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Simultaneous approximation on generalized Bernstein–Durrmeyer operators

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Abstract In the present paper, we study some theorems on approximation of the r-th derivative of a given function f by corresponding r-th derivative of the generalized Bernstein operator.

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1 Introduction

Very recently Deo et al. [4] introduced modified Bernstein operator B_n defined as:

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$p_{n,k}(x) = \left(1+\frac{1}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k} \quad \text{and} \quad x \in \left[0, 1-\frac{1}{n+1}\right].$$

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In this context Deo [1] has studied direct as well as converse results for the Beta operators and in [2,3] Deo has given Voronovskaya type results for exponential operators.

To approximate Lebesgue integrable functions on the interval [0, 1], Durrmeyer [6] first proposed integrated Bernstein polynomial. Later Derriennic [5], Gupta [7], Gupta and Srivastava [8] and Heilmann [9] studied so called Bernstein–Durrmeyer operators in detail and established many interesting properties of these operators. We study the following Durrmeyer variant of the operator (1.1) as:

$$(M_n f)(x) = n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t)dt, \tag{1.2}$$

where $p_{n,k}(x)$ is defined in (1.1) above. In the operators (1.2), the interval of the definition of integrability of function has been contracted from class [0, 1] to $[0, 1-\frac{1}{n+1}]$.

In this paper we prove some theorems on the approximation of r-th derivative of a function f by the corresponding operators $(M_n^{(r)})$.

2 Auxiliary results

In this section, we shall mention some definitions and certain lemmas to prove our main theorems.

Lemma 2.1 *If f is differentiable r times on $[0, 1-\frac{1}{n+1}]$, then we get*

$$\left(M_n^{(r)} f\right)(x) = \frac{(n+1)^{2(r+1)}}{n^{2r+1}} \frac{(n!)^2}{(n-r)!(n+r)!} \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^{\frac{n}{n+1}} p_{n+r,k+r}(t) f^{(r)}(t)dt. \tag{2.1}$$

Proof We have by Leibnitz's theorem

$$\begin{aligned}\left(M_n^{(r)} f\right)(x) &= \frac{(n+1)^{n+2}}{n^{n+1}} \sum_{i=0}^{r} \sum_{k=0}^{n-r+i} \binom{r}{i} \frac{(-1)^{r-i} n! x^{k-i}}{(k-i)!(n-k-r+i)!} \\ &\quad \cdot \left(\frac{n}{n+1} - x\right)^{n-k-r+i} \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t)dt \\ &= \frac{(n+1)^{r+2}}{n^{r+1}} \sum_{k=i}^{n-r+i} \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r-i} n!}{(n-r)!} p_{n-r,k-i}(x) \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t)dt \\ &= \frac{(n+1)^{r+2}}{n^{r+1}} \frac{n!}{(n-r)!} \sum_{k=0}^{n-r} (-1)^r p_{n-r,k}(x) \int_0^{\frac{n}{n+1}} \sum_{i=0}^{r} \binom{r}{i} (-1)^i p_{n,k+i}(t) f(t)dt.\end{aligned}$$

Again using Leibnitz's theorem

$$\frac{d^r}{dt^r}p_{n+r,k+r}(t)=\sum_{i=0}^{r}\binom{r}{i}(-1)^i\frac{(n+r)!}{n!}\left(\frac{n}{n+1}\right)^r p_{n,k+i}(t)$$

$$\left(M_n^{(r)}f\right)(x)=\frac{(n+1)^{2(r+1)}}{n^{2r+1}}\frac{(n!)^2}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}p_{n-r,k}(x)\int_0^{\frac{n}{n+1}}(-1)^r p_{n+r,k+r}^{(r)}(t)f(t)dt.$$

Further integrating by parts r times, we get the required result. □

Lemma 2.2 *Let $r,m\in N^0$ (the set of non-negative integers), $n\in N$ and $x\in[0,\infty)$. Let the m-th order moments are defined by if*

$$\mu_{n,m}(x)=(n+r+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n-r}p_{n-r,k}(x)\int_0^{\frac{n}{n+1}}p_{n+r,k+r}(t)(t-x)^m dt,$$

then we get

$$\mu_{n,0}(x)=1,\quad \mu_{n,1}(x)=\frac{(1+r)\{n-2x(n+1)\}}{(n+1)(n+r+2)},\tag{2.2}$$

$$\mu_{n,2}(x)=\frac{(r+1)(r+2)\{n-2x(n+1)\}^2}{(n+1)^2(n+r+2)(n+r+3)}+\frac{2x\,\{n-x(n+1)\}}{(n+r+2)(n+r+3)}\tag{2.3}$$

and

$$(m+n+r+2)\mu_{n,m+1}(x)=(1+m+r)\left(\frac{n}{n+1}-2x\right)\mu_{n,m}(x)$$
$$+2mx\left(\frac{n}{n+1}-x\right)\mu_{n,m-1}(x)+x\left(\frac{n}{n+1}-x\right)\mu'_{n,m}(x).\tag{2.4}$$

Consequently,

(i) *$\mu_{n,m}(x)$ is a polynomial in x of degree $\leq m$;*
(ii) *$\mu_{n,m}(x)=O(n^{-[\frac{m+1}{2}]})$, where $[\alpha]$ denotes the integral part of α.*

Proof The values of $\mu_{n,0}$ and $\mu_{n,1}$ can easily follows from the definition. We prove the recurrence relation as follows:

$$\mu'_{n,m}(x)=(n+r+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n-r}p'_{n-r,k}(x)\int_0^{\frac{n}{n+1}}p_{n+r,k+r}(t)(t-x)^m dt-m\mu_{n,m-1}(x)$$

Using the following relation

$$x\left(\frac{n}{n+1}-x\right)p'_{n,k}(x)=n\left(\frac{k}{n+1}-x\right)p_{n,k}(x),\tag{2.5}$$

then we get

$$
\begin{aligned}
& x\left(\frac{n}{n+1}-x\right)\left\{\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right\} \\
& = (n+r+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n-r} p_{n-r,k}(x)\int_0^{\frac{n}{n+1}}\left\{\frac{nk}{n+1}-(n-r)x\right\} p_{n+r,k+r}(t)(t-x)^m dt \\
& = (n+r+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n-r} p_{n-r,k}(x)\int_0^{\frac{n}{n+1}} t\left(\frac{n}{n+1}-t\right) p'_{n+r,k+r}(t)(t-x)^m dt \\
& \quad -r\left(\frac{n}{n+1}-2x\right)\mu_{n,m}(x)+(n+r)\mu_{n,m+1}(x) \\
& = (n+r+1)\left(1+\frac{1}{n}\right)\sum_{k=0}^{n-r} p_{n-r,k}(x)\int_0^{\frac{n}{n+1}}\left[-(t-x)^{m+2}+\left(\frac{n}{n+1}-2x\right)(t-x)^{m+1}\right. \\
& \quad \left. +x\left(\frac{n}{n+1}-x\right)(t-x)^m\right] p'_{n+r,k+r}(t)-r\left(\frac{n}{n+1}-2x\right)\mu_{n,m}(x)+(n+r)\mu_{n,m+1}(x) \\
& = (m+2)\mu_{n,m+1}(x)-\left(\frac{n}{n+1}-2x\right)(m+1)\mu_{n,m}(x)-mx\left(\frac{n}{n+1}-x\right)\mu_{n,m-1}(x) \\
& \quad -r\left(\frac{n}{n+1}-2x\right)\mu_{n,m}(x)+(n+r)\mu_{n,m+1}(x).
\end{aligned}
$$

This completes the proof of the recurrence relation. The values of $\mu_{n,1}(x)$ and $\mu_{n,2}(x)$ can be easily obtained from the above recurrence relation. □

3 Main results

In this section we shall prove the following main results.

Theorem 3.1 *If* $f^{(r)}$ *is a bounded and integrable in* $[0, 1-\frac{1}{n+1}]$ *and admits* $(r+2)$*-th derivative at a point* $x \in [0, 1-\frac{1}{n+1}]$*, then*

$$
\begin{aligned}
& \lim_{n\to\infty} n\left[\frac{n^{2r}(n-r)!(n+r+1)!}{(n+1)^{2r+1}(n!)^2}\left(M_n^{(r)}f\right)(x)-f^{(r)}(x)\right] \\
& = (1-2x)(1+r)f^{(r+1)}(x)+x(1-x)f^{(r+2)}(x). \qquad (3.1)
\end{aligned}
$$

Proof By Taylor's formula, we have

$$
f^{(r)}(t)=f^{(r)}(x)+(t-x)f^{(r+1)}(x)+\frac{(t-x)^2}{2}f^{(r+2)}(x)+\frac{(t-x)^2}{2}\zeta(t-x), \quad (3.2)
$$

where $\zeta(u)\to 0$ as $u\to 0$ and ζ is a bounded and integrable function on $[-x, 1-x]$.

Now, using (3.2) and by Lemma 2.2, we get

$$\frac{n^{2r}(n-r)!(n+r+1)!}{(n+1)^{2r+1}(n!)^2}\left(M_n^{(r)}f\right)(x)-f^{(r)}(x)$$
$$=\left\{\frac{(1+r)\{n-2x(n+1)\}}{(n+1)(n+r+2)}\right\}f^{(r+1)}(x)$$
$$+\frac{1}{2}\left\{\frac{(r+1)(r+2)\{n-2x(n+1)\}^2}{(n+1)^2(n+r+2)(n+r+3)}+\frac{2x\{n-x(n+1)\}}{(n+r+2)(n+r+3)}\right\}f^{(r+2)}(x)+R_{n,r}(x),$$

where

$$R_{n,r}(x)=\frac{1}{2}\sum_{k=0}^{n-r}p_{n-r,k}(x)\int_0^{\frac{n}{n+1}}p_{n+r,k+r}(t)(t-x)^2\zeta(t-x)dt.$$

Now we have to show that $nR_{n,r}\to 0$ as $n\to\infty$. Let $K=\sup_{u\in[-x,1-x]}|\zeta(u)|$ and let $\varepsilon>0$. Choose $\delta>0$ such that $|\zeta(u)|<\varepsilon$ when $|u|\le\delta$. So for all $t\in[0,1-\frac{1}{n+1}]$, we have $|\zeta(t-x)|<\varepsilon+K\frac{(t-x)^2}{\delta^2}$. Clearly

$$|nR_{n,r}(x)|<\frac{n\varepsilon}{2}M_n^{(r)}(t-x)^2(x)+\frac{Kn}{2\delta^2}M_n^{(r)}(t-x)^4(x)$$
$$=\frac{n\varepsilon}{2}\left\{\frac{(r+1)(r+2)\{n-2x(n+1)\}^2}{(n+1)^2(n+r+2)(n+r+3)}+\frac{2x\{n-x(n+1)\}}{(n+r+2)(n+r+3)}\right\}$$
$$+\frac{K}{2\delta^2}O\left(\frac{1}{n}\right),$$

since $\varepsilon>0$ is arbitrary, this implies that $|nR_{n,r}(x)|\to 0$ as $n\to\infty$. Thus as $n\to\infty$, we get the required result from (3.2). This completes the proof of the theorem. □

Theorem 3.2 *If* $f^{(r+1)}\in C[0,1-\frac{1}{n+1}]$ *and let* $\omega\left(f^{(r+1)};.\right)$ *be the moduli of continuity of* $f^{(r+1)}$. *Then for* $n\ge r,\ (r=0,1,2,....)$, *we have*

$$\left\|M_n^{(r)}-f^{(r)}\right\|\le\left\|f^{(r+1)}\right\|+\frac{1}{2\sqrt{n}}\left\{\sqrt{\lambda r}+\frac{\lambda r}{2}\right\}\omega\left(f^{(r+1)};\frac{1}{\sqrt{n}}\right),\tag{3.3}$$

where the norm is sup-norm over $[0,1-\frac{1}{n+1}]$, *and* $\lambda r=1+\frac{r}{2}$.

Proof Following [10] and by the Taylor formula

$$f^{(r)}(t)-f^{(r)}(x)=(t-x)f^{(r+1)}(x)+\int_x^t\left\{(f^{(r+1)}(y)-f^{(r+1)}(x)\right\}.$$

Now, applying (2.1) to the above and using the inequality

$$\left|f^{(r+1)}(y)-f^{(r+1)}(x)\right|\le\left\{1+\frac{|y-x|}{\delta}\right\}\omega\left(f^{(r+1)};\delta\right),$$

and the results (2.2) and (2.3), we have

$$\begin{aligned}
&\left|\left(M_n^{(r)} f\right)(x) - f^{(r)}(x)\right| \\
&\leq \left|f^{(r+1)}(x)\right| \left|M_n^{(r)}(t-x)(x)\right| + \omega\left(f^{(r+1)};\delta\right) M_n^{(r)}\left[\left|\int_x^t 1 + \frac{|y-x|}{\delta} dy\right|\right](x), \\
&\leq \left|f^{(r+1)}(x)\right| \left|M_n^{(r)}(t-x)(x)\right| + \omega\left(f^{(r+1)};\delta\right) \\
&\quad \times \left\{\sqrt{M_n^{(r)}(t-x)^2(x)} + \frac{1}{2\delta} M_n^{(r)}(t-x)^2(x)\right\}.
\end{aligned}$$

Choosing $\delta = \frac{1}{\sqrt{n}}$ and using the result (2.3), we get the required result (3.3). This completes the proof. □

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