Applied Mathematics and Computation 273 (2016) 281–289

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Stancu–Kantorovich operators based on inverse Pólya–Eggenberger distribution

Naokant Deo[a], Minakshi Dhamija[a,*], Dan Miclăuş[b]

[a] Delhi Technological University Formerly Delhi College of Engineering, Department of Applied Mathematics Bawana Road, Delhi 110042, India
[b] Technical University of Cluj-Napoca North University Center at Baia Mare Department of Mathematics and Computer Science Victoriei 76, 430122 Baia Mare, Romania

ARTICLE INFO

MSC:
41A25
41A36

Keywords:
Stancu operators
Baskakov operators
Kantorovich operators
Pólya–Eggenberger distribution

ABSTRACT

The purpose of this paper is to investigate approximation properties of Stancu–Kantorovich operators based on inverse Pólya–Eggenberger distribution. For these new operators we establish some approximation properties including uniform convergence, asymptotic formula and degree of approximation.

1. Introduction

In 1923, Eggenberger and Pólya [5] considered an urn model contains w white balls and b black balls. A ball is drawn at random and then replaced together with s balls of the same color. This procedure is repeated n times and noting the distribution of the random variable X representing the number of times a white ball is drawn. The distribution of X is given by

$$Pr(X=k)=\binom{n}{k}\frac{w(w+s)\cdot\ldots\cdot(w+\overline{k-1}s)b(b+s)\cdot\ldots\cdot(b+\overline{n-k-1}s)}{(w+b)(w+b+s)\cdot\ldots\cdot(w+b+\overline{n-1}s)}, \tag{1.1}$$

for $k=0,1,\ldots,n$ and $\overline{k-1}s=(k-1)s$. The distribution (1.1) is known as Pólya–Eggenberger distribution with parameters (n,w,b,s) and contains binomial, respectively hypergeometric distribution as particular cases.

The inverse Pólya–Eggenberger distribution is defined by

$$Pr(N=n+k)=\binom{n+k-1}{k}\frac{w(w+s)\cdot\ldots\cdot(w+\overline{n-1}s)b(b+s)\cdot\ldots\cdot(b+\overline{k-1}s)}{(w+b)(w+b+s)\cdot\ldots\cdot(w+b+\overline{n+k-1}s)}, \tag{1.2}$$

for $k=0,1,\ldots$ and is the distribution of the number N of drawings needed to obtain n white balls. More details about Pólya–Eggenberger distributions (1.1) and (1.2) can be found in [10].

Based on Pólya–Eggenberger distribution (1.1), Stancu [15] introduced a new class of positive linear operators associated to a real-valued function $f:[0,1]\to\mathbb{R}$, given by

$$P_n^{[\alpha]}(f;x)=\sum_{k=0}^{n}p_{n,k}(x,\alpha)f\left(\frac{k}{n}\right)=\sum_{k=0}^{n}\binom{n}{k}\frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}f\left(\frac{k}{n}\right), \tag{1.3}$$

* Corresponding author. Tel.:+919996512123.
E-mail addresses: dr_naokant_deo@yahoo.com (N. Deo), minakshidhamija11@gmail.com (M. Dhamija), danmiclausrz@yahoo.com (D. Miclăuş).

http://dx.doi.org/10.1016/j.amc.2015.10.008
0096-3003/

where α is a non-negative parameter which may depend only on the natural number n and $t^{[n,h]} = t(t-h)(t-2h)\cdot\ldots\cdot(t-\overline{n-1}h)$, $t^{[0,h]} = 1$ represents the factorial power of t with increment h.

In the case when $\alpha = 0$ operators (1.3) reduce, obviously, to the original Bernstein operators [2] and for $\alpha = \frac{1}{n}$ we get a special case

$$P_n^{\left[\frac{1}{n}\right]}(f;x) = \sum_{k=0}^{n}\binom{n}{k}\frac{x^{\left[k,-\frac{1}{n}\right]}(1-x)^{\left[n-k,-\frac{1}{n}\right]}}{1^{\left[n,-\frac{1}{n}\right]}}f\left(\frac{k}{n}\right), \tag{1.4}$$

introduced by Lupaş and Lupaş [11]. Concerning operators (1.3) and (1.4), the reader is invited to see a recent paper [12], where some results of the recalled operators are revised.

Using inverse Pólya–Eggenberger distribution (1.2), Stancu [16] introduced a generalization of the Baskakov operators for a real-valued function bounded on $[0,+\infty)$, given by

$$V_n^{[\alpha]}(f;x) = \sum_{k=0}^{\infty} v_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{1^{[n,-\alpha]}x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}f\left(\frac{k}{n}\right). \tag{1.5}$$

The operators (1.5) include as a special case ($\alpha = 0$), the Baskakov operators [1]

$$V_n(f,x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right). \tag{1.6}$$

In 1989, Stancu–Kantorovich operators based on Pólya–Eggenberger distribution had been given by Razi [14] and he studied its convergence properties as well as degree of approximation. Recently, some modifications in connection with Stancu type operators are given by Gupta et al. [8], [9] and Deo et al. [3].

Now, for any bounded and integrable function f defined on $[0,+\infty)$ we introduce Stancu–Kantorovich operators based on inverse Pólya–Eggenberger distribution (1.2), given by

$$K_n^{[\alpha]}(f;x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x,\alpha)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt. \tag{1.7}$$

The aim of this paper is to establish some approximation properties for Stancu–Kantorovich operators (1.7). We start our study with some auxiliary results, which we will use in what follows. Taking into account these auxiliary results we derive some basic approximation properties including uniform convergence, a Voronvskaja type asymptotic formula and direct estimates in terms of moduli of continuity, respectively K-functional.

2. Auxiliary results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. We recall that the monomials $e_k(x) = x^k$, for $k \in \mathbb{N}_0$ called also test functions, play an important role in uniform approximation by linear positive operators. The computation of the images of test functions by Stancu–Baskakov operators (1.5) was done in [16], in order to prove a theorem concerning the uniform approximation.

Lemma 2.1 ([16]). *For the Stancu–Baskakov operators (1.5) hold*

$$V_n^{[\alpha]}(e_0;x) = 1, \qquad V_n^{[\alpha]}(e_1;x) = \frac{x}{1-\alpha},$$
$$V_n^{[\alpha]}(e_2;x) = \frac{1}{(1-\alpha)(1-2\alpha)}\left[x^2 + \frac{x(x+1)}{n} + \alpha\left(1-\frac{1}{n}\right)x\right].$$

In the sequel we shall use the following rules for computation with factorial powers

$$t^{[-n,h]} = \frac{1}{t^{[n,h]}}, \qquad t^{[-n,-h]} = \frac{1}{t^{[n,-h]}}, \qquad t^{[n,h]} = t^{[j,h]}\cdot(t-jh)^{[n-j,h]},$$

if, of course, the denominators are different from zero, n and j being natural numbers such that $n > j$. We remark that Stancu–Baskakov operators (1.5) could be written in the following form

$$V_n^{[\alpha]}(f;x) = \frac{1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{n^{[k,-1]}\cdot\left(\frac{x}{\alpha}\right)^{[k,-1]}}{\left(\frac{1+x}{\alpha}+n\right)^{[k,-1]}}f\left(\frac{k}{n}\right), \tag{2.1}$$

respectively Stancu–Kantorovich operators (1.7) could be written in the following form

$$K_n^{[\alpha]}(f;x) = \frac{(n-1)\cdot 1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{n^{[k,-1]}\cdot\left(\frac{x}{\alpha}\right)^{[k,-1]}}{\left(\frac{1+x}{\alpha}+n\right)^{[k,-1]}}\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt. \tag{2.2}$$

Remark 2.1. The proof of the Lemma 2.1 was done using hypergeometric series

$$ {}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}\cdot\frac{a^{[k,-1]}\cdot b^{[k,-1]}}{c^{[k,-1]}}, \tag{2.3}$$

where the parameters a, b, c satisfy the conditions $a, b > 0$, respectively $a + b < c$. Next, taking $z = 1$ the following representation for the hypergeometric series in terms of Gamma function

$$ {}_2F_1(a,b;c;1)=\frac{\Gamma(c)\cdot\Gamma(c-a-b)}{\Gamma(c-a)\cdot\Gamma(c-b)} \tag{2.4}$$

was used.

Lemma 2.2. *For the Stancu–Kantorovich operators (1.7) hold*

$$K_n^{[\alpha]}(e_0;x)=1,\quad K_n^{[\alpha]}(e_1;x)=\frac{nx}{(n-1)(1-\alpha)}+\frac{1}{2(n-1)},$$

$$K_n^{[\alpha]}(e_2;x)=\frac{n^2}{(1-\alpha)(1-2\alpha)(n-1)^2}\left[x(x+\alpha)+\frac{x(x+1)}{n}+\frac{(1-3\alpha)x}{n}\right]+\frac{1}{3(n-1)^2}.$$

Proof. Using the representation (2.2) for the Stancu–Kantorovich operators, it follows

$$\begin{aligned}
K_n^{[\alpha]}(e_0;x)&=\frac{(n-1)\cdot 1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{n^{[k,-1]}\cdot\left(\frac{x}{\alpha}\right)^{[k,-1]}}{\left(\frac{1+x}{\alpha}+n\right)^{[k,-1]}}\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}dt\\
&=\frac{1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{n^{[k,-1]}\cdot\left(\frac{x}{\alpha}\right)^{[k,-1]}}{\left(\frac{1+x}{\alpha}+n\right)^{[k,-1]}}=V_n^{[\alpha]}(e_0;x)=1.\\
K_n^{[\alpha]}(e_1;x)&=\frac{(n-1)\cdot 1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{n^{[k,-1]}\cdot\left(\frac{x}{\alpha}\right)^{[k,-1]}}{\left(\frac{1+x}{\alpha}+n\right)^{[k,-1]}}\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}t\,dt\\
&=\frac{(n-1)\cdot 1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{2k+1}{2(n-1)^2}\cdot\frac{n^{[k,-1]}\cdot\left(\frac{x}{\alpha}\right)^{[k,-1]}}{\left(\frac{1+x}{\alpha}+n\right)^{[k,-1]}}\\
&=\frac{n}{n-1}V_n^{[\alpha]}(e_1;x)+\frac{1}{2(n-1)}V_n^{[\alpha]}(e_0;x)=\frac{nx}{(n-1)(1-\alpha)}+\frac{1}{2(n-1)}.\\
K_n^{[\alpha]}(e_2;x)&=\frac{(n-1)\cdot 1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{n^{[k,-1]}\cdot\left(\frac{x}{\alpha}\right)^{[k,-1]}}{\left(\frac{1+x}{\alpha}+n\right)^{[k,-1]}}\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}t^2dt\\
&=\frac{(n-1)\cdot 1^{[n,-\alpha]}}{(1+x)^{[n,-\alpha]}}\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{3k^2+3k+1}{3(n-1)^3}\cdot\frac{n^{[k,-1]}\cdot\left(\frac{x}{\alpha}\right)^{[k,-1]}}{\left(\frac{1+x}{\alpha}+n\right)^{[k,-1]}}\\
&=\frac{n^2}{(n-1)^2}V_n^{[\alpha]}(e_2;x)+\frac{n}{(n-1)^2}V_n^{[\alpha]}(e_1;x)+\frac{1}{3(n-1)^2}V_n^{[\alpha]}(e_0;x)\\
&=\frac{n^2}{(1-\alpha)(1-2\alpha)(n-1)^2}\left[x(x+\alpha)+\frac{x(x+1)}{n}+\frac{(1-3\alpha)x}{n}\right]+\frac{1}{3(n-1)^2}.
\end{aligned}$$

□

Corollary 2.1. *The computation of the moments up to the second order, for Stancu–Kantorovich operators (1.7) is given by the following equalities*

$$\begin{aligned}
K_n^{[\alpha]}(e_1-x;x)&=\frac{((n-1)\alpha+1)x}{(n-1)(1-\alpha)}+\frac{1}{2(n-1)},\\
K_n^{[\alpha]}\big((e_1-x)^2;x\big)&=\frac{n^2}{(1-\alpha)(1-2\alpha)(n-1)^2}\left[x(x+\alpha)+\frac{x(x+1)}{n}+\frac{(1-3\alpha)x}{n}\right]\\
&\quad-\frac{n}{(1-\alpha)(n-1)}\left[(1+\alpha)x^2+\frac{(1-\alpha)x(x+1)}{n}\right]+\frac{1}{3(n-1)^2}.
\end{aligned}$$

Proof. Taking Lemma 2.2 into account, it follows the above equalities. □

Remark 2.2. The Stancu–Kantorovich operators could be reduced to the classical Baskakov–Kantorovich operators

$$K_n(f;x) = (n-1)\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(t)dt, \tag{2.5}$$

if we take $\alpha = 0$ in relation (1.7).

Lemma 2.3. *Let be $\alpha > 0$ and $x \in (0,+\infty)$, then we can present also the following representation for the Stancu–Kantorovich operators (1.7), given by*

$$K_n^{[\alpha]}(f;x) = \left(B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)\right)^{-1}\cdot\int_0^{\infty}\frac{t^{\frac{x}{\alpha}-1}}{(1+t)^{\frac{1+x}{\alpha}}}\cdot K_n(f;t)dt, \tag{2.6}$$

where K_n are the Baskakov–Kantorovich operators (2.5).

Proof. Using the relationship between Euler's functions

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

where $B(p,q)$ and $\Gamma(r)$ is Beta function of second kind, respectively Gamma function defined by

$$B(p,q) = \int_0^{\infty}\frac{u^{p-1}}{(1+u)^{p+q}}du, \qquad \Gamma(r) = \int_0^{\infty}u^{r-1}e^{-u}du,$$

with $\Gamma(r+n) = r(r+1)\cdot\ldots\cdot(r+n-1)\,\Gamma(r)$, for natural number n, then we get

$$\begin{aligned} B\left(\frac{x}{\alpha}+k,\frac{1}{\alpha}+n\right) &= \frac{\Gamma\left(\frac{x}{\alpha}+k\right)\Gamma\left(\frac{1}{\alpha}+n\right)}{\Gamma\left(\frac{1+x}{\alpha}+n+k\right)} \\ &= \frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\cdot\ldots\cdot\left(\frac{x}{\alpha}+k-1\right)\Gamma\left(\frac{x}{\alpha}\right)\cdot\frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)\cdot\ldots\cdot\left(\frac{1}{\alpha}+n-1\right)\Gamma\left(\frac{1}{\alpha}\right)}{\left(\frac{1+x}{\alpha}\right)\left(\frac{1+x}{\alpha}+1\right)\cdot\ldots\cdot\left(\frac{1+x}{\alpha}+n+k-1\right)\Gamma\left(\frac{1+x}{\alpha}\right)} \\ &= v_{n,k}(x,\alpha)\binom{n+k-1}{k}^{-1}B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right). \end{aligned}$$

Hence

$$v_{n,k}(x,\alpha) = \binom{n+k-1}{k}\left(B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)\right)^{-1}B\left(\frac{x}{\alpha}+k,\frac{1}{\alpha}+n\right) \tag{2.7}$$

and it follows

$$\begin{aligned} K_n^{[\alpha]}(f;x) &= \frac{(n-1)}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}\cdot\sum_{k=0}^{\infty}\binom{n+k-1}{k}B\left(\frac{x}{\alpha}+k,\frac{1}{\alpha}+n\right)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}f(t)dt \\ &= \frac{(n-1)}{B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}\cdot\sum_{k=0}^{\infty}\binom{n+k-1}{k}\int_0^{\infty}\frac{t^{\frac{x}{\alpha}+k-1}}{(1+t)^{\frac{1+x}{\alpha}+n+k}}dt\cdot\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}f(t)dt \\ &= \left(B\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)\right)^{-1}\cdot\int_0^{\infty}\frac{t^{\frac{x}{\alpha}-1}}{(1+t)^{\frac{1+x}{\alpha}}}\cdot K_n(f;t)dt. \end{aligned}$$

□

Remark 2.3. The computation of images of test functions by Stancu–Kantorovich operators (1.7), could be also derived using relation (2.6) and knowing previously the computation of the images of test functions by Baskakov–Kantorovich operators (2.5).

Proposition 2.1. *Let f be a bounded function defined on $[0,+\infty)$, with $\|f\| = \sup_{x\in[0,+\infty)}|f(x)|$, then*

$$\left|K_n^{[\alpha]}(f;x)\right| \le \|f\|.$$

Proof. Taking into account the definition of Stancu–Kantorovich operators (1.7) and Lemma 2.2, it follows

$$\begin{aligned} \left|K_n^{[\alpha]}(f;x)\right| &= \left|(n-1)\cdot\sum_{k=0}^{\infty}v_{n,k}(x,\alpha)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}f(t)dt\right| \\ &\le (n-1)\cdot\sum_{k=0}^{\infty}v_{n,k}(x,\alpha)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}|f(t)|dt \le \|f\|\cdot K_n^{[\alpha]}(e_0;x) = \|f\|. \end{aligned}$$

□

3. Direct results

The new Stancu–Kantorovich operators (1.7) based on functions defined on $[0, +\infty)$, which are bounded on each compact subset of $[0, +\infty)$ become an approximation process in approximating unbounded functions of the unbounded infinite interval $[0, +\infty)$. Since an immediate analog of the Bohman–Korovkin–Popoviciu theorem does not hold in the unbounded interval, some restrictions are needed. We give these restrictions and notations.

Let $B[0, +\infty)$ be the space of all functions f defined on the $[0, +\infty)$ satisfying the inequality $|f(x)| \le M_f(1 + x^2)$, where M_f is a positive constant depending only on the function f. Introduce

$$C_B[0, +\infty) = B[0, +\infty) \cap C[0, +\infty)$$

and

$$C^*[0, +\infty) = \left\{ f \in C_B[0, +\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1 + x^2} = K_f < \infty \right\}.$$

Endowed these spaces with the norm

$$\|f\| = \sup_{x\ge 0} \frac{|f(x)|}{1 + x^2}.$$

As it follows from the Gadzhiev papers [6], [7], the Korovkin type theorem for positive linear operators does not hold in the space $C_B[0, +\infty)$, but holds in the space $C^*[0, +\infty)$ and has the following form:

Theorem 3.1. *A sequence of positive linear operators L_n which satisfy the conditions*

$$\lim_{n\to\infty} \left\| L_n(e_i; x) - x^i \right\| = 0, \quad i = 0, 1, 2,$$

we get that for any function $f \in C^*[0, +\infty)$

$$\lim_{n\to\infty} \|L_n f - f\| = 0.$$

Theorem 3.2. *Let $f \in C^*[0, +\infty)$ and α being a non-negative parameter, which may depend only on the natural number n, with $\alpha \to 0$ when $n \to \infty$, then we have*

$$\lim_{n\to\infty} \left\| K_n^{[\alpha]} f - f \right\| = 0.$$

Proof. Taking Lemma 2.2 into account, it follows

$$\lim_{n\to\infty} \left\| K_n^{[\alpha]}(e_i; x) - x^i \right\| = 0, \quad i = 0, 1, 2.$$

Hence, applying Theorem 3.1, we get the desired result. □

Now, we present the asymptotic behavior of the Stancu–Kantorovich operators.

Theorem 3.3. *Let f be a bounded and integrable function on $[0, +\infty)$. If there exists first and second derivative of the function f in a fixed point $x \in [0, +\infty)$, then*

$$\lim_{n\to\infty} \left(K_n^{[\alpha]}(f; x) - f(x) \right) = \frac{\alpha x}{1 - \alpha} f'(x) + \frac{\alpha x(1 + x + 2\alpha x)}{2(1 - \alpha)(1 - 2\alpha)} f''(x).$$

Proof. Using Taylor's expansion of the function f, we can write

$$f(t) = f(x) + (t - x) f'(x) + \frac{1}{2!}(t - x)^2 f''(x) + \varepsilon(t, x)(t - x)^2,$$

where $\varepsilon(t, x) := \varepsilon(t - x)$ is a bounded function and $\lim_{t\to x} \varepsilon(t, x) = 0$. By linearity of Stancu–Kantorovich operators (1.7), it follows

$$K_n^{[\alpha]}(f; x) - f(x) = K_n^{[\alpha]}(e_1 - x; x) f'(x) + \frac{1}{2} K_n^{[\alpha]}\big((e_1 - x)^2; x\big) f''(x) + K_n^{[\alpha]}\big(\varepsilon(t, x) \cdot (e_1 - x)^2; x\big).$$

Taking Corollary 2.1 into account, we get

$$\lim_{n\to\infty} \left(K_n^{[\alpha]}(f; x) - f(x) \right) = \frac{\alpha x}{1 - \alpha} f'(x) + \frac{\alpha x(1 + x + 2\alpha x)}{2(1 - \alpha)(1 - 2\alpha)} f''(x) + \lim_{n\to\infty} \left(K_n^{[\alpha]}\big(\varepsilon(t, x) \cdot (e_1 - x)^2; x\big) \right), \tag{3.1}$$

then applying Cauchy–Schwarz inequality, it follows

$$K_n^{[\alpha]}\left(\varepsilon(t,x)(t-x)^2;x\right) \le \sqrt{K_n^{[\alpha]}(\varepsilon^2(t,x);x)}\sqrt{K_n^{[\alpha]}((e_1-x)^4;x)}. \tag{3.2}$$

Because $\varepsilon^2(x,x)=0$ and $\varepsilon^2(\cdot,x)\in C^*[0,+\infty)$, using the convergence from Theorem 3.2, we get

$$\lim_{n\to\infty} K_n^{[\alpha]}\left(\varepsilon^2(t,x);x\right) = \varepsilon^2(x,x) = 0. \tag{3.3}$$

Therefore, from (3.2) and (3.3) yields

$$\lim_{n\to\infty} K_n^{[\alpha]}\left(\varepsilon(t,x)\cdot(e_1-x)^2;x\right) = 0$$

and using (3.1) we obtain the asymptotic behavior of the Stancu–Kantorovich operators. □

The main tools to measure the approximation degree of linear positive operators towards the identity operators are moduli of continuity.

Definition 3.1. Let $f\in C_B[0,+\infty)$ be given and $\delta\ge 0$. The modulus of continuity of the function f is defined by

$$\omega(f,\delta) := \sup\{\left|f(x)-f(y)\right| \,:\, x,y\in[0,+\infty),\ |x-y|\le\delta\}, \tag{3.4}$$

where $C_B[0,+\infty)$ is the space of all real-valued functions continuous and bounded on $[0,+\infty)$.

Definition 3.2. For $f\in C[0,+\infty)$ and $\delta\ge 0$

$$\omega_1(f,\delta) := \sup\{|f(x+h)-f(x)| \,:\, x, x+h\in[0,+\infty),\ 0\le h\le\delta\} \tag{3.5}$$

and

$$\omega_2(f,\delta) := \sup\{|f(x+h)-2f(x)+f(x-h)| \,:\, x, x\pm h\in[0,+\infty),\ 0\le h\le\delta\} \tag{3.6}$$

are the moduli of smoothness of first, respectively second order.

Definition 3.3. Let be f from the space $C_B[0,+\infty)$ endowed with the norm $\|f\| = \sup_{x\in[0,+\infty)}|f(x)|$ and let us consider Peetre's K-functional

$$K_2(f,\delta) = \inf_{g\in W_\infty^2}\left\{\|f-g\| + \delta\left\|g''\right\|\right\}, \tag{3.7}$$

where $\delta>0$ and $W_\infty^2 = \left\{g\in C_B[0,+\infty) : g', g''\in C_B[0,+\infty)\right\}$. There exists an absolute constant $C>0$, such that

$$K_2(f,\delta) \le C\omega_2\left(f,\sqrt{\delta}\right), \tag{3.8}$$

conformable ([4], p. 177, Theorem 2.4).

In the following, we get direct estimates in terms of moduli of continuity and Peetre's K-functional.

Theorem 3.4. *If $f\in C_B[0,+\infty)$, then for any $x\in[0,+\infty)$ and $\delta>0$, it follows*

$$\left|K_n^{[\alpha]}(f;x)-f(x)\right| \le 2\cdot\omega(f,\delta),\ \ \text{with}\ \ \delta = \left(K_n^{[\alpha]}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}.$$

Proof. Knowing that Stancu–Kantorovich operators (1.7) preserve constants and using the well-known property of modulus of continuity

$$|f(x)-f(y)| \le \omega(f,|x-y|) \le \left(1+\frac{1}{\delta}|x-y|\right)\cdot\omega(f,\delta),$$

it follows

$$\begin{aligned}\left|K_n^{[\alpha]}(f;x)-f(x)\right| &\le (n-1)\sum_{k=0}^{\infty} v_{n,k}(x,\alpha)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} |f(t)-f(x)|dt \\ &\le \left(1+\frac{1}{\delta}(n-1)\sum_{k=0}^{\infty} v_{n,k}(x,\alpha)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} |t-x|dt\right)\omega(f,\delta).\end{aligned}$$

Applying Cauchy–Schwarz inequality for integral, we get

$$\left|K_n^{[\alpha]}(f;x)-f(x)\right| \le \left[1+\frac{1}{\delta}(n-1)\sum_{k=0}^{\infty} v_{n,k}(x,\alpha)\left(\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} dt\right)^{1/2}\left(\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} (t-x)^2 dt\right)^{1/2}\right]\omega(f,\delta).$$

Next, applying Cauchy–Schwarz inequality for sum, it follows

$$\left|K_n^{[\alpha]}(f;x)-f(x)\right| \le \left[1+\frac{1}{\delta}\left((n-1)\sum_{k=0}^{\infty}v_{n,k}(x,\alpha)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}dt\right)^{1/2}\right.$$
$$\left.\times\left((n-1)\sum_{k=0}^{\infty}v_{n,k}(x,\alpha)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}}(t-x)^2dt\right)^{1/2}\right]\omega(f,\delta)$$
$$=\left[1+\frac{1}{\delta}\left(K_n^{[\alpha]}(e_0;x)\right)^{1/2}\left(K_n^{[\alpha]}((e_1-x)^2;x)\right)^{1/2}\right]\omega(f,\delta)=2\cdot\omega(f,\delta),$$

with $\delta := (K_n^{[\alpha]}((e_1-x)^2;x))^{1/2}$. □

Theorem 3.5. *If f is a differentiable function on* $[0,+\infty)$ *and* $f'\in C_B[0,+\infty)$, *then for any* $x\in[0,+\infty)$ *and* $\delta>0$, *it follows*

$$\left|K_n^{[\alpha]}(f;x)-f(x)\right| \le \left|\frac{2((n-1)\alpha+1)x+1-\alpha}{(n-1)(1-\alpha)}\right|\cdot|f'(x)|+2\delta\cdot\omega(f',\delta),$$

with $\delta := (K_n^{[\alpha]}((e_1-x)^2;x))^{\frac{1}{2}}$.

Proof. Starting with the identity

$$f(t)-f(x)=f'(x)(t-x)+f(t)-f(x)-f'(x)(t-x), \tag{3.9}$$

we get for ξ between t and x

$$|f(t)-f(x)-f'(x)(t-x)|=|f'(\xi)-f'(x)|\cdot|t-x|,$$

using the Lagrange mean value theorem ($f(t)-f(x)=f'(\xi)(t-x)$, with ξ between t and x). Because $|\xi-x|\le|t-x|$, it follows

$$|f'(\xi)-f'(x)|\le\omega(f',|t-x|)\le\left(1+\frac{1}{\delta}|t-x|\right)\cdot\omega(f',\delta)$$

and

$$|f(t)-f(x)-f'(x)(t-x)|\le\left(|t-x|+\frac{1}{\delta}(t-x)^2\right)\cdot\omega(f',\delta).$$

Applying the linear positive Stancu–Kantorovich operators (1.7) to the inequality

$$|f(t)-f(x)|\le|f'(x)(t-x)|+\left(|t-x|+\frac{1}{\delta}(t-x)^2\right)\cdot\omega(f',\delta),$$

obtained from (3.9) and the above relations, it follows

$$\left|K_n^{[\alpha]}(f;x)-f(x)\right|\le|f'(x)|\cdot\left|K_n^{[\alpha]}(e_1-x;x)\right|+\left(K_n^{[\alpha]}(|e_1-x|;x)+\frac{1}{\delta}K_n^{[\alpha]}((e_1-x)^2;x)\right)\cdot\omega(f',\delta).$$

The Cauchy–Schwarz inequality for linear positive operators leads to

$$K_n^{[\alpha]}(|e_1-x|;x)\le\left(K_n^{[\alpha]}(e_0;x)\right)^{\frac{1}{2}}\cdot\left(K_n^{[\alpha]}((e_1-x)^2;x)\right)^{\frac{1}{2}}. \tag{3.11}$$

Using the relation (3.11) and the results presented in Corollary 2.1, the inequality (3.10) become

$$\left|K_n^{[\alpha]}(f;x)-f(x)\right|\le\left|\frac{2((n-1)\alpha+1)x+1-\alpha}{2(n-1)(1-\alpha)}\right|\cdot|f'(x)|+2\delta\cdot\omega(f',\delta),$$

with $\delta := (K_n^{[\alpha]}((e_1-x)^2;x))^{\frac{1}{2}}$. □

Estimates using combinations of first and second order modulus of smoothness are more refined than estimates using only the modulus of continuity.

Theorem 3.6. *If* $f\in C[0,+\infty)$, *then for any* $x\in[0,+\infty)$ *and* $\delta>0$, *it follows*

$$\left|K_n^{[\alpha]}(f;x)-f(x)\right|\le\omega_1(f,\delta)+\frac{3}{2}\cdot\omega_2(f;\delta),$$

with $\delta := (K_n^{[\alpha]}((e_1-x)^2;x))^{\frac{1}{2}}$.

Proof. Using the result of Păltănea [13] established for a linear positive operator L

$$|L(f;x)-f(x)| \leq |L(e_0;x)-1|\cdot|f(x)| + \frac{1}{\delta}|L(e_1-x;x)|\cdot\omega_1(f,\delta) + \left(L(e_0;x)+\frac{1}{2\delta^2}L((e_1-x)^2;x)\right)\cdot\omega_2(f,\delta),$$

we get for $L := K_n^{[\alpha]}$ the estimate

$$\left|K_n^{[\alpha]}(f;x)-f(x)\right| \leq \left|K_n^{[\alpha]}(e_0;x)-1\right|\cdot|f(x)| + \frac{1}{\delta}\left|K_n^{[\alpha]}(e_1-x;x)\right|\cdot\omega_1(f,\delta)$$
$$+\left(K_n^{[\alpha]}(e_0;x)+\frac{1}{2\delta^2}K_n^{[\alpha]}((e_1-x)^2;x)\right)\cdot\omega_2(f,\delta).$$

Taking into account the results of Lemma 2.2, respectively Corollary 2.1 and choosing $\delta = (K_n^{[\alpha]}((e_1-x)^2;x))^{\frac{1}{2}}$ we get the desired result, using previously again the Cauchy–Schwarz inequality

$$\left|K_n^{[\alpha]}(e_1-x;x)\right| = K_n^{[\alpha]}(|e_1-x|;x) \leq \left(K_n^{[\alpha]}(e_0;x)\right)^{\frac{1}{2}}\cdot\left(K_n^{[\alpha]}(e_1-x)^2;x)\right)^{\frac{1}{2}}.$$

□

Using Peetre's K-functional we give an estimate for the approximation error by Stancu–Kantorovich operators (1.7).

Theorem 3.7. *Let be $f \in C[0,+\infty)$, then for any $x \in [0,+\infty)$ yields*

$$\left|K_n^{[\alpha]}(f;x)-f(x)\right| \leq M\omega_2\left(f,\frac{1}{2}\delta_n(x,\alpha)\right)+\omega(f,\delta_\omega),$$

where M is an absolute constant and

$$\delta_n(x,\alpha) = \left(K_n^{[\alpha]}\left((e_1-x)^2;x\right)+\left(K_n^{[\alpha]}(e_1-x;x)\right)^2\right)^{\frac{1}{2}},\quad \delta_\omega = \left|K_n^{[\alpha]}(e_1-x;x)\right|.$$

Proof. For $x \in [0,+\infty)$ we define the operators

$$\tilde{K}_n^{[\alpha]}(f;x) = K_n^{[\alpha]}(f;x) - f\left(\frac{2nx+1-\alpha}{2(n-1)(1-\alpha)}\right)+f(x). \tag{3.12}$$

We remark that $\tilde{K}_n^{[\alpha]}(e_0;x)=1$ and $\tilde{K}_n^{[\alpha]}(e_1;x)=x$, i.e. the operators $\tilde{K}_n^{[\alpha]}$ preserve constants and linear functions. Therefore

$$\tilde{K}_n^{[\alpha]}(e_1-x;x)=0. \tag{3.13}$$

Let $g \in W_\infty^2$ and $x,t \in [0,+\infty)$. By Taylor's expansion, we have

$$g(t) = g(x)+(t-x)g'(x)+\int_x^t (t-u)g''(u)du$$

Applying $\tilde{K}_n^{[\alpha]}$ on both sides of the above equation, we get

$$\tilde{K}_n^{[\alpha]}(g;x)-g(x) = g'(x)\cdot\tilde{K}_n^{[\alpha]}(e_1-x;x)+\tilde{K}_n^{[\alpha]}\left(\int_x^t (t-u)g''(u)du;x\right)$$
$$= K_n^{[\alpha]}\left(\int_x^t (t-u)g''(u)du;x\right) - \int_x^{\frac{2nx+1-\alpha}{2(n-1)(1-\alpha)}}\left(\frac{2nx+1-\alpha}{2(n-1)(1-\alpha)}-u\right)g''(u)du.$$

On the other hand

$$\left|\int_x^t (t-u)g''(u)\right| \leq (t-x)^2\cdot\|g''\|,$$

then

$$\left|\tilde{K}_n^{[\alpha]}(g;x)-g(x)\right| \leq \left(K_n^{[\alpha]}((e_1-x)^2;x)+\left(K_n^{[\alpha]}(e_1-x;x)\right)^2\right)\cdot\|g''\|.$$

Using the definition (3.12) of the operators $\tilde{K}_n^{[\alpha]}$ and Proposition 2.1, it follows

$$\left|K_n^{[\alpha]}(f;x)-f(x)\right| \leq \left|\tilde{K}_n^{[\alpha]}(f-g;x)\right| + \left|\tilde{K}_n^{[\alpha]}(g;x)-g(x)\right|$$
$$+|g(x)-f(x)| + \left|f\left(\frac{2nx+1-\alpha}{2(n-1)(1-\alpha)}\right)-f(x)\right|$$
$$\leq 4\|f-g\|+\delta_n^2(x,\alpha)\left\|g''\right\|+\omega(f,\delta_\omega),$$

with $\delta_n^2(x,\alpha) = K_n^{[\alpha]}((e_1-x)^2;x)+(K_n^{[\alpha]}(e_1-x;x))^2$ and $\delta_\omega = |K_n^{[\alpha]}(e_1-x;x)|$.

Now, taking infimum on the right-hand side over all $g \in W_\infty^2$ and using the relation (3.8), we get

$$\left|K_n^{[\alpha]}(f;x) - f(x)\right| \leq 4K_2\left(f, \frac{{\delta_n}^2(x,\alpha)}{4}\right) + \omega(f, \delta_\omega)$$
$$\leq M\omega_2\left(f, \frac{1}{2}\delta_n(x,\alpha)\right) + \omega(f,\delta_\omega). \qquad \square$$

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