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Naokant Deo & Minakshi Dhamija

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RESEARCH PAPER

Generalized Positive Linear Operators Based on PED and IPED

Naokant Deo[1] · **Minakshi Dhamija**[2]

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Abstract

The paper deals with generalized positive linear operators based on Pólya–Eggenberger distribution (PED) as well as inverse Pólya–Eggenberger distribution (IPED). Initially, we give the moments using Stirling numbers of second kind and then establish direct results for proposed operators.

Keywords Generalized operators · Pólya–Eggenberger distribution · Modulus of continuity · Weighted approximation

Mathematics Subject Classification 41A25 · 41A36

1 Introduction

In the year 1968, Stancu Stancu (1968) introduced a new class of positive linear operators based on Pólya–Eggenberger distribution (PED) and associated with a real-valued function on [0,1] as:

$$B_n^{(\alpha)}(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}} f\left(\frac{k}{n}\right), \qquad (1)$$

where α is a non-negative parameter which may depend only on the natural number n and $t^{[n,h]} = t(t-h)(t-2h)\cdot\dots\cdot(t-\overline{n-1}h)$, $t^{[0,h]} = 1$ represents the factorial power of t with increment h.

Later on, Stancu Stancu (1970) introduced a generalized form of the Baskakov operators based on inverse Pólya–Eggenberger distribution (IPED) for a real-valued function bounded on $[0,\infty)$, given by

$$V_n^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1^{[n,-\alpha]}x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}} f\left(\frac{k}{n}\right). \qquad (2)$$

Now we consider new positive linear operators $L_n^{(\alpha)}$, for each f, real-valued function bounded on interval I, as:

$$L_n^{(\alpha)}(f;x) = \sum_{k} w_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \quad x \in I,\ n = 1,2,\ldots, \qquad (3)$$

where $\alpha = \alpha(n) \to 0$ when n $\to \infty$, p and k are nonnegative integers and for $\lambda = -1, 0$, we have

$$\begin{aligned}\omega_{n,k}^{(\alpha)}(x) &= \frac{n+p}{n+p+\overline{\lambda+1}k}\binom{n+p+\overline{\lambda+1}k}{k} \\ &\quad \frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{i=0}^{n+p+\lambda k-1}(1+\lambda x+i\alpha)}{\prod_{i=0}^{n+p+\overline{\lambda+1}k-1}(1+\overline{\lambda+1}x+i\alpha)} \\ &= \frac{n+p}{n+p+\overline{\lambda+1}k}\binom{n+p+\overline{\lambda+1}k}{k} \\ &\quad \frac{x^{[k,-\alpha]}(1+\lambda x)^{[n+p+\lambda k,-\alpha]}}{(1+\overline{\lambda+1}x)^{[n+p+\overline{\lambda+1}k,-\alpha]}},\end{aligned}$$

using the notation $\overline{m-r}\alpha = (m-r)\alpha$. Operator (3) is the generalized form of above two operators (1) and (2) and associated with PED and IPED (Eggenberger and Pólya 1923).

Kantorovich form of Stancu operators (1) had been given by Razi (1989) and he studied its convergence properties and degree of approximation. Ispir et al. (2015) also discussed the Kantorovich form of operators (1) and they estimated the rate of convergence for absolutely

✉ Minakshi Dhamija
minakshidhamija11@gmail.com

Naokant Deo
naokantdeo@dce.ac.in; dr_naokant_deo@yahoo.com

[1] Department of Applied Mathematics, Delhi Technological University, Formerly Delhi College of Engineering, Bawana Road, Delhi 110042, India

[2] Department of Mathematics, Shaheed Rajguru College of Applied Sciences for Women, University of Delhi, Vasundhara Enclave, Delhi 110096, India

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continuous functions having a derivative coinciding a.e. with a function of bounded variation. Recently, Miclăuş (2014) established some approximation results for the Stancu operators (1) and its Durrmeyer type integral modification was studied by Gupta and Rassias (2014) and obtained some direct results which include an asymptotic formula, local and global approximation results for these operators in terms of modulus of continuity.

Very recently, Durrmeyer type modification of generalized Baskakov operators (2) associated with IPED was introduced by Dhamija and Deo (2016) and studied the moments with the help of Vandermonde convolution formula and then gave approximation properties of these operators which include uniform convergence and degree of approximation. Deo et al. (2016) also investigated approximation properties of Kantorovich variant of operators (2) and they established uniform convergence, asymptotic formula and degree of approximation. Various Durrmeyer type modifications and then their local approximation along with some other approximation behaviour have been discussed by many authors, e.g. Deo (2012), Jung et al. (2014) and Gupta and Agarwal (2014).

The main object of this paper is to find moments of proposed operators up to order 4, with the help of Stirling numbers of the second kind (see Miclăuş 2012c); however, we will use only moments till order 2 to estimate the rate of convergence of operators (3) via local approximation and rest moments are an open gate for future research to obtain other approximation properties of same operators as well as for their different modifications.

Throughout this paper, we consider interval $I=[0,\infty)$ for $\lambda=0$ and $I=[0,1]$ for $\lambda=-1$.

2 Special Cases

It is easy to understand that the special cases of operators (3) are as follows:

(1) For $\lambda=-1$; we have

(i) When $\alpha\neq 0\neq p$

$$L_n^{(\alpha)}(f;x)=\sum_{k=0}^{n+p}\binom{n+p}{k}\frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{i=0}^{n+p-k-1}(1-x+i\alpha)}{\prod_{i=0}^{n+p-1}(1+i\alpha)}f\left(\frac{k}{n}\right).$$

This leads to Schurer type Stancu operators.

(ii) When $\alpha\neq 0, p=0$ we get alternate form of operators (1) as:

$$L_n^{(\alpha)}(f;x)=\sum_{k=0}^{n}\binom{n}{k}\frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{i=0}^{n-k-1}(1-x+i\alpha)}{\prod_{i=0}^{n-1}(1+i\alpha)}f\left(\frac{k}{n}\right).$$

Particular case: when $\alpha=1/n, p=0$, we obtain Lupaş and Lupaş (1987) operators as:

$$L_n^{(\alpha)}(f;x)=\frac{2(n!)}{(2n)!}\sum_{k=0}^{n}\binom{n}{k}\prod_{i=0}^{k-1}(nx+i)\prod_{i=0}^{n-k-1}(\overline{1-x}n+i)f\left(\frac{k}{n}\right).$$

(iii) When $\alpha=0, p\neq 0$, we have Bernstein–Schurer operators (Schurer 1962) as:

$$L_n^{(\alpha)}(f;x)=\sum_{k=0}^{n+p}\binom{n+p}{k}x^k(1-x)^{n+p-k}f\left(\frac{k}{n}\right).$$

(iv) When $\alpha=0, p=0$ we obtain original Bernstein operators (Bernstein 1912) as:

$$L_n^{(\alpha)}(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right).$$

(2) For $\lambda=0$, we obtain the following operators:

(i) When $\alpha\neq 0\neq p$, we have Stancu–Schurer operators as:

$$L_n^{(\alpha)}(f;x)=\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{i=0}^{n+p-1}(1+i\alpha)}{\prod_{i=0}^{n+p+k-1}(1+x+i\alpha)}f\left(\frac{k}{n}\right),$$

(ii) When $\alpha\neq 0, p=0$, we obtain alternate form of operators (2) as:

$$L_n^{(\alpha)}(f;x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{i=0}^{n-1}(1+i\alpha)}{\prod_{i=0}^{n+k-1}(1+x+i\alpha)}f\left(\frac{k}{n}\right),$$

(iii) When $\alpha=0, p\neq 0$, we get Baskakov–Schurer operators as:

$$L_n^{(\alpha)}(f;x)=\sum_{k=0}^{\infty}\binom{n+p+k-1}{k}\frac{x^k}{(1+x)^{n+p+k}}f\left(\frac{k}{n}\right),$$

(iv) When $\alpha = 0, p = 0$, we obtain classical Baskakov operators (Baskakov 1957) as:
$$L_n^{(\alpha)}(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right).$$

3 Preliminary Results

In 1730, Stirling (1730) introduced an important concept of numbers, useful in various branches of mathematics like number theory, calculus of Bernstein polynomials, etc., known as Stirling numbers of first kind and afterwards Stirling numbers of second kind. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{N}$, a collection of natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $x \in \mathbb{R}$ and $i, j \in \mathbb{N}_0$, let $S(j, i)$ denote the Stirling numbers of second kind, then
$$x^j = \sum_{i=1}^{j} S(j,i)(x)_i,$$
with alternate form
$$x^j = \sum_{i=0}^{j-1} S(j,j-i)(x)_{j-i}, \tag{4}$$
such that
$$(x)_i := \prod_{n=0}^{i-1} (x-n)$$
is falling factorial. Also, these numbers have the following properties:
$$S(j,i) := \begin{cases} 1, & \text{if } j=i=0; j=i \text{ or } j>1, i=1 \\ 0, & \text{if } j>0, i=0 \\ 0, & \text{if } j<i \\ iS(j-1,i)+S(j-1,i-1), & \text{if } j,i>1. \end{cases} \tag{5}$$

These Stirling numbers of second kind are very useful in calculating the moments of linear positive operators, especially for higher order moments. Miclăuş (2012a, b) obtained higher order moments for Bernstein type operators using these numbers. In what follows, we shall find the moments of $L_n^{(\alpha)}$ given by (3)with the help of same. Let us recall the monomials $e_j(x) = x^j, j \in \mathbb{N}_0$ be the test functions.

Lemma 1 *For the monomial t^j, where $j \in \mathbb{N}$, and $t, x \in I$, we have*
$$L_n^{(\alpha)}(t^j;x) = \frac{1}{n^j} \sum_{i=0}^{j-1} S(j,j-i) \phi_{n+p}^{j-i} \frac{x^{[j-i,-\alpha]}}{1^{[j-i+(\lambda+1),-\alpha]}},$$
where
$$\phi_{n+p}^{j-i} = \begin{cases} (n+p)_{(j-i)}, & \lambda = -1 \\ (n+p)^{(j-i)}, & \lambda = 0 \end{cases},$$
$(y)_n := \prod_{i=0}^{n-1}(y-i), (y)_0 := 1$ and $(y)^n := \prod_{i=0}^{n-1}(y+i), (y)^{(0)} := 1$ are, respectively, the falling factorial and rising factorial with $y \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proof For $\lambda = -1$, we have
$$\begin{aligned} L_n^{(\alpha)}(t^j;x) &= \sum_{k=0}^{n+p} w_{n,k}^{(\alpha)}(x) \frac{k^j}{n^j} \\ &= \frac{1}{n^j} \sum_{k=0}^{n+p} w_{n,k}^{(\alpha)}(x) \sum_{i=0}^{j-1} S(j,j-i)(k)_{j-i} \\ &= \frac{1}{n^j} \sum_{i=0}^{j-1} \Bigg[S(j,j-i)(n+p)_{j-i} \\ &\quad \times \sum_{k=j-i}^{n+p} \binom{n+p-j+i}{k-j+i} \frac{x^{[j-i,-\alpha]}(x+(j-i)\alpha)^{[k-j+i,-\alpha]}(1-x)^{[n+p-k,-\alpha]}}{1^{[j-i,-\alpha]}(1+(j-i)\alpha)^{[n+p-j+i,-\alpha]}} \Bigg] \\ &= \frac{1}{n^j} \sum_{i=0}^{j-1} S(j,j-i)(n+p)_{j-i} \frac{x^{[j-i,-\alpha]}(1+(j-i)\alpha)^{[n+p-j+i,-\alpha]}}{1^{[j-i,-\alpha]}(1+(j-i)\alpha)^{[n+p-j+i,-\alpha]}} \\ &= \frac{1}{n^j} \sum_{i=0}^{j-1} S(j,j-i)(n+p)_{j-i} \frac{x^{[j-i,-\alpha]}}{1^{[j-i,-\alpha]}} \end{aligned}$$

And for $\lambda = 0$, we have
$$\begin{aligned} L_n^{(\alpha)}(t^j;x) &= \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) \frac{k^j}{n^j} \\ &= \frac{1}{n^j} \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) \sum_{i=0}^{j-1} S(j,j-i)(k)_{j-i} \\ &= \frac{1}{n^j} \sum_{i=0}^{j-1} \Bigg[S(j,j-i) \sum_{k=0}^{\infty} (k)_{j-i} \binom{n+p+k-1}{k} \\ &\quad \times \frac{x^{[j-i,-\alpha]}(x+(j-i)\alpha)^{[k-j+i,-\alpha]} 1^{[j-i,-\alpha]}(1+(j-i)\alpha)^{[n+p-j+i,-\alpha]}}{(1+x)^{[n+p+k,-\alpha]}} \Bigg] \\ &= \frac{1}{n^j} \sum_{i=0}^{j-1} \Bigg[S(j,j-i)(n+p)^{(j-i)} \sum_{k=j-i}^{\infty} \binom{n+p+k-1}{n+p+j-i-1} \\ &\quad \times \frac{x^{[j-i,-\alpha]}(x+(j-i)\alpha)^{[k-j+i,-\alpha]} 1^{[j-i,-\alpha]}(1+(j-i)\alpha)^{[n+p-j+i,-\alpha]}}{(1+x)^{[n+p+k,-\alpha]}} \Bigg] \\ &= \frac{1}{n^j} \sum_{i=0}^{j-1} S(j,j-i)(n+p)^{j-i} \frac{x^{[j-i,-\alpha]}}{1^{[j-i+1,-\alpha]}} \end{aligned}$$

By combining both cases $(\lambda = -1 \text{ and } 0)$, we get the required result. □

Lemma 2 *For the generalized positive linear operators, (3) holds*
$$L_n^{(\alpha)}(1;x) = 1, \quad L_n^{(\alpha)}(t;x) = \left(\frac{n+p}{n}\right) \frac{x}{(1-\overline{\lambda+1}\alpha)},$$
$$L_n^{(\alpha)}(t^2;x) = \left(\frac{n+p}{n^2}\right) \frac{1}{(1-\lambda\alpha)(1-\overline{\lambda+1}\alpha)} \times \left[\frac{(n+p+\lambda+1)x(x+\alpha)}{1-2\overline{\lambda+1}\alpha} + x(1+\lambda x)\right],$$

$$L_n^{(\alpha)}(t^3;x) \left[\frac{(n+p+2\lambda+1)\left(n+p+2\overline{2\lambda+1}\right)(x+\alpha)(x+2\alpha)}{\left(1-\overline{3\lambda+2}\alpha\right)\left(1-\overline{5\lambda+3}\alpha\right)} + \frac{3(n+p+2\lambda+1)(x+\alpha)}{\left(1-\overline{3\lambda+2}\alpha\right)}+1\right],$$

and

$$L_n^{(\alpha)}(t^4;x) = \frac{(n+p)x}{n^4\left(1-\overline{\lambda+1}\alpha\right)} \times \left[\frac{(n+p+2\lambda+1)\left(n+p+2\overline{2\lambda+1}\right)\left(n+p+3\overline{2\lambda+1}\right)(x+\alpha)(x+2\alpha)(x+3\alpha)}{\left(1-\overline{3\lambda+2}\alpha\right)\left(1-\overline{5\lambda+3}\alpha\right)\left(1-\overline{7\lambda+4}\alpha\right)} + \frac{6(n+p+2\lambda+1)\left(n+p+2\overline{2\lambda+1}\right)(x+\alpha)(x+2\alpha)}{\left(1-\overline{3\lambda+2}\alpha\right)\left(1-\overline{5\lambda+3}\alpha\right)} + \frac{7(n+p+2\lambda+1)(x+\alpha)}{\left(1-\overline{3\lambda+2}\alpha\right)}+1\right].$$

Proof From the definition of operators (3), we can obtain the moment for $j=0$, i. e., $L_n^{(\alpha)}(1;x)=1$.

Also by the application of Lemma 1 for $j=1,2,3,4$ and taking into account the relation (5), we can follow the values of remaining moments. □

Further, to obtain the central moments of generalized positive operators (3), we use the following result:

Lemma 3 (Gonska et al. 2006) *Let V be any linear operators then*

$$V\left((t-x)^j;x\right) = V\left(t^j;x\right) - \sum_{i=0}^{j-1}\binom{j}{i}x^{j-i}V\left((t-x)^i;x\right),$$

and in the case when $V(t^j;x)=x^j$, *for* $j=0,1$, *then we get*

$$V\left((t-x)^3;x\right) = V\left(t^3;x\right) - x^3 - 3xV\left((t-x)^2;x\right),$$

and

$$V\left((t-x)^4;x\right) = V\left(t^4;x\right) - x^4 - 4xV\left((t-x)^3;x\right) + 6x^2V\left((t-x)^2;x\right).$$

Lemma 4 *The generalized linear positive operators* (3) *satisfy*

$$L_n^{(\alpha)}(t-x;x) = \frac{\left(p+n\overline{\lambda+1}\alpha\right)x}{n\left(1-\overline{\lambda+1}\alpha\right)}, \tag{6}$$

$$L_n^{(\alpha)}\left((t-x)^2;x\right) = \frac{n+p}{n(1-\lambda\alpha)\left(1-\overline{\lambda+1}\alpha\right)} \left[(1-\lambda\alpha)\left(1-\overline{\lambda+1}\alpha\right)\frac{nx^2}{n+p} + \frac{(n+p+\lambda+1)x(x+\alpha)}{n\left(1-2\overline{\lambda+1}\alpha\right)}\right] + \frac{x(1+\lambda x)}{n} - 2(1-\lambda\alpha)x^2 \tag{7}$$

$$L_n^{(\alpha)}\left((t-x)^3;x\right) = \frac{(n+p)x(x+\alpha)}{} \left[+ \frac{3(n+p+2\lambda+1)}{n\left(1-\overline{3\lambda+2}\alpha\right)} - \frac{3(n+p+\lambda+1)x}{(1-\lambda\alpha)\left(1-2\overline{(\lambda+1)}\alpha\right)}\right] + \frac{(n+p)x}{n\left(1-\overline{\lambda+1}\alpha\right)}\left[\frac{1}{n^2} - 3x\left(\frac{px+\overline{\lambda+1}\alpha}{(n+p)}\right) - \frac{3}{(1-\lambda\alpha)}\frac{x(1+\lambda x)}{n}\right] -2x^3\left[2-\frac{3(n+p)}{n\left(1-\overline{\lambda+1}\alpha\right)}\right], \tag{8}$$

$$L_n^{(\alpha)}\left((t-x)^4;x\right) = \frac{(n+p)x(x+\alpha)}{n^2\left(1-\overline{\lambda+1}\alpha\right)} \times \left[\frac{(n+p+2\lambda+1)\left(n+p+2\overline{2\lambda+1}\right)\left(n+p+3\overline{2\lambda+1}\right)(x+2\alpha)(x+3\alpha)}{n^2\left(1-\overline{3\lambda+2}\alpha\right)\left(1-\overline{5\lambda+3}\alpha\right)\left(1-\overline{7\lambda+4}\alpha\right)} + \frac{2(3-2nx)(n+p+2\lambda+1)\left(n+p+2\overline{2\lambda+1}\right)(x+2\alpha)}{n^2\left(1-\overline{3\lambda+2}\alpha\right)\left(1-\overline{5\lambda+3}\alpha\right)} + \frac{(7-12nx)(n+p+2\lambda+1)}{n^2\left(1-\overline{3\lambda+2}\alpha\right)} + \frac{6(n+p+\lambda+1)x^2}{(1-\lambda\alpha)\left(1-2\overline{(\lambda+1)}\alpha\right)}\right] + \frac{(n+p)x}{n\left(1-\overline{\lambda+1}\alpha\right)}\left[\frac{1-4nx}{n^3} + 8x^2\left(\frac{px+\overline{\lambda+1}\alpha}{(n+p)}\right)\frac{6x^2(1+\lambda x)}{n(1-\lambda\alpha)}\right] + 3x^4\left[3-\frac{4(n+p)}{n\left(1-\overline{\lambda+1}\alpha\right)}\right]. \tag{9}$$

Proof The combined use of Lemma 2 and 3 will follow the proof. □

Lemma 5 *For positive linear operators* (3), *there holds*

$$\left|L_n^{(\alpha)}(f;x)\right| \leqslant \|f\|,$$

where

$$\|f\| = \sup_{x\in I}|f(x)|.$$

Proof From operators (3) and the fact that $\left|L_n^{(\alpha)}(1;x)\right|=1$, we have

$$\left|L_n^{(\alpha)}(f;x)\right| = \left|\sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x)f\left(\frac{k}{n}\right)\right| \leqslant \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x)\left|f\left(\frac{k}{n}\right)\right| \leqslant \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x)\sup|f(x)| = \|f\|.$$

□

4 Direct Results

Let $C_B(I)$ be the space of all the real-valued continuous and bounded functions f on the interval I, endowed with the norm $\|f\|$.

For $f \in C_B(I)$, the Peetre's $K-$functional is defined by

$$K_2(f;\delta) := \inf_{g\in C_B^2(I)} \{\|f-g\| + \delta\|g''\|\},\ \delta > 0,$$

where $C_B^2(I) = \{g \in C_B(I) : g', g'' \in C_B(I)\}$. By DeVore and Lorentz (1993, p.177. Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leqslant C\omega_2\left(f;\sqrt{\delta}\right), \tag{10}$$

where $\omega_2\left(f;\sqrt{\delta}\right)$ is the second-order modulus of continuity defined by

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{0<h<\sqrt{\delta}} \sup_{x\in I} |f(x+2h) - 2f(x+h) + f(x)|.$$

Also the first-order modulus of smoothness(or simply modulus of continuity) is given by

$$\omega\left(f;\sqrt{\delta}\right) = \sup_{0<h\leqslant\sqrt{\delta}} \sup_{x\in I} |f(x+h) - f(x)|.$$

Theorem 1 *For $f \in C_B(I)$, we have*

$$\left|L_n^{(\alpha)}(f;x) - f(x)\right| \leqslant \omega\left(f, \frac{px + nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right) + C\omega_2\left(f, \frac{\sqrt{\psi_{n,\lambda}^{(\alpha)}(x)}}{2}\right),$$

where C is a positive constant and

$$\psi_{n,\lambda}^{(\alpha)}(x) = L_n^{(\alpha)}\left((t-x)^2;x\right) + \left\{\frac{px + nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right\}^2.$$

Proof First, we consider auxiliary operators

$$\hat{L}_n^{(\alpha)}(f;x) = L_n^{(\alpha)}(f;x) + f(x) - f\left(\frac{n+p}{n}.\frac{x}{1-(\lambda+1)\alpha}\right). \tag{11}$$

In view of first and second formula in Lemma 2, we observed that for all $x \in I$, operators $\hat{L}_n^{(\alpha)}(f;x)$ are linear such that

$$\hat{L}_n^{(\alpha)}(1;x) = 1 \ \ and \ \ \hat{L}_n^{(\alpha)}(t;x) = x,$$

i. e., preserve linear functions. Therefore

$$\hat{L}_n^{(\alpha)}(t-x;x) = 0. \tag{12}$$

Let $g \in C_B^2(I)$ and $t, x \in I$ then Taylor's theorem implies

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)\mathrm{d}u,$$

we can write

$$\begin{aligned}\hat{L}_n^{(\alpha)}(g;x) - g(x) &= g'(x)\hat{L}_n^{(\alpha)}((t-x);x) + \hat{L}_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\\ &= \hat{L}_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)du;x\right)\\ &= L_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right) - \int_x^{\frac{n+p}{n}\left(\frac{x}{1-(\lambda+1)\alpha}\right)} \left(\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha} - u\right) g''(u)\mathrm{d}u\end{aligned}.$$

Hence, we have

$$\left|\hat{L}_n^{(\alpha)}(g;x) - g(x)\right| \leqslant L_n^{(\alpha)}\left(\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right|;x\right) + \left|\int_x^{\frac{n+p}{n}\left(\frac{x}{1-(\lambda+1)\alpha}\right)} \left(\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha} - u\right)g''(u)\mathrm{d}u\right|. \tag{13}$$

Since $\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right| \leqslant (t-x)^2\|g''\|$ and

$$\left|\int_x^{\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha}} \left(\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha} - u\right)g''(u)\mathrm{d}u\right| \leqslant \left\{\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha} - x\right\}^2\|g''\|.$$

Therefore, (13) implies that

$$\begin{aligned}\left|\hat{L}_n^{(\alpha)}(g;x) - g(x)\right| &\leqslant \left[L_n^{(\alpha)}\left((t-x)^2;x\right) + \left\{\frac{n+p}{n}\left(\frac{x}{1-(\lambda+1)\alpha}\right) - x\right\}^2\right]\|g''\|\\ &\leqslant \left[L_n^{(\alpha)}\left((t-x)^2;x\right) + \left\{\frac{px+nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right\}^2\right]\|g''\|.\end{aligned}$$

Take $\psi_{n,\lambda}^{(\alpha)}(x) = \left[L_n^{(\alpha)}\left((t-x)^2;x\right) + \left\{\frac{px+nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right\}^2\right]$. Therefore

$$\left|\hat{L}_n^{(\alpha)}(g;x) - g(x)\right| \leqslant \psi_{n,\lambda}^{(\alpha)}(x)\|g''\| \tag{14}$$

Again using definition of auxiliary operators and note Lemma 5, we have

$$\left|\hat{L}_n^{(\alpha)}((f-g);x)\right| \leqslant 3\|f-g\|$$

Thus, we get

$$\begin{aligned}\left|L_n^{(\alpha)}(f;x)-f(x)\right| &\leqslant \left|\hat{L}_n^{(\alpha)}((f-g);x)\right|+|g(x)-f(x)| \\ &+\left|\hat{L}_n^{(\alpha)}(g;x)-g(x)\right| \\ &+\left|f\left(\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha}\right)-f(x)\right| \\ &\leqslant 4\|f-g\|+\psi_{n,\lambda}^{(\alpha)}(x)\|g''\|+ \\ &\omega\left(f;\frac{px+nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right).\end{aligned}$$

Taking infimum on both the sides over $g \in C_B^2(I)$,

$$\begin{aligned}\left|L_n^{(\alpha)}(f;x)-f(x)\right| \leqslant &4K_2\left(f;\frac{\psi_{n,\lambda}^{(\alpha)}(x)}{4}\right) \\ &+\omega\left(f;\frac{px+nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right).\end{aligned}$$

Hence by (10), we get

$$\begin{aligned}\left|L_n^{(\alpha)}(f;x)-f(x)\right| &\leqslant C\omega_2\left(f;\frac{\sqrt{\psi_{n,\lambda}^{(\alpha)}(x)}}{2}\right) \\ &+\omega\left(f;\frac{px+nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right).\end{aligned}$$

□

We consider the following Lipschitz-type space (see Özarslan and Duman 2010)

$$Lip_M^*(\beta) := \left\{f \in C_B(I) : |f(y)-f(x)| \leqslant M\frac{|y-x|^\beta}{(x+y)^{\beta/2}}; x,y \in (0,\infty)\right\},$$

where M is a positive constant and $0<\beta\leqslant 1$.

Theorem 2 *For all $x \in I$ and $f \in Lip_M^*(\beta),\ 0<\beta \in (0,1]$, we get*

$$\left|L_n^{(\alpha)}(f;x)-f(x)\right| \leqslant M\left(\frac{\phi_n^{(\alpha)}(x)}{x}\right)^{\beta/2}, \tag{15}$$

where $\phi_n^{(\alpha)}(x)=L_n^{(\alpha)}\left((t-x)^2;x\right)$.

Proof Assume that $\beta=1$. Then, for $f \in Lip_M^*(1)$, we have

$$\begin{aligned}\left|L_n^{(\alpha)}(f;x)-f(x)\right| &\leqslant \left|\sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}(x)f\left(\frac{k}{n}\right)-f(x)\right| \\ &\leqslant \sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right| \\ &\leqslant M\sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}(x)\frac{\left|\frac{k}{n}-x\right|}{\left(\frac{k}{n}+x\right)^{1/2}}.\end{aligned}$$

Applying Cauchy–Schwarz inequality for sum and $\frac{1}{\sqrt{\frac{k}{n}+x}} \leqslant \frac{1}{\sqrt{x}}$ along with (3) as well as linearity of $L_n^{(\alpha)}(f;x)$, we have

$$\begin{aligned}\left|L_n^{(\alpha)}(f;x)-f(x)\right| &\leqslant \frac{M}{\sqrt{x}}\sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}(x)\left\{\left(\frac{k}{n}-x\right)^2\right\}^{1/2} \\ &\leqslant \frac{M}{\sqrt{x}}\left\{\sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}(x)\right\}^{1/2} \\ &\left\{\sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}\left(\frac{k}{n}-x\right)^2\right\}^{1/2} \\ &\leqslant \frac{M}{\sqrt{x}}\left\{L_n^{(\alpha)}(1;x)\right\}^{1/2}\left\{L_n^{(\alpha)}\left((t-x)^2;x\right)\right\}^{1/2} \\ &= M\left\{\frac{\phi_n^{(\alpha)}(x)}{x}\right\}^{1/2}.\end{aligned}$$

Therefore, the result is true for $\beta=1$.

Now, we prove the required result for $0<\beta<1$. Consider $f \in Lip_M^*(\beta)$

$$\begin{aligned}\left|L_n^{(\alpha)}(f;x)-f(x)\right| &\leqslant \sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right| \leqslant \\ &M\sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}(x)\frac{\left|\frac{k}{n}-x\right|^\beta}{\left(\frac{k}{n}+x\right)^{\beta/2}}.\end{aligned}$$

Using Holder's inequality for sum with $p=2/\beta, q=2/(2-\beta)$ and inequality $\frac{1}{\sqrt{\frac{k}{n}+x}} \leqslant \frac{1}{\sqrt{x}}$, we have

$$\begin{aligned}\left|L_n^{(\alpha)}(f;x)-f(x)\right| &\leqslant \frac{M}{x^{\beta/2}}\sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}(x)\left\{\left(\frac{k}{n}-x\right)^2\right\}^{\beta/2} \\ &\leqslant \frac{M}{x^{\beta/2}}\left\{\sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}(x)\left(\frac{k}{n}-x\right)^2\right\}^{\beta/2} \\ &\left\{\sum_{k=0}^{\infty}\omega_{n,k}^{(\alpha)}(x)\right\}^{\frac{2-\beta}{2}} \\ &\leqslant M\left\{\frac{L_n^{(\alpha)}\left((t-x)^2;x\right)}{x}\right\}^{\beta/2} \\ &= M\left\{\frac{\phi_n^{(\alpha)}(x)}{x}\right\}^{\beta/2}.\end{aligned}$$

□

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