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Faster rate of convergence on Srivastava–Gupta operators

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ABSTRACT

In this paper, we consider a modification of Srivastava–Gupta operators, which is a general sequence of summation-integral operators. We are able to achieve faster convergence for our modified operators over the well known Srivastava–Gupta operators. Our results include some approximation properties, which include rate of convergence and Voronovskaya kind results. In the last section of this paper we give Stancu variant of these operators.

1. Introduction

It is obvious that all classical operators (Bernstein, Szász and Baskakov) reproduce constant as well as linear function possess these properties i.e. $L_n(e_i;x) = e_i(x)$ where $e_i(x) = x^i (i = 0, 1)$.

In the year 2003, King [12] introduced a method so as the Bernstein operators reproduce quadratic function so that using this method in the Bernstein operators would not reproduced the linear function. In the same year, Srivastava and Gupta [15] introduced a general sequence of linear positive operators $G_{n,c}$ and investigated as well as estimated the rate of convergence of the general sequence of operators $G_{n,c}$ by means of the decomposition technique for functions of bounded variation. Later Ispir and Yuksel [11] introduced the Bézier variant of these operators and estimated the rate of convergence for function of bounded variation. It is observed that the Srivastava–Gupta operators reproduce only constant function. So here we modify the Srivastava–Gupta operators so that they may be capable to reproduce constant as well as linear function.

For $f \in C_\gamma[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leqslant M(1+t)^\gamma \text{ for some } M > 0,\ \gamma > 0\}$, Srivastava and Gupta defined a sequence of linear positive operators $G_{n,c}$ as:

$$G_{n,c}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x;c) \int_0^\infty p_{n+c,k-1}(t;c)f(t)dt + p_{n,0}(x;c)f(0), \tag{1.1}$$

where

$$p_{n,k}(x;c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x)$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0 \\ (1+cx)^{-n/c}, & c = 1,2,3,\ldots \end{cases}$$

Here $\{\phi_{n,c}(x)\}_{n=1}^{\infty}$ is a sequence of functions, defined on $[0,b],\ b > 0$, and satisfy the following properties: for each $n \in \mathbb{N}$ and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

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(i) $\phi_{n,c} = C^{\infty}([a,b])(b > a \geqslant 0)$;
(ii) $\phi_{n,c}(0) = 1$;
(iii) $\phi_{n,c}(x)$ is completely monotone so that $(-1)^k \phi_{n,c}^{(k)} \geqslant 0$ $(0 \leqslant x \leqslant b)$;
(iv) there exists an integer c such that

$$\phi_{n,c}^{(k+1)}(x) = -n\phi_{n+c,c}^{(k)}(x), \quad n > \max\{0,-c\};\ x \in [a,b].$$

It is easy to see that, when $c = 0$ with $f \in C[0,\infty)$, the operators $G_{n,c}$ reduce to the Phillips operators [14],

$$G_{n,0}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-1}(t)f(t)dt + e^{-nx}f(0), \quad 0 \leqslant x < \infty,$$

where $p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$.

When $c = 1$ with $f \in C[0,\infty)$, the operators $G_{n,c}$ reduce to summation-integral type operators, which were studied by Gupta et al. [9],

$$G_{n,1}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n+1,k-1}(t)f(t)dt + (1+x)^{-n}f(0), \quad 0 \leqslant x < \infty,$$

where $p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1-x)^{n+k}}$.

A slightly modified form of $G_{n,1}$ represented another summation-integral type operators, which were defied by Agrawal and Thamar [1],

$$G_{n,1}^{*}(f;x) = (n-1)\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-1}(t)f(t)dt + (1+x)^{-n}f(0), \quad 0 \leqslant x < \infty.$$

When $c = -1$ with a view to approximating some Lebesgue-integrable function f on the closed interval $[0,1]$ the operators $G_{n,c}$ reduce to Gupta and Maheshwari [10],

$$G_{n,-1}(f;x) = n\sum_{k=1}^{n} p_{n,k}(x)\int_0^{1} p_{n-1,k-1}(t)f(t)dt + (1-x)^{-n}f(0), \quad 0 \leqslant x \leqslant 1,$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$.

The better error estimation for Meyer-König and Zeller operators was first introduced by Ozarslan and Duman [13] and for Szász-Mirakyan operators by Duman and Ozarslan [5]. After these many researchers have given faster convergence for different well-known operators like Modified Baskakov operators [2,3], Meyer-König and Zeller operators [4], Szász–Mirakian-Beta operators [6], Szász–Mirakjan–Kantorovich operators [7], Szász–Mirakyan operators [8]. The goal of this article is to construct and investigate a variant of Srivastava–Gupta operators [15], which preserve the functions e_0 and e_2. We study approximation properties viz. Voronovskaya-kind asymptotic formula and better rate of convergence of this modified operators (3.1) (see Section 3).

Throughout the paper, we consider $0 \leqslant x \leqslant 1$ and $f \in [0,1]$ for $c = -1$ and for other cases $c = 0, 1$ we consider $0 \leqslant x < \infty$ and $f \in C[0,\infty)$ for the operators $G_{n,c}$.

2. Auxiliary results

Now the following lemmas follow from [15], for the operators $G_{n,c}$ mentioned by (1.1).

Lemma 2.1 [15]. *Let $e_i(x) = x^i,\ i = 0,1,2$. Then, for each $x \geqslant 0$ and $n > 2c$, we have*

(i) $G_{n,c}(e_0;x) = 1$,
(ii) $G_{n,c}(e_1;x) = \frac{nx}{n-c}$,
(iii) $G_{n,c}(e_2;x) = \frac{nx\{(n+c)x+2\}}{(n-c)(n-2c)}$.

Lemma 2.2 [15]. *For each $x \geqslant 0,\ n > 2c$ and $\phi_x(t) = t - x$, we have*

(i) $G_{n,c}(\varphi_x;x) = \frac{cx}{n-c}$,
(ii) $G_{n,c}(\varphi_x^2;x) = \frac{x(1+cx)(2n-c)+cx(1+3cx)}{(n-c)(n-2c)}$,
(iii) $G_{n,c}(\varphi_x^m;x) = O\left(n^{-[(m+1)/2]}\right)$.

3. Construction of the operators

In this section, we modify the operators (1.1) such that the linear functions are preserved. First we transform the operators defined at (1.1) in order to preserve the function e_1. By defining the functions

$$r_n(x) = \frac{(n-c)x}{n}, \quad x \geqslant 0,$$

we replace x by $r_n(x)$ in (1.1). So, let $\{r_n(x)\},\ [0,\infty)$ into itself, be a sequence of continuous functions for any $n \in \mathbb{N}$. Then we have the following modification of the operators $G_{n,c}$:

$$\hat{G}_{n,c}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(r_n(x);c)\int_0^{\infty} p_{n+c,k-1}(t;c)f(t)dt + p_{n,0}(r_n(x);c)f(0), \tag{3.1}$$

for $f \in C_\gamma[0,\infty),\ \gamma > 0$ and $x \geqslant 0$. Hence, in the special case $r_n(x) = x,\ n = 1,2,\ldots$, reduce to original operators (1.1).

Alternatively the operators (3.1) may be written as

$$\hat{G}_{n,c}(f;x) = \int_0^{\infty} W_n(x,t;c)f(t)dt,$$

where the kernel $W_n(x,t;c)$ is defined by

$$W_n(x,t;c) = n\sum_{k=1}^{\infty} p_{n,k}(r_n(x);c)p_{n+c,k-1}(t;c) + p_{n,0}(\mathrm{r}_n\mathrm{x});c)f(0).$$

It is clear that $\hat{G}_{n,c}$ are positive and linear operators. **For the case** $c \geq 0, x \in [0,\infty)$ **and for** $c = -1, x \in [0,\frac{1}{2}]$ **otherwise the function is assumed to be zero.**

By simple computation we have the following Lemmas for moments.

Lemma 3.1. *The operators defined at (3.1) verify the following identities*

(i) $\hat{G}_{n,c}(e_0;x) = 1$,
(ii) $\hat{G}_{n,c}(e_1;x) = x$,
(iii) $\hat{G}_{n,c}(e_2;x) = \frac{(n^2-c^2)x^2+2nx}{n(n-2c)}$.

Lemma 3.2. *For each* $x \geqslant 0,\ n > 2c$ *and* $\phi_x(t) = t - x$, *we have*

(i) $\hat{G}_{n,c}(\varphi_x;x) = 0$,
(ii) $\hat{G}_{n,c}(\varphi_x^2;x) = \frac{(2n-c)cx^2+2nx}{n(n-2c)}$,
(iii) $\hat{G}_{n,c}(\varphi_x^m;x) = O\left(n^{-[(m+1)/2]}\right)$.

Lemma 3.1 show that our operators $\hat{G}_{n,c}$ preserve the linear functions, that is, for $h(t) = at + b$ any real constants, we get $\hat{G}_{n,c}(h;x) = h(x)$.

4. Error estimations

Let $f \in C_B[0,\infty)$ and $x \geqslant 0$, then the modulus of continuity of f denoted by $\omega(f,\delta)$ is defined to be

$$\omega(f,\delta) = \sup_{x-\delta\leqslant t\leqslant x+\delta;\, t\in[0,\infty)} |f(t) - f(x)|.$$

Then we have the following results.

Theorem 4.1. *For every* $f \in C_B[0,\infty),\ x \in [0,\infty)$ *and for* $n \in \mathbb{N}$, *we have*

$$\left|\hat{G}_{n,c}(f;x) - f(x)\right| \leqslant C\omega(f,\delta_{n,x}), \quad n > 2c, \tag{4.1}$$

where

$$\delta_{n,x} = \sqrt{\frac{(2n-c)cx^2 + 2nx}{n(n-2c)}}.$$

Proof. Let $f \in C_B[0,\infty)$ and $x \in [0,\infty)$. Using linearity and monotonicity of $\hat{G}_{n,c}$, we easily obtain, for every $\delta > 0$ and $n \in \mathbb{N}$, that

$$\left|\hat{G}_{n,c}(f;x)-f(x)\right| \leqslant \omega(f,\delta)\left\{1+\frac{1}{\delta}\sqrt{\hat{G}_{n,c}(\varphi_x^2;x)}\right\}.$$

Using Lemma 3.2(ii) and choosing $\delta=\delta_{n,x}$ the proof is completed. □

Remark 4.2. For the original Srivastava–Gupta operator $G_{n,c}$, we may write that, for every $f\in C_B[0,\infty),\ x\in[0,\infty)$ and $n\in\mathbb{N}$,

$$\left|G_{n,c}(f;x)-f(x)\right| \leqslant 2\omega(f,v_{n,x}), \tag{4.2}$$

where

$$v_{n,x}=\sqrt{\frac{x(1+cx)(2n-c)+cx(1+3cx)}{(n-c)(n-2c)}},\quad n>2c,$$

and $\omega(f,v_{n,x})$ is the modulus of continuity of f. We claim that the error estimation in Theorem 4.1 is better than that of (4.2) for $f\in C_B[0,\infty)$ and $x\in[0,\infty)$. Indeed, in order to get this better estimation we must show that $\delta_{n,x}\leqslant v_{n,x}$. We can obtain that

$$\begin{aligned}\delta_{n,x}\leqslant v_{n,x} &\iff \frac{2nx+(2n-c)cx^2}{n(n-2c)}\leqslant\frac{x(1+cx)(2n-c)+cx(1+3cx)}{(n-c)(n-2c)}\\ &\iff \frac{2nx+(2n-c)cx^2}{n}\leqslant\frac{x(1+cx)(2n-c)+cx(1+3cx)}{(n-2c)}\\ &\iff 5nc^2x^2+2ncx-c^3x^2\geqslant 0\iff(5n-c)cx+2n\geqslant 0,\end{aligned}$$

which hold true, thus we have $\delta_{n,x}\leqslant v_{n,x}$, which corrects our claim.

Theorem 4.3. *Let f be bounded and integrable on $[0,\infty)$ and admitting second derivative at a point $x\in[0,\infty)$, then*

$$\lim_{n\to\infty} n\left[\hat{G}_{n,c}(f;x)-f(x)\right]=x(1+cx)f''(x). \tag{4.3}$$

Proof. By Taylor's expansion of f, we have

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2!}f''(x)+\varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$.

Now

$$n\left[\hat{G}_{n,c}(f;x)-f(x)\right]=n\frac{(2n-c)cx^2+2nx}{n(n-2c)}f''(x)+E(n,x),$$

where

$$E(n,x)=n\int_0^\infty W_n(x,t;c)\varepsilon(t,x)(t-x)^2dt.$$

To complete the proof of the theorem it is sufficient to show that $E(n,x)\to 0$ as $n\to\infty$.

Since $\varepsilon(t,x)\to 0$ as $t\to x$, hence for a given $\varepsilon>0$ there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$ whenever $|t-x|<\delta$.

Next

$$E(n,x)\leqslant n\left[\int_{|t-x|<\delta}W_n(x,t;c)|\varepsilon(t,x)|(t-x)^2dt+\int_{|t-x|\geqslant\delta}W_n(x,t;c)|\varepsilon(t,x)|(t-x)^2dt\right]=I_1+I_2,\quad(\text{say}).$$

Application of Lemma 3.2, leads us to

$$I_1\leqslant\varepsilon n\int_0^\infty W_n(x,t;c)(t-x)^2dt,$$

and

$$I_2\leqslant Kn\int_0^\infty W_n(x,t;c)\frac{(t-x)^{2s}}{\delta^{2s-2}}dt,\quad s\geqslant 2=O(n^{-s+1}),\quad \text{where } K=\sup_{t\in[-x,\infty)}|\varepsilon(t,x)|.$$

Due to arbitrariness of $\varepsilon>0,\ E(n,x)\to$ for sufficiently large n. This completes the proof of the theorem. □

Remark 4.4. We may note that under the same conditions of Theorem 4.3, for the original Srivastava and Gupta operator $G_{n,c}$, we have

$$\lim_{n\to\infty} n[G_{n,c}(f;x) - f(x)] = cxf'(x) + x(1+cx)f''(x). \tag{4.4}$$

5. Rate of convergence

Now we compute rate of convergence of the operators of $\hat{G}_{n,c}$ by means of the Lipschitz class $Lip_M(\alpha),\ (0 < \alpha \leqslant 1)$. As usual, we say that $f \in C_B[0,\infty)$ belongs to $Lip_M(\alpha)$ if the inequality

$$|f(t) - f(x)| \leqslant M|t-x|^{\alpha}, \quad x,t \in [0,\infty), \tag{5.1}$$

holds.

Theorem 5.1. *If $f \in Lip_M(\alpha),\ x \in [0,\infty)$ and $n > 2c$, we have*

$$\left|\hat{G}_{n,c}(f;x)(x) - f(x)\right| \leqslant M\left[\frac{c\{2n + (2n-c)x^2\}}{n(n-2c)}\right]^{\alpha/2}.$$

Proof. Since $f \in Lip_M(\alpha)$ and $x \geqslant 0$, from inequality (5.1) and applying the Hölder inequality with $p = \frac{2}{\alpha},\ q = \frac{2}{2-\alpha}$, we get the require result. □

Remark 5.2. If using Lemma 2.2, for the original operator $G_{n,c}$, then we get the following result

$$\left|G_{n,c}(f;x) - f(x)\right| \leqslant M\left\{\frac{x(1+cx)(2n-c) + cx(1+3cx)}{(n-c)(n-2c)}\right\}^{\alpha/2},$$

for every $f \in Lip_M(\alpha),\ x \geqslant 0$ and $n > 2c$.

6. Stancu variant

For η and γ positive numbers with condition $0 \leqslant \eta \leqslant \gamma$ and any non negative integer n,

$$f \in C[0,\infty) \to G_{n,c}^{(\eta,\gamma)} f,$$

we consider Stancu type Srivastava–Gupta operators

$$G_{n,c}^{(\eta,\gamma)}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^{\infty} p_{n+c,k-1}(t;c) f\left(\frac{nt+\eta}{n+\gamma}\right)dt + p_{n,0}(x;c)f(0). \tag{6.1}$$

For $\eta = \gamma = 0$ these operators become Srivastava–Gupta operators $G_{n,c}^{(\eta,\gamma)}(f;x) = G_{n,c}(f;x)$.

Now we first give the following lemma.

Lemma 6.1. *Let $e_i(x) = x^i,\ i = 0,1,2$. Then, for each $x \geqslant 0$ and $n > 2c$, we have*

(i) $G_{n,c}^{(\eta,\gamma)}(e_0;x) = 1$,
(ii) $G_{n,c}^{(\eta,\gamma)}(e_1;x) = \frac{n^2x + \eta(n-c)}{(n+\gamma)(n-c)}$,
(iii) $G_{n,c}^{(\eta,\gamma)}(e_2;x) = \frac{n^3x\{(n+c)x+2\} + \eta^2(n-c)(n-2c) + 2\eta n^2 x(n-2c)}{(n+\gamma)^2(n-c)(n-2c)}$.

Proof. It is easy to verify (i) by using (6.1). For the proof of others, we proceed as follows:

$$G_{n,c}^{(\eta,\gamma)}(e_1;x) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^{\infty} p_{n+c,k-1}(t;c)\left(\frac{nt+\eta}{n+\gamma}\right)dt = \left(\frac{n^2}{n+\gamma}\right)\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^{\infty} p_{n+c,k-1}(t;c)t\,dt + \frac{\eta}{(n+\gamma)}$$
$$= \frac{n^2x}{(n+\gamma)(n-c)} + \frac{\eta}{(n+\gamma)} = \frac{n^2x + \eta(n-c)}{(n+\gamma)(n-c)},$$

and

$$G_{n,c}^{(\eta,\gamma)}(e_2;x) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^{\infty} p_{n+c,k-1}(t;c)\left(\frac{nt+\eta}{n+\gamma}\right)^2 dt = \frac{1}{(n+\gamma)^2}\left[n^2 G_{n,c}(e_2;x) + \eta^2 + 2\eta n G_{n,c}(e_1;x)\right]$$
$$= \frac{1}{(n+\gamma)^2}\left[\frac{n^3x\{(n+c)x+2\}}{(n-c)(n-2c)} + \eta^2 + 2\eta n\frac{nx}{n-c}\right]$$
$$= \frac{n^3x\{(n+c)x+2\} + \eta^2(n-c)(n-2c) + 2\eta n^2 x(n-2c)}{(n+\gamma)^2(n-c)(n-2c)}. \qquad \square$$

From Lemma 6.1 we immediately have the following result.

Lemma 6.2. *For each* $x \in [0,\infty),\ n \in \mathbb{N},\ n > 2c$ *and with* $\phi_x(t) = t - x$*, we have*

(i) $G_{n,c}^{(\eta,\gamma)}(\varphi_x;x) = \frac{n^2x + \{\eta - x(n+\gamma)\}(n-c)}{(n+\gamma)(n-c)}$,
(ii) $G_{n,c}^{(\eta,\gamma)}(\varphi_x^2;x) = \frac{n^3x\{(n+c)x+2\} + \eta^2(n-c)(n-2c) + 2\eta n^2x(n-2c)}{(n+\gamma)^2(n-c)(n-2c)} - 2x\left[\frac{n^2x + \eta(n-c)}{(n+\gamma)(n-c)}\right] + x^2$.

We are led to the following asymptotic formula by using Lemma 6.2.

Theorem 6.3. *Let f be bounded and integrable on* $[0,\infty)$ *and admitting second derivative at a point* $x \in [0,\infty)$*, then we have*

$$\lim_{n\to\infty} n[(G_{n,c}^{(\eta,\gamma)})(f,x) - f(x)] = \{\eta + (c-\gamma)x\}f'(x) + x(1+cx)f''(x).$$

The proof follows along the lines of Theorem 4.3.

The analogous results of the Theorem 4.1 and Theorem 5.1 can easily be obtained for Stancu variant of Srivastava–Gupta operators as the methods are similar so we omit the details.

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