloc = find(localweight == 0);
localweight(zeroloc) = nan;
localweight
[asize1, asize2] = size(localweight);
%finding global weight
```

```
for j = 1:asize2
 for i = 1:asize1
 nlocalweight(i,j) = localweight(i,j)/localweight(pivotcriloc(j),j);
 end
end
snlocalweight = 0;
for j = 1:asize2
    for i =  1:asize1
        if(isnan(nlocalweight(i,j)) == 0)
            snlocalweight = snlocalweight + nlocalweight(i,j);
        end
    end
end
for j = 1:asize2
  for i =  1:asize1
    if(nlocalweight(i,j) ~= nan)
     globalweight(i,j) = nlocalweight(i,j)/[snlocalweight-k+1];
      end
  end
end
globalweight
```

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List of Publications

1. **Mamata Sahu**, Anjana Gupta and Aparna Mehra; *Hierarchical clustering of interval-valued intuitionistic fuzzy relations and its application to elicit criteria weights in MCDM problems*, Opsearch, springer, **54**, 388–416, (2017).

2. **Mamata Sahu**, Anjana Gupta and Aparna Mehra; *Acceptably consistent incomplete interval-valued intuitionistic multiplicative preference relations*, Soft Computing, Springer, **22**, 7463–7477 (2018).

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4. **Mamata Sahu** and Anjana Gupta; *New transitivity property of intuitionistic fuzzy multiplicative preference relation and its application in missing value estimation*, Annals of Fuzzy Mathematics and Informatics **16**, (1) 71–86 (2018).

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