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Charlier–Szász–Durrmeyer type positive linear operators

Naokant Deo[1] · **Minakshi Dhamija**[1]

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Abstract In the present paper, we study modified Szász–Durrmeyer positive linear operators involving Charlier polynomials, one of the discrete orthogonal polynomials which are generalization of Szász Durrmeyer operators. Also, King type modification of these operators is given. We obtain uniform convergence of our operators with the help of Korovkin theorem, asymptotic formula and the order of approximation by using classical modulus of continuity.

Keywords Szász–Durrmeyer operators · Charlier polynomials · Modulus of continuity

Mathematics Subject Classification 41A25 · 41A36

1 Introduction

In the year 1950, for $f \in C[0,\infty)$, Szász [11] introduced the well-known Szász operators on half axis as:

$$S_n(f;x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in [0,\infty). \tag{1.1}$$

Jakimovski and Leviatan [6] introduced a Favard–Szász type operators by using Appell polynomials. Later, Ciupa [1] studied the following Durrmeyer type integral modification of these operators:

$$P_n(f;x) = \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)dt, \quad x \in [0,\infty), \tag{1.2}$$

✉ Minakshi Dhamija
minakshidhamija11@gmail.com

Naokant Deo
naokantdeo@dce.ac.in

1 Department of Applied Mathematics, Delhi Technological University (Formerly Delhi College of Engineering), Bawana Road, Delhi 110042, India

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where $p_k(x) \geq 0$ is Appell polynomial defined as:

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k$$

and $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function in the disk $|z| < R(R > 1)$ and $g(1) \neq 0$.

Ismail [5] defined the generating function of Poisson–Charlier polynomials $C_k(u; a)$ by

$$\sum_{k=0}^{\infty} C_k(u; a)\frac{t^k}{k!} = e^t\left(1 - \frac{t}{a}\right)^u, \quad |t| < a, \tag{1.3}$$

where

$$C_k(u; a) = \sum_{r=0}^{k} \binom{k}{r} (-u)_r \left(\frac{1}{a}\right)^r$$

and $(\alpha)_k$ is the Pochhammer's symbol given as:

$$(\alpha)_0 = 1, (\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \alpha(\alpha + 1)...(\alpha + k - 1), \quad k = 1, 2, ...$$

These Poisson–Charlier polynomials are positive for $a > 0$ and $u \leq a$.

Recently, Varma and Taşdelen [15] introduced positive linear operators involving Charlier polynomials, one of the discrete orthogonal polynomials which were generalization of Szász operators:

$$L_n(f; x, a) = e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k(-(a-1)nx; a)}{k!} f\left(\frac{k}{n}\right), \quad a > 1, \tag{1.4}$$

and they derive some direct results concerning uniform convergence and degree of approximation by using classical modulus of continuity.

Motivated from [1,15], for $f \in C[0, \infty)$, we consider Durrmeyer type Charlier–Szász positive linear operators

$$V_n(f; x, a) := e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k(-(a-1)nx; a)}{k!} \frac{n^{\lambda+k+1}}{\Gamma(\lambda + k + 1)} \times \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)dt, \tag{1.5}$$

where $a > 1, \lambda \geq 0,\ x \in [0, \infty)$ and Γ is gamma function. As $a \to \infty$ and $\lambda = 0$, we obtain again Szász–Durrmeyer operators introduced by Mazhar and Totik [8]. Work done by various researchers for Durrmeyer operators as well as operators associated with some polynomial can be seen in [2,3,9,10,12–14,16].

The aim of this paper is to present the above Durrmeyer variant i.e. operators (1.5) alongwith its King type modification, studying in each case uniform convergence, asymptotic formula and degree of approximation by using modulus of continuity. In the last section we give only the definition and moments of king type modification of operators (1.4) and we let an open gate for further research.

2 Auxiliary results and degree of approximation

In this section, first we give some basic definitions and lemmas. With the help of well-known Korovkin theorem, we state our main theorem and then calculate the rate of convergence by classical modulus of continuity, second modulus of continuity and Peetre's K-functional.

Definition 2.1 For $\delta > 0$ and $f \in \tilde{C}[0,\infty)$, the modulus of continuity $\omega(f;\delta)$ of the function f is defined by

$$\omega(f;\delta) := \sup_{\substack{x,y\in[0,\infty) \\ |x-y|\leq\delta}} |f(x) - f(y)|, \tag{2.1}$$

where $\tilde{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$. Then, for any $\delta > 0$ and each $x \in [0,\infty)$, it is well known that one can write

$$|f(x) - f(y)| \leqslant \omega(f;\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{2.2}$$

Definition 2.2 For $f \in \tilde{C}_B[0,\infty)$, the second modulus of continuity of 'f' is defined by

$$\omega_2(f;\delta) := \sup_{0<t\leq\delta} \|f(x+2t) - 2f(x+t) + f(x)\|_{\tilde{C}_B}, \tag{2.3}$$

where $\tilde{C}_B[0,\infty)$ is the class of real valued bounded and uniformly continuous functions defined on $[0,\infty)$ and $\|f\|_{\tilde{C}_B} = \sup_{x\in[0,\infty)} |f(x)|$.

Definition 2.3 Peetre's K-functional for the function $f \in \tilde{C}_B[0,\infty)$ is defined by

$$K(f;\delta) := \inf_{g\in\tilde{C}_B^2[0,\infty)} \left\{\|f-g\|_{\tilde{C}_B} + \delta\|g\|_{\tilde{C}_B^2}\right\}, \tag{2.4}$$

where

$$\tilde{C}_B^2[0,\infty) = \left\{g \in \tilde{C}_B[0,\infty) : g', g'' \in \tilde{C}_B[0,\infty)\right\}$$

and norm $\|g\|_{\tilde{C}_B^2} := \|g\|_{\tilde{C}_B} + \|g'\|_{\tilde{C}_B} + \|g''\|_{\tilde{C}_B}$. From the above definitions, we can conclude the following inequality:

$$K(f;\delta) \leq M\left\{\omega_2\left(f;\sqrt{\delta}\right) + \min(1,\delta)\|f\|_{\tilde{C}_B}\right\}, \quad \forall\delta > 0 \tag{2.5}$$

The constant M does not depend upon f and δ.

Let us define the class H as follows:

$$H := \left\{f : x \in [0,\infty), \frac{f(x)}{1+x^2} \textit{ is convergent as } x \to \infty\right\}.$$

Lemma 2.4 [15] *The operators given by* (1.4) *satisfy the following equalities:*

$$L_n(1;x,a) = 1,$$
$$L_n(t;x,a) = x + \frac{1}{n},$$
$$L_n\left(t^2;x,a\right) = x^2 + \frac{x}{n}\left(3 + \frac{1}{a-1}\right) + \frac{2}{n^2},$$

where $x \geq 0$.

Lemma 2.5 *The operators V_n satisfy the following equalities*

$$V_n\,(1; x, a) = 1, \tag{2.6}$$

$$V_n\,(t; x, a) = x + \frac{(\lambda+2)}{n}, \tag{2.7}$$

$$V_n\left(t^2; x, a\right) = x^2 + \frac{x}{n}\left\{\frac{1}{a-1} + 2\,(\lambda+3)\right\} + \frac{1}{n^2}\left\{\lambda^2 + 5\lambda + 7\right\}. \tag{2.8}$$

Proof We can easily obtain above three assertions with the help of Charlier polynomials (1.3). □

Remark 2.6 For $V_n\,(f; x, a)$ operators, verify that

$$V_n\,(t-x; x, a) = \frac{(\lambda+2)}{n}$$

and

$$V_n\left((t-x)^2; x, a\right) = \frac{x}{n}\left(\frac{2a-1}{a-1}\right) + \frac{1}{n^2}(\lambda^2 + 5\lambda + 7).$$

Proof Using Lemma 2.5 and linearity of operators $V_n\,(f; x, a)$, it is easy to prove the above result. □

Lemma 2.7 [4] *Let $g \in C^2\,[0, b]$ and for $n \geq 1$, $\{L_n\,(g; x)\}$ be a sequence of positive linear operators with the property $L_n\,(1; x) = 1$. Then*

$$|L_n\,(g; x) - g\,(x)| \leq \left\|g'\right\| \sqrt{L_n\left((t-x)^2; x\right)} + \frac{1}{2}\left\|g''\right\| L_n\left((t-x)^2; x\right). \tag{2.9}$$

Lemma 2.8 [17] *For $f \in C[a, b]$ and $h \in (0, \frac{b-a}{2})$, let f_h be the second order Steklov function attached to the function f. Then we have the following inequalities:*

$$\|f_h - f\| \leq \frac{3}{4}\omega_2\,(f; h)\,, \tag{2.10}$$

$$\left\|f_h''\right\| \leq \frac{3}{2h^2}\omega_2\,(f; h)\,. \tag{2.11}$$

Theorem 2.9 *Let $f \in C[0, \infty) \bigcap H$, then we have*

$$\lim_{n\to\infty} V_n\,(f; x, a) = f\,(x)$$

uniformly on each compact subset of $[0, \infty)$.

Proof By using Lemma 2.5, we have

$$\lim_{n\to\infty} V_n\left(t^i; x, a\right) = x^i, \quad i = 0, 1, 2$$

uniformly on each compact subset of $[0, \infty)$.
Hence, by applying well known Korovkin theorem, we obtain the desired results. □

We establish the asymptotic behavior of operators (1.5) by giving a Voronovskaja type theorem.

Theorem 2.10 *Let f be a bounded and integrable function on $[0, \infty)$. If there exists first and second derivative of the function f at a fixed point $x \in [0, \infty)$, then*

$$\lim_{n\to\infty} n\left(V_n\left(f; x, a\right) - f\left(x\right)\right) = (\lambda + 2)\, f'\left(x\right) + \frac{1}{2}x\left(\frac{2a-1}{a-1}\right) f''(x).$$

Proof Using Taylor's expansion formula of function f, it follows

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2!}(t-x)^2 f''(x) + \varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x) := \varepsilon(t-x)$ is a bounded function and $lim_{t\to x}\varepsilon(t,x) = 0$. Taking into account the linearity of modified Szász Durrmeyer operators and then apply the operators V_n on both sides of above equation, we get

$$V_n\left(f; x, a\right) - f(x) = V_n\left(t-x; x, a\right) f'(x) + \frac{1}{2} V_n\left((t-x)^2; x, a\right) f''(x)$$
$$+V_n\left(\varepsilon(t,x)\cdot(t-x)^2; x, a\right).$$

Therefore using Remark 2.6, we get

$$\lim_{n\to\infty} n\left(V_n\left(f; x, a\right) - f\left(x\right)\right) = (\lambda + 2)\, f'\left(x\right) + \frac{1}{2}x\left(\frac{2a-1}{a-1}\right) f''(x)$$
$$+ \lim_{n\to\infty} n\left(V_n\left(\varepsilon(t,x)\cdot(t-x)^2; x, a\right)\right). \tag{2.12}$$

We estimate the last term on the right hand side of the above equality, applying the Cauchy–Schwarz inequality, such that

$$V_n\left(\varepsilon(t,x)\cdot(t-x)^2; x, a\right) \le \sqrt{V_n\left(\varepsilon^2(t,x); x, a\right)}\sqrt{V_n\left((t-x)^4; x, a\right)}. \tag{2.13}$$

Because $\varepsilon^2(x,x) = 0$ and $\varepsilon^2(\cdot, x) \in C[0,\infty)\bigcap H$, using the convergence from Theorem 2.9, we get

$$\lim_{n\to\infty} V_n\left(\varepsilon^2(t,x); x, a\right) = \varepsilon^2(x,x) = 0. \tag{2.14}$$

Therefore, from (2.13) and (2.14) yields

$$\lim_{n\to\infty} n\left(V_n\left(\varepsilon(t,x)\cdot(e_1-x)^2; x, a\right)\right) = 0$$

and using (2.12) we obtain the asymptotic behavior of operators (1.5). □

Now, we find the rate of convergence by following theorems:

Theorem 2.11 *Let $f \in \tilde{C}[0,\infty)\bigcap H$, then the operators V_n satisfy:*

$$|V_n\left(f; x, a\right) - f(x)| \le 2\omega\left(f; \sqrt{\gamma_n\left(x\right)}\right),$$

where

$$\gamma_n(x) = V_n\left((t-x)^2; x, a\right) = \frac{x}{n}\left(\frac{2a-1}{a-1}\right) + \frac{1}{n^2}(\lambda^2 + 5\lambda + 7). \tag{2.15}$$

Proof By using Lemma 2.5 and property of modulus of continuity, we have

$$|V_n(f;x,a)-f(x)| \le e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k(-(a-1)nx;a)}{k!} \times \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}|f(t)-f(x)|\,dt$$
$$\le \left\{1+\frac{1}{\delta}e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k(-(a-1)nx;a)}{k!} \times \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}|t-x|\,dt\right\}\omega(f;\delta).$$

Now applying Cauchy–Schwarz inequality for the integral, we get

$$|V_n(f;x,a)-f(x)| \le \left\{1+\frac{1}{\delta}e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k(-(a-1)nx;a)}{k!} \times \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\left(\int_0^{\infty} e^{-nt}t^{\lambda+k}dt\right)^{1/2}\left(\int_0^{\infty} e^{-nt}t^{\lambda+k}(t-x)^2dt\right)^{1/2}\right\}\omega(f;\delta).$$

Once again using Cauchy–Schwarz inequality for sum, we obtain

$$|V_n(f;x,a)-f(x)| \le \left\{1+\frac{1}{\delta}\left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k(-(a-1)nx;a)}{k!} \times \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}dt\right)^{1/2} \times \left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k(-(a-1)nx;a)}{k!} \times \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}(t-x)^2dt\right)^{1/2}\right\}\omega(f;\delta)$$
$$=\left\{1+\frac{1}{\delta}(V_n(1;x,a))^{1/2}\left(V_n\left((t-x)^2;x,a\right)\right)^{1/2}\right\}\omega(f;\delta).$$

Choose $\delta=\sqrt{\gamma_n(x)}$ and using Lemma 2.5, we have

$$|V_n(f;x,a)-f(x)| \le 2\omega\left(f;\sqrt{\gamma_n(x)}\right),$$

which is the desired result. □

Theorem 2.12 *If* $f\in\tilde{C}_B^2[0,\infty)$ *then we have*

$$|V_n(f;x,a)-f(x)| \le \frac{1}{n}\mu(x)\,\|f\|_{\tilde{C}_B^2},$$

where

$$\mu(x) = \frac{2a-1}{2(a-1)}x + \frac{1}{2}\left(\lambda^2 + 7\lambda + 11\right).$$

Proof By Taylor's formula

$$f(t) = f(x) + f'(x)(t-x) + f''(\eta)\frac{(t-x)^2}{2}, \quad \eta \in (x,t).$$

The linearity property of the operators V_n and (2.6) gives that

$$V_n(f;x,a) - f(x) = f'(x)V_n((t-x);x,a) + \frac{f''(\eta)}{2}V_n\left((t-x)^2;x,a\right).$$

Using Lemma 2.5 in above result, we have

$$\begin{aligned}|V_n(f;x,a) - f(x)| &\le \left(\frac{\lambda+2}{n}\right)\|f'\|_{\tilde{C}_B} \\ &\quad + \frac{1}{2}\left[\frac{x}{n}\left(\frac{2a-1}{a-1}\right) + \frac{1}{n^2}\left(\lambda^2+5\lambda+7\right)\right]\|f''\|_{\tilde{C}_B} \\ &\le \frac{1}{n}\left[\frac{1}{2}\left(\frac{2a-1}{a-1}\right)x + \left(\lambda^2+6\lambda+9\right)\right]\|f\|_{\tilde{C}_B^2} \\ &= \frac{1}{n}\mu(x)\|f\|_{\tilde{C}_B^2},\end{aligned}$$

which is required result. □

Theorem 2.13 *Let $f \in \tilde{C}_B[0,\infty)$, then*

$$|V_n(f;x,a) - f(x)| \le 2M\left(\omega_2(f,h) + \min\left(1,h^2\right)\right)\|f\|_{\tilde{C}_B},$$

where

$$h := \sqrt{\frac{\mu(x)}{2n}}.$$

Proof Let $g \in \tilde{C}_B^2[0,\infty)$ then by using Theorem 2.12, we have

$$\begin{aligned}|V_n(f;x,a) - f(x)| &\le |V_n(f-g;x,a)| \\ &\quad + |V_n(g;x,a) - g(x)| + |g(x) - f(x)| \\ &\le 2\left[\|f-g\|_{\tilde{C}_B} + \frac{1}{2n}\left\{\frac{1}{2}\left(\frac{2a-1}{a-1}\right)x \right.\right. \\ &\quad \left.\left. + \left(\lambda^2+6\lambda+9\right)\right\}\|g\|_{\tilde{C}_B^2}\right].\end{aligned}$$

Since L.H.S of above inequality does not depend upon g, so by choosing $h := \sqrt{\frac{\mu(x)}{2n}}$, we get

$$|V_n(f;x,a) - f(x)| \le 2K\left(f,h^2\right).$$

Now, using the relation between Peetre's K-functional and second modulus of continuity, we get desired result. □

3 King type modification

In this section, we discuss better convergence rates by King type operators. King [7] studied an interesting class of Bernstein operators which preserves e_0 and e_2. For $f \in C[0,1]$

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} r_n^k(x)(1-r_n(x))^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where $\{r_n(x)\}$ is a sequence of continuous functions on $[0,1]$ with $0 \leqslant r_n(x) \leqslant 1$. For $r_n(x) = x,\ n \in \mathbb{N}$; operators B_n become classical Bernstein operators.

Using this King's idea, now we assume that $\{s_n\}$ be a sequence of continuous functions on $[0,\infty)$ such that after replacing $s_n(x)$ by x in $V_n(f;x,a)$, we get modified Durrmeyer type Charlier–Szász positive linear operators:

$$\hat{V}_n(f;x,a) := e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)ns_n(x)} \sum_{k=0}^{\infty} \frac{C_k(-(a-1)ns_n(x);a)}{k!} \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \times \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)dt, \tag{3.1}$$

where $s_n(x) = \frac{nx-\lambda-2}{n}$, $a>1$ and $x \geq 0$.

Next, we give basic lemmas for moments which are helpful to present main theorems on the same line as in case of operators (1.5).

Lemma 3.1 *The operators $\hat{V}_n(f;x,a)$ satisfy following equalities:*

$$\hat{V}_n(1;x,a) = 1,$$
$$\hat{V}_n(t;x,a) = x,$$
$$\hat{V}_n\left(t^2;x,a\right) = x^2 + \frac{x}{n}\left(\frac{2a-1}{a-1}\right) - \frac{1}{n^2}\left\{\frac{(4a-3)\lambda+(a+1)}{a-1}\right\}.$$

Proof It is very easy to prove above equalities by using Lemma 2.5. □

Remark 3.2 For $\hat{V}_n(f;x,a)$ operators, verify that

$$\hat{V}_n\left((t-x)^2;x,a\right) = \frac{x}{n}\left(\frac{2a-1}{a-1}\right) - \frac{1}{n^2}\left\{\frac{(4a-3)\lambda+(a+1)}{a-1}\right\}.$$

Proof Using Lemma 3.1 and linearity of operators $\hat{V}_n(f;x,a)$, it is easy to prove the above result. □

Theorem 3.3 *Let $f \in C[0,\infty) \bigcap H$, then*

$$\lim_{n\to\infty} \hat{V}_n(f;x,a) = f(x),$$

uniformly on each compact subset of $[0,\infty)$.

Proof By using Lemma 3.1, we have

$$\lim_{n\to\infty} \hat{V}_n\left(t^i;x,a\right) = x^i, \quad i = 0,1,2$$

uniformly on each compact subset of $[0,\infty)$.

Hence, by applying Korovkin theorem, we get the desired result. □

Theorem 3.4 *Let f be a bounded and integrable function on $[0,\infty)$ having first and second derivative of the function f at a fixed point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n\left(\hat{V}_n(f;x,a) - f(x)\right) = \frac{1}{2}x\left(\frac{2a-1}{a-1}\right)f''(x).$$

Theorem 3.5 *Let $f \in \tilde{C}[0,\infty)\bigcap H$, then the operators $\hat{V}_n$ satisfy:*

$$\left|\hat{V}_n(f;x,a) - f(x)\right| \le 2\omega\left(f;\sqrt{\alpha_n(x)}\right),$$

where

$$\alpha_n(x) = \frac{x}{n}\left(\frac{2a-1}{a-1}\right) - \frac{1}{n^2}\left\{\frac{(4a-3)\lambda + (a+1)}{a-1}\right\}. \tag{3.2}$$

Theorem 3.6 *If $f \in \tilde{C}_B^2[0,\infty)$ then we have*

$$\left|\hat{V}_n(f;x,a) - f(x)\right| \le \frac{1}{n}\beta(x)\,\|f\|_{\tilde{C}_B^2},$$

where

$$\beta(x) = \frac{2a-1}{2(a-1)}x + \frac{1}{2}\left\{\frac{(4a-3)\lambda + (a+1)}{a-1}\right\}.$$

Theorem 3.7 *Let $f \in \tilde{C}_B[0,\infty)$, then*

$$\left|\hat{V}_n(f;x,a) - f(x)\right| \le 2M\left(\omega_2(f,h) + \min\left(1,h^2\right)\right)\|f\|_{\tilde{C}_B},$$

where

$$h := \sqrt{\frac{\beta(x)}{2n}}.$$

Remark 3.8 Now we claim that the error estimation obtained in (3.2) is better than (2.15) for $f \in \tilde{C}[0,\infty)\bigcap H$, $a > 1$, $\lambda \ge 0$, $x \in [0,\infty)$, and $n \in N$. To prove this claim we must show that $\alpha_n(x) \le \gamma_n(x)$.

Now

$$\begin{aligned}\alpha_n(x) \le \gamma_n(x) \Leftrightarrow &\frac{x}{n}\left(\frac{2a-1}{a-1}\right) - \frac{1}{n^2}\left\{\frac{(4a-3)\lambda + (a+1)}{a-1}\right\}\\ &\le \frac{x}{n}\left(\frac{2a-1}{a-1}\right) + \frac{1}{n^2}(\lambda^2 + 5\lambda + 7)\\ &\Leftrightarrow \lambda^2 + \left(\frac{9a-8}{a-1}\right)\lambda + \frac{8a-6}{a-1} \ge 0,\end{aligned}$$

which is true as $\lambda \ge 0,\ a > 1$.

Remark 3.9 Now we define a King type modification of L_n operators and give basic lemma for this operator:

$$\hat{L}_n(f;x,a) = e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_n(x)} \sum_{k=0}^{\infty} \frac{C_k(-(a-1)nr_n(x);a)}{k!} f\left(\frac{k}{n}\right),$$

where $r_n(x) = \frac{nx-1}{n}$, $a > 1$ and $x \ge 0$.

Lemma 3.10 *The operators* $\hat{L}_n(f; x, a)$ *satisfy following equalities:*

$$\hat{L}_n(1; x, a) = 1,$$
$$\hat{L}_n(t; x, a) = x,$$
$$\hat{L}_n\left(t^2; x, a\right) = x^2 + \frac{x}{n}\left(\frac{a}{a-1}\right) - \frac{1}{n^2}\left\{\frac{1}{a-1}\right\}.$$

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