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Approximation by Kantorovich form of modified Szász–Mirakyan operators

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ABSTRACT

In the present article, we consider the Kantorovich type generalized Szász–Mirakyan operators based on Jain and Pethe operators [32]. We study local approximation results in terms of classical modulus of continuity as well as Ditzian–Totik moduli of smoothness. Further we establish the rate of convergence in class of absolutely continuous functions having a derivative coinciding a.e. with a function of bounded variation.

1. Introduction

In 1977, Jain and Pethe [32] generalized the well-known Szász–Mirakyan operators [40] as:

$$\begin{aligned} S_n^{[\alpha]}(f;x) &= (1+n\alpha)^{-\frac{x}{\alpha}} \sum_{k=0}^{\infty} \left(\alpha + \frac{1}{n}\right)^{-k} \frac{x^{(k,-\alpha)}}{k!} f\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^{\infty} s_{n,k}^{[\alpha]}(x) f\left(\frac{k}{n}\right), \end{aligned} \tag{1.1}$$

where

$$s_{n,k}^{[\alpha]}(x) = (1+n\alpha)^{-\frac{x}{\alpha}} \left(\alpha + \frac{1}{n}\right)^{-k} \frac{x^{(k,-\alpha)}}{k!},$$

$x^{(k,-\alpha)} = x(x+\alpha)\dots(x+(k-1)\alpha)$, $x^{(0,-\alpha)} = 1$ and f is any function of exponential type such that

$$|f(t)| \le Ke^{At} \quad (t \ge 0),$$

for some finite constants $K, A > 0$. Here $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ is such that

$$0 \le \alpha_n \le \frac{1}{n}.$$

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The operators $S_n^{[\alpha]}$ have also been considered by Stancu [39], Mastroianni [36], Della Vecchia and Kocic [17] and Finta [26,27].

Abel and Ivan [1] gave the following alternate form of operators (1.1) (by putting $c = \frac{1}{n\alpha}$):

$$S_{n,c}(f;x) = \sum_{k=0}^{\infty} \left(\frac{c}{1+c}\right)^{ncx} \binom{ncx+k-1}{k} (1+c)^{-k} f\left(\frac{k}{n}\right), \qquad x \geq 0, \tag{1.2}$$

where $c = c_n \geq \beta (n = 0, 1, 2, ...)$ for certain constant $\beta > 0$. Also, for a particular case $\alpha = \frac{1}{n}$, the operators (1.1) reduce to another form, which was considered by Agratini [2] as follows:

$$S_n^{[\frac{1}{n}]}(f;x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \tag{1.3}$$

where

$$(nx)_k = nx(nx+1)\dots(nx+k-1), \qquad k \geq 1,$$

and $(nx)_0 = 1$. These operators (1.3) are special cases of Lupaş operators [35]. The operators (1.3) have also been studied in [25] and [37].

Agratini [3] modified the operators (1.3) into integral form in Kantorovich sense as:

$$T_n(f;x) = n \sum_{k=0}^{\infty} 2^{-nx} \frac{(nx)_k}{2^k k!} \int_{k/n}^{(k+1)/n} f(t)dt, \tag{1.4}$$

and studied some approximation properties. Very recently, Deo et al. considered generalized positive linear operators based on Pólya–Eggenberger and inverse Pólya–Eggenberger distribution in [21] and furthermore, they gave Kantorovich variant of these generalized operators in [18]. Several researchers have given some interesting results on Kantorovich variant of various operators (see [7–16,30,38]). Motivated by above works, for any bounded and integrable function f defined on $[0, \infty)$, we also modify the operators (1.1) in Kantorovich form:

$$L_n^{[\alpha]}(f;x) = n \sum_{k=0}^{\infty} s_{n,k}^{[\alpha]}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt. \tag{1.5}$$

Special cases:

(1) For $\alpha = 0$ in (1.5), we get Szász–Kantorovich operators given by Totik in [41].
(2) For $\alpha = \frac{1}{n}$ in (1.5), we obtain another Kantorovich operators considered by Agratini [3].

The focus of this paper is to study the approximation properties of modified Kantorovich operators (1.5). First we obtain local approximation formula via modulus of continuity of second order then we use Ditzian–Totik moduli of smoothness to discuss the rate of convergence of our operators. Finally, we establish the rate of convergence for functions having derivatives of bounded variation. The properties discussed in this article can be found in some recent papers like [4,6,19,20,24,28,29,31,33].

2. Auxiliary results

In order to prove the main convergence properties of operators (1.5), we need the following basic results:

Lemma 2.1 *[39]. For the generalized Szász–Mirakyan operators (1.1) hold*

$$S_n^{[\alpha]}(1;x) = 1, \qquad S_n^{[\alpha]}(t;x) = x,$$

and

$$S_n^{[\alpha]}\left(t^2;x\right) = x^2 + \left(\alpha + \frac{1}{n}\right)x.$$

Proposition 2.1. *For the operators (1.1), there hold the following higher order moments:*

$$S_n^{[\alpha]}\left(t^3;x\right) = x^3 + 3\left(\alpha + \frac{1}{n}\right)x^2 + \left(2\alpha^2 + \frac{3\alpha}{n} + \frac{1}{n^2}\right)x,$$

and

$$S_n^{[\alpha]}\left(t^4;x\right) = x^4 + 6\left(\alpha + \frac{1}{n}\right)x^3 + \left(11\alpha^2 + \frac{18\alpha}{n} + \frac{7}{n^2}\right)x^2 + \left(6\alpha^3 + \frac{12\alpha^2}{n} + \frac{7\alpha}{n^2} + \frac{1}{n^3}\right)x.$$

Proof. By definition we can write

$$S_n^{[\alpha]}\left(t^3;x\right) = (1+\alpha n)^{-\frac{x}{\alpha}} \sum_{k=1}^{\infty} \frac{x(x+\alpha)\dots(x+(k-1)\alpha)n^k}{k!(1+\alpha n)^k} \frac{k^3}{n^3}$$

$$= \frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=1}^{\infty}\frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\dots\left(\frac{x}{\alpha}+k-1\right)\alpha^k n^k}{(k-1)!(1+\alpha n)^k}(k(k-1)+k)$$

$$= \frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=2}^{\infty}\frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\dots\left(\frac{x}{\alpha}+k-1\right)\alpha^k n^k}{(k-2)!(1+\alpha n)^k}k+\frac{1}{n}S_n^{[\alpha]}\left(t^2;x\right)$$

$$= \frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=3}^{\infty}\frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\dots\left(\frac{x}{\alpha}+k-1\right)\alpha^k n^k}{(k-3)!(1+\alpha n)^k}$$

$$+\frac{2(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=2}^{\infty}\frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\dots\left(\frac{x}{\alpha}+k-1\right)\alpha^k n^k}{(k-2)!(1+\alpha n)^k}+\frac{1}{n}S_n^{[\alpha]}\left(t^2;x\right)$$

$$= \frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{\alpha}+k+2\right)!\alpha^{k+3}n^{k+3}}{k!\left(\frac{x}{\alpha}-1\right)!(1+\alpha n)^{k+3}}$$

$$+\frac{2(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=0}^{\infty}\frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\dots\left(\frac{x}{\alpha}+k+1\right)}{k!}\left(\frac{\alpha n}{1+\alpha n}\right)^{k+2}+\frac{1}{n}S_n^{[\alpha]}\left(t^2;x\right)$$

$$= \frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{\alpha}+k+2\right)!\alpha^{k+3}n^{k+3}}{k!\left(\frac{x}{\alpha}-1\right)!(1+\alpha n)^{k+3}}$$

$$+\frac{2(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\sum_{k=0}^{\infty}\frac{\left(\frac{x}{\alpha}+k+1\right)!}{k!\left(\frac{x}{\alpha}+1\right)!}\left(\frac{\alpha n}{1+\alpha n}\right)^{k+2}+\frac{1}{n}S_n^{[\alpha]}\left(t^2;x\right)$$

$$= \frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{\alpha}+k+2\right)!\alpha^{k+3}n^{k+3}}{k!\left(\frac{x}{\alpha}-1\right)!(1+\alpha n)^{k+3}}$$

$$+\frac{2(1+\alpha n)^{-\frac{x}{\alpha}}}{n}\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\alpha^2\left(1-\frac{\alpha n}{1+\alpha n}\right)^{-\left(\frac{x}{\alpha}+2\right)}+\frac{1}{n}S_n^{[\alpha]}\left(t^2;x\right)$$

$$= \frac{(1+\alpha n)^{-\frac{x}{\alpha}}}{n^3}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{\alpha}+k+2\right)!\alpha^{k+3}n^{k+3}}{k!\left(\frac{x}{\alpha}-1\right)!(1+\alpha n)^{k+3}}+\frac{2}{n}\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\alpha^2+\frac{1}{n}\left[x^2+\left(\alpha+\frac{1}{n}\right)x\right]$$

$$= x^3+3\left(\alpha+\frac{1}{n}\right)x^2+\left(2\alpha^2+\frac{3\alpha}{n}+\frac{1}{n^2}\right)x.$$

Similarly, we can prove the expression for $S_n^{[\alpha]}\left(t^4;x\right)$. □

Lemma 2.2. *For Kantorovich operators (1.5), we have*

$$L_n^{[\alpha]}(1;x)=1,\qquad L_n^{[\alpha]}(t;x)=x+\frac{1}{2n},$$

$$L_n^{[\alpha]}\left(t^2;x\right)=x^2+\left(\alpha+\frac{2}{n}\right)x+\frac{1}{3n^2},$$

$$L_n^{[\alpha]}\left(t^3;x\right)=x^3+3\left(\alpha+\frac{3}{2n}\right)x^2+\left(2\alpha^2+\frac{9\alpha}{2n}+\frac{7}{2n^2}\right)x+\frac{1}{4n^3},$$

$$L_n^{[\alpha]}\left(t^4;x\right)=x^4+\left(6\alpha+\frac{8}{n}\right)x^3+\left(11\alpha^2+\frac{24\alpha}{n}+\frac{15}{n^2}\right)x^2$$

$$+\left(6\alpha^3+\frac{16\alpha^2}{n}+\frac{15\alpha}{n^2}+\frac{6}{n^3}\right)x+\frac{1}{5n^4}.$$

Proof. Taking into account of Lemma 2.1 and Proposition 2.1, we can easily get the desired result. □

Remark 2.1. By simply applying Lemma 2.2, we have

$$L_n^{[\alpha]}(t-x;x)=x+\frac{1}{2n}-x=\frac{1}{2n},$$

$$L_n^{[\alpha]}\left((t-x)^2;x\right)=L_n^{[\alpha]}\left(t^2;x\right)-2xL_n^{[\alpha]}(t;x)+x^2L_n^{[\alpha]}(1;x)$$

$$=x^2+\left(\alpha+\frac{2}{n}\right)x+\frac{1}{3n^2}-2x\left(x+\frac{1}{2n}\right)+x^2$$

$$=\left(\alpha+\frac{1}{n}\right)x+\frac{1}{3n^2},$$

$$
\begin{aligned}
L_n^{[\alpha]}\big((t-x)^3;x\big) &= L_n^{[\alpha]}\big(t^3;x\big) - 3xL_n^{[\alpha]}\big(t^2;x\big) + 3x^2L_n^{[\alpha]}(t;x) - x^3L_n^{[\alpha]}(1;x)\\
&= x^3 + 3\Big(\alpha + \frac{3}{2n}\Big)x^2 + \Big(2\alpha^2 + \frac{9\alpha}{2n} + \frac{7}{2n^2}\Big)x + \frac{1}{4n^3}\\
&\quad - 3x\Big(x^2 + \Big(\alpha + \frac{2}{n}\Big)x + \frac{1}{3n^2}\Big) + 3x^2\Big(x + \frac{1}{2n}\Big) - x^3\\
&= \frac{3x^2}{n} + \Big(2\alpha^2 + \frac{9\alpha}{2n} + \frac{5}{2n^2}\Big)x + \frac{1}{4n^3},
\end{aligned}
$$

and

$$
\begin{aligned}
L_n^{[\alpha]}\big((t-x)^4;x\big) &= L_n^{[\alpha]}\big(t^4;x\big) - 4xL_n^{[\alpha]}\big(t^3;x\big) + 6x^2L_n^{[\alpha]}\big(t^2;x\big)\\
&\quad - 4x^3L_n^{[\alpha]}(t;x) + x^4L_n^{[\alpha]}(1;x)\\
&= x^4 + \Big(6\alpha + \frac{8}{n}\Big)x^3 + \Big(11\alpha^2 + \frac{24\alpha}{n} + \frac{15}{n^2}\Big)x^2\\
&\quad + \Big(6\alpha^3 + \frac{16\alpha^2}{n} + \frac{15\alpha}{n^2} + \frac{6}{n^3}\Big)x + \frac{1}{5n^4}\\
&\quad - 4x\Big(x^3 + 3\Big(\alpha + \frac{3}{2n}\Big)x^2 + \Big(2\alpha^2 + \frac{9\alpha}{2n} + \frac{7}{2n^2}\Big)x + \frac{1}{4n^3}\Big)\\
&\quad + 6x^2\Big(x^2 + \Big(\alpha + \frac{2}{n}\Big)x + \frac{1}{3n^2}\Big) - 4x^3\Big(x + \frac{1}{2n}\Big) + x^4\\
&= \Big(3\alpha^2 + \frac{6\alpha}{n} + \frac{3}{n^2}\Big)x^2 + \Big(6\alpha^3 + \frac{16\alpha^2}{n} + \frac{15\alpha}{n^2} + \frac{5}{n^3}\Big)x + \frac{1}{5n^4}.
\end{aligned}
$$

Lemma 2.3. *Let f be a bounded function defined on $[0, \infty)$ with $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$, then*

$$\Big|L_n^{[\alpha]}(f;x)\Big| \le \|f\|.$$

Lemma 2.4. *For $n \in \mathbb{N}$, we have*

$$L_n^{[\alpha]}\big((t-x)^2;x\big) \le \frac{C}{n}\delta_n^2(x),$$

where $\delta_n^2(x) = \varphi^2(x) + \frac{1}{n}$ and $\varphi^2(x) = x$.

Now we can write the operators (1.5) in other form as:

$$L_n^{[\alpha]}(f;x) = \int_0^\infty K_n^{[\alpha]}(x,t)f(t)dt, \tag{2.1}$$

where

$$K_n^{[\alpha]}(x,t) = n\sum_{k=0}^{\infty} s_{n,k}^{[\alpha]}(x)\chi_{n,k}(t),$$

and $\chi_{n,\,k}(t)$ is the characteristic function of the interval $\big[\frac{k}{n}, \frac{k+1}{n}\big]$ w.r.t $[0, \infty)$.

Lemma 2.5. *For $x \in (0, \infty)$ and sufficiently large n, we have*

(i) *Since $0 \le y < x$, therefore*

$$\beta_n(x,y) = \int_0^y K_n^{[\alpha]}(x,t)dt \le \frac{C\delta_n^2(x)}{n(x-y)^2}.$$

(ii) *If $x < z < \infty$ then we get*

$$1 - \beta_n(x,z) = \int_z^\infty K_n^{[\alpha]}(x,t)dt \le \frac{C\delta_n^2(x)}{n(z-x)^2}.$$

3. Direct results

Using the well-known Bohman–Korovkin–Popoviciu theorem (see [34]) we get the uniform convergence of the operators (1.5).

Theorem 3.1. *Let $f\in C[0,\infty)\cap E$ and $\alpha(n)$ be such that $\alpha\to 0$ as $n\to\infty$, then we have*

$$\lim_{n\to\infty} L_n^{(\alpha)}(f;x)=f(x)$$

uniformly on each compact subset of $[0,\infty)$, where $C[0,\infty)$ is the space of all real-valued continuous functions on $[0,\infty)$ and

$$E:=\left\{f:x\in[0,\infty),\ \frac{f(x)}{1+x^2}\text{is convergent as }\ x\to\infty\right\}$$

Proof. Taking Lemma 2.2 into the account and the fact that $\alpha\to 0$ as $n\to\infty$, it is clear that

$$\lim_{n\to\infty} L_n^{(\alpha)}(e_i;x)=x^i,\quad i=0,1,2$$

uniformly on each compact subset of $[0,\infty)$. Hence, applying the well-known Korovkin-type theorem [5] regarding the convergence of a sequence of positive linear operators, we get the desired result. □

Now we present local approximation formula via modulus of continuity of second order. For this let us start by recalling the following definitions:

Definition 3.1. *Let $f\in C_B[0,\infty)$, the space of all real-valued continuous and bounded functions on $[0,\infty)$ then Peetre's K-functional is defined as*

$$K_2(f,\delta)=\inf_{g\in W_\infty^2}\left\{\|f-g\|+\delta\left\|g''\right\|\right\},\tag{3.1}$$

where $\delta>0$, $\|f\|=\sup_{x\in[0,\infty)}|f(x)|$ and $W_\infty^2=\left\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\right\}$. Recall that, from ([22], p. 177, Theorem 2.4), there exists a positive constant M such that

$$K_2(f,\delta)\le M\omega_2\left(f,\sqrt{\delta}\right),\tag{3.2}$$

where $\omega_2\left(f,\sqrt{\delta}\right)$ is second order modulus of continuity given by

$$\omega_2(f,\delta):=\sup\{|f(x+h)-2f(x)+f(x-h)|\ :\ x,x\pm h\in[0,\infty),\ 0\le h\le\delta\}.\tag{3.3}$$

Moreover, we define the usual (first order) modulus of continuity as follows:

Definition 3.2. *Let $f\in C_B[0,\infty)$ then*

$$\omega(f,\delta):=\sup\{|f(x)-f(y)|\ :\ x,y\in[0,\infty),\ |x-y|\le\delta\},\tag{3.4}$$

where $\delta>0$.

Theorem 3.2. *Let be $f\in C_B[0,\infty)$, then for any $x\in[0,\infty)$ it follows*

$$\left|L_n^{[\alpha]}(f;x)-f(x)\right|\le M\omega_2\left(f,\tfrac{1}{2}\delta_n(x)\right)+\omega(f,\beta_n),$$

where M is an absolute constant and

$$\delta_n(x)=\left(L_n^{[\alpha]}\left((t-x)^2;x\right)+\left(L_n^{[\alpha]}(t-x;x)\right)^2\right)^{\frac{1}{2}},\quad \beta_n=L_n^{[\alpha]}(t-x;x)$$

such that both terms δ_n and β_n tends to zero as $n\to\infty$.

Proof. For $x\in[0,\infty)$, consider the operators

$$\hat{L}_n^{[\alpha]}(f;x)=L_n^{[\alpha]}(f;x)-f\left(x+\frac{1}{2n}\right)+f(x).\tag{3.5}$$

Note that $\hat{L}_n^{[\alpha]}(1;x)=1$ and $\hat{L}_n^{[\alpha]}(t;x)=x$, i.e. constants and linear functions are preserved by the operators $\hat{L}_n^{[\alpha]}$. Therefore,

$$\hat{L}_n^{[\alpha]}(t-x;x)=0.\tag{3.6}$$

Let $g\in W_\infty^2$ and $x,\ t\in[0,\infty)$. By Taylor's expansion, we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)du.$$

Applying $\hat{L}_n^{[\alpha]}$ on both sides of the above Taylor's expansion, we get

$$\hat{L}_n^{[\alpha]}(g;x)-g(x)=g'(x)\cdot\hat{L}_n^{[\alpha]}(t-x;x)+\hat{L}_n^{[\alpha]}\left(\int_x^t(t-u)g''(u)du;x\right)$$
$$=L_n^{[\alpha]}\left(\int_x^t(t-u)g''(u)du;x\right)-\int_x^{x+\frac{1}{2n}}\left(x+\frac{1}{2n}-u\right)g''(u)du.$$

Observe that

$$\left|\int_x^t(t-u)g''(u)\right|\le(t-x)^2\cdot\|g''\|.$$

Thus

$$\left|\hat{L}_n^{[\alpha]}(g;x)-g(x)\right|\le\left(L_n^{[\alpha]}((t-x)^2;x)+\left(L_n^{[\alpha]}(t-x;x)\right)^2\right)\cdot\|g''\|.$$

Making use of Definition (3.5) of the operators $\hat{L}_n^{[\alpha]}$ and Lemma 2.3, we have

$$\left|L_n^{[\alpha]}(f;x)-f(x)\right|\le\left|\hat{L}_n^{[\alpha]}(f-g;x)\right|+\left|\hat{L}_n^{[\alpha]}(g;x)-g(x)\right|$$
$$+|g(x)-f(x)|+\left|f\left(x+\frac{1}{2n}\right)-f(x)\right|$$
$$\le 4\|f-g\|+\delta_n^2(x)\left\|g''\right\|+\omega(f,\beta_n),$$

with $\delta_n^2(x)=L_n^{[\alpha]}\left((t-x)^2;x\right)+\left(L_n^{[\alpha]}(t-x;x)\right)^2$ and $\beta_n=L_n^{[\alpha]}(t-x;x)$.

Now taking infimum on the right-hand side over all $g\in W_\infty^2$ and using the relation (3.2), we get

$$\left|L_n^{[\alpha]}(f;x)-f(x)\right|\le 4K_2\left(f,\frac{{\delta_n}^2(x)}{4}\right)+\omega(f,\beta_n)$$
$$\le M\omega_2\left(f,\tfrac{1}{2}\delta_n(x)\right)+\omega(f,\beta_n).$$

Hence the proof. □

Now we obtain the convergence with the help of Ditzian–Totik moduli of smoothness which is defined as

$$\omega_{\varphi^\lambda}^2(f,t)=\sup_{0<h\le t}\sup_{x\pm h\varphi^\lambda\in[0,\infty)}\left|\Delta_{h\varphi^\lambda}^2f(x)\right|,\tag{3.7}$$

where

$$\varphi(x)=\sqrt{x}\qquad\text{and}\qquad 0\le\lambda\le 1,$$

$$\Delta_{h\varphi^\lambda}^2f(x)=f\left(x+h\varphi^\lambda(x)\right)-2f(x)+f\left(x-h\varphi^\lambda(x)\right),$$

and corresponding K-functional is

$$K_{\varphi^\lambda}\left(f,t^2\right)=\inf_{g\in D_\lambda^2}\left\{\|f-g\|+t^2\left\|\varphi^{2\lambda}g''\right\|\right\},\tag{3.8}$$

with

$$D_\lambda^2=\left\{f\in C[0,\infty),\ f'\in A.C_{loc},\ \left\|\varphi^{2\lambda}f''\right\|<\infty\right\}.$$

We have following relation:

$$\omega_{\varphi^\lambda}^2(f,t)\sim K_{\varphi^\lambda}\left(f,t^2\right).\tag{3.9}$$

Theorem 3.3. *Let $f\in C_B[0,\infty)$, then for any $x\in[0,\infty)$ we have*

$$\left|L_n^{[\alpha]}(f;x)-f(x)\right|\le C\omega_2^{\varphi^\lambda}\left(f,\frac{\delta_n^{(1-\lambda)}(x)}{\sqrt{n}}\right)+\omega\left(f,\frac{1}{2n}\right),$$

where C is an absolute constant and

$$\delta_n(x)=\left(L_n^{[\alpha]}\left((t-x)^2;x\right)+\left(L_n^{[\alpha]}(t-x;x)\right)^2\right)^{\frac{1}{2}}.$$

Proof. Consider the operators defined by (3.5)

$$\hat{L}_n^{[\alpha]}(f;x) = L_n^{[\alpha]}(f;x) + f(x) - f\left(x + \frac{1}{2n}\right) \tag{3.10}$$

For above considered operators, we can write $\hat{L}_n^{[\alpha]}(1;x) = 1$ and $\hat{L}_n^{[\alpha]}(t;x) = x$.

Therefore, Definition (3.10), Lemma 2.3 and Lemma 2.4 gives

$$\hat{L}_n^{[\alpha]}(t-x;x) = 0, \qquad \hat{L}_n^{[\alpha]}\left((t-x)^2;x\right) \le \frac{C}{n}\delta_n^2(x) \tag{3.11}$$

and
$$\left\|\hat{L}_n^{[\alpha]}(f;x)\right\| \le 3\|f\|.$$

Again, from ([23], p. 141), for $t < u < x$, we have

$$\frac{|t-u|}{\varphi^{2\lambda}(u)} \le \frac{|t-x|}{\varphi^{2\lambda}(x)} \quad \text{and} \quad \frac{|t-u|}{\delta_n^{2\lambda}(u)} \le \frac{|t-x|}{\delta_n^{2\lambda}(x)}. \tag{3.12}$$

Now

$$\begin{aligned} \left|\hat{L}_n^{[\alpha]}(f;x) - f(x)\right| &\le \left|\hat{L}_n^{[\alpha]}(f-g;x)\right| + \left|\hat{L}_n^{[\alpha]}(g;x) - g(x)\right| \\ &\quad + |f(x) - g(x)| \\ &\le 4\|f-g\| + \left|\hat{L}_n^{[\alpha]}(g;x) - g(x)\right|. \end{aligned} \tag{3.13}$$

For $g \in D_\lambda^2$ and $t, x \in [0, \infty)$, using Taylor's expansion with integral remainder,

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Operating $\hat{L}_n^{[\alpha]}$ and using (3.11) and (3.12), we get

$$\begin{aligned} \left|\hat{L}_n^{[\alpha]}(g;x) - g(x)\right| &= \left|\hat{L}_n^{[\alpha]}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\ &\le \left|L_n^{[\alpha]}\left(\int_x^t (t-u)g''(u)du;x\right)\right| + \left|\int_x^{x+\frac{1}{2n}} \left(x + \frac{1}{2n} - u\right)g''(u)du\right| \\ &\le \left\|\delta_n^{2\lambda}g''\right\| L_n^{[\alpha]}\left(\frac{(t-x)^2}{\delta_n^{2\lambda}(x)};x\right) + \left\|\delta_n^{2\lambda}g''\right\| \frac{\left(\frac{1}{2n}\right)^2}{\delta_n^{2\lambda}(x)} \\ &= \delta_n^{-2\lambda}(x)\left\|\delta_n^{2\lambda}g''\right\| L_n^{[\alpha]}\left((t-x)^2;x\right) + \left\|\delta_n^{2\lambda}g''\right\| \delta_n^{-2\lambda}(x)\frac{1}{(2n)^2} \\ &\le C\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\delta_n^{2\lambda}g''\right\|\right) + \frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\delta_n^{2\lambda}g''\right\| \\ &\le C\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\delta_n^{2\lambda}g''\right\|. \end{aligned} \tag{3.14}$$

From (3.13), (3.14) and then using definition of K-functional (corresponding to Ditzian–Totik) along with the relation (3.9),

$$\begin{aligned} \left|\hat{L}_n^{[\alpha]}(f;x) - f(x)\right| &\le 4\|f-g\| + C\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\varphi^{2\lambda}g''\right\| \\ &\le C\omega_{\varphi^\lambda}^2\left(f, \frac{\delta_n^{(1-\lambda)}(x)}{\sqrt{n}}\right). \end{aligned}$$

Hence

$$\begin{aligned} \left|L_n^{[\alpha]}(f;x) - f(x)\right| &\le \left|\hat{L}_n^{[\alpha]}(f;x) - f(x)\right| + \left|f\left(x + \frac{1}{2n}\right) - f(x)\right| \\ &\le C\omega_{\varphi^\lambda}^2\left(f, \frac{\delta_n^{(1-\lambda)}(x)}{\sqrt{n}}\right) + \omega\left(f, \frac{1}{2n}\right). \end{aligned}$$

Thus the proof is complete. □

3.1. Rate of convergence:

Here we estimate the rate of convergence of the operators (1.5) in the class $DBV[0,\infty)$, the class of all absolutely continuous functions f defined on $[0,\infty)$ having a derivative coinciding a.e. with a function of bounded variation on $[0,\infty)$. It can be observed that for $f\in DBV[0,\infty)$, we can write

$$f(x)=\int_0^x g(t)dt+f(0),$$

where $g(t)$ is a function of bounded variation on each finite subinterval of $[0,\infty)$.

Theorem 3.4. *Let $f\in DBV(0,\infty)$ then for all $x\in(0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}\left|L_n^{[\alpha]}(f;x)-f(x)\right| &\le \frac{1}{4n}\left|f'(x+)+f'(x-)\right|+\frac{1}{2}\sqrt{\frac{C}{n}}\left|f'(x+)-f'(x-)\right|\\ &+\frac{C\delta_n^2(x)}{nx^2}\left|f(2x)-f(x)-xf'(x+)\right|\\ &+\frac{x}{\sqrt{n}}\vee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}\left(f_x{}'\right)+\frac{C\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\vee_{x-\frac{x}{k}}^{x+\frac{x}{k}}\left(f_x{}'\right)\\ &+M(\gamma,r,x)+\frac{|f(x)|}{nx^2}C\delta_n^2(x)+\sqrt{\frac{C}{n}}\delta_n(x)f'(x+),\end{aligned} \tag{3.15}$$

where $\vee_a^b f(x)$ denotes the total variation of f on $[a,b]$, f_x is an auxiliary operator given by

$$f_x(t)=\begin{cases} f(t)-f(x-), & 0\le t<x\\ 0, & t=x\\ f(t)-f(x+), & x<t<\infty\end{cases}$$

and

$$M(\gamma,r,x)=M2^{\gamma}\left(\int_0^\infty (t-x)^{2r}K_n^{[\alpha]}(x,t)dt\right)^{\frac{\gamma}{2r}}$$

Proof. Because $L_n^{[\alpha]}(1;x)=1$, therefore for all $x\in(0,\infty)$, we obtain

$$\begin{aligned}L_n^{[\alpha]}(f;x)-f(x)&=\int_0^\infty (f(t)-f(x))K_n^{[\alpha]}(x,t)dt\\ &=\int_0^\infty K_n^{[\alpha]}(x,t)\int_x^t f'(u)dudt.\end{aligned} \tag{3.16}$$

For $f\in DBV(0,\infty)$, we may write

$$\begin{aligned}f'(u)&=\frac{1}{2}\left(f'(x+)+f'(x-)\right)+f_x{}'(u)+\frac{1}{2}\left(f'(x+)-f'(x-)\right)\mathrm{sgn}(u-x)\\ &+\delta_x(u)\left(f'(u)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right),\end{aligned} \tag{3.17}$$

where

$$\delta_x(u)=\begin{cases}1, & u=x\\ 0, & u\neq x.\end{cases}$$

It is easy to write that

$$\int_0^\infty\left(\int_x^t\left(f'(u)-\frac{1}{2}\left(f'(x+)+f'(x-)\right)\right)\delta_x(u)du\right)K_n^{[\alpha]}(x,t)dt=0. \tag{3.18}$$

Using (2.1), we obtain

$$\begin{aligned}&\int_0^\infty\left(\int_x^t\frac{1}{2}\left(f'(x+)+f'(x-)\right)du\right)K_n^{[\alpha]}(x,t)dt\\ &=\frac{1}{2}\left(f'(x+)+f'(x-)\right)L_n^{[\alpha]}((t-x);x).\end{aligned} \tag{3.19}$$

Moreover,

$$\int_0^\infty\left(\int_x^t\frac{1}{2}\left(f'(x+)-f'(x-)\right)\mathrm{sgn}(u-x)du\right)K_n^{[\alpha]}(x,t)dt$$

$$
\begin{aligned}
&= \int_0^\infty \frac{1}{2}\big(f'(x+)-f'(x-)\big)(t-x)K_n^{[\alpha]}(x,t)dt\\
&\le \frac{1}{2}\big|f'(x+)-f'(x-)\big|\int_0^\infty |t-x|K_n^{[\alpha]}(x,t)dt\\
&= \frac{1}{2}\big|f'(x+)-f'(x-)\big|L_n^{[\alpha]}(|t-x|;x)\\
&\le \tfrac{1}{2}|f'(x+)-f'(x-)|\Big(L_n^{[\alpha]}\big((t-x)^2;x\big)\Big)^{1/2}.
\end{aligned}
\tag{3.20}
$$

Using Eqs. (3.16)–(3.20) with Lemma 2.4, we have

$$
\begin{aligned}
L_n^{[\alpha]}(f;x)-f(x) &\le \frac{1}{2}\big(f'(x+)+f'(x-)\big)L_n^{[\alpha]}((t-x);x)\\
&\quad+\frac{1}{2}\big|f'(x+)-f'(x-)\big|\Big(L_n^{[\alpha]}\big((t-x)^2;x\big)\Big)^{1/2}\\
&\quad+\int_0^\infty\left(\int_x^t f_x{}'(u)du\right)K_n^{[\alpha]}(x,t)dt\\
&\le \frac{1}{4n}\big(f'(x+)+f'(x-)\big)+\frac{1}{2}\sqrt{\frac{C}{n}}\delta_n(x)\big|f'(x+)-f'(x-)\big|\\
&\quad+\int_0^\infty\left(\int_x^t f_x{}'(u)du\right)K_n^{[\alpha]}(x,t)dt.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\Big|L_n^{[\alpha]}(f;x)-f(x)\Big| &\le \frac{1}{4n}\big|f'(x+)+f'(x-)\big|+\frac{1}{2}\sqrt{\frac{C}{n}}\delta_n(x)\big|f'(x+)-f'(x-)\big|\\
&\quad+A_{nx}+B_{nx},
\end{aligned}
\tag{3.21}
$$

where

$$
A_{nx}=\left|\int_0^x\left(\int_x^t f_x{}'(u)du\right)K_n^{[\alpha]}(x,t)dt\right|,
$$

and

$$
B_{nx}=\left|\int_x^\infty\left(\int_x^t f_x{}'(u)du\right)K_n^{[\alpha]}(x,t)dt\right|.
$$

Applying Lemma 2.5, integrating by parts and taking $y=x-\frac{x}{\sqrt{n}}$, we obtain

$$
\begin{aligned}
A_{nx}&=\left|\int_0^x\left(\int_x^t f_x{}'(u)du\right)d_t\beta_n(x,t)\right|=\left|\int_0^x\beta_n(x,t)f_x{}'(t)dt\right|\\
&\le \int_0^y|\beta_n(x,t)|\big|f'{}_x(t)\big|dt+\int_y^x|\beta_n(x,t)|\big|f'{}_x(t)\big|dt\\
&=\int_0^{x-\frac{x}{\sqrt{n}}}\beta_n(x,t)\big|f'{}_x(t)\big|dt+\int_{x-\frac{x}{\sqrt{n}}}^x\beta_n(x,t)\big|f'{}_x(t)\big|dt.
\end{aligned}
$$

Since $f_x{}'(x)=0$ and $\beta_n(x,\,t)\le 1$, it follows

$$
\begin{aligned}
\int_{x-\frac{x}{\sqrt{n}}}^x\big|f_x{}'(t)\big|\beta_n(x,t)dt&=\int_{x-\frac{x}{\sqrt{n}}}^x\big|f_x{}'(t)-f_x{}'(x)\big|\beta_n(x,t)dt\\
&\le\int_{x-\frac{x}{\sqrt{n}}}^x\vee_t^x\big(f_x{}'\big)dt\le\frac{x}{\sqrt{n}}\vee_{x-\frac{x}{\sqrt{n}}}^x\big(f_x{}'\big).
\end{aligned}
$$

Again using Lemma 2.5 and substituting $t=x-\frac{x}{u}$,

$$
\begin{aligned}
\int_0^{x-\frac{x}{\sqrt{n}}}\big|f_x{}'(t)\big|\beta_n(x,t)dt&\le\frac{C\delta_n^2(x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{\big|f_x{}'(t)\big|}{(x-t)^2}dt\\
&\le\frac{C\delta_n^2(x)}{nx}\int_1^{\sqrt{n}}\vee_{x-\frac{x}{u}}^x\big(f_x{}'\big)du
\end{aligned}
$$

$$\leq \frac{C\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]} \vee_{x-\frac{x}{k}}^{x}\left(f_x{}'\right).$$

Thus, $A_{nx} \leq \frac{x}{\sqrt{n}} \vee_{x-\frac{x}{\sqrt{n}}}^{x}\left(f_x{}'\right) + \frac{C\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]} \vee_{x-\frac{x}{k}}^{x}\left(f_x{}'\right)$.

Now we can write

$$B_{n,x} \leq \left|\int_x^{2x}\left(\int_x^t f_x{}'(u)du\right)K_n^{[\alpha]}(x,t)dt\right| + \left|\int_{2x}^{\infty}\left(\int_x^t f_x{}'(u)du\right)K_n^{[\alpha]}(x,t)dt\right|$$

Also from part (ii) of Lemma 2.5, we have

$$K_n^{[\alpha]}(x,t) = d_t(1-\beta_n(x,t)) \quad \text{for} \quad t > x$$

Thus,

$$B_{nx} \leq B_{1,n,x} + B_{2,n,x},$$

where

$$B_{1,n,x} = \left|\int_x^{2x}\left(\int_x^t f_x{}'(u)du\right)d_t(1-\beta_n(x,t))\right|$$

and

$$B_{2,n,x} = \left|\int_{2x}^{\infty}\left(\int_x^t f_x{}'(u)du\right)K_n^{[\alpha]}(x,t)dt\right|.$$

Applying integration by parts as well as using Lemma 2.5, (3.17), $1-\beta_n(x,t) \leq 1$ and putting $t = x + \frac{x}{u}$ successively,

$$\begin{aligned}
B_{1,n,x} &= \left|\int_x^{2x} f_x{}'(u)du(1-\beta_n(x,2x)) - \int_x^{2x} f_x{}'(t)(1-\beta_n(x,t))dt\right| \\
&\leq \left|\int_x^{2x}\left(f'(u) - f'(x+)\right)du\right||1-\beta_n(x,2x)| \\
&\quad + \int_x^{2x}\left|f_x{}'(t)\right||1-\beta_n(x,t)|dt \\
&\leq \frac{C\delta_n^2(x)}{nx^2}\left|f(2x) - f(x) - xf'(x+)\right| \\
&\quad + \int_x^{x+\frac{x}{\sqrt{n}}}\left|f_x{}'(t)\right||1-\beta_n(x,t)|dt + \int_{x+\frac{x}{\sqrt{n}}}^{2x}\left|f_x{}'(t)\right||1-\beta_n(x,t)|dt \\
&\leq \frac{C\delta_n^2(x)}{nx^2}\left|f(2x) - f(x) - xf'(x+)\right| \\
&\quad + \frac{C\delta_n^2(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{\vee_x^t\left(f_x{}'\right)}{(t-x)^2}dt + \int_x^{x+\frac{x}{\sqrt{n}}}\vee_x^t\left(f_x{}'\right)dt \\
&\leq \frac{C\delta_n^2(x)}{nx^2}\left|f(2x) - f(x) - xf'(x+)\right| \\
&\quad + \frac{C\delta_n^2(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{\vee_x^t\left(f_x{}'\right)}{(t-x)^2}dt + \frac{x}{\sqrt{n}}\vee_x^{x+\frac{x}{\sqrt{n}}}\left(f_x{}'\right) \\
&\leq \frac{C\delta_n^2(x)}{nx^2}\left|f(2x) - f(x) - xf'(x+)\right| \\
&\quad + \frac{C\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\vee_x^{x+\frac{x}{k}}\left(f_x{}'\right) + \frac{x}{\sqrt{n}}\vee_x^{x+\frac{x}{\sqrt{n}}}\left(f_x{}'\right).
\end{aligned}$$

Finally, Remark 2.1 implies

$$\begin{aligned}
B_{2,n,x} &= \left|\int_{2x}^{\infty}\left(\int_x^t\left(f'(u) - f'(x+)\right)du\right)K_n^{[\alpha]}(x,t)dt\right| \\
&\leq \int_{2x}^{\infty}|f(t) - f(x)|K_n^{[\alpha]}(x,t)dt + \int_{2x}^{\infty}|t-x|f'(x+)K_n^{[\alpha]}(x,t)dt
\end{aligned}$$

$$\leq M\int_{2x}^{\infty} t^{\gamma}K_n^{[\alpha]}(x,t)dt + |f(x)|\int_{2x}^{\infty} K_n^{[\alpha]}(x,t)dt + \sqrt{\frac{C}{n}}\delta_n(x)f'(x+).$$

As it is obvious that $t \leq 2(t-x)$ and $x \leq t-x$ when $t \geq 2x$, applying Holder's inequality, we get

$$B_{2,n,x} \leq M2^{\gamma}\left(\int_0^{\infty}(t-x)^{2r}K_n^{[\alpha]}(x,t)dt\right)^{\frac{\gamma}{2r}} + \frac{C\delta_n^2(x)|f(x)|}{nx^2} + \sqrt{\frac{C}{n}}\delta_n(x)f'(x+) = M(\gamma,r,x) + \frac{C\delta_n^2(x)|f(x)|}{nx^2} + \sqrt{\frac{C}{n}}\delta_n(x)f'(x+).$$

Estimates of $B_{1,\,n,\,x}$ and $B_{2,\,n,\,x}$ results

$$B_{n,x} \leq \frac{C\delta_n^2(x)}{nx^2}\left|f(2x)-f(x)-xf'(x+)\right| + \frac{C\delta_n^2(x)}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{k}}\left(f_x'\right) + \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}\left(f_x'\right) + M(\gamma,r,x) + \frac{C\delta_n^2(x)|f(x)|}{nx^2} + \sqrt{\frac{C}{n}}\delta_n(x)f'(x+).$$

Hence values of $A_{n,\,x}$ and $B_{n,\,x}$ in (3.21), we get the required result. □

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